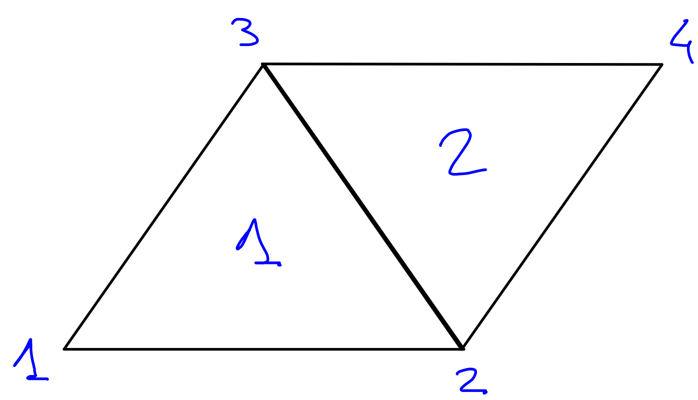
LECTURE - 18

EXAMPLE ON LINEAR ELASTICITY ELEMENTS



MESH

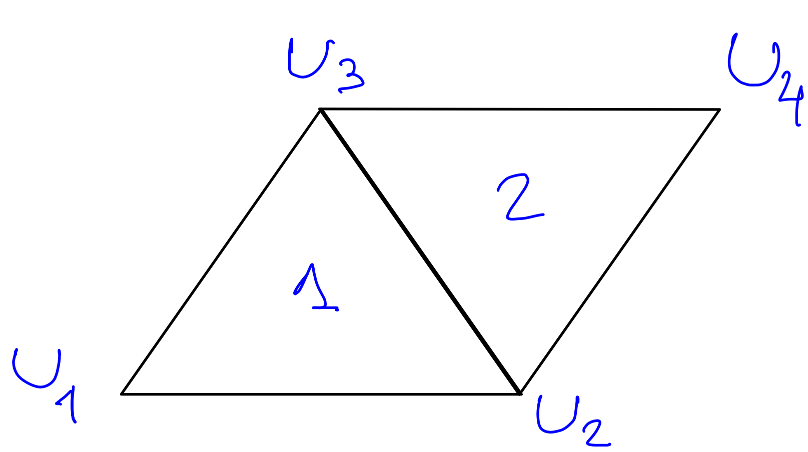
SCALAR:

$$W_h = \text{span}\{N_1, N_2, N_3, N_4\}$$

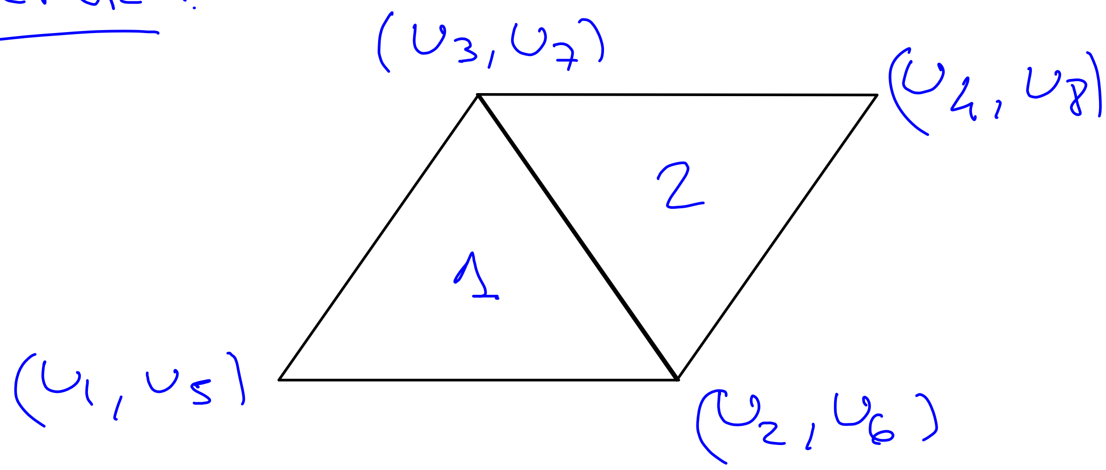
$$LG = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 3 \end{bmatrix}$$

$$\mathcal{V}^e = \text{span}\{N_1^e, N_2^e, N_3^e\} \qquad e = 1, 2$$

$$u_h = U_1 N_1 + U_2 N_2 + U_3 N_3 + U_4 N_4$$



VECTOR :



