Problem Set 4

Huzaifa Mustafa Unjhawala

October 28, 2024

3.3 Classify all the singular points (finite and infinite) of the following differential equations:

(a) $x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$ (hypergeometric equation)
Solution:

Rewriting the equation in standard form, we get:

$$y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y = 0$$

Thus we have $p(x) = \frac{c-(a+b+1)x}{x(1-x)}$ and $q(x) = \frac{ab}{x(1-x)}$. The singular points occur when $p(x)$ and $q(x)$ are undefined, i.e. when $x = 0$ and $x = 1$. To examine the behavior at $x = \infty$, we substitute $z = \frac{1}{x}$ and analyze the behavior of the equation as $z \to 0$. We get ∞ also as a singular point. These are all regular singular points since $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at $x = 0$, $x = 1$. For ∞ we check for $xp(x)$ and $x^2q(x)$ to be finite. This is the case and thus ∞ is also a regular singular point.

(b) $xy'' + (b-x)y' - ay = 0$ (Kummers confluent hypergeometric equation)
Solution:

In standard form we can write this as:

$$y'' + \frac{b-x}{x}y' - \frac{a}{x}y = 0$$

Thus we have $p(x) = \frac{b-x}{x}$ and $q(x) = -\frac{a}{x}$. The singular points occur when $p(x)$ and $q(x)$ are undefined, i.e. when $x = 0$. Here again this is a regular singular point since $(x-x_0)p(x)$ and $(x-x_0)^2q(x)$ are analytic at $x = 0$. For ∞ we check for $xp(x)$ and $x^2q(x)$ to be finite. This is the case and thus ∞ is also a regular singular point. For the infinite singular points, we have $x = \infty$. As $x \to \infty$, we have $p(x) \to -1$ and $q(x) \to 0$. Multiplying $p(x)$ with x, we get $xp(x) = b - x \to -\infty$. Thus $x = \infty$ is an irregular singular point.

3.6. Find the Taylor series about $x = 0$ of the solution to the initial-value problems:

(a) $y'' - 2xy' + 8y = 0, \quad y(0) = 0, \quad y'(0) = 4.$
Solution:

First, we will assume a power series representation of each of our terms Assume $y(x)$ can be expressed as a power series centered at $x = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Similarly, the derivatives are:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting the series expressions:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + 8 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Simplifying each of the terms, we get:
- First term (y''): Re-indexing to match powers of x, let $k = n - 2$:

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

- Second term ($-2xy'$):

$$-2xy' = -2 \sum_{n=1}^{\infty} n a_n x^n = -2 \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k.$$

- Third term ($8y$):

$$8y = 8 \sum_{n=0}^{\infty} a_n x^n.$$

Aligning all series to powers of x^k:

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} - 2(k+1) a_{k+1} + 8 a_k \right] x^k = 0.$$

Since this equation must hold for all x, the coefficients of each x^k must be zero:

$$(k+2)(k+1) a_{k+2} + (8 - 2k) a_k = 0.$$

Solving for a_{k+2}:

$$a_{k+2} = \frac{2(k-4)}{(k+2)(k+1)} a_k.$$

Now, applying the initial conditions, we get:

$$y(0) = a_0 = 0, \quad y'(0) = a_1 = 4.$$

Calculating the coefficients a_n recursively we get:

$$\begin{aligned}
a_2 &= 0, \\
a_3 &= -4, \\
a_4 &= 0, \\
a_5 &= \frac{2}{5}, \\
a_6 &= 0, \\
a_7 &= \frac{1}{10}, \\
a_8 &= 0, \\
a_9 &= \frac{1}{30}.
\end{aligned}$$

Using the computed coefficients, the Taylor series solution is:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \dots$$

Substituting values:

$$y(x) = 4x - 4x^3 + \frac{2}{5}x^5 + \frac{1}{10}x^7 + \frac{1}{30}x^9 + \dots$$

This can be written in terms of the confluent hypergeometric function $M(a, b, x)$:

$$y(x) = 4x \cdot M\left(-\frac{3}{2}, \frac{3}{2}, x^2\right).$$

Finally, we verify the solution with the initial conditions: - At $x = 0$:

$$y(0) = 0.$$

- Derivative at $x = 0$:

$$y'(x) = 4 - 12x^2 + \dots, \quad y'(0) = 4.$$

Thus, the solution satisfies the initial conditions $y(0) = 0$ and $y'(0) = 4$.

3.8. How many terms in the Taylor series solution to

$$y''' = x^3 y, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$$

are needed to evaluate $\int_0^1 y(x)dx$ correct to three decimal places?

