Differential Equations of Equilibrium:

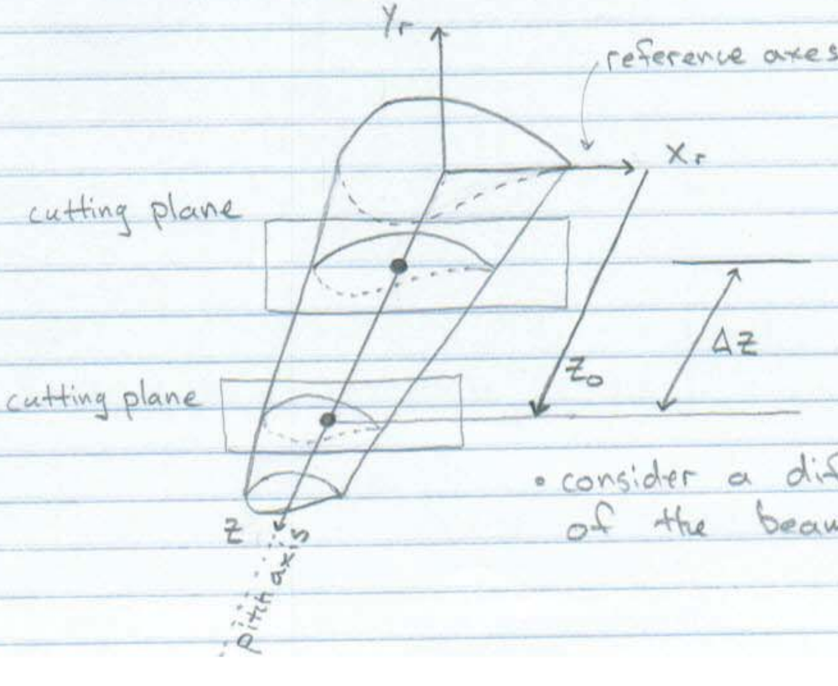
- To include coupling between extension, bending, and torsion, re-derive differential eqns. of equilibrium to account for offsets of the inertial, centroidal, and elastic axes from the reference (pitch) axis.

- resultant forces and moments (V_x, V_y, V_z and M_x, M_y, M_z) at the centroidal axis (locus of tension centers: $x_{tc}(z), y_{tc}(z)$)

