

HW 4

...

Due Wednesday, May 10, 2023

Constructing Some FE Spaces (35)

Consider a mesh of Lagrange P_k -elements (see Example 1.65 in the notes) with $n_{\text{el}} = 3$ elements of equal length in the interval $[0, 3]$. Elements are numbered consecutively from 1 to n_{el} from left to right (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

(a) (5)

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(b) (5)

$$LG = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

1.

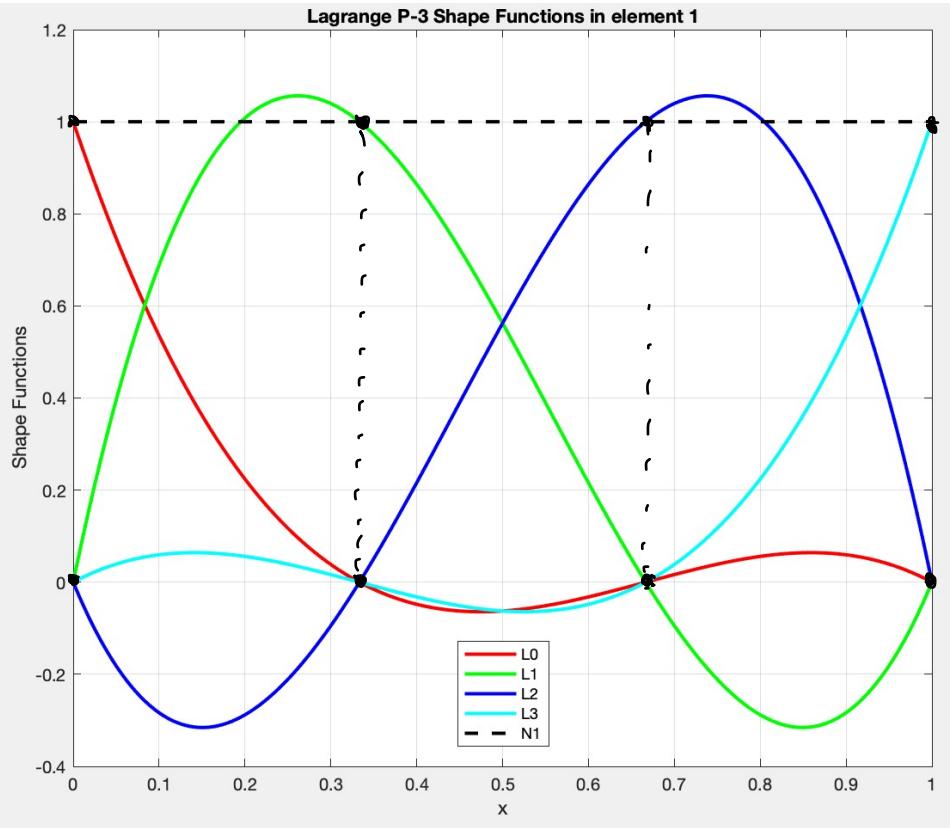
(a) Observing LG Map, we know there are 3 global basis functions N_1, N_2, N_3 . The dimension of the finite element space is 3.

For each element, there are 4 local degree of freedom. We can use Lagrange P_3 elements defined as:

$$N_a^e(x) = \frac{\prod_{b=1, b \neq a}^4 (x - x_b^e)}{\prod_{b=1, b \neq a}^4 (x_a^e - x_b^e)}$$

Consider element 1 for example. All N_a^e only contribute to N^1 , we have

$$N(x) = N_1^1(x) + N_2^1(x) + N_3^1(x) + N_4^1(x) = 1$$

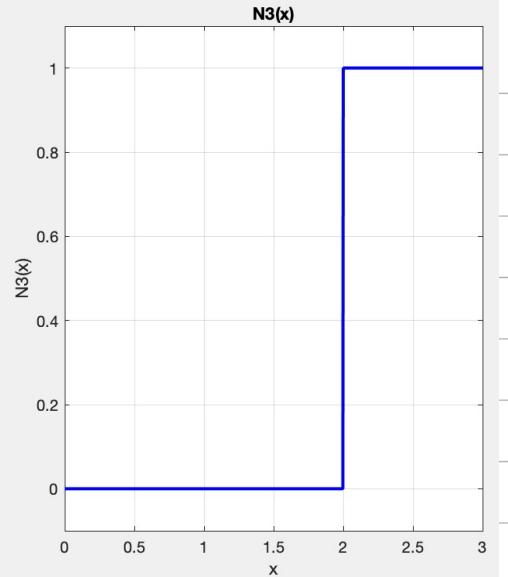
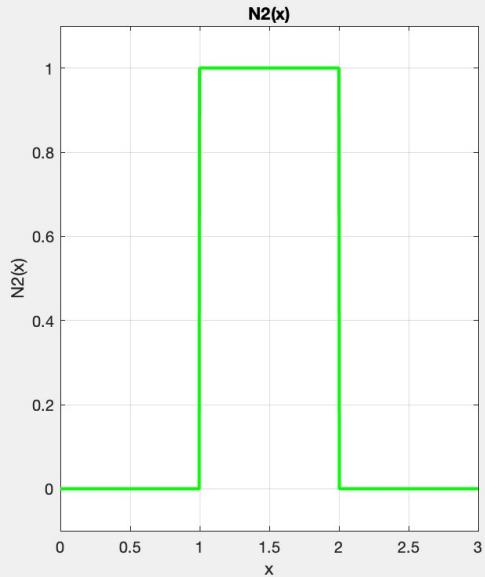
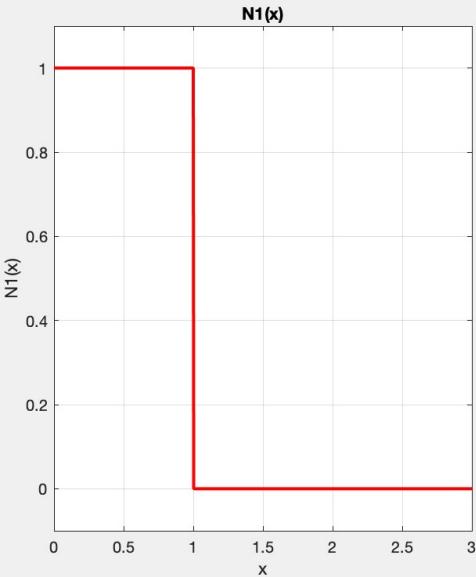


Therefore, $N^1(x) = 1$ (dash line) in element 1.

