MATH1851

Notes for HKU · Spring 2024

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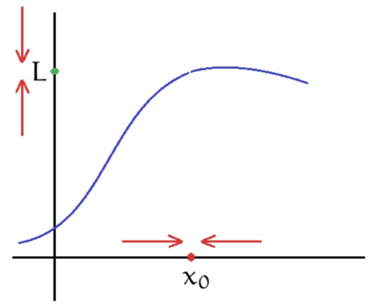
Credit: Heavily inspired by Paul's Calculus I Notes.

1 Limits and Continuity

1.1 Introduction to the concept of limit



We can conceptualize that the limit of a function $f(x)$ is L as x approaches c, given that we can make $f(x)$ *as close to L as we want* for all x sufficiently close to a, from both sides, *without actually letting x be a*. We can write this as:

$$\lim_{x \to a} f(x) = L$$

1.2 One-sided limits

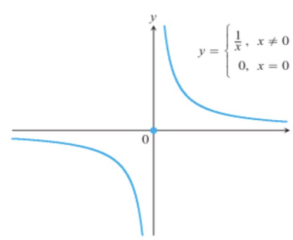
There are two sides that x can tend to a number. We can write it as $x \to n^-$ and $x \to n^+$, which represents from the negative (left) / positive (right) side.

1.3 Existence of limits

Condition for limit to exist

The limit for a function $f(x)$ only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$



For this example, when $x \to 0^-, y \to -\infty$.
Similarly, as $x \to 0^+, y \to +\infty$.
Hence, we can conclude that the limit for this function as $x \to 0$ doesn't exist.

1.4 Continuity

Continuity

A function $f(x)$ is *continuous* at $x = a$ if:

$$\lim_{x \to a} f\left(x\right) = f\left(a\right)$$

Continuity properties

For f, g continuous at c, the following are also continuous at c:

1. $f \pm g$
2. kf
3. fg
4. $\frac{f}{g}$, given that $g(c) \neq 0$

Intermediate value theorem

If a function f is continuous on $[a, b]$, there is a number c in $[a, b]$ where $f(c)$ in $[f(a), f(b)]$.

1.5 Computing limits

Indeterminate forms

Indeterminate forms are forms that cannot be solved by simply substituting the value of x into the function. They are:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad \infty^0, \quad 1^\infty, \quad \infty^0$$

Using the limit laws

For functions f, g and using $\lim_{x \to a} = \bigcirc$ for simpler notation:

1. $lim_{x \to a} c = c$
2. $\bigcirc(f \pm g) = \bigcirc f \pm \bigcirc g$
3. $\bigcirc(k \cdot f) = k \cdot \bigcirc L$
4. $\bigcirc(f^n) = (\bigcirc f)^n$
5. $\bigcirc(fg) = \bigcirc f \bigcirc g$
6. $\bigcirc(\frac{f}{g}) = \frac{\bigcirc f}{\bigcirc g}$, given that $\bigcirc g \neq 0$. *This strict condition prevents indeterminate forms.*

We can use these laws to break a limit into separate limits, and compute that way. Also note that:

7. $\bigcirc f(g) = f(\bigcirc g)$, given that f is **continuous** at $\bigcirc g$

Limit of a polynomial

For the limit of a polynomial $p(x)$:

$$lim_{x \to a} p(x) = p(a)$$

This can be proven easily with the limit laws above.

Techniques to compute limits

To solve for limits, we have to get the expression to the right form - a polynomial, for us to substitute our limit value into the function.
To do this, often we have to **factorize** or **rationalize**.

Example 1.1 *Indeterminate forms by substitution*

This applies limit law #5. As substituting into the function directly gives 0/0, we have to change it into a form such that we could apply the limit laws directly.

$$lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \frac{(x-2)(x+6)}{(x-2)x}$$
$$= \frac{x+6}{x}$$
$$\textit{Substituting 2 gives} = 4$$

The squeeze / sandwich theorem

Suppose $f(x) \leq g(x) \leq h(x)$ in the range $[a, b]$, for c in $[a, b]$:

$$\lim_{x\to c} f \leq \lim_{x\to c} g \leq \lim_{x\to c} h$$

We will make use of the fact that the limits can be equal to solve for the limit of $g(x)$.

