ME 335A
Finite Element Analysis
Instructor: Adrian Lew
Problems Set #6

June 11, 2023

Due Wednesday, May 31, 2023

On Norms and Convergence (55)

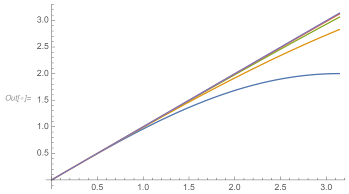
In this problem we would like to play with the convergence, norms, and membership of sequence of functions in different spaces. To this end, let $I = (0, \pi)$, and recall (see Appendix A in the notes) that a function $f\colon I \to \mathbb{R}$ is a member of the following spaces if

$$
\begin{aligned}
f \in L^2(I) &\Leftrightarrow \|f\|_{0,2} = \left(\int_0^\pi f^2 \, dx\right)^{1/2} < \infty \\
f \in L^\infty(I) &\Leftrightarrow \|f\|_{0,\infty} = \max_{x \in I} |f(x)| < \infty \\
f \in H^1(I) &\Leftrightarrow \|f\|_{1,2} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2\right)^{1/2} < \infty.
\end{aligned}
$$

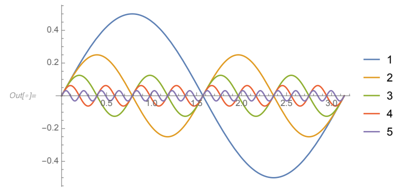
Consider the sequences of functions for $n = 1, 2 \ldots$:

$$
\begin{aligned}
f_n(x) &= n \sin\left(\frac{x}{n}\right), \quad & f_\infty(x) &= x, \\
g_n(x) &= \frac{1}{n} \sin(nx), \quad & g_\infty(x) &= 0, \\
h_n(x) &= \frac{1}{1 + nx}, \quad & h_\infty(x) &= 0.
\end{aligned}
$$

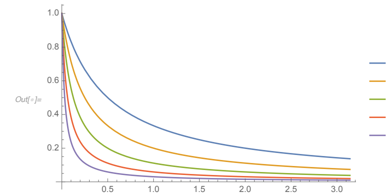
1. (5) Plot $f_{2^n}, g_{2^n}, h_{2^n}$ and $f'_{2^n}, g'_{2^n}, h'_{2^n}$ in $(0, \pi)$ for $n = 1, \ldots, 5$.
   Solution:



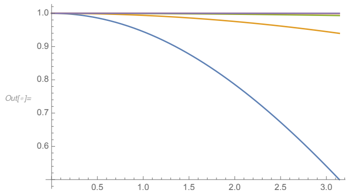
f_{2n}



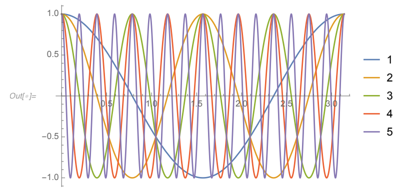
g_{2n}



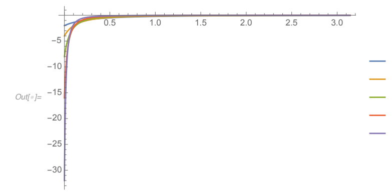
h_{2n}



f'_{2n}



g'_{2n}



h'_{2n}.

2. (10) Does $f_\infty \in L^2(I)$? Does $f_\infty \in L^\infty(I)$? Does $f_\infty \in H^1(I)$? Justify.

   Solution: To see if f_∞ any of these spaces, its norm in the space should be less than ∞. For the L^2 and H^1 norms, we compute their squares, since it is a simpler expression.

$$\|f_\infty\|_{0,2}^2 = \int_0^\pi x^2\, dx = \pi^3/3 < \infty$$
$$\|f_\infty\|_{1,2}^2 = \|f_\infty\|_{0,2}^2 + \int_0^\pi 1\, dx = \pi^3/3 + \pi < \infty$$
$$\|f_\infty\|_{0,\infty} = \sup_{x\in(0,\pi)} |x| = \pi < \infty.$$