Similarly, we have $N^2(x) = N_1^2(x) + N_2^2(x) + N_3^2(x)$
 $+ N_4^2(x) = 1$.

$$N^3(x) = N_1^3(x) + N_2^3(x) + N_3^3(x) + N_4^3(x) = 1$$

So, the global basis functions N_1, N_2, N_3 are



(b). We have $\mathcal{L}G = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 5 & 6 \end{bmatrix}$. We know there are 6 global basis functions.

The dimension of FE space is 6.

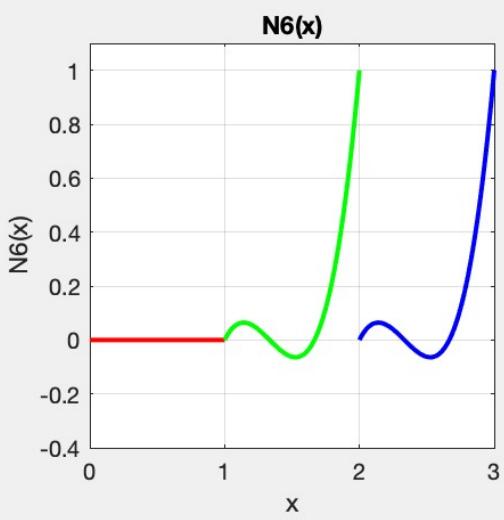
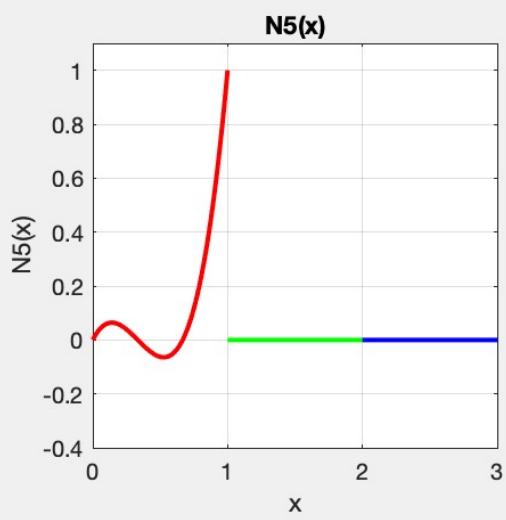
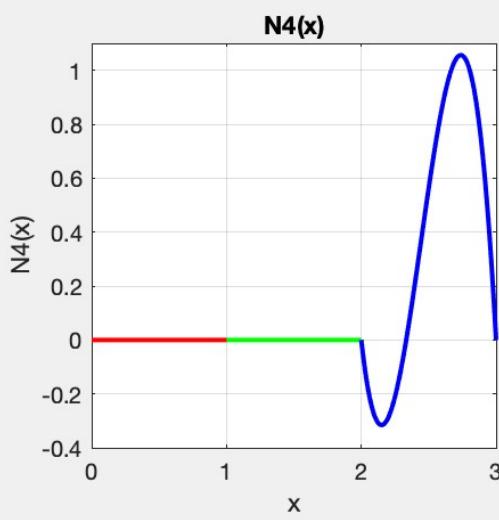
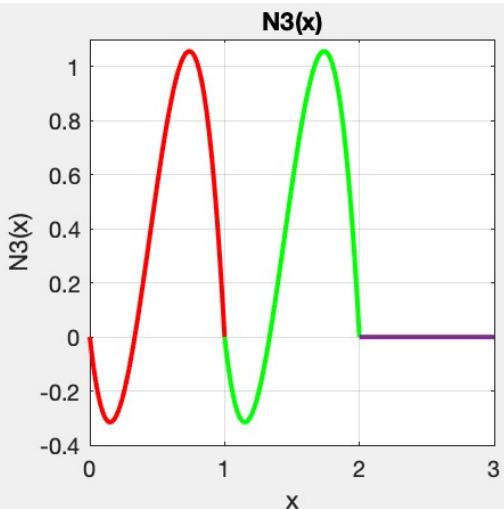
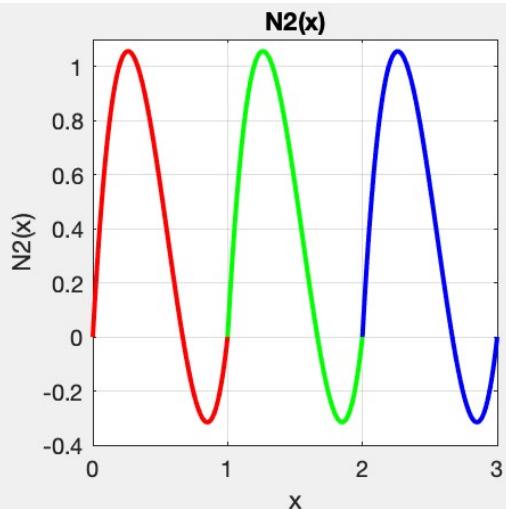
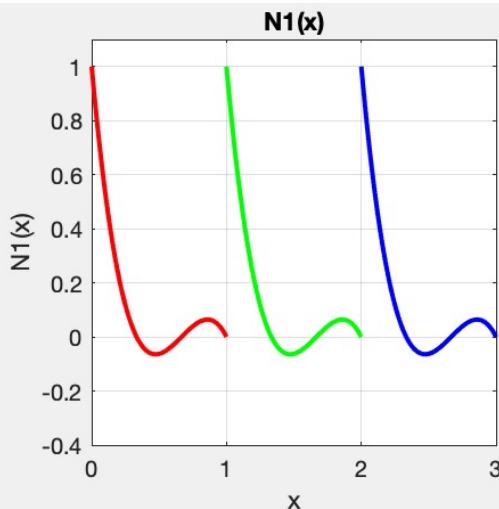
Observing first row, we know N_1^1, N_1^2, N_1^3 contribute to N_1 . Here, shape functions $N_\alpha^e(x)$ are defined as P_3 lagrange functions.

