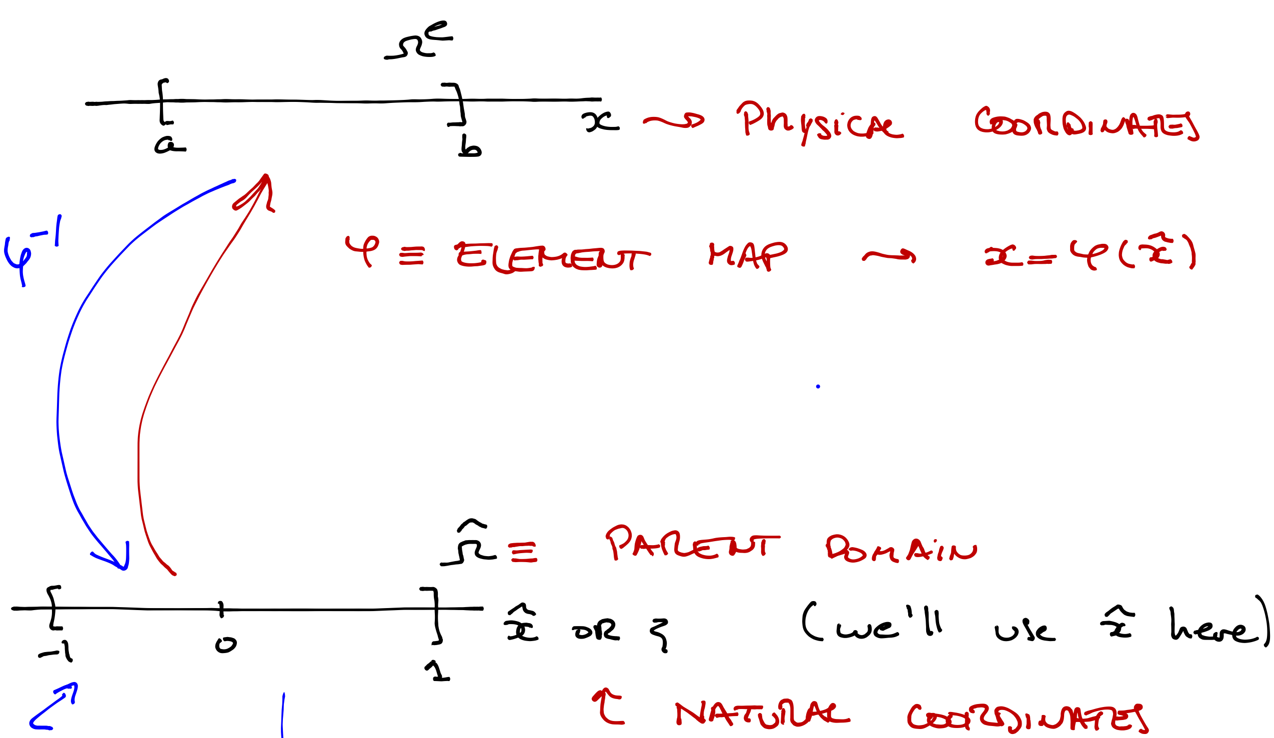
REFERENCE ELEMENT (ALSO KNOWN AS "PARENT ELEMENT")

IN PRACTICE, PARTICULARLY IN 2D AND 3D, IT IS CONVENIENT TO CONSTRUCT A "REFERENCE ELEMENT" OVER WHICH ALL INTEGRALS, SHAPE FUNCTIONS AND THEIR DERIVATIVES ARE COMPUTED. LATER ON, THIS REFERENCE ELEMENT PLAYS A CRUCIAL PART IN DEFINING ELEMENTS WITH CURVED BOUNDARIES.

IDEA :



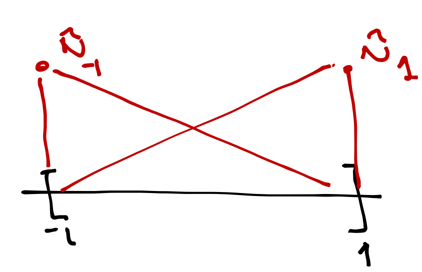
BUILD SHAPE FUNCTIONS AND COMPUTE INTEGRALS HERE !

WE WILL FIRST USE "AFFINE MAPS" ONLY:

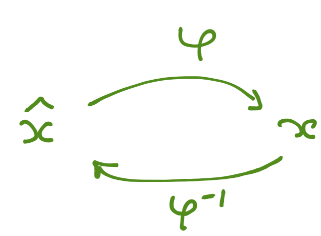
- $\varphi(\hat{x}) = a\,\hat{N}_1(\hat{x}) + b\,\hat{N}_2(\hat{x})$

$$= a\,\frac{(1-\hat{x})}{2} + b\,\frac{(\hat{x}+1)}{2}$$

$$= \frac{a+b}{2} + \hat{x}\,\frac{(b-a)}{2}$$



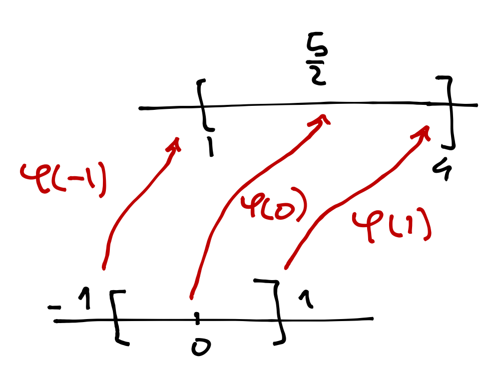
- $\varphi^{-1}(x) = \frac{2}{(b-a)}\left[x - \frac{(a+b)}{2}\right]$



TO REMEMBER THE MAPS

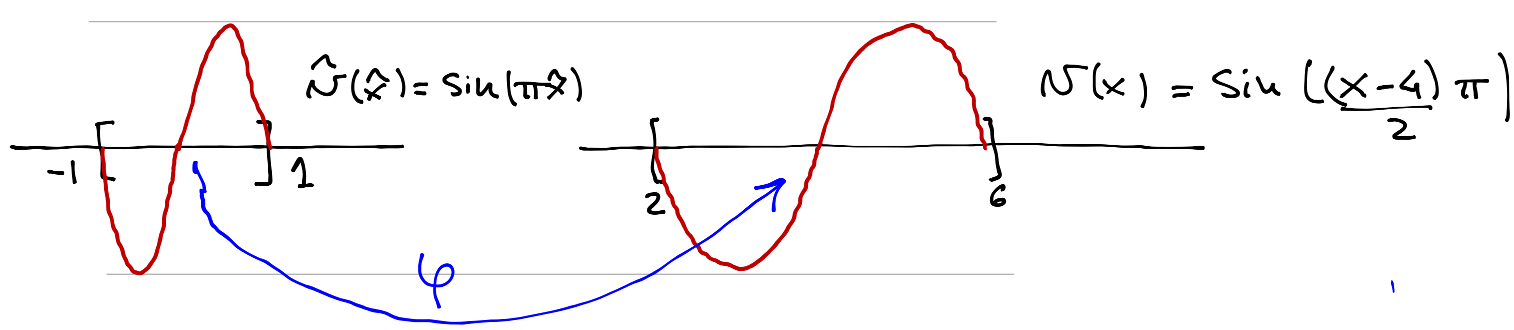
EXAMPLE : $a = 1$ $b = 4$

$$\varphi(\hat{x}) = \frac{5}{2} + \hat{x}\,\frac{3}{2}$$



Given $v(x)$, we define the "pullback" of v as $\hat{v}(\hat{x}) = v(\varphi(\hat{x}))$

It is v evaluated at corresponding points in $\hat{\Omega}$



WE DEFINE THE PARENT ELEMENT: $\left([-1,1],\ \{\hat{N}_1, \hat{N}_2\}\right)$

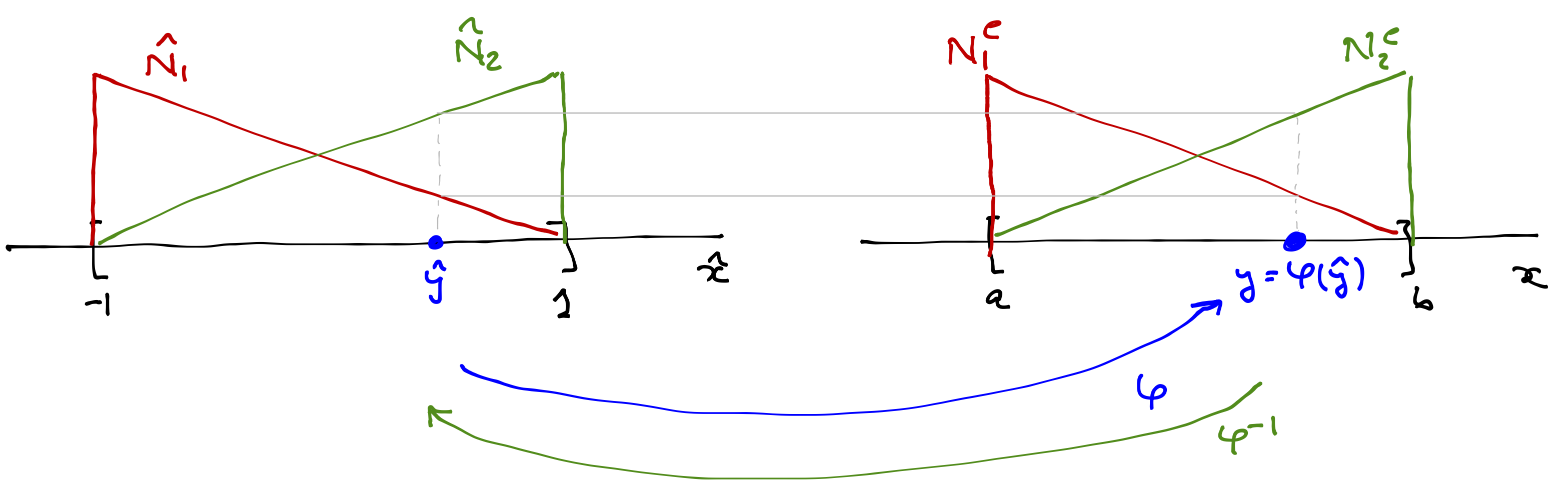
$$\hat{N}_1(\hat{x}) = \frac{1}{2}(1-\hat{x})$$

$$\hat{N}_2(\hat{x}) = \frac{1}{2}(1+\hat{x})$$

Note then that:

$$N_1^e(\varphi(\hat{x})) = \hat{N}_1(\hat{x})$$

$$N_2^e(\varphi(\hat{x})) = \hat{N}_2(\hat{x})$$



Check:

$$\left.\begin{aligned} N_1^e(x) &= \frac{b-x}{b-a} \\ \varphi(\hat{x}) &= \frac{a}{2}(1-\hat{x}) + \frac{b}{2}(1+\hat{x}) \end{aligned}\right\} \Rightarrow N_1^e(\varphi(\hat{x})) = \frac{b - a/2(1-\hat{x}) - b/2(1+\hat{x})}{b-a}$$

$$= \frac{(b-a)/2 - \hat{x}(b-a)/2}{b-a}$$

$$= \frac{1}{2}(1-\hat{x}) = \hat{N}_1(\hat{x}) \checkmark$$

SAME FOR $N_2^e, \hat{N}_2$.

why?

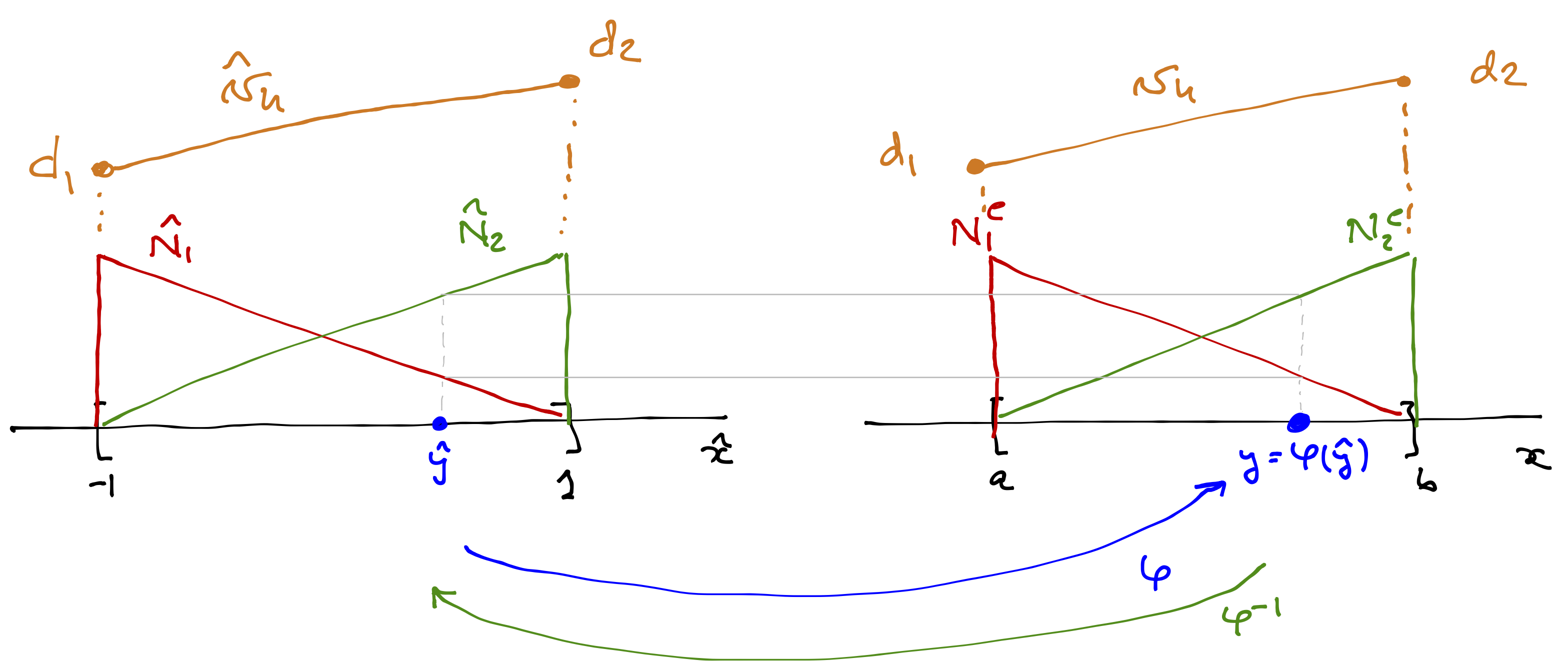
composition of affine functions is affine $\Rightarrow N_1^e \circ \varphi$ is affine, and equal to 1 at $\hat{x}=-1$ AND 0 AT $\hat{x}=1$.

More generally, for

$$v_h(x) = d_1 N_1^e(x) + d_2 N_2^e(x)$$

$$v_h(\varphi(\hat{x})) = \hat{v}_h(\hat{x}) = d_1 \hat{N}_1(\hat{x}) + d_2 \hat{N}_2(\hat{x})$$

SAME COMPONENTS!



WANT TO EVALUATE v AT x? SIMPLY EVALUATE $\hat{v}$ AT $\hat{x}$.

