HW-2

On Vector Spaces of Functions (40)

For this problem section titled "1.1.3 Sets of Functions" in the notes has a discussion about the the notation used in this part.

Let

$$W = \{u\colon [-1,1] \to \mathbb{R} \text{ smooth}\}.$$

1. Are the following sets vector spaces of functions? Explain.
   (a) (5) $V_1 = \{u \in W \mid u(0) = 0\}$.
   (b) (5) $V_2 = \{u \in W \mid u''(0) = 0\}$.
   (c) (5) $V_3 = \{u \in W \mid u(x) \neq 0 \quad \forall x \in [-1,1]\}$.
   (d) (5) $V_4 = \{u \in W \mid \int_{-1}^{1} u''(x)\, dx = 0\}$.
   (e) (5) $V_5 = \{u \in W \mid \int_{-1}^{1} x^2 u(x)\, dx = 1\}$.
   (f) (5) $V_6 = \{u \in W \mid u(0) = -5\}$.
2. (5) The set V_6 is an affine subspace of W. What is its direction? You do not need to prove it, just state it.
3. (5) Is $\ell\colon V_1 \to \mathbb{R}$ a linear functional, where

$$\ell(u) = \int_{-1}^{1} u''(x)\, dx? \tag{1}$$

1. (a). V_1 is a vector space of function.
Since $V_1 \subset W$, V_1 inherites commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

i Closure: $u, v \in V_1$ and $\alpha \in \mathbb{R}$, then we know
$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$
This is because u and v is smooth function. with $u(0)=0$ $v(0)=0$. and $u+v$ is also smooth and $(u+v)(0) = 0$.

Similarly, αu is smooth function and $\alpha u(0) = 0$.

ii. Identity: $\exists\, z(x) = 0 \ \forall x \in [-1,1]$.
$$u(x) \in V_1 \quad \forall x \in [-1,1]$$
We know:
$$u(x) + z(x) = u(x)$$

iii. additive inverse:

if $u \in V_1$, then $-u \in V_2$ as $u(0) = -u(0) = 0$.
and $-u(0) \in W$

1.(b) V_2 is a vector space of function.
Since $V_2 \subset W$, V_2 inherites commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

i Closure: $u, v \in V_1$ and $\alpha \in \mathbb{R}$, then we know

$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$

This is because u and v is smooth function. with $u''(0)=0$ $v''(0)=0$. and $u+v$ is also smooth and $(u+v)''(0) = (u''(0)+v''(0))$
$= 0$.

Similarly, αu is smooth function and $\alpha u''(0)=0$.

ii. Identity: $\exists z(x) \mid z(x)=0$ and $z''(x)=0 \quad \forall x \in [-1,1]$

$u(x) \in V_1 \quad \forall x \in [-1,1]$

We know:

$$u(x) + z(x) = u(x)$$

$$u''(0) + z''(0) = 0$$

iii. additive inverse:

if $u \in V_1$, then $-u \in V_2$ since $u''(0) = -u''(0) = 0$.

(c). V_3 is not a vector space of function, since closure does not hold.

Let $u, v \in V_3 \mid u(x) = a$ and $v(x) = -a$

$(u + v)(0) = 0$ which violates the constrain of V_3.

(d). V_4 is a vector space of function.

Since $V_4 \subset W$, V_4 inherites commutativity, associativity, and distributivity. We check closure, identity and additive inverse.

i Closure: $u, v \in V_4$ and $\alpha \in \mathbb{R}$, then we know

$$u + v \in V_1 \quad \text{and} \quad \alpha u \in V_1$$

This is because u and v is smooth function. with

① $\int_{-1}^{1} u''(x)dx = 0$ and $\int_{-1}^{1} v''(x)dx = 0$

$\Rightarrow$

$$\int_{-1}^{1} (u(x) + v(x))'' dx = \int_{-1}^{1} (u''(x) + v''(x)) dx = 0.$$

⑤ $\int_{-1}^{1} \alpha u''(x) dx = \alpha \int_{-1}^{1} u''(x) dx = 0$

(e) V_5 is not a vector space of function.

