Chapter 4

Linear Elasticity

In Chapters 1 and 2 we have followed a certain path of which the starting point was always the strong form of the problem, that is, the differential equation and the boundary conditions. From there, we inferred a weak form and finally to the Galerkin method, which is simply the restriction of a problem in weak form to a finite-dimensional subspace. The Finite Element Method is the combination of the Galerkin method (and variants thereof) with some specially crafted subspaces, the Finite Element spaces.

In this chapter dedicated to linear elasticity we will follow another presentation path, known as *variational* path. This path does not start from a differential equation, but from a *variational principle*. The variational principle that governs the behavior of many mechanical systems is the **minimization of the energy**. The static deformation of an elastic body, in particular, minimizes the **potential energy**, as we will soon see. The displacement of each tiny piece, be it located near the application of the load or far away from it, is dictated by this principle.

From a variational principle it is possible to deduce a weak form of the problem. It is even easier than when we start from the differential equation. Once we arrive at the weak form, the rest of the procedure is as before: Formulate the Galerkin method and propose Finite Element spaces for it.

4.1 The Variational Problem of Linear Elasticity

The displacement field. Consider the problem sketched in Fig. 4.1. A solid body occupies the domain Ω, with boundary $\partial\Omega$. Along the **Dirichlet** part of the boundary ($\partial\Omega_D$) a displacement $\mathbf{g}$ is imposed to the particles of the body, while along the **Neumann** part ($\partial\Omega_N$) a distribution of forces $\mathbf{H}$ is applied. In addition, a body force $\mathbf{b}$ loads the body.

Under these conditions, the body will deform. The material particles will change position. This is expressed mathematically by a **displacement field** $\mathbf{u}(\mathbf{x})$, so that the displacement induced by the load on the particle at $\mathbf{x}$ is

$$\mathbf{x} \quad \mapsto \quad \mathbf{x} + \mathbf{u}(\mathbf{x}) \,. \tag{4.1}$$

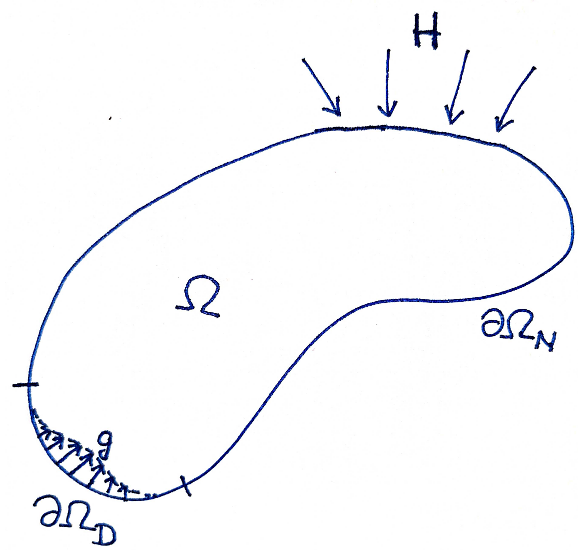

Figure 4.1 A sketch of an elasticity problem. The Dirichlet and Neumann boundaries, $\partial\Omega_D$ and $\partial\Omega_N$, are indicated.

Our quest is to determine the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$. The unknown is thus a vector field over Ω, which can also be seen as d (two) unknown functions $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$, since $\mathbf{u} = (u_1, u_2)^T$ or, equivalently,

$$\mathbf{u}(\mathbf{x}) = u_1(\mathbf{x})\,\mathbf{e}_1 + u_2(\mathbf{x})\,\mathbf{e}_2 \ .$$

The space $\mathscr{W}$ over which we will seek the physically valid solution $\mathbf{u}$ is, thus,

$$\mathscr{W} = \{\mathbf{w} : \Omega \to \mathbb{R}^d \mid \mathbf{w} \text{ is a smooth vector field}\} \ . \tag{4.2}$$

The minimum smoothness required will be discussed further along.