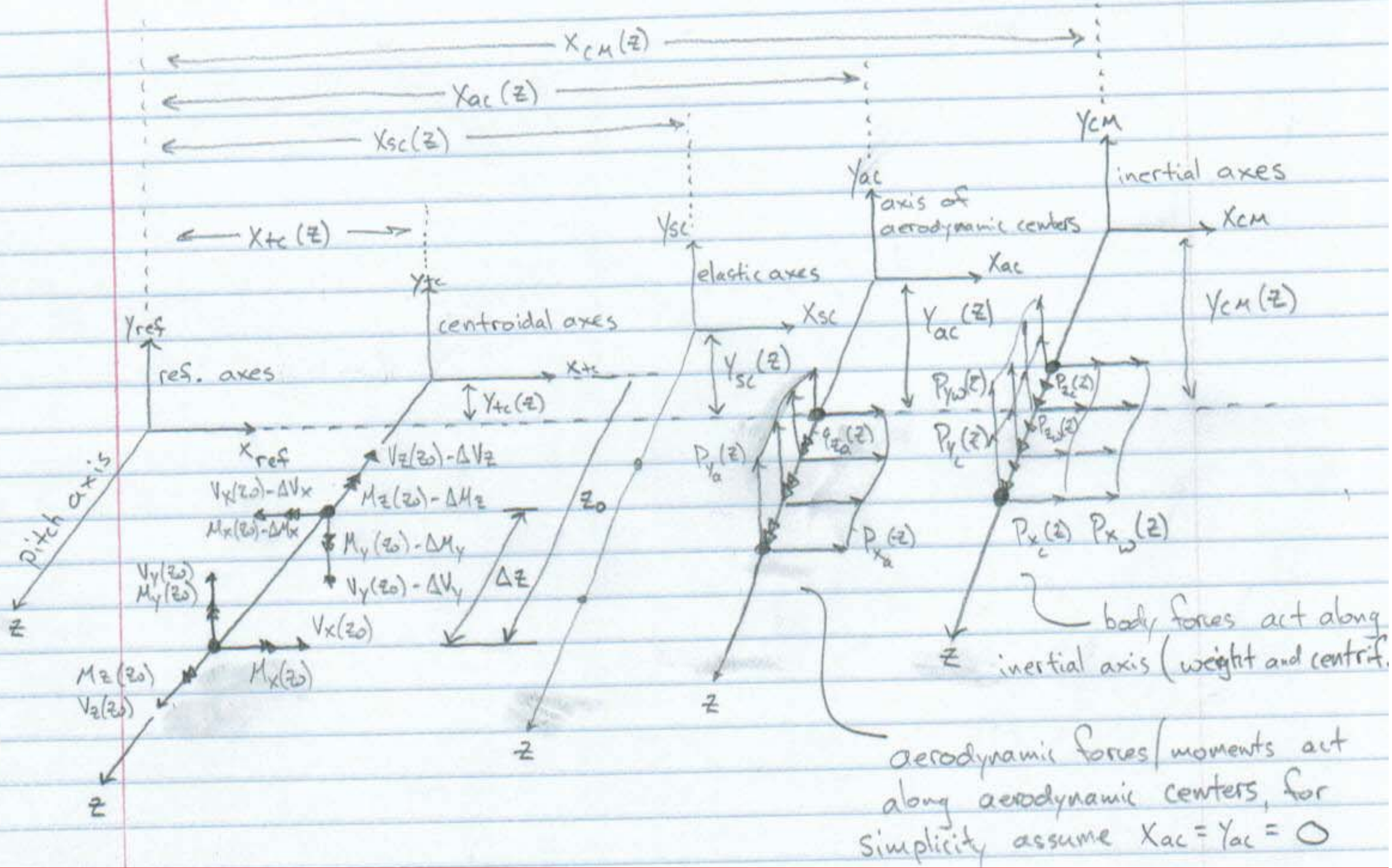
- applied aerodynamic forces per unit length and aerodynamic moment per unit length ($p_{x_a}(z), p_{y_a}(z), q_{z_a}(z)$) act at aerodynamic axis (the locus of aerodynamic centers: $x_{ac}(z), y_{ac}(z)$). For simplicity, assume $x_{ac} = y_{ac} = 0$ such that aero forces/moments act along pitch axis

- self weight (including buoyancy) and centrifugal forces per unit length ($p_{x_w}(z), p_{y_w}(z), p_{z_w}(z)$ and $p_{x_c}(z), p_{y_c}(z), p_{z_c}(z)$) act at inertial axis (the locus of mass centers: $x_{cm}(z), y_{cm}(z)$)

- torsion causes the beam cross sections to rotate about the shear center. The elastic axis denoted by the locus of shear centers: $x_{sc}(z), y_{sc}(z)$



- consider a differential element of the beam (Δz)



$X_{CM}(z)$
$X_{ac}(z)$
$X_{sc}(z)$
$X_{tc}(z)$
Y_{CM}
inertial axes
X_{CM}
Y_{ac}
axis of aerodynamic centers
X_{ac}
Y_{sc}
elastic axes
X_{sc}
Y_{tc}
centroidal axes
X_{tc}
Y_{ref}
ref. axes
X_{ref}
pitch axis
$Y_{tc}(z)$
$Y_{sc}(z)$
$Y_{ac}(z)$
$Y_{CM}(z)$
$V_z(z_0)-\Delta V_z$
$V_x(z_0)-\Delta V_x$
$M_z(z_0)-\Delta M_z$
$M_x(z_0)-\Delta M_x$
$M_y(z_0)-\Delta M_y$
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$M_x(z_0)$
z_0
Δz
$P_{y_a}(z)$
$q_{z_a}(z)$
$P_{x_a}(z)$
$P_{y_w}(z)$
$P_{y_c}(z)$
$P_{z_c}(z)$
$P_{z_w}(z)$
$P_{x_c}(z)$
$P_{x_w}(z)$
z
body forces act along inertial axis (weight and centrif.
aerodynamic forces/moments act along aerodynamic centers, for simplicity assume $X_{ac} = Y_{ac} = 0$

