Chapter 4

Linear Elasticity

In Chapters 1 and 2 we have followed a certain path of which the starting point was always the differential equation and the boundary conditions of a problem. From there, we inferred a variational equation and finally a variational numerical method. The finite element method is the combination of the variational method with some specially crafted subspaces, the finite element spaces.

In this chapter dedicated to linear elasticity we will follow another presentation path, known as a *variational* path. This path does not start from a differential equation, but from a *variational principle*. The variational principle that governs the behavior of many mechanical systems is the **minimization of the energy**. The static deformation of an elastic body, in particular, minimizes the **potential energy**, as we will soon see. The displacement of each tiny piece, be it located near the application of the load or far away from it, is dictated by this principle.

From a variational principle it is possible to deduce a *weak form* of a problem. It is even easier than when we start from the differential equation. Once we arrive at the weak form, the rest of the procedure is as before: Formulate the variational method and propose finite element spaces for it.

4.1 The Variational Problem of Linear Elasticity

The displacement field. Consider the problem sketched in Fig. 4.1. A solid body occupies the domain $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$. Along the **Dirichlet** part of the boundary ($\partial\Omega_D$) a displacement $\mathbf{g}$ is imposed to the particles of the body, while along the **Neumann** part ($\partial\Omega_N$) a distribution of forces $\mathbf{H}$ is applied. In addition, a body force $\mathbf{b}$ loads the body.

Under these conditions, the body will deform. The material particles will change position. This is expressed mathematically by a **displacement field** $\mathbf{u}(\mathbf{x})$, so that the displacement induced by the load on the particle at $\mathbf{x}$ is

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{u}(\mathbf{x}) . \tag{4.1}$$

The underformed domain Ω is called the **reference configuration** of the body, while the deformed domain $\{\mathbf{x}+\mathbf{u}(\mathbf{x}) \mid \mathbf{x} \in \Omega\}$ is called the **deformed configuration**.

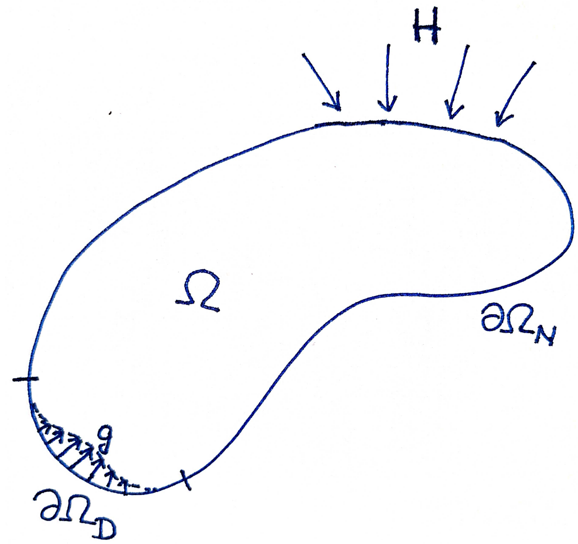

Figure 4.1 A sketch of an elasticity problem. The Dirichlet and Neumann boundaries, $\partial\Omega_D$ and $\partial\Omega_N$, are indicated.

Our quest is to determine the displacement field $\mathbf{u}:\Omega\to\mathbb{R}^2$. The unknown is thus a vector field over Ω, which can also be seen as two unknown functions $u_1(x_1,x_2)$ and $u_2(x_1,x_2)$, since $\mathbf{u}=(u_1,u_2)^T$ or, equivalently,

$$\mathbf{u}(\mathbf{x}) = u_1(\mathbf{x})\,\mathbf{e}_1 + u_2(\mathbf{x})\,\mathbf{e}_2 \,.$$

We will seek a physically valid solution $\mathbf{u}$ among displacement fields in a space

$$\mathscr{W} = \{\mathbf{w}:\Omega\to\mathbb{R}^2 \mid \mathbf{w} \text{ is a smooth vector field}\} \,. \tag{4.2}$$

The minimum smoothness required will be discussed further along.

Notice that we already know the value of $\mathbf{u}$ ($=\mathbf{g}=(g_1,g_2)^T$) along $\partial\Omega_D$, but we do not know how this displacement is "distributed" over Ω, or along $\partial\Omega_N$. We define the **trial space** $\mathscr{S}$ as

$$\mathscr{S} = \{\mathbf{w}\in\mathscr{W} \mid \mathbf{w}=\mathbf{g} \text{ on } \partial\Omega_D\} \,. \tag{4.3}$$

The Principle of Minimum Potential Energy. Instead of stating a differential equation to determine $\mathbf{u}$, we will invoke a **variational principle**, in particular, the **principle of minimum potential energy**. The solid is modeled as endowed with an **internal energy** U (also called **strain energy**) which depends solely on the displacement field $\mathbf{u}$. Then the **potential energy** V is defined as

$$V(\mathbf{u}) = U(\mathbf{u}) - \int_\Omega \mathbf{b}\cdot\mathbf{u}\,d\Omega - \int_{\partial\Omega_N} \mathbf{H}\cdot\mathbf{u}\,d\partial\Omega \,. \tag{4.4}$$

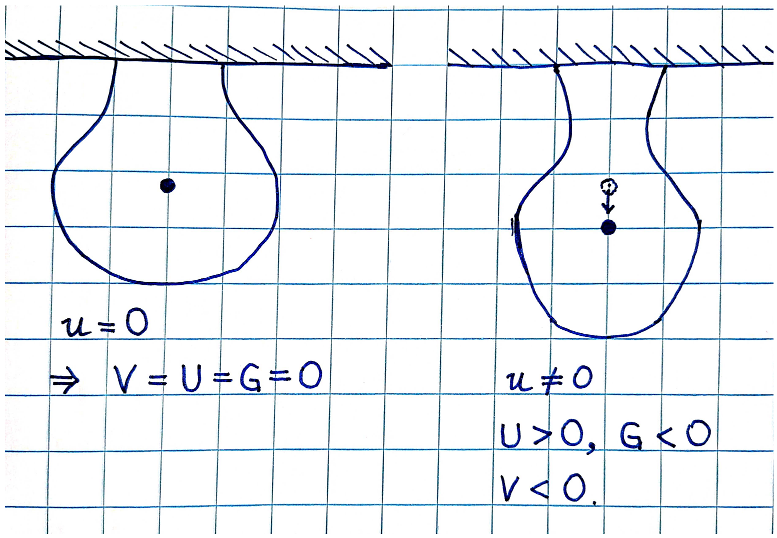

Figure 4.2 An elastic body hanging from the ceiling. On the left we have the reference configuration, which would correspond to gravity being "turned off". On the right we have the equilibrium configuration, which minimizes the potential energy V. Under the load of gravity the body deforms, increasing its strain energy, but this increase is more than compensated by the decrease in gravitational energy, evidenced by the lowering of the center of mass. The equilibrium configuration has less potential energy than the reference one.

The potential energy considers both the strain energy and the work done by the external loads.

The principle reads: *The* **equilibrium solution u** *minimizes the potential energy V among all smooth displacement fields* **w** *that are equal to* **g** *on* $\partial\Omega_D$. In mathematical terms,

$$V(\mathbf{u}) \leq V(\mathbf{w}), \qquad \text{for all } \mathbf{w} \in \mathscr{S}. \tag{4.5}$$

As an example, consider an elastic body which has a part of its boundary attached to the ceiling (see Fig. 4.2). The body is hanging from its upper fixation. The only applied load is the body's own weight $\mathbf{b} = -\rho\, g\, \mathbf{e}_2$, thus

$$V(\mathbf{u}) = U(\mathbf{u}) + \rho\, g \int_\Omega u_2 \, d\Omega\,.$$

The actual solution results from a compromise. The second term (which is nothing but the gravitational energy, denoted by G) decreases as the particles of the body displace downwards, but since the body is fixed at the top any downward displacement generates **strains** that make the internal energy increase. The strain energy function U depends on the material of the body, and so does the solution (usually termed **equilibrium solution**).

The small deformation hypothesis (SDH). We adopt here the SDH, which

$\frac{1}{2}(\nabla\vec{u}+\nabla u^T)$

$= \frac{1}{2}\left(\begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} + \begin{pmatrix} \partial_1 u_1 & \partial_1 u_2 \\ \partial_2 u_1 & \partial_2 u_2 \end{pmatrix}\right)$

assumes that the data **g**, **H** and **b** are small enough that **u** and its gradient

$\nabla u = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix}$

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix} \tag{4.6}$$

are very very small at all points of Ω. So small that the deformed domain essentially coincides with Ω.

$\varepsilon(\nabla u) = \begin{pmatrix} \partial_1 u_1 & \frac{1}{2}(\partial_2 u_1 + \partial_1 u_2) \\ \frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) & \partial_2 u_2 \end{pmatrix}$

Why bother to compute **u** if by hypothesis it is negligible at all points? It happens that solid materials are quite stiff, so that with just tiny displacements they can generate forces that equilibrate significant loads. As an example, consider a 1-m long steel wire with cross-sectional area of 10^{-4} m^2. If it is loaded with 1000 N the maximum displacement is (assuming a Young modulus $E = 2 \times 10^{11}$ Pa)

$$\Delta\ell = \frac{P\ell}{AE} = \frac{1000\,\mathrm{N} \times 1\,\mathrm{m}}{10^{-4}\,\mathrm{m}^2 \times 2 \times 10^{11}\,\mathrm{Pa}} = 5 \times 10^{-5}\,\mathrm{m} = 50\,\mathrm{microns}\,.$$

Consider the decomposition of $\nabla\mathbf{u}$ into symmetric (ε) and anti-symmetric (ω) parts, i.e.,

$$\nabla \mathbf{u} = \varepsilon(\nabla\mathbf{u}) + \omega(\nabla\mathbf{u}) = \boxed{\frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)} + \frac{1}{2}\left(\nabla\mathbf{u} - \nabla\mathbf{u}^T\right)\,. \tag{4.7}$$

Under the SDH, it can be shown that the tensor (or matrix) ε measures the **local deformation** (or **strain**) of the solid, while ω measures the **local rotation**.

The strain energy of a linearly elastic body under the SDH. If the material is **isotropic**, the strain energy takes the form

$$U(\mathbf{u}) = \int_\Omega \frac{E}{2(1+\nu)}\left(\varepsilon(\nabla\mathbf{u}) : \varepsilon(\nabla\mathbf{u}) + \frac{\nu}{1-2\nu}(\operatorname{div}\mathbf{u})^2\right)\, d\Omega\,, \tag{4.8}$$

where E is Young's modulus, ν is Poisson's ratio, $\operatorname{div}\mathbf{u} = \partial_1 u_1 + \partial_2 u_2 = \varepsilon_{11} + \varepsilon_{22}$, and ":" stands to the double contraction of two tensors, or equivalently the Frobenius product of two matrices:

$$\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}\,.$$

The value of Young's modulus is always a positive number, and Poisson's ratio spans the range $-1 \leq \nu \leq 0.5$. For example, metals typically have Poisson's ratios between 0.1 and 0.4. The integrand in (4.8) is the **strain energy density** at each point **x**, which is non-negative, and it is zero if and only if $\nabla\mathbf{u}(\mathbf{x})$ is antisymmetric. Therefore, the strain energy is always non-negative, and it is minimal if $\nabla\mathbf{u}(\mathbf{x})$ is anti-symmetric for all $\mathbf{x} \in \Omega$, or a small rigid body rotation.

If the material is not isotropic the corresponding expression is less appealing. The constitutive coefficients E and ν turn into a fourth-order array of coefficients C_{ijkl}, and then

$$U(\mathbf{u}) = \frac{1}{2}\int_\Omega \sum_{i,j,k,l} C_{ijkl}\, \varepsilon_{ij}(\nabla\mathbf{u})\, \varepsilon_{kl}(\nabla\mathbf{u})\, d\Omega\,. \tag{4.9}$$

In the following we will restrict the discussion to isotropic materials, so that the two coefficients E and ν (which may depend on **x**) characterize the elastic properties of the material.

Problem 4.1 ("Primal" variational form of the isotropic linear elasticity problem)**.** *Given the domain* Ω *and the data* E, ν, $\mathbf{b}$, $\mathbf{g}$ *and* $\mathbf{H}$, *determine the smooth vector field* $\mathbf{u} \in \mathscr{S}$ *that minimizes* V *over* $\mathscr{S}$, *where for a generic* $\mathbf{w} \in \mathscr{W}$ *the potential energy function is*

$$V(\mathbf{w}) = \int_\Omega \frac{E}{2(1+\nu)} \left(\varepsilon(\nabla \mathbf{w}) : \varepsilon(\nabla \mathbf{w}) + \frac{\nu}{1-2\nu} (div\ \mathbf{w})^2 \right) d\Omega$$
$$- \int_\Omega \mathbf{b} \cdot \mathbf{w}\, d\Omega - \int_{\partial\Omega_N} \mathbf{H} \cdot \mathbf{w}\, d\partial\Omega\,. \quad (4.10)$$

By replacing the first term above by (4.9) one obtains the primal variational form for anisotropic materials.

Example 4.1 A sphere under pressure. What is the deformation of a spherical homogeneous elastic body when a radial force $\mathbf{H} = -p\,\check{\mathbf{e}}_r$ is applied to its surface? The displacement will be radially symmetric, i.e.,

$$\mathbf{u}(\mathbf{x}) = \varphi(r)\,\check{\mathbf{e}}_r.$$

From the expression of the gradient in spherical coordinates we know that

$$\varepsilon(\nabla \mathbf{u}) : \varepsilon(\nabla \mathbf{u}) = \varphi'(r)^2 + 2\frac{\varphi(r)^2}{r^2}$$

and that

$$\operatorname{div} \mathbf{u} = \varphi'(r) + 2\frac{\varphi(r)}{r}\,.$$

The physical restriction $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ translates into $\varphi(0) = 0$. The equilibrium displacement of the sphere will thus be given by the function φ that minimizes, over the set of smooth functions

$$\mathscr{S} = \mathscr{V} = \left\{ \varphi : [0,R] \to \mathbb{R} \mid \varphi(0) = 0 \right\},$$

the potential energy

$$V(\varphi) = \frac{E}{2(1+\nu)} \int_0^R \left(\varphi'(r)^2 + 2\frac{\varphi(r)^2}{r^2} + \frac{\nu}{1-2\nu}\left(\varphi'(r) + 2\frac{\varphi(r)}{r}\right)^2 \right) 4\pi r^2 dr\ + $$
$$+4\pi p R^2 \varphi(R)\,. \quad (4.11)$$

It turns out that the exact minimizer is of the form $\varphi(r) = A\,r$ for some $A \in \mathbb{R}$ that depends on p. If we assume that $\mathscr{S}$ only contains such functions, then V becomes a function of A, that is

$$V = \gamma A^2 + 4\pi p R^3 A, \qquad \text{with} \qquad \gamma = \frac{2\pi E R^3}{1+\nu}\left(1 + \frac{3\nu}{1-2\nu}\right) = \frac{2\pi E R^3}{1-2\nu}\,.$$

The minimum takes place for

$$A = -\frac{4\pi p R^3}{2\gamma} = -\frac{1-2\nu}{E}\, p\,. \tag{4.12}$$

We have thus our first solution of a linear elastic problem. The displacement field is

$$\mathbf{u}(\mathbf{x}) = A\, r\, \check{\mathbf{e}}_r = A\mathbf{x} = -\frac{1-2\nu}{E}\, p\, \mathbf{x}.$$

Notice that if $\nu = \frac{1}{2}$ then $A = 0$ and thus $\mathbf{u} = \mathbf{0}$ at all points. The sphere does not contract under applied pressure. The limit $\nu \to \frac{1}{2}$ is called the "incompressible limit."

We cannot yet prove that this is the exact solution, but we can at least confirm that the polynomial $A\,r$ minimizes V over all **quadratic** polynomials. For this, we take $\varphi(r) = A\,r + B\,r^2$, with unknown coefficients A and B. Inserting this φ into (4.11) we obtain V as a function of A and B, namely

$$V(A,B) = \frac{2\pi E}{1-2\nu}\left[R^3 A^2 + 2R^4 A B + \frac{6+4\nu}{5+5\nu} R^5 B^2\right] + 4\pi R^3 p\,(A + R\,B)$$

By equating $\partial V/\partial A = \partial V/\partial B = 0$ you can check that the minimum takes place when A takes the value given in (4.12) and $B = 0$.