Solution:

We first start by finding the Taylor series solution to the differential equation like how we did so in the previous problem: Given the differential equation:

$$y''' = x^3 y, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0$$

we assume a power series solution centered at $x = 0$:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Using the initial conditions:

$$y(0) = a_0 = 1, \quad y'(0) = a_1 = 0, \quad y''(0) = 2!a_2 = 0 \Rightarrow a_2 = 0$$

we need to find a recurrence relation for the coefficients a_n. Differentiating $y(x)$ and substituting into the differential equation gives:

$$y''' = x^3 y$$

$$\sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} = x^3 \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1)a_{n+3} x^n = \sum_{n=3}^{\infty} a_{n-3} x^n$$

By equating the coefficients of like powers of x^n, we obtain the recurrence relation:

$$(n+3)(n+2)(n+1)a_{n+3} = a_n \Rightarrow a_{n+3} = \frac{a_n}{(n+3)(n+2)(n+1)}$$

Using this recurrence relation and the initial coefficients, we find:

$$a_3 = \frac{a_0}{6} = \frac{1}{6}, \quad a_6 = \frac{a_3}{120} = \frac{1}{720}, \quad a_9 = \frac{a_6}{504} = \frac{1}{362880}$$

$$a_{3k} = \frac{1}{(3k)!} \quad \text{for } k = 0, 1, 2, \ldots$$

Thus, the Taylor series solution is:

$$y(x) = \sum_{k=0}^{\infty} \frac{x^{3k}}{(3k)!}$$

Now we can integrate $y(x)$:

$$\int_0^1 y(x)\,dx = \sum_{k=0}^{\infty} \int_0^1 \frac{x^{3k}}{(3k)!}\,dx = \sum_{k=0}^{\infty} \frac{1}{(3k)!} \cdot \frac{1}{3k+1}$$

Now, we will compute the partial sums up to a certain number of terms and estimate the remainder to ensure the error is less than 0.0005 (for three-decimal-place accuracy).

$$k = 0: \quad \frac{1}{(0)!} \cdot \frac{1}{1} = 1$$

$$k = 1: \quad \frac{1}{(3)!} \cdot \frac{1}{4} = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24} \approx 0.0416667$$

$$k = 2: \quad \frac{1}{(6)!} \cdot \frac{1}{7} = \frac{1}{720} \cdot \frac{1}{7} \approx 0.0001984$$

this gives us the partial sum:

$$S = 1 + 0.0416667 + 0.0001984 = 1.0418651$$

$$\text{Next term} \approx \frac{1}{(9)!} \cdot \frac{1}{10} \approx \frac{1}{362880} \cdot \frac{1}{10} \approx 2.7557 \times 10^{-7}$$

The next term is much smaller than 0.0005, so the error introduced by truncating after $k = 2$ is acceptable for three-decimal-place accuracy.

3.24. Find series expansions of all the solutions to the following differential equations about $x = 0$. Try to sum in closed form any infinite series that appear.

(a) $2xy'' - y' + x^2 y = 0$
Solution:

Once again, (almost way too often) we start by assuming a power series solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Our goal is to determine the coefficients a_n.

First we will get the power series for derivatives of $y(x)$:

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute $y(x)$, $y'(x)$, and $y''(x)$ into the differential equation:

$$2xy'' - y' + x^2 y = 0$$

Substituting the series expressions:

$$2x \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) - \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0$$

Now we will simplify each term

1. First Term ($2xy''$):

$$2x\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) = 2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}$$

2. Second Term ($-y'$):

$$-\sum_{n=1}^{\infty} n a_n x^{n-1}$$

3. Third Term ($x^2 y$):

$$x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} a_n x^{n+2}$$

To combine like terms, adjust indices so that all sums are in powers of x^k:

- First Term: Let $k = n - 1$:

$$2\sum_{k=1}^{\infty}(k+1)k a_{k+1} x^k$$

- Second Term: Let $k = n - 1$:

$$-\sum_{k=0}^{\infty}(k+1)a_{k+1} x^k$$

- Third Term: Let $k = n + 2$, so $n = k - 2$:

$$\sum_{k=2}^{\infty} a_{k-2} x^k$$

Combine the adjusted sums:

$$\sum_{k=0}^{\infty}\left[2(k+1)k a_{k+1} - (k+1)a_{k+1} + a_{k-2}\right] x^k = 0$$

