CP-3.

The weak form of the isotropic linear Elasticity problem can be given as:

Given the domain Ω and the following spaces:

$$\mathcal{W} = \{ \vec{w} : \Omega \to \mathbb{R}^2 \mid \vec{w} \text{ is smooth vector field} \}$$

$$S = \{ \vec{w} \in \mathcal{W} \mid \vec{w} = \vec{g} \text{ on } \partial\Omega_D \}$$

$$\mathcal{V} = \{ \vec{v} \in \mathcal{W} \mid \vec{v} = 0 \text{ on } \partial\Omega_D \}$$

Find $\vec{w} \in S$ such that

$$a(\vec{w}, \vec{v}) = \ell(\vec{v}) \qquad \forall \vec{v} \in \mathcal{V}$$

where

$$a(\vec{w}, \vec{v}) = \int_\Omega \frac{E}{1+\nu} \left(\varepsilon(\nabla\vec{w}) : \varepsilon(\nabla\vec{v}) + \frac{\nu}{1-2\nu} \operatorname{div}\vec{w} \operatorname{div}\vec{v} \right)$$

$$\ell(\vec{v}) = \int_\Omega \vec{b} \cdot \vec{v} \, d\Omega + \int_{\partial\Omega_N} \vec{H} \cdot \vec{v} \, d\Omega.$$

1. See Matlab implementation

2.

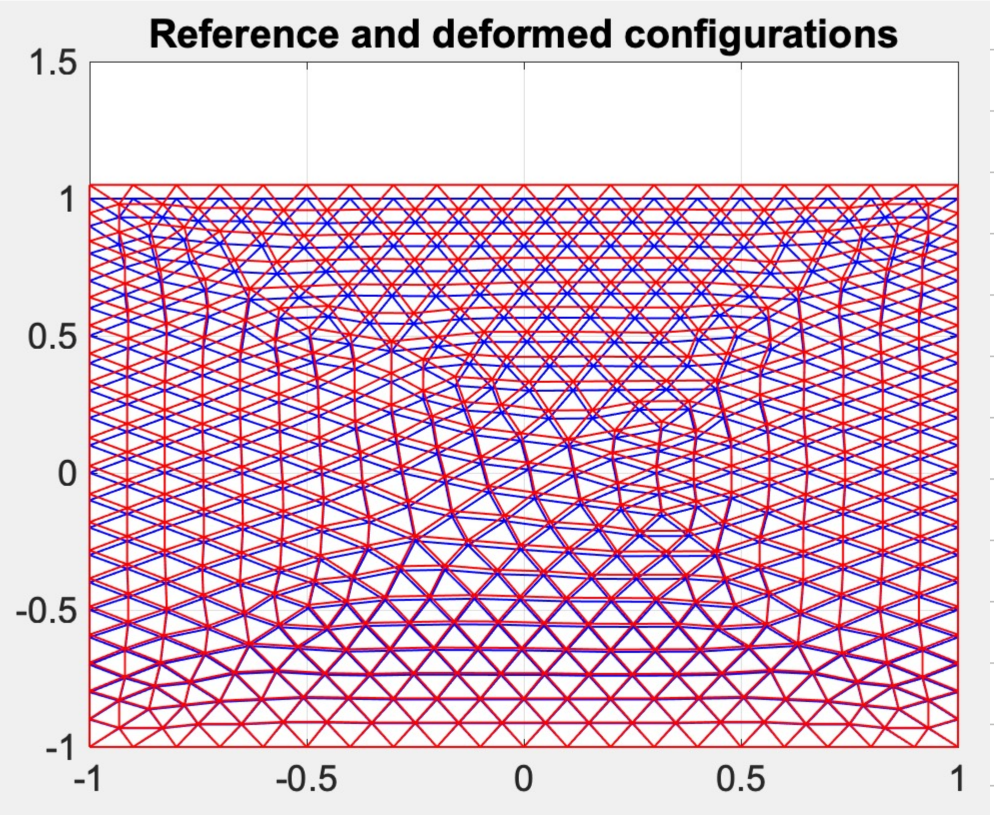
2. (10) To test your code, we will begin by setting `eccentricity`=0,so that we have a square with no holes. We will also set

$$\begin{aligned} \nu &= 0, \\ E &= 1\text{MPa} \\ g &= 0 \qquad \text{no gravity} \\ g_{2y} &= 0.05. \end{aligned}$$

