ME 335A
Finite Element Analysis
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Problems Set #4

Due Wednesday, May 10, 2023

Constructing Some FE Spaces (35)

Consider a mesh of Lagrange P_k-elements (see Example 1.65 in the notes) with $n_{\text{el}} = 3$ elements of equal length in the interval $[0, 3]$. Elements are numbered consecutively from 1 to n_{el} from left to rigth (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

   (a) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

   (b) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

2. For each one of the following finite elements, write the local-to-global map so that the basis functions are continuous and have minimal support (each basis function should be zero in the maximum number of elements).

   (a) (5) For P_3-elements.

   (b) (5) For P_4-elements.

3. (5) Plot the basis functions that are non-zero in the second element for each of the cases in part 2 of this problem.

4. (5) For a problem with periodic boundary conditions of the form $u(0) = u(3)$, it is convenient to build a finite element space in which all functions in the space satisfy this periodicity constraint. If $k = 1$, write the required local-to-global map. What is the dimension of the finite element space?

5. (5) Let $f(x) = \sin(nx)$. Plot f, $\mathcal{I}f$ and $f - \mathcal{I}f$ when all elements in the mesh are P_3-elements, for $n = 2, 4, 6$.

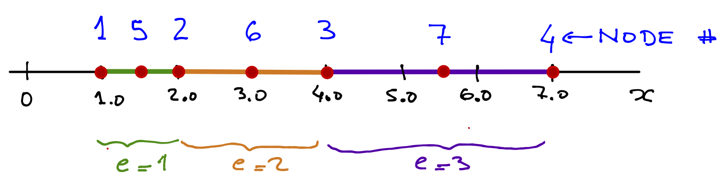
(Manual) Assembly (80)

Consider the domain $\Omega = [1, 7]$, and the convection-diffusion problem: Find $u\colon \Omega \to \mathbb{R}$ such that

$$-\varepsilon u_{,xx} + c u_{,x} = f \qquad x \in \mathring{\Omega}, \tag{1a}$$
$$-\varepsilon u_{,x}(7) = h, \tag{1b}$$
$$u(1) = g, \tag{1c}$$

where $\mathring{\Omega} = (1, 7)$ indicates the *interior* of the set Ω, $c \in \mathbb{R}$ is the convection velocity, $\varepsilon > 0$ is the diffusion coefficient, $h, g \in \mathbb{R}$ are boundary conditions, and $f\colon \Omega \to \mathbb{R}$ is a source. We would like to construct a finite element approximation of the solution of this problem.

1. (10) Find a variational equation according to the recipe in the notes, and identify natural and essential boundary conditions.
2. (5) Consider the nodes 1 to 7 with positions $\{1, 2, 4, 7, 1.5, 3, 5.5\}$, respectively; see figure. These nodes form P_2 elements 1,2, and 3, whose domains are $\Omega^1 = [1, 2]$, $\Omega^2 = [2, 4]$, $\Omega^3 = [4, 7]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.



3. (5) Plot the global basis functions N_3, N_6, and N_4.
4. (5) Let $\mathcal{P}^2$ denote the element space for element 2. Find a function $v_h \in \mathcal{P}^2$ that attains the value 1.5 at node 2, -1.5 at node 6, and 3 at node 3. Express it as a linear combination of shape functions in the element, and as an explicit function of x.
5. (5) Let $\mathcal{W}_h$ be the finite element space of continuous functions over the given mesh. Find a function $v_h \in \mathcal{W}_h$ that is equal to 2 on odd nodes, and to 3 on even nodes. Express it in terms of the global basis functions $\{N_A\}_{A=1,\ldots,7}$.
6. (10) State the finite element method for this problem using the variational equation obtained in part 1, identifying the spaces $\mathcal{V}_h$ and $\mathcal{S}_h$, and a basis for $\mathcal{V}_h$. Identify active and constrained indices and $\overline{u}_h \in \mathcal{S}_h$.
7. In the following, assume that $f(x) = x$, $h = -20$, $\varepsilon = 1$, $c = 1$, $g = 2$.
   (a) (10) Compute the element matrices of the three elements.
   (b) (10) Compute the element load vectors for the three elements. Do not forget the natural boundary condition.
   (c) (10) Assemble the stiffness matrix and the load vector.

(d) (10) Find the finite element solution, and plot it. If you want to compare, the exact solution of this problem is

$$u(x) = -12e^{-6} + 12e^{x-7} + \frac{1}{2}(1+x)^2.$$