HW 3.

In this exercise, we consider the following problem in its weak from:

Problem (Weak Form). *Given $f\colon [0,1] \to \mathbb{R}$, find $u \in \mathcal{V} = \{w\colon [0,1] \to \mathbb{R}$ smooth $\mid w(0) = 0\}$ such that*

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \qquad \text{(W)}$$

for any $v \in \mathcal{V}$.

We set $f(x) = \cos(4\pi x)$, and will examine the use of a variational method to approximate the solution to this problem that uses variational equation (W).

1. (10) Is $u(x) = -\frac{\sin(2\pi x)^2}{8\pi^2}$ the exact solution to this problem ? Justify.

   Hint: It is convenient to replace in the weak form and integrate by parts to verify it is satisfied for any $v \in \mathcal{V}$. Alternatively, you can compute the Euler-Lagrange equations and see if $u(x)$ satisfies them.

2. Consider the set of functions $\mathcal{W}_h = \{w = \sum_{k=0}^{n} c_k i P_k(x) \mid (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}\}$. We will denote with ${}^n u_h$ the solution of the variational method for a given n.

   (a) (15) Define $\mathcal{V}_h, \mathcal{S}_h, a(\cdot,\cdot), \ell(\cdot), \eta_a$ and η_g. State also the equations that define the solution of the variational method, compute the entries in the stiffness matrix, and the entries in the load vector

   (b) (15) Compute ${}^1u_h$, ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^1u_h$, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

   (c) (5) Is ${}^n u_h$ "visually converging" to u as n grows?

3. Consider next the set of functions $\mathcal{W}_h = \{w = \sum_{k=2}^{n} c_k i P_k(x) \mid (c_2, \ldots, c_n) \in \mathbb{R}^{n+1}\}$ (notice that k begins at 2 instead of at 0).

   (a) (15) Compute ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

   (b) (5) Is ${}^n u_h$ "visually converging" to u as n grows? Do you think that your answer to this question will change if we keep increasing n beyond 7?

1. The variational equation is:

$$\int_0^1 u'(x)\,v'(x)\,dx - \int_0^1 f(x)\,v(x)\,dx = 0.$$

We ~~can~~ eliminate $v'(x)$ ~~by~~ integrating by parts.

$$[u'(x)\,v(x)]\Big|_0^1 - \int_0^1 u''(x)\,v(x)\,dx - \int_0^1 f(x)\,v(x)\,dx = 0$$

$u'(1)v(1) - u'(0)\cancel{v(0)} - \int_0^1 (u''(x) + f(x))\, v(x)\, dx = 0.$

$\forall v \in V$, the equation above holds, we can have the following Euler-Lagrange equations:

$$u''(x) + f(x) = 0, \qquad x \in [0, 1]$$

$$u'(1) = 0$$

The function solution must satisfy the EL above. Obviously, we can check:

$$u(x) = \frac{-\sin(2\pi x)^2}{8\pi^2} \qquad f(x) = \cos(4\pi x)$$

$$u''(x) + f(x) = -\cos^2(2\pi x) + \sin^2(2\pi x) + \cos(4\pi x)$$
$$= -\cos(4\pi x) + \cos(4\pi x)$$
$$= 0$$

$$u'(1) = -\frac{\cos(2\pi x)\sin(2\pi x)}{2\pi}\Big|_1 = -\frac{\cos(2\pi)\sin(2\pi)}{2\pi} = 0$$

Satisfying $u'(1) = 0$, and $u''(x) + f(x) = 0$.

2(a) We define:

$$S_n = \{ w_n \in W_n \mid w_n(0) = 0 \}$$
$$= \{ w = \textstyle\sum_{k=0}^{n} c_k i P_k(x) \mid (c_0, \dots, c_n) \in \mathbb{R}^{n+1} \}.$$
$$V_n = S_n = \{ v_n \in W_n \mid v_n(0) = 0 \}.$$
$$= \{ v = \textstyle\sum_{k=0}^{n} c_k i P_k(x) \mid (c_0, \dots, c_n) \in \mathbb{R}^{n+1} \}.$$

$D_a = \{c_0, \dots, c_n\}$

D_g is an empty set.