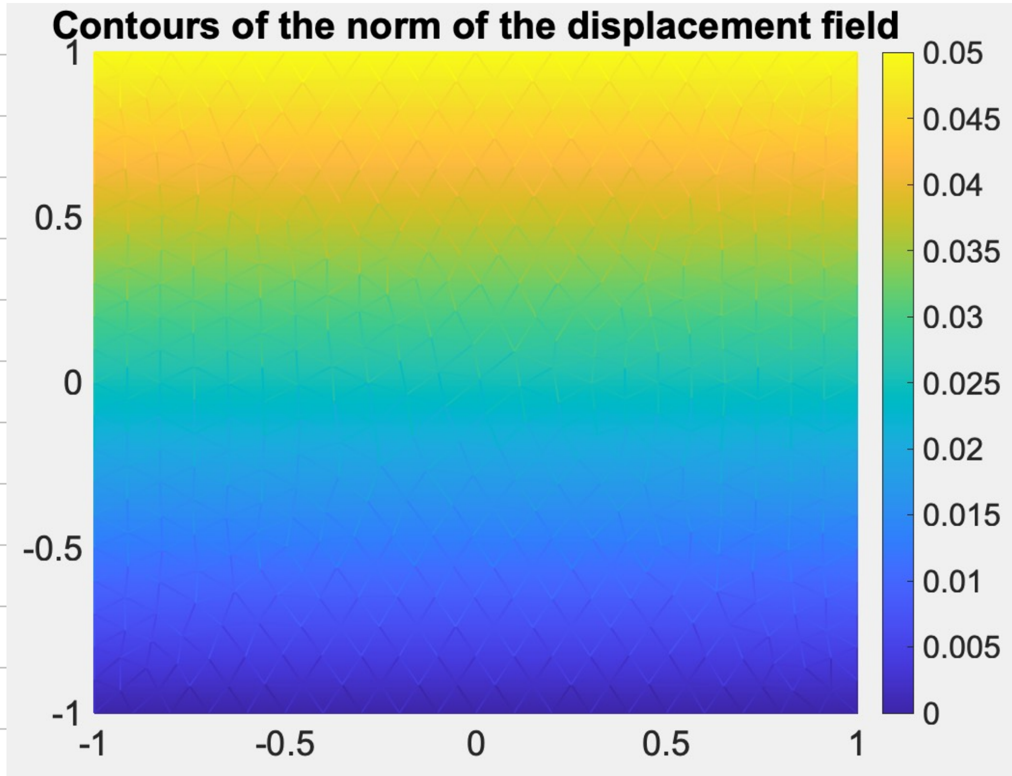
The exact solution of this problem is that both the strain and the stresses should constant in the domain, and equal to

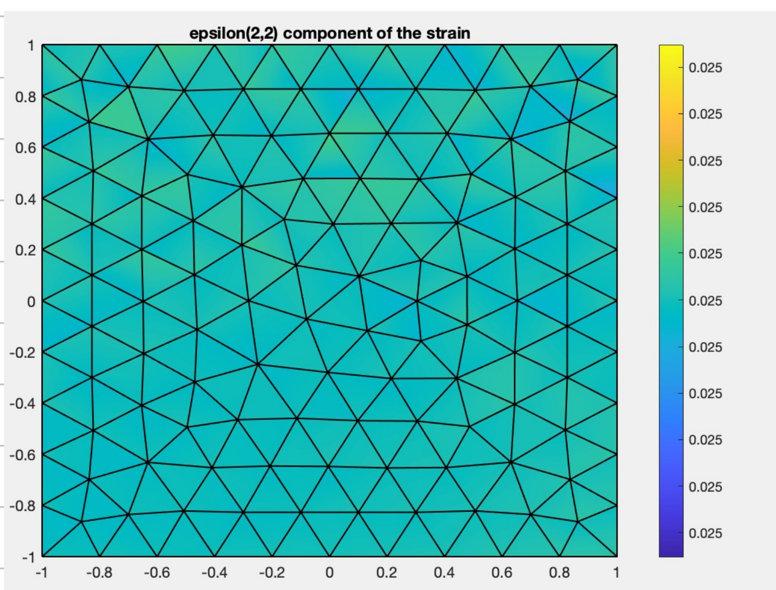
$$\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & 0.025 \end{bmatrix}, \qquad \sigma = \begin{bmatrix} 0 & 0 \\ 0 & 0.025\text{MPa} \end{bmatrix}.$$

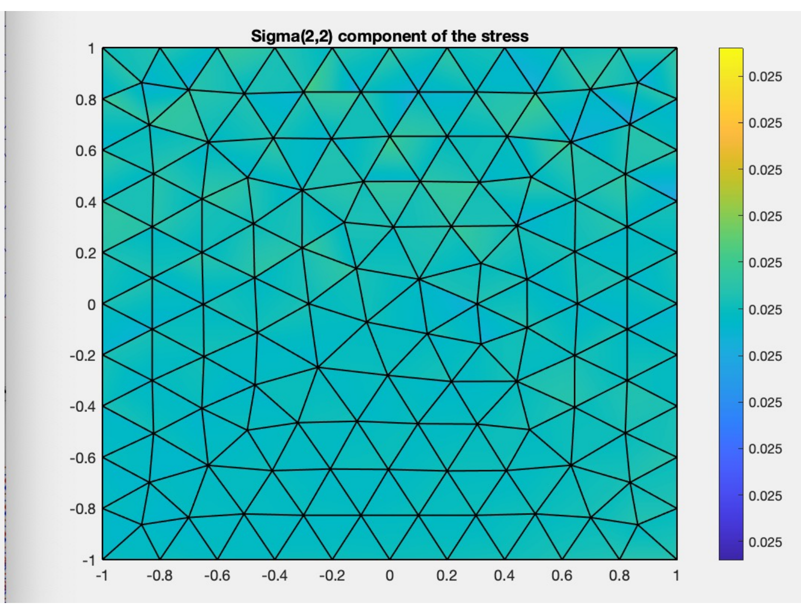
Please verify that you code can reproduce this deformation. This is similar to what is known as the *patch test* in the finite element literature. Show the plot of ε_{22} and σ_{22} over the mesh (as output from the code, with minor tweaks) for `HMax`=0.2, as well as a plot of the reference and deformed configurations, as output by the code.



