as we do when finding the Euler-Lagrange equations, we integrate by parts the left hand side of (1.97) to eliminate derivatives over the test function v_h:

$$
\begin{aligned}
F(u, v_h) &= \int_0^1 u'(x)\, v_h'(x)\, dx - \int_0^1 v_h(x)\, dx = \\
&= \underbrace{u'(1)\, v_h(1)}_{=0,\text{ due to (1.96c)}} - \underbrace{u'(0)\, v_h(0)}_{=0,\, v_h \in \mathscr{V}_h} - \int_0^1 u''(x)\, v_h(x)\, dx - \int_0^1 v_h(x) dx \\
&= -\int_0^1 \underbrace{(u''(x) + 1)}_{=0,\text{ due to (1.96a)}}\, v_h(x)\, dx \\
&= 0.
\end{aligned}
$$

The test space $\mathscr{V}$ is commonly defined so that it already includes continuous finite element functions. This is convenient but not needed.

Therefore, we can state that

$$F(u, v) = 0 \qquad \forall\, v \in \mathscr{V} + \mathscr{V}_h,$$

where $\mathscr{V} + \mathscr{V}_h = \{w = v + v_h \mid v \in \mathscr{V}, v_h \in \mathscr{V}_h\}$.

The key ingredients to prove consistency here were that functions are smooth inside each element, and continuous across them, since in this case the integration by parts formula 1.2 holds.

1.4.2 What is a Finite Element?

We proceed now to describe how we construct finite element spaces beyond the span of hat functions that we introduced in §1.4.1. The construction of finite element spaces is done in two steps: (1) definition of vector spaces of functions over each element, and (2) adding functions defined over different elements to form functions whose domain is the entire interval $[c, d]$. We describe the first step next, and the second step in the next section, §1.4.3.

We begin by introducing the definition of a finite element.

Definition 1.11 (Finite Element)**.** *A finite element is a pair $e = (\Omega_e, \mathscr{N}^e)$ of an element domain Ω_e and a finite set of basis functions $\mathscr{N}^e = \{N_1^e, \dots, N_k^e\}$ defined over Ω_e.*

Given a finite element $e = (\Omega_e, \mathscr{N}_e)$ with element domain Ω_e and a set $\mathscr{N}^e = \{N_1^e, \dots, N_k^e\}$ of linearly independent functions $N_i^e\colon \Omega_e \to \mathbb{R}$, the space of functions $\mathscr{P}^e$ over Ω_e is defined as

$$\mathscr{P}^e = \operatorname{span}\{N_1^e, \dots, N_k^e\} \tag{1.102}$$

for $k \geq 1$, and it is called the **element space**. The set $\mathscr{N}^e$ is a basis for $\mathscr{P}^e$. Functions in $\mathscr{N}^e$ are known as **shape functions**. The number of shape functions k, or dimension of $\mathscr{P}^e$, is the **number of degrees of freedom** of the element. The **degrees of freedom** of the element are the components $\{\phi_1^e, \dots, \phi_k^e\}$ of functions in this basis. Each one of the components ϕ_i^e, $i = 1, \dots, k$, is a variable that can take any real value, and hence the k-tuple $(\phi_1^e, \ldots, \phi_k^e)$ can take any value in $\mathbb{R}^k$. For each such value, a unique function $f^e \in \mathscr{P}^e$, $f^e : \Omega_e \to \mathbb{R}$, is defined through

$$f^e(x) = \phi_1^e N_1^e(x) + \ldots + \phi_k^e N_k^e(x) = \sum_{a=1}^{k} \phi_a N_a^e(x). \tag{1.103}$$

The symbol e will be used interchangeably to denote an element or an **element index**, often a natural number, given that the index is another way to identify what element we are referring to. It is also common to use the word element in lieu of element domain; for example, wording such as ... *integrating over an element...*, as a way to say integrating over Ω_e.

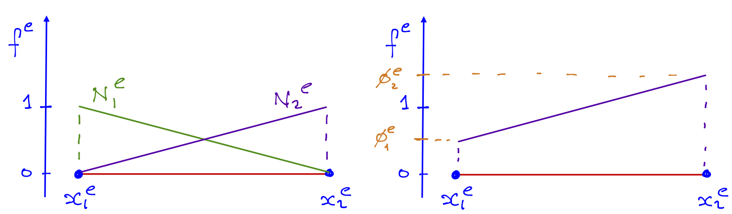
For the following examples we will consider a generic element with element domain $\Omega_e = [x_1^e, x_2^e]$, with vertex 1 at x_1^e and vertex 2 at x_2^e.

Examples:

1.60 P_1**-element.** One of the simplest element spaces is generated by the basis functions

$$\begin{aligned} N_1^e(x) &= \frac{x - x_2^e}{x_1^e - x_2^e}, \\ N_2^e(x) &= \frac{x - x_1^e}{x_2^e - x_1^e}, \end{aligned} \tag{1.104}$$

which satisfy that $N_a^e(x_b^e) = \delta_{ab}$.

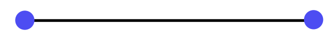


To see that the two are linearly independent, let

$$f^e(x) = \phi_1^e \frac{x - x_2^e}{x_1^e - x_2^e} + \phi_2^e \frac{x - x_1^e}{x_2^e - x_1^e}$$

and assume that $f^e(x) = 0$ for all $x \in \Omega_e$. In particular, $f^e(x_1^e) = \phi_1^e = 0$, and similarly, $f^e(x_2^e) = \phi_2^e = 0$. Therefore, this is a set of linearly independent functions.