How about derivatives?

$$\frac{d}{d\hat{x}}\hat{N}_u(\hat{x}) = \frac{d}{d\hat{x}}\left[N_u(\varphi(\hat{x}))\right] = N_u'(\varphi(\hat{x}))\cdot\varphi'(\hat{x}) = N_u'(\varphi(\hat{x}))\; j_e(\hat{x})$$

$$\boxed{j_e(\hat{x}) = \varphi'(\hat{x}) \equiv \text{JACOBIAN OF THE MAP AT } \hat{x}}$$

$$\Rightarrow \boxed{N_u'(\varphi(\hat{x})) = \hat{N}_u'(\hat{x})\; j_e(\hat{x})^{-1}}$$

($N_u'(\varphi(\hat{x}))$ is the derivative of $N_u(x)$ AT $\varphi(\hat{x})$)

WANT TO COMPUTE V' AT x? SIMPLY COMPUTE $\hat{V}'$ AND j_e AT $\hat{x}$.

IN PARTICULAR, FOR AN AFFINE MAP, $j_e = \frac{b-a}{2} = \frac{|\Omega|}{2} = \frac{h_e}{2}$

AND

$$N_1^{e\,\prime}(\varphi(\hat{x})) = \hat{N}_1'(\hat{x})\; j_e(\hat{x})^{-1} = -\frac{1}{2}\cdot\frac{2}{h_e} = -\frac{1}{h_e}$$

$$N_2^{e\,\prime}(\varphi(\hat{x})) = \hat{N}_2'(\hat{x})\; j_e(\hat{x})^{-1} = \frac{1}{2}\cdot\frac{2}{h_e} = \frac{1}{h_e}$$

COMPUTATION OF INTEGRALS :

$$\boxed{\int_{\Omega^e} f(x)\, dx = \int_{\hat{\Omega}} f(\varphi(\hat{x}))\, j_e(\hat{x})\, d\hat{x}}$$

EXAMPLE:

- $\varphi(\hat{x}) = (\hat{x}+2)^2 \qquad \Omega^e = [\varphi(-1), \varphi(1)] = [1, 9] \qquad f(x) = x^3$

$$j_e(\hat{x}) = \varphi'(\hat{x}) = 2(\hat{x}+2)$$

$$\int_{\Omega^e} f(x)\, dx = \int_1^9 x^3\, dx = \left.\frac{x^4}{4}\right|_1^9 = \frac{9^4}{4} - \frac{1}{4}$$

$$\int_{\hat{\Omega}} f(\varphi(\hat{x}))\, j_e(\hat{x})\, d\hat{x} = \int_{-1}^{1} \left[(\hat{x}+2)^2\right]^3 2(\hat{x}+2)\, d\hat{x} = \left.\frac{1}{4}(2+\hat{x})^8\right|_{-1}^{1}$$

$$= \frac{1}{4}3^8 - \frac{1}{4}1^8 = \frac{9^4}{4} - \frac{1}{4} \checkmark$$

- ELEMENT MATRIX (for example)

$$K^e_{ab} = \int_{\Omega^e} p(x)\, N^{e\prime}_a(x)\, N^{e\prime}_b(x)\, dx$$

$$= \int_{\hat{\Omega}} p(\varphi(\hat{x}))\, N^{e\prime}_a(\varphi(\hat{x}))\, N^{e\prime}_b(\varphi(\hat{x}))\, j_e(\hat{x})\, d\hat{x}$$

$$= \int_{\hat{\Omega}} \hat{p}(\hat{x})\, \frac{\hat{N}_a'(\hat{x})}{j_e(\hat{x})}\, \frac{\hat{N}_b'(\hat{x})}{j_e(\hat{x})}\, j_e(\hat{x})\, d\hat{x}$$

$$\boxed{K^e_{ab} = \int_{-1}^{1} \hat{p}(\hat{x})\, \hat{N}_a'(\hat{x})\, \hat{N}_b'(\hat{x})\, j_e(\hat{x})^{-1}\, d\hat{x}}$$

Can compute this in the parent element.

- ELEMENT LOAD VECTOR:

$$(f_f)^e_a = \int_{\Omega^e} f(x)\, N^e_a(x)\, dx$$

$$= \int_{\hat{\Omega}} f(\varphi(\hat{x}))\, N^e_a(\varphi(\hat{x}))\, j_e(\hat{x})\, d\hat{x}$$

$$\boxed{(f_f)^e_a = \int_{\hat{\Omega}} \hat{f}(\hat{x})\, \hat{N}_a(\hat{x})\, j_e(\hat{x})\, d\hat{x}}$$

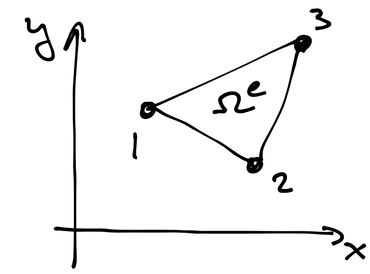
Can compute this in the parent element.

THE CONSTANT STRAIN TRIANGLE (CST), OR P_1-ELEMENT

REF: P. PINSKY CH. 7.

ELEMENT

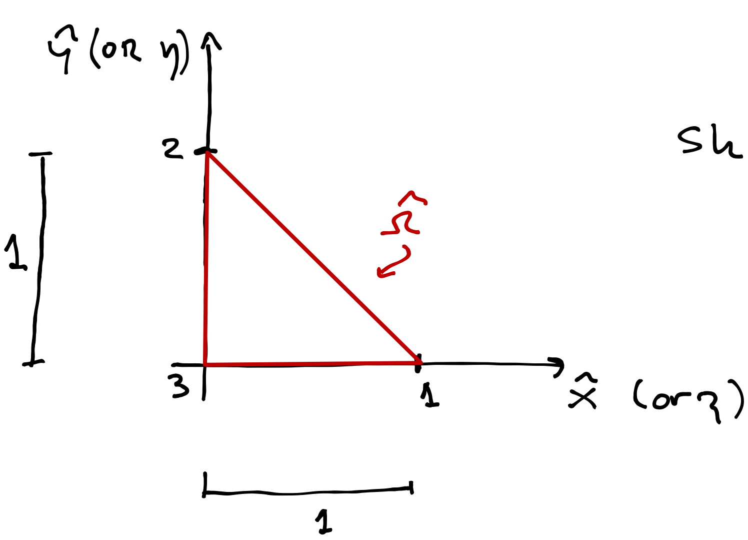
- $\Omega_e \equiv$ TRIANGLE IN 2D WITH NON-ZERO AREA
- $P_e = \text{span}\{1, x, y\}$



$\Rightarrow \quad p_h \in P \Rightarrow \quad p_h = c_1 + c_2 x + c_3 y,$
for $c_1, c_2, c_3 \in \mathbb{R}$.

$\bar{x}_1, \bar{x}_2, \bar{x}_3 \equiv$ POSITIONS OF THE NODES

REFERENCE ELEMENT.



Shape functions:

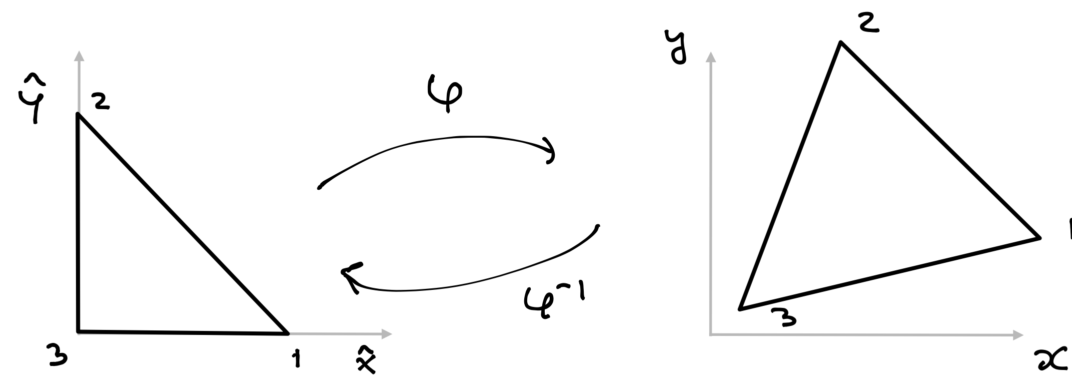
$\hat{N}_1(\hat{x},\hat{y}) = \hat{x} \qquad = \lambda_1$

$\hat{N}_2(\hat{x},\hat{y}) = \hat{y} \qquad = \lambda_2$

$\hat{N}_3(\hat{x},\hat{y}) = 1 - \hat{x} - \hat{y} \qquad = \lambda_3$

HOW DO WE CALCULATE N_a^e IN GENERAL?

BUILD AFFINE ELEMENT MAP $\varphi : \hat{\Omega} \to \Omega^e$



NUMBER NODES IN A COUNTER-CLOCKWISE DIRECTION.

$\bar{x}_a = (x_a, y_a)$

$\equiv$ COORDINATES OF NODE a.

$$\varphi(\hat{x},\hat{y}) = \sum_{a=1}^{3} \bar{x}_a \hat{N}_a(\hat{x},\hat{y})$$

$$\Rightarrow \begin{cases} \varphi_x(\hat{x},\hat{y}) = \sum_{a=1}^{3} x_a \hat{N}_a(\hat{x},\hat{y}) = x_1 \hat{x} + x_2 \hat{y} + x_3(1-\hat{x}-\hat{y}) \\ \varphi_y(\hat{x},\hat{y}) = \sum_{a=1}^{3} y_a \hat{N}_a(\hat{x},\hat{y}) = y_1 \hat{x} + y_2 \hat{y} + y_3(1-\hat{x}-\hat{y}) \end{cases}$$

φ_x, φ_y ARE LINEAR POLYNOMIALS

$\rightsquigarrow$ AFFINE FUNCTIONS

$\rightsquigarrow$ φ IS AN AFFINE MAP

Check:

- $\varphi_x(0,0) = x_3$
- $\varphi_y(0,0) = y_3$
- IN GENERAL: $\varphi(\hat{\bar{x}}_a) = \bar{x}_a \qquad a = 1,2,3$
- $\varphi(1/2, 1/2) = \left(\frac{x_1}{2} + \frac{x_2}{2}, \frac{y_1}{2} + \frac{y_2}{2}\right) = \frac{\bar{x}_1 + \bar{x}_2}{2}$

CAN WRITE: $\varphi(\hat{x},\hat{y}) = J^e \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} + \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad J^e = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix}$

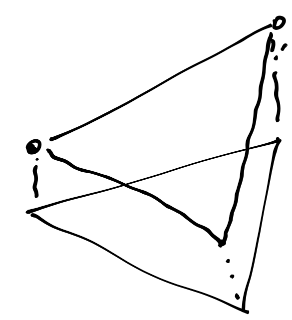
so

$$\varphi^{-1}(x,y) = (J^e)^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_3 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

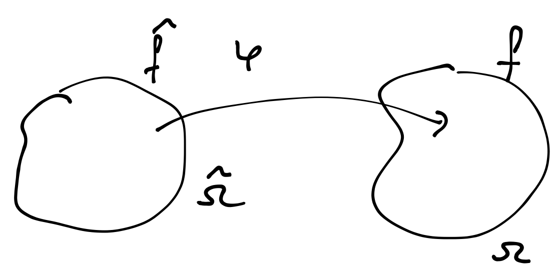
SHAPE FUNCTIONS: $N_a^e(\bar{x}) = \hat{N}_a(\hat{\bar{X}}_a) = \hat{N}_a(\varphi^{-1}(\bar{x}))$

WE DO NOT CONSTRUCT N_a^e explicitly, but evaluate them at corresponding points in $\hat{\Omega}$, AND THE SAME FOR DERIVATIVES.

N_a^e is linear along element edges



DERIVATIVES:



$\hat{f}$ is pullback of f

$f(\varphi(\hat{x})) = \hat{f}(\hat{x}) \Rightarrow \dfrac{\partial \hat{f}}{\partial \hat{x}_I}(\varphi(\hat{x})) = \dfrac{\partial f}{\partial x_i}(\varphi(\hat{x}))\, \dfrac{\partial \varphi_i}{\partial \hat{x}_I}(\hat{x})$ (sum over i)

$J_{iI}(\hat{x}) = \dfrac{\partial \varphi_i}{\partial \hat{x}_I}$ JACOBIAN MATRIX OF φ OR GRADIENT OF φ

$\Rightarrow \dfrac{\partial \hat{f}}{\partial \hat{x}_I} = J_{iI} \dfrac{\partial f}{\partial x_i}$ OR $\nabla \hat{f}(\hat{x}) = \nabla f(\varphi(\hat{x})) . J(\hat{x})$

$\Rightarrow \boxed{\nabla f(\bar{x}) = \nabla \hat{f}(\varphi^{-1}(\bar{x})) . J^{-1}(\varphi^{-1}(\bar{x}))}$

IN OUR CASE: $\varphi(\hat{x}_1, \hat{x}_2) = (J_e)^T$

IN OUR CASE:

$$\varphi_x(\hat{x},\hat{y}) = x_1\,\hat{x} + x_2\,\hat{y} + x_3\,(1-\hat{x}-\hat{y})$$

$$\varphi_y(\hat{x},\hat{y}) = y_1\,\hat{x} + y_2\,\hat{y} + y_3\,(1-\hat{x}-\hat{y})$$

$$J(\hat{x},\hat{y}) = \begin{bmatrix} \frac{\partial \varphi_x}{\partial \hat{x}} & \frac{\partial \varphi_x}{\partial \hat{y}} \\ \frac{\partial \varphi_y}{\partial \hat{x}} & \frac{\partial \varphi_y}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{bmatrix} = J^e \text{ ?}$$

CONSTANT OVER Ω^e ?

So:

$$\boxed{\nabla N_a^e(\bar{x}) = \nabla \hat{N}_a(\hat{x}).(J^e)^{-1}}$$

$$(J^e)^{-1} = \begin{bmatrix} y_2 - y_3 & -x_2 + x_3 \\ -y_1 + y_3 & x_1 - x_3 \end{bmatrix} \frac{1}{\det J^e}$$

$$\det J^e = (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3)$$

what about the earlier formula,

∇_λ , not $\nabla_{(\hat{x},\hat{y})}$

$$\boxed{\nabla N_a^e = \nabla \hat{N}_a^e \cdot dN^T}$$

$$dN = \begin{bmatrix} x_2^2 - x_2^3 & x_2^3 - x_2^1 & x_2^1 - x_2^2 \\ x_1^3 - x_1^2 & x_1^1 - x_1^3 & x_1^2 - x_1^1 \end{bmatrix} \frac{1}{2A^e}$$

we wrote $\hat{N}_a^e(\lambda_1, \lambda_2, \lambda_3) = \lambda_a$

instead write $\hat{N}_1^e = \lambda_1$, $\hat{N}_2^e = \lambda_2$, $\hat{N}_3^e = 1 - \lambda_1 - \lambda_2$

$\uparrow \hat{x}$ $\qquad$ $\uparrow \hat{y}$

INTEGRALS OVER Ω^e

$$\int_{\Omega^e} f(x)\, d\Omega = \int_{\hat{\Omega}} f(\varphi(\hat{x}))\ \det J\ d\hat{\Omega}$$

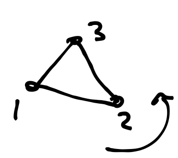
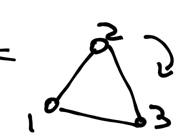
$\det J \equiv$ JACOBIAN OF φ

IN OUR CASE: $\det J = \det J^e$ CONSTANT OVER $\hat{\Omega}$

$$A^e = \int_{\Omega} d\Omega = \int_{\hat{\Omega}} \det J^e\, d\hat{\Omega} = \det J^e \int_{\hat{\Omega}} d\hat{\Omega} = \det J_e\, A^{\hat{e}} = \frac{\det J^e}{2}$$

The $\det J^e$ is 2 times the area of Ω^e.

