Design and Analysis of Algorithm (DAA)

Dynamic Programming

(Optimal Binary Search Tree)

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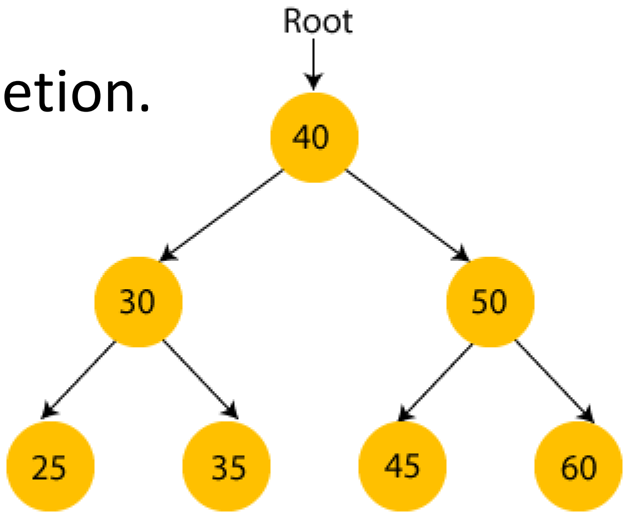
Binary Search Tree (BST)

A tree where each node has a value, and all nodes in the left subtree of a node have smaller values, while nodes in the right subtree have larger values.

left_subtree (value) < node (value) < right_subtree (value)

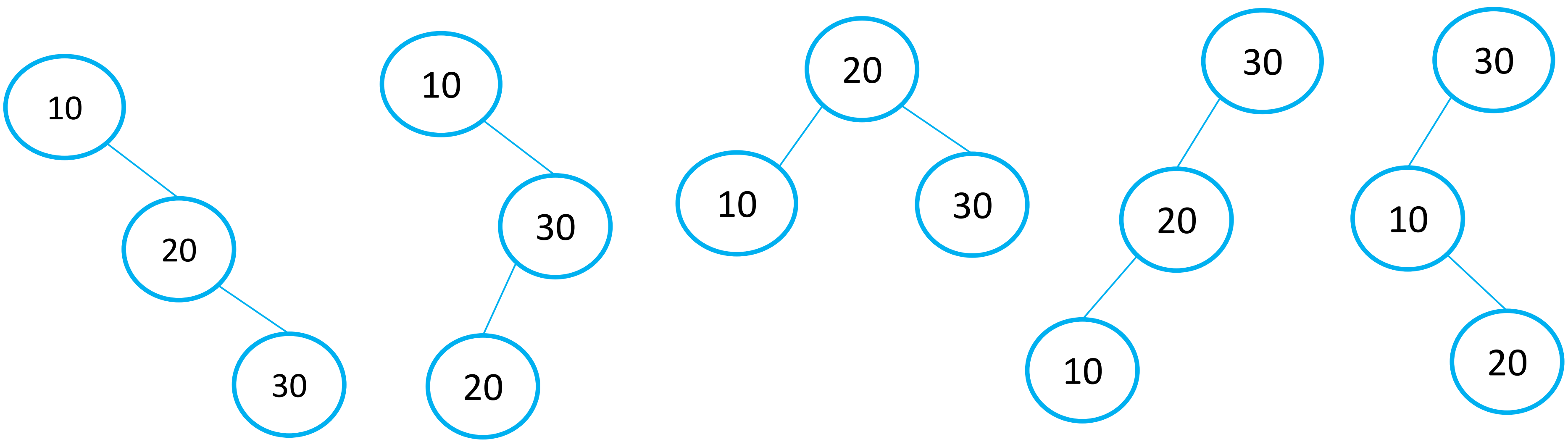
Data structure for fast search, insertion, deletion.



Binary Search Tree (BST)

