Chapter 3

Numerical Analysis of the FEM for Elliptic Problems

3.1 A Short Recapitulation

Let u be the **exact solution** of the problem of our interest, posed over a domain Ω. For our analysis we can assume u to be smooth (e.g., C^∞) and thus a solution of the corresponding strong form. The problem can be a second or fourth order elliptic problem in one or more dimensions, with suitable boundary conditions.

In the previous chapters, by multiplying the differential equation by an arbitrary smooth function v and integrating over Ω, we arrived at a bilinear form $a(\cdot,\cdot)$ and a linear form $\ell(\cdot)$ such that

$$a(u,v) = \ell(v)\ , \tag{3.1}$$

which holds for any smooth v that satisfies **homogeneous essential boundary conditions**. In other words, v satisfies the **essential boundary conditions** with value **zero**. The identification of which boundary conditions are essential was part of the process of deriving (3.1).

Our next step was to introduce a **finite element space** $\mathscr{W}_h$ defined on a mesh over Ω. The space $\mathscr{W}_h$ was in fact introduced by providing a set of **basis functions**, $\{\mathscr{N}_a, a = 1,2,\ldots,n\}$, such that

$$w_h \in \mathscr{W}_h \quad \Leftrightarrow \quad w_h(x) = \sum_{a=1}^{n} c_a N_a(x)\ .$$

This is the way finite element spaces are specified in practice.

We defined the **trial space** $\mathscr{S}_h$ and the **test space** $\mathscr{V}_h$ as

$$\begin{aligned}\mathscr{S}_h &= \{w_h \in \mathscr{W}_h \mid w_h \text{ satisfies essential boundary conditions}\}\\ \mathscr{V}_h &= \text{Direction of } \mathscr{V}_h.\end{aligned}$$

Remark: The possibility of taking $\mathscr{S}_h$ and $\mathscr{V}_h$ totally independent of one another certainly exists and has been explored in the literature (we called the Petrov-Galerkin methods). The methods we consider in this book, for which $\mathscr{V}_h$ is the direction of $\mathscr{S}_h$ and both are contained in an encompassing $\mathscr{W}_h$, remain the most popular choice.

Remark: When the domain is one-dimensional the essential boundary conditions consist of the value of u and/or u' at the boundary points. They can always be imposed exactly. In two or more dimensions the boundary values are arbitrary functions specified over lines or surfaces. If the boundary values are not piecewise polynomials, we assume that $\mathscr{S}_h$ satisfies an approximate (interpolated) version of the boundary values. It can be rigorously justified that the approximation of the boundary values does not hinder the convergence of the method. We will come back to this issue later on.

The last step in our construction was to define the **finite element solution** $u_h \in \mathscr{S}_h$ by

$$a(u_h, v_h) = \ell(v_h) \, , \qquad \forall \, v_h \in \mathscr{V}_h \, . \tag{3.2}$$

At this point, other than knowing that u_h and u satisfy (at least approximately) the same essential boundary conditions, we only know that u_h satisfies, over $\mathscr{V}_h$, the same variational equation that u satisfies for **smooth** functions (functions in $\mathscr{V}$). Is this sufficient to guarantee that u_h approximates u in some sense? What are the conditions on $\mathscr{W}_h$, $a(\cdot,\cdot)$ and $\ell(\cdot)$ for u_h to converge to u as the mesh is refined?

These questions concern the **numerical analysis** of the FEM, which deals with the mathematical properties of the method and is the subject of this chapter. We will make a brief excursion into the most important theorems and definitions, so as to provide a basis for understanding the specialized literature.

3.2 The Fundamental Approximation Result

The following is the fundamental theorem on which most of the analysis is based.

