1. Prove by induction.

(a) Base case: $n = 1$. $1^2 = \frac{1(1+1)(2\cdot1+1)}{6} \Leftrightarrow 1 = \frac{1\cdot2\cdot3}{6} \Leftrightarrow 1 = 1$. Base case is true.
Inductive step: suppose $1^2+2^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$ and prove $1^2+2^2+\cdots+(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$.

$$
\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{2k^3 + 3k^2 + k}{6} + \frac{6(k^2 + 2k + 1)}{6} = \frac{2k^3 + 9k^2 + 13k + 6}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}
$$

Inductive step is true. Therefore, the equality is true for all $n \in \mathbb{N}$.

(b) Base case: $n = 4$. $4! > 2^4 \Leftrightarrow 24 > 16$. Base case is true.
Inductive step: suppose $k! > 2^k, k \geq 4$ and prove $(k+1)! > 2^{k+1}$.

$$
\begin{aligned}
(k+1)! &= 1 \cdot 2 \cdot \ldots \cdot k \cdot (k+1) = k!(k+1) \\
&> 2^k(k+1) > 2^k \cdot k > 2^k \cdot 2 = 2^{k+1}
\end{aligned}
$$

Inductive step is true. Therefore, the inequality is true for all $n \geq 4$.

2.

(a) $$\sum_{i=1}^{5}(2i-5) = (2\cdot1-5) + (2\cdot2-5) + (2\cdot3-5) + (2\cdot4-5) + (2\cdot5-5)$$
$$= -3 + (-1) + 1 + 3 + 5 = 5$$

(b) $$\sum_{j=-2}^{2} 2^j = 2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 = \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 = \frac{31}{4}$$

(c) $$\sum_{k=0}^{3} \frac{k!}{2} = \frac{0!}{2} + \frac{1!}{2} + \frac{2!}{2} + \frac{3!}{2} = \frac{1}{2} + \frac{1}{2} + \frac{2}{2} + \frac{6}{2} = 5$$

(d) $$\sum_{t=0}^{99} \frac{(-1)^t}{3} = \frac{(-1)^0}{3} + \frac{(-1)^1}{3} + \frac{(-1)^2}{3} + \cdots + \frac{(-1)^{99}}{3} = \frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{3} = 0$$

3. $2 + 6 + 10 + \cdots + (4n-2) = \sum_{i=1}^{n}(4i-2)$
Base case: $n = 1$. $\sum_{i=1}^{1}(4i-2) = 2\cdot1^2 \Leftrightarrow 4\cdot1-2 = 2\cdot1^2 \Leftrightarrow 2 = 2$. Base case is true.

Inductive step: suppose $\sum_{i=1}^{k}(4i-2)=2k^2$ and prove $\sum_{i=1}^{k+1}(4i-2)=2(k+1)^2$.

$$\sum_{i=1}^{k+1}(4i-2)=\sum_{i=1}^{k}(4i-2)+4(k+1)-2=2k^2+4k+2=2(k+1)^2$$

Inductive step is true. Therefore, $\sum_{i=1}^{k}(4i-2)=2n^2\ \forall\ n\in\mathbb{N}$.

4. (a) $A\subseteq A, A\subseteq C, B\subseteq B, B\subseteq C, C\subseteq C$.

   (b)

$$\begin{aligned}
A\cup B &= (0,1] \\
A\cap B &= \varnothing \\
A\cup C &= [0,1] \\
C-B &= \{0,1\} \\
A-C &= \varnothing \\
B\cap C &= (0,1) \\
\overline{B} &= (-\infty,0]\cup[1,\infty) \\
\overline{A} &= (-\infty,1)\cup(1,\infty)
\end{aligned}$$

5. (a) $X\in P(X)$ T (b) $\{\varnothing\}\in P(X)$ F (c) $a\in P(X)$ F (d) $\{a\}\in X$ F

   (e) $a\in X$ T (f) $X\subseteq P(X)$ F (g) $a\subseteq P(X)$ F (h) $\{X\}\subseteq P(X)$ T

6. $A=D$ only.

7. Consider the universes $U_1=\mathbb{R}, U_2=\mathbb{R}\setminus\{0\}$, and the operations $+,-,\cdot,/$.

   (a) $+,-,\cdot$ are closed on U_1, since $\forall\ x,y\in U_1$ we have $x+y, x-y, xy\in U_1$. $/$ is not, since $\frac{x}{0}\notin U_1$. $\cdot,/$ are closed on U_2, since $\forall\ x,y\in U_2$ we have $xy,\frac{x}{y}\in U_2$. $+,-$ are not, since $1+(-1)=0$ and $1-1=0$, which is not in U_2.

