

On Vector Spaces of Functions (40)

For this problem section titled “1.1.3 Sets of Functions” in the notes has a discussion about the notation used in this part.

Let

$$W = \{u: [-1, 1] \rightarrow \mathbb{R} \text{ smooth}\}.$$

1. Are the following sets vector spaces of functions? Explain.

- (a) (5) $V_1 = \{u \in W \mid u(0) = 0\}$.
 - (b) (5) $V_2 = \{u \in W \mid u''(0) = 0\}$.
 - (c) (5) $V_3 = \{u \in W \mid u(x) \neq 0 \quad \forall x \in [-1, 1]\}$.
 - (d) (5) $V_4 = \{u \in W \mid \int_{-1}^1 u''(x) dx = 0\}$.
 - (e) (5) $V_5 = \{u \in W \mid \int_{-1}^1 x^2 u(x) dx = 1\}$.
 - (f) (5) $V_6 = \{u \in W \mid u(0) = -5\}$.
2. (5) The set V_6 is an affine subspace of W . What is its direction? You do not need to prove it, just state it.
3. (5) Is $\ell: V_1 \rightarrow \mathbb{R}$ a linear functional, where

$$\ell(u) = \int_{-1}^1 u''(x) dx? \tag{1}$$

i. (a). V_1 is a vector space of function.

Since $V_1 \subset W$, V_1 inherits commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

i. Closure: $u, v \in V_1$ and $\alpha \in \mathbb{R}$, then we know

$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$

This is because u and v is smooth function with $u(0)=0$, $v(0)=0$. and $u+v$ is also smooth and $(u+v)(0)=0$.

Similarly, αu is smooth function and $\alpha u(0)=0$.

ii. Identity: $\exists z(x)=0 \quad \forall x \in [-1, 1]$.

$$u(x) \in V_1, \quad \forall x \in [-1, 1]$$

We know:

$$u(x) + z(x) = u(x)$$

iii, additive inverse:

if $u \in V_1$, then $-u \in V_2$ as $U(0) = -U(0) = 0$.
and $-U(0) \in W$

1.(b) V_2 is a vector space of function.

Since $V_2 \subset W$, V_2 inherits commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

i Closure: $u, v \in V_1$ and $\alpha \in \mathbb{R}$, then we know

$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$

This is because u and v is smooth function. with $U''(0) = 0$
 $V''(0) = 0$. and $u+v$ is also smooth and $(u+v)''(0) = (U''(0) + V''(0)) = 0$.

Similarly, αu is smooth function and $\alpha U''(0) = 0$.

ii. Identity: $\exists z(x) | z(x)=0$ and $z''(x)=0 \quad \forall x \in [-1, 1]$
 $u(x) \in V_1, \quad \forall x \in [-1, 1]$

We know:

$$u(x) + z(x) = u(x)$$

$$U''(0) + z''(0) = 0$$

iii, additive inverse:

if $u \in V_1$, then $-u \in V_2$ since $U''(0) = -U''(0) = 0$.

(c). V_3 is not a vector space of function, since closure does not hold.

Let $u, v \in V_3 | u(x) = a$ and $v(x) = -a$

$(u + v)(0) = 0$ which violates the constraint of V_3 .

(d). V_4 is a vector space of function.

Since $V_4 \subset W$, V_4 inherits commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

↪ Closure: $u, v \in V_4$ and $\alpha \in \mathbb{R}$, then we know

$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$

This is because u and v is smooth function with

$$\textcircled{1} \quad \int_{-1}^1 u''(x) dx = 0 \quad \text{and} \quad \int_{-1}^1 v''(x) dx = 0$$

$$\Rightarrow \int_{-1}^1 (u(x) + v(x))'' dx = \int_{-1}^1 (u''(x) + v''(x)) dx = 0.$$

$$\textcircled{2} \quad \int_{-1}^1 \alpha u''(x) dx = \alpha \int_{-1}^1 u''(x) dx = 0$$

(e) V_5 is not a vector space of function.

