HW-6.

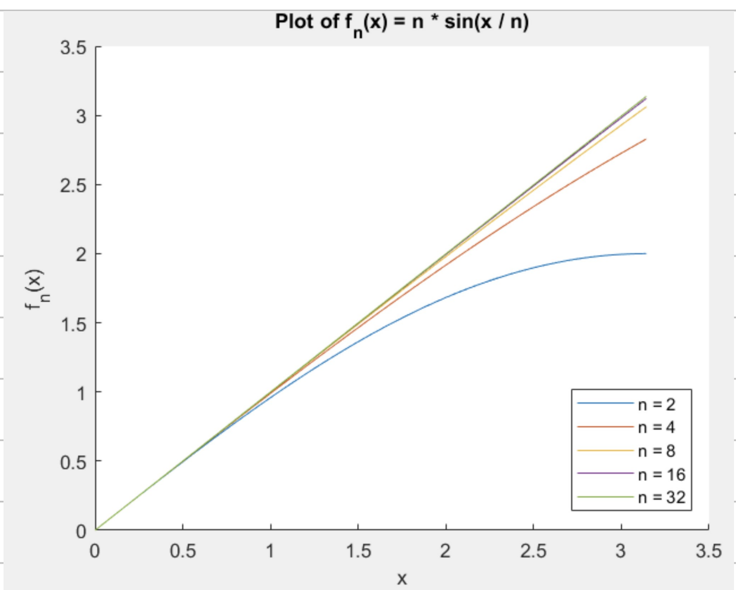
In this problem we would like to play with the convergence, norms, and membership of sequence of functions in different spaces. To this end, let $I = (0, \pi)$, and recall (see Appendix A in the notes) that a function $f\colon I \to \mathbb{R}$ is a member of the following spaces if

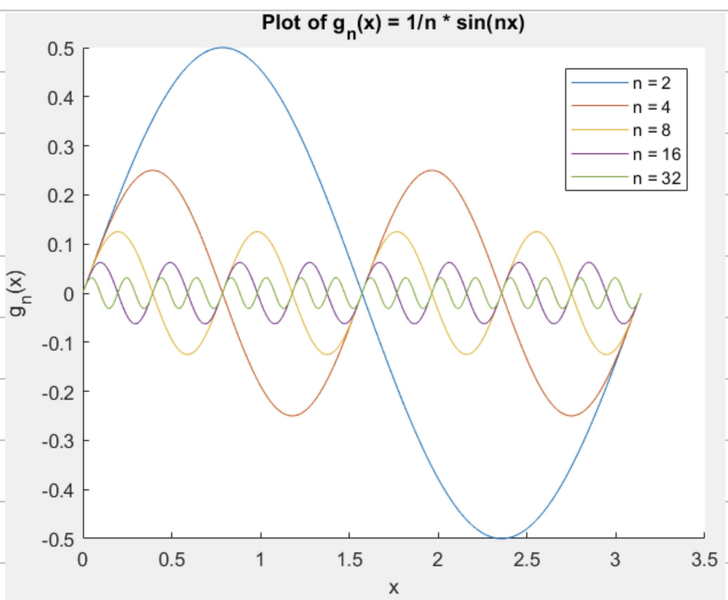
$$f \in L^2(I) \Leftrightarrow \|f\|_{0,2} = \left(\int_0^\pi f^2\,dx\right)^{1/2} < \infty$$
$$f \in L^\infty(I) \Leftrightarrow \|f\|_{0,\infty} = \max_{x\in I} |f(x)| < \infty$$
$$f \in H^1(I) \Leftrightarrow \|f\|_{1,2} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2\right)^{1/2} < \infty.$$

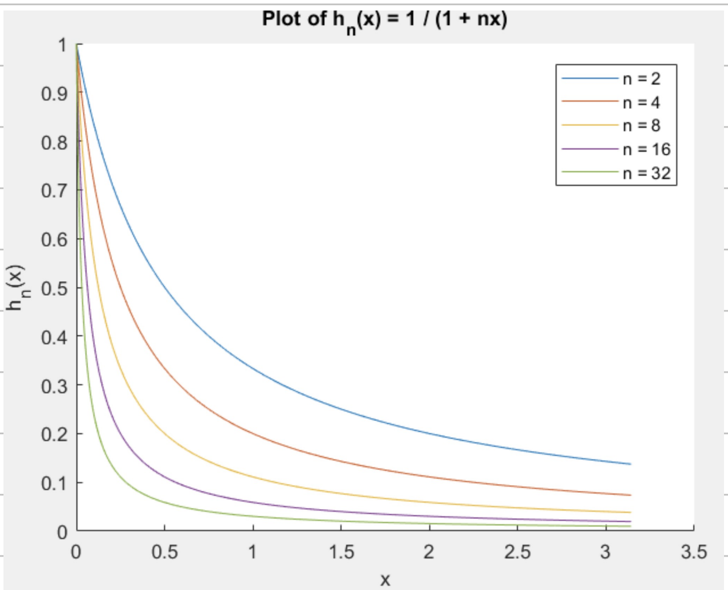
Consider the sequences of functions for $n = 1, 2\ldots$:

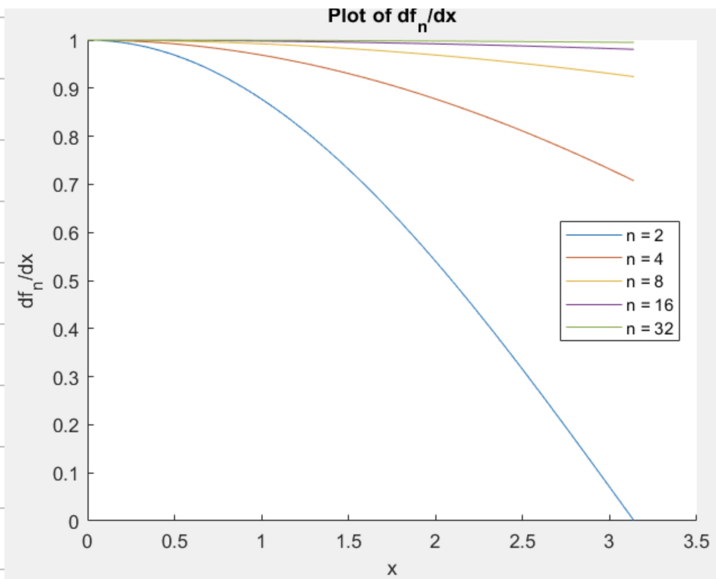
$$f_n(x) = n\sin\left(\frac{x}{n}\right), \quad f_\infty(x) = x,$$
$$g_n(x) = \frac{1}{n}\sin(nx), \quad g_\infty(x) = 0,$$
$$h_n(x) = \frac{1}{1+nx}, \qquad h_\infty(x) = 0.$$

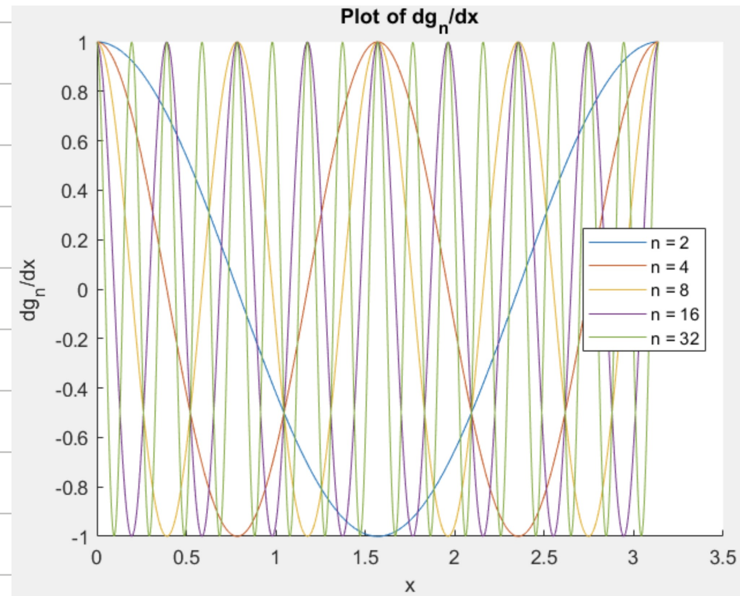
1.

