MATH1851

Notes for HKU · Spring 2024

Author: Jax

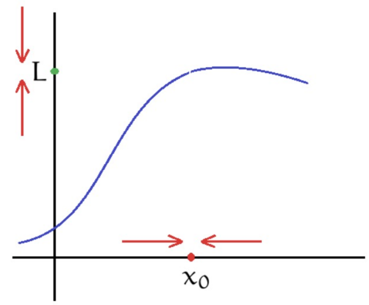
Contact: enhanjax@connect.hku.hk

MORE notes on my website!

Credit: Heavily inspired by Paul's Calculus I Notes.

1 Limits and Continuity

1.1 Introduction to the concept of limit



We can conceptualize that the limit of a function $f(x)$ is L as x approaches c, given that we can make $f(x)$ *as close to* L *as we want* for all x sufficiently close to a, from both sides, *without actually letting* x *be* a. We can write this as:

$$\lim_{x \to a} f(x) = L$$

1.2 One-sided limits

There are two sides that x can tend to a number. We can write it as $x \to n^-$ and $x \to n^+$, which represents from the negative (left) / positive (right) side.

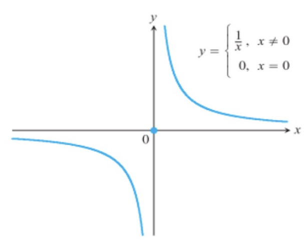
1.3 Existence of limits

Condition for limit to exist

The limit for a function $f(x)$ only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

WARNING: If the **limit is** ∞ it **doesn't exist**.



For this example, when $x \to 0^-, y \to -\infty$.
Similarly, as $x \to 0^+, y \to +\infty$.
Hence, we can conclude that the limit for this function as $x \to 0$ doesn't exist.

1.4 Continuity

Continuity

A function $f(x)$ is *continuous* at $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

Continuity properties

For f, g continuous at c, the following are also continuous at c:

1. $f \pm g$
2. kf
3. fg
4. $\frac{f}{g}$, given that $g(c) \neq 0$

Intermediate value theorem (IVT)

If a function f is continuous on $[a, b]$, there is a number c in $[a, b]$ where $f(c)$ in $[f(a), f(b)]$.

To prove that there is a root, we can use the IVT by showing there is a **change of sign** between the interval given.
To show that there's *only one solution*, check if $f'(x) > 0$ or < 0 (strict $>$) in the interval.

1.5 Computing limits

Indeterminate forms

Indeterminate forms are forms that cannot be solved by simply substituting the value of x into the function. They are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 1^\infty, \quad \infty^0$$

Note that $\infty + \infty = \infty$. Related: L'Hopital's rule

Using the limit laws

For functions f, g and using $\lim_{x \to a} = \bigcirc$ for simpler notation:

1. $lim_{x \to a} c = c$
2. $\bigcirc(f \pm g) = \bigcirc f \pm \bigcirc g$
3. $\bigcirc(k \cdot f) = \bigcirc k \cdot \bigcirc f$
4. $\bigcirc(f^n) = (\bigcirc f)^n$
5. $\bigcirc(fg) = \bigcirc f \bigcirc g$
6. $\bigcirc(\frac{f}{g}) = \frac{\bigcirc f}{\bigcirc g}$, given that $\bigcirc g \neq 0$. *This strict condition prevents indeterminate forms.*

We can use these laws to break a limit into separate limits, and compute that way. Also note that:

7. $\bigcirc f(g) = f(\bigcirc g)$, given that f is **continuous** at $\bigcirc g$

Limit of a polynomial

For the limit of a polynomial $p(x)$:

$$lim_{x \to a} p(x) = p(a)$$

This can be proven easily with the limit laws above.

Techniques to compute limits

To solve for limits, we have to get the expression to the right form - a polynomial, for us to substitute our limit value into the function.
To do this, often we have to **factorize** or **rationalize**.

