Appendix A

Normed Spaces

To be able to assess convergence, or how close our approximations are to the exact solution, we need to define a way to measure distances in a vector space. The most common way to do this is through a *norm*.

Definition A.1 (Norm)**.** *Let V be a vector space. A norm is a function $\|\cdot\|\colon V \to \mathbb{R}$ such that for $v, u \in V$ and $\alpha \in \mathbb{R}$:*

1. **N.1.** $\|v\| \geq 0$*, and* $\|v\| = 0$ *if and only* $v = 0$.
2. **N.2.** $\|\alpha v\| = |\alpha|\|v\|$.
3. **N.3.** $\|v + u\| \leq \|v\| + \|u\|$ *(triangle inequality).*

The typical norm that you are familiar with is the "Euclidean norm" in $\mathbb{R}^3$. For example, if $x = (x_1, x_2, x_3)$, then $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Clearly if $\|x\| \geq 0$, and if $\|x\| = 0$, then $x = 0$. The second condition, N.2, is also simple to verify, and the triangle inequality is the common statement that the sum of the lengths of two sides of a triangle is always greater or equal than the length of the third. These three conditions are intuitive to understand in the case of $\mathbb{R}^n$, and the fact that the Euclidean norm satisfies them is easy to see. Defining a norm for vector spaces of functions is more delicate, and less intuitive. Let's look at some examples.

Examples:

A.1 For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the function $g(x) = \sqrt{2x_1^2 + 3x_2^2 + 4x_3^2}$ is also a norm in $\mathbb{R}^3$. We will not prove this.

The importance of this example is to illustrate that we can endow the same set of vectors with different norms. A simple way to think about this example is that we are using different units to measure distances in each coordinate direction.

A.2 For $v \in V_1 = \{f\colon [a, b] \to \mathbb{R} \text{ smooth }\}$, we define the L^∞-*norm*

$$\|v\|_{0,\infty} = \max_{x \in [a,b]} |v(x)|. \tag{A.1}$$

Let's check the conditions for this to be a norm, since it is simple in this case. For N.1, since $|v(x)| \geq 0$ for all $x \in [a,b]$, then $\|v\|_{0,\infty} \geq 0$. Also, if $0 = \|v\|_{0,\infty} = \max_{x\in[a,b]} |v(x)| \geq |v(x)| \geq 0$ for any $x \in [a,b]$, then $v = 0$. For N.2,

$$\|\alpha v\|_{0,\infty} = \max_{x\in[a,b]} |\alpha v(x)| = \max_{x\in[a,b]} |\alpha||v(x)| = |\alpha| \max_{x\in[a,b]} |v(x)| = |\alpha|\|v\|_{0,\infty}.$$

Finally, for N.3,

$$\begin{aligned}\|u+v\|_{0,\infty} &= \max_{x\in[a,b]} |u(x)+v(x)| \leq \max_{x\in[a,b]} |u(x)|+|v(x)| \\ &\leq \max_{x\in[a,b]} |u(x)| + \max_{x\in[a,b]} |v(x)| = \|u\|_{0,\infty} + \|v\|_{0,\infty}.\end{aligned}$$

For instance, let $[a,b] = [0,\pi]$, then:

i. If $v(x) = \cos(x)$, then $\|v\|_{0,\infty} = 1$.
ii. If $v(x) = x(x-\pi)$, then $\|v\|_{0,\infty} = -v(\pi/2) = \pi^2/4$.

A.3 For $v \in V_1$ from Example A.2, we define the L^2*-norm*

$$\|v\|_{0,2} = \left[\int_a^b v(x)^2\, dx\right]^{1/2}. \tag{A.2}$$

We will not check that this is a norm, but just state it. For $[a,b] = [0,\pi]$:

i. If $v(x) = \cos(x)$, then $\|v\|_{0,2} = \left[\int_0^\pi \cos(x)^2\, dx\right]^{1/2} = \sqrt{\pi/2}$.
ii. If $v(x) = x(x-\pi)$, then $\|v\|_{0,2} = \left[\int_0^\pi x^2(x-\pi)^2\, dx\right]^{1/2} = \pi^{5/2}/\sqrt{30}$.

A.4 For $v \in V_2 = \{w \in V_1 \mid w(a) = w(b) = 0\}$, we define the H^1*-seminorm*

$$|v|_{1,2} = \left[\int_a^b v'(x)^2\, dx\right]^{1/2}. \tag{A.3}$$

The fact that this is a norm requires a discussion of condition N.1: Why does it hold? To answer this, notice that if $|v|_{1,2} = 0$, we can conclude that $v'(x) = 0$ for all $x \in [0,1]$, since the integrand $v'(x)^2 \geq 0$ everywhere. Therefore, $v(x)$ is a constant function. Since $v(a) = 0$, then $v(x) = 0$ for $x \in [a,b]$.

