Chapter 1

Finite Element Methods for Elliptic Problems in 1D

1.1 Second-Order Problems

1.1.1 The Differential Equation

Linear, second-order elliptic differential equations in one dimension have the general form

$$-\left(k(x)\,u'(x)\right)' + b(x)\,u'(x) + c(x)\,u(x) = f(x)\,, \tag{1.1}$$

where x is the **independent variable**, typically a space variable, k, b, c and f are the **coefficients** of the equation, which may depend on x, and u is the relevant function being studied.

A differential equation is a condition for the function u that must be fulfilled at every point of the domain. At each $x \in \Omega$, the function u must **(a)** be smooth enough for all terms in the equation to be computable and **(b)** satisfy the algebraic equation (1.1). In this case, we say that the function u is a **solution** of the differential equation 1.1.

A multitude of physical problems can be modeled with Equation (1.1). We see some of them next, together with examples of solutions u to 1.1 in particular cases.

Examples:

1.1 Consider a vertical cylindrical column of uniform cross sectional area and height H, Young modulus $E(x)$ and density $\rho(x)$, x being the vertical coordinate. Then the vertical displacement $u(x)$ of the cross section at height x must satisfy the equilibrium equation

$$-\left(E(x)\,u'(x)\right)' = -\rho(x)\,g\,, \tag{1.2}$$

where g is the magnitude of the acceleration of gravity. Clearly, it reduces to (1.1) by taking $k = E$, $b = 0$, $c = 0$ and $f = -\rho\,g$. The vertical stress (force per unit area) on the cross-section is given by

$$\sigma = E\,u' \,. \tag{1.3}$$

1.2 When heat is flowing through a wall, x being the through-the-wall coordinate, the temperature u obeys the **diffusion equation**

$$-\left(k(x)\,u'(x)\right)' = f(x)\,, \tag{1.4}$$

where $k(x)$ is the **thermal conductivity** of the wall material at point x and f is a volumetric heat source (due for example to γ-radiation).
In the case of a homogeneous wall without volumetric sources, the temperature is affine in x. The temperature gradient, and thus also the **heat flux** $-ku'$, are constant.

1.3 **Steady-state convection-diffusion-reaction equation:** Let $u(x)$ denote $C(x) - C_{\text{eq}}$, where $C(x)$ the concentration of a species in a mobile one-dimensional reactive medium that moves with **velocity** $b(x)$ and C_{eq} is the equilibrium value. Then $u(x)$ satisfies (1.1), with k being the **diffusion coefficient**, c the **reaction coefficient** and f a possible volumetric source of the species.
In this case the first term is called **diffusion term** and consists of the spatial derivative of the **diffusive flux**

$$J_{\text{diff}} = -k\,u' \,.$$

It expresses the differential mass balance due to molecular diffusion. The second term is called **advection term** (sometimes **convection term**). It models the mass balance due to the transport of the species by the movement of the ambient medium. In fact, it is the *material time derivative* of the concentration in steady-state conditions (when the concentration does not depend on time), namely

$$\frac{Du}{Dt} = bu'.$$

In fact, the **advective flux** is

$$J_{\text{adv}} = b\,u \,.$$

The term $c(x)\,u(x)$ models the reaction of the species with the medium towards equilibrium. The coefficient c is proportional to the kinetic constant of the reaction. As c grows the local concentration is increasingly driven towards equilibrium and thus u gets closer to zero.

1.4 If we consider now a horizontal membrane under tension which is subject to vertical loads f, the vertical displacement u satisfies

$$-\,T\,u''(x) = f(x)\,, \tag{1.5}$$

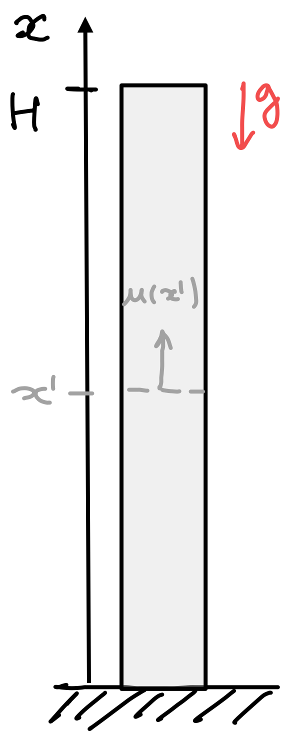

Figure 1.1 Sketch for Example 1.1.

where T is the **membrane tension**. This equation models the equilibrium of vertical forces on each arbitrary part of the membrane, which in this one-dimensional setting should better be visualized as an elastic string.

We hope the reader is by now convinced of the usefulness of models governed by (1.1). Of course the actual field $u(x)$ that arises for given coefficients (k, b, c and f) depends on the **boundary conditions**. In one dimension, the domain of analysis Ω is usually an interval, which we take as $0 < x < L$, i.e.,

$$\Omega = (0, L) \ .$$

In the physical systems considered in the previous examples, as in many other models of mathematical physics, the relevant field u arises as the solution of a **boundary value problem**. This means that *one* additional condition is imposed at $x = 0$ and *a second one* at $x = L$, and that these two conditions are necessary and sufficient to fully determine u. This is an intrinsic property of **second-order elliptic problems**: To fully define the solution u of the differential equation, it is necessary and sufficient to specify one condition at all points of the boundary. If there is some part of the boundary where no condition is specified, then there exist infinitely many solutions. In the one-dimensional setting considered here the boundary (denoted in general by $\partial\Omega$) consists of just the extreme points of the interval, i.e.

$$\partial\Omega = \{0, L\} \ .$$

We will also refer to the **closure** of Ω, defined for this domain as

$$\overline{\Omega} = \Omega \cup \partial\Omega = [0, L].$$

The most popular boundary conditions are

- the **Dirichlet condition**, which imposes the value of u (for example, $u(0) = g_0$ or $u(L) = g_L$),
- and the **Neumann condition**, which imposes the value of u' (for example $u'(0) = d_0$ or $u'(L) = d_L$).

The problem in strong form that we introduce now has a Dirichlet condition at $x = 0$ and a Neumann condition at $x = L$. Other possibilities of boundary conditions will be discussed later.

Problem 1.1. *Given the coefficients k, b, c and f (as functions of x), together with the boundary constants g_0 and d_L, find a continuous function $u : \Omega \to \mathbb{R}$ satisfying*

$$-\big(k(x)\,u'(x)\big)' + b(x)\,u'(x) + c(x)\,u(x) = f(x) \qquad \forall\, x \in \Omega \tag{1.6a}$$

$$u(0) = g_0 \tag{1.6b}$$

$$u'(L) = d_L \tag{1.6c}$$

Examples:

1.5 Consider a purely diffusive ($b = c = 0$), homogeneous ($k(x) = k_0$, $\forall\, x$) case without source ($f = 0$). The constants g_0 and d_L remain arbitrary. Then the solution of the problem in strong form must be continuous and satisfy

$$-u''(x) = 0, \quad \forall\, x \in (0, L), \qquad u(0) = g_0, \qquad \text{and} \;\; u'(L) = d_L$$

Polynomials of degree ≤ 1, i.e., of the form $c_1 + c_2 x$, have vanishing second derivative in $(0, L)$, and they are continuous functions of x. They thus satisfy condition (1.12a) above, and hence they are solution of the differential equation. Further, choosing $c_1 = g_0$ and $c_2 = d_L$ we identify the only polynomial solution to Problem 1.1, namely

$$u(x) = g_0 + d_L\, x\,.$$

Further, it is known that the *only* functions that have zero second derivative in an interval are polynomials of degree 1. Thus the function $u(x)$ above is the *unique* solution to Problem 1.1.