Let $u, v \in V_5$ such that $\int_{-1}^1 x^2 u dx = 1$ and $\int_{-1}^1 x^2 v dx = 1$

$$u + v = \int_{-1}^1 x^2 (u + v) dx = 2 \notin V_5$$

Therefore, closure doesn't hold.

(f). V_6 is not a vector space of functions.

Similarly, let $u, v \in V_6$ such that $u(0) = -5$
 $v(0) = -5$

$$(u + v)(0) = u(0) + v(0) = -10 \notin V_6$$

Therefore, closure doesn't hold.

2. The direction of V_0 is:

$$V_0 = \{u \in W \mid u(0) = 0\}$$

3.

Using definition, for any $u, v \in V$ and $\alpha \in \mathbb{R}$

$$\begin{aligned} f(u + \alpha v) &= \int_{-1}^1 (u + \alpha v)''(x) dx \\ &= \int_{-1}^1 (u''(x) + \alpha v''(x)) dx \\ &= f(u) + \alpha f(v) \end{aligned}$$

which proves f is a linear functional.

On Bases for Vector Spaces of Functions (20)

For $x \in \mathbb{R}$, define $g(x) = 1$ and

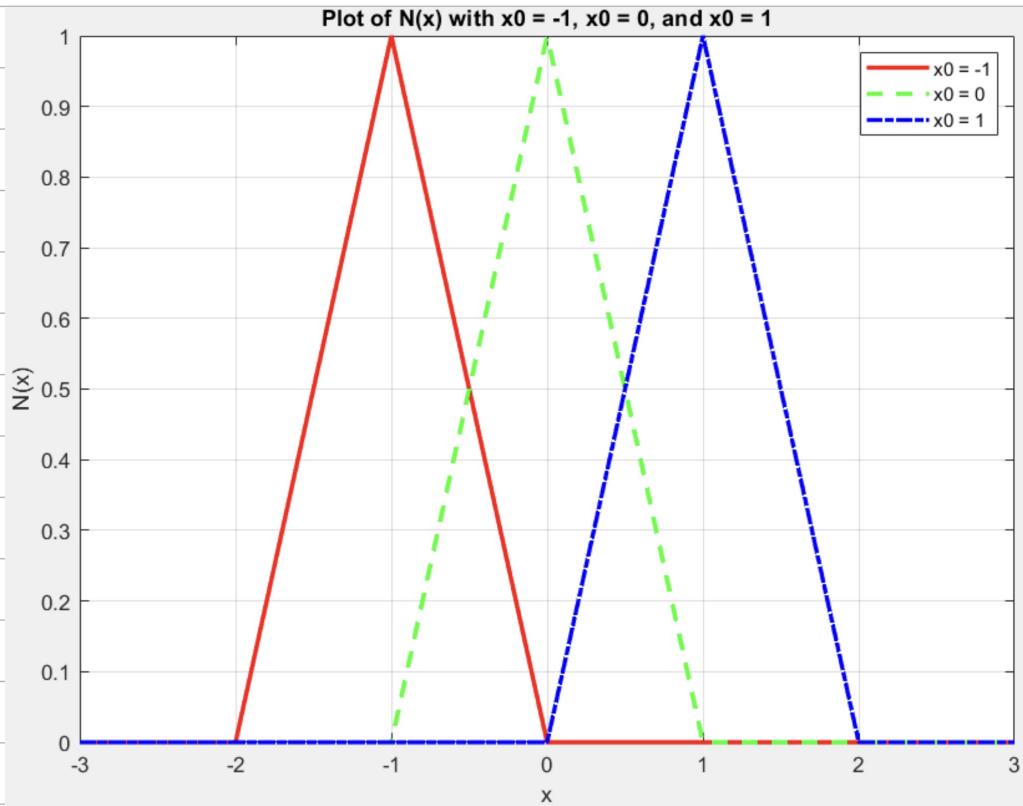
$$N_{x_0}(x) = \max(1 - |x - x_0|, 0).$$

1. (5) Plot the functions N_{-1} , N_0 , and N_1 over the interval $(-3, 3)$.

