ME 335A
Finite Element Analysis
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Problems Set #3

Due Wednesday, April 19, 2023

This problem set contains two problems that involve some serious numerical calculations. Please utilize software to help you perform them. The calculations here are only pathways to exercise and learn some concepts, so please utilize a tool that will allow you to focus on the concepts and not on the calculations.

On Convergence (50)

As mentioned in class, approximating the solution of a boundary value problem involves the construction of a method to obtain numerical solutions (approximate solutions) that are as close as we want to the exact solution of the problem. Let's illustrate here two cases that illuminate the wonders and the perils that we may find in walking this path.

In this problem we will take advantage of Legendre polynomials over the interval $[0,1]$. For $n = 0, 1, \ldots$, the Legendre polynomial of order n can be written as

$$P_n(x) = \sqrt{2n+1}(-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-x)^k. \tag{1}$$

Here

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2}$$

We can also define their integrals

$$iP_n(x) = \int_0^x P_n(y)\, dy = \sqrt{2n+1}(-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{(-x)^{k+1}}{k+1}. \tag{3}$$

Legendre Polynomials satisfy the following "orthogonality" condition:

$$\int_0^1 P_n(x) P_m(x)\, dx = \delta_{mn}. \tag{4}$$

Here δ_{mn} is the Kronecker delta, whose values are

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases} \tag{5}$$

For this problem, we strongly suggest you use Matlab, Mathematica, or Maple to perform all integrals and calculations, as well as to plot the results.

In this exercise, we consider the following problem in its weak from:

Problem (Weak Form)**.** *Given $f\colon [0,1] \to \mathbb{R}$, find $u \in \mathcal{V} = \{w\colon [0,1] \to \mathbb{R}$ smooth $\mid w(0) = 0\}$ such that*

$$\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \tag{W}$$

for any $v \in \mathcal{V}$.

We set $f(x) = \cos(4\pi x)$, and will examine the use of a variational method to approximate the solution to this problem that uses variational equation (W).

1. (10) Is $u(x) = -\frac{\sin(2\pi x)^2}{8\pi^2}$ the exact solution to this problem ? Justify.

   Hint: It is convenient to replace in the weak form and integrate by parts to verify it is satisfied for any $v \in \mathcal{V}$. Alternatively, you can compute the Euler-Lagrange equations and see if $u(x)$ satisfies them.

2. Consider the set of functions $\mathcal{W}_h = \{w = \sum_{k=0}^{n} c_k i P_k(x) \mid (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}\}$. We will denote with ${}^n u_h$ the solution of the variational method for a given n.

   (a) (15) Define $\mathcal{V}_h, \mathcal{S}_h, a(\cdot,\cdot), \ell(\cdot), \eta_a$ and η_g. State also the equations that define the solution of the variational method, compute the entries in the stiffness matrix, and the entries in the load vector

   (b) (15) Compute ${}^1u_h$, ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^1u_h$, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

   (c) (5) Is ${}^n u_h$ "visually converging" to u as n grows?

3. Consider next the set of functions $\mathcal{W}_h = \{w = \sum_{k=2}^{n} c_k i P_k(x) \mid (c_2, \ldots, c_n) \in \mathbb{R}^{n+1}\}$ (notice that k begins at 2 instead of at 0).

   (a) (15) Compute ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

   (b) (5) Is ${}^n u_h$ "visually converging" to u as n grows? Do you think that your answer to this question will change if we keep increasing n beyond 7?

Euler-Lagrange Equations (30)

For this problem section "1.3.3 The Euler-Lagrange Equations" of the notes provides an explanation of the steps.

Consider the weak form: Find $u \in \mathcal{V} = \{u\colon (0,1) \to \mathbb{R}$ smooth $\mid u(0) = 0\}$ such that $a(u,w) = \ell(w)$ for all $w \in \mathcal{V}$, where

$$a(u,w) = \int_0^1 (w_{,x}u_{,x} + \lambda wu)\,dx + w(1)u(1)$$
$$\ell(w) = \int_0^1 wx^2\,dx + w(1),$$

and $\lambda > 0$.

