Problem Set 3

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Problem 1: Elastic Rod with Gravity

For the vertically-oriented elastic rod under the influence of gravity, with a fixed end at $x = 0$ and a free end at $x = 1$, the equilibrium condition is given by:

$$\frac{dw(x)}{dx} = -f, \quad w(1) = 0$$

where $w(x)$ is the internal force on the rod and f is the gravitational force per unit length. The constitutive relation is:

$$w(x) = C(x)\frac{du(x)}{dx}$$

where $u(x)$ is the deformation from equilibrium in the absence of gravity, with the boundary condition $u(0) = 0$. $C(x)$ is a positive function with dimensions of force.

(a) Minimize the functional:

$$L(u, w) = \int_0^1 \frac{w^2(x)}{2C(x)}\,dx + \int_0^1 u(x)\left(\frac{dw(x)}{dx} + f(x)\right)dx.$$

(b) Show that at the minimizing w, $L(u, w)$ is equal to $-P(u)$, where $P(u)$ is the potential energy defined in class, thus demonstrating the duality principle.

Solution:

(a) First, taking term $\int_0^1 u(x)\left(\frac{dw(x)}{dx}\right)dx$ and integrating by parts, we get:

$$\int_0^1 u(x)\left(\frac{dw(x)}{dx}\right)dx = [u(x)w(x)]_0^1 - \int_0^1 w(x)\frac{du(x)}{dx}dx$$

Given the boundary conditions, the first term vanishes, and we are left with:

$$\int_0^1 u(x)\left(\frac{dw(x)}{dx}\right)dx = -\int_0^1 w(x)\frac{du(x)}{dx}dx$$

The functional then becomes:

$$L(u,w) = \int_0^1 \frac{w^2(x)}{2C(x)} - w(x)\frac{du(x)}{dx} + u(x)f(x)\,dx \tag{1}$$

Now, we take small variations in u and w, and compute the first variation of L:

$$\delta L = \int_0^1 \left(\frac{w(x)}{C(x)}\delta w(x) - \frac{du(x)}{dx}\delta w(x) - w(x)\frac{d(\delta u(x))}{dx} + \delta u(x)f(x)\right) dx$$

Again, to make use of our boundary conditions, we integrate by parts the term involving $w(x)\frac{d(\delta u(x))}{dx}$:

$$\int_0^1 w(x)\frac{d(\delta u(x))}{dx}dx = [w(x)\delta u(x)]_0^1 - \int_0^1 \frac{dw(x)}{dx}\delta u(x)dx$$

Poof, the boundary term vanishes again. Now we are left with:

$$\delta L = \int_0^1 \left(\left(\frac{w(x)}{C(x)} - \frac{du(x)}{dx}\right)\delta w(x) + \left(\frac{dw(x)}{dx} + f(x)\right)\delta u(x)\right) dx$$

For $\delta L = 0$ for all δu and δw, the coefficients of δu and δw must vanish separately. Variation with respect to w gives us the constitutive relation:

$$\frac{w(x)}{C(x)} - \frac{du(x)}{dx} = 0 \quad \Rightarrow \quad w(x) = C(x)\frac{du(x)}{dx}$$

Variation with respect to u gives us the equilibrium equation:

$$\frac{dw(x)}{dx} + f(x) = 0 \quad \Rightarrow \quad C(x)\frac{d^2u(x)}{dx^2} + f(x) = 0$$

Thus, minimizing $L(u,w)$ gives us the physics equations of the problem.

(b) We know that $P(u)$ is the potential energy of the system, which is given by:

$$P(u) = \int_0^1 \frac{1}{2}C(x)\left(\frac{du(x)}{dx}\right)^2 - u(x)f(x)dx$$

At Minimizing $w(x) = C(x)\frac{du(x)}{dx}$, we have (below equation is from Eq. 1):

$$\begin{aligned}
L(u,w) &= \int_0^1 \frac{\left(C(x)\frac{du(x)}{dx}\right)^2}{2C(x)} - C(x)\frac{du(x)}{dx}\frac{du(x)}{dx} + u(x)f(x)\,dx \\
&= \int_0^1 -\frac{1}{2}\left(C(x)\frac{du(x)}{dx}\right)^2 + u(x)f(x)dx
\end{aligned}$$

Therefore, $L(u,w) = -P(u)$, demonstrating the duality principle.

Problem 2: Minimal Surface Problem

In a 3D Cartesian coordinate system, consider a surface $z = u(x, y)$. The area of the surface is given by:

$$A = \int_S \left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| dx\, dy$$

where $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + u(x, y)\hat{k}$, and $|\mathbf{v}|$ denotes the magnitude of vector $\mathbf{v}$.

Find the equation for the minimal surface with $u = u_0(x, y)$ on the boundary, similar to the case of a soap film attached to a wire rim.

Hint: See Strang for a reference.

