1

Friday, February 11, 2022 1:32 PM

Assembly

given : $u : \Omega \to \mathbb{R}$

$-\varepsilon u_{,xx} + c u_{,x} = f$ $\quad x \in \Omega \quad \longrightarrow$ (1a)

$-\varepsilon u_{,x}(7) = h,$ $\quad x$ on $\partial\Omega_N$

$u(1) = g,$ $\quad x$ on $\partial\Omega_D$

Weak form

multiply (1a) by test function $v(x)$ & integrate over domain

$$\int_\Omega -\varepsilon u_{,xx} v + c u_{,x} v = \int_\Omega f v$$

domain is 1 to 7

$$\int_1^7 \left(\underbrace{-\varepsilon u_{,xx} v}_{\text{By parts}} + c u_{,x} v\right) dx = \int_1^7 f v \, dx$$

$$\underbrace{-\varepsilon u_{,x} v \Big|_1^7}_{\text{Substitute equation (1b)}} + \int_1^7 \varepsilon u_{,x} v_{,x} \, dx + \int_1^7 c u_{,x} v \, dx = \int_1^7 f v \, dx$$

$$h v(7) + \varepsilon u_{,x}(1) v(1) + \int_1^7 (\varepsilon u_{,x} v_{,x} + c u_{,x} v) dx = \int_1^7 f v \, dx$$

$$\Rightarrow \int_1^7 u_{,x}(\varepsilon v_{,x} + c v) \, dx = \int_1^7 f v \, dx - h v(7) - \varepsilon u_{,x}(1) v(1)$$

$\Rightarrow$ $\qquad - \varepsilon\, u_{,x}(1)\, v(1)$

we will request $v(1) = 0$ & $u(1) = g$ becomes our essential boundary condition

where as $-\varepsilon\, u_{,x}(7) = h$ is Natural boundary condition

$$\mathcal{S} = \{ w : [1,7] \to \mathbb{R} \text{ smooth} \mid w(1) = g \}$$ Trial space

$$\mathcal{V} = \{ w : [1,7] \to \mathbb{R} \text{ smooth} \mid w(1) = 0 \}$$ test space

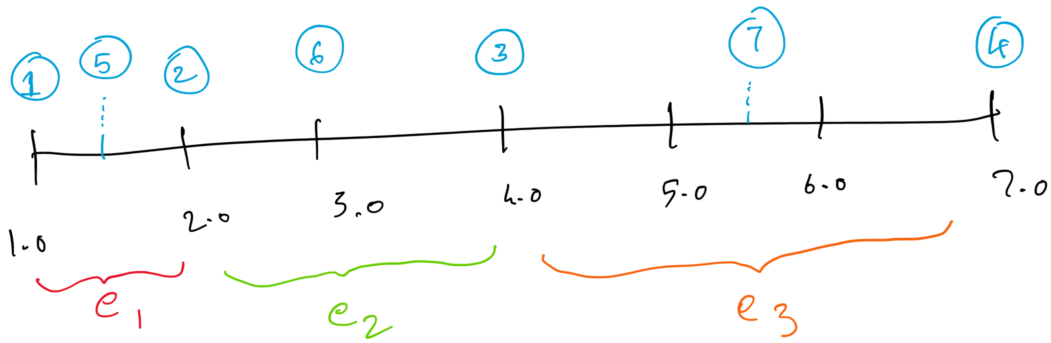
$$a(u,v) = \int_1^7 u_{,x} \left(\varepsilon v_{,x} + c\, v \right) dx$$

$$l(v) = \int_1^7 f v \, dx - h\, v(7)$$

2

Friday, February 11, 2022 1:44 PM

Our problem is



For the continuous basis function over LG Map should look like

$$
LG = \begin{array}{c} \begin{array}{ccc} \color{red}{e_1} & \color{green}{e_2} & \color{orange}{e_3} \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix} \end{array}
$$

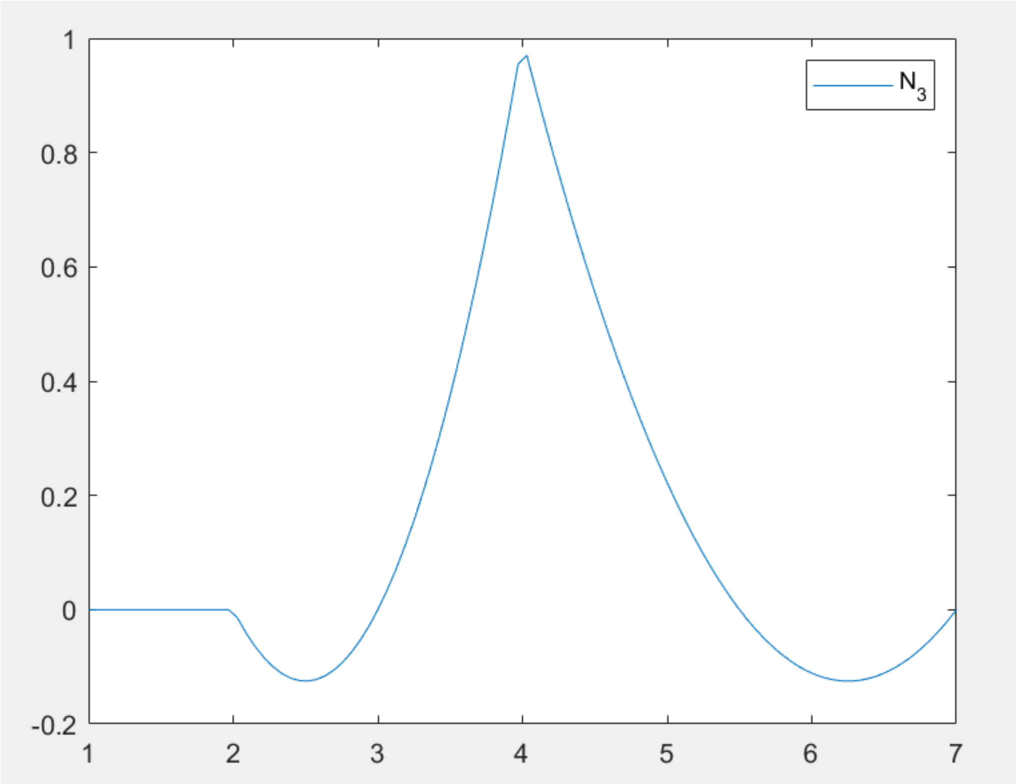
here we see both nodes 2 & 3 basis functions are non zero in multiple elements.