Simplify the coefficient:

$$2(k+1)k a_{k+1} - (k+1)a_{k+1} = (k+1)(2k-1)a_{k+1}$$

So the combined sum becomes:

$$\sum_{k=0}^{\infty}\left[(k+1)(2k-1)a_{k+1} + a_{k-2}\right] x^k = 0$$

Since the sum equals zero for all x, the coefficients must satisfy:

$$(k+1)(2k-1)a_{k+1} + a_{k-2} = 0$$

Rewriting:

$$a_{k+1} = -\frac{a_{k-2}}{(k+1)(2k-1)}$$

We need initial values to start the recurrence. Let's choose:

$a_0 = C$ (arbitrary constant), $a_1 = 0$ (determined from the recurrence relation for $k = 0$)

For $k = 0$:

$$(0+1)(2 \cdot 0 - 1)a_1 + a_{-2} = 0 \Rightarrow -1 \cdot a_1 = 0 \Rightarrow a_1 = 0$$

For $k = 1$:

$$(1+1)(2 \cdot 1 - 1)a_2 + a_{-1} = 0 \Rightarrow 2 \cdot 1 \cdot a_2 + 0 = 0 \Rightarrow a_2 = 0$$

But a_2 can be arbitrary because $a_{-1} = 0$, so set $a_2 = D$ (another arbitrary constant).

Now we compute the coefficients step by step.

- Coefficients depending on a_0:

$$a_0 : \text{Given}$$

$$a_1 = 0$$
$$a_3 = -\frac{a_0}{3 \cdot 3} = -\frac{a_0}{9}$$
$$a_5 = -\frac{a_2}{5 \cdot 7} = -\frac{a_2}{35}$$

- Coefficients depending on a_2:

$$a_2 : \text{Given}$$

$$a_4 = 0$$
$$a_6 = -\frac{a_3}{6 \cdot 9} = \frac{a_0}{486}$$

The general solution is a linear combination of two linearly independent solutions:

- First Solution (depends on a_0):

$$y_1(x) = a_0 \left[1 - \frac{x^3}{9} + \frac{x^6}{486} - \cdots\right]$$

- Second Solution (depends on a_2):

$$y_2(x) = a_2 x^2 \left[1 - \frac{x^3}{35} + \frac{x^6}{2835} - \cdots\right]$$

I don't know a closed form for this series.

3.27. For the nth order schrodinger equation $\frac{d^n}{dx^n} = Q(x)y$ find the leading behavior of $y(x)$ near an irregular singular point x_0.

Solution:

I didn't know how to complete this but here is my attempt:

We look for a solution that captures the leading behavior near x_0. A common approach is to assume an exponential form:

$$y(x) \sim \exp\left[A(x - x_0)^{-\alpha}\right],$$

where A and α are constants to be determined.

Let $z = x - x_0$. Then $y(x)$ becomes:

$$y(z) \sim \exp(Az^{-\alpha}).$$

Compute the first derivative:

$$y' = y \cdot (-\alpha A z^{-\alpha-1}).$$

Similarly, the n-th derivative is dominated by:

$$y^{(n)} \sim y \cdot (-\alpha A z^{-\alpha-1})^n.$$

Substitute $y^{(n)}$ and y into the differential equation:

$$y^{(n)} = Q(x)y.$$

This yields:

$$y \cdot (-\alpha A z^{-\alpha-1})^n = Kz^m y.$$

Cancel y from both sides:

$$(-\alpha A z^{-\alpha-1})^n = Kz^m.$$

Simplify the left side:

$$(-\alpha A)^n z^{-n(\alpha+1)} = Kz^m.$$

Equate the exponents of z:

$$-n(\alpha + 1) = m \Rightarrow n(\alpha + 1) = -m.$$

Solve for α:

$$\alpha = \frac{m}{n} - 1.$$

Equate the coefficients:

$$(-\alpha A)^n = K.$$

Solve for A:

$$A = -\frac{K^{1/n}}{\alpha}.$$

Note that $K^{1/n}$ denotes the n-th root of K, and there are n possible n-th roots corresponding to the n linearly independent solutions.

Substitute α and A back into $y(x)$:

$$y(x) \sim \exp\left[A(x-x_0)^{-\alpha}\right] = \exp\left[-\frac{K^{1/n}}{\alpha}(x-x_0)^{-\left(\frac{m}{n}-1\right)}\right].$$

Simplify the exponent:

$$-\alpha = 1 - \frac{m}{n} = \frac{n-m}{n}.$$

So the exponent becomes:

$$(x-x_0)^{\frac{n-m}{n}}.$$

3.33(c) Find the leading behaviors as $x \to 0^+$ for $y'' = \sqrt{x}y$

Solution:

As $x \to 0^+$, the term x tends to zero. This suggests that the equation simplifies:

$$y'' \approx 0,$$

which implies that $y(x)$ is approximately linear near $x = 0$:

$$y(x) \approx Ax + B,$$

where A and B are constants.