Theorem 3.1. *(***Céa's Lemma***) Assume that the exact solution satisfies the discrete problem (***exact consistency***):*

$$a(u, v_h) = \ell(v_h)\,, \qquad \forall\, v_h \in \mathcal{V}_h\,. \tag{3.3}$$

Assume, further, that there is a **norm** $\|\cdot\|$ *(defined on functions $f : \Omega \to \mathbb{R}$), such that:*

1. **Domain of the Norm:** $\|u\| < +\infty$, $\|w_h\| < +\infty$, $\forall\, w_h \in \mathcal{W}_h$.
2. **Continuity:** *There exist $M > 0$ and $m > 0$ such that*

$$a(u - w_h, v_h) \le M\,\|u - w_h\|\,\|v_h\|, \qquad \forall\, v_h \in \mathcal{V}_h, \forall\, w_h \in \mathcal{S}_h \tag{3.4}$$

$$\ell(v_h) \le m\,\|v_h\|, \qquad \forall\, v_h \in \mathcal{V}_h \tag{3.5}$$

3. **Coercivity:** *There exists $\alpha > 0$ such that*

$$a(v_h, v_h) \ge \alpha\,\|v_h\|^2\,, \qquad \forall\, v_h \in \mathcal{V}_h\,. \tag{3.6}$$

Then,

a) *the finite element solution* **exists***, is* **unique***, and satisfies the* **stability** *estimate*

$$\|u_h\| \le \frac{m}{\alpha} + \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|w_h\|\,; \tag{3.7}$$

b) *the following* **a priori approximation result** *holds:*

$$\|u - u_h\| \le \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|u - w_h\| \tag{3.8}$$

Let us briefly recall what a **norm** is.

Definition 3.1 (Norm)**.** *Let V be a vector space. A norm is a function $\|\cdot\| : V \to \mathbb{R}$ such that for $v, u \in V$ and $\beta \in \mathbb{R}$:*

1. **N.1.** $\|v\| \ge 0$, *and* $\|v\| = 0$ *if and only if* $v = 0$.
2. **N.2.** $\|\beta v\| = |\beta|\,\|v\|$.
3. **N.3.** $\|v + u\| \le \|v\| + \|u\|$ *(triangle inequality).*

The typical norm that you are familiar with is the "Euclidean norm" in $\mathbb{R}^3$: If $x = (x_1, x_2, x_3)$, then $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Clearly, N.1 holds, since $\|x\| \ge 0$, and if $\|x\| = 0$, then $x = 0$. The second condition, N.2, is also simple to verify, and the triangle inequality is the common statement that the sum of the lengths of two sides of a triangle is always greater or equal than the length of the third. These three conditions are intuitive to understand in the case of $\mathbb{R}^n$, and the fact that the Euclidean norm satisfies them is easy to see. Defining a norm for vector spaces of functions and proving N.1-N.3 is a little bit more complex, but can be done. Some such norms will be introduced soon. A more extensive discussion with examples can be found in §A.

Going back to Thm. 3.1, it is easy to see why this theorem, assuming its hypotheses are met, provides useful information about the numerical solution u_h:

- **Conclusion a)** guarantees that u_h exists and is unique. Since u_h is defined algorithmically as the solution of the algebraic system

  $$\text{stiffness matix} \quad \times \quad \text{unknown vector} \quad = \quad \text{load vector}$$

  this conclusion guarantees that the stiffness matrix is non-singular.

- Further, this conclusion tells us what to expect of the norm of u_h as the mesh is refined. Assume that α does not depend on the mesh size h, and also accept the technical hypothesis that the minimum in (3.7) is bounded independently of h. Then there exists a constant C such that

  $$\|u_h\| \le C$$

  for all mesh sizes $h > 0$. The discrete solution is uniformly bounded (in the norm $\|\cdot\|$ of course) for arbitrarily refined meshes.

  Remark: The value of $A_h = \min_{w_h \in \mathscr{S}_h} \|w_h\|$ depends on how “nice” the essential boundary conditions are. If, for example, the boundary is a curve (2D problem) and the imposed boundary value is discontinuous along the curve, then A_h could (or will) grow to infinity as the mesh is refined. If the boundary data are continuous then A_h remains bounded as $h \to 0$.

  Remark: In a first reading, it may prove useful to consider that the essential boundary conditions are zero. This makes $\mathscr{S}_h = \mathscr{V}_h$ and thus $0 \in \mathscr{S}_h$, implying that $A_h = 0$.

- **Conclusion b)** tells us how the error $u - u_h$ behaves. Let us define a function $w_h^* \in \mathscr{S}_h$ that realizes the minimum in (3.8). This function is thus, as measured by the norm $\|\cdot\|$, the **closest approximation** to u from the finite element space $\mathscr{S}_h$. It is computable *if the exact solution u is known* by solving a minimization problem.

