

MATH255: Mathematics for Computing

Tutorial Sheet Week 4 - Autumn 2023

Note. Questions 2 and 3 are your Tutorial Preparation exercises for this week. They must be completed and handed in on Moodle as a pdf before the start of your tutorial.

1. Prove by induction.

(a) $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(b) $n! > 2^n, n \geq 4$.

2. Evaluate.

(a) $\sum_{i=1}^5 (2i - 5)$ (b) $\sum_{j=-2}^2 2^j$ (c) $\sum_{k=0}^3 \frac{k!}{2}$ (d) $\sum_{t=0}^{99} \frac{(-1)^t}{3}$

3. Express the sum $2 + 6 + 10 + \dots + (4n - 2)$ using sigma notation and prove by induction that the sum equals $2n^2$ for all $n \in \mathbb{N}$.

4. Let $A = \{1\}, B = (0, 1), C = [0, 1]$.

(a) Which sets are subsets of which?

(b) With universe \mathbb{R} , find $A \cup B, A \cap B, A \cup C, C - B, A - C, B \cap C, \overline{B}, \overline{A}$.

5. Let $X = \{a, b, c, d, e, f\}$. Determine whether the following power set statements are true or false.

(a) $X \in P(X)$ (b) $\{\emptyset\} \in P(X)$ (c) $a \in P(X)$ (d) $\{a\} \in X$

(e) $a \in X$ (f) $X \subseteq P(X)$ (g) $a \subseteq P(X)$ (h) $\{X\} \subseteq P(X)$

6. Which of the following sets are equal?

$A = \{0, 1, 2\}, B = \{x \in \mathbb{R} : -1 \leq x < 3\}, C = \{x \in \mathbb{R} : -1 < x < 3\},$
 $D = \{x \in \mathbb{Z} : -1 < x < 3\}, E = \{x \in \mathbb{N} : -1 < x < 3\}$

7. Consider the universes $U_1 = \mathbb{R}, U_2 = \mathbb{R} \setminus \{0\}$, and the operations $+, -, \cdot, /$.

(a) Which operations are closed on U_1 ? On U_2 ?

(b) Which operations have identities on U_1 ? On U_2 ?

(c) Which are the invertible elements, if any, under each operation on U_1 ? On U_2 ?

8. On \mathbb{Q} , define the operation $\#$ by $a \# b = ab + b$. Is $\#$ closed, is there an identity and are there invertible elements?

9. Demonstrate by counterexample that subtraction is not commutative, not associative and not distributive over multiplication on \mathbb{R} .

10. Let U be the universe that contains sets A, B, C . By using the set algebra laws, simplify the following.

(a) $(C \cap U) \cup \overline{C}$ (b) $\overline{(A \cap U) \cup \overline{A}}$ (c) $\overline{(\overline{C \cup \emptyset}) \cup C}$ (d) $(A \cap B) \cap \overline{A}$

11. Prove or disprove.

(a) $\overline{A} - \overline{B} = B - A$ (b) $A - (B - C) = (A - B) - C$

12. Use Venn diagrams to show the following.

- (a) $A \subseteq B \rightarrow A \cap B = A$
- (b) $A \subseteq B \rightarrow A \cap C \subseteq B \cap C$
- (c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- (d) $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- (e) $(A \subseteq B \wedge A \subseteq C) \rightarrow A \subseteq (B \cap C)$