

HW-6.

In this problem we would like to play with the convergence, norms, and membership of sequence of functions in different spaces. To this end, let $I = (0, \pi)$, and recall (see Appendix A in the notes) that a function $f: I \rightarrow \mathbb{R}$ is a member of the following spaces if

$$f \in L^2(I) \Leftrightarrow \|f\|_{0,2} = \left(\int_0^\pi f^2 dx \right)^{1/2} < \infty$$

$$f \in L^\infty(I) \Leftrightarrow \|f\|_{0,\infty} = \max_{x \in I} |f(x)| < \infty$$

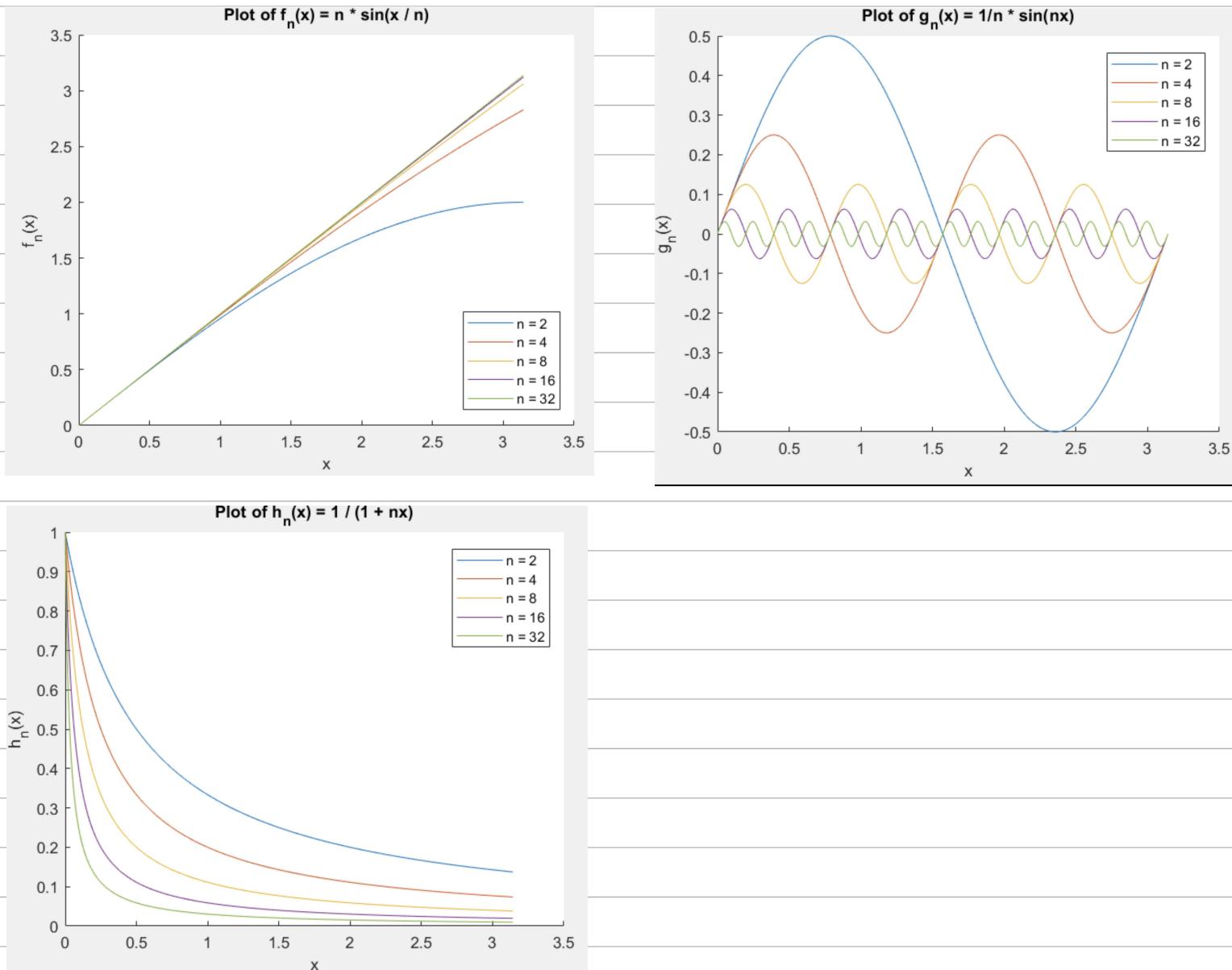
$$f \in H^1(I) \Leftrightarrow \|f\|_{1,2} = (\|f\|_{0,2}^2 + \|f'\|_{0,2}^2)^{1/2} < \infty.$$

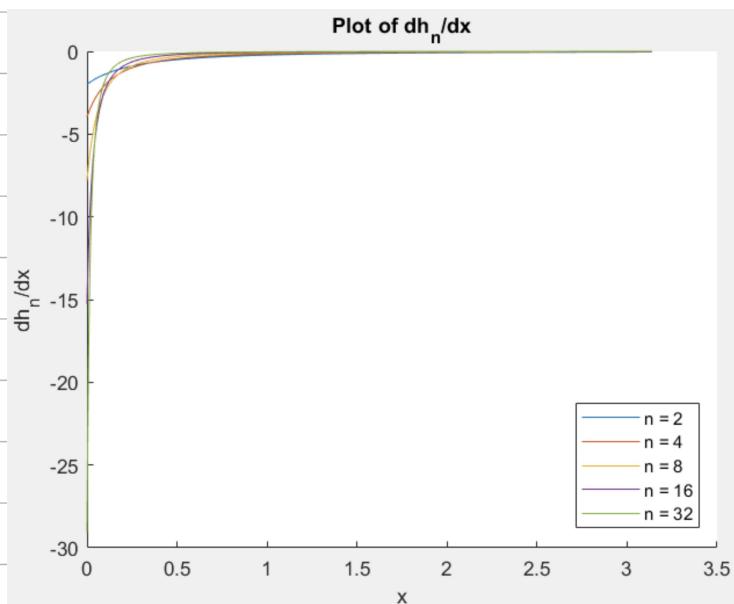
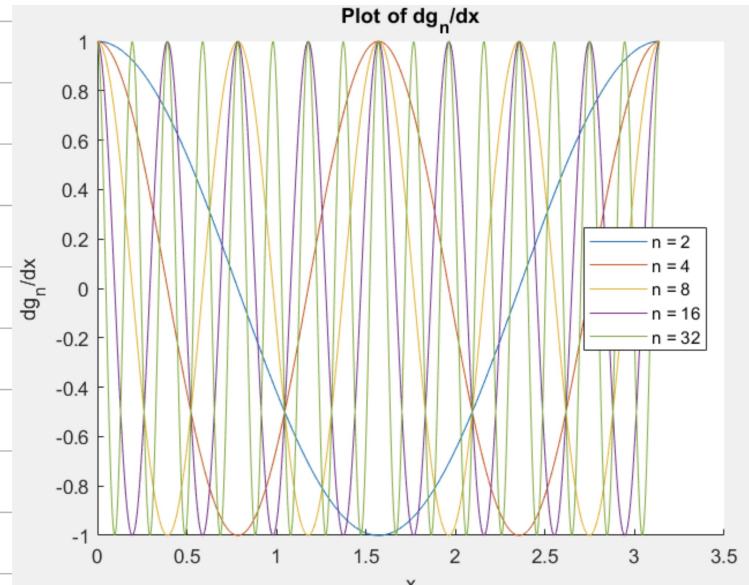
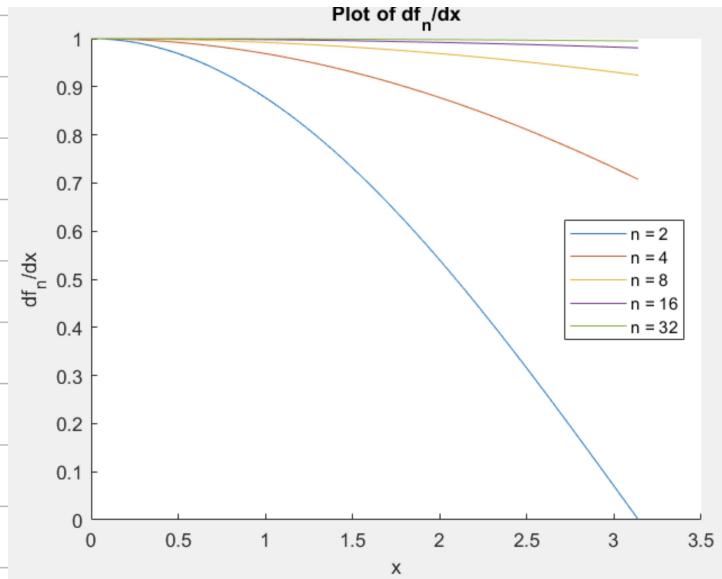
Consider the sequences of functions for $n = 1, 2, \dots$:

$$f_n(x) = n \sin\left(\frac{x}{n}\right), \quad f_\infty(x) = x,$$

$$g_n(x) = \frac{1}{n} \sin(nx), \quad g_\infty(x) = 0,$$

$$h_n(x) = \frac{1}{1+nx}, \quad h_\infty(x) = 0.$$





2) Given $f_{ab}(x) = x$, $\|f_{ab}\|_{0,2} = \left(\int_0^{\pi} x^2 dx \right)^{1/2}$.

$$= \left(\frac{\pi^3}{3} \right)^{1/2} < \infty.$$

Hence, $f_{ab} \in L^2(\mathbb{T})$.

