Strong form:

Find $u \in C^2(\Omega)$ such that:

$$-k\Delta u = f \text{ in } \Omega$$
$$u = g_1 \text{ on } \Gamma_{g_1}$$
$$u = g_2 \text{ on } \Gamma_{g_2}$$
$$k\nabla u \cdot \check{n} = h_1 \text{ on } \Gamma_{h_1}$$
$$k\nabla u \cdot \check{n} = h_2 \text{ on } \Gamma_{h_2}$$
$$k\nabla u \cdot \check{n} = h_3 \text{ on } \Gamma_{h_3}$$
$$k\nabla u \cdot \check{n} = h_4 \text{ on } \Gamma_{h_4}$$

Define

$S = \{u : \Omega \to \mathbb{R}$ smooth such that $u(x) = g_1$ for $x \in \Gamma_{g_1}$; $u(x) = g_2$ for $x \in \Gamma_{g_2}\}$.

$V = \{v : \Omega \to \mathbb{R}$ smooth $\mid v(x) = 0$ for $x \in \Gamma_{g_1} \cup \Gamma_{g_2}\}$

Find $u \in S$ such that:

$$a(u, v) = l(v) \quad \text{for all } v \in V$$

where $a(u,v) = k\int_\Omega \nabla u \cdot \nabla v \, d\Omega$

$$l(v) = \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} H v \, d\Gamma$$

where $\partial\Omega_N = \{\Gamma_{h_1} \cup \Gamma_{h_2} \cup \Gamma_{h_3} \cup \Gamma_{h_4}\}$

$$H = \begin{cases} h_1 & x \in \Gamma_{h_1} \\ h_2 & x \in \Gamma_{h_2} \\ h_3 & \vdots \\ h_4 & \end{cases}$$

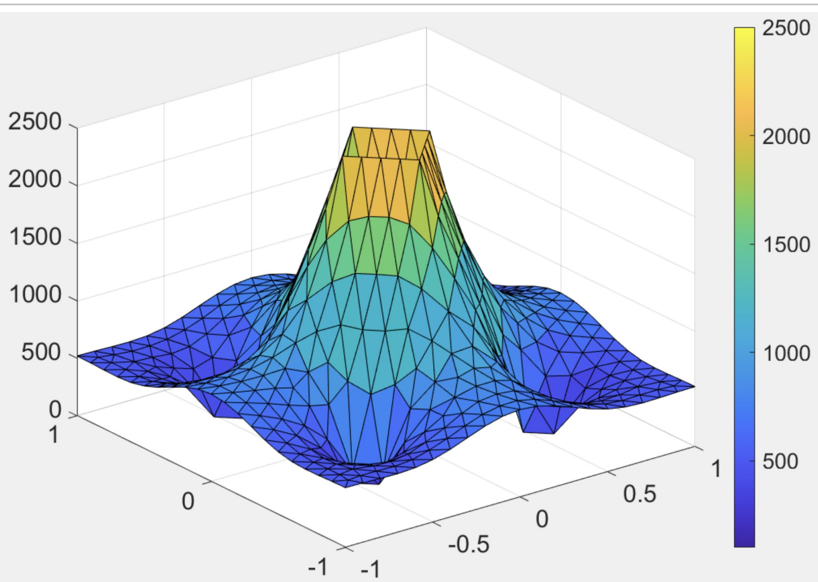
Discretized Form:

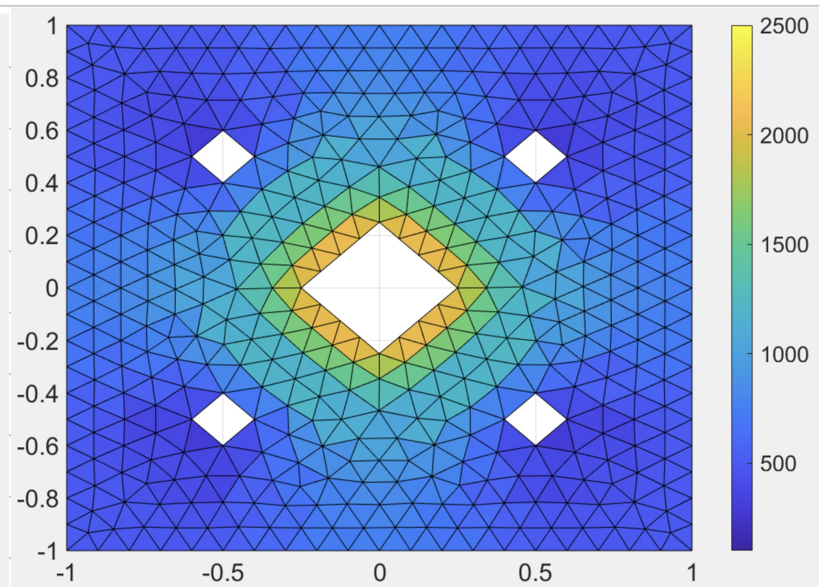
Let $\mathcal{W}_h = \text{span}(\{N_1, \cdots, N_{el+1}\})$.