1.6 **Problems without solution.** Problem 1.1 does not always have a solution. Consider a case with constant diffusivity ($k = 1$), with no convection or source ($b = f = 0$) and with $c(x) = 1/x^2$. So, the equation reads

$$-\,u''(x) + \frac{u(x)}{x^2} = 0, \qquad \forall\, x \in (0, L)\,. \tag{1.7}$$

It can be checked by substitution that any function of the form

$$u(x) = c_1\, x^{(1+\sqrt{5})/2} + c_2\, x^{(1-\sqrt{5})/2}$$

satisfies (1.7). In fact, it is known that these functions are the *only* solutions of (1.7). However, notice that the exponent of x in the second term is *negative*, so that if $c_2 \ne 0$ the solution is $\pm\infty$ at $x = 0$ and thus different from g_0. So, for the problem to have a solution, $c_2 = 0$. But now, since the exponent of x in the first term is *positive*, the value of u at $x = 0$ is zero for any value of c_1! Unless the given value of g_0 is zero, there is no solution. Just for completeness, in the particular case $g_0 = 0$ there is indeed a unique solution, and the constant c_1 can be computed so that $u'(L) = d_L$.

1.7 **Deformed Column.** Consider again Example 1.1 of a vertical cylindrical column of uniform cross sectional area, height H, Young modulus $E(x)$ and density $\rho(x)$, with x being the vertical coordinate, see Fig. 1.1. Assume that the column is unloaded on its top face, and supported on a rigid foundation at its base. These define the boundary conditions of the problem by

$$u(0) = 0, \qquad \sigma(H) = E(H)\,u'(H) = 0,$$

which are of the Dirichlet and Neumann type, respectively. These equations, together with (1.2), define the strong form of this problem. To solve it, we integrate (1.3) over a slice of the column, from $x = h_1$ to $x = h_2$ to get

$$\sigma(h_1) - \sigma(h_2) = -g \int_{h_1}^{h_2} \rho(x)\, dx$$

so that the difference in σ between two positions equals the weight of the slice (per unit area). Since the column is unloaded on its top face, $\sigma(H) = 0$, and hence

$$\sigma(x) = -g \int_x^H \rho(\xi)\, d\xi.$$

Therefore, from (1.3), the displacement of the column follows as

$$u(x) - u(0) = \int_0^x \frac{\sigma(\xi)}{E(\xi)}\, d\xi = -\int_0^x \frac{g}{E(\xi)} \int_\xi^H \rho(y)\, dy. \tag{1.8}$$

Since the foundation is rigid, $u(0) = 0$. We can verify next that function $u(x)$ defined in (1.8) is a solution of the differential equation (1.2). First, we can compute $(E(x)u'(x))'$ for any point x by using the fundamental lemma of calculus, so it is smooth enough for all terms in the equation to be computable (condition (**a**) above), and these terms satisfy the algebraic equation (1.2) at any point x.

In the particular case in which $E(x) = E$ and $\rho(x) = \rho$, both constants through the length of the column, the solution u is

$$u(x) = -x(2H - x)\frac{g\rho}{2E}.$$

What about the solution of Problem 1.1? Does it exist at all? Is it unique? The answer to this question is derived from the general theory of linear ordinary differential equations, and the answer is **yes**, but of course conditional to some hypotheses. The hypotheses we consider here correspond to *elliptic* problems, and as we have seen allow us to model a wide variety of physical problems.

In general, sound physical models lead to well-posed mathematical problems, that is, problems for which a unique solution exists, and the solution changes smoothly when the coefficients of the equation or the boundary conditions do. However, theorems are helpful references to go to in case of doubt. We state here an existence and uniqueness theorem that covers most applications.

Theorem 1.1 (Existence and Uniqueness of Solutions)**.** *Assume that $k(x)$, $b(x)$, $c(x)$ and $f(x)$ are smooth and bounded, and also that $k(x) \geq k_0 > 0$. Further, let $c_0 = \min_x c(x)$ and assume that $c_0 \geq 0$. Then Problem 1.1 has a unique solution.*

This is more than what we need to know at this point about the strong form of the elliptic second-order boundary-value problem that we are set to analyze.

1.1.2 Variational Equations

The finite element method is based on the observation that the solution of Problem 1.1 satisfies a **variational equation**. A variational equation is not only a staple of finite element methods, but it is always a puzzling and welcome surprise to those who are introduced to it for the first time. Let's see what a variational equation for Problem 1.1 looks like.

Let u be the solution of Problem 1.1. Then, u satisfies that

$$\int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\,dx \\ - k(L)d_L v(L) = \int_\Omega f(x)v(x)\,dx. \quad (1.9a)$$

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon [0,L] \to \mathbb{R} \text{ smooth } \mid w(0) = 0\}. \quad (1.9b)$$

The *variational equation* is (1.9a), while (1.9b) defines the **test space** $\mathcal{V}$. A functions $v \in \mathcal{V}$ is called a **test function**. The name variational equation originates in the appearance of equations of this form in the Calculus of Variations, or the general theory of extreme values (see, e.g., [6]).

For now, we will understand a function to be **smooth** in the definition of $\mathcal{V}$ as a function in which all derivatives exists and are continuous in $[0,L]$. Later on, we will expand this definition to include more functions, so that some commonly used functions used in the finite element method are also included in $\mathcal{V}$.

Equation (1.9a) is called a variational equation because it has the form

$$F(u,v) = 0 \qquad \forall v \in \mathcal{V}, \quad (1.10)$$

for a scalar-valued function F and a test space $\mathcal{V}$. In our example,

$$F(u,v) = \int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\,dx \\ - k(L)d_L v(L) - \int_\Omega f(x)v(x)\,dx.$$

We will have the opportunity to more precisely define what a variational equation is later.

Notice that a variational equation is a statement of a potentially infinite set of equations, since each function in $\mathcal{V}$ defines a condition that u needs to satisfy. If there are enough functions in $\mathcal{V}$, or enough conditions, a variational equation may be able to define u as the unique function that can satisfy it.

Example 1.8 Let's illustrate that variational equation (1.9a) is satisfied for a solution of Problem (1.1) and some choice of test function.