② Given $f_{ab}(x) = x$, $\|f_{ab}\|_{0,\infty} = \max_{x \in \mathbb{T}} |f(x)|$

$$= \pi < \infty$$

$f_{00}(x) \in L^{\infty}(\mathbb{I})$.

$$\textcircled{3}. \text{ Given } f_{00}(x) = x, \|f_{00}\|_{1,2} = (\|f\|_{0,2}^2 + \|f'_{00}\|_{0,2}^2)^{\frac{1}{2}}$$

$$f'_{00}(x) = \lim_{n \rightarrow \infty} \cos\left(\frac{x}{n}\right) = 1; \|f'_{00}\|_{0,2}^2 = 1.$$

$$\|f_{00}\|_{0,2}^2 = \frac{\pi^3}{3}.$$

$$\text{Hence, } \|f_{00}\|_{1,2} = \left(\frac{\pi^3}{3} + 1\right)^{\frac{1}{2}} < \infty$$

$f_{00} \in H^1(\mathbb{I})$.

3.

$L^2(\mathbb{I})$ for f_n is as follows:

$$\begin{aligned} \|f\|_{0,2} &= \left(\int_0^{\pi} \left(n \sin\left(\frac{x}{n}\right)\right)^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_0^{\pi} n^2 \sin^2\left(\frac{x}{n}\right) dx \right)^{\frac{1}{2}} \\ &= \left[\frac{\pi n^2}{2} - \underbrace{n^3 \sin\left(\frac{2\pi}{n}\right)}_{4} \right]^{\frac{1}{2}} \end{aligned}$$

$H^1(\mathbb{I})$ for f_n is similar

$$\|f\|_{1,2} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2 \right)^{\frac{1}{2}}$$

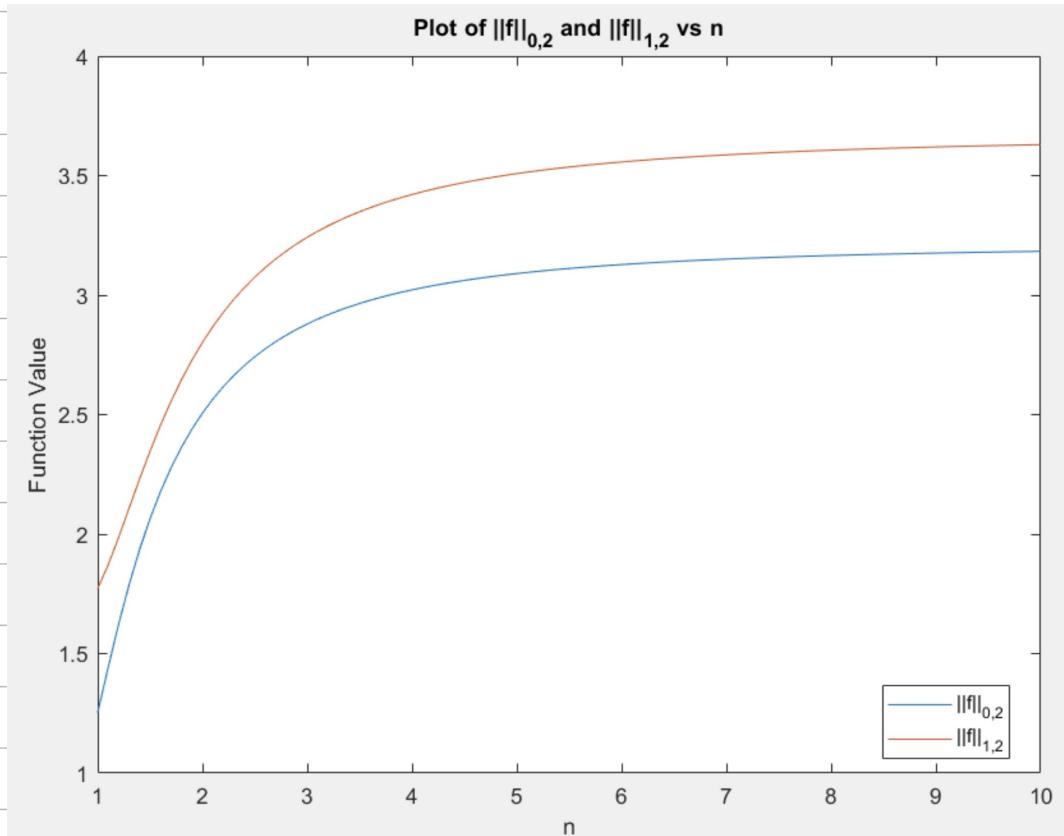
Calculating the second part in the parenthesis

$$\|f'\|_{0,2} = \left(\int_0^{\pi} \left(\cos\left(\frac{x}{n}\right)\right)^2 dx \right)^{\frac{1}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{n \sin\left(\frac{2\pi}{n}\right)}{4} \right]^{\frac{1}{2}}$$

$$\|f\|_{1,2} = \left(\frac{\pi n^2}{2} - \frac{n^3 \sin\left(\frac{2\pi}{n}\right)}{4} + \frac{\pi}{2} + \frac{n \sin\left(\frac{2\pi}{n}\right)}{4} \right)^{\frac{1}{2}}$$

The $\|f\|_{0,2}$ and $\|f\|_{1,2}$ vs n are as follows:



Q.

$$\textcircled{1}. \quad f_n(x) = n \sin\left(\frac{x}{n}\right); \quad f_{00}(x) = x.$$

$$\|f_n(x) - f_{00}(x)\|_{0,2} = \left(\int_0^{\pi} (n \sin\left(\frac{x}{n}\right) - x)^2 dx \right)^{\frac{1}{2}}$$

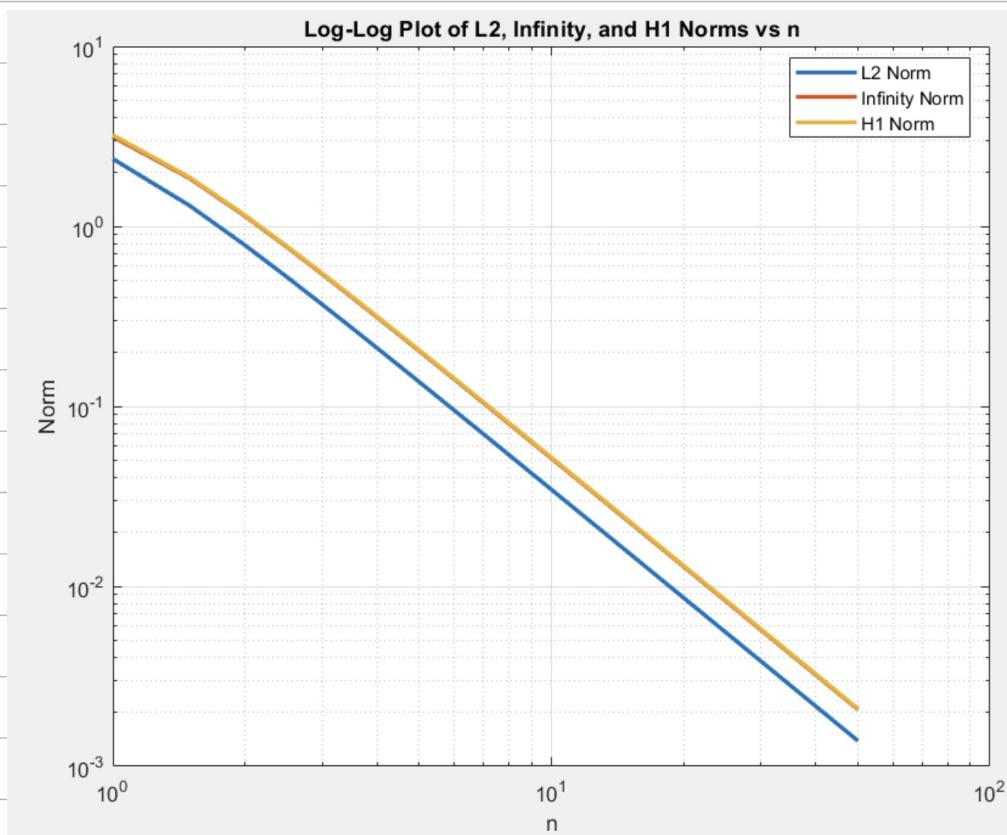
$$= \left[\int_0^{\pi} \left(n^2 \sin^2\left(\frac{x}{n}\right) - 2n \sin\left(\frac{x}{n}\right)x + x^2 \right) dx \right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi n^2}{2} - 2n^3 \sin\left(\frac{\pi}{n}\right) + \frac{\pi^3}{3} + 2n^2 \pi \cos\left(\frac{\pi}{n}\right) - \frac{n^3 \cos\left(\frac{\pi}{n}\right) \sin\left(\frac{\pi}{n}\right)}{2} \right]^{\frac{1}{2}}$$