Notice that we already know the value of $\mathbf{u}$ ($= \mathbf{g} = (g_1, g_2)^T$) along $\partial\Omega_D$, but we do not know how this displacement is "distributed" over Ω, or along $\partial\Omega_N$. We define the **trial space** $\mathscr{S}$ as

$$\mathscr{S} = \{\mathbf{w} \in \mathscr{W} \mid \mathbf{w} = \mathbf{g} \text{ on } \partial\Omega_D\} \ . \tag{4.3}$$

The Principle of Minimum Potential Energy. Instead of stating a differential equation to determine $\mathbf{u}$, we will invoke a **variational principle**, in particular, the **principle of minimum potential energy**. The solid is modeled as endowed with an **internal energy** U (also called **strain energy**) which depends solely on the displacement field $\mathbf{u}$. Then the **potential energy** V is defined as

$$V(\mathbf{u}) = U(\mathbf{u}) - \int_\Omega \mathbf{b} \cdot \mathbf{u} \, d\Omega - \int_{\partial\Omega_N} \mathbf{H} \cdot \mathbf{u} \, d\partial\Omega \ . \tag{4.4}$$

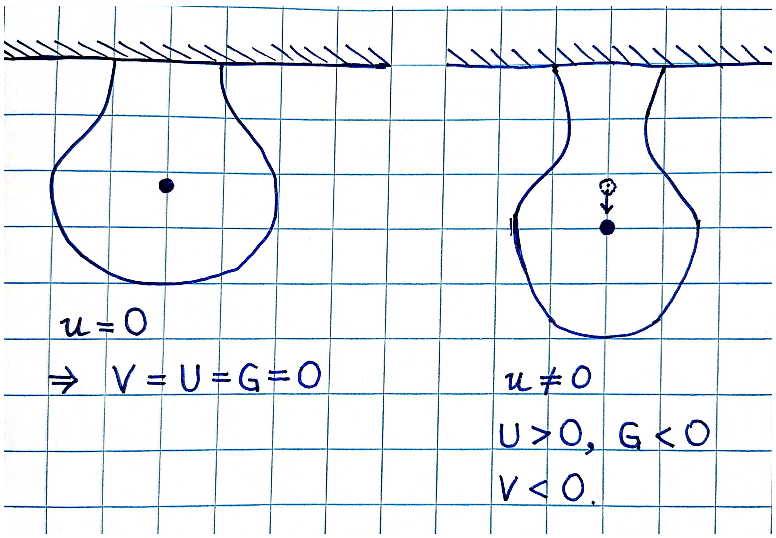

Figure 4.2 An elastic body hanging from the ceiling. On the left we have the reference configuration, which would correspond to gravity being "turned off". On the right we have the equilibrium configuration, which minimizes the potential energy V. Under the load of gravity the body deforms, increasing its strain energy, but this increase is more than compensated by the decrease in gravitational energy, evidenced by the lowering of the center of mass. The equilibrium configuration has less potential energy than the reference one.

The potential energy considers both the strain energy and the work done by the external loads.

The principle reads: *The* **equilibrium solution u** *minimizes the potential energy V among all smooth displacement fields* **w** *that are equal to* **g** *on* $\partial\Omega_D$. In mathematical terms,

$$V(\mathbf{u}) \leq V(\mathbf{w}), \qquad \text{for all } \mathbf{w} \in \mathscr{S},\ \mathbf{w} \neq \mathbf{u}. \tag{4.5}$$

As an example, consider an elastic body which has a part of its boundary attached to the ceiling (see Fig. 4.2). The body is hanging from its upper fixation. The only applied load is the body's own weight $\mathbf{b} = -\rho\, g\, \mathbf{e}_2$, thus

$$V(\mathbf{u}) = U(\mathbf{u}) + \rho\, g \int_\Omega u_2 \, d\Omega .$$

The actual solution results from a compromise. The second term (which is nothing but the gravitational energy, denoted by G) decreases as the particles of the body displace downwards, but since the body is fixed at the top any downward displacement generates **strains** that make the internal energy increase. The strain energy function U depends on the material of the body, and so does the solution (usually termed **equilibrium solution**).

The small deformation hypothesis (SDH). We adopt here the SDH, which assumes that the data **g**, **H** and **b** are small enough that **u** and its gradient

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} \tag{4.6}$$

are very very small at all points of Ω. So small that the deformed domain essentially coincides with Ω.

