We set $\overline{u}_h(x) = 1 + (\exp(-\pi) - 1)\sin x$, so that $\overline{u}_h \in \mathscr{S}_h$, $u_1 = 1$ and $u_2 = \exp(-\pi) - 1$. The load vector is

$$F = \begin{bmatrix} 1 \\ \exp(-\pi) - 1 \\ \ell(N_3) \\ \ell(N_4) \end{bmatrix} = \begin{bmatrix} 1 \\ \exp(-\pi) - 1 \\ 5(1 + \exp(-\pi))/4 \\ \exp(-\pi) - 1 \end{bmatrix}.$$

The components of the solution are obtained from $U = K^{-1}F$, or

$$U = \begin{bmatrix} 1 \\ \exp(-\pi) - 1 \\ \frac{\exp(-\pi)(-e^{\pi}(1216+765\pi)-9945\pi+1216)}{5(256+765\pi^2)} \\ \frac{12\exp(-\pi)(e^{\pi}(4-19\pi)+19\pi+52)}{256+765\pi^2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.96 \\ -0.13 \\ -0.08 \end{bmatrix}.$$

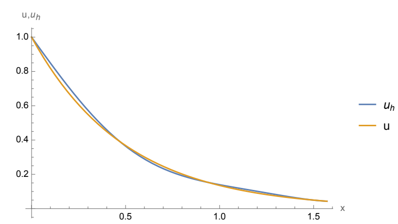

Figure 1.7

The solution is then

$$1 - 0.96\sin x - 0.13\sin 2x - 0.08\sin 4x,$$

and it is plotted in Fig. 1.7.

1.3.3 The Euler-Lagrange Equations

Imagine that a friend described a variational equation to you that a function u should satisfy. Could you find out what differential equation it corresponds to, if any, and what boundary conditions the variational equation requires u to satisfy, if any? In other words, does the fact that u satisfies a variational equation imply that it should also satisfy a differential equation and/or some boundary conditions? By definition, such boundary conditions are what we have termed natural boundary conditions for u.

For example, in engineering many problems require finding the minimizer of a functional, the most well-known one is minimizing the potential energy of a physical system. Minimizing a functional often leads to a variational equation that the minimizer needs to satisfy, and from here to a differential equation and/or boundary conditions.

You have already been in this situation: In examples 1.14, 1.15, and 1.16 we stated variational equations (1.31), (1.34) and (1.38) which we did not derive. How do we check if they imply what is stated in the example? For example, that a boundary condition is natural or essential. The process by which we answer this question will also be useful in §**??** when we perform analysis of the convergence of the finite element method.

The first question to ask is how could the variational equation define a differential equation or boundary conditions for u. An intuitive argument could be made based on the finite dimensional case, or the variational method. In that case, by testing with all the basis functions in $\mathscr{V}_h$, we could determine u_h from

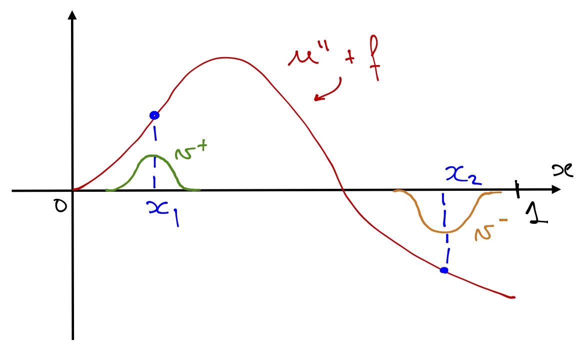

Figure 1.8 Potential choice of weighting functions to show that a weak solution is a strong solution.

the variational equation. Imagine now that we test with all functions in $\mathcal{V}$, can we determine a function u, up to perhaps the essential boundary conditions? We will see that often this will be the case; the variational equation can define a differential equation that the function u should satisfy.