Let $N_a = i P_a(x)$ we can express $u_h = \Sigma_{a=0}^{n} c_a N_a$

$v_h = \Sigma_{a=0}^{n} c_b N_b$

Bilinear term is:

$$\int_0^1 u_h' v_h' = \Sigma_{a=0}^{n} c_a a(N_a', N_b')$$

We ~~know~~ $N_a' = \left(\int_0^x P_a(y)\,dy\right)' = P_a(x)$

Since $\int_0^1 P_n(x) P_m(x)\,dx = \int_0^1 N_n'(x) N_m'(x)\,dx = a(N_n', N_m')$

and $\int_0^1 P_n(x) P_m(x)\,dx = \delta_{nm}$, we can simplify

bilinear term as:

$$k_{ab} = a(N_a', N_b') = \delta_{ab} \quad , \forall a = \{0, 1, \dots n\}$$
$$b = \{0, 1, \dots, n\}.$$

For linear functional $l(\cdot)$, we set $f(x) = \cos(4\pi x)$.

$$\int_0^1 f(x) N_b = \int_0^1 \cos(4\pi x) \cdot i P_a(x). \quad \forall a = \{0, 1, \dots, n\}$$

The discretized version of weak solution is:

$$\sum_{b=0}^{n} a(N_b, N_a) u_b = l(N_a) \qquad a = \{0, \dots, n\}$$

Therefore, we can form: $KU = F$

where $K_{ab} = \delta_{ab}$; $f_a = \int_0^1 \cos(4\pi x) i P_a(x)$

(b) ① When $k = 1$.

$$F = \begin{bmatrix} 0 \\ 0.0219 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

② When $k = 3$

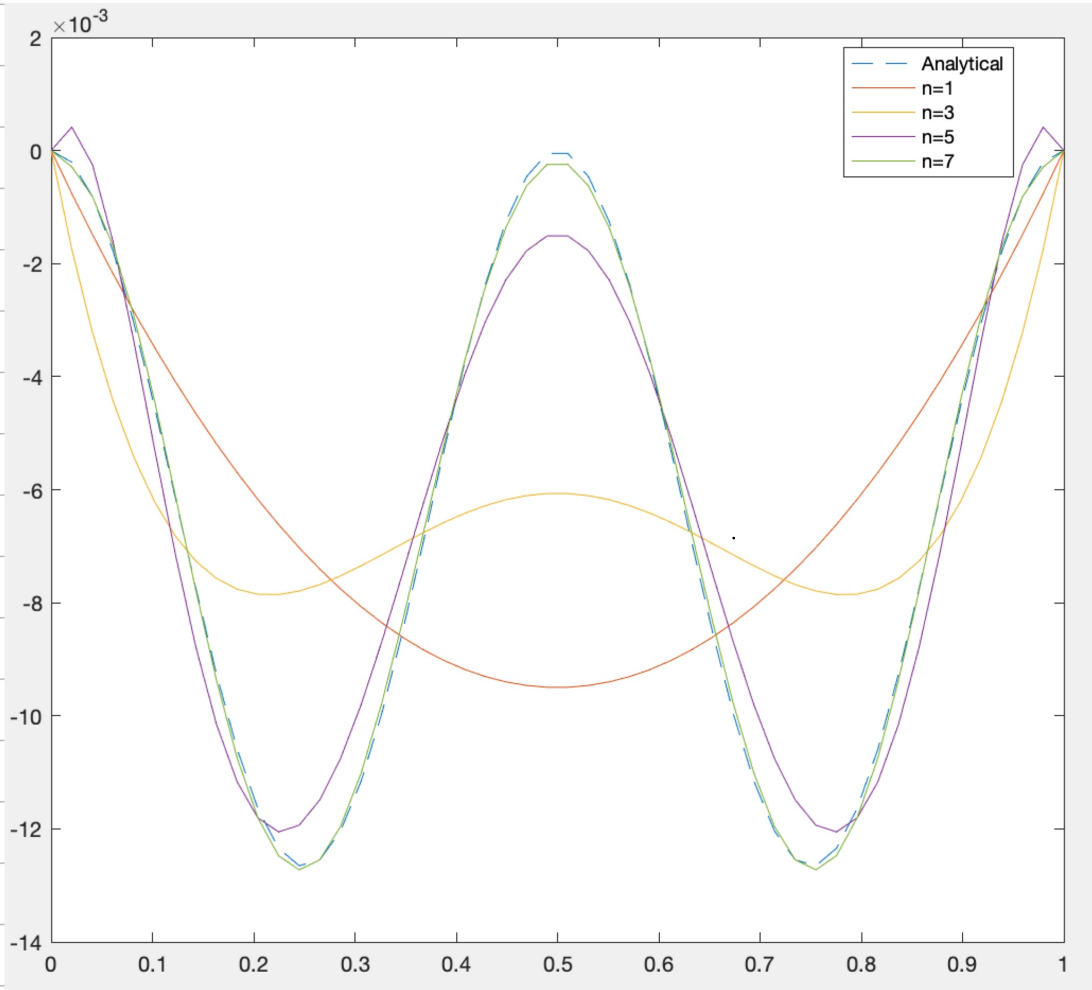
$$F = \begin{bmatrix} 0 \\ 0.0219 \\ 0 \\ 0.0208 \end{bmatrix} \qquad K = I_4$$

