CP-3.

The weak form of the isotropic linear Elasticity problem can be given as:

Given the domain Ω and the following spaces:

$$\mathcal{W} = \{ \vec{w} : \Omega \to \mathbb{R}^2 \mid \vec{w} \text{ is smooth vector field} \}$$

$$S = \{ \vec{w} \in \mathcal{W} \mid \vec{w} = \vec{g} \text{ on } \partial\Omega_D \}$$

$$\mathcal{V} = \{ \vec{v} \in \mathcal{W} \mid \vec{v} = 0 \text{ on } \partial\Omega_D \}$$

Find $\vec{w} \in S$ such that

$$a(\vec{w}, \vec{v}) = l(\vec{v}) \qquad \forall \vec{v} \in \mathcal{V}$$

where

$$a(\vec{w}, \vec{v}) = \int_\Omega \frac{E}{1+\nu} \left(\varepsilon(\nabla \vec{w}) : \varepsilon(\nabla \vec{v}) + \frac{\nu}{1-2\nu} \operatorname{div} \vec{w} \operatorname{div} \vec{v} \right)$$

$$l(\vec{v}) = \int_\Omega \vec{b} \cdot \vec{v} \, d\Omega + \int_{\partial\Omega_N} \vec{H} \cdot \vec{v} \, d\Omega.$$

1. See Matlab implementation

2.

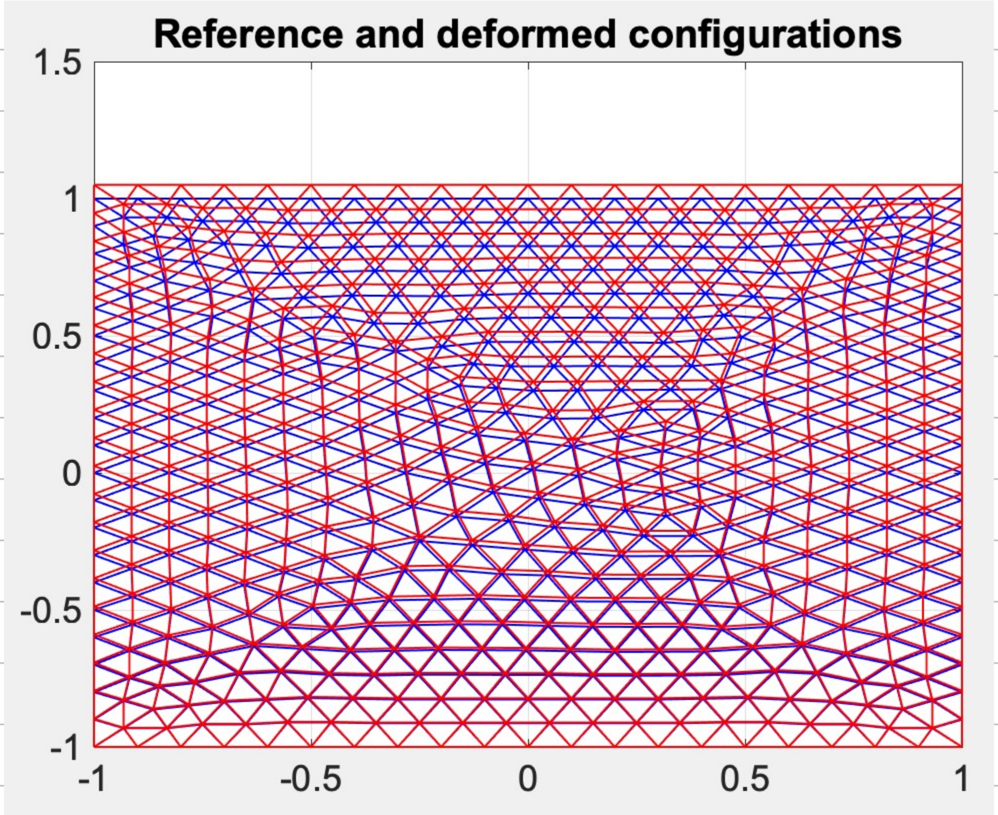
2. (10) To test your code, we will begin by setting `eccentricity`=0,so that we have a square with no holes. We will also set

$$\nu = 0,$$
$$E = 1\text{MPa}$$
$$g = 0 \qquad \text{no gravity}$$
$$g_{2y} = 0.05.$$