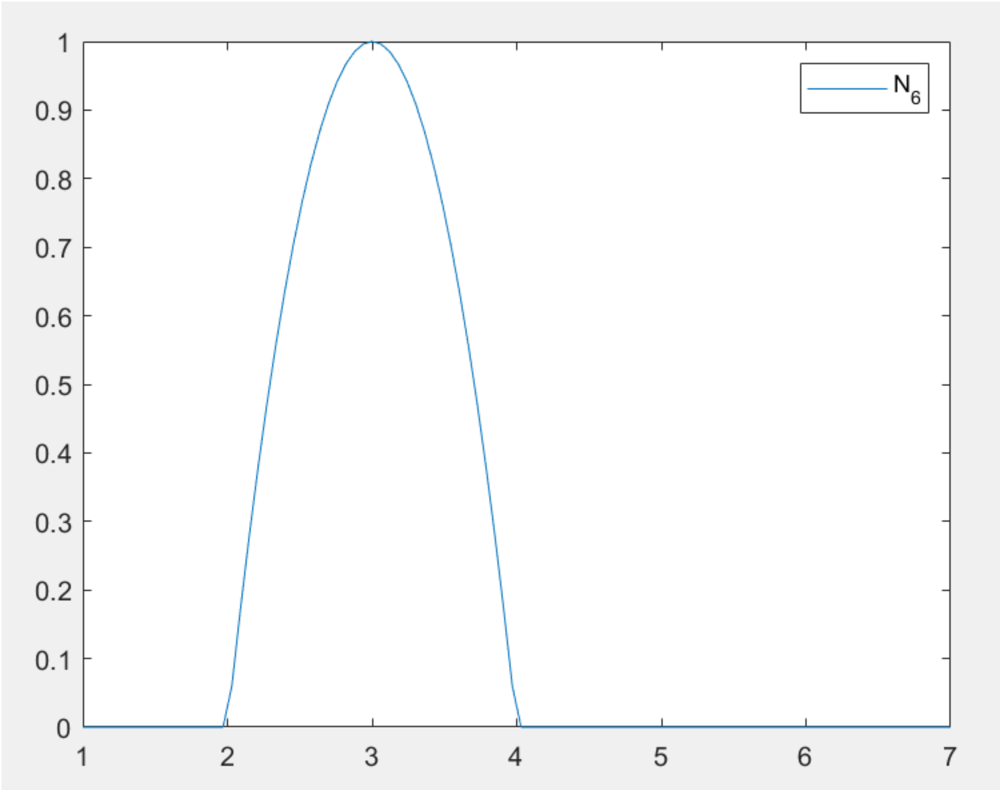
Also these are $\mathbb{P}_2$ elements.

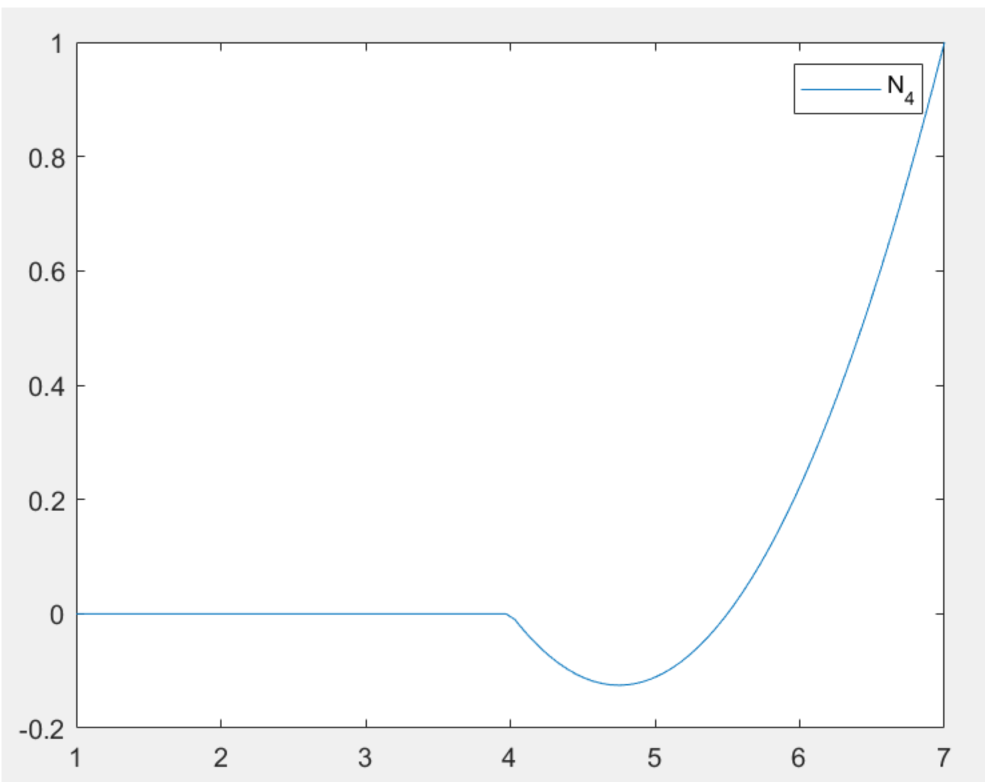
3

Friday, February 11, 2022 3:23 PM

All three basis functions are plotted below,
N3, N6, N4




1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
N_6


1
0.8
0.6
0.4
0.2
0
-0.2
1
2
3
4
5
6
7
N_4

4

Friday, February 11, 2022 3:23 PM

We have value of v_h at nodes 2, 6, 3 as

$[2.0, 3.0, 4.0]$

$$\left.\begin{aligned} v_h(2.0) &= 1.5 = w_2 N_2 + w_6 N_6 + w_3 N_3 \quad (2.0) \\ v_h(3.0) &= -1.5 = w_2 N_2 + w_6 N_6 + w_3 N_3 \quad (3.0) \\ v_h(4.0) &= 3 = w_2 N_2 + w_6 N_6 + w_3 N_3 \quad (4.0) \end{aligned}\right\}$$

The N_a functions evaluated at these points

$$N w = V$$

$$[\quad]_{3\times 3} \, [\quad]_{3\times 1} = [\quad]_{3\times 1}$$

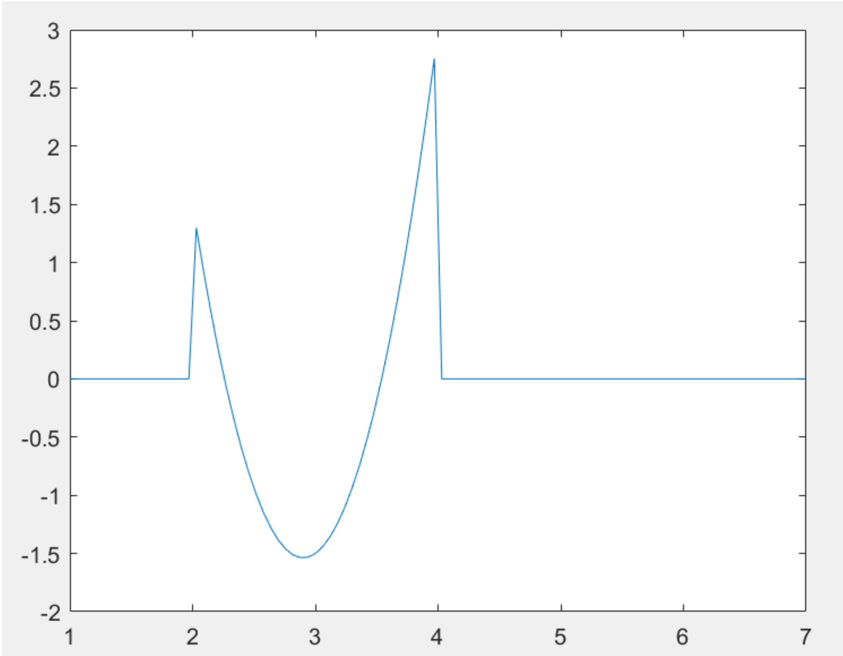
at 2.0 N_3 & $N_6 = 0$ | 3.0 N_2 & $N_3 = 0$