The space $\mathscr{P}^e$ has 2 degrees of freedom, it is the space $\mathbb{P}_1(\Omega_e)$ of all polynomials of degree 1 or less over Ω_e. To see this, notice that $N_1^e(x) + N_2^e(x) = 1$ for all x, and $x_1^e N_1^e(x) + x_2^e N_2^e(x) = x$, so $\{1, x\} \in \mathscr{P}^e$. The degrees of freedom here are the values of f^e at x_1^e and x_2^e; this is the interpretation of ϕ_1^e and ϕ_2^e. Thus, we say that this element has a node at x_1^e and a node at x_2^e, and indicated them with a filled disc as follows



1.61 A variation of the P_1-element has the basis

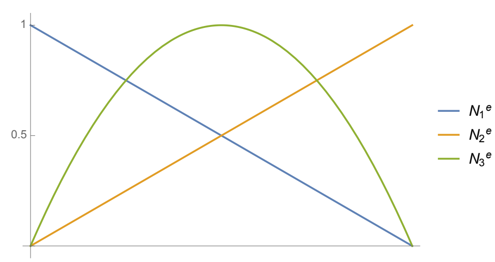
$$\begin{aligned} N_1^e(x) &= 1 \\ N_2^e(x) &= x. \end{aligned} \tag{1.105}$$

The space $\mathrm{span}\{N_1^e, N_2^e\}$ is still $\mathbb{P}_1(\Omega_e)$. However, the degrees of freedom in this case do not always lend themselves to be interpreted as pointwise values of the function $f^e = \phi_1^e\, 1 + \phi_2^e x$ somewhere in the element. There is no standard graphical depiction of this element.

1.62 P_1**-element+bubble.** Next, consider the basis functions

$$\begin{aligned} N_1^e(x) &= \frac{x - x_2^e}{x_1^e - x_2^e}, \\ N_2^e(x) &= \frac{x - x_1^e}{x_2^e - x_1^e}, \\ N_3^e(x) &= 4N_1^e(x)N_2^e(x). \end{aligned} \tag{1.106}$$

Their plot is



It is simple to check that this is a set of linearly independent functions. A function $f^e \in \mathscr{P}^e$ has the form

$$f^e(x) = \phi_1^e \frac{x - x_2^e}{x_1^e - x_2^e} + \phi_2^e \frac{x - x_1^e}{x_2^e - x_1^e} + \phi_3^e 4 \frac{(x - x_1^e)(x_2^e - x)}{(x_1^e - x_2^e)^2},$$

and one example of such function is shown in Fig. 1.12.

The space $\mathscr{P}^e$ has 3 degrees of freedom, and it is the space $\mathbb{P}_2(\Omega_e)$ of all polynomials of degree 2 or less over Ω_e. To see this, notice that $1, x \in \mathscr{P}^e$ from example 1.60, and that $x^2 = (x_2^e)^2 N_2^e(x) + (x_1^e)^2 N_1^e(x) - (x_2^e - x_1^e)^2/4\, N_3^e(x)$, so $x^2 \in \mathscr{P}^e$.

The degrees of freedom of this element do not all have a simple interpretation: ϕ_1^e and ϕ_2^e are the values of f^e at x_1^e and x_2^e, but ϕ_3^e lacks one. The name *bubble* comes from the shape of N_3^e, which is zero at the two boundaries of the element.

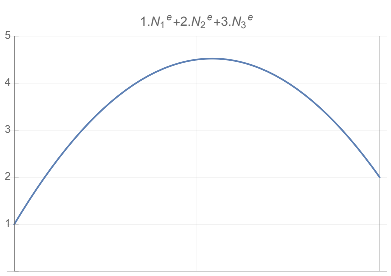


Figure 1.12 A function in the P_1-element+bubble.

What is a node? In general, whenever a degree of freedom of an element is the value of the function f^e or one of its derivatives at a location $\bar{x}$, we say that the element has a *node* at $\bar{x}$. When the degree of freedom is the value of the function, we depict it with a filled disk at $\bar{x}$. The symbol to depict the value of a derivative as a degree of freedom will be introduced later.

As a counterexample, degree of freedom ϕ_3^e in the P_1-element+bubble (Example 1.62) does not always correspond to the value of a function in the space at the midpoint between x_1^e and x_2^e, so such degree of freedom cannot be indicated by a node, c.f. Fig. 1.12.

The pictorial depiction of nodes in an element is a way to graphically indicate the degrees of freedom of an element, and it is commonly used in the finite element literature.

It is important to retain a strict distinction between the vertices of an element, which are used to define the geometry of the element domain, and the degrees of freedom indicated by the nodes, which are used to define functions over the element domain.

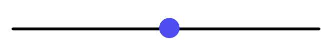
Examples:

1.63 P_0**-element.** The simplest element space is that of a single constant function over the domain of the element Ω_e, or in a fancy way, a polynomial in $\mathbb{P}_0(\Omega_e)$. A single basis function is needed,

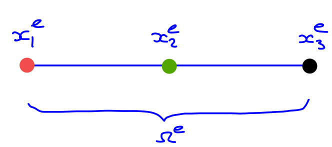
$$N_1^e(x) = 1, \tag{1.107}$$

so the space $\mathscr{P}^e$ has one degree of freedom.

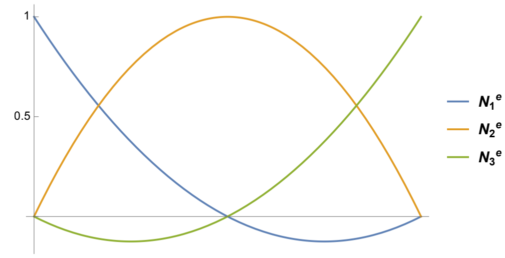
It can be represented with a node at the center of the element, or elsewhere.



1.64 P_2**-element.** The second most common element has the following basis functions over an element domain $\Omega_e = [x_1^e, x_3^e]$, with $x_2^e = (x_1^e + x_3^e)/2$,



$$\begin{aligned}
N_1^e(x) &= \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)}, \\
N_2^e(x) &= \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}, \\
N_3^e(x) &= \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}.
\end{aligned} \tag{1.108}$$

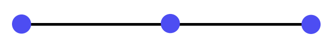


This is a linearly independent set of functions, a fact that follows from similar arguments to those for the P_1-element.