Because of this discussion, $|\cdot|_{1,2}$ is not a norm in the space V_1 in Example A.2, since functions therein need not be zero at the boundaries, and hence condition N.1 is not satisfied. All we would be able to say if $|v|_{1,2} = 0$ is that v is a constant function. For example, let $[a,b] = [0,1]$,

i. If $v(x) = \sin(\pi x)$, $v \in V_2$, then $|v|_{1,2} = \left[\int_0^1 (\pi\cos(\pi x))^2\, dx\right]^{1/2} = \frac{\pi}{\sqrt{2}}$.
ii. If $v(x) = 3$, $v \notin V_2$, then $|v|_{1,2} = \left[\int_0^1 0\, dx\right]^{1/2} = 0$.

A.5 For $v \in V_1$ from Example A.2, we define the H^1*-norm*

$$\begin{aligned}\|v\|_{1,2} &= \left[\int_a^b v(x)^2\,dx + \int_a^b v'(x)^2\,dx\right]^{1/2} \\ &= \left[\|v\|_{0,2}^2 + |v|_{1,2}^2\right]^{1/2}.\end{aligned} \tag{A.4}$$

In contrast to what happens with $|v|_{1,2}$ in Example A.4, condition N.1 is satisfied in this case, since it is satisfied for $\|v\|_{0,2}$.

Notice that we talked about three different norms for space V_1 above: We defined the L^∞-norm, the L^2-norm and the H^1-norm. The three norms measure distance differently, emphasizing different aspects of the functions.

We can now define the notion of a normed space.

Definition A.2 (Normed Space.)**.** *A vector space V with a norm defined over it $\|\cdot\|: V \to \mathbb{R}$ is called a* **normed space***, and denoted by* $(V, \|\cdot\|)$.

Examples:

A.6 The space $\mathbb{R}^n$, $n \in \mathbb{N}$, with the Euclidean norm $\|\cdot\|$ is a normed space $(\mathbb{R}^n, \|\cdot\|)$, since the norm is defined for every element of $\mathbb{R}^n$.

A.7 Consider the space $V_1 = \{f: [a,b] \to \mathbb{R} \text{ smooth}\}$ with the L^∞-norm $\|\cdot\|_{0,\infty}$. Since all smooth functions are bounded in $[a,b]$, the L^∞-norm is well defined for every function in V_1, and hence $(V_1, \|\cdot\|_{0,\infty})$ is a normed space.

A.8 Consider again the space $V_1 = \{f: [a,b] \to \mathbb{R} \text{ smooth}\}$ with the L^2-norm $\|\cdot\|_{0,2}$. Since all smooth functions are bounded in $[a,b]$, the integrals needed to compute the L^2-norm are well defined for every function in V_1, and hence $(V_1, \|\cdot\|_{0,2})$ is a normed space. It is, however, a different normed space than $(V_1, \|\cdot\|_{0,\infty})$, since functions that are close in one may not be close in the other, as we shall see.

A.9 Consider the space $V_2 = \{f: (a,b) \to \mathbb{R} \text{ smooth}\}$ (notice the open interval) with the L^2-norm $\|\cdot\|_{0,2}$. The function $f(x) = 1/(x-a)$ is in V_2, since it is smooth in (a,b), but

$$\|f\|_{0,2}^2 = \int_a^b \frac{1}{(x-a)^2}\,dx = +\infty,$$

so the norm is not defined for f. Therefore, $(V_2, \|\cdot\|_{0,2})$ is *not* a normed space.

A.10 Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For such domain Ω, the norm $\|v\|_{0,2}$ of $v: \Omega \to \mathbb{R}$ is defined as

$$\|v\|_{0,2} = \left[\int_\Omega v(x)^2\,d\Omega\right]^{1/2}. \tag{A.5}$$

The set

$$L^2(\Omega) = \{v\colon \Omega \to \mathbb{R} \mid \|v\|_{0,2} < \infty\} \tag{A.6}$$

is called the $L^2(\Omega)$ **space**, and $(L^2(\Omega), \|\cdot\|_{0,2})$ is a normed space. The space $L^2(\Omega)$ is said to contain all *square-integrable* functions, and these need not be smooth. For example, if $\Omega = [-1,1]$, it contains the Heaviside function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0. \end{cases}$$

In contrast, $H(x) \notin L^2(\mathbb{R})$, since $\|H\|_{0,2} = \infty$ in this case.