ALL INTEGRALS ARE COMPUTED OVER $\hat{\Omega}$.

IF  THEN $\det J^e > 0$. IF  THEN $\det J^e < 0$.

(This is why)

LECTURE #19 03/02/20

ISOPARAMETRIC ELEMENTS

$\mathbb{P}_2$ ELEMENT AND ISOPARAMETRIC MAP (ISOPARAMETRIC $\mathbb{P}_2$ ELEMENT)

GET WHAT WE GET HERE $\rightarrow$ $(\Omega^e, \mathcal{N}) \xleftarrow{\varphi} (\hat{\Omega}, \hat{\mathcal{N}})$ $\leftarrow$ DEFINE IT HERE

DEF: (SPACES OF POLYNOMIALS IN $\mathbb{R}^d$)

$\mathbb{P}_k(\Omega)$ = POLYNOMIALS OF DEGREE $\leq k$ OVER $\Omega \subset \mathbb{R}^d$

$$p(x_1, \dots, x_d) \in \mathbb{P}_k(\Omega) \Rightarrow p(x_1, \dots, x_d) = \sum_{\substack{i_1 + \dots + i_d \leq k \\ i_j \geq 0 \ \forall j}} a_{i_1 \dots i_d} \, x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$$

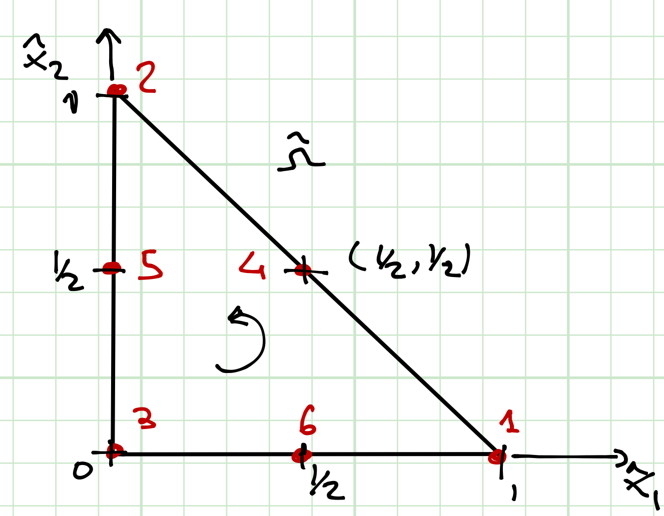
$$\mathbb{P}_k(\Omega) = \text{span} \{ x_1^{i_1} \cdots x_d^{i_d} \}_{\substack{i_1 + \dots + i_d \leq k \\ i_j \geq 0 \ \forall j}}$$

EXAMPLE: $p(x) \in \mathbb{P}_2(\Omega)$, $\Omega \subset \mathbb{R}^2$

$$p(x_1, x_2) = a_{00} + a_{10} x_1 + a_{01} x_2 + a_{20} x_1^2 + a_{11} x_1 x_2 + a_{02} x_2^2$$

REFERENCE ELEMENT:

- $\hat{\Omega}$ = TRIANGLE
- $\mathcal{N} = \{ \hat{N}_1, \hat{N}_2, \hat{N}_3, \hat{N}_4, \hat{N}_5, \hat{N}_6 \}$



SHAPE FUNCTIONS:

$$\hat{N}_a(\xi_1,\xi_2,\xi_3) = 2\xi_a(\xi_a - 1/2) \qquad a=1,2,3$$

$$\hat{N}_4(\xi_1,\xi_2,\xi_3) = 4\xi_1\xi_2$$

$$\hat{N}_5(\xi_1,\xi_2,\xi_3) = 4\xi_2\xi_3$$

$$\hat{N}_6(\xi_1,\xi_2,\xi_3) = 4\xi_3\xi_1$$

SEE PLOTS IN MATHEMATICA FILE

DEF: (LAGRANGE FINITE ELEMENT) $(\Omega, \mathcal{N})$ SUCH THAT FOR ALL $N_a \subset \mathcal{N}$, $N_a(\bar{x}_b) = \delta_{ab}$ FOR $\{\bar{x}_1, \dots, \bar{x}_k\} \subset \Omega$, $\bar{x}_a \neq \bar{x}_b$ if $a \neq b$.

DEF: (ISOPARAMETRIC MAP)

LET $(\hat{\Omega}, \hat{\mathcal{N}})$ BE A LAGRANGE FINITE ELEMENT. AN ELEMENT MAP

$$\varphi(\hat{x}) = \sum \bar{x}_a \hat{N}_a(\hat{x})$$

is CALLED AN ISOPARAMETRIC MAP.

WE NEED φ TO BE ONE-TO-ONE, AND NEED $\det \nabla\varphi > 0$ in $\hat{\Omega}$

EXAMPLE:

a) THE AFFINE MAP IS AN ISOPARAMETRIC MAP FOR CST OR P_1-ELEMENTS

b) ISOPARAMETRIC MAP FOR P_2 ELEMENTS

$$\varphi(\hat{x}) = \sum_{a=1}^{6} \bar{x}_a \hat{N}_a(\hat{x}) \longrightarrow \varphi_x, \varphi_y \text{ ARE GENERALLY quadratic polynomials in } \hat{x}_1, \hat{x}_2.$$

in components:

$$\varphi_i(\hat{x}) = \sum_{a=1}^{6} x_{ai} \hat{N}_a(\hat{x})$$

$$\text{If } \begin{cases} \bar{x}_4 = \dfrac{\bar{x}_1 + \bar{x}_2}{2} \\ \bar{x}_5 = \dfrac{\bar{x}_2 + \bar{x}_3}{2} \\ \bar{x}_6 = \dfrac{\bar{x}_3 + \bar{x}_1}{2} \end{cases} \Rightarrow \varphi(\hat{x}) \text{ is AFFINE AGAIN}$$

SEE MATHEMATICA FILE

ELEMENT DEFINITION: GIVEN THE ISOPARAMETRIC MAP φ, AN ISOPARAMETRIC $\mathbb{P}_2$ ELEMENT IS DEFINED AS:

a) $\Omega = \varphi(\hat{\Omega})$

b) $\mathcal{N} = \{N_1, \ldots, N_k\}$ with $N_a(x) = \hat{N}_a(\varphi^{-1}(x)) \quad \forall x \in \Omega.$

Why is this special? Because if φ is not affine, $\mathcal{P} \neq \mathbb{P}_2(\Omega)$, i.e., functions in $\mathcal{P}$ are not necessarily polynomials

See Mathematica file.

DERIVATIVES: WE WILL NEED $J^e(\hat{x})$.

$$J^e_{iI}(\hat{x}) = \frac{\partial \varphi_i(\hat{x})}{\partial \hat{x}_I} = \sum_{a=1}^{6} x_{ai} \frac{\partial \hat{N}_a}{\partial \hat{x}_I}$$

Then:

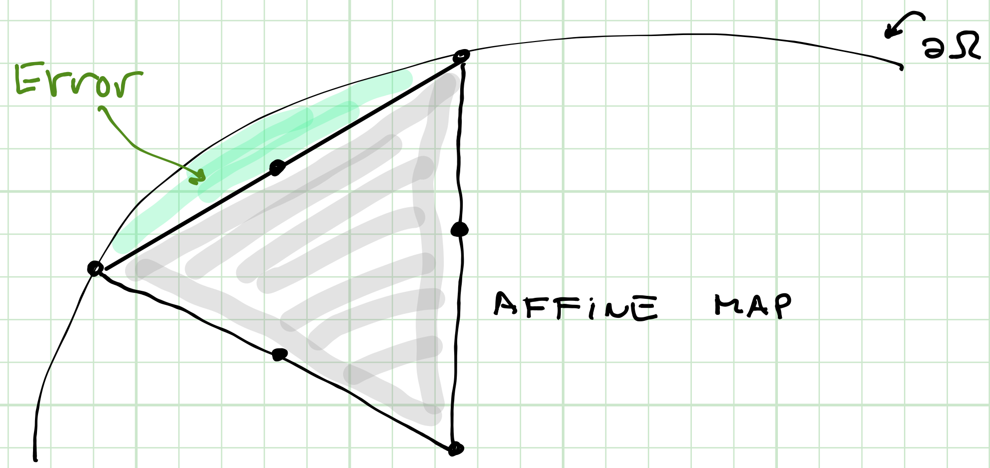
$$\nabla N_a(\bar{x}) = \nabla \hat{N}_a(\varphi^{-1}(x)) \cdot (J^e)^{-1}$$

$$N_{a,i}(\bar{x}) = \hat{N}_{a,I}(\hat{x}) \, (J^e)^{-1}_{Ii}$$

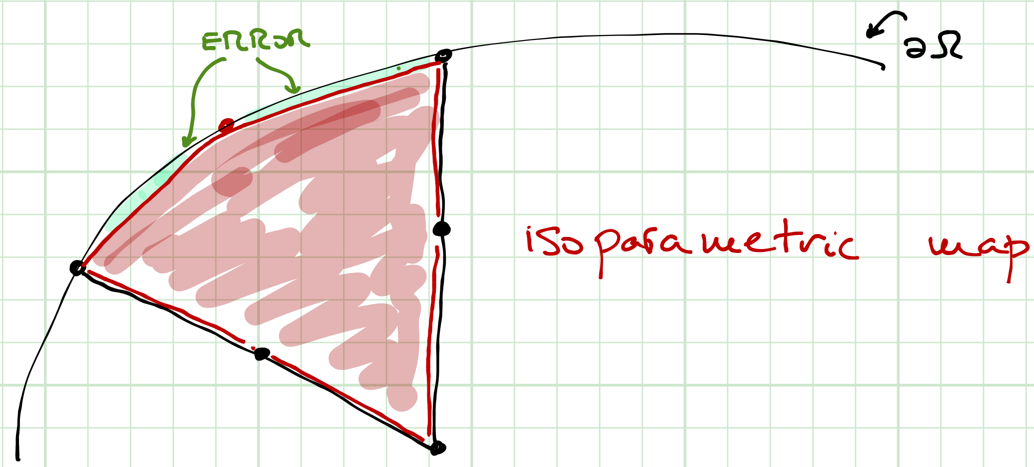
SEE MATHEMATICA FILE FOR AN EXAMPLE

why complicate our life with non-affine maps?

1. Isoparametric mappings make the error introduced in approximating curved boundaries decrease at a faster speed as the mesh size is refined.



FIND A POINT on $\partial\Omega$ where to place a middle node



2. To BUILD A GENERAL CLASS OF QUADRILATERAL AND HEXAHEDRAL ELEMENTS

QUADS AND HEXES (TENSOR PRODUCT ELEMENTS)

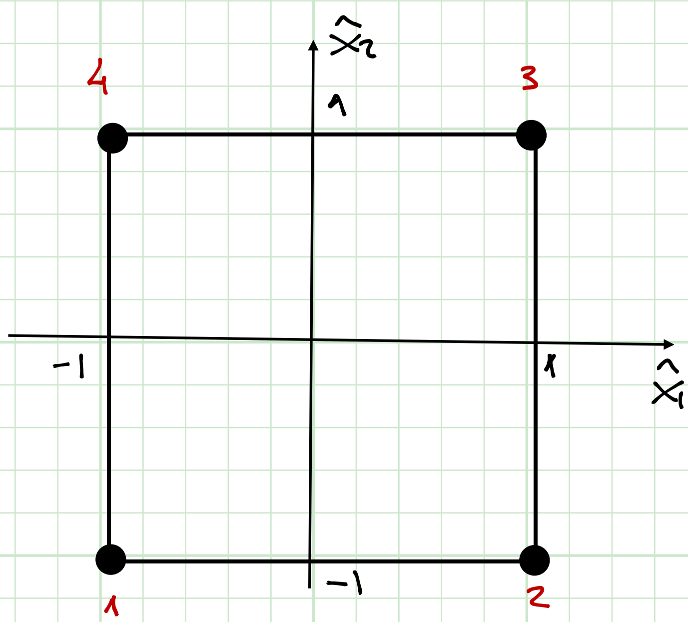
REFERENCE ELEMENTS:

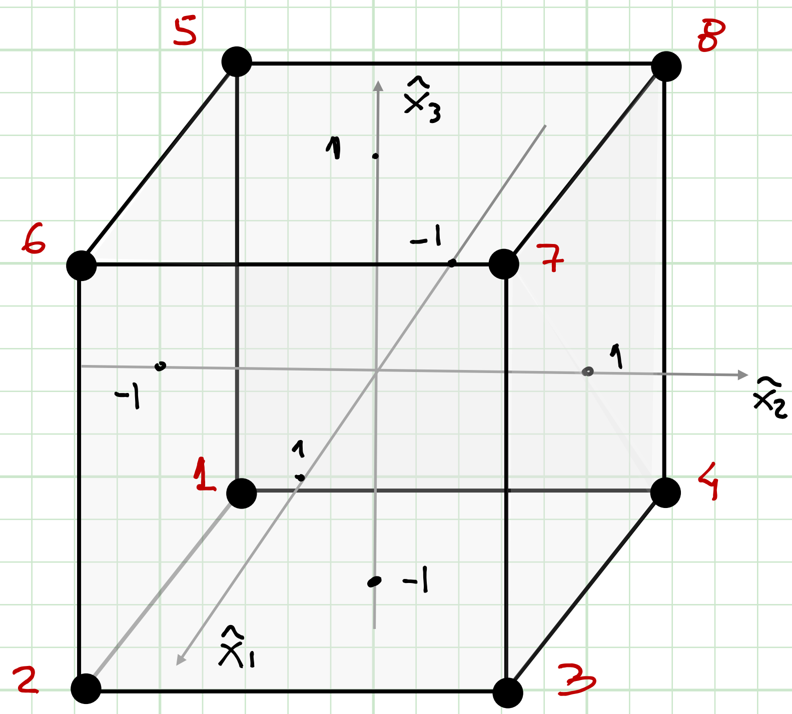
Q_1 - QUAD

- $\hat{\Omega} = [-1,1] \times [-1,1]$
- $\hat{P} = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1\hat{x}_2\}$
  BILINEAR FUNCTIONS
- $\hat{\mathcal{N}} = \{\hat{N}_1, \hat{N}_2, \hat{N}_3, \hat{N}_4\}$

Q_1 - HEX

- $\hat{\Omega} = [-1,1]^3$
- $\hat{P} = \text{span}\{1, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_1\hat{x}_2, \hat{x}_1\hat{x}_3, \hat{x}_2\hat{x}_3, \hat{x}_1\hat{x}_2\hat{x}_3\}$
  TRILINEAR FUNCTIONS
- $\hat{\mathcal{N}} = \{\hat{N}_1, \hat{N}_2, \dots, \hat{N}_8\}$