Blue mesh is the reference and red mesh is the deformed one.





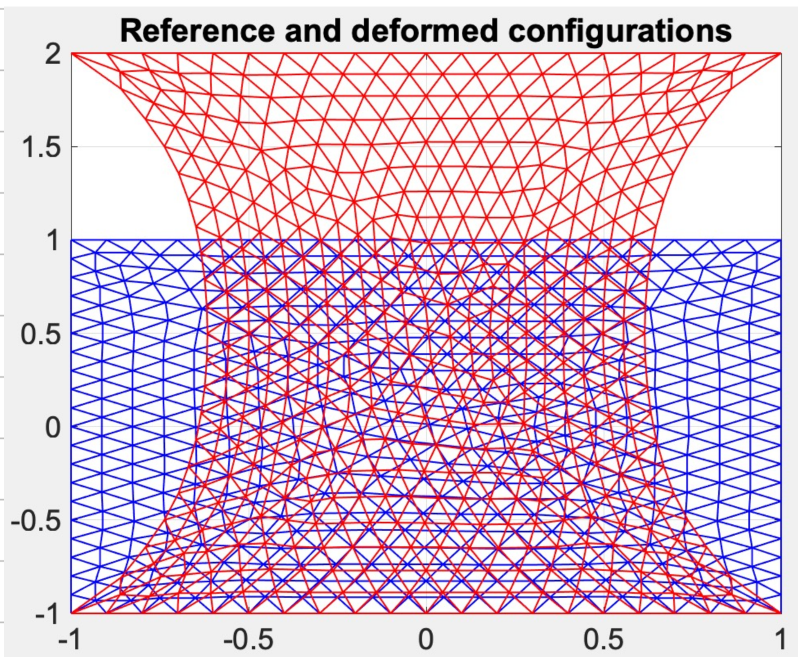


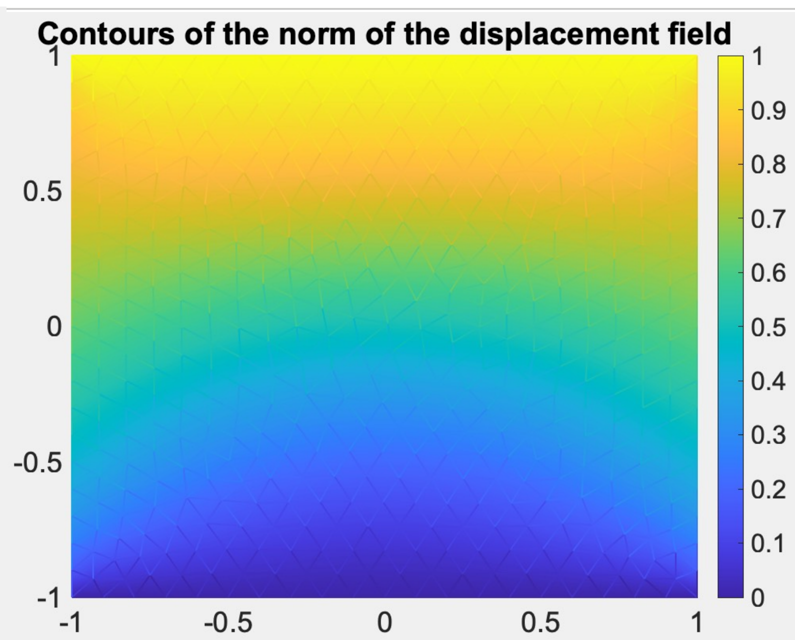
Clearly, our code passes the patch test, where constant σ_{22} and ε_{22} are accurately captured.