4.0 N_6 & $N_2 = 0$

$\therefore \; w_2 = 1.5, \quad w_6 = -1.5, \quad w_3 = 3$

The function is explicitly represented as: where xa is our x co-ordinate
Also this function is only valid in interval xa = {2, 4}

```
(3*(xa - 2)*(xa - 3))/2 + (3*(xa - 2)*(xa - 4))/2 + (3*(xa - 3)*(xa - 4))/4
```

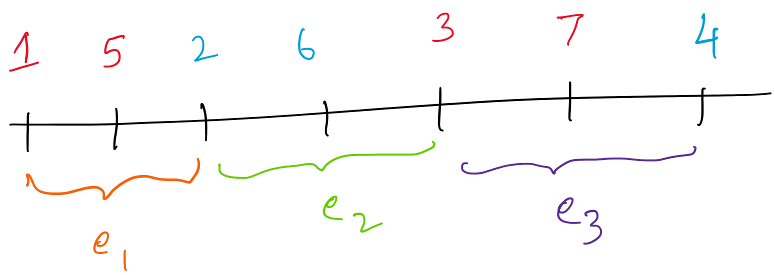

3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
1
2
3
4
5
6
7

5

Friday, February 11, 2022 3:49 PM

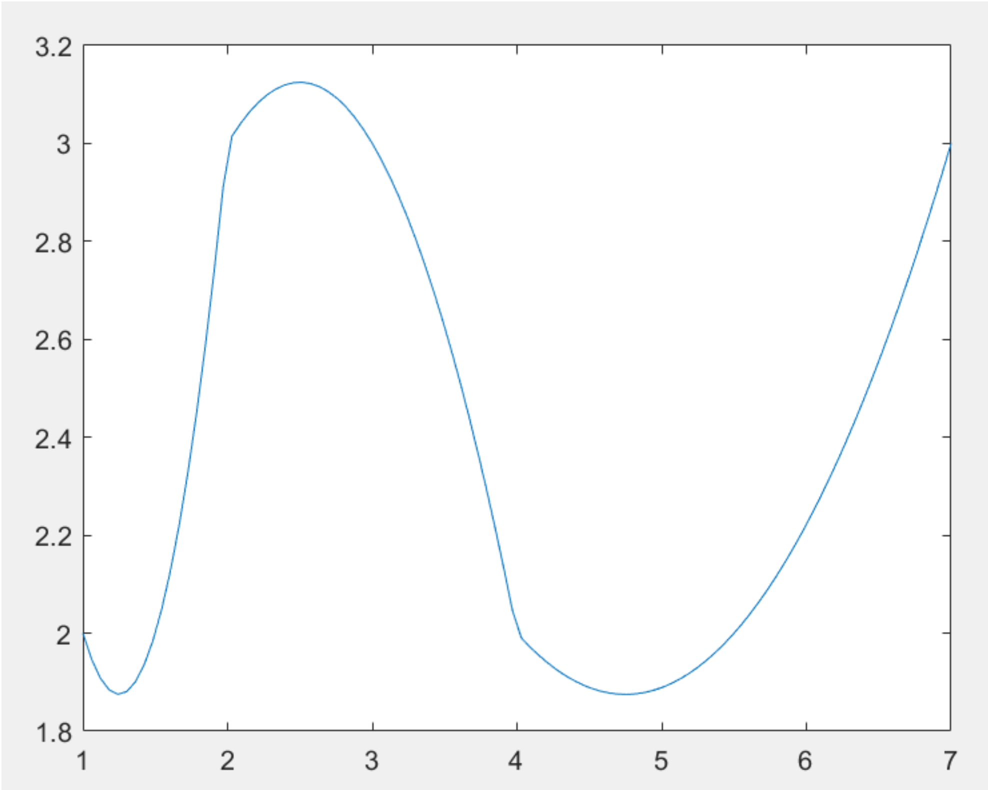
2 at odd nodes & 3 at even nodes.

blue - 3
red - 2



$$v_n = 2N_1 + 2N_5 + 3N_2 + 3N_6 + 2N_3 + 2N_7 + 3N_4$$

```
N_1
4*(xa - 2)*(xa - 3/2)

N_5
-8*(xa - 1)*(xa - 2)

N_2
6*(xa - 1)*(xa - 3/2) + (3*(xa - 3)*(xa - 4))/2

N_6
-3*(xa - 2)*(xa - 4)

N_3
(xa - 2)*(xa - 3) + (4*(xa - 7)*(xa - 11/2))/9

N_7
-(8*(xa - 4)*(xa - 7))/9

N_4
(2*(xa - 4)*(xa - 11/2))/3
```


3.2
3
2.8
2.6
2.4
2.2
2
1.8
1
2
3
4
5
6
7

Friday, February 11, 2022 4:50 PM

In the nodes / Indices we have 1 is constrained & 2 to 7 form the active indices

$$\eta_a = \{2, \dots, 7\}$$

$$\eta_g = \{1\}$$

let $\mathcal{W}_h \subset \mathcal{W}$

$\mathcal{S}_h = \mathcal{S} \cap \mathcal{W}_h$ and

$\mathcal{V}_h = \mathcal{V} \cap \mathcal{W}_h$

$$\mathcal{V}_h = \{ w_h \in \mathcal{W}_h \mid w_h(1) = 0 \}$$

$$= \text{span}(N_2, N_3, N_4, N_5, N_6, N_7)$$

$$= \sum_{i=2}^{7} c_i N_i$$

$$\mathcal{S}_h = \{ w_h \in \mathcal{W}_h \mid w_h(1) = g \}$$

$$= \text{Span}\left(\bigcup_{i=1}^{7} N_i \right)$$

$$= \sum_{i=1}^{7} c_1 N_1 \quad \text{where } c_1 = g$$

$$= \{ w_h = g N_1 + v_h \mid v_h \in \mathcal{V}_h \}$$

Friday, February 11, 2022 5:03 PM

(a) Element stiffness matrices

given $f(x) = x$, $h = -20$, $E = 1$, $c = 1$, $g = 2$

$$a(u,v) = \int_1^7 u_{,x} \left(v_{,x} + v \right) dx$$

$$l(v) = \int_1^7 x\, v(x)\, dx + 20\, v(7)$$

for element 1

$$\begin{bmatrix} 1 & 0 & 0 \\ -3.33 & 5.33 & -2 \\ 0.5 & -3.33 & 2.83 \end{bmatrix}$$

element 2

$$\begin{bmatrix} 0.667 & -0.667 & 0 \\ -2.00 & 2.667 & -0.667 \\ 0.33 & -2.0 & 1.667 \end{bmatrix}$$

$$\begin{bmatrix} 0.667 & -0.667 & 0 \\ -2.00 & 2.667 & -0.667 \\ 0.33 & -2.0 & 1.667 \end{bmatrix}$$

element 3

$$\begin{bmatrix} 0.278 & -0.22 & -0.055 \\ -1.556 & 1.778 & -0.22 \\ 0.2778 & -1.5556 & 1.2778 \end{bmatrix}$$

