ME 335A
Finite Element Analysis
Instructor: Adrian Lew
Problems Set #3– Solutions

Due Wednesday, April 19, 2023

This problem set contains two problems that involve some serious numerical calculations. Please utilize software to help you perform them. The calculations here are only pathways to exercise and learn some concepts, so please utilize a tool that will allow you to focus on the concepts and not on the calculations.

On Convergence (50)

As mentioned in class, approximating the solution of a boundary value problem involves the construction of a method to obtain numerical solutions (approximate solutions) that are as close as we want to the exact solution of the problem. Let's illustrate here two cases that illuminate the wonders and the perils that we may find in walking this path.

In this problem we will take advantage of Legendre polynomials over the interval $[0, 1]$. For $n = 0, 1, \ldots$, the Legendre polynomial of order n can be written as

$$P_n(x) = \sqrt{2n+1}(-1)^n \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-x)^k. \tag{1}$$

Here

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2}$$

We can also define their integrals

$$iP_n(x) = \int_0^x P_n(y)\, dy = \sqrt{2n+1}(-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{(-x)^{k+1}}{k+1}. \tag{3}$$

Legendre Polynomials satisfy the following "orthogonality" condition:

$$\int_0^1 P_n(x)P_m(x)\, dx = \delta_{mn}. \tag{4}$$

Here δ_{mn} is the Kronecker delta, whose values are

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases} \tag{5}$$

For this problem, we strongly suggest you use Matlab, Mathematica, or Maple to perform all integrals and calculations, as well as to plot the results.

In this exercise, we consider the following problem in its weak from:

Problem (Weak Form)**.** *Given $f\colon [0,1] \to \mathbb{R}$, find $u \in \mathcal{V} = \{w\colon [0,1] \to \mathbb{R}$ smooth $\mid w(0) = 0\}$ such that*

$$\int_0^1 u'(x)v'(x)\, dx = \int_0^1 f(x)v(x)\, dx \tag{W}$$

for any $v \in \mathcal{V}$.

We set $f(x) = \cos(4\pi x)$, and will examine the use of a variational method to approximate the solution to this problem that uses variational equation (W).

1. (10) Is $u(x) = -\frac{\sin(2\pi x)^2}{8\pi^2}$ the exact solution to this problem ? Justify.

   Hint: It is convenient to replace in the weak form and integrate by parts to verify it is satisfied for any $v \in \mathcal{V}$. Alternatively, you can compute the Euler-Lagrange equations and see if $u(x)$ satisfies them.

   Solution: To show that u is a solution, we must show that $u \in \mathcal{V}$ and that u satisfies **Problem**.

   We first note that $\sin(2\pi x)$ is infinitely differentiable and continuous, and $u(0) = -\frac{\sin(0)^2}{8\pi^2} = 0$. Thus, we have $u \in \mathcal{V}$.

   Next, we apply the integration by parts to the left-hand-side of (W). For any $v \in \mathcal{V}$:

   $$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = u'(1)v(1) - u'(0)v(0) - \int_0^1 u''(x)v(x)\,\mathrm{d}x.$$

   Let us calculate u' and u''. By the formula $\sin(x)^2 = (1 - \cos(2x))/2$, we have

   $$\begin{aligned} u(x) &= \frac{\cos(4\pi x) - 1}{16\pi^2} \\ u'(x) &= \frac{-\sin(4\pi x)}{4\pi} \\ u''(x) &= -\cos(4\pi x). \end{aligned}$$

   Thus, $u'(0) = u'(1) = 0$, and hence for any $v \in \mathcal{V}$

   $$\int_0^1 u'(x)v'(x)\,\mathrm{d}x = \int_0^1 \cos(4\pi x)v(x)\,\mathrm{d}x.$$

   In other words, $u(x)$ solves the weak problem.

2. Consider the set of functions $\mathcal{W}_h = \{w = \sum_{k=0}^{n} c_k i P_k(x) \mid (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}\}$. We will denote with ${}^n u_h$ the solution of the variational method for a given n.