  We cannot expect (though in some cases it happens!) that the finite element solution, which is computed *without knowing u*, coincides with the best possible approximation w_h^*. But conclusion b) tells us that $\|u - u_h\|$ is **at most** a factor $(1 + M/\alpha)$ larger that $\|u - w_h^*\|$.

  Further, **if the constants M and α do not depend on** h, we have that

$$\text{if } \|u - w_h^*\| \to 0 \text{ as the mesh is refined} \quad \Rightarrow \quad \|u - u_h\| \to 0$$

In other words: “If the finite element trial space **can** approximate the exact solution, it (eventually) **will**”.

Mathematically speaking, one can create a **sequence** of successively refined meshes of sizes $h_k \to 0$, obtain the corresponding FEM solutions u_{h_k}, and this sequence will **converge** to the exact solution u in the norm $\|\cdot\|$.

As we see, if a finite element method satisfies the hypotheses of Theorem 3.1 we have substantial guarantees with respect to its numerical performance. The mathematical analysis of a method usually begins by determining the appropriate norm and checking the hypotheses of the Theorem. Though the details depend on the specific problem and norm, it is worth making some general comments about each of the hypotheses.

Domain of the Norm. To be able to use the norm to measure distances among the exact solution u and/or any of the functions in the approximation space $\mathscr{W}_h$, the norm has to be defined on u and each function $w_h \in \mathscr{W}_h$. This is what this hypothesis requires.

Exact consistency. The consistency of the discrete formulation depends on the adopted variational equation and on the finite element space $\mathscr{W}_h$. In the previous chapters we have seen finite element formulations for second and fourth-order elliptic problems in one or more dimensions. For each formulation, we introduced a finite element space that “works,” and we verified that it works properly through numerical tests. For **second order problems** we selected finite element spaces consisting of C^0 functions, while for **fourth order problems** they consisted of C^1 functions. As we have seen, the main reason for this is that by enforcing such continuity on $\mathscr{W}_h$ we obtain a method that satisfies (3.3).

Condition (3.3) can be equivalently expressed as the so-called **Galerkin orthogonality**,

$$a(u - u_h, v_h) = 0\,, \qquad \forall\, v_h \in \mathscr{V}_h\,. \tag{3.9}$$

To check for consistency, one considers the **consistency residual**

$$r(u, v_h) = a(u, v_h) - \ell(v_h)\,.$$

If it is identically zero, as said, the method is **exactly consistent**. If it is not identically zero, then the theorem cannot be applied and, in general, u_h does not converge to u. There are exceptions, however. **Approximately consistent** formulations can still yield convergent results, but they are outside of our scope at this point.

Continuity. Inequality (3.4) is the definition of the continuity of a bilinear form, with respect to both its arguments. The continuity with respect to the first argument, in particular, guarantees that, if $\|u - w_h\|$ is small, then $a(u - w_h, \cdot)$ is also small. It can also be rewritten as

$$|a(u, v_h) - a(w_h, v_h)| \leq M \|u - w_h\| \, \|v_h\| \ ,$$

which brings forth that, if $\|u - w_h\|$ is small, then the difference between $a(u, \cdot)$ and $a(w_h, \cdot)$ is also small, as applied on any v_h.

Similarly, (3.5) implies that $|\ell(v_h) - \ell(w_h)| = |\ell(v_h - w_h)| \leq m\|v_h - w_h\|$ for all w_h and v_h in $\mathscr{W}_h$, which again shows that if $\|v_h - w_h\|$ is small then the difference between $\ell(v_h)$ and $\ell(w_h)$ is small.

In fact, since $\mathscr{W}_h$ is finite dimensional, such constants M and m always exist (because linear or bilinear functions are always continuous in finite dimensional spaces). However, in most cases of interest the continuity hypothesis holds in a **uniform** sense, i.e., with M and m **independent of the mesh size** h.

Coercivity. Inequality (3.6) defines the so-called **strong coercivity** of the bilinear form $a(\cdot, \cdot)$ over the space $\mathscr{V}_h$. It implies that

$$a(v_h, v_h) > 0 \ , \qquad \forall v_h \in \mathscr{V}_h, \ v_h \neq 0 \ . \tag{3.10}$$

Testing the strong coercivity of a bilinear form is thus easy, one simply plugs a generic v_h of the test space in both arguments of $a(\cdot, \cdot)$ and checks that the result is zero if and only if $v_h = 0$. If there exists $v_h \neq 0$ such that $a(v_h, v_h) = 0$, then (3.6) does not hold and the theorem cannot be applied.

