

HW 5

A Variational Method with a almost Spectral Basis (53)

Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R^2\}$ for $R = 1$, $\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the problem: Find $u \in \Omega \rightarrow \mathbb{R}$ such that

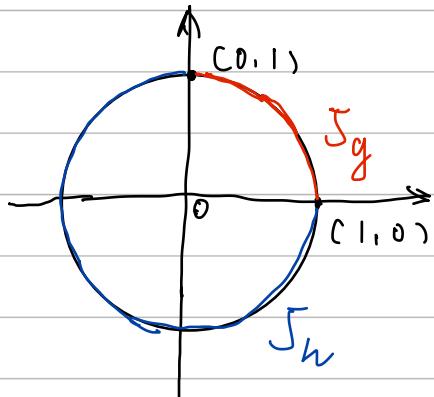
$$-\frac{1}{2}\Delta u = \frac{2}{R^2} \quad \text{in } \Omega \tag{1a}$$

$$u = 0 \quad \text{on } \Gamma_g \tag{1b}$$

$$\frac{1}{2}\nabla u \cdot \vec{n} = -\frac{1}{R} \quad \text{on } \Gamma_h. \tag{1c}$$

1. (10) Construct a variational equation that u satisfies, following the standard recipe.

1. The domain Ω has nonoverlapping boundaries, Γ_g and Γ_h as follows:



$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R^2\} \text{ for } R=1$$

$$\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$$

$$\Gamma_h = \partial\Omega \setminus \Gamma_g$$

Define $W = \{w \in \Omega \rightarrow \mathbb{R} \text{ smooth}\}$.

Test space $V = \{v \in W \mid v(\vec{x}) = 0 \text{ for all } \vec{x} \in \Gamma_g\}$.

Trial space $U = \{u \in W \mid u(\vec{x}) = 0 \text{ for all } \vec{x} \in \Gamma_h\}$.

$$V = V$$

Multiply (1a) with v and integrate over Ω for all $v \in V$

$$\int_{\Omega} -\frac{1}{2}\Delta u v = \int_{\Omega} \frac{2}{R^2} v \quad \text{for all } v \in V$$

Integrate by parts:

$$\frac{1}{2} \int_D \nabla u \cdot \nabla v - \frac{1}{2} \int_{\partial D} \nabla u \cdot v \vec{n} = \int_D \frac{\nabla v}{R^2}$$

$$\frac{1}{2} \int_D \nabla u \cdot \nabla v = \frac{1}{2} \int_{J_g} \nabla u \cdot v \vec{n} + \frac{1}{2} \int_{J_h} \nabla u \cdot v \vec{n} + \int_D \frac{\nabla v}{R^2}$$

$$\frac{1}{2} \int_D \nabla u \cdot \nabla v = - \int_{J_h} \frac{1}{R} v + \int_D \frac{\nabla v}{R^2}$$

The weak form reads:

Find $u \in S$ such $a(u, v) = l(v)$ for all $v \in V$.

$$a(u, v) = \int_D \nabla u \cdot \nabla v$$

$$l(v) = \int_D \frac{\nabla v}{R^2} - \int_{J_h} \frac{v}{R}$$

2. Essential boundary: $u=0$ over J_g

Natural boundary: $\frac{1}{2} \nabla u \cdot \vec{n} = -\frac{1}{R}$ over J_h

3.

3. Consider the approximation space

$$\mathcal{W}_h = \text{span} \left(\sin \left(\pi (x_1^2 + x_2^2) \right), \cos \left(\frac{\pi}{2} (x_1^2 + x_2^2) \right), 1 \right).$$

- (a) (10) Identify test and trial spaces, and active and constrained indices, naming the basis functions with indices in the order they appear above.
- (b) (5) In this problem, there is a possibility of selecting a smaller space \mathcal{W}_h without changing the results. What is this smaller space \mathcal{W}_h ? Identify active and constrained indices in this new space.
- (c) (15) Using the smaller space \mathcal{W}_h , compute the stiffness matrix and load vector.
- (d) (5) Find the numerical approximation. Plot it together with the exact solution.
- (e) (5) Do you think the numerical approximation would change if we change the boundary condition on the Neumann boundary to

$$\frac{1}{2} \nabla u \cdot \vec{n} = -\frac{2}{R}?$$

3(a).