$$S_h = \{ u_h \in \mathcal{W}_h \mid u_h(x_a^e) = g(x_a^e) \ \forall x_a^e \in \Gamma_g \}$$

$$\mathcal{V}_h = \{ v_h \in \mathcal{W}_h \mid v_h(x_a^e) = 0 \ \forall x_a^e \in \Gamma_g \}$$

1.





2.&3

See implementations of function uValue.

function P1Function.

4.

At $(0, 0.9)$,

$$u_h = 769.4303 \ ^\circ C$$

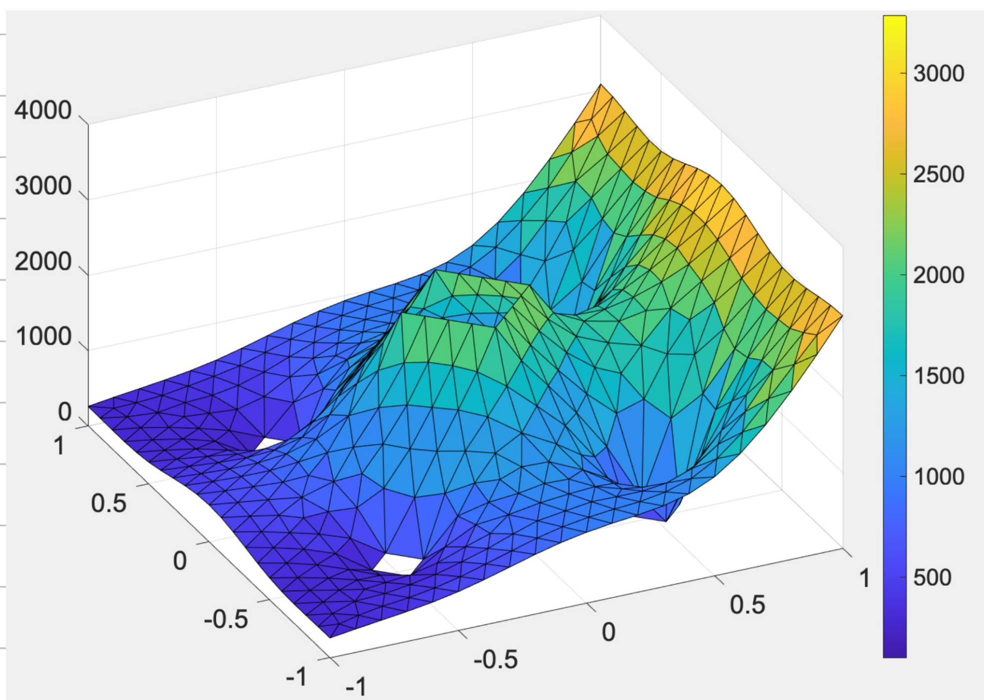
$$\frac{\partial u_h}{\partial x_1} = -0.5502 \ ^\circ C/ft \ ; \ \frac{\partial u_h}{\partial x_2} = -337.7738 \ ^\circ C/ft$$