$$\|f_n(x) - f_\infty(x)\|_{0,\infty} = \max_{x \in I} |n \sin(\frac{x}{n}) - x|$$

$$\|f_n(x) - f_\infty(x)\|_{1,2} = (\|f\|_{0,2}^2 + \|f'\|_{0,2}^2)^{\frac{1}{2}}$$

Using Matlab Symbolic Calculation, we have:



The figure shows that $f_n(x) \rightarrow f_\infty$ in $L^2(I)$

$f_n(x) \rightarrow f_\infty$ in $H^1(I)$.

$f_n(x) \rightarrow f_\infty$ in $L^\infty(I)$.

This also reflects the observations in Part 1, as larger n leads to a better approximation for $f(x)$ and $f'(x)$.

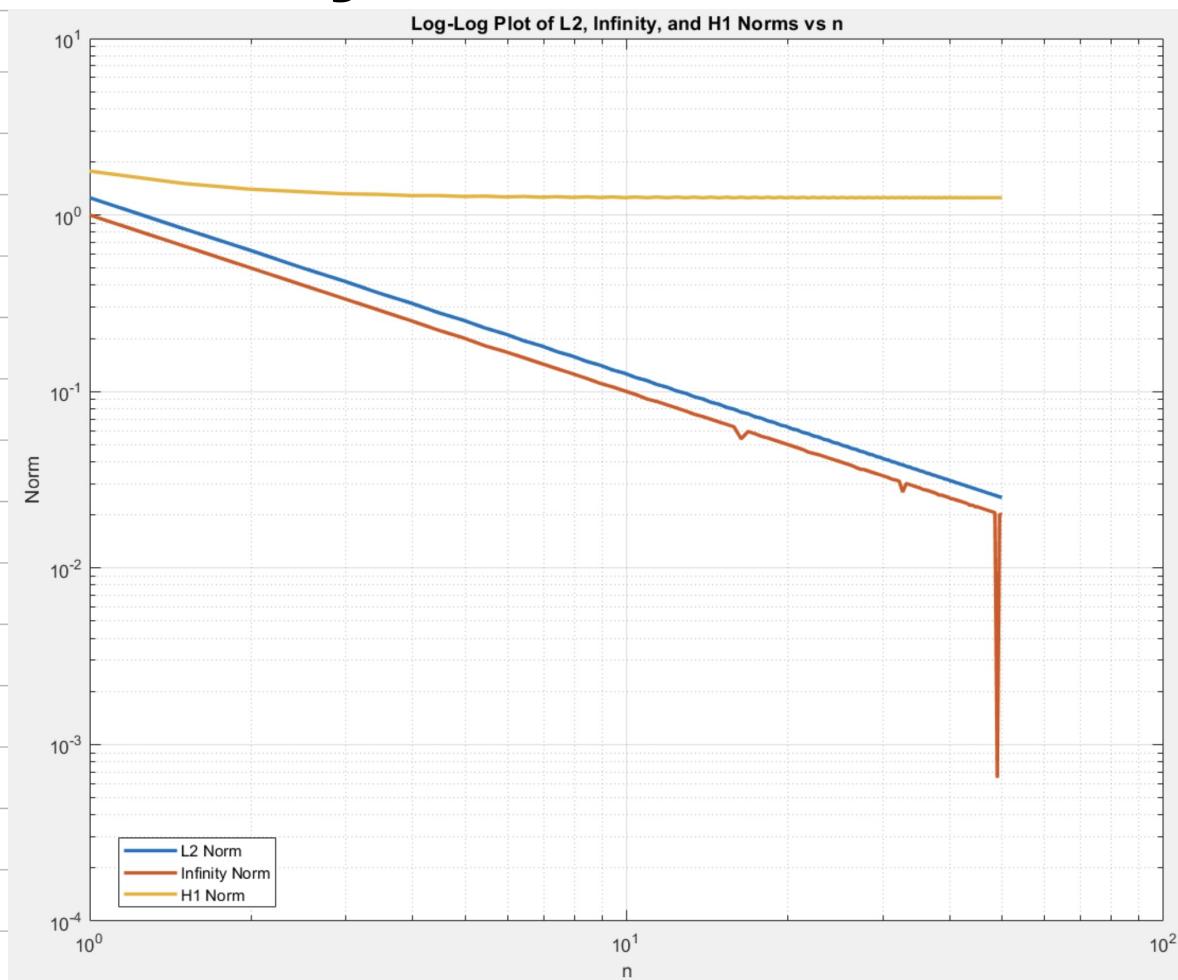
$$\textcircled{2} \quad g_n = \frac{1}{n} \sin(nx) , \quad g_{\infty}(x) = 0 .$$

$$\|g_n(x) - g_{\infty}(x)\|_{0,2} = \left[\int_0^{\pi} \left(\frac{1}{n} \sin(nx) - 0 \right)^2 dx \right]^{\frac{1}{2}}$$

$$\|g_n(x) - g_{\infty}(x)\|_{0,\infty} = \max_{x \in I} \left| \frac{1}{n} \sin(nx) - 0 \right|$$

$$\|g_n(x) - g_{\infty}(x)\|_{1,\infty} = (\|f\|_{0,2}^2 + \|f'\|_{0,2}^2)^{\frac{1}{2}}$$

Following the integration procedure above, we can plot norms vs n. for g_n as follows:



We have observed that

$$f_n(x) \rightarrow f_0 \text{ in } L^2(I)$$

$$f_n(x) \rightarrow f_0 \text{ in } L^\infty(I).$$

Bwt, $f_n(x)$ does not converge in $H^1(I)$.

This is because the analytical expression of $H^1(I)$ is as follows:

$$\left[\frac{\pi}{2} + \underbrace{\left[\frac{\pi n}{2} - \frac{\sin(2n\pi)}{4} \right]}_{n^3} + \underbrace{\frac{n^2 \sin(2\pi n)}{4} \right] \right]^{\frac{1}{2}}$$
$$\rightarrow \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \approx 1.2533.$$

No matter how large the n 's, f_n never converge in H^1 Norm space. This phenomenon can be explained from the plot of $f'(x)$. As the $f'(x)$ is sinusoidal by nature and $f'_0(x)$ is 1, $\frac{f'(x)}{n}$ can not approximate $f'_0(x)$.

$$(3) \quad h_n(x) = \frac{1}{1+nx}, \quad h_{00}(x) = 0.$$

$$\|h_n(x) - h_{00}(x)\|_{0,2} = \left[\int_0^\pi \left(\frac{1}{1+nx} \right)^2 dx \right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi}{n\pi+1} \right]^{\frac{1}{2}}$$