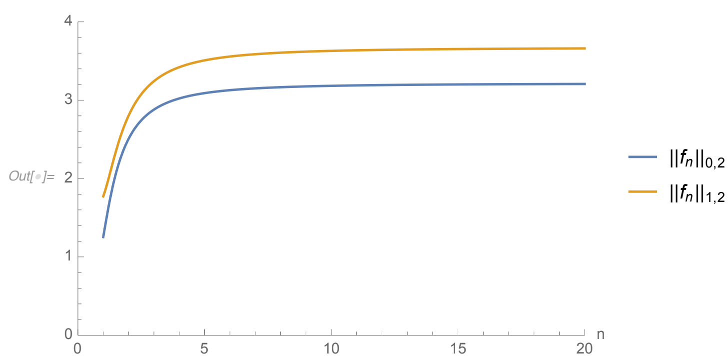
   So f_∞ is in the three spaces.

3. (10) Compute the $L^2(I)$ and $H^1(I)$ norms for f_n for $n < \infty$, and plot them as a function of n.

   Solution:

$$\|f_n\|_{0,2}^2 = \int_0^\pi \left[n \sin\left(\frac{x}{n}\right)\right]^2\, dx = -\frac{n^2}{4}\left(n \sin\left(\frac{2\pi}{n}\right) - 2\pi\right) < \infty$$
$$\|f_n\|_{1,2}^2 = \|f_n\|_{0,2}^2 + \int_0^\pi \left[\cos\left(\frac{x}{n}\right)\right]^2\, dx = \|f_n\|_{0,2}^2 + \frac{\pi}{2} + \frac{n}{4}\sin\left(\frac{2\pi}{n}\right) < \infty$$

   The corresponding plots of two of these norms are:



4. You can check if you want, but f_n, g_n and h_n for $n = 1, \ldots, \infty$ belong to the three spaces $L^2(I)$, $L^\infty(I)$ and $H^1(I)$. Next, we say that a sequence z_n converges to z as $n \to \infty$ in a normed

space V if $\|z_n - z\| \to 0$ as $n \to \infty$. For the following calculations, we encourage you to use Mathematica, Maple, or Matlab to either perform integrals analytically, or numerically for each value of n and plot the resulting trends. For the L^∞-cases below, you may plot the absolute value of the error for a few values of n, and argue based on the plots.

This problem is a little bit laborious, but it will show you three different sequences that converge is some spaces and not in others, so it is an instructive exercise.

(a) (10) Evaluate if $f_n \to f_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$.

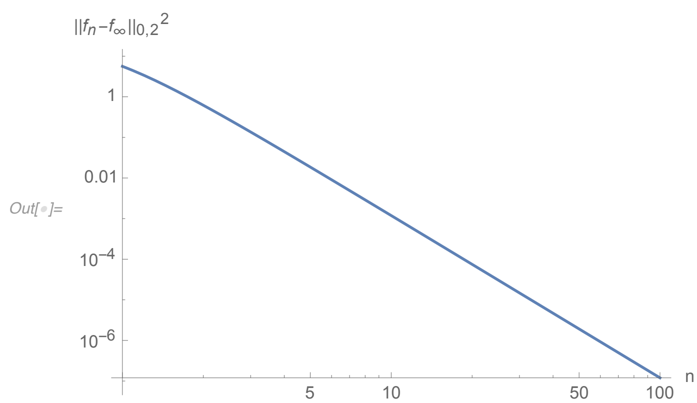
Solution: We need to compute

$$\begin{aligned}\|f_n - f_\infty\|_{0,2}^2 &= \int_0^\pi \left[n \sin\left(\frac{x}{n}\right) - x\right]^2 dx \\ &= \frac{\pi^3}{3} + \frac{n^2}{2}\pi\left(1 + 4\cos\left(\frac{\pi}{n}\right)\right) - \frac{n^3}{4}\left(8\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\right)\end{aligned}$$

You can compute the limit directly in one of the software programs above, but we will do this one here just to show. To this end, we will use the Taylor expansions of sin and cos near 0, since these are values of their arguments for n large enough.