1

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2. (5) For functions whose domain is \mathbb{R} , is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain.
Hint: Find inspiration in Example 1.32 in the notes.
3. (5) For functions whose domain is $(-1, 1)$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent?
Explain.
4. (5) Consider functions whose domain is $(-1, 1)$, and let $f(x) = 2x + 1$. Does $f \in \text{span}(N_{-1}, N_0, N_1)$?
If so, what are its components?



2 We can define the following function
 $P(x) = C_1 N_1(x) + C_2 N_0(x) + C_3 N_1(x) + C_4 g$

We can evaluate $P(x)$ at arbitrary x values, so that we can compute coefficients.

$$P(2), P(0), P(-\frac{1}{2}), P(\frac{1}{2}).$$

$$\left. \begin{aligned} P(2) &= C_1 \cdot 0 + C_2 \cdot 0 + C_3 \cdot 0 + C_4 \\ P(0) &= C_1 \cdot 0 + C_2 \cdot 1 + C_3 \cdot 0 + C_4 \\ P(-1) &= C_1 \left(\frac{1}{2}\right) + C_2 \left(\frac{1}{2}\right) + C_3 \cdot 0 + C_4 \\ P(-2) &= C_1 \cdot (0) + C_2 \left(\frac{1}{2}\right) + C_3 \left(\frac{1}{2}\right) + C_4 \end{aligned} \right\} \quad \dots \quad (1)$$

Let D all equal to 0, we have

$$\begin{bmatrix} 0 & 0 & 0 & | & c_1 \\ 0 & 1 & 0 & | & c_2 \\ \frac{1}{2} & \frac{1}{2} & 0 & | & c_3 \\ 0 & \frac{1}{2} & \frac{1}{2} & | & c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant is non-zero

The only solution to this system is the trivial one.

$$c_1 = c_2 = c_3 = c_4 = 0.$$

These functions are linearly independent.

$$3. \text{ Over } (-1, 1) \quad N_{-1} = \begin{cases} -x, & (-1, 0) \\ 0, & (0, 1) \end{cases}$$

$$N_0 = \begin{cases} 1+x, & (-1, 0) \\ -x, & (0, 1) \end{cases}$$

$$N_1 = \begin{cases} 0, & (-1, 0) \\ x, & (0, 1) \end{cases}$$

$$g(x) = 1$$

Define the span as :

$$P(x) = C_1 N_{-1} + C_2 N_0 + C_3 N_1 + C_4 g$$

Evaluate $P(x)$ at $x=0$, $x=-\frac{1}{2}$, $x=\frac{1}{2}$, $x=\frac{3}{4}$

$$C_1 \cdot 0 + C_2 \cdot 1 + C_3 \cdot (0) + C_4 \cdot (1) = 0.$$

$$C_1 \cdot \frac{1}{2} + C_2 \cdot (\frac{1}{2}) + C_3 (0) + C_4 (1) = 0.$$

$$C_1 (0) + C_2 (\frac{1}{2}) + C_3 (\frac{1}{2}) + C_4 (1) = 0.$$

$$C_1 (0) + C_2 (\frac{1}{4}) + C_3 (\frac{3}{4}) + C_4 (1) = 0.$$

$$\begin{cases} C_4 = C_2 \\ \frac{1}{2}(C_1 + C_2) + C_4 = 0. \\ \frac{1}{2}(C_2 + C_3) + C_4 = 0 \\ \frac{1}{4}C_2 + \frac{3}{4}C_3 + C_4 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A C b

Such a linear system has a determinant of zero. which means there are many solutions satisfying $AC = 0$.

We can find that $g = N_{-1}(x) + N_0(x) + N_1(x) \quad \forall x \in [-1, 1]$
 therefore, $\{g, N_{-1}, N_0, N_1\}$ are linearly dependent.