The space $\mathscr{P}^e$ has 3 degrees of freedom, it is also the space $\mathbb{P}_2(\Omega_e)$ of all polynomials of degree 2 or less over Ω_e. This is a set of three linearly independent quadratic polynomials, precisely the dimension of $\mathbb{P}_2(\Omega_e)$, and hence they need to span $\mathbb{P}_2(\Omega_e)$.

In this element we have that $N_1^e(x)+N_2^e(x)+N_3^e(x)=1$ for any $x\in\Omega_e$. To see this, let $f^e(x)=N_1^e(x)+N_2^e(x)+N_3^e(x)$, and notice that $f^e(x)=1$ for $x\in\{x_1^e,x_2^e,x_3^e\}$ and that $f^e\in\mathscr{P}^e$. Thus, $f^e(x)-1$ is a quadratic polynomial that is equal to 0 at these three points. We conclude then that $f^e(x)-1=0$ for all $x\in\Omega_e$. You can also check this by simply adding the three expressions in (1.108).

Since the basis functions satisfy that $N_a^e(x_b^e)=\delta_{ab}$, the degrees of freedom in the space are the values of a function $f^e\in\mathscr{P}^e$ at x_1^e, x_2^3 and x_3^e. Because these three spatial locations have the value of a function therein as a degree of freedom, the element has three nodes, each one represented with a filled disc, one at each vertex and one at x_3^e, to wit:



1.65 **P_k-element, for $k=1,\ldots$.** The Lagrange P_k-elements, often known simply as the P_k-elements, are a generalization of the P_0, P_1 and P_2 elements to any positive integer k. To simplify notation, we will denote the position of the vertices of the element by $z_1<z_2$, so that $\Omega_e=[z_1,z_2]$. Additionally, we introduce $k+1$ nodes at locations

$$x_a^e = z_1 + (a-1)\frac{(z_2-z_1)}{k}$$

for $a=1,\ldots,k+1$. The basis functions for this element are

$$N_a^e(x) = \frac{\Pi_{b=1,b\neq a}^{k+1}(x-x_b^e)}{\Pi_{b=1,b\neq a}^{k+1}(x_a^e-x_b^e)} \tag{1.109}$$

for $a=1,\ldots,k+1$. Each of these functions is a polynomial of degree k, and as will see next, they form a linearly independent set of $k+1$ functions. Therefore, $\mathscr{P}^e=\text{span}(N_1^e,\ldots,N_{k+1}^e)=\mathbb{P}_k(\Omega_e)$, or the set of all polynomials of degree less or equal than k over Ω_e, since the number

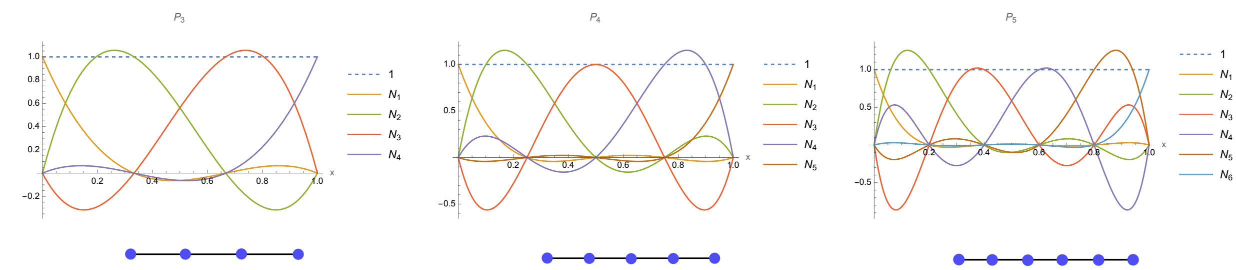


Figure 1.13 Shape functions for elements P_3, P_4 and P_5 over $\Omega_e = [0, 1]$, together with the constant function $f(x) = 1$ for comparison. The graphical depiction of each element is shown as well.

of linearly independent vectors functions is equal to the dimension of $\mathbb{P}_k(\Omega_e)$. The plots of these basis functions for $k = 3, 4, 5$ are shown in Fig. 1.13.

The first noteworthy feature of this set of functions is that

$$N_a^e(x_b^e) = \delta_{ab}, \tag{1.110}$$

so that the degrees of freedom of this element are the values of a function at $\{x_1^e, \ldots, x_{k+1}^e\}$. Therefore, this element has nodes at these locations. To see (1.110), notice that if $a \neq b$, then x_b^e is a zero of the numerator of N_a^e. Instead, if $a = b$, then the numerator and denominator of (1.110) are equal, and hence $N_a^e(x_a^e) = 1$.

To see that this is a basis, consider $(\phi_1^e, \ldots, \phi_{k+1}^e) \in \mathbb{R}^{k+1}$ such that

$$f(x) = \phi_1^e N_1^e(x) + \ldots + \phi_{k+1}^e N_{k+1}^e(x) = 0 \qquad \forall x \in \Omega_e.$$

Then, for any $a = 1, \ldots, k+1$, $f(x_a^e) = \phi_a^e N_a^e(x_a^e)$, because $N_b^e(x_a^e) = 0$ for $a \neq b$, from where $\phi_a^e = 0$. It then follows that this is a basis.

The final interesting property of this basis is that if $f \in \mathbb{P}_k(\Omega_e)$, then

$$f(x) = f(x_1^e)N_1^e(x) + \ldots + f(x_{k+1}^e)N_{k+1}^e(x) \qquad \forall x \in \Omega_e. \tag{1.111}$$

In particular, if $f(x) = 1$, then $N_1^e(x) + \ldots + N_{k+1}^e(x) = 1$ for all $x \in \Omega_e$. To prove (1.111), let $g(x) = f(x_1^e)N_1^e(x) + \ldots + f(x_{k+1}^e)N_{k+1}^e(x) - f(x)$. Notice then that $g(x_a^e) = 0$ for $a = 1, \ldots, k+1$, and that $g(x)$ is a polynomial of degree less or equal than k that is equal to zero at $k+1$ distinct points. This can only happen if $g(x) = 0$ for all $x \in \Omega_e$, from where (1.111) follows.