$$\begin{aligned}\lim_{n\to\infty} \|f_n - f_\infty\|_{0,2}^2 &= \lim_{n\to\infty} \frac{\pi^3}{3} + \frac{n^2}{2}\pi\left(1 + 4 - 2\frac{\pi^2}{n^2} + \mathcal{O}(n^{-4})\right) \\ &\qquad - \frac{n^3}{4}\left(8\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \mathcal{O}(n^{-4})\right) + \frac{2\pi}{n} - \frac{4\pi^3}{3n^3} + \mathcal{O}(n^{-4})\right) \\ &= \lim_{n\to\infty} \mathcal{O}(n^{-4}) = 0.\end{aligned}$$

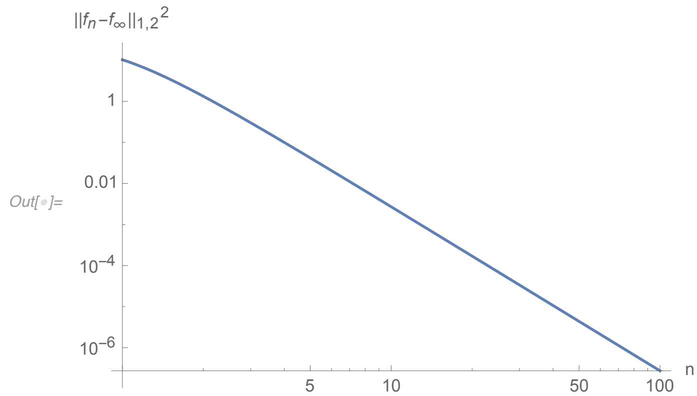
Alternatively, we can plot the norm of the difference as a function of n. We do so in logarithmic scales in both axes, so that we a straight line whose slope indicates the power at which it decreases. In this case, the error square decreases as the fourth power of n.



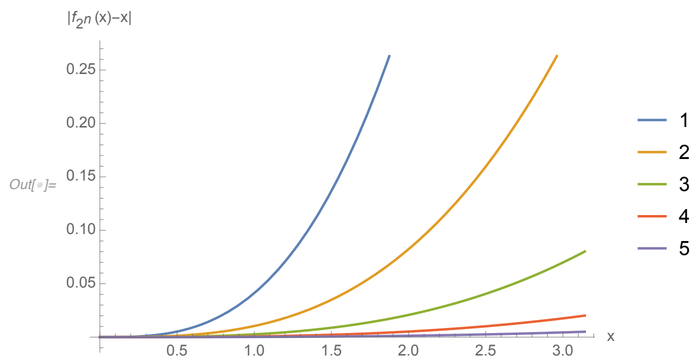
Similarly,

$$\begin{aligned}\|f_n - f_\infty\|_{1,2}^2 &= \|f_n - f_\infty\|_{0,2}^2 + \int_0^\pi \left[\cos\left(\frac{x}{n}\right) - 1\right]^2 dx \\ &= \|f_n - f_\infty\|_{0,2}^2 + \frac{3\pi}{2} - 2n\sin\left(\frac{\pi}{n}\right) + \frac{n}{4}\sin\left(\frac{2\pi}{n}\right),\end{aligned}$$

and $\lim_{n\to\infty} \|f_n - f_\infty\|_{1,2}^2 = \lim_{n\to\infty} \mathcal{O}(n^{-4}) = 0$. Its graph is



Finally, let's plot $|f_{2^n} - f_\infty|$ for $n = 1, 2, 3, 4, 5$:



Clearly the supremum of $|f_{2^n} - f_\infty|$ over $(0, \pi)$ decreases to 0 as $n \to \infty$, so $\|f_n - f_\infty\|_{0,\infty} \to 0$. Therefore, $f_n \to f_\infty$ in the three spaces.

(b) (10) Evaluate if $g_n \to g_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$. Reflect on what you see if you want, by writing one or two sentences about it.

