ME 335A
Finite Element Analysis
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Problems Set #4– Solutions

Due Wednesday, May 10, 2023

For this problem set, we are going to play with the interpolant of a function on a finite element mesh of Lagrange finite elements. Given a domain Ω and a mesh over it with m elements with $k+1$ nodes each, the *local interpolant* of a function f over an element e is a function $\mathcal{I}_e f:\ \Omega_e \to \mathbb{R}$ defined by

$$\mathcal{I}_e f(x) = \sum_{a=1}^{k+1} f(x_a^e) N_a^e(x),$$

where x_a^e is the location of the *a-th* node in the element e, and $N_a^e(x)$ are the basis functions of each element.

The *global interpolant* $\mathcal{I}f:\ \Omega \to \mathbb{R}$ is a function defined by patches over each element, that is,

$$\mathcal{I}f\,\Big|_{\Omega_e} = \mathcal{I}_e f,$$

or

$$\mathcal{I}f(x) = \mathcal{I}_e f \qquad \text{whenever } x \in \Omega_e.$$

So, to find the value of the global interpolant at a point x, we need to find the element to which x belongs, and evaluate the local interpolant there.

Constructing Some FE Spaces (35)

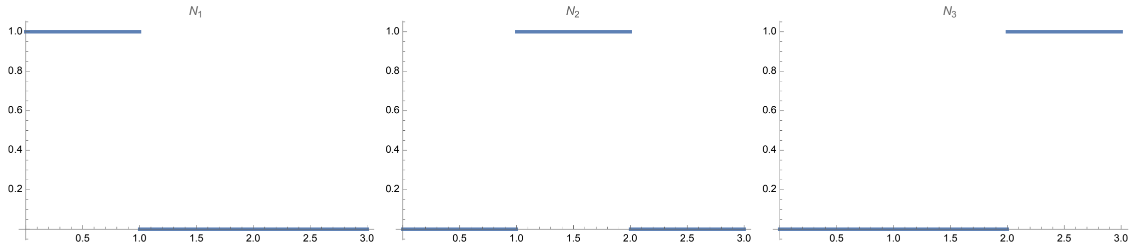
Consider a mesh of Lagrange P_k-elements (see Example 1.65 in the notes) with $n_{\text{el}} = 3$ elements of equal length in the interval $[0,3]$. Elements are numbered consecutively from 1 to n_{el} from left to rigth (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

   (a) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

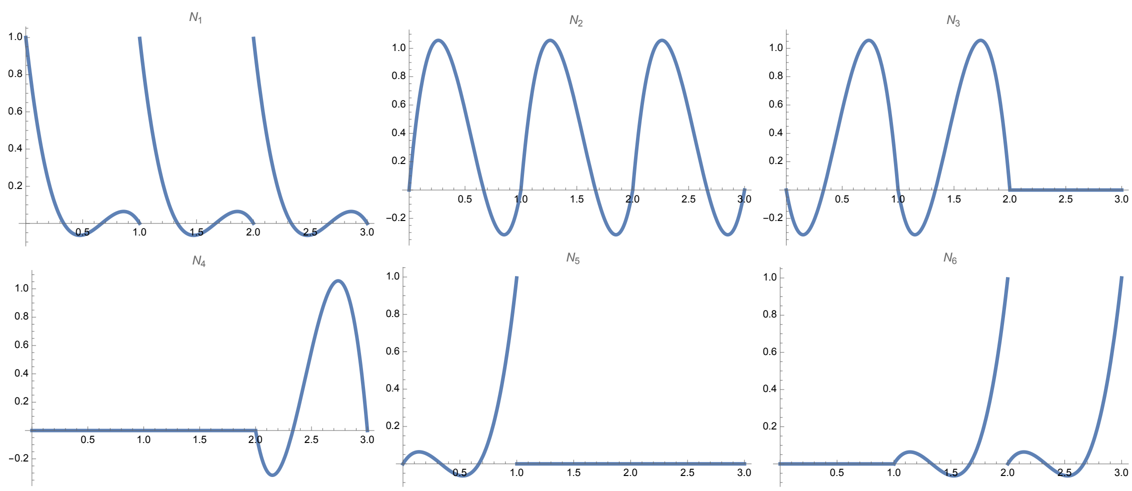
Solution: In this case, all shape functions of the same element need to be added up. Their sum is equal to 1 in each element. There is a total of 3 basis functions, and hence the dimension of the space is 3. Therefore, the basis functions are



(b) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

Solution: In this case, there are 6 basis functions, and hence the dimension of the space is 6. The basis functions are



2. For each one of the following finite elements, write the local-to-global map so that the basis functions are continuous and have minimal support (each basis function should be zero in the maximum number of elements).

