Today's p

- LU factorization
- Reference: [Solomon] Chapter 3.5

Motivat

- In many applications, we need to solve a s
  equations:

$$x^{1)} = b^{1)}, \quad x^{2)} = b^{2)},$$

where the coefficient matrices are the sam

- Assume A is nonsingular.
- Gaussian elimination for the r systems has
- Can we do better?

Motivat

- Idea: preprocessing A (writing A in a nice f
A to solve the systems $x^{\,1)} = b^{\,1)}, \quad x^{\,2)}$

- What form is nice? $\begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix}$

  - Factorize A as a lower-triangular matrix

  - A system in an upper-triangular matrix c
  substitution in time $O\ n^2)$ $<< n^3$

  - Similarly, a system in a lower-triangular
  substitution in time $O\ n^2)$ $\begin{pmatrix} * & & 0 \\ \vdots & \ddots & \\ * & \cdots & * \end{pmatrix}$
  $<< n^3$

Motivat

- Idea of solving $x^{1)} = b^{1)}, \quad x^{2)} = b^{2)},$
  - Factorize A = LU where

    LU -

    - L is a lower-triangular matrix
    - U is an upper-triangular matrix
  - Solve $LUx^{1)} = b^{1)}, LUx^{2)} = b^{2)}, \ldots$

    runtime $O(n^2)$

    total time $O(r \cdot n^2)$

Solve LU

- Since A is nonsingular => L, U must be no
- Solving LUx = b
  - Solve Ly = b in time $O\ n^{2})$
  - Solve Ux = y in time $O\ n^{2})$
- Total time: $O\ n^{2})$
- x satisfies Ax = b: $x = LUx = Ly = b$

LU factorization via Gau

- Assume no pivoting (that is, no row permut
- Forward-substitution converts A to an uppe
  - $M_k M_{k-1} \cdots M_2 M_1 \quad = U$
  - U is an upper-triangular matrix
  - Each $M_i \in \mathbb{R}^{n\ \ n}$ encodes a row operatio matrix or an elimination matrix
  - Each M_i is nonsingular.

LU factorization via Gau

$M_k M_{k-1} \cdots M_1 A = U$

- Claim: $= M_1^{-1} M_2^{-1} \cdots M_k^{-1} U$

$\underbrace{M_k^{-1} M_k}_{I} M_{k-1} \cdots M_1 A = M_k^{-1} U$

$\Rightarrow M_{k-1} \cdots M_1 A = M_k^{-1} U$

$\Rightarrow \underbrace{M_{k-1}^{-1} M_{k-1}}_{I} \cdots M_1 A = M_{k-1}^{-1}$

$\Rightarrow M_{k-2} \cdots M_1 A = M_{k-1}^{-1} M$

$\vdots$

$\Rightarrow A = \underbrace{(M_1^{-1} M_2^{-1} \cdots M_k^{-1})}_{\hat{=} L} U =$

LU factorization via Gau

- Claim: $= M_1^{-1} M_2^{-1} \cdots M_k^{-1} U$
- Let $L := M_1^{-1} M_2^{-1} \cdots M_k^{-1}$.
- Claim: L is a lower-triangular matrix. $\Rightarrow$
  - Prove this claim by the following two clai
  - Claim 1: Each M_i^{-1} is a lower-triangular m
    for $i = 1, 2, \cdots,$
  - Claim 2: The multiplication of two lower-tr
    triangular matrix.

LU factorization via Ga

- Claim 1: Each M_i^{-1} is a lower-triangular m

Pf. ① M_i is a row scal

Suppose $M_i A$ scales the

$$M_i = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & c & \\ & 0 & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}$$ jth diag

$$M_i A = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & c & \\ & 0 & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots \\ \vdots & \\ a_{m1} & \cdots \end{pmatrix}$$

$$M_i^{-1} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \frac{1}{c} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

lower-

② M_i is an elimination m

$$\begin{array}{r} \\ j\text{th row} \leftarrow \\ l\text{th row} \leftarrow \\ \\ \\ \end{array} \left(\begin{array}{ccccc|c} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & & \vdots & \vdots \\ 0 & * & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & & \\ 0 & * & * & \cdots & * & * \end{array}\right)$$