Example 1.1 *Indeterminate forms by substitution*

This applies limit law #5. As substituting into the function directly gives $0/0$*, we have to change it into a form such that we could apply the limit laws directly.*

$$\begin{aligned} lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \frac{(x-2)(x+6)}{(x-2)x} \\ &= \frac{x+6}{x} \\ \textit{Substituting 2 gives} &= 4 \end{aligned}$$

The squeeze / sandwich theorem

Suppose $f(x) \leq g(x) \leq h(x)$ in the range $[a, b]$, for c in $[a, b]$:

$$\lim_{x\to c} f \leq \lim_{x\to c} g \leq \lim_{x\to c} h$$

We will make use of the fact that the limits can be equal to solve for the limit of $g(x)$.

Example 1.2 *Squeezing a function*

When we can't seem to factorize a function, we can try squeezing it between two other functions.

$$\lim_{x\to 0} x^2 \cos\frac{1}{x}$$

We know the limits of the function $\cos\frac{1}{x}$*, so we can start from there.*

$$\textit{Given that } x \neq 0, -1 \leq \cos\frac{1}{x} \leq 1$$

$$\textit{Multiplying } x^2 \textit{ on both sides}, -x^2 \leq \cos x^2 \frac{1}{x} \leq x^2$$

$$\textit{As } \lim_{x\to 0} \pm x^2 = 0, \quad \textit{we can conclude that } \lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$$

1.6 Infinite limits

Determining infinite limits

If $f(x)$ gets (negatively) arbitrarily large when x approaches a, we can say:

$$\lim_{x\to a} f(x) = (-)\infty$$

After we know that the limit *may* be infinity, we then have to make sure that *the limit is the same from both sides*, so that the limit is actually ∞. We can do so by plugging numbers which are approaching the limit from both sides.

Example 1.3 *Infinite limit exists*

$$\lim_{x\to 0} \frac{6}{x^2}$$

Consider both $\lim_{x\to 0^-} \frac{6}{x^2}, \lim_{x\to 0^+} \frac{6}{x^2}$:

$$\lim_{x\to 0} \frac{6}{x^2} = \infty$$

Example 1.4 *Infinite limit doesn't exist*

$$\lim_{x\to 4} \frac{3}{(4-x)^3}$$

Checking both sides, we can conclude that the limit doesn't exist, as:

$$\lim_{x\to 4^+} \frac{3}{(4-x)^3} = -\infty, \quad \lim_{x\to 4^-} \frac{3}{(4-x)^3} = \infty$$

1.7 Limits at infinity

Infinity operations

Note the following operations:

1. $\infty + k = \infty$
2. For $k < 0$, $k\infty = -\infty$

Determining limits of infinity

It is not hard to see that, for rational numbers n:

$$\lim_{x\to\pm\infty} \frac{k}{x^n} = 0$$

The easiest way to determine the limit would be to *factorize* the function so that we can use the fact above.

Determining limits of infinity of polynomials

Using the above fact, we can see that for a polynomial $p(x)$ with degree n and largest coefficient a_n:

$$\lim_{x\to\pm\infty} p(x) = a_n x^n$$

Which means we can *only consider the largest term in a polynomial* for limits of infinity.

Example 1.5 *Indeterminate forms by substitution of infinity*

Substituting ∞ *into the function gives* $\infty - \infty - \infty$*, which is indeterminate. Hence, we must factorize it.*

$$\begin{aligned}\lim_{x\to\infty} 2x^4 - x^2 - 8x &= \lim_{x\to\infty} [x^4(2 - \frac{1}{x^2} - \frac{8}{x^3})] \\ &= \infty \times 2 \\ &= \infty\end{aligned}$$

Or we can just simply use the theorem above and consider $\lim_{x\to\infty} 2x^4$ *only to give* ∞.

Example 1.6 *Factor polynomials limit to infinity*

We can simply consider the largest terms on each side and give the final answer easily.