$$\|h_n(x) - h_{00}(x)\|_{0,2} = \max_{x \in I} \left| \frac{1}{1+nx} \right|$$

$$= 1$$

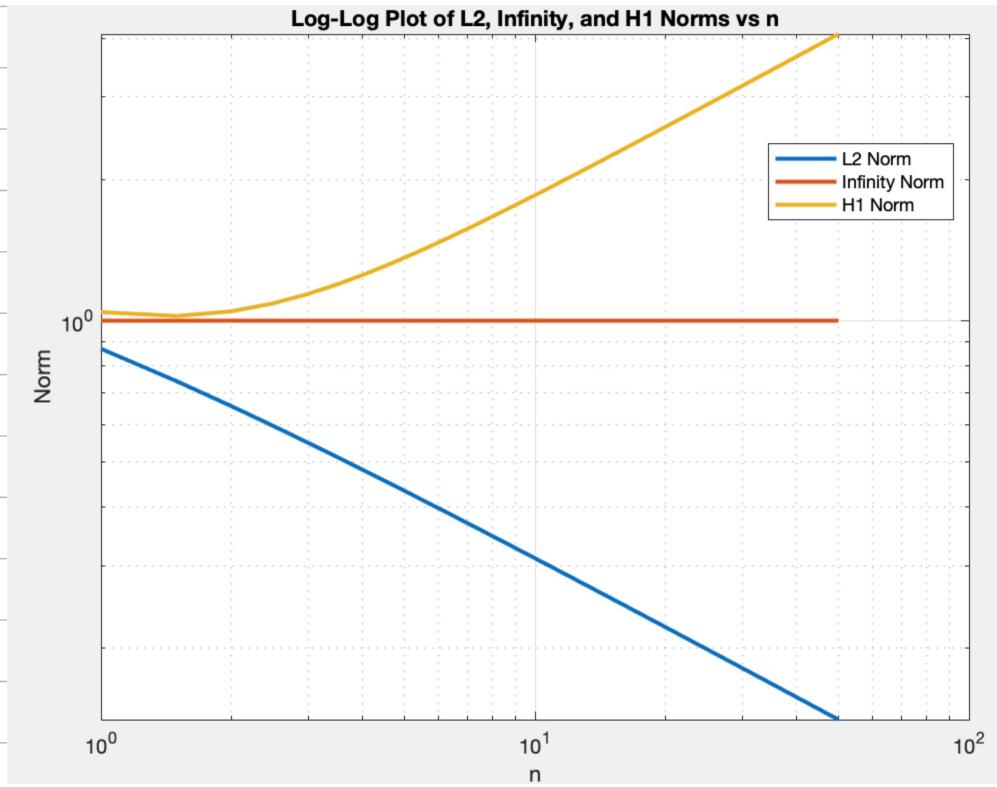
$$\|h_n(x) - h_{00}(x)\|_{1,2} = (\|f\|_{0,2}^2 + \|f'\|_{0,2}^2)^{\frac{1}{2}}.$$

$$= \left[\left[\frac{1}{n^2x+n} \right]_0^\pi + \int_0^\pi \left(\frac{n}{(nx+1)^2} \right)^2 dx \right]^{\frac{1}{2}}$$

$$= -\frac{1}{n^2\pi+n} + \frac{1}{n} + \left[-\frac{1}{3(n\pi+1)^3} \right]_0^\pi$$

$$= \frac{\pi}{n\pi+1} - \frac{n}{3(n\pi+1)^3} + \frac{n}{3}$$

$\lim_{n \rightarrow \infty} \|h_n(x) - h_{00}(x)\|_{1,2} \rightarrow 0$. $h_n(x)$ does not converge to $h_{00}(x)$.



We observe that $h_n \rightarrow h_0$ in L_2 Norm. But it doesn't converge with infinity norm and H_1 Norm. The func even diverges in the H_1 Norm.

On Interpolation Errors (70)

Consider the interval $\Omega = [-1, 1]$, and a mesh of $n_{\text{el}} \in \mathbb{N}$ equally long P_k -elements on it, for $k = 1, 2, 4$. for P_k -elements is shown in Example 3.24 in the notes, while The Lagrange finite element interpolant Iu is constructed through (3.24) in the notes.

For $\omega \in \mathbb{R}$, consider the functions

$$v_\omega(x) = \cos(\omega x)$$

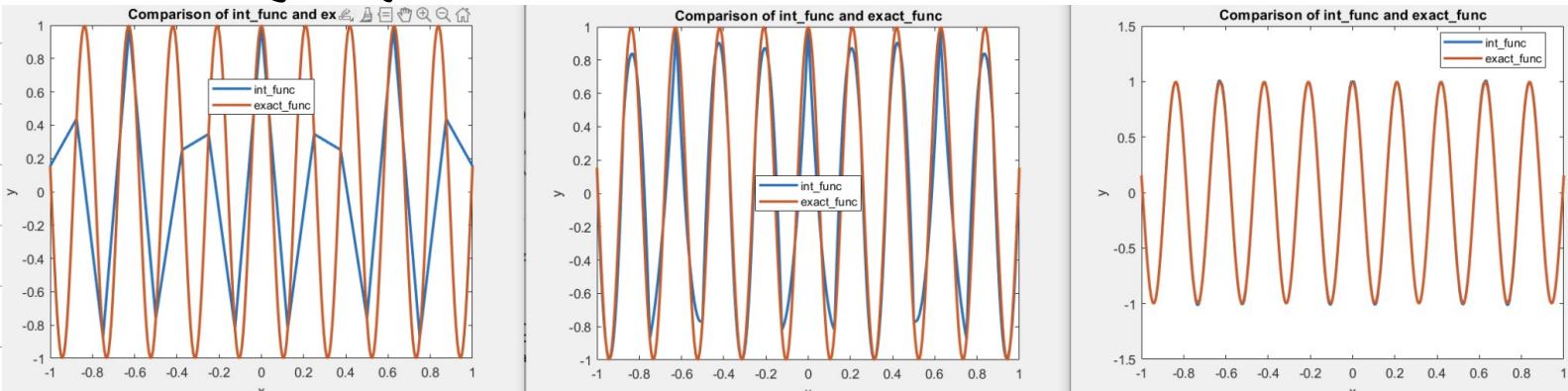
$$w_\omega(x) = \begin{cases} 0 & x < 0 \\ x^\omega & x \geq 0. \end{cases}$$

The Lagrange finite element interpolation is defined as:

$$Iu = \sum_{a=1}^m u(a) N_a^e.$$

$$\text{where } N_a^e = \frac{\prod_{b=1, b \neq a}^{k+1} (x - x_b^e)}{\prod_{b=1, b \neq a}^{k+1} (x_a^e - x_b^e)}.$$

1. when $u = v_{20}$, using 2³ points, we can approximate u very Iu built above. Clearly, higher k leads to better approximation.



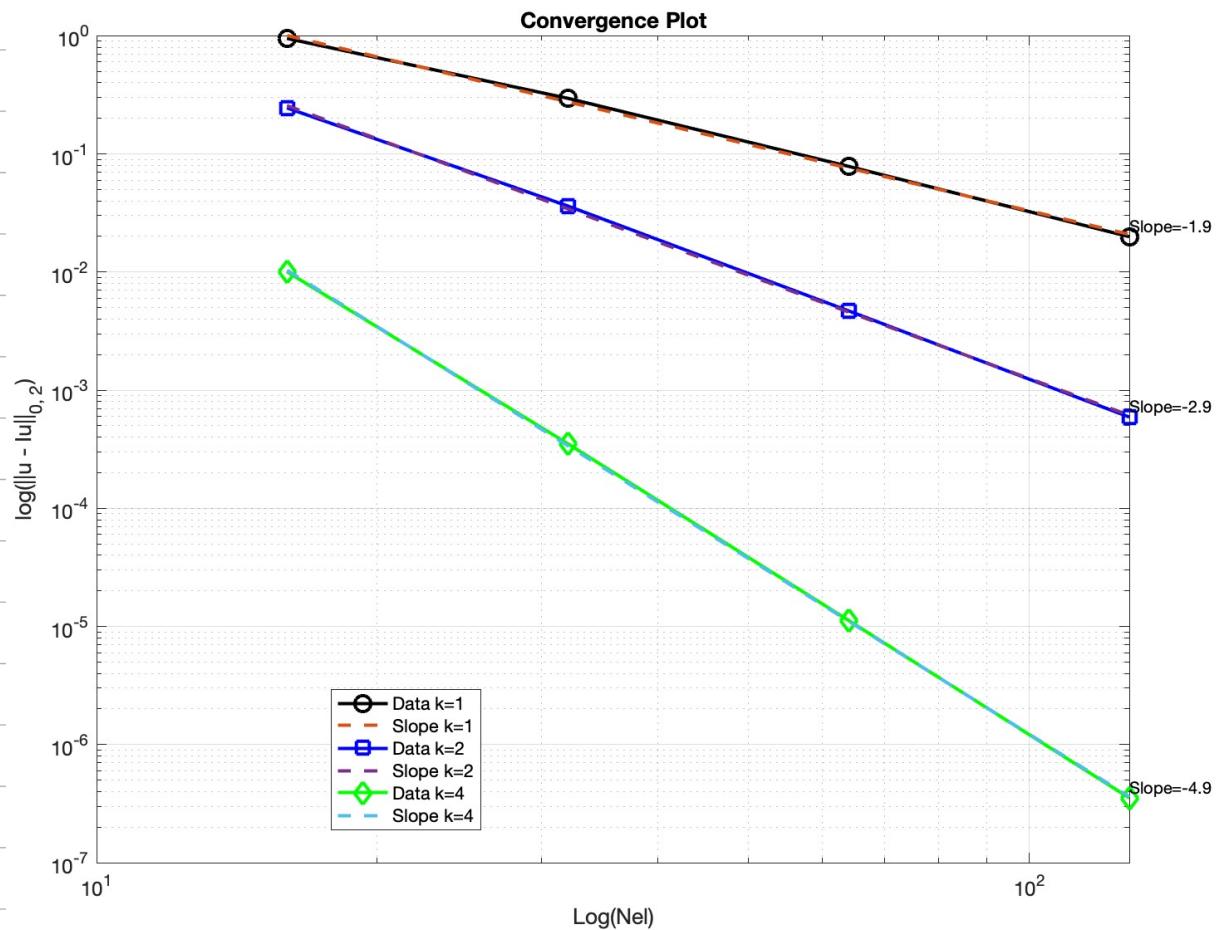
$k=1$

$k=2$

$k=4$.