Second row tells N_2^1, N_2^2, N_2^3 contribute to N_2

N_3^1, N_3^2 contribute to N_3 . and N_3^3 contribute to N_4

N_4^1 contribute to N_5 ; N_4^2 and N_4^3 contribute to N_6

Therefore, the plots of all basis functions are:



2. The following local-to-global map renders the basis functions to be continuous and have minimal support.

(a). For P_3 -elements.

Each element have 4 shape functions.

FG should be continuous with and across element boundaries. One possible LG map is:

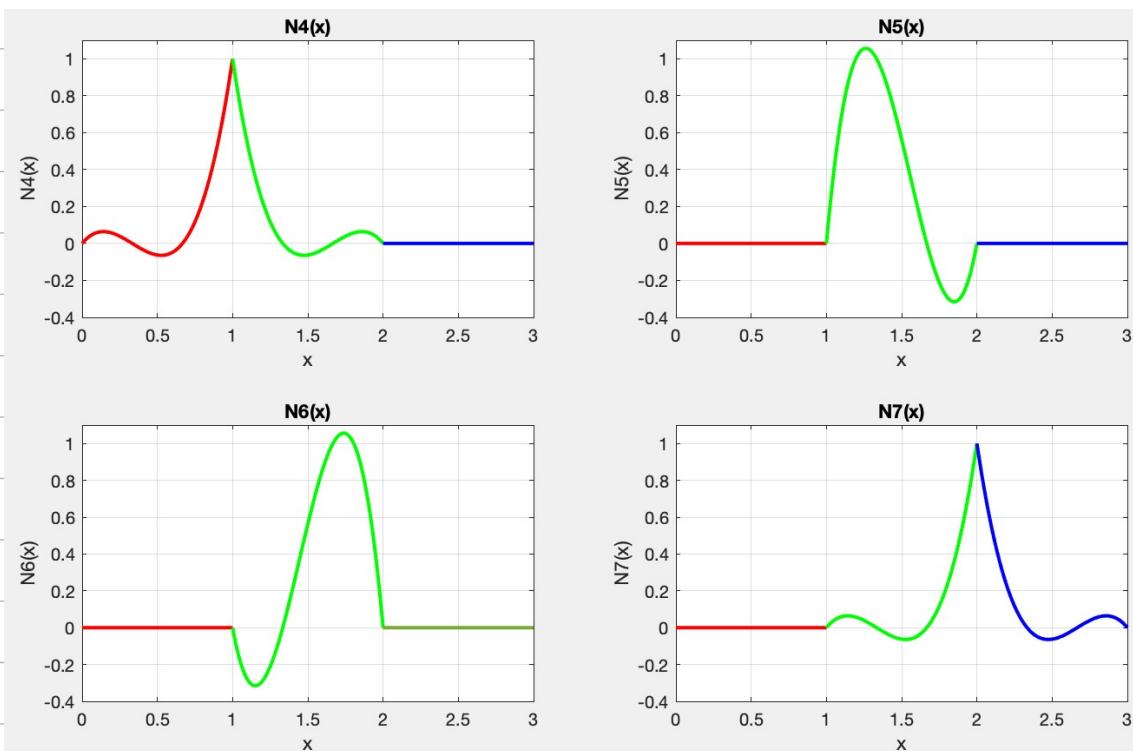
$$LG = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

Therefore, there are 10 global basis functions.

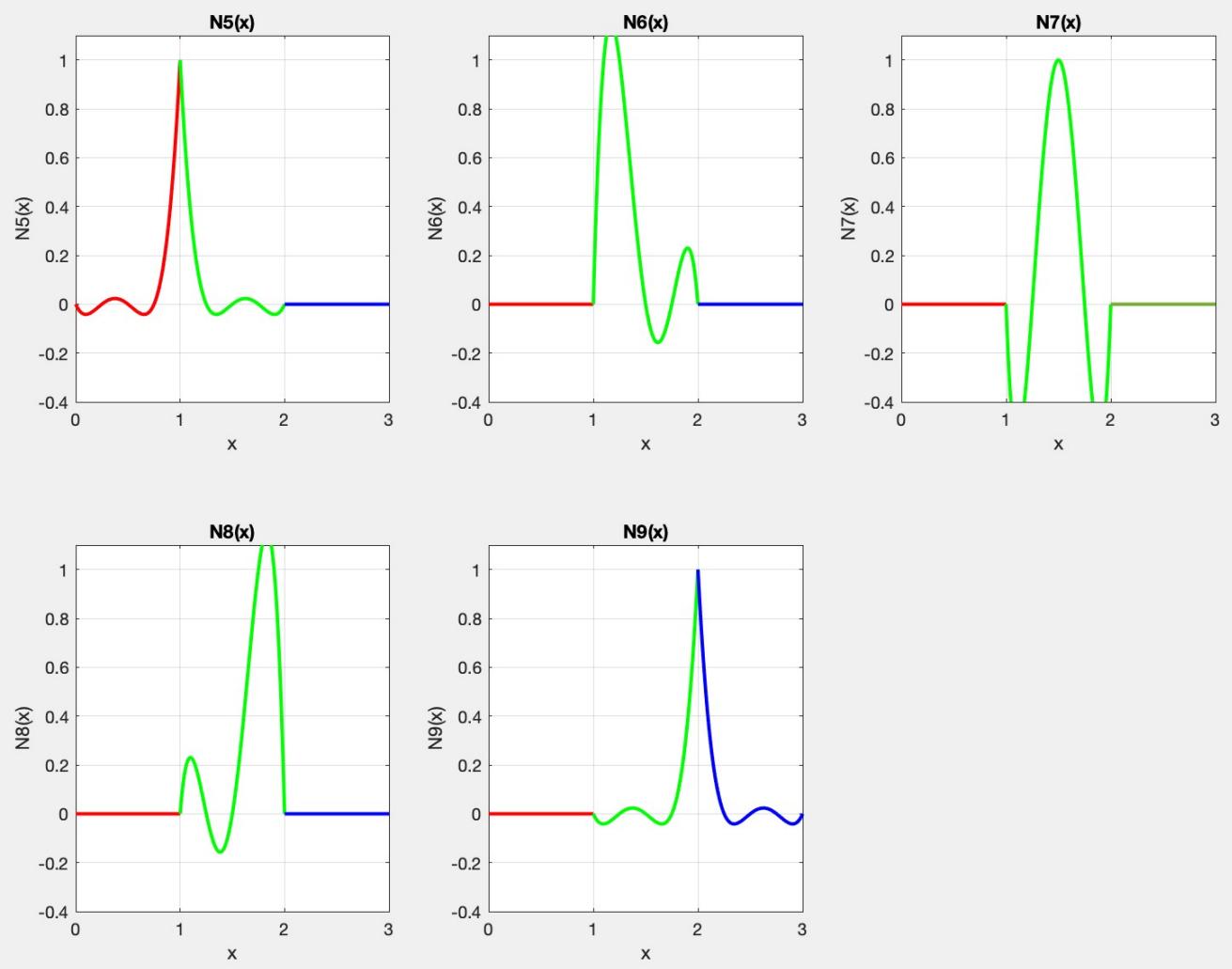
(b). Similarly, for P_4 -element, we have 5 local basis functions. The LG can be written as:

$$LG = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{bmatrix}$$

3.① For P_3 element



② For P_0 element



4. To enforce $u(0) = u(3)$, we could make

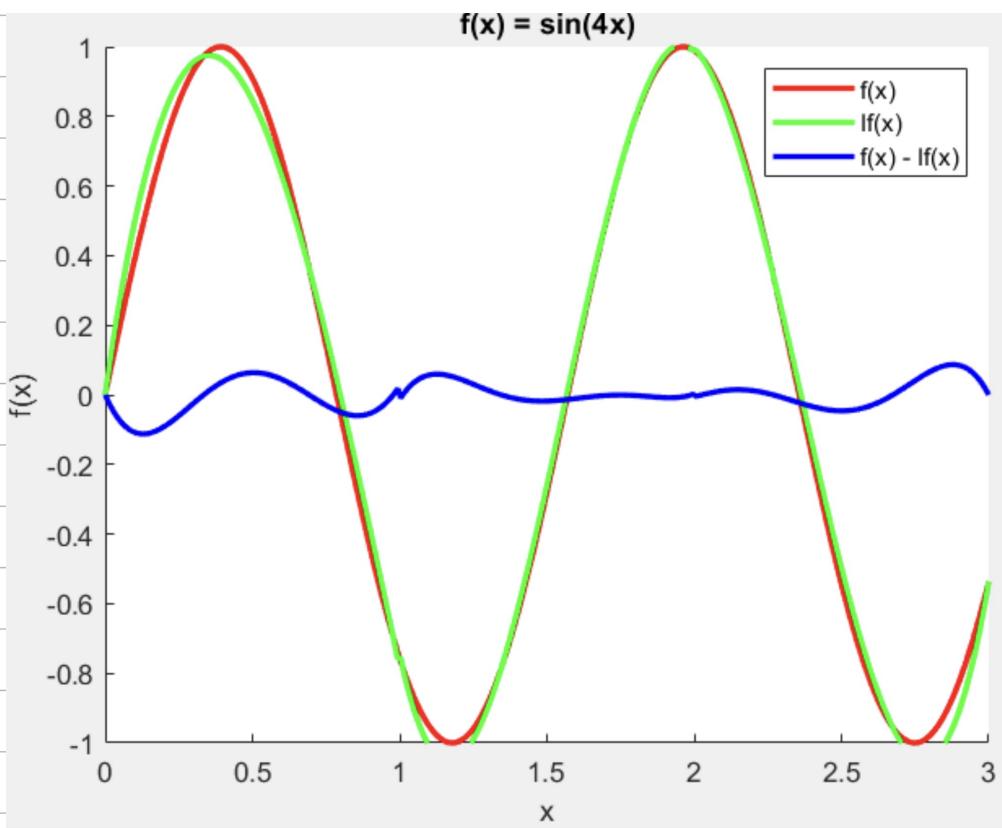
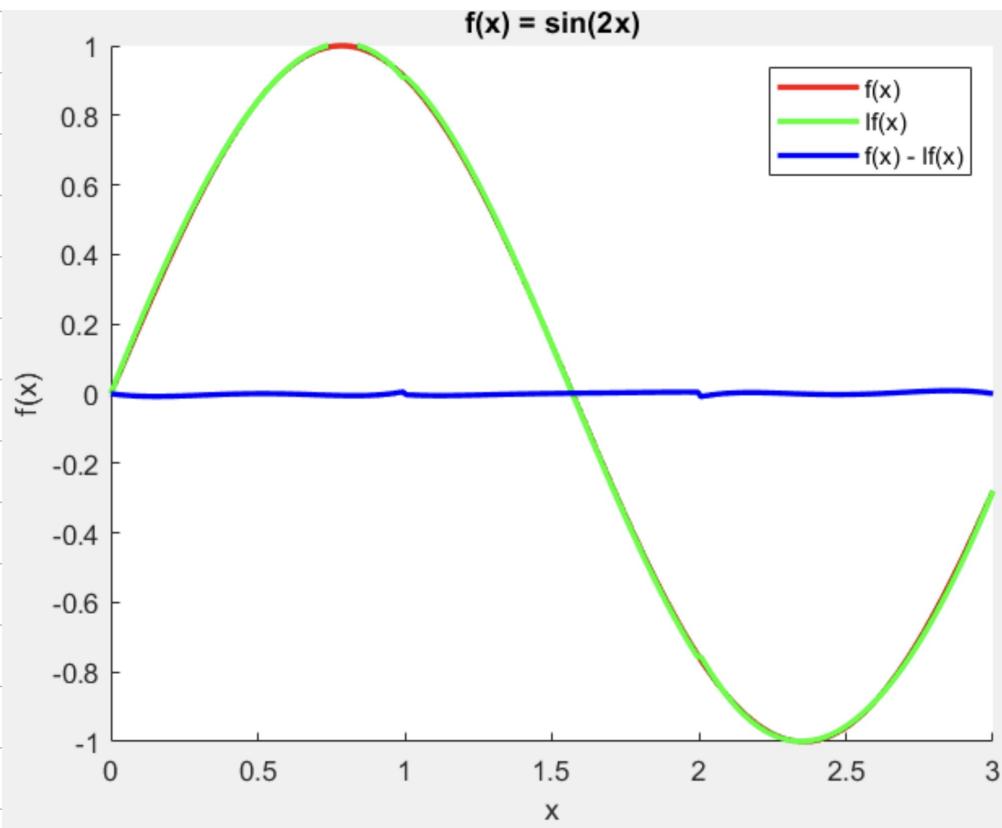
$$\sum_{i=1}^a N_i(x=0) = \sum_{i=1}^a N_i(x=3).$$