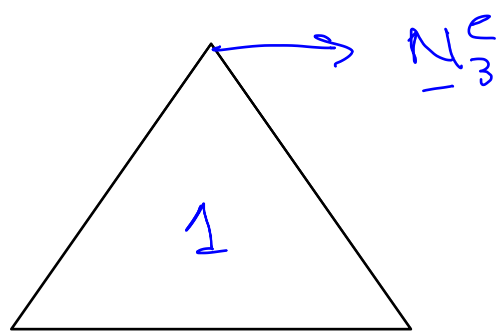
$$\underline{W}_u = \left\{ \begin{pmatrix} N_1 \\ 0 \end{pmatrix}, \begin{pmatrix} N_2 \\ 0 \end{pmatrix}, \begin{pmatrix} N_3 \\ 0 \end{pmatrix}, \begin{pmatrix} N_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ N_1 \end{pmatrix}, \begin{pmatrix} 0 \\ N_2 \end{pmatrix}, \begin{pmatrix} 0 \\ N_3 \end{pmatrix}, \begin{pmatrix} 0 \\ N_4 \end{pmatrix} \right\}$$

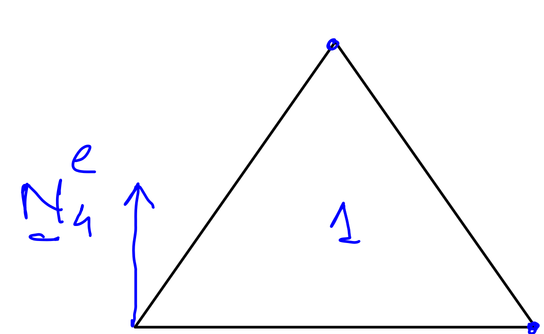
$$\underline{P}^e = \left\{ \begin{pmatrix} N_1^e \\ 0 \end{pmatrix}, \begin{pmatrix} N_2^e \\ 0 \end{pmatrix}, \begin{pmatrix} N_3^e \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ N_1^e \end{pmatrix}, \begin{pmatrix} 0 \\ N_2^e \end{pmatrix}, \begin{pmatrix} 0 \\ N_3^e \end{pmatrix} \right\}$$

$\underline{N}_1^e$

$\underline{N}_4^e$

$$LG = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 3 \\ 1+4 & 2+4 \\ 2+4 & 4+4 \\ 3+4 & 3+4 \end{bmatrix}$$





$$\sigma(\varepsilon) = \lambda \, tr(\varepsilon) \, I + 2\mu \, \varepsilon$$

$$a^e(\underline{N}_a, \underline{N}_b) = \int_{\Omega^e} \sigma(\varepsilon(\nabla \underline{N}_a)) : \varepsilon(\nabla \underline{N}_b) \, d\Omega$$

$$l^e(\underline{N}_b) = \int_{\Omega} \underline{b} \cdot \underline{N}_b \, d\Omega$$

P_2 TRIANGULAR ELEMENTS

OUTLINE:

- TOOLS THAT WE NEED FOR CP-3 (HOW TO DO)
  - P_2 TRIANGLE
  - QUADRATURE OVER P_2 TRIANGLE
  - ELEMENT COMPUTATION WITH QUADRATURE.

