Project III: Robot-Arm Kinematics

MiniBOT6-R — Report

Charles Friley

May 1, 2025

Project Overview

This report addresses Questions 1 and 2 of the *Robot-Arm Kinematics* project. Question 3 (MuJoCo validation) was not attempted. The goals are:

- **Q1a.** Derive the position-level forward kinematics (FK) using the Denavit–Hartenberg (DH) convention and evaluate the end-effector pose for a specified joint configuration.
- **Q1b.** Obtain the closed-form inverse-kinematics (IK) solutions and compute all joint-angle sets with a given target pose.
- **Q2.** Derive the spatial Jacobian and solve for the joint rates $\dot{\mathbf{q}}$ that achieve a desired end-effector twist $\boldsymbol{\nu}$ at one of the IK configurations.

1 Question 1 — Position-Level Kinematics

1.1 DH Parameters

Using the DH convention, the following coordinate systems were constructed. (see fig 1 below)

The DH Table can be constructed using fig 1:

Link i	α_i (rad)	a_i (mm)	d_i (mm)	θ_i (variable)	Comment
1	$\frac{\pi}{2}$	27.5	339	θ_1	
2	0	250	0	θ_2	
3	$\frac{\pi}{2}$	70	0	θ_3	
4	$-\frac{\pi}{2}$	0	250	θ_4	
5	$\frac{\pi}{2}$	0	0	θ_5	
6	0	0	95	θ_6	

The homogeneous transformation of link i with respect to $i-1$ is

$$\mathbf{A}_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad c_{\theta_i} = \cos\theta_i,\ s_{\theta_i} = \sin\theta_i. \tag{1}$$