Example 1.2 *Squeezing a function*

When we can't seem to factorize a function, we can try squeezing it between two other functions.

$$\lim_{x\to 0} x^2 \cos\frac{1}{x}$$

We know the limits of the function $\cos\frac{1}{x}$*, so we can start from there.*

$$\textit{Given that } x \neq 0, -1 \leq \cos\frac{1}{x} \leq 1$$

$$\textit{Multiplying } x^2 \textit{ on both sides}, -x^2 \leq \cos x^2\frac{1}{x} \leq x^2$$

$$\textit{As } \lim_{x\to 0} \pm x^2 = 0, \quad \textit{we can conclude that } \lim_{x\to 0} x^2 \cos\frac{1}{x} = 0$$

1.6 Infinite limits

Determining infinite limits

If $f(x)$ gets (negatively) arbitrarily large when x approaches a, we can say:

$$\lim_{x\to a} f(x) = (-)\infty$$

After we know that the limit *may* be infinity, we then have to make sure that *the limit is the same from both sides*, so that the limit actually exists. We can do so by plugging numbers which are approaching the limit from both sides.

Example 1.3 *Infinite limit exists*

$$\lim_{x\to 0}\frac{6}{x^2}$$

$$\textit{Consider both } \lim_{x\to 0^-}\frac{6}{x^2}, \lim_{x\to 0^+}\frac{6}{x^2}:$$

$$\lim_{x\to 0}\frac{6}{x^2} = \infty$$

Example 1.4 *Infinite limit doesn't exist*

$$\lim_{x\to 4}\frac{3}{(4-x)^3}$$

Checking both sides, we can conclude that the limit doesn't exist, as:

$$\lim_{x\to 4^+}\frac{3}{(4-x)^3} = -\infty, \quad \lim_{x\to 4^-}\frac{3}{(4-x)^3} = \infty$$

1.7 Limits at infinity

Infinity operations

Note the following operations:

1. $\infty + k = \infty$
2. For $k < 0,\ k\infty = -\infty$

Determining limits of infinity

It is not hard to see that, for rational numbers n:

$$\lim_{x\to\pm\infty} \frac{k}{x^n} = 0$$

The easiest way to determine the limit would be to *factorize* the function so that we can use the facts above.

Determining limits of infinity of polynomials

Using the above fact, we can produce the fact, for a polynomial $p(x)$ with degree n and largest coefficient a_n:

$$\lim_{x\to\pm\infty} p(x) = a_n x^n$$

Which means we can *only consider the largest term in a polynomial* for limits of infinity.

Example 1.5 *Indeterminate forms by substitution of infinity*

Substituting ∞ into the function gives $\infty - \infty - \infty$, which is indeterminate. Hence, we must factorize it.

$$\begin{aligned}\lim_{x\to\infty} 2x^4 - x^2 - 8x &= \lim_{x\to\infty}[x^4(2 - \frac{1}{x^2} - \frac{8}{x^3})] \\ &= \infty \times 2 \\ &= \infty\end{aligned}$$

Or we can just simply use the theorem above and consider $\lim_{x\to\infty} 2x^4$ only to give ∞.

Example 1.6 *Factor polynomials limit to infinity*

We can simply consider the largest terms on each side and give the final answer easily.

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{3x^2+6}}{5-2x} &= \lim_{x\to\infty} \frac{\sqrt{3x^2}}{-2x} \\ &= \lim_{x\to\infty} \frac{\sqrt{3}|x|}{-2x} \\ &= \frac{-\infty\sqrt{3}}{-2\times\infty} \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Note that, as we are considering the negative limit of infinity, we need to add - to the abs sign on line 3.

2 Derivatives

First principle

The first principle is the definition of the derivative of a function $f(x)$ at $x = a$:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.1 Differentiation formulas and rules

Basic formulas

- We can differentiate individual items: $(f \pm g)' = f' \pm g'$
- We can factor out a multiplicative constant: $(cf)' = cf'$
- Derivative of a constant is 0: $\frac{d}{dx}k = 0$
- Power rule: $\frac{d}{dx}x^n = nx^{n-1}$

Chain rule

Shorthand: **d**1x2 + **d**2x1

$$(u(v))' = u'(v)v' \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Product rule

Shorthand: **d** from outside to inside

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient rule

Shorthand: move lower **d** upper **-** **d** lower x upper, lower square

$$(\frac{u}{v})' = \frac{vu' - uv'}{v^2} \qquad \text{or} \qquad \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

	$f(x)$	$f'(x)$
1.	a^x	$\ln a \cdot a^x$
2.	e^{kx}	ke^{kx}
3.	$\ln kx$	x^{-1}
4.	$\sin kx$	$k\cos kx$
5.	$\cos kx$	$-k\sin kx$
6.	$\lvert x\rvert$	$\frac{\lvert x\rvert}{x}$
7.	$\tan kx$	$k\sec^2 kx$
8.	$\csc x$	$-\csc x \cot x$
9.	$\sec x$	$\sec x \tan x$
10.	$\cot x$	$-\csc^2 x$

Implicit differentiation

Differentiate all xy, add $\frac{dy}{dx}$ behind all differentiations of y.