When we talk about *conditions* imposed by a variational equation, we mean the following: For $u\colon \Omega \to \mathbb{R}$ that satisfies variational equation $F(u,v) = 0$ for all $v \in \mathcal{V}$, find a functional EL such that

$$F(u,v) = 0 \quad \forall v \in \mathcal{V} \qquad \Longleftrightarrow \qquad EL(u,x) = 0 \quad \forall x \in \omega \subseteq \overline{\Omega}, \tag{1.82}$$

Here ω is a subset of the closure of Ω, and as such, it may include points in both the interior and the boundary of Ω.

In the context of the Calculus of Variations, $EL(u,x) = 0$ for $x \in \omega$ is the Euler-Lagrange equation of the variational principle.

So, not only do we want to find conditions implied by the variational equation (left-to-right implication), but we want enough conditions that together imply that u satisfies the variational equation (right-to-left implication). Equation $EL(u,x) = 0 \quad \forall x \in \omega$ is called the **Euler-Lagrange** equation of the variational equation.

Let's illustrate how we do this through the simplest example. Assume that u satisfies variational equation (1.16). First, we integrate by parts the left hand side of (1.16) to remove the derivative of v', to get

$$0 = (u'(L) - d_L)\,v(L) - u'(0)\,v(0) - \int_0^L (u''(x) + f(x))\,v(x)\,dx. \tag{1.83}$$

Next, we use that we can choose any $v \in \mathcal{V}$. Referring to Fig. 1.8, we proceed by contradiction and assume that $u''(x_1) + f(x_1) > 0$ for some $x_1 \in (0,L)$. Then we can choose $v^+ \in \mathcal{V}$ as sketched in the figure[5]. In this case, using that $v^+(L) = v^+(0) = 0$,

[5] For example, we can set

$$v^+(x) = \begin{cases} 0 & \text{if } |x - x_1| > \epsilon \\ \left[1 - \left(\frac{x - x_1}{\epsilon}\right)^2\right]^3 & \text{if } |x - x_1| \le \epsilon, \end{cases}$$

where $\epsilon > 0$ is the "half-width" of v^+, and can be chosen as small as desired.

(1.83) reads

$$0 = -\int_0^L (u''(x) + f(x))\, v^+(x)\, dx. \tag{1.84}$$

But $v^+(x)$ is not negative anywhere, it is zero wherever $u''(x) + f(x)$ is negative, and both $u''(x) + f(x)$ and v^+ are positive in a neighborhood of x_1, so the integral in the right hand side of (1.84) needs to be positive, a contradiction. We conclude then that $u''(x) + f(x)$ cannot be positive anywhere. A similar argument can be made around a point in which $u'' + f$ is negative by selecting a weighting function v^-, as in the figure. We can then conclude that $u'' + f$ is neither positive nor negative anywhere in $(0, L)$, so $u''(x) + f(x) = 0$ for any $x \in (0, L)$. The function u then needs to satisfy (1.12a).

☞ It emerges from the discussion here that the set $\mathcal{V}$ should at least contain all functions v^+ and v^- for this argument to be made; the precise conditions for what functions $\mathcal{V}$ should include take the form of the "Fundamental lemma of the calculus of variations," see e.g. [2].

An important detail that often gets lost in a first view of the last argument is that each one of the different v's described above imposes a different condition on u, and that the only way for u to satisfy them all is by satisfying the differential equation (1.12a). Therefore, even though we may not have considered *all* functions $v \in \mathcal{V}$, we considered enough of them to conclude that u satisfies the differential equation.

☞ Often the following question is asked: "Wait... shouldn't u satisfy (1.12a) *only* when we consider those functions $v \in \mathcal{V}$ in Fig. 1.8? This question does not merit further consideration after realizing that whether u satisfies the differential equation or not depends on u only, and is independent of v.