③ When $k = 5$

$$F = \begin{bmatrix} 0 \\ 0.0219 \\ 0 \\ 0.0208 \\ 0 \\ -0.0442 \end{bmatrix} \qquad K = I_6$$

④. When $k = 7$

$$F = \begin{bmatrix} 0 \\ 0.0219 \\ 0 \\ 0.0208 \\ 0 \\ -0.0442 \\ 0 \\ 0.0169 \end{bmatrix} \qquad K = I_8$$



(c). observing the figure above shows with more polynomials, the numerical solution approaches the exact solution. For example, ${}^{7}u_n$ is close to $u(x)$.

${}^{n}u_n$ is visually approaching $u(x) = \frac{-\sin(2\pi x)^2}{8\pi}$.

3. consider $W_n = \{ w = \Sigma_{k=2}^{n} c_k \cdot i P_k(x) \mid (c_2, \dots, c_n) \in \mathbb{R}^{n+1} \}$.

$$S_n = \{ w_n \in W_n \mid w_n(0) = 0 \}$$

$$= \{ w = \Sigma_{k=2}^{n} c_k i P_k(x) \mid (c_2, \dots, c_n) \in \mathbb{R}^{n+1}) \}.$$

$$V_n = S_n = \{ v_n \in W_n \mid v_n(0) = 0 \}.$$

$$= \{ v = \Sigma_{k=2}^{n} c_k i P_k(x) \mid (c_2, \dots, c_n) \in \mathbb{R}^{n+1}) \}$$

Still, we solve

$$\sum_{b=2}^{n} a(N_b, N_a) u_b = l(N_a) \qquad a = \{2, \cdots, n\}$$

(b) ① When $k = 3$

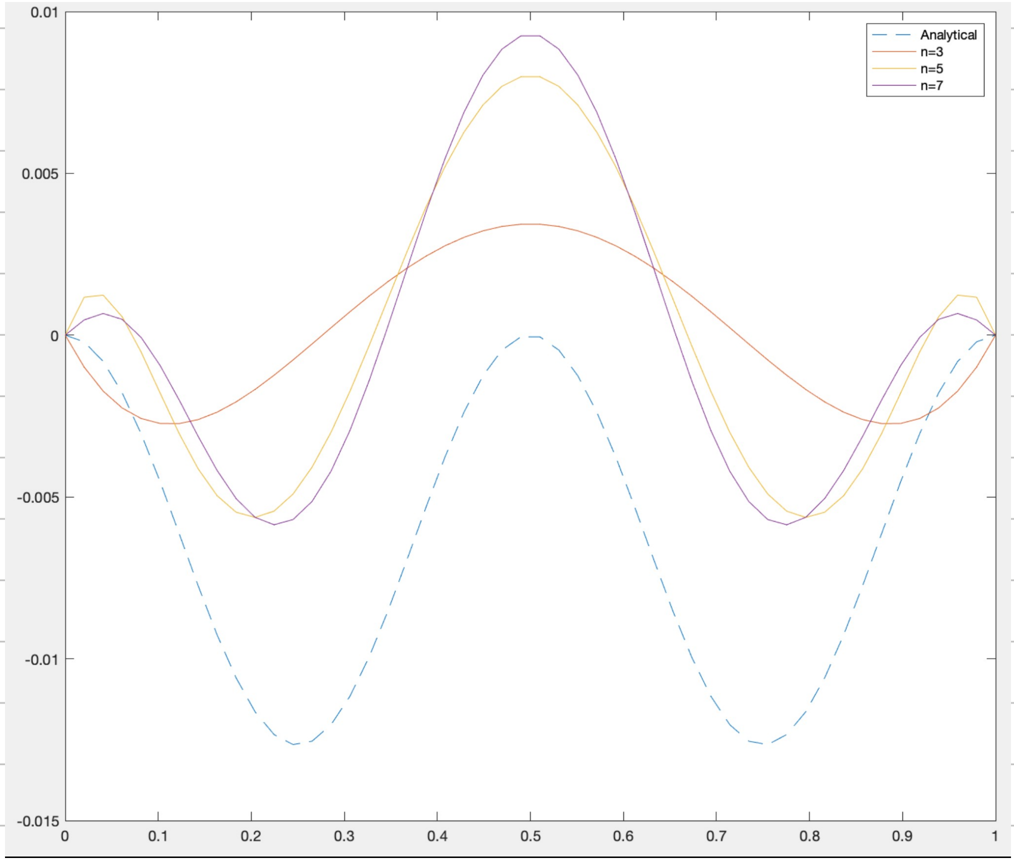
$$F = \begin{bmatrix} 0 \\ 0.0208 \end{bmatrix} \qquad K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

