Chapter 2

Diffusion Problems in 2D

We now turn to consider diffusion problems in two dimensions, which govern heat conduction in solids, electrostatics and some mass transfer situations. We follow the same methodology as in Chapter 1, beginning with the partial differential equation of the mathematical problem and deriving a variational equation in the standard way, discussing the novelties brought by the higher dimensionality of the domain, and finally introducing the simplest finite element method to compute an approximate solution.

2.1 The Partial Differential Equation

Consider a general **diffusion equation**

$$-\operatorname{div}(K\nabla u) = f\ , \tag{2.1}$$

which should be satisfied by a function $u\colon \Omega \to \mathbb{R}$ in a domain $\Omega \subset \mathbb{R}^2$. It is convenient to denote the Cartesian coordinates by $x = (x_1, x_2)^T$, and the partial derivatives of a function $u(x_1, x_2)$ by $\partial_1 u$ and $\partial_2 u$. The components of a vector v in a basis to be specified will be likewise denoted by $(v_1, v_2)^T$.

Above, div is the divergence operator which applied to a vector field $v\colon \Omega \to \mathbb{R}^2$ yields

$$\operatorname{div} v = \partial_1 v_1 + \partial_2 v_2,$$

K is a positive-definite symmetric matrix (all eigenvalues are positive), ∇u is the gradient vector

$$\nabla u = (\partial_1 u\ , \partial_2 u\ )^T$$

and $f\colon \Omega \to \mathbb{R}$ is a source density (per unit area). When u represents the temperature of a solid, (2.1) is known as the **heat conduction equation**.

The diffusion equation (2.1) can be recast as $\operatorname{div} J = f$, where

$$J = -K\nabla u\ , \tag{2.2}$$

is the **diffusive flux vector** (or **heat flux vector** in the thermal setting). In the context of the heat conduction equation, relationship (2.2) is called **Fourier's law**.

In the context of mass transport, u is the concentration of mass and (2.2) is called **Fick's law**.

All the previous expressions have been written in operator form, which is a concise way of writing formulae involving partial derivatives. They can of course be rewritten in (Cartesian) coordinates, as

$$-\sum_{i=1}^{2} \partial_i \left(\sum_{j=1}^{2} K_{ij} \partial_j u \right) = f\,, \tag{2.3}$$

$$J_i = -\sum_{j=1}^{2} K_{ij} \partial_j u. \tag{2.4}$$

Notice that we have not adopted notation that indicates if a symbol is a scalar, a vector, or a matrix; the nature of a symbol will be interpreted from the context. For additional background material on the divergence operator, including its definition for domains in three dimensions, we refer the reader to [5, Ch. 2].

Example 2.1 (The Poisson equation) When K is a multiple of the identity matrix,

$$K(x) = k(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we say that the diffusive medium is **isotropic**. If we further assume that k is independent of x we have the case in which (2.1) is a **Poisson equation**. In such a case the expressions simplify considerably. Since $K_{ij} = k\delta_{ij}$, we have

$$K = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \tag{2.5}$$

$$J = -K\nabla u = -k\nabla u = \begin{pmatrix} -k\partial_1 u \\ -k\partial_2 u \end{pmatrix}, \tag{2.6}$$

and thus (2.1) reads

$$\partial_{11}^2 u + \partial_{22}^2 u = -\frac{f}{k}\,. \tag{2.7}$$

The notation of the second partial derivatives is

$$\partial_{ij}^2 u = \frac{\partial^2 u}{\partial x_i \partial x_j}$$

and one can recognize in the left-hand side of (2.7) the **Laplacian** of u, namely

$$\Delta u = \left(\partial_{11}^2 + \partial_{22}^2\right) u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\,. \tag{2.8}$$

There exist many analytical solutions of $\Delta u = -f/k$ even if $f = c$ is a constant. In the case $c = 0$ the following functions are solutions for any α, β and γ real numbers,

$$u(x_1, x_2) = \alpha + \beta x_1 + \gamma x_2\,. \tag{2.9}$$

In other words, all *affine functions* are solutions to $\Delta u = 0$. This is also the case in 1D, where all functions of the form $u(x) = \alpha + \beta x$ are solutions to $u'' = 0$. There is a crucial difference though: In 1D the affine functions are *all possible solutions* to $u'' = 0$, while in 2D there are infinitely many linearly independent functions that satisfy $\Delta u = 0$, also called **harmonic functions**. For example, the function

$$u(x_1, x_2) = \ln\left((x_1 - X_1)^2 + (x_2 - X_2)^2\right) \tag{2.10}$$

defined for all $x \neq X$ satisfies $\Delta u = 0$ in $\mathbb{R}^2 \setminus X$, for *any choice of* X. These functions are smooth in Ω if $X \notin \overline{\Omega}$.[1] Two such functions with different choices of X are linearly independent, so that the set of harmonic functions has infinite dimensions.