Let $u, v \in V_5$ such that $\int_{-1}^{1} x^2 u\, dx = 1$ and $\int_{-1}^{1} x^2 v\, dx = 1$

$$u + v = \int_{-1}^{1} x^2 (u + v) dx = 2 \notin V_5$$

Therefore, closure doesn't hold.

(f). V_6 is not a vector space of functions.

Similarly, Let $u, v \in V_6$ such that $u(0) = -5$
$v(0) = -5$

$$(u+v)(0) = u(0) + v(0) = -10 \notin V_6$$

Therefore, closure doesn't hold.

2. The direction of V_δ is:

$$\mathcal{V}_T = \{ u \in \mathcal{V} \mid u(0) = 0 \}$$

3.

Using definition, for any $u, v \in V$ and $\alpha \in \mathbb{R}$

$$\ell(u + \alpha v) = \int_{-1}^{1} (u + \alpha v)''(x)\, dx$$

$$= \int_{-1}^{1} (u''(x) + \alpha v''(x))\, dx$$

$$= \ell(u) + \alpha \ell(v)$$

Which proves ℓ is a linear functional.

On Bases for Vector Spaces of Functions (20)

For $x \in \mathbb{R}$, define $g(x) = 1$ and

$$N_{x_0}(x) = \max(1 - |x - x_0|, 0).$$

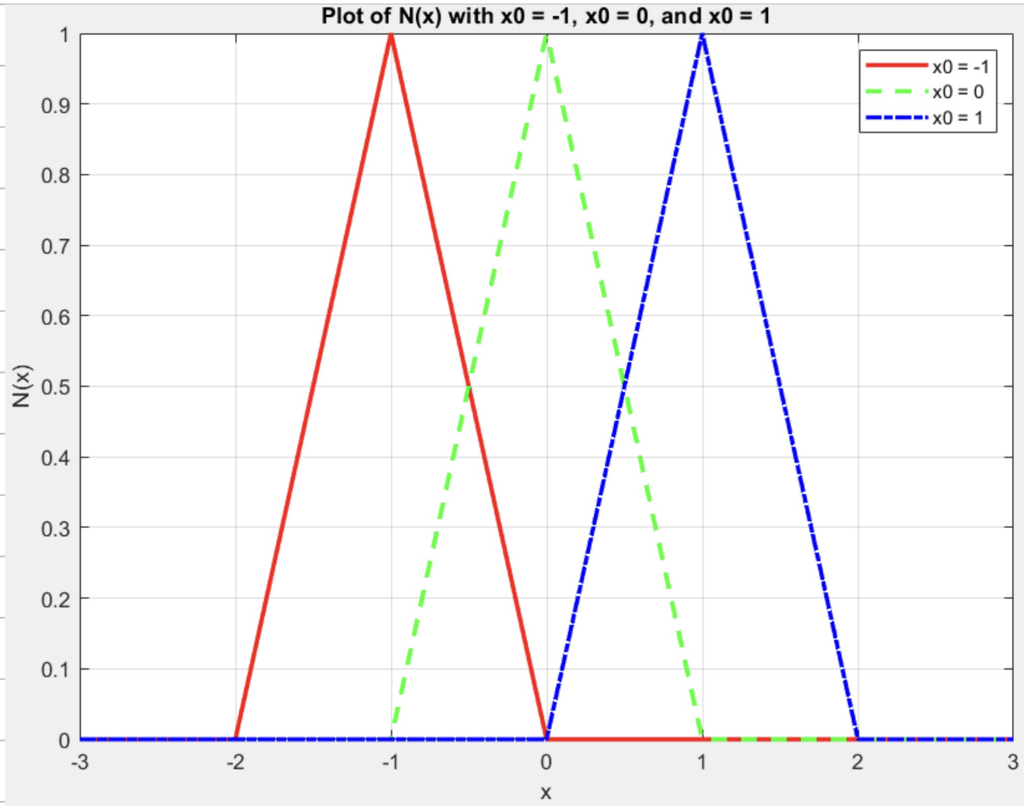
1. (5) Plot the functions N_{-1}, N_0, and N_1 over the interval $(-3, 3)$.

