ME 335A
Finite Element Analysis
Instructor: Adrian Lew
Problems Set #1 – Solutions

Due Wednesday, April 12, 2023

Remark: For this problem set, and generally for this class, consider using a symbolic math program to help with your problem calculations – Mathematica, Matlab and Maple are good choices. Here is an example using Matlab's symbolic toolbox to compute integrals. Let's compute

$$\int_{-1}^{1} (1+x)(1-x)dx = \frac{4}{3} \tag{1}$$

The commands in Matlab for such computation would be:

```
>> syms x
% Define a symbolic variable x
>> int((1 + x) * (1 - x), -1, 1)
% Compute the integral in [-1, 1]
ans =
4/3
```

If the integral involves other symbolic variables, i.e.

$$\int_{-1}^{1} (1+ax)(1-bx)dx = 2 - \frac{2}{3}ab \tag{2}$$

the Matlab commands will be

```
>> syms x a b
>> int((1 + a * x) * (1 - b * x),x, -1, 1)
ans =
2 - (2*a*b)/3
```

Other useful functions for symbolic tools includes: `diff`, `subs`, `simplify`, `solve`, etc. Use `help` command (e.g. `help diff`) to learn more about them.

On PDEs and Variational Equations (51)

For this problem section "1.1.2.3 A Recipe to Obtain Variational Equations" of the notes provides a more detailed explanation of the steps.

1. (15) Given $f\colon (0,1) \to \mathbb{R}$ continuous and a constant $\lambda > 0$, consider the differential equation

$$-u_{,xx} + \lambda u_{,x} = f \qquad x \in (0,1). \tag{3}$$

with boundary conditions $u(0) = 0.1$ and $u'(1) = 1$.

Let the test space be

$$\mathcal{V} = \{v\colon [0,1] \to \mathbb{R} \text{ smooth } \mid v(0) = 0\}$$

Obtain a variational equation for the problem following the steps in §1.1.2.3. Identify essential and natural boundary conditions.

Solution: Form the residual function

$$-u_{,xx} + \lambda u_{,x} - f = r$$

We then proceed to multiply this equation by a function $v \in \mathcal{V}$ and integrate over (0,1), where $\mathcal{V}$ is some set of smooth enough functions over (0,1).

$$\int_0^1 (-u_{,xx} + \lambda u_{,x} - f)v \, dx = 0$$

for all v $\in \mathcal{V}$.

Integrating by parts,

$$\int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x} - vf)dx - vu_{,x}\Big|_0^1 = 0,$$

and expanding the boundary terms,

$$\int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x} - vf)dx - v(1)u_{,x}(1) + v(0)u_{,x}(0) = 0$$

In this case, we know the value of $u_{,x}(1)$, i.e. $u_{,x}(1) = 1$, and we can replace it in the last equation. However, we do not know anything about the value of $u_{,x}(0)$; we only know about $u(0)$. So we request the value of the accompanying test function to be zero. If $v \in \mathcal{V}$ then $v(0) = 0$. For any such v,

$$\int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x} - vf)dx - v(1) = 0.$$

2. (10) Transform the last variational equation for this problem so that it takes advantage of Nitsche's method; see Example 1.14 in the notes.

Solution: As in Example 1.14, we begin by stating the following two variational equations that u satisfies

$$(0.1 - u(0))v'(0) = 0,$$
$$\mu(u(0) - 0.1)v(0) = 0,$$

for some $\mu > 0$. For example, we can take $\mu = 1$.

The variational equation that takes advantage of Nitsche's method is that u satisfies that

$$0 = \int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x} - vf)dx + u'(0)v(0) - v(1) + (0.1 - u(0))v'(0) + (u(0) - 0.1)v(0)$$

for all $v \in \mathcal{V} = \{v\colon [0,1] \to \mathbb{R} \text{ smooth}\}$, or

$$\int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x})dx + u'(0)v(0) - u(0)v'(0) + u(0)v(0) = \int_0^1 vf dx + v(1) - 0.1v'(0) + 0.1v(0). \quad (4)$$

3. Assume that $f(x) = x$.

   (a) (5) Find the general solution of (3)?