Why bother to compute **u** if by hypothesis it is negligible at all points? It happens that solid materials are quite stiff, so that with just tiny displacements they can generate forces that equilibrate significant loads. As an example, consider a 1-m long steel wire with cross-sectional area of 10^{-4} m^2. If it is loaded with 1000 N the maximum displacement is (assuming a Young modulus $E = 2 \times 10^{11}$ Pa)

$$\Delta \ell = \frac{P\ell}{AE} = \frac{1000\,\mathrm{N} \times 1\,\mathrm{m}}{10^{-4}\,\mathrm{m}^2 \times 2 \times 10^{11}\,\mathrm{Pa}} = 5 \times 10^{-5}\,\mathrm{m} = 50\,\mathrm{microns}\ .$$

Consider the decomposition of $\nabla \mathbf{u}$ into symmetric (ε) and anti-symmetric (ω) parts, i.e.,

$$\nabla \mathbf{u} = \varepsilon(\nabla \mathbf{u}) + \omega(\nabla \mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) + \frac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right)\ . \tag{4.7}$$

Under the SDH, it can be shown that the tensor (or matrix) ε measures the **local deformation** (or **strain**) of the solid, while ω measures the **local rotation**.

The strain energy of a linearly elastic body under the SDH. If the material is **isotropic**, the strain energy takes the form

$$U(\mathbf{u}) = \int_\Omega \frac{E}{2(1+\nu)} \left(\varepsilon(\nabla \mathbf{u}) : \varepsilon(\nabla \mathbf{u}) + \frac{\nu}{1-2\nu} (\operatorname{div} \mathbf{u})^2 \right) d\Omega\ , \tag{4.8}$$

where E is Young's modulus, ν is the Poisson coefficient, and ":" stands to the double contraction of two tensors, or equivalently the Frobenius product of two matrices:

$$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}\ .$$

The strain energy always is minimal (and equal to zero) when the displacement is equal to zero everywhere. The integrand in (4.8) is the **strain energy density** at each point **x**, which is zero if and only if $\nabla \mathbf{u}(\mathbf{x})$ is antisymmetric.

If the material is not isotropic the corresponding expression is less appealing. The constitutive coefficients E and ν turn into a fourth-order array of coefficients C_{ijkl}, and then

$$U(\mathbf{u}) = \frac{1}{2} \int_\Omega \sum_{i,j,k,l} C_{ijkl}\, \varepsilon_{ij}(\nabla \mathbf{u})\, \varepsilon_{kl}(\nabla \mathbf{u})\, d\Omega\ . \tag{4.9}$$

In what follows we will restrict to isotropic materials, so that the two coefficients E and ν (which may depend on **x**) characterize the elastic properties of the material.

Problem 4.1 ("Primal" Variational Form of the Isotropic Linear Elasticity Problem)**.** *Given the domain Ω and the data E, ν, $\mathbf{b}$, $\mathbf{g}$ and $\mathbf{H}$, determine the smooth vector field $\mathbf{u} \in \mathscr{S}$ that minimizes V over $\mathscr{S}$, where for a generic $\mathbf{w} \in \mathscr{W}$ the potential energy function is*

$$V(\mathbf{w}) = \int_\Omega \frac{E}{2(1+\nu)} \left(\varepsilon(\nabla \mathbf{w}) : \varepsilon(\nabla \mathbf{w}) + \frac{\nu}{1-2\nu} (div\ \mathbf{w})^2\right) d\Omega - \int_\Omega \mathbf{b} \cdot \mathbf{w}\, d\Omega - \int_{\partial\Omega_N} \mathbf{H} \cdot \mathbf{w}\, d\partial\Omega\,. \tag{4.10}$$

By replacing the first term above by (4.9) one obtains the primal variational form for anisotropic materials.

Example 4.1 A sphere under pressure. What is the deformation of a spherical homogeneous elastic body when a radial force $\mathbf{H} = -p\,\check{\mathbf{e}}_r$ is applied to its surface? The displacement will be radially symmetric, i.e.,

$$\mathbf{u}(\mathbf{x}) = \varphi(r)\,\check{\mathbf{e}}_r.$$

From the expression of the gradient in spherical coordinates we know that

$$\varepsilon(\nabla \mathbf{u}) : \varepsilon(\nabla \mathbf{u}) = \varphi'(r)^2 + 2\frac{\varphi(r)^2}{r^2}$$

and that

$$\operatorname{div} \mathbf{u} = \varphi'(r) + 2\,\frac{\varphi(r)}{r}\,.$$

The physical restriction $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ translates into $\varphi(0) = 0$. The equilibrium displacement of the sphere will thus be given by the function φ that minimizes, over the set of smooth functions

$$\mathscr{S} = \mathscr{V} = \left\{\varphi : [0,R] \to \mathbb{R} \mid \varphi(0) = 0\right\},$$

the potential energy

$$\begin{aligned} V(\varphi) \quad &= \quad \frac{E}{2(1+\nu)} \int_0^R \left(\varphi'(r)^2 + 2\frac{\varphi(r)^2}{r^2} + \frac{\nu}{1-2\nu}\left(\varphi'(r) + 2\,\frac{\varphi(r)}{r}\right)^2\right) 4\pi r^2 dr \ + \\ &\quad + 4\pi p R^2 \varphi(R)\,. \end{aligned} \tag{4.11}$$

