ME 335A
Finite Element Analysis
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Problems Set #2 – Solutions

Due Wednesday, April 19, 2023

On Vector Spaces of Functions (40)

For this problem section titled "1.1.3 Sets of Functions" in the notes has a discussion about the the notation used in this part.

Let

$$W = \{u\colon [-1,1] \to \mathbb{R} \text{ smooth}\}.$$

1. Are the following sets vector spaces of functions? Explain.

   (a) (5) $V_1 = \{u \in W \mid u(0) = 0\}$.
   Solution: Yes, because: (a) $u + v$ is smooth and αu is smooth for $u, v \in V_1$ and $\alpha \in \mathbb{R}$; and (b) $u(0) + v(0) = 0$ and $\alpha u(0) = 0$. So $u + v \in V_1$ and $\alpha u \in V_1$, and the closure condition is satisfied. The rest of the conditions are trivially satisfied.

   (b) (5) $V_2 = \{u \in W \mid u''(0) = 0\}$.
   Solution: Yes, because: (a) $u + v$ is smooth and αu is smooth for $u, v \in V_2$ and $\alpha \in \mathbb{R}$; and (b) $u''(0) + v''(0) = 0$ and $\alpha u''(0) = 0$. Then $u + v \in V_2$ and $\alpha u \in V_2$, and the closure condition is satisfied. The rest of the conditions are trivially satisfied.

   (c) (5) $V_3 = \{u \in W \mid u(x) \neq 0 \quad \forall x \in [-1,1]\}$.
   Solution: No, it is not a vector space, because the function $u(x) = 0$ for all $x \in [-1,1]$ is not in V_3, or $0 \notin V_3$, and hence the identity requirement is not satisfied.

   (d) (5) $V_4 = \{u \in W \mid \int_{-1}^{1} u''(x)\, dx = 0\}$.
   Solution: Yes, because: (a) $u + v$ is smooth and αu is smooth for $u, v \in V_4$ and $\alpha \in \mathbb{R}$; and $\int_{-1}^{1} u''(x) + v''(x)\, dx = 0$ and $\int_{-1}^{1} \alpha u''\, dx = 0$. Then $u + v \in V_4$ and $\alpha u \in V_4$, and the closure condition is satisfied. The rest of the conditions are trivially satisfied.

   (e) (5) $V_5 = \{u \in W \mid \int_{-1}^{1} x^2 u(x)\, dx = 1\}$.
   Solution: No, because the closure condition is not satisfied. If $u, v \in V_5$ and $1 \neq \alpha \in \mathbb{R}$, then $\int_{-1}^{1} x^2(u(x) + v(x))\, dx = 2$ and $\int_{-1}^{1} x^2 \alpha u(x)\, dx = \alpha$, and hence $u + v, \alpha u \notin V_5$.

   (f) (5) $V_6 = \{u \in W \mid u(0) = -5\}$.
   Solution: No, because the closure condition is not satisfied. If $u, v \in V_6$, then $u(0)+v(0) = -10$, and hence $u + v \notin V_6$.

2. (5) The set V_6 is an affine subspace of W. What is its direction? You do not need to prove it, just state it.

   Solution: The direction is the set V_1. To see this, for $v_1 \in V_6$, let

$$V = \{v_2 - v_1 \mid v_2 \in V_6\}.$$

   Since for any $v_2 \in V_6$, $v_2 - v_1 \in V_1$, we have that $V \subseteq V_1$. Alternatively, for any $v \in V_1$, $v_2 = v_1 + v \in V_6$, or $v_2 - v_1 = v$. Hence, $V \supseteq V_1$ (any $v \in V_1$ can be written as the difference between two functions in V_6). From here, $V = V_1$, and hence V_1 is the direction of V_6.

3. (5) Is $\ell \colon V_1 \to \mathbb{R}$ a linear functional, where

$$\ell(u) = \int_{-1}^{1} u''(x)\, dx? \tag{1}$$

   Solution: Yes, because ℓ can be computed and returns a finite value for any $u \in V_1$, and because it is linear; thas is for any $u, v \in V_1$ and any $\alpha \in \mathbb{R}$

$$\ell(u + \alpha v) = \int_{-1}^{1} (u(x) + \alpha v(x))''\, dx = \int_{-1}^{1} u''(x)\, dx + \alpha \int_{-1}^{1} v''(x)\, dx = \ell(u) + \alpha \ell(v).$$