(a) (5) For P_3-elements.
Solution:

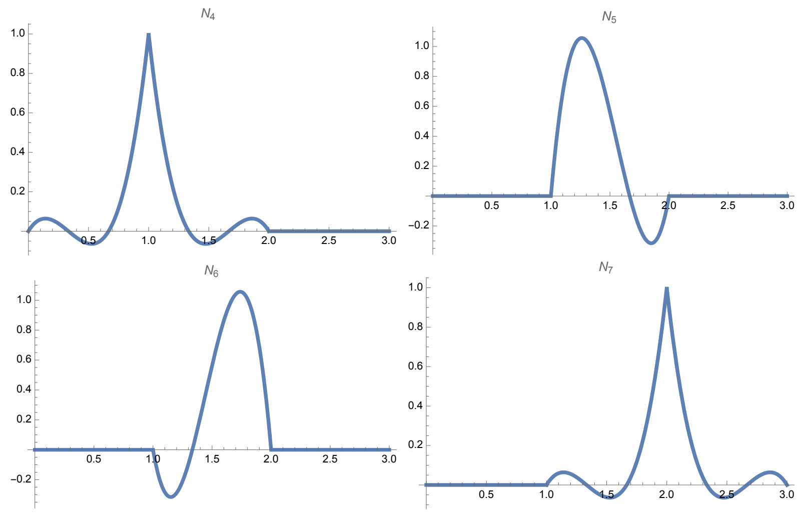
$$\mathsf{LG} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

(b) (5) For P_4-elements.
Solution:

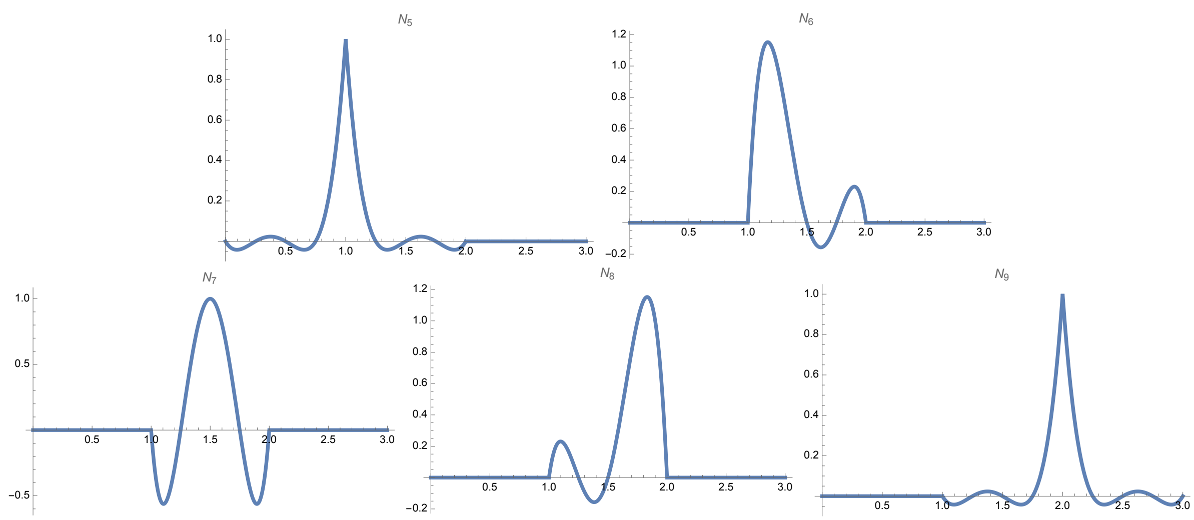
$$\mathsf{LG} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{bmatrix}$$

3. (5) Plot the basis functions that are non-zero in the second element for each of the cases in part 2 of this problem.