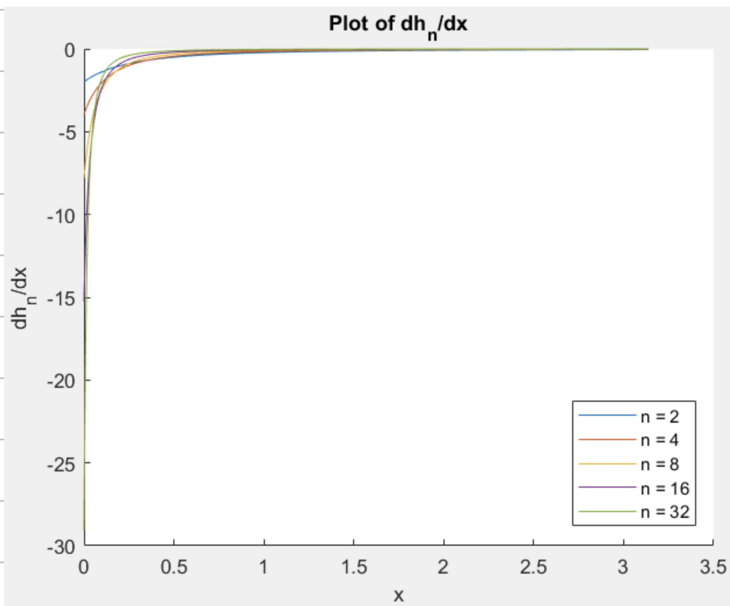












2① Given $f_\infty(x) = x$, $\|f_\infty\|_{0,2} = \left(\int_0^\pi x^2\,dx\right)^{\frac{1}{2}}$

$$= \left(\frac{\pi^3}{3}\right)^{\frac{1}{2}} < \infty.$$

Hence, $f_\infty \in L^2(I)$.

②. Given $f_\infty(x) = x$, $\|f_\infty\|_{0,\infty} = \max_{x\in I} |f(x)|$

$$= \pi < \infty$$

$f_\infty(x) \in L^\infty(I)$.

③. Given $f_\infty(x) = x$, $\|f_\infty\|_{1,2} = (\|f_\infty\|^2_{0,2} + \|f'_\infty\|^2_{0,2})^{1/2}$

$f'_\infty(x) = \lim_{n\to\infty} \cos(\frac{x}{n}) = 1$; $\|f'_\infty\|^2_{0,2} = 1$.

$\|f_\infty\|^2_{0,2} = \frac{\pi^3}{3}$.

Hence, $\|f_\infty\|_{1,2} = (\frac{\pi^3}{3}+1)^{\frac{1}{2}} < \infty$

$f_\infty \in H^1(I)$.

3.

$L^2(I)$ for f_n is as follows:

$$\|f\|_{0,2} = \left(\int_0^\pi (n \sin(\tfrac{x}{n}))^2 dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^\pi n^2 \sin^2(\tfrac{x}{n})\, dx\right)^{\frac{1}{2}}$$

$$= \left[\frac{\pi n^2}{2} - \frac{n^3 \sin(\frac{2\pi}{n})}{4}\right]^{\frac{1}{2}}$$

$H^1(I)$ for f_n is similiar

$$\|f\|_{1,2} = (\|f\|^2_{0,2} + \|f'\|^2_{0,2})^{1/2}$$

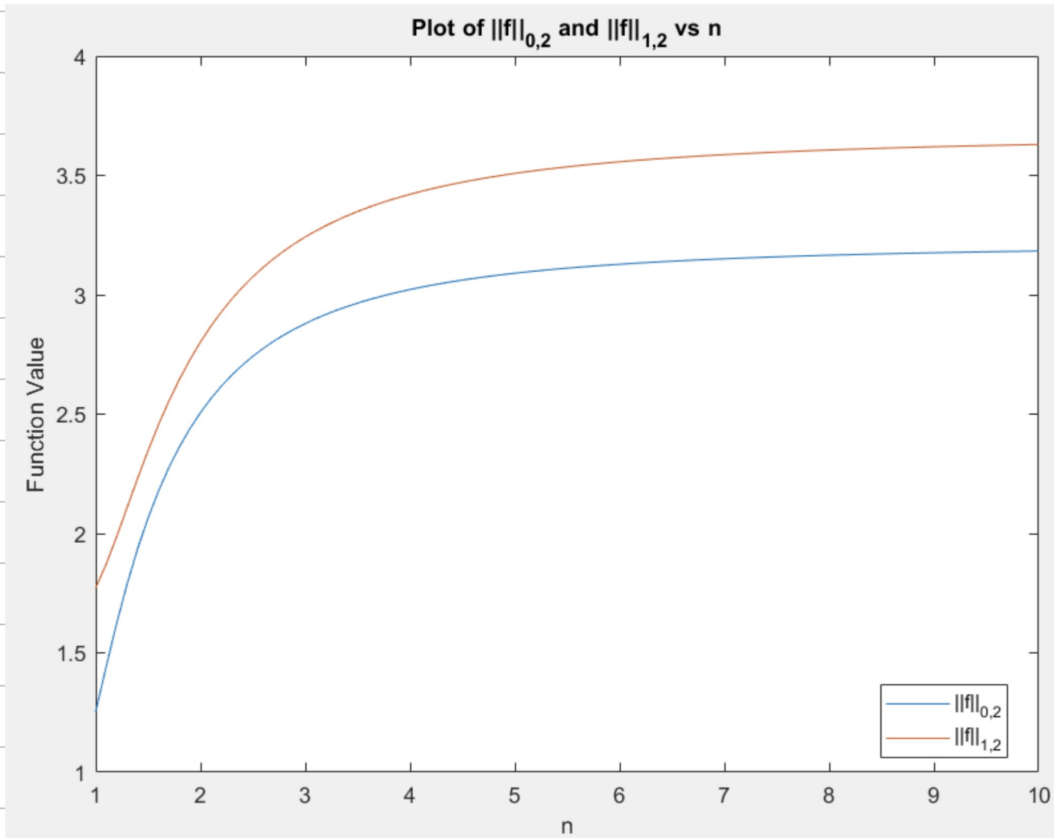
Calculately the second part in the parenthesis

$$\|f'\|_{0,2} = \left[\int_0^\pi (\cos(\tfrac{x}{n}))^2 dx\right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{n\sin(\frac{2\pi}{n})}{4}\right]^{\frac{1}{2}}$$

$$\|f\|_{1,2} = \left(\frac{\pi n^2}{2} - \frac{n^3\sin(\frac{2\pi}{n})}{4} + \frac{\pi}{2} + \frac{n\sin(\frac{2\pi}{n})}{4}\right)^{\frac{1}{2}}$$

The $\|f\|_{0,2}$ and $\|f\|_{1,2}$ vs n are as follows:



Q4.

①. $f_n(x) = n\sin(\frac{x}{n})$; $f_\infty(x) = x$.

$$\|f_n(x) - f_\infty(x)\|_{0,2} = \left(\int_0^\pi \left(n\sin(\tfrac{x}{n}) - x\right)^2\right)^{\frac{1}{2}}$$

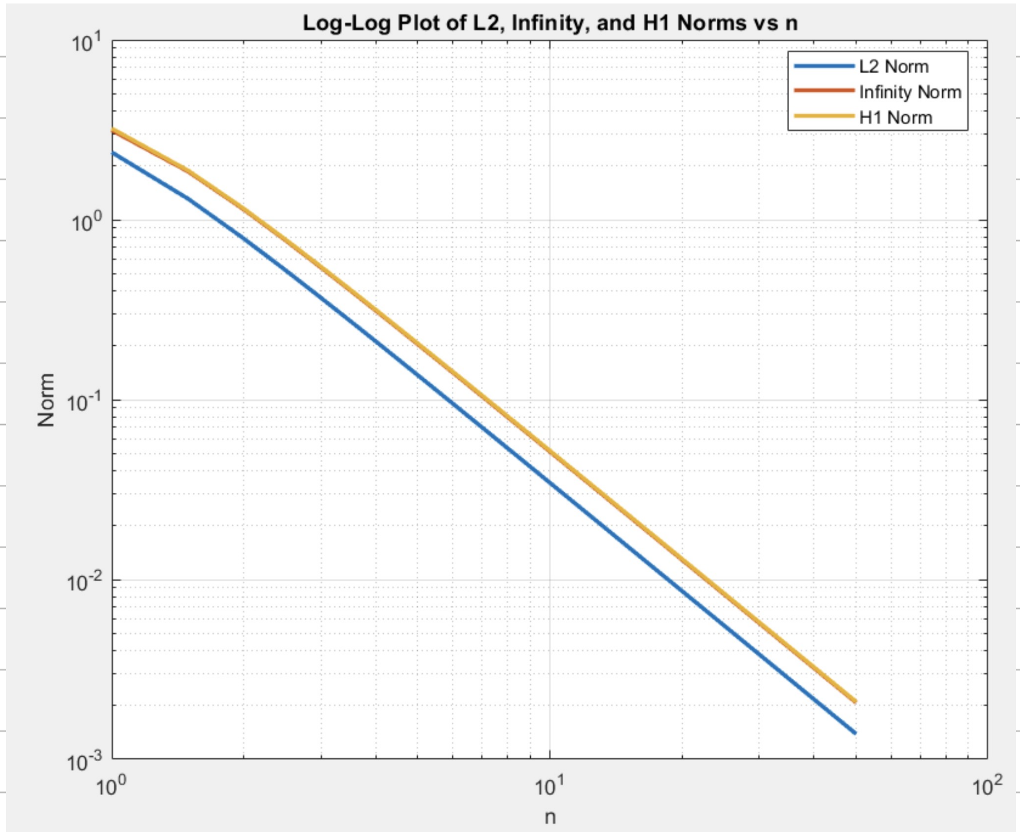
$$= \left[\int_0^\pi \left(n^2\sin^2(\tfrac{x}{n}) - 2n\sin(\tfrac{x}{n})x + x^2\right)dx\right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi n^2}{2} - 2n^3\sin(\tfrac{\pi}{n}) + \frac{\pi^3}{3} + 2n^2\pi\cos(\pi/n) - \frac{n^3\cos(\frac{\pi}{n})\sin(\frac{\pi}{n})}{2}\right]^{\frac{1}{2}}$$

$$\| f_n(x) - f_\infty(x) \|_{0,\infty} = \max_{x \in I} \left| n \sin\left(\frac{x}{n}\right) - x \right|$$

$$\| f_n(x) - f_\infty(x) \|_{1,2} = \left(\| f \|_{0,2}^2 + \| f' \|_{0,2}^2 \right)^{\frac{1}{2}}$$

