Topic: Measuring Uncertainty with Probability

Bayes' Theorem - Exercise

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Probability Rules - Summary

If A and B represent any two events then,

Conditional $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Multiplicative $P(A \cap B) = P(A|B)P(B)$

Law of Total Probability $P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A|B_i)P(B_i)$

Bayes' Theorem

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots P(A|B_k)P(B_k)}$$

Exercise.: Bayes' Theorem

In a certain factory, Machines 1, 2, and 3 are all producing springs of the same length. Machines 1, 2, and 3 produce 1%, 4% and 2% defective springs, respectively. Of the total production of springs in the factory, Machine 1 produces 30%, Machine 2 produces 25%, and Machine 3 produces 45%.

Let D be the event that the spring is defective.

Let M_i be the event that a spring is produced Machine $(i = 1, 2, 3)$. M_1 M_2 M_3.

a. If one spring is selected at random from the total springs produced in a given day, determine the probability that it is defective.

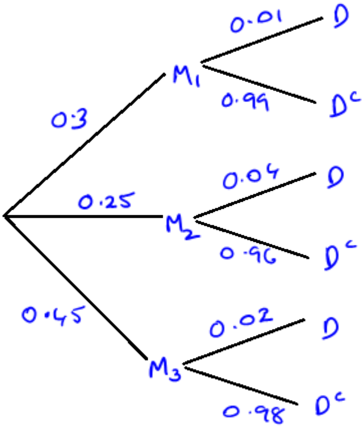
b. Given that the ~~defective~~ spring is defective, find the conditional probability that it was produced by Machine 2.

c. Further, determine $P(M_1|D)$ and $P(M_3|D)$ and demonstrate that $\sum_{i=1}^{k} P(M_i|D) = 1$.

Ref: From Hogg, McKeon, Craig, (2013) Introduction to Mathematical Statistics. p28 Ex 1.4.8

Exercise cont.: Apply the Law of Total Probability

a. Draw the tree diagram



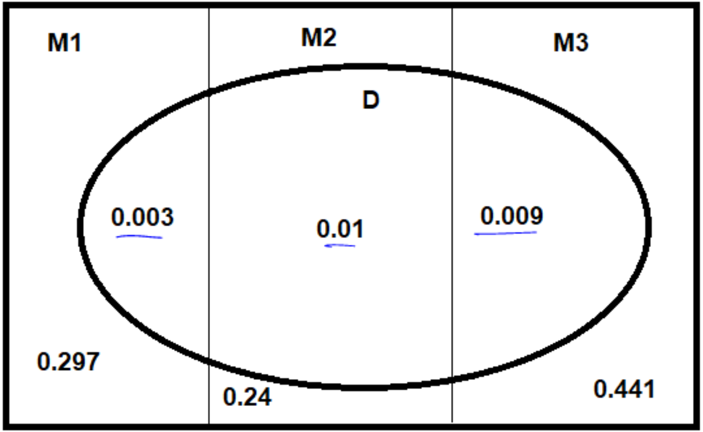
$* \quad P(D \cap M_1) = 0.003$

$* \quad P(D \cap M_2) = 0.01$

$* \quad P(D \cap M_3) = 0.009$

Exercise cont.: Apply the Law of Total Probability

The Venn diagram showing all probabilities (previously determined).



$P(D) = 0.003 + 0.010 + 0.009 = 0.022$

or 2.2%

Exercise cont.: Apply Bayes' Theorem

b. Given that the ~~defective~~ spring is defective, find the conditional probability that it was produced by Machine 2.

$$P(M_2 | D) = \frac{P(D|M_2)P(M_2)}{P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_3)} \leftarrow P(D).$$

$$= \frac{0.04 \times 0.25}{0.022} \leftarrow (a).$$

$$= \frac{0.010}{0.022}$$

$$= \frac{10}{22} = \frac{5}{11}.$$

Exercise cont.: Apply Bayes' Theorem

c. Determine $P(M_1|D)$ and $P(M_3|D)$ and demonstrate that $\sum_{i=1}^{3} P(M_i|D) = 1$.

$$P(M_1|D) = \frac{P(M_1 \cap D)}{P(D)} = \frac{0.003}{0.022} = \frac{3}{22}.$$

$$P(M_3|D) = \frac{P(M_3 \cap D)}{P(D)} = \frac{0.009}{0.022} = \frac{9}{22}.$$

Exercise cont.: Apply Bayes' Theorem

$$\sum_{i=1}^{3} P(M_i \mid D) = P(M_1 \mid D) + P(M_2 \mid D) + P(M_3 \mid D)$$

$$= \frac{3}{22} + \frac{10}{22} + \frac{9}{22}$$

$$= \frac{22}{22}$$

$$= 1 \qquad \checkmark \ \text{required.}$$