It turns out that the exact minimizer is of the form $\varphi(r) = A\,r$ for some $A \in \mathbb{R}$ that depends on p. If we assume that $\mathscr{S}$ only contains such functions, then V becomes a function of A, that is

$$V = \gamma A^2 + 4\pi p R^3 A, \qquad \text{with} \qquad \gamma = \frac{2\pi E R^3}{1+\nu}\left(1 + \frac{3\nu}{1-2\nu}\right) = \frac{2\pi E R^3}{1-2\nu}\,.$$

The minimum takes place for

$$A = -\frac{4\pi p R^3}{2\gamma} = -\,\frac{1-2\nu}{E}\,p\,. \tag{4.12}$$

We have thus our first solution of a linear elastic problem. The displacement field is

$$\mathbf{u}(\mathbf{x}) = A\,r\,\check{\mathbf{e}}_r = A\mathbf{x} = -\frac{1-2\nu}{E}\,p\,\mathbf{x}.$$

Notice that if $\nu = \frac{1}{2}$ then $A = 0$ and thus $\mathbf{u} = \mathbf{0}$ at all points. The sphere does not contract under applied pressure. The limit $\nu \to \frac{1}{2}$ is called the "incompressible limit."

We cannot yet prove that this is the exact solution, but we can at least confirm that the polynomial $A\,r$ minimizes V over all **quadratic** polynomials. For this, we take $\varphi(r) = A\,r + B\,r^2$, with unknown coefficients A and B. Inserting this φ into (4.11) we obtain V as a function of A and B, namely

$$V(A,B) = \frac{2\pi E}{1-2\nu}\left[R^3\,A^2 + 2R^4\,A\,B + \frac{6+4\nu}{5+5\nu}\,R^5\,B^2\right] + 4\pi R^3\,p\,(A + R\,B)$$

By equating $\partial V/\partial A = \partial V/\partial B = 0$ you can check that the minimum takes place when A takes the value given in (4.12) and $B = 0$.

Remark: If we consider a 2D sphere (a circle), then

$$\varepsilon(\nabla\mathbf{u}) : \varepsilon(\nabla\mathbf{u}) = \varphi'(r)^2 + \frac{\varphi(r)^2}{r^2} \qquad \text{and} \qquad \operatorname{div}\mathbf{u} = \varphi'(r) + \frac{\varphi(r)}{r}\,.$$

This modifies the expression of $V(A)$ to

$$V = \frac{\pi E R^2}{(1+\nu)(1-2\nu)}\,A^2 + 2\pi p R^2\,A,$$

and the value of A that corresponds to the minimum is

$$A = -\frac{(1+\nu)(1-2\nu)}{E}\,p\,. \tag{4.13}$$

Under the same pressure, a circle deforms less than a sphere.

4.2 From the Variational Form to the Weak Form

How to obtain a weak form when starting from a variational principle? It is quite straightforward. The procedure is based on the following rather abstract theorem.

Theorem 4.1. *Let $\mathscr{W}$ be a* **vector space** *(it could be of functions, of vector fields, etc.), and let $\mathscr{S}$ be an affine subspace of $\mathscr{W}$. The* **direction** *of $\mathscr{S}$ is denoted by $\mathscr{V}$. Assume that:*

a) *An energy function V is defined on $\mathscr{W}$ which can be written as*

$$V(w) = \frac{1}{2}a(w,w) - \ell(w) \tag{4.14}$$

where a is a symmetric bilinear form satisfying that

$$a(v,v) > 0 \qquad \forall\, v \in \mathscr{V}, \quad v \neq 0,$$

and ℓ is a linear form.