Example 2.2 (The Elastic Membrane) The diffusion equation (2.1) also appears when modeling small deformations of planar elastic membranes under tension and subjected to loads normal to the undeformed surface [9, §93], see Fig. 2.1. Specifically, in this case the membrane occupies a domain $\Omega \subset \mathbb{R}^2$, and the unknown function u is the vertical displacement normal to the membrane. The membrane is "pulled" with a force per unit length of magnitude T normal to its boundary, and pressurized with a pressure p along the vertical direction, while the boundary or part of the boundary $\partial\Omega$ is prevented from moving. The problem consists in determining the deformed shape of the membrane $(x, y, u(x, y)) \in \mathbb{R}^3$, where $(x, y) \in \Omega$. The equation that defines u is

$$p = -\mathrm{div}(T\nabla u), \tag{2.11}$$

Example 2.3 (Torsion of a Prismatic Bar) Consider a prismatic bar B of cross-section Ω and length L, so that the bar occupies the set of points $B = \Omega \times [0, L]$ in $\mathbb{R}^3$, see Fig. 2.2. The top surface of the bar, $\Omega \times \{L\}$, is rotated by an angle θL relative to the bottom surface $\Omega \times \{0\}$. The bar is made of a linear elastic material, and the shear stress components (τ_1, τ_2) on any cross section $\Omega \times \{x_3\}$, $x_3 \in [0, L]$, is computed from the so-called stress function $\phi\colon \Omega \to \mathbb{R}$ [9, §90], which does not depend on x_3, and satisfies the Poisson equation

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = -2\mu\theta$$

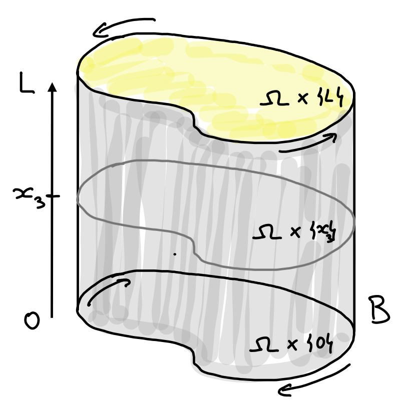

Figure 2.2 Torsion of a prismatic bar B with cross section Ω.

[1] The set $\overline{\Omega}$ is the **closure** of Ω, or the set of points in Ω and all those points that can be reached as limits of sequences of points in Ω. We have seen this concept for one-dimensional domains in §1, and in the discussion on the sparsity of the stiffness matrix in §1.4.4.

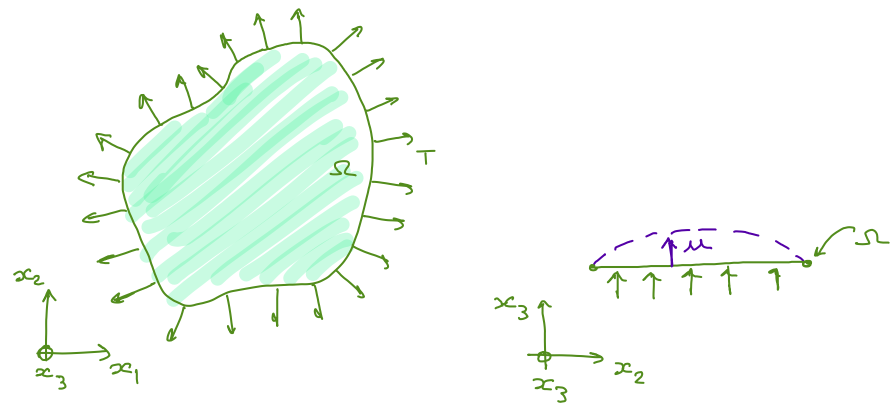

Figure 2.1 A planar elastic membrane under an isotropic tension T, pressurized from the bottom. The shape of the elastic membrane is shown on the left, as seen from above, while a view parallel to the membrane, on the right, shows the pressure loading p at the bottom, and the vertical displacement u of each point in the membrane that defines its deformed shape.

in Ω, where μ is the shear modulus. The shear stress components follow from ϕ as

$$\begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} = \begin{bmatrix} \partial_2\phi \\ -\partial_1\phi \end{bmatrix}.$$

Two-dimensional domains. The diffusion equation (2.1) is assumed to hold at all points x of the **two-dimensional domain** Ω. In 1D the domains could be intervals, or at most groups of intervals. The diversity of domains in 2D is much larger. The shape of the domain usually comes from the geometry of the physical system under study. The theory and methods we describe below hold for **bounded** domains that do not have cusps or cracks. A visual guide of the **admissible domains** is given in Fig. 2.3. To avoid unnecessary technical discussions at this stage of the learning, however, we restrict our attention to **polygonal domains**, with the possibility of them having one or several polygonal holes. The **boundary** of a domain Ω, denoted $\partial\Omega$, is formed by one or more polygonal lines. We will have an opportunity to consider curved domains later.