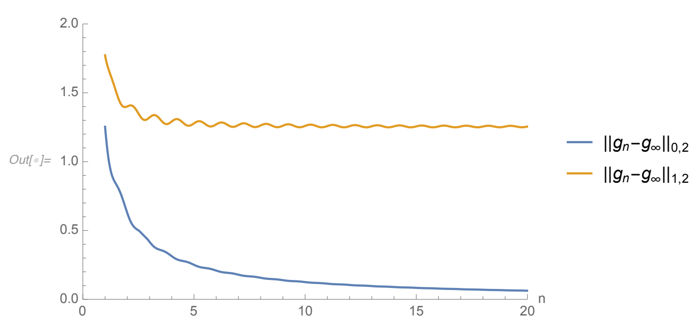
Solution: We compute

$$\lim_{n\to\infty} \|g_n - g_\infty\|_{0,2}^2 = \lim_{n\to\infty} \int_0^\pi \left[\frac{1}{n}\sin(nx)\right]^2 dx = \lim_{n\to\infty} \frac{\pi}{2n^2} = 0$$

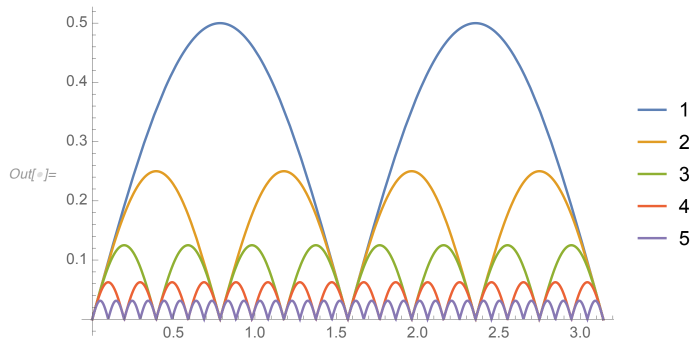
$$\lim_{n\to\infty} \|g_n - g_\infty\|_{1,2}^2 = \lim_{n\to\infty} \|g_n - g_\infty\|_{0,2}^2 + \lim_{n\to\infty} \int_0^\pi [\cos(nx)]^2 dx = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

$$\lim_{n\to\infty} \|g_n - g_\infty\|_{0,\infty} = \lim_{n\to\infty} \sup_{x\in(0,\pi)} \left|\frac{1}{n}\sin(nx)\right| \le \lim_{n\to\infty} \sup_{x\in(0,\pi)} \left|\frac{1}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

The corresponding plots of the L^2 and H^1 norms are:



The plot of the absolute value of the error for a few values of n is:



The conclusion is then that $g_n \to g_\infty$ in $L^2(I)$ and $L^\infty(I)$, but not in $H^1(I)$. This is because the size of the derivatives of g_n does not decrease as n grows, as shown in the figure in part 1 for g'_{2^n}.

(c) (10) Evaluate if $h_n \to h_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$. Does $h_n(0) \to h_\infty(0)$?

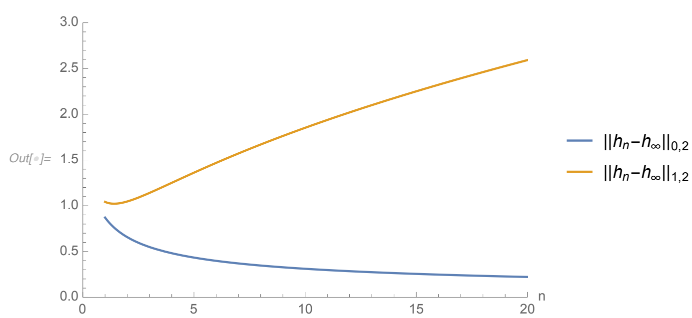
Solution: We compute

$$\lim_{n\to\infty} \|h_n - h_\infty\|_{0,2}^2 = \lim_{n\to\infty} \int_0^\pi \frac{1}{(1+nx)^2}\, dx = \lim_{n\to\infty} \frac{\pi}{1+n\pi} = 0$$

