HW5

A Variational Method with an almost Spectral Basis (53)

Let $\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < R^2\}$ for $R = 1$, $\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the problem: Find $u \in \Omega \to \mathbb{R}$ such that

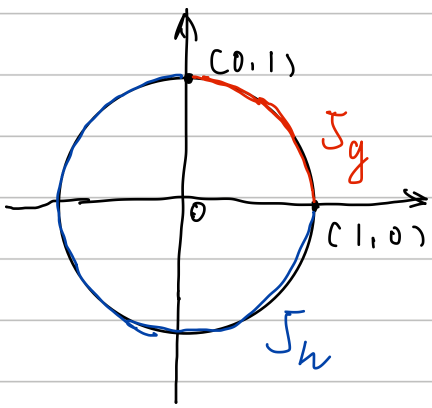
$$-\frac{1}{2}\Delta u = \frac{2}{R^2} \qquad \text{in } \Omega \qquad (1a)$$

$$u = 0 \qquad \text{on } \Gamma_g \qquad (1b)$$

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{1}{R} \qquad \text{on } \Gamma_h. \qquad (1c)$$

1. (10) Construct a variational equation that u satisfies, following the standard recipe.

1. The domain has nonoverlapping boundaries, Γ_g and Γ_h as follow:



$\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < R^2\}$ for $R = 1$

$\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$

$\Gamma_h = \partial\Omega \setminus \Gamma_g$

Define $W = \{w : \Omega \to \mathbb{R} \text{ smooth}\}$.

Test space $V = \{v \in W \mid v(\vec{x}) = 0 \text{ for all } x \in \Gamma_g\}$.

Trial space $U = \{u \in W \mid u(\vec{x}) = 0 \text{ for all } x \in \Gamma_h\}$.

$U = V$

Multiple (1a) with v and integrate over Ω for all $v \in V$

$$\int_\Omega -\frac{1}{2}\Delta u \, v = \int_\Omega \frac{2}{R^2} v \qquad \text{for all } v \in V$$

Integrate by parts:

$$\frac{1}{2}\int_{\Omega} \nabla u\cdot\nabla v - \frac{1}{2}\int_{\partial\Omega} \nabla u\cdot v\vec{n} = \int_{\Omega} \frac{2v}{R^2}$$

$$\frac{1}{2}\int_{\Omega} \nabla u\cdot\nabla v = \frac{1}{2}\int_{\Gamma_g} \underbrace{\cancel{\nabla u\cdot v\vec{n}}}_{0} + \frac{1}{2}\int_{\Gamma_h} \nabla u\cdot v\vec{n} + \int_{\Omega}\frac{2v}{R^2}$$

$$\frac{1}{2}\int_{\Omega} \nabla u\cdot\nabla v = -\int_{\Gamma_h}\frac{1}{R}v + \int_{\Omega}\frac{2v}{R^2}$$

The weak form reads:

Find $u \in S$ such $a(u,v) = l(v)$ for all $v \in V$.

$$a(u,v) = \frac{1}{2}\int_{\Omega} \nabla u\cdot\nabla v$$

$$l(v) = \int_{\Omega}\frac{2v}{R^2} - \int_{\Gamma_h}\frac{v}{R}$$

2. Essential boundary: $u=0$. over Γ_g

Natural boundary: $\frac{1}{2}\nabla u\cdot\vec{n} = -\frac{1}{R}$ over Γ_h

3.

3. Consider the approximation space

$$\mathcal{W}_h = \text{span}\left(\sin\left(\pi\left(x_1^2+x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right), 1\right).$$

(a) (10) Identify test and trial spaces, and active and constrained indices, naming the basis functions with indices in the order they appear above.

(b) (5) In this problem, there is a possibility of selecting a smaller space $\mathcal{W}_h$ without changing the results. What is this smaller space $\mathcal{W}_h$? Identify active and constrained indices in this new space.

(c) (15) Using the smaller space $\mathcal{W}_h$, compute the stiffness matrix and load vector.

(d) (5) Find the numerical approximation. Plot it together with the the exact solution.