Boundary conditions. Boundary conditions are necessary to uniquely identify a solution of (2.1). A salient feature of elliptic second-order problems is that at all points in $\partial\Omega$ one (and only one) piece of information is needed about the solution u. We assume that $\partial\Omega$ is decomposed into two parts, depending on the available boundary information.

- If at $x \in \partial\Omega$ we know the **value** of u, we say that x belongs to the **Dirichlet boundary** $\partial\Omega_D$.
- On the other hand, if at x we know the value of the **normal flux** $J \cdot \check{n}$, with $\check{n}$ the **exterior unit normal to** $\partial\Omega$ at x, we say that x belongs to the **Neumann boundary** $\partial\Omega_N$.

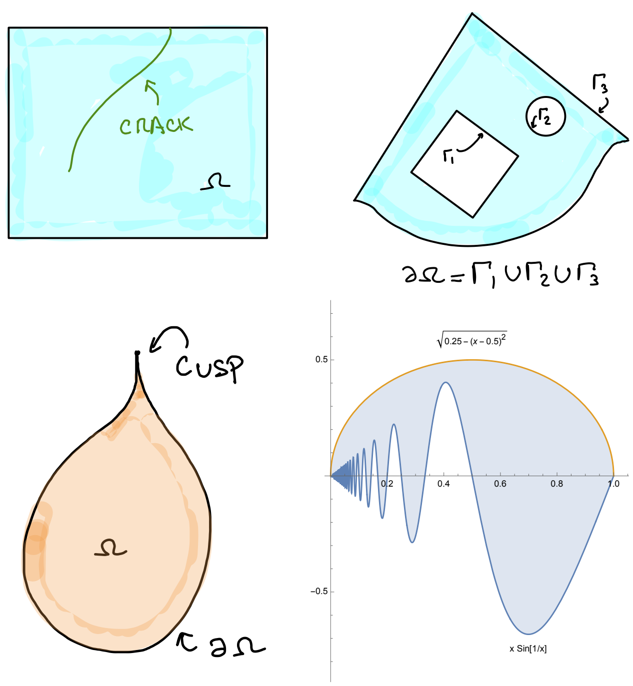

Figure 2.3 Domains that we will consider can contain smooth cracks, holes, and have a boundary made of a finite number of smooth curves, each of finite length, such as polygonal domains, as shown in the top row. Domains that we will not consider have boundaries that form cusps (vertices with the same tangent line on each side), have boundaries that have infinite length, or that do not have normal to the boundary defined at almost every point of the boundary (e.g., by replacing the function $x \sin(1/x)$ for a fractal curve, such as the Weierstrass function [4]), as illustrated in the bottom row. Typical engineering domains are idealized to be of the type in the top row.

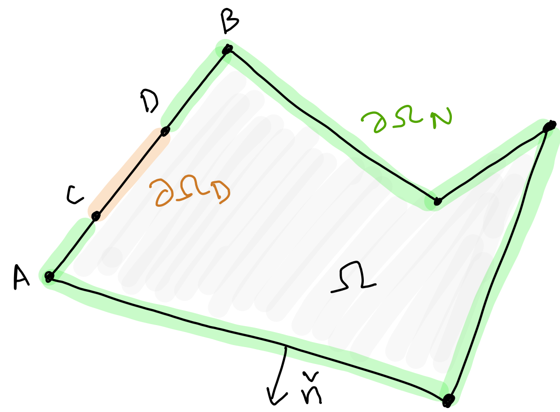

Figure 2.4 Sketch of the domain for Problem 2.1

- There exist other possibilities, such as **Robin boundary conditions**, in which we know the value of a linear combination of the function and the normal flux.

We assume that both $\partial\Omega_D$ and $\partial\Omega_N$ consist of a subset of edges of $\partial\Omega$, which is a polygonal line (or several).

The problem whose solution we aim to approximate reads as follows.

Problem 2.1 (Strong Form of the 2D Diffusion Problem)**.** *Given the coefficients K and f as functions of $x \in \Omega$, and given a real function g defined in $\partial\Omega_D$ and another real function H defined in $\partial\Omega_N$, determine a function $u : \Omega \to \mathbb{R}$ satisfying*

$$-div\,(K\nabla u) = f(x) \qquad \forall\, x \in \Omega \tag{2.12a}$$

$$u = g \qquad \forall\, x \in \partial\Omega_D \tag{2.12b}$$

$$(K\nabla u)\cdot \check{n} = H \qquad \forall\, x \in \partial\Omega_N \tag{2.12c}$$

Problem 2.1 admits one and only one solution under sufficient smoothness of the data (in particular, g must be continuous) plus the two essential hypotheses:

H1) the thermal conductivity K is everywhere a bounded, positive definite matrix, with all eigenvalues greater than some $\kappa_0 > 0$, and

H2) the length of $\partial\Omega_D$ is strictly positive.