   (a) (15) Define $\mathcal{V}_h, \mathcal{S}_h, a(\cdot,\cdot), \ell(\cdot), \eta_a$ and η_g. State also the equations that define the solution of the variational method, compute the entries in the stiffness matrix, and the entries in the load vector

   Solution: In this case we can identify the bilinear form and linear functional by

   $$a(u, v) = \int_0^1 u'(x)v'(x)\, dx, \qquad \ell(v) = \int_0^1 \cos(4\pi x)v(x)\, dx.$$

The boundary condition at $x = 0$ is essential, so we the trial space for the method is

$$\mathcal{S}_h = \{w_h \in \mathcal{W}_h \mid w_h(0) = 0\},$$

and its direction is the same space, namely, $\mathcal{V}_h = \mathcal{S}_h$.
Because $iP_n(0) = 0$ for any $n \in \{0, 1, \ldots\}$, $\{iP_0, \ldots, iP_n\}$ is a basis for $\mathcal{V}_h$. We also have that $\eta = \eta_a = \{0, 1, \ldots, n\}$ and $\eta_g = \emptyset$.
The variational method for the problem reads: Find $u_h \in \mathcal{V}_h$ such that $a(u_h, v_h) = \ell(v_h)$ for all $v_h \in \mathcal{V}_h$, so the equations that define its solution are

$$a(u_h, iP_a) = \ell(iP_a)$$

for $a = 0, \ldots, n$.
Let now $u_h, v_u \in \mathcal{V}_h$, with

$$u_h(x) = \sum_k c_k iP_k(x) \qquad v_h(x) = \sum_k d_k iP_k(x).$$

Then

$$u_h'(x) = \sum_k c_k P_k(x) \qquad v_h'(x) = \sum_k d_k P_k(x).$$

The stiffness matrix in this case is then computed as

$$\begin{aligned} K_{ab} &= a(iP_b, iP_a) \\ &= \int_0^1 iP_a'(x) iP_b'(x)\, dx \\ &= \int_0^1 P_a(x) P_b(x)\, dx \\ &= \delta_{ab}. \end{aligned} \tag{6}$$

So, the stiffness matrix is the identity matrix.
The load vector follows as

$$\begin{aligned} f_a &= \ell(iP_a) \\ &= \int_0^1 iP_a(x) \cos(4\pi x)\, dx. \end{aligned} \tag{7}$$

Because the stiffness matrix is the identity, the column matrix of components of u_h follows as

$$U = K^{-1} F = F,$$

and hence

$$u_h = \sum_{a=0}^{n} f_a iP_a(x). \tag{8}$$

Since functions $iP_k(x)$ are polynomials with order $k + 1$, and we are looking for solutions up to $n = 7$, we must calculate

$$\int_0^1 \cos(4\pi x) x^k \, \mathrm{d}x$$

for $k = 0 \ldots 8$. These are

$$\int_0^1 \cos(4\pi x)x^0 \,\mathrm{d}x = 0$$
$$\int_0^1 \cos(4\pi x)x^1 \,\mathrm{d}x = 0$$
$$\int_0^1 \cos(4\pi x)x^2 \,\mathrm{d}x = \frac{1}{8\pi^2}$$
$$\int_0^1 \cos(4\pi x)x^3 \,\mathrm{d}x = \frac{3}{16\pi^2}$$
$$\int_0^1 \cos(4\pi x)x^4 \,\mathrm{d}x = \frac{8\pi^2 - 3}{32\pi^4}$$
$$\int_0^1 \cos(4\pi x)x^5 \,\mathrm{d}x = \frac{20\pi^2 - 15}{64\pi^4}$$
$$\int_0^1 \cos(4\pi x)x^6 \,\mathrm{d}x = \frac{45 - 120\pi^2 + 32\pi^4}{256\pi^6}$$
$$\int_0^1 \cos(4\pi x)x^7 \,\mathrm{d}x = \frac{315 - 420\pi^2 + 224\pi^4}{512\pi^6}$$
$$\int_0^1 \cos(4\pi x)x^8 \,\mathrm{d}x = \frac{-315 + 840\pi^2 - 672\pi^4 + 256\pi^6}{512\pi^8}$$