(e) (5) Do you think the numerical approximation would change if we change the boundary condition on the Neumann boundary to

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{2}{R}?$$

3(a).

Given $W_h = \text{span}\left(\sin(\pi(x_1^2+x_2^2)), \cos(\frac{\pi}{2}(x_1^2+x_2^2)), 1\right)$

Let $N_1 = \sin(\pi(x_1^2+x_2^2))$, $N_2 = \cos(\frac{\pi}{2}(x_1^2+x_2^2))$, $N_3 = 1$ as global basis functions.

Define test space V_h:

$$V_h = \{ v_1 N_1 + v_2 N_2 \mid (v_1, v_2) \in \mathbb{R} \}.$$

Trial space S_h:

$$S_h = \{ V_h + v_3 g \mid v_3 \in \mathbb{R} \}.$$

Using the notation defined above, active indices are:

$$\eta_a = \{ v_1, v_2 \}$$

constrained indices are:

$$\eta_g = \{ v_3 \}.$$

3(b) One possibility is only using trigonometric functions. This space will work for the problem, because the dirichlet boundary is zero on Γ_g, automatically met by the following space:

$$W_h = \mathrm{Span}\left(\sin(\pi(x_1^2+x_2^2)),\ \cos(\tfrac{\pi}{2}(x_1^2+x_2^2))\right)$$

$$v_1 \sin(\pi(x_1^2+x_2^2)) + v_2 \cos(\tfrac{\pi}{2}(x_1^2+x_2^2)) \text{ for } (x_1,x_2) \in \Gamma_g$$
$$\Rightarrow v_1 \cdot 0 + v_2 \cdot 0 = 0.$$

$$\eta_a = \{v_1, v_2\}$$

$$\eta_g = \{\phi\} \text{ an empty set.}$$

3(c). Given

$$V_h = \{v_1 N_1 + v_2 N_2\}$$

$$S_h = \{v_1 N_1 + v_2 N_2\}.$$

We know $a(u,v)$ and $\ell(v)$ are defined as follow:

$$a(u,v) = \frac{1}{2}\int_\Omega \nabla u \cdot \nabla v$$

$$\ell(v) = \int_\Omega \frac{\partial v}{\cancel{R}} - \int_{\Gamma_h} \frac{v}{R}$$

we can compute the stiffness matrix as:

$$a(N_1, N_1) = \frac{1}{2}\int_\Omega \nabla(\sin(\pi(x_1^2+x_2^2))) \cdot \nabla(\sin(\pi(x_1^2+x_2^2)))$$

$$= \frac{1}{2}\int_\Omega \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} \end{bmatrix} d\vec{x}$$

Using matlab symbolic calculation, we can have

$$a(N_1,N_1) = \frac{\pi^3}{2} \approx 15.5031$$

Similarly, we can compute $a(N_1, N_2)$

$$a(N_1, N_2) = \frac{1}{2}\int_\Omega \nabla(\sin(\pi(x_1^2+x_2^2)))\cdot\nabla(\cos(\pi(x_1^2+x_2^2)))$$

$$= \frac{1}{2}\int_\Omega \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial x_1} \\ \frac{\partial N_2}{\partial x_2} \end{bmatrix} d\vec{x}$$

$$= \frac{1}{2}\int_\Omega \frac{\partial N_1}{\partial x_1}\frac{\partial N_2}{\partial x_1} + \frac{\partial N_1}{\partial x_2}\frac{\partial N_2}{\partial x_2} \, d\vec{x}$$

$$a(N_1, N_2) = a(N_2, N_1) = \frac{20\pi}{9} \approx 6.9813$$

Finally, $a(N_2, N_2)$ reads:

$$a(N_2, N_2) = \frac{1}{2}\int_\Omega \nabla(\cos(\pi(x_1^2+x_2^2)))\cdot\nabla(\cos(\pi(x_1^2+x_2^2)))\, d\vec{x}$$

$$= \frac{1}{2}\int_\Omega \begin{bmatrix} \frac{\partial N_2}{\partial x_1} & \frac{\partial N_2}{\partial x_1} \end{bmatrix} \begin{bmatrix} \frac{\partial N_2}{\partial x_1} \\ \frac{\partial N_2}{\partial x_1} \end{bmatrix} d\vec{x}$$