2. (5) For functions whose domain is $\mathbb{R}$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain. Hint: Find inspiration in Example 1.32 in the notes.

3. (5) For functions whose domain is $(-1, 1)$, is the set $\{N_{-1}, N_0, N_1, g\}$ linearly independent? Explain.

4. (5) Consider functions whose domain is $(-1, 1)$, and let $f(x) = 2x+1$. Does $f \in \text{span}(N_{-1}, N_0, N_1)$? If so, what are its components?

1.



2. It's easy to see the $\{N_{-1}, N_0, N_1, g\}$ is linearly dependent over $\mathbb{R}$.

For example,

$$c_1 N_{-1}(10) + c_2 N_0(10) + c_3 N_1(10) + c_4 g = 0$$

where c_1, c_2, c_3 can be any real number as long as c_4 is zero.

3. Over $(-1, 1)$

$$N_{-1} = \begin{cases} -x, & (-1, 0] \\ 0, & (0, 1) \end{cases}$$

$$N_0 = \begin{cases} 1+x, & (-1, 0] \\ 1-x & (0, 1) \end{cases}$$

$$N_1 = \begin{cases} 0, & (-1, 0] \\ x, & (0, 1) \end{cases}$$

$$g(x) = 1$$

Define the span as :

$$P(x) = C_1 N_{-1} + C_2 N_0 + C_3 N_1 + C_4 g$$

Evalute $P(x)$ at $x=0$, $x=-\frac{1}{2}$, $x=\frac{1}{2}$, $x=\frac{3}{4}$

$$C_1 \cdot 0 + C_2 \cdot (0) + C_3 \cdot (0) + C_4 \cdot (1) = 0.$$

$$C_1 \cdot \frac{1}{2} + C_2 \cdot (\frac{1}{2}) + C_3 (0) + C_4 (1) = 0.$$

$$C_1 (0) + C_2 (\frac{1}{2}) + C_3 (\frac{1}{2}) + C_4 (1) = 0.$$

$$C_1 (0) + C_2 \cdot (\frac{1}{4}) + C_3 (\frac{3}{4}) + C_4 (1) = 0.$$

$$\Rightarrow \begin{cases} C_4 = 0. \\ \frac{1}{2}(C_1 + C_2) + C_4 = 0. \\ \frac{1}{2}(C_2 + C_3) + C_4 = 0 \\ \frac{1}{4} C_2 + \frac{3}{4} C_3 + C_4 = 0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

Such a linear system has a determiant of zero. which means there are many solutions satisfying $AC = 0$.
Therefore, $\{N_{-1}, N_0, N_1, g\}$ is linearly dependent.

4. The span $\{N_{-1}, N_0, N_1\}$ is

$$P(x) = C_1 N_{-1} + C_2 N_0 + C_3 N_1$$

$$= \begin{cases} -C_1 x + (1+x) \cdot C_2, & (-1, 0] \\ C_2 (1-x) + C_3 x & (0, 1) \end{cases}$$

$f(x) = 2x+1$ can be expressed by $\{N_{-1}, N_0, N_1\}$

The components are:

$$C_1 = -1, \quad C_2 = 1, \quad C_3 = 3.$$

A Simple Variational Method Example (35)

1. (15) Consider the problem: Find $u\colon [0,1] \to \mathbb{R}$ continuous such that

$$\begin{aligned} (1+x^2)u_{,xx} + xu_{,x} + x^2 u &= 0 \\ u_{,x}(1) - 3u(1) &= 0 \\ u(0) &= 1 \end{aligned}$$

   Find the variational equation of the problem using the recipe from the notes, with

$$\mathcal{V} = \{w\colon [0,1] \to \mathbb{R} \quad \text{smooth} \ \mid w(0) = 0\}.$$

   Identify essential and natural boundary conditions.