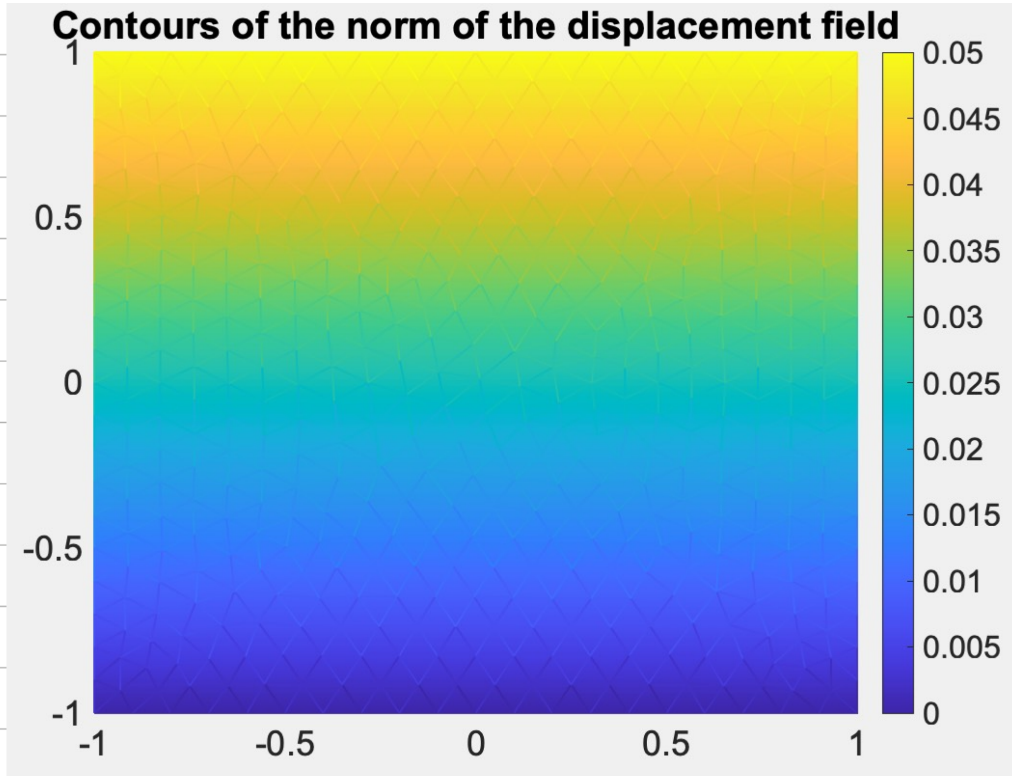
The exact solution of this problem is that both the strain and the stresses should constant in the domain, and equal to

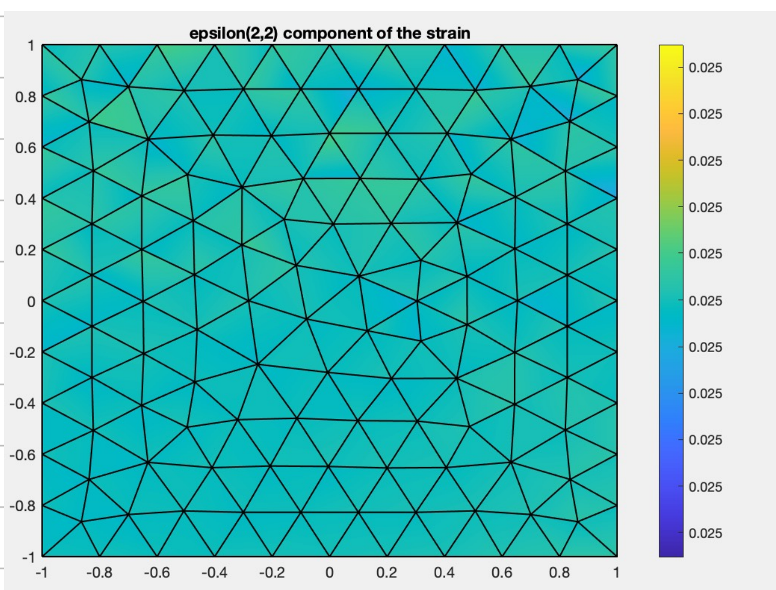
$$\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & 0.025 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0.025\text{MPa} \end{bmatrix}.$$