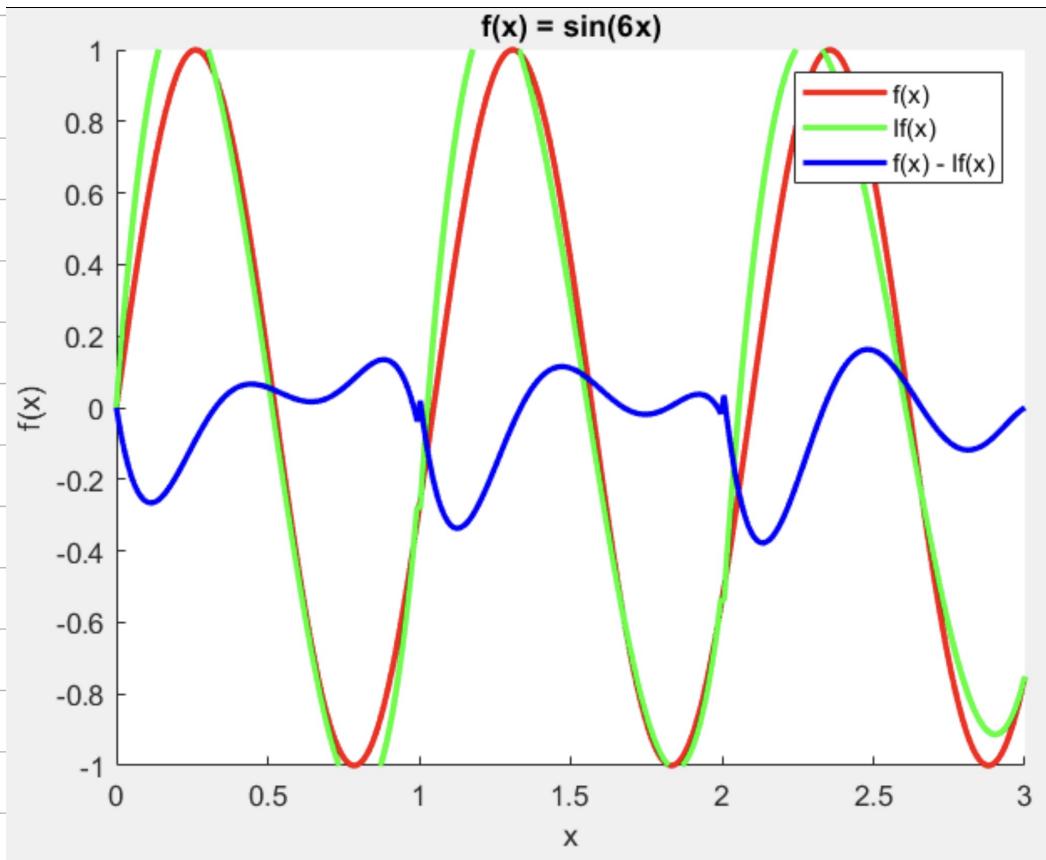
If we assume using P_1 element and continuous everywhere; we can write Lf map as:

$$Lf = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

The dimension of FE space is 3.

5.





(Manual) Assembly (80)

Consider the domain $\Omega = [1, 7]$, and the convection-diffusion problem: Find $u: \Omega \rightarrow \mathbb{R}$ such that

$$-\varepsilon u_{xx} + cu_x = f \quad x \in \overset{\circ}{\Omega}, \quad (1a)$$

$$-\varepsilon u_x(7) = h, \quad (1b)$$

$$u(1) = g, \quad (1c)$$

where $\overset{\circ}{\Omega} = (1, 7)$ indicates the *interior* of the set Ω , $c \in \mathbb{R}$ is the convection velocity, $\varepsilon > 0$ is the diffusion coefficient, $h, g \in \mathbb{R}$ are boundary conditions, and $f: \Omega \rightarrow \mathbb{R}$ is a source. We would like to construct a finite element approximation of the solution of this problem.

1. (10) Find a variational equation according to the recipe in the notes, and identify natural and essential boundary conditions.

1. Define solution space and test space as follows:

$$\begin{aligned} S &= \{u: [1, 7] \rightarrow \mathbb{R} \text{ Smooth } | u(1) = g\} \\ V &= \{v: [1, 7] \rightarrow \mathbb{R} \text{ Smooth } | v(1) = 0\} \end{aligned}$$

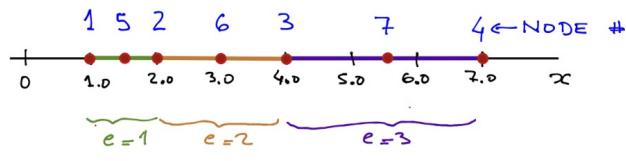
The variational form of the strong form can be derived as:

$$\begin{aligned} \int_1^7 (-\varepsilon u_{xx} + cu_x) v \, dx &= \int_1^7 f v \, dx, \\ \int_1^7 \varepsilon v_{xx} - \int_1^7 (\varepsilon u_{xx})' + \int_1^7 c u_{xx} v = \int_1^7 f v \\ -\varepsilon v(7) u_x(7) + \varepsilon v(1) u_x(1) + \int_1^7 (u_{xx} + cu_x) v \, dx &= \int_1^7 f v \\ \int_1^7 (\varepsilon u_{xx} + cu_x) v \, dx &= \int_1^7 f v \, dx - v(7) h \end{aligned}$$

Natural B.C.: $-\varepsilon u_x(7) = h$

Essential B.C.: $u(1) = g$

2. (5) Consider the nodes 1 to 7 with positions $\{1, 2, 4, 7, 1.5, 3, 5.5\}$, respectively; see figure. These nodes form P_2 elements 1, 2, and 3, whose domains are $\Omega^1 = [1, 2]$, $\Omega^2 = [2, 4]$, $\Omega^3 = [4, 7]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.