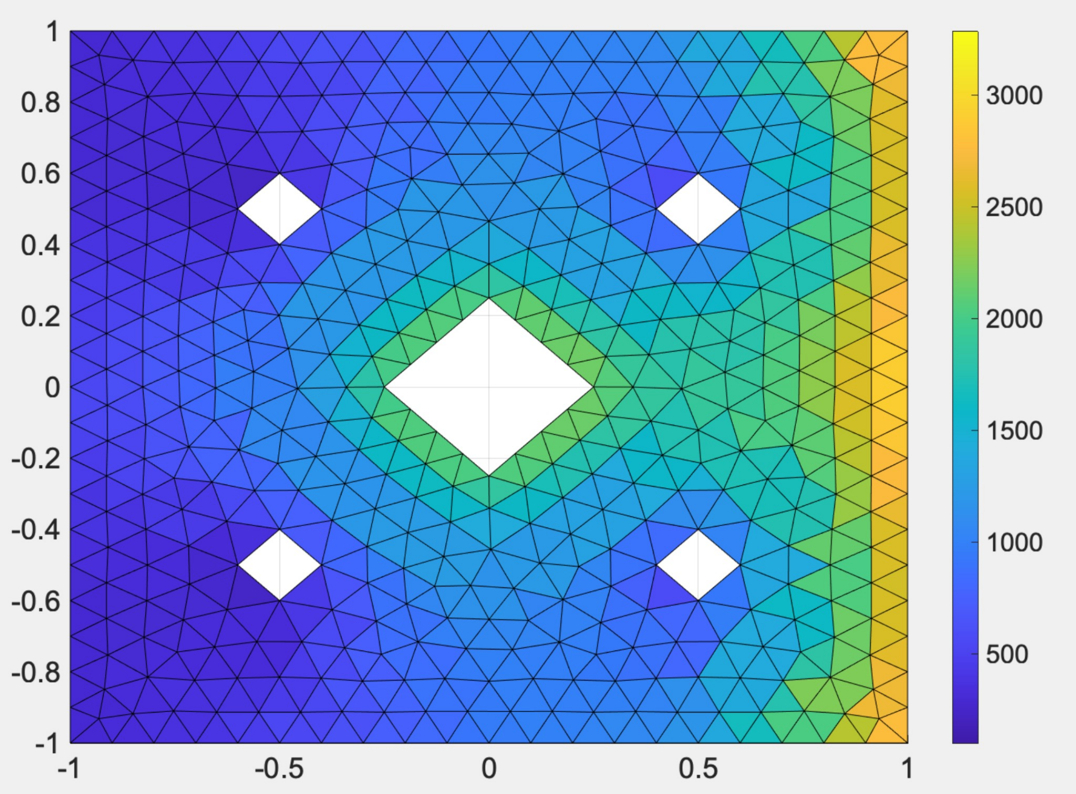
At $(0.25, 0.25)$,

$$u_h = 1.5811e3 \ ^\circ C$$

$$\frac{\partial u_h}{\partial x_1} = -3.4063e3 \ ^\circ C/ft \ ; \ \frac{\partial u_h}{\partial x_2} = -3.4060e3 \ ^\circ C/ft$$

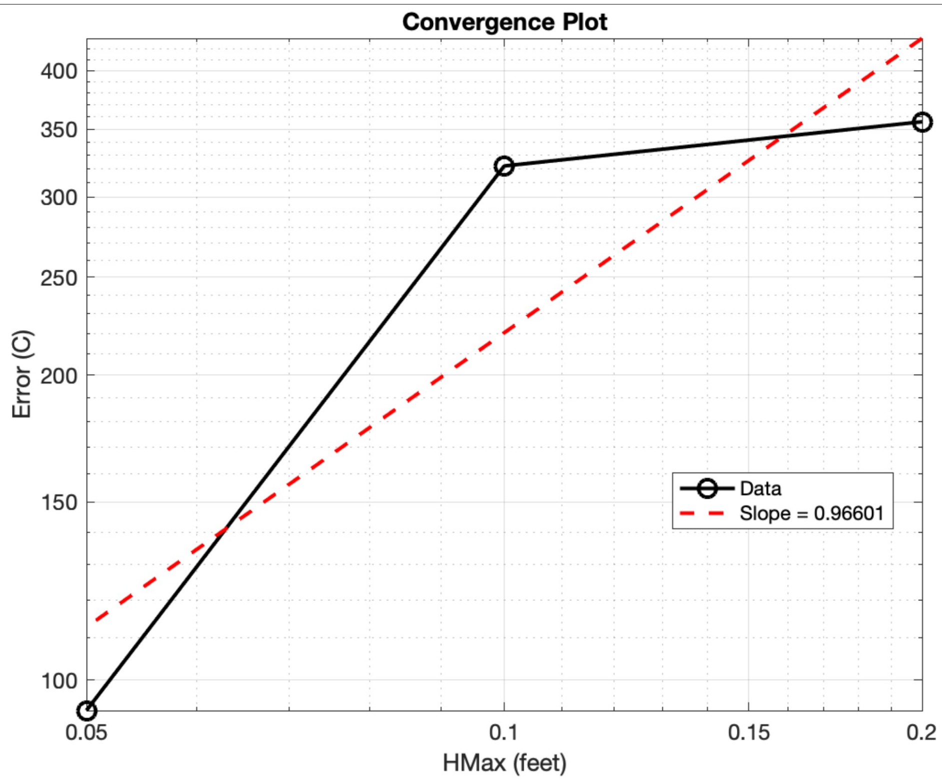
5.