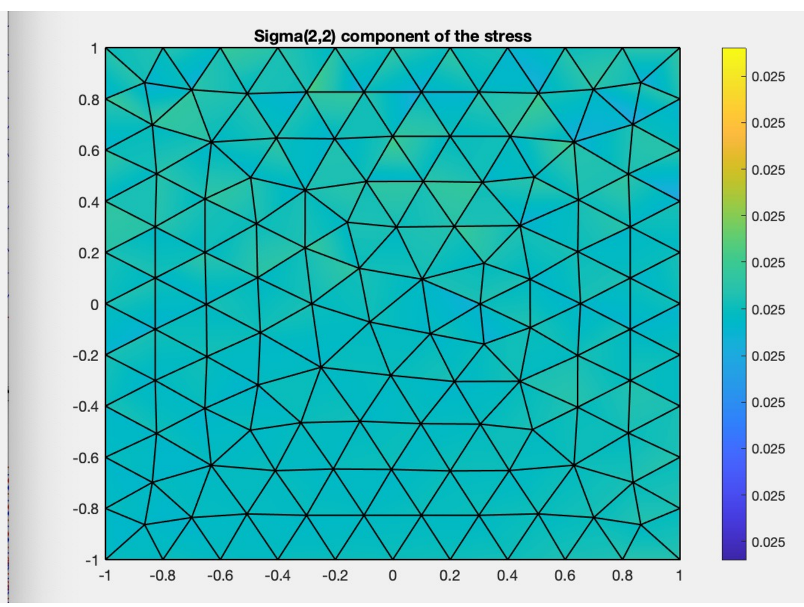
Please verify that you code can reproduce this deformation. This is similar to what is known as the *patch test* in the finite element literature. Show the plot of ε_{22} and σ_{22} over the mesh (as output from the code, with minor tweaks) for `HMax`=0.2, as well as a plot of the reference and deformed configurations, as output by the code.



Blue mesh is the reference and red mesh is the deformed one.