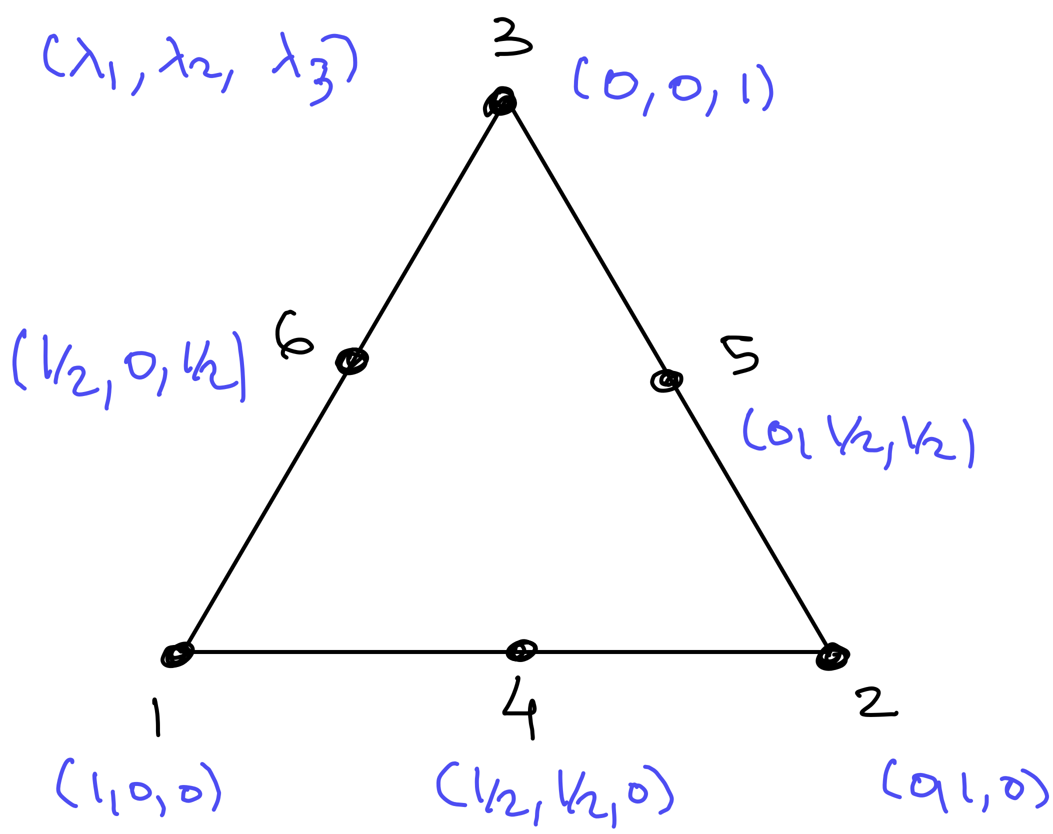
TOOLS:

- P_2 TRIANGLE $(\Omega_e, \mathcal{N}_e)$

$\Omega_e \equiv$ TRIANGLE

$\mathcal{N}_e \equiv$ LAGRANGE ELEMENT
WITH 6 NODES
AND QUADRATIC POLYNOMIALS

$$\mathcal{P}_e = \text{span}(\mathcal{N}_e) = \mathbb{P}_2(\Omega_e)$$



3 NODES AT THE VERTICES

3 NODES AT THE MIDPOINTS OF THE EDGES

$\underline{x}_a \equiv$ POSITION OF NODE a

HOW DO WE BUILD THE SHAPE FUNCTIONS?

$\lambda_i(x_1, x_2)$ ARE AFFINE POLYNOMIALS

$\Rightarrow$ $\lambda_i \lambda_j$ ARE QUADRATIC!

$$N_a(\lambda_1, \lambda_2, \lambda_3) = 2\lambda_a(\lambda_a - 1/2) \qquad a = 1, 2, 3$$

$$N_4(\lambda_1, \lambda_2, \lambda_3) = 4\lambda_1 \lambda_2$$

$$N_5(\lambda_1, \lambda_2, \lambda_3) = 4\lambda_2 \lambda_3$$

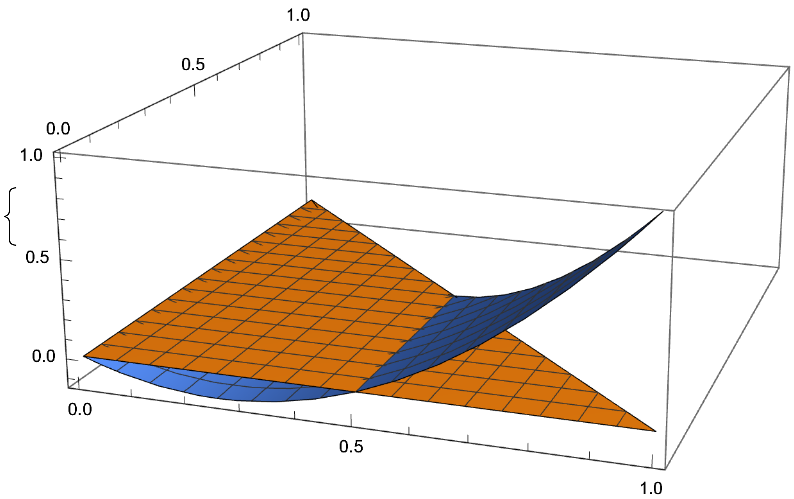
$$N_6(\lambda_1, \lambda_2, \lambda_3) = 4\lambda_3 \lambda_1$$

REPLACING BY THE FORMULAS WE HAVE FOR $\lambda_1, \lambda_2, \lambda_3$ IN TERMS OF (x_1, x_2).

