Binary Test

New Team Binary

October 28, 2024

1 Version 1

Suppose we have two particles 1 and 2. Their masses, positions and velocities are m_1 and m_2, $\vec{r}_1$ and $\vec{r}_2$, $\vec{v}_1$ and $\vec{v}_1$. The relative position and velocity are $\vec{r} = \vec{r}_1 - \vec{r}_2$ $\vec{v} = \vec{v}_1 - \vec{v}_2$.

We calculate their reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1}$$

The position of the center of mass is

$$\vec{R}_{\mathrm{C}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}. \tag{2}$$

We choose the coordinate where the center of mass is static. The total energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}. \tag{3}$$

Here $v = |\vec{v}|$ and $r = |\vec{r}|$.

The angular momentum is

$$L = \vec{r} \times \mu \vec{v}. \tag{4}$$

The major axis is

$$a = -\frac{G m_1 m_2}{2E}. \tag{5}$$

The period is

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}. \tag{6}$$

The eccentricity is

$$e = \sqrt{1 + \frac{2EL^2}{\mu (G m_1 m_2)^2}}. \tag{7}$$

The maximum and minimum distance between the two particles are $r_{\max} = a(1+e)$ and $r_{\min} = a(1-e)$.

The initial conditions we choose are as follows: $m_1 = 10^6 M_\odot$,$m_2 = 10^6 M_\odot$; $\vec{r}_1 = (0,0,0)$, $\vec{r}_2 = (10^{-5}\,\mathrm{kpc}, 0, 0)$; $\vec{v}_1 = (0,0,0)$, $\vec{v}_2 = (0, 10^{-6}\,\mathrm{kpc/s}, 0)$.

2 Version 2

We generate masses for two particles as m_1 and m_2, relative distance r, the eccentricity $e = \frac{c}{a} \in [0.10, 0.40)$. Here we choose the coordinate at which the center of mass locates at $(0, 0, 0)$,

$$\vec{R}_\text{C} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{8}$$
$$= 0 \tag{9}$$

Thus,

$$\vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1, \tag{10}$$

and

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{11}$$
$$= \frac{m_1 + m_2}{m_2}\vec{r}_1. \tag{12}$$

Thus,

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}, \tag{13}$$
$$\vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}. \tag{14}$$

If we make C.O.M. (center of mass) to be static,

$$\vec{V}_\text{C} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \tag{15}$$
$$= 0. \tag{16}$$

Similarly,

$$\vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1, \tag{17}$$

and

$$\vec{v} = \vec{v}_1 - \vec{v}_2 \tag{18}$$
$$= \frac{m_1 + m_2}{m_2}\vec{v}_1. \tag{19}$$

Thus,

$$\vec{v}_1 = \frac{m_2}{m_1 + m_2}\vec{v}, \tag{20}$$
$$\vec{v}_2 = -\frac{m_1}{m_1 + m_2}\vec{v}. \tag{21}$$

According to Eq. (3)) and Eq. (7), we have

$$E = \frac{1}{2}\mu v^2 - \frac{Gm_1m_2}{r} \tag{22}$$
$$= \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}v^2 - \frac{Gm_1m_2}{r}, \tag{23}$$

and

$$e^2 = 1 + \frac{2EL^2}{\mu(Gm_1m_2)^2}. \tag{24}$$

Suppose $\vec{r} \perp \vec{v}$ initially which means the two particles are separating from each other most or least, we can derive total energy from eccentricity,

$$\begin{aligned}
E &= \frac{\mu\left(Gm_1m_2\right)^2\left(e^2-1\right)}{2L^2} &(25)\\
&= \frac{\mu\left(Gm_1m_2\right)^2\left(e^2-1\right)}{2(r\mu v)^2} &(26)\\
&= \frac{(Gm_1m_2)^2(e^2-1)}{2\mu(rv)^2} &(27)\\
&= \frac{G^2m_1m_2(m_1+m_2)(e^2-1)}{2(rv)^2}. &(28)
\end{aligned}$$

Combine Eq. (23) and Eq. (28), we have

$$\frac{1}{2}\frac{m_1m_2}{m_1+m_2}v^2 - \frac{Gm_1m_2}{r} = \frac{G^2m_1m_2(m_1+m_2)(e^2-1)}{2(rv)^2}, \tag{29}$$

$$\begin{aligned}
\frac{1}{2}\frac{m_1m_2}{m_1+m_2}v^4 - \frac{Gm_1m_2}{r}v^2 - \frac{G^2m_1m_2(m_1+m_2)(e^2-1)}{2r^2} &= 0, &(30)\\
v^4 - \frac{2G(m_1+m_2)}{r}v^2 - \frac{G^2(m_1+m_2)^2(e^2-1)}{r^2} &= 0. &(31)
\end{aligned}$$

$$\begin{aligned}
v^2 &= \frac{\frac{2G(m_1+m_2)}{r} \pm \sqrt{\left[\frac{2G(m_1+m_2)}{r}\right]^2 + 4\frac{G^2(m_1+m_2)^2(e^2-1)}{r^2}}}{2} &(32)\\
&= \frac{G(m_1+m_2)}{r}(1\pm e) &(33)
\end{aligned}$$

Here, '+' and '−' correspond the smallest and largest initial r respectively.

For given m_1, m_2, r and e, here we choose

$$v = \sqrt{\frac{G(m_1+m_2)}{r}(1-e)}. \tag{34}$$

Then total energy E is determined by Eq. (28) and v. For simplicity, we set positions and velocities at $x-y$ plane. Then

$$\begin{aligned}
\vec{r}_1 &= (\frac{m_2}{m_1+m_2}r,\ 0,\ 0), &(35)\\
\vec{r}_2 &= (-\frac{m_1}{m_1+m_2}r,\ 0,\ 0), &(36)\\
\vec{v}_1 &= (0,\ -\frac{m_2}{m_1+m_2}v,\ 0), &(37)\\
\vec{v}_2 &= (0,\ \frac{m_1}{m_1+m_2}v,\ 0). &(38)
\end{aligned}$$