4. The span $\{N_{-1}, N_0, N_1\}$ is

$$P(x) = C_1 N_{-1} + C_2 N_0 + C_3 N_1$$

$$= -C_1 x + (1+x) \cdot C_2, \quad (-1, 0]$$

$$C_2(1-x) + C_3 x \quad (0, 1)$$

$f(x) = 2x + 1$ can be expressed by $\{N_1, N_0, N_1\}$

The components are:

$$C_1 = -1, \quad C_2 = 1, \quad C_3 = 3.$$

A Simple Variational Method Example (35)

1. (15) Consider the problem: Find $u: [0, 1] \rightarrow \mathbb{R}$ continuous such that

$$\begin{aligned} (1+x^2)u_{xx} + xu_x + x^2u &= 0 \\ u_x(1) - 3u(1) &= 0 \\ u(0) &= 1 \end{aligned}$$

Find the variational equation of the problem using the recipe from the notes, with

$$\mathcal{V} = \{w: [0, 1] \rightarrow \mathbb{R} \text{ smooth } | w(0) = 0\}.$$

Identify essential and natural boundary conditions.

2. (2) Identify the bilinear form and the linear functional of the problem so that the variational equation can be written as $a(u, v) = \ell(v)$. Is a symmetric?
3. Consider a subspace of functions $\mathcal{W}_h = \text{span}\{1, x, x^2, x^3\}$. We want to formulate a variational method with the variational equation in 1 and find its solution.
 - (5) What are the spaces trial and test spaces \mathcal{S}_h and \mathcal{V}_h ? What are the sets of active and constrained indices?
 - (2) Is the method consistent?
 - (7) Find the stiffness matrix and load vector.
 - (3) Find the solution to the variational method, and plot it.
- Hint: Recall that the stiffness matrix $K_{ab} = a(N_b, N_a)$; the order is important for non-symmetric bilinear forms.
4. (1) Is the natural boundary condition satisfied exactly by the solution of the variational method?

1. (a). Form residual function:

$$r(x) = (1+x^2)u_{xx} + xu_{x} + x^2u$$

(b) Multiply by a test function and integrate:

$$\text{Given } V = \{w : [0, 1] \rightarrow \mathbb{R} \text{ smooth} \mid w(0) = 0\}.$$

$$S = \{u : [0, 1] \rightarrow \mathbb{R} \text{ smooth} \mid u(0) = 1\}.$$

The weak form is given as:

$$\int_0^1 r(x)w(x)dx = 0 \quad \forall w \in V$$

$\forall w \in V$

$$\int_0^1 ((1+x^2)u_{xx} + xu_x + x^2u)w(x) dx = 0.$$

$$\Rightarrow \int_0^1 ((1+x^2)u_{xx} - xu_x + x^2u)w(x) dx = 0$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{xx}w_{xx} + \left[(1+x^2)u_{xx}w\right]_0^1 - \int_0^1 xu_xw \\ + \int_0^1 x^2uw = 0.$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{xx}w_{xx} + (1+1)u_{xx}(1)w(1) - \int_0^1 xu_xw \\ + \int_0^1 x^2uw = 0.$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{xx}w_{xx} + 2u_{xx}(1)w(1) - \int_0^1 xu_xw + \int_0^1 x^2uw = 0$$

$$\text{B.C. } u_{xx}(1) = 3u(1)$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{xx}w_{xx} + 6u(1)w(1) - \int_0^1 xu_xw + \int_0^1 x^2uw = 0.$$

$$\Rightarrow \int_0^1 -(1+x^2)u_{xx}w_{xx} - xu_xw + x^2uw dx + 6u(1)w(1) = 0,$$

$$\Rightarrow \int_0^1 (1+x^2)u_{xx}w_{xx} + xu_xw - x^2uw dx - 6u(1)w(1) = 0.$$

The variational form is:

$$\int_0^1 ((+x^2)U_{,x}W_{,x} + xU_{,x}W) - x^2 UW \, dx - b U(1) W(1) = 0 \quad \forall W \in V$$

Nonzero boundary $U_{,x}(1) = 3U(1)$

Essential boundary $U(0) = 1$

2. We have the variational form:

$$\int_0^1 ((+x^2)U_{,x}W_{,x} + xU_{,x}W) - x^2 UW \, dx - b U(1) W(1) = 0 \quad \forall W \in V$$

a. The bilinear form $a(U, W) = \int_0^1 ((+x^2)U_{,x}W_{,x} + xU_{,x}W) - x^2 UW \, dx - b U(1) W(1)$