3. (10) Next, set

$$\nu = 0.45$$
$$E = 1\text{MPa}$$
$$g = 0 \qquad \text{no gravity}$$
$$g_{2y} = 1,$$

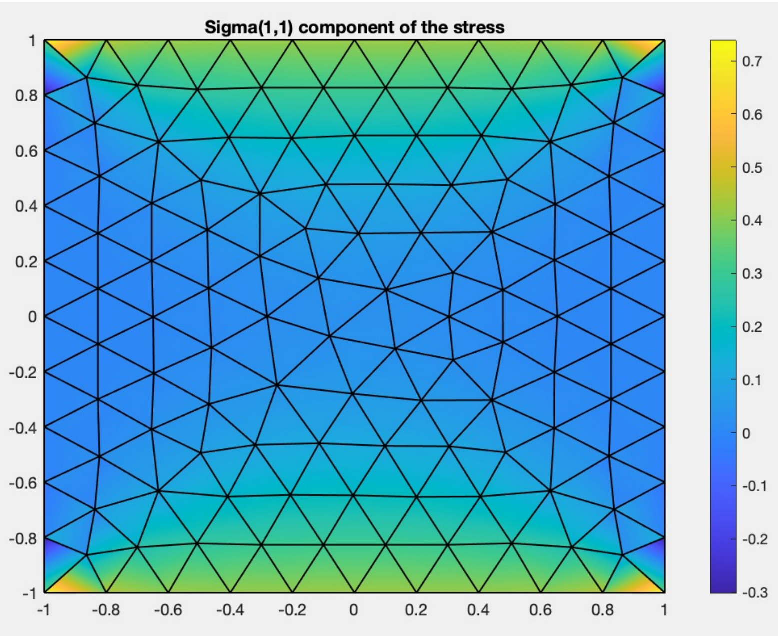
and `eccentricity` $= 0$. This will give you a deformation beyond small strains and small displacements, but it is a interesting to see. Please plot the reference and deformed configurations, and the stress components σ_{22}, σ_{12} and σ_{11}. Are the traction free boundary conditions on Γ_{h1} and Γ_{h3} well imposed? Comment on this.

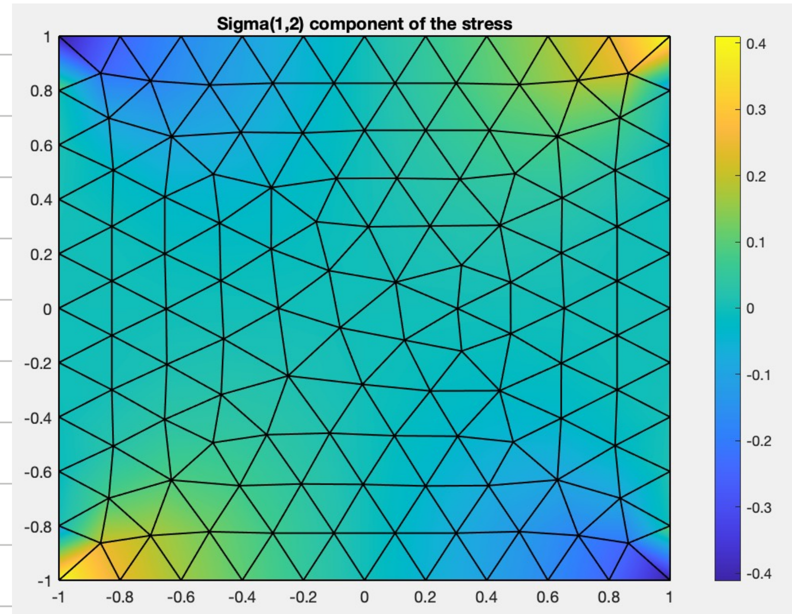


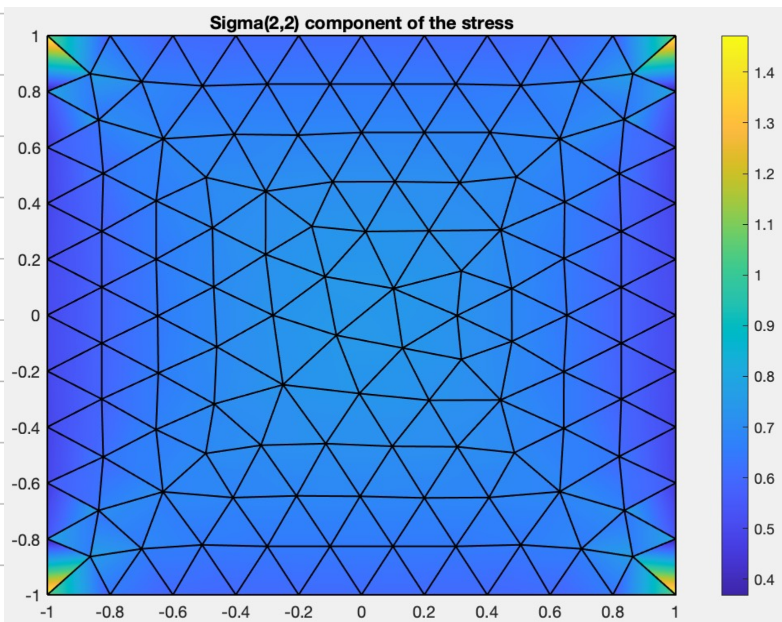


The left figure above show the deformed mesh after the displacement is imposed. We can see there the deformation is so large that the deformed shape is not close to the reference one