1. (5) Is $a(u, v)$ a bilinear form? Justify.
2. (5) Is $a(u, v)$ symmetric? Justify.
3. (5) Is $\ell(v)$ a linear functional? Justify.
4. (10) Obtain the Euler-Lagrange equations.
5. (5) Identify natural and essential boundary conditions.

Your "First" Finite Element Approximation (45)

We want to construct a piecewise linear approximation to a function u that satisfies the variational equation $a(u, w) = \ell(w)$ for all $w \in \mathcal{V} = \{u : [0, 1] \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$, where

$$\begin{aligned} a(u, w) &= \int_0^1 (w_{,x}\, u_{,x} + \lambda w u)\, dx + w(1)u(1) \\ \ell(w) &= \int_0^1 w x^2\, dx + w(1), \end{aligned}$$

and $\lambda > 0$.

To this end, we will partition $[0, 1]$ into four equal intervals, and build a finite element approximation with continuous functions that are lineal polynomials over each interval.

1. (5) Identify the location of all the vertices in the mesh, and number them.
2. (5) Number and sketch all hat functions $\{N_a\}$ over $[0, 1]$.
3. (5) Let $\mathcal{W}_h = \text{span}(\{N_1, \dots, N_5\})$. What are the trial space $\mathcal{S}_h$ and test space $\mathcal{V}_h$ for the variational method, and the constrained and active index sets? Select $\overline{u}_h \in \mathcal{S}_h$.
4. (5) State the variational method for this problem.
5. (10) Compute the stiffness matrix, assuming that $\lambda = 2$.
6. (10) Compute the load vector.
7. (5) Find the approximate solution u_h.

On Integration by Parts (Optional, not graded)

A version of the fundamental lemma of calculus states that if $u \in C^1([a, b])$, then

$$\int_a^b u'(x)\, dx = u(b) - u(a). \tag{6}$$

A version of of the integration by parts theorem states that if $u, v \in C^1([a, b])$, then

$$\int_a^b u'(x)v(x)\, dx = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\, dx \tag{7}$$

We want to play with the theorem in situations in which one of the functions is discontinuous, to see how it fails. A discusion on this topic can be found in §1.1.4 in the notes. To this end, we will make use of the sign function

$$sgn(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0. \end{cases} \tag{8}$$

Consider the function $w(x) = 0$ for all $x \in \mathbb{R}$. The function $w(x)$ is equal to the derivative of $sgn(x)$ for every $x \neq 0$. The derivative of $sgn(x)$ is not defined at $x = 0$, but since $w(x) = sgn'(x)$ for all $x \neq 0$, $w(x)$ is called the *classical derivative* or *pointwise derivative* of $sgn(x)$. In the following, please use $w(x)$ for $sgn'(x)$ any time you need it.

Caution: Please do not use a relationship of the form $sgn'(x) = 2\delta(x)$, where $\delta(x)$ is a Dirac delta (if you are familiar with it), since $\delta(x)$ is not really a function.

In the following, we set $a = -1$, $b = 1$, and let $u(x) = sgn(x) + x$, and $v(x) = 1 + x$.

1. (5) Plot u, v. Is $u \in C^1([a, b])$? Is $v \in C^1([a, b])$?

2. (5) Let's first check if the fundamental lemma of calculus works for u. Is it true that

$$\int_{-1}^{1} u'(x)\, dx = u(1) - u(-1)? \tag{9}$$

3. (5) Is it true that

$$\int_{-1}^{1} u'(x)\, dx + [\![u(0)]\!] = u(1) - u(-1)? \tag{10}$$

   where $[\![u(0)]\!] = \lim_{x\to 0^+} u(x) - \lim_{x\to 0^-} u(x)$.

4. (10) Now let's check the integration by parts formula. Compute the left hand side and the right hand side of (7). Does formula (7) hold?

5. (5) Does it hold that

$$\int_a^b u'(x)v(x)\, dx + [\![u(0)]\!]v(0) = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\, dx? \tag{11}$$

6. (5) Next, consider the function $N(x) = 1 - |x|$, a type of functions we will see often along this quarter. Let $\mathcal{T} = \{w \in C^1([-1, 1]) \mid w(0) = 1, w(1) = w(-1) = 0\}$. What is the value of

$$\int_{-1}^{1} N'(x)v'(x)\, dx \tag{12}$$

   for any $v \in \mathcal{T}$?