NOTICE THAT $N^e_a(x^e_b) = \delta_{ab}$.

SEE ATTACHMENTS FOR PLOTS.

Out[•]=



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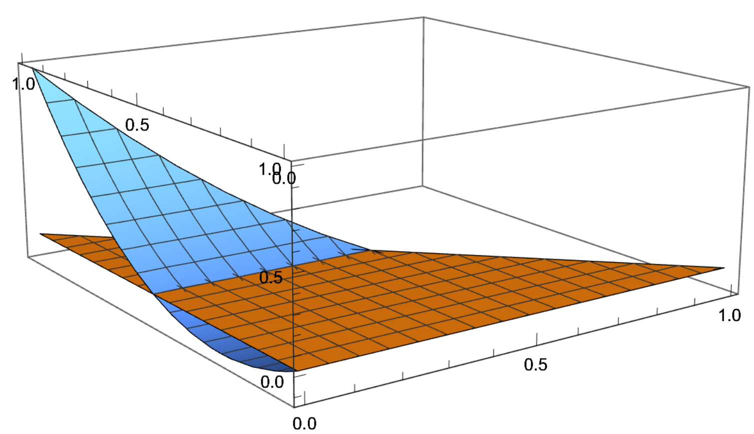

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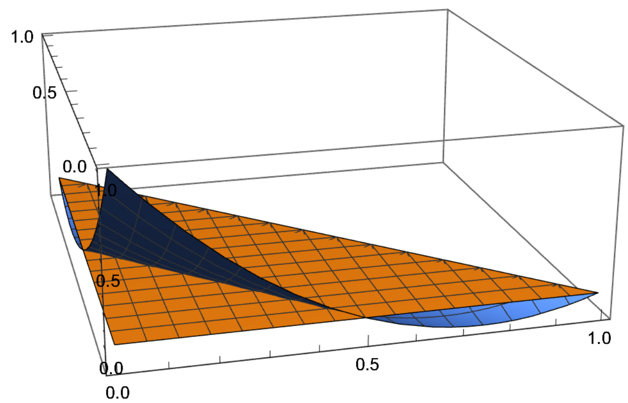
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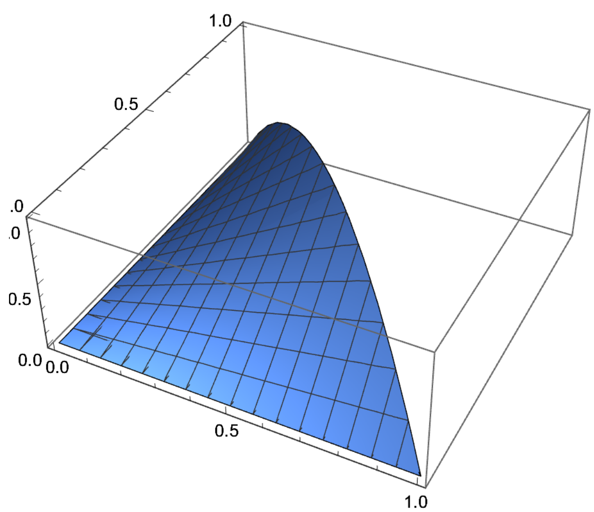
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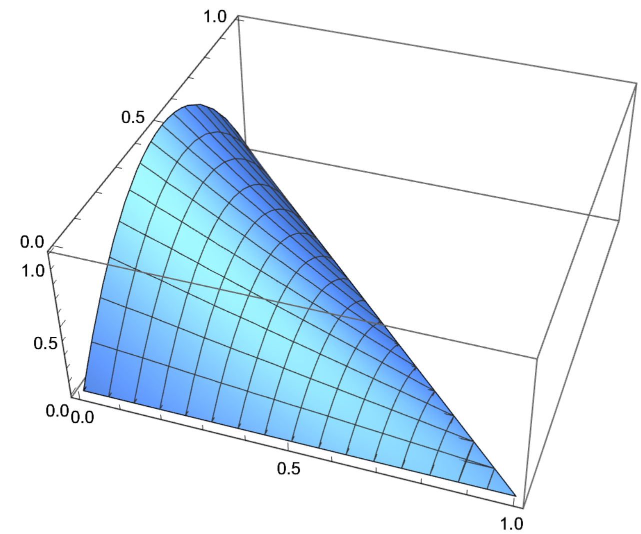
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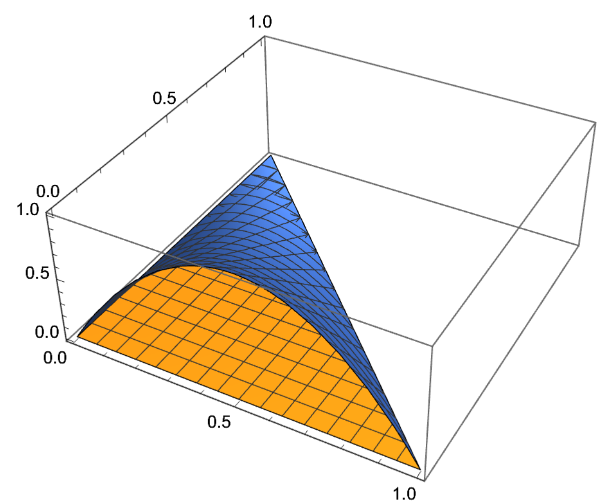
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AND ∇N_a? BY THE CHAIN RULE