Elements in which all the degrees of freedom are values of the function at predefined locations in the element are called **Lagrange elements**. For example, the P_k-element is a Lagrange element, while the P_1-element+bubble is not.

1.4.3 Construction of Finite Element Spaces

Once we define a mesh over the interval Ω and element spaces on each element, we have what is called a **finite element mesh**. A vector space $\mathscr{W}_h$ of functions over the interval Ω can be constructed by defining a basis for it using the shape functions in each finite element.

What type of basis functions for $\mathscr{W}_h$ can we construct with the shape functions in each element? Let's look at some examples (we do not specify the entire basis yet). To this end, we consider a mesh of three P_1-elements in Fig. 1.14.

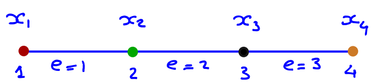
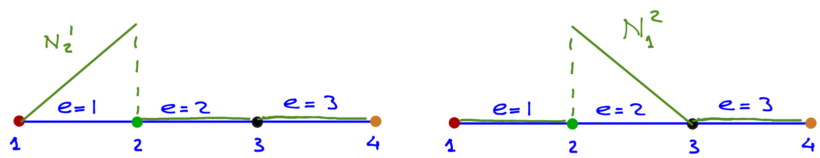


Figure 1.14

Examples:

1.66 Function N_2^1 could be a basis function, if we define it as equal to zero for points outside element $e = 1$. The function is discontinuous at x_2, so has two one-sided limits, $\lim_{x\to x_2^-} N_2^1(x) = 1$, and $\lim_{x\to x_2^+} N_2^1(x) = 0$. Similarly, N_1^2 could be a basis function, if we define it as equal to zero outside element $e = 2$, and it is discontinuous at x_2 as well. These two functions are sketched next:



1.67 The hat function N_2 can be constructed as the sum of N_1^2 and N_2^1, $N_2 = N_2^1 + N_1^2$, when each of them is defined as equal to zero outside elements $e = 1$ and $e = 2$, respectively, see Fig. 1.15. Because N_2^1 and N_1^2 are discontinuous at x_2, the value of N_2 at x_2 depends on what values each one of them takes at x_2. If we defined $N_2^1(x_2) = N_1^2(x_2) = 1$, then $N_2(x_2) = 2$, and N_2 would be discontinuous at x_2.

Instead, it is convenient to define

$$N_2(x) = N_2^1(x) + N_1^2(x) \qquad \text{for } x \neq x_2,$$

and define the value of N_2 at x_2 only if the two one-sided limits are the same. In this case they are, so

$$N_2(x_2) = \lim_{x\to x_2^-} N_2(x) = \lim_{x\to x_2^+} N_2(x) = \lim_{x\to x_2} N_2(x) = 1.$$

We will not consider cases in which the two one-sided limits are not equal. But if they were, the value of $N_2(x_2)$ need not be defined, and instead the method would use the values of the two one-sided limits. This is what is typically done in *Discontinuous Galerkin Methods*.

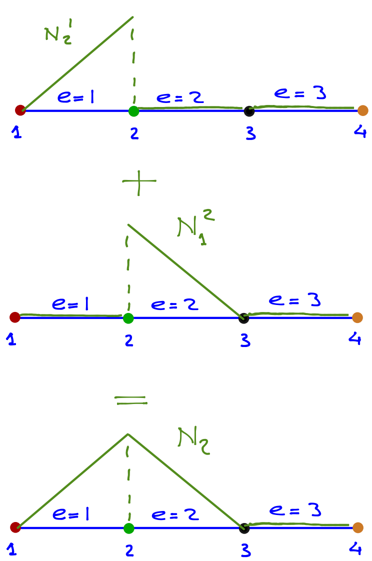


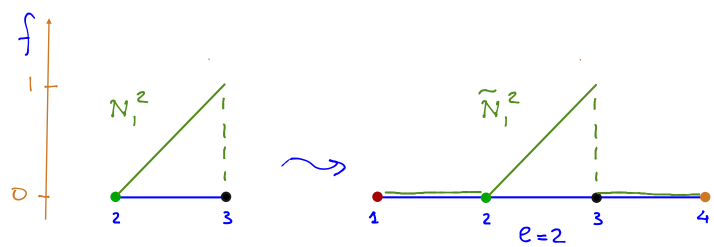
Figure 1.15

In general, to build a basis for $\mathscr{W}_h$, we proceed as follows:

1. **Extend Shape Functions by Zero.** For each element e in the mesh, we extend each basis function in the element by zero outside the element. More precisely, if we denote by $\tilde{N}^e_a\colon \Omega \to \mathbb{R}$ the extension-by-zero of the function $N^e_a\colon \Omega_e \to \mathbb{R}$, we can write

$$\tilde{N}^e_a(x) = \begin{cases} N^e_a(x) & x \in \Omega_e \\ 0 & x \notin \Omega_e. \end{cases} \tag{1.112}$$

See the example below.



In the following, we will not make an explicit distinction between $\tilde{N}^e_a$ and N^e_a, and simply use N^e_a for both.

2. **Define a Local-to-Global Map.** We will define *every* basis functions N_A for $\mathscr{W}_h$, with $A \in \{1,\dots,m\}$, by *adding* shape functions from one or more elements, with the condition that

each shape function in an element contributes to exactly one basis function in $\mathscr{W}_h$.