2. (2) Identify the bilinear form and the linear functional of the problem so that the variational equation can be written as $a(u,v) = \ell(v)$. Is a symmetric?

3. Consider a subspace of functions $\mathcal{W}_h = \text{span}\{1, x, x^2, x^3\}$. We want to formulate a variational method with the variational equation in 1 and find its solution.

   (a) (5) What are the spaces trial and test spaces $\mathcal{S}_h$ and $\mathcal{V}_h$? What are the sets of active and constrained indices?

   (b) (2) Is the method consistent?

   (c) (7) Find the stiffness matrix and load vector.

   (d) (3) Find the solution to the variational method, and plot it.

   Hint: Recall that the stiffness matrix $K_{ab} = a(N_b, N_a)$; the order is important for non-symmetric bilinear forms.

4. (1) Is the natural boundary condition satisfied exactly by the solution of the variational method?

1. (a). Form residual function:

$$r(x) = (1+x^2)u_{,xx} + x\,u_{,x} + x^2 u$$

(b) Multiply by a test function and integrate:

Given : $V = \{ w : [0,1] \to \mathbb{R} \text{ smooth} \mid w(0)=0 \}$.

$S = \{ u : [0,1] \to \mathbb{R} \text{ smooth} \mid u(0)=1 \}$.

The weak form is given as:

$$\int_0^1 r(x)\,w(x)\,dx = 0 \qquad \forall w \in V$$

$\forall w \in V$

$$\int_0^1 ((1+x^2)u_{,xx} + x u_{,x} + x^2 u)\,w(x)\,dx = 0.$$

$$\Rightarrow \int_0^1 (((1+x^2)u_{,x})' - x u_{,x} + x^2 u)\,w(x)\,dx = 0$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{,x} w_{,x} + [(1+x^2)u_{,x} w]_0^1 - \int_0^1 x u_{,x} w + \int_0^1 x^2 u w = 0.$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{,x} w_{,x} + (1+1)u_{,x}(1)w(1) - 0 - \int_0^1 x u_{,x} w + \int_0^1 x^2 u w = 0.$$

$$\Rightarrow -\int_0^1 (1+x^2)u_{,x} w_{,x} + 2u_{,x}(1)w(1) - \int_0^1 x u_{,x} w + \int_0^1 x^2 u w = 0$$

B.C. $u_{,x}(1) = 3u(1)$

$$\Rightarrow -\int_0^1 (1+x^2)u_{,x} w_{,x} + 6u(1)w(1) - \int_0^1 x u_{,x} w + \int_0^1 x^2 u w = 0.$$

$$\Rightarrow \int_0^1 -(1+x^2)u_{,x} w_{,x} - x u_{,x} w + x^2 u w\,dx + 6u(1)w(1) = 0.$$

$$\Rightarrow \int_0^1 (1+x^2)u_{,x} w_{,x} + x u_{,x} w - x^2 u w\,dx - 6u(1)w(1) = 0.$$

The variational form is:

$$\int_0^1 (1+x^2)u_{,x}w_{,x} + xu_{,x}w - x^2uw \, dx - 6u(1)w(1) = 0 \; \forall w \in V$$

Natural boundary $u_{,x}(1) = 3u(1)$

Essential boundary $u(0) = 1$

2. We have the variational form:

$$\int_0^1 (1+x^2)u_{,x}w_{,x} + xu_{,x}w - x^2uw \, dx - 6u(1)w(1) = 0 \; \forall w \in V$$

a. The bilinear form $a(u,w) = \int_0^1 (1+x^2)u_{,x}w_{,x} + xu_{,x}w - x^2uw\,dx$
$- 6u(1)w(1)$