However, (3.10) being true does not automatically imply that (3.6) holds **with a constant** α **independent of** h. This condition needs to be checked mathematically and in fact guides the selection of the appropriate norm $\|\cdot\|$.

It is now time to apply Theorem 3.1 to the different finite element methods that we built in the previous chapter. This is done in the following sections. Below we provide a proof of Céa's lema for the sake of completeness.

Proof. (Proof of Thm. 3.1) Let w_h be an arbitrary function belonging to $\mathscr{S}_h$ and let $u_h \in \mathscr{S}_h$ satisfy

$$a(u_h, v_h) = \ell(v_h) \ , \qquad \forall v_h \in \mathscr{V}_h \ . \tag{3.11}$$

Noticing that $u_h - w_h$ belongs to $\mathscr{V}_h$, we have from the coercivity hypothesis that

$$\|u_h - w_h\|^2 \quad \leq \quad \frac{1}{\alpha} \, a(u_h - w_h, u_h - w_h)$$

and, from the definition of u_h,

$$a(u_h - w_h, u_h - w_h) = \ell(u_h - w_h) - a(w_h, u_h - w_h) \ .$$

Combining the two equations above and using the hypotheses of continuity of $a(\cdot, \cdot)$ and $\ell(\cdot)$ we get

$$\|u_h - w_h\|^2 \leq \frac{1}{\alpha} \left(m \|u_h - w_h\| + M \|w_h\| \, \|u_h - w_h\| \right)$$

and thus

$$\|u_h - w_h\| \le \frac{m}{\alpha} + \frac{M}{\alpha}\|w_h\|, \quad \forall w_h \in \mathscr{S}_h\,.$$

Applying the triangle inequality $\|u_h\| \le \|u_h - w_h\| + \|w_h\|$ and taking the minimum over $\mathscr{S}_h$ completes the proof of (3.7).

In fact, (3.7) implies the uniqueness of u_h. To see this, notice first that u_h results from a *square linear system of equations*

$$\mathsf{KU} = \mathsf{F} \tag{3.12}$$

which has the dimension of $\mathscr{W}_h$. From linear algebra we know that, if the solution is not unique, there exists a vector Z such that $\mathsf{U} + \beta\mathsf{Z}$ satisfies the linear system, for all $\beta \in \mathbb{R}$. Alternatively, there exists a non-zero vector Z such that $\mathsf{KZ} = 0$.

We have already discussed at length the construction of the linear system (3.12) starting from the weak form and a basis $\{N_a\}$ of $\mathscr{W}_h$. It is clear that, if $\mathsf{U} + \beta\mathsf{Z}$ satisfies (3.12), then $u_h + \beta z_h$ belongs to $\mathscr{S}_h$ and satisfies (3.11), where $z_h = \sum_a Z_a N_a$.

We can thus apply inequality (3.7), that we already proved, to infer that

$$\|u_h + \beta z_h\| \le C\,, \qquad \forall \alpha \in \mathbb{R}$$

with $C = m/\alpha + (1 + M/\alpha)\min_{w_h}\|w_h\|$. But this can only be satisfied for all β if $z_h = 0$,[1] which implies $\mathsf{Z} = 0$ and thus that U and u_h are unique.

To finish the proof of a) it only remains to prove the existence of U (and thus of u_h). However, it is known from elementary linear algebra (the rank-nullity theorem) that for square systems *uniqueness implies existence*. Alternatively, if the system $\mathsf{KZ} = 0$ has only the trivial solution $\mathsf{Z} = 0$, then K is invertible and the solution exists and is computed as $\mathsf{U} = \mathsf{K}^{-1}\mathsf{F}$.