We can then use these to construct u_h. For example, for $n = 3$,

$$^3u_h = \frac{1}{8\pi^2} iP_2(x) + \frac{8\pi^2 - 3}{32\pi^4} iP_3(x).$$

Alternative, more laborious path: In the solution above we directly used the way to solve the variational method from the notes. Here, we proceed directly from the bilinear form:
Plugging into (W) gives

$$\int_0^1 u_h'(x)v_h'(x)\,\mathrm{d}x = \int_0^1 \cos(4\pi x)v_h(x)\,\mathrm{d}x.$$

For the left-hand-side, we expand:

$$\begin{aligned}
\int_0^1 u_h'(x)v_h'(x)\,\mathrm{d}x &= \int_0^1 \left[\sum_k c_k P_k(x)\right]\left[\sum_l d_l P_l(x)\right]\mathrm{d}x \\
&= \sum_k \sum_l c_k d_l \int_0^1 P_k(x)P_l(x)\,\mathrm{d}x \\
&= \sum_k \sum_l c_k d_l \delta_{kl} \\
&= \sum_k c_k d_k.
\end{aligned}$$

In the above, because the sums are finite, we are justified in bringing them outside of the integral. We also used the orthogonality of the Legendre polynomials to reduce the double sum to a single sum. For the right-hand-side, we have

$$\begin{aligned}\int_0^1 \cos(4\pi x) v_h(x)\,\mathrm{d}x &= \int_0^1 \cos(4\pi x)\left[\sum_k d_k iP_k(x)\right]\mathrm{d}x\\ &= \sum_k d_k \int_0^1 \cos(4\pi x) iP_k(x)\,\mathrm{d}x.\end{aligned}$$

Thus, our final equation is to find real coefficients $\{c_k\}_k$ such that

$$\sum_k c_k d_k = \sum_k d_k \int_0^1 \cos(4\pi x) iP_k(x)\,\mathrm{d}x$$

for any real numbers $\{d_k\}_k$. In particular, if we select $\{d_k\}_k$ such that only coefficient $d_l = 1$ and all others are zero, we get

$$c_l = \int_0^1 \cos(4\pi x) iP_l(x)\,\mathrm{d}x.$$

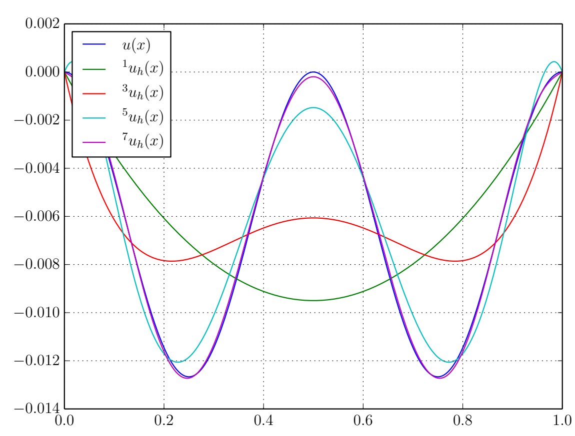
We note that each coefficient may be calculated independently of the other coefficients. This is due to the fact that the basis functions are orthogonal ($\int_0^1 P_m P_n\,\mathrm{d}x = \delta_{mn}$). Applying a similar strategy for the other coefficients gives the final result:

$$u_h(x) = \sum_k \left[\int_0^1 \cos(4\pi x) iP_k(x)\,\mathrm{d}x\right] iP_k(x).$$