$$\begin{aligned}\lim_{n\to\infty} \|h_n - h_\infty\|_{1,2}^2 &= \lim_{n\to\infty} \|h_n - h_\infty\|_{0,2}^2 + \lim_{n\to\infty} \int_0^\pi \left[-\frac{n}{(1+nx)^2}\right]^2 dx \\ &= 0 + \lim_{n\to\infty} \frac{n}{3}\left(1 - \frac{1}{(1+n\pi)^3}\right) = \infty\end{aligned}$$

$$\lim_{n\to\infty} \|h_n - h_\infty\|_{0,\infty} = \lim_{n\to\infty} \sup_{x\in(0,\pi)} \left|\frac{1}{1+nx}\right| = \lim_{n\to\infty} 1 = 1.$$

The corresponding plots of the L^2 and H^1 norms are:



The plot of the absolute value of the error for a few values of n is already shown in part 1, plot h^{2n}.

The conclusion is then that $h_n \to h_\infty$ in $L^2(I)$, but not in $L^\infty(I)$ or $H^1(I)$.

The non-convergence in $L^\infty(I)$ is because $h_n(0) = 1$ for all n, and hence $1 = h_n(0) \not\to h_\infty(0) = 0$ as $n \to \infty$.

As in the case for g_n, the derivatives do not converge in $L^2(I)$, so h_n does not converge in $H^1(I)$.

On Interpolation Errors (70)

Consider the interval $\Omega = [-1, 1]$, and a mesh of $n_{\text{el}} \in \mathbb{N}$ equally long P_k-elements on it, for $k = 1, 2, 4$. The Lagrange finite element interpolant $\mathcal{I}u$ with these elements is constructed through (3.24) in the notes.

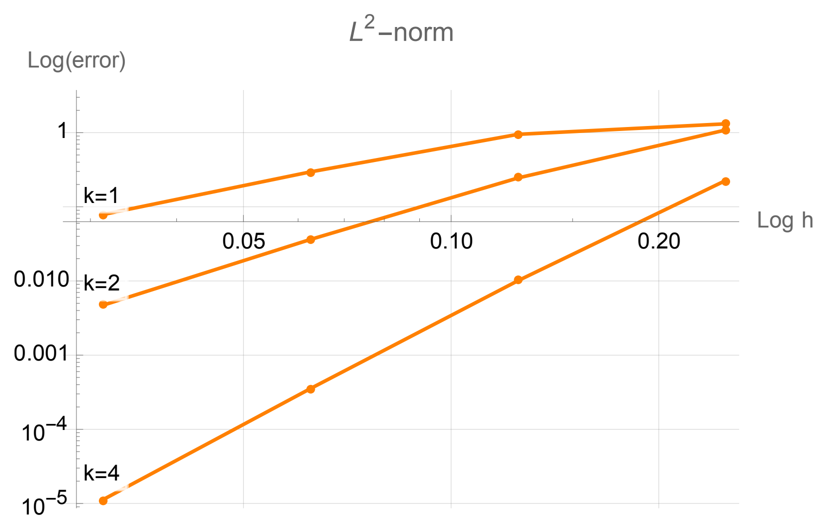
For $\omega \in \mathbb{R}$, consider the functions

$$v_\omega(x) = \cos(\omega x)$$
$$w_\omega(x) = \begin{cases} 0 & x < 0 \\ x^\omega & x \geq 0. \end{cases}$$

1. For $u = v_{30}$, $k = 1, 2, 4$, and $n_{\text{el}} = 2^i$ with $i = 3, 4, 5, 6$.

   (a) (10) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be? Explain.