② When $k = 5$

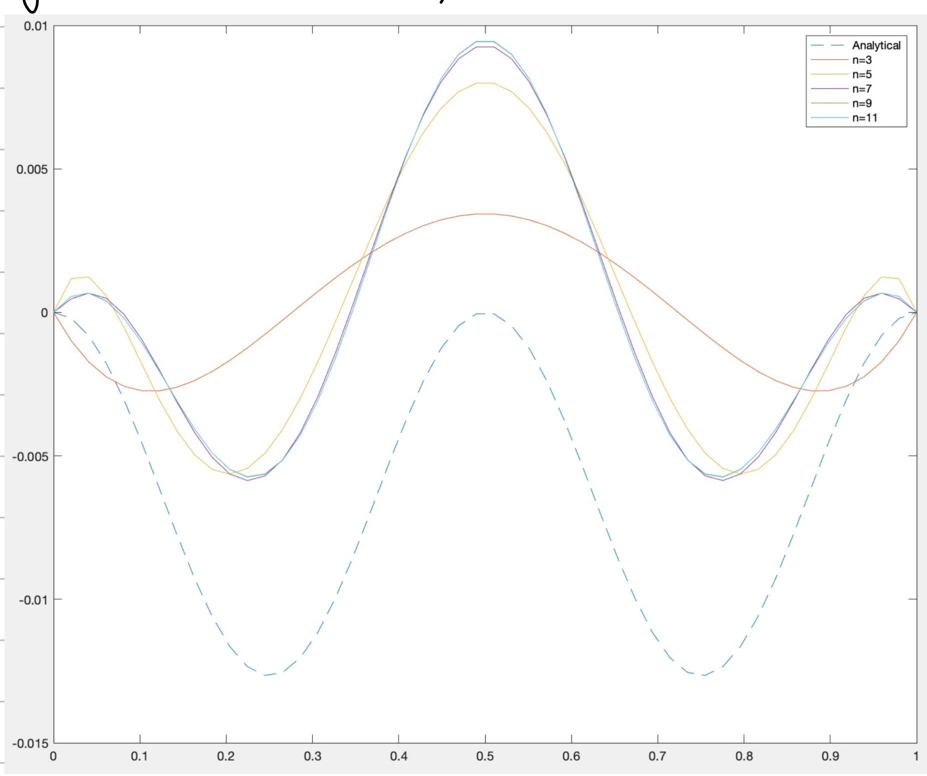
$$F = \begin{bmatrix} 0 \\ 0.0208 \\ 0 \\ -0.0442 \end{bmatrix} \qquad K = I_4$$

③ When $k = 7$

$$F = \begin{bmatrix} 0 \\ 0.0208 \\ 0 \\ -0.0442 \\ 0 \\ 0.0169 \end{bmatrix} \qquad K = I_6$$



(b). ${}^{n}u_n$ is not converging to u as n grows. After increasing n to 9 and 11, solutions still diverge to true solution.



I expect the numerical solution will not converge when increasing n. This is because we do not have a complete series of basis functions to approximate $u(x)$.

Euler-Lagrange Equations (30)

For this problem section "1.3.3 The Euler-Lagrange Equations" of the notes provides an explanation of the steps.