To this end, let $\Omega = (0,1)$, $k(x) = b(x) = c(x) = 1$ and $f(x) = -5\exp(-2x)$ for all $x \in \Omega$, $g_0 = 1$, and $d_L = -2\exp(-2)$. Equations (1.6) from the strong form of Problem 1.1 become

$$-u''(x) + u'(x) + u(x) = -5\exp(-2x), \qquad \forall\, x \in \Omega, \tag{1.11a}$$
$$u(0) = 1, \tag{1.11b}$$
$$u'(1) = -2\exp(-2). \tag{1.11c}$$

The exact solution of this problem is $u(x) = \exp(-2x)$.

As a function in the test space, consider the function $v(x) = \sin(x)$, which satisfies that $v(0) = 0$, so $v \in \mathcal{V}$. The left hand side of variational equation (1.9a) then reads

$$\begin{aligned}
&\int_0^1 (\exp(-2x))' \sin'(x) + (\exp(-2x))' \sin(x) + \exp(-2x)\sin(x)\, dx \\
&\qquad + 2\exp(-2)\sin(1) = \\
&\int_0^1 -2\exp(-2x)\cos(x) - 2\exp(-2x)\sin(x) + \exp(-2x)\sin(x)\, dx \\
&\qquad + 2\exp(-2)\sin(1) = \\
&\qquad\qquad \cos(1)\exp(-2) - 1 + 2\exp(-2)\sin(1).
\end{aligned}$$

The right hand side of the same equation is

$$-\int_0^1 5\exp(-2x)\sin(x)\ \ dx \;=\; -1 \,+\, \cos(x)\exp(-2) \,+\, 2\sin(1)\exp(-2).$$

Since both sides have the same value, (1.9a) is satisfied. A similar result would follow for any $v \in \mathcal{V}$.

Instead, if $v \not\in V$, (1.9a) may not be satisfied. For example, if $v(x) = \cos(x)$, then the left hand side equals $\left[1 + \exp(-2)\,(9\cos(1) - 2\sin(1))\right]/5$, and the right hand side differs from it with the value $-1 + \exp(-2)\,(\cos(1) + 2\sin(1))$.

1.1.2.1 Derivation of a Variational Equation

How do we obtain a variational equation that the solution of Problem 1.1 satisfies? To illustrate it, we begin with the simplest possible case, and set $k(x) = 1$, $b(x) = c(x) = 0$ for all $x \in \Omega = (0, L)$, so that

$$-u''(x) = f(x) \qquad x \in \Omega, \tag{1.12a}$$
$$u(0) = g_0, \tag{1.12b}$$
$$u'(L) = d_L. \tag{1.12c}$$

The variational equation is obtained from Problem 1.1 by following three steps:

1. Multiply the partial differential equation (1.6a) by an arbitrary smooth function $v\colon \Omega \to \mathbb{R}$ and integrate over the interval $[0,L]$ to obtain

$$0 = u''(x)v(x) + f(x)v(x) \Rightarrow 0 = \int_0^L u''(x)v(x) + f(x)v(x)\,dx \tag{1.13}$$

2. Integrate by parts[1] the second derivative of $u''(x)$, to pass the derivative to v, to get:

$$0 = u'(L)v(L) - u'(0)v(0) - \int_0^L u'(x)v'(x)\,dx + \int_0^L f(x)v(x)\,dx. \tag{1.14}$$

3. Use boundary condition (1.12c) to replace the value of $u'(L) = d_L$ in (1.14), and since we know nothing about the value of $u'(0)$, we will require $v(0) = 0$. Thus,

$$0 = \underbrace{u'(L)}_{=d_L,\text{ due to }(1.12b)} v(L) - u'(0) \underbrace{v(0)}_{=0,\text{ require it from }v\in\mathcal{V}} - \int_0^L u'(x)v'(x)\,dx + \int_0^L f(x)v(x)\,dx, \tag{1.15}$$

from where it follows that

$$\int_0^L u'(x)v'(x)\,dx - d_L v(L) = \int_0^L f(x)v(x)\,dx. \tag{1.16}$$

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon [0,L] \to \mathbb{R} \text{ smooth} \mid v(0) = 0\}.$$

Equation (1.16) is a variational equation that the solution u of Problem 1.1 satisfies. It is the particularization of (1.9a) for the values of k, b, c and Ω in this example.

Thus, in obtaining variational equation (1.16) we also introduced conditions that test functions $v \in \mathcal{V}$ should satisfy. What if we did not require $v(0) = 0$ for $v \in \mathcal{V}$? In this case, we would need to retain the second term on the left hand side in step 3 above, (1.15), and would have obtained a different variational equation that u satisfies, namely,

$$\int_0^L u'(x)v'(x)\,dx + u'(0)v(0) - d_L v(L) = \int_0^L f(x)v(x)\,dx. \tag{1.17}$$

[1] For two smooth functions w and v, $(wv)' = w'v + wv'$ by the product formula, and integrating on both sides we get the integration by parts formula:

$$\lim_{x\to b^-} w(x)v(x) - \lim_{x\to a^+} w(x)v(x) = \int_a^b w'(x)v(x)\,dx + \int_a^b w(x)v'(x)\,dx.$$

We use it by setting $w = u'$.

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon [0, L] \to \mathbb{R} \text{ smooth}\}.$$

We may go even further and question what if we did not integrate by parts in step 2 above. In that case, we obtain yet another variational equation that u satisfies (from (1.13)):

$$0 = \int_0^L u''(x)v(x) + f(x)v(x)\,dx \tag{1.18}$$

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon [0, L] \to \mathbb{R}\}, \tag{1.19}$$

where we do not require smoothness for functions in $\mathcal{V}$ since we are not integrating by parts and computing derivatives of test functions; we just need the integrals we are performing to be computable.

This discussion shows that u satisfies different variational equations, each one for a potentially different set of test functions. In fact, there are an infinite number of variational equations that u satisfies. The most remarkable difference between the variational equations we listed is the way that u, v and their derivatives are involved, and the boundary conditions required from functions in the test space. In (1.16), only the first derivative of u and v in Ω is needed; in (1.14) we additionally need the value of the first derivative of u on part of the boundary of Ω, but no boundary conditions on test functions; and in (1.18) two derivatives of u are required, while no derivative of test functions are needed, nor boundary conditions on them.

Each variational equation could serve as the basis for the formulation of a finite element method. For reasons that we will have the chance to discuss later, the most commonly adopted variational equation is (1.16), because it has the same number of derivatives required from both u and v, and no evaluation of derivatives of u on the boundary of the domain.

1.1.2.2 Essential and Natural Boundary Conditions

In obtaining variational equation (1.16) we incorporated boundary condition (1.12c) into it when we replaced $u'(L)$ by d_L. In contrast, boundary condition $u(0) = g_0$, (1.12b), was not used at all. Because of this, *any* function that satisfies partial differential equation (1.12a) and boundary condition (1.12c) also satisfies variational equation (1.16), even if they do not satisfy boundary condition (1.12b).