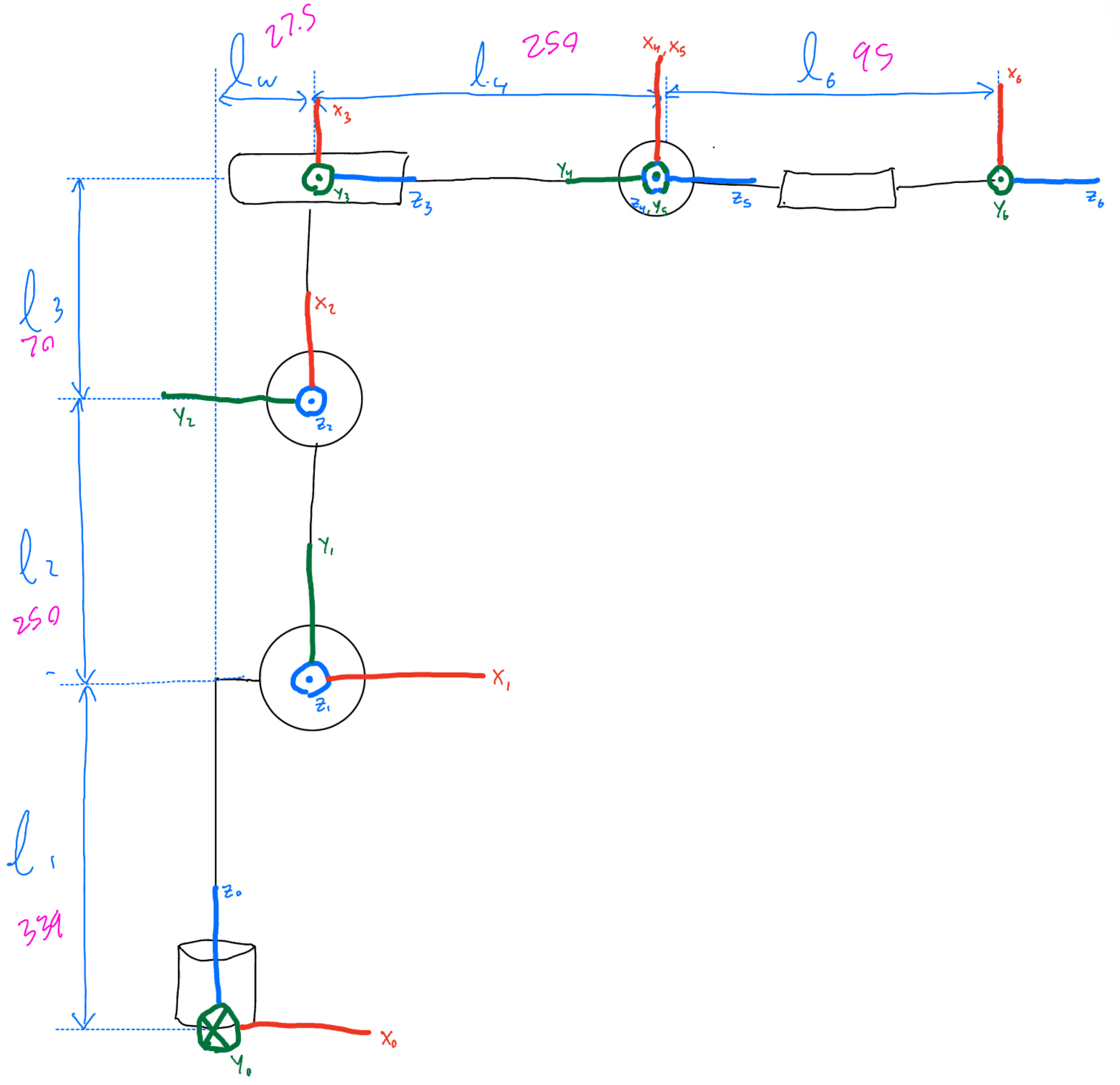

Figure 1: Robot Arm DH Frames

1.2 Forward Kinematics Evaluation

For the configuration $\boldsymbol{\theta}^{(1)} = \big[0,\ 90°,\ 0,\ 0,\ -90°,\ 0\big]$, Python evaluation (`robotics-toolbox`) produced the pose and Euler ZYX angles.

$$
{}^{0}T_6(\boldsymbol{\theta}^{(1)}) = \begin{bmatrix} 1 & 0 & 0 & 277.5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 564 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{\xi}^{(1)} = \begin{bmatrix} 277.5 & 0 & 564 & \pi & 0 & 0 \end{bmatrix}^{\top}. \tag{2}
$$

Under the hood, the robotics toolbox is simply constructing all homogeneous transformations for each joint and multiplying them together.

1.3 Closed-Form Inverse Kinematics

A geometric approach was used:

1. Compute the wrist-centre $\mathbf{p}_{\mathrm{c}} = \mathbf{p}_{\mathrm{e}} - d_6\mathbf{z}_{\mathrm{e}}$.
2. Solve θ_1 from the projection of $\mathbf{p}_{\mathrm{c}}$ on the base xy-plane. (see fig 2)
3. Treating links 2–3–4 as a planar 2-DoF mechanism in the r–z half-plane, intersect two circles to find θ_2 and θ_3 (elbow-up/down, shoulder-left/right). (see fig 3, 4, and 5)
4. Recover $\theta_4, \theta_5, \theta_6$ from ${}^{3}R_6 = {}^{0}R_3^{\top}\,{}^{0}R_6$ via ZYZ Euler angles.

The intersecting circles method works as follows (see fig 5):