   (b) The additive identity is 0, which is in U_1 but not in U_2, so $+$ has an identity on U_1 because $x+0=x$ and $0+x=x\ \forall\ x\in U_1$.
   There is no subtractive identity. Remember that $x-e=x$ and $e-x=x$ would both be required.
   The multiplicative identity is 1, which is in U_1 and U_2, andit's true that $1\cdot x=x\cdot 1=x\ \forall\ x$ in both universes, so U_1 and U_2 have multiplicative identities.
   There is no divisive identity, as it would have to be true that $\frac{x}{e}=x$ and $\frac{e}{x}=x\ \forall\ x$.

(c) On U_1, every element x is invertible and the inverse element is $-x$.
On U_1, every element $x \neq 0$ is invertible and has inverse element $\frac{1}{x}$. 0 is the only non-invertible element. On U_2, every element x is invertible and has inverse element $\frac{1}{x}$.

8. Let $a, b \in \mathbb{Q}$. Then $\exists\ c, d, e, f \in \mathbb{Z}, d, f \neq 0$ such that $a = \frac{c}{d}$ and $b = \frac{e}{f}$. Then $a\#b = ab + b = \frac{c}{d}\frac{e}{f} + \frac{e}{f} = \frac{ce}{df} + \frac{de}{df} = \frac{ce+de}{df} \in \mathbb{Q}$, so $\#$ is closed on $\mathbb{Q}$.
If there is an identity e, then $a\#e = a$ and $e\#a = a\ \forall\ a \in \mathbb{Q}$.

$$a\#e = a \to ae + e = a \to e(a+1) = a \to e = \frac{a}{a+1}\ \forall\ a \in \mathbb{Q}$$

There is clearly no one number e that satisfies the above for all $a \in \mathbb{Q}$, since e depends on a. There is no identity element. Thus, there are no invertible elements.

9. Let $a = 1, b = 2$. Then $a - b = 1 - 2 = -1$ and $b - a = 2 - 1 = 1$, so $a - b \neq b - a$. Subtraction is not commutative.
Let $a = 1, b = 2, c = 3$. Then $(a - b) - c = (1 - 2) - 3 = -1 - 3 = -4$ and $1 - (2 - 3) = 1 - (-1) = 2$, so $(a - b) - c \neq a - (b - c)$. Subtraction is not associative.
Let $a = 1, b = 2, c = 3$. Then $a - bc = (a - b)(a - c) \leftrightarrow 1 - 2 \cdot 3 = (1 - 2)(1 - 3) \leftrightarrow -5 = 2$, a contradiction. Subtraction is not distributive over multiplication.

10. (a)

$$(C \cap U) \cup \overline{C} = C \cup \overline{C} = U$$

(b)

$$\overline{(A \cap U)} \cup \overline{A} = (\overline{A} \cup \overline{U}) \cup \overline{A} = (\overline{A} \cup \varnothing) \cup \overline{A} = \overline{A} \cup \overline{A} = \overline{A}$$

(c)

$$\overline{\overline{(C \cup \varnothing)} \cup C} = \overline{\overline{(C \cup \varnothing)}} \cap \overline{C} = (C \cup \varnothing) \cap \overline{C} = C \cap \overline{C} = \varnothing$$

(d)

$$(A \cap B) \cap \overline{A} = (A \cap \overline{A}) \cap B = \varnothing \cap B = \varnothing$$

11. (a)

$\overline{A} - \overline{B} = \overline{A} \cap \overline{\overline{B}} = \overline{A} \cap B = B \cap \overline{A} = B - A$ TRUE, (Property 8, Set Algebra Laws)

(b)

$$A - (B - C) = A \cap \overline{(B - C)} = A \cap \overline{B \cap \overline{C}} = A \cap (\overline{B} \cup C)$$
$$(A - B) - C = (A \cap \overline{B}) - C = (A \cap \overline{B}) \cap \overline{C} = A \cap (\overline{B} \cap \overline{C})$$

These are different, as an element of $B \cap C$ is in $\overline{B} \cup C$ but not in $\overline{B} \cap \overline{C}$, so this equality is FALSE. Draw the Venn diagrams to convince yourself.