Then, We plot $\|u - Iu\|_{0,2}$ with different orders of Iu :

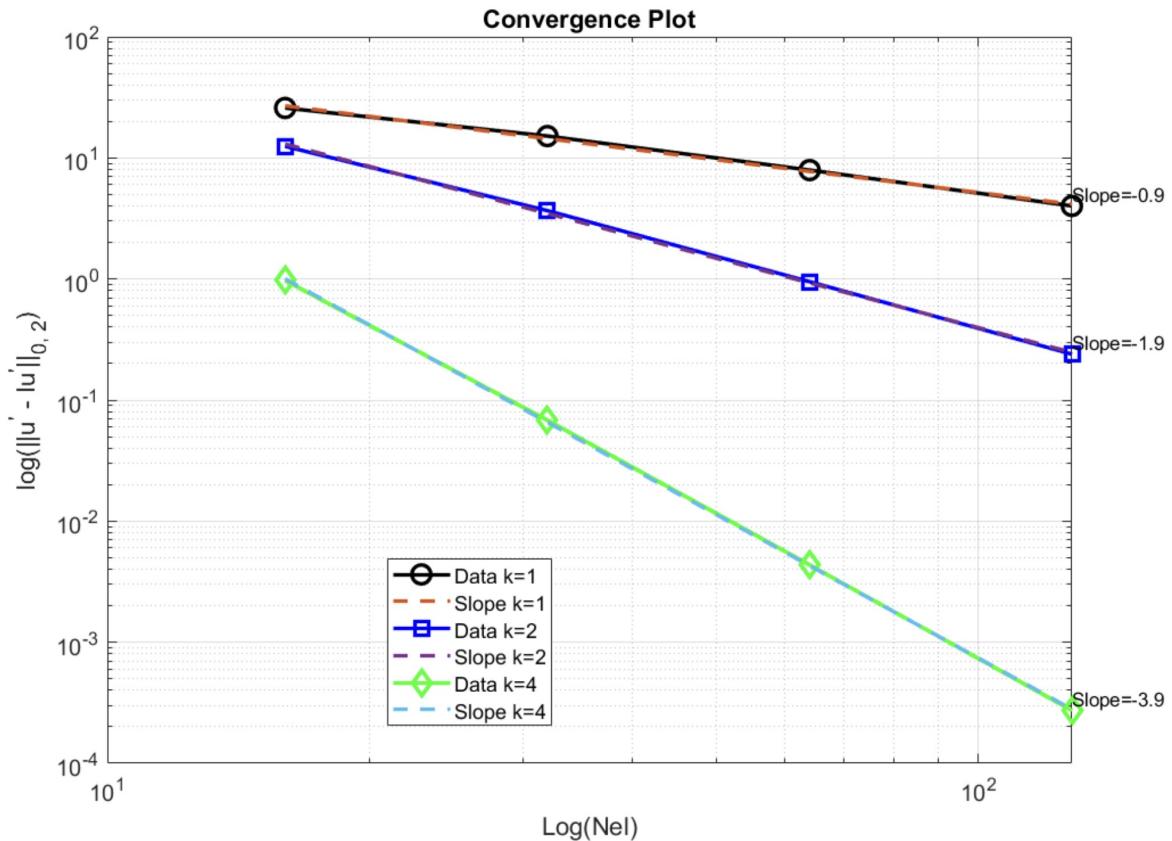
(a). $\|u - Iu\|_{0,2,\kappa}$ for $\kappa = 1, 2, 4$.



The figure above shows how the order of interpolation affects the convergence behavior of approximation solutions.

Clearly, the convergence rates are around $k+1$, which corresponds to the theoretical ones.

The $\|u' - J_u'\|_{0,2}$ for $k=1, 2, 4$ are also given.

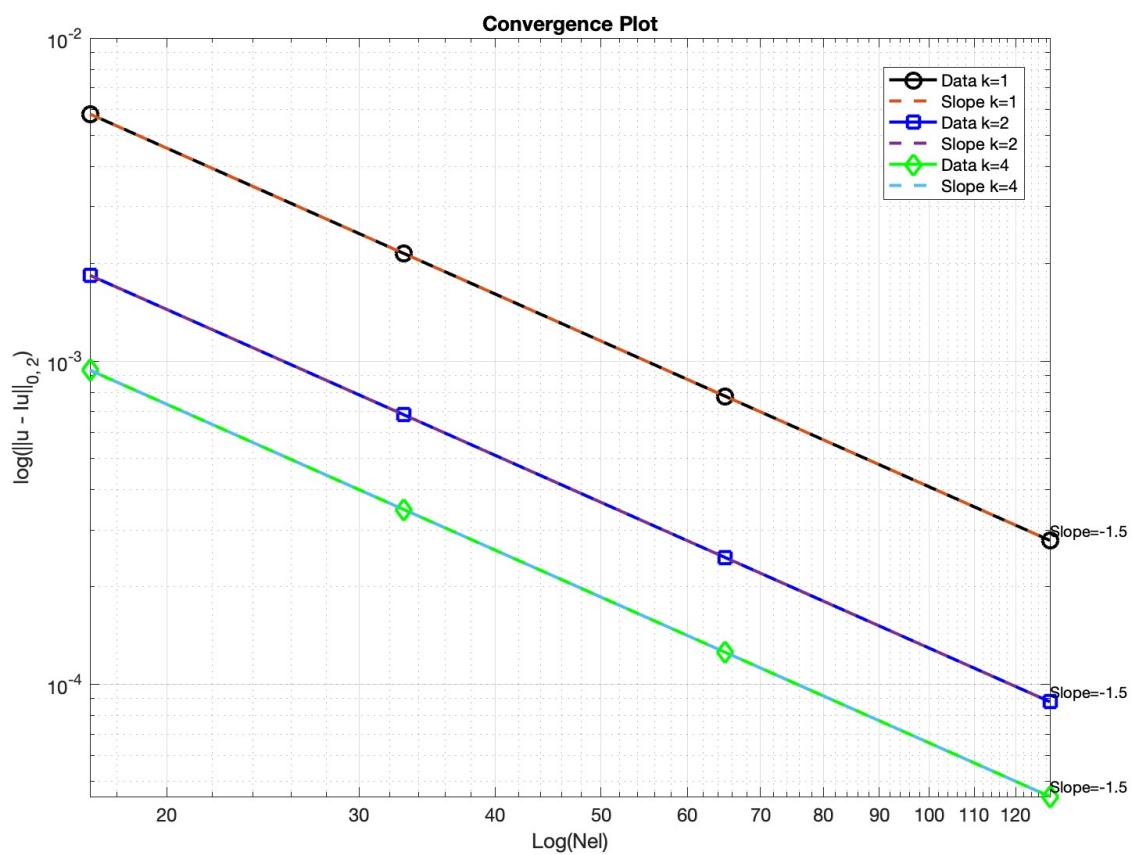
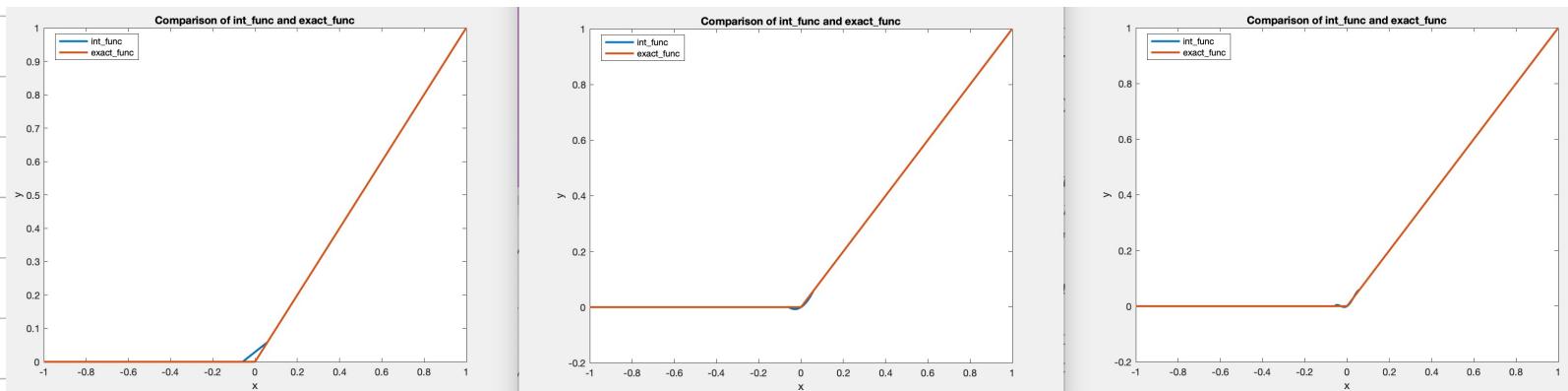


The figure above shows how the order of interpolant affects the convergence behavior of approximating $u'(x)$.