Using Matlab Symbolic calculation, we have:



The figure shows that $f_n(x) \to f_\infty$ in $L^2(I)$

$f_n(x) \to f_\infty$ in $H^1(I)$.

$f_n(x) \to f_\infty$ in $L^\infty(I)$.

This also reflects the observations in part 1, as larger n leads to a better approximation for $f(x)$ and $f'(x)$.

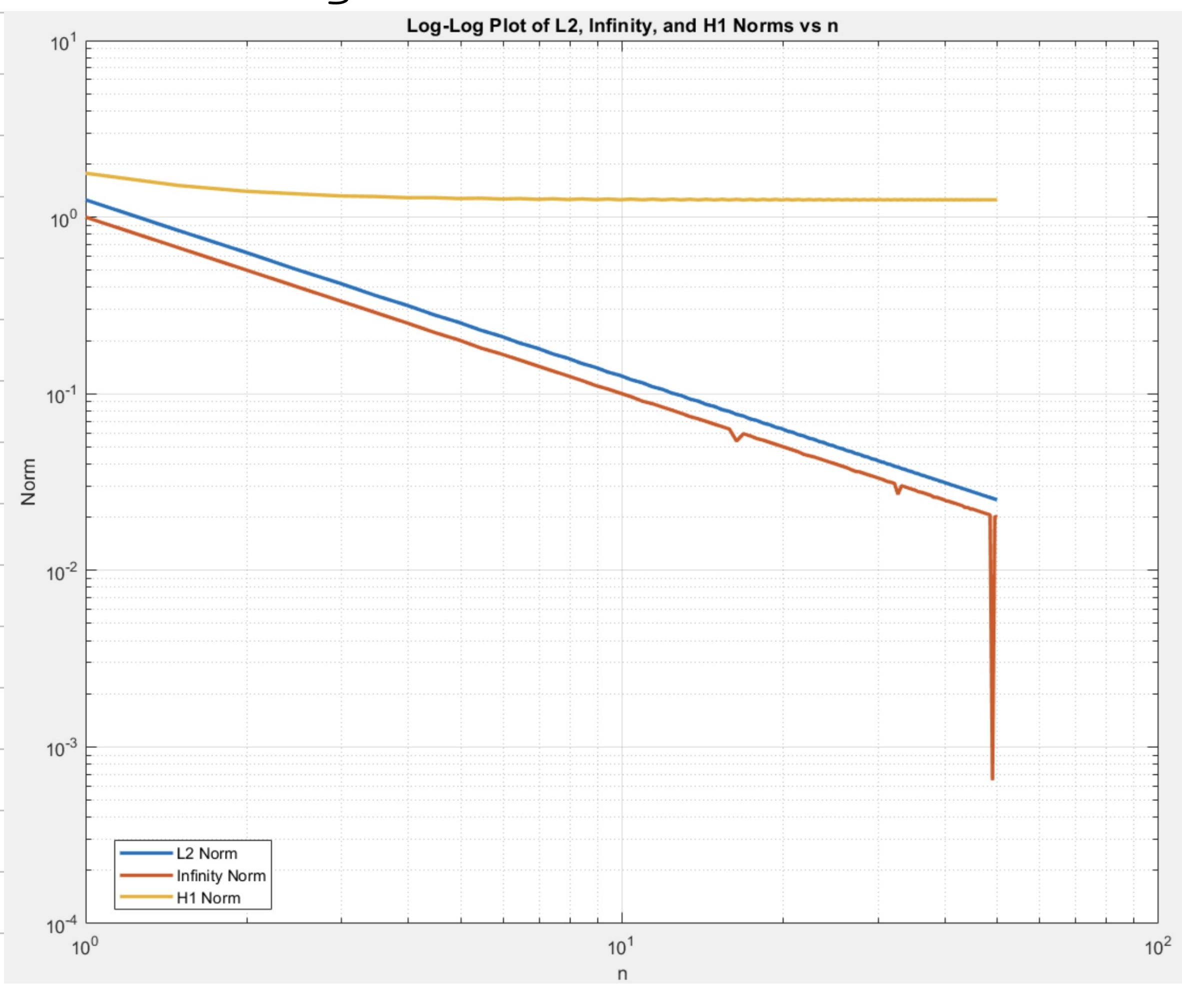
② $g_n = \frac{1}{n}\sin(nx)$, $g_\infty(x) = 0$.

$$\| g_n(x) - g_\infty(x) \|_{0,2} = \left[\int_0^{\pi} \left(\frac{1}{n}\sin(nx) - 0 \right)^2 dx \right]^{\frac{1}{2}}$$

$$\| g_n(x) - g_\infty(x) \|_{0,\infty} = \max_{x \in I} \left| \frac{1}{n}\sin(nx) - 0 \right|$$

$$\| g_n(x) - g_\infty(x) \|_{1,\infty} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2 \right)^{\frac{1}{2}}$$

Following the integration procedure above, we can plot norms vs n. ~~of~~ g_n as follows:



We have observed that

$$f_n(x) \to f_\infty \text{ in } L^2(I)$$

$$f_n(x) \to f_\infty \text{ in } L^\infty(I).$$

But, $f_n(x)$ does not converge in $H^1(I)$.

This is because the analytical expression of $H^1(I)$ is as follows:

$$\left[\frac{\pi}{2} + \frac{\left[\frac{\pi n}{2} - \frac{\sin(2n\pi)}{4} + \frac{n^2 \sin(2n\pi)}{4} \right]}{n^3} \right]^{\frac{1}{2}}$$

$$\to \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \approx 1.2533.$$

No matter how large the n is, f_n never converge in H_1 Norm space. This phenomenon can be explained from the plot of $f'(x)$. As the $f'(x)$ is sinodial by nature and $f'_\infty(x)$ is 1, $f'_n(x)$ can not approximate $f'_\infty(x)$.

③ $h_n(x) = \frac{1}{1+nx}$, $h_\infty(x) = 0$.

$$\| h_n(x) - h_\infty(x) \|_{0,2} = \left[\int_0^{\pi} \left(\frac{1}{1+nx} \right)^2 dx \right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi}{n\pi+1} \right]^{\frac{1}{2}}$$

$$\| h_n(x) - h_\infty(x) \|_{0,2} = \max_{x \in I} \left| \frac{1}{1+nx} \right|$$

$$= 1$$

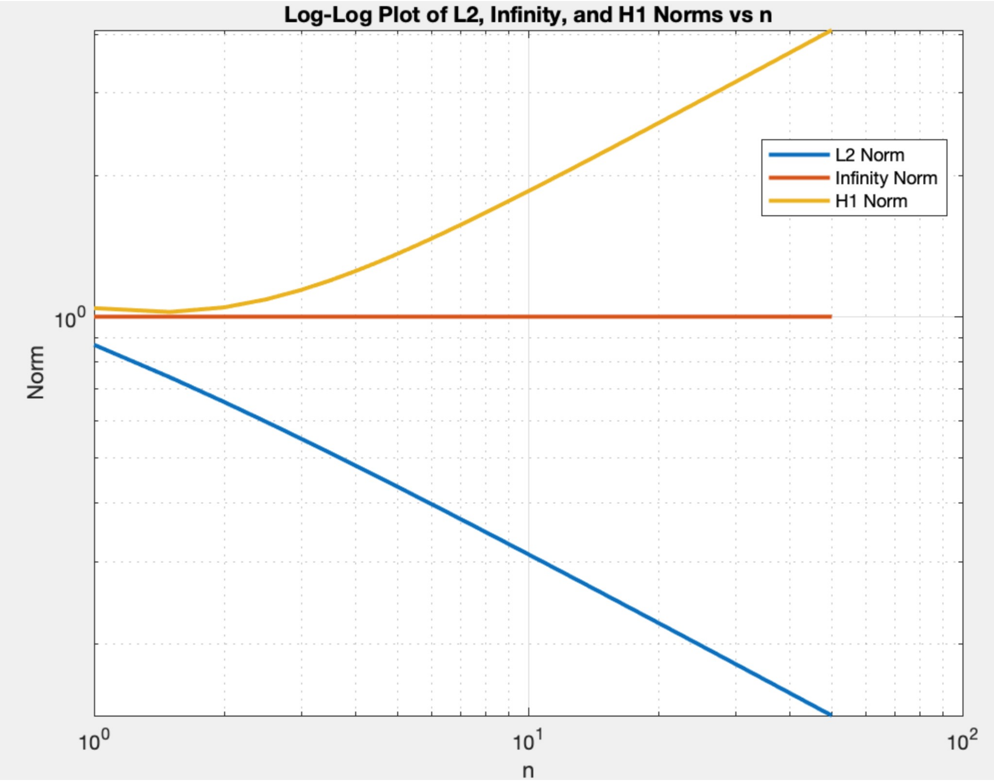
$$\| h_n(x) - h_\infty(x) \|_{1,2} = \left(\| f \|_{0,2}^2 + \| f' \|_{0,2}^2 \right)^{1/2}.$$

$$= \left[\left[- \frac{1}{n^2x+n} \right]_0^{\pi} + \int_0^{\pi} \left(- \frac{n}{(nx+1)^2} \right)^2 dx \right]^{\frac{1}{2}}$$

$$= - \frac{1}{n^2\pi+n} + \frac{1}{n} + \left[- \frac{n}{3(nx+1)^3} \right]_0^{\pi}$$

$$= \frac{\pi}{n\pi+1} - \frac{n}{3(n\pi+1)^3} + \frac{n}{3}$$

$\lim_{n \to \infty} \| h_n(x) - h_\infty(x) \|_{1,2} \to \infty$. $h_n(x)$ does not converge to $h_\infty(x)$.



Log-Log Plot of L2, Infinity, and H1 Norms vs n
L2 Norm
Infinity Norm
H1 Norm
Norm
10^0
10^0
10^1
10^2
n

We observe that $h_n \to h_\infty$ in L_2 Norm. But it doesn't converge with infinity norm and H_1 Norm. The $f_n(x)$ even diverges in the H_1 Norm.

On Interpolation Errors (70)

Consider the interval $\Omega = [-1, 1]$, and a mesh of $n_{\mathrm{el}} \in \mathbb{N}$ equally long P_k-elements on it, for $k = 1, 2, 4$. for P_k-elements is shown in Example 3.24 in the notes, while The Lagrange finite element interpolant $\mathcal{I}u$ is constructed through (3.24) in the notes.

For $\omega \in \mathbb{R}$, consider the functions

$$v_\omega(x) = \cos(\omega x)$$

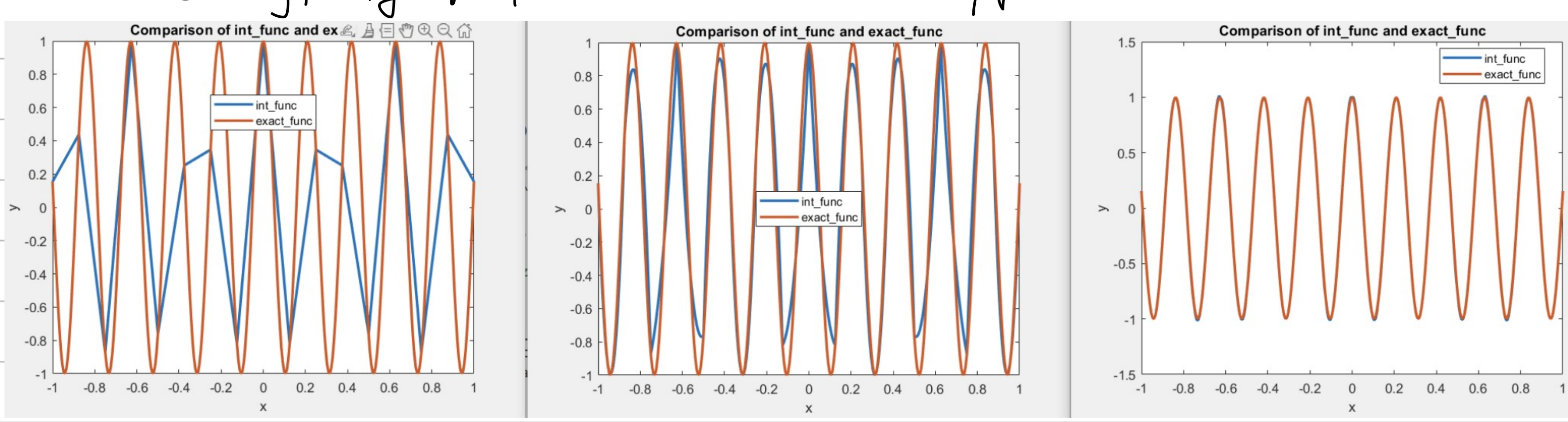
$$w_\omega(x) = \begin{cases} 0 & x < 0 \\ x^\omega & x \geq 0. \end{cases}$$

The Lagrange finite element interpolant is defined as:

$$\mathcal{I}u = \sum_{a=1}^{m} u(x_a) N_a^e .$$

$$\text{where } N_a^e = \frac{\prod_{b=1,\, b\neq a}^{k+1} (x - x_b^e)}{\prod_{b=1,\, b\neq a}^{k+1} (x_a^e - x_b^e)}.$$

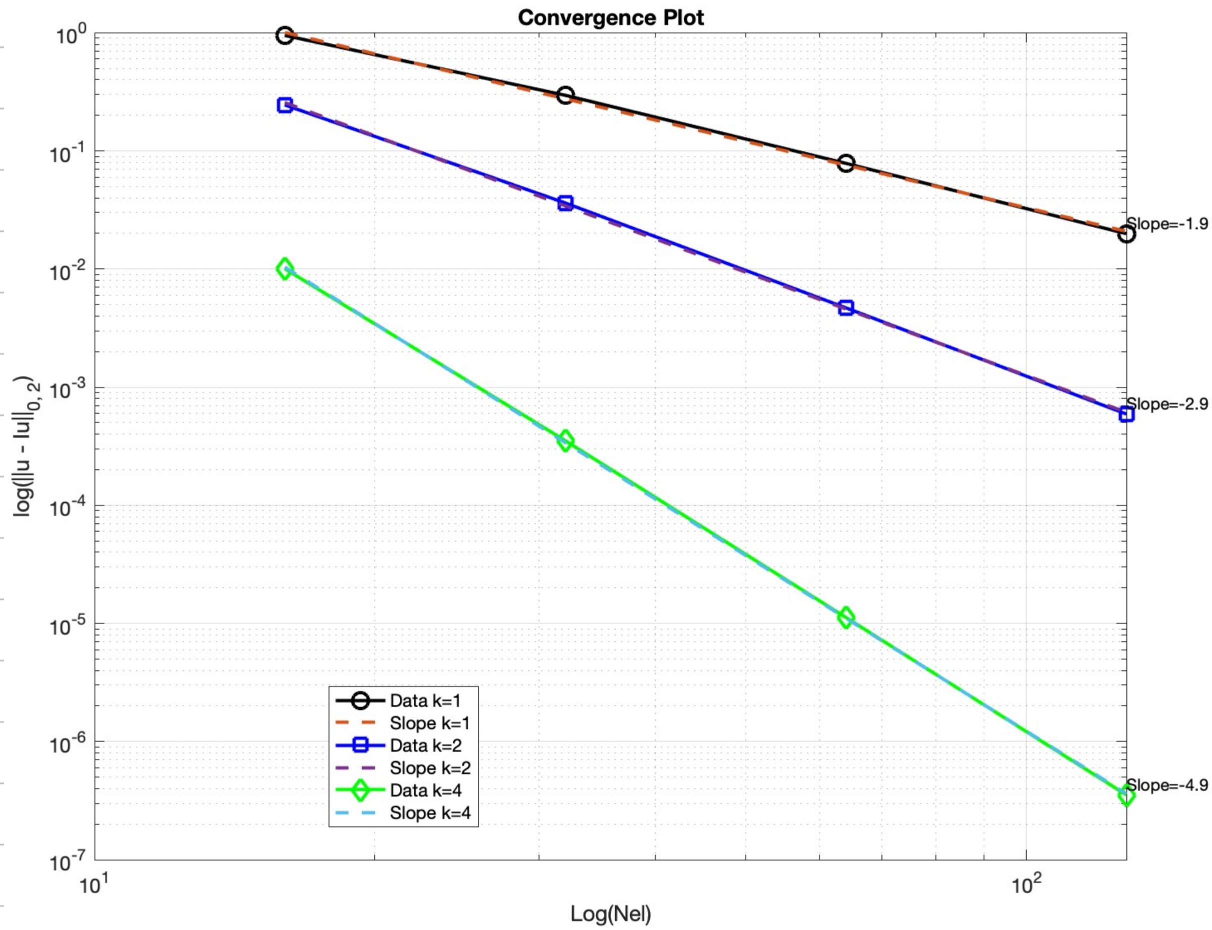
1. when $u = v_{30}$, using 2^3 points, we can approximate u using $\mathcal{I}u$ built above. Clearly, higher k leads to better approximation.



$k = 1$ $\qquad$ $k = 2$ $\qquad$ $k = 4$.

Then, we plot $\|u - \mathcal{I}u\|_{0,2}$ with difference orders of $\mathcal{I}u$:

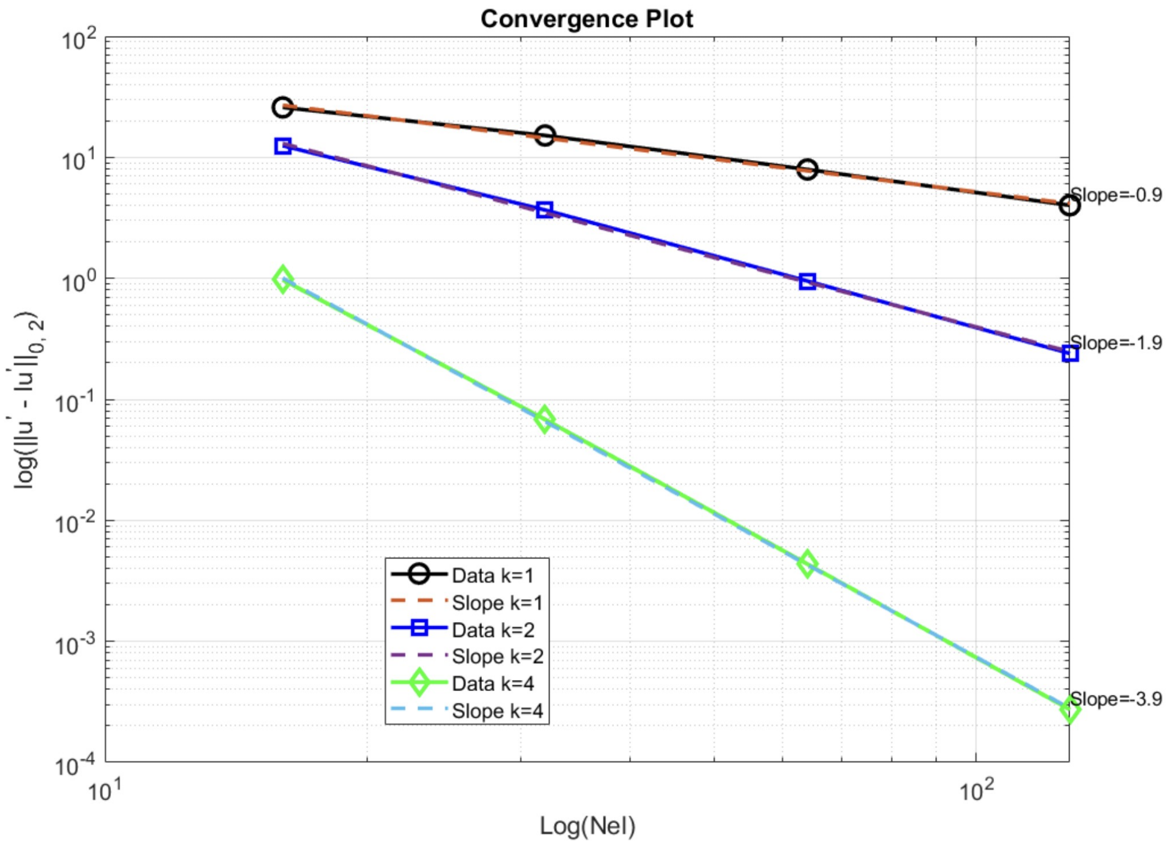
(a). $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $k = 1, 2, 4$.



The figure above shows how the order of interpolant affects the convergence behavior of approximation solutions.

Clearly, the convergence rates are around $k+1$, which corresponds to the theoretical ones.

The $\|u' - \mathcal{I}u'\|_{0,2}$ for $k = 1, 2, 4$ are also given.

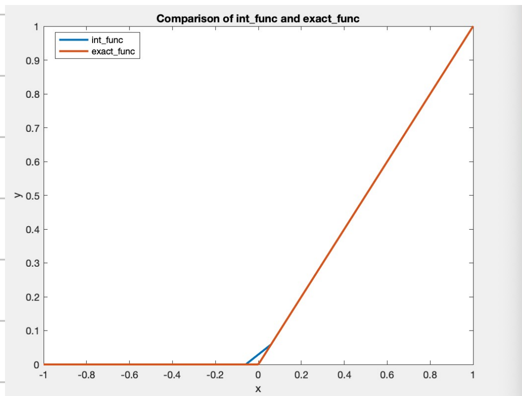


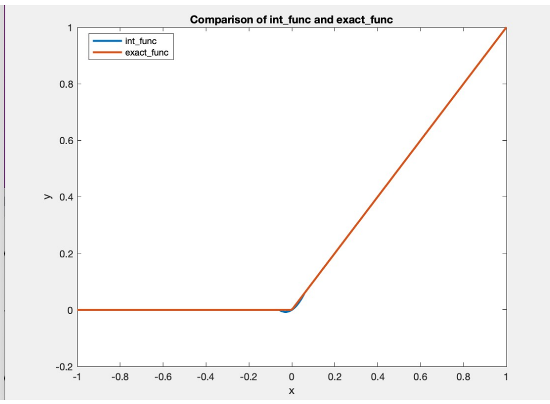
The figure above shows how the order of interpolant ~~affects~~ affects the convergence behavior of approximating $u'(x)$.

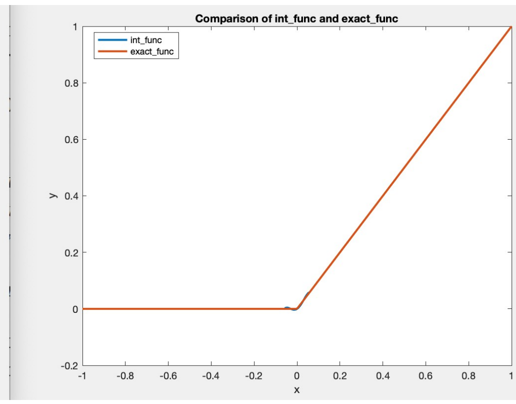
Clearly, the convergence rates are around k.
This is the case for the derivative of function.

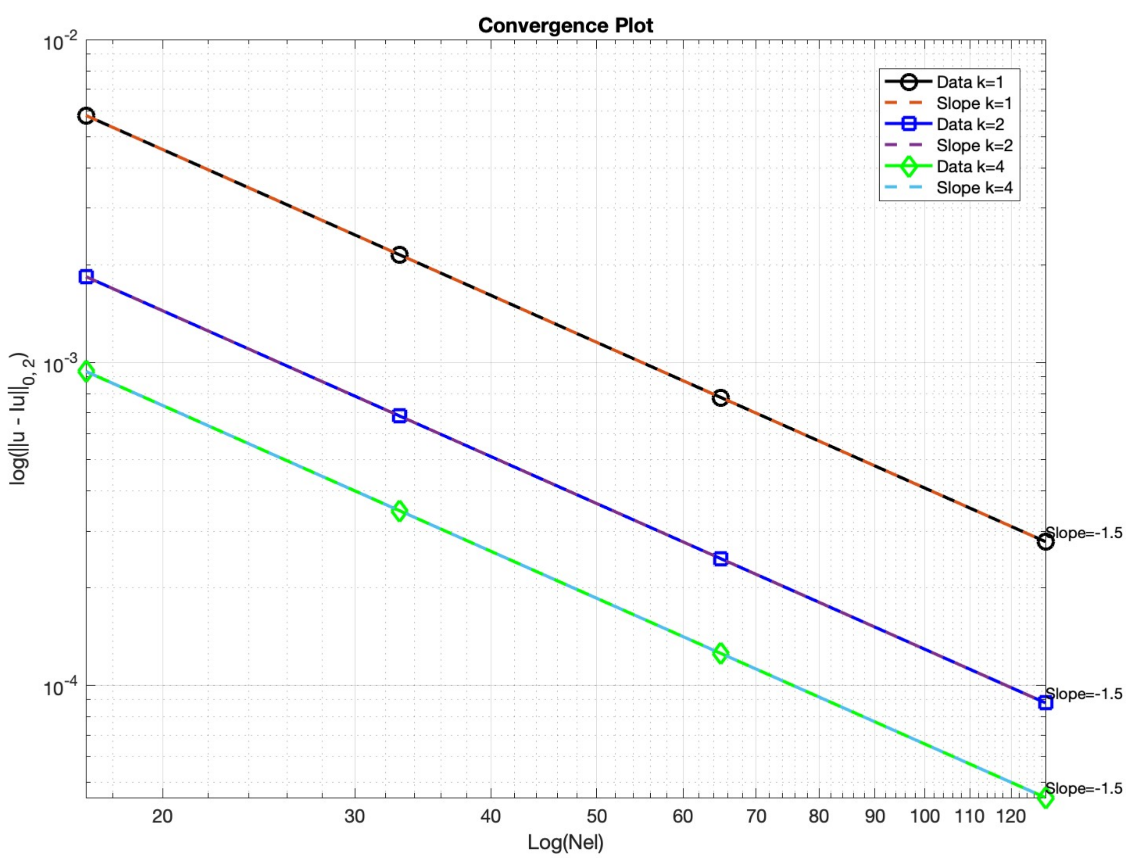
2. $u = w_1$, $\quad w_1 = \begin{cases} 0, & x < 0 \\ x^1, & x \geq 0 \end{cases}$

We use nel $= 2^{\varphi}+1$, three different orders of P_k to approximate w_1. Clearly, higher order can approximate the discontinuity of derivative at the origin.

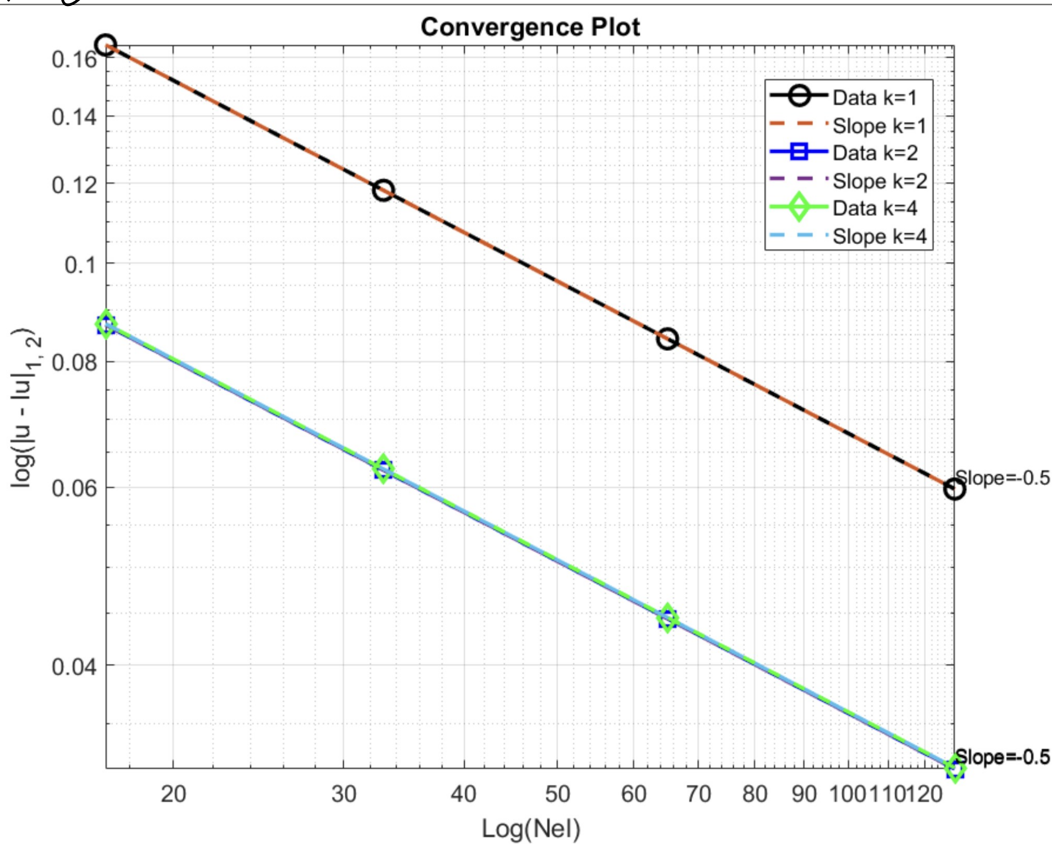






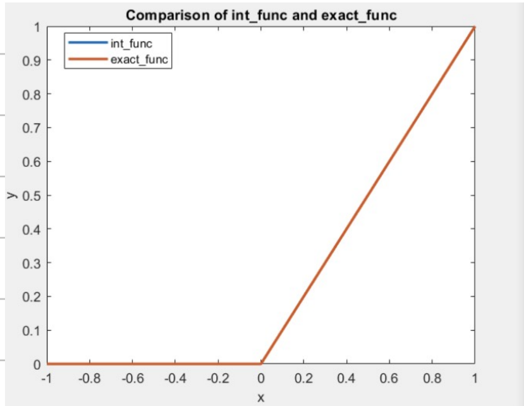


The convergence rates of three cases for u is around the same. This could be explained by the fact that Iu needs to approximate origin with the element when Nel is odd. However, lagrange interpolant element has continuous derivative within any element.

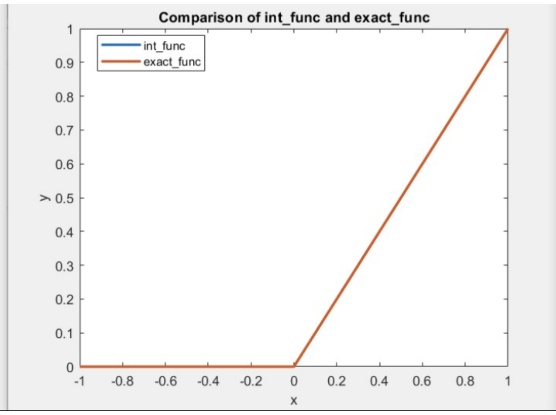


The convergence rates of three cases for u' remain the same. The reason is the same with the one explained above: there is a jump at origin for u', whereas Iu has continuous u' over the element where origin resides.

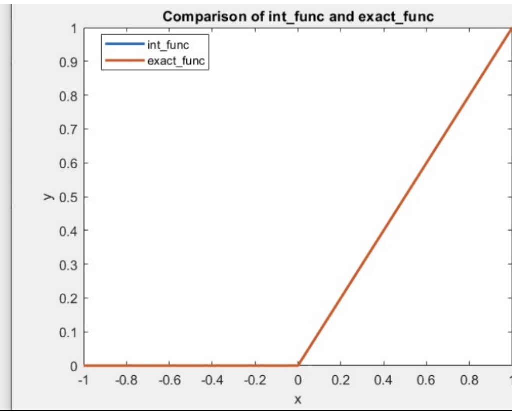
(c). When element number is even, we first look the Iu.



K=1



K=2

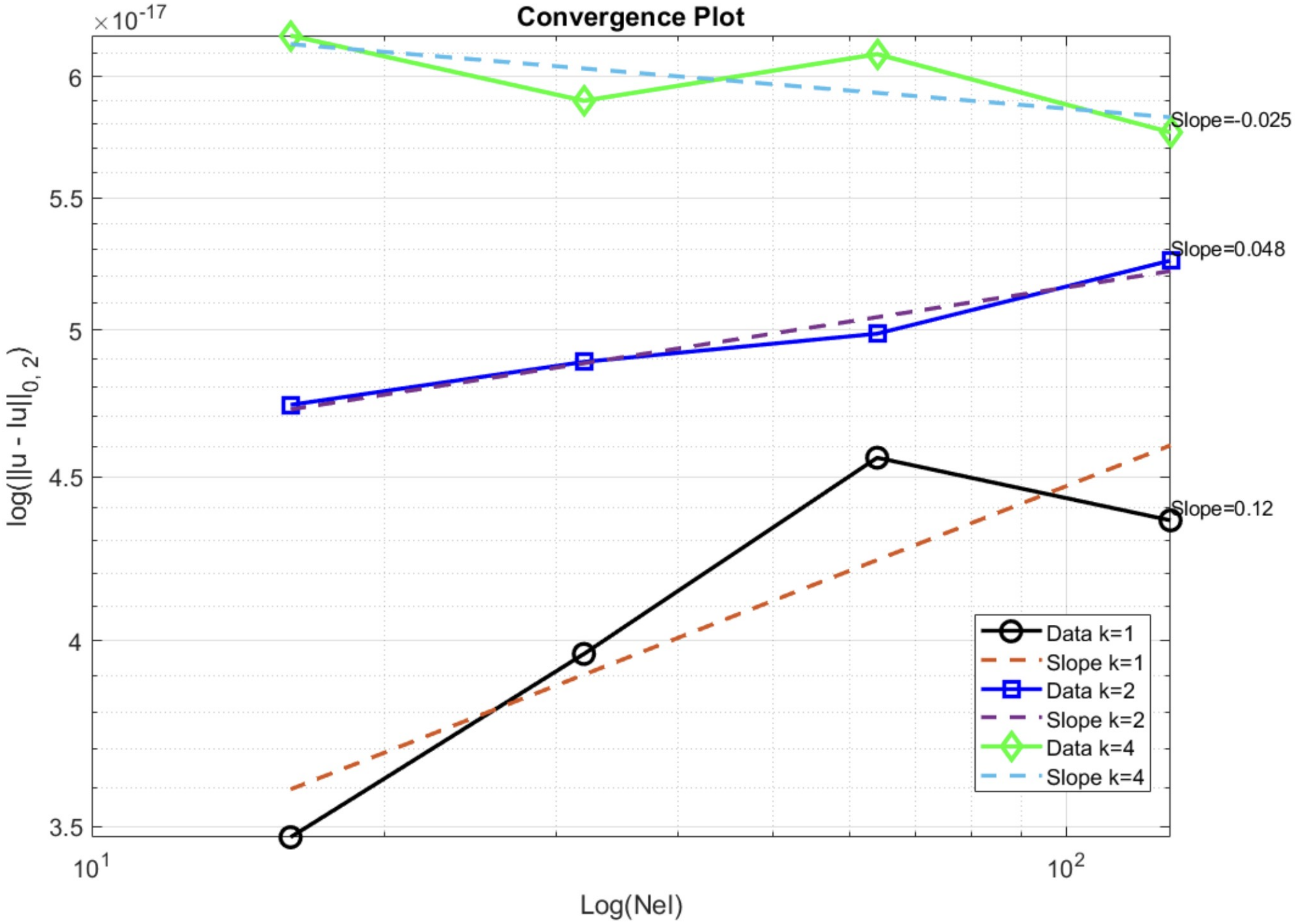


K=4

Clearly. Iu with even nel approximates u much better than In with odd nel. This is because when nel is odd, there is one element, indexing by $\frac{nel}{2}$, used for capturing the origin point within the element. This will lead to large, since inside an element, the derivative is continuous.

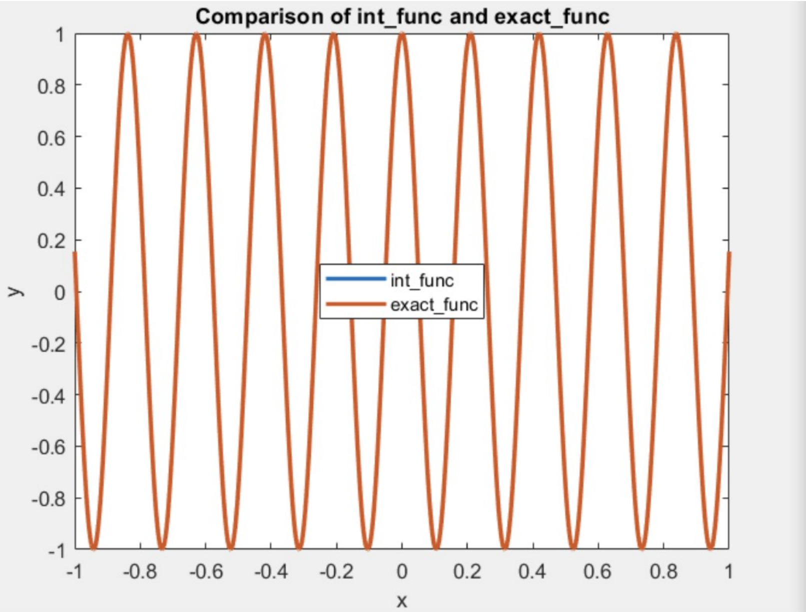
But, when Nel is Even, we can treat the discontinuous derivative point (origin) at the element boundary.

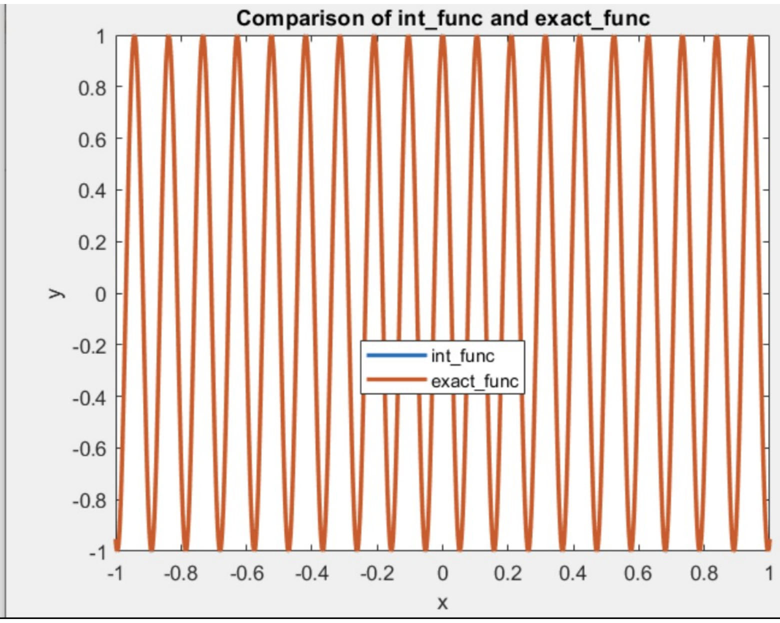
The following figure gives $\| u - Iu \|_{0,2}$ when nel is even. Strikingly, Iu can approximate u to a machine precision.



This case is different from previous two because even nel results treating origin at the boundary of element, while odd case will approximates such a point within the element.

3. $\|u - I_h u\|_{0,2}$ for $u = v_{30}$ and $u = v_{60}$ for $k=2$ and nel $= 200$.





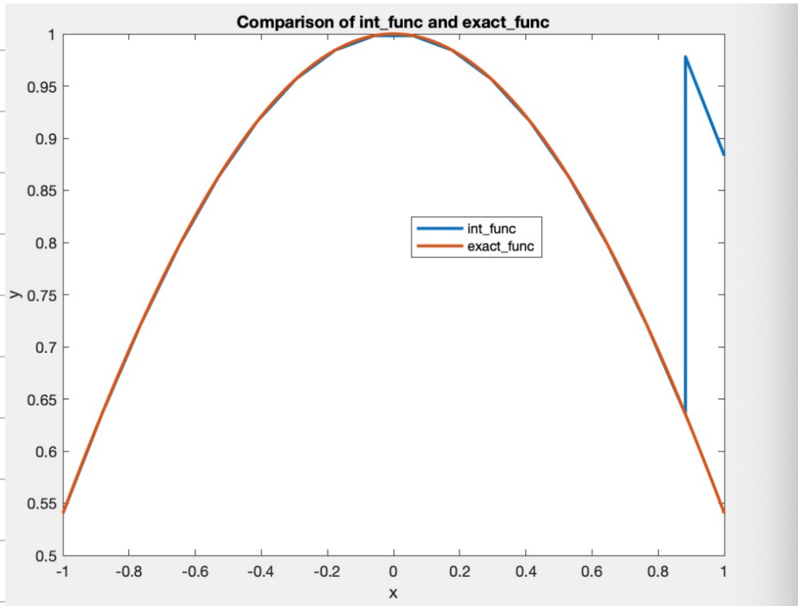
Using $k=2$ interpolant gives visually close close approximation of u.

The $\|u - I_h u\|_{0,2}$ is listed as follow:

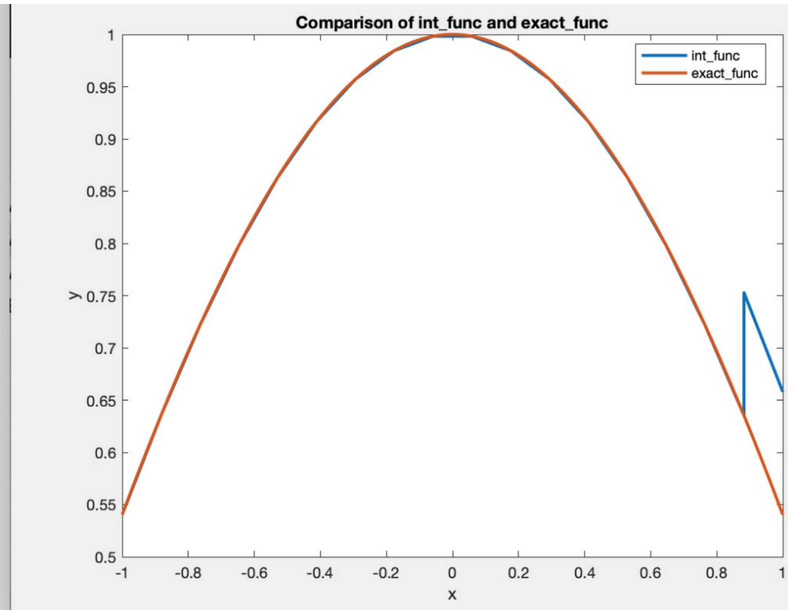
	v_{30}	v_{60}
$\|u - I_h u\|_{0,2}$	1.5595e-4	0.0012

Approximating v_{60} results in a larger L2 Norm. This is because with a fixed low order interpolant, the higher the frequency the function, the less accurate the interpolant can approximate.