   Solution: The general solution is

   $$u(x) = \frac{x}{\lambda^2} + \frac{x^2}{2\lambda} + c_1\frac{\exp(\lambda x)}{\lambda} + c_2$$

   for any $c_1, c_2 \in \mathbb{R}$.

   (b) (5) Consider a test function v so that $v(0) = v'(0) = 0$ and $v(1) = 0$. What terms of the variational equation you found in part 2 are guaranteed to be zero for such a choice, regardless of u?

   Solution: The following terms are equal to zero for any u:

   $$\int_0^1 (v_{,x}u_{,x} + v\lambda u_{,x})dx + \cancel{u'(0)v(0)} - \cancel{u(0)v'(0)} + \cancel{u(0)v(0)} = \int_0^1 vxdx + \cancel{v(1)} - \cancel{0.1v'(0)} + \cancel{0.1v(0)}.$$

   (c) (5) Select a test function v so that $v(0) = v'(0) = 0$ and $v(1) = 0$. Using the general solution, test the variational equation you found in part 2 with the test function you selected. Is the variational equation satisfied by the exact solution for such v?

   Solution: Choose $v(x) = x^2(x-1)$. Replacing in variational equation (4), computing and evaluating, we conclude that the variational equation is satisfied by the exact solution and this choice of a test function.

   (d) (5) Using the general solution, test the variational equation you found in part 2 by selecting a test function v so that $v(0) = v'(0) = 0$, but such that $v(1) \neq 0$. What can you conclude about $u'(1)$?

   Solution: Choose $v(x) = x^2$. Replacing in variational equation (4), evaluating and simplifying, we find that it implies that

   $$-1 + \frac{1}{\lambda^2} + \frac{1}{\lambda} + \exp(\lambda)c_1 = 0$$

   Solving for c_1 we find that

   $$c_1 = \frac{\exp(-\lambda)}{\lambda^2}\left(\lambda^2 - \lambda - 1\right).$$

   Replacing in the general solution, we obtain

   $$u(x) = \frac{x}{\lambda^2} + \frac{x^2}{2\lambda} + \frac{\exp(\lambda(x-1))}{\lambda^3}\left(\lambda^2 - \lambda - 1\right) + c_2 \quad (5)$$

   Evaluating u', we obtain $u'(1) = 1$.

(e) (5) The last part should have allowed you to narrow the set of possible general solutions that satisfy the variational equation. Using this smaller set of general solutions, test the variational equation by selecting another test function for which $v(0) \neq v'(0)$. What can you conclude about $u(0)$?

Solution: Choose $v(x) = x$. Replacing u by (5) and the selected v in variational equation (4), we obtain that

$$0 = 0.1 - \frac{\exp(-\lambda)}{\lambda^3}\left(\lambda^2 - \lambda - 1\right) - c_2,$$

from where we conclude that

$$c_2 = 0.1 - \frac{\exp(-\lambda)}{\lambda^3}\left(\lambda^2 - \lambda - 1\right).$$

Replacing in (5), we get

$$u(x) = \frac{x}{\lambda^2} + \frac{x^2}{2\lambda} + \frac{\exp(\lambda(x-1))}{\lambda^3}\left(\lambda^2 - \lambda - 1\right) + 0.1 - \frac{\exp(-\lambda)}{\lambda^3}\left(\lambda^2 - \lambda - 1\right) \tag{6}$$

It trivially follows that $u(0) = 0.1$.

(f) (1) Based on the work you've done so far, what is the exact solution of the problem then?

Solution: The exact solution is (6).

To test, choose the simplest function that you can imagine and that is not identically zero, so that the integrals are simpler. Play with linear, quadratic and cubic polynomials to build them.

You are advised to use one of the programs we mentioned at the beginning to perform all integrals in this problem and to find the solution. This problem shares some traits with Examples 1.9 and 1.10 in the notes.