$a(U, V)$ can be computed for any $U \in S$, $V \in V$. The bilinearity of $a(U, V)$ can be verified as:

$$\begin{aligned} a(U + \alpha V, W) &= \int_0^1 ((+x^2)(U_{,x} + \alpha V_{,x})W_{,x} + \int_0^1 x(U_{,x} + \alpha V_{,x})W \\ &\quad - \int_0^1 x^2 (U + \alpha V)W - b U(1)V(1)) \\ &= \int_0^1 ((+x^2)U_{,x}W_{,x} + xU_{,x}W) - x^2 UW + \\ &\quad [\alpha \int_0^1 ((+x)^2 V_{,x}W_{,x} + xV_{,x}W) - x^2 V W] - b U(1)V(1) \\ \Rightarrow a(U, W) + \alpha a(V, W) \end{aligned}$$

But, this bilinear term is not symmetric. To wit,

$$a(W, U) = \int_0^1 ((+x^2)W_{,x}U_{,x} + xW_{,x}U) - x^2 UW \, dx - b U(1)W(1)$$

$$a(u, w) = \int_0^1 (1+x^2) u_{,x} w_{,x} + x u_{,x} w - x^2 w u \, dx - 6 u(1) w(1)$$

Generally, $\int_0^1 x w_{,x} u \neq \int_0^1 x u_{,x} w$, therefore

$$a(w, u) \neq a(u, w).$$

b. The linear form $f(v)$ can be expressed as

$$f(v) = \int_0^1 \phi f(v) \, dx$$

Obviously, $f(u+\alpha w) = f(u)$ since

$$\int_0^1 \phi f(u+\alpha w) \, dx = 0.$$

$$a(u, w) = \int_0^1 (1+x^2) u_{,x} w_{,x} + x u_{,x} w - x^2 w u \, dx$$

$$- 6 u(1) w(1)$$

$$f(u) = \int_0^1 \phi f(u) \, dx$$

3.(a) We know $W_h = \text{Span}\{1, x, x^2, x^3\}$.

if $w_h \in W_h$, $w_h = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

We know $S_n \subset W_h$ and satisfy direct boundary.

$$S_n = \{u_n \in W_h \mid u_n(0) = 1\}.$$

$$= \{c_0 + c_1 x + c_2 x^2 + c_3 x^3 \mid (c_1, c_2, c_3) \in \mathbb{R}^3\}.$$

c_1, c_2, c_3 are active indices.

c_0 are constrained indices. c_0 should be 1. so that $u_n(0) = 1$.

For V_h , we find the directions of S_h to get

$$V_h = \{ v_h \in W_h \mid v_h(0) = 0 \}.$$

$$= \{ c_1 x + c_2 x^2 + c_3 x^3 \mid (c_1, c_2, c_3) \in \mathbb{R}^3 \}.$$

c_1, c_2, c_3 are active indices,

(b). Because functions in V_h are smooth;

$$V_h \subset V = \{ w : [0, 1] \rightarrow \mathbb{R} \text{ smooth} \mid w(0) = 0 \}.$$

and the method is consistent.

$$(C). \quad W_h = \text{Span}(1, x, x^2, x^3)$$

$$V_h = \text{Span}(x, x^2, x^3)$$

$$S_h = \{ 1 + v_h \mid v_h \in V_h \}.$$

We set $N_1(x) = x$, $N_2(x) = x^2$, $N_3(x) = x^3$, $N_4(x) = 1$.