Remark: If we consider a 2D sphere (a circle), then

$$\varepsilon(\nabla\mathbf{u}) : \varepsilon(\nabla\mathbf{u}) = \varphi'(r)^2 + \frac{\varphi(r)^2}{r^2} \quad \text{and} \quad \operatorname{div}\mathbf{u} = \varphi'(r) + \frac{\varphi(r)}{r}\,.$$

This modifies the expression of $V(A)$ to

$$V = \frac{\pi E R^2}{(1+\nu)(1-2\nu)} A^2 + 2\pi p R^2 A,$$

and the value of A that corresponds to the minimum is

$$A = -\frac{(1+\nu)(1-2\nu)}{E}\, p\,. \tag{4.13}$$

Under the same pressure, a circle deforms less than a sphere.

4.2 From the Variational Form to the Weak Form

How to obtain a weak form when starting from a variational principle? It is quite straightforward. The procedure is based on the following rather abstract theorem.

Theorem 4.1. *Let $\mathscr{W}$ be a* **vector space** *(it could be of functions, of vector fields, etc.), and let $\mathscr{S}$ be an affine subspace of $\mathscr{W}$. The* **direction** *of $\mathscr{S}$ is denoted by $\mathscr{V}$. Assume that:*

a) *An energy function V is defined on $\mathscr{W}$ which can be written as*

$$V(w) = \frac{1}{2}a(w,w) - \ell(w) \tag{4.14}$$

where a is a symmetric bilinear form satisfying that

$$a(v,v) > 0 \qquad \forall v \in \mathscr{V}, \quad v \neq 0,$$

and ℓ is a linear form.

b) *There exists a minimizer u of V over $\mathscr{S}$. Precisely, there exists $u \in \mathscr{S}$ satisfying*

$$V(u) \leq V(w), \qquad \forall w \in \mathscr{S}\,. \tag{4.15}$$

Then, u is the unique minimizer in $\mathscr{S}$ (i.e., $V(u) < V(w)$, $\forall w \neq u$). In addition, u is also the unique element of $\mathscr{S}$ satisfying

$$a(u,v) = \ell(v) \qquad \forall v \in \mathscr{V}\,. \tag{4.16}$$

Proof. First let us prove that u necessarily satisfies (4.16). For this, assume that there exists some particular $0 \neq v \in \mathscr{V}$ for which

$$a(u,v) - \ell(v) = \beta \neq 0\,.$$

We will show that then u is not a minimizer of V over $\mathscr{S}$. Let us define $\alpha = a(v,v) > 0$ (because of hypothesis (a)) and

$$w = u - \frac{\beta}{\alpha}\,v\,.$$

Using the linearity of $a(\cdot,\cdot)$ and ℓ and the symmetry of $a(\cdot,\cdot)$ we see that

$$\begin{aligned} V(w) &= \frac{1}{2}a\left(u - \frac{\beta}{\alpha}\,v, u - \frac{\beta}{\alpha}\,v\right) - \ell\left(u - \frac{\beta}{\alpha}\,v\right) \\ &= \frac{1}{2}a(u,u) - \ell(u) - \frac{\beta}{\alpha}\underbrace{(a(u,v) - \ell(v))}_{\beta} + \frac{\beta^2}{2\alpha^2}\underbrace{a(v,v)}_{\alpha} \\ &= V(u) - \frac{\beta^2}{2\alpha} \\ &< V(u)\,. \end{aligned}$$

This proves (4.16). Now assume that there exists another element of $\mathscr{S}$, let us call it $\overline{u}$, that also satisfies $a(\overline{u},v) = \ell(v)$ for all $v \in \mathscr{V}$. Then,

$$\begin{aligned} a(u-\overline{u}, u-\overline{u}) &= a(u, \underbrace{u-\overline{u}}_{\in\mathscr{V}}) - a(\overline{u}, \underbrace{u-\overline{u}}_{\in\mathscr{V}}) \\ &= 0 \;-\; 0 \qquad = 0\,. \end{aligned}$$

According to hypothesis (a) this implies $u - \overline{u} = 0$ and thus $u = \overline{u}$.

This last argument also proves that u is the unique minimizer, since the existence of two minimizers would imply the existence of two solutions of (4.16) and that has been shown to be impossible. □

This theorem, though elementary, has interesting consequences. Notice that very little is said about the space $\mathscr{W}$. It could be finite or infinite dimensional, for example. The bilinear and linear forms are not required to be continuous. The hypotheses on the spaces involved are very weak, but on the other hand we make the strong assumption that a unique minimizer exists. Sometimes physical reasons make us believe that a unique minimum exists, though a mathematical proof could be unavailable. Under this assumption, the theorem provides us with **a weak form of the problem**, namely $a(u,v) = \ell(v), \quad \forall v \in \mathscr{V}$.

Applying this theorem to Problem (4.1) we obtain:

Problem 4.2 (Weak form of the Isotropic Linear Elasticity Problem)**.** *Given the domain* Ω *and the data* E*,* ν*,* $\mathbf{b}$*,* $\mathbf{g}$ *and* $\mathbf{H}$*, determine the smooth vector field* $\mathbf{u} \in \mathscr{S}$ *(i.e.,* $\mathbf{u} = \mathbf{g}$ *on* $\partial\Omega_D$*) such that*

$$a(\mathbf{u},\mathbf{v}) = \ell(\mathbf{v}) \tag{4.17}$$

for all $\mathbf{v} \in \mathscr{V}$*, where*

$$a(\mathbf{w},\mathbf{v}) = \int_\Omega \frac{E}{1+\nu}\left(\varepsilon(\nabla\mathbf{w}) : \varepsilon(\nabla\mathbf{v}) + \frac{\nu}{1-2\nu} div\, \mathbf{w}\; div\, \mathbf{v}\right)\, d\Omega, \tag{4.18}$$

$$\ell(\mathbf{v}) = \int_\Omega \mathbf{b}\cdot\mathbf{v}\, d\Omega + \int_{\partial\Omega_N} \mathbf{H}\cdot\mathbf{v}\, d\partial\Omega\,. \tag{4.19}$$

and

$$\mathscr{V} = \{\mathbf{v} \in \mathscr{W} \mid \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\}\,. \tag{4.20}$$

We should check that the hypotheses of Theorem 4.1 indeed hold true. It is readily seen that $V(\mathbf{w}) = \frac{1}{2}a(\mathbf{w},\mathbf{w}) - \ell(\mathbf{w})$. A little more subtle is to prove that $a(\mathbf{v},\mathbf{v}) > 0$ for all $\mathbf{v} \in \mathscr{V}$. We do this below when we study its coercivity, and it is true for $E > 0$ and $-1 \le \nu \le 0.5$, the range of values of these elastic moduli. It also requires that $\mathscr{V}$ does not contain any *rigid mode* (infinitesimal translations/rotations, which have $\varepsilon = 0$); to be seen during the study of coercivity as well.

Then, the theorem tells us that Problem 4.2 has **as unique solution** the displacement field $\mathbf{u}$ that minimizes the potential energy V (assumed to exist).

The stress field. The integrand in (4.18) can also be written as

$$\sigma(\nabla\mathbf{u}) : \varepsilon(\nabla\mathbf{v})$$

where σ is the **Cauchy stress tensor**, or simply **stress tensor**,

$$\sigma = \frac{E}{1+\nu}\,\varepsilon(\nabla\mathbf{u}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\,(\underbrace{\mathrm{div}\,\mathbf{u}}_{=\mathbf{I}:\,\nabla\mathbf{u}})\,\mathbf{I}\,. \tag{4.21}$$

In many cases the elasticity problem is solved mainly looking for the stress field over the body, since too high stresses may lead to the failure of the material.

Example 4.2 The stress field inside a sphere under pressure. We saw in Example 4.1 that the equilibrium displacement field of a 2D sphere (circle) under uniform pressure is

$$\mathbf{u}(\mathbf{x}) = A\,\mathbf{x}, \quad \text{where} \quad A = -\frac{(1+\nu)(1-2\nu)p}{E}$$

and $\mathbf{x}$ has the center of the circle as origin. This equation is intrinsic, valid for all coordinate systems. We can thus express it in **Cartesian** coordinates $x_1 - x_2$ so that $\mathbf{x} = (x_1, x_2)^T$. Then $u_1 = Ax_1$ and $u_2 = Ax_2$ and thus

$$\nabla\mathbf{u} = \varepsilon(\nabla\mathbf{u}) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = A\mathbf{I}, \quad \omega(\nabla\mathbf{u}) = 0, \quad \text{div }\mathbf{u} = 2A.$$

Inserting these values into (4.21) we obtain the corresponding stress field:

$$\sigma = \frac{EA}{1+\nu}\mathbf{I} + \frac{2E\nu A}{(1+\nu)(1-2\nu)}\mathbf{I} = \frac{EA}{(1+\nu)(1-2\nu)}\mathbf{I} = -p\mathbf{I} = \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix}$$

The stress field is **homogeneous**. It does not depend on $\mathbf{x}$. It is called **spherical** (or **hydrostatic**) because at all points it is a multiple of the identity tensor/matrix.

The Euler-Lagrange equations. The weak form is all we need to set up a variational method to approximate the exact solution $\mathbf{u}$. Let us however briefly talk about the Euler-Lagrange equations of the problem, and formulate the strong form. As usual, it is obtained by integrating by parts the weak form. Since σ is symmetric, it holds that

$$\int_\Omega \sigma : \varepsilon(\nabla\mathbf{v}) = \int_\Omega \sigma : \nabla\mathbf{v} = \int_{\partial\Omega} (\sigma\cdot\check{\mathbf{n}})\cdot\mathbf{v}\,d\partial\Omega - \int_\Omega (\text{div }\sigma)\cdot\mathbf{v}\,d\Omega\,.$$

From this, remembering that $\mathbf{v} = 0$ on $\partial\Omega_D$, we arrive at

$$0 = a(\mathbf{u},\mathbf{v}) - \ell(\mathbf{v}) = \int_\Omega (-\text{div }\sigma - \mathbf{b})\cdot\mathbf{v}\,d\Omega + \int_{\partial\Omega_N} (\mathbf{H} - \sigma\cdot\check{\mathbf{n}})\cdot\mathbf{v}\,d\partial\Omega\,,$$

for all $\mathbf{v} \in \mathcal{V}$. From this we conclude that $\mathbf{u}$ is also the solution of the following differential problem.

Problem 4.3 (Strong form of the Isotropic Linear Elasticity Problem)**.** *Given the same data as in Problems 4.1 and 4.2, find a smooth vector field* $\mathbf{u}$ *satisfying*

$$\begin{aligned} \mathit{div}\,\sigma(\nabla\mathbf{u}) + \mathbf{b} &= 0 && \mathit{in}\ \Omega, && (4.22)\\ \mathbf{u} &= \mathbf{g} && \mathit{on}\ \partial\Omega_D, && (4.23)\\ \sigma(\nabla\mathbf{u})\cdot\check{\mathbf{n}} &= \mathbf{H} && \mathit{on}\ \partial\Omega_N, && (4.24) \end{aligned}$$

where σ *is defined by (4.21).*

Equation (4.22) is also known as **equation of static equilibrium**. It expresses the local equilibrium of forces at each point of the domain, irrespective of the material being linearly elastic or not. Of course, if the material is not linearly elastic the expression for σ is different from (4.21).

It is important to internalize that Problems 4.1, 4.2 and 4.3 are essentially equivalent. Each one of them totally determines $\mathbf{u}$.

Examples:

4.3 **The exact solution of the problem of a sphere under pressure.** It is easy to verify that the stress field $\sigma = -p\mathbf{I}$ computed in Example 4.2 is a solution to Problem 4.3. In fact, since σ is independent of $\mathbf{x}$, div $\sigma = 0$, which is consistent with (4.22) because $\mathbf{b} = 0$. Also, since the boundary condition is $\mathbf{H} = -p\check{\mathbf{e}}_r$ and $\check{\mathbf{n}} = \check{\mathbf{e}}_r$, it follows that $\sigma \cdot \check{\mathbf{n}} = \mathbf{H}$ all over the boundary. Then the displacement field $\mathbf{u}$ calculated in Example 4.1 is indeed the unique solution (up to a rigid motion) of the problem of an isotropic linear elastic sphere under pressure.

4.4 **The exact solution of the problem of a rectangle under pressure.** Assume that the body subject to a uniform pressure p over its surface is the rectangle $\Omega = [-\frac{W}{2}, \frac{W}{2}] \times [-\frac{H}{2}, \frac{H}{2}]$. The force imposed by the pressure is $\mathbf{H} = -p\check{\mathbf{n}}$ at all points, so that $\mathbf{H}$ takes the value $-p\check{\mathbf{e}}_1$ at the east boundary, $p\check{\mathbf{e}}_1$ at the west one, $-p\check{\mathbf{e}}_2$ at the north one, and $p\check{\mathbf{e}}_2$ at the south one. The same displacement and stress fields (though now defined in the rectangular domain Ω) that solve the elastic problem for the sphere also solve it for the rectangle, any rectangle. In fact, **for any shape**.

The space of functions for elasticity. The space of candidate displacement fields for elasticity $\mathscr{W}$ should satisfy some minimal conditions. First, the potential energy $V(\mathbf{u})$ of any displacement field $u \in \mathscr{W}$ should be finite. Second, unless we consider situations in which fractures may appear (and additional modeling is needed), the displacement fields should not be discontinuous across curves with positive length in the domain of the problem (in 3D it should be across surfaces with positive area); pointwise discontinuities are still acceptable.

A space in which all of these requirements are met is

$$\mathscr{W} = \mathbf{H}^1(\Omega) = \left\{\mathbf{w} = (w_1, w_2)^T : \Omega \to \mathbb{R}^2 \mid w_1 \in H^1(\Omega),\ w_2 \in H^1(\Omega)\right\}, \tag{4.25}$$

where $H^1(\Omega)$ is the space of functions over Ω with a finite H^1-norm[1] (see A.11), namely, if $w \in H^1(\Omega)$ then

$$\|w\|_{1,2} = \left[\int_\Omega w^2 + |\nabla w|^2 \, d\Omega\right]^{1/2} < +\infty. \tag{4.26}$$

[1]This is almost the definition of H^1; we also need to require that the function should have *weak derivatives*.

So, $\mathscr{W}$ is a space of vector fields in which each component is a function in $H^1(\Omega)$. A norm on $\mathscr{W}$ is

$$\|\mathbf{w}\|_{1,2} = \sqrt{\|w_1\|_{1,2}^2 + \|w_2\|_{1,2}^2}\,. \tag{4.27}$$

In fact, $a(\cdot,\cdot)$ has the necessary properties of continuity in $\mathbf{H}^1(\Omega)$, and coercivity on $\mathscr{V}$, for $-1 < \nu < 0.5$.

Continuity of the Bilinear Form. For $-1 < \nu < 0.5$, the bilinear form satisfies that

$$|a(\mathbf{w},\mathbf{v})| \le M, \|\mathbf{w}\|_{1,2}\ \|\mathbf{v}\|_{1,2} \tag{4.28}$$

for any two elements $\mathbf{w}$ and $\mathbf{v}$ of $\mathscr{W}$, where

$$M = \sup_{\Omega} \max\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\}. \tag{4.29}$$

The constant M tends to infinity when E tends to infinity, or ν tends to $\frac{1}{2}$, somewhere over the domain.

Continuity of the bilinear form, (4.28)

To see the continuity, you can verify that for any $\mathbf{v},\mathbf{w} \in \mathscr{W}$,

$$\frac{E}{(1+\nu)}\left(\varepsilon(\nabla\mathbf{w}) : \varepsilon(\nabla\mathbf{v}) + \frac{\nu}{1-2\nu}\operatorname{div}\mathbf{w}\operatorname{div}\mathbf{v}\right) = \bar{\varepsilon}(\nabla\mathbf{w})^{\mathsf{T}} \cdot D \cdot \bar{\varepsilon}(\nabla\mathbf{v}) \tag{4.30}$$

where,

$$\bar{\varepsilon}(\nabla\mathbf{v}) = \begin{bmatrix} \varepsilon_{11}(\nabla\mathbf{v}) \\ \varepsilon_{22}(\nabla\mathbf{v}) \\ \varepsilon_{33}(\nabla\mathbf{v}) \\ \sqrt{2}\varepsilon_{12}(\nabla\mathbf{v}) \\ \sqrt{2}\varepsilon_{13}(\nabla\mathbf{v}) \\ \sqrt{2}\varepsilon_{23}(\nabla\mathbf{v}) \end{bmatrix}, \qquad D = \frac{E}{1+\nu}\begin{bmatrix} 1+\frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & 1+\frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 1+\frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix D is positive definite, since its eigenvalues are

$$\frac{E}{1+\nu}\left\{\frac{1+\nu}{1-2\nu}, 1, 1, 1, 1, 1\right\}. \tag{4.31}$$

Therefore, denoting the eigenvalues by $\lambda_1, \dots, \lambda_6$, and by $\|\varepsilon\| = \sqrt{\varepsilon : \varepsilon}$ the Frobenius norm, we have

$$
\begin{aligned}
|\bar{\varepsilon}(\nabla \mathbf{v})^{\mathsf{T}} \cdot D \cdot \bar{\varepsilon}(\nabla \mathbf{w})| &= \left|\sum_{i=1}^{6} \bar{\varepsilon}_i(\nabla \mathbf{v})\bar{\varepsilon}_i(\nabla \mathbf{w})\lambda_i\right| \\
&\le \sum_{i=1}^{6} |\bar{\varepsilon}_i(\nabla \mathbf{v})||\bar{\varepsilon}_i(\nabla \mathbf{w})||\lambda_i| \\
&\le \max_i |\lambda_i| \sum_{i=1}^{6} |\bar{\varepsilon}(\nabla \mathbf{v})||\bar{\varepsilon}(\nabla \mathbf{w})| \\
&\le \max_i |\lambda_i| \|\bar{\varepsilon}(\nabla \mathbf{v})\| \|\bar{\varepsilon}(\nabla \mathbf{w})\| && \text{Cauchy-Schwartz.} \\
&= \max\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\} \|\varepsilon(\nabla \mathbf{v})\| \|\varepsilon(\nabla \mathbf{w})\| && \text{since } \|\bar{\varepsilon}\| = \|\varepsilon\| \\
&\le \max\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\} \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| && \text{Triangle inequality.}
\end{aligned}
\tag{4.32}
$$

In the last line we used that

$$\|\varepsilon(\nabla \mathbf{w})\| = \frac{1}{2}\|\nabla \mathbf{w} + \nabla \mathbf{w}^T\| \le \|\nabla \mathbf{w}\|$$

from the triangle inequality. The version of Cauchy-Schwartz inequality we used here is one for vectors in $\mathbb{R}^n$, instead of functions: if $a, b \in \mathbb{R}^n$, then

$$|a \cdot b| \le \|a\| \|b\|. \tag{4.33}$$

Then,

$$
\begin{aligned}
|a(\mathbf{w}, \mathbf{v})| \quad &= \quad \int_\Omega \frac{E}{1+\nu}\left(\varepsilon(\nabla \mathbf{w}) : \varepsilon(\nabla \mathbf{v}) + \frac{\nu}{1-2\nu} \operatorname{div} \mathbf{w} \operatorname{div} \mathbf{v}\right) d\Omega, \\
&\le \quad \int_\Omega \max\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\} \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\| d\Omega \\
&\le \quad \sup_\Omega \max\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\} \|\nabla \mathbf{w}\|_{0,\Omega} \|\nabla \mathbf{v}\|_{0,\Omega}, \\
&\le \quad M \|\mathbf{w}\|_{1,2} \|\mathbf{v}\|_{1,2}.
\end{aligned}
$$

Coercivity. Well-posed elasticity problems have Dirichlet boundary conditions that preclude rigid motions. In fact, **rigid motions are automatically excluded from $\mathcal{V}$ if the length (measure) of $\partial\Omega_D$ is strictly positive.** Under this assumption, it is not difficult to prove that $a(\cdot, \cdot)$ is coercive on $\mathcal{V}$, i.e.,

Lemma 4.1. *There exists $c_{\mathcal{V}} > 0$ such that*

$$a(\mathbf{v}, \mathbf{v}) \ge c_{\mathcal{V}} \|\mathbf{v}\|^2_{\mathcal{W}} \qquad \forall\, \mathbf{v} \in \mathcal{V}. \tag{4.34}$$

Coercivity of the bilinear form, Lemma 4.1

To prove coercivity, we need to appeal to a classical result known as **Korn's inequality**. The version that we use here is:

Korn's inequality: There exists $C_K > 0$ such that, for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ satisfying $\mathbf{v} = 0$ on $\partial\Omega_D$ (which has to have positive length/measure),

$$\int_\Omega \|\varepsilon(\nabla \mathbf{v})\|^2 \, d\Omega \;\geq\; C_K \, \|\mathbf{v}\|_{1,2}^2. \tag{4.35}$$

We can now proceed with the proof of Lemma 4.1. Using (4.30) and the eigenvalues of the matrix D, (4.31), we can write

$$\begin{aligned}
\bar{\varepsilon}(\nabla \mathbf{v})^\top \cdot D \cdot \bar{\varepsilon}(\nabla \mathbf{v}) &\geq \min\left\{\frac{E}{1+\nu}, \frac{E}{1-2\nu}\right\} \bar{\varepsilon}(\nabla \mathbf{v})^\top \cdot \bar{\varepsilon}(\nabla \mathbf{v})^\top \\
&\geq \frac{E}{3} \bar{\varepsilon}(\nabla \mathbf{v})^\top \cdot \bar{\varepsilon}(\nabla \mathbf{v})^\top \\
&= \frac{E}{3} \varepsilon(\nabla \mathbf{v}) : \varepsilon(\nabla \mathbf{v}),
\end{aligned}$$

Remembering that $\|\varepsilon\| = \sqrt{\varepsilon : \varepsilon}$, we have that

$$\begin{aligned}
a(\mathbf{v}, \mathbf{v}) &= \int_\Omega \frac{E}{(1+\nu)} \left(\varepsilon(\nabla \mathbf{v}) : \varepsilon(\nabla \mathbf{v}) + \frac{\nu}{1-2\nu} (\operatorname{div} \mathbf{v}))^2\right) \, d\Omega \\
&\geq \frac{E}{3} \int_\Omega \|\varepsilon(\nabla \mathbf{v})\|^2 \, d\Omega \\
&\geq \frac{E}{3} C_K \|\mathbf{v}\|_{1,2}^2 && \text{Korn's inequality,(4.35).}
\end{aligned}$$

4.3 Variational Numerical Method

The variational numerical method to approximate $\mathbf{u}$ is built directly from Problem 4.2, as we have done in all previous chapters. Selecting a finite dimensional space $\mathscr{W}_h$ and constructing $\mathscr{S}_h$ and $\mathscr{V}_h$ as earlier, it reads:

Variational numerical method: *Find* $\mathbf{u}_h \in \mathscr{S}_h$ *such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) = \ell(\mathbf{v}_h) \tag{4.36}$$

for all $\mathbf{v}_h \in \mathscr{V}_h$*, where* $a(\cdot,\cdot)$ *and* $\ell(\cdot)$ *are given by (4.17) and (4.19), respectively.*

Expressed in this abstract form the only difference with our previous encounters with the variational methods is the bold characters used to denote the solution $\mathbf{u}_h$ and the test function $\mathbf{v}_h$. This is just a purely notational innovation we have introduced to remind us that $\mathscr{S}_h$ and $\mathscr{V}_h$ are spaces of **vector fields**.

For consistency, again, functions in $\mathscr{V}_h$ should be continuous vector fields, and hence we set $\mathscr{W}_h$ to be a space of continuous vector fields.

Before describing suitable finite element spaces and bases for the Galerkin approximation, let us illustrate how it works when using a basis of **global polynomials**, which certainly are continuous.

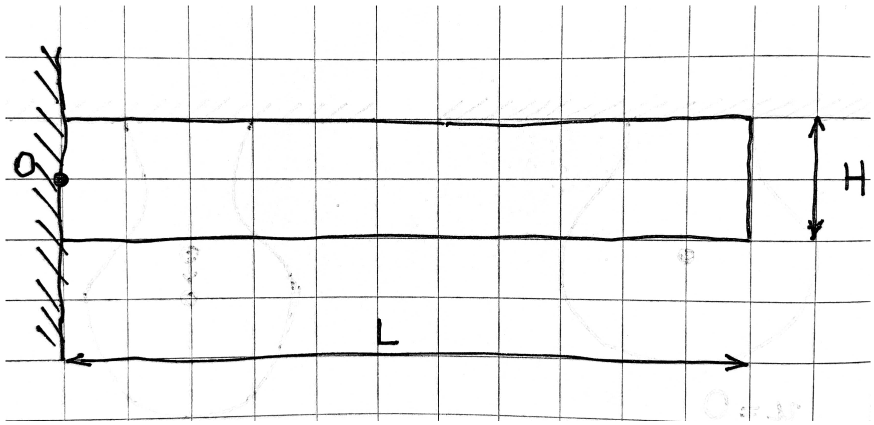

Figure 4.3 Sketch of a 2D cantilever elastic bar under its own weight.

Example 4.5 (A cantilever rectangular bar under its own weight) Consider the 2D rectangular bar of Figure 4.3 ($\Omega = [0, L] \times [-\frac{H}{2}, \frac{H}{2}]$), which is fixed to the rigid wall along $x_1 = 0$ and subject to its own body weight $\mathbf{b} = (0, -\rho g)^{\mathsf{T}}$. There are no other loads.

As $\mathcal{W}_h$ we take polynomials (in the two variables x_1 and x_2) of degree up to k_1 for u_1 and up to k_2 for u_2. In other words,

$$\mathcal{W}_h = \left\{\mathbf{w}_h : \Omega \to \mathbb{R}^2 \mid w_{h1} \in \mathbb{P}_{k_1},\ w_{h2} \in \mathbb{P}_{k_2}\right\}$$

Any $\mathbf{w}_h \in \mathcal{W}_h$, taking $k_1 = 1$, $k_2 = 2$ as an example, can be written as

$$\mathbf{w}_h = \begin{pmatrix} c_1 + c_2 x_1 + c_3 x_2 \\ \\ c_4 + c_5 x_1 + c_6 x_2 + c_7 x_1^2 + c_8 x_1 x_2 + c_9 x_2^2 \end{pmatrix}. \tag{4.37}$$

By varying the 9 coefficients over $\mathbb{R}^9$ the vector field $\mathbf{w}_h$ spans $\mathcal{W}_h$, with each set of coefficients corresponding to exactly one vector field and vice versa. The dimension of $\mathcal{W}_h$ is certainly 9. The next step is to build a basis of 9 vector fields for $\mathcal{W}_h$.

When one encounters a vector space defined by a linear expression involving a set of arbitrary real coefficients $c_1, \ldots, c_m$, one can always build a basis of the space by setting the coefficients to the values of the canonical basis of $\mathbb{R}^m$, that is,

$$(1, 0, \ldots, 0)^T, \quad (0, 1, 0, \ldots, 0)^T, \quad \ldots$$

This leads to the following basis of $\mathcal{V}_h$ (we adopt $\mathbf{N}_k$ as notation for the elements of the basis to emphasize that they are vector fields over Ω):

$$\mathbf{N}_1(x_1, x_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{N}_2(x_1, x_2) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \qquad \mathbf{N}_3(x_1, x_2) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix},$$

$$\mathbf{N}_4(x_1,x_2)=\begin{pmatrix}0\\1\end{pmatrix}, \qquad \mathbf{N}_5(x_1,x_2)=\begin{pmatrix}0\\x_1\end{pmatrix}, \qquad \mathbf{N}_6(x_1,x_2)=\begin{pmatrix}0\\x_2\end{pmatrix},$$

$$\mathbf{N}_7(x_1,x_2)=\begin{pmatrix}0\\x_1^2\end{pmatrix}, \qquad \mathbf{N}_8(x_1,x_2)=\begin{pmatrix}0\\x_1x_2\end{pmatrix}, \qquad \mathbf{N}_9(x_1,x_2)=\begin{pmatrix}0\\x_2^2\end{pmatrix}.$$

Remark: One could use any other basis of $\mathbb{R}^m$, which would lead to a different basis of $\mathscr{W}_h$.

The choice of the basis is not irrelevant. Remember that the space $\mathscr{V}_h$ is defined, as usual, as

$$\mathscr{V}_h=\{\mathbf{v}_h\in\mathscr{W}_h \mid v_{h1}(x_1=0,x_2)=0,\ v_{h2}(x_1=0,x_2)=0\}\ ,$$

and we need to take a basis of $\mathscr{W}_h$ **of which a subset is a basis of** $\mathscr{V}_h$.

Fortunately, our judicious choice of coordinates and location of the origin makes the basis $\mathbf{N}_1$ above an adequate one. The only fields in $\mathscr{W}_h$ that are zero at the left boundary are those belonging to

$$\mathscr{V}_h=\left\{\mathbf{v}_h\in\mathscr{W}_h \mid \mathbf{v}_h=\left(c_2x_1\ ,\ c_5x_1+c_7x_1^2+c_8x_1x_2\right)^T\right\}\ . \tag{4.38}$$

Thus, the functions $\mathbf{N}_2$, $\mathbf{N}_5$, $\mathbf{N}_7$ and $\mathbf{N}_8$ are a basis of $\mathscr{V}_h$, **which in this case coincides with** $\mathscr{S}_h$ because the Dirichlet conditions are all zero.

Once we have established a suitable basis, the next step is as usual to build the stiffness matrix entries $\mathsf{K}_{ij}=a(\mathbf{N}_j,\mathbf{N}_i)$ and the load vector entries $\mathsf{f}_i=\ell(\mathbf{N}_i)$, for $i\in\eta\setminus\eta_g=\{2,5,7,8\}$. The entries with $i\in\eta_g$ will be taken from the identity matrix and the imposed boundary condition, so as to enforce, in this specific case,

$$c_1=c_3=c_4=c_6=c_9=0\ .$$

The computation of the stiffness matrix is easier to understand if we introduce a notation for the strain, divergence and stress fields induced by each basis function, i.e.,

$$B^i=\varepsilon(\nabla\mathbf{N}_i), \qquad D^i=\operatorname{div}\mathbf{N}_i, \qquad \Sigma^i=\sigma(\nabla\mathbf{N}_i)\ . \tag{4.39}$$

Explicitly,

$$B^1=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad B^2=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad B^3=\begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix},$$

$$B^4=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad B^5=\begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix},\quad B^6=\begin{pmatrix}0&0\\0&1\end{pmatrix},$$

$$B^7=\begin{pmatrix}0&x_1\\x_1&0\end{pmatrix},\quad B^8=\begin{pmatrix}0&\frac{x_2}{2}\\\frac{x_2}{2}&x_1\end{pmatrix},\quad B^9=\begin{pmatrix}0&0\\0&2x_2\end{pmatrix}.$$

The trace of B^i equals D^i, so

$$D^1 = 0, \quad D^2 = 1, \quad D^3 = 0, \quad D^4 = 0, \quad D^5 = 0,$$

$$D^6 = 1, \quad D^7 = 0, \quad D^8 = x_1, \quad D^9 = 2x_2.$$

The matrix expression for Σ^i is

$$\Sigma^i = \underbrace{\frac{E}{1+\nu}}_{2\mu} B^i + \underbrace{\frac{E\nu}{(1+\nu)(1-2\nu)}}_{\lambda} D^i \mathbf{I},$$

where we have introduced the Lamé coefficients μ and λ. We thus have

$$\Sigma^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 2\mu+\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix},$$

$$\Sigma^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^5 = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad \Sigma^6 = \begin{pmatrix} 0 & 0 \\ 0 & 2\mu+\lambda \end{pmatrix},$$

$$\Sigma^7 = \begin{pmatrix} 0 & 2\mu x_1 \\ 2\mu x_1 & 0 \end{pmatrix}, \quad \Sigma^8 = \begin{pmatrix} \lambda x_1 & \mu x_2 \\ \mu x_2 & (2\mu+\lambda)x_1 \end{pmatrix}, \quad \Sigma^9 = \begin{pmatrix} 2\lambda x_2 & 0 \\ 0 & (4\mu+2\lambda)x_2 \end{pmatrix}.$$

In terms of these matrices, it holds that

$$a(\mathbf{N}_j, \mathbf{N}_i) = \int_\Omega \Sigma^j : B^i \, d\Omega. \tag{4.40}$$

These integrals are easy to calculate, for example

$$\begin{aligned} \mathsf{K}_{87} &= a(\mathbf{N}_7, \mathbf{N}_8) \\ &= \int_0^L \int_{-\frac{H}{2}}^{\frac{H}{2}} \Sigma^7 : B^8 \, dx_1 dx_2 \\ &= \int_0^L \int_{-\frac{H}{2}}^{\frac{H}{2}} 2\mu x_1 x_2 \, dx_1 dx_2 \\ &= 0, \end{aligned}$$

or

$$\begin{aligned} \mathsf{K}_{77} &= a(\mathbf{N}_7, \mathbf{N}_7) \\ &= \int_0^L \int_{-\frac{H}{2}}^{\frac{H}{2}} \Sigma^7 : B^7 \, dx_1 dx_2 \\ &= \int_0^L \int_{-\frac{H}{2}}^{\frac{H}{2}} 4\mu x_1^2 \, dx_1 dx_2 \\ &= \frac{4\mu H L^3}{3}. \end{aligned}$$

After computing all necessary integrals we arrive at the stiffness matrix

$$\mathsf{K}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & (2\mu+\lambda)HL & 0 & 0 & 0 & 0 & 0 & \frac{\lambda HL^2}{2} & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mu HL & 0 & \mu HL & 0 & \mu HL^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \mu HL^2 & 0 & \mu HL^2 & 0 & \frac{4\mu HL^3}{3} & 0 & 0\\ 0 & \frac{\lambda HL^2}{2} & 0 & 0 & 0 & \frac{(2\mu+\lambda)HL^2}{2} & 0 & \frac{\mu H^3 L}{4} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The load vector F has

$$\mathsf{F}_1=\mathsf{F}_3=\mathsf{F}_4=\mathsf{F}_6=\mathsf{F}_9=0$$

because the corresponding coefficients are imposed as zero. For $i\in\{2,5,7,8\}$, we have to compute

$$\mathsf{F}_i=\ell(\mathbf{N}_i)=-\int_\Omega \rho\, g\, N_2^i(x_1,x_2)\;dx_1dx_2$$

so,

$$\begin{aligned}
\mathsf{F}_2 &= -\int_0^L\int_{-\frac{H}{2}}^{\frac{H}{2}} \rho\, g\, 0\; dx_1dx_2 \;=\; 0\\
\mathsf{F}_5 &= -\int_0^L\int_{-\frac{H}{2}}^{\frac{H}{2}} \rho\, g\, x_1\; dx_1dx_2 \;=\; -\frac{\rho g H L^2}{2}\\
\mathsf{F}_7 &= -\int_0^L\int_{-\frac{H}{2}}^{\frac{H}{2}} \rho\, g\, x_1^2\; dx_1dx_2 \;=\; -\frac{\rho g H L^3}{3}\\
\mathsf{F}_8 &= -\int_0^L\int_{-\frac{H}{2}}^{\frac{H}{2}} \rho\, g\, x_1 x_2\; dx_1dx_2 \;=\; 0
\end{aligned}$$

We are now in a position to solve the linear system

$$\mathsf{K}\,\mathsf{U}=\mathsf{F}\,,$$

to obtain the unknown vector of coefficients

$$\mathsf{U}=(c_1,c_2,\ldots,c_9)^T\;.$$

We adopt, as an example, the values $H = 0.1$ m, $L = 1$ m, $E = 2 \times 10^{11}$ Pa, $\nu = 0.3$, $\rho = 8000$ kg/m^3, $g = 9.8$ m/s^2. The corresponding Lamé coefficients are $\mu = 7.692 \times 10^{10}$ Pa and $\lambda = 1.154 \times 10^{11}$ Pa.

The solution of the linear system is

$$\mathsf{U} = \left(0\,,\, 0\,,\, 0\,,\, 0\,,\, -1.0192 \times 10^{-6}\,,\, 0\,,\, 0.5096 \times 10^{-6}\,,\, 0\,,\, 0 \right)^T .$$

Equivalently, the unknown coefficients turn out to be

$$c_2 = 0, \qquad c_5 = -1.0192 \times 10^{-6}, \qquad c_7 = 0.5096 \times 10^{-6}\mathrm{m}^{-1}, \qquad c_9 = 0.$$

This means that the Galerkin approximation to the displacement field is given by

$$\mathbf{u}_h(x_1, x_2) = \begin{pmatrix} 0 \\ -1.0192 \times 10^{-6} \times x_1 + 0.5096 \times 10^{-6}\mathrm{m}^{-1} \times x_1^2 \end{pmatrix} .$$

The deformed geometry in which each point is moved according to

$$\mathbf{x} \mapsto \mathbf{x} + \alpha\, \mathbf{u}_h(\mathbf{x}),$$

where $\alpha = 10^5$ is a scaling factor added to render the deformation visible, is shown in Fig. 4.4. Notice that the maximum displacement occurs at $x_1 = L$ (which is logical), and the value is

$$\mathbf{u}_h(x_1, x_2 = L) = \begin{pmatrix} 0 \\ -0.5096\,\text{microns} \end{pmatrix} .$$

The associated strain and stress fields are

$$\epsilon(\mathbf{u}_h) = \sum_{j=1}^{9} c_j\, \varepsilon(\mathbf{N}_j) = \sum_{j=1}^{9} c_j\, B^j = \begin{pmatrix} 0 & \frac{1}{2}c_5 + c_7\, x_1 \\ \text{symm} & 0 \end{pmatrix}$$

$$\sigma(\mathbf{u}_h) = \sum_{j=1}^{9} c_j\, \sigma(\mathbf{N}_j) = \sum_{j=1}^{9} c_j\, \Sigma^j = \begin{pmatrix} 0 & \left(\frac{1}{2}c_5 + c_7\, x_1\right)\mu \\ \text{symm} & 0 \end{pmatrix}$$

The maximum shear strain and stress occurs at $x_1 = 0$. The values are 0.5096×10^{-6} and 39200 Pa.

A salient feature of the approximate solution obtained with the basis functions $\{\mathbf{N}_1, \ldots, \mathbf{N}_9\}$ is that it is very wrong.

The space $\mathscr{S}_h$, which is given by (4.38), keeps the vertical planes vertical (because v_{h1} is independent of x_2). This space is not suitable for representing the typical bending deformations, characterized by the rotation of the vertical planes with the (small) angle varying with x_1.

To improve the result one needs to enlarge the space $\mathscr{S}_h$, which is accomplished by enlarging the encompassed space $\mathscr{W}_h$. More specifically, **let us add three aditional basis functions** $\mathbf{N}_{10}$, $\mathbf{N}_{11}$ and $\mathbf{N}_{12}$ so as to complete the quadratic polynomials in the horizontal component:

$$\mathbf{N}_{10}(x_1,x_2) = \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}, \quad \mathbf{N}_{11}(x_1,x_2) = \begin{pmatrix} x_1 x_2 \\ 0 \end{pmatrix}, \quad \mathbf{N}_{12}(x_1,x_2) = \begin{pmatrix} x_2^2 \\ 0 \end{pmatrix}.$$

The crucial one, in view of what was discussed above, is $\mathbf{N}_{11}$. From them we can compute the associated strain and stress fields,

$$B^{10} = \begin{pmatrix} 2x_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^{11} = \begin{pmatrix} x_2 & \frac{x_1}{2} \\ \frac{x_1}{2} & 0 \end{pmatrix}, \quad B^{12} = \begin{pmatrix} 0 & x_2 \\ x_2 & 0 \end{pmatrix}.$$

$$D^{10} = 2x_1, \quad D^{11} = x_2, \quad D^{12} = 0.$$

$$\Sigma^{10} = \begin{pmatrix} (4\mu+2\lambda)x_1 & 0 \\ 0 & 2\lambda x_1 \end{pmatrix}, \quad \Sigma^{11} = \begin{pmatrix} (2\mu+\lambda)x_2 & \mu x_1 \\ \mu x_1 & \lambda x_2 \end{pmatrix},$$

$$\Sigma^{12} = \begin{pmatrix} 0 & 2\mu x_2 \\ 2\mu x_2 & 0 \end{pmatrix}.$$

From this it is straightforward to enlarge the previous 9×9 stiffness matrix into the 12×12 new one. Notice that c_{12} **must be zero** to satisfy the boundary conditions. The new 12×1 load vector array is just the previous one with 3 zeros added at the end (why?).