If K is a multiple of the identity matrix, i.e., $K(x) = k(x)\mathbf{I}_{2\times 2}$, then H1 requires that $k(x) > \kappa_0 > 0$ for all $x \in \Omega$. Concerning H2, it requires that u is known not just at a point or a finite set of points of $\partial\Omega$, the condition $u = g$ must hold all along the full length of an edge of $\partial\Omega$. Notice that if $\partial\Omega_D$ is just a segment $\overline{CD}$ within a larger edge $\overline{AB}$, it is possible to redefine the polygon incorporating C and D as vertices, so that $\partial\Omega_D$ is a full edge, c.f. Fig. 2.4.

Example 2.4 (A uniformly heated rod) Consider the circular cross section of a homogeneous and isotropic rod, in which heat is generated uniformly at rate f. The domain is thus $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < R^2\}$.

Let $g \in \mathbb{R}$ be the temperature at the rod's surface, assumed uniform. We are interested in the temperature field **inside** the rod.

The corresponding differential equation is

$$\Delta u = -\frac{f}{k},$$

so, we are looking for $u(x, y)$, satisfying $u = g$ on the circle $x_1^2 + x_2^2 = R^2$ and having (constant) Laplacian equal to $-f/k$ in the enclosed region. The Dirichlet boundary $\partial\Omega_D$ is the whole boundary $\partial\Omega$ of the domain, and thus $\partial\Omega_N$ is empty. The constant k is assumed positive, so that both H1 and H2 are satisfied.

It is easy to check that

$$u(x, y) = g - \frac{f}{4k}\left(x_1^2 + x_2^2 - R^2\right)$$

satisfies these conditions and is thus the unique solution to the boundary value problem.

By differentiating u we can compute the heat flux

$$J = -k\nabla u = -k\left(-\frac{f x_1}{2k}, -\frac{f x_2}{2k}\right)^T = \frac{f}{2}x$$

and see that it is constant along the boundary circle, pointing outwards and with magnitude $fR/2$.

The maximum temperature takes place at the center (if $f > 0$), with value $g + fR^2/(4k)$.

Important: The same solution $u(x_1, x_2)$ also satisfies the **Neumann** boundary value problem, in which the **normal flux** is specified as

$$(k\nabla u) \cdot \check{n} = -\frac{fR}{2}$$

over the boundary circle. However, this problem **does not satisfy H2** (because $\partial\Omega_D$ is empty) and in facts admits not just u as solution but also any function $v = u + C$, with C an arbitrary real constant.

2.2 A Variational Equation

As already discussed in the 1D case, the finite element method is built upon a *variational equation* for the problem under consideration, which at present is Problem 2.1. Getting to a variational formulation often involves integration by parts, so let us recall a useful result.

Theorem 2.1. **(Integration by parts in 2D or 3D)** *Let w be a smooth vector field in Ω (an admissible domain), and v a smooth scalar function. Then,*

$$\int_\Omega v \operatorname{div} w \; d\Omega = \int_{\partial\Omega} v\, w \cdot \check{n} \; d\Gamma - \int_\Omega w \cdot \nabla v \; d\Omega \,. \tag{2.13}$$

It is also convenient to illustrate how the divergence theorem, (2.13) is written in Cartesian components in $\mathbb{R}^d$, where d is the dimension of the space. The components of w are $(w_1, \ldots, w_d)$. Then:

$$\sum_{i=1}^{d} \left[\int_\Omega v \partial_i w_i \; d\Omega \right] = \sum_{i=1}^{d} \left[\int_{\partial\Omega} v\, w_i \check{n}_i \; d\Gamma - \int_\Omega w_i \partial_i v \; d\Omega. \right] \tag{2.14}$$

Then, a useful mnemonic rule to apply integration by parts in higher dimensions is to move the "index" i from indicating a partial derivative of the component w_i to the partial derivative of the function v, or viceversa.

Example 2.5 Let's consider an example of the use of the theorem. Let $\Omega = [0,1]^3$ and $w(x) = x_i e_i$, where x_i are the components of the point x in a Cartesian basis e_i, and $v(x) = 1$, a constant function. Then, div $w = 3$, and hence

$$\int_\Omega \operatorname{div} \, w \; dV = 3. \tag{2.15}$$

Alternatively, let's compute

$$\int_{\partial\Omega} w \cdot \check{n} \; d\Gamma - \int_\Omega w \cdot \nabla v \; d\Omega. \tag{2.16}$$

We have that in the face where $x_1 = 0$

$$\int_{\{x_1=0\}\cap\partial\Omega} w \cdot \check{n} \; d\Gamma = \int_{\{x_1=0\}\cap\partial\Omega} (x_2 e_2 + x_3 e_3) \cdot (-e_1) \; d\Gamma = 0, \tag{2.17}$$

and the same happens in the faces defined by $x_2 = 0$ and $x_3 = 0$. Alternatively, when $x_1 = 1$, we have

$$\int_{\{x_1=1\}\cap\partial\Omega} w \cdot \check{n} \; d\Gamma = \int_{\{x_1=0\}\cap\partial\Omega} (e_1 + x_2 e_2 + x_3 e_3) \cdot e_1 \; d\Gamma = 1, \tag{2.18}$$

and the same thing happens in the faces defined by $x_2 = 1$ and $x_3 = 1$. Since $\nabla v = 0$, we verified that

$$\int_{\partial\Omega} w \cdot \check{n} \; d\Gamma - \int_\Omega w \cdot \nabla v \; d\Omega = 3, \tag{2.19}$$

as the theorem states.