b) *There exists a minimizer u of V over $\mathscr{S}$. Precisely, there exists $u \in \mathscr{S}$ satisfying*

$$V(u) \leq V(w), \qquad \forall\, w \in \mathscr{S}\,. \tag{4.15}$$

Then, u is the unique minimizer in $\mathscr{S}$ (i.e., $V(u) < V(w)$, $\forall\, w \neq u$). In addition, u is also the unique element of $\mathscr{S}$ satisfying

$$a(u,v) = \ell(v) \qquad \forall\, v \in \mathscr{V}\,. \tag{4.16}$$

Proof. First let us prove that u necessarily satisfies (4.16). For this, assume that there exists some particular $0 \neq v \in \mathscr{V}$ for which

$$a(u,v) - \ell(v) = \beta \neq 0\,.$$

We will show that then u is not a minimizer of V over $\mathscr{S}$. Let us define $\alpha = a(v,v) > 0$ (because of hypothesis (a)) and

$$w = u - \frac{\beta}{\alpha}\, v\,.$$

Using the linearity of $a(\cdot,\cdot)$ and ℓ and the symmetry of $a(\cdot,\cdot)$ we see that

$$\begin{aligned}
V(w) &= \frac{1}{2} a\left(u - \frac{\beta}{\alpha}\, v, u - \frac{\beta}{\alpha}\, v\right) - \ell\left(u - \frac{\beta}{\alpha}\, v\right) \\
&= \frac{1}{2} a(u,u) - \ell(u) - \frac{\beta}{\alpha} \underbrace{(a(u,v) - \ell(v))}_{\beta} + \frac{\beta^2}{2\alpha^2} \underbrace{a(v,v)}_{\alpha} \\
&= V(u) - \frac{\beta^2}{2\alpha} \\
&< V(u)\,.
\end{aligned}$$

This proves (4.16). Now assume that there exists another element of $\mathscr{S}$, let us call it $\overline{u}$, that also satisfies $a(\overline{u}, v) = \ell(v)$ for all $v \in \mathscr{V}$. Then,

$$\begin{aligned}
a(u - \overline{u}, u - \overline{u}) &= a(u, \underbrace{u - \overline{u}}_{\in \mathscr{V}}) - a(\overline{u}, \underbrace{u - \overline{u}}_{\in \mathscr{V}}) \\
&= 0 \; - \; 0 \qquad = 0\,.
\end{aligned}$$

According to hypothesis (a) this implies $u - \overline{u} = 0$ and thus $u = \overline{u}$.

This last argument also proves that u is the unique minimizer, since the existence of two minimizers would imply the existence of two solutions of (4.16) and that has been shown to be impossible. □

This theorem, though elementary, has interesting consequences. Notice that very little is said about the space $\mathscr{W}$. It could be finite or infinite dimensional, for example. It may or not be a normed space. The bilinear and linear forms are not required to be continuous. The hypotheses on the spaces involved are very weak, but on the other hand we make the strong assumption that a unique minimizer exists. Sometimes physical reasons make us believe that a unique minimum exists, though a mathematical proof could be unavailable. Under this assumption, the theorem provides us with **a weak form of the problem**, namely $a(u,v) = \ell(v), \quad \forall v \in \mathscr{V}$.

Applying this theorem to Problem (4.1) we obtain:

Problem 4.2 (Weak form of the Isotropic Linear Elasticity Problem)**.** *Given the domain* Ω *and the data* E, ν, $\mathbf{b}$, $\mathbf{g}$ *and* $\mathbf{H}$, *determine the smooth vector field* $\mathbf{u} \in \mathscr{S}$ *(i.e.,* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega_D$*) such that*

$$a(\mathbf{u},\mathbf{v}) = \ell(\mathbf{v}) \tag{4.17}$$

for all $\mathbf{v} \in \mathscr{V}$, *where*

$$a(\mathbf{w},\mathbf{v}) = \int_\Omega \frac{E}{1+\nu}\left(\varepsilon(\nabla\mathbf{w}) : \varepsilon(\nabla\mathbf{v}) + \frac{\nu}{1-2\nu} div\, \mathbf{w}\; div\, \mathbf{v}\right) d\Omega, \tag{4.18}$$

$$\ell(\mathbf{v}) = \int_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\partial\Omega_N} \mathbf{H}\cdot\mathbf{v}\, d\partial\Omega\,. \tag{4.19}$$

We should check that the hypotheses of Theorem 4.1 indeed hold true. It is readily seen that $V(\mathbf{w}) = \frac{1}{2}a(\mathbf{w},\mathbf{w}) - \ell(\mathbf{w})$. A little more subtle is to prove that $a(\mathbf{v},\mathbf{v}) > 0$ for all $\mathbf{v} \in \mathscr{V}$. This requires that $E > 0$ and $0 < \nu < \frac{1}{2}$, which is true for all materials. It also requires that $\mathscr{V}$ does not contain any *rigid mode* (infinitesimal translations/rotations, which have $\varepsilon = 0$).