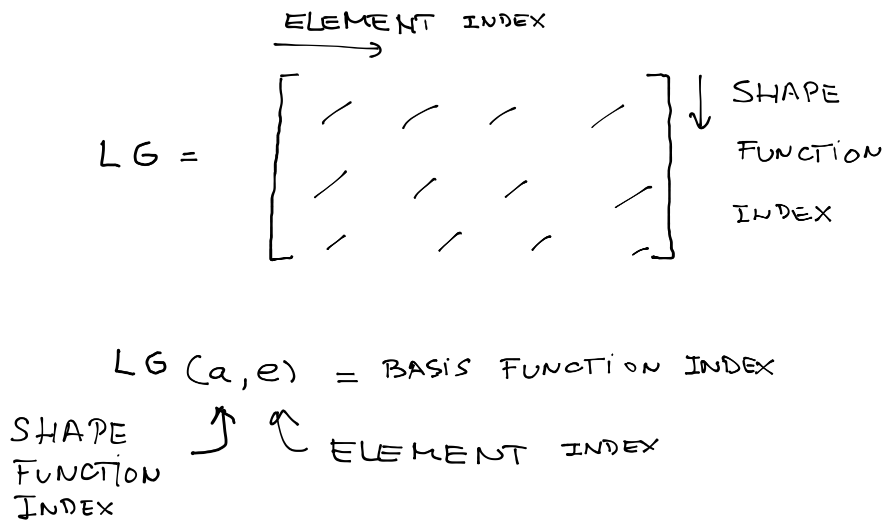
Then, $\mathscr{W}_h = \text{span}(N_1,\dots,N_m)$ and its dimension is m. Symbolically, if N_A is the result of adding $r \geq 1$ shape functions, we can write

$$N_A = N^{e_1}_{a_1} + \ldots + N^{e_r}_{a_r},$$

and each shape function N^e_a appears in exactly one of such sums.

We specify what basis function N_A a shape function N^e_a contributes to through a **local-to-global map**.

In this class, the local-to-global map is indicated with an $n_{\text{el}} \times k$ matrix termed LG, so that $A = \mathsf{LG}(a,e)$ is the entry in row a and column e in LG. Graphically:



The entry $\mathsf{LG}(a,e)$ in the matrix is a number in $\{1,\dots,m\}$ that defines that shape function N_a^e should be added when constructing basis function N_A. Alternatively, function N_A is obtained by adding all shape functions with index A in the matrix LG. Because each shape function contributes to exactly one basis function, every entry in the LG matrix is well-defined (there is no ambiguity on what value should go in entry of LG). The range of LG needs to be $\{1,\dots,m\}$, so that all basis functions in $\mathscr{W}_h$ are constructed in this way.

The name local-to-global map originates in the fact that it maps the indices of shape functions in each element, defined only locally over the domain of the element to form the element space, to indices of basis functions whose domain is the entire interval Ω to form the space $\mathscr{W}_h$.

3. **Add Shape Functions.** Because we are adding functions that are potentially discontinuous at the interfaces between neighboring elements, some care is needed in the definition of the basis functions for $\mathscr{W}_h$, as in Example 1.67.

   With the local-to-global map, we define the basis functions for $\mathscr{W}_h$. For any $A \in \{1,\dots,m\}$, let $N_A\colon \Omega \to \mathbb{R}$ be given by

$$N_A(x) = \sum_{\{(a,e) \mid \mathsf{LG}(a,e)=A\}} N_a^e(x), \tag{1.113a}$$

   for $x \neq x_i$ and

$$N_A(x_i) = \lim_{x \to x_i} N_A(x), \tag{1.113b}$$

   for all $i \in \{1,\dots,n_{\mathrm{el}}+1\}$.

   The set

$$\{(a,e) \mid \mathsf{LG}(a,e) = A\} \tag{1.114}$$

   says that we should seek all pairs (a,e) of shape function index a and element number e that are mapped to basis function index A. In other words, we should add all shape functions that contribute to basis function N_A. Because each basis function N_A is the sum of some shape functions, the set in 1.114 is never empty. Finally, if the limit in (1.113b) is not defined, the value of $N_A(x_i)$ is left undefined.

To remember that in performing this special sum we add the values everywhere except at the nodes as in (1.113a), and evaluate limits to find their values at the nodes as in (1.113b), we introduce a special name and symbol for it. We call it the **broken sum**, $\mathring{+}\colon \mathscr{W}_h \times \mathscr{W}_h \to \mathscr{W}_h$, so that for $f_h, g_h \in \mathscr{W}_h$,

$$(f_h \mathring{+} g_h)(x) = f_h(x) + g_h(x), \qquad x \neq x_i, \tag{1.115a}$$

(1.115b)

and

$$(f_h \mathring{+} g_h)(x_i) = \lim_{x \to x_i} (f_h \mathring{+} g_h)(x). \tag{1.115c}$$

Consistently with the new symbol, the broken summation sum is $\mathring{\sum}$.

With this notation, we can write

$$N_A = \mathring{\sum_{\{(a,e)|\mathsf{LG}(a,e)=A\}}} N_a^e. \tag{1.116}$$

Notice that we are now regularly referring to different sets of basis functions in the same context: the basis functions for $\mathscr{W}_h$ and the basis functions for the element spaces $\mathscr{P}^e$, or shape functions. To distinguish them, the basis functions for $\mathscr{W}_h$, whose domain is the entire interval, are called **global basis functions**. Conversely, basis functions for the element spaces $\mathscr{P}^e$, defined only over the element domain, are called **local basis functions**.

☞ Basis functions for the finite element space are called *global basis functions*, and basis functions for an element space are called *local basis functions*, or shape functions. Global basis functions are defined in the entire domain of the problem, while local basis functions are defined only over an element.

As a convention and whenever possible, uppercase letters will be used for indices of global degrees of freedom or basis functions, while local degrees of freedom or basis functions will use indices that are lowercase letters.

