Topic: Measuring Uncertainty with Probability

Law of Total Probability

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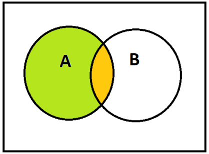
Rewriting $P(A)$

For any events A and B, we can rewrite $P(A)$ as:

$$P(A) = \underbrace{P(A \cap B)}_{} + \underbrace{P(A \cap B^c)}_{} \qquad (1)$$

orange green



We know from the Multiplicative rule that

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ P(A \cap B^c) &= P(A|B^c)P(B^c) \end{aligned}$$

So we can rewrite $P(A)$ from (1) as

orange green.

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) \qquad (2)$$

✓

We have **rewritten or separated** the event A into two parts:

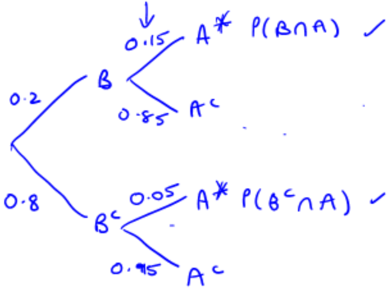
- the part which intersects with B, and
- the part which intersects with B^c

Example: finding $P(A)$

Example 1: Let $A =$ event a person has lung cancer, and $B =$ the event a person is a smoker, and given $P(A|B) = 0.15$. $P(A|B^c) = 0.05$, $P(B) = 0.2$, find the proportion of people with lung cancer.

$P(B^c) = 0.8$

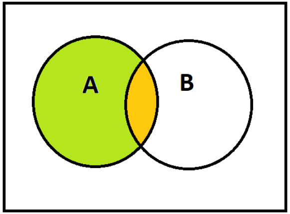
$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c) \\ &= 0.15 \times 0.2 + 0.05 \times 0.8 \\ &= 0.03 + 0.04 \\ &= 0.07 \end{aligned}$$



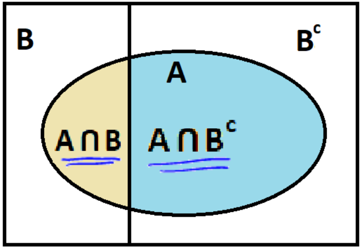
Partitioning the Sample Space S

From (1) and (2) we have:

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^c) \\ &= P(A|B)P(B) + P(A|B^c)P(B^c) \end{aligned}$$



Now consider that S **is partitioned** such that $S = B \cup B^c$ then we could redraw our diagram as:

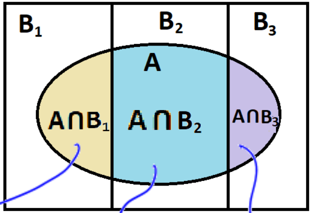


The result above for $P(A)$ still holds.

Partitioning the Sample Space S

Now consider that S **is partitioned** such that

- $S = B_1 \cup B_2 \cup B_3$ and
- $B_1 \cap B_2 = \varnothing$; and
  $B_1 \cap B_3 = \varnothing$; and
  $B_2 \cap B_3 = \varnothing$



Then A is decomposed as: $A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3)$

and $P(A)$ is given by

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \end{aligned}$$

Law of Total Probability

More generally, we say that the collection of sets $\{B_1, \ldots, B_k\}$ is said to be **a partition of S** when

- for some positive integer k,

overall union

$$B_1 \cup B_2 \cup \cdots \cup B_k = \bigcup_{i=1}^{k} B_i = S, \qquad i = 1, \ldots k$$

- $\{B_1, \ldots, B_k\}$ are non-overlapping such that $B_i \cap B_j = \varnothing$ for all $i \neq j$

In this case,

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i) = \sum_{i=1}^{k} P(A|B_i)P(B_i) \qquad (3)$$

This is called the **Law of Total Probability**