For “n” nodes, No. of Binary Search Tree: $\frac{2n_{C_n}}{n+1}$

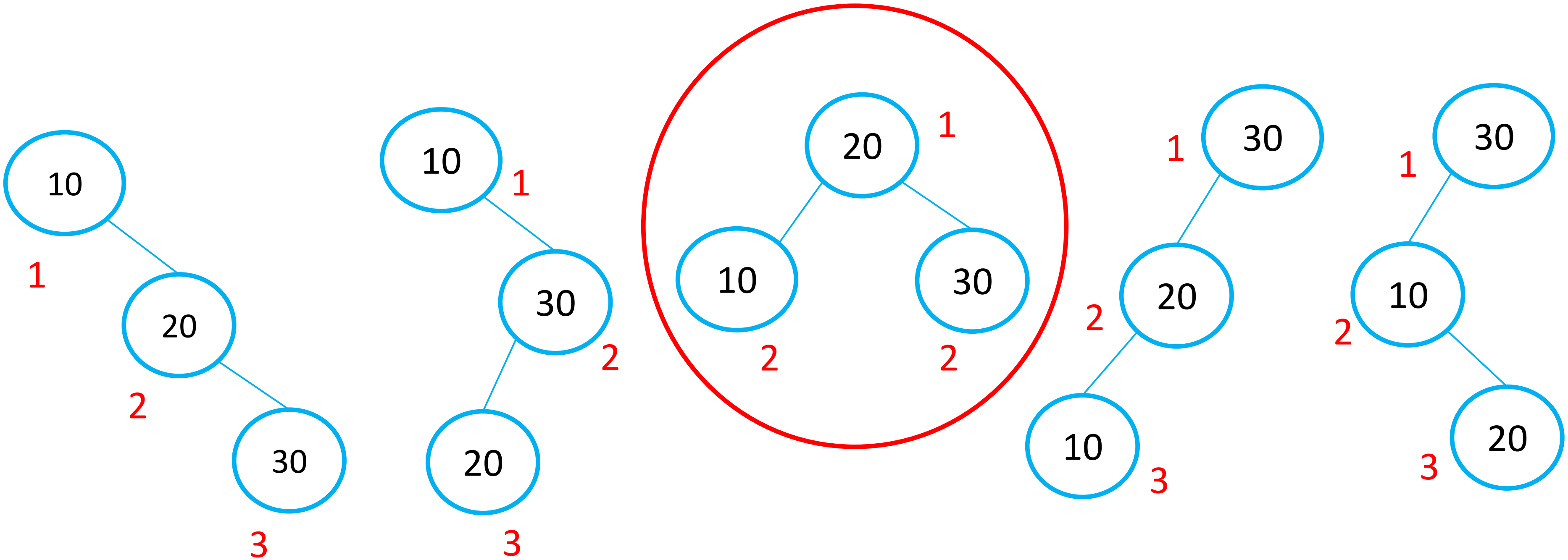
Let n =3, Keys = [10, 20, 30]



No. of Binary Search Tree: $\frac{6_{C_3}}{3+1} = 5$

Binary Search Tree (BST)

No of Comparison:



No of Comparison = (1+2+3) = 6

(1+2+3) = 6

(1+2+2) = 5

(1+2+3) = 6

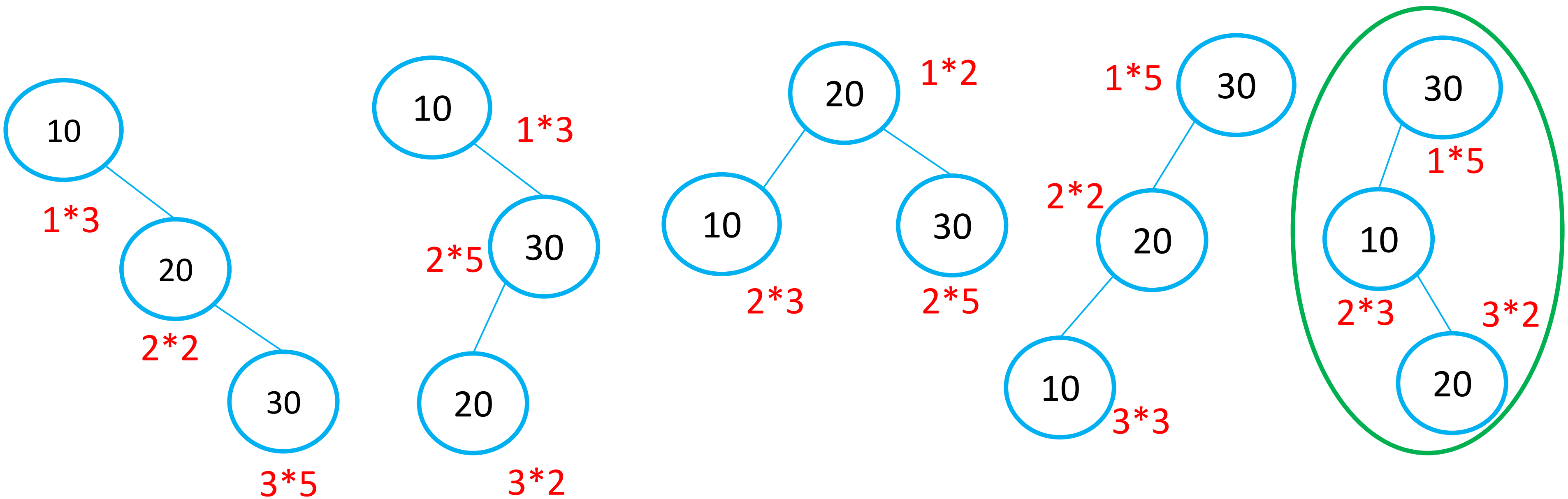
(1+2+3) = 6

The search cost of a node in a BST is proportional to its level (depth).