For dirichlet B.C., we choose $\bar{u}_h(x) = 1 \cdot N_4(x) \in S_h$

With those choices, the equations imposed by the variational methods are.

$$a(bu_h, N_1) = f(N_1) ; \quad a(u_h, N_2) = f(N_2)$$

$$a(u_h, N_3) = f(N_3) \quad \text{and} \quad u_4 = 1$$

Therefore, the stiffness matrix is:

$$K = \begin{bmatrix} a(N_1, N_1) & a(N_2, N_1) & a(N_3, N_1) & a(N_4, N_1) \\ a(N_1, N_2) & a(N_2, N_2) & a(N_3, N_2) & a(N_4, N_2) \\ a(N_1, N_3) & a(N_2, N_3) & a(N_3, N_3) & a(N_4, N_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N_1^1 = 1, \quad N_2^1 = 2x, \quad N_3^1 = 3x^2, \quad N_4^1 = 0$$

$$a(N_1, N_1) = \int_0^1 (1+x^2) N_1^1 N_1^1 + x N_1^1 N_1 - x^2 N_1 N_1 dx$$

$$= \int_0^1 (1+x^2) + x^2 - x^4 dx - 6 N_1(1) N_1(1)$$

$$a(N_2, N_1) = \int_0^1 (1+x^2) 2x \cdot 1 + 2x^3 - x^5 dx - 6 N_2(1) N_1(1)$$

$$a(N_3, N_1) = \int_0^1 3x^2(x^2+1) + 3x^4 - x^6 dx - 6 N_3 N_1$$

$$a(N_4, N_1) = \int_0^1 -x^3 dx - 6 N_4(1) N_1(1)$$

Similarly, we can get all entries of stiffness matrix K .

Then, we can use Matlab to evaluate the integration

$$K = \begin{bmatrix} -68/15, & -25/6, & -138/35, & -25/4 \\ -53/12, & -379/105, & -25/8, & -31/5 \\ -152/35, & -79/24, & -818/315, & -37/6 \\ 0, & 0, & 0, & 1 \end{bmatrix}$$

Float Version:

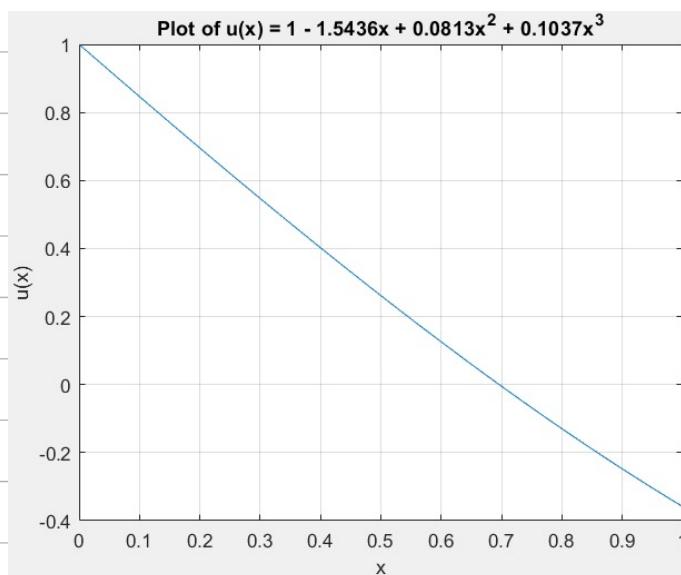
$$\begin{array}{cccc} -4.5333 & -4.1667 & -3.9429 & -6.2500 \\ -4.4167 & -3.6095 & -3.1250 & -6.2000 \\ -4.3429 & -3.2917 & -2.5968 & -6.1667 \\ 0 & 0 & 0 & 1.0000 \end{array}$$

$$F = \begin{bmatrix} \ell(N_1) \\ \ell(N_2) \\ \ell(N_3) \\ u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U = K^{-1}F = \begin{bmatrix} -1.5436 \\ 0.0813 \\ 0.1037 \\ 1.000 \end{bmatrix}$$

$$u(x) = 1 - 1.5436x + 0.0813x^2 + 0.1037x^3$$

The plot looks like:



Q.

$$u_{1x}(1) = -1.5436 + 0.0813 \cdot 28 + 0.1037 \cdot 38^2 \\ = -1.0700$$

$$3u(1) = -1.0758$$

$$|u_{1x}(1) - 3u(1)| = 0.0058$$

Therefore, natural b. c. is not exactly satisfied.