Examples:

1.9 In (1.12), let $L = \pi/2$, $f(x) = \cos(x)$, $g_0 = 1$ and $d_L = -1$. Then, the function $u(x) = \cos(x)$ is the solution of (1.12). As such, it satisfies variational equation (1.16) for any smooth v such that $v(0) = 0$. For example, for $v(x) = \sin(x)$,

$$\underbrace{\int_0^{\pi/2} -\sin(x)\cos(x)\,dx}_{=-1/2} + 1\sin(\pi/2) - \underbrace{\int_0^{\pi/2} \cos(x)\sin(x)\,dx}_{=-1/2} = 0.$$

Consider then $u_1(x) = u(x) + 1 = \cos(x) + 1$, which satisfies (1.12a) and (1.12c), but does not satisfy (1.12b), since $u_1(0) = 2 \neq g_0$. In spite of this, function u_1 also satisfies variational equation (1.16), as it can be easily inferred from the fact that $u' = u_1'$.

In contrast, consider $\hat{u}_2(x) = u(x) + x = \cos(x) + x$, which satisfies partial differential equation (1.12a) but *not* boundary condition (1.12c), since $u_2'(\pi/2) = 0 \neq -1$. This function does not satisfy variational equation (1.16): for $v(x) = \sin(x)$,

$$\underbrace{\int_0^{\pi/2} (1-\sin(x))\cos(x)\,dx}_{=1/2} + 1\sin(\pi/2) - \underbrace{\int_0^{\pi/2} \cos(x)\sin(x)\,dx}_{=-1/2} = 1 \neq 0.$$

1.10 For $f(x) = \cos(x)$ and $L = \pi/2$, the general solution of the differential equation (1.12a) is $u(x) = c_1 + c_2 x + \cos(x)$, for any $c_1, c_2 \in \mathbb{R}$. The values of the two constants c_1, c_2 can be determined so as to satisfy the boundary conditions (1.12b) and (1.12c), respectively. In fact,

$$u(0) = c_1 + 1, \qquad \text{and} \qquad u'(\pi/2) = c_2 - 1.$$

Because we are discussing about variational equations, we will see now that we can determine the value of c_2 from variational equation (1.16) by selecting a suitable test function v. In other words, we will see that the variational equation defines the value of the Neumann boundary condition.

As a first case, let's choose $v \in \mathcal{V}$ such that $v(L) = v(\pi/2) = 0$. For example, we can choose $v(x) = x(x - \pi/2)$, a quadratic polynomial that has zeros at $x = 0$ and $x = \pi/2$. Replacing in the left hand side of the variational equation, we obtain

$$\int_0^{\pi/2} (c_2 - \sin(x))(2x - \pi/2)\,dx = (\pi - 4)/2.$$

Computing the right hand side we obtain the same value, namely,

$$\int_0^{\pi/2} \cos(x)x(x - \pi/2)\,dx = (\pi - 4)/2,$$

so the variational equation is satisfied for this choice of $v \in \mathcal{V}$, *regardless* of the values of c_1 and c_2. This choice of test function does not add any condition to the two free constants.

Instead, let's choose $v(x) = x$ as a test function, which is different than zero at $x = \pi/2$; $v(\pi/2) = \pi/2$. In this case, the left hand side evaluates to

$$\int_0^{\pi/2} (c_2 - \sin(x))\, dx - d_L\pi/2 = \frac{\pi}{2}(c_2 - d_L) - 1.$$

The right hand side takes the value

$$\int_0^{\pi/2} \cos(x)x\, dx = \pi/2 - 1.$$

Equating both sides, we obtain an equation that involves c_2, namely,

$$\frac{\pi}{2}(c_2 - d_L) - 1 = \pi/2 - 1,$$

and solving it for c_2 we conclude that $c_2 = 1 + d_L$.

In summary, by testing with $v(x) = x(x - L)$ the variational equation did not impose any condition on the values of c_1 and c_2. However, by testing with another element of $\mathcal{V}$, $v(x) = x$, we obtained a necessary condition for the variational equation to be satisfied: $c_2 = 1 + d_L$.

This means that not all solutions of (1.12a) satisfy variational equation (1.16). A necessary condition is that c_2 acquire a specific value. Therefore, the general solution of (1.12a) that could satisfy the variational equation is

$$u(x) = c_1 + (1 + d_L)x + \cos(x).$$

Since $u'(\pi/2) = 1 + d_L - \sin(\pi/2) = d_L$, u satisfies the Neumann boundary condition (1.12c) for any value of c_1.

These examples illustrate the more general fact that for a function to satisfy a variational equation, it may need to satisfy some boundary conditions as well. It also illustrates that not all boundary conditions in a problem may need to be satisfied for the solution to satisfy a variational equation.

Therefore, given a variational equation that the solution of a problem such as Problem 1.1 needs to satisfy, we can classify the boundary conditions of the problem into two types:

Natural Boundary Conditions (NBC): Boundary conditions that any function that satisfies the variational equation needs to satisfy.

Essential Boundary Conditions (EBC): Any boundary condition of the problem that is not a natural boundary condition.

If we derive the variational equation, as we did earlier, then all boundary conditions that we incorporate into it during the derivation will be natural boundary conditions. All other boundary conditions will be essential. Instead, if we are given a variational equation, we will learn later how to infer what natural boundary conditions it requires.

Examples:

1.11 For variational equation (1.16), $u(0) = g_0$ is an essential boundary condition, and $u'(L) = d_L$ is a natural boundary condition. This is also true for variational equation (1.17).

1.12 For variational equation (1.18), both $u(0) = g_0$ and $u'(L) = d_L$ are essential boundary condition, since none of them was incorporated in the variational equation during its derivation.

1.1.2.3 A Recipe to Obtain Variational Equations

In the following we describe a recipe to obtain variational equations from a partial differential equation and boundary conditions. We illustrate each step of the recipe with Problem 1.1 and derive (1.9), but the recipe works for a large class of problems. The steps of the recipe are:

1. **Form the residual:** Begin by forming the residual of (1.6a): subtract the right hand side from the left hand side of the equality (or vice versa). That is, for a function u we define a function $r\colon [0,L] \to \mathbb{R}$ as

$$r = -(k(x)u'(x))' + b(x)u'(x) + c(x)u(x) - f(x). \tag{1.20}$$

   Then, according to (1.6a), for the solution u of the strong form we should have

$$r(x) = 0 \qquad x \in (0,L). \tag{1.21}$$

2. **Multiply by a test function and integrate.** We then proceed and multiply this equation by a function $v \in \mathcal{V}$ and integrate over $(0,L)$, where $\mathcal{V}$ is some set of smooth functions over $(0,L)$ that we shall specify later. As aforementioned, functions $v \in \mathcal{V}$ are called *test* functions, but are also labeled **weight** functions. For any such $v \in \mathcal{V}$, we have

$$\int_0^L r(x)v(x)\,dx = 0. \tag{1.22}$$

   Again, we are replacing the requirement $r(x) = 0$ for all $x \in (0,L)$, for (1.22) to be satisfied for all functions in $\mathcal{V}$. Because the residual functions are multiplied by the weight functions, this form of formulating the problem is also called the *Method of Weighted Residuals* (MWR)[1].