(b) (15) Compute ${}^1u_h$, ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^1u_h$, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.
Solution: Using the above, we calculate the coefficients (to 6 decimal places):

$$\begin{aligned}c_0 &= 0\\ c_1 &= 0.021937\\ c_2 &= 0\\ c_3 &= 0.020777\\ c_4 &= 0\\ c_5 &= -0.044246\\ c_6 &= 0\\ c_7 &= 0.016923\end{aligned}$$

We plot the functions:



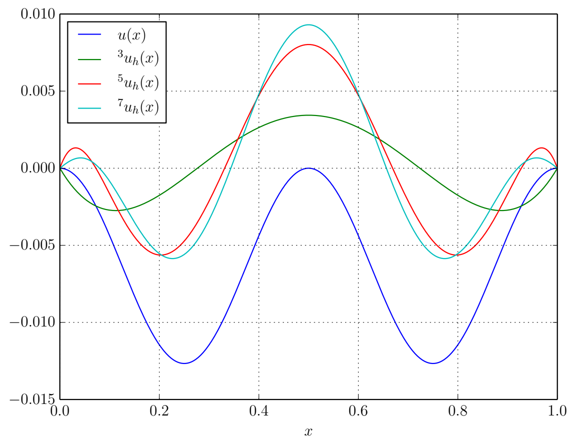
(c) (5) Is ${}^n u_h$ "visually converging" to u as n grows?

Solution: Yes. We observe convergence of the solutions ${}^n u_h$ to u as $n \to \infty$.

3. Consider next the set of functions $\mathcal{W}_h = \{w = \sum_{k=2}^n c_k iP_k(x) \mid (c_2, \ldots, c_n) \in \mathbb{R}^{n+1}\}$ (notice that k begins at 2 instead of at 0).

(a) (15) Compute ${}^3 u_h$, ${}^5 u_h$, ${}^7 u_h$. In the same plot, plot u, ${}^3 u_h$, ${}^5 u_h$ and ${}^7 u_h$.

Solution: Here, ${}^n u_h(x) = \sum_{k=2}^n c_k iP_k(x)$. We plot the functions:



(b) (5) Is ${}^n u_h$ "visually converging" to u as n grows? Do you think that your answer to this question will change if we keep increasing n beyond 7?

Solution: Here, we do *not* observe "visual convergence" as n grows. Our answer would not change if we increased n beyond 7. For any n, let ${}^n u_h^{(2)}$ be as in Part 2(a) and let ${}^n u_h^{(3)}$ as in Part 3(a). Then

$$ {}^n u_h^{(2)} = {}^n u_h^{(3)} + {}^1 u_h^{(2)}. $$

This is due to the fact that the computation of coefficient c_k is independent of the other coefficients. If ${}^n u_h^{(2)}$ approaches u as $n \to \infty$, then

$$ {}^n u_h^{(3)} \to u - {}^1 u_h^{(2)} $$

as $n \to \infty$. Because ${}^1 u_h^{(2)} \neq 0$, this will certainly never converge to u.

Euler-Lagrange Equations (30)

For this problem section "1.3.3 The Euler-Lagrange Equations" of the notes provides an explanation of the steps.

Consider the weak form: Find $u \in \mathcal{V} = \{u\colon (0,1) \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$ such that $a(u, w) = \ell(w)$ for all $w \in \mathcal{V}$, where

$$ \begin{aligned} a(u, w) &= \int_0^1 (w_{,x} u_{,x} + \lambda w u)\, dx + w(1)u(1) \\ \ell(w) &= \int_0^1 w x^2\, dx + w(1), \end{aligned} $$

and $\lambda > 0$.

1. (5) Is $a(u,v)$ a bilinear form? Justify.