The story is not over, as we see next. Since we just concluded that u needs to satisfy (1.12a), we may use it to simplify the variational equation, which reads

$$0 = (u'(L) - d_L)\, v(L) - u'(0)\, \underbrace{v(0)}_{=0} - \int_0^L \underbrace{(u''(x) + f(x))}_{=0}\, v(x)\, dx = (u'(L) - d_L)\, v(L)$$

for *any* $v \in \mathcal{V}$. In this case we can choose any $v \in \mathcal{V}$ that satisfies $v(L) \neq 0$. This implies that we need $u'(L) - d_L = 0$. So, the variational equation implies that u needs to satisfy (1.12c).

So far we proved the left-to-right implication in (1.82), that is, if $F(u, v) = 0$, then

$$0 = EL(u, x) = u''(x) + f(x) \qquad x \in (0, L) \tag{1.85a}$$

$$0 = EL(u, L) = u'(L) - d_L \tag{1.85b}$$

These are precisely (1.12a) and (1.12c).

Notice now that if u satisfies (1.85), then it satisfies (1.83) for any $v \in \mathcal{V}$, as it follows from replacing in (1.83). That is, we concluded the right-to-left implication in (1.82). Hence,

$$EL(u, x) = 0 \quad \forall x \in \omega = (0, L] \Longleftrightarrow F(u, v) = 0 \quad \forall v \in \mathcal{V}.$$

Notice that ω does not include $x = 0$. The variational equation does not require u to satisfy the condition $u(0) = g_0$, since regardless of the value of $u(0)$, u would satisfy the variational equation as long as it satisfies (1.85). Boundary condition $u(0) = g_0$ needs to be explicitly required from u, it does not arise from satisfying the variational equation, so it is an essential boundary condition.

1.3.3.1 General Steps to Obtain the Euler-Lagrange Equations

We can summarize the general steps to find the Euler-Lagrange equations next, illustrating them with variational equation (1.9a), namely,

$$\int_\Omega \left[k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x)\right]\,dx \\ - k(L)d_L v(L) = \int_\Omega f(x)v(x)\,dx. \quad (1.86)$$

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon [0,L] \to \mathbb{R} \text{ smooth } \mid w(0) = 0\}.$$

We proceed as follows:

1. **Integrate the variational equation by parts to eliminate all derivatives from the test function.** In this case, we want to eliminate all derivatives that appear on v, so we integrate by parts as many times as needed to do that. For our example in (1.9a),

$$\begin{aligned} 0 &= \int_0^L k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx \\ &\quad - k(L)d_L v(L) \\ &= \int_0^L -k(x)u''(x)v(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx \\ &\quad + (k(L)u'(L) - k(L)d_L)\,v(L) - k(0)u'(0)v(0) \end{aligned}$$

2. **Group terms with v at the same location, and use conditions in $\mathcal{V}$.** A number of boundary terms will appear as a result of the integration by parts. Since functions in $\mathcal{V}$ often need to satisfy conditions at the boundary, use such conditions at this point. Then, gather all terms that involve the same values of the test function v, or its derivatives, in the domain and at the boundary. For our example, $v(0) = 0$, and we collect the terms containing $v(L)$ and $v(x)$:

$$0 = \int_0^L \left[-k(x)u''(x) + b(x)u'(x) + c(x)u(x) - f(x)\right] v(x)\,dx \\ + k(L)(u'(L) - d_L)\,v(L). \quad (1.87)$$

3. **Obtain the differential equation and potential boundary conditions.** We use the fact that the resulting expression should be valid for any $v \in \mathcal{V}$. Again, appealing to a simple and formal argument, this means that every term that multiplies a test function v at some point x should be equal to zero at that point, since the value of $v(x)$ can be chosen arbitrarily. For our example, this means that

$$0 = -k(x)u''(x) + b(x)u'(x) + c(x)u(x) - f(x) \qquad x \in (0,L) \quad (1.88a)$$

$$0 = k(L)(u'(L) - d_L), \quad (1.88b)$$

since they multiply $v(x)$ for $x \in (0, L)$ and $v(L)$, respectively. Notice that if both equations in (1.88) are satisfied, then variational equation (1.86) holds for all $v \in \mathcal{V}$.