To find $\frac{dy}{dx}$ for $y^2 = x^2 + \sin(xy)$:

$$\begin{aligned}
y^2 &= x^2 + \sin(xy) \\
2y\frac{dy}{dx} &= \frac{d}{dx}(\sin(xy)) \\
2y\frac{dy}{dx} &= 2x + \cos(xy)(x\frac{dy}{dx} + y) \\
\text{Then we simply collect terms of } \frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}$$

2.2 Extremum points

Critical points

A critical point is a point where $f'(x) = 0$ or $f'(x)$ is undefined, or the end-points of the domain.

Absolute max/minimum points

The absolute maximum/minimum points are the points where the function has the largest/smallest value in the entire domain. It can be written as:

$$\max_{x\in D} f(x) \qquad \text{or} \qquad \min_{x\in D} f(x)$$

Local max/minimum points

The local maximum/minimum points are the points where the function has the largest/smallest value in a small interval around the point.

Extreme value theorem

For $f(x)$ that is *continuous* in $[a, b]$, there must be both a abs maximum and minimum point.

To find the absolute extremas of $f(x)$ in $[a, b]$, we can:

1. Verify that $f(x)$ is continuous in $[a, b]$
2. Find all critical points in $[a, b]$
3. Evaluate the critical points as well as the end points of the interval

Rolle's theorem

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b), if $f(a) = f(b)$, there must be a critical point c in the interval.

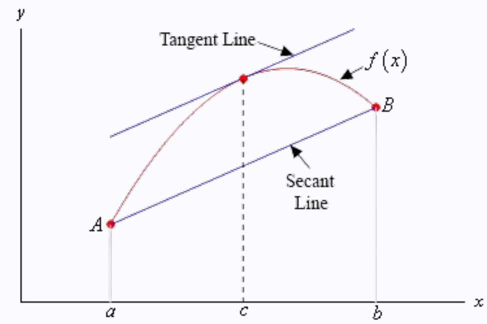
2.3 Other cool theorems

Mean value theorem

For $f(x)$ that is *continuous* in $[a, b]$ and *differentiable* in (a, b), there must be a point c in (a, b) where:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The theorem tells us that, in describe conditions, there must be a point c where the slope of the *tangent line* is equal to the slope of the line from $a \to b$ (secant line).



Source: https://tutorial.math.lamar.edu/Classes/CalcI/MeanValueTheorem.aspx

L'Hopital's rule

For any $a \in [\mathbb{R}, \pm\infty]$, if $\lim_{x\to a}(\frac{f}{g})$ is in indeterminate form after substitution, we can conclude:

$$\lim_{x\to a}(\frac{f}{g}) = \lim_{x\to a}(\frac{f'}{g'})$$

To find $\lim_{x\to -\infty} xe^x$, we first check if the limit is indeterminate, then we can apply the rule:

$$\begin{aligned}
\lim_{x\to -\infty} xe^x &\implies \infty \times 0 \\
\lim_{x\to -\infty} xe^x &= \lim_{x\to -\infty} \frac{x}{e^{-x}} \\
&= \lim_{x\to -\infty} \frac{1}{-e^{-x}} \text{ (rule applied)} \\
&= 0
\end{aligned}$$

2.4 Shape of graph

graph feature	f(x)	f'(x)	f''(x)	Notes
rising (L to R)	slope > 0	+		
falling (L to R)	slope < 0	-		
extrema: maximum	slope = 0	= 0 + on L - on R	- at x_{max}	derivative may not exist at a max or min, e.g. or
extrema: minimum	slope = 0	= 0 - on L + on R	+ at x_{min}	
inflection pt.	Curvature changes: ↔		= 0 potential inflection point	Check f''(x) on either side of a potential inflection point.
concave up		- | +	+	
concave down		+ | -	-	

Source: *https://zaktly.com/CurveSketchSecondDeriv.html*

3 Integrals

3.1 Area and Estimating with Finite Sums

3.2 Definite Integrals

3.3 The Fundamental Theorem of Calculus

3.4 theorem of Natural Logarithms

3.5 Interlude - Hyperbolic Functions

4 Parametric Equations

5 Polar Coordinates

6 Ordinary Differential Equations

6.1 1st Order Linear ODEs & Integrating Factors

6.2 Bernoulli Equations & Riccati Equations