$l > j$

$$M_i = I + c\, e_l e_j^T$$

$$M_i^{-1} = I - c e_l e_j^T$$

$$M_i M_i^{-1} = (I + c e_l e_j^T)(I - c e$$

$$= I + c^2 e_l \underbrace{(e_j^T e_l)}_{=0} e_j^T$$

$(\because$

$$e_l e_j^T = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (0 \cdots 0 \, 1 \, 0 \cdots \cdots 0)$$

$l > j$

lth entry

jth entry

lt

ro

$$I - c e_l e_j^T = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ c \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{array}{cc} & \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 1 & -c & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \\ \ell\text{th row} & \\ & j\text{th col.} \end{array}$$

LU factorization via Ga

- Claim 2: The multiplication of two lower-tr
  triangular matrix.

Pf. $A, B \in \mathbb{R}^{n \times n}$, lower

Let $C = AB \in \mathbb{R}^{n \times n}$

$$A = \begin{pmatrix} a_{11} & \cdots & & a_{1n} \\ \vdots & a_{22} & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \; 0 \qquad B =$$

$$C = \begin{pmatrix} c_{11} & \cdots & & c_{1n} \\ \vdots & c_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ c_{n1} & \cdots & & c_{nn} \end{pmatrix}$$

went to

$$c_{ij} = (\underline{a_{i1}} \cdots \underline{\underline{a_{ii}}} \; \underbrace{\boxed{\underline{\underline{a_{i,i+1}}} \cdots}}_{} $$

$$= 0$$

$\Rightarrow$ C is lower-triangular

LU factorization via Gau

- What if we need partial pivoting during Ga
- Partial pivoting -> row permutation -> mul
- PA = LU where P is a permutation matrix
- We solve PA = Pb, that is, LU x = Pb in time

Implementing LU

$O(n^3)$

- Gaussian elimination $M_k M_{k-1} \cdots M_2 M_1 \quad =$
- Construct $L = M_1^{-1} M_2^{-1} \cdots M_k^{-1}$ in $O\ n^3)$
- Runtime for LU factorization: $O\ n^3)$
- Total runtime for solving $\quad x\ ^{1)} = b\ ^{1)}, \quad x$

$$O\ n^3) + O\ r \cdot$$

construct $L = M_1^{-1} M_2^{-1} \cdots M_k^{-1}$

M_1^{-1} ↓ row scaling matrix

$\underbrace{M_2^{-1} \cdots M_k^{-1}}$ elimination for the

multiply 2 matrices for eliminat

$$(I - c_l e_l e_i^T)(I - c_p e_p e_i^T)$$

$$= I - c_l e_l e_i^T - c_p e_p e_i^T$$

$$+ c_l c_p e_l \underbrace{(e_i^T e_p)}_{=0} e_i^T$$

$$= I - \underline{c_l e_l e_i^T} - \underline{c_p e_p e_i^T}$$

$$\begin{array}{cc} & \\ & \\ l\text{th row} & \\ & \\ & \end{array}\begin{pmatrix} 0 & & & O \\ \vdots & \ddots & & \\ & & \ddots & \\ & c_l & & \\ 0 & & & 0 \end{pmatrix} \begin{array}{cc} & \\ & \\ p\text{th row} & \\ & \end{array}\begin{pmatrix} 0 & & & O \\ \vdots & \ddots & & \\ & & \ddots & \\ 0 & c_p & \cdots\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} \text{ith col.} \qquad\qquad \text{ith col.} \end{array}$$

$$= \begin{array}{c} \\ \\ l\text{th row} \\ p\text{th row} \end{array}\begin{pmatrix} 1 & & & O \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ & -c_l & & \\ 0 & -c_p & 0 & 1 \end{pmatrix}$$

$$\text{ith col}$$

multiply all the elimination matrices for the ith variable

$O(n)$

multiplying M to another

total time for n variables:

Implementing LU

- For any nonsingular diagonal matrix $D \in$
alternative LU factorization.

- We can choose D such that all the diagon

- Store L,U using a single $n \quad n$ matrix: