ME 335 A

1. (15) Given $f: (0,1) \to \mathbb{R}$ continuous and a constant $\lambda > 0$, consider the differential equation

$$-u_{,xx} + \lambda u_{,x} = f \qquad x \in (0,1). \tag{3}$$

with boundary conditions $u(0) = 0.1$ and $u'(1) = 1$.

Let the test space be

$$\mathcal{V} = \{v \colon [0,1] \to \mathbb{R} \text{ smooth } \mid v(0) = 0\}$$

Obtain a variational equation for the problem following the steps in §1.1.2.3. Identify essential and natural boundary conditions.

1. Form the residual:
for a function u, we define a function $r : [0, 1.] \to \mathbb{R}$ as:

$$r = -u_{,xx} + \lambda u_{,x} - f \qquad \cdots\ (1.1)$$

The solution of u gives the following:

$$r(x) = 0, \quad x \in (0,1).$$

2. Multiply by a test function and integrate:
Given: $V = \{ v : [0, 1] \to \mathbb{R} \text{ smooth} \mid v(0) = 0\}$.
The weak form is given as:

$$\int_0^1 r(x)\, v(x)\, dx = 0 \qquad \cdots\ (1.2)$$

$$\int_0^1 (-u_{,xx} + \lambda u_{,x} - f)\, v\, dx = 0. \qquad \text{for all } v \in V.$$

3. Integrate by parts:

$$\int_0^1 (u_{,x} v_{,x})\, dx - \int_0^1 (u_{,x} v)'\, dx + \int_0^1 (\lambda u_{,x} v - f v)\, dx = 0.$$

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v - f v)\, dx - [u_{,x}(1)\, v(1) - u_{,x}(0)\, v(0)] = 0$$

$$\cdots\ (1.3)$$

4. Use boundary condition:

$$u(0) = 0.1 \; ; \; u'(1) = 1$$

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v - f v) dx - [1 \cdot v(1) - 0] = 0.$$

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v - f v) dx - v(1) = 0. \quad \cdots (1.4)$$

5. State the variational boundary:

The solution of u satisfies:

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v) dx = \int_0^1 f v \, dx + v(1)$$

for all $v \in \mathcal{V}$, where

$$\mathcal{V} = \{ v : [0,1] \to \mathbb{R} \text{ smooth} \mid v(0) = 0 \}. \quad \cdots (1.5)$$

Essential conditions: $u(0) = 0.1$

Natural conditions: $u'(1) = 1.$

2. (10) Transform the last variational equation for this problem so that it takes advantage of Nitsche's method; see Example 1.14 in the notes.

To construct Nitsche's Method, we use a new test space:

$$V = \{ v : [0,1] \to \mathbb{R} \text{ smooth} \}.$$

for all $v \in V$, we have:

$$(g_0 - u(0)) v'(0) = 0 \quad \cdots (2.1)$$

$$\mu (u(0) - g_0) v(0) = 0$$

where $g_0 = 0.1$; $\mu > 0$.

Add (2.1) to (1.3), we have the following form:

$$\int_0^1 (u_{,x} v_x + \lambda u_{,x} v - f v)\,dx - [u_{,x}(1)\,v(1) - u_{,x}(0)\,v(0)]$$
$$+ (g_0 - u(0))\,v'(0) + \mu(u(0) - g_0)\,v(0) = 0.$$

$$\Rightarrow \int_0^1 (u_{,x} v_x + \lambda u_{,x} v)\,dx + u_{,x}(0)\,v(0) - u(0)\,v'(0) + \mu u(0)\,v(0) =$$
$$\int_0^1 (f v)\,dx + v(1) - 0.1\,v'(0) + 0.1\,\mu\,v(0)$$

for all $v \in \mathcal{V} = \{v : [0, L] \to \mathbb{R} \text{ smooth}\}$.

3. Assume that $f(x) = x$.

(a) (5) Find the general solution of (3)?

(b) (5) Consider a test function v so that $v(0) = v'(0) = 0$ and $v(1) = 0$. What terms of the variational equation you found in part 2 are guaranteed to be zero for such a choice, regardless of u?

(c) (5) Select a test function v so that $v(0) = v'(0) = 0$ and $v(1) = 0$. Using the general solution, test the variational equation you found in part 2 with the test function you selected. Is the variational equation satisfied by the exact solution for such v?

(d) (5) Using the general solution, test the variational equation you found in part 2 by selecting a test function v so that $v(0) = v'(0) = 0$, but such that $v(1) \neq 0$. What can you conclude about $u'(1)$?

(e) (5) The last part should have allowed you to narrow the set of possible general solutions that satisfy the variational equation. Using this smaller set of general solutions, test the variational equation by selecting another test function for which $v(0) \neq v'(0)$. What can you conclude about $u(0)$?

(f) (1) Based on the work you've done so far, what is the exact solution of the problem then?

To test, choose the simplest function that you can imagine and that is not identically zero, so that the integrals are simpler. Play with linear, quadratic and cubic polynomials to build them.

You are advised to use one of the programs we mentioned at the beginning to perform all integrals in this problem and to find the solution. This problem shares some traits with Examples 1.9 and 1.10 in the notes.

(a). We have a 2nd order ODE.

1. Homogenone eqn: $-u'' + \lambda u' = 0$.

Let solution in the form $u = e^{(\gamma x)}$

So;
$$-\frac{d^2(e^{\gamma x})}{dx^2} + \lambda \frac{d(e^{\gamma x})}{dx} = 0$$

$$-\gamma^2 e^{\gamma x} + \lambda \gamma e^{\gamma x} = 0$$

$$\gamma(\gamma - \lambda) = 0.$$

$$\gamma_1 = 0 \quad ; \quad \gamma_2 = \lambda$$

$$u_c(x) = c_1 + c_2 e^{\lambda x}$$

2. Particular solution:

As x is a linear function, we can make the following guess:

$$u_p(x) = c_3 x^2 + c_4 x + c_5.$$

Plug it into ODE:

$$-2c_3 + \lambda(2c_3 x + c_4) = x$$

$$\begin{cases} 2\lambda c_3 = 1 \\ -2c_3 + \lambda c_4 = 0. \end{cases} \Rightarrow \begin{cases} c_3 = \frac{1}{2\lambda} \\ c_4 = \frac{1}{\lambda^2} \end{cases}$$

$$u_p(x) = \frac{x^2}{2\lambda} + \frac{x}{\lambda}$$

general solution: $u_p(x) + u_c(x) = c_1 + c_2 e^{\lambda x} + \frac{x^2}{2\lambda} + \frac{x}{\lambda^2}$

$$u(x) = c_1 + c_2 e^{\lambda x} + \frac{x^2}{2\lambda} + \frac{x^2}{\lambda^2}$$

(b). $v(0) = v'(0) = 0$ and $v(1) = 0$. ;

$$\Rightarrow \int_0^1 (u_{,x} v_x + \lambda u_{,x} v)\,dx + u_{,x}(0)v(0) - u(0)v'(0) + \mu u(0)v(0) =$$
$$\int_0^1 (fv)\,dx + v(1) - 0.1 v'(0) + 0.1\mu \cdot v(0)$$

$u_{,x}(0)v(0) = 0$; $-u(0)v'(0) = 0$; $\mu u(0)v(0) = 0$
$v(1) = 0$; $-0.1 v'(0) = 0$; $0.1\mu v(0) = 0$.

So, we have :

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v)\,dx = \int_0^1 fv\,dx. \quad \cdots (3.b).$$

(c). Let. $v(x)$ be a general cubic polynomial:

$$v(x) = ax^3 + bx^2 + cx + d.$$

Apply constraints., we have:

$$v'(0) \Rightarrow 3a \cdot 0 + 2b \cdot 0 + c = 0$$
$$v(0) \Rightarrow d = 0.$$
$$v(1) \Rightarrow a + b + c + d = 0. \Rightarrow a+b=0.$$

So, we have a undertermined system, which means there are infinitely many functions, satisfying our constraints. We can choose:

$$v(x) = x^3 - x^2$$

The general solution of $u(x)$ is given as:

$$u(x) = C_1 + C_2 e^{\lambda x} + \frac{x}{\lambda^2} + \frac{x^2}{2\lambda}$$

$$v(x) = x^3 - x^2$$

Plug $u(x)$ and $v(x)$ into

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v) dx = \int_0^1 x v \, dx$$

Using symbolic calculation in matlab, we can calculate:

$$LHS = \int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v) dx = -\frac{1}{20}$$

$$RHS = \int_0^1 x v \, dx = -\frac{1}{20}$$

So the variational equation is satisfied by the exact solution.