NOTICE THAT

$$N_a(x_1, x_2) = \hat{N}_a(\lambda_1(x_1,x_2), \lambda_2(x_1,x_2), \lambda_3(x_1,x_2))$$

THEN

$$\frac{\partial N_a}{\partial x_i} = \sum_{\alpha=1}^{3} \frac{\partial \hat{N}_a}{\partial \lambda_\alpha} \frac{\partial \lambda_\alpha}{\partial x_i}$$ CHAIN RULE

BUT $\frac{\partial \lambda_\alpha}{\partial x_i} = dN_{i\alpha}$ IN THE NOTES (GRADIENTS OF P_1 SHAPE FUNCTIONS)

SO, IN MATRIX FORM

$$\boxed{\nabla N_a = \nabla \hat{N}_a \cdot dN^T}$$

$$dN = \begin{bmatrix} x_2^2 - x_2^3 & x_2^3 - x_2^1 & x_2^1 - x_2^2 \\ x_1^3 - x_1^2 & x_1^1 - x_1^3 & x_1^2 - x_1^1 \end{bmatrix} \frac{1}{2A^e}$$

CONSTANT IN Ω^e

THEN, FOR EXAMPLE,

$$\nabla N_1 = \begin{bmatrix} \frac{\partial N_1}{\partial \lambda_1} & \frac{\partial N_1}{\partial \lambda_2} & \frac{\partial N_1}{\partial \lambda_3} \end{bmatrix} dN^T$$

$$= \begin{bmatrix} 4\lambda_1 - 1 & 0 & 0 \end{bmatrix} dN^T$$

$$= (4\lambda_1 - 1) \begin{bmatrix} x_2^2 - x_2^3 \\ x_1^3 - x_1^2 \end{bmatrix}$$

- QUADRATURE OVER P_2-TRIANGLE

INTEGRALS THROUGH QUADRATURE (Ref. P. Pinsky Ch. 5.7)

"QUADRATURE": THE PROCESS OF COMPUTING AREAS BY CONSTRUCTING A RECTANGLE BY CUTTING AND PASTING A SHAPE (e.g., A CIRCLE).

THIS IS WHERE THE PHRASE "SQUARING THE CIRCLE" COMES FROM, AS A METAPHOR FOR AN IMPOSSIBLE TASK.

HERE WE WILL USE QUADRATURE TO APPROXIMATE THE VALUES OF INTEGRAL, OR IN MANY CASES COMPUTE THEM EXACTLY.

IDEA:

$$\int_\Omega g(x)\,dx \approx \sum_{q=1}^{n_q} w_q\, g(x_q)$$

$\{x_1, \dots, x_q\} \subset \Omega$ $\equiv$ COORDINATES OF QUADRATURE POINTS.

