

Topic: Measuring Uncertainty with Probability

Bayes' Theorem

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Recall: Conditional probability

We know

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

and similarly

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

Rearranging

$$P(A \cap B) = P(A|B) \times P(B)$$

and

$$P(A \cap B) = P(B|A) \times P(A)$$

So equating these two expressions for $P(A \cap B)$ we can see

$$P(A|B)P(B) = P(B|A)P(A)$$

Then dividing both sides by $P(A)$ we get

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

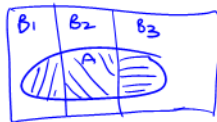
Eqn. (1)

Recall: Law of Total Probability

Recall: that the collection of sets $\{B_1, \dots, B_k\}$ is said to be **a partition of S** when

- for some positive integer k ,

$$B_1 \cup B_2 \cup \dots \cup B_k = \bigcup_{i=1}^k B_i = S, \quad i = 1, \dots, k$$



- where $\{B_1, \dots, B_k\}$ are non-overlapping such that $B_i \cap B_j = \emptyset$ for all $i \neq j$

The **Law of Total Probability**:

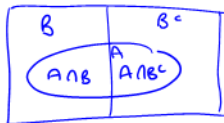
$$P(A) = \sum_{i=1}^k P(A \cap B_i) = \sum_{i=1}^k P(A|B_i)P(B_i) \quad (2)$$

(Note: A blue checkmark is drawn under $P(A)$ and a blue underline is drawn under the summation term $\sum_{i=1}^k P(A \cap B_i)$ in the original image.)

Recall: Conditional probability

So let's apply the Law of Total Probability if we have two partitions: B and B^c :

$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$



Then substitute for $P(A)$ into Eqn(1): we obtain **Bayes' Theorem**

$$\underline{P(B|A)} = \frac{\underline{P(A|B)}P(B)}{\underline{P(A|B)}P(B) + \underline{P(A|B^c)}P(B^c)} \leftarrow P(A)$$

Thus, if we know $P(A|B)$, we can determine the $P(B|A)$

Bayes' Theorem cont.

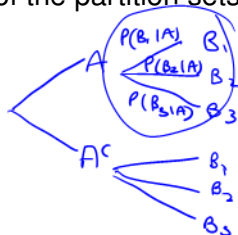
This result can be generalised: If the sets B_1, B_2, \dots, B_k constitute a partition of S , then **Bayes' Theorem** may be written as

$$\underline{P(B_i|A)} = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + \cancel{P(A|B_k)P(B_k)}} \leftarrow \text{11A,}$$

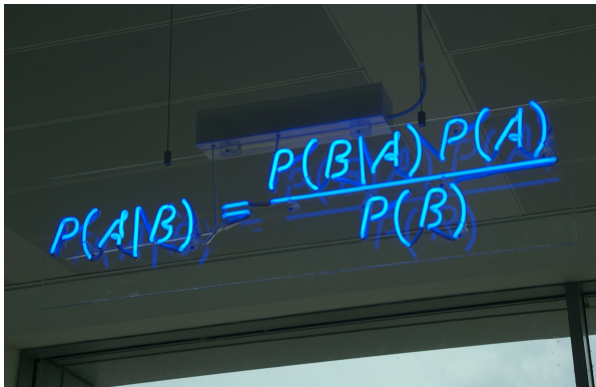
The denominator uses the **Law of Total Probability** from Eqn2

Notice that the sum of these conditional probabilities of the partition sets, given A , is 1.

$$\sum_{i=1}^k P(B_i|A) = 1$$



Bayes' Theorem in Neon Lights!

A photograph of a neon sign in a dark room. The sign is made of blue neon tubing and displays the formula for Bayes' Theorem: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. The sign is mounted on a wall and is illuminated from below, casting a glow on the surface it's attached to. The background is dark, making the bright blue neon stand out. The sign is slightly tilted and has some faint, illegible markings on it, possibly from previous signs or reflections.

Bayes' Theorem has many applications in computer science.

Video: Prof Saharmi from Stanford University <http://www.youtube.com/watch?v=MSIoBqvTKOY>

Bayes' Rule

Bayes' Theorem may be conveniently presented in table

^① $P(B_i)$	^② $P(A B_i)$	^③ $P(B_i)P(A B_i) = P(B_i \cap A)$	^④ $\frac{P(B_i)P(A B_i)}{P(A)} = P(B_i A)$
x	y	$x \times y$	$\frac{x \times y}{P(A)}$
1		<u>$\sum_i P(B_i)P(A B_i) = P(A)$</u>	1

- The first two columns usually given information
- The 3rd column is product of first two, and the sum is $P(A)$
- The 4th column is the 3rd divided by $P(A)$
these are by Bayes' theorem $P(B_i|A)$