Now, applying the same recipe as in 1D, §1.1.2.3, we multiply the differential equation (2.12a) by a smooth $v : \Omega \to \mathbb{R}$ and integrate over Ω, to get

$$-\int_\Omega \operatorname{div}(K\nabla u)\, v\, d\Omega = \int_\Omega f\, v\, d\Omega\,.$$

Using (2.13) with $w = K\nabla u$ for the left-hand side and decomposing the integral over $\partial\Omega$ into the sum of $\int_{\partial\Omega_D}$ and $\int_{\partial\Omega_N}$, we arrive to

$$\int_\Omega (K\nabla u)\cdot\nabla v\, d\Omega = \int_\Omega f\, v\, d\Omega + \int_{\partial\Omega_N} v\, K\nabla u\cdot\check{n}\, d\Gamma + \int_{\partial\Omega_D} v\, K\nabla u\cdot\check{n}\, d\Gamma\,. \quad (2.20)$$

In the integral over $\partial\Omega_N$, $K\nabla u\cdot\check{n}$ can be replaced by H, since (2.12c) holds. This makes Neumann boundary conditions to be **natural** boundary conditions. On the other hand, in the integral over $\partial\Omega_D$ we have no way to know $K\nabla u\cdot\check{n}$, so we require $v = 0$ there. The **test space** is then given by

$$\mathscr{V} = \{v\colon \Omega \to \mathbb{R} \text{ smooth } \mid v(x) = 0 \text{ for all } x \in \partial\Omega_D\}\,, \quad (2.21a)$$

so that the last integral in (2.20) is zero.

The solution u, which is assumed to be smooth, thus satisfies the following variational equation:

$$a(u, v) = \ell(v) \qquad \forall v \in \mathscr{V}\,, \quad (2.21b)$$

where the bilinear and linear forms are given by

$$\begin{aligned} a(u, v) &= \int_\Omega (K\nabla u)\cdot\nabla v\, d\Omega\,, && (2.21c)\\ \ell(v) &= \int_\Omega f\, v\, d\Omega + \int_{\partial\Omega_N} H\, v\, d\Gamma\,. && (2.21d)\end{aligned}$$

For completeness, the weak form of the problem reads

Problem 2.2. *(Weak Form of the 2D Diffusion Problem) Let*

$$\mathscr{S} = \left\{v\colon \Omega \to \mathbb{R} \textit{ smooth } \mid v(x) = g(x) \textit{ for all } x \in \partial\Omega_D\right\}\,. \quad (2.22)$$

Find $u \in \mathscr{S}$ such that $a(u, v) = \ell(v)$ for all $v \in \mathscr{V}$.

As you can see, the variational equation involves just first-order derivatives (the gradient ∇), but instead of having the pointwise requirement that $\operatorname{div}(K\nabla u) + f = 0$ at all points we have integral expressions in two dimensions that must hold for all $v \in \mathscr{V}$.

2.2.1 Other Variational Equations

Just as we did in §1.1.2.4, we can obtain a new variational equation that the solution u satisfies through linear combinations of other variational equations it satisfies. Let's see this in this case.

Example 2.6 Nitsche's Method. The solution u of Problem 2.1 satisfies variational equation (2.20) for any $v \in \mathscr{V} = \{w \colon \Omega \to \mathbb{R} \text{ smooth}\}$, and after replacing $K\nabla u \cdot \check{n}$ by H, it reads

$$\int_\Omega (K\nabla u) \cdot \nabla v \, d\Omega - \int_{\partial\Omega_D} vK\nabla u \cdot \check{n} \, d\Gamma = \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} vH \, d\Gamma. \tag{2.23a}$$

Additionally, u also satisfies the following variational equations

$$\int_{\partial\Omega_D} (g-u)K\nabla v \cdot \check{n} \, d\Gamma = 0 \tag{2.23b}$$

$$\int_{\partial\Omega_D} \mu(u-g) v \, d\Gamma = 0. \tag{2.23c}$$

for all $v \in \mathscr{V}$, where $\mu > 0$ is a positive real number. Adding the three equations in (2.23), we obtain the following variational equation that u also satisfies

$$\begin{aligned}\int_\Omega (K\nabla u) \cdot \nabla v \, d\Omega - \int_{\partial\Omega_D} (vK\nabla u + uK\nabla v) \cdot \check{n} \, d\Gamma + \int_{\partial\Omega_D} \mu u v \, d\Gamma \\ = \int_\Omega f v \, d\Omega + \int_{\partial\Omega_N} vH \, d\Gamma - \int_{\partial\Omega_D} gK\nabla v \cdot \check{n} \, d\Gamma + \int_{\partial\Omega_D} \mu g v \, d\Gamma\end{aligned} \tag{2.24}$$

for all $v \in \mathscr{V}$. Computing the Euler-Lagrange equations would show that both the Dirichlet and Neumann boundary conditions are natural boundary conditions for this variational equation.