$\{w_1, \dots, w_q\}$ $\equiv$ QUADRATURE WEIGHTS.

$n_q \in \mathbb{N}$ $\equiv$ NUMBER OF QUADRATURE POINTS.

$\bigcup_{q=1}^{n_q} (x_q, w_q)$ $\equiv$ QUADRATURE RULE

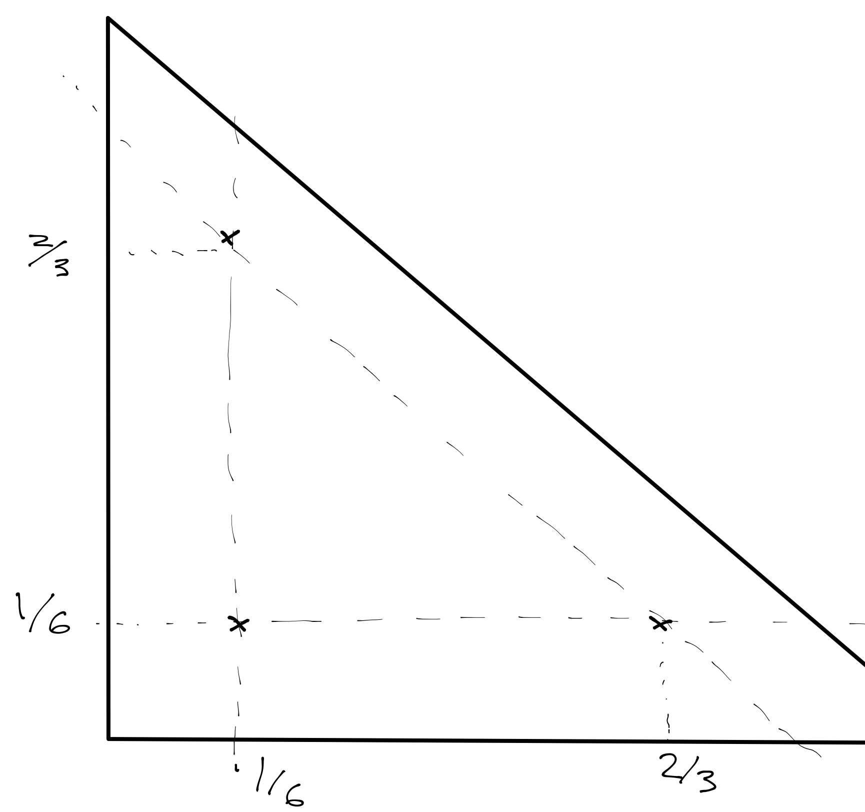
FOR A P_2 TRIANGLE, WE HAVE (WE'LL SEE WHY LATER)

3-POINT QUADRATURE

BARYCENTRIC COORDS (λ^q)	WEIGHTS
1/6, 1/6, 2/3	1/3 S
1/6, 2/3, 1/6	1/3 S
2/3, 1/6, 1/6	1/3 S

S = AREA TRIANGLE

$$x_q = \sum_{a=1}^{3} \lambda_a X^a$$



EXAMPLE:

$$K^e_{12} = \int_{\Omega^e} K(x) \nabla N^e_1(x) \cdot \nabla N^e_2(x)\, dx$$

$$\cong \sum_{q=1}^{3} w_q\, K(x_q)\, \nabla N^e_1(x_q) \cdot \nabla N^e_2(x_q)$$