Example 1.59 Let's find the Euler-Lagrange equations for the variational equation in Nitsche's method, c.f. Example 1.31. In this case, the function u satisfies variational equation (1.31):

$$\int_0^L u'(x)v'(x)\,dx + u'(0)v(0) - u(0)v'(0) + \mu u(0)v(0) = \\ \int_0^L f(x)v(x)\,dx + d_L v(L) - g_0 v'(0) + \mu g_0 v(0) \quad (1.89)$$

for all $v \in \mathcal{V} = \{v\colon [0, L] \to \mathbb{R} \text{ smooth}\}$.

To obtain the differential equation and boundary conditions it implies, we first eliminate the derivative on v by integrating by parts, to get (the integrated-by-parts first term on the left hand side is on the first line):

$$u'(L)v(L) - u'(0)v(0) - \int_0^L u''(x)v(x)\,dx + u'(0)v(0) \\ - u(0)v'(0) + \mu u(0)v(0) = \int_0^L f(x)v(x)\,dx + d_L v(L) - g_0 v'(0) + \mu g_0 v(0)$$

Of course, we cannot eliminate the derivative from the term that contains $v'(0)$, so we will keep it. Notice also that a term that appeared when integrating by parts cancel with an existing term of the variational equation (the ones in color). Collecting terms with v evaluated at the same points, we obtain

$$\int_0^L (u''(x) + f(x))v(x)\,dx = \\ (u'(L) - d_L)v(L) + (g_0 - u(0))v'(0) + \mu(u(0) - g_0)v(0). \quad (1.90)$$

At this point we can consider subsets of test functions to reach conclusions from each term. For $v \in \mathcal{V}$ be such that $v(0) = v(L) = v'(0) = 0$, we conclude that

$$\int_0^L (u''(x) + f(x))v(x)\,dx = 0.$$

Therefore, using the same rationale as in the first example, we conclude that

$$u''(x) + f(x) = 0 \qquad x \in (0, L). \quad (1.91)$$

Therefore u needs to satisfy this differential equation to satisfy the variational equation. For such u then, the first term in (1.90) is identically zero for *all* $v \in \mathcal{V}$ (not only for those that satisfy $v(0) = v(L) = v'(0) = 0$), and hence

$$0 = (u'(L) - d_L)\, v(L) + (g_0 - u(0))\, v'(0) + \mu(u(0) - g_0)\, v(0)$$

for all $v \in \mathcal{V}$. If we select $v \in \mathcal{V}$ such that $v'(0) = v(0) = 0$ and $v(L) = 1$ (for example, $v(x) = x^2/L^2$, then we conclude that $(u'(L) - d_L) = 0$, or that $u'(L) = d_L$, the Neumann boundary condition. We can again use this information, and assert that for *any* $v \in \mathcal{V}$,

$$0 = (g_0 - u(0))\, v'(0) + \mu(u(0) - g_0)\, v(0).$$

Here we have options for how to test. We could test with $v \in \mathcal{V}$ such that $v(0) = 1$ and $v'(0) = 0$ (for example, $v(x) = 1 - x^2$). This leads us to conclude that the first term $(g_0 - u(0))\, v'(0)$ is equal to zero, and that from there the second term implies that $u(0) = g_0$. Alternatively, we could have selected a function for which $v(0) = 0$ and $v'(0) = 1$ (for example, $v(x) = x^2/2$, and conclude that the second term is identically zero, and that the first term implies that $u(0) = g_0$.

At this point, we can list the Euler-Lagrange equations we obtained as

$$u''(x) + f(x) = 0 \qquad x \in (0, L) \tag{1.92a}$$
$$u(0) = g_0 \tag{1.92b}$$
$$u'(L) = d_L. \tag{1.92c}$$

If (1.92) holds, then (1.90) is satisfied for all $v \in \mathcal{V}$. Hence there are no other Euler-Lagrange equations, since if u satisfies these three conditions, we can guarantee that variational equation (1.89) holds for all functions in $\mathcal{V}$.