(b) Element load vectors

Element 1	Element 2	Element 3
2.0000	0.6667	2.0000
1.0000	4.0000	11.0000
0.3333	1.3333	23.5000

Part C

Stiffness matrix (Global)

7x7 double

	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	0.5000	3.5000	0	0	-3.3333	-0.6667	0
3	0	0.3333	1.9444	-0.0556	0	-2	-0.2222
4	0	0	0.2778	1.2778	0	0	-1.5556
5	-3.3333	-2	0	0	5.3333	0	0
6	0	-2	-0.6667	0	0	2.6667	0
7	0	0	-1.5556	-0.2222	0	0	1.7778

Load Vector Global

7x1 double

	1
1	2
2	1
3	3.3333
4	23.5000
5	1
6	4
7	11

Pard D (FE solution)

U vector is

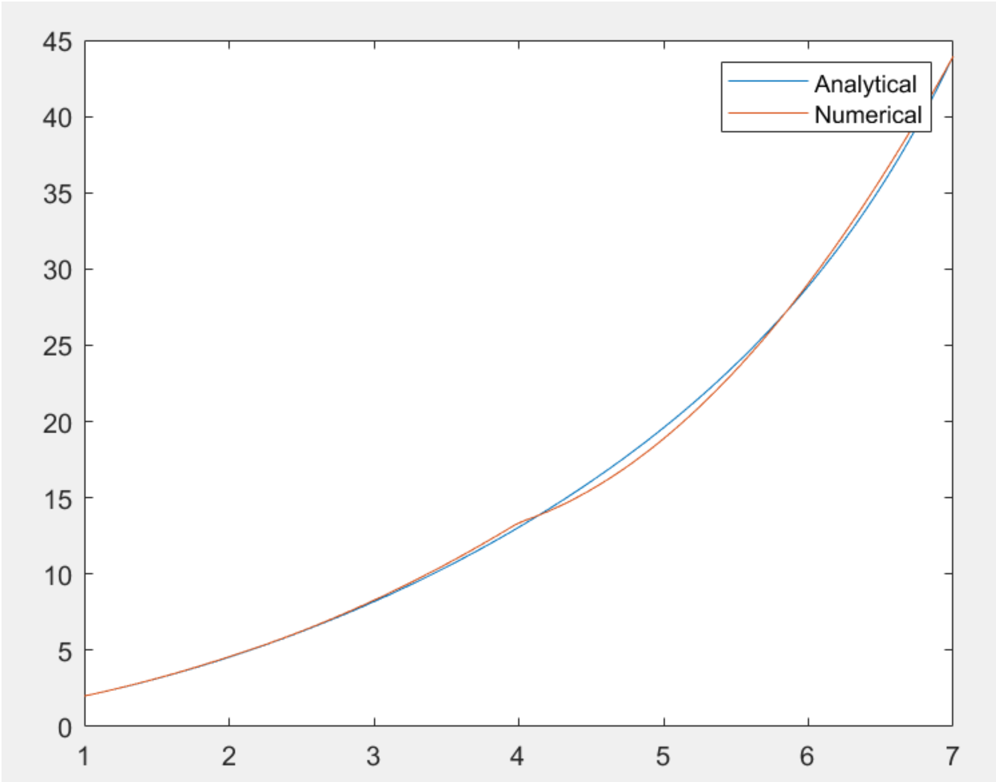
7x1 double

	1
1	2.0000
2	4.5833
3	13.3745
4	43.9514
5	3.1562
6	8.2811
7	23.3841

Where it is arranged as node indices, so the linear combination will be

2*N1 + 4.583*N2 + 13.3745*N3 + 43.9514*N4 + 3.1562*N5 + 8.2811*N6 + 23.3841*N7

The plot of numerical FE result and analytical solution


Analytical
Numerical
45
40
35
30
25
20
15
10
5
0
1
2
3
4
5
6
7

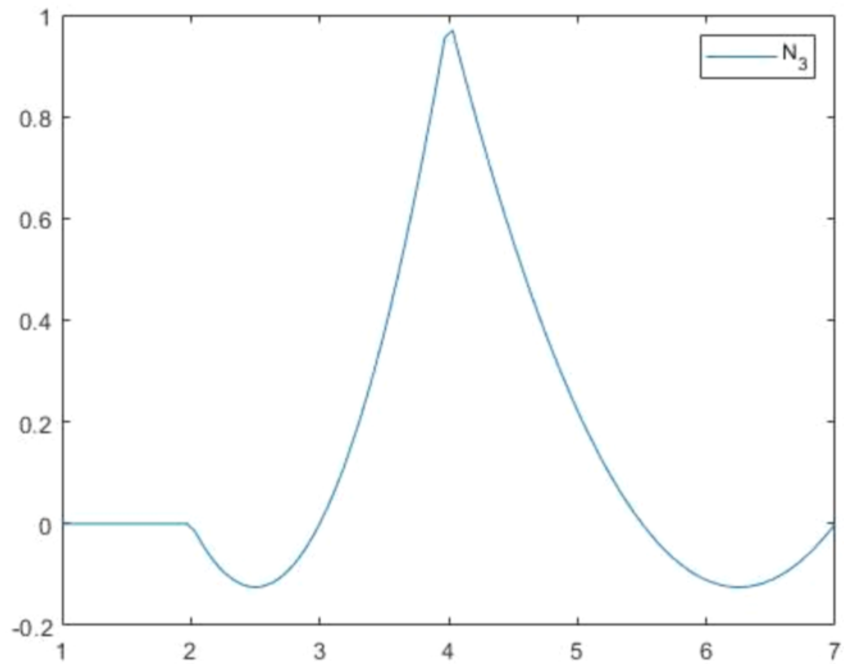
Code 3, 4, 5

12 February 2022 19:11

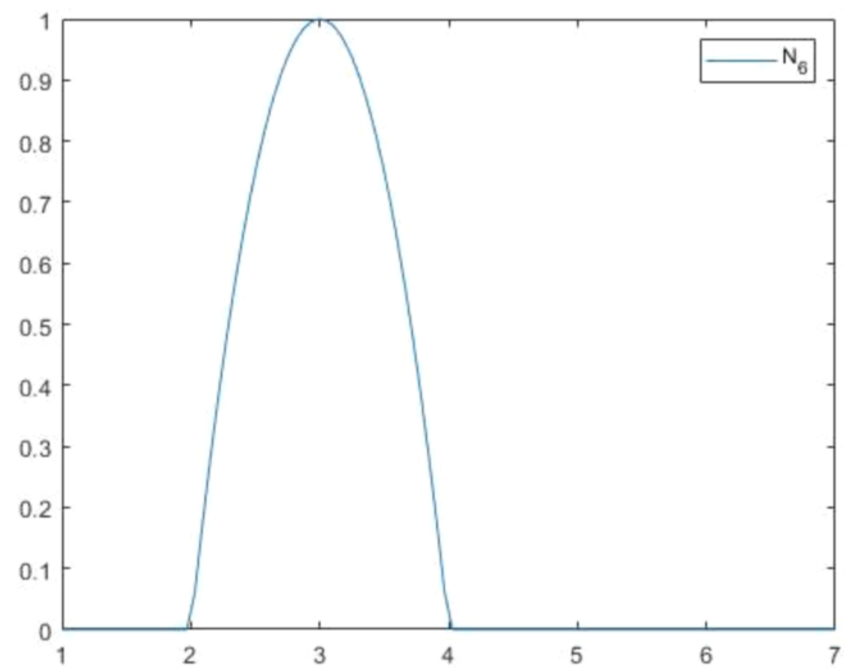
```
x = linspace(1, 7, 100);
y3 = P2elements(x, 4.0, 3.0, 2.0) + P2elements(x, 4.0, 5.5, 7.0);
y6 = P2elements(x, 3.0, 4.0, 2.0);
y4 = P2elements(x, 7.0, 5.5, 4.0);

figure(1);
plot(x, y3(1,:), 'DisplayName','N_3')
legend
figure(2);
plot(x, y6(1,:), 'DisplayName','N_6')
legend
figure(3);
plot(x, y4(1,:), 'DisplayName','N_4')
legend

x_test = x;

y_test = 1.5*P2elements(x_test, 2.0, 3.0, 4.0) - 1.5*P2elements(x_test, 3.0, 2.0, 4.0) ...
        + 3*P2elements(x_test, 4.0, 3.0, 2.0) ;
%y_test = P2elements(x_test, 2.0, 3.0, 4.0) + P2elements(x_test, 3.0, 2.0, 4.0) ...
     %   + P2elements(x_test, 4.0, 3.0, 2.0) ;

syms xa
y_a = 1.5*P2elemNew(xa, 2.0, 3.0, 4.0) - 1.5*P2elemNew(xa, 3.0, 2.0, 4.0) ...
        + 3*P2elemNew(xa, 4.0, 3.0, 2.0) ;
f(xa) = y_a;
f(xa)

figure(4);