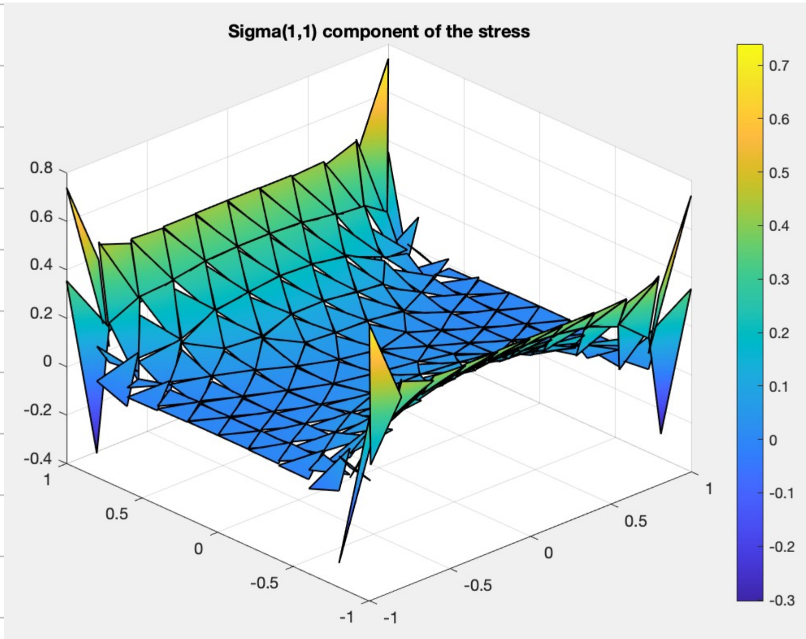
The right figure shows the displacement contour of the object, where the upper boundary has 1. m deforma

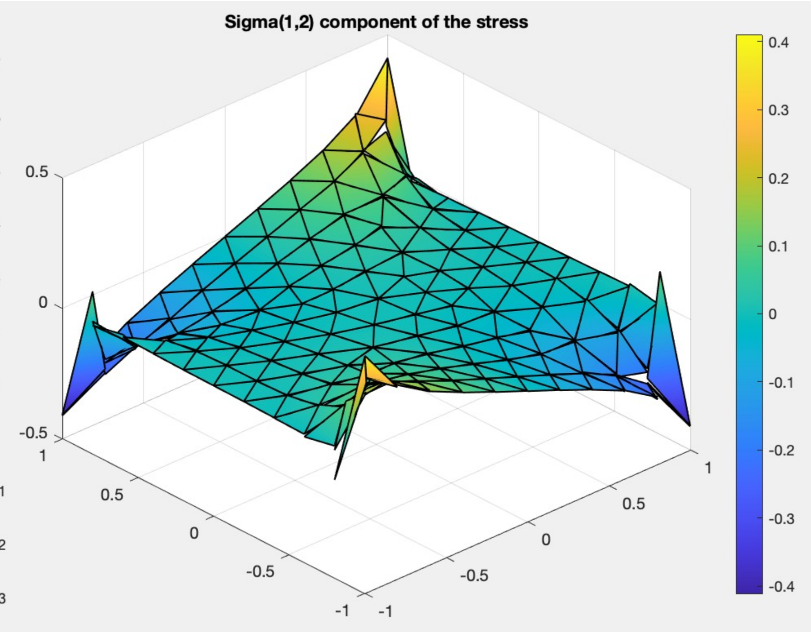


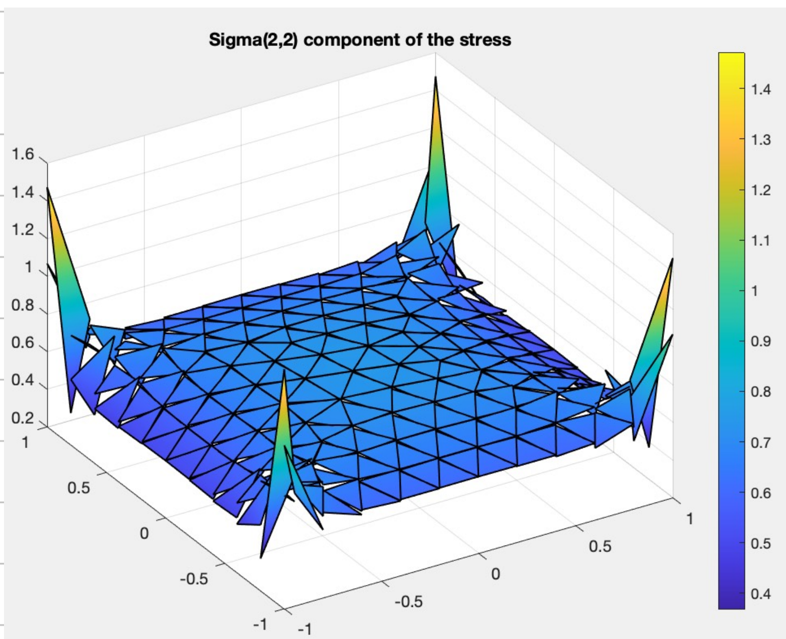




The figures above show the σ_{11}, σ_{12}, σ_{22} contours over the referenced mesh. The 3D plots of σ_{11}, σ_{12}, σ_{22} are shown to highlight the stresses distribution near the boundary, as follows:


Sigma(1,1) component of the stress
0.8
0.6
0.4
0.2
0
-0.2
-0.4
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3

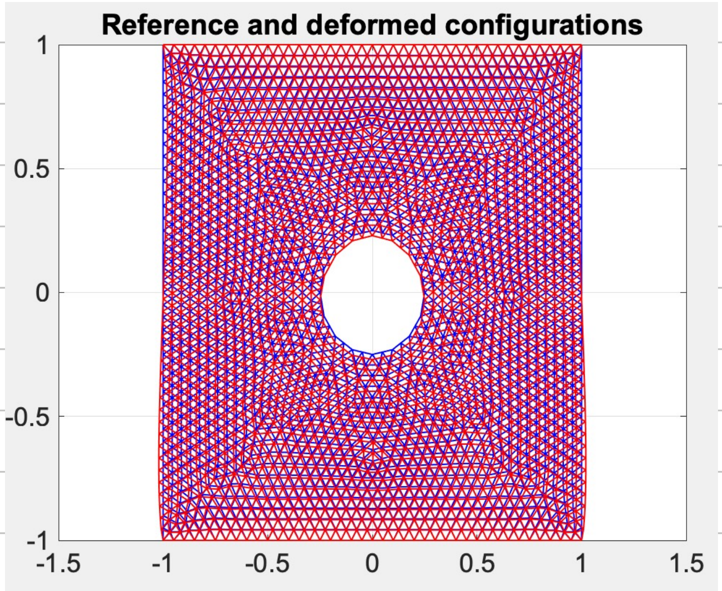

Sigma(1,2) component of the stress
0.5
0
-0.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4

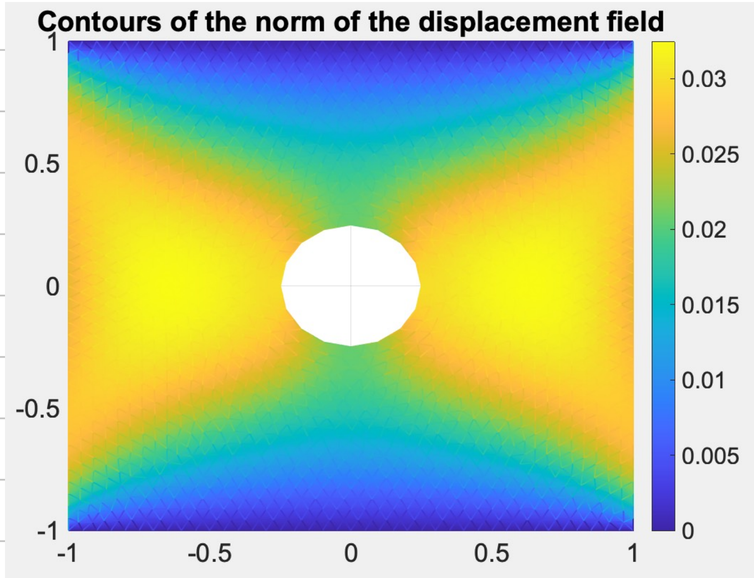

Sigma(2,2) component of the stress
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4

4. (10) Now we will set `eccentricity`=1, and

$$\nu = 0.45$$
$$E = 0.001\text{MPa}$$
$$g = 9.81\text{m/s}^2$$
$$g_{2y} = 0,$$

In this part it is important to pay attention to the units. To get a consistent result, you will need to use all SI units, since otherwise we may be adding Newtons with MegaNewtons. Plot the reference and deformed configurations for `HMax`=0.1. As a reference, the maximum displacement of a point on the left edge should be ≈ -0.15m.