Let us now turn to prove conclusion b). Again, let w_h be an arbitrary element of $\mathscr{S}_h$. Then,

$$\begin{aligned}
\alpha\|u_h - w_h\|^2 &\le a(u_h - w_h, u_h - w_h) && \text{coercivity, (3.6)}\\
&= \underbrace{a(u_h - u, u_h - w_h)}_{=0} + a(u - w_h, u_h - w_h) && \text{add and subtract } u\\
&= a(u - w_h, u_h - w_h) && \text{Galerkin orthogonality, (3.9)}\\
&\le M\|u - w_h\|\|u_h - w_h\| && \text{continuity, (3.4).}
\end{aligned}$$

Notice that it is possible to use Galerkin orthogonality above because $w_h - u_h \in \mathscr{V}_h$, or because $\mathscr{V}_h$ is the direction of $\mathscr{S}_h$. We can then conclude that

$$\|u_h - w_h\| \le \frac{M}{\alpha}\|u - w_h\|.$$

[1] Specifically, by the triangle inequality, $\beta\|z_h\| \le \|u_h + \beta z_h\| + \|-u_h\| \le C + \|-u_h\|$, and hence $0 \le \|z_h\| \le (C + \|-u_h\|)/\beta$ for any β. This means that $\|z_h\|$ is smaller than any positive number by selecting β large enough, and hence $z_h = 0$.

Application of the triangle inequality leads to (3.8) because, for all w_h,

$$\begin{aligned}\|u-u_h\| &\leq \|u-w_h\| + \|w_h-u_h\| \\ &\leq \|u-w_h\| + \frac{M}{\alpha}\|u-w_h\| = \left(1+\frac{M}{\alpha}\right)\|u-w_h\|.\end{aligned}$$

□

3.3 Second Order Problems in One Dimension

Let us proceed to show how Theorem 3.1 can be applied to analyze finite element methods for second order elliptic problems in one dimension.

Let $u:[0,L]\to\mathbb{R}$ be the exact solution of Problem 1.1 with $b=0$, i.e., u satisfies $u(0)=g_0$, $u'(L)=d_L$ and

$$-\left(k(x)u'(x)\right)' + c(x)u(x) = f(x)\,, \qquad \forall\, x \in (0,L)\,. \tag{3.13}$$

Physical considerations allow us to consider that the coefficients k and c satisfy the bounds

- From below: $k(x) \geq k_{\min} > 0$ and $c(x) \geq 0$.
- From above: $k(x) \leq k_{\max} < +\infty$ and $c(x) \leq c_{\max} < +\infty$.

For $f(x)$ we assume that it is square-integrable, i.e., that the L^2-norm of f

$$\|f\|_0 = \left(\int_0^L f(x)^2\, dx\right)^{\frac{1}{2}} \tag{3.14}$$

is finite.

A variational equation for this problem was obtained in §1.1.2.3. It was shown that

- the corresponding bilinear form and linear functional are
$$a(u,v) = \int_0^L (ku'v' + cuv)\, dx\,, \qquad \ell(v) = \int_0^L fv\, dx\, +\, k(L)d_L v(L)\,,\ \text{and}$$
- the **essential** boundary condition is $u(0)=g_0$, while the condition $u'(L)=d_L$ is a **natural** boundary condition (it is incorporated into $\ell(v)$, see above).

To approximate u, we choose a finite element space $\mathcal{W}_h$ from which we define the trial and test spaces as

$$\mathcal{S}_h = \{w_h \in \mathcal{W}_h | w_h(0) = g_0\}\,, \qquad \mathcal{V}_h = \{w_h \in \mathcal{W}_h | w_h(0) = 0\}\,.$$

and then compute the **finite element solution** $u_h \in \mathcal{S}_h$ by solving

$$a(u_h, v_h) = \ell(v_h)\,, \qquad \forall\, v_h \in \mathcal{V}_h\,. \tag{3.15}$$

We will assume that functions in the finite element space $\mathcal{W}_h$ are C^1 in each closed element. We now study whether u_h indeed provides an approximation of u that becomes closer and closer as the mesh is refined.

3.3.1 Approximation Result

Let us analyze the validity of each one of the hypotheses of Theorem 3.1.

3.3.1.1 Exact Consistency

The consistency residual is given by

$$r(u, v_h) = \int_0^L (ku'v_h' + cuv_h)\,dx - \int_0^L f v_h\,dx - k(L)d_L v_h(L)\ .$$

The exact solution u is assumed to be a smooth function, and v_h by construction is smooth inside each element K in the mesh $\mathcal{T}_h$. We can thus integrate by parts (Theorem 1.2) in each K to get

$$\begin{aligned} r(u, v_h) \quad = \ & \int_0^L [-(ku')' + bu' + cu - f]\,v_h\,dx \\ & + k(L)(u'(L) - d_L)\,v_h(L) \\ & + k(0)\,u'(0)\,v_h(0) \\ & + \sum_z k(z)u'(z)\,(v_h(z^-) - v_h(z^+))\ , \end{aligned}$$

where z runs over all interelement boundaries and $v_h(z^-) - v_h(z^+)$ is the jump of v_h across z.