Consider the weak form: Find $u \in \mathcal{V} = \{u \colon (0,1) \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$ such that $a(u,w) = \ell(w)$ for all $w \in \mathcal{V}$, where

$$a(u,w) = \int_0^1 (w_{,x}u_{,x} + \lambda wu)\, dx + w(1)u(1)$$
$$\ell(w) = \int_0^1 wx^2\, dx + w(1),$$

and $\lambda > 0$.

1. (5) Is $a(u,v)$ a bilinear form? Justify.
2. (5) Is $a(u,v)$ symmetric? Justify.
3. (5) Is $\ell(v)$ a linear functional? Justify.
4. (10) Obtain the Euler-Lagrange equations.
5. (5) Identify natural and essential boundary conditions.

1. $a(u,v)$ is a bilinear form.
Using the definition, we can see:

$$a(u+\alpha v, w) = \int_0^1 (w_{,x}(u+\alpha v)_{,x} + \lambda w(u+\alpha v))\,dx + w(1)(u+\alpha v)(1).$$
$$= \int_0^1 w_{,x}u_{,x} + \lambda wu\, dx + u(1)w(1) +$$
$$\alpha\int_0^1 w_{,x}v_{,x} + \lambda wv\,dx + \alpha\, w(1)v(1)$$
$$= a(u,w) + \alpha\, a(v,w)$$

Similarly, we can have:

$$a(u, w+\alpha v) = \int_0^1 \left((w+\alpha v)_{,x} u_{,x} + \lambda (w+\alpha v) u \right) dx + (w+\alpha v)(1) u(1)$$

$$= \int_0^1 w_{,x} u_{,x} + \lambda w u \, dx + w(1) u(1) +$$

$$\alpha \int_0^1 v_{,x} u_{,x} + \lambda v u \, dx + \alpha \, v(1) u(1)$$

$$= a(u, w) + \alpha \, a(u, v)$$

Therefore, both arguemenres are linear. $a(u, w)$ is a billinear term.

2. $a(u, v)$ is a symmetric term. We can show that $a(u,v) = a(v,u)$

$$a(v, u) = \int_0^1 (u_{,x} v_{,x} + \lambda u v) dx + u(1) v(1)$$

$$= \int_0^1 u_{,x} v_{,x} \, dx + \lambda \int_0^1 u v \, dx + u(1) v(1)$$

Because $u_{,x} v_{,x} = v_{,x} u_{,x}$; $uv = vu$; $u(1) v(1) = v(1) u(1)$

we can rewrite it as:

$$\Rightarrow \int_0^1 v_{,x} u_{,x} dx + \lambda \int_0^1 v u \, dx + v(1) u(1)$$

$$= a(u, v)$$

Therefore, $a(v, u) = a(u, v)$.

3. Using the definition, for any $w, v \in V$, we can show:

$$l(w + \alpha v) = \int_0^1 (w+\alpha v) x^2 dx + (w+\alpha v)(1)$$

$$= \int_0^1 w x^2 dx + w(1) + \alpha \int_0^1 v x^2 dx + v(1)$$

$$= l(w) + \alpha l(v).$$

Therefore, $l(w)$ is a linear functional.

4. The variational equation is: Find $u \in \mathcal{V} = \{u : (0,1) \to \mathbb{R}$ smooth $u(0)=0\}$ such that $a(u,w) = \ell(w)$ for $w \in \mathcal{V}$. Such that

$$a(u,w) = \ell(w)$$

① Integration by parts to eliminate $w_{,x}$

$$\underline{\int_0^1 (w_{,x} u_{,x}} + \lambda w u)\,dx + w(1)u(1) = \int_0^1 w x^2\,dx + w(1)$$

$$[w u_{,x}]\Big|_0^1 - \int_0^1 w\, u_{,xx}\,dx + \lambda \int_0^1 w u\,dx + w(1)u(1) - \int_0^1 w x^2\,dx - w(1) = 0$$

$$w(1)\,u_{,x}(1) - \cancel{w(0)}\,u_{,x}(0) + \int_0^1 (-u_{,xx} + \lambda u - x^2)\,w\,dx + w(1)u(1) - w(1) = 0$$

$$w(1)(u_{,x}(1) + u(1) - 1) + \int_0^1 (-u_{,xx} + \lambda u - x^2)\,w\,dx = 0$$

This should hold true for any w in $\mathcal{V}$

We can have the equation:

$$\begin{cases} -u_{,xx} + \lambda u - x^2 = 0. & x \in (0,1) \\ u_{,x}(1) + u(1) - 1 = 0 \end{cases}$$

5. Natural boundary is: $u_{,x}(1) = 1 - u(1)$

Essential boundary is: $u(0) = 0$.