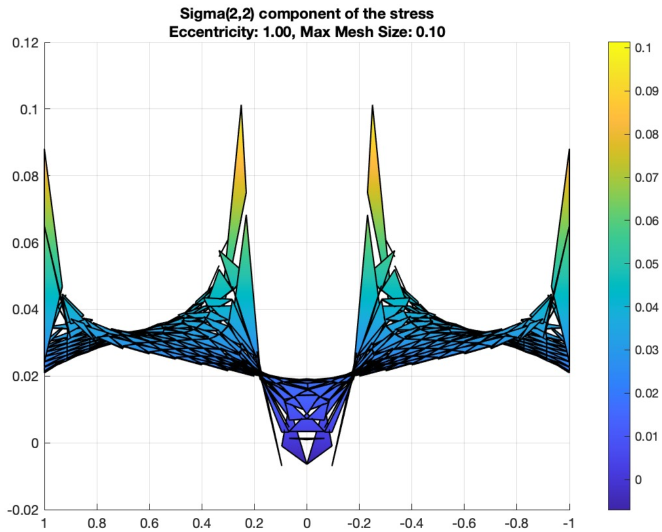


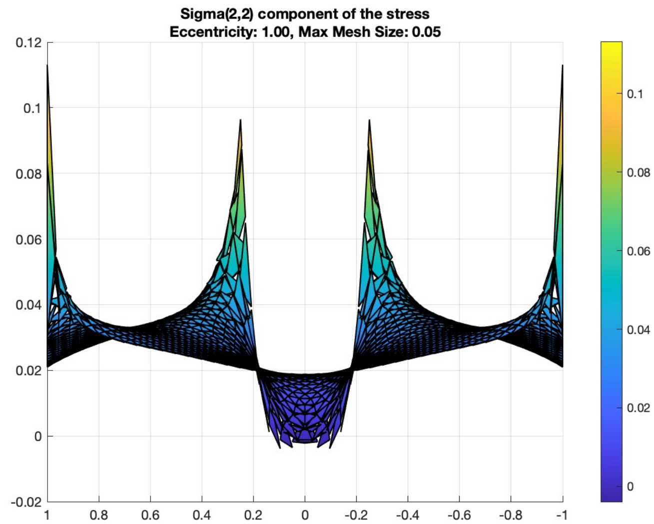


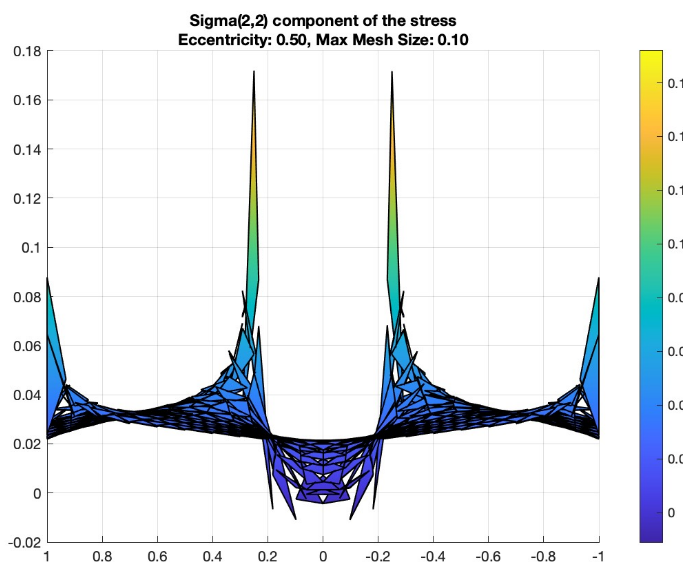
5. (10) Finally, we will study how the stresses change as we pull from a body with an elliptical hole, as the hole become flatter and flatter approximating a crack (Irwin's solution). To this end, set

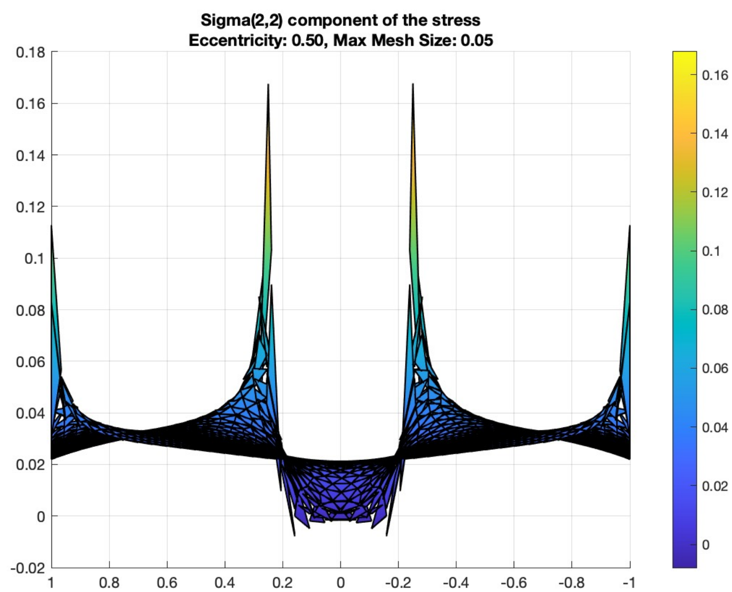
$$\nu = 0.45$$
$$E = 1\text{MPa}$$
$$g = 0$$
$$g_{2y} = 0.05,$$