Remembering that u is the exact solution to (3.13), one automatically has that the first and second terms of the consistency residual are zero.

The third and fourth terms, on the other hand, are not guaranteed *by the solution u* to be identically zero because in general u' is not zero at $x = 0$ or at the element boundaries. Thus, **for exact consistency to hold automatically, the space $\mathcal{V}_h$ must consist of continuous functions that are zero at $x = 0$.** This makes $v_h(0)$ and the jumps in v_h identically zero, and thus $r(u, v_h) = 0$ for all v_h.

The satisfaction of the consistency condition requires finite element spaces $\mathcal{V}_h$ that consist of continuous functions that are zero at the essential boundary. Because $\mathcal{V}_h$ is the direction of $\mathcal{S}_h$, it is convenient to set $\mathcal{S}_h$ to be continuous as well , or directly, to build a space $\mathcal{W}_h$ of continuous functions.

This results is a generalization of the results in §1.4.1.1, in a little more detail.

Even if $\mathcal{V}_h$ contains only continuous functions, $\mathcal{S}_h$ could have *all* discontinuous functions if, for example, we set $\mathcal{S}_h = \{w_h = v_h + s_h \mid v_h \in \mathcal{V}_h\}$ where s_h is a discontinuous function.

3.3.1.2 Coercivity and the H^1-Norm

We look for a norm such that there exists $\alpha > 0$ (preferrably independent of h) satisfying

$$a(v_h, v_h) = \int_0^L [k(x)\,v_h'(x)^2 + c(x)\,v_h(x)^2]\ dx \geq \alpha\,\|v_h\|^2 \tag{3.16}$$

for all $v_h \in \mathcal{V}_h$. Indeed, we will prove that a norm that satisfies this is the so-called H^1-norm,

$$\|v_h\|_1 = \left[\int_0^L \left(v_h'(x)^2 + v_h(x)^2\right)\ dx\right]^{\frac{1}{2}} \ = \ \left(\|v_h\|_0^2 + \|v_h'\|_0^2\right)^{\frac{1}{2}}. \tag{3.17}$$

The proof is very easy if we add the hypothesis $c(x) \geq c_{\min} > 0$ because in such a case

$$a(v_h, v_h) = \int_0^L [k(x)\, v_h'(x)^2 + c(x)\, v_h(x)^2]\, dx$$
$$\geq \min\{k_{\min}, c_{\min}\} \underbrace{\int_0^L [v_h'(x)^2 + v_h(x)^2]\, dx}_{\|v_h\|_1^2}$$

and (3.16) holds with $\alpha = \min\{k_{\min}, c_{\min}\}$. Notice that α is totally independent of the mesh.

However, it is not unfrequent to deal with problems in which $c(x)$ is either zero everywhere (a pure diffusion problem) or at some region of the domain. In such a case coercivity in the H^1-norm still holds, but the proof is slightly different and relies on v_h being zero at $x = 0$ (which was not used above). Details are given as complementary material.

3.3.1.3 Domain of the Norm

Once coercivity has been verified, we need to check that the norm is finite for u and for any $w_h \in \mathscr{W}_h$. Fortunately, this is immediate, since it was assumed that u is smooth (in the *closed* interval $[0, L]$) and that w_h is continuous in $[0, L]$ and C^1 in each close element. The latter guarantees that the derivatives are continuous up to the boundary of each element, and hence bounded, so the $\|\cdot\|_1$ is finite.