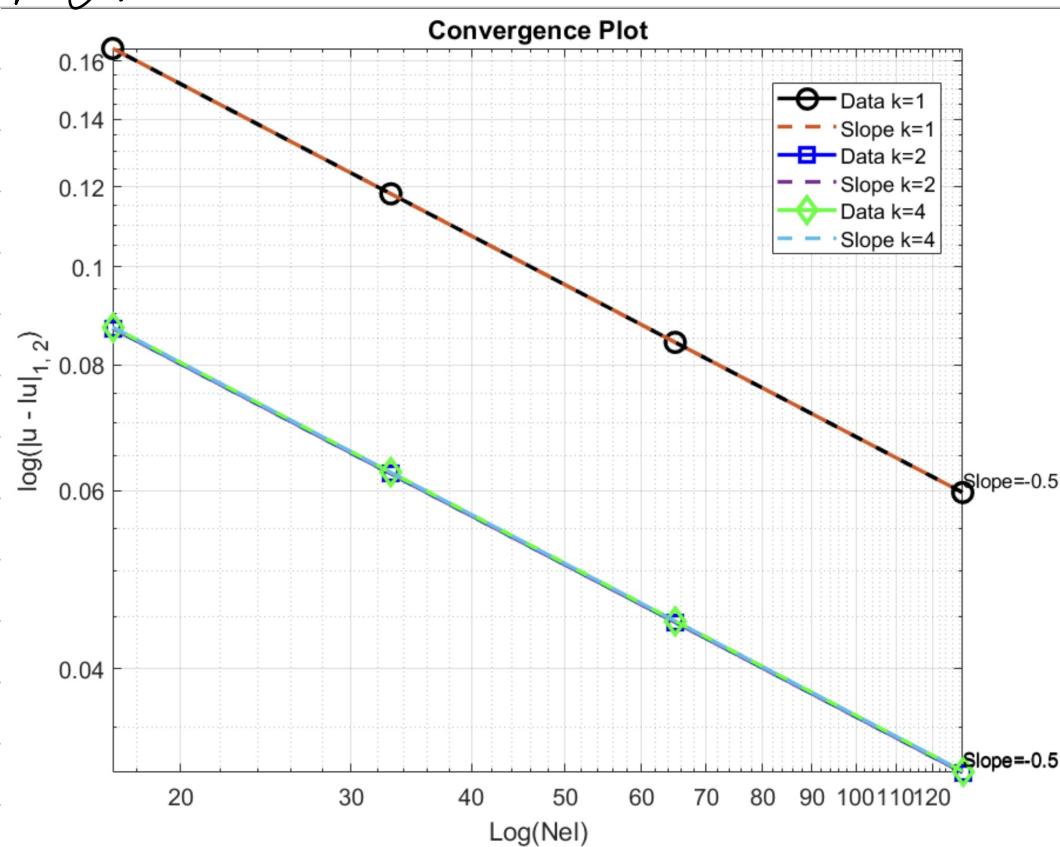
Clearly, the convergence rates are around k . This is the case for the derivative of functions.

$$2. \quad u = w_1, \quad w_1 = \begin{cases} 0 & , x < 0 \\ x^1, & x \geq 0 \end{cases}$$

We use $Nel = 2^k + 1$, three different orders of P_k to approximate w_1 . Clearly, higher order can approximate the discontinuity of derivative at the origin.

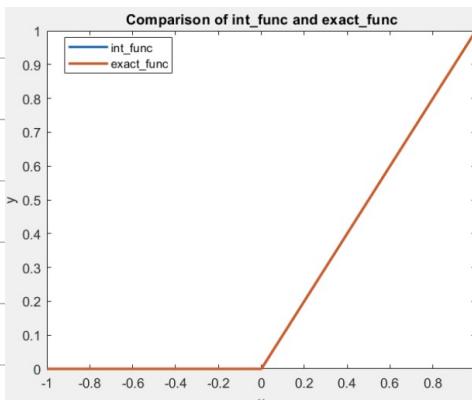


The convergence rates of three cases for u is around the same. This could be explained by the fact that u needs to approximate origin with the element when Nel is odd. However, lagrange interpolant element has continuous derivative within any element.

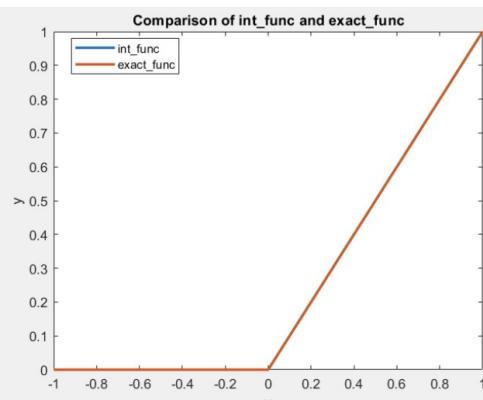


The convergence rates of three cases for u' remain the same. The reason is the same with the one explained above: there is a jump at origin for u' , whereas u has continuous u' over the elements where origin resides.

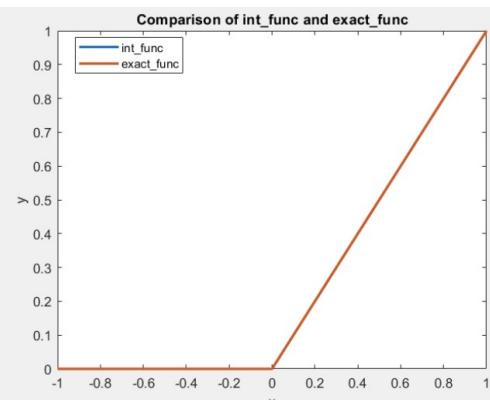
(c). When element number is even, we first look the J_n .