plot(x_test, y_test)

y_part5 = 2*P2elements(x, 1.0, 1.5, 2.0) + 2*P2elements(x, 1.5, 1.0, 2.0)... % 1 and 5
    + 3*(P2elements(x, 2.0, 1.5, 1.0) + P2elements(x_test, 2.0, 3.0, 4.0))... % 2
    + 3*P2elements(x_test, 3.0, 2.0, 4.0)... % 6
    + 2*(P2elements(x, 4.0, 5.5, 7.0) + P2elements(x_test, 4.0, 3.0, 2.0))... % 3
    + 2*P2elements(x, 5.5, 4.0, 7.0) + 3*P2elements(x, 7.0, 4.0, 5.5); % 7 and 4
disp('N_1')
disp(2*P2elemNew(xa, 1.0, 1.5, 2.0))
disp('N_5')
disp(2*P2elemNew(xa, 1.5, 1.0, 2.0))
disp('N_2')
disp(3*(P2elemNew(xa, 2.0, 1.5, 1.0) + P2elemNew(xa, 2.0, 3.0, 4.0)))
disp('N_6')
disp(3*P2elemNew(xa, 3.0, 2.0, 4.0))
disp('N_3')
disp(2*(P2elemNew(xa, 4.0, 5.5, 7.0) + P2elemNew(xa, 4.0, 3.0, 2.0)))
disp('N_7')
disp(2*P2elemNew(xa, 5.5, 4.0, 7.0))
disp('N_4')
disp(3*P2elemNew(xa, 7.0, 4.0, 5.5))
% X_d = [str2sym(2*P2elemNew(xa, 1.0, 1.5, 2.0)),' N_1 \n',...
%      str2sym(2*P2elemNew(xa, 1.5, 1.0, 2.0)),' N_5 \n',...
%      str2sym(3*(P2elemNew(xa, 2.0, 1.5, 1.0) + P2elemNew(xa, 2.0, 3.0, 4.0))),' N_2 \n',...
%      str2sym(3*P2elemNew(xa, 3.0, 2.0, 4.0)),' N_6 \n',...
%      str2sym(2*(P2elemNew(xa, 4.0, 5.5, 7.0) + P2elemNew(xa, 4.0, 3.0, 2.0))),' N_3 \n',...
%      str2sym(2*P2elemNew(xa, 5.5, 4.0, 7.0)),' N_7 \n',...
%      str2sym(3*P2elemNew(xa, 7.0, 4.0, 5.5)),' N_4'];
```

```
figure(5);
plot(x_test, y_part5)

function [y]=P2elemNew(x, x1, x2, x3)
    y = (x - x2)*(x - x3)/((x1 - x2)*(x1 - x3));
end
```

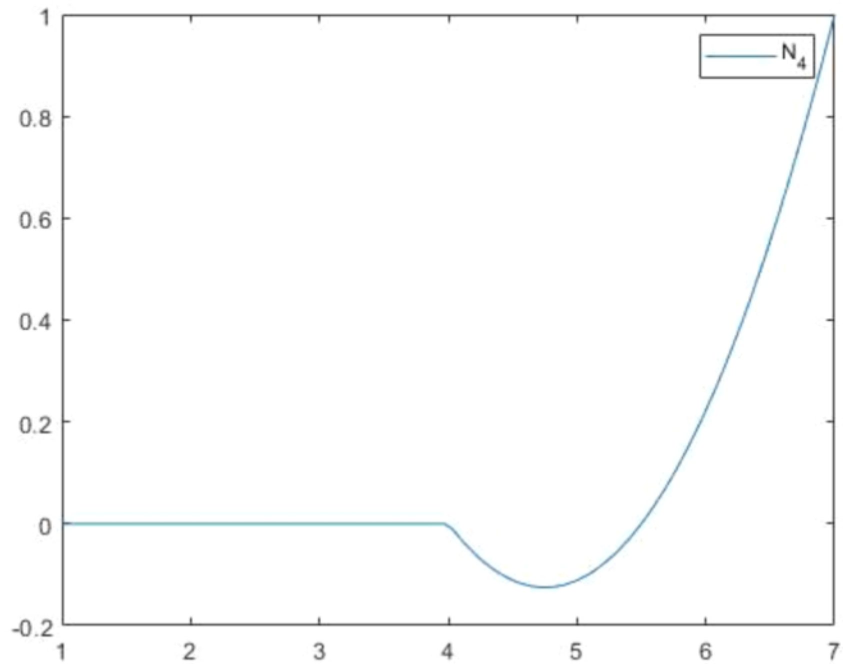
```
ans =

(3*(xa - 2)*(xa - 3))/2 + (3*(xa - 2)*(xa - 4))/2 + (3*(xa - 3)*(xa - 4))/4

N_1
4*(xa - 2)*(xa - 3/2)

N_5
-8*(xa - 1)*(xa - 2)

N_2
6*(xa - 1)*(xa - 3/2) + (3*(xa - 3)*(xa - 4))/2

N_6
-3*(xa - 2)*(xa - 4)

N_3
(xa - 2)*(xa - 3) + (4*(xa - 7)*(xa - 11/2))/9

N_7
-(8*(xa - 4)*(xa - 7))/9

N_4
(2*(xa - 4)*(xa - 11/2))/3
```