The algebraic solution obtained with 12 basis functions ($k_1 = k_2 = 2$) is

$$\mathsf{U} = 10^{-6} \times (\,0\,,\,0\,,\,0\,,\,0\,,\,-1.0192\,,\,0\,,\,-28.61\,,\,0\,,\,0\,,\,0\,,\,58.24\,,\,0\,)^T\,.$$

With respect to the previous solution, one observes a significant change in c_7 while c_5 remains the same. The c_{11} coefficient, which did not exist previously, takes a relatively large value, while c_{10} is zero. A deformation mode that combines the functions $\mathbf{N}_7$ and $\mathbf{N}_{11}$ is activated by the load (this mode was not possible when $\mathbf{N}_{11}$ was not there).

The new Galerkin approximation to $\mathbf{u}$ is

$$\mathbf{u}_h(x_1,x_2) = \begin{pmatrix} 58.24 \times 10^{-6}\mathrm{m}^{-1} \times x_1 \times x_2 \\ -1.0192 \times 10^{-6} \times x_1 - 28.61 \times 10^{-6}\mathrm{m}^{-1} \times x_1^2 \end{pmatrix}.$$

It is shown in Fig. 4.5, but notice that the scaling factor α has been reduced to 10^4 because the displacements are much larger. The solution now makes physical sense. A typical bending deformation takes place. The vertical displacement of the tip of the bar is -29.63 microns (almost 60 times larger in magnitude than with the previous space). The computation of the corresponding strain and stress fields is straightforward.

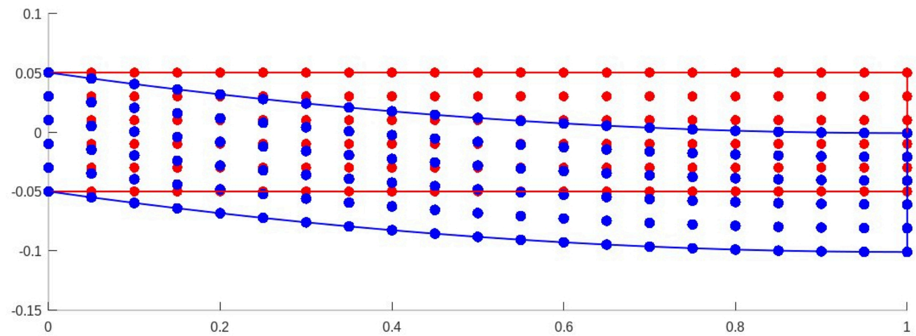

Figure 4.4 Deformed geometry (with the displacement scaled by a factor of 10^5 to render it visible) obtained with the variational method when using $k_1 = 1$ and $k_2 = 2$ (9 basis functions). Shown are the reference and deformed positions of a set of sampling points (in red and blue, respectively). Also shown are the boundaries of the reference and deformed configurations. This solution is physically unrealistic.

> *Food for thought:* The solution to the variational method only "uses" the basis functions $\mathbf{N}_5$, $\mathbf{N}_7$ and $\mathbf{N}_{11}$. If we had chosen simply $\mathscr{S}_h = \mathscr{V}_h = \text{span}\,\{\mathbf{N}_5, \mathbf{N}_7, \mathbf{N}_{11}\}$ (with just 3 unknown coefficients!), **would we have obtained the same $\mathbf{u}_h$?** The answer is **yes** (why?).
>
> **Summary of this long example:** This example illustrated the application of a variational method in elasticity problems with spaces of global polynomials. This is only possible for some domains and boundary conditions. We followed the same procedure as in Example 2.7, with the necessary adaptations brought by the vector character of the unknown $\mathbf{u}$.
>
> During the exposition we showed that the solution of the variational method may be very inaccurate if the selected space is unable to approximate the dominant deformation modes of the exact solution.

Invertibility of the Stiffness Matrix. If the space $\mathscr{V}_h$ is made of continuous functions, then $\mathscr{V}_h \subset \mathscr{V}$, and we have that (4.34) is in particular satisfied for all $\mathbf{v}_h \in \mathscr{V}_h$ and thus $a(\cdot,\cdot)$ **is coercive on** $\mathscr{V}_h$. As with scalars problems in Theorem 3.1, this implies that **the stiffness matrix is invertible**.

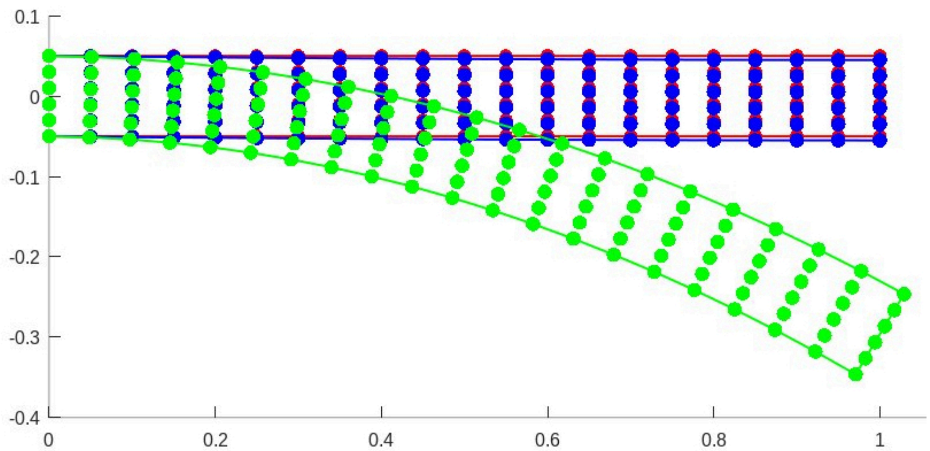

Figure 4.5 In green we show the deformed geometry (with the displacement scaled by a factor of 10^4 to render it visible) obtained with the variational method when using $k_1 = k_2 = 2$ (12 basis functions). In red and blue we show the same reference and approximate deformation as in Fig. 4.4, but now they are almost superposed because α has been decreased to 10^4.

4.4 Finite Element Spaces for Multifield problems in 2D

In Chapter 2 we introduced the P_1 finite element space in two space dimensions. The P_1 finite element provides basis functions that are **continuous scalar functions** which, by refining the mesh, can approximate any function $u \in H^1(\Omega)$.

Though they have not yet been discussed in this notes, there exist P_k finite elements that provide continuous scalar functions which are piecewise polynomials of any order $k \geq 1$ in the variables $x_1 - x_2$.

None of these finite element spaces can *per se* approximate problems with several unknowns such as the 2D elasticity problem, in which the unknown is a vector field **u** which, upon selecting a suitable coordinate frame, becomes **a pair** of scalar functions (u_1, u_2). How to build a finite element space that can approximate **u**? This is an example of one among many **multifield** problems in Engineering.

Multifield problems. These are problems in which there exist $p > 1$ coupled unknowns, $u_1, \dots, u_p$. Typical examples are thermoelasticity (temperature and displacement), thermal convection (temperature and fluid velocity and pressure), electrodynamics (electric and magnetic fields), etc. The solution is thus a p-tuple $(u_1, u_2, \dots, u_p)$, which will be organized as a column array for convenience, that belongs to the space

$$\mathscr{W} = \left\{ \underline{w} = (w_1, w_2, \dots, w_p)^T \mid w_1 \in \mathscr{W}_1,\ w_2 \in \mathscr{W}_2, \dots,\ w_p \in \mathscr{W}_p \right\}. \tag{4.41}$$

The space defined above is called the **product space** $\mathscr{W}_1 \times \mathscr{W}_2 \times \ldots \times \mathscr{W}_p$ of the individual solution spaces.