$$K=1$$



$$K=2$$

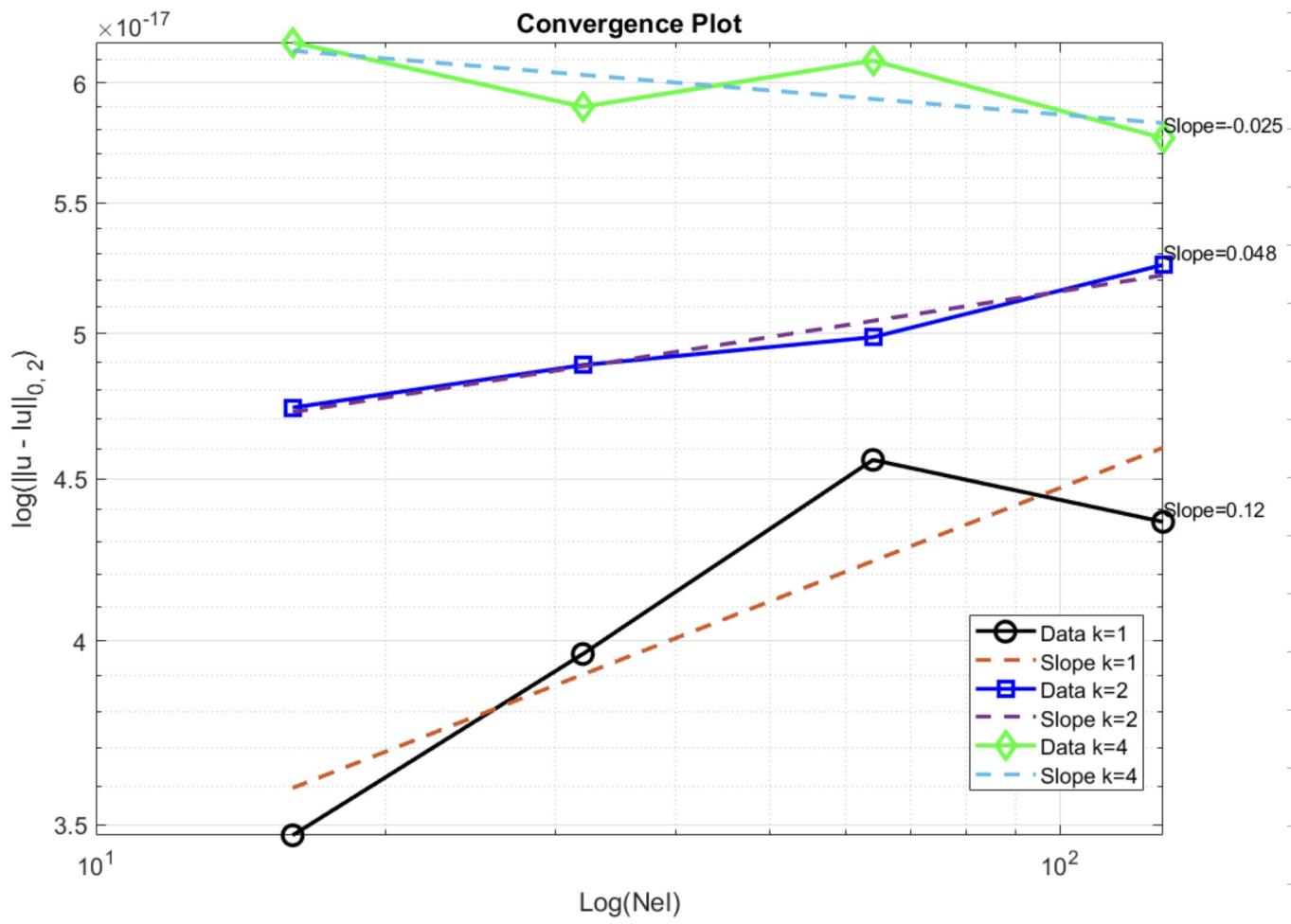


$$K=4$$

Clearly, J_n with even n_e approximates it much better than J_n with odd n_e . This is because when n_e is odd, there is one element, indexing by $\frac{n_e}{2}$, used for capturing the origin point within the element. This will lead to large, since inside an element, the derivative is continuous.

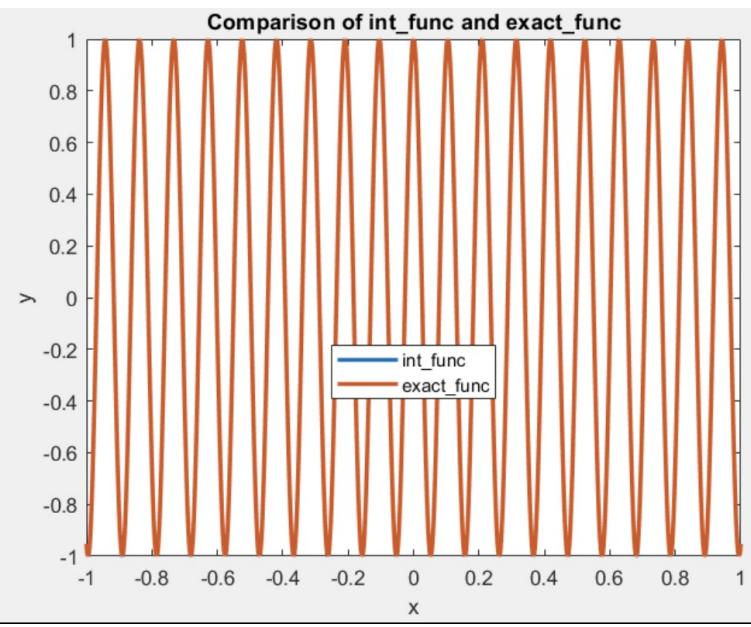
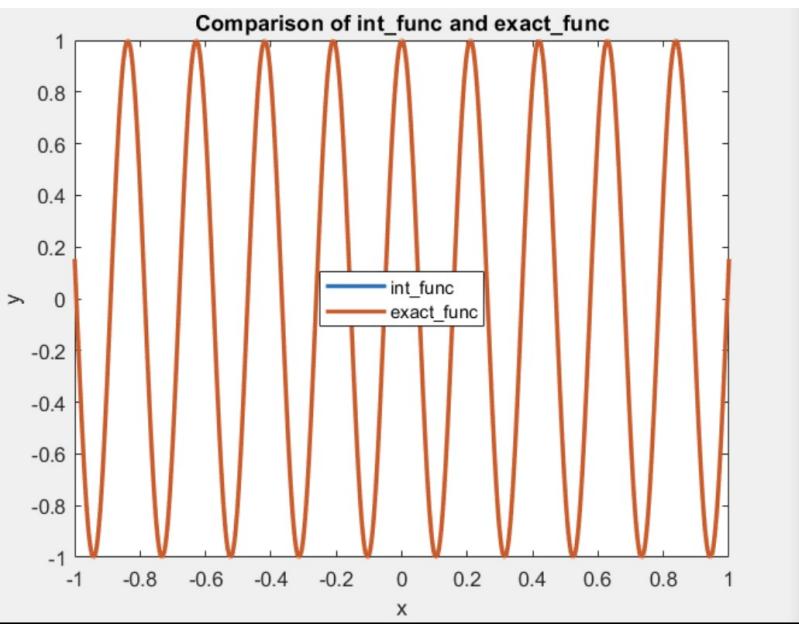
BUT, when N_e is Even, we can treat the discontinuous derivative point (origin) at the element boundary.

The following figure gives $\|u - J_u\|_{0,2}$ when Nel is even. Strikingly, J_u can approximate u to a machine precision.



This case is different from previous two because even Nel results treating origin at the boundary of element, while odd case will approximates such a point within the element.

3. $\|u - I_u\|_{0,2}$ for $u = V_{30}$ and $u = V_{60}$ for $k=2$
and $nel = 200$.



Using $k=2$ interpolate gives visually close approximation of u .

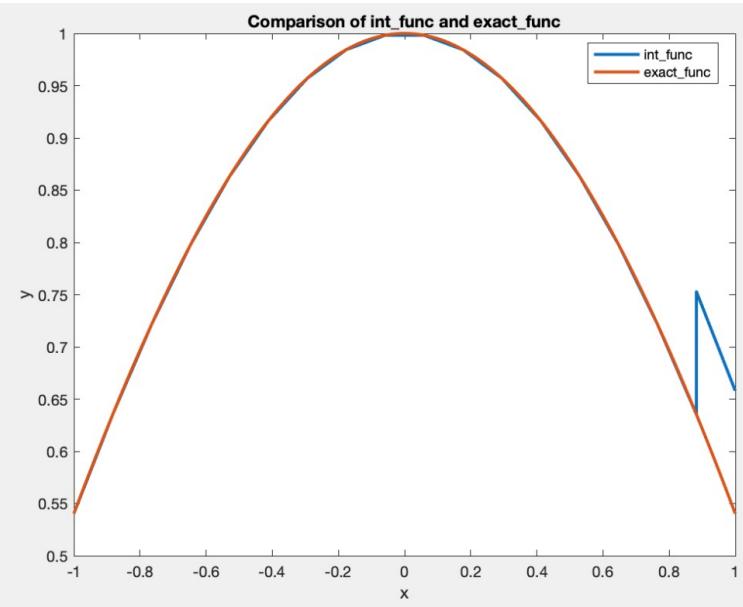
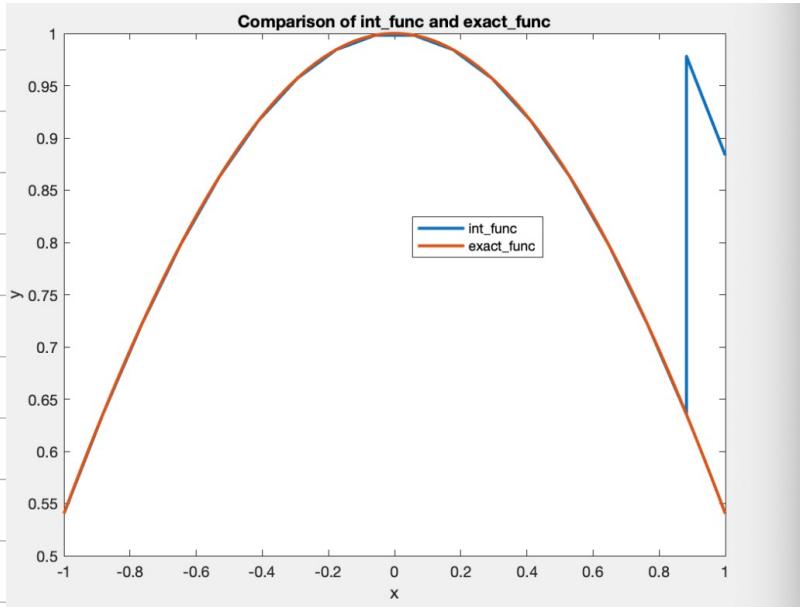
The $\|u - I_u\|_{0,2}$ is listed as follow:

	V_{30}	V_{60}
$\ u - I_u\ _{0,2}$	$1.5545e-4$	0.0012

Approximating V_{60} results in a larger L2 norm.