- force balances on differential element

$\Sigma F_x = 0$

$$V_x(z_0) - \left[V_x(z_0) - \Delta V_x\right] + \int_{z_0-\Delta z}^{z_0} P_{xa}(z) + P_{xw}(z) + P_{xc}(z)\, dz = 0$$

$$\Delta V_x = -\int_{z_0-\Delta z}^{z_0} P_{xa}(z) + P_{xw}(z) + P_{xc}(z)\, dz \qquad \Big| \cdot \text{divide by } \Delta z$$

$$\frac{\Delta V_x}{\Delta z} = -\frac{1}{\Delta z}\int_{z_0-\Delta z}^{z_0} P_{xa}(z) + P_{xw}(z) + P_{xc}(z)\, dz \qquad \Big| \cdot \lim \text{ as } \Delta z \to 0$$

note: as $\Delta z \to 0$ the integrand of $\int_{z_0-\Delta z}^{z_0} (\cdot)\, dz$ becomes constant, so $\int_{z_0-\Delta z}^{z_0} (\cdot)\, dz \to (\cdot)\,\Delta z$

[1] $$\boxed{\frac{dV_x}{dz} = -\left(P_{xa}(z) + P_{xw}(z) + P_{xc}(z)\right)}$$

$\Sigma F_y = 0$

$$V_y(z_0) - \left[V_y(z_0) - \Delta V_y\right] + \int_{z_0-\Delta z}^{z_0} P_{ya}(z) + P_{yw}(z) + P_{yc}(z)\, dz = 0$$

[2] $$\boxed{\frac{dV_y}{dz} = -\left(P_{ya}(z) + P_{yw}(z) + P_{yc}(z)\right)}$$

$\Sigma F_z = 0$

$$V_z(z_0) - \left[V_z(z_0) - \Delta V_z\right] + \int_{z_0-\Delta z}^{z_0} P_{zw}(z) + P_{zc}(z)\, dz = 0$$

[3] $$\boxed{\frac{dV_z}{dz} = -\left(P_{zw}(z) + P_{zc}(z)\right)}$$

- moment balance on the differential element

$$\sum M_x = 0$$

$$M_x(z_0) - \left[M_x(z_0) - \Delta M_x\right] - \left[V_y(z_0) - \Delta V_y\right]\Delta z \ \ldots$$

$$+ \int_{z_0-\Delta z}^{z_0} \left(p_{y_a}(z) + p_{y_w}(z) + p_{y_c}(z)\right)\alpha(z)\,\Delta z\, dz \ \ldots$$

$$+ \int_{z_0-\Delta z}^{z_0} \left(p_{zw}(z) + p_{zc}(z)\right)\left(y_{cm}(z) - y_{tc}(z)\right) dz = 0 \quad , \text{where } 0 \le \alpha(z) \le 1$$

| · divide by Δz

$$\frac{\Delta M_x}{\Delta z} = V_y(z_0) - \Delta V_y - \frac{1}{\Delta z}\int_{z_0-\Delta z}^{z_0} \left(p_{y_a}(z) + p_{y_w}(z) + p_{y_c}(z)\right)\alpha(z)\,\Delta z\, dz \ldots$$

$$- \frac{1}{\Delta z}\int_{z_0-\Delta z}^{z_0} \left(p_{zw}(z) + p_{zc}(z)\right)\left(y_{cm}(z) - y_{tc}(z)\right) dz$$

| · $\lim \Delta z \to 0$

[4] $$\boxed{\frac{dM_x}{dz} = V_y(z) - \left(p_{zw}(z) + p_{zc}(z)\right)\left(y_{cm}(z) - y_{tc}(z)\right)}$$

$$\sum M_y = 0$$

$$M_y(z_0) - \left[M_y(z_0) - \Delta M_y\right] + \left[V_x(z_0) - \Delta V_x\right]\Delta z \ \ldots$$

$$- \int_{z_0-\Delta z}^{z_0} \left(p_{xa}(z) + p_{xw}(z) + p_{xc}(z)\right)\alpha(z)\,\Delta z\, dz \ \ldots$$

$$- \int_{z_0-\Delta z}^{z_0} \left(p_{zw}(z) + p_{zc}(z)\right)\left(x_{cm}(z) - x_{tc}(z)\right) dz = 0$$