Given $\mathcal{W}_h = \text{span} (\sin(\pi(x_1^2 + x_2^2)), \cos(\frac{\pi}{2}(x_1^2 + x_2^2)), 1)$

Let $N_1 = \sin(\pi(x_1^2 + x_2^2))$, $N_2 = \cos(\frac{\pi}{2}(x_1^2 + x_2^2))$, $N_3 = 1$. as global basis functions.

Define test space V_h :

$$V_h = \{ v_1 N_1 + v_2 N_2 \mid (v_1, v_2) \in \mathbb{R}^2 \}.$$

Trial space S_h :

$$S_h = \{ v_h + v_3 g \mid v_3 \in \mathbb{R} \}.$$

Using the notation defined above, active indices are:

$$D_a = \{ v_1, v_2 \}$$

Constrained indices are:

$$D_g = \{ v_3 \}.$$

3cb) One possibility is Only using trigonometric functions

This space will work for the problem, because the Dirichlet boundary is zero on Γ_g , automatically met by the following space:

$$W_h = \text{Span}(\sin(\pi(x_1^2 + x_2^2)), \cos(\pi(x_1^2 + x_2^2)))$$

$$\forall v_1 \sin(\pi(x_1^2 + x_2^2)) + v_2 \cos(\frac{\pi}{2}(x_1^2 + x_2^2)) \text{ for } (x_1, x_2) \in \Omega$$

$$\Rightarrow v_1 \cdot 0 + v_2 \cdot 0 = 0.$$

$$D_h = \{v_1, v_2\}$$

$$\Omega_g = \emptyset \text{. an empty set.}$$

3(c). Given

$$V_h = \{v_1 N_1 + v_2 N_2\}$$

$$S_h = \{v_1 N_1 + v_2 N_2\}.$$

We know $a(u, v)$ and $f(v)$ are defined as follow:

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$$

$$f(v) = \int_{\Omega} \frac{\partial v}{\partial \vec{n}} - \int_{\Gamma_h} \frac{v}{\vec{n}}$$

We can compute the stiffness matrix as:

$$a(N_1, N_1) = \int_{\Omega} (\sin(\pi(x_1^2 + x_2^2))) \cdot \nabla (\sin(\pi(x_1^2 + x_2^2)))$$

$$= \int_{\Omega} \left[\begin{array}{cc} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} \end{array} \right] \left[\begin{array}{c} \frac{\partial N_1}{\partial x_1} \\ \frac{\partial N_1}{\partial x_2} \end{array} \right] d\vec{x}$$

Using matlab symbolic calculation, we can have

$$a(N_1, N_1) = \frac{\pi^3}{2} \approx 15.503$$

Similarly, we can compute $a(N_1, N_2)$

$$\begin{aligned} a(N_1, N_2) &= \frac{1}{2} \int_D D(\sin(\pi(x_1^2 + x_2^2))) \cdot D(\cos(\pi(x_1^2 + x_2^2))) d\vec{x} \\ &= \frac{1}{2} \int_D \left[\frac{\partial N_1}{\partial x_1} \frac{\partial N_1}{\partial x_2} \right] \left[\begin{array}{c} \frac{\partial N_2}{\partial x_1} \\ \frac{\partial N_2}{\partial x_2} \end{array} \right] d\vec{x} \\ &= \frac{1}{2} \int_D \frac{\partial N_1}{\partial x_1} \frac{\partial N_2}{\partial x_1} + \frac{\partial N_1}{\partial x_2} \frac{\partial N_2}{\partial x_2} d\vec{x} \end{aligned}$$

$$a(N_1, N_2) = a(N_2, N_1) = \frac{20\pi}{9} \approx 6.9813$$

Finally, $a(N_2, N_2)$ reads:

$$\begin{aligned} a(N_2, N_2) &= \frac{1}{2} \int_D D(\cos(\pi(x_1^2 + x_2^2))) \cdot D(\cos(\pi(x_1^2 + x_2^2))) d\vec{x} \\ &= \frac{1}{2} \int_D \left[\frac{\partial N_2}{\partial x_1} \frac{\partial N_2}{\partial x_1} \right] \left[\begin{array}{c} \frac{\partial N_2}{\partial x_1} \\ \frac{\partial N_2}{\partial x_2} \end{array} \right] d\vec{x} \\ &= \frac{1}{2} \int_D \frac{\partial N_2}{\partial x_1} \frac{\partial N_2}{\partial x_1} + \frac{\partial N_2}{\partial x_2} \frac{\partial N_2}{\partial x_2} d\vec{x} \\ &= \underbrace{\left(\pi * (\pi^2 + 4) \right)}_{8} \approx 5.4466 \end{aligned}$$

Stiffness matrix reads:

$$K = \begin{bmatrix} a(N_1, N_1) & a(N_1, N_2) \\ a(N_2, N_1) & a(N_2, N_2) \end{bmatrix} = \begin{bmatrix} \frac{\pi^3}{2} & \frac{20\pi}{9} \\ \frac{20\pi}{9} & \frac{\pi^3 + 4\pi}{8} \end{bmatrix}$$

$$\text{Load vector: } \ell(v) = \int_D \frac{\partial v}{\partial x} - \int_{\Gamma_h} \frac{v}{R}$$

$$\ell(N_1) = \int_D \frac{\partial}{\partial x} N_1 - \int_{\Gamma_h} \frac{N_1}{R}$$

= 4.00.

$$\ell(N_2) = \int_D \frac{\partial}{\partial y} N_2 - \int_{\Gamma_h} \frac{N_2}{R}$$

= 4.00.