(d).

Now the test function satisfies $v(0) = v'(0) = 0$.

We can choose $v(x) = x^2$ that has $v(0) = v'(0) = 0$ and $v(1) = 1$

Plug it into the variational form:

$$\int_0^1 (u_{,x} v_{,x} + \lambda u_{,x} v) dx = \int_0^1 x v \, dx + u'(1) v(1)$$

Still, $u(x) = C_1 + C_2 e^{\lambda x} + \frac{x}{\lambda^2} + \frac{x^2}{2\lambda}$

$$LHS = \frac{(\lambda(2C_2 - 2C_2 e^{\lambda} + 2/3) + 1)}{\lambda^2} + 2C_2 e^{\lambda} + \dots$$

$$\frac{1}{3\lambda} - C_2 \lambda^2 \left(\frac{2}{\lambda^3} - \left(\frac{e^{\lambda} * (\lambda^2 - 2\lambda + 2)}{\lambda^3}\right)\right) + \frac{1}{4}.$$

$$RHS = 5/4$$

Equating LHS and RHS, and solving c_2 in terms of λ, we have:

$$C_2 = -(e^{(-\lambda)} - \lambda^2 e^{(-\lambda)} + \lambda e^{(-\lambda)})/\lambda^3.$$

We have found C_2

$$u'(1) = \frac{1}{\lambda} + \frac{1}{\lambda^2} - \left[\frac{e^{\lambda}(e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda})}{\lambda^2}\right]$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda^2} - \left(\frac{1}{\lambda^2} - 1 + \frac{1}{\lambda}\right)$$

$$= 1$$

So, we have found $C_2 = -\dfrac{(e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda})}{\lambda^3}$

$$u'(1) = 1$$

(e). We can choose $v(x) = x$ such that $v(0) \neq v'(0)$

The general solution of $u(x)$ has the form:

$$u(x) = c_1 + c_2 e^{\lambda x} + \frac{x}{\lambda^2} + \frac{x^2}{2\lambda}$$

where $c_2 = -\dfrac{(e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda})}{\lambda^3}$

$$\int_0^1 (u_{,x} v_x + \lambda u_{,x} v - x v)\,dx - [u_{,x}(1)\, v(1) - u_{,x}(0)\, v(0)] + (g_0 - u(0))\, v'(0) + \mu(u(0) - g_0)\, v(0) = 0.$$

$$\int_0^1 (u_{,x} v_x + \lambda u_{,x} v)\,dx - v(1) + (g_0 - u(0)) + \cancel{\mu(u(0) - g_0)\cdot 0} = 0$$

$$\int_0^1 (u_{,x} v_x + \lambda u_{,x} v)\,dx = \int_0^1 x v\,dx + v(1) - g_0 + u(0)$$

Use $g_0 = 0.1$ and $u(0) = c_1 + c_2 e^0$

$$\int_0^1 (u_{,x} v_x + \lambda u_{,x} v)\,dx = \int_0^1 x v\,dx + 1 - 0.1 + c_1 + c_2$$

where $c_2 = -\dfrac{(e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda})}{\lambda^3}$

Using symbolic calculation, we can back solve c_1 as

$$c_1 = \frac{(10e^{-\lambda} - 10\lambda^2 e^{-\lambda} + \lambda^3 + 10\lambda e^{-\lambda})}{10\lambda^3}$$

Therefore, $u(0) = c_1 + c_2 e^0$

$$= \cancel{\frac{(10e^{-\lambda} - 10\lambda^2 e^{-\lambda} + 10\lambda e^{-\lambda})}{10\lambda^3}} + \frac{1}{10} - \cancel{\frac{(e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda})}{\lambda^3}}$$

$$= \frac{1}{10}.$$

We have found that $u(0) = \frac{1}{10}$.

(f). Using the coefficients c_1, c_2 found above, we have the exact solution as:

$$u(x) = \left(\frac{e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda}}{\lambda^3}\right) + \frac{1}{10} - \left(\frac{e^{-\lambda} - \lambda^2 e^{-\lambda} + \lambda e^{-\lambda}}{\lambda^3}\right) e^{-\lambda x}$$

$$+ \frac{x}{\lambda^2} + \frac{x^2}{2\lambda} \quad , \lambda > 0$$