- Dual Basis:

$$\hat{N}_1 = \frac{1}{4}(1-\hat{x}_1)(1-\hat{x}_2)$$

$$\hat{N}_2 = \frac{1}{4}(1+\hat{x}_1)(1-\hat{x}_2)$$

$$\hat{N}_3 = \frac{1}{4}(1+\hat{x}_1)(1+\hat{x}_2)$$

$$\hat{N}_4 = \frac{1}{4}(1-\hat{x}_1)(1+\hat{x}_2)$$

- Dual Basis:

$$\hat{N}_1 = \frac{1}{8}(1-\hat{x}_1)(1-\hat{x}_2)(1-\hat{x}_3)$$

$$\hat{N}_2 = \frac{1}{8}(1+\hat{x}_1)(1-\hat{x}_2)(1-\hat{x}_3)$$

$$\hat{N}_3 = \frac{1}{8}(1+\hat{x}_1)(1+\hat{x}_2)(1-\hat{x}_3)$$

$$\hat{N}_4 = \frac{1}{8}(1-\hat{x}_1)(1+\hat{x}_2)(1-\hat{x}_3)$$

$$\hat{N}_5 = \frac{1}{8}(1-\hat{x}_1)(1-\hat{x}_2)(1+\hat{x}_3)$$

$$\hat{N}_6 = \frac{1}{8}(1+\hat{x}_1)(1-\hat{x}_2)(1+\hat{x}_3)$$

$$\hat{N}_7 = \frac{1}{8}(1+\hat{x}_1)(1+\hat{x}_2)(1+\hat{x}_3)$$

$$\hat{N}_8 = \frac{1}{8}(1-\hat{x}_1)(1+\hat{x}_2)(1+\hat{x}_3)$$

SUCCINTLY:

$$\hat{N}_a(\hat{x}_1,\hat{x}_2) = \frac{1}{4}(1+\xi_{1a}\hat{x}_1)(1+\xi_{2a}\hat{x}_2) \qquad \hat{N}_a(\hat{x}_1,\hat{x}_2,\hat{x}_3) = \frac{1}{8}(1+\xi_{1a}\hat{x}_1)(1+\xi_{2a}\hat{x}_2)(1+\xi_{3a}\hat{x}_3)$$

FOR SUITABLE VALUES OF $(\xi_{1a}, \xi_{2a}, \xi_{3a})$ (see e.g. TJRH 3.2)

WHEN MOVING ONE VARIABLE KEEPING ALL OTHER VARIABLES FIXED, THE FUNCTION IS AFFINE WRT THE MOVING VARIABLE

GENERAL ELEMENT THROUGH THE ISOPARAMETRIC MAP:

Q_1 — QUAD

$$\varphi(\hat{x}_1, \hat{x}_2) = \sum_{a=1}^{4} \bar{X}_a \hat{N}_a(\hat{x}_1, \hat{x}_2)$$

Q_1 — HEX

$$\varphi(\hat{x}_1, \hat{x}_2) = \sum_{a=1}^{8} \bar{X}_a \hat{N}_a(\hat{x}_1, \hat{x}_2)$$

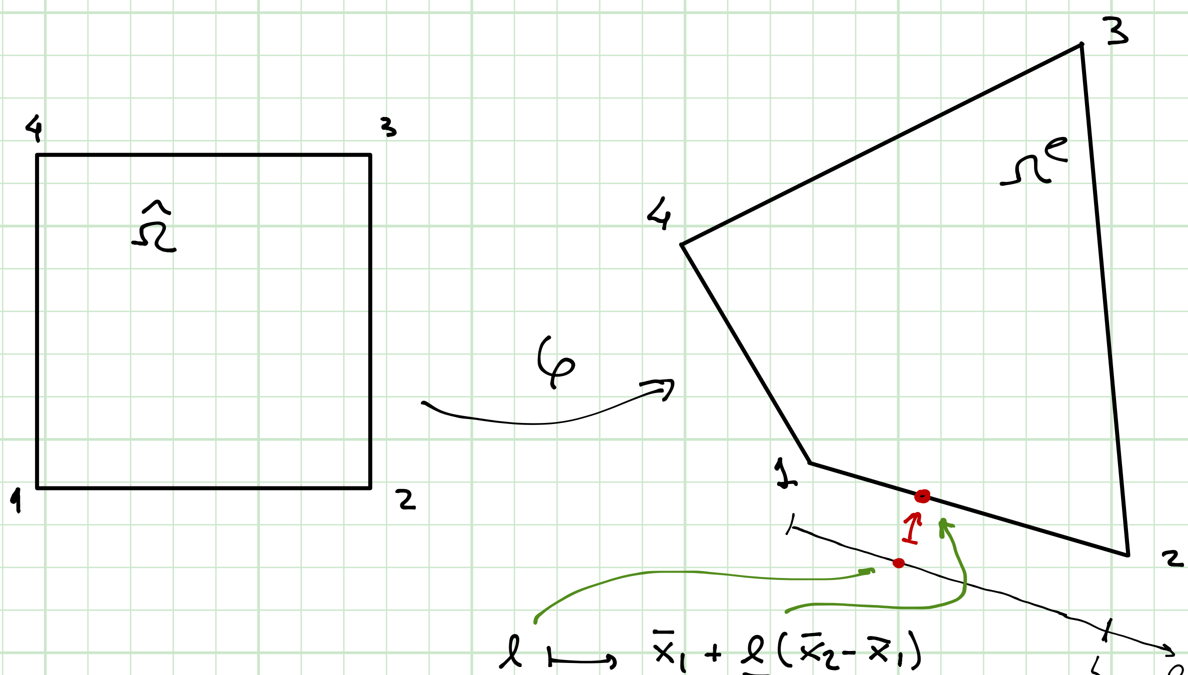
EXAMPLES IN MATHEMATICA FILE

<u>SHAPE FUNCTIONS:</u>

- $N_a^e(x_1, x_2) = \hat{N}_a(\varphi^{-1}(x_1, x_2))$ for $(x_1, x_2) \in \varphi(\hat{e})$
- $N_a^e(x_1, x_2, x_3) = \hat{N}_a(\varphi^{-1}(x_1, x_2, x_3))$ for $(x_1, x_2, x_3) \in \varphi(\hat{e})$

THESE FUNCTIONS ARE ALSO AFFINE ALONG THE EDGES OR FACES OF THE ELEMENT.

<u>EXAMPLE:</u> ALONG THE 1-2 EDGE IN A QUAD, $\hat{x}_2 = -1$



$$\ell \longmapsto \bar{x}_1 + \frac{\ell}{L}(\bar{x}_2 - \bar{x}_1)$$

$$\Rightarrow \varphi(\hat{x}_1, -1) = \bar{x}_1 \hat{N}_1(\hat{x}_1, -1) + \bar{x}_2 \hat{N}_2(\hat{x}_2, -1)$$

$$= \bar{x}_1 \frac{1}{2}(1-\hat{x}_1) + \bar{x}_2 \frac{1}{2}(1+\hat{x}_1)$$

$$= \bar{x}_1 + \frac{1}{2}(1+\hat{x}_1)(\bar{x}_2 - \bar{x}_1) \rightsquigarrow \frac{\ell}{L} = \frac{1}{2}(1+\hat{x}_1)$$

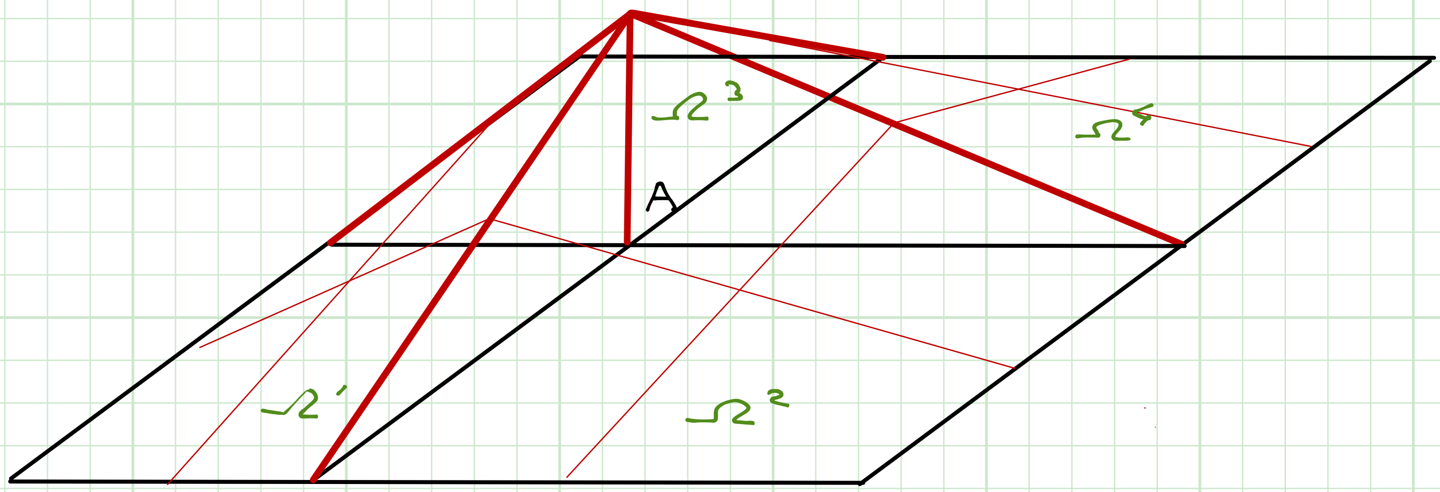
$$\Rightarrow \hat{x}_1(\ell) = 2\frac{\ell}{L} - 1$$

$$N_1^e(\ell) = \hat{N}_1(\hat{x}_1(\ell), -1) = \frac{1}{4}(1-\hat{x}_1(\ell))\, 2$$

$$= \frac{1}{2}\left(1 - \left(2\frac{\ell}{L} - 1\right)\right) = 1 - \frac{\ell}{L}$$

AFFINE
ALONG
THE SIDE

Global BASIS FUNCTIONS:



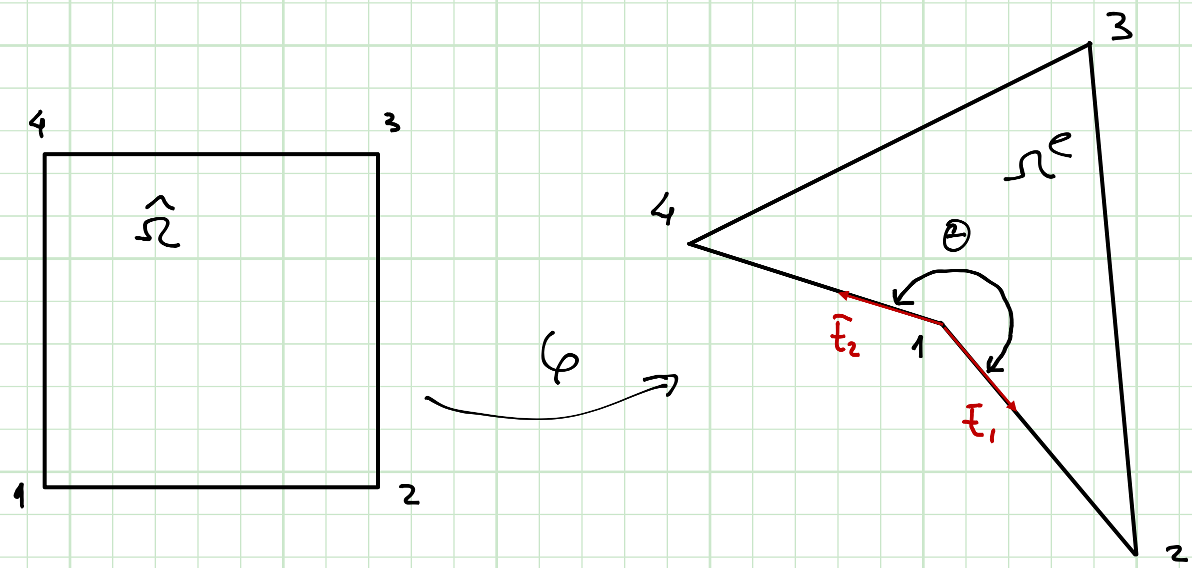
element →

$$LV = \begin{array}{c} n \\ o \\ d \\ e \\ \downarrow \end{array} \begin{pmatrix} A_1 & & \\ A_2 & \cdots & \\ A_3 & & \\ A_4 & & \end{pmatrix} = L6$$

A global shape function then is 1 AT 1 NODE, AND ZERO AT ALL OTHERS.

BECAUSE FUNCTIONS ARE AFFINE ALONG EACH EDGE, THE Global shape function is continuous and in $H^1(\Omega)$.

Problem: If $\Theta \geq \pi$, $\det J^e(\hat{x}_1, \hat{x}_2) \leq 0$ for some $(\hat{x}_1, \hat{x}_2) \in \hat{\Omega}$.



IN GENERAL: $\det J^e(\hat{x}_1, \hat{x}_2) = \left(\frac{\partial \varphi}{\partial x_1} \times \frac{\partial \varphi}{\partial x_2} \right) \cdot \bar{e}_3$ UNIT VECTOR NORMAL TO $\hat{\Omega}$, $\bar{e}_3 = \bar{e}_1 \times \bar{e}_2$

AT 1: $\frac{\partial \varphi}{\partial x_1} \parallel \bar{t}_1 \quad \frac{\partial \varphi}{\partial x_2} \parallel \bar{t}_2$

$$(\bar{t}_1 \times \bar{t}_2) \cdot \bar{e}_z < 0$$

IF $\Theta = \pi \Rightarrow (\bar{t}_1 \times \bar{t}_2) \cdot \bar{e}_z = 0 = \det J_e|_{\hat{x} = \bar{x}_1}$.

Why is this an issue? The derivatives of the shape functions are not defined where $\det J^e = 0$.

I DO NOT RECOMMEND USING ELEMENTS WITH $\Theta \geq \pi$.

EXAMPLES:

- P_k AND Q_k elements are complete to order K.

Lagrange quadrilaterals

$$
\begin{array}{c}
1 \\
\xi \quad \eta \\
\xi^2 \quad \xi\eta \quad \eta^2 \\
\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \\
\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4 \\
\cdots \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \cdots \\
\xi^m \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \eta^m \\
\xi^m\eta \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi\eta^m \\
\xi^m\eta^2 \quad \xi^4\eta^4 \quad \xi^2\eta^m \\
\xi^m\eta^3 \quad \xi^3\eta^m \\
\xi^m\eta^4 \quad \xi^4\eta^m \\
\cdots \quad \cdots \\
\xi^m\eta^m
\end{array}
$$

$K=4$

INCLUDED, BUT DO NOT ENHANCE CONVERGENCE

Serendipity quadrilaterals

$$
\begin{array}{c}
1 \\
\xi \quad \eta \\
\xi^2 \quad \xi\eta \quad \eta^2 \\
\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \\
\xi^4 \quad \xi^3\eta \quad \xi\eta^3 \quad \eta^4 \\
\cdots \quad \xi^4\eta \quad \xi\eta^4 \quad \cdots \\
\xi^m \quad \cdots \quad \cdots \quad \eta^m \\
\xi^m\eta \quad \xi\eta^m
\end{array}
$$

Triangles (internal nodal functions included)

$$
\begin{array}{c}
1 \\
\xi \quad \eta \\
\xi^2 \quad \xi\eta \quad \eta^2 \\
\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3 \\
\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4 \\
\cdots \\
\xi^m \quad \xi^{m-1}\eta \quad \xi^{m-2}\eta^2 \quad \cdots \quad \xi^2\eta^{m-2} \quad \xi\eta^{m-1} \quad \eta^m
\end{array}
$$

ALL FUNCTIONS IMPORTANT FOR ORDER OF CONVERGENCE

Figure 3.7.8. Pascal triangles for standard two-dimensional element families.

