ME 335A
Finite Element Analysis
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Problems Set #6

May 30, 2023

Due Wednesday, May 31, 2023

On Norms and Convergence (55)

In this problem we would like to play with the convergence, norms, and membership of sequence of functions in different spaces. To this end, let $I = (0, \pi)$, and recall (see Appendix A in the notes) that a function $f\colon I \to \mathbb{R}$ is a member of the following spaces if

$$
\begin{aligned}
f \in L^2(I) &\Leftrightarrow \|f\|_{0,2} = \left(\int_0^\pi f^2 \, dx\right)^{1/2} < \infty \\
f \in L^\infty(I) &\Leftrightarrow \|f\|_{0,\infty} = \max_{x\in I} |f(x)| < \infty \\
f \in H^1(I) &\Leftrightarrow \|f\|_{1,2} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2\right)^{1/2} < \infty.
\end{aligned}
$$

Consider the sequences of functions for $n = 1, 2 \ldots$:

$$
\begin{aligned}
f_n(x) &= n \sin\left(\frac{x}{n}\right), \quad & f_\infty(x) &= x, \\
g_n(x) &= \frac{1}{n} \sin(nx), \quad & g_\infty(x) &= 0, \\
h_n(x) &= \frac{1}{1+nx}, \quad & h_\infty(x) &= 0.
\end{aligned}
$$

1. (5) Plot $f_{2^n}, g_{2^n}, h_{2^n}$ and $f'_{2^n}, g'_{2^n}, h'_{2^n}$ in $(0, \pi)$ for $n = 1, \ldots, 5$.
2. (10) Does $f_\infty \in L^2(I)$? Does $f_\infty \in L^\infty(I)$? Does $f_\infty \in H^1(I)$? Justify.
3. (10) Compute the $L^2(I)$ and $H^1(I)$ norms for f_n for $n < \infty$, and plot them as a function of n.
4. You can check if you want, but f_n, g_n and h_n for $n = 1, \ldots, \infty$ belong to the three spaces $L^2(I)$, $L^\infty(I)$ and $H^1(I)$. Next, we say that a sequence z_n converges to z as $n \to \infty$ in a normed space V if $\|z_n - z\| \to 0$ as $n \to \infty$. For the following calculations, we encourage you to use Mathematica, Maple, or Matlab to either perform integrals analytically, or numerically for each

value of n and plot the resulting trends. For the L^∞-cases below, you may plot the absolute value of the error for a few values of n, and argue based on the plots.

This problem is a little bit laborious, but it will show you three different sequences that converge is some spaces and not in others, so it is an instructive exercise.

(a) (10) Evaluate if $f_n \to f_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$.

(b) (10) Evaluate if $g_n \to g_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$. Reflect on what you see if you want, by writing one or two sentences about it.

(c) (10) Evaluate if $h_n \to h_\infty$ in $L^2(I)$, in $L^\infty(I)$ and/or in $H^1(I)$. Does $h_n(0) \to h_\infty(0)$?

On Interpolation Errors (70)

Consider the interval $\Omega = [-1, 1]$, and a mesh of $n_{\text{el}} \in \mathbb{N}$ equally long P_k-elements on it, for $k = 1, 2, 4$. The Lagrange finite element interpolant $\mathcal{I}u$ with these elements is constructed through (3.24) in the notes.

For $\omega \in \mathbb{R}$, consider the functions

$$v_\omega(x) = \cos(\omega x)$$

$$w_\omega(x) = \begin{cases} 0 & x < 0 \\ x^\omega & x \geq 0. \end{cases}$$

1. For $u = v_{30}$, $k = 1, 2, 4$, and $n_{\text{el}} = 2^i$ with $i = 3, 4, 5, 6$.

   (a) (10) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be? Explain.

   (b) (10) Plot $\|u' - (\mathcal{I}u)'\|_{0,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be? Explain.

2. For $u = w_1$, $k = 1, 2, 4$, and $n_{\text{el}} = 2^i + 1$ with $i = 3, 4, 5, 6$.

   (a) (10) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be?

   (b) (10) Plot $|u - \mathcal{I}u|_{1,2,\Omega}$ for $k = 1, 2, 4$ in the same plot. Compute the convergence rate for each k for i large enough. Are they approximately what you would expect them to be?

   (c) (5) What is the value of $\|u - \mathcal{I}u\|_{0,2,\Omega}$ when n_{el} is even? Can you elaborate on the reasons behind the differences between the last two questions and this one?

3. (10) Compare $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $u = v_{30}$ and $u = v_{60}$ for $k = 2$ and $n_{\text{el}} = 200$. Which one is larger? Why?

4. Let's now examine what happens with the convergence rate if we do not exactly satisfy the essential boundary conditions, as we generally have to do in 2D and 3D problems. To this end, let's construct an interpolant $\mathcal{I}u$ of $u = v_1$ that at $x = 1$ has the value $\cos(1) + h^m$ for $m = 1/2, 1, 2, 4$, instead of being $\mathcal{I}u(1) = v_1(1) = \cos(1)$, as the standard interpolant would. For $k = 1$ and $n_{\text{el}} = 2^i + 1$ with $i = 3, 4, 5, 7$:

(a) (5) Plot $\|u - \mathcal{I}u\|_{0,2,\Omega}$ for $m = 1, 2, 4$ in the same plot. Compute the convergence rate for each m for i large enough.

(b) (5) Plot $\|u' - (\mathcal{I}u)'\|_{0,2,\Omega}$ for $m = 1, 2, 4$ in the same plot. Compute the convergence rate for each m for i large enough.

(c) (5) Reflecting on the convergence result in the fundamental approximation theorem (Cea's lemma), which states that

$$\|u - u_h\|_1 \leq \left(1 + \frac{M}{\alpha}\right) \min_{w_h \in \mathcal{S}_h} \|u - w_h\|_1,$$

explain how the results that you observed in part 4a and/or 4b could affect the convergence rate.