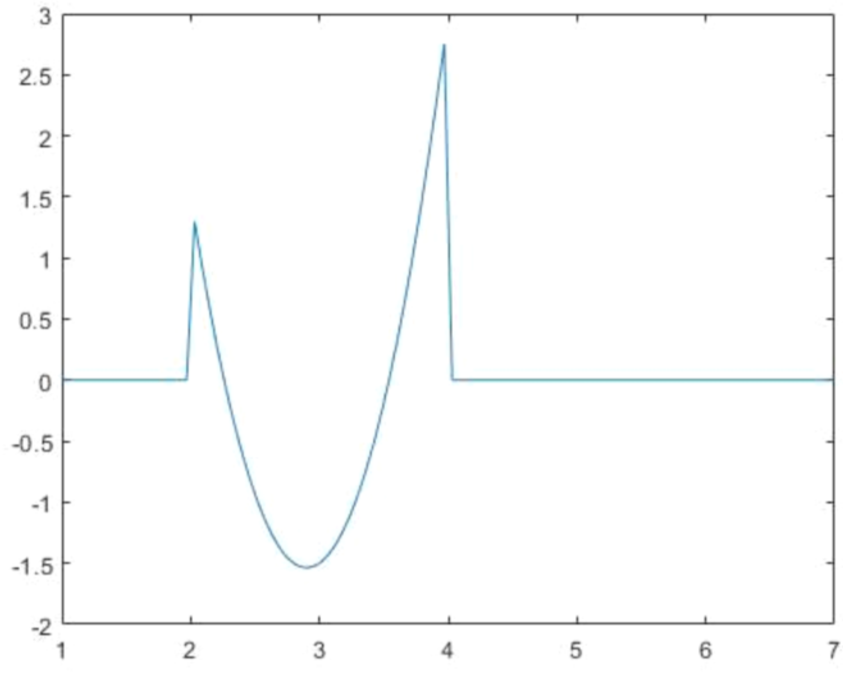


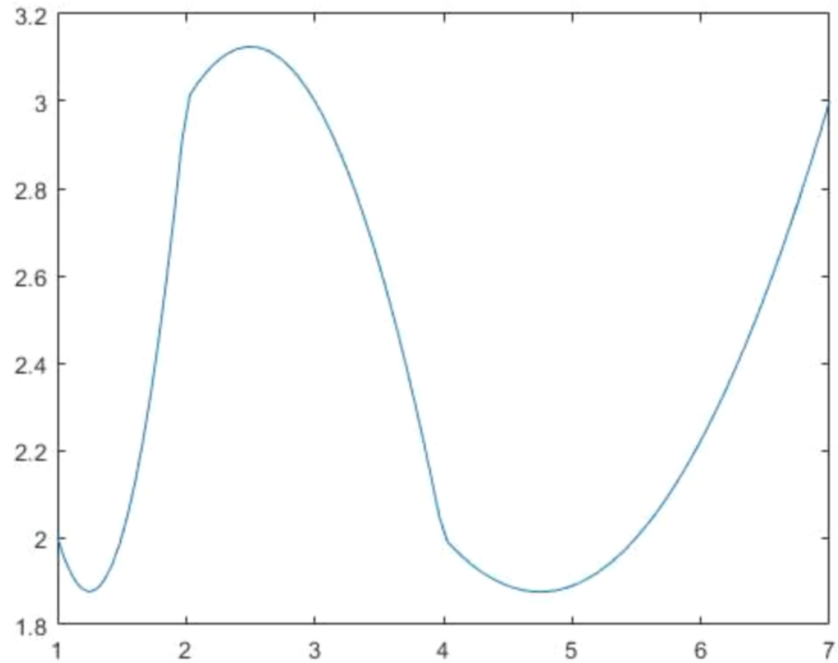
N3
1
0.8
0.6
0.4
0.2
0
-0.2
1
2
3
4
5
6
7



N6
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7


N_4
1
0.8
0.6
0.4
0.2
0
-0.2
1
2
3
4
5
6
7


3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
1
2
3
4
5
6
7



Published with MATLAB® R2021b

Editor - P2elements.m

Asign1.m | CP1_p6.m | Asig2.m | P2elements.m | Prob3.m | Prob7.m

```
function [y]=P2elements(x, x1, x2, x3)
    t = size(x);
    y = zeros([size(x)]);
    for ii=1:t(2)
        if x(ii) >= min([x1, x2, x3]) && x(ii) <= max([x1, x2, x3])
            y(ii) = (x(ii) - x2)*(x(ii) - x3)/((x1 - x2)*(x1 - x3));
        else
            y(ii) = 0;
        end
    end
end
```