The essential boundary is required by the solution space.

Your "First" Finite Element Approximation (45)

We want to construct a piecewise linear approximation to a function u that satisfies the variational equation $a(u, w) = \ell(w)$ for all $w \in \mathcal{V} = \{u \colon [0,1] \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$, where

$$a(u, w) = \int_0^1 (w_{,x}\, u_{,x} + \lambda w u)\, dx + w(1)u(1)$$
$$\ell(w) = \int_0^1 w x^2 \, dx + w(1),$$

and $\lambda > 0$.

To this end, we will partition $[0, 1]$ into four equal intervals, and build a finite element approximation with continuous functions that are lineal polynomials over each interval.

1. (5) Identify the location of all the vertices in the mesh, and number them.
2. (5) Number and sketch all hat functions $\{N_a\}$ over $[0, 1]$.
3. (5) Let $\mathcal{W}_h = \operatorname{span}(\{N_1, \ldots, N_5\})$. What are the trial space $\mathcal{S}_h$ and test space $\mathcal{V}_h$ for the variational method, and the constrained and active index sets? Select $\overline{u}_h \in \mathcal{S}_h$.
4. (5) State the variational method for this problem.
5. (10) Compute the stiffness matrix, assuming that $\lambda = 2$.
6. (10) Compute the load vector.
7. (5) Find the approximate solution u_h.

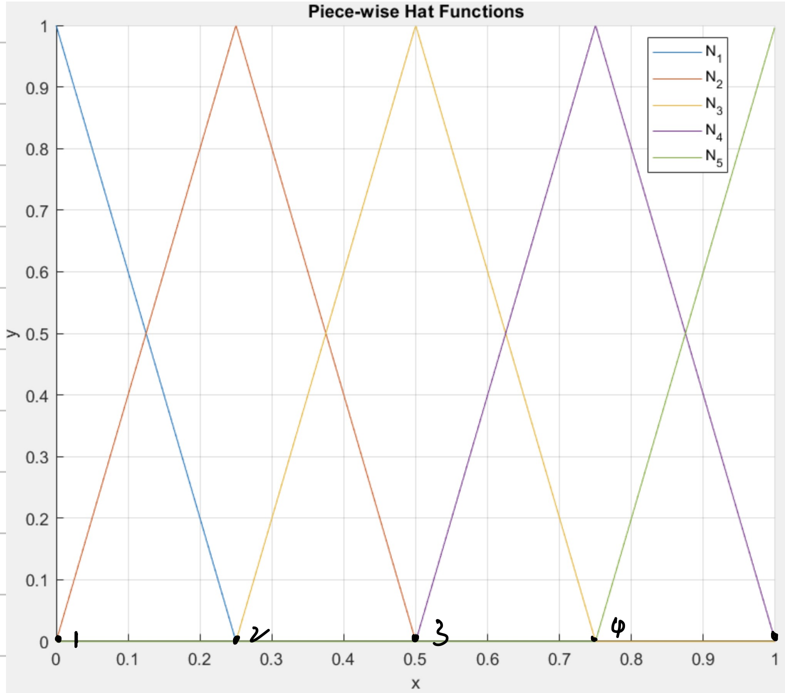
1. We partition the domain into $n_{el} \in N$ interval by selecting vertices $\{x_i\}$ $i = 1, 2, 3, 4, 5$ such that.

$$x_1 = 0, \quad x_2 = 0.25, \quad x_3 = 0.5, \quad x_4 = 0.75, \quad x_5 = 1.0$$

2. We define $\{N_a\}$ $a = 1, 2, 3, 4, 5$ as follow.

$$N_a = \begin{cases} 0 & x < x_{a-1} \\ \dfrac{x - x_{a-1}}{x_a - x_{a-1}} & x_{a-1} \le x < x_a \\ 1 & x = x_a \\ \dfrac{x_{a+1} - x}{x_{a+1} - x_a} & x_a < x \le x_{a+1} \\ 0 & x_a < x. \end{cases} = \max\left[0, \min\left[\frac{x - x_{a-1}}{x_a - x_{a-1}}, \frac{x - x_{a+1}}{x_a - x_{a+1}}\right]\right]$$

$$N_1 = \begin{cases} \dfrac{x_2 - x}{x_2 - x_1}, & x_1 \le x \le x_2 \\ 0, & x_2 < x \end{cases}, \quad N_5 = \begin{cases} 0, & x < x_4 \\ \dfrac{x - x_4}{x_5 - x_4}, & x_4 \le x \le x_5. \end{cases}$$