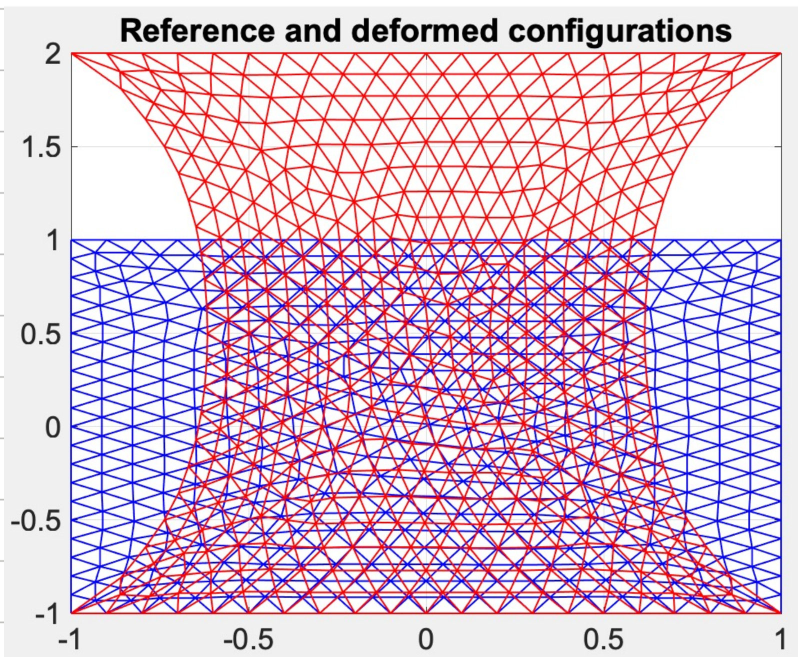


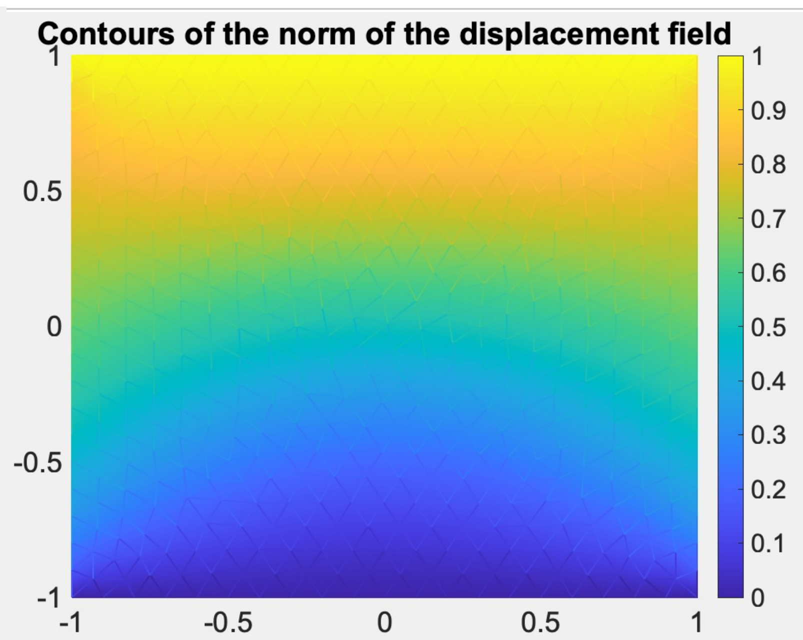
Clearly, our code passes the patch test, where constant σ_{22} and ε_{22} are accurately captured.

3. (10) Next, set

$$\nu = 0.45$$
$$E = 1\text{MPa}$$
$$g = 0 \qquad \text{no gravity}$$
$$g_{2y} = 1,$$

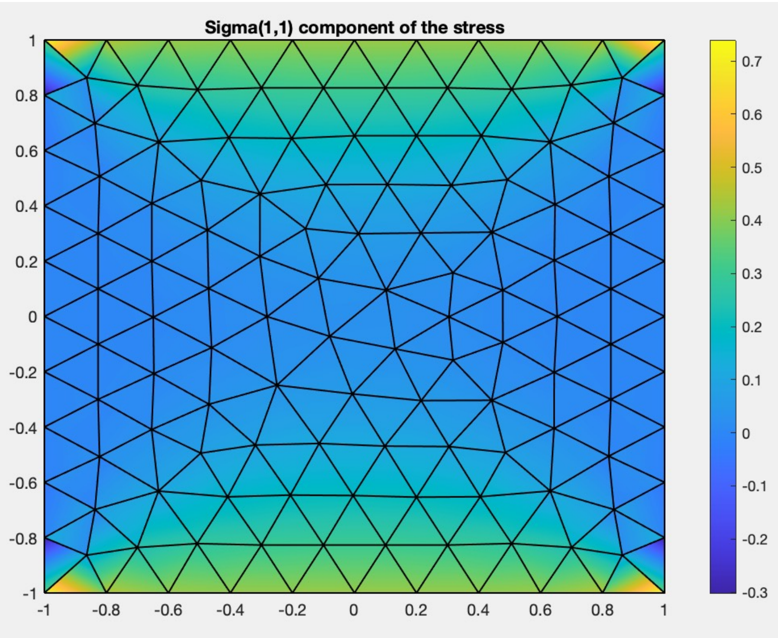
and `eccentricity` = 0. This will give you a deformation beyond small strains and small displacements, but it is a interesting to see. Please plot the reference and deformed configurations, and the stress components σ_{22}, σ_{12} and σ_{11}. Are the traction free boundary conditions on Γ_{h1} and Γ_{h3} well imposed? Comment on this.