   Solution: Yes. Let $u, v, w \in \mathcal{V}$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned} a(u+\alpha w, v) &= \int_0^1 (u,_x + \alpha w,_x)v,_x + \lambda(u+\alpha w)v\, dx + (u(1)+\alpha w(1))v(1) \\ &= \int_0^1 u,_x v,_x + \lambda uv\, dx + \alpha \int_0^1 w,_x v,_x + \lambda wv\, dx + u(1)v(1) + \alpha w(1)v(1) \\ &= a(u,v) + \alpha a(w,v) \end{aligned}$$

   and similarly, $a(u, v+\alpha w) = a(u,v) + \alpha a(v,w)$.

2. (5) Is $a(u,v)$ symmetric? Justify.

   Solution: Yes, since it is simple to see that $a(u,v) = a(v,u)$.

3. (5) Is $\ell(v)$ a linear functional? Justify.

   Solution: Yes. Since for $v, u \in \mathcal{V}$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} \ell(v+\alpha u) &= \int_0^1 (v+\alpha u)x^2\, dx + (v(1)+\alpha u(1)) \\ &= \int_0^1 vx^2\, dx + v(1) + \alpha \left(\int_0^1 ux^2\, dx + u(1) \right) \\ &= \ell(v) + \alpha \ell(u). \end{aligned}$$

4. (10) Obtain the Euler-Lagrange equations.

   Solution: We proceed by computing $a(u,v) - \ell(v)$ and integrating by parts to eliminate all derivatives of the test function, and group all terms containing the test function at the same point:

$$\begin{aligned} a(u,v) - \ell(v) &= \int_0^1 (v,_x u,_x + \lambda vu)\, dx + v(1)u(1) - \int_0^1 vx^2\, dx - v(1) \\ &= v(1)u,_x(1) - \underbrace{v(0)}_{=0} u,_x(0) + \int_0^1 (-u,_{xx} + \lambda u)v\, dx + v(1)u(1) - \int_0^1 vx^2\, dx - v(1) \\ &= \int_0^1 (-u,_{xx} + \lambda u - x^2)v\, dx + v(1)(u,_x(1) + u(1) - 1) \end{aligned}$$

   Since this should hold for all $v \in \mathcal{V}$, it follows that

$$0 = -u,_{xx} + \lambda u - x^2 \qquad x \in (0,1) \tag{9}$$
$$1 = u,_x(1) + u(1). \tag{10}$$

   The strong form is then: Find $u\colon (0,1) \to \mathbb{R}$, a continuous function, such that (9) and (21) are satisfied, and $u(0) = 0$.

5. (5) Identify natural and essential boundary conditions.

   Solution: The essential boundary condition is $u(0) = 0$, and the natural boundary condition is (21).

Your "First" Finite Element Approximation (45)

We want to construct a piecewise linear approximation to a function u that satisfies the variational equation $a(u,w) = \ell(w)$ for all $w \in \mathcal{V} = \{u\colon [0,1] \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$, where

$$a(u,w) = \int_0^1 (w_{,x}\, u_{,x} + \lambda w u)\, dx + w(1)u(1)$$
$$\ell(w) = \int_0^1 w x^2\, dx + w(1),$$

and $\lambda > 0$.

To this end, we will partition $[0,1]$ into four equal intervals, and build a finite element approximation with continuous functions that are lineal polynomials over each interval.

1. (5) Identify the location of all the vertices in the mesh, and number them. **Solution:** There are m=4 equal intervals, the element size is h= 1/4, where the locations of the vertices are $x_i = (i-1)/m$ for $i = 1 \ldots 5$. then,

2. (5) Number and sketch all hat functions $\{N_a\}$ over $[0,1]$.