Whether height balanced tree is Optimal Binary Search Tree?

Binary Search Tree (BST)

Consider Keys = [10, 20, 30] with search frequency = [3, 2, 5], resp.



No of Comparison
= (3+4+15) = 22

19

18

18

17

Optimal Binary Search Tree

OBST Problem

Given:

Keys: $k_1, k_2 \ldots, k_n$ sorted in increasing order.

Probabilities: Each key k_i has a probability p_i of being accessed (search frequency).

Goal: Build a BST where the total search cost (expected search time) is minimized.

Understanding Cost in OBST

Search Cost: The search cost of a node in a BST is proportional to its level(depth).

Expected Search Cost: Assuming only successful search

$$Expected\ Cost = \sum_{i=1}^{n} p_i * level(k_i)$$

DP Approach to OBST

Dynamic Programming can be used to find the OBST by breaking down the problem into smaller subproblems, which are combined to obtain the final solution.

Steps:

1. Define Subproblems:

Let C[i][j] represent the minimum search cost for the subtree containing keys ki to kj.

W[i][j] is the sum of probabilities from 'i' to 'j'. $W[i][j] = \sum_{k=i}^{j} p_k$

R[i][j] stores the root of the optimal subtree containing keys ki to kj.

DP Approach to OBST

2. Base Case Initialization:

For single nodes (When i==j): C$[i][j] = \ p_i$ and $\mathrm{R}[i][i] = i$

For nodes beyond the range (i.e., empty subtree): C$[i][i-1] = 0$

DP Approach to OBST

3. **Recursive Formula to Compute Optimal Cost:**

For each range of keys k_i to k_j (for all $j \geq i$):

$$C[i][j] = \min_{r=i\ to\ j} \{ C[i][r-1] + C[r+1][j] + W[i][j] \}$$

Explanation: Choose each key k_r as the root, then recursively find the cost of left and right subtrees.

Add $W[i][j]$ to account for the cumulative search cost when all nodes from i to j are in subtree.

DP Approach to OBST

4. Constructing the DP Table:

Fill $\mathrm{C}[i][j]$ in increasing order of length $L = j - i + 1$

For each length L compute $\mathrm{C}[i][j]$ for all valid i and j.

Keep track of the root in $\mathrm{R}[i][j]$ for each subtree to construct the Optimal BST.

5. Extract the Result:

The minimum cost of the OBST for all keys is stored in $\mathrm{C}[1][n]$

Example

Keys (k_i)	10	20	30
Frequency or Probabilities (p_i)	4	2	6

Algorithm

Algorithm OBST(keys, p)

```
n = length(keys)

// Initialize DP tables C, W, and R
for i = 1 to n do
    C[i][i] = p[i]          // Cost of a single key as its own subtree
    W[i][i] = p[i]          // Weight is the probability itself
    R[i][i] = i             // Root of a single key is the key itself
    C[i][i-1] = 0           // Base case for empty subtree
```

```
// Fill DP tables for increasing lengths of subtrees
for L = 2 to n do                    // L is the length of the subtree
   for i = 1 to n - L + 1 do      // Start of the subtree
      j = i + L - 1              // End of the subtree

      // Compute W[i][j] as the sum of probabilities from i to j
      W[i][j] = W[i][j-1] + p[j]

      // Initialize C[i][j] with a large value for comparison
      C[i][j] = INF

      // Calculate the minimum cost for each possible root in the range i to j
      for r = i to j do            // Try each key as the root
         cost = C[i][r-1] + C[r+1][j] + W[i][j]
         if cost < C[i][j] then
            C[i][j] = cost       // Update minimum cost for this range
            R[i][j] = r          // Store the root of the optimal subtree

// The minimum cost of the entire OBST is stored in C[1][n]
return C[1][n], R
```

Complexity Analysis

Time Complexity: O (n^3)

Nested loops for length and root choice.

Space Complexity: O (n^2)

for C[i][j], W[i][j], R[i][j]

"Each of your actions will have an impact on your future.

Thank you