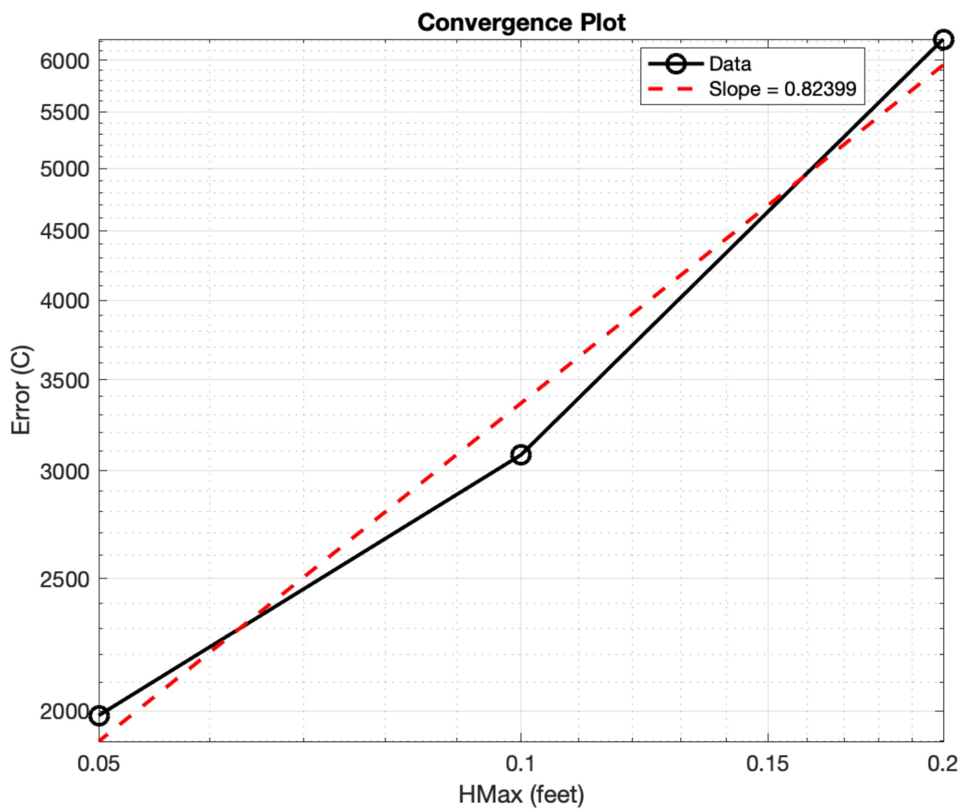
6. (a).

$$e_{0,2}(u) = \left[\sum_{i,j=0}^{N} u(X_{ij}) \right]^{1/2}$$



The figure above shows the log log plot of $e_{0,2}(U_i - U_{i+1})$ vs HMax. We can clearly observe that the convergence order is around 1 (0.96601) for this problem with P1 element.