Solution:

First computing the individual terms $\frac{\partial \mathbf{r}}{\partial x}$ and $\frac{\partial \mathbf{r}}{\partial y}$:

$$\frac{\partial \mathbf{r}}{\partial x} = \hat{\imath} + u_x\hat{k}, \quad \frac{\partial \mathbf{r}}{\partial y} = \hat{\jmath} + u_y\hat{k}$$

Then, we can take the cross product:

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \hat{\imath}\,(u_y) - \hat{\jmath}\,(u_x) + \hat{k}\,(1)$$

The magnitude of this vector is:

$$\left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| = \sqrt{u_x^2 + u_y^2 + 1}$$

Thus, the area functional is:

$$A = \int_S \sqrt{u_x^2 + u_y^2 + 1}\, dx\, dy$$

Now, we are tasked with extremizing this functional. We can thus use the Euler-Lagrange equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = \frac{\partial F}{\partial u}$$

where $F(u_x, u_y) = \sqrt{u_x^2 + u_y^2 + 1}$. It thus does not depend on u, so $\frac{\partial F}{\partial u} = 0$. Computing each of the partial derivatives, we get:

$$\frac{\partial F}{\partial u_x} = \frac{u_x}{\sqrt{u_x^2 + u_y^2 + 1}}, \quad \frac{\partial F}{\partial u_y} = \frac{u_y}{\sqrt{u_x^2 + u_y^2 + 1}}$$

Then computing the divergence, we get:

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2 + 1}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2 + 1}}\right) = 0$$

Which is the minimal surface equation. This can be further simplified to:

$$u_{xx}(1+u_y^2)+u_{yy}(1+u_x^2)-2u_xu_yu_{xy}=0$$

Note, this solution must still satisfy the boundary conditions $u=u_0(x,y)$ on the boundary.

Problem 3: Sturm-Liouville Problems

In this problem, we review some major results regarding regular Sturm-Liouville problems. The general equation is:

$$L[y(x)]=\lambda\sigma(x)y(x), \quad L=-\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x), \quad a<x<b$$

with boundary conditions:

$$\beta_1y(a)+\beta_2y'(a)=0, \quad \beta_3y(b)+\beta_4y'(b)=0,$$

where $p(x)$, $p'(x)$, $q(x)$, and $\sigma(x)$ are real and continuous in $a\le x\le b$, and $p(x)$, $\sigma(x)>0$ in $a\le x\le b$. At least one of β_1, β_2 is nonzero, and at least one of β_3, β_4 is nonzero.

You are asked to prove the following results:

(1) The operator L together with the separated boundary conditions is symmetric.

(2) The eigenvalues λ are real.

(3) For each eigenvalue, there is only one linearly independent eigenfunction.

(4) The eigenfunctions may be chosen to be real.

(5) The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function $\sigma(x)$, i.e., for different eigenvalues $\lambda_1\neq\lambda_2$ with corresponding eigenfunctions $\phi_1(x)$, $\phi_2(x)$, we have:

$$\int_a^b \sigma(x)\phi_1(x)\phi_2(x)\,dx=0.$$

Solution:

1. We need to show, for real-valued functions u and v satisfying the boundary conditions, that:

$$\int_a^b vL[u]dx=\int_a^b uL[v]dx$$

Taking $\int_a^b uL[v]dx$, and expanding $L[v]$, we get:

$$\int_a^b uL[v]dx = \int_a^b (u(pv')' + uqv)\,dx$$

Taking integration by parts on both the terms, we get:

$$\int_a^b uL[v]dx = [upv']_a^b - \int_a^b u'pv'dx + \int_a^b uqvdx$$

Taking integration by parts again on both the terms, we get:

$$\int_a^b uL[v]dx = [upv']_a^b - [u'pv]_a^b + \int_a^b v(pu')'dx + \int_a^b uqvdx$$

$$\int_a^b uL[v]dx = \int_a^b vL[u]dx + p\,(uv' - u'v)\Big|_a^b$$

Rewriting this, we get:

$$\int_a^b (uL[v] - vL[u])\,dx = \int_a^b \frac{d}{dx}\,(p\,(uv' - u'v))\,dx$$

This gives us:

$$uL[v] - vL[u] = \frac{d}{dx}\,(p\,(uv' - u'v))$$

Now, assume that u and v satisfy the boundary conditions. Then we have $L[u] = -\lambda\sigma u$ and $L[v] = -\lambda\sigma v$. Additionally the boundary conditions can be written as $u'(a) = \frac{-\beta_1}{\beta_2}u(a)$ and $v'(a) = \frac{-\beta_1}{\beta_2}v(a)$ and similarly for b. Thus, we have:

$$uL[v] - vL[u] = \lambda\sigma\,(uv - vu) = 0$$

Thus $\int_a^b (uL[v] - vL[u])\,dx = 0$, and L is symmetric.