   In our example, this means:

$$\int_0^L \big(-(k(x)u'(x))' + b(x)u'(x) + c(x)u(x) - f(x)\big)\,v\,dx = 0 \tag{1.23}$$

   for all $v \in \mathcal{V}$.

3. **Integrate the residual by parts.** Assuming that both u and v are smooth enough for the integration by parts formula to hold, integrate by parts terms in the residual $m \geq 0$ times, with m less or equal than the maximum number of derivatives of u in the residual. For each value of m a *different* variational equation and hence weak form is obtained. Typically, we integrate by parts until the number of derivatives of u is equal or only one higher than the number of derivatives of v in each term. In other words, use integration by parts to "transfer" derivatives from u to v, until u has either an equal number of or one more derivative than v in any term in the resulting expression. In our example, $m = 1$, and this leads to

$$\int_0^L k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx \\ - k(L)u'(L)v(L) + k(0)u'(0)v(0) = 0 \quad (1.24)$$

In this example, we only integrated by parts the first term of the residual, since the second term already has the desired difference in the order of the derivatives between u and v.

4. **Use boundary conditions and identify conditions for $\mathcal{V}$.** Out of the boundary terms that appear from integrating by parts, identify those for which the value has been provided or can be solved for from the boundary conditions, and replace them in the boundary terms. As with the integration by parts, there could be some ambiguity here, leading to different weak formulations. In this case, (1.6c) gives the value of $u'(L)$. Replacing in our example,

$$\int_0^L k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx \\ - k(L)d_L v(L) + k(0)u'(0)v(0) = 0 \quad (1.25)$$

However, we do not know anything about the value of $u'(0)$; we only know about $u(0)$. In this case, we request the value of the accompanying test function to be zero through the definition of $\mathcal{V}$. In a general case, we proceed similarly with any other boundary term for which we do not have any boundary condition, and require the value of the accompanying (derivative of the) test function to be zero in the definition of $\mathcal{V}$. This is the process to identify conditions that we need to impose for functions in $\mathcal{V}$.

In our example, we are going to request that if $v \in \mathcal{V}$, then $v(0) = 0$. For any such v,

$$\int_0^L k(x)u'(x)v'(x) + b(x)u'(x)v(x) + c(x)u(x)v(x) - f(x)v(x)\,dx \\ - k(L)d_L v(L) = 0 \quad (1.26)$$

Those boundary conditions that we are able to incorporate into the variational equation by replacing some of the boundary terms are the natural boundary conditions for the problem. The remaining boundary conditions need to be explicitly requested for u to satisfy; they are the essential boundary conditions. Hence, for our example, boundary condition (1.12b) is an essential boundary condition.

If we do not request $v(0) = 0$ in the definition of $\mathcal{V}$, we would arrive to a *different* variational equation, still satisfied by the exact solution, with a different set of essential and natural boundary conditions. Similarly, we would obtain an alternative variational equation for each number of derivatives we decide to transfer from u to v.

5. **State the variational equation.** For our example, this leads to (1.9), namely, u satisfies that

$$\int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\,dx \\ - k(L)d_L v(L) = \int_\Omega f(x)v(x)\,dx. \quad (1.27a)$$

for any $v \in \mathcal{V}$, where

$$\mathcal{V} = \{w\colon \Omega \to \mathbb{R} \text{ smooth } \mid w(0) = 0\}. \quad (1.27b)$$

Also, list the essential boundary conditions of the problem, since these will be used later on when we formulate a finite element method.

As remarked in step 3, it would also be possible to choose a different distribution of derivatives between u and v than the guideline provided above, leading to a different variational equation. For example, we could transfer all derivatives to v, or leave all derivatives in u. In finite element analysis, we are generally interested in variational equations with minimal smoothness requirements for u and v, since it is simpler to build spaces to approximate solutions in this case.

Example 1.13 A variational equation for a third-order problem. Given $f\colon [a,b] \to \mathbb{R}$, find $u\colon [a,b] \to \mathbb{R}$ such that

$$u_{,xxx} = f \qquad x \in (a,b) \quad (1.28a)$$
$$u(a) = 1 \quad (1.28b)$$
$$u_{,x}(b) = 2 \quad (1.28c)$$
$$u_{,xx}(a) = 3. \quad (1.28d)$$

Here the notation $u_{,x}$ denotes the derivative, $u_{,xx}$ the second derivative, etc. The exact solution of this problem is obtained by repeated integration of

(1.28a):

$$\int_a^x f(y)\,dy = \int_a^x u,_{yyy}(y)\,dy$$
$$= u,_{xx}(x) - u,_{xx}(a) = u,_{xx}(x) - 3$$
$$\int_b^x \left[\int_a^z f(y)\,dy\right] dz = \int_b^x u,_{zz}(z) - 3\,dz$$
$$= u,_x(x) - u,_x(b) - 3(x-b)$$
$$= u,_x(x) - 2 - 3(x-b)$$
$$\int_a^x \left[\int_b^w \left[\int_a^z f(y)\,dy\right] dz\right] dw = \int_a^x u,_w(w) - 2 - 3(w-b)\,dw$$
$$= u(x) - \underbrace{u(a)}_{=1} - 2(x-a) - \frac{3}{2}(x^2-a^2)$$
$$+ 3b(x-a).$$

Therefore,

$$u(x) = 1 + (2-3b)(x-a) + \frac{3}{2}(x^2-a^2) + \int_a^x \left[\int_b^w \left[\int_a^z f(y)\,dy\right] dz\right] dw. \tag{1.29}$$

To identify the variational equation, we proceed as above:

(a) *Form the residual*:

$$r = u,_{xxx} - f.$$

(b) *Multiply by a test function and integrate*:

$$\int_a^b (u,_{xxx} - f)\,v\,dx = 0$$

for all v that is smooth enough.

(c) *Integrate the residual by parts*: In this case, we will integrate by parts only once,

$$u,_{xx}(b)\,v(b) - u,_{xx}(a)\,v(a) - \int_a^b u,_{xx}\,v,_x + f\,v\,dx = 0$$

for all v that is smooth enough.

(d) *Use boundary conditions and identify conditions for $\mathcal{V}$*: Here we know the value of $u,_{xx}(a)$, so we need to request $v(b) = 0$. For such v we have

$$-3v(a) - \int_a^b u,_{xx}\,v,_x + f\,v\,dx = 0$$

Then, $u,_{xx}(a) = 3$ is a natural boundary condition.

(e) *State the variational equation*: The solution u of (1.28) satisfies that

$$-\int_a^b u,_{xx}\, v,_x \; dx = \int_a^b f\, v \; dx - 3v(a).$$

for all $v \in \mathcal{V}$, where

$$\mathcal{V} = \{v\colon [a,b] \to \mathbb{R} \text{ smooth } \mid v(b) = 0\}.$$

The essential boundary conditions are then $u(a) = 1$ and $u,_x(b) = 2$.