$a(u,v)$ can be computed for any $u \in S$, $v \in V$. The bilinearity of $a(u,v)$ can be verified as:

$$a(u+\alpha v, w) = \int_0^1 (1+x^2)(u_{,x}+\alpha v_{,x})w_{,x} + \int_0^1 x(u_{,x}+\alpha v_{,x})w$$
$$-\int_0^1 x^2(u+\alpha v)w - 6u(1)w(1)$$
$$= \int_0^1 (1+x^2)u_{,x}w_{,x} + xu_{,x}w - x^2uw +$$
$$\left[\alpha\int_0^1 (1+x)^2 v_{,x}w_{,x} + xv_{,x}w - x^2vw\right] - 6u(1)w(1)$$
$$\Rightarrow \quad a(u,w) + \alpha(v,w)$$

But, this bilinear term is not symmetrical. To wit,

$$a(w,u) = \int_0^1 (1+x^2)w_{,x}u_{,x} + xw_{,x}u - x^2wu\,dx - 6u(1)w(1)$$

$$a(u,w) = \int_0^1 (1+x^2)\, u_{,x}\, w_{,x} + x u_{,x} w - x^2 w u \, dx - 6u(1)w(1)$$

Generally, $\int_0^1 x w_{,x} u \neq \int_0^1 x u_{,x} w$, therefore

$$a(w,u) \neq a(u,w).$$

b. The linear form $l(v)$ can be expressed as

$$l(v) = \int_0^1 0 f(v)\, dx$$

Obviously, $l(u+\alpha w) = l(u)$ since

$$\int_0^1 0 f(u+\alpha w)\, dx = 0.$$

$$a(u,w) = \int_0^1 (1+x^2) u_{,x} w_{,x} + x u_{,x} w - x^2 w u \, dx - 6u(1)w(1)$$

$$l(u) = \int_0^1 0 f(u)\, dx$$

3.(a) We ~~know~~ $W_h = \mathrm{span}\{1, x, x^2, x^3\}$.

if $w_h \in W_h$, $w_h = c_4 + c_1 x + c_2 x^2 + c_3 x^3$

We know $S_h \subset W_h$ and satisfy direchlee boundary.

$$S_h = \{u_h \in W_h \mid u_h(0) = 1\}.$$

$$= \{c_4 + c_1 x + c_2 x^2 + c_3 x^3 \mid (c_1, c_2, c_3) \in \mathbb{R}^3\}.$$

c_1, c_2, c_3 are active indices,

c_4 are constrained indices. c_4 should be 1. so that $u_h(0)=1$

For V_h, we find the direction of S_h to get

$$V_h = \{ v_h \in W_h \mid v_h(0) = 0 \}.$$
$$= \{ c_1 x + c_2 x^2 + c_3 x^3 \mid (c_1, c_2, c_3) \in \mathbb{R}^3 \}.$$

c_1, c_2, c_3 are active indices.

(b). Because functions in V_h are smooth:

$$V_h \subset V = \{ w : [0,1] \to \mathbb{R} \text{ smooth} \mid w(0) = 0 \}.$$

and the method is consistent.

(c). $W_h = \text{span}(1, x, x^2, x^3)$

$V_h = \text{span}(x, x^2, x^3)$

$S_h = \{ 1 + v_h \mid v_h \in V_h \}.$

We set $N_1(x) = x$, $N_2(x) = x^2$, $N_3(x) = x^3$, $N_4(x) = 1$.
For direchilet B.C., we choose $\bar{u}_h(x) = 1 N_4(x) \in S_h$

With those choices, the equations imposed by the variational methods are.