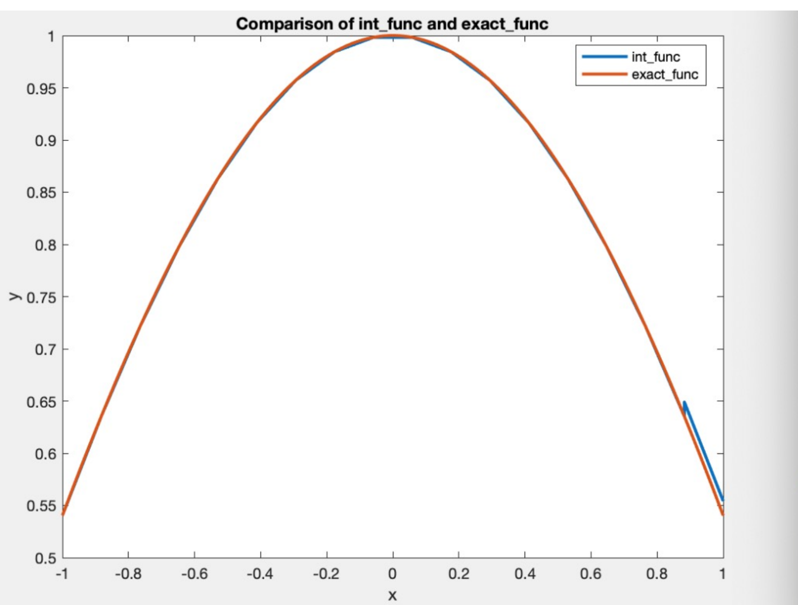
4. First, we plot I_u and $u = v_1$ with the inexact imposition of boundary condition as: $\cos(1) + h^m$ at $x = 1$.



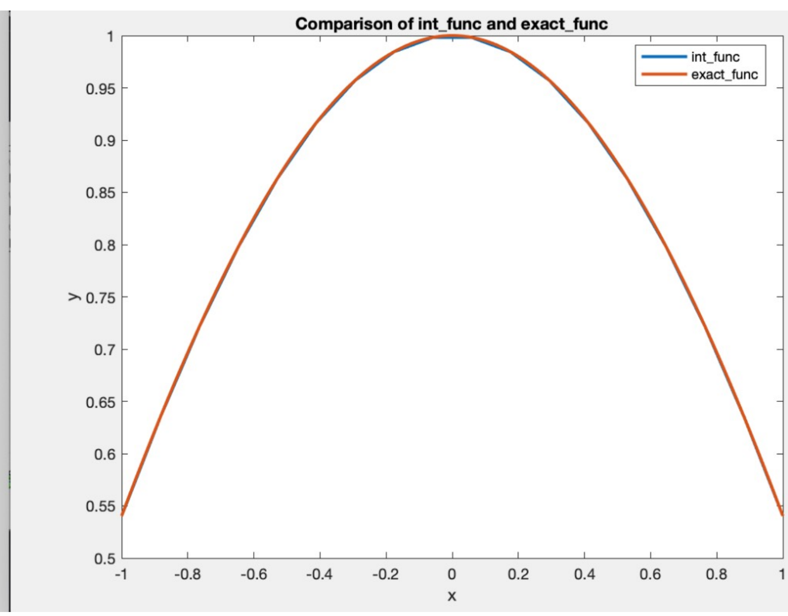
$m = 0.5$



$m = 1.0$



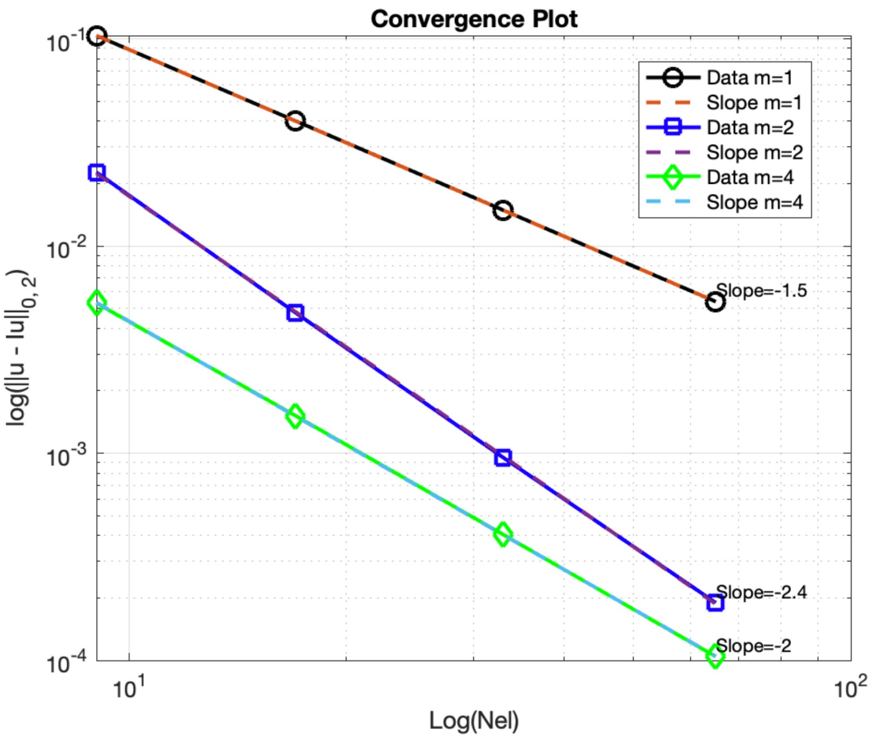
$m = 2.0$



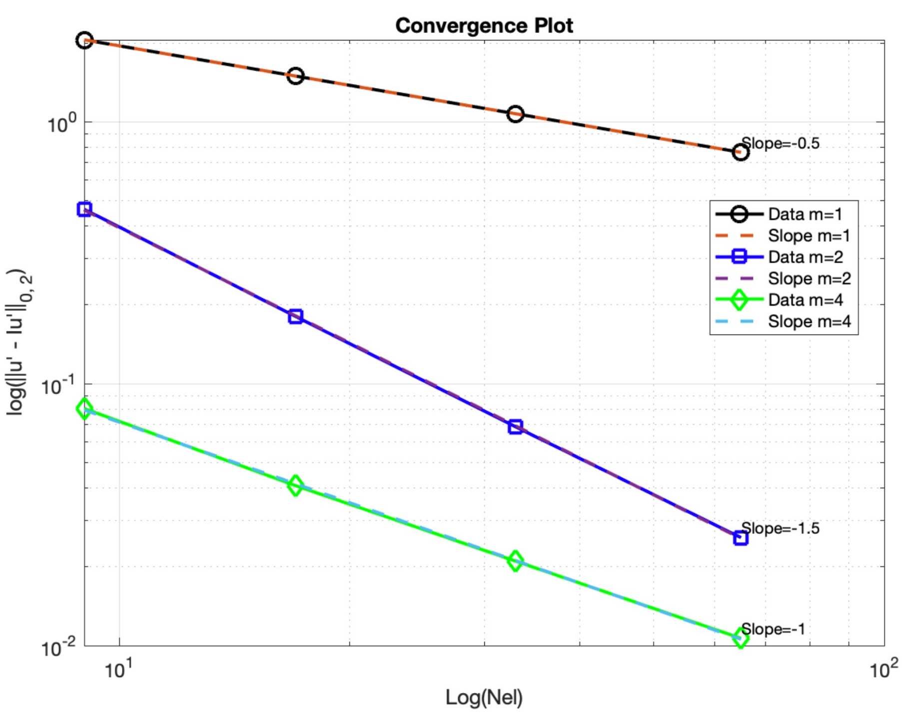
$m = 4.0$

Higher m will result in a faster decrease in the difference between I_u and u.

(a). $\|u - Iu\|_{0,2}$



(b). $\|u' - (Iu)'\|_{0,2}$



(c)

$$\|u-u_h\|_1 \le \left(1+\frac{M}{\alpha}\right) \min_{w_h \in S_h} \|u-w_h\|_1,$$

The Ceas' lemma tells us the prior error is bounded by the minmuma of interpolance error, $\|u-w_h\|_1$. But, the convergence rate of inexact boundary condition shows that when $m=1$, $\|u'-Iu'\|_0$ is smaller than 1, which means the interpolant error will dominate the error. However, when we increase the order of m, $\|u'-Iu'\|$ become around 1. This means the interpolance error will not affect the error too much. The numerical solution will still converge to the theoritical ones, h^k for H_1 Norm h^{k+1} for L_2 Norm.