2.3 Variational Numerical Methods

As in the 1D case, a linear variational method for finding an approximation to the exact solution u consists of finding u_h that is the solution to Problem 1.2. There are, however, significant differences with the one-dimensional case:

- The spaces $\mathscr{S}_h$ and $\mathscr{V}_h$ must now be composed of functions that take values over a two-dimensional domain.
- The domain boundary is a closed line, which we assumed to be a polygon for simplicity. The construction of the basis of $\mathscr{V}_h$ must ensure that the functions are zero on the boundary $\partial\Omega_D$.
- As we will have the chance to see, a variational method based on variational equation 2.21 is *consistent* if functions in the test space $\mathscr{V}_h$ are continuous. Because $\mathscr{V}_h$ is the direction of $\mathscr{S}_h$, we will set $\mathscr{S}_h$ and $\mathscr{W}_h$ to be spaces of continuous functions. So, if the domain is subdivided into element domains, continuity of the global basis functions between elements must be enforced. This continuity, contrary to the 1D case, must hold not just at the nodes but along all edges of the subdivision.

All functions in $\mathscr{S}_h$ are obtained as $\overline{u}_h + v_h$ for all $v_h \in \mathscr{V}_h$. If all functions in $\mathscr{V}_h$ are continuous, and $\overline{u}_h$ is discontinuous somewhere, then *all* functions in $\mathscr{S}_h$ are discontinuous. While it would be possible to construct approximations to the exact solution u in this way, it is not necessary nor convenient at this stage.

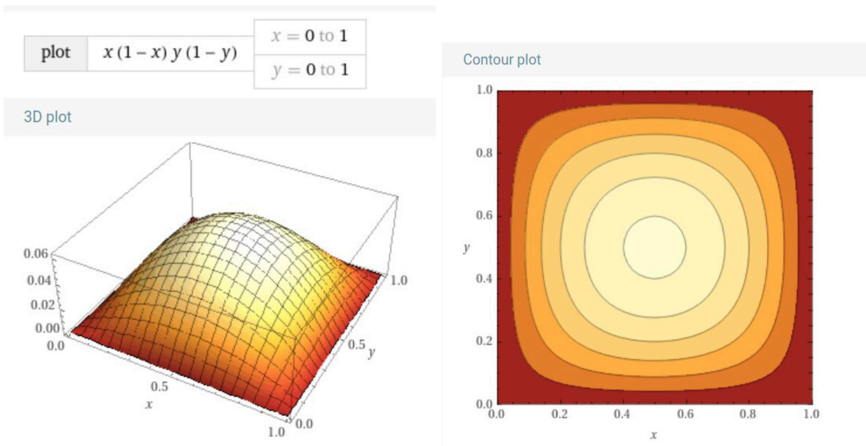

Figure 2.5 The function $N_1(x_1, x_2)$ when $L = 1$. Its maximum value (at $(1/2, 1/2)$) is $1/16$.

Let us begin by discussing an example of using a global (instead of piecewise) polynomial basis on Ω, which is possible when Ω is a square.

Example 2.7 (Uniformly heated square rod) Let us revisit Example 2.4 but now considering a square geometry, so that $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, 0 < x_1 < L, 0 < x_2 < L\}$. The surface temperature is $g \in \mathbb{R}$ and the governing equation is $\Delta u = -f/k$ as before, where the heat source is $f \in \mathbb{R}$.

We want to select the space $\mathscr{W}_h$ as a subset of

$$\mathbb{P}_r(\Omega) = \{\text{polynomials of degree} \leq r \text{ in two variables in } \Omega\}\,. \tag{2.25}$$

If $r = 2$ this space consists of functions of the form

$$v(x_1, x_2) = c_1 + c_2 x_1 + c_3 x_2 + c_4 x_1^2 + c_5 x_1 x_2 + c_6 x_2^2\,,$$

if $r = 3$ the following terms are added

$$\ldots + c_7 x_1^3 + c_8 x_1^2 x_2 + c_9 x_1 x_2^2 + c_{10} x_2^3\,,$$

and so on. We will use this example to illustrate that the choice of a basis for $\mathscr{W}_h$ that can easily accommodate the constraints on $\mathscr{V}_h$ and $\mathscr{S}_h$ is not always trivial.