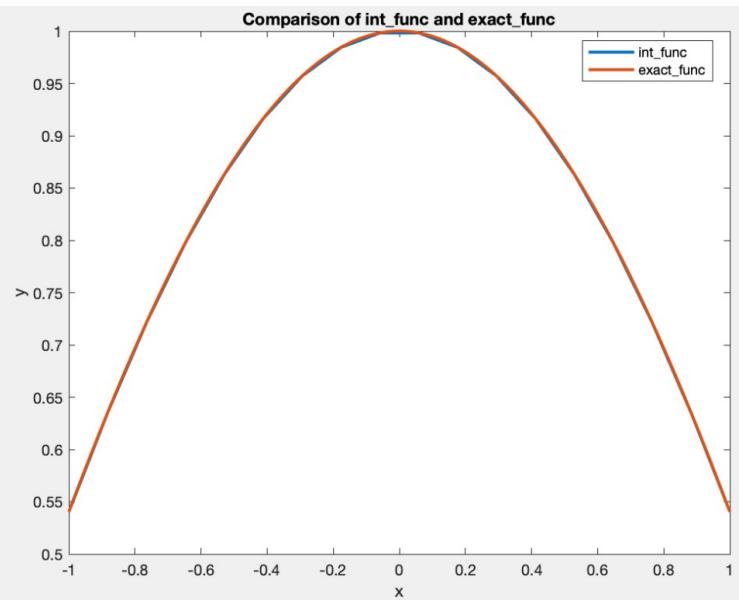
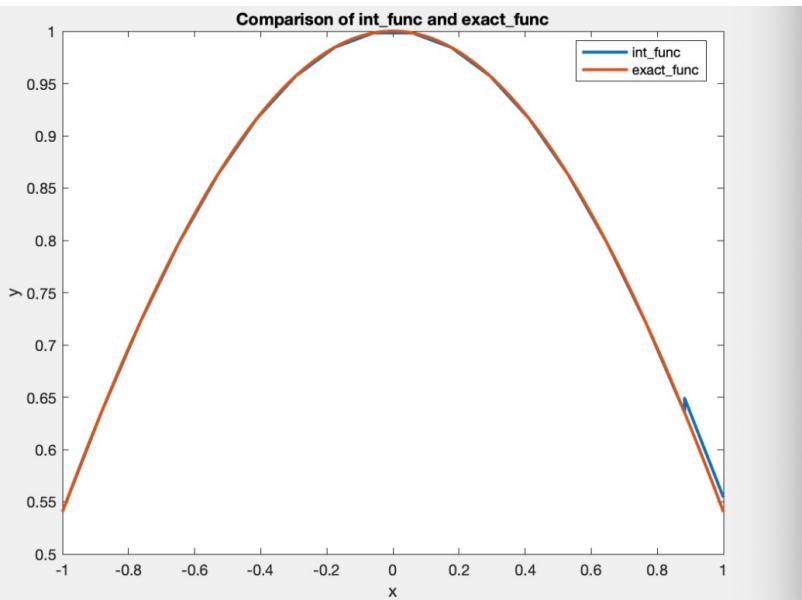
This is because with a fixed order interpolate, the higher the frequency the function, the less accurate the interpolant can approximate.

4. First, we plot U_n and $U = \gamma_1$ with the inexact imposition of boundary condition as: $\cos(1) \approx h^m$ w.r.t $x = 1$.



$$m = 0.5$$

$$m = 1.0$$

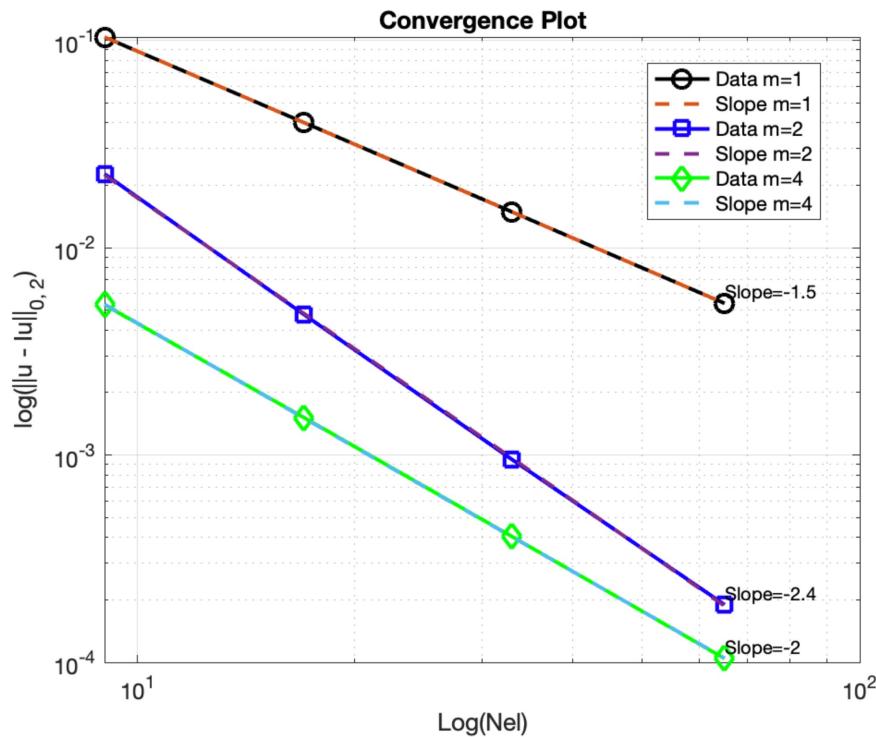


$$m = 2.0$$

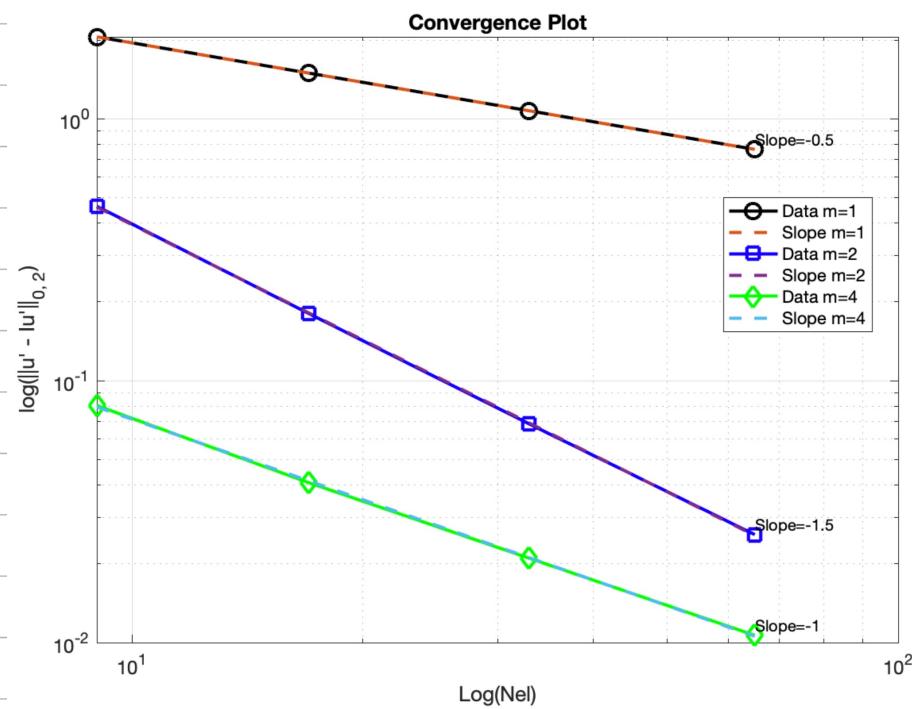
$$m = 4.0$$

Higher m will result in a faster decrease in the difference between \mathbf{J}_n and \mathbf{u} .

$$(a). \|\mathbf{u} - \mathbf{J}_n\|_{0,2}$$



$$(b). \|\mathbf{u}' - (\mathbf{J}\mathbf{u})\|_{0,2}$$



(C)