1.1.2.4 Other Variational Equations

Not all variational equations follow from the recipe in §1.1.2.3. For example, if a function u satisfies variational equations

$$F(u,v) = 0, \qquad G(u,v) = 0$$

for all $v \in \mathcal{V}$ for scalar-valued functions F and G, then u satisfies the variational equation

$$\alpha F(u,v) + \beta G(u,v) = 0$$

for all $v \in \mathcal{V}$, for any $\alpha, \beta \in \mathbb{R}$.

This enables us to construct a variety of variational equations. Each variational equation could give rise to a different finite element method, as we will have the opportunity to see later. With this in mind, the following are examples of variational equations that give rise to popular finite element methods. We label each example of a variational equation with the method it finds use for.

Example 1.14 Nitsche's Method. A solution u of Problem 1.1 satisfies the following variational equations that can be obtained after multiplying the Dirichlet boundary condition (1.12b) by the value of a test function or its derivative on the boundary

$$(g_0 - u(0))\, v'(0) = 0 \tag{1.30a}$$

$$\mu(u(0) - g_0)\, v(0) = 0 \tag{1.30b}$$

for all $v \in \mathcal{V} = \{v\colon \Omega \to \mathbb{R} \text{ smooth }\}$, where $\mu > 0$ is a positive real number.

If we add the two equations in (1.30) to variational equation (1.17), which has the same test space $\mathcal{V}$, we obtain the following variational equation that u also satisfies:

$$\int_0^L u'(x)v'(x)\; dx + u'(0)\,v(0) - u(0)\,v'(0) + \mu u(0)\,v(0) = \int_0^L f(x)\,v(x)\; dx + d_L\, v(L) - g_0\, v'(0) + \mu g_0\, v(0) \tag{1.31}$$

for all $v \in \mathcal{V} = \{v\colon [0,L] \to \mathbb{R} \text{ smooth }\}$. This variational equation is used to formulate the so-called Nitsche's finite element method for Problem 1.1, in which the Dirichlet boundary condition (1.12b) for u is *also a natural boundary condition* of the problem. There are no essential boundary conditions. We will have the opportunity to discuss this last part in detail in §XXX.

This is a general technique that can be used in a variety of problems. When it is used, we say that we use Nitsche's method for such problem.

Example 1.15 Residual-Stabilized Method. The solution u of Problem 1.1 satisfies the following variational equation

$$0 = \int_{\Omega_E} (u''(x) - f(x))\, v(x)\, dx \tag{1.32}$$

for all $v \in \mathcal{V}_E = \{v\colon \Omega \to \mathbb{R}\}$, where $\Omega_E \subset \Omega$ is a subset of Ω. For example, $\Omega_E = (L/4, L/2)$. When $\Omega_E = \Omega$, we recover (1.18).

We can combine this variational equation with (1.16), namely,

$$\int_0^L u'(x)\, v'(x)\, dx - d_L\, v(L) = \int_0^L f(x)\, v(x)\, dx \tag{1.33}$$

for all $v \in \mathcal{V} = \{v\colon \Omega \to \mathbb{R} \text{ smooth} \mid v(0) = 0\}$, to get that u should satisfy the variational equation

$$\begin{aligned}\int_0^L u'(x)\, v'(x)\, dx + \mu \int_{\Omega_E} (u''(x) - f(x))\, v(x)\, dx \\ - d_L\, v(L) = \int_0^L f(x)\, v(x)\, dx\end{aligned} \tag{1.34}$$

for any $v \in \mathcal{V}$ and some $\mu > 0$. Notice that the test spaces for variational equations (1.32) and (1.33) are different. However, $\mathcal{V} \subset \mathcal{V}_E$, so (1.32) is in particular valid for all $v \in \mathcal{V}$. This is why we can combine them to form (1.34). This variational equation can be used to formulate some of the so-called residual-stabilized finite element methods.

In this case, the Dirichlet boundary condition (1.12b) is an essential boundary condition, and the Neumann boundary condition (1.12c) is a natural one.

Example 1.16 Interior Penalty Methods. The following variational equation is stated on a domain that is split into two or more parts. For simplicity, let's split $\Omega = \Omega_1 \cup \Omega_2$, where for this example, $\Omega_1 = (0, L/2)$ and $\Omega_2 = (L/2, L)$.

The solution u of Problem 1.1 is continuous across the boundary of the two domains, in this case at $x = L/2$, and hence it satisfies that

$$0 = [\![u]\!]|_{x=L/2} = \lim_{x\to L/2^-} u(x) - \lim_{x\to L/2^+} u(x) = u^-(L/2) - u^+(L/2), \qquad (1.35)$$

which just states that the "jump" $[\![u]\!]|_{x=L/2}$ of u at $x = L/2$ should be equal to zero, since the function is continuous there. For convenience, we also introduced the notation $u^+(x) = \lim_{y\to x^+} u(y)$ and $u^-(x) = \lim_{y\to x^-} u(y)$.

For this variational equation, we introduce the test space

$$\mathcal{V} = \{v\colon \Omega \to \mathbb{R} \mid v(0) = 0,\ v \text{ is smooth in } \Omega_1 \text{ and } v \text{ is smooth in } \Omega_2\}.$$

Functions in $\mathcal{V}$ can be discontinuous at $x = L/2$, the common boundary between Ω_1 and Ω_2. Then, the solution u of Problem (1.1) satisfies the following variational equations

$$-(u'v)^-(L/2) + \int_0^{L/2} u'(x)v'(x) = \int_0^{L/2} f(x)v(x)\,dx \qquad (1.36a)$$

$$-d_L v(L) + (u'v)^+(L/2) + \int_{L/2}^{L} u'(x)v'(x) = \int_{L/2}^{L} f(x)v(x)\,dx \qquad (1.36b)$$

$$[\![u]\!]|_{x=L/2}[\![v]\!]|_{x=L/2} = 0 \qquad (1.36c)$$

for all $v \in \mathcal{V}$. The first and second equations, (1.36a) and (1.36b), are obtained by multiplying by v, integrating by parts in Ω_1 and Ω_2, respectively, and using that $v(0) = 0$ and the Neumann boundary condition 1.12c to simplify two of the boundary terms that appear. The third equation, (1.36c), is obtained by multiplying (1.35) by the jump of v at $x = L/2$.

We will combine the three variational equations in (1.38) to form the one we are interested in. But before doing that, it is convenient to state a useful identity. To this end, we define the "average" of a function u at a point x_0 as

$$\{u\}|_{x=x_0} = \frac{u^+(x_0) + u^-(x_0)}{2}.$$

With it, a simple algebraic manipulation of the right hand side of the next identity shows that

$$(u'v)^-(L/2) - (u'v)^+(L/2) = \big([\![u']\!]\{v\} + \{u'\}[\![v]\!]\big)_{x=L/2}. \qquad (1.37)$$

Finally, we obtain a variational equation of the type used in Interior Penalty finite element methods by adding (1.36a), (1.36b) and $\mu > 0$ times (1.36c), and replacing the two boundary terms at $x = L/2$ with (1.37), to get

$$\begin{aligned}\sum_{i=1}^{2}\int_{\Omega_i} u'(x)v'(x)\,dx - d_L v(L)&\\ - \big([\![u]\!]\{v'\} + \{u'\}[\![v]\!] - \mu[\![u]\!][\![v]\!]\big)|_{x=L/2} &= \sum_{i=1}^{2}\int_{\Omega_i} f(x)v(x)\,dx \qquad (1.38)\end{aligned}$$

for all $v \in \mathcal{V}$.