Isoparametric P2 Element

Shape Functions

Transformation from triangular to natural coordinates in reference element

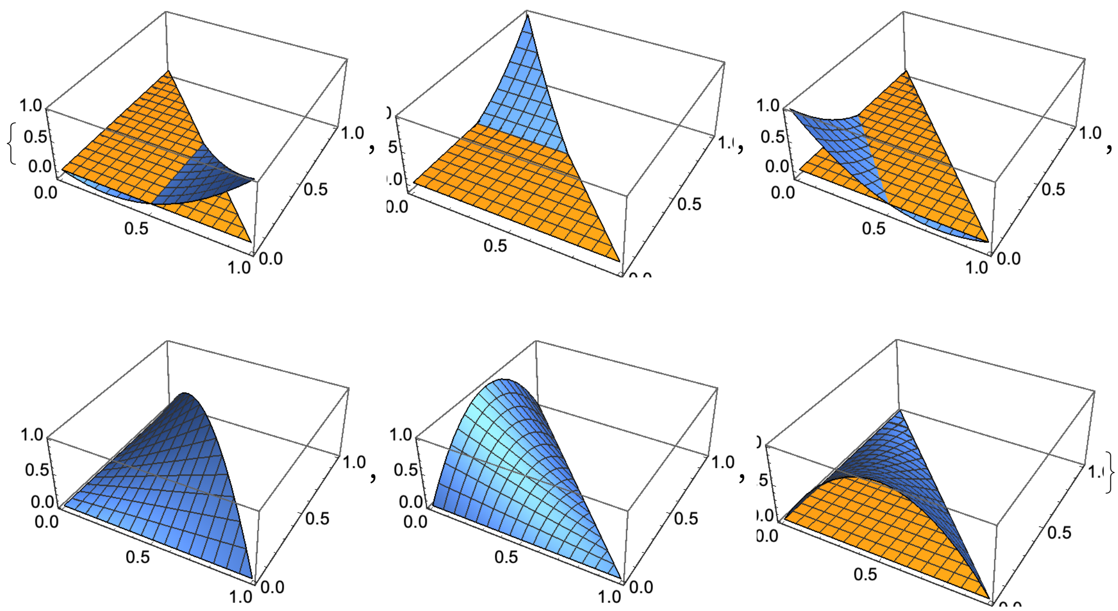
```
In[∘]:= transf = {ξ1 → x1, ξ2 → x2, ξ3 → 1 - x1 - x2};
```

Shape functions in terms of natural coordinates

```
In[∘]:= NN = {2 ξ1 (ξ1 - 1 / 2), 2 ξ2 (ξ2 - 1 / 2), 2 ξ3 (ξ3 - 1 / 2), 4 ξ1 ξ2, 4 ξ2 ξ3, 4 ξ1 ξ3} /. transf
```

Out[∘]=
$$\left\{2\left(-\frac{1}{2}+x1\right)x1,\ 2\left(-\frac{1}{2}+x2\right)x2,\ 2\left(\frac{1}{2}-x1-x2\right)(1-x1-x2),\ 4\,x1\,x2,\ 4\,(1-x1-x2)\,x2,\ 4\,x1\,(1-x1-x2)\right\}$$

```
In[∘]:= Table[Plot3D[{0, NN[[i]]}, {x1, 0, 1}, {x2, 0, 1 - x1}, PlotRange → All], {i, 1, 6}]
```

Out[∘]=


Isoparametric Map

```
In[∘]:= xxcorners = {{1, 1}, {2, 2}, {0, 3}}
```

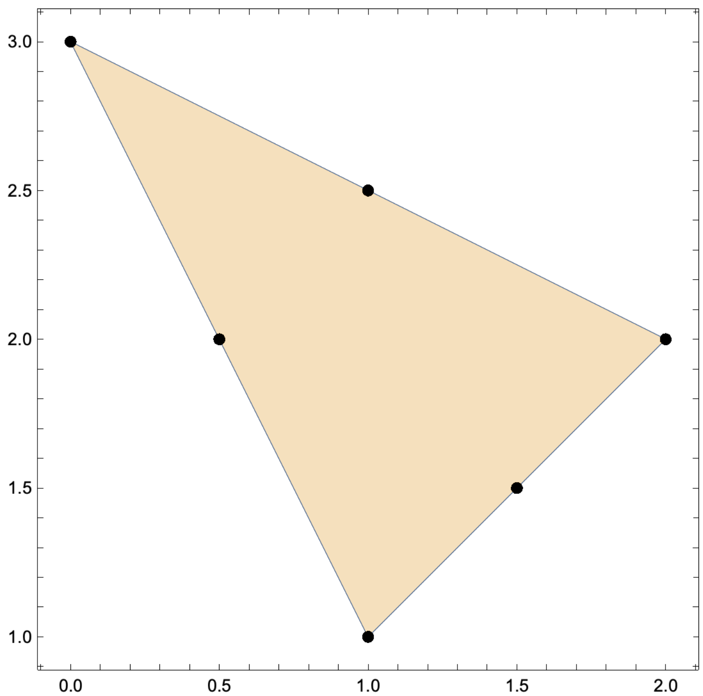
```
Out[∘]= {{1, 1}, {2, 2}, {0, 3}}
```

```
In[∘]:= xxmiddle = {(xxcorners[[1]] + xxcorners[[2]]) / 2,
   (xxcorners[[2]] + xxcorners[[3]]) / 2, (xxcorners[[3]] + xxcorners[[1]]) / 2}
```

Out[∘]=
$$\left\{\left\{\frac{3}{2},\frac{3}{2}\right\},\left\{1,\frac{5}{2}\right\},\left\{\frac{1}{2},2\right\}\right\}$$

In[∘]:=
```
xx = Join[xxcorners, xxmiddle]
```
Out[∘]= $\left\{\{1, 1\}, \{2, 2\}, \{0, 3\}, \left\{\frac{3}{2}, \frac{3}{2}\right\}, \left\{1, \frac{5}{2}\right\}, \left\{\frac{1}{2}, 2\right\}\right\}$

In[∘]:=
```
φ = Simplify[Sum[xx[[i]] × NN[[i]], {i, 1, 6}]]
```
Out[∘]= {x1 + 2 x2, 3 - 2 x1 - x2}

In[∘]:=
```
pl1 = ParametricPlot[φ, {x1, 0, 1}, {x2, 0, 1 - x1}];
pl2 = Table[Graphics[Disk[xx[[i]], 0.02]], {i, 1, 6}];
Show[pl1, pl2]
```
Out[∘]=


Now, let’s curve the edges

In[∘]:=
```
shifts = {{-0.1, 0.1}, {0.1, 0.1}, {0, -0.1}};
α = 3;
(*α=-20;*)
```

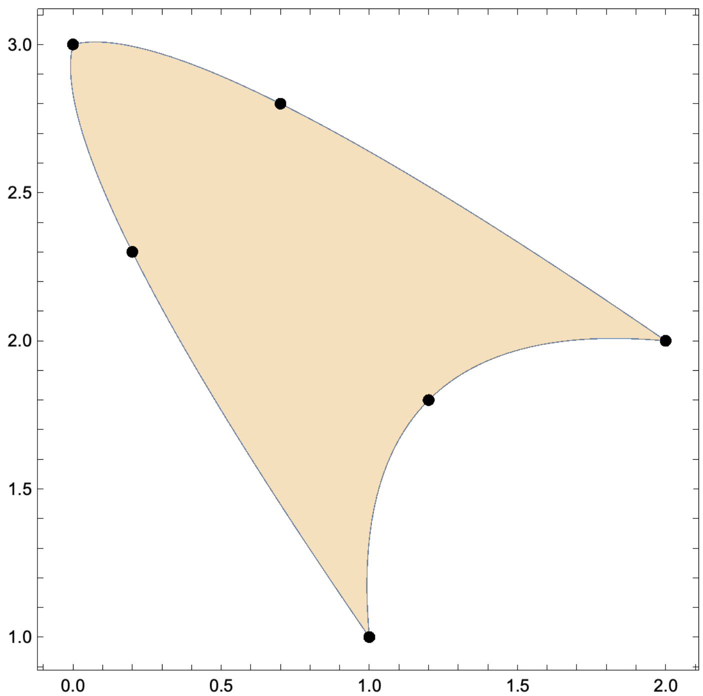
In[∘]:=
```
xxmiddle = {(xxcorners[[1]] + xxcorners[[2]]) / 2 + α shifts[[1]],
   (xxcorners[[2]] + xxcorners[[3]]) / 2 + α shifts[[1]],
   (xxcorners[[3]] + xxcorners[[1]]) / 2 + α shifts[[1]]}
```
Out[∘]= {{1.2, 1.8}, {0.7, 2.8}, {0.2, 2.3}}

In[∘]:=
```
xx = Join[xxcorners, xxmiddle]
```
Out[∘]= {{1, 1}, {2, 2}, {0, 3}, {1.2, 1.8}, {0.7, 2.8}, {0.2, 2.3}}

In[∘]:=
```
φ = Simplify[Sum[xx[[i]] × NN[[i]], {i, 1, 6}]]
```

Out[∘]=
$$\{1.2\,x1^2 + x1\,(-0.2 + 1.2\,x2) + x2\,(0.8 + 1.2\,x2),\ 3. - 1.2\,x1^2 + x1\,(-0.8 - 1.2\,x2) + 0.2\,x2 - 1.2\,x2^2\}$$

In[∘]:=
```
pl1 = ParametricPlot[φ, {x1, 0, 1}, {x2, 0, 1 - x1}];
pl2 = Table[Graphics[Disk[xx[[i]], 0.02]], {i, 1, 6}];
Show[pl1, pl2]
```

Out[∘]=


Shape functions over the element

Compute inverse map

In[∘]:=
```
φinv = Simplify[Solve[{z1, z2} == φ, {x1, x2}]]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[∘]=
$$\{\{x1 \to 1.41667 - 0.5\,z1 - 0.5\,z2 - 0.0833333\sqrt{-47. + 92.\,z1 - 12.\,z1^2 + 52.\,z2 - 24.\,z1\,z2 - 12.\,z2^2},\ x2 \to -1.58333 + 0.5\,z1 + 0.5\,z2 - 0.0833333\sqrt{-47. + 92.\,z1 - 12.\,z1^2 + 52.\,z2 - 24.\,z1\,z2 - 12.\,z2^2}\},$$
$$\{x1 \to 0.0833333\left(17. - 6.\,z1 - 6.\,z2 + \sqrt{-47. + 92.\,z1 - 12.\,z1^2 + 52.\,z2 - 24.\,z1\,z2 - 12.\,z2^2}\right),$$
$$x2 \to 0.0833333\left(-19. + 6.\,z1 + 6.\,z2 + \sqrt{-47. + 92.\,z1 - 12.\,z1^2 + 52.\,z2 - 24.\,z1\,z2 - 12.\,z2^2}\right)\}\}$$

Check which one is the good solution (it'll be #2)

In[∘]:=
```
φinv /. z1 → 1 /. z2 → 1
```

Out[∘]=
```
{{x1 → -0.166667, x2 → -1.16667}, {x1 → 1., x2 → 0.}}
```

Shape functions over the element by composition

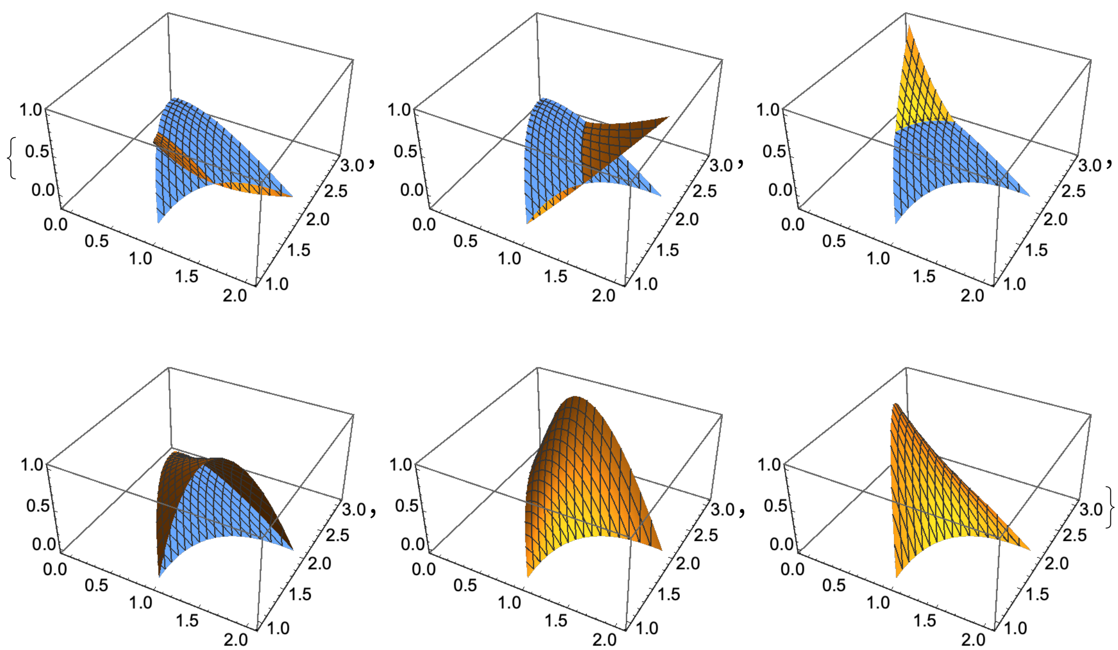
In[∘]:=
```
NNe = FullSimplify[NN /. φinv[[2]]];
NNe // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.0138889\left(17.-6.\,z1-6.\,z2+\sqrt{-47.-12.\,z1^2+z1\,(92.-24.\,z2)+(52.-12.\,z2)\,z2}\right)\left(11.\right. \\ 0.0138889\left(-19.+6.\,z1+6.\,z2+\sqrt{-47.-12.\,z1^2+z1\,(92.-24.\,z2)+(52.-12.\,z2)\,z2}\right)\left(-25\right. \\ -1.05556-0.666667\,z1^2+z1\,(5.11111-1.33333\,z2)+(2.88889-0.666667\,z2)\,z2- \\ -10.2778-1.33333\,z1^2+z1\,(8.55556-2.66667\,z2)+(7.44444-1.33333\,z2)\,z2-0 \\ -0.0555556\left(-19.+6.\,z1+6.\,z2+\sqrt{-47.-12.\,z1^2+z1\,(92.-24.\,z2)+(52.-12.\,z2)\,z\right.\right. \\ -0.0555556\left(17.-6.\,z1-6.\,z2+\sqrt{-47.-12.\,z1^2+z1\,(92.-24.\,z2)+(52.-12.\,z2)\,z2\right.\right. \end{pmatrix}$$

Plot the shape functions over the element

In[∘]:=
```
Table[ParametricPlot3D[{{φ[[1]], φ[[2]], NN[[i]]}, {φ[[1]], φ[[2]], 0}},
  {x1, 0, 1}, {x2, 0, 1 - x1}, PlotRange → All], {i, 1, 6}]
```