Both spaces $\mathscr{V}_h$ and $\mathscr{S}_h$ require functions to be constant along the boundary of the domain, that is, whenever x_1 or x_2 are either equal to 0 or equal to L. If, for example, we choose $r = 1$, then $p(x_1, x_2) = c_1 + c_2 x_1 + c_3 x_2$ for any $(c_1, c_2, c_3) \in \mathbb{R}^3$. The fact that p is constant at $x_2 = 0$ implies that

$\partial^i p/\partial x_1^i(x_1, 0) = 0$ for all $i \in \mathbb{N}$ and any $x_1 \in [0, L]$. A similar argument can be made for the boundary conditions at $x_2 = L$, $x_1 = 0$ and $x_1 = L$. Hence,

$$\begin{aligned}0 &= \frac{\partial p}{\partial x_1}(x_1, 0) = c_2,\\ 0 &= \frac{\partial p}{\partial x_1}(x_1, L) = c_2,\\ 0 &= \frac{\partial p}{\partial x_2}(0, x_2) = c_3,\\ 0 &= \frac{\partial p}{\partial x_2}(L, x_2) = c_3,\end{aligned}$$

and in this case we only need to evaluate the first derivative. As a result, we conclude that functions in $\mathcal{V}_h$ or $\mathcal{S}_h$ need $c_2 = c_3 = 0$, so they can only be constant functions. Then, if $r = 1$, $\mathcal{V}_h = \{0 \cdot N_2\}$ and $\mathcal{S}_h = \{g \cdot N_2\}$, where $N_2(x_1, x_2) = 1$ constant for all $x \in \Omega$. These spaces contain a single function each, so they are poor choices for any approximation.

In a similar way, it can be verified that, if $r < 4$, $\mathcal{V}_h = \{0 \cdot N_2\}$ and $\mathcal{S}_h = \{g \cdot N_2\}$. For $r \geq 4$, due to the simplicity of the geometry, we have that the set of function in $\mathbb{P}_r(\Omega)$ that are constant on $\partial\Omega$ is (see the explanation after the example)

$$\mathcal{W}_h = \{v(x_1, x_2) = c_1 + \underbrace{x_1(L - x_1)x_2(L - x_2)}_{=0 \text{ on } \partial\Omega}\, p(x_1, x_2) \mid p \in \mathbb{P}_{r-4}, c_1 \in \mathbb{R}\},$$

and this will be our choice for $\mathcal{W}_h$. **Take** $r = 4$, which is the simplest case. Then $p(x_1, x_2)$ is a constant and $\mathcal{V}_h$ **has dimension 1**. Define the basis function

$$N_1(x_1, x_2) = x_1(L - x_1)x_2(L - x_2), \quad \text{so that } \nabla N_1 = \begin{pmatrix} (L - 2x_1)x_2(L - x_2) \\ x_1(L - x_1)(L - 2x_2) \end{pmatrix}.$$

Therefore, we can write

$$\mathcal{W}_h = \operatorname{span}(\{N_1, N_2\}).$$

In particular, functions in $\mathcal{V}_h$ are zero on $\partial\Omega$, so

$$\mathcal{V}_h = \{v_1 N_1 \mid v_1 \in \mathbb{R}\}$$

and functions in $\mathcal{S}_h$ are equal to g on $\partial\Omega$, so

$$\mathcal{S}_h = \{v_1 N_1 + g N_2 \mid v_1 \in \mathbb{R}\}.$$

We then have $\eta_a = \{1\}$, $\eta_g = \{2\}$, we can choose $\overline{u}_h = g N_2$, and

$$u_h(x_1, x_2) = u_1 N_1(x_1, x_2) + g N_2(x_1, x_2) = g + u_1 N_1(x_1, x_2).$$

For u_h to satisfy the variational method with $a(\cdot,\cdot)$ given by (2.21c) and $\ell(\cdot)$ given by (2.21d) it must hold that

$$\int_\Omega k\nabla(g+u_1 N_1)\cdot\nabla N_1 \; dx_1 dx_2 = \int_\Omega f N_1 \; dx_1 dx_2 \,.$$

Here we directly used that $u_2 = g$, so we do not need to add an equation for the constrained index.

Noticing that $\nabla g = 0$ and taking u_1 out of the integral by linearity, the final equation to compute $U = [u_1]$ is

$$KU = F$$

where the 1×1 stiffness matrix and load vector are

$$K = \int_\Omega k\nabla N_1\cdot\nabla N_1 \; dx_1 dx_2, \qquad F = \int_\Omega f N_1 \; dx_1 dx_2 \,.$$

Performing the double integrals we obtain

$$K = \frac{kL^8}{45}\,, \quad F = \frac{fL^6}{36}\,, \quad \text{and thus} \quad u_1 = \frac{5f}{4kL^2}\,.$$

This means that the solution of the variational method is

$$u_h(x_1,x_2) = g + \frac{5f}{4kL^2} N_1(x_1,x_2) = g + \frac{5f}{4kL^2} x_1(L-x_1)x_2(L-x_2)\,.$$

The maximum temperature takes place at the center (if $f>0$), with value $g+5fL^2/(64k)$. The temperature contours are shown in Fig. 2.6, where we compare the numerical solution (obtained by solving an equation with just one unknown!) with the exact solution. They are qualitatively very similar. The maximum difference is, in fact, less than 6%.

