1. (a) Every nonzero real number is either positive or negative. True.
   (b) There is a natural number whose square root is a natural number. True.

2. (a) $xy = 0 \to x, y = 0$. False.
   (b) $\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{Z}\ s.t.\ x \leq y$. True.

3. (a) True. Since $1 > 0$, any number greater than 1 is also greater than 0.
   (b) True. One example is enough for proof, for instance $x = 3$.
   (c) True. For instance, for any x, choose $y = x^2$.
   (d) False. For example, $x = y = 0$.

4. (a) $\exists\ \varepsilon > 0\ s.t.\ \forall\ x \neq 0, |x| \geq \varepsilon$. The negation is false.
   (b) $\forall\ y \in \mathbb{R},\ \exists\ x \in \mathbb{R}\ s.t.\ y \geq x^2$. The negation is false.
   (c) $\exists\ x, y \in \mathbb{R}\ s.t.\ x < y\ \wedge \left(x \geq \frac{x+y}{2} \vee \frac{x+y}{2} \geq y\right)$. The negation is false.

5. (i) p: I go to the movies. q: I carry my phone. r: I carry my 3D glasses.

$$\begin{aligned} &p \to q \vee r \\ &q \wedge \sim r \\ \therefore\ &p \end{aligned}$$

p	q	r	$q \vee r$	$p \to (q \vee r)$	$\sim r$	$q \wedge \sim r$
T	T	T	T	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	F
T	F	F	F	F	T	T
F	T	T	T	T	F	T
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	F	T	T	T

There are some rows in which the assumptions are true but the conclusion is false. Therefore, the argument is not valid.

   (ii) p: I buy a new bike. q: I buy a used car. r: I need a loan.

$$\begin{aligned} &p\ \vee q \\ (p \wedge q) &\to r \\ &q \wedge \sim r \\ \therefore\ &\sim p \end{aligned}$$

p	q	r	$p \vee q$	$p \wedge q$	$(p \wedge q) \to r$	$\sim r$	$q \wedge \sim r$	$\sim p$
T	T	T	T	T	T	F	F	F
T	T	F	T	T	F	T	T	F
T	F	T	T	F	T	F	F	F
T	F	F	T	F	T	T	F	F
F	T	T	T	F	T	F	F	T
F	T	F	T	F	T	T	T	T
F	F	T	F	F	T	F	F	T
F	F	F	F	F	T	T	F	T

In all rows in which the three assumptions are true, the conclusion is true. Therefore, the argument is valid.

6. (i) False. For instance, $n = 29$ gives $n^2 + n + 29 = 29^2 + 29 + 29$, which is obviously a multiple of 29 and thus not prime. There are others, such as $n = 2 \Rightarrow n^2+n+29 = 35$.
   (ii) True, with the choice $x = 0$ only.
   (iii) False. For instance, $a = b = 1$ yields $(a + b)^2 = 4$ and $a^2 + b^2 = 2$.
   (iv) False. For instance, the average of 3 and 5 is 4.

7. Find the mistakes in the following proofs.

   (i) The mistake is trying to prove a universal statement with just one example. It must be proved for all elements of the domain, or else disproved with one counterexample. A valid proof might be to observe that $k^2 + 2k + 1 = (k + 1)^2 = (k + 1)(k + 1)$, so composite for all k.

   (ii) The mistake is in using k for both numbers. To say $n = 2k + 1$ is fine, but then m doesn't have to be $2k$; rather it is $2p$ for some other $p \in \mathbb{Z}$. Then $n-m = 2(k-p)+1$, which is odd.

8. (i) Let $x \in \mathbb{R}$. Then

$$\begin{aligned} x^2 + 1 \geq 2x &\Leftrightarrow x^2 - 2x + 1 \geq 0 \\ &\Leftrightarrow (x - 1)^2 \geq 0. \end{aligned}$$

   Since the last statement is true for any $x \in \mathbb{R}$, all previous statements are as well. Therefore, $x^2 + 1 \geq 2x \ \forall \ x \in \mathbb{R}$.

   (ii) Let n be odd. Then $n = 2k + 1$ for some $k \in \mathbb{N}$. We have

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \quad \text{and} \quad 2k^2 + 2k \in \mathbb{N},$$

   thus n^2 is odd. Therefore, n is odd $\rightarrow n^2$ is odd $\forall \ n \in \mathbb{N}$.

   (iii) Let a, b be any two odd numbers. Then $a = 2p+1$ and $b = 2q+1$ for some $p, q \in \mathbb{Z}$. We have

$$a + b = (2p + 1) + (2q + 1) = 2p + 2q + 2 = 2(p + q + 1) \quad \text{and} \quad p + q + 1 \in \mathbb{Z},$$

   thus $a + b$ is even.

   (iv) Let the sum of two angles equal the third angle of a triangle: $a + b = c$. Since the sum of all three angles of any triangle is 180 degrees, $a + b + c = 180$. Then

$$a + b + c = 180 \rightarrow (a + b) + c = 180 \rightarrow c + c = 180 \rightarrow c = 90,$$

   and we have that c is a right angle. Therefore, if two angles sum to the third angle of a triangle, it is a right triangle.

9. Prove each of the following statements using a proof by contradiction.

   (i) Let n^2 be odd. Suppose that n is even. Then $n = 2k, k \in \mathbb{Z}$, so $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ with $2k^2 \in \mathbb{Z}$ and we have n^2 is even, a contradiction. Therefore, n^2 is odd $\rightarrow n$ is odd $\forall n \in \mathbb{N}$.

   (ii) Suppose that there is a smallest positive real number, say $p > 0$. Then $\frac{p}{2} > 0$ and $\frac{p}{2} < p$, a contradiction. Therefore, there is no smallest positive real number.

10. Prove each of the following statements using a proof by cases.

(i) Case 1: $x = 4$. Then $x^2 - 3x + 21 = 4^2 - 3 \cdot 4 + 21 = 16 - 12 + 21 = 25 \neq 4$.
Case 2: $x = 5$. Then $x^2 - 3x + 21 = 5^2 - 3 \cdot 5 + 21 = 25 - 15 + 21 = 31 \neq 5$.
Case 3: $x = 6$. Then $x^2 - 3x + 21 = 6^2 - 3 \cdot 6 + 21 = 36 - 18 + 21 = 39 \neq 6$.
Therefore, if $x \in \{4, 5, 6\}$, then $x^2 - 3x + 21 \neq x$.

(ii) Case 1: let $x < 0$. Then $2^x < 1$, since it has the form $\frac{1}{2^p}$ with $p > 1$. So $2^x + 3 < 4$.
Case 2: let $x > 0$. Since 1 is the smallest such x, we have $2^x + 3 \geq 2^1 + 3 = 5 > 4$.
Therefore, $\forall\ x \in \mathbb{Z}, x \neq 0 \Rightarrow 2^x + 3 \neq 4$.