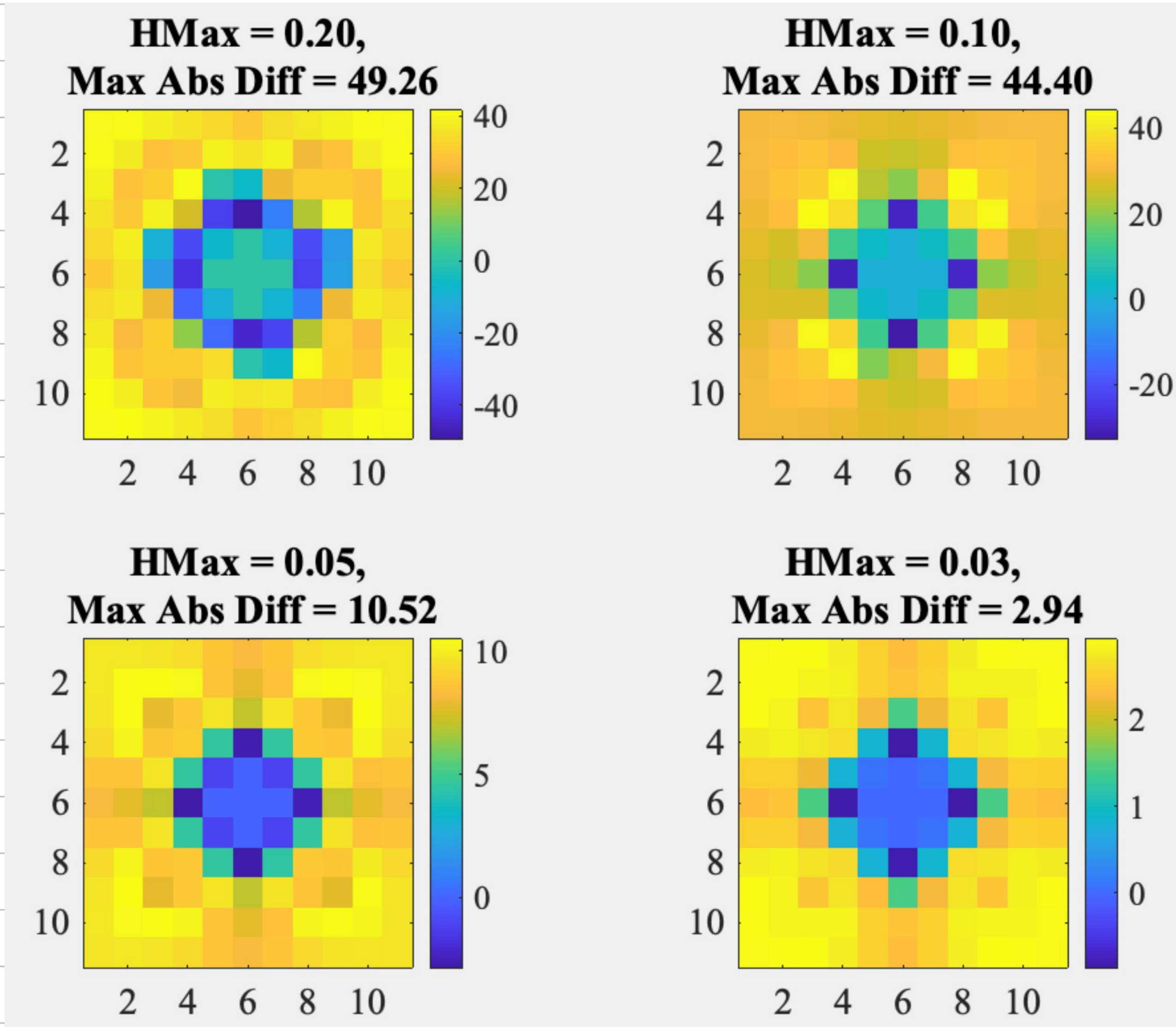
$$e_{1,2}(u) = \left[\sum_{i,j=0}^{N} \|\nabla u(X_{ij})\|^2 \right]^{1/2}$$



The figure above shows the log log plot of $e_{1,2}(U_i - U_{i+1})$ vs HMax. For the gradient of $u(x)$, we observed a slightly smaller convergence rate compared with $u(x)$. This is as expected as we are using P1 elements that have constant derivative.

(b).

$u_j(X) - u_{j+1}(X)$



For u(X), the mesh of Hmax = 0.05 will result in a max absolute difference between two consecutive solutions to be less than 30°C.

$\nabla u_j(X) - \nabla u_{j+1}(X)$