$$= \frac{1}{2}\int_\Omega \frac{\partial N_2}{\partial x_1}\frac{\partial N_2}{\partial x_1} + \frac{\partial N_2}{\partial x_1}\frac{\partial N_2}{\partial x_1} \, d\vec{x}$$

$$= \frac{(\pi * (\pi^2 + 4))}{8} \approx 5.4466$$

Stiffness matrix reads:

$$K = \begin{bmatrix} a(N_1, N_1) & a(N_2, N_1) \\ a(N_1, N_2) & a(N_2, N_2) \end{bmatrix} = \begin{bmatrix} \frac{\pi^3}{2} & \frac{20\pi}{9} \\ \frac{20\pi}{9} & \frac{\pi^3+4\pi}{8} \end{bmatrix}$$

Load vector: $l(v) = \int_{\Omega} \frac{2v}{R^2} - \int_{\Gamma_N} \frac{v}{R}$

$$l(N_1) = \int_{\Omega} \frac{2}{R^2} N_1 - \int_{\Gamma_N} \frac{N_1}{R}$$

$$= 4.00.$$

$$l(N_2) = \int_{\Omega} \frac{2}{R^3} N_2 - \int_{\Gamma_N} \frac{N_2}{R}$$

$$= 4.00.$$

$$F = \begin{bmatrix} 4.00 \\ 4.00 \end{bmatrix}$$

(d). $U = K \backslash F = \begin{bmatrix} -0.1720 \\ 0.9508 \end{bmatrix}$

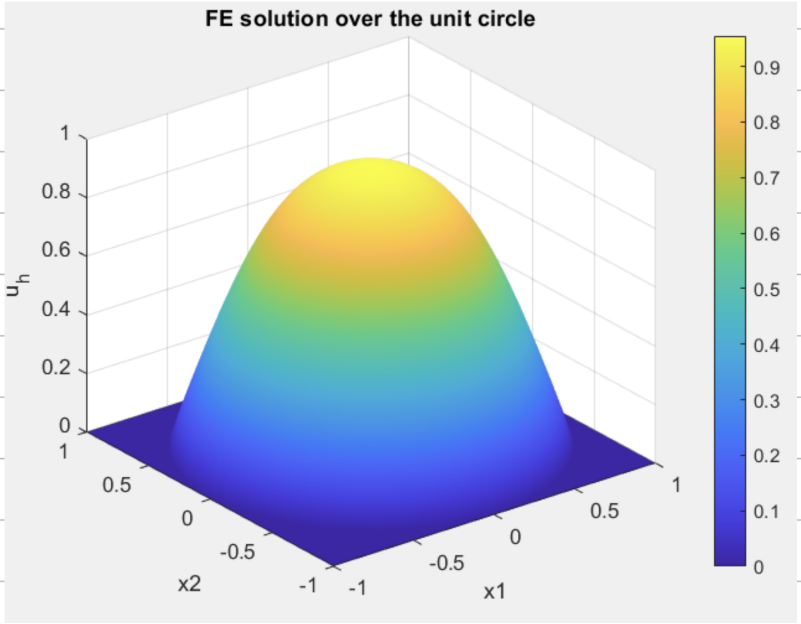
$$u(\vec{x}) = -0.1720 \sin(\pi(x_1^2 + x_2^2)) + 0.9508 \cos(\tfrac{\pi}{2}(x_1^2 + x_2^2))$$