Then, the theorem tells us that Problem 4.2 has **as unique solution** the displacement field $\mathbf{u}$ that minimizes the potential energy V (assumed to exist).

The stress field. The integrand in (4.18) can also be written as

$$\sigma(\nabla\mathbf{u}) : \varepsilon(\nabla\mathbf{v})$$

where σ is the **Cauchy stress tensor**, or simply **stress tensor**,

$$\sigma = \frac{E}{1+\nu}\,\varepsilon(\nabla\mathbf{u}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\,(\underbrace{\mathrm{div}\,\mathbf{u}}_{=\mathbf{I}:\,\nabla\mathbf{u}})\,\mathbf{I}\,. \tag{4.20}$$

In many cases the elasticity problem is solved mainly looking for the stress field over the body, since too high stresses may lead to the failure of the material.

> **Example 4.2 The stress field inside a sphere under pressure.** We saw in Example 4.1 that the equilibrium displacement field of a 2D sphere (circle) under uniform pressure is
>
> $$\mathbf{u}(\mathbf{x}) = A\,\mathbf{x}, \quad \text{where} \quad A = -\frac{(1+\nu)(1-2\nu)\,p}{E}$$

and **x** has the center of the circle as origin. This equation is intrinsic, valid for all coordinate systems. We can thus express it in **Cartesian** coordinates $x_1 - x_2$ so that $\mathbf{x} = (x_1, x_2)^T$. Then $u_1 = Ax_1$ and $u_2 = Ax_2$ and thus

$$\nabla\mathbf{u} = \varepsilon(\nabla\mathbf{u}) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = A\mathbf{I}, \quad \omega(\nabla\mathbf{u}) = 0, \quad \text{div } \mathbf{u} = 2A.$$

Inserting these values into (4.20) we obtain the corresponding stress field:

$$\sigma = \frac{EA}{1+\nu}\mathbf{I} + \frac{2E\nu A}{(1+\nu)(1-2\nu)}\mathbf{I} = \frac{EA}{(1+\nu)(1-2\nu)}\mathbf{I} = -p\mathbf{I} = \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix}$$

The stress field is **homogeneous**. It does not depend on **x**. It is called **spherical** (or **hydrostatic**) because at all points it is a multiple of the identity tensor/matrix.

The strong form. The weak form is all we need to set up a Galerkin method to approximate the exact solution **u**. Let us however briefly talk about the strong form of the problem. As usual, it is obtained by integrating by parts the weak form. Since σ is symmetric, it holds that

$$\int_\Omega \sigma : \varepsilon(\nabla\mathbf{v}) = \int_\Omega \sigma : \nabla\mathbf{v} = \int_{\partial\Omega} (\sigma \cdot \check{\mathbf{n}}) \cdot \mathbf{v} d\partial\Omega - \int_\Omega (\text{div } \sigma) \cdot \mathbf{v}\, d\Omega\,.$$

From this, remembering that $\mathbf{v} = 0$ on $\partial\Omega_D$, we arrive at

$$0 = a(\mathbf{u}, \mathbf{v}) - \ell(\mathbf{v}) = \int_\Omega (-\text{div } \sigma - \mathbf{b}) \cdot \mathbf{v}\, d\Omega + \int_{\partial\Omega_N} (\mathbf{H} - \sigma \cdot \check{\mathbf{n}}) \cdot \mathbf{v}\, d\partial\Omega\,,$$

for all $\mathbf{v} \in \mathcal{V}$. From this we conclude that **u** is also the solution of the following differential problem.