2. Given we are using P_2 Lagrange element, the shape functions are defined as:

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)}$$

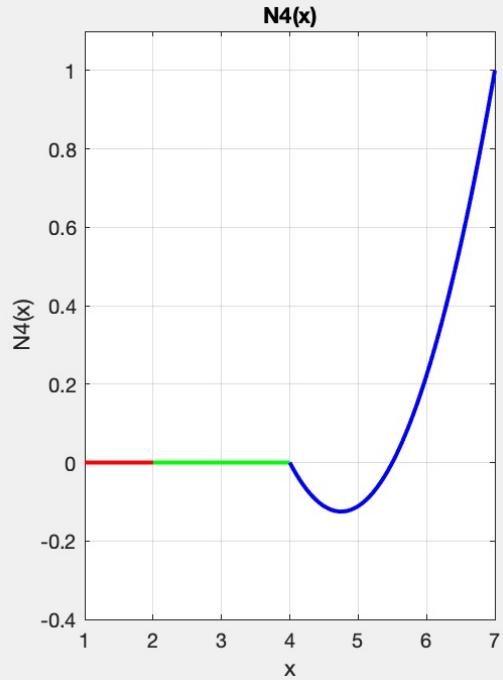
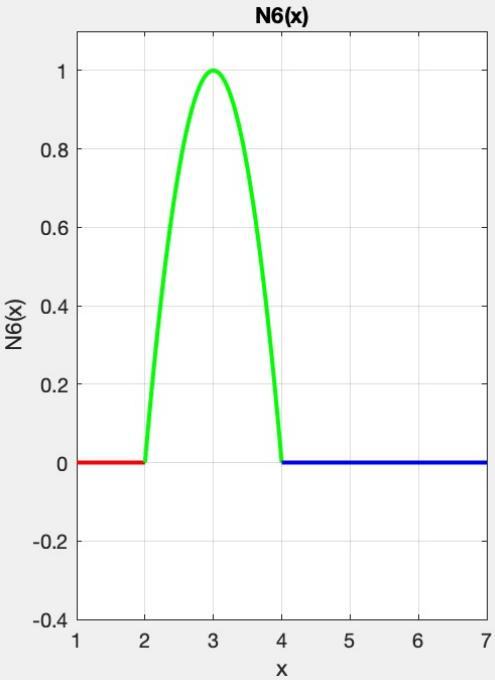
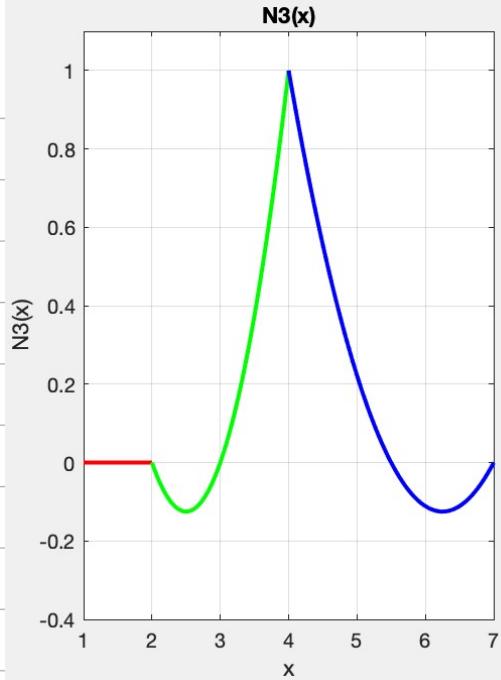
$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}$$

$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}$$

According to the global DOFs defined in the figure and continuous function requirement, we can write LG as:

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

3. Plot the global Basis functions N_3 , N_6 , and N_4 .



4.

So $V_h \in \mathbb{P}^2$ can be written as:

$$V_h = 1.5 N_1^2(x) + (-1.5) N_2^2(x) + 3 N_3^2(x)$$

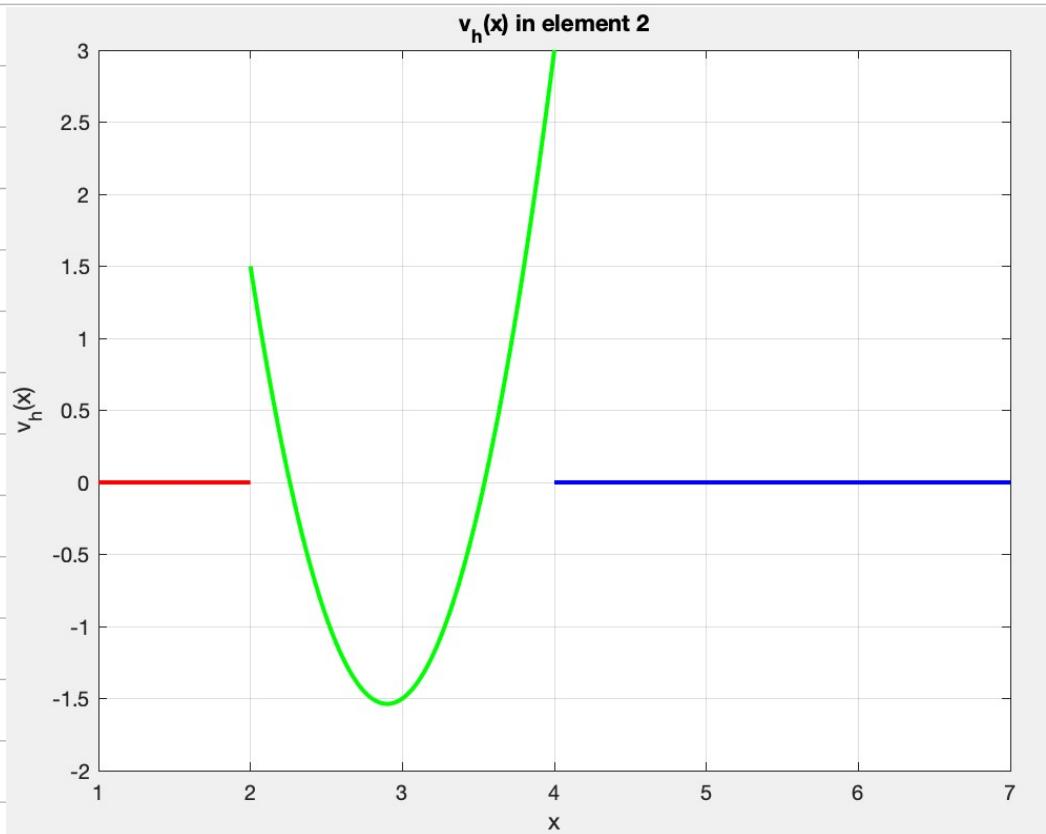
$$\text{We know } N_1^2(x) = \frac{(x-2)(x-4)}{(-1) \cdot (-2)} = \frac{(x-2)(x-4)}{2}$$