Out[∘]= 

Jacobian Matrix and Jacobian

In[∘]:=
```
Je = Transpose[{D[φ, x1], D[φ, x2]}];
MatrixForm[Je]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -0.2 + 2.4\,x1 + 1.2\,x2 & 0.8 + 1.2\,x1 + 2.4\,x2 \\ -0.8 - 2.4\,x1 - 1.2\,x2 & 0.2 - 1.2\,x1 - 2.4\,x2 \end{pmatrix}$$

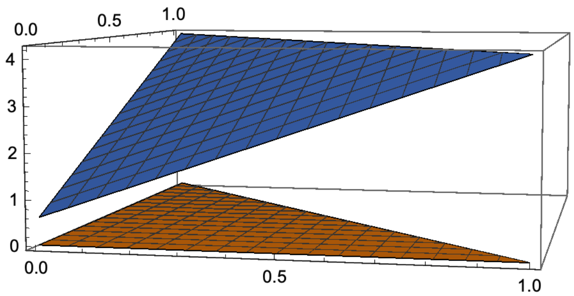
In[∘]:=
```
detJe = Simplify[Det[Je]]
```

Out[∘]= $0.6 + 3.6\,x1 + 1.77636 \times 10^{-15}\,x1^2 + 3.6\,x2 - 1.77636 \times 10^{-15}\,x2^2$

Check it is not equal to zero over the element

In[∘]:=
```
Plot3D[{0, detJe}, {x1, 0, 1}, {x2, 0, 1 - x1}]
```

Out[∘]=



Derivatives of the shape functions

In[∘]:=
```
DNN = Transpose[{D[NN, x1], D[NN, x2]}];
MatrixForm[DNN]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 2\left(-\frac{1}{2} + x1\right) + 2\,x1 & 0 \\ 0 & 2\left(-\frac{1}{2} + x2\right) + 2\,x2 \\ -2\left(\frac{1}{2} - x1 - x2\right) - 2\,(1 - x1 - x2) & -2\left(\frac{1}{2} - x1 - x2\right) - 2\,(1 - x1 - x2) \\ 4\,x2 & 4\,x1 \\ -4\,x2 & 4\,(1 - x1 - x2) - 4\,x2 \\ -4\,x1 + 4\,(1 - x1 - x2) & -4\,x1 \end{pmatrix}$$

In[∘]:=
```
Chop[Inverse[Je]]
```

Out[∘]=

$$\left\{\left\{\frac{0.2 - 1.2\,x1 - 2.4\,x2}{0.6 + 3.6\,x1 + 3.6\,x2}, \frac{-0.8 - 1.2\,x1 - 2.4\,x2}{0.6 + 3.6\,x1 + 3.6\,x2}\right\}, \left\{\frac{0.8 + 2.4\,x1 + 1.2\,x2}{0.6 + 3.6\,x1 + 3.6\,x2}, \frac{-0.2 + 2.4\,x1 + 1.2\,x2}{0.6 + 3.6\,x1 + 3.6\,x2}\right\}\right\}$$

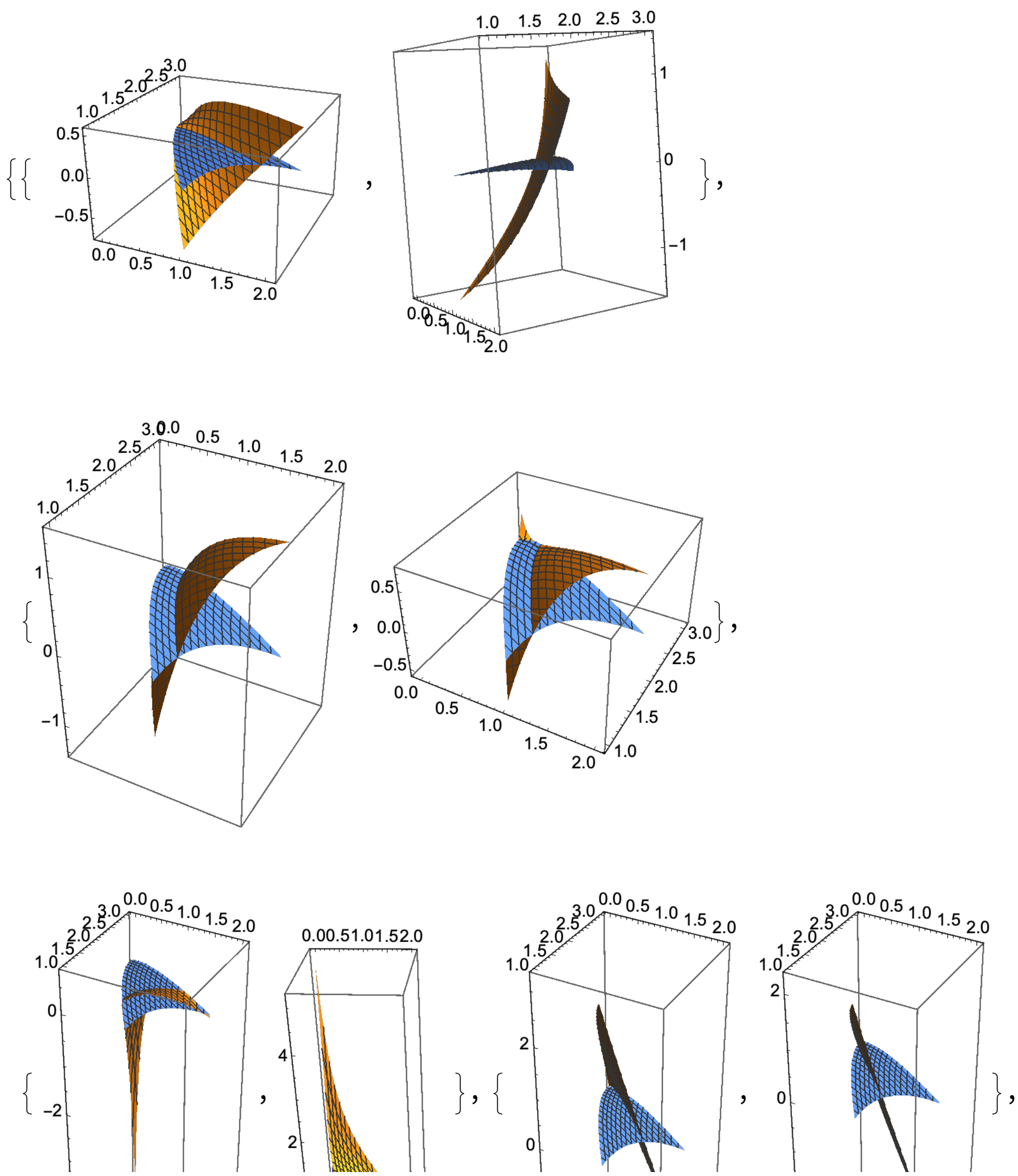
In[•]:=
```
DNNe = Simplify[DNN.Inverse[Je]];
MatrixForm[Chop[DNNe]]
```

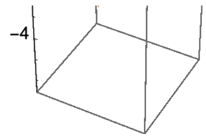
Out[•]//MatrixForm=

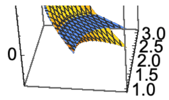
$$\begin{pmatrix}
-\frac{4.8\,(-0.25+1.\,x1)\,(-0.166667+1.\,x1+2.\,x2)}{0.6+3.6\,x1+3.6\,x2} & -\frac{4.8\,(-0.25+1.\,x1)\,(0.666667+1.\,x1+2.\,x2)}{0.6+3.6\,x1+3.6\,x2} \\
\frac{9.6\,(0.333333+1.\,x1+0.5\,x2)\,(-0.25+1.\,x2)}{0.6+3.6\,x1+3.6\,x2} & \frac{9.6\,(-0.0833333+1.\,x1+0.5\,x2)\,(-0.25+1.\,x2)}{0.6+3.6\,x1+3.6\,x2} \\
\frac{-3.+0.4\,x1+4.8\,x1^2+7.6\,x2-4.8\,x2^2}{0.6+3.6\,x1+3.6\,x2} & \frac{3.-7.6\,x1+4.8\,x1^2-0.4\,x2-4.8\,x2^2}{0.6+3.6\,x1+3.6\,x2} \\
\frac{3.2\,x1+9.6\,x1^2+0.8\,x2-9.6\,x2^2}{0.6+3.6\,x1+3.6\,x2} & \frac{-0.8\,x1+9.6\,x1^2-3.2\,x2-9.6\,x2^2}{0.6+3.6\,x1+3.6\,x2} \\
\frac{3.2+6.4\,x1-9.6\,x1^2-2.4\,x2-19.2\,x1\,x2}{0.6+3.6\,x1+3.6\,x2} & \frac{-0.8-9.6\,x1^2+x1\,(10.4-19.2\,x2)+9.6\,x2}{0.6+3.6\,x1+3.6\,x2} \\
\frac{0.8-9.6\,x1-10.4\,x2+19.2\,x1\,x2+9.6\,x2^2}{0.6+3.6\,x1+3.6\,x2} & \frac{-3.2-6.4\,x2+9.6\,x2^2+x1\,(2.4+19.2\,x2)}{0.6+3.6\,x1+3.6\,x2}
\end{pmatrix}$$

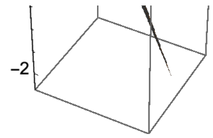
In[•]:=
```
Table[ParametricPlot3D[{{φ[[1]], φ[[2]], DNNe[[i]][[j]]}, {φ[[1]], φ[[2]], 0}},
  {x1, 0, 1}, {x2, 0, 1 - x1}, PlotRange → All], {i, 1, 6}, {j, 1, 2}]
```

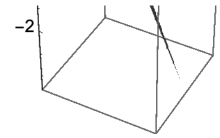
Out[•]=



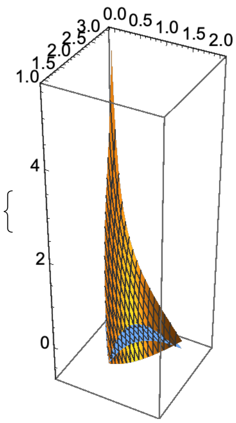

−4


0
3.0
2.5
2.0
1.5
1.0

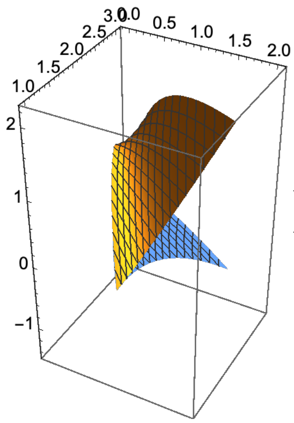

−2


−2

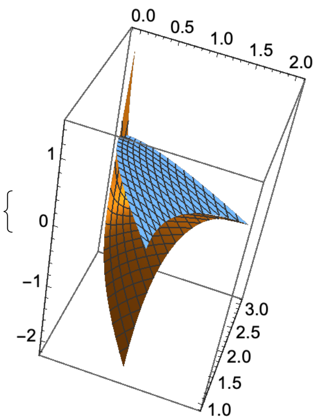
{


1.0 1.5 2.0 2.5 3.0 0.0 0.5 1.0 1.5 2.0
4
2
0

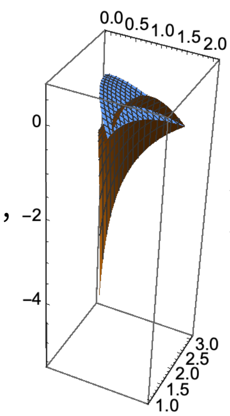
,


1.0 1.5 2.0 2.5 3.0 0.0 0.5 1.0 1.5 2.0
2
1
0
−1

}, {


0.0 0.5 1.0 1.5 2.0
1
0
−1
−2
3.0 2.5 2.0 1.5 1.0

,


0.0 0.5 1.0 1.5 2.0
0
−2
−4
3.0 2.5 2.0 1.5 1.0

}}

Q1 Elements

Quads

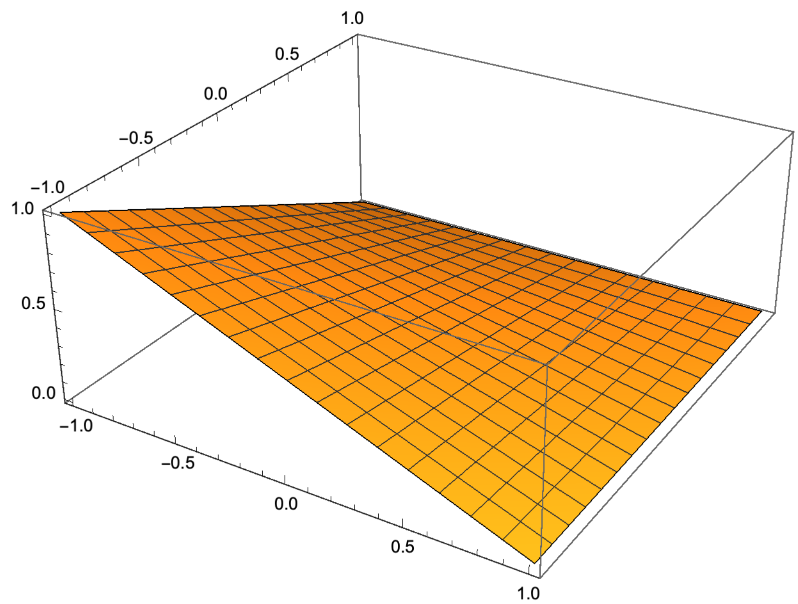
Shape functions in the Reference element

```
In[113]:= NN = {(1 - x1) (1 - x2) / 4, (1 + x1) (1 - x2) / 4, (1 + x1) (1 + x2) / 4, (1 - x1) (1 + x2) / 4}
```

Out[113]= $\left\{\frac{1}{4}(1-x1)(1-x2), \frac{1}{4}(1+x1)(1-x2), \frac{1}{4}(1+x1)(1+x2), \frac{1}{4}(1-x1)(1+x2)\right\}$

```
In[114]:= Plot3D[NN[[1]], {x1, -1, 1}, {x2, -1, 1}]
```

Out[114]=



Derivatives of shape functions in the reference element, each one an affine function of either x1 or x2.

```
In[115]:= DNN = Transpose[{D[NN, x1], D[NN, x2]}];
MatrixForm[DNN]
```

Out[116]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4}(-1+x2) & \frac{1}{4}(-1+x1) \\ \frac{1-x2}{4} & \frac{1}{4}(-1-x1) \\ \frac{1+x2}{4} & \frac{1+x1}{4} \\ \frac{1}{4}(-1-x2) & \frac{1-x1}{4} \end{pmatrix}$$

General Element

```
In[∘]:= VV = {{0, 0}, {3, -1}, {4, 3}, {-1, 2}}

In[∘]:= {{0, 0}, {3, -1}, {4, 3}, {-1, 2}}

Out[∘]= {{0, 0}, {3, -1}, {4, 3}, {-1, 2}}
```