For this variational equation, the Dirichlet boundary condition (1.12b) is an essential boundary condition, while the Neumann boundary condition (1.12c) is a natural boundary condition. As will have the opportunity to discuss in §XXX, this variational equation imposes the condition $[\![u]\!]|_{x=L/2}$, so it is possible to think about the continuity of u as a natural boundary condition for this problem, and not require u to be continuous a priori. Interior Penalty finite element methods are an example of the so-called Discontinuous Galerkin methods, characterized by imposing the continuity of the solution as a natural boundary condition, as in this example.

1.1.3 Sets of Functions*

Part of starting a variational equation is defining a set of test functions for which we require an equation such as (1.9a) to hold (the statement "for all functions v"). Let's describe some common sets of functions and the notation we use to specify them. We proceed by examining some examples.

Examples:

1.17 The set $C^0(I)$ is the set of continuous scalar(real)-valued functions over the interval $I \subset \mathbb{R}$. For example:

   i. Let $f(x) = \sin x$. Then, if $I = [0,1]$ we have that $f \in C^0([0,1])$, since $\sin x$ assigns a real value to each point in the interval $[0,1]$, and f is continuous over that interval. Moreover, we have that $f \in C^0(\mathbb{R})$, in which we set $I = \mathbb{R}$, since $\sin x$ is continuous over the entire real line.

   ii. The *Heaviside step function* is defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0, \end{cases}$$

   for $x \in \mathbb{R}$. Then, $H \in C^0((0,1))$ because it is continuous in the open interval $(0,1)$, but $H \notin C^0([-1,1])$, because H is discontinuous at $x = 0$.

1.18 The set $C^k(I)$, for $k \in \mathbb{N}$, is the set of continuous functions with k continuous derivatives over the interval $I \subset \mathbb{R}$.
The set $C^\infty(I)$ is the set of functions in which *all* derivatives are continuous. For example:

   i. Let $f(x) = \cos x$. Then, $f \in C^2(\mathbb{R})$ since it has two continuous derivatives anywhere in the real line. Moreover, $f \in C^\infty(\mathbb{R})$, since all derivatives of f are continuous.

ii. Let $g(x) = |x|$, the absolute value function, whose derivative exists anywhere but at $x = 0$. Then, $g \in C^2((0,1])$, $g \in C^0([-1,1])$, but $g \notin C^1([-1,1])$ since g' is discontinuous at $x = 0$, i.e., $\lim_{x\to 0^-} g'(x) \neq \lim_{x\to 0^+} g'(x)$.

1.19 The set $\mathbb{P}_k(I)$, for $k \in \mathbb{N} \cup \{0\}$, is the set of all polynomials of degree less or equal than k over the interval $I \subset \mathbb{R}$. For example:

i. Let $f(x) = x^3 + 1$, then $f \in \mathbb{P}_k(\mathbb{R})$ for any $k \geq 3$.

ii. Let $f(x) = (x-2)^{10}$, then $f \in \mathbb{P}_{10}([0,1])$ for any $k \geq 10$.

A set of functions often contains an infinite number of functions, and it is impossible to enumerate all members of the set. Nevertheless, it is possible to test whether a given function belongs to the set, as we did in the above examples. Additionally, sets of functions are often defined by imposing additional conditions for a function to belong to a set. For example, we could define a set by writing

$$V_1 = \left\{ f \in C^0([0,1]) \mid f(0) = 2 \right\},$$

which indicates the set of all continuous functions over the interval $[0,1]$ whose value at $x = 0$ is 2.

We introduced new notation here, which we proceed to explain: The curly brackets $\{\cdot\}$ indicate that what is inside describes the members of the set, and the separator "|" should be read as "*such that*." So, if we write $V = \{f \in C^0([0,1])\}$ we are saying that the set contains all functions f that are in $C^0([0,1])$; f stands for a generic member of the set. It is equivalent to writing $V = C^0([0,1])$. The expression that defines V_1 above should be read as "*all functions f in* $C^0([0,1])$ *such that* $f(0) = 2$." The "|" serves the function of allowing us to add conditions for a function to belong to a set, and we do so by indicating the conditions on the generic member of the set f.

Examples:

1.20 Let $f(x) = x^2$ and $g(x) = x^2 + 2$. Then $f \notin V_1$ and $g \in V_1$.

1.21 Let $V_2 = \left\{ g \in C^2([-1,1]) \mid g(-1) = 1, g'(1) = 2 \right\}$. Then, $x^2 \in V_2$ but $h(x) = x^4 \notin V_2$, since $h'(1) = 4 \neq 2$.

1.22 Let $V_3 = \left\{ h \in C^2([0,L]) \mid h(0) = 0, h(L) = 0 \right\}$. Then $V_3 \subset C^2([0,L])$, that is, the set $C^2([0,L])$ contains all functions in V_3. This is a trivial statement, since in the definition of V_3 we are requesting functions to be in $C^2([0,L])$ as one of the conditions they should satisfy to belong to V_3. However, in the next section we will use the idea that the set $C^2([0,L])$ contains all functions in V_3, so it is a good idea to become familiar with this now.

1.23 Let $V_4 = \{h\colon [a,b] \to \mathbb{R} \text{ smooth} \mid h(a) = 0\}$. In this example, functions in V_4 take a real value for each point in the interval $[a,b]$, and the word smooth indicates that they should have as many continuous derivatives as required by the problem or the manipulation of expressions we perform; we will talk about what this precisely means later. For example, $\sin(x-a) \in V_4$, since all derivatives exists and are continuous, but the membership of $|x-a|$ will depend on the specifics of the problem.

1.1.4 Integration by Parts of Piecewise Smooth Functions*

Up to now we have been looking at examples in which all functions and a their derivatives are continuous, and we used the integration-by-parts formula on these functions to obtain variational equations. There are incentives, however, to expand the class of functions we consider so as to include functions in which either the function or some of its relevant derivatives are discontinuous. In particular, the finite element method provides a way to construct sets of functions, and the less continuity requirements functions in a set have, the easier it is to construct the set of finite element functions. This is particularly true in two and three spatial dimensions, and over domains that cannot smoothly be mapped to a cube (curved domains, domains with holes, etc.). The integration by parts formula needs to be modified for functions with discontinuities, and this is the focus of the forthcoming discussion.