1. First, draw a circle c_1 around the wrist center with a radius of $h = \sqrt{l_4^2 + l_3^2}$. This circle perimeter shows the possible locations for joint 3 (l_i values defined in fig 1).
2. Second, draw a circle c_2 around joint 2 at origin 1 with a radius of l_2. This circle perimeter shows the possible locations for joint 3.
3. Where these two circles c_1 and c_2 intersect represents the only two locations that joint 3 can be located. This will later be used to find our two possible θ_2 values. (Note: there are two more possible θ_2 values once you consider the other possible θ_1 value.
4. Draw a circle c_3 and c_4 about each possible location for Joint 3 with a radius of l_3. This circle perimeter shows the possible locations for joint 4.
5. Draw a circle c_5 about the wrist center with a radius of l_4. this circle perimeter shows the possible locations for joint 4.
6. Where c_3 and c_4 intersect with c_5 shows the four possible locations for joint 4. This will later be used to find our four possible θ_3 values. (Note: there are four more possible θ_3 values once you consider the other possible θ_1 value.
7. Once all the possible locations are known for every single point, finding the corresponding joint angles is simple geometry. (see fig 4)

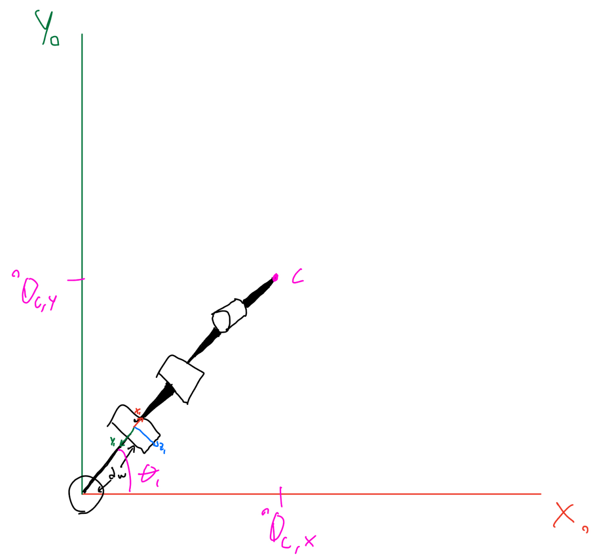

Figure 2: Projection on x_0 and y_0 plane.

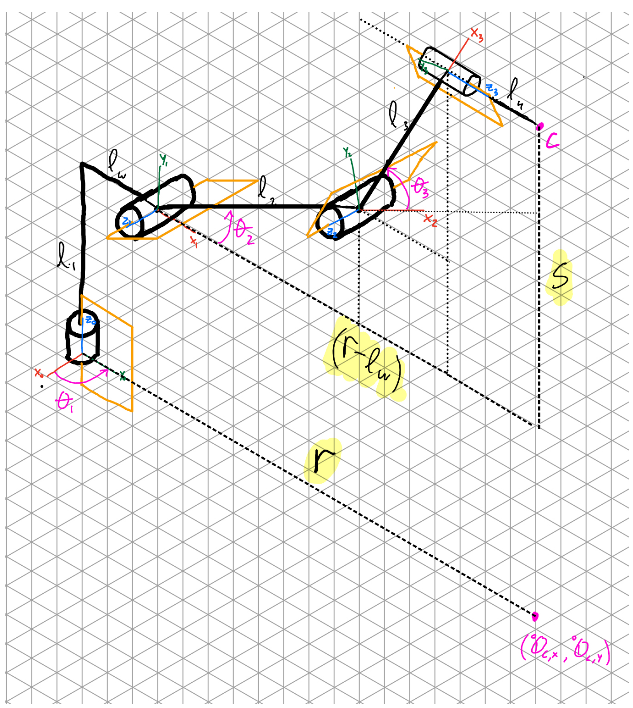

Figure 3: 3D view of Manipulator from Base Frame to Wrist Center

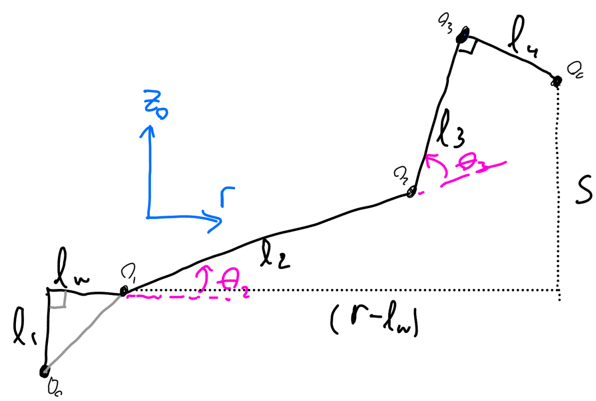

Figure 4: Projection from Base Frame onto $r - z_0$ frame

Eight candidate solutions were generated for each $\theta_1 - \theta_6$ value:

$$\begin{bmatrix} 0.524 & 1.047 & -0.000 & 2.356 & 0.873 & -2.749 \\ 0.524 & 1.047 & -2.596 & 0.597 & 1.842 & 0.002 \\ 0.524 & -0.279 & 2.596 & 2.531 & 1.903 & 2.739 \\ 0.524 & -0.279 & 0.000 & 0.833 & 0.822 & -0.821 \\ 3.665 & 1.047 & -0.000 & -2.562 & 1.423 & -0.275 \\ 3.665 & 1.047 & -2.596 & -0.605 & 1.262 & -3.113 \\ 3.665 & -0.279 & 2.596 & -2.191 & 2.413 & 0.629 \\ 3.665 & -0.279 & 0.000 & -1.675 & 0.576 & -1.625 \end{bmatrix} \text{rad.}$$

Verifying each set via FK revealed two exact matches :

$$\boxed{\mathbf{q}^{(a)} = [0.524,\ 1.047,\ -0.000,\ 2.356,\ 0.873,\ -2.749]^\top}, \qquad \boxed{\mathbf{q}^{(b)} = [0.524,\ -0.279,\ 2.596,\ 2.531,\ 1.903,\ 2.739]^\top}.$$

Numerical Validation Levenberg–Marquardt (LM), Newton–Raphson (NR) and Gauss–Newton (GN) solvers, implemented with built in functions in robotics toolbox, converged to the same two solutions, confirming correctness.

2 Question 2 — Velocity-Level Kinematics

2.1 Spatial Jacobian

For a revolute joint i with axis $\mathbf{z}_{i-1}$ and origin $\mathbf{o}_{i-1}$, the i-th column of the spatial Jacobian is[1] $\mathbf{J}_i = \left[\mathbf{z}_{i-1} \times (\mathbf{o}_6 - \mathbf{o}_{i-1});\ \mathbf{z}_{i-1}\right]$. Stacking $\mathbf{J}_i$ yields $\mathbf{J}(\mathbf{q}) \in \mathbb{R}^{6\times 6}$.

2.2 Joint Rates for a Desired Twist

Given the target twist $\boldsymbol{\nu} = \left[100,\ -200,\ -300,\ 2,\ -1,\ 0.5\right]^\top$(mm/s, rad/s) and the two IK configurations above, the solved joint rates were

[1] All vectors are expressed in the base frame.