$$F = \begin{bmatrix} 4.00 \\ 4.00 \end{bmatrix}$$

$$(d). \quad U = K^{-1} F = \begin{bmatrix} -0.1720 \\ 0.9548 \end{bmatrix}$$

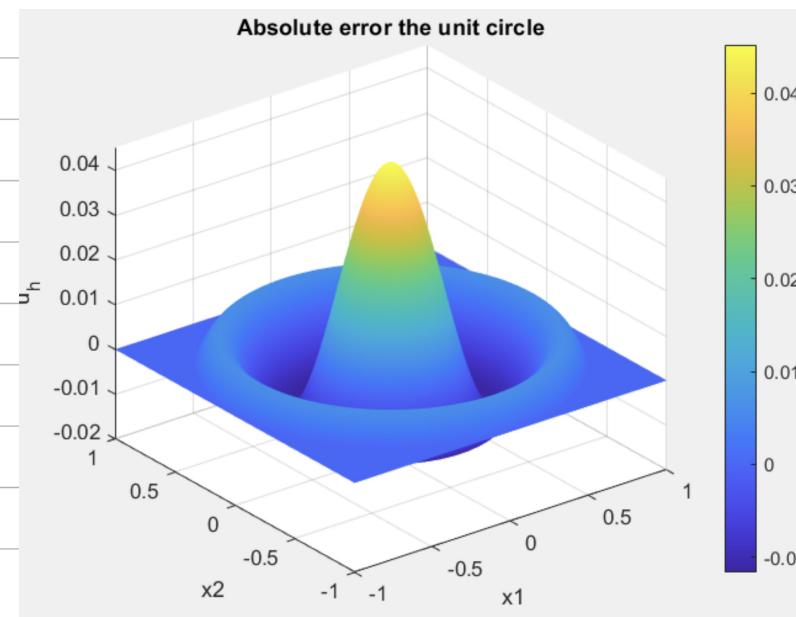
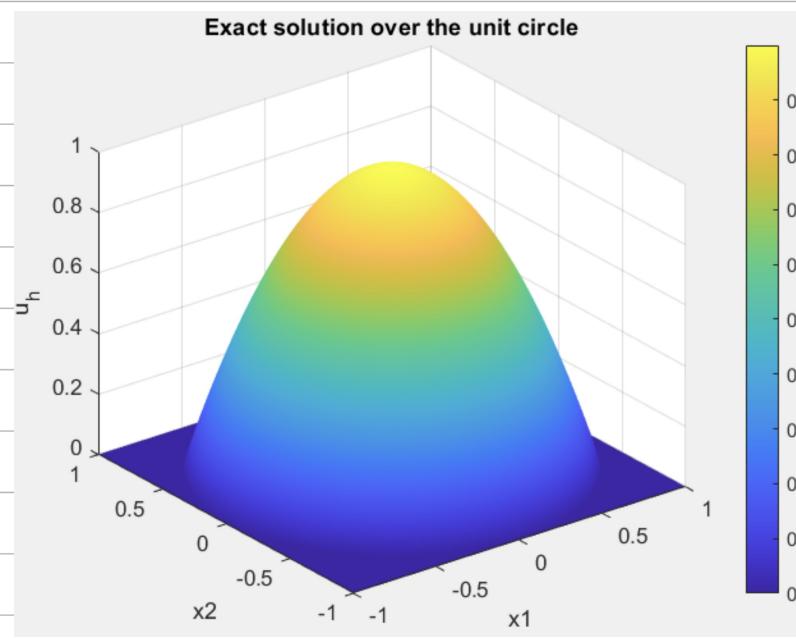
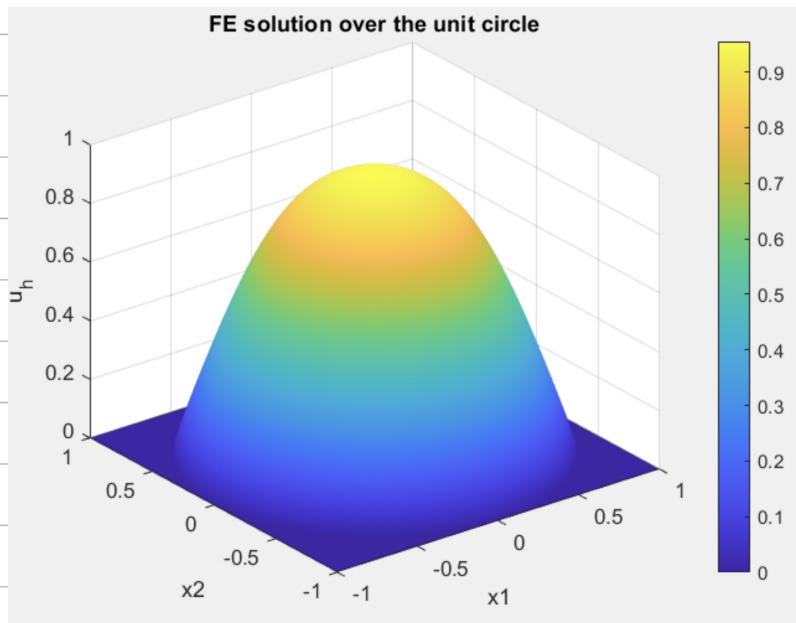
$$U(\vec{x}) = -0.1720 \sin(\pi(x_1^2 + x_2^2)) + 0.9548 \cos(\pi(x_1^2 + x_2^2))$$

The exact solution for a unit disk is:

$$U(x, y) = g - \frac{f}{4K} (x_1^2 + x_2^2 - R^2)$$

where $R = 1$, $K = \frac{1}{2}$, $f = -2$, $g = 0$.

$$U(x, y) = 1 - x_1^2 - x_2^2$$



(e). Given $\frac{1}{2} \nabla u \cdot \vec{n} = -\frac{2}{R}$, the load vector will change.

Load vector: $f(v) = \int_D \frac{2v}{R^2} - \int_{S_h} \frac{2v}{R}$

For N_1 , $f(N_1) = \int_D \frac{2}{R^2} N_1 - \int_{S_h} \frac{2N_1}{R}$

But, $\int_{S_h} \frac{2N_1}{R} = 0$ so is $\int_{S_h} \frac{2N_2}{R} = 0$

Therefore, the change in Newman B.C. will not change the solution.