Code 6, 7

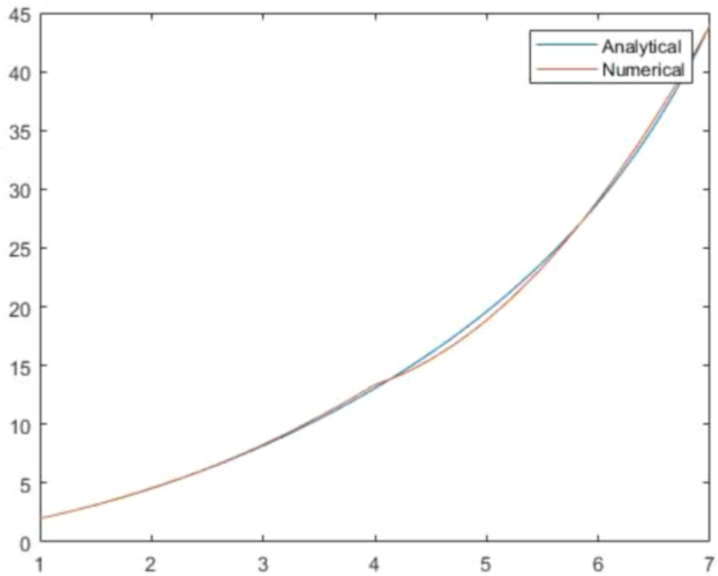
12 February 2022 19:12

```
syms x
% N_all = [P2elemNew(x, 1.0, 1.5, 2.0), (P2elemNew(x, 2.0, 1.5, 1.0) + P2elemNew(x, 2.0, 3.0, 4.0)),...
%     (P2elemNew(x, 4.0, 5.5, 7.0) + P2elemNew(x, 4.0, 3.0, 2.0)),  P2elemNew(x, 7.0, 4.0, 5.5), ...
%     P2elemNew(x, 1.5, 1.0, 2.0), P2elemNew(x, 3.0, 2.0, 4.0), P2elemNew(x, 5.5, 4.0, 7.0)];

N_all = [P2elemNew(x, 1.0, 1.5, 2.0), P2elemNew(x, 2.0, 3.0, 4.0), P2elemNew(x, 4.0, 5.5, 7.0);...
    P2elemNew(x, 1.5, 1.0, 2.0), P2elemNew(x, 3.0, 2.0, 4.0), P2elemNew(x, 5.5, 4.0, 7.0);...
    P2elemNew(x, 2.0, 1.5, 1.0), P2elemNew(x, 4.0, 3.0, 2.0), P2elemNew(x, 7.0, 4.0, 5.5)];

% N_all{1} = @(x) P2elements(x, 1.0, 1.5, 2.0);
% N_all{2} = @(x) (P2elements(x, 2.0, 1.5, 1.0) + P2elements(x, 2.0, 3.0, 4.0));
% N_all{3} = @(x) (P2elements(x, 4.0, 5.5, 7.0) + P2elements(x, 4.0, 3.0, 2.0));
% N_all{4} = @(x) P2elements(x, 7.0, 4.0, 5.5);
% N_all{5} = @(x) P2elements(x, 1.5, 1.0, 2.0);
% N_all{6} = @(x) P2elements(x, 3.0, 2.0, 4.0);
% N_all{7} = @(x) P2elements(x, 5.5, 4.0, 7.0);
f(x) = N_all(4);
nel = 3;
n_nodes = 7;
x_n = [1; 2; 4; 7; 1.5; 3; 5.5].';
LG=[1, 2, 3; 5, 6, 7; 2, 3, 4];
EtaG=[1];
g_bound = 2;
K = zeros(7, 7);
F = zeros(7, 1);
for iel=1:nel
    nodes_el = LG(:, iel);
    xe = x_n(nodes_el);
    [Ke, Fe] = elementKandF(xe,N_all,nodes_el,g_bound, iel);
    Ke, Fe
    for ii=1:3
        for jj=1:3
            K(nodes_el(ii), nodes_el(jj)) = K(nodes_el(ii), nodes_el(jj)) + Ke(ii, jj);
        end
        F(nodes_el(ii)) = F(nodes_el(ii)) + Fe(ii);
    end
end
U=K\F;
x_plt = linspace(1,7,100);
u_anl = analyticalSol(x_plt);
u_num = U(1)*P2elements(x_plt, 1.0, 1.5, 2.0) + U(2)*(P2elements(x_plt, 2.0, 1.5, 1.0) + P2elements(x_plt, 2.0, 3.0, 4.0))...
    +U(3)*(P2elements(x_plt, 4.0, 5.5, 7.0) + P2elements(x_plt, 4.0, 3.0, 2.0)) + U(4)*P2elements(x_plt, 7.0, 4.0, 5.5)...
    +U(5)*P2elements(x_plt, 1.5, 1.0, 2.0) + U(6)*P2elements(x_plt, 3.0, 2.0, 4.0) + U(7)*P2elements(x_plt, 5.5, 4.0, 7.0);
plot(x_plt,u_anl, 'DisplayName','Analytical');
hold on
plot(x_plt,u_num, 'DisplayName','Numerical');
legend
norm(u_num - u_anl)


function [Ke, Fe]=elementKandF(xe,N_all,nodes_el,g_bound, iel)
  % <<-------------------------->>
  % Complete with the values of Ke and Fe
  min_val = min(xe);
  max_val = max(xe);
  Ke = zeros(3, 3);
  Fe = zeros(3, 1);
  for ii=1:3
      i_a = nodes_el(ii);
      if i_a~=1
          for jj=1:3
              syms x
              i_b = nodes_el(jj);
%               diff(N_all(i_a))
%               diff(N_all(i_a))*(diff(N_all(i_b)) + N_all(i_b))
              f = difdif(N_all(ii,iel),N_all(jj,iel));
%               int(f, x, min_val, max_val)
              Ke(ii,jj) = int(f, x, min_val, max_val);
```

```
            end
            syms x
            f(x) = N_all(ii,iel);
            if i_a==4
                Fe(ii,:) = lEvaluate(f(x),x, min_val, max_val) + 20*f(7);
            else
                Fe(ii,:) = lEvaluate(f(x),x, min_val, max_val);
            end

        else
            Ke(ii,ii) = 1.0;
            Fe(ii,:) = g_bound;
        end
    end
    % <<-------------------------->>
end

function [auv]=aEvaluate(u,v, min_val, max_val)
    syms x
    auv = int((diff(u)*(diff(v) + v)), min_val, max_val);
end

function [lv]=lEvaluate(v,x, min_val, max_val)
    syms x
    lv = int((x * v), min_val, max_val);
end

function [ing]=difdif(v,u)
    ing = diff(u)*(diff(v) + v);
end

function [u]=analyticalSol(x)
    u = -12*exp(-6) + 12*exp(x-7) + 0.5*((1+x).^2);
end

function [y]=P2elemNew(x, x1, x2, x3)
    y = (x - x2)*(x - x3)/((x1 - x2)*(x1 - x3));
end
```

```
Ke =

    1.0000         0         0
   -3.3333    5.3333   -2.0000
    0.5000   -3.3333    2.8333


Fe =

    2.0000
    1.0000
    0.3333


Ke =

    0.6667   -0.6667         0
   -2.0000    2.6667   -0.6667
    0.3333   -2.0000    1.6667


Fe =

    0.6667
    4.0000
    1.3333


Ke =
```

```
    0.2778   -0.2222   -0.0556
   -1.5556    1.7778   -0.2222
    0.2778   -1.5556    1.2778


Fe =

    2.0000
   11.0000
   23.5000


ans =

    3.3958
```



Published with MATLAB® R2021b

Saturday, February 12, 2022 5:45 PM

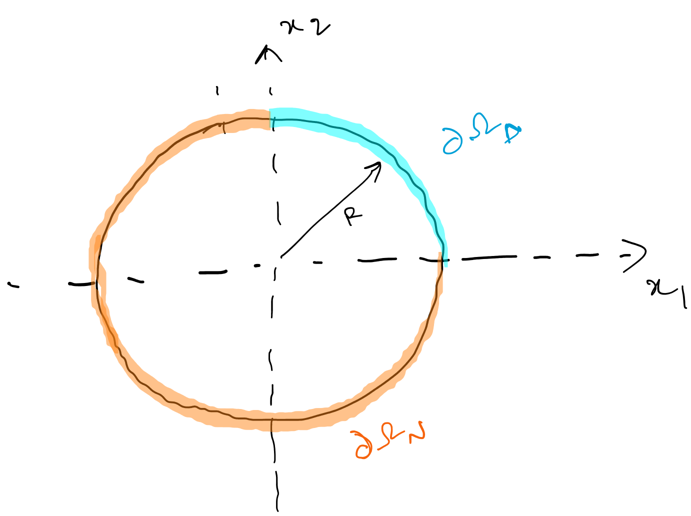
Given $\Omega = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R^2 \}$

for $R > 0$

& $\partial\Omega_D = \partial\Omega \cap \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 , x_2 \geq 0 \}$

