ME 335A
Finite Element Analysis
Instructor: Adrian Lew
Computing Project #1

April 22, 2023

Due Wednesday, May 3, 2023

Consider the two-point BVP: Given $f(x) = -2$; $g = 1$ and $h = -1$; find $u\colon [0,1] \to \mathbb{R}$ such that,

$$-u_{,xx} + \lambda u = f \qquad x \in (0,1) \tag{1}$$
$$u(1) = g \tag{2}$$
$$-u_{,x}(0) = h \tag{3}$$

The parameter λ can be positive, negative or zero. The objective of this assignment is to complete a Matlab code and employ the code to solve the above problem using piecewise linear finite elements. Some parts of the supplied code are missing and you are required to program the missing parts.

The program CP-1.m can be downloaded from Canvas. It is a single file with a script and two functions. The script contains the flow of the finite element program, and the two functions are:

1. `elementKandF()`: The function that returns the element stiffness matrix and load vector.
2. `P1()`: Returns the values of the shape functions, for plotting.

We will use a similar structure in future assignments, with appropriate modifications. It is not intended to be a general purpose code, nor to be the most efficient implementation, but rather, to be compact, concise and clear to read.

The goal is to construct an approximation of the solution of the above problem with a finite element space of continuous, piecewise affine functions. In other words, we will construct finite element meshes with P_1 elements and a finite element space $\mathcal{W}_h$ of continuous functions.

A completed assignment consists of:

- (60) The submission of the completed code, which should run and provide the solutions requested. Please send the code by email to me335-staff@lists.stanford.edu.
- (50) The answer to Questions 6 and 7 below, through Gradescope.

Question 1 The exact solution of the model problem $u(x)$ has been given for three cases as[1]

$$u(x) = A\exp(\sqrt{\lambda}x) + B\exp(-\sqrt{\lambda}x) + C; \qquad \lambda > 0 \tag{4}$$

$$\text{where } A = \frac{\lambda + \sqrt{\lambda}e^{-\sqrt{\lambda}} + 2}{\lambda(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})}, B = \frac{\lambda - \sqrt{\lambda}e^{\sqrt{\lambda}} + 2}{\lambda(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})}, C = -\frac{2}{\lambda}$$

$$u(x) = x^2 + x - 1; \qquad \lambda = 0 \tag{5}$$

$$u(x) = A\sin(\sqrt{-\lambda}x) + B\cos(\sqrt{-\lambda}x) + C; \qquad \lambda < 0 \tag{6}$$

$$\text{where } A = \frac{1}{\sqrt{-\lambda}}, B = \frac{\lambda + \sqrt{-\lambda}\sin\sqrt{-\lambda} + 2}{\lambda\cos\sqrt{-\lambda}}, C = -\frac{2}{\lambda}$$

On lines `123-126`, use (4), (5) (6) to compute the vectors `uep` and `duep`, which represent the exact solution and its derivative evaluated at the points defined by the vector `xp`.

Question 2 In the function `P1` there are missing lines of code which are used to compute the values of the shape functions and its derivative over an element with domain $\Omega_e = [0, 1]$. You are asked to fill in these missing lines of code.

Question 3 In the function `elementKandF` there are missing lines of code which are used to compute the element load vector and the element stiffness matrix. You are asked to fill in these missing lines of code.

Question 4 On lines `43-45`, please define the local-to-global map.

Question 5 On lines `72-44` and `80-82`, complete with the assembly to construct the stiffness matrix and load vector.

Note that once you have completed this step, the code will be ready for testing. Check to make sure that your solution provides a good approximation of the exact solution and its derivative.

Question 6 Once your code is working, vary the number of elements used to compute the solution (the value of `nel`). Generate figures of the FE solution and its derivative for `nel =1, 10 and 100`, for each one of the three cases $\lambda = -10, 0$ and 10.

Question 7 In line `89` the code to plot the results begins. Arrays `up` and `dup` contain the values of u_h and u_h', respectively, at 1000 equispaced points, while `uep` and `duep` should have the values of u and u', respectively, at the same points.

For each value of $\lambda = -10, 10$ and each `nel`$=10, 100, 500$, compute the following error measures:

$$\text{error}_u = \sqrt{\sum_{i=1}^{1000} |u(x_i) - u_h(x_i)|^2}, \qquad \text{error}_{u'} = \sqrt{\sum_{i=1}^{1000} |u'(x_i) - u_h'(x_i)|^2}.$$

Report the errors for each case in a table.

For each λ, they should behave approximately as $\sim h^k$, where $h = 1/\text{nel}$ is the mesh size and k is some real number.

Find and report what k is for each λ, for both the function and its derivative. To this end, it is convenient to plot log(`nel`) – log(error) and fit a straight line through them. The slope is $-k$. Please show these plots, and the fitted line.

Comment on your results, paying close attention to the accuracy of the solution, its derivative, and how that accuracy changes with the number of elements.

[1] You can use the `dsolve` command by Matlab to obtain the analytical solution of the given equation.