Solution: For P_3,



For P_4,



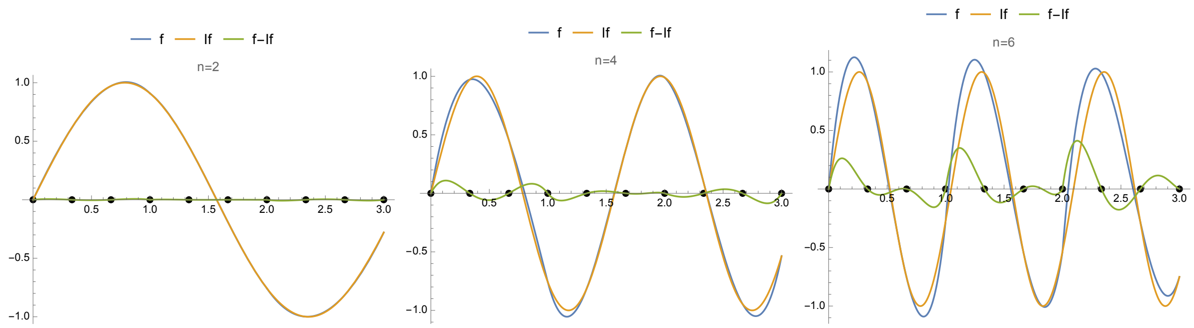
4. (5) For a problem with periodic boundary conditions of the form $u(0) = u(3)$, it is convenient to build a finite element space in which all functions in the space satisfy this periodicity constraint. If $k = 1$, write the required local-to-global map. What is the dimension of the finite element space?

Solution: The space has dimension 3, and the local-to-global map is

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

5. (5) Let $f(x) = \sin(nx)$. Plot f, $\mathcal{I}f$ and $f - \mathcal{I}f$ when all elements in the mesh are P_3-elements, for $n = 2, 4, 6$.

Solution:



(Manual) Assembly (80)

Consider the domain $\Omega = [1, 7]$, and the convection-diffusion problem: Find $u \colon \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
-\varepsilon u,_{xx} + c u,_{x} &= f & x \in \mathring{\Omega}, \quad &\text{(1a)}\\
-\varepsilon u,_{x}(7) &= h, & &\text{(1b)}\\
u(1) &= g, & &\text{(1c)}
\end{aligned}
$$

where $\mathring{\Omega} = (1, 7)$ indicates the *interior* of the set Ω, $c \in \mathbb{R}$ is the convection velocity, $\varepsilon > 0$ is the diffusion coefficient, $h, g \in \mathbb{R}$ are boundary conditions, and $f \colon \Omega \to \mathbb{R}$ is a source. We would like to construct a finite element approximation of the solution of this problem.

1. (10) Find a variational equation according to the recipe in the notes, and identify natural and essential boundary conditions.

   Solution: Following the steps in the notes:

$$0 = \int_1^7 (\varepsilon u,_{xx} - c u,_{x} + f) v dx \tag{2}$$

$$0 = \varepsilon u,_{x} v|_1^7 + \int_1^7 (-\varepsilon u,_{x} v,_{x} - c v u,_{x} + f v) dx \tag{3}$$

$$0 = -h v(7) + \int_1^7 (-\varepsilon u,_{x} v,_{x} - c v u,_{x} + f) v dx \qquad \text{use (1b), set } v(1) = 0 \tag{4}$$

$$0 = \int_1^7 (\varepsilon u,_{x} v,_{x} + c v u,_{x}) dx - \left[\int_1^7 f v dx - h v(7) \right] \tag{5}$$

   The variational equations that u satisfies is:

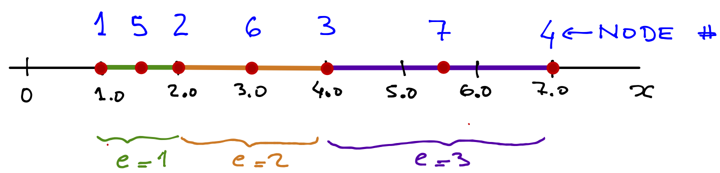
$$a(u, v) = \ell(v) \ \forall v \in \mathcal{V} = \{ v \colon \Omega \to \mathbb{R} \mid v(1) = 0 \} \tag{6}$$

$$a(u, v) = \int_1^7 (\varepsilon u,_{x} v,_{x} + c v u,_{x}) dx \tag{7}$$

$$\ell(v) = \int_1^7 f v dx - h v(7) \tag{8}$$

   The essential boundary condition is $u(1) = g$, and the natural boundary condition is $-\varepsilon u,_{x}(7) = h$.