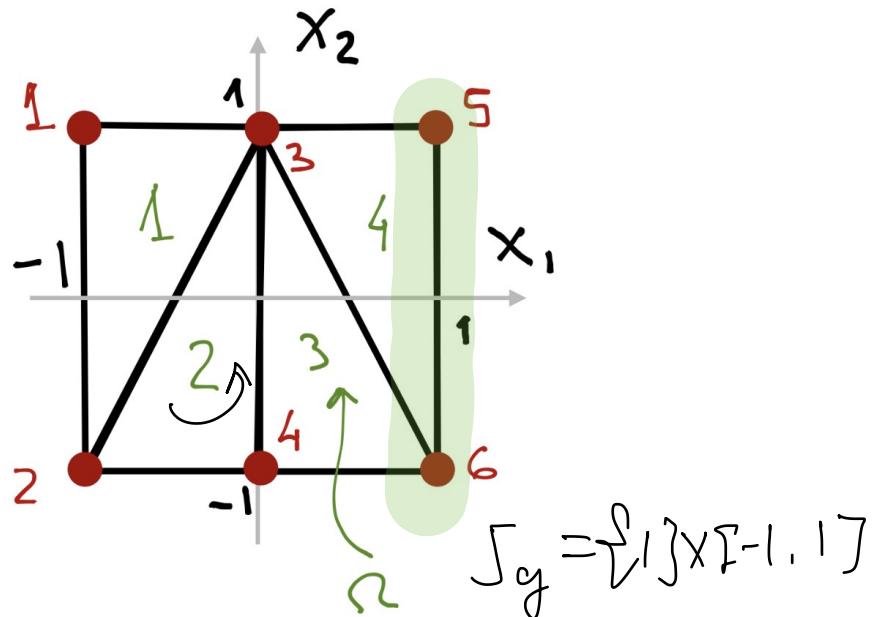
Manual Assembly, Once More (60)

Let $\Omega = [-1, 1] \times [-1, 1]$, $\Gamma_g = \{1\} \times [-1, 1]$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the variational equation that $u: \Omega \rightarrow \mathbb{R}$ satisfies:

$$\int_{\Omega} \nabla u \cdot \nabla v + uv \, d\Omega = \int_{\Omega} (x_1 + x_2)v \, d\Omega + \int_{\Gamma_h} (x_2^2 - 1)v \, d\Gamma$$

for all $v \in \mathcal{V} = \{v: \Omega \rightarrow \mathbb{R} \mid v = 0 \text{ on } \Gamma_g\}$, where (x_1, x_2) are the Cartesian coordinates in Ω . The function u satisfies the essential boundary condition $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in \Gamma_g$.

Consider then the mesh shown in the figure, made of all P_1 elements:



1. (5) What is the local-to-global map for the mesh?
2. (5) Identify S_h and \mathcal{V}_h by providing the general expression for their functions in terms of the P_1 basis functions of the mesh. Identify active and constrained indices.
3. (10) Evaluate the shape function N_3^2 and its derivative ∇N_3^2 at $(x_1, x_2) = (-0.5, -0.1)$.

1. Assuming the vertices are ordered counter-clockwise, the local-to-global map for conforming mesh is:

$$LG = \begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 2 & 6 & 6 \\ 3 & 4 & 3 & 5 \end{bmatrix}$$

2. The vertices belonging to Γ_g are 5 and 6.

Let $N_i(x_1, x_2)$ be the global basis functions. We

require $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in \mathcal{T}_g$. Therefore,
Test spaces:

$$\mathcal{V}_h = \text{Span}(N_1, N_2, N_3, N_4)$$

Trial spaces:

$$S_h = \{ V_h + x_2 N_5 + x_2 N_6 \mid x_2 \in \mathbb{R} \}.$$

active set:

$$\mathcal{J}_a = \{ 3, 4, 5, 6 \}.$$

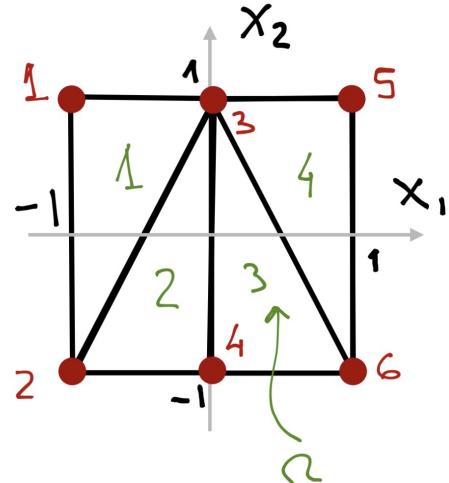
Constrained set:

$$\mathcal{J}_g = \{ 1, 2 \}.$$

We select $\bar{u} = x_2$

3. The local shape function N_3^2 can be expressed as:

$$N_3^2 = \lambda_3(x_1, x_2) = \frac{1}{2A} [- (x_2 - x_3)(x_1 - x'_1) + (x_2 - x'_1) \\ (x_2 - x'_3)]$$



$$2A = (x_1^2 - x_1^1)(x_1^3 - x_1^1) - (x_2^2 - x_2^1)(x_2^3 - x_2^1)$$

From the figure, we know:

$$x^1 = (0, 1).$$

$$x^2 = (-1, -1)$$

$$x^3 = (0, -1)$$

$$N_3^2 = \frac{1}{2} \cdot [(1 - (-1)) (x_1 - 0) + (-1 - 0)(x_2 - 1)] \\ = \frac{1}{2} [2x_1 - (x_2 - 1)]$$

$$\nabla N_3^2 = \frac{1}{2A} \begin{pmatrix} x_2' - x_2 \\ x_1' - x_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

Therefore

$$N_3^2(-0.5, -0.1) = 0.05$$

$$\nabla N_3^2(-0.5, -0.1) = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Q. Using the trial / test spaces defined above, we can formulate the discrete weak form as:
find $u_h \in S_h$ such that

$$a_h(u_h, v_h) = l(v_h) \quad \text{for all } v_h \in V_h$$

$$a_h(u_h, v_h) = \int_{\Omega} \nabla u_h \cdot \nabla v_h + u_h v_h \, d\Omega$$

$$l(v_h) = \int_{\Omega} (x_1 + x_2) v_h \, d\Omega + \int_{\Gamma_h} (x_2^2 - 1) v \, ds$$

Since we are using P1 element $\nabla u_h, \nabla v_h$ are constants, we can simplify the local stiffness matrix as:

$$k_{ab}^e = \int_{\Omega^e} \nabla N_b^e \cdot \nabla N_a^e + N_b^e N_a^e d\Omega^e$$

$$= A_e \nabla N_b^e \cdot \nabla N_a^e + \int_{\Omega^e} N_b^e N_a^e d\Omega^e$$

$$k'_{ii} = A_i \left[\underbrace{\begin{bmatrix} \frac{\partial N'_i}{\partial x_1} & \frac{\partial N'_i}{\partial x_2} \end{bmatrix}}_{(dN'_i)^T} \right] \left[\begin{bmatrix} \frac{\partial N'_i}{\partial x_1} \\ \frac{\partial N'_i}{\partial x_2} \end{bmatrix} \right] + \int_{\Omega^e} N'_i N'_i d\Omega^e$$

$$dN'_i = \begin{bmatrix} x_2^2 - x_2^3 \\ x_1^3 - x_1^2 \end{bmatrix} \quad N'_i = \frac{1}{2A} (dN'_i(1) \cdot (x_1 - x_1^2) + dN'_i(2) \cdot (x_2 - x_2^3)).$$

For load term, we have

$$f_a^e = \int_{\Omega^e} (x_1 + x_2) N_i d\Omega + \int_{J_h} (x_2^2 - 1) N_i d\Omega$$

Special attentions are given to the treatment of element boundary. As the integrand is $(x_2^2 - 1)$.

its value is non zero on the $J_{h \in \{1, 2\}} = \{-1\} \times [-1, 1]$.

Therefore, we only need to evaluate the integration

$$\text{of } \int_{J_h} (x_2^2 - 1) N_1 \text{ and } \int_{J_h} (x_2^2 - 1) N_2$$

By following the same idea, we can compute the local stiffness matrix and load vector.