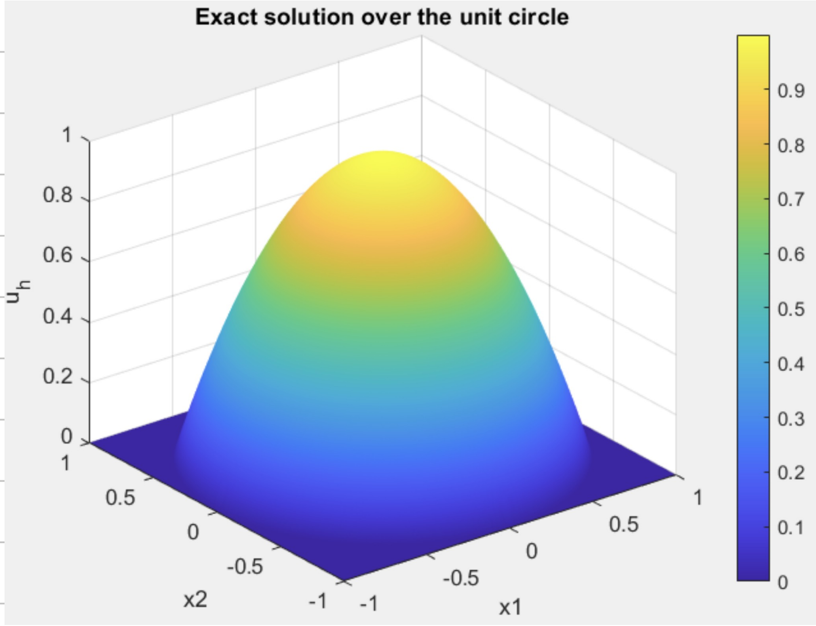
The exact solution for a unit disk is:

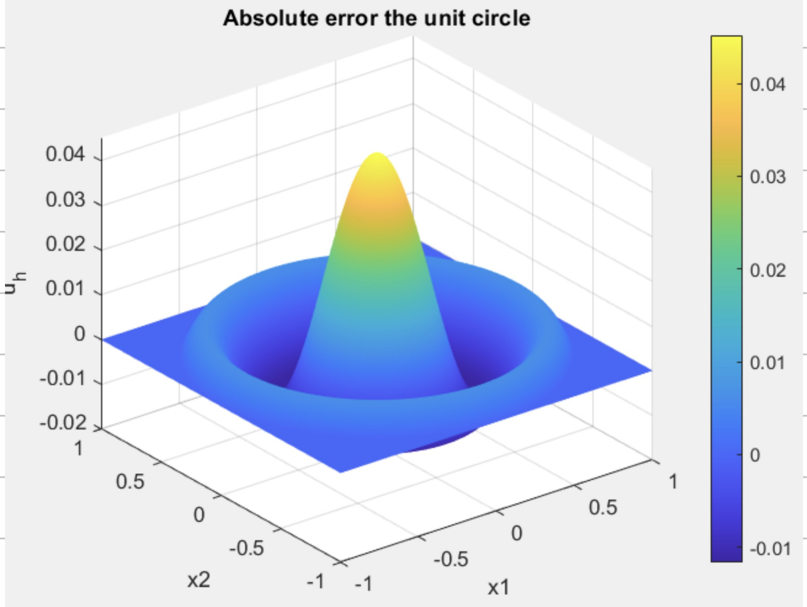
$$u(x, y) = g - \frac{f}{4K}(x_1^2 + x_2^2 - R^2)$$

where $R = 1$, $K = \frac{1}{2}$, $f = -2$, $g = 0$.

$$u(x, y) = 1 - x_1^2 + x_2^2$$


FE solution over the unit circle
u_h
x1
x2


Exact solution over the unit circle
u_h
x1
x2


Absolute error the unit circle
u_h
x1
x2

(e). Given $\frac{1}{2}\nabla u \cdot \vec{n} = -\frac{2}{R}$, the load vector will change.

load vector: $\ell(v) = \int_{\Omega} \frac{2v}{R^2} - \int_{\Gamma_h} \frac{2v}{R}$

For N_1, $\ell(N_1) = \int_{\Omega} \frac{2}{R^2} N_1 - \int_{\Gamma_h} \frac{2N_1}{R}$

But, $\int_{\Gamma_h} \frac{2N_1}{R} = 0$ so is $\int_{\Gamma_h} \frac{2N_2}{R} = 0$

Therefore, the change in Neumann B.C. will not change the solution.