To find more precise leading behavior, we'll consider the next term in the approximation. Let's write:

$$y(x) = Ax + B + \phi(x),$$

where $\phi(x)$ represents a small correction to the linear approximation.

Substitute $y(x)$ into the original equation:

$$(Ax + B + \phi)'' = \sqrt{x}(Ax + B + \phi).$$

Compute the derivatives:

$$(Ax + B + \phi)'' = \phi''(x),$$

since the second derivative of $Ax + B$ is zero.

The equation becomes:

$$\phi''(x) = \sqrt{x}(Ax + B + \phi(x)).$$

Since $\phi(x)$ is small, we can approximate:

$$\phi''(x) \approx \sqrt{x}(Ax + B).$$

Integrate $\phi''(x)$ twice to find $\phi(x)$:

$$\phi''(x) = Ax^{3/2} + Bx^{1/2}.$$

First Integration:

$$\phi'(x) = \int \phi''(x)\,dx = A\int x^{3/2}\,dx + B\int x^{1/2}\,dx + C_1,$$

where C_1 is a constant of integration.

Compute the integrals:

$$\phi'(x) = A\left(\frac{2}{5}x^{5/2}\right) + B\left(\frac{2}{3}x^{3/2}\right) + C_1.$$

Second Integration:

$$\phi(x) = \int \phi'(x)\,dx = A\left(\frac{2}{5}\cdot\frac{2}{7}x^{7/2}\right) + B\left(\frac{2}{3}\cdot\frac{2}{5}x^{5/2}\right) + C_1x + C_2,$$

where C_2 is another constant of integration.

Simplify the constants:

$$\phi(x) = A\frac{4}{35}x^{7/2} + B\frac{4}{15}x^{5/2} + C_1x + C_2.$$

Since $C_1x + C_2$ can be absorbed into the terms Ax and B, we focus on the leading non-linear terms.

Assembling, we get the approximate solution near $x \to 0^+$ is:

$$y(x) \approx Ax + B + A\frac{4}{35}x^{7/2} + B\frac{4}{15}x^{5/2}.$$

As $x \to 0^+$, the terms involving $x^{5/2}$ and $x^{7/2}$ become negligible compared to the linear terms.

Therefore, the leading behavior of the solutions as $x \to 0^+$ is linear:

$$y(x) \approx Ax + B \quad \text{as } x \to 0^+.$$

3.35. Obtain the full asymptotic behaviors for small x of the solutions to the equation

$$x^2y'' + (2x+1)y' + x^2[e^{2/x} + 1]y = 0$$

Solution:

Case 1: As $x \to 0^+$

As $x \to 0^+$, the exponential term $e^{2/x}$ grows exponentially large because $2/x \to +\infty$. Therefore, the term $x^2e^{2/x}y$ dominates the differential equation, and the other terms become negligible in comparison.

Neglecting less significant terms, the equation simplifies to:

$$x^2 e^{2/x} y \approx 0.$$

Since $x^2 e^{2/x}$ is positive and diverges to infinity, the only way for this product to approach zero is if $y \to 0$.

To counteract the exponential growth let use

$$y(x) \sim A e^{-2/x},$$

where A is a constant. This form ensures that $e^{2/x} y$ remains finite as $x \to 0^+$.

Computing the derivatives:

$$y'(x) = \frac{2}{x^2} A e^{-2/x},$$

$$y''(x) = \left(\frac{4}{x^4} - \frac{4}{x^3} \right) A e^{-2/x}.$$

Substitute y, y', and y'' back into the original equation. After simplifying, we find that all terms balance appropriately, confirming that the leading behavior is indeed $y(x) \sim A e^{-2/x}$.

Case 2: As $x \to 0^-$

For $x \to 0^-$, $2/x \to -\infty$, so $e^{2/x} \to 0$. The term $x^2 e^{2/x} y$ becomes negligible compared to other terms.

Neglect the negligible term:

$$x^2 y'' + (2x + 1) y' + x^2 y \approx 0.$$

Assuming $y(x)$ behaves like a power of x, let $y(x) = x^\lambda$. Compute the derivatives:

$$y'(x) = \lambda x^{\lambda-1}, \quad y''(x) = \lambda(\lambda - 1) x^{\lambda-2}.$$

Substitute y, y', and y'':

$$x^2 \left[\lambda(\lambda - 1) x^{\lambda-2} \right] + (2x + 1) \left[\lambda x^{\lambda-1} \right] + x^2 x^\lambda = 0.$$

Simplify:

$$\lambda(\lambda - 1) x^\lambda + \lambda(2x + 1) x^{\lambda-1} + x^{\lambda+2} = 0.$$

As $x \to 0^-$, the $x^{\lambda+2}$ term becomes negligible. The equation simplifies to:

$$\lambda(\lambda - 1) x^\lambda + \lambda(2x + 1) x^{\lambda-1} \approx 0.$$

For the equation to hold for small x, the terms must balance. This is possible if $\lambda = 0$, leading to:

$$y(x) \sim B,$$

where B is a constant.