$$N_2^2(x) = \frac{(x-2)(x-4)}{(-1)(-1)} = -\frac{(x-2)(x-4)}{1}$$

$$N_3^2(x) = \frac{(x-2)(x-3)}{(2)(1)} = \frac{(x-2)(x-3)}{2}$$

$$\begin{aligned} V_h &= \frac{3}{4} (x-2)(x-4) + \frac{3}{2} (x-2)(x-4) + 3 (x-2)(x-3) \\ &= \frac{15}{4} x^2 - \frac{45}{2} x + 30 \end{aligned}$$

The plot of V_h looks like:

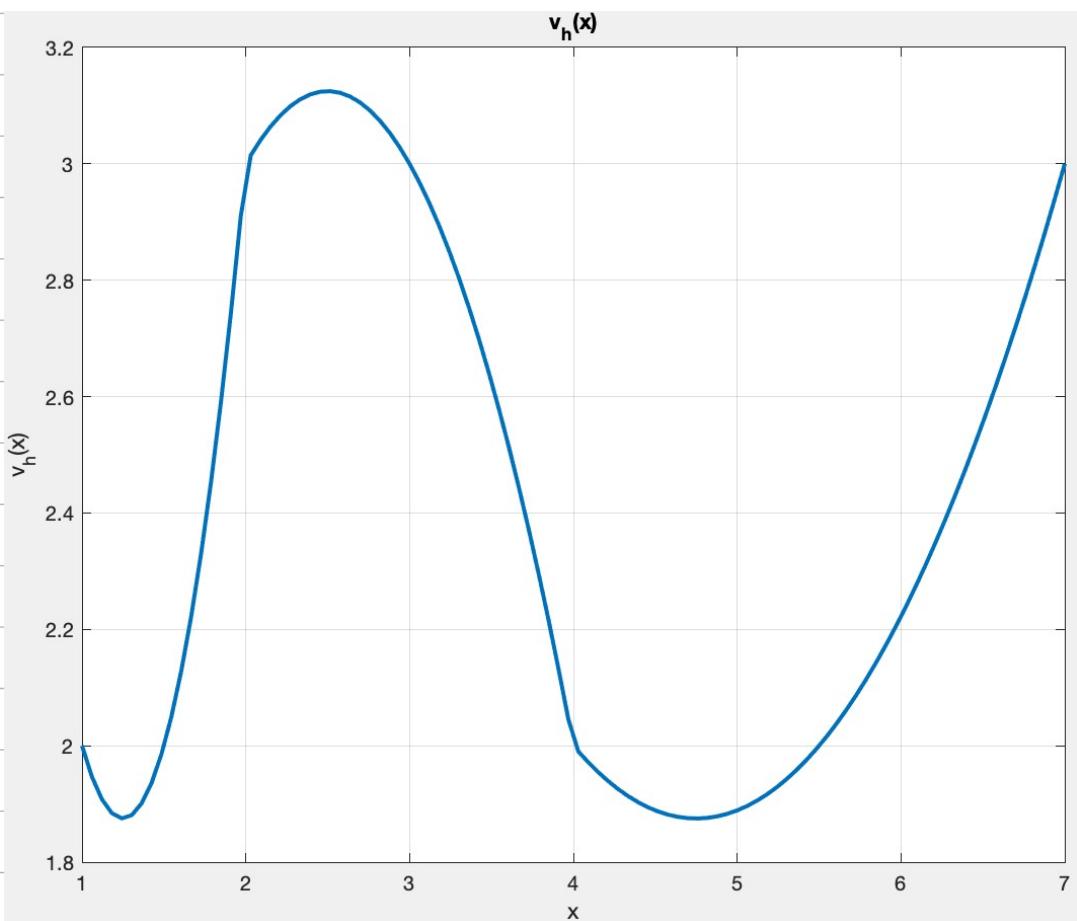


5. We know the LG map, which defines the relationship between local shape functions and global basis functions.

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

Given \mathcal{V}_h is the FE space of continuous functions, the function $v_h \in \mathcal{V}_h$ that is equal to 2 on odd nodes, to 3 on even nodes. can be written as :

$$\begin{aligned} v_h = & 2N_1 + 2N_5 + 3N_2 + 3N_6 + 2N_3 \\ & + 2N_7 + 3N_4 \end{aligned}$$



6.

$$\text{Let } \mathcal{W}_n = \text{span}\{N_1, \dots, N_7\}$$

$$\mathcal{V}_n = \{v_n \in \mathcal{W}_n \mid v_n(1) = 0\}.$$

$$S_n = \{v_n + g \mid v_n \in \mathcal{V}_n, g \in \mathcal{W}_n \mid u_n(1) = g\}.$$

A basis for \mathcal{V}_n is obtained by setting aside index
set $\mathcal{I}_x = \{2, 3, 4, 5, 6, 7\}$, and $\mathcal{I}_g = \{1\}$. So,

$$\mathcal{V}_n = \text{span}(N_2, N_3, N_4, N_5, N_6, N_7)$$

$$S_n = \mathcal{V}_n + \bar{U}_n, \text{ where } \bar{U}_n = g.$$

Find $u_n \in S_n$ such that

$$a(u_n, v_n) = f(v_n) \text{ for all } v_n \in \mathcal{V}_n$$

where

$$a(u, v) = \int_1^7 v_{,x} u_{,x} + c u_{,x} v \, dx$$

$$f(v) = \int_1^7 f v \, dx - v(T) h$$

T. Assume that $f(x) = x$, $h = -20$, $\epsilon = 1$, $c = 1$, $g = 2$

Plugging these parameters into the weak form above, we have:

Find $u_n \in S_h$ such that

$$a(u_n, v_n) = f(v_n) \text{ for all } v_n \in V_h$$

where

$$a(u, v) = \int_1^T (v_{,x} u_{,x} + u_{,x} v) dx$$

$$f(v) = \int_1^T x v dx + 20v(T)$$

If we assume a natural ordering (namely from 1 to 7) for dofs, the local stiffness matrix follows as:

For element 1, we have: K_{ab}^e where e is element #
 a is the index of basis functions \uparrow test space,
 b is the index of local basis functions in solution
 space.

$$K_{11} = 1, \quad K_{12} = K_{15} = 0.$$

$$K_{21} = \int_1^2 N_{3,x}^1 N_{1,x}^1 + N_{1,x}^1 N_3^1 dx$$

$$K_{22} = \int_1^2 N_{3,x}^1 N_{3,x}^1 + N_{3,x}^1 N_3^1 dx.$$

$$K_{25} = \int_1^2 N_{3,x}^1 N_{2,x}^1 + N_{2,x}^1 N_3^1 dx$$

$$K_{51} = \int_1^2 N'_{2,x} N'_{1,x} + N'_{1,x} \underline{N'_{2,x}} dx$$

$$K_{52} = \int_1^2 N'_{2,x} N'_{3,x} + N'_{3,x} N'_{2,x} dx$$

$$K_{53} = \int_1^2 N'_{2,x} N'_{2,x} + N'_{2,x} N'_{3,x} dx$$

Ke =

1.0000	0	0
-3.3333	5.3333	-2.0000
0.5000	-3.3333	2.8333

$$f_1 = \int_1^2 f N'_1 dx = \int_1^2 x N'_1 dx$$

$$f_2 = \int_1^2 x N'_3 dx$$

$$f_3 = \int_1^2 x N'_2 dx$$

Fe =

2.0000
1.0000
0.3333

For element 2.

$$K_{22} = \int_2^4 N_{1,x}^2 N_{1,x}^2 + N_{1,x}^2 N_1^2 dx$$

$$K_{26} = \int_2^4 N_{1,x}^2 N_{2,x}^2 + N_{2,x}^2 N_1^2 dx$$

$$K_{23} = \int_2^4 N_{1,x}^2 N_{3,x}^2 + N_{3,x}^2 N_1^2 dx$$

$$K_{62} = \int_2^4 N_{2,x}^2 N_{1,x}^2 + N_{1,x}^2 N_2^2 dx$$

$$K_{66} = \int_2^4 N_{2,x}^2 N_{2,x}^2 + N_{2,x}^2 N_2^2 dx$$

$$K_{63} = \int_2^4 N_{2,x}^2 N_{3,x}^2 + N_{3,x}^2 N_2^2 dx$$

$$K_{32} = \int_2^4 N_{3,x}^2 N_{1,x}^2 + N_{1,x}^2 N_3^2 dx$$

$$K_{36} = \int_2^4 N_{3,x}^2 N_{2,x}^2 + N_{2,x}^2 N_3^2 dx$$

$$K_{33} = \int_2^4 N_{3,x}^2 N_{3,x}^2 + N_{3,x}^2 N_3^2 dx$$

$$f_2 = \int_2^4 x N_1^2 dx$$

$$f_6 = \int_2^4 x N_2^2 dx$$

$$f_3 = \int_2^4 x N_3^2 dx$$

Ke =

0.6667	-0.6667	0
-2.0000	2.6667	-0.6667
0.3333	-2.0000	1.6667

Fe =

0.6667
4.0000
1.3333

For element 3.

$$K_{33} = \int_0^T N_{1,x}^3 N_{1,x}^3 + N_{1,x}^3 N_1^3 dx$$

$$K_{37} = \int_0^T N_{1,x}^3 N_{2,x}^3 + N_{2,x}^3 N_1^3 dx$$

$$K_{34} = \int_0^T N_{1,x}^3 N_{3,x}^3 + N_{3,x}^3 N_1^3 dx$$

$$K_{73} = \int_0^T N_{2,x}^3 N_{1,x}^3 + N_{1,x}^3 N_2^3 dx$$

$$K_{77} = \int_0^T N_{2,x}^3 N_{2,x}^3 + N_{2,x}^3 N_2^3 dx$$

$$K_{74} = \int_0^T N_{2,x}^3 N_{3,x}^3 + N_{3,x}^3 N_2^3 dx$$

$$K_{43} = \int_0^T N_{3,x}^3 N_{1,x}^3 + N_{1,x}^3 N_3^3 dx$$

$$K_{47} = \int_0^T N_{3,x}^3 N_{2,x}^3 + N_{2,x}^3 N_3^3 dx$$

$$K_{44} = \int_0^T N_{3,x}^3 N_{3,x}^3 + N_{3,x}^3 N_3^3 dx$$

$$f_3 = \int_0^T x N_1^3 dx$$

$$f_7 = \int_0^T x N_2^3 dx$$

$$f_4 = \int_0^T x N_3^3 dx + 20 N_3^3 (CT)$$

Ke =

0.2778	-0.2222	-0.0556
-1.5556	1.7778	-0.2222
0.2778	-1.5556	1.2778

Fe =

2.0000
11.0000
23.5000

(c) Assembled matrix:

1	2	3	4	5	6	7
1	1	0	0	0	0	0
2	0.5000	3.5000	0	0	-3.3333	-0.6667
3	0	0.3333	1.9444	-0.0556	0	-2
4	0	0	0.2778	1.2778	0	0
5	-3.3333	-2	0	0	5.3333	0
6	0	-2	-0.6667	0	0	2.6667
7	0	0	-1.5556	-0.2222	0	0

Assembled Local Vector

1	2
1	2
2	1
3	3.3333
4	23.5000
5	1
6	4
7	11

$$(d). \quad U = K^T F$$

U can be calculated using direct solver. The solution follows as:

	1
1	2.0000
2	4.5833
3	13.3745
4	43.9514
5	3.1562
6	8.2811
7	23.3841

$$U(x) = 2N_1(x) + 4.5833N_2(x) + 13.3745N_3(x) \\ + 43.9514N_4(x) + 3.1562N_5(x) + 8.2811N_6(x) \\ + 23.3841N_7(x).$$