2. Suppose that λ is an eigenvalue and complex and $y(x)$ is a corresponding complex eigenfunction. Then, we have, $\lambda = \mu + i\nu$, and $y(x) = u(x) + iv(x)$. Taking complex conjugate of $L[y] = \lambda\sigma y$, we get:

$$L[\bar{y}] = \bar{\lambda}\sigma\bar{y}$$

Since we proved symmetry in part (a), we have:

$$\int_a^b \bar{y}L[y]dx = \int_a^b yL[\bar{y}]dx$$

Substituting $L[\bar{y}] = \bar{\lambda}\sigma\bar{y}$ and comparing both sides, we get $\bar{\lambda} = \lambda$, thus λ is real. Plugging in the complex conjugate we get:

$$\int_a^b \bar{y}\,(\lambda\sigma y)\,dx = \int_a^b y\left(\bar{\lambda}\sigma\bar{y}\right)dx$$

Now, if we apply $y\bar{y} = u^2 + v^2$ and $\sigma > 0$, we get:

$$\left(\lambda - \bar{\lambda}\right) \int_a^b \sigma y\bar{y} dx = 0$$

Since $\sigma > 0$ and $y\bar{y} > 0$, we must have $\lambda = \bar{\lambda}$, and λ is real.

3. We can prove this by contradiction. Suppose that λ has two eigenfunctions $y_1(x)$ and $y_2(x)$. Then, we have:

$$L[y_1] = -\lambda\sigma y_1, \quad L[y_2] = -\lambda\sigma y_2$$

Thus $\lambda = -\frac{L[y_1]}{y_1}$. Using this, we can write:

$$L[y_2] = -\frac{L[y_1]}{y_1} y_2$$

$$y_1 L[y_2] - y_2 L[y_1] = 0$$

By the lagrange identity we derived in part (a), we have:

$$y_1 L[y_2] - y_2 L[y_1] = \frac{d}{dx}\left(p\left(y_1 y_2' - y_2 y_1'\right)\right)$$

Therefore $\frac{d}{dx}\left(p\left(y_1 y_2' - y_2 y_1'\right)\right) = 0$, and $p\left(y_1 y_2' - y_2 y_1'\right) = C$, for some constant C. For the boundary conditions to be satisfied, we must have $C = 0$, and thus $y_1 y_2' = y_2 y_1'$. This implies that y_1 and y_2 are linearly dependent. Thus a single eigenvalue has a unique linearly independent eigenfunction.

4. We can prove this by contradiction. Suppose that $y(x)$ is a complex eigenfunction corresponding to eigenvalue λ. Then, we have:

$$L[y] = -\lambda\sigma y$$

Writing $y(x) = u(x) + iv(x)$, we get:

$$L[y] = L[u] + iL[v] = -\lambda\sigma(u + iv)$$

Equating real and imaginary parts, we get:

$$L[u] = -\lambda\sigma u, \quad L[v] = -\lambda\sigma v$$

Thus the real and complex parts of a complex eigenfunction are both eigenfunctions that satisfy Strum-Liouville. Now we show whether they satisfy the boundary conditions:

$$\beta_1(u(a) + iv(a)) + \beta_2(u'(a) + iv'(a)) = 0$$

$$\beta_1(u(b) + iv(b)) + \beta_2(u'(b) + iv'(b)) = 0$$

Hence:

$$\beta_1 u(a) + \beta_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0$$

$$\beta_3 v(a) + \beta_4 v'(a) = 0, \quad \beta_3 v(b) + \beta_4 v'(b) = 0$$

Thus $u(x)$ and $v(x)$ satisfy also the boundary conditions. They also have the same eigenvalue λ, and thus they must be linearly dependent. Therefore, $v = cu$. Then, $y = u + icu = (1 + ic)u = c_0 u$. Thus $y(x)$ is real which contradicts our assumption. Therefore, $y(x)$ must be real.

5. Lets consider two eigenfunctions $y_1(x)$ and $y_2(x)$ corresponding to eigenvalues λ_1 and λ_2. Then, we have:

$$L[y_1] = -\lambda_1 \sigma y_1, \quad L[y_2] = -\lambda_2 \sigma y_2$$

Multiplying the first equation by y_2 and the second by y_1 and subtracting, we get:

$$y_1 L[y_2] - y_2 L[y_1] = \lambda_1 \sigma y_1 y_2 - \lambda_2 \sigma y_2 y_1 = (\lambda_1 - \lambda_2)\sigma y_1 y_2$$

Using derivation from part (a), we have:

$$0 = (\lambda_1 - \lambda_2) \int_a^b \sigma y_1 y_2 dx$$

Since $\lambda_1 \neq \lambda_2$, we must have $\int_a^b \sigma y_1 y_2 dx = 0$. Therefore the eigenfunctions are orthogonal with respect to the weight function $\sigma(x)$.