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{3x^2+6}}{5-2x} &= \lim_{x\to-\infty} \frac{\sqrt{3x^2}}{-2x} \\
&= \lim_{x\to-\infty} \frac{\sqrt{3}|x|}{-2x} \leftarrow \sqrt{x^2} = |x| \\
&= \lim_{x\to-\infty} \frac{-\sqrt{3}}{-2} \leftarrow |c|, c<0 = -c \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

Note that, as we are considering the negative limit of infinity, we need to add - to the abs sign on line 3.

1.8 Asymptotes

Vertical asymptotes

f will have v-asymptotes at $x = a$ if any $\pm$ is true:

$$\lim_{x\to a^{\pm}} f(x) = \pm\infty$$

Horizontal asymptotes

f will have h-asymptotes at $y = L$ if any $\pm$ is true:

$$\lim_{x\to\pm\infty} f(x) = L$$

Related: Graphing functions

2 Trigonometry review

Trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$
2. $\tan x = \frac{\sin x}{\cos x}$
3. $\csc x = \frac{1}{\sin x}$
4. $\sec x = \frac{1}{\cos x}$
5. $\cot x = \frac{1}{\tan x}$
6. $\sin 2x = 2\sin x \cos x$
7. $\cos 2x = \cos^2 x - \sin^2 x$
8. $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
9. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
10. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
11. $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
12. $1 + tan^2x = sec^2x$
13. $1 + cot^2x = csc^2x$

Related: Techniques of integration

3 Derivatives

First principle

The first principle is the definition of the derivative of a function $f(x)$ at $x = a$:

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Differentiability

The function is differentiable at $x = a$ if:

$$\lim_{x \to a^-} f' = \lim_{x \to a^+} f'$$

3.1 Differentiation formulas and rules

Basic formulas

- We can differentiate individual items: $(f \pm g)' = f' \pm g'$
- We can factor out a multiplicative constant: $(cf)' = cf'$
- Derivative of a constant is 0: $\frac{d}{dx} k = 0$
- Power rule: $\frac{d}{dx} x^n = nx^{n-1}$

Chain rule

Shorthand: **d**1x2 + **d**2x1

$$(u(v))' = u'(v)v' \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product rule

Shorthand: **d** from outside to inside

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule

Shorthand: move lower **d** upper **-** **d** lower x upper, lower square

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2} \qquad \text{or} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

	$f(x)$	$f'(x)$
1.	a^x	$\ln a \cdot a^x$
2.	e^{kx}	ke^{kx}
3.	$\ln kx$	x^{-1}
4.	$\lvert x \rvert$	$\frac{\lvert x \rvert}{x}$
5.	$\sin kx$	$k \cos kx$
6.	$\cos kx$	$-k \sin kx$
7.	$\tan kx$	$k \sec^2 kx$
8.	$\csc x$	$-\csc x \cot x$
9.	$\sec x$	$\sec x \tan x$
10.	$\cot x$	$-\csc^2 x$
11.	$\sin^{-1} x$	$(\sqrt{1-x^2})^{-1}$
12.	$\cos^{-1} x$	$-(\sqrt{1-x^2})^{-1}$
13.	$\tan^{-1} x$	$(1+x^2)^{-1}$

Note that the trigo derivatives can be extended to hyperbolic trigo functions, with the except of $\frac{d}{dx}\text{sech}u = -\text{sech}u \tanh u$.

3.2 Techniques of differentiation

Parametric differentiation

For a parametric equation $y = f(t)$ and $x = g(t)$:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

Derivative of Inverse functions

At c, we first find $f'(x)$ and the value of $f^{-1}(c)$, then we apply the formula to find the value of $f^{-1'}(c)$:

$$f^{-1'}(c) = \frac{1}{f'(f^{-1}(c))}$$

$f^{-1}(c)$ can be found by solving $f(x) = c$.

Implicit differentiation

Differentiate all xy, add y' behind all differentiations of y.