3.3.1.4 Continuity

It only remains to prove continuity of $a(\cdot,\cdot)$ and $\ell(\cdot)$ in the H^1-norm.

$$\begin{aligned}
|a(u-w_h, v_h)| &= \\
&= \left|\int_0^L \left[k(x)(u'(x) - w_h'(x))\, v_h'(x) + c(x)(u(x) - w_h(x))\, v_h(x)\right] dx\right| \\
&\leq \int_0^L \left|k(x)(u'(x) - w_h'(x))\, v_h'(x) + c(x)(u(x) - w_h(x))\, v_h(x)\right| dx && \text{Triangle inequality for integrals} \\
&\leq \int_0^L \left(k(x)|u'(x) - w_h'(x)||v_h'(x)| + c(x)|u(x) - w_h(x)||v_h(x)|\right) dx && \text{Triangle inequality} \\
&\leq k_{\max} \int_0^L |u'(x) - w_h'(x)||v_h'(x)| dx + c_{\max} \int_0^L |u(x) - w_h(x)||v_h(x)| dx \\
&\leq k_{\max} \|u' - w_h'\|_0 \|v_h'\|_0 + c_{\max} \|u - w_h\|_0 \|v_h\|_0 && \text{Cauchy-Schwartz, (3.19)} \\
&\leq M \|u - w_h\|_1 \|v_h\|_1. && \|g\|_0, \|g'\|_0 \leq \|g\|_1, \forall g
\end{aligned}$$

with $M = k_{\max} + c_{\max}$.

To prove the continuity of the linear form, on the other hand, we consider

$$\ell(v_h) = \int_0^L f(x)\, v_h(x)\, dx + k(L)\, d_L\, v_h(L)\,. \tag{3.18}$$

The first term is easily proved to be bounded in the H^1-norm when $f \in L^2([0,L])$, since

$$\int_0^L f(x)\, v_h(x)\, dx \le \|f\|_0 \, \|v\|_0 \le \|f\|_0 \, \|v\|_1 \,.$$

Here we have used the **Cauchy-Schwartz inequality for integrals** which states that, if two functions v and w have finite L^2-norm, then

$$\int_0^L v(x)w(x)\; dx \le \left(\int_0^L v^2(x)\; dx\right)^{\frac{1}{2}} \left(\int_0^L w^2(x)\; dx\right)^{\frac{1}{2}} = \|v\|_0 \, \|w\|_0 \,. \tag{3.19}$$

The integral $\int_0^L v(x)w(x)\; dx$ is referred to as the L^2 **inner product** of v and w.

The second term in the right-hand side of (3.18) is trickier but still can be proved to be continuous without using that $v_h(0) = 0$ (so that it holds for Neumann boundary conditions too). We use the fact that v_h is *continuous* in space. As a consequence, there exists a point $\overline{x} \in [0,L]$ such that $v_h(\overline{x})$ equals the mean of v_h, i.e.,

$$v_h(\overline{x}) = \frac{\int_0^L v_h(x)\; dx}{L} \,.$$

From this,

$$\begin{aligned}
v_h(L) &= v_h(\overline{x}) + \int_{\overline{x}}^L v_h'(x)\, dx \\
&= \frac{1}{L}\int_0^L v_h(x)\; dx + \int_{\overline{x}}^L v_h'(x)\, dx \\
&\le \frac{1}{L}\|1\|_0 \, \|v_h\|_0 + \int_0^L |v_h'(x)|\, dx \\
&\le \frac{1}{L}\|1\|_0 \, \|v_h\|_0 + \|1\|_0 \|v_h'\|_0 \\
&\le \left(\frac{1}{L} + L\right)^{\frac{1}{2}} \|v_h\|_1 \,,
\end{aligned}$$

where we have used that $\|1\|_0 = \sqrt{L}$. We have thus proved that, if f is square-integrable and L is finite, then $\ell(\cdot)$ is continuous in the H^1-norm, i.e.,

$$\ell(v_h) \le m \, \|v_h\|_1 \,, \qquad \forall\, v_h \in \mathscr{W}_h \,,$$

where $m = \|f\|_0 + k(L)\, d_L \left(\frac{1}{L} + L\right)^{\frac{1}{2}}$. **It is again important to notice that both M and m do not depend on the mesh.**

We have thus checked all the hypothesis of Thm. 3.1 and can thus infer the following theorem:

Theorem 3.2. *If the finite element space $\mathscr{W}_h$ consists of* **continuous functions** *that in addition are C^1 in each closed element, then the finite element solution u_h defined by* (3.15) *exists, is unique, and satisfies*

$$\|u - u_h\|_1 \le C \min_{w_h \in \mathscr{S}_h} \|u - w_h\|_1 \tag{3.20}$$

with the constant C independent of the adopted mesh.