$\partial\Omega_N = \partial\Omega / \partial\Omega_D$

which can be represented as



strong form

$$-\frac{1}{2} \Delta u = \frac{2}{R^2} \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega_D$$

$$\frac{1}{2} \nabla u \cdot \hat{n} = -\frac{1}{R} \quad \text{on } \partial\Omega_N$$

our problem is analogous to heat flux problem

where $k = \frac{1}{2}$, $f = \frac{2}{R^2}$

& we also have $q = 0$, $\mathbb{H} = -\frac{fR}{2} = -\frac{2}{R^2} \times \frac{R}{2} = -\frac{1}{R}$

& we also have $g = 0$, $H = \frac{-fR}{2} = \frac{-2}{R^2} \times \frac{R}{2} = \frac{-1}{R}$

solution to this general problem is

$$u(x_1, x_2) = g - \frac{f}{4k}\left(x_1^2 + x_2^2 - R^2\right)$$

$$= \frac{-\cancel{2}}{R^2} \times \frac{\cancel{2}}{\cancel{4}}\left(x_1^2 + x_2^2 - R^2\right)$$

$$= 1 - \left(\frac{x_1^2 + x_2^2}{R^2}\right)$$

using this sotion in above equations i.e in strong form

$$\frac{-1}{2}\Delta u = \frac{2}{R^2}$$

$$\Rightarrow \frac{-1}{2}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) - \frac{2}{R^2} = 0$$

$$\Rightarrow \frac{-1}{2} \times \left(\frac{-2}{R^2} - \frac{2}{R^2}\right) - \frac{2}{R^2}$$

$$\Rightarrow \frac{2}{R^2} - \frac{2}{R^2} = 0$$ $\therefore$ satisfied

Neumam Boundary

$$\frac{1}{2}\nabla u \cdot \hat{n} + \frac{1}{R} = 0$$

$$\frac{1}{2} \nabla u \cdot \hat{n} + \frac{1}{R} = 0$$

$$\Rightarrow \frac{1}{2} \times \left(-\frac{2x_1}{R^2}, -\frac{2x_2}{R^2} \right) \cdot \left(\frac{x_1}{R}, \frac{x_2}{R} \right) + \frac{1}{R}$$

$$\Rightarrow \frac{1}{2} \times \left(-\frac{2x_1^2}{R^3} - \frac{2x_2^2}{R^3} \right) + \frac{1}{R}$$

$$\Rightarrow -\frac{1}{2} \left(\frac{2 \times (x_1^2 + x_2^2)}{R^3} \right) + \frac{1}{R}$$

we know $x_1^2 + x_2^2 = R^2$

$$\Rightarrow -\frac{1}{\cancel{2}} \times \cancel{2} \times \frac{1}{R} + \frac{1}{R} = 0 \quad \therefore \text{proved}$$

Saturday, February 12, 2022 6:04 PM

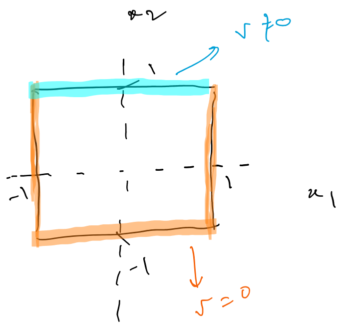
Given domain $\Omega = [-1,1] \times [-1,1]$ &

weak form

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = -\int_\Omega 4 v(x_1,x_2)\, dx + \int_{-1}^{1} 2v(x_1,1)\, dx_1$$

$\forall \; v \in \{ w: \Omega \to \mathbb{R} \text{ smooth} \mid w(x_1,x_2) = 0 \text{ if } x_1,x_2 \in \partial\Omega \;\&\; x_2 \neq 1 \}$

the solution we have is $u(x_1,x_2) = x_1^2 + x_2^2$



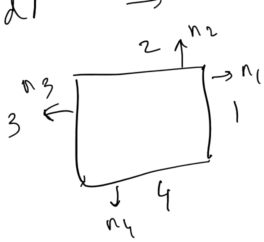
From the weak form Using byparts 2.14.5

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_{\partial\Omega} v . \nabla u . \hat{n} \, d\Gamma - \int_\Omega v \, \mathrm{div}(\nabla u) \, d\Omega$$

plug this

$$\int_{\partial\Omega} v . \nabla u . \hat{n} \, d\Gamma - \int_\Omega v \, \mathrm{div}(\nabla u) \, d\Omega = -\int_\Omega 4v \, d\Omega + \int_{-1}^{1} 2v(x_1,1)\, dx_1 \quad - (2a)$$

the term $\int_{\partial\Omega} v . \nabla u . \hat{n} \, d\Gamma \rightarrow$ can be split in 4 sections



$$\Rightarrow \int^{1} v(1,x_2) \, \nabla u(1,x_2)^{n_1} dx_2 + \int_{-1}^{1} v(-1,x_2) \, \nabla u(-1,x_2)^{n_3} dx_2$$

n4

$$\Rightarrow \int_{-1} v(1, x_2)\ \nabla u(\ldots$$

$$+ \int_{-1}^{1} v(x_1, 1)\ \nabla u(x_1, 1)\cdot n_2 \, dx_1 + \int_{-1}^{1} v(x_1, -1)\ \nabla u(x_1, -1)\cdot n_4 \, dx_2$$

Using $\nabla u = [2x_1, 2x_2]$ & $n_1 = 1, 0$

$n_2 = 0, 1$

$\Rightarrow$ $n_3 = -1, 0$

$n_4 = 0, -1$

we get

$$\Rightarrow \int_{-1}^{1} \cancel{v(1, x_2)} \times 2 \, dx_2 + \int_{-1}^{1} \cancel{v(-1, x_2)} \times 2 \, dx_2$$

$$+ \int_{-1}^{1} v(x_1, 1) \times 2 \, dx_1 + \int_{-1}^{1} \cancel{v(x_1, -1)} \times 2 \, dx_1$$

here $v(1, x_2) = v(-1, x_2) = v(x_1, -1) = 0$ as stated in space of $\mathcal{V}$

$\Rightarrow$ Using this in equation 2a

$$\int_{-1}^{1} 2 \cancel{v(x_1, 1)} \, dx_1 - \int_{\Omega} v \operatorname{div}(\nabla u) \, d\Omega + \int_{\Omega} 4 v \, d\Omega - \int_{-1}^{1} 2 \cancel{v(x_1, 1)} \, dx_1 = 0$$

$$\Rightarrow \int_{\Omega} v \left[- \operatorname{div}(\nabla u) + 4 \right] d\Omega = 0$$

we know $\operatorname{div}(\nabla u) = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 2 + 2 = 4$

$$\int_{\Omega} v \left[\underbrace{-4 + 4}_{0} \right] d\Omega = 0 \quad \therefore \text{ proved that}$$

$$\therefore \int_{\Omega} v [0] \, d\Omega$$

$u(x_1, x_2) = x_1^2 + x_2^2$ satisfies

$\therefore \int_{\Omega}$

$u(x_1, x_2) = x_1^2 + x_2^2$ satisfies this weak form.