MATH255: Mathematics for Computing
Tutorial Sheet Week 3 - Autumn 2023

Note. Question 4 is you Tutorial Preparation exercise for this week. It must be completed and handed in on Moodle as a pdf before the start of your tutorial.

1. Write each statement in words. Write down whether you think the statement is true or false.

   (a) $\forall\ x \in \mathbb{R}, x \neq 0 \to (x > 0\ \vee\ x < 0)$

   (b) $\exists\ n \in \mathbb{N}\ s.t.\ \sqrt{n} \in \mathbb{N}$

2. Write each statement in logic notation. Write down whether you think the statement is true or false.

   (a) If the product of two numbers is zero, then both numbers are zero.

   (b) Each real number is less than or equal to some integer.

3. Are the following statements true or false? Explain briefly why.

   (a) $\forall\ x \in \mathbb{R}, x > 1 \to x > 0$

   (b) $\exists\ x \in \mathbb{R}\ s.t.\ x > 1 \to \frac{x}{x^2+1} < \frac{1}{3}$

   (c) $\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R}\ s.t.\ x^2 < y + 1$

   (d) $\forall\ x, y \in \mathbb{R}, x^2 + y^2 = 9$

4. Write the negation of the following statements and make your best guess whether the statement or its negation is false.

   (a) $\forall\ \varepsilon > 0,\ \exists\ x \neq 0\ s.t.\ |x| < \varepsilon$

   (b) $\exists\ y \in \mathbb{R}\ s.t.\ \forall\ x \in \mathbb{R}, y < x^2$

   (c) $\forall\ x, y \in \mathbb{R}, x < y \to x < \frac{x+y}{2} < y$

5. Rewrite each of the following arguments into logical form, then use a truth table to check the validity of the argument.

   (i) If I go to the movies, then I will carry my phone or my 3D glasses. I am carrying my phone but not my 3D glasses. Therefore, I will go to the movies.

   (ii) I will buy a new bike or a used car. If I buy both a new bike and a used car, I will need a loan. I bought a used car and I didn't need a loan. Therefore, I didn't buy a new bike.

6. Prove or disprove the following statements.

   (i) For all natural numbers n, the expression $n^2 + n + 29$ is prime.

   (ii) $\exists\ x \in \mathbb{Q}$ s.t. $\forall\ y \in \mathbb{Q},\ xy \neq 1$.

   (iii) $\forall\ a, b \in \mathbb{R},\ (a + b)^2 = a^2 + b^2$.

   (iv) The average of any two odd integers is odd.

7. Find the mistakes in the following proofs.

   (i) *Result:* $\forall\ k \in \mathbb{Z},\ (k > 0\ \Rightarrow k^2 + 2k + 1$ is not prime).
   Proof: For $k = 2$, $k^2 + 2k + 1 = 9$, which is not prime. Therefore, the result is true.

(ii) *Result:* The difference between any odd integer and any even integer is odd.
Proof: Let n be any odd integer, and m any even integer. By definition of odd, $n = 2k+1$ for some $k \in \mathbb{Z}$, and by definition of even, $n = 2k$ for some $k \in \mathbb{Z}$. Then $n - m = (2k+1) - 2k = 1$. But 1 is odd. Therefore, the result holds.

8. Prove each of the following results using a direct proof.

   (i) $\forall\, x \in \mathbb{R},\ x^2 + 1 \geq 2x$.

   (ii) For any $n \in \mathbb{N}$, if n is odd, then n^2 is odd.

   (iii) The sum of any two odd integers is even.

   (iv) If the sum of two angles of a triangle is equal to the third angle, then the triangle is a right triangle.

9. Prove each of the following statements using a proof by contradiction.

   (i) If n^2 is odd then n is odd.

   (ii) There is no smallest positive real number.

10. Prove each of the following statements using a proof by cases.

   (i) If $x \in \{4, 5, 6\}$, then $x^2 - 3x + 21 \neq x$.

   (ii) $\forall\, x \in \mathbb{Z}, x \neq 0 \Rightarrow 2^x + 3 \neq 4$.