To find y' for $y^2 = x^2 + \sin(xy)$:

$$\begin{aligned} y^2 &= x^2 + \sin(xy) \\ 2yy' &= \frac{d}{dx}(\sin(xy)) \\ 2yy' &= 2x + \cos(xy)(xy' + y) \\ \text{Then we simply collect terms of } y' & \\ y' &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)} \end{aligned}$$

To find the second derivative y'', differentate the expression and substitute y' back in.

Logarithmic differentiation

For $y = f(x)$, $\quad y' = f(x) \times (\ln f(x))'$. (Takes natural log for both sides)

To find $\frac{dy}{dx}$ for $y = x^x$:

$$\begin{aligned} y &= x^x \\ \ln y &= x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\ \frac{dy}{dx} &= x^x(\ln x + 1) = x^x(x \ln x)' \end{aligned}$$

L'Hopital's rule

For any $a \in [\mathbb{R}, \pm\infty]$, if $\lim_{x\to a}(\frac{f}{g})$ is in indeterminate form after substitution, we can conclude:

$$\lim_{x\to a}(\frac{f}{g}) = \lim_{x\to a}(\frac{f'}{g'})$$

To find $\lim_{x\to -\infty} xe^x$, we first check if the limit is indeterminate, then we can apply the rule:

$$\begin{aligned}
\lim_{x\to -\infty} xe^x &\implies \infty \times 0 \\
\lim_{x\to -\infty} xe^x &= \lim_{x\to -\infty} \frac{x}{e^{-x}} \\
&= \lim_{x\to -\infty} \frac{1}{-e^{-x}} \text{ (rule applied)} \\
&= 0
\end{aligned}$$

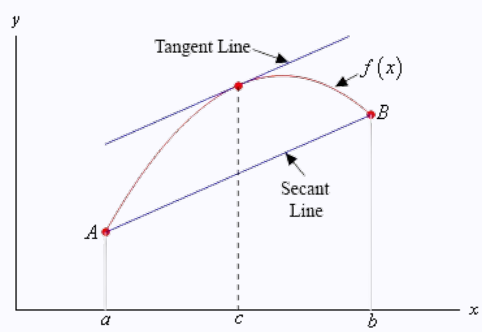
3.3 Important theorems

Mean value theorem (MVT)

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b):

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad c \in (a, b)$$

The theorem tells us that, in described conditions, there must be a point c where the slope of the *tangent line* is equal to the slope of the line from $a \to b$ (secant line).

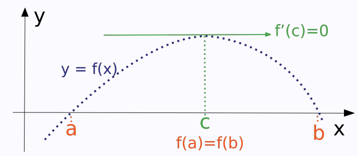


Source: https://tutorial.math.lamar.edu/Classes/CalcI/MeanValueTheorem.aspx

Rolle's theorem

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b):

$$\forall\ f(a) = f(b) \text{ there exists } f'(c) = 0, \quad c \in (a, b)$$



For $F' = f$, if F has 4 roots, then f has 3 roots.

3.4 Extremum points

Critical and inflection points

A **critical point** is a point where $f'(x) = 0$ or undefined, or the end-points of the domain if inclusive. An **inflection point** is a point where $f''(x) = 0$ or undefined, that the concavity of the function changes.

Max/minimum points

The **absolute maximum/minimum points** are the points where the function has the largest/smallest value in the entire domain.
The **local maximum/minimum points** are the points where the function has the largest/smallest value in a small interval around the point.

3.5 Determining shape of graph

Concavity

Concavity is the direction of the curve, and can be described by the values of f' and f'':

f''	$-$	$+$	$+$	$-$
f'	$-$	$+$	$-$	$+$
f	$\searrow$	$\swarrow$	$\nwarrow$	$\nearrow$

Note: Arrows goes clock-wise.