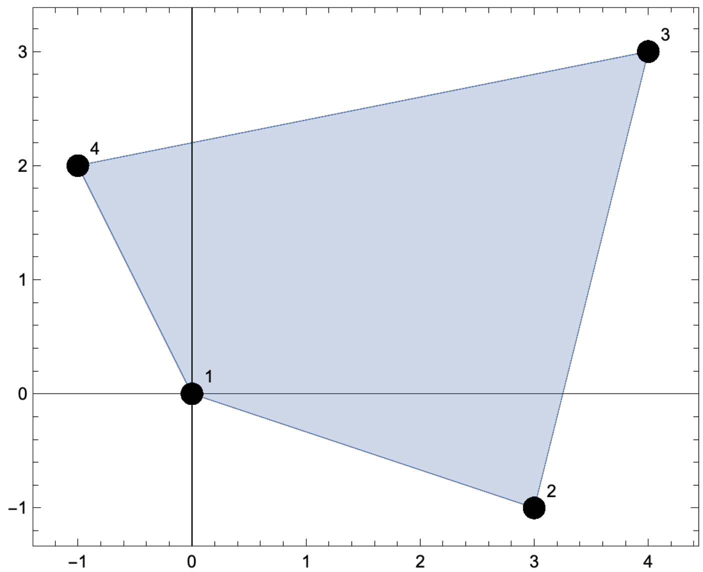
In[∘]:=
```
φ[x1_, x2_] = Sum[VV[[i]] × NN[[i]], {i, 1, 4}];
MatrixForm[φ[x1, x2]]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{3}{4}(1+x1)(1-x2) - \frac{1}{4}(1-x1)(1+x2) + (1+x1)(1+x2) \\ -\frac{1}{4}(1+x1)(1-x2) + \frac{1}{2}(1-x1)(1+x2) + \frac{3}{4}(1+x1)(1+x2) \end{pmatrix}$$

In[∘]:=
```
pl1 = ParametricPlot[φ[x1, x2], {x1, -1, 1}, {x2, -1, 1}];
pl2 = Graphics[Table[
    {Disk[VV[[i]], 0.1], Text[ToString[i], VV[[i]] + {0.15, 0.15}]}, {i, 1, 4}]];
Show[{pl1, pl2}, PlotRange -> All]
```

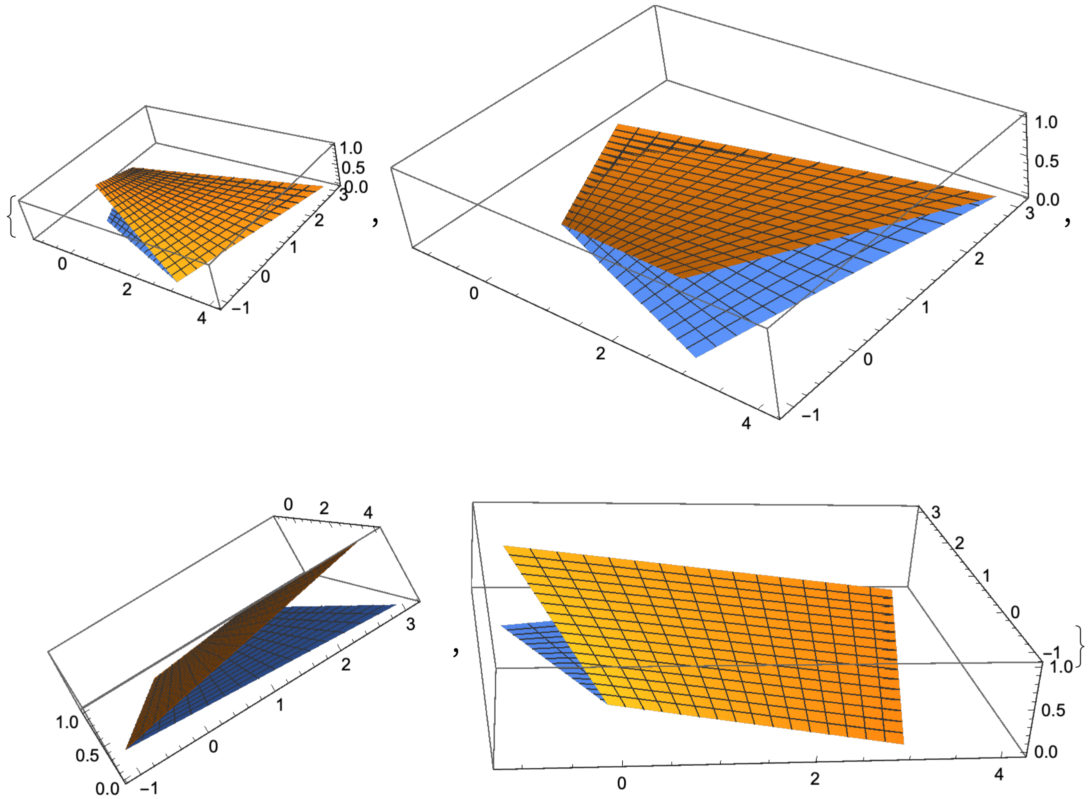
Out[∘]=



Shape functions over the general element

In[∘]:=
```
Table[ParametricPlot3D[
  {{φ[x1, x2][[1]], φ[x1, x2][[2]], NN[[i]]}, {φ[x1, x2][[1]], φ[x1, x2][[2]], 0}},
  {x1, -1, 1}, {x2, -1, 1}, PlotRange → All], {i, 1, 4}]
```

Out[∘]=



Jacobian Matrix and Jacobian

In[∘]:=
```
Je = Transpose[{D[φ[x1, x2], x1], D[φ[x1, x2], x2]}];
MatrixForm[Je]
```

Out[∘]//MatrixForm=

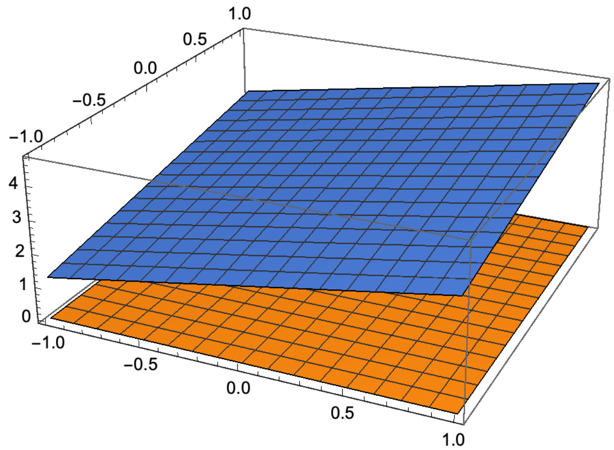
$$\begin{pmatrix} 1 + \frac{3\,(1-x2)}{4} + x2 + \frac{1+x2}{4} & 1 + \frac{1}{4}\,(-1 + x1) + x1 - \frac{3\,(1+x1)}{4} \\ \frac{1}{2}\,(-1 - x2) + \frac{1}{4}\,(-1 + x2) + \frac{3\,(1+x2)}{4} & 1 + \frac{1-x1}{2} + x1 \end{pmatrix}$$

In[∘]:=
```
DetJe = Det[Je]
```

Out[∘]= $3 + x1 + \frac{3\,x2}{4}$

In[∘]:=
```
Plot3D[{0, DetJe}, {x1, -1, 1}, {x2, -1, 1}]
```

Out[∘]=



Derivatives of shape functions over the general element

In[∘]:=
```
DNNe = Simplify[DNN.Inverse[Je]];
MatrixForm[DNNe]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{3+x1-4\,x2}{24+8\,x1+6\,x2} & -\frac{4-5\,x1+x2}{24+8\,x1+6\,x2} \\ \frac{3+x1-2\,x2}{24+8\,x1+6\,x2} & -\frac{4+5\,x1+x2}{24+8\,x1+6\,x2} \\ \frac{3+x1+2\,x2}{24+8\,x1+6\,x2} & \frac{4+3\,x1+x2}{24+8\,x1+6\,x2} \\ -\frac{3+x1+4\,x2}{24+8\,x1+6\,x2} & \frac{4-3\,x1+x2}{24+8\,x1+6\,x2} \end{pmatrix}$$

For example, d N4/dX2 is DNNe[[4]][[2]]

In[∘]:=
```
DNNe[[4]][[2]]
```

Out[∘]=

$$\frac{4-3\,x1+x2}{24+8\,x1+6\,x2}$$

These shape functions do not even have affine derivatives.

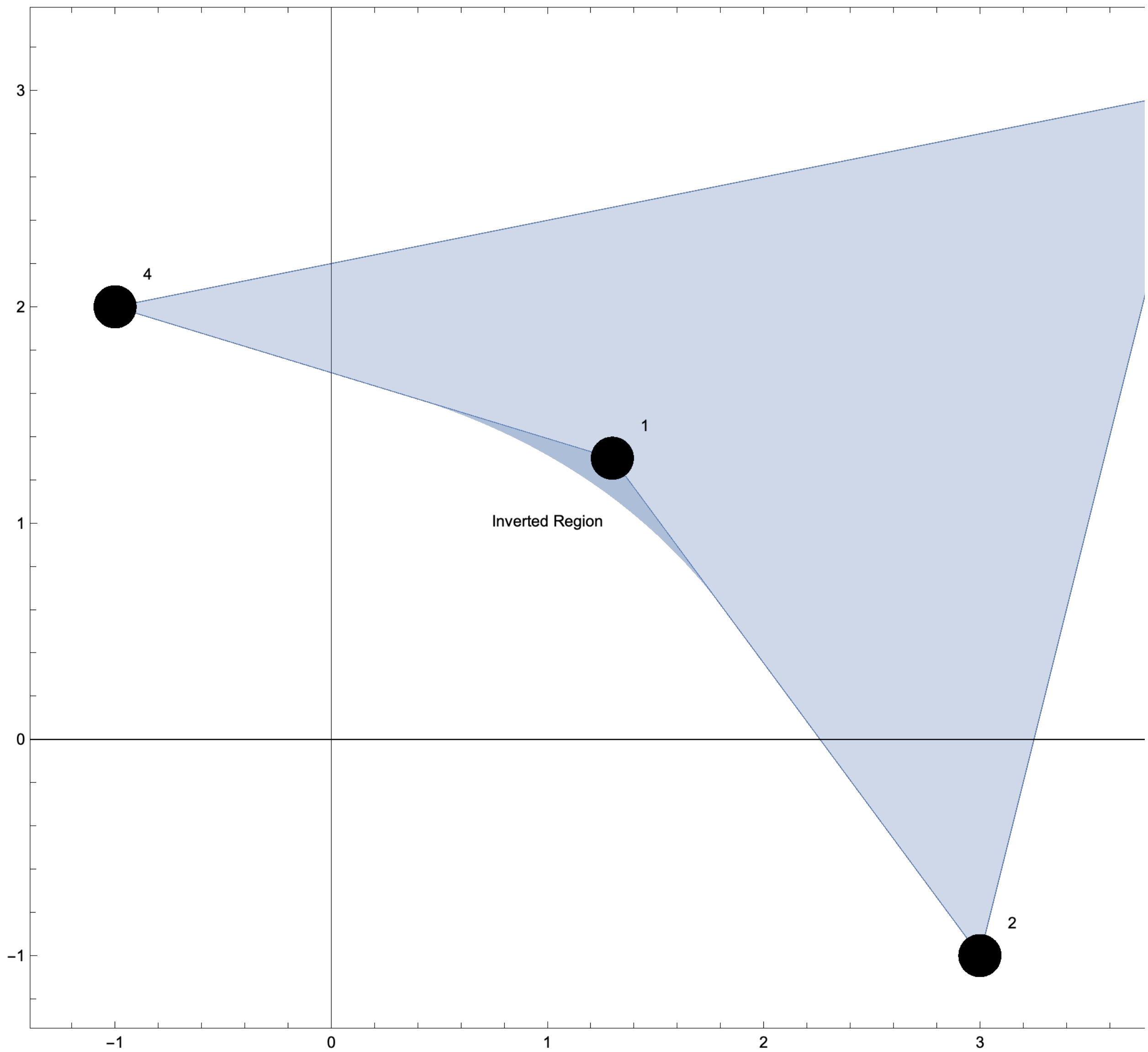
Example of an element with a degenerate Jacobian Matrix

In[∘]:=
```
VV = {{1.3, 1.3}, {3, -1}, {4, 3}, {-1, 2}};
φ[x1_, x2_] = Sum[VV[[i]] × NN[[i]], {i, 1, 4}];
MatrixForm[φ[x1, x2]]
pl1 = ParametricPlot[φ[x1, x2], {x1, -1, 1}, {x2, -1, 1}];
pl2 = Graphics[Table[
    {Disk[VV[[i]], 0.1], Text[ToString[i], VV[[i]] + {0.15, 0.15}]}, {i, 1, 4}]];
pl3 = Graphics[Text["Inverted Region", VV[[1]] - {0.3, 0.3}]];
Show[{pl1, pl2, pl3}, PlotRange -> All]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.325\,(1-x1)\,(1-x2)+\frac{3}{4}\,(1+x1)\,(1-x2)-\frac{1}{4}\,(1-x1)\,(1+x2)+(1+x1)\,(1+x2) \\ 0.325\,(1-x1)\,(1-x2)-\frac{1}{4}\,(1+x1)\,(1-x2)+\frac{1}{2}\,(1-x1)\,(1+x2)+\frac{3}{4}\,(1+x1)\,(1+x2) \end{pmatrix}$$

Out[∘]=



In[•]:=
```
Je = Transpose[{D[φ[x1, x2], x1], D[φ[x1, x2], x2]}];
MatrixForm[Je]
```

Out[•]//MatrixForm=

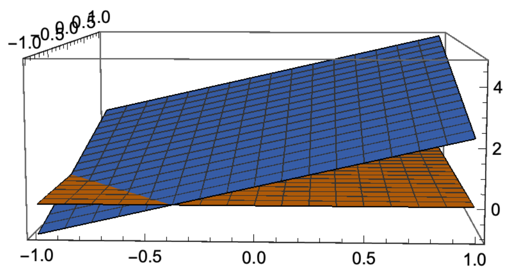
$$\begin{pmatrix} 1+0.425\,(1-x2)+x2+\frac{1+x2}{4} & 1-0.325\,(1-x1)+\frac{1}{4}\,(-1+x1)+x1-\frac{3\,(1+x1)}{4} \\ \frac{1}{2}\,(-1-x2)-0.325\,(1-x2)+\frac{1}{4}\,(-1+x2)+\frac{3\,(1+x2)}{4} & 1+0.175\,(1-x1)+x1 \end{pmatrix}$$

In[•]:=
```
DetJe = Det[Je]
```

Out[•]= $1.8625 + 1.65\,x1 + 1.2375\,x2$

In[•]:=
```
Plot3D[{0, DetJe}, {x1, -1, 1}, {x2, -1, 1}]
```

Out[•]=



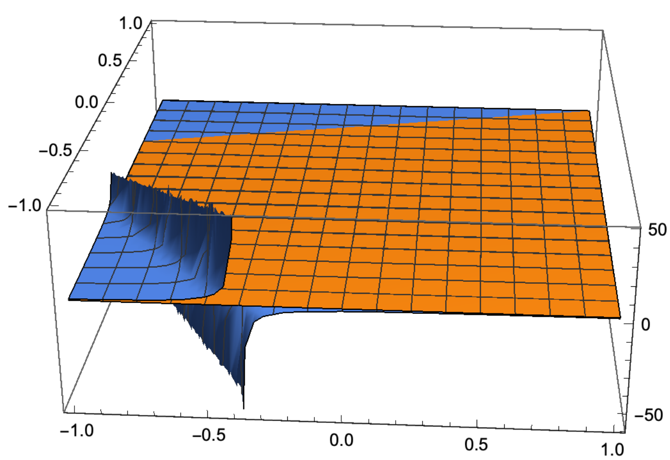
In[•]:=
```
DNNe = Simplify[DNN. Inverse[Je]];
MatrixForm[DNNe]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \frac{-0.227273-0.0757576\,x1+0.30303\,x2}{1.12879+1.\,x1+0.75\,x2} & \frac{-0.30303+0.378788\,x1-0.0757576\,x2}{1.12879+1.\,x1+0.75\,x2} \\ \frac{0.128788+0.0757576\,x1-0.0530303\,x2}{1.12879+1.\,x1+0.75\,x2} & \frac{-0.204545-0.378788\,x1-0.174242\,x2}{1.12879+1.\,x1+0.75\,x2} \\ \frac{0.227273+0.174242\,x1+0.0530303\,x2}{1.12879+1.\,x1+0.75\,x2} & \frac{0.30303+0.128788\,x1+0.174242\,x2}{1.12879+1.\,x1+0.75\,x2} \\ \frac{-0.128788-0.174242\,x1-0.30303\,x2}{1.12879+1.\,x1+0.75\,x2} & \frac{0.204545-0.128788\,x1+0.0757576\,x2}{1.12879+1.\,x1+0.75\,x2} \end{pmatrix}$$