   Solution:



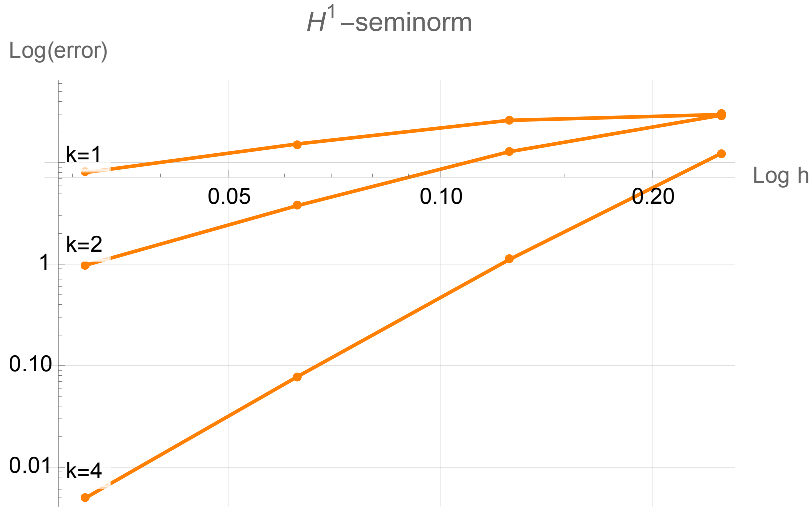
The rates of convergence are

k	rate
1	1.91341
2	2.94906
4	4.98123

These agree with the expected rates for a function in $C^5([-1,1])$, in which $k = 1, 2, 4$ and $m = 0$, so $k + 1 - m = 2, 3, 5$. Since $v_{30} \in C^5([-1,1])$, it is also in $C^2([-1,1])$ and $C^3([-1,1])$, the requirements of the theorem for $k = 1, 2$, respectively.

   (b) (10) Plot $\|u' - (\mathcal{I}u)'\|_{0,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be? Explain.

   Solution:



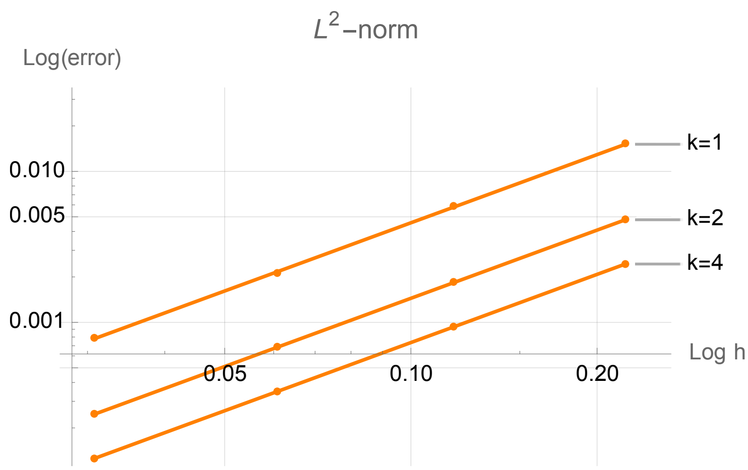
The rates of convergence are

k	rate
1	0.934715
2	1.95015
4	3.96831

These agree with the expected rates for a function in $C^5([-1,1])$, in which $k = 1,2,4$ and $m = 1$, so $k+1-m = 1,2,4$. Since $v_{30} \in C^5([-1,1])$, it is also in $C^2([-1,1])$ and $C^3([-1,1])$, the requirements of the theorem for $k = 1,2$, respectively.

2. For $u = w_1$, $k = 1,2,4$, and $n_{\text{el}} = 2^i + 1$ with $i = 3,4,5,6$.

   (a) (10) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $k = 1,2,4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be?
   Solution:



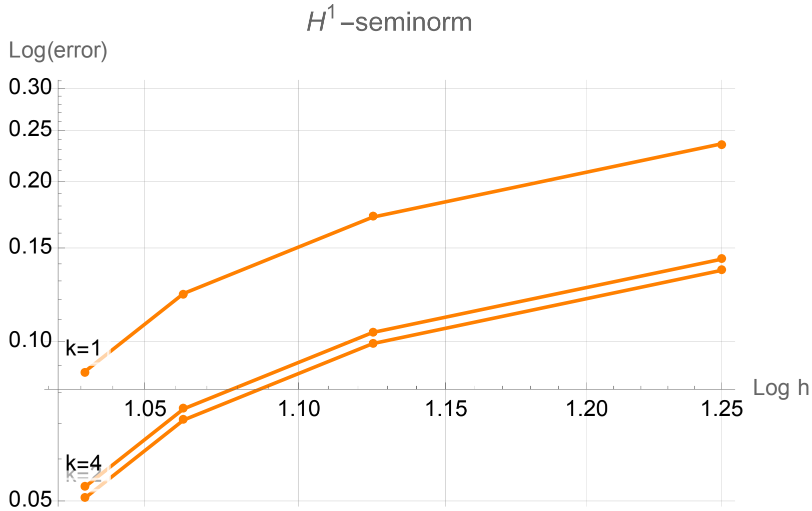
The rates of convergence are

k	rate
1	1.5
2	1.5
4	1.5

The rate of convergence is the same for all orders, and it reflects the fact that $w_1 \in C^1([-1,1])$, but $w_1 \notin C^2([-1,1])$, because of the discontinuity of the w_1' at $x = 0$. The theorem cannot be applied as stated, and the understanding of why the rate of convergence is 1.5 does not emerge directly from the theorem.

(b) (10) Plot $|u - \mathcal{I}u|_{1,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be?

Solution:



The rates of convergence are (computed for the last two pairs of points)

k	rate
1	0.5
2	0.5
4	0.5

As before, these rates are not what we expect for smooth functions in the theorem, and they reflect that $w_1 \in C^1([-1, 1])$, but $w_1 \notin C^2([-1, 1])$.

(c) (5) What is the value of $\|u - \mathcal{I}u\|_{0,2,\Omega}$ when n_{el} is even? Can you elaborate on the reasons behind the differences between the last two questions and this one?

Solution: When n_{el} is even, a node of the mesh falls precisely at $x = 0$. The function w_1 is the precisely in the space of P_k-continuous functions in $[-1, 1]$ over any of the meshes, and hence, it is interpolated *exactly*. Therefore, the error is exactly zero.

3. (10) Compare $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $u = v_{30}$ and $u = v_{60}$ for $k = 2$ and $n_{\text{el}} = 200$. Which one is larger? Why?

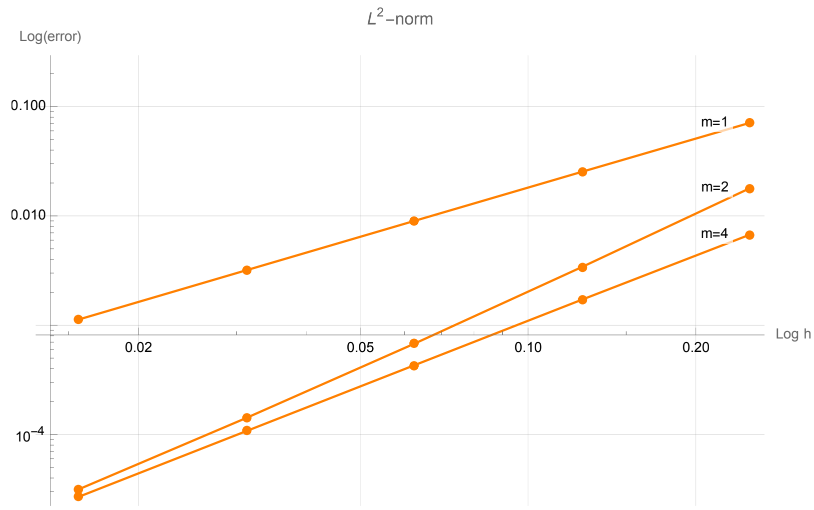
   Solution: For $u = v_{30}$, $\|u - \mathcal{I}u\|_{0,2,\Omega} = 0.000155472$, and for $u = v_{60}$, $\|u - \mathcal{I}u\|_{0,2,\Omega} = 0.00123277$.

   The value of the error for v_{60} is larger simply because $|v_{60}|_{2,2,\Omega} > |v_{30}|_{2,2,\Omega}$, that is, because the second derivative is larger. This agrees with the error estimate that says that for the same mesh, and hence h, the large error should appear for the function with the largest value of the H^2-seminorm. In this case, $|v_{60}|_{2,2,\Omega} = 14428$ and $|v_{30}|_{2,2,\Omega} = 898$.