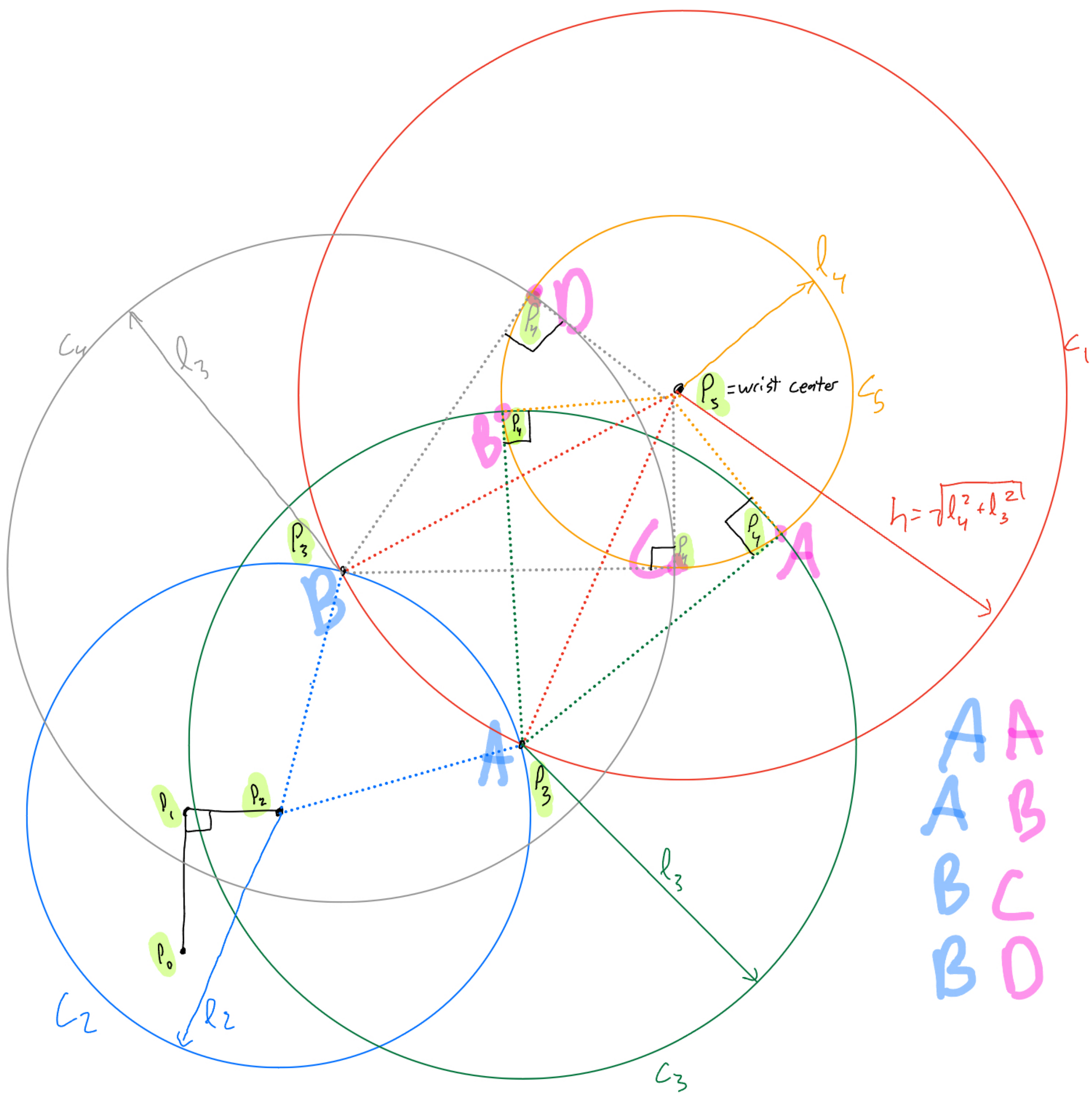

Figure 5: Intersecting Circles Method to find Joint Angles θ_2 and θ_3

Joint	$\dot{\mathbf{q}}^{(a)}$ (rad/s)	$\dot{\mathbf{q}}^{(b)}$ (rad/s)
$\dot{\theta}_1$	-0.815	-0.815
$\dot{\theta}_2$	0.425	-1.407
$\dot{\theta}_3$	-1.779	1.779
$\dot{\theta}_4$	-0.460	2.056
$\dot{\theta}_5$	-3.518	-1.299
$\dot{\theta}_6$	1.352	0.792

The reconstruction error $\|\mathbf{J}\dot{\mathbf{q}} - \boldsymbol{\nu}\|$ was below 10^{-6} for both cases, confirming correctness.

Conclusions

The report presented a complete analytical FK and IK solution for the miniBOT6-R, verified numerically, and derived the manipulator Jacobian to compute the required joint rates for a specified spatial twist. All results satisfied the accuracy criteria defined in the project brief.