Examples:

1.68 **A space with discontinuous functions.** The simplest basis to define a space $\mathscr{W}_h$ over the mesh in Fig. 1.14 is the one in which each shape function defines a single global basis function. In this case, $m = k \times n_{\text{el}} = 2 \times 3 = 6$, and a local-to-global map is a 2×3 matrix that can be defined as

$$\mathsf{LG} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}. \tag{1.117}$$

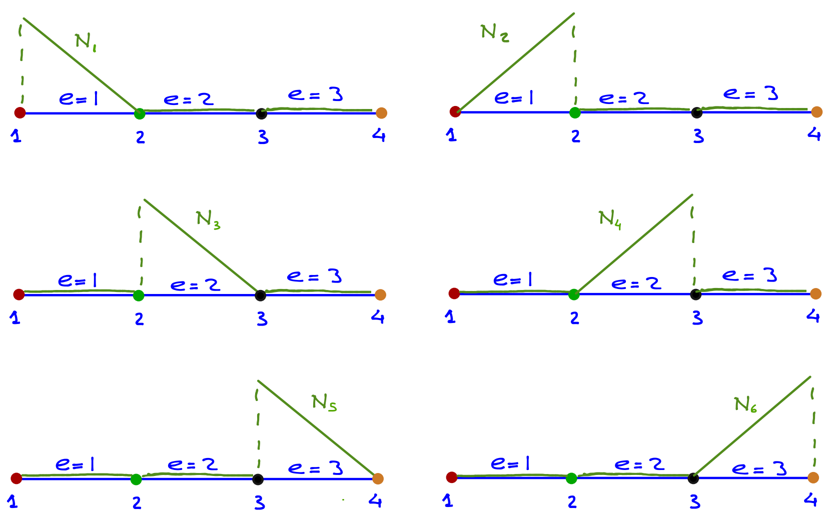
For example,

$$\mathsf{LG}(2,1) = 2,$$
$$\mathsf{LG}(1,3) = 5,$$
$$\mathsf{LG}(2,3) = 6.$$

With this local-to-global map, each global basis function index appears only once, so the basis functions for $\mathscr{W}_h$ can be written as

$$\begin{aligned} N_1 &= N_1^1, \quad N_2 = N_2^1 \\ N_3 &= N_1^2, \quad N_4 = N_2^2 \\ N_5 &= N_1^3, \quad N_6 = N_2^3. \end{aligned}$$

A sketch of the global basis functions is



☞ As defined by the broken sum, values at the discontinuities are left undefined, since the limit in (1.115c) does not exist. The two one-sided limits do exist. We will not need to worry about this, since we are going to require functions in the finite element space to be continuous.

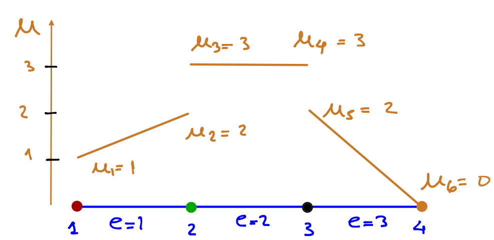
An example of a function defined on this space is

$$u = 1\ N_1 \mathring{+} 2\ N_2 \mathring{+} 3\ N_3 \mathring{+} 3\ N_4 \mathring{+} 2\ N_5 \mathring{+} 0\ N_6,$$

with components in this basis

$$U = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

These components can be interpreted as the one-sided limits of the function at the nodes, as sketched next:



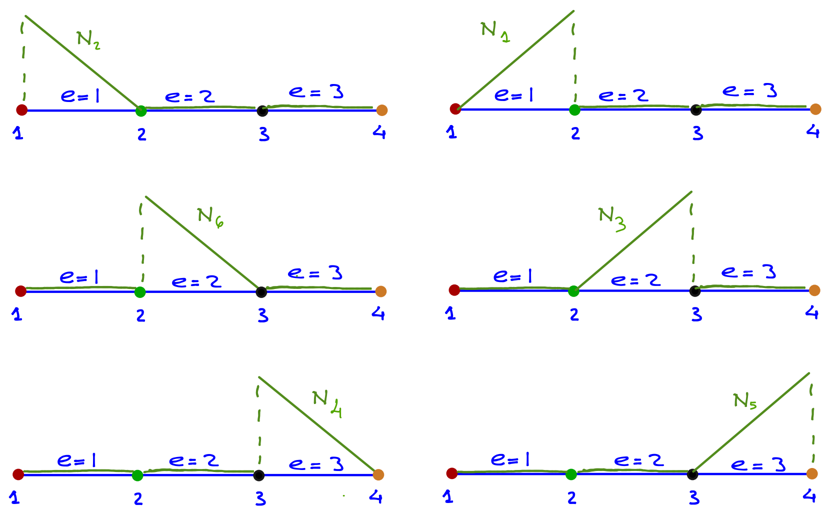
The index we assigned to each basis function of $\mathscr{W}_h$ is immaterial, since $\mathscr{W}_h$ does not change upon altering the name of each basis function. For example, we could have used the following local-to-global map

$$\mathsf{LG} = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 5 \end{bmatrix}. \tag{1.118}$$

In this case, the global basis functions can be written as

$$\begin{aligned} N_1 &= N_2^1, \quad N_2 = N_1^1 \\ N_3 &= N_2^2, \quad N_4 = N_1^3 \\ N_5 &= N_2^3, \quad N_6 = N_1^2. \end{aligned}$$

A sketch of the global basis functions with this new label is



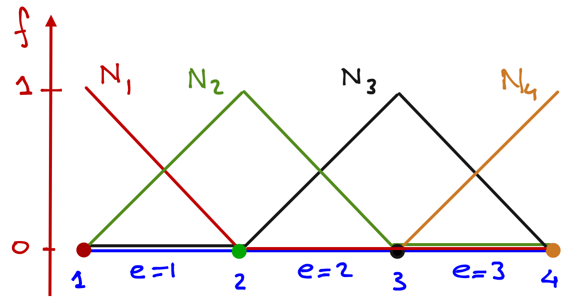
The function u is now written as

$$\begin{aligned} u &= 1\ N_2 \mathring{+} 2\ N_1 \mathring{+} 3\ N_6 \mathring{+} 3\ N_3 \mathring{+} 2\ N_4 \mathring{+} 0\ N_5 \\ &= 2\ N_1 \mathring{+} 1\ N_2 \mathring{+} 3\ N_3 \mathring{+} 2\ N_4 \mathring{+} 0\ N_5 \mathring{+} 3\ N_6, \end{aligned}$$

and its components are

$$U = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \\ 0 \\ 3 \end{bmatrix}.$$