[5] $$\boxed{\frac{dM_y}{dz} = -V_x(z) + \left(p_{zw}(z) + p_{zc}(z)\right)\left(x_{cm}(z) - x_{tc}(z)\right)}$$

$$\sum M_z = 0$$

$$M_z(z_0) - \left[M_z(z_0) - \Delta M_z \right] + \int_{z_0-\Delta z}^{z_0} q_{za}(z)\,dz \quad \ldots$$

$$+ \int_{z_0-\Delta z}^{z_0} p_{Y_a}(z)\left(X_{ac}(z) - X_{sc}(z) \right) dz + \int_{z_0-\Delta z}^{z_0} \left(p_{Y_w}(z) + p_{Y_c}(z) \right)\left(X_{CM}(z) - X_{sc}(z) \right) dz \quad \ldots$$

$$- \int_{z_0-\Delta z}^{z_0} p_{X_a}(z)\left(Y_{ac}(z) - Y_{sc}(z) \right) dz - \int_{z_0-\Delta z}^{z_0} \left(p_{X_w}(z) + p_{X_c}(z) \right)\left(Y_{CM}(z) - Y_{sc}(z) \right) dz = 0$$

[6]
$$\boxed{\begin{aligned} \frac{dM_z}{dz} = & -q_{za}(z) \quad \ldots \\ & - p_{Y_a}(z)\left(X_{ac}(z) - X_{sc}(z) \right) - \left(p_{Y_w}(z) - p_{Y_c}(z) \right)\left(X_{CM}(z) - X_{sc}(z) \right) \quad \ldots \\ & + p_{X_a}(z)\left(Y_{ac}(z) - Y_{sc}(z) \right) + \left(p_{X_w}(z) + p_{X_c}(z) \right)\left(Y_{CM}(z) - Y_{sc}(z) \right) \end{aligned}}$$

- eqns. [1-6] are 1st order linear ODEs and can be integrated directly

· integration of ODE's [1-6] yields:

[1] $$V_x(z_1) = V_x(z_2) + \int_{z_1}^{z_2} p_{x_a}(z) + p_{x_\omega}(z) + p_{x_c}(z)\, dz$$

[2] $$V_y(z_1) = V_y(z_2) + \int_{z_1}^{z_2} p_{y_a}(z) + p_{y_\omega}(z) + p_{y_c}(z)\, dz$$

[3] $$V_z(z_1) = V_z(z_2) + \int_{z_1}^{z_2} p_{z_\omega}(z) + p_{z_c}(z)\, dz$$

[4] $$M_x(z_1) = M_x(z_2) - \int_{z_1}^{z_2} V_y(z) - \left(p_{z_\omega}(z) + p_{z_c}(z)\right)\left(y_{cm}(z) - y_{tc}(z)\right) dz$$

[5] $$M_y(z_1) = M_y(z_2) - \int_{z_1}^{z_2} -V_x(z) + \left(p_{z_\omega}(z) + p_{z_c}(z)\right)\left(x_{cm}(z) - x_{tc}(z)\right) dz$$

[6] $$M_z(z_1) = M_z(z_2) - \int_{z_1}^{z_2} \Big\{ -q_{z_a}(z) \ \ldots$$

$$- p_{y_a}(z)\left(x_{ac}(z) - x_{sc}(z)\right) - \left(p_{y_\omega}(z) + p_{y_c}(z)\right)\left(x_{cm}(z) - x_{sc}(z)\right) \ldots$$