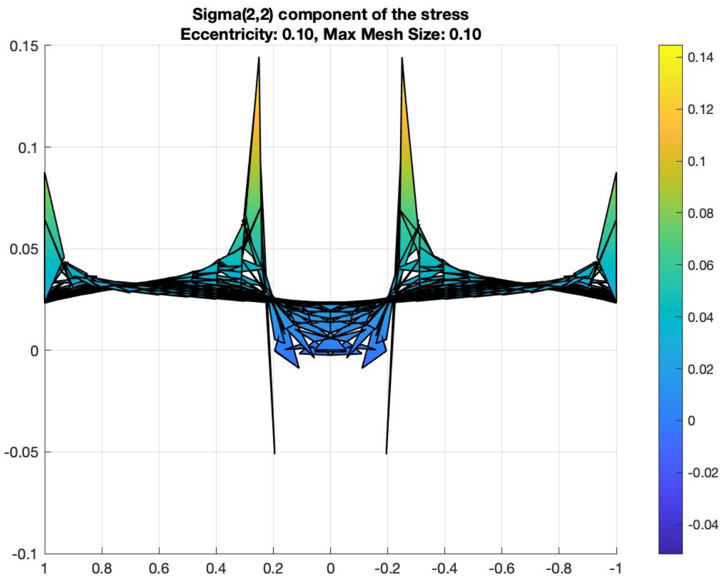
and `eccentricity`$\in \{1, 0.5, 0.1\}$. We will also consider `HMax`$\in \{0.1, 0.05\}$, to visually compare the convergence of solutions. To this end, for each case please plot σ_{22} with the x_2-axis coming out of the page, so that you can appreciate the height of the stress component (use `view([180, 0])` in MATLAB). Would you trust any of the solutions, or what would you do? Explain.

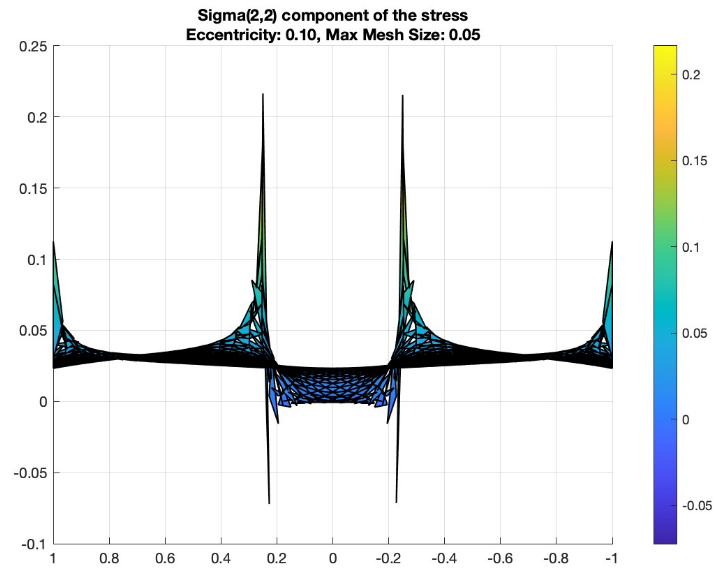












We have observed a higher eccentricity leads ~~to~~ a flatter crack, the stress concentration near the crack tip is growing increasingly.

According to Irwin's solution, σ_{22} near the tip of a crack is given as:

$$\sigma = K/\sqrt{2\pi r}.$$

where. K is the stress intensity factor, which depends on boundary conditions, crack geometry. K is constant for all cases considered here. r is the distance from the tip. So, when r approaches 0, we will have a singular solution at the tip.

This nature of Irwin solution causes high-gradient of stress near crack area, which can not be approximated properly by our P2-element.

As we refine the mesh, we will have better approximations. But, stress concentration will continue growing, as mesh refines.

$$\frac{dN_1}{dx_1} = \sum_{i=1}^{3} \frac{\partial N_1}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial x_1}$$

$$\frac{d\lambda_i}{dx_i} = \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_1} & \frac{\partial \lambda_2}{\partial x_1} & \frac{\partial \lambda_3}{\partial x_1} \\ \frac{\partial \lambda_1}{\partial x_2} & \frac{\partial \lambda_2}{\partial x_2} & \frac{\partial \lambda_3}{\partial x_2} \end{bmatrix}$$

$$dN_1 = \begin{bmatrix} \frac{\partial N_1}{\partial \lambda_1} & \frac{\partial N_1}{\partial \lambda_2} & \frac{\partial N_1}{\partial x_3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \lambda_1}{\partial x_1} & \frac{\partial \lambda_1}{\partial x_2} \\ \frac{\partial \lambda_2}{\partial x_1} & \frac{\partial \lambda_2}{\partial x_2} \\ \frac{\partial \lambda_3}{\partial x_1} & \frac{\partial \lambda_3}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x_1} + \frac{\partial N_1}{\partial x_1} + \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} + \frac{\partial N_1}{\partial x_2} + \frac{\partial N_1}{\partial x_2} \\ 0 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} N_1 & \cdots & N_6 \end{array}$$

$$\begin{bmatrix} \frac{\partial N_1}{\partial x_1} & \frac{\partial N_1}{\partial x_2} & \frac{\partial N_6}{\partial x_1} & \frac{\partial N_6}{\partial x_2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\varepsilon(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$= \left[\begin{matrix} \frac{\partial N_1}{\partial x_1} & \frac{1}{2}\frac{\partial N_1}{\partial x_2} \\ 0 & 0 \end{matrix} \middle| \begin{matrix} \frac{\partial N_2}{\partial x_1} & \frac{1}{2}\frac{\partial N_2}{\partial x_2} \\ 0 & 0 \end{matrix} \middle| \cdots \right.$$

ε_{11}