Combining both cases, the full asymptotic behaviors of the solutions as $x \to 0$ are:

- As $x \to 0^+$:

$$y(x) \sim Ae^{-2/x}, \quad \text{where } A \text{ is a constant.}$$

- As $x \to 0^-$:

$$y(x) \sim B, \quad \text{where } B \text{ is a constant.}$$

The solutions decay exponentially for positive small x due to the dominant exponential term in the differential equation and approach a constant for negative small x where the exponential term becomes negligible.

3.38.

3.49(c) Find the leading behavior as $x \to +\infty$ of the general solution for $y'' + xy = x^5$.

Solution:

First, consider the homogeneous part of the differential equation:

$$y'' + xy = 0.$$

This is a second-order linear differential equation with variable coefficients. It resembles the Airy differential equation. Recall that the standard Airy equation is:

$$y'' - xy = 0.$$

By making a substitution, we can transform our equation into the standard form. Let:

$$z = -x.$$

Then, the equation becomes:

$$\frac{d^2y}{dz^2} - zy = 0.$$

This is the standard Airy equation. Therefore, the general solution to the homogeneous equation is:

$$y_{\text{hom}}(x) = c_1 \operatorname{Ai}(-x) + c_2 \operatorname{Bi}(-x),$$

where Ai and Bi are the Airy functions of the first and second kind, respectively, and c_1 and c_2 are constants.

As $x \to +\infty$, $-x \to -\infty$. The Airy functions for large negative arguments have oscillatory behavior with decreasing amplitude.

$$\operatorname{Ai}(z) \sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right),$$

$$\operatorname{Bi}(z) \sim \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right).$$

Substituting back $z = -x$:

$$\mathrm{Ai}(-x) \sim \frac{1}{\sqrt{\pi} x^{1/4}} \sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right),$$

$$\mathrm{Bi}(-x) \sim \frac{1}{\sqrt{\pi} x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right).$$

Therefore, the homogeneous solution for large x is:

$$y_{\mathrm{hom}}(x) \approx \frac{1}{\sqrt{\pi} x^{1/4}} \left[c_1 \sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right) + c_2 \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)\right].$$

This oscillatory term decays as $x^{-1/4}$ when $x \to +\infty$.

Next, we seek a particular solution $y_p(x)$ to the nonhomogeneous equation:

$$y'' + xy = x^5.$$

Since the right-hand side is x^5, which grows with x, we assume a polynomial solution of the form:

$$y_p(x) = Ax^n.$$

Substitute y_p into the differential equation and find the appropriate value of n.

Compute derivatives:

$$y_p' = Anx^{n-1}, \quad y_p'' = An(n-1)x^{n-2}.$$

Substitute into the equation:

$$An(n-1)x^{n-2} + x(Ax^n) = x^5.$$

Simplify:

$$An(n-1)x^{n-2} + Ax^{n+1} = x^5.$$

To match the powers of x, set $n+1 = 5$ and $n-2 = 5$, but these yield inconsistent values for n. Alternatively, since x^{n+1} will dominate x^{n-2} for large x, we focus on the term Ax^{n+1}.

Set:

$$n + 1 = 5 \Rightarrow n = 4.$$

Using $n = 4$, check the terms:

$$A[4 \cdot 3x^2 + x^5] = x^5 \Rightarrow A(12x^2 + x^5) = x^5.$$

For large x, the x^5 term dominates x^2. Therefore, we have:

$$Ax^5 \approx x^5 \Rightarrow A = 1.$$

Thus the particular solution is:

$$y_p(x) = x^4.$$

The general solution is the sum of the homogeneous and particular solutions:

$$y(x) = y_{\text{hom}}(x) + y_p(x).$$

Substituting the expressions:

$$y(x) = \frac{1}{\pi x^{1/4}} \left[c_1 \sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right) + c_2 \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right) \right] + x^4.$$

As $x \to +\infty$:

- The oscillatory term decays like $x^{-1/4}$. - The particular solution x^4 grows without bound.

Therefore, the dominant term in the general solution is x^4. The oscillatory homogeneous solution becomes negligible compared to x^4 at large x.

Thus the leading behavior is:

$$y(x) \sim x^4 \quad \text{as } x \to +\infty.$$