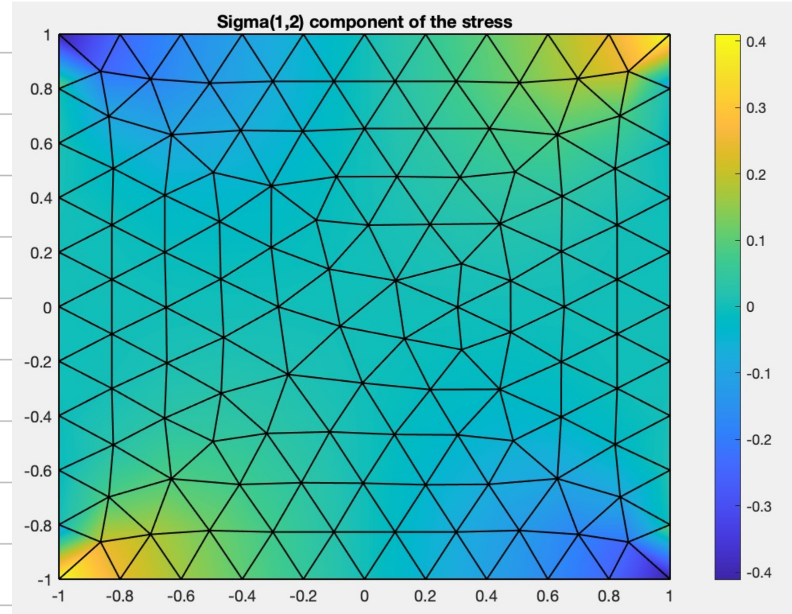


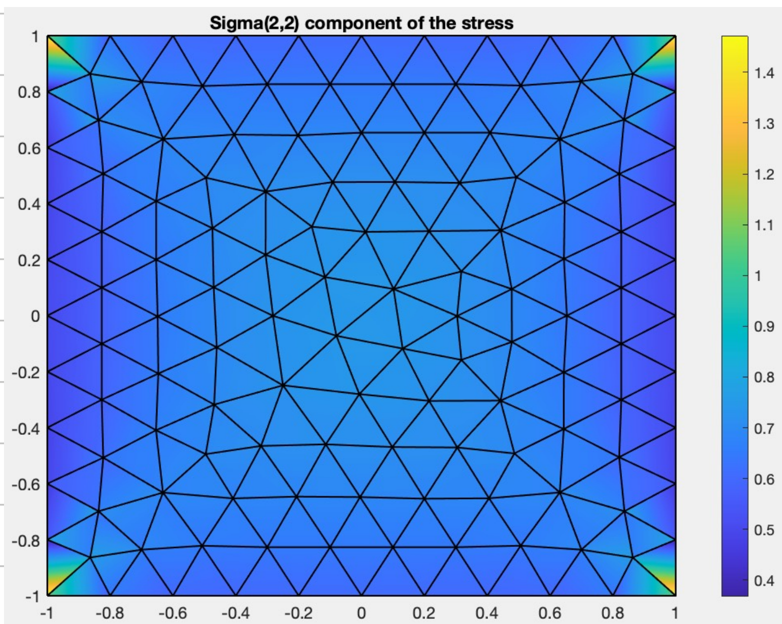


The left figure above show the deformed mesh after the displacement is imposed. We can see there the deformation is so large that the deformed shape is not close to the reference one

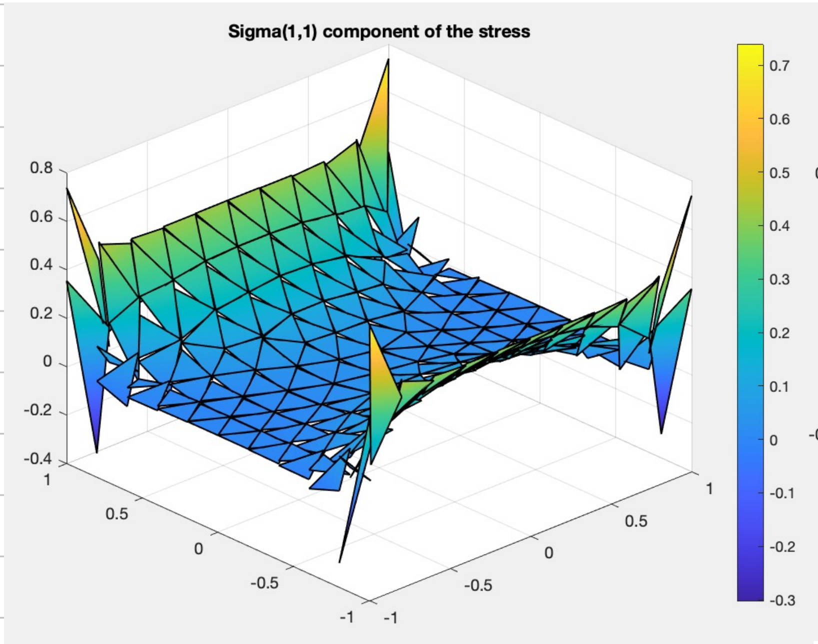
The right figure shows the displacement contour of the object, where the upper boundary has 1. m deforma