The type of functions we want to consider are illustrated by the following two (see Fig. 1.2):

- The hat function $N\colon \mathbb{R} \to \mathbb{R}$,

$$N(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & |x| \geq 1. \end{cases} \tag{1.39}$$

- The function $M\colon \mathbb{R} \to \mathbb{R}$

$$M(x) = N(x) + \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{1.40}$$

The hat function N is continuous, while M is not continuous at $x=0$. They have the same derivative[2], it is

$$N'(x) = M'(x) = \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & 1 < x \end{cases}$$

[2]Precisely, they have the same **classical derivative**. For a function $N\colon \mathbb{R} \to \mathbb{R}$, the classical derivative at a point x is computed as

$$N'(x) = \lim_{h \to 0} \frac{N(x+h) - N(x)}{h}. \tag{1.41}$$

The classical derivative is defined wherever this limit is.

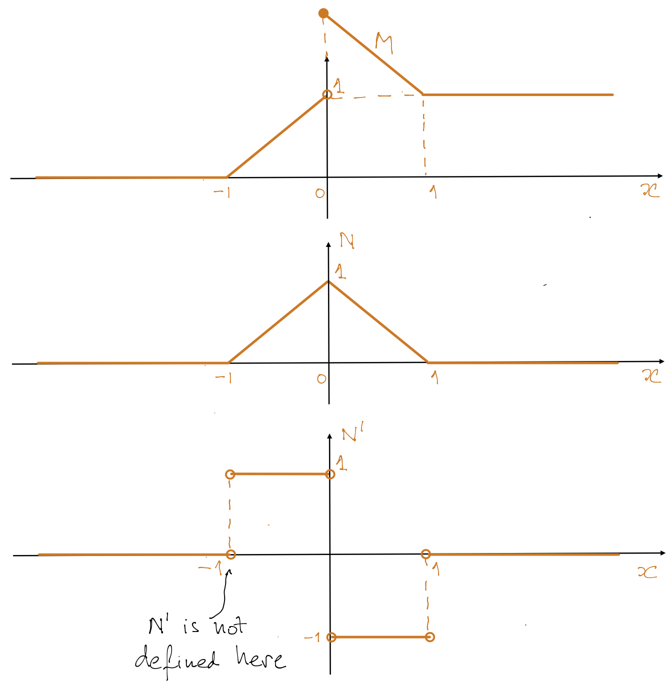

Figure 1.2 A hat function (middle), another function (top), and the common classical derivative (bottom).

and it is not defined at $x \in \{-1, 0, 1\}$. So, the domain of $N'(x)$ and $M'(x)$ is $\mathbb{R} \setminus \{-1, 0, 1\}$.

Consider then the following question. Let v be a smooth function over $\mathbb{R}$, such as $v(x) = \sin x$, how do we apply the integration-by-parts formula to the following integrals?

$$\int_{-1}^{1} N(x)\, v'(x)\, dx, \qquad \int_{-1}^{1} M(x)\, v'(x)\, dx. \tag{1.42}$$

Neither N nor M are smooth over the interval $(-1, 1)$, so we need to proceed with caution. Notice, however, that both M and N are smooth over the intervals $(-1, 0)$ and $(0, 1)$, so we can proceed as follows, using u for either N or M,

$$\begin{aligned}
\int_{-1}^{1} u(x)\, v'(x)\, dx &= \int_{-1}^{0} u(x)\, v'(x)\, dx + \int_{0}^{1} u(x)\, v'(x)\, dx \\
&= \lim_{x \to 0^-} u(x)\, v(x) - u(-1)\, v(-1) - \int_{-1}^{0} u'(x)\, v(x)\, dx \\
&\qquad + u(1)\, v(1) - \lim_{x \to 0^+} u(x)\, v(x) - \int_{0}^{1} u'(x)\, v(x)\, dx \\
&= u(1)\, v(1) - u(-1)\, v(-1) - \int_{-1}^{1} u'(x)\, v(x)\, dx \\
&\qquad + v(0) \left(\lim_{x \to 0^-} u(x) - \lim_{x \to 0^+} u(x) \right)
\end{aligned} \tag{1.43}$$

The value

$$[\![u]\!]_{x=c} = \lim_{x\to c^-} u(x) - \lim_{x\to c^+} u(x) \tag{1.44}$$

is called the **jump** of u at $x = c \in \mathbb{R}$. Its value is equal to the jump discontinuity of u at $x = c$, so it is zero when u is continuous at $x = c$, and different than zero otherwise. For example, $[\![M]\!]_{x=0} = -1$. Hence, (1.43) for N and M is

$$\begin{aligned}
\int_{-1}^{1} N(x)v'(x)\,dx &= -\int_{-1}^{1} N'(x)v(x)\,dx \\
&= -\int_{-1}^{0} v(x)\,dx + \int_{0}^{1} v(x)\,dx \\
\int_{-1}^{1} M(x)v'(x)\,dx &= v(1)M(1) + v(0)[\![M]\!]_{x=0} - \int_{-1}^{1} M'(x)v(x)\,dx \\
&= v(1) - v(0) - \int_{-1}^{0} v(x)\,dx + \int_{0}^{1} v(x)\,dx.
\end{aligned}$$

So, the integration-by-parts formula applies as we know it for N over the interval $(-1,1)$, but not to M because of the discontinuity it has at $x = 0$. We generalize this observation next.

Functions N and M are called piecewise smooth, since each one of them has derivatives of any order in the open intervals $(-1,0)$ and $(0,1)$, but not on the entire real line. More generally, in the context of these notes, we say that a function $f\colon (a,b)\subset\mathbb{R}\to\mathbb{R}$ is **piecewise smooth** over (a,b) if there are $k\in\mathbb{N}$ points $a = c_0 \le \ldots \le c_k = b$ such that f is smooth in each interval (c_i, c_{i+1}) for $i = 0, k-1$.

☞ If both functions u and v are continuous and piecewise smooth, then the integration-by-parts formula used for smooth functions holds.

Theorem 1.2 (Integration by Parts Formula for Piecewise Smooth Functions)**.** *Let $(a,b)\subset\mathbb{R}$, and u, v be piecewise smooth functions. Let $c_0 = a \le \ldots \le c_k = b$ for $k\in\mathbb{N}$ be such that both u and v are smooth in each interval (c_i, c_{i+1}) for $i = 0,\ldots,k-1$. Then,*

$$\int_a^b u'(x)v(x)\,dx = \sum_{i=0}^{k} [\![u(x)v(x)]\!]_{x=c_i} - \int_a^b u(x)v'(x)\,dx, \tag{1.45}$$

where

$$\begin{aligned}
[\![u(x)v(x)]\!]_{x=c_0} &= -\lim_{x\to a^+} u(x)v(x) \\
[\![u(x)v(x)]\!]_{x=c_k} &= \lim_{x\to b^-} u(x)v(x).
\end{aligned}$$

The proof of this theorem is simple, and it follows the ideas we used in (1.43). It consists of decomposing the integral over (a,b) into a sum of integrals over (c_i, c_{i+1}) for $i = 0,\ldots,k-1$, and then integrating by parts in each one of these intervals, in which the two functions are smooth.

It follows from Thm. 1.2 that *if both u and v are continuous in (a,b) and piecewise smooth, then the same integration-by-parts used for smooth functions holds.*