1.69 **The simplest space of continuous functions.** The basis made of hat functions over the mesh in Fig. 1.14 is:



We can then build each basis function of $\mathscr{W}_h$ as a sum of shape functions as follows:

$$\begin{aligned} N_1 &= N_1^1 \\ N_2 &= N_2^1 \overset{\circ}{+} N_1^2 \\ N_3 &= N_2^2 \overset{\circ}{+} N_1^3 \\ N_4 &= N_2^2. \end{aligned} \tag{1.119}$$

This space has dimension $m = 4$, and the local-to-global map is defined by the following 2×3 matrix:

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}. \tag{1.120}$$

What is then the set

$$\{(a, e) \mid \mathsf{LG}(a, e) = 2\}?$$

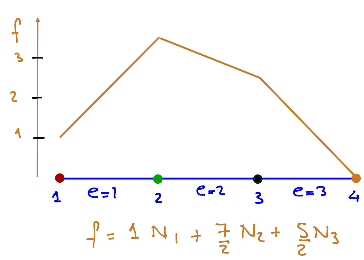
It is $\{(2, 1), (1, 2)\}$, that is, the rows and columns of the two entries equal to 2 in the LG matrix in (1.120).

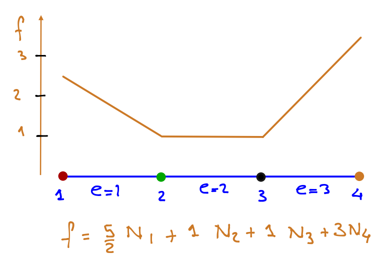
What about the set

$$\{(a, e) \mid \mathsf{LG}(a, e) = 4\}?$$

It is the set $\{(2, 3)\}$. You can now check that (1.113a) reduces to (1.119) for this example.

Examples of functions in this space are shown next:





1.70 **A space of continuous piecewise quadratic functions.** Consider the mesh of Fig. 1.14 with P_2-elements, c.f. (1.108). A space $\mathscr{W}_h$ of continuous functions that are polynomials of degree less or equal than 2 over each element can be built with the basis functions sketched in Fig. 1.17. These basis functions can be written as

$$
\begin{aligned}
N_1 &= N_1^1, \\
N_2 &= N_3^1 \overset{\circ}{+} N_1^2 \\
N_3 &= N_3^2 \overset{\circ}{+} N_1^3 \\
N_4 &= N_3^3 \\
N_5 &= N_2^1, \\
N_6 &= N_2^2, \\
N_7 &= N_2^3.
\end{aligned}
$$

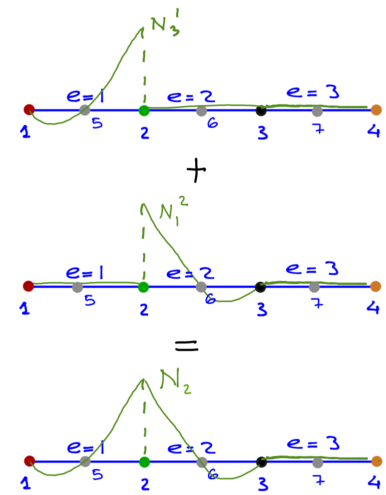


Figure 1.16

For example, the construction of N_2 is sketched in Fig. 1.16.

The space has dimension $m = 7$, and it is obtained from the local-to-global map given by the $k \times n_{\rm el} = 3 \times 3$ matrix

$$
\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}.
$$

For example,

$$
\mathsf{LG}(2,2) = 6, \qquad \mathsf{LG}(3,1) = 2.
$$

It is clear from Examples 1.68 and 1.69 that different local-to-global maps can be defined over the same finite element mesh, in this case a mesh of 3 P_1-elements. Each combination of a finite element mesh and a local-to-global map defines a (potentially different) space $\mathscr{W}_h$. This is a very general and flexible framework, which starting from the definition of finite elements enables the construction of very rich and varied vector spaces of functions.

Similarly to the basis functions, the degrees of freedom of the element space $\mathscr{P}^e$ are labeled **local degrees of freedom**, while those of $\mathscr{W}_h$ are called **global degrees of freedom**. We see next that, by construction, *the local-to-global map is also a map from the index of the local degree of freedom to the index of the global degree of freedom.* Specifically, consider a function $u \in \mathscr{W}_h$ defined by the values $(u_1, \dots, u_m)$ of the global degrees of freedom, namely,

$$
u = \sum_{A=1}^{m} u_A N_A,
$$

then

$$
u = \overset{\circ}{\sum_{e=1}^{n_{\rm el}}} \overset{\circ}{\sum_{a=1}^{k}} u_{\mathsf{LG}(a,e)} N_a^e.
$$