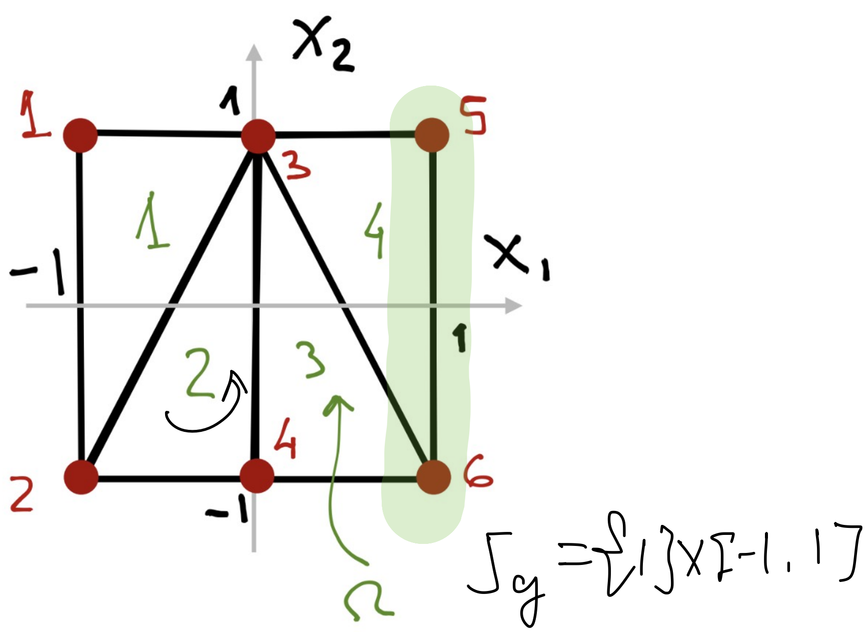
Manual Assembly, Once More (60)

Let $\Omega = [-1,1] \times [-1,1]$, $\Gamma_g = \{1\} \times [-1,1]$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the variational equation that $u\colon \Omega \to \mathbb{R}$ satisfies:

$$\int_\Omega \nabla u \cdot \nabla v + uv \; d\Omega = \int_\Omega (x_1 + x_2) v \; d\Omega + \int_{\Gamma_h} (x_2^2 - 1) v \; d\Gamma$$

for all $v \in \mathcal{V} = \{v\colon \Omega \to \mathbb{R} \mid v = 0 \text{ on } \Gamma_g\}$, where (x_1, x_2) are the Cartesian coordinates in Ω. The function u satisfies the essential boundary condition $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in \Gamma_g$.

Consider then the mesh shown in the figure, made of all P_1 elements:



1. (5) What is the local-to-global map for the mesh?
2. (5) Identify $\mathcal{S}_h$ and $\mathcal{V}_h$ by providing the general expression for their functions in terms of the P_1 basis functions of the mesh. Identify active and constrained indices.
3. (10) Evaluate the shape function N_3^2 and its derivative ∇N_3^2 at $(x_1, x_2) = (-0.5, -0.1)$.

1. Assuming the vertices are ordered counterclockwise, the local-to-global map for conforming mesh is:

$$LG = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 2 & 6 & 6 \\ 3 & 4 & 3 & 5 \end{bmatrix}$$

2. The vertices belonging to Γ_g are 5 and 6.

Let $N_i(x_1, x_2)$ be the global basis functions. We require $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in J_g$. Therefore,

Test spaces:

$$V_h = \text{span}(N_1, N_2, N_3, N_4)$$

Trial spaces:

$$S_h = \{ V_h + x_2 N_5 + x_2 N_6) \mid x_2 \in \mathbb{R} \}.$$

active set:

$$\mathcal{J}_a = \{ 3, 4, 5, 6 \}.$$

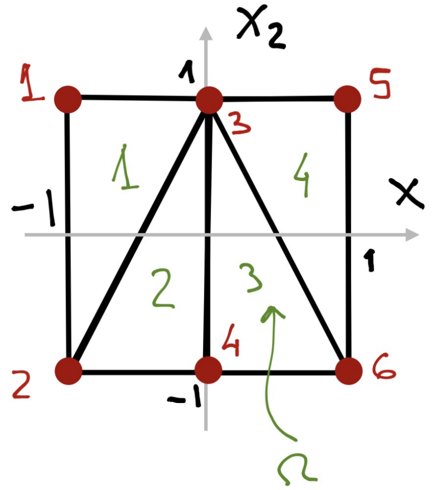
Constrained set:

$$\mathcal{J}_g = \{ 1, 2 \}.$$

We select $\bar{u} = x_2$

3. The local shape function N_3^2 can be expressed as:

$$N_3^2 = \lambda_3(x_1, x_2) = \frac{1}{2A}\left[-(x_2^2 - x_2^1)(x_1 - x_1^1) + (x_1^2 - x_1^1)(x_2 - x_2^1)\right]$$



$$2A = (x_1^2 - x_1^1)(x_1^3 - x_1^1) - (x_2^2 - x_2^1)(x_2^3 - x_2^1)$$

From the figure, we know:

$x^1 = (0, 1)$.

$x^2 = (-1, -1)$

$x^3 = (0, -1)$

$$N_3^2 = \frac{1}{2}\cdot\left[(1-(-1))(x_1-0) + (-1-0)(x_2-1)\right]$$

$$= \frac{1}{2}\left[2x_1 - (x_2-1)\right]$$

$$\nabla N_3^2 = \frac{1}{2A}\begin{pmatrix} x_2^1 - x_2^2 \\ x_1^2 - x_1^1 \end{pmatrix} = \frac{1}{2}\cdot\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

Therefore

$$N_3^2(-0.5, -0.1) = 0.05$$

$$\nabla N_3^2(-0.5, -0.1) = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

4. Using the ~~trial~~ / test spaces defined above, we can formulate the discrete weak form as:

find $u_h \in S_h$ such that

$$a_h(u_h, v_h) = l(v_h) \quad \text{for all } v_h \in V_h$$

$$a_h(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h + u_h v_h \, d\Omega$$

$$l(v_h) = \int_\Omega (x_1 + x_2) v_h \, d\Omega + \int_{\Gamma_h} (x_2^2 - 1) v \, d\Gamma$$

Since we are using P1 element $\nabla u_h, \nabla v_h$ are constants, we can simplify the local stiffness matrix as:

$$K_{ab}^e = \int_{\Omega^e} \nabla N_b^e \cdot \nabla N_a^e + N_b^e N_a^e \, d\Omega^e$$

$$= A_e \nabla N_b^e \cdot \nabla N_a^e + \int_{\Omega^e} N_b^e N_a^e \, d\Omega^e$$

$$K_{11}^1 = A_1 \underbrace{\begin{bmatrix} \frac{\partial N_1^1}{\partial x_1} & \frac{\partial N_1^1}{\partial x_2} \end{bmatrix}}_{(dN_1^1)^T} \underbrace{\begin{bmatrix} \frac{\partial N_1^1}{\partial x_1} \\ \frac{\partial N_1^1}{\partial x_2} \end{bmatrix}}_{dN_1^1} + \int_{\Omega^e} N_1^1 N_1^1 \, d\Omega^e$$

$$dN_1^1 = \begin{bmatrix} x_2^2 - x_2^3 \\ x_1^3 - x_1^2 \end{bmatrix} \qquad N_1^1 = \frac{1}{2A}\left(dN_1^1(1)\cdot(x_1 - x_1^2) + dN_1^1(2)\cdot(x_2 - x_2^2)\right).$$

For load term, we have

$$F_a^e = \int_{\Omega^e} (x_1 + x_2) N_1 \, d\Omega + \int_{\Gamma_h} (x_2^2 - 1) N_1 d\Omega$$

Special attentions are given to the ~~treatmore~~ of neumment boundary. As the integrand is $(x_2^2 - 1)$. its value is non zero on the $\Gamma_{h\{1,2\}} = \{-1\} \times [-1, 1]$.

Therefore, we only need to evaluate the integration of $\int_{\Gamma_h} (x_2^2 - 1) N_1$ and $\int_{\Gamma_h} (x_2^2 - 1) N_2$

By following the same idea, we can compute the local stiffness matrix and load vector.