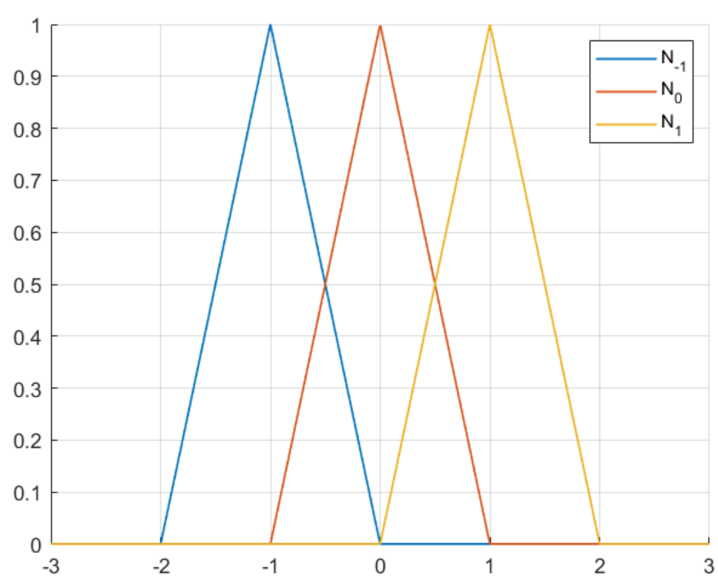
On Bases for Vector Spaces of Functions (20)

For $x \in \mathbb{R}$, define $g(x) = 1$ and

$$N_{x_0}(x) = \max(1 - |x - x_0|, 0).$$

1. (5) Plot the functions N_{-1}, N_0, and N_1 over the interval $(-3, 3)$.

   Solution:



2. (5) For functions whose domain is $\mathbb{R}$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain. Hint: Find inspiration in Example 1.32 in the notes.

   Solution: Yes. To see this, consider a linear combination

   $$v(x) = c_{-1}N_{-1}(x) + c_0 N_0(x) + c_1 N_1(x) + c_g g(x), \tag{2}$$

   for real values c_{-1}, c_0, c_1, c_g. Then, we need to show that $v(x) = 0$ for all x implies that $c_{-1} = c_0 = c_1 = c_g = 0$. To see this, it is enough to evaluate v at four selected different points, such as $x = -1, 0, 1$ and, for example, $x = 2$. To wit,

   $$\begin{aligned} 0 = v(-1) &= c_{-1} + c_g \\ 0 = v(0) &= c_0 + c_g \\ 0 = v(1) &= c_1 + c_g \\ 0 = v(2) &= c_g. \end{aligned}$$

   It is clear that the solution of this system is $c_{-1} = c_0 = c_1 = c_g = 0$. Therefore, this is a linearly independent set.

3. (5) For functions whose domain is $(-1, 1)$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain.

   Solution: No, it is not. To see this, it is enough to notice that $N_0(x)+N_1(x)+N_2(x) = 1 = g(x)$ for $x \in (-1, 1)$, so this is a linearly dependent set.

4. (5) Consider functions whose domain is $(-1, 1)$, and let $f(x) = 2x+1$. Does $f \in \operatorname{span}(N_{-1}, N_0, N_1)$? If so, what are its components?

   Solution: You can check by plotting that

   $$f(x) = -N_{-1}(x) + N_0(x) + 3N_1(x),$$

   and hence $f \in \operatorname{span}(\{N_{-1}, N_0, N_1)$, and its components are $(-1, 1, 3)$.

A Simple Variational Method Example (35)

1. (15) Consider the problem: Find $u\colon [0, 1] \to \mathbb{R}$ continuous such that

   $$\begin{aligned} (1 + x^2)u,_{xx} + xu,_x + x^2 u &= 0 \\ u,_x(1) - 3u(1) &= 0 \\ u(0) &= 1 \end{aligned}$$

   Find the variational equation of the problem using the recipe from the notes, with

   $$\mathcal{V} = \{w\colon [0, 1] \to \mathbb{R} \quad \text{smooth} \ \mid w(0) = 0\}.$$

   Identify essential and natural boundary conditions.

   Solution: We follow the steps from the notes. Let $v \in \mathcal{V}$. Then,

   $$\begin{aligned} 0 &= \int_0^1 ((1 + x^2)u,_{xx} + xu,_x + x^2 u)v \, dx \\ &= (1 + x^2)u,_x v|_0^1 - \int_0^1 ((1 + x)^2 v),_x u,_x \, dx + \int_0^1 xu,_x v + x^2 uv \, dx \\ &= 2u,_x(1)v(1) - u,_x(0)v(0) + \int_0^1 -((1 + x^2)v),_x u,_x + xu,_x v + x^2 uv \, dx \\ &= 2u,_x(1)v(1) - u,_x(0)v(0) + \int_0^1 -(1 + x^2)v,_x u,_x - xu,_x v + x^2 uv \, dx \end{aligned}$$

We can now use the boundary conditions we have to get

$$0 = 6u(1)v(1) + \int_0^1 -(1+x^2)v_{,x}\, u_{,x} - xu_{,x}\, v + x^2 uv \, dx$$

The variational equation that u satisfies is:

$$a(u, v) = 6u(1)v(1) + \int_0^1 -(1+x^2)v_{,x}\, u_{,x} - xu_{,x}\, v + x^2 uv \, dx = 0 \tag{3}$$

for all $v \in \mathcal{V}$. As a side remark, the linear functional in this case is $\ell(v) = 0$.

Here $u(0) = 1$ is an essential boundary condition, and $u_{,x}(1) - 3u(1) = 0$ is a natural one.