4. Let's now examine what happens with the convergence rate if we do not exactly satisfy the essential boundary conditions, as we generally have to do in 2D and 3D problems. To this end, let's construct an interpolant $\mathcal{I}u$ of $u = v_1$ that at $x = 1$ has the value $\cos(1) + h^m$ for $m = 1/2, 1, 2, 4$, instead of being $\mathcal{I}u(1) = v_1(1) = \cos(1)$, as the standard interpolant would. For $k = 1$ and $n_{\text{el}} = 2^i + 1$ with $i = 3, 4, 5, 7$:

(a) (5) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $m = 1, 2, 4$ in the same plot. Compute the convergence rate for each m for i large enough.

Solution:

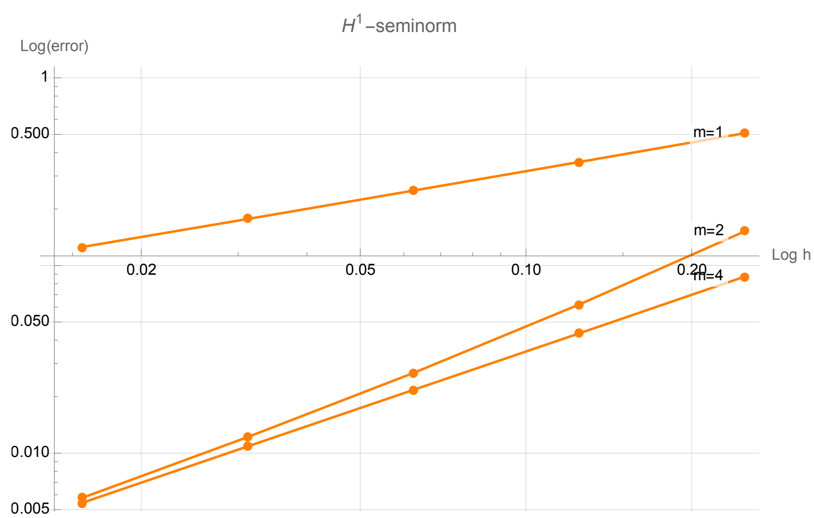


The rates of convergence are (computed for the last two pairs of points)

m	rate
1	1.5
2	2.17
4	2.00

(b) (5) Plot $\|u' - (\mathcal{I}u)'\|_{0,2,\Omega}$ for $m = 1, 2, 4$ in the same plot. Compute the convergence rate for each m for i large enough.

Solution:



The rates of convergence are (computed for the last two pairs of points)

m	rate
1	0.5
2	1.1
4	1.0

(c) (5) Reflecting on the convergence result in the fundamental approximation theorem (Cea's lemma), which states that

$$\|u - u_h\|_1 \leq \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|u - w_h\|_1,$$

explain how the results that you observed in part 4a and/or 4b could affect the convergence rate.

Solution: For $m = 1$, $\|u' - (\mathcal{I}u)'\|_{0,2,\Omega}$ converges as $h^{0.5}$, so $\|u' - (\mathcal{I}u)'\|_1$ converges with order 0.5. This rate differs from the order 1 we expect when u is interpolated exactly. This observation guarantees that $\min_{w_h \in \mathcal{S}_h} \|u - w_h\|_1$ converges with order 0.5 or higher, since the w_h that minimizes $\|u - w_h\|_1$ in $\mathcal{S}_h$ may converge with a faster rate than the interpolant that we constructed. However, this observation prevents us from being able to conclude the the finite element solution will converge with order 1 at least. In practice, the finite element solution also converges with order 0.5.

For $m = 2, 4$, the error committed in approximating u at $x = 1$ decreases fast enough to not affect the order of convergence of the interpolant, guaranteeing that the finite element solution will converge with at least the same expected rate. In other word, if the error in the boundary conditions decreases fast enough with h, the error of convergence of the finite element solution is not be affected. It was affected for $m = 1$ because it did not decrease fast enough.