For element 1; the ke reads:

$$k_e^1 = \begin{bmatrix} 1.4167 & -0.1667 & -0.9167 \\ -0.1667 & 0.4167 & 0.0833 \\ -0.9167 & 0.0833 & 1.1667 \end{bmatrix}$$

$$F_e^1 = \begin{bmatrix} -0.0833 \\ -0.2500 \\ 0.00 \end{bmatrix}$$

For element 2, the ke reads:

$$k_e^2 = \begin{bmatrix} 0.4167 & 0.0833 & -0.1667 \\ 0.0833 & 1.1667 & -0.9167 \\ -0.1667 & -0.9167 & 1.4167 \end{bmatrix}$$

$$F_e^2 = \begin{bmatrix} -0.0833 \\ -0.3333 \\ 0.00 \end{bmatrix}$$

For element 3, the Ke reads:

$$K_e^3 = \begin{bmatrix} 1.4167 & -0.9167 & -0.1667 \\ -0.9167 & 1.1667 & 0.0833 \\ -0.1667 & 0.0833 & 0.4167 \end{bmatrix}$$

$$F_e^3 = \begin{bmatrix} -0.0833 \\ -0.07 \\ 0.0833 \end{bmatrix}$$

for element 4, the Ke reads:

$$K_e^4 = \begin{bmatrix} 1.1667 & 0.0833 & -0.9167 \\ 0.0833 & 0.4167 & -0.1667 \\ -0.9167 & -0.1667 & 1.4167 \end{bmatrix}$$

$$F_e^4 = \begin{bmatrix} 0.3333 \\ 0.2500 \\ 0.4167 \end{bmatrix}$$

$$5. \quad S_g = \{1\} \times [-1, 1].$$

$S_g = \{5, 6\}$. containing vertex 5 and 6.

$$S_h = \partial D / S_g = \{-1\} \times [-1, 1] \cup \\ [-1, 1] \times \{1\} \cup \\ [-1, 1] \times \{-1\}.$$

For natural B.C.

$$F_a = \int_{S_h} (x_2^2 - 1) N_a dA$$

Special attentions are given to the treatment of hennment boundary. As the integrand is $(x_2^2 - 1)$.

its value is nonzero only on the $S_{h, \{1, 2\}} = \{-1\} \times [-1, 1]$.

Therefore, we only need to evaluate the integration

$$\text{of } \int_{S_h} (x_2^2 - 1) N_1 \text{ and } \int_{S_h} (x_2^2 - 1) N_2$$

$$F_{n,1} = \int_{S_h} (x_2^2 - 1) N_1 = \frac{1}{2A} \int_{-1}^1 (x_2^2 - 1) \cdot (x_1^3 - x_1^2) \cdot (x_2 - x_2^2) dx_2$$

$$= -0.667$$

$$F_{n,2} = \int_{S_h} (x_2^2 - 1) N_2 = \frac{1}{2A} \int_{-1}^1 (x_2^2 - 1) \cdot (x_1^3 - x_1^2) \cdot (x_2 - x_2^2) dx_2$$

$$= -0.667$$

$$F_n = [-0.666], [-0.666], [0, 0, 0, 0, 0]$$

6. Use local to global map to assemble the global stiffness matrix, we have:

	1	2	3	4	5	6
1	1.4167	-0.1667	-0.9167	0	0	0
2	-0.1667	1.5833	0.1667	-0.9167	0	0
3	-0.9167	0.1667	3.1667	-0.3333	-0.9167	0.1667
4	0	-0.9167	-0.3333	2.8333	0	-0.9167
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Load Vector is:

	1
1	-0.7500
2	-1.2500
3	0.3333
4	-0.3333
5	1
6	-1

	1
1	-0.5067
2	-1.3643
3	0.2831
4	-0.8493
5	1
6	-1

(T).

$$u_h = -0.5067 N_1 - 1.3643 N_2 + 0.2831 N_3 \\ - 0.8493 N_4 + N_5 - N_6.$$