Problem 4.3 (Strong form of the Isotropic Linear Elasticity Problem)**.** *Given the same data as in Problems 4.1 and 4.2, find a smooth vector field* **u** *satisfying*

$$\begin{aligned} \mathit{div}\, \sigma(\nabla\mathbf{u}) + \mathbf{b} &= 0 \quad \mathit{in}\ \Omega, && (4.21)\\ \mathbf{u} &= \mathbf{g} \quad \mathit{on}\ \partial\Omega_D, && (4.22)\\ \sigma(\nabla\mathbf{u}) \cdot \check{\mathbf{n}} &= \mathbf{H} \quad \mathit{on}\ \partial\Omega_N, && (4.23)\end{aligned}$$

where σ *is defined by (4.20).*

Equation (4.21) is also known as **equation of static equilibrium**. It expresses the local equilibrium of forces at each point of the domain, irrespective of the material being linearly elastic or not. Of course, if the material is not linearly elastic the expression for σ is different from (4.20).

It is important to internalize that Problems 4.1, 4.2 and 4.3 are essentially equivalent. Each one of them totally determines **u**.

Example 4.3 The exact solution of the problem of a sphere under pressure. It is easy to verify that the stress field $\sigma = -p\mathbf{I}$ computed in Example 4.2 is a solution to Problem 4.3. In fact, since σ is independent of $\mathbf{x}$, div $\sigma = 0$, which is consistent with (4.21) because $\mathbf{b} = 0$. Also, since the boundary condition is $\mathbf{H} = -p\check{\mathbf{e}}_r$ and $\check{\mathbf{n}} = \check{\mathbf{e}}_r$, it follows that $\sigma \cdot \check{\mathbf{n}} = \mathbf{H}$ all over the boundary. Then the displacement field $\mathbf{u}$ calculated in Example 4.1 is indeed the unique solution (up to a rigid motion) of the problem of an isotropic linear elastic sphere under pressure.

The exact solution of the problem of a rectangle under pressure. Assume that the body subject to a uniform pressure p over its surface is the rectangle $\Omega = [-\frac{W}{2}, \frac{W}{2}] \times [-\frac{H}{2}, \frac{H}{2}]$. The force imposed by the pressure is $\mathbf{H} = -p\check{\mathbf{n}}$ at all points, so that $\mathbf{H}$ takes the value $-p\check{\mathbf{e}}_1$ at the East boundary, $p\check{\mathbf{e}}_1$ at the West one, $-p\check{\mathbf{e}}_2$ at the North one, and $p\check{\mathbf{e}}_2$ at the South one. The same displacement and stress fields (though now defined in the rectangular domain Ω) that solve the elastic problem for the sphere also solve it for the rectangle, any rectangle. In fact, **for any shape**.

4.3 Galerkin method

The Galerkin method to approximate $\mathbf{u}$ is built directly from Problem 4.2, as we have done in all previus chapters. Assuming that a finite dimensional space $\mathscr{W}_h \subset \mathscr{W}$ has been selected, it reads:

Galerkin discrete problem: *Find* $\mathbf{u}_h \in \mathscr{S}_h \subset \mathscr{S}$ *such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) = \ell(\mathbf{v}_h) \tag{4.24}$$

for all $\mathbf{v}_h \in \mathscr{V}_h \subset \mathscr{V}$*, where* $a(\cdot,\cdot)$ *and* $\ell(\cdot)$ *are given by (4.17) and (4.19, respectively.*

Expressed in this abstract form the only difference with our previous encounters with the Galerkin method is the bold characters used to denote the solution $\mathbf{u}_h$ and the test function $\mathbf{v}_h$. This is just a purely notational innovation we have introduced to remind us that $\mathscr{S}_h$ and $\mathscr{V}_h$ are spaces of **vector fields**.

The requirement $\mathscr{W}_h \subset \mathscr{W}$ makes it necessary to better characterize the space $\mathscr{W}$. In terms of Sobolev spaces, it is not difficult to see that

$$\mathscr{W} = \mathbf{H}^1(\Omega) = \left\{\mathbf{w} = (w_1, w_2)^T : \Omega \to \mathbb{R}^2 \mid w_1 \in H^1(\Omega),\ w_2 \in H^1(\Omega)\right\} \tag{4.25}$$

which is a normed space with

$$\|\mathbf{w}\|_{\mathscr{W}} = \sqrt{\|w_1\|^2_{H^1(\Omega)} + \|w_2\|^2_{H^1(\Omega)}}\,. \tag{4.26}$$

In fact, $a(\cdot,\cdot)$ has the necessary properties of continuity and coercivity in $\mathbf{H}^1(\Omega)$.