2. (5) Consider the nodes 1 to 7 with positions $\{1, 2, 4, 7, 1.5, 3, 5.5\}$, respectively; see figure. These nodes form P_2 elements 1,2, and 3, whose domains are $\Omega^1 = [1, 2]$, $\Omega^2 = [2, 4]$, $\Omega^3 = [4, 7]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.

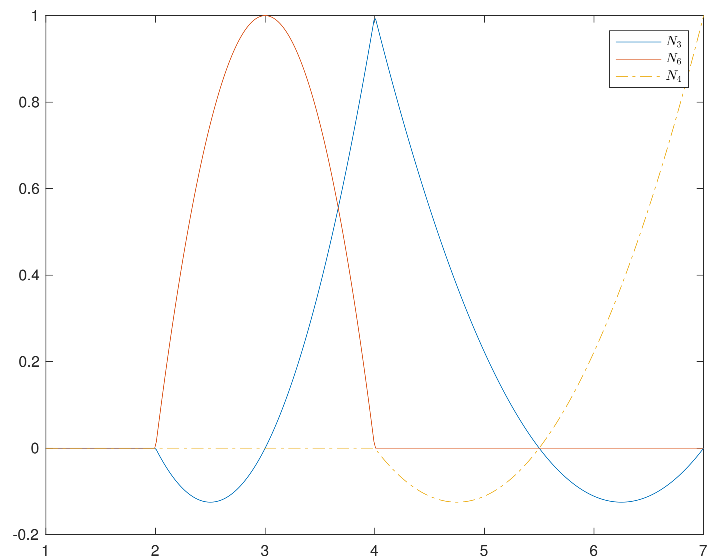


Solution:

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

3. (5) Plot the global basis functions N_3, N_6, and N_4.

Solution:



4. (5) Let $\mathcal{P}^2$ denote the element space for element 2. Find a function $v_h \in \mathcal{P}^2$ that attains the value 1.5 at node 2, -1.5 at node 6, and 3 at node 3. Express it as a linear combination of shape functions in the element, and as an explicit function of x.

Solution:

$$v_h(x) = 1.5N_1^2(x) - 1.5N_2^2(x) + 3N_3^2(x) \tag{9}$$

$$= \frac{15}{4}x^2 - \frac{87}{4}x + 30 \tag{10}$$

5. (5) Let $\mathcal{W}_h$ be the finite element space of continuous functions over the given mesh. Find a function $v_h \in \mathcal{W}_h$ that is equal to 2 on odd nodes, and to 3 on even nodes. Express it in terms of the global basis functions $\{N_A\}_{A=1,\ldots,7}$.

   Solution:

$$v_h = 2N_1 + 3N_2 + 2N_3 + 3N_4 + 2N_5 + 3N_6 + 2N_7$$

6. (10) State the finite element method for this problem using the variational equation obtained in part 1, identifying the spaces $\mathcal{V}_h$ and $\mathcal{S}_h$, and a basis for $\mathcal{V}_h$. Identify active and constrained indices and $\overline{u}_h \in \mathcal{S}_h$.

   Solution:
   The spaces $\mathcal{V}_h$ and $\mathcal{S}_h$ are described as

$$\mathcal{V}_h = \{v_h \in \mathcal{W}_h \mid v_h(1) = 0\}$$
$$\mathcal{S}_h = \{v_h \in \mathcal{W}_h \mid v_h(1) = g\}.$$

   For $w_h \in \mathcal{W}_h$, we need only to set $w_h(1) = 0$, and this is true if and only if $w_1 = 0$. Hence, we can write

$$\mathcal{V}_h = \operatorname{span}(N_2, N_3, N_4, N_5, N_6, N_7).$$

   The active and constrained indices are $\eta_a = \{2, 3, 4, 5, 6, 7\}$ and constrained indices are $\eta_g = \{1\}$. We can then choose

$$\overline{u}_h(x) = gN_1(x). \tag{11}$$

   The finite element method is then stated as: Find $w_h \in \mathcal{S}_h$ such that

$$a(v_h, w_h) = \ell(v_h) \text{ for all } v_h \in \mathcal{V}_h \tag{12}$$
$$a(v, w) = \int_1^7 (\varepsilon w_{,x} v_{,x} + cvw_{,x})dx \tag{13}$$
$$\ell(v) = \int_1^7 fvdx - hv(7) \tag{14}$$