It is important to remark that the numerical solution u_h is **not** an exact solution of the differential equation. To check this, simply compute

$$\Delta u_h(x_1,x_2) = -\frac{5f}{2kL^2}\left[x_1(L-x_1)+x_2(L-x_2)\right]$$

and compare to the exact equation $\Delta u = -f/k$. The **residual of the differential equation**, evaluated on the numerical solution, is

$$r_h = \Delta u_h(x_1,x_2) + \frac{f}{k} = \frac{f}{k}\left(1 - \frac{5}{2}\,\frac{x_1(L-x_1)+x_2(L-x_2)}{L^2}\right).$$

Its average value over the domain is $f/(6k)$. The residual is maximum at the vertices, where its value is f/k. At the center the value is $-1/4$. The residual is plotted in Fig. 2.7. It must not be mistaken for the actual **error of the approximate solution** $e_h = u - u_h$, which in this case we can compute because the exact solution is known and is also shown in the same figure.

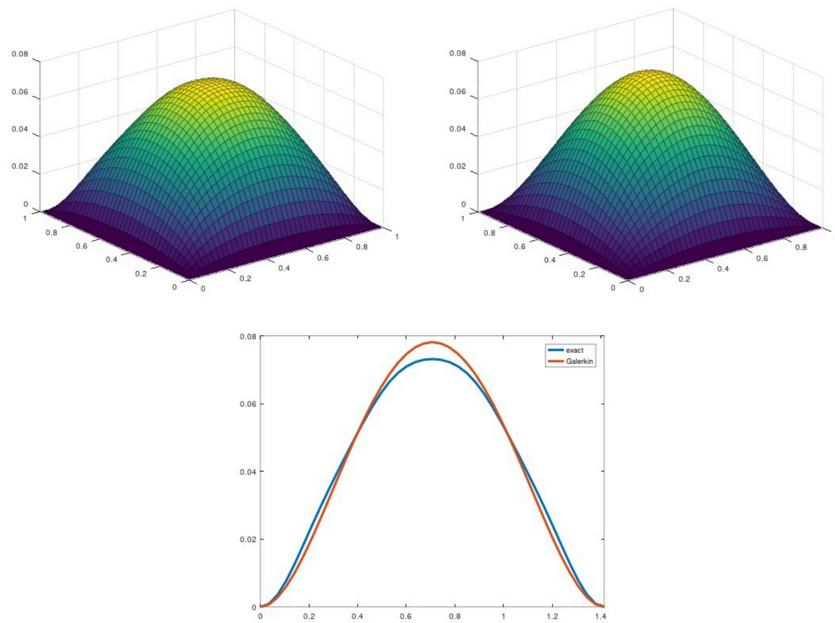

Figure 2.6 Exact solution, variational method solution, and comparison along the diagonal.

Functions in $\mathbb{P}_r(\Omega)$, $r \geq 4$ that are constant on $\partial\Omega$, where $\Omega = [0, L] \times [0, L]$

To obtain the result, we will repeatedly use the following observation. Let $q \in \mathbb{P}_r(\Omega)$, and subtract a constant c_1 so that $q_r(x_1, x_2) = q(x_1, x_2) - c_1$ is equal to zero on $\partial\Omega$. Let $h(x_1, x_2) = h_0 + h_1 x_1 + h_2 x_2$ for $h_0, h_1, h_2 \in \mathbb{R}$ such that either h_1 or h_2 are not zero, and such that if $h(\bar{x}_1, \bar{x}_2) = 0$ then $q(\bar{x}_1, \bar{x}_2) = 0$. Then,

$$q_r(x_1, x_2) = h(x_1, x_2) q_{r-1}(x_1, x_2), \tag{2.26}$$

where $q_{r-1}(x_1, x_2)$ is a polynomial of degree $r - 1$. To see this, without loss of generality assume that $h_1 \neq 0$, and let $z = h(x_1, x_2)$, so that $x_1 = g(z, x_2) = (z - h_2 x_2 - h_0)/h_1$. Consider then the polynomial $\hat{q}_r(z, x_2) = q_r(g(z, x_2), x_2)$, which satisfies that $\hat{q}_r(0, x_2) = 0$ for any x_2. Then, it admits a factorization of the form

$$\hat{q}_r(z, x_2) = z \hat{q}_{r-1}(z, x_2)$$

for a polynomial $\hat{q}_{r-1}$ of degree $r - 1$. Defining $q_{r-1}(x_1, x_2) = \hat{q}_{r-1}(h(x_1, x_2), x_2)$, we arrive to (2.26).

We can apply this to our case, by sequentially considering $h(x_1, x_2)$ equal to $x_1, L - x_1, x_2$ and $L - x_2$. It then follows that

$$q_r(x_1, x_2) = x_1(L - x_1) x_2 (L - x_2) q_{r-4}(x_1, x_2), \tag{2.27}$$

where q_{r-4} is a polynomial of degree $r - 4$.