It is evident from here that both boundary conditions are natural boundary conditions in this case.

1.3.4 The Weak and the Strong Forms

Consider the following problem:

Problem 1.3 (A Weak Form for Problem 1.1)**.** *Let*

$$\mathcal{S} = \left\{w \colon \Omega \to \mathbb{R} \textit{ smooth} \mid w(0) = g_0\right\}, \tag{1.93a}$$
$$\mathcal{V} = \{w \colon \Omega \to \mathbb{R} \textit{ smooth} \mid w(0) = 0\}. \tag{1.93b}$$

Find $u \in \mathcal{S}$ *such that for any functions* $v \in \mathcal{V}$

$$\int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\, dx - k(L)d_L v(L) = \int_\Omega f(x)v(x)\, dx. \tag{1.94}$$

This is called a **weak form** for Problem 1.1. A solution of Problem 1.1 is called a **weak solution**. Conversely, Problem 1.1 is called the **strong form**, and its solution is called the **strong solution**. The affine space $\mathscr{S}$ is the trial space, and $\mathscr{V}$ is the test space.

A solution u of Problem 1.1 is a solution of Problem 1.3, since it satisfies variational equation (1.94). Conversely, a solution of Problem 1.3 is a solution of Problem 1.1, since:

- A weak solution satisfies variational equation (1.94), and therefore it satisfies its Euler-Lagrange equations. Based on the discussion in §1.3.3.1, the Euler-Lagrange equations are the differential equation (1.6a) and the Neumann boundary condition (1.6c), namely,
$$-\left(k(x)u'(x)\right)' + b(x)u'(x) + c(x)u(x) = f(x) \qquad \forall\, x \in \Omega$$
$$u'(L) = d_L$$
- A weak solution to belongs to $\mathscr{S}$, so it also satisfies the Dirichlet boundary condition (1.6b).
$$u(0) = g_0.$$

Notice that variational equation (1.94) involves only the first derivative of u, while the strong form requires the second derivative of u to be defined as well. So, the conditions for a function to satisfy variational equation (1.94) are *weaker* than for the strong form. This means that, potentially, a functions that does not have well-defined second derivative could satisfy the variational equation.

If we expanded the trial space $\mathscr{S}$ to include more functions, functions that may have a first derivative but not a continuous second derivative, we could potentially find weak solutions[6] for which the strong form does not make sense, since a discontinuous second derivative implies that the differential equation in the strong form may not be defined at some points.

These same smoothness requirements need to be considered when stating the equivalence between a variational equation and its Euler-Lagrange equations, as in §1.3.3. If the function u that satisfies variational equation $F(u, v) = 0$ for all $v \in \mathscr{V}$ is not smooth enough for the Euler-Lagrange equation $EL(u, x) = 0$ to be defined for all x in the domain, then we cannot talk about equivalence.

This discussion is included here only for the reader to be aware of potential caveats. In this first approach to the subject, and to avoid delving into more tools in mathematics[7], we do not believe this is necessary.

We conclude this section by stating a type of abstract weak forms:

Problem 1.4 (Abstract Weak Form)**.** *Let $\mathscr{W}$ be a vector space, $a\colon \mathscr{W} \times \mathscr{W} \to \mathbb{R}$ be a bilinear form, and $\ell\colon \mathscr{W} \to \mathbb{R}$ be a linear functional. Let the trial space $\mathscr{S}$ be an affine subspace of $\mathscr{W}$, and let the test space $\mathscr{V} \subset \mathscr{W}$ be the direction of $\mathscr{S}$.*

$$\text{Find } u \in \mathscr{S} \text{ such that } a(u, v) = \ell(v) \text{ for all } v \in \mathscr{V}. \tag{1.95}$$

[6] If, for example, $f(x)$ is discontinuous.
[7] For example, weak derivatives and Hilbert spaces.