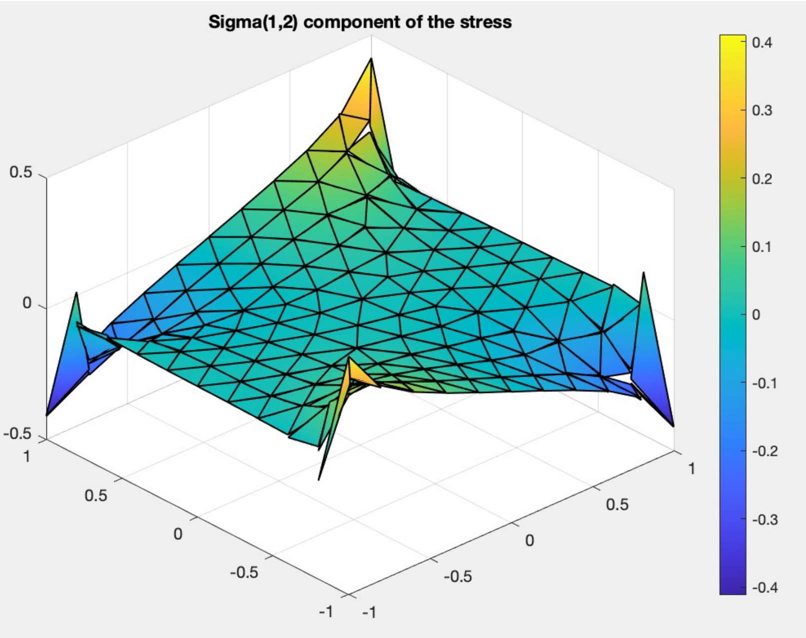


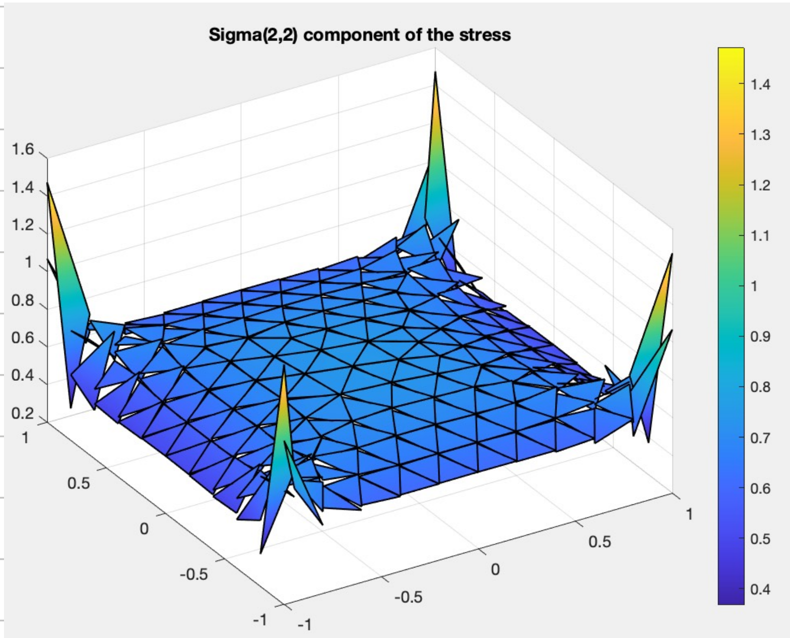




The figures above show the σ_{11}, σ_{12}, σ_{22} contours over the referenced mesh. The 3D plots of σ_{11}, σ_{12}, σ_{22} are shown to highlight the stresses distribution near the boundary, as follows:


Sigma(1,1) component of the stress

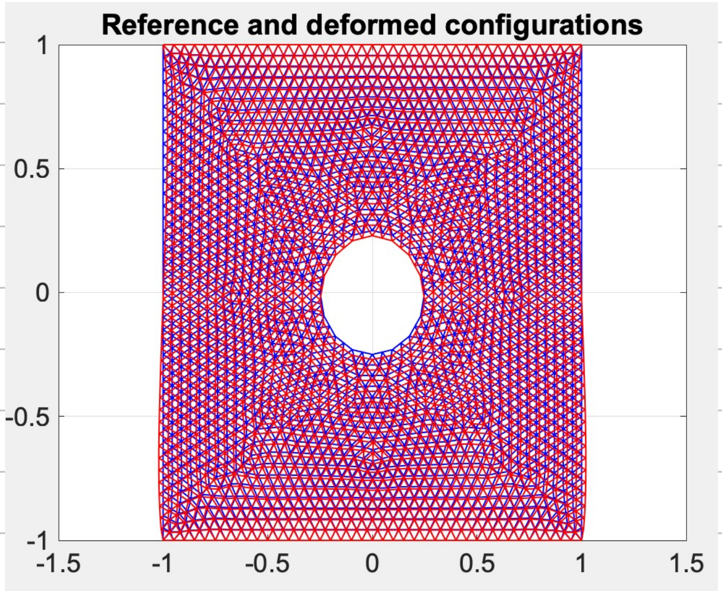

Sigma(1,2) component of the stress

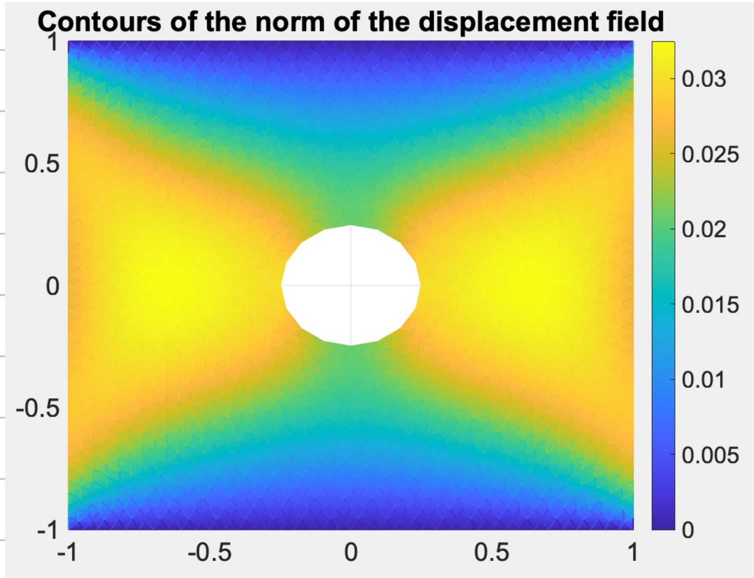

Sigma(2,2) component of the stress

4. (10) Now we will set `eccentricity`=1, and

$$\nu = 0.45$$
$$E = 0.001\text{MPa}$$
$$g = 9.81\text{m/s}^2$$
$$g_{2y} = 0,$$

In this part it is important to pay attention to the units. To get a consistent result, you will need to use all SI units, since otherwise we may be adding Newtons with MegaNewtons. Plot the reference and deformed configurations for `HMax`=0.1. As a reference, the maximum displacement of a point on the left edge should be ≈ -0.15m.





$$\frac{dN_1}{dx_1} = \sum_{i=1}^{3} \frac{\partial N_1}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial x_1}$$

$$\frac{d\lambda_i}{dx_i} = \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_1} & \frac{\partial \lambda_2}{\partial x_1} & \frac{\partial \lambda_3}{\partial x_1} \\ \frac{\partial \lambda_1}{\partial x_2} & \frac{\partial \lambda_2}{\partial x_2} & \frac{\partial \lambda_3}{\partial x_2} \end{bmatrix}$$

$$dN_1 = \begin{bmatrix} \frac{\partial N_1}{\partial \lambda_1} & \frac{\partial N_1}{\partial \lambda_2} & \frac{\partial N_1}{\partial x_3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_1} & \frac{\partial \lambda_1}{\partial x_2} \\ \frac{\partial \lambda_2}{\partial x_1} & \frac{\partial \lambda_2}{\partial x_2} \\ \frac{\partial \lambda_3}{\partial x_1} & \frac{\partial \lambda_3}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x_1} + \frac{\partial N_1}{\partial x_1} + \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} + \frac{\partial N_1}{\partial x_2} + \frac{\partial N_1}{\partial x_2} \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} N_1 & & \cdots & N_6 \end{array}$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} & & \frac{\partial N_6}{\partial x_1} & \frac{\partial N_6}{\partial x_2} \\ 0 & 0 & & 0 & 0 \end{bmatrix}$$

$$\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$= \left[\begin{array}{cc|cc|} \frac{\partial N_1}{\partial x_1} & \frac{1}{2}\frac{\partial N_1}{\partial x_2} & \frac{\partial N_2}{\partial x_1} & \frac{1}{2}\frac{\partial N_2}{\partial x_2} \\ 0 & 0 & 0 & 0 \end{array} \right. \cdots$$

$$\varepsilon_{11}$$