	Step	Expression
1.	Determine domain of function	
2.	Special points without continuity?	
3.	Axis intercepts	$(f(x) = 0, 0)$ $(0, f(0))$
4.	Critical points	$f'(x) = 0$ or undefined
5.	Point maxima?	$+f''(x)$ or $-f''(x)$
6.	Inflection points	$f''(x) = 0$ or undefined
7.	Horizontal asymptotes	$\lim_{x\to\pm\infty} f(x) = n$?
8.	Vertical asymptotes	$\lim_{x\to a^\pm} f(x) = \pm\infty$?
9.	Area strictly increasing/decreasing?	$f'(x) > 0$ or $f'(x) < 0$
10.	Area concavity?	$- + + - / - + - +/ \searrow\swarrow\nwarrow\nearrow$

Related: Definition of asymptotes

Higher derivatives

Derivatives of higher order (e.g. $f''(x), f'''(x)$) can be expressed as $f^{(n)}(x)$

4 Integrals

Definition of natural logs

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0$$

4.1 Definite integrals

Signed areas

Signed area is the area between the curve and the x-axis, where the area above the x-axis is positive and below is negative.

Properties of definite intergrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. $\int_a^a f(x)dx = 0$
3. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
5. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

4.2 Fundamental theorem of calculus

Fundamental theorem of calculus (FTC)

If $f(x)$ is continuous in $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where $F(x)$ is the definite integral of $f(x)$.

Second fundamental theorem of calculus

If $f(x)$ is continuous in $[a, b]$, then:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Find $g'(x)$ for $g(x) = \int_1^{x^2} \cos t \; dt$:

We first define $G(u) = \int_1^u \cos t \; dt$, then we can apply the chain rule:

$$\begin{aligned} g'(x) &= (G(x^2))' \\ &= G'(x^2) \cdot 2x \\ &= \cos x^2 \cdot 2x \end{aligned}$$

Cases of area 0

For $\int_a^b f(x)dx$:

- If $b = a$, then the area is 0 (no width)
- If $f(x) = f(-x)$ and $a = -b$, then the area is 0 (symmetry)

4.3 Techniques of integration

Overview

- By **substitution**: "sub $g(x) \to u$, find du and replace all functions of $x(dx)$ with $u(du)$"
- By **part**: "$\int ab = a \int b - \int (a' \times \int b)$"
- By **joining recurring parts**: "$\int ab = f(ab) + \int ab \to 2\int ab = f(ab)$"
- By **partial fractions**: "$\frac{k}{f(x)g(x)} = \frac{A}{f(x)} + \frac{B}{g(x)}$"
- By **trigonometric identities**
- By **multiplying fractions by one**: "Multiplying $\frac{\sec^2 x}{\sec^2 x}$ to fit trigo form"

Integration by substitution

For $\int f(g)g'\,dx$:

1. Let $u = g$
2. Find $du = g'dx$
3. Change limits in terms of u if definite
4. Replace all $g \to u$ and $g'dx \to du$
5. Integrate, then subsitute $u \to g$

Integration by parts

$$\int ab = a\int b - \int(a' \times \int b)$$

Tips: Let a (differentiating term) to the first term you see on the list:

1. **L**ogarithmic
2. **I**nverse trigo
3. **A**lebratic (polynomial)
4. **T**rigo
5. **E**xponential

Deriving the IBP formula

The formula is derived from the product rule of differentiation:

$$\begin{aligned}(ab)' &= a'b + b'a \\ b'a &= (ab)' - a'b \\ \int b'a &= ab - \int a'b \\ \int ab &= a\int b - \int(a' \times \int b)\end{aligned}$$

Trigo-identities substitution tips

Use the following trigonometric identities to simplify the integral:

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1-\cos 2x}{2}$
- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $2\sin x \cos x = \sin 2x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\sec^2 x = 1 + \tan^2 x$
- $\csc^2 x = 1 + \cot^2 x$

Use the following substitutions for similar expressions:

$$\text{Expressions of the form} \begin{cases} a^2 - f(x)^2 & \rightarrow f(x) = a\sin\theta \\ a^2 + f(x)^2 & \rightarrow f(x) = a\tan\theta \\ f(x)^2 - a^2 & \rightarrow f(x) = a\sec\theta \end{cases}$$

Remember that this uses *integration by substitution*, so we need to find dx in terms of θ.