$a(u_h, N_1) = \ell(N_1)$; $a(u_h, N_2) = \ell(N_2)$
$a(u_h, N_3) = \ell(N_3)$ and $u_4 = 1$

Therefore, the stiffness matrix is:

$$K = \begin{bmatrix} a(N_1, N_1) & a(N_2, N_1) & a(N_3, N_1) & a(N_4, N_1) \\ a(N_1, N_2) & a(N_2, N_2) & a(N_3, N_2) & a(N_4, N_2) \\ a(N_1, N_3) & a(N_2, N_3) & a(N_3, N_3) & a(N, N_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$N_1' = 1, \quad N_2' = 2x, \quad N_3' = 3x^2, \quad N_4' = 0$

$$a(N_1, N_1) = \int_0^1 (1+x^2) N_1' N_1' + x N_1' N_1 - x^2 N_1 N_1 \, dx$$

$$= \int_0^1 (1+x^2) + x^2 - x^4 \, dx - 6 N_1(1) N_1(1)$$

$$a(N_2, N_1) = \int_0^1 (1+x^2) 2x \cdot 1 + 2x^3 - x^5 \, dx - 6 N_2 N_1(1)$$

$$a(N_3, N_1) = \int_0^1 3x^2(x^2+1) + 3x^4 - x^6 \, dx - 6 N_3 N_1$$

$$a(N_4, N_1) = \int_0^1 -x^3 \, dx - 6 N_4(1) N_1(1)$$

Similarly, we can get all entries of stiffness matrix K.
Then, we can use Matlab to evaluate the integration

$$K = \begin{bmatrix} -68/15, & -25/6, & -138/35, & -25/4 \\ -53/12, & -379/105, & -25/8, & -31/5 \\ -152/35, & -79/24, & -818/315, & -37/6 \\ 0, & 0, & 0, & 1 \end{bmatrix}$$

Float version:

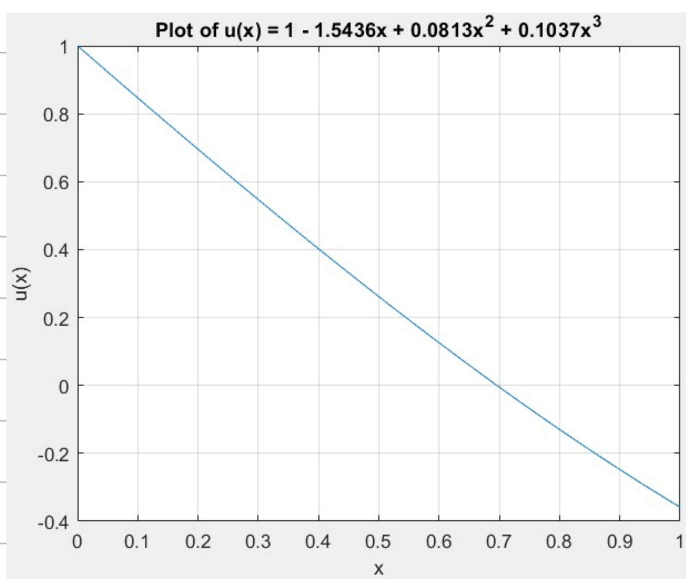
$$\begin{matrix} -4.5333 & -4.1667 & -3.9429 & -6.2500 \\ -4.4167 & -3.6095 & -3.1250 & -6.2000 \\ -4.3429 & -3.2917 & -2.5968 & -6.1667 \\ 0 & 0 & 0 & 1.0000 \end{matrix}$$

$$F = \begin{bmatrix} \ell(N_1) \\ \ell(N_2) \\ \ell(N_3) \\ u_\varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$U = K^{-1} F = \begin{bmatrix} -1.5436 \\ 0.0813 \\ 0.1037 \\ 1.000 \end{bmatrix}$$

$$u(x) = 1 - 1.5436x + 0.0813x^2 + 0.1037x^3$$

The plot looks like:



4. $u_{,x}(1) = -1.5436 + 0.0813 \cdot 2x + 0.1037 \cdot 3x^2$

$= -1.0700$

$3u(1) = -1.0758$

$|u_{,x}(1) - 3u(1)| = 0.0058$

Therefore, natural b.c. is not exactly satisfied.