7. In the following, assume that $f(x) = x$, $h = -20$, $\varepsilon = 1$, $c = 1$, $g = 2$.

   (a) (10) Compute the element matrices of the three elements.
   Solution:

$$K^e_{ab} = a^e(N^e_b, N^e_a) = \int_{\Omega^e} (\varepsilon N^e_{a,x} N^e_{b,x} + cN^e_a N^e_{b,x})dx \tag{15}$$

$$K^1 = \begin{bmatrix} \frac{11}{6} & -2 & \frac{1}{6} \\ -\frac{10}{3} & \frac{16}{3} & -2 \\ \frac{1}{2} & -\frac{10}{3} & \frac{17}{6} \end{bmatrix}$$

$$K^2 = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & 0 \\ -2 & \frac{8}{3} & -\frac{2}{3} \\ \frac{1}{3} & -2 & \frac{5}{3} \end{bmatrix}$$

$$K^3 = \begin{bmatrix} \frac{5}{18} & -\frac{2}{9} & -\frac{1}{18} \\ -\frac{14}{9} & \frac{16}{9} & -\frac{2}{9} \\ \frac{5}{18} & -\frac{14}{9} & \frac{23}{18} \end{bmatrix}$$

(b) (10) Compute the element load vectors for the three elements. Do not forget the natural boundary condition.

Solution:

$$F_a^e = \int_{\Omega^e} f N_a^e dx \tag{16}$$

$$F_a^3 = \int_{\Omega^e} f N_a^3 dx - h N_a^3(7) \tag{17}$$

$$F^1 = \begin{bmatrix} \frac{1}{6} \\ 1 \\ \frac{1}{3} \end{bmatrix}$$

$$F^2 = \begin{bmatrix} \frac{2}{3} \\ 4 \\ \frac{4}{3} \end{bmatrix}$$

$$F^3 = \begin{bmatrix} 2 \\ 11 \\ \frac{7}{2} + 20 \end{bmatrix}$$

(c) (10) Assemble the stiffness matrix and the load vector.

Solution:

To assemble the stiffness matrix, K_{ab}^e should be added to $K(LG(a,e), LG(b,e))$ if $LG(a,e) \in \eta_a$, and F_a^e should be added to $F(LG(a,e))$ if $LG(a,e) \in \eta_a$. Then we need to set the values for indices in η_g, in this case $K_{11} = 1$ and $F_1 = g = 2$. The stiffness matrix and load vectors are:

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{10}{3} & \frac{16}{3} & -2 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{10}{3} & \frac{7}{2} & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & \frac{8}{3} & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & -2 & \frac{35}{18} & -\frac{2}{9} & -\frac{1}{18} \\ 0 & 0 & 0 & 0 & -\frac{14}{9} & \frac{16}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & 0 & \frac{5}{18} & -\frac{14}{9} & \frac{23}{18} \end{bmatrix}$$

The load vector is

$$F = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 4 \\ \frac{10}{3} \\ 11 \\ \frac{47}{2} \end{bmatrix}$$

(d) (10) Find the finite element solution, and plot it. If you want to compare, the exact solution of this problem is

$$u(x) = -12e^{-6} + 12e^{x-7} + \frac{1}{2}(1+x)^2.$$

Solution: We solve the linear system

$$KU = F$$

The solution U is

$$U = \begin{bmatrix} 2 \\ 3.1562 \\ 4.5833 \\ 8.2811 \\ 13.3745 \\ 23.3841 \\ 43.9514 \end{bmatrix}$$

$$\begin{aligned} u_h(x) =& 2N_1(x) + 4.5833N_2(x) + 13.3745N_3(x) + 43.9514N_4(x) + \\ & 3.1562N_5(x) + 8.2811N_6(x) + 23.3841N_7(x) \end{aligned} \tag{18}$$