4.4 Solids of revolution

Volume by disk

Used to find volume by rotating a **curve** around an axis.

Volume by x-axis: $(y : x \rightarrow y)$

$$V = \pi \int_a^b f^2(x)dx$$

Where $f(x)$ is the function of the curve, and a, b are the limits of the curve.

Volume by Washers

Used to find volume by rotating a **region bounded by 2 curves** around an axis.

Volume by x-axis $(y : x \rightarrow y)$

$$V = \pi \int_a^b R^2(x) - r^2(x)\, dx$$

Where $R(x)$ is the outer curve, $r(x)$ is the inner curve.

Volume by Cylindrical shells

Used to find volume by rotating a **curve** around a **vertical line** $x = n$.

Volume is given by:

$$V = 2\pi \int_a^b (x-n)f(x)dx$$

Where n is the vertical line.

4.5 Arcs and surfaces

Arc length

The *arc length* of a curve $y = f(x)$ from $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + f'^2(x)}dx$$

And for parametized equations $x = f(t)$, $y = g(t)$:

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)}dt$$

Area surfaces of revolution

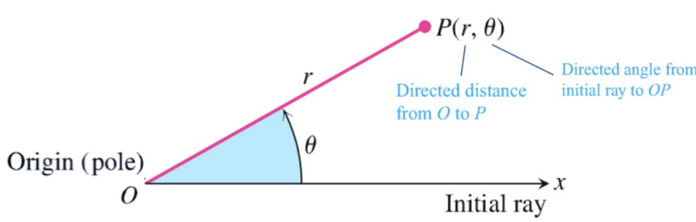
Used to find the *surface area* generated by rotating a curve along an axis.
Surface area by x-axis ($y : x \to y$):

$$A = 2\pi \int_a^b f(x)\sqrt{1 + f'^2(x)}dx$$

And for parametized equations $x = f(t)$, $y = g(t)$:

$$A = 2\pi \int_a^b g(t)\sqrt{f'^2(t) + g'^2(t)}dt$$

5 Polar Coordinates



Equations relating Rolar and Cartesian coordinates

We can convert between polar and Cartesian coordinates using:

$$x = r\cos\theta, \quad y = r\sin\theta$$

We can also see that:

$$r^2 = x^2 + y^2. \quad \tan\theta = \frac{y}{x}$$

The polar equation for the circle $r^2 = x^2 + y^2$ is simply $r = a$.

Polar equations of ellipse

In Cartesian form, an ellipse can be expressed as:

$$\frac{(x+c)^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } c = \sqrt{a^2+b^2}$$

Then we can write in polar form:

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \quad \text{where } e = \frac{\sqrt{a^2+b^2}}{a}$$

Slope of polar curves

The slope of a polar curve at a point is given by:

$$\frac{dy}{dx} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta}$$

To the the slope of the tangent to the unit circle at Cartesian coordinates $(\frac{1}{2}, \frac{\sqrt{3}}{2})$:
We can find the the polar coordinates to be $(1, \frac{\pi}{3})$ using $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$, then we can use the formula to find the slope:

$$\frac{dy}{dx} = \frac{1\cos\frac{\pi}{3} + 1'\sin\frac{\pi}{3}}{-1\sin\frac{\pi}{3} + 1'\cos\frac{\pi}{3}} = -\frac{1}{\sqrt{3}}$$

Graphing

We can easily graph any polar formula $r = f(\theta)$ following the following steps:

1. Plot $r - \theta$ graph. If the function involves trigo functions, we can simply translate its graph. The height at a certain point is the distance from the origin.
2. Then we can simply trace the graph from the left, and plot the points according to the distance from the origin at each angle.
3. If the graph is symmetric, we can simply copy the other half.

6 Ordinary Differential Equations

6.1 1st Order Linear ODEs & Integrating Factors

6.2 Bernoulli Equations & Riccati Equations