For element 1; the ke reads:

$$k_e^1 = \begin{bmatrix} 1.4167 & -0.1667 & -0.9167 \\ -0.1667 & 0.4167 & 0.0833 \\ -0.9167 & 0.0833 & 1.1667 \end{bmatrix}$$

$$F_e^1 = \begin{bmatrix} -0.0833 \\ -0.2500 \\ 0.00 \end{bmatrix}$$

For element 2, the ke reads:

$$k_e^2 = \begin{bmatrix} 0.4167 & 0.0833 & -0.1667 \\ 0.0833 & 1.1667 & -0.9167 \\ -0.1667 & -0.9167 & 1.4167 \end{bmatrix}$$

$$F_e^2 = \begin{bmatrix} -0.0833 \\ -0.3333 \\ 0.00 \end{bmatrix}$$

For element 3, the Ke reads:

$$K_e^3 = \begin{bmatrix} 1.4167 & -0.9167 & -0.1667 \\ -0.9167 & 1.1667 & 0.0833 \\ -0.1667 & 0.0833 & 0.4167 \end{bmatrix}$$

$$F_e^3 = \begin{bmatrix} -0.0833 \\ -0.00 \\ 0.0833 \end{bmatrix}$$

For element 4, the Ke reads:

$$K_e^4 = \begin{bmatrix} 1.1667 & 0.0833 & -0.9167 \\ 0.0833 & 0.4167 & -0.1667 \\ -0.9167 & -0.1667 & 1.4167 \end{bmatrix}$$

$$F_e^4 = \begin{bmatrix} 0.3333 \\ 0.2500 \\ 0.4167 \end{bmatrix}$$

5. $\Gamma_g = \{1\} \times [-1, 1]$.

$\Gamma_g = \{5, 6\}$ containing vertex 5 and 6.

$\Gamma_h = \partial\Omega / \Gamma_g = \{-1\} \times [-1, 1] \cup$

$[-1, 1] \times \{1\} \cup$

$[-1, 1] \times \{-1\}$.

For natural B.C.

$$F_a = \int_{\Gamma_h} (x_2^2 - 1) N_a d\Omega$$

Special attentions are given to the ~~treatmore~~ of neumann boundary. As the integrand is $(x_2^2 - 1)$. its value is nonzero only on the $\Gamma_{h\{1,2\}} = \{-1\} \times [-1, 1]$. Therefore, we only need to evaluate the integration of $\int_{\Gamma_h} (x_2^2 - 1) N_1$ and $\int_{\Gamma_h} (x_2^2 - 1) N_2$

$$F_{n,1} = \int_{\Gamma_{h_1}} (x_2^2 - 1) N_1 = \frac{1}{2A} \int_{-1}^{1} (x_2^2 - 1) \cdot (x_1^3 - x_1^2) \cdot (x_2 - x_2^2) dx_2$$

$$= -0.6667$$

$$F_{n2} = \int_{\Gamma_{h_2}} (x_2^2 - 1) N_2 = \frac{1}{2A} \int_{-1}^{1} (x_2^2 - 1) \cdot (x_1^3 - x_2^2)(x_2 - x_2^2) dx_2$$

$$= -0.6667$$

$F_n = [-0.6667, -0.6667, 0, 0, 0, 0.]$

6. Use local to global map to assemble the global stiffness matrix, we have:

	1	2	3	4	5	6
1	1.4167	-0.1667	-0.9167	0	0	0
2	-0.1667	1.5833	0.1667	-0.9167	0	0
3	-0.9167	0.1667	3.1667	-0.3333	-0.9167	0.1667
4	0	-0.9167	-0.3333	2.8333	0	-0.9167
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Load vector is:

F=

	1
1	-0.7500
2	-1.2500
3	0.3333
4	-0.3333
5	1
6	-1

$U = K \backslash F =$

	1
1	-0.5067
2	-1.3643
3	0.2831
4	-0.8493
5	1
6	-1

(T).

$$u_h = -0.5067 N_1 - 1.3643 N_2 + 0.2831 N_3 - 0.8493 N_4 + N_5 - N_6.$$