3. $W_h = \mathrm{span}(\{N_1, \dots, N_5\})$

$$S_h = \{u_h \in W_h \mid u_h(0) = \bar{u}_h\}.$$

$$= \{\bar{u}_h N_1 + C_2 N_2 + C_3 N_3 + C_4 N_4 + C_5 N_5 \mid (C_2, C_3, C_4, C_5) \in \mathbb{R}\}$$

$$V_h = \{v_h \in W_h \mid v_h(0) = 0\}.$$

$$= \{C_2 N_2 + C_3 N_3 + C_4 N_4 + C_5 N_5 \mid (C_2, C_3, C_4, C_5) \in \mathbb{R}\}$$

$$S_h = \{v_h + \bar{u}_h N_1 \mid v_h \in V_h\},$$

$$\eta_a = \{2, 3, 4, 5\}$$

$$\eta_g = \{1\}.$$

4. Find $u_h \in S_h$ such that

$$a(u_h, v_h) = \ell(v_h) \quad \text{for all } v_h \in V_h$$

where

$$a(u, v) = \int_0^1 (v_{,x} u_{,x} + \lambda v u)\, dx + v(1)u(1)$$

$$\ell(v) = \int_0^1 v x^2\, dx + v(1).$$

5. We know $\lambda = 2$.

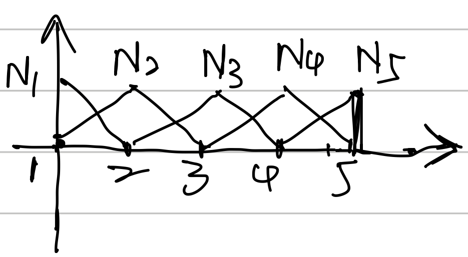
$K_{11} = \delta_{11} = 1$. $\quad K_{12} = \delta_{12} = 0$

$$K_{21} = a(N_1, N_2) = \int_0^1 (N_1' N_2' + 2 N_2 N_1)\, dx + N_2(1) N_1(1)$$

$$= \int_0^{0.25} \left[-\frac{1}{0.25}\cdot\frac{1}{0.25} + 2\cdot\frac{x}{0.25}\left(\frac{0.25-x}{0.25}\right)\right] dx + 0$$

$$= -\frac{47}{12}$$

$$K_{22} = a(N_2, N_2) = \int_0^1 (N_2' N_2' + 2 N_2 N_2)\, dx + 0.$$



$$= \int_0^{0.25} \frac{1}{0.25}\cdot\frac{1}{0.25} + 2\cdot\left(\frac{x}{0.25}\right)^2 dx$$

$$+ \int_{0.25}^{0.5} -\frac{1}{0.25}\cdot -\frac{1}{0.25} + 2\left(\frac{0.5-x}{0.25}\right)^2 dx$$

$$= 25/3.$$

$$K_{23} = a(N_3, N_2) = \int_0^1 (N_3' N_2' + 2 N_3 N_2)\, dx + 0$$

$$= \int_{0.25}^{0.5} \left(\frac{1}{0.25} \cdot -\frac{1}{0.25} + 2\left(\frac{x-0.25}{0.25} \cdot \frac{0.5-x}{0.25} \right) \right) dx$$

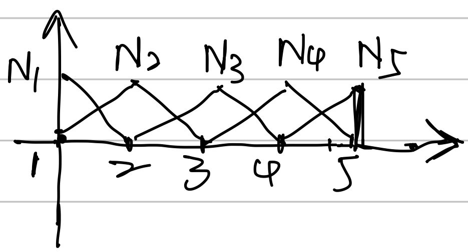
$$= -\frac{47}{12}$$

$$K_{24} = 0.$$

$$K_{25} = 0$$

$$K_{32} = K_{23}$$

$$K_{33} = a(N_3, N_3) = \int_0^1 (N_3' N_3' + 2 N_3 N_3)\, dx + 0.$$



$$= \int_{0.25}^{0.5} \frac{1}{0.25} \cdot \frac{1}{0.25} + 2 \cdot \left(\frac{x-0.25}{0.25} \right)^2 dx$$

$$+ \int_{0.5}^{0.75} \frac{1}{0.25} \cdot -\frac{1}{0.25} + 2 \left(\frac{0.75-x}{0.25} \right)^2 dx$$