In[•]:=
```
Plot3D[{0, DNNe[[1]][[1]]}, {x1, -1, 1}, {x2, -1, 1}, PlotRange → All]
```

Out[•]=



Global Shape Functions

In[•]:=
```
VMesh = Table[{i 2, 2 j}, {i, -1, 1}, {j, -1, 1}]
```

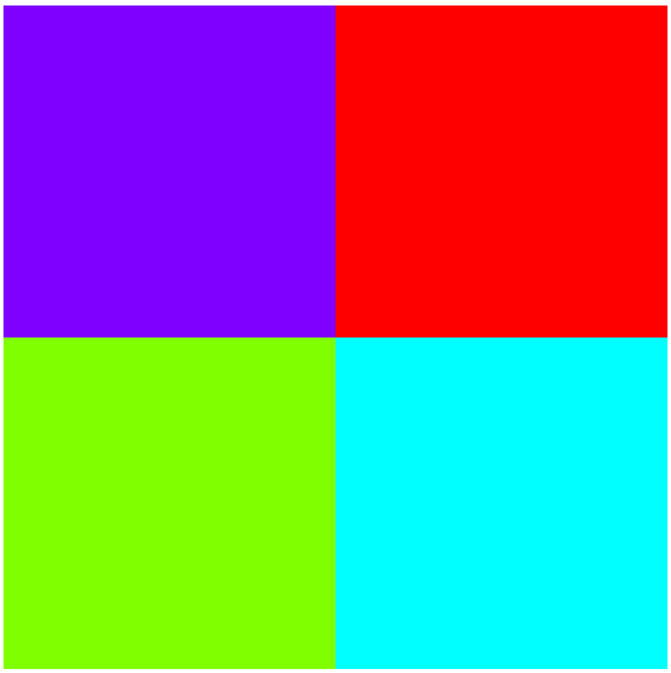
Out[•]= {{{-2, -2}, {-2, 0}, {-2, 2}}, {{0, -2}, {0, 0}, {0, 2}}, {{2, -2}, {2, 0}, {2, 2}}}

In[∘]:=
```
NodalCoordinatesByElement = {
  {VMesh[[1]][[1]], VMesh[[2]][[1]], VMesh[[2]][[2]], VMesh[[1]][[2]]},
  {VMesh[[2]][[1]], VMesh[[3]][[1]], VMesh[[3]][[2]], VMesh[[2]][[2]]},
  {VMesh[[1]][[2]], VMesh[[2]][[2]], VMesh[[2]][[3]], VMesh[[1]][[3]]},
  {VMesh[[2]][[2]], VMesh[[3]][[2]], VMesh[[3]][[3]], VMesh[[2]][[3]]}
 }
```

Out[∘]=
```
{{{-2, -2}, {0, -2}, {0, 0}, {-2, 0}}, {{0, -2}, {2, -2}, {2, 0}, {0, 0}},
 {{-2, 0}, {0, 0}, {0, 2}, {-2, 2}}, {{0, 0}, {2, 0}, {2, 2}, {0, 2}}}
```

In[∘]:=
```
Graphics[Table[{Hue[i / 4], Polygon[NodalCoordinatesByElement[[i]]]}, {i, 1, 4}]]
```

Out[∘]=



In[∘]:=
```
φMaps =
  Table[Sum[NodalCoordinatesByElement[[e]][[i]] × NN[[i]], {i, 1, 4}], {e, 1, 4}];
```

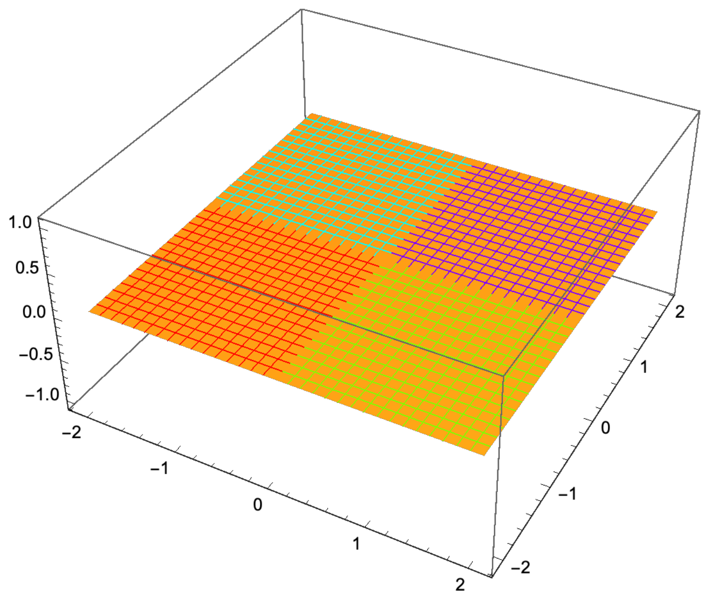
In[∘]:=
```
φMaps // MatrixForm
```

Out[∘]//MatrixForm=

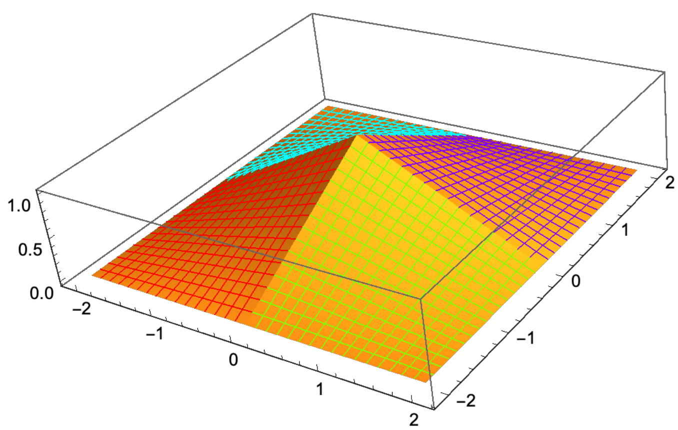
$$\begin{pmatrix} -\frac{1}{2}(1-x1)(1-x2)-\frac{1}{2}(1-x1)(1+x2) & -\frac{1}{2}(1-x1)(1-x2)-\frac{1}{2}(1+x1)(1-x2) \\ \frac{1}{2}(1+x1)(1-x2)+\frac{1}{2}(1+x1)(1+x2) & -\frac{1}{2}(1-x1)(1-x2)-\frac{1}{2}(1+x1)(1-x2) \\ -\frac{1}{2}(1-x1)(1-x2)-\frac{1}{2}(1-x1)(1+x2) & \frac{1}{2}(1-x1)(1+x2)+\frac{1}{2}(1+x1)(1+x2) \\ \frac{1}{2}(1+x1)(1-x2)+\frac{1}{2}(1+x1)(1+x2) & \frac{1}{2}(1-x1)(1+x2)+\frac{1}{2}(1+x1)(1+x2) \end{pmatrix}$$

In[∘]:=
```
ElePlot = Table[ParametricPlot3D[{φMaps[[i]][[1]], φMaps[[i]][[2]], 0}, {x1, -1, 1},
    {x2, -1, 1}, MeshStyle → {Hue[(i - 1) / 4], Hue[(i - 1) / 4]}], {i, 1, 4}];
CIE = {3, 4, 2, 1};(*Index of where the node with coordinate {0,0} is*)
NNCenterPlot =
  Table[ParametricPlot3D[{φMaps[[i]][[1]], φMaps[[i]][[2]], NN[[CIE[[i]]]]},
    {x1, -1, 1}, {x2, -1, 1}, MeshStyle → {Hue[(i - 1) / 4], Hue[(i - 1) / 4]}], {i, 1, 4}];
```

In[∘]:=
```
Show[ElePlot]
```

Out[∘]=


In[∘]:=
```
Show[NNCenterPlot]
```

Out[∘]=


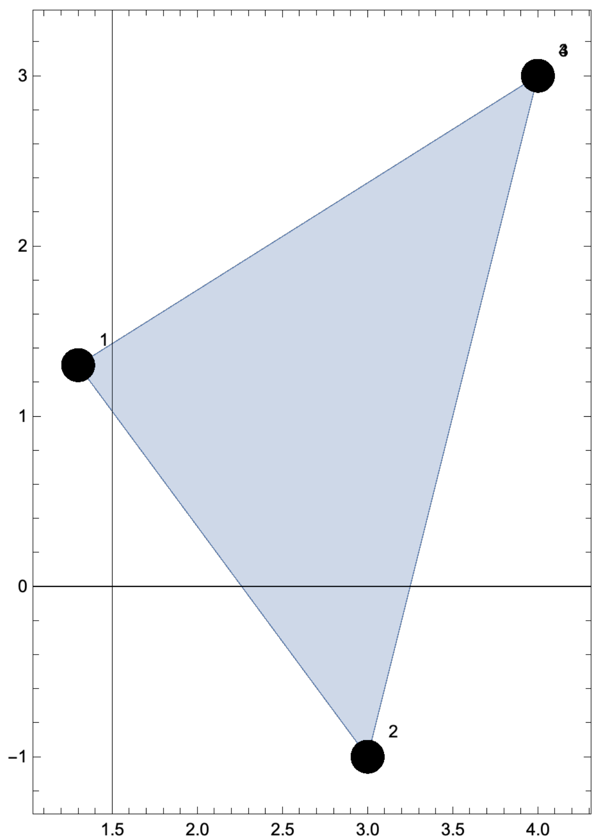
Triangles from Squares

In[123]:=
```
VV = {{1.3, 1.3}, {3, -1}, {4, 3}, {4, 3}};
φ[x1_, x2_] = Sum[VV[[i]] × NN[[i]], {i, 1, 4}];
MatrixForm[φ[x1, x2]]
pl1 = ParametricPlot[φ[x1, x2], {x1, -1, 1}, {x2, -1, 1}];
pl2 = Graphics[Table[
    {Disk[VV[[i]], 0.1], Text[ToString[i], VV[[i]] + {0.15, 0.15}]}, {i, 1, 4}]];
Show[{pl1, pl2}, PlotRange -> All]
```

Out[125]//MatrixForm=

$$\begin{pmatrix} 0.325\,(1-x1)\,(1-x2)+\frac{3}{4}\,(1+x1)\,(1-x2)+(1-x1)\,(1+x2)+(1+x1)\,(1+x2) \\ 0.325\,(1-x1)\,(1-x2)-\frac{1}{4}\,(1+x1)\,(1-x2)+\frac{3}{4}\,(1-x1)\,(1+x2)+\frac{3}{4}\,(1+x1)\,(1+x2) \end{pmatrix}$$

Out[128]=



Compute Jacobian

In[131]:=
```
Je = Simplify[Transpose[{D[φ[x1, x2], x1], D[φ[x1, x2], x2]}]];
MatrixForm[Je]
```

Out[132]//MatrixForm=

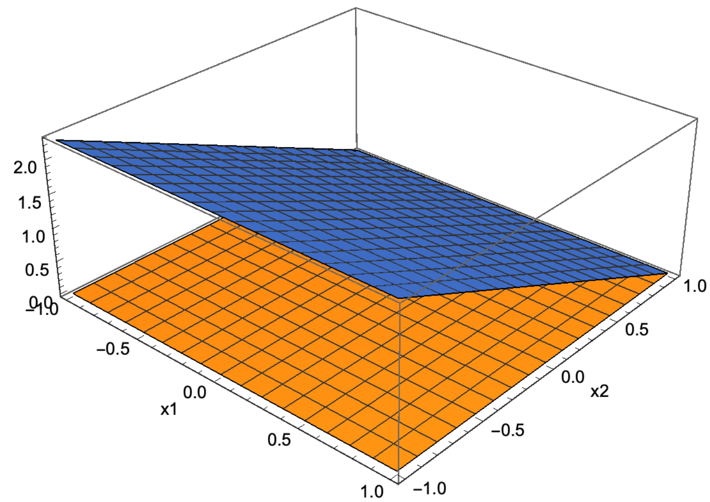
$$\begin{pmatrix} 0.425-0.425\,x2 & 0.925-0.425\,x1 \\ -0.575+0.575\,x2 & 1.425+0.575\,x1 \end{pmatrix}$$

In[133]:=
```
DetJe = Det[Je]
```

Out[133]= 1.1375 - 1.1375 x2

In[135]:=
```
Plot3D[{0, DetJe}, {x1, -1, 1}, {x2, -1, 1}, AxesLabel → {"x1", "x2"}]
```

Out[135]=



See that we recover the shape functions over the triangle

In[147]:=
```
solz1z2 = Simplify[Solve[φ[x1, x2] == {z1, z2}, {x1, x2}]];
φinv[z1_, z2_] = {x1, x2} /. solz1z2[[1]]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

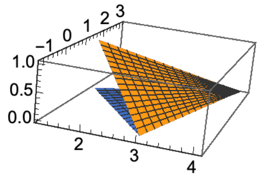
Out[148]= $\left\{\frac{117. - 57.\ z1 + 37.\ z2}{-143. + 23.\ z1 + 17.\ z2},\ -2.14286 + 0.505495\ z1 + 0.373626\ z2\right\}$

In[155]:=
```
NNPQ = Simplify[NN /. solz1z2]
```

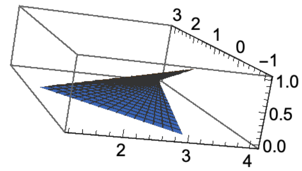
Out[155]= $\left\{\left\{1.42857 - 0.43956\ z1 + 0.10989\ z2,\ 0.142857 + 0.186813\ z1 - 0.296703\ z2,\ -\frac{0.186813\ (0.764706 + 1.\ z1 - 1.58824\ z2)\ (-2.26087 + 1.\ z1 + 0.73913\ z2)}{-6.21739 + 1.\ z1 + 0.73913\ z2},\ \frac{0.43956\ (-3.25 + 1.\ z1 - 0.25\ z2)\ (-2.26087 + 1.\ z1 + 0.73913\ z2)}{-6.21739 + 1.\ z1 + 0.73913\ z2}\right\}\right\}$

In[159]:=
```
Table[ParametricPlot3D[
  {{φ[x1, x2][[1]], φ[x1, x2][[2]], NN[[i]]}, {φ[x1, x2][[1]], φ[x1, x2][[2]], 0}},
  {x1, -1, 1}, {x2, -1, 1}, PlotRange → All], {i, 1, 4}]
```

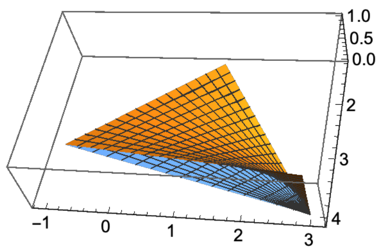
Out[159]= {



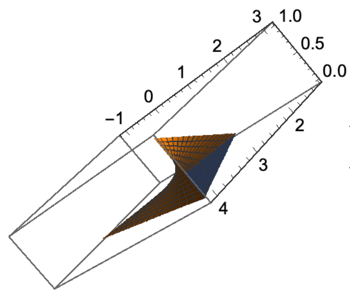
,



,



,



}