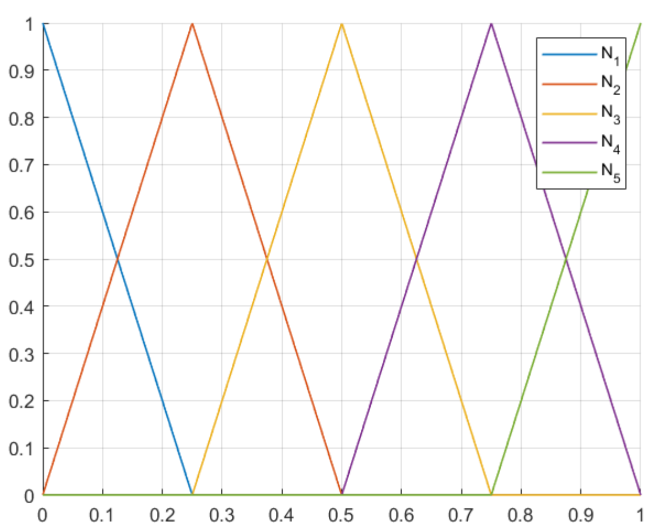
   Solution: The hat functions for integer $a \in \{1,2,3,4\}$, $x_a = \frac{a-1}{4}$. For $a = 2,3,4$,

$$N_a = \begin{cases} \frac{x - x_{a-1}}{x_a - x_{a-1}}, & x_{a-1} \le x < x_a \\ \frac{x_{a+1} - x}{x_{a+1} - x_a}, & x_a \le x \le x_{a+1} \\ 0, & \text{otherwise,} \end{cases}$$

   and

$$N_1 = \begin{cases} \frac{x_2 - x}{x_2 - x_1}, & x_1 \le x \le x_2 \\ 0, & x_2 < x \end{cases}, \qquad N_5 = \begin{cases} 0, & x < x_4 \\ \frac{x - x_4}{x_5 - x_4}, & x_4 \le x \le x_5 \end{cases}$$

   The hat functions are sketched in Fig. 2



3. (5) Let $\mathcal{W}_h = \text{span}(\{N_1, \ldots, N_5\})$. What are the trial space $\mathcal{S}_h$ and test space $\mathcal{V}_h$ for the variational method, and the constrained and active index sets? Select $\overline{u}_h \in \mathcal{S}_h$. **Solution:** For this problem, we require $u(0) = 0$ and $w(0) = 0$. Therefore, we need

$$\mathcal{V}_h = \mathcal{S}_h = \{v_h \in \mathcal{W}_h \mid v_h(0) = 0\}. \tag{11}$$

A basis for $\mathcal{V}_h$ is obtained by setting $\eta_a = \{2, 3, 4, 5\}$ and $\eta_g = \{1\}$. So,

$$\mathcal{V}_h = \mathcal{S}_h = \text{span}(N_2, N_3, N_4, N_5).$$

Also, if $\overline{u}_h \in \mathcal{S}_h$, then $\overline{u}_1 = 0$. We can select $\overline{u}_h = 0$, for example.

4. (5) State the variational method for this problem.

Solution: The Galerkin Method is: Find $u_h \in \mathcal{S}_h$ such that

$$a(u_h, w_h) = l(w_h) \text{ for all } w_h \in \mathcal{V}_h. \tag{12}$$

where

$$a(u, w) = \int_0^1 (w_{,x}\, u_{,x} + \lambda w u)\, dx + w(1)u(1) \tag{13}$$

$$\ell(w) = \int_0^1 w x^2\, dx + w(1) \tag{14}$$

5. (10) Compute the stiffness matrix, assuming that $\lambda = 2$. **Solution:**

The entries of the stiffness matrix for $a \in \eta_a$ and for any $b \in \eta$ are

$$K_{ba} = a(N_a, N_b) = \int_0^1 (N_{a,x} N_{b,x} + 2N_a N_b)\, dx + N_a(1)N_b(1), \tag{15}$$

while for $a \in \eta_g$ and any $b \in \eta$ we have

$$K_{ab} = \delta_{ab}.$$

Computing,

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 & 0 \\ 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 \\ 0 & 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} \\ 0 & 0 & 0 & -\frac{47}{12} & \frac{31}{6} \end{bmatrix}$$

6. (10) Compute the load vector.

