ME 335A
Finite Element Analysis
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Problems Set #2

Due Wednesday, April 19, 2023

On Vector Spaces of Functions (40)

For this problem section titled "1.1.3 Sets of Functions" in the notes has a discussion about the the notation used in this part.

Let

$$W = \{u \colon [-1,1] \to \mathbb{R} \text{ smooth}\}\,.$$

1. Are the following sets vector spaces of functions? Explain.
   (a) (5) $V_1 = \{u \in W \mid u(0) = 0\}$.
   (b) (5) $V_2 = \{u \in W \mid u''(0) = 0\}$.
   (c) (5) $V_3 = \{u \in W \mid u(x) \neq 0 \quad \forall x \in [-1,1]\}$.
   (d) (5) $V_4 = \{u \in W \mid \int_{-1}^{1} u''(x)\,dx = 0\}$.
   (e) (5) $V_5 = \{u \in W \mid \int_{-1}^{1} x^2 u(x)\,dx = 1\}$.
   (f) (5) $V_6 = \{u \in W \mid u(0) = -5\}$.
2. (5) The set V_6 is an affine subspace of W. What is its direction? You do not need to prove it, just state it.
3. (5) Is $\ell \colon V_1 \to \mathbb{R}$ a linear functional, where

$$\ell(u) = \int_{-1}^{1} u''(x)\,dx? \tag{1}$$

On Bases for Vector Spaces of Functions (20)

For $x \in \mathbb{R}$, define $g(x) = 1$ and

$$N_{x_0}(x) = \max(1 - |x - x_0|, 0).$$

1. (5) Plot the functions N_{-1}, N_0, and N_1 over the interval $(-3,3)$.

2. (5) For functions whose domain is $\mathbb{R}$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain. Hint: Find inspiration in Example 1.32 in the notes.

3. (5) For functions whose domain is $(-1, 1)$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain.

4. (5) Consider functions whose domain is $(-1, 1)$, and let $f(x) = 2x+1$. Does $f \in \operatorname{span}(N_{-1}, N_0, N_1)$? If so, what are its components?

A Simple Variational Method Example (35)

1. (15) Consider the problem: Find $u\colon [0,1] \to \mathbb{R}$ continuous such that

$$\begin{aligned} (1+x^2)u_{,xx} + xu_{,x} + x^2 u &= 0 \\ u_{,x}(1) - 3u(1) &= 0 \\ u(0) &= 1 \end{aligned}$$

   Find the variational equation of the problem using the recipe from the notes, with

$$\mathcal{V} = \{w\colon [0,1] \to \mathbb{R} \quad \text{smooth} \ \mid w(0) = 0\}.$$

   Identify essential and natural boundary conditions.

2. (2) Identify the bilinear form and the linear functional of the problem so that the variational equation can be written as $a(u, v) = \ell(v)$. Is a symmetric?

3. Consider a subspace of functions $\mathcal{W}_h = \operatorname{span}\{1, x, x^2, x^3\}$. We want to formulate a variational method with the variational equation in 1 and find its solution.

   (a) (5) What are the spaces trial and test spaces $\mathcal{S}_h$ and $\mathcal{V}_h$? What are the sets of active and constrained indices?

   (b) (2) Is the method consistent?

   (c) (7) Find the stiffness matrix and load vector.

   (d) (3) Find the solution to the variational method, and plot it.

   Hint: Recall that the stiffness matrix $K_{ab} = a(N_b, N_a)$; the order is important for non-symmetric bilinear forms.

4. (1) Is the natural boundary condition satisfied exactly by the solution of the variational method?