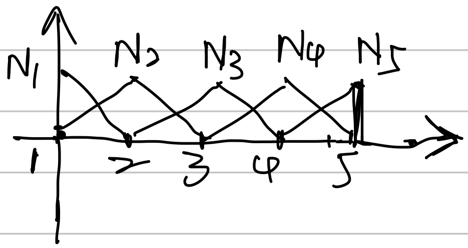
$$= \frac{25}{3}$$

$$K_{34} = a(N_4, N_3) = \int_{0.5}^{0.75} \frac{1}{0.25} \cdot -\frac{1}{0.25} + 2 \cdot \left(\frac{x-0.5}{0.25} \right) \cdot \left(\frac{0.75-x}{0.25} \right) dx$$

$$= -\frac{47}{12}.$$

$$K_{43} = K_{34} = -\frac{47}{12}$$

$$K_{44} = a(N_4, N_4) = \int_0^1 (N_4' N_4' + 2 N_4 N_4)\,dx + 0.$$



$$= \int_{0.25}^{0.5} \frac{1}{0.25}\cdot\frac{1}{0.25} + 2\cdot\left(\frac{x-0.5}{0.25}\right)^2 dx$$

$$+ \int_{0.5}^{0.75} -\frac{1}{0.25}\cdot -\frac{1}{0.25} + 2\left(\frac{1.0-x}{0.25}\right)^2 dx$$

$$= \frac{25}{3}.$$

$$K_{45} = K_{54} = -\frac{47}{12}$$

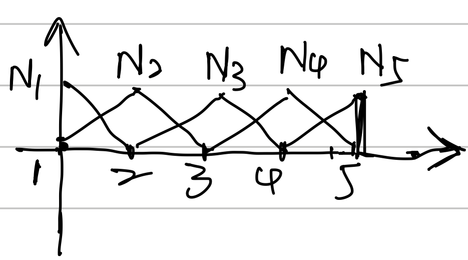
$$K_{55} = a(N_5, N_5) = \int_0^1 (N_5' N_5' + 2 N_5 N_5)\,dx + 1$$

$$= \int_{0.75}^{1} \frac{1}{0.25}\cdot\frac{1}{0.25} + 2\cdot\left(\frac{x-0.75}{0.25}\right)^2 dx$$

$$= \frac{31}{6}.$$

The stiffness matrix is:

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 & 0 \\ 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 \\ 0 & 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} \\ 0 & 0 & 0 & -\frac{47}{12} & \frac{31}{6} \end{bmatrix}$$



6. The load vector can be computed as:

$$\ell(w) = \int_0^1 w x^2 \, dx + w(1)$$

$$F(N_1) = 0$$

$$F(N_2) = \int_0^{0.25} \frac{x}{0.25} \cdot x^2 \, dx + \int_{0.25}^{0.5} \frac{0.5-x}{0.25} x^2 \, dx + N_2(1)$$

$$= 7/384$$

$$F(N_3) = \int_{0.25}^{0.5} \frac{x-0.25}{0.25} x^2 \, dx + \int_{0.5}^{0.75} \frac{0.75-x}{0.25} x^2 \, dx + 0.$$

$$= \frac{25}{384}.$$

$$F(N_4) = \int_{0.5}^{0.75} \frac{x-0.5}{0.25} \cdot x^2 \, dx + \int_{0.75}^{1.0} \frac{(1.0-x)}{0.25} x^2 \, dx + 0.$$

$$= \frac{55}{384}.$$

$$F(N_5) = \int_{0.75}^{1.0} \frac{x-0.75}{0.25} x^2 \, dx + 1 = \frac{283}{256}$$

$$F = \begin{bmatrix} 0 \\ \frac{7}{384} \\ \frac{25}{384} \\ \frac{55}{384} \\ \frac{283}{256} \end{bmatrix}$$

7. Solve the linear system:

$$KV = F \Rightarrow V = K^{-1}F$$

$$V = \begin{bmatrix} 0.0 \\ 0.1044 \\ 0.2175 \\ 0.3417 \\ 0.4730 \end{bmatrix}$$

$$u_h(x) = 0.1044 N_2 + 0.2175 N_3 + 0.3417 N_4 + 0.4730 N_5$$