Solution:
For $a \in \eta_a$,

$$f_a = \ell(N_a) = \int_0^1 N_a x^2\, dx + N_a(1) \tag{16}$$

and

$$f_1 = \overline{u}_1 = 0.$$

Then,

$$f = \begin{bmatrix} 0 \\ \frac{7}{384} \\ \frac{25}{384} \\ \frac{55}{384} \\ \frac{283}{256} \end{bmatrix}$$

7. (5) Find the approximate solution u_h.

Solution:
After obtaining K and f, we can solve the linear system $Kd = f$

$$d \approx \begin{bmatrix} 0 \\ 0.1044 \\ 0.2175 \\ 0.3417 \\ 0.4730 \end{bmatrix}$$

The approximate solution u_h is

$$u_h = 0.1044N_2 + 0.2175N_3 + 0.3417N_4 + 0.4730N_5 \tag{17}$$

On Integration by Parts (Optional, not graded)

A version of the fundamental lemma of calculus states that if $u \in C^1([a, b])$, then

$$\int_a^b u'(x)\, dx = u(b) - u(a). \tag{18}$$

A version of of the integration by parts theorem states that if $u, v \in C^1([a, b])$, then

$$\int_a^b u'(x)v(x)\, dx = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\, dx \tag{19}$$

We want to play with the theorem in situations in which one of the functions is discontinuous, to see how it fails. A discusion on this topic can be found in §1.1.4 in the notes. To this end, we will make use of the sign function

$$sgn(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases} \tag{20}$$

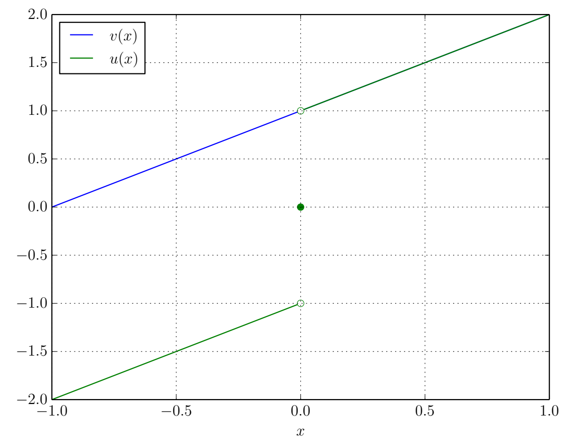
Consider the function $w(x) = 0$ for all $x \in \mathbb{R}$. The function $w(x)$ is equal to the derivative of $sgn(x)$ for every $x \neq 0$. The derivative of $sgn(x)$ is not defined at $x = 0$, but since $w(x) = sgn'(x)$ for all $x \neq 0$, $w(x)$ is called the *classical derivative* or *pointwise derivative* of $sgn(x)$. In the following, please use $w(x)$ for $sgn'(x)$ any time you need it.

Caution: Please do not use a relationship of the form $sgn'(x) = 2\delta(x)$, where $\delta(x)$ is a Dirac delta (if you are familiar with it), since $\delta(x)$ is not really a function.

In the following, we set $a = -1$, $b = 1$, and let $u(x) = sgn(x) + x$, and $v(x) = 1 + x$.

1. (5) Plot u, v. Is $u \in C^1([a,b])$? Is $v \in C^1([a,b])$?

   Solution: We plot u and v below:



   Because v is a polynomial, it is clear that $v \in C^1([a,b])$. On the other hand, $u \notin C^1([a,b])$, as u is not continuous at $x = 0$.

2. (5) Let's first check if the fundamental lemma of calculus works for u. Is it true that

$$\int_{-1}^{1} u'(x)\, dx = u(1) - u(-1)? \tag{21}$$

   Solution: No. We can calculate directly. For the left-hand-side:

$$\int_{-1}^{1} u'(x)\,\mathrm{d}x = \int_{-1}^{1} w(x) + 1\,\mathrm{d}x = \int_{-1}^{1} 1\,\mathrm{d}x = 2.$$

   For the right-hand-side:

$$u(1) - u(-1) = (sgn(1) + 1) - (sgn(-1) - 1) = 2 - (-2) = 4.$$

   Hence, the fundamental lemma of calculus does *not* work for u.