2. (2) Identify the bilinear form and the linear functional of the problem so that the variational equation can be written as $a(u, v) = \ell(v)$. Is a symmetric?

   Solution: The bilinear form in this case is in (3), and the linear functional is $\ell(v) = 0$ for any v. The bilinear form is not symmetric, because of the term $u_{,x}v$ inside the integral.

3. Consider a subspace of functions $\mathcal{W}_h = \text{span}\{1, x, x^2, x^3\}$. We want to formulate a variational method with the variational equation in 1 and find its solution.

   (a) (5) What are the spaces trial and test spaces $\mathcal{S}_h$ and $\mathcal{V}_h$? What are the sets of active and constrained indices?

   Solution: Let's label the basis functions for $\mathcal{W}_h$ as $N_1(x) = x$, $N_2(x) = x^2$, $N_3(x) = x^3$, $N_4(x) = 1$. So, for $w_h \in \mathcal{W}_h$ we can write

   $$w_h(x) = w_1 x + w_2 x^2 + w_3 x^3 + w_4.1.$$

   To satisfy the essential boundary condition $u(0) = 1$, we need to require that

   $$\mathcal{S}_h = \{1 + v_h \mid v_h \in \mathcal{V}_h\}.$$

   It direction is

   $$\mathcal{V}_h = \text{span}(N_1, N_2, N_3).$$

   We can then define the set of active indices as $\eta_a = \{1, 2, 3\}$ and the set of constrained indices as $\eta_g = \{4\}$, and

   (b) (2) Is the method consistent?

   Solution: Yes, since $\mathcal{V}_h \subset \mathcal{V}$, and hence the exact solution u satisfies the variational equation of the method for all $v \in \mathcal{V}_h$.

   (c) (7) Find the stiffness matrix and load vector.

   Solution: Let's compute the stiffness matrix first. Its expression is

   $$K = \begin{bmatrix} a(x,x) & a(x^2,x) & a(x^3,x) & a(1,x) \\ a(x,x^2) & a(x^2,x^2) & a(x^3,x^2) & a(1,x^2) \\ a(x,x^3) & a(x^2,x^3) & a(x^3,x^3) & a(1,x^3) \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{68}{15} & \frac{25}{6} & \frac{138}{35} & \frac{25}{4} \\ \frac{53}{12} & \frac{379}{105} & \frac{25}{8} & \frac{31}{5} \\ \frac{152}{35} & \frac{79}{24} & \frac{818}{315} & \frac{37}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

   The load vector is

   $$F = \begin{bmatrix} \ell(x) \\ \ell(x^2) \\ \ell(x^3) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5}$$

(d) (3) Find the solution to the variational method, and plot it.

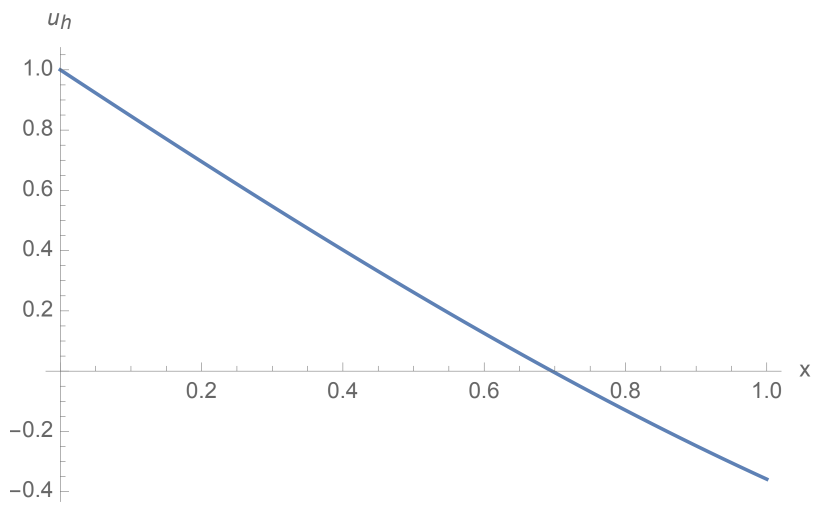
Solution: We have

$$U = K^{-1}F = \begin{bmatrix} -1.54358 \\ 0.0813227 \\ 0.103656 \\ 1 \end{bmatrix}, \tag{6}$$

from where the approximation to u follows as

$$u_h(x) = 1 - 1.54358x + 0.0813227x^2 + 0.103656x^3. \tag{7}$$

A plot of the result is



Hint: Recall that the stiffness matrix $K_{ab} = a(N_b, N_a)$; the order is important for non-symmetric bilinear forms.

4. (1) Is the natural boundary condition satisfied exactly by the solution of the variational method?

Solution: The value we are seeking is

$$(u_h)_{,x}(1) - 3u_h(1) = 0.00582901,$$

which is quite close to 0 relative to $3u_h(1) = -1.07579$, so the natural boundary condition is satisfied quite well.