12. (a)

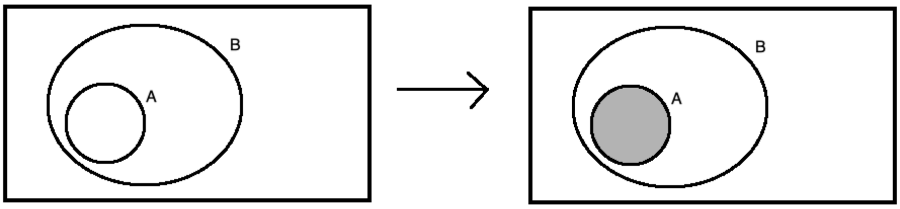

Figure 1: Left: $A \subseteq B$. Right: $A \cap B = A$

(b)

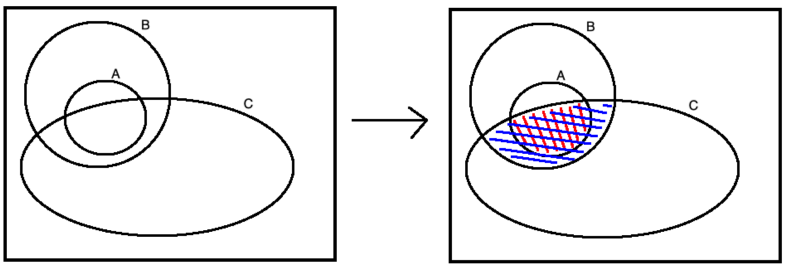

Figure 2: Left: $A \subseteq B$. Right: $A \cap C$ is red, $B \cap C$ is blue, red is contained in blue.

(c)

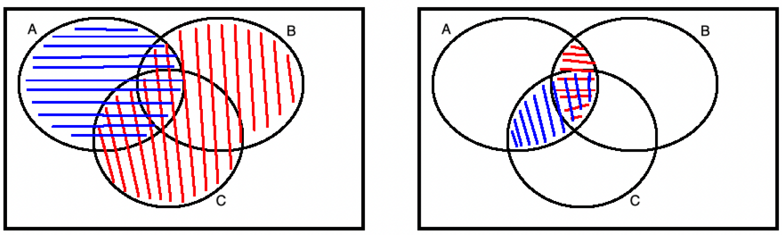

Figure 3: Left: $B \cup C$ is red, A is blue, so $A \cap (B \cup C)$ is shaded with both colours. Right: $A \cap B$ is red, $A \cap C$ is blue, so $(A \cap B) \cup (A \cap C)$ is the entire shaded region.

(d)

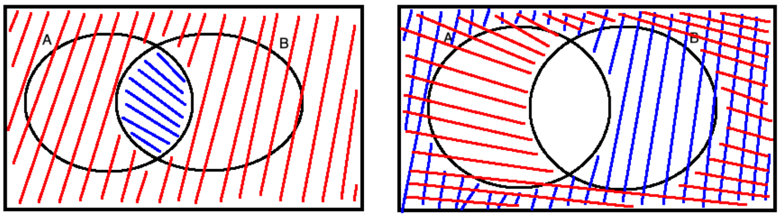

Figure 4: Left: $A \cap B$ is blue, so $\overline{A \cap B}$ is red. Right: $\overline{A}$ is blue, $\overline{B}$ is red, so $\overline{A} \cup \overline{B}$ is the entire shaded region.

(e)

$(A \subseteq B \wedge A \subseteq C) \rightarrow A \subseteq (B \cap C)$

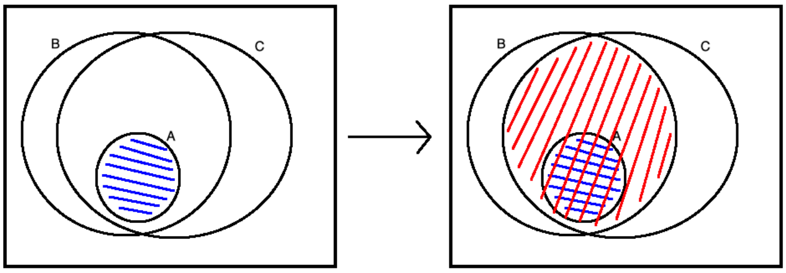

Figure 5: Left: A is blue, it is contained in B and also contained in C. Right: $B \cap C$ is red, blue is contained in red.