3. (5) Is it true that

$$\int_{-1}^{1} u'(x)\, dx + [\![u(0)]\!] = u(1) - u(-1)? \tag{22}$$

   where $[\![u(0)]\!] = \lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x)$.

   Solution: Yes. We previously calculated all terms in (22) except for $[\![u(0)]\!]$. Let us calculate:

$$[\![u(0)]\!] = \lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x) = 1 - (-1) = 2.$$

   Hence

$$\int_{-1}^{1} u'(x)\, dx + [\![u(0)]\!] = 4 = u(1) - u(-1),$$

   and (22) holds.

4. (10) Now let's check the integration by parts formula. Compute the left hand side and the right hand side of (19). Does formula (19) hold?

   Solution: No. We directly calculate each term in (19):

$$\begin{aligned}
\int_a^b u'(x)v(x)\,\mathrm{d}x &= \int_{-1}^1 (w(x)+1)v(x)\,\mathrm{d}x = \int_{-1}^1 v(x) = 2 \\
u(b)v(b) &= 4 \\
u(a)v(a) &= 0 \\
\int_a^b u(x)v'(x)\,\mathrm{d}x &= \int_{-1}^1 (sgn(x)+x)1\,\mathrm{d}x = 0.
\end{aligned}$$

   Because the left- and right-hand-sides do not match, we have that (19) does *not* hold.

5. (5) Does it hold that

$$\int_a^b u'(x)v(x)\;dx + [\![u(0)]\!]v(0) = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\;dx? \tag{23}$$

   Solution: Yes. Let us calculate the additional term:

$$[\![u(0)]\!]v(0) = 2,$$

   and hence

$$\int_a^b u'(x)v(x)\,\mathrm{d}x + [\![u(0)]\!]v(0) = 4 = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\,\mathrm{d}x.$$

   Thus, (23) holds.

6. (5) Next, consider the function $N(x) = 1 - |x|$, a type of functions we will see often along this quarter. Let $\mathcal{T} = \{w \in C^1([-1,1]) \mid w(0) = 1, w(1) = w(-1) = 0\}$. What is the value of

$$\int_{-1}^1 N'(x)v'(x)\;dx \tag{24}$$

   for any $v \in \mathcal{T}$?

   Solution: We note that

$$N'(x) = -sgn(x)$$

   for any $x \neq 0$. As with the relationship between $w(x)$ and $sgn'(x)$, we will say that $-sgn(x)$ is the classical derivative of $N(x)$. We now apply (23), rearranging and taking $u = N'(x)$

$$\int_{-1}^1 N'(x)v'(x)\,\mathrm{d}x = N'(1)v(1) - N'(-1)v(-1) - [\![N'(0)]\!]v(0) - \int_{-1}^1 N''(x)v(x)\,\mathrm{d}x.$$

   Since $N'(x) = -sgn(x)$, we have $N''(x) = -w(x)$, and the last term on the right-hand-side vanishes. Because $v \in \mathcal{T}$, we have $v(1) = v(-1) = 0$, so the first two terms on the right-hand-side vanish. Thus, because $v(0) = 1$,

$$\begin{aligned}
\int_{-1}^{1} N'(x)v'(x)\,\mathrm{d}x &= -[\![N'(0)]\!]v(0) \\
&= -[\![N'(0)]\!] \\
&= -\left[\lim_{x\to 0+} N'(x) - \lim_{x\to 0-} N'(x)\right] \\
&= -\left[\lim_{x\to 0+} (-sgn(x)) - \lim_{x\to 0-} (-sgn(x))\right] \\
&= -\left[-1-1\right] \\
&= 2.
\end{aligned}$$