Therefore, for any element e, the function $f^e : \Omega_e \to \mathbb{R}$ defined by

$$
f^e(x) = u(x) \qquad x \in \Omega_e,
$$

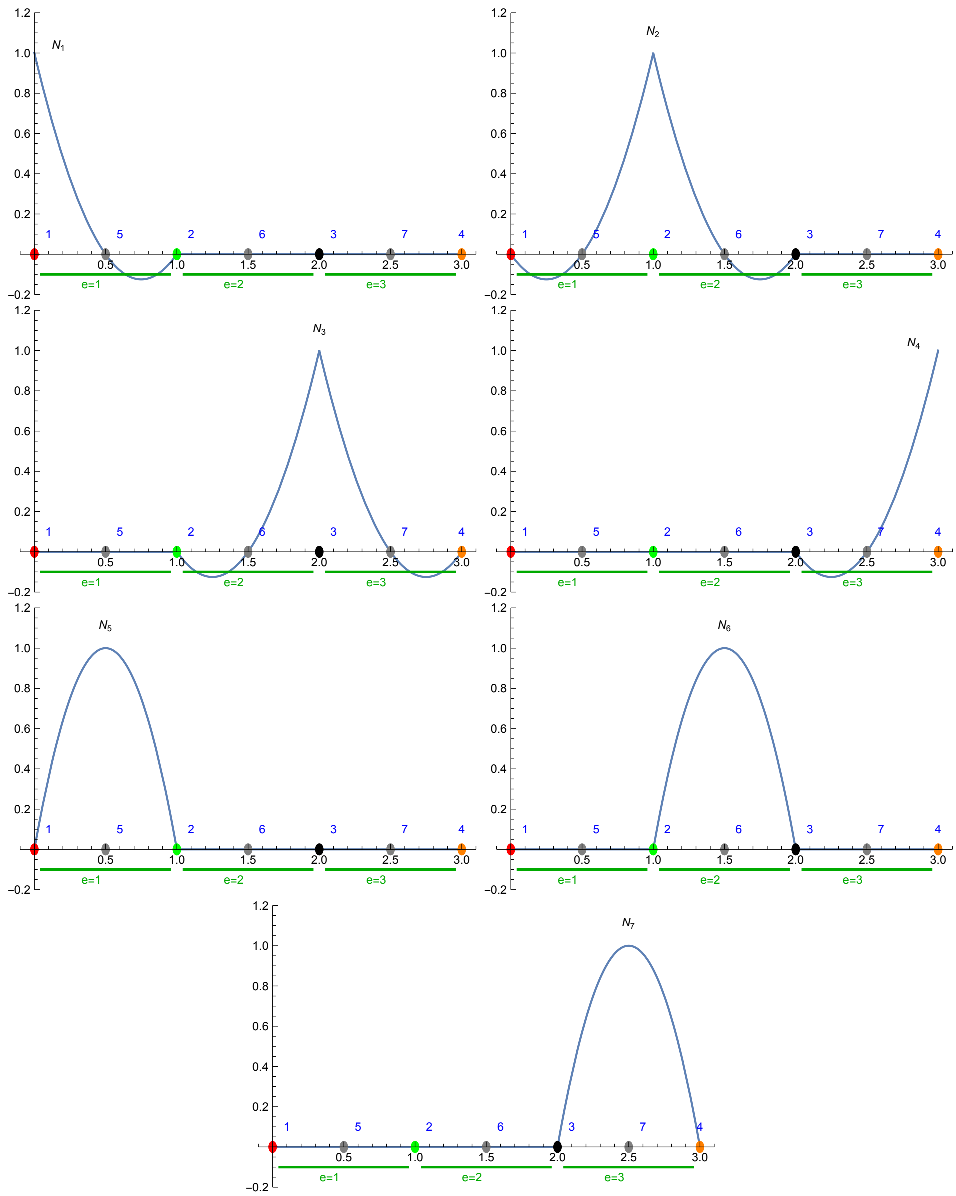


Figure 1.17 Basis functions for a finite element space of 3 P_2-elements.

belongs to $\mathscr{P}^e$. Its components in the local basis are

$$\phi^e_a = u_{\mathsf{LG}(a,e)}. \tag{1.121}$$

The function f^e is called the **restriction** of u to element e.

So each set of values for the global degrees of freedom define a set of values for the local degrees of freedom to describe the same function over an element. The process of obtaining the local degrees of freedom from the global ones through (1.121) is called **localization**.

Map between local and global degrees of freedom (1.121)**.**

Consider $u \in \mathscr{W}_h$, then

$$\begin{aligned}
u(x) &= \sum_{A=1}^{m} u_A N_A(x) \\
&= \sum_{A=1}^{m} u_A \mathop{\mathring{\sum}}_{\{(a,e)|\mathsf{LG}(a,e)=A\}} N^e_a(x) && \text{from (1.116)} \\
&= \sum_{A=1}^{m} \mathop{\mathring{\sum}}_{\{(a,e)|\mathsf{LG}(a,e)=A\}} u_{\mathsf{LG}(a,e)} N^e_a(x) && \text{from definition of } \mathsf{LG} \\
&= \mathop{\mathring{\sum}}_{e=1}^{n_{\text{el}}} \mathop{\mathring{\sum}}_{a=1}^{k} u_{\mathsf{LG}(a,e)} N^e_a(x) && \text{see below}
\end{aligned}$$

The last step uses the fact that in spanning all values of A with the first sum, the two sums together effectively guarantee that all pairs (a,e) will be added exactly once, since $\{1,\dots,m\}$ is precisely the range of LG, so its pre-image is the entire domain. This is again a consequence of the fact that every shape function contributes to exactly one global basis function, and all global basis functions are built in this way. It is also a consequence of defining global basis functions as sums of shape functions. Had global basis functions been defined as more general linear combinations of shape functions, each local degree of freedom would not be directly equal to a global degree of freedom.

So, because of the construction of the basis functions, it follows that: (a) the function u restricted to element e belongs to $\mathscr{P}^e$, (b) the values of the degrees of freedom of u restricted to element e are $\phi^e_a = u_{\mathsf{LG}(a,e)}$, so the local-to-global map also maps the local degrees of freedom to local ones.

1.4.4 Assembly of the Stiffness Matrix and Load Vector

The computation of the stiffness matrix and load vector generally involves the calculation of integrals over the domain, such as those involved in the bilinear form and linear functional. For example, for model Problem 1.3 with $b(x) = 0$,

$$a(u,v) = \int_\Omega \left[k(x)u'(x)v'(x) + c(x)u(x)v(x)\right] dx, \tag{1.122a}$$

$$\ell(v) = k(L)d_L v(L) + \int_\Omega f(x)v(x)\, dx. \tag{1.122b}$$