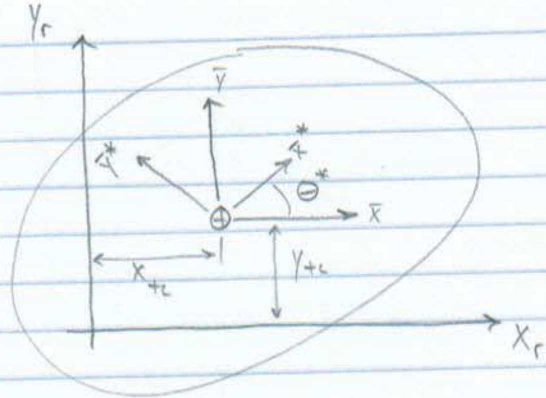
$$+ p_{x_a}(z)\left(y_{ac}(z) - y_{sc}(z)\right) + \left(p_{x_\omega}(z) + p_{x_c}(z)\right)\left(y_{cm}(z) - y_{sc}(z)\right) \Big\} dz$$

· for a cantilever beam, if z_2 is the free end

$$\left\{V_x = V_y = V_z = M_x = M_y = M_z\right\}\Big|_{z = z_2} = 0$$

$$z_1 \le z \le z_2$$

Deflections and Stress



(X_r, Y_r) - reference x-y axes
$(\bar{x}, \bar{y})$ - centroidal x-y axes (located at tension center)
$(\bar{x}^*, \bar{y}^*)$ - principal centroidal axes (located at tension center @ θ^*)

1) compute location of centroid (tension center)

$$A^* = \sum_{i=1}^{N} \frac{E_i A_i}{E_r}, \quad X_{tc} = \frac{1}{A^*} \sum_{i=1}^{N} \frac{E_i A_i X_{cen_i}}{E_r}, \quad Y_{tc} = \frac{1}{A^*} \sum_{i=1}^{N} \frac{E_i A_i Y_{cen_i}}{E_r}$$

2) compute second area moments of inertia (w.r.t. X_r-Y_r axes)

$$I_{x_r} = \sum_{i=1}^{N} \frac{E_i}{E_r} I_{x_{r_i}}$$

$$I_{y_r} = \sum_{i=1}^{N} \frac{E_i}{E_r} I_{y_{r_i}}$$

$$I_{x_r y_r} = \sum_{i=1}^{N} \frac{E_i}{E_r} I_{x_r y_{r_i}}$$

3) compute centroidal second area moments of inertia (w.r.t. X_{tc}-Y_{tc})

$$I_{\bar{x}} = I_{x_r} - Y_{tc}^2 A^*$$

$$I_{\bar{y}} = I_{y_r} - X_{tc}^2 A^*$$

$$I_{\bar{x}\bar{y}} = I_{x_r y_r} - X_{tc} Y_{tc} A^*$$

3) compute principle centroidal second area moments of inertia

$$I_{\bar{x}^*} = \frac{I_{\bar{y}} + I_{\bar{x}}}{2} - \sqrt{\left(\frac{I_{\bar{y}} - I_{\bar{x}}}{2}\right)^2 + I_{\bar{x}\bar{y}}^2}$$

$$I_{\bar{y}^*} = \frac{I_{\bar{y}} + I_{\bar{x}}}{2} + \sqrt{\left(\frac{I_{\bar{y}} - I_{\bar{x}}}{2}\right)^2 + I_{\bar{x}\bar{y}}^2}$$