A.11 Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For such domain Ω, we define the H^1*-norm* as

$$\|v\|_{1,2} = \left[\|v\|_{0,2}^2 + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{0,2}^2 \right]^{1/2}.$$

With it, we can define the $H^1(\Omega)$**-space** as

$$H^1(\Omega) = \left\{v\colon \Omega \to \mathbb{R} \mid \|v\|_{1,2} < \infty\right\}, \tag{A.7}$$

and $(H^1(\Omega), \|\cdot\|_{1,2})$ is normed space. Functions in $H^1(\Omega)$ contain all functions in which both the function and each one of its partial derivatives is square integrable. Alternatively, the function and each one of its partial derivatives is in $L^2(\Omega)$. Therefore, if a funtion $v \in H^1(\Omega)$, then $v \in L^2(\Omega)$. For example: Let $\Omega = [-1,1] \times [-1,1]$, then

☞ The definition of H^1 here is incomplete. We will have an opportunity to complete the definition later.

i. The function $v(x_1, x_2) = x_1^2 + x_2^3 \in H^1(\Omega)$, since

$$\begin{aligned} \|v\|_{1,2}^2 &= \int_{-1}^{1}\int_{-1}^{1} (x_1^2 + x_2^3)^2 \, dx_1 dx_2 + \int_{-1}^{1}\int_{-1}^{1} (2x_1)^2 \, dx_1 dx_2 \\ &\quad + \int_{-1}^{1}\int_{-1}^{1} (3x_2^2)^2 \, dx_1 dx_2 = \frac{292}{21} < \infty. \end{aligned}$$

ii. The function $v(x_1, x_2) = \ln(1+x_1) + \ln(1+x_2) \notin H^1(\Omega)$, but $v \in L^2(\Omega)$, since

$$\begin{aligned} \|v\|_{0,2}^2 &= \int_{-1}^{1}\int_{-1}^{1} (\ln(1+x_1) + \ln(1+x_2))^2 \, dx_1 dx_2 \\ &= 24 + 8\ln(4)(\ln(2) - 2) < \infty. \\ \|v\|_{1,2}^2 &= \|v\|_{0,2}^2 \\ &\quad + \int_{-1}^{1} \frac{1}{(1+x_1)^2} \, dx_1 dx_2 + \int_{-1}^{1} \frac{1}{(1+x_2)^2} \, dx_1 dx_2 = \infty. \end{aligned}$$

A.12 Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$. For $v \in H^1(\Omega)$ we can define the H^1*-seminorm* as

$$|v|_{1,2} = \left[\int_{\Omega} \|\nabla v\|^2 \, d\Omega \right]^{1/2}. \tag{A.8}$$

Notice that

$$|v|_{1,2}^2 = \int_\Omega \|\nabla v\|^2 \, d\Omega = \int_\Omega \sum_{i=1}^n \left|\frac{\partial v}{\partial x_i}\right|^2 \, d\Omega = \sum_{i=1}^n \int_\Omega \left|\frac{\partial v}{\partial x_i}\right|^2 \, d\Omega$$
$$= \sum_{i=1}^n \left\|\frac{\partial v}{\partial x_i}\right\|_{0,2}^2 .$$

This allows us to write the H^1-norm as

$$\|v\|_{1,2}^2 = \|v\|_{0,2}^2 + |v|_{1,2}^2. \tag{A.9}$$

A.13 Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. The H^m-seminorm of a function $u\colon \Omega \to \mathbb{R}$ is defined as

$$|u|_{m,2}^2 = \sum_{\substack{i_1,\ldots,i_n \geq 0 \\ i_1+\ldots+i_n=m}} \left\| \frac{\partial^m u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} \right\|_{0,2}^2 , \tag{A.10}$$

where $\partial^m u / \partial x_k^0 = u$.
For example, the H^2-seminorm in $\mathbb{R}^2$ is

$$|u|_{2,2}^2 = \left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{0,2}^2 + 2\left\|\frac{\partial^2 u}{\partial x_1 \partial x_2}\right\|_{0,2}^2 + \left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{0,2}^2 ,$$

and the H^0-seminorm is directly the L^2-norm.
The H^m-norm is then defined as

$$\|u\|_{m,2}^2 = \sum_{i=0}^m |u|_{i,2}^2 \tag{A.11}$$

With it, we can define the $H^m(\Omega)$**-space** as

$$H^m(\Omega) = \left\{v\colon \Omega \to \mathbb{R} \mid \|u\|_{m,2} < \infty\right\}. \tag{A.12}$$