PSEUDO-CODE FOR ELEMENT ARRAYS

$K^e = 0$, $F^e = 0$

FOR $q = 1, \ldots, n_q$

  FOR $a = 1, \ldots, k$

    FOR $b = 1, \ldots, k$

      $K^e_{ab} \mathrel{+}= w_q \nabla N_a(x_q) \cdot \nabla N_b(x_q)\, K(x_q)$

    END FOR

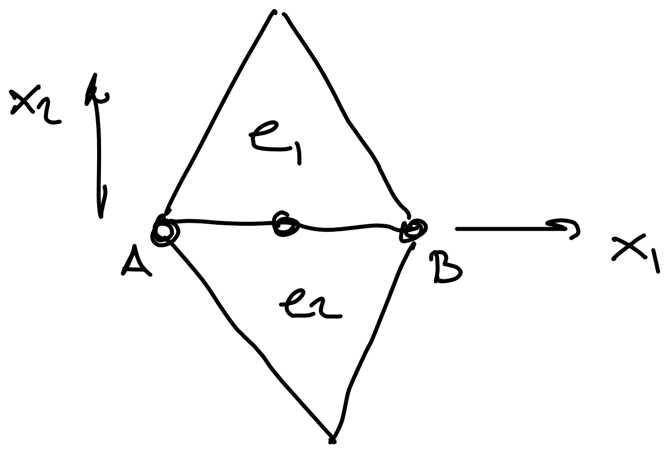
    $F^e_a \mathrel{+}= w_q\, f(x_q)\, N_a(x_q)$ $\rightsquigarrow \int_{\Omega^e} f N_a$

  END FOR

END FOR

```matlab
%% Example element computation with quadrature
function [Ke Fe]=ElementKandF(xe)
dN=[xe(2,2)-xe(2,3),xe(2,3)-xe(2,1),xe(2,1)-xe(2,2);...
    xe(1,3)-xe(1,2),xe(1,1)-xe(1,3),xe(1,2)-xe(1,1)];
Ae2=dN(2,3)*dN(1,2)-dN(1,3)*dN(2,2);
dN=dN/Ae2;
% Do not assume that the integrands are constant inside the element, and
% use quadrature
Ke=zeros(3,3);
Fe=zeros(3,1);
nq=3;
lambdaq=[1/6, 1/6, 2/3; 1/6, 2/3, 1/6; 2/3, 1/6, 1/6];
wq=[1/3, 1/3, 1/3]*Ae2/2;
for q=1:nq
    NN=lambdaq(q,:);
    xq=xe*NN';
    % As an example, say k(x)=x1;
    Ke=Ke+wq(q)*xq(1)*dN'*dN;
    % As an example, say f(x)=x1^2+x2^2
    Fe=Fe+wq(q)*(xq(1)^2+xq(2)^2)*NN';
end
end
```

• LOCAL-TO-GLOBAL MAP FOR $\mathcal{P}_2$



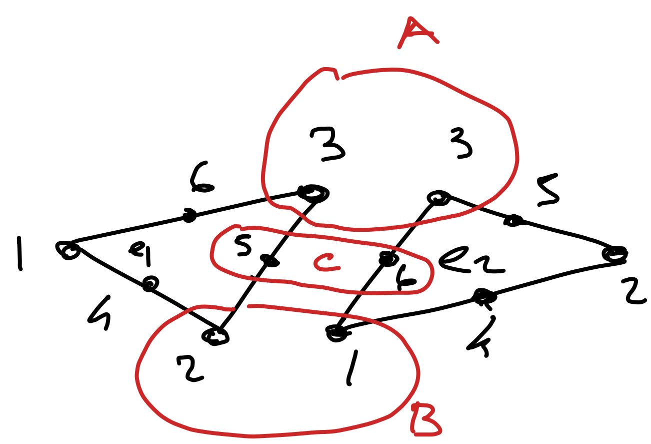
ON EDGE AB,

$$\sum_{a=1}^{6} \phi_a N_a^e(x) \qquad x \in \overline{AB}$$

is A QUADRATIC POLYNOMIAL OF x_1 ONLY, FOR $e=e_1$ AND $e=e_2$.

Since they coincide at the 3 NODES, THEY ARE THE SAME POLYNOMIAL

So



$$N_A = N_3^{e_2} \overset{\circ}{+} N_3^{e_1} + \text{others}$$

$$N_B = N_2^{e_1} \overset{\circ}{+} N_1^{e_2} + \text{others}$$

$$N_C = N_5^{e_1} \overset{\circ}{+} N_6^{e_2}$$

THEY ARE CONTINUOUS

IN A MESH OF QUADRATIC TRIANGLES, THE COORDINATES OF ALL NODES ARE PROVIDED

$X(i,A) \equiv$ COORDINATES OF ALL NODES

$LV(a,e) \equiv$ INDICES OF ALL NODES IN ELEMENT e

$$LV(:,e) = [A_1 \ A_2 \ A_3 \ A_4 \ A_5 \ A_6]^T \Rightarrow LG = LV \ !$$