by definition $I_{\bar{x}\bar{y}}^* = 0$

4) define stiffnesses

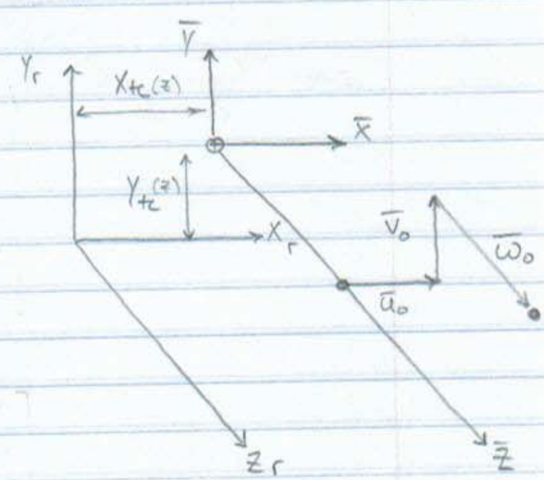
axial stiffness $S = E_r A^*$

bending stiffness
$H_x = E_r I_x$
$H_y = E_r I_y$
$H_{xy} = E_r I_{xy}$

centroidal bending stiffness
$H_{\bar{x}} = E_r I_{\bar{x}}$
$H_{\bar{y}} = E_r I_{\bar{y}}$
$H_{\bar{x}\bar{y}} = E_r I_{\bar{x}\bar{y}}$

principle centroidal bending stiffness
$H_{\bar{x}^*} = E_r I_{\bar{x}^*}$ (flapwise)
$H_{\bar{y}^*} = E_r I_{\bar{y}^*}$ (edgewise)

5) compute centroidal deflections



$$\boxed{\begin{aligned} \frac{d\bar{\omega}_o}{d\bar{z}} &= \frac{V_z}{S} \\ \frac{d^2\bar{u}_o}{d\bar{z}^2} &= \frac{M_y H_{\bar{x}} + M_x H_{\bar{x}\bar{y}}}{H_{\bar{x}}H_{\bar{y}} - H_{\bar{x}\bar{y}}^2} \\ \frac{d^2\bar{v}_o}{d\bar{z}^2} &= \frac{-M_x H_{\bar{y}} - M_y H_{\bar{x}\bar{y}}}{H_{\bar{x}}H_{\bar{y}} - H_{\bar{x}\bar{y}}^2} \end{aligned}}$$

subject to B.C.

$$\frac{d\bar{\omega}_o(0)}{d\bar{z}} = \frac{d\bar{u}_o(0)}{d\bar{z}} = \frac{d\bar{v}_o(0)}{d\bar{z}} = \bar{\omega}_o(0) = \bar{u}_o(0) = \bar{v}_o(0) = 0$$

6) compute axial stress

$$\sigma_{zz}(\bar{x},\bar{y}) = E(\bar{x},\bar{y})\left[\frac{V_z}{S} - \left(\frac{M_y H_{\bar{x}} + M_x H_{\bar{x}\bar{y}}}{H_{\bar{x}}H_{\bar{y}} - H_{\bar{x}\bar{y}}^2}\right)\bar{x} + \left(\frac{M_x H_{\bar{y}} + M_y H_{\bar{x}\bar{y}}}{H_{\bar{x}}H_{\bar{y}} - H_{\bar{x}\bar{y}}^2}\right)\bar{y}\right]$$

where
$$\bar{x} = x - x_{tc}(z)$$
$$\bar{y} = y - y_{tc}(z)$$

$$\Delta\bar{H} = H_{\bar{x}}H_{\bar{y}} - H_{\bar{x}\bar{y}}^2$$

$$\sigma_{zz}(x,y) = E\left[\frac{V_z}{S} - \left(\frac{\bar{x}H_{\bar{x}\bar{y}} - \bar{y}H_{\bar{y}}}{\Delta\bar{H}}\right)M_x - \left(\frac{\bar{x}H_{\bar{x}} - \bar{y}H_{\bar{x}\bar{y}}}{\Delta\bar{H}}\right)M_y\right]$$

7) compute twist angle, Φ_z, about the z-axis (twist due to torsion)

$$\boxed{H_{\bar{z}} \frac{d^2 \Phi_z}{dz^2} = \frac{dM_z}{dz}}$$

subject to B.C. $\frac{d\Phi_z}{dz}(z = L) = 0$ and $\Phi_z(0) = 0$

- each section of the beam undergoes a rigid body rotation of magnitude $\Phi_z(z)$ <u>about the shear center</u> $(x_{sc}(z), y_{sc}(z))$

torsional stiffness $H_{\bar{z}} = \frac{M_z}{k_z^{[i]}}$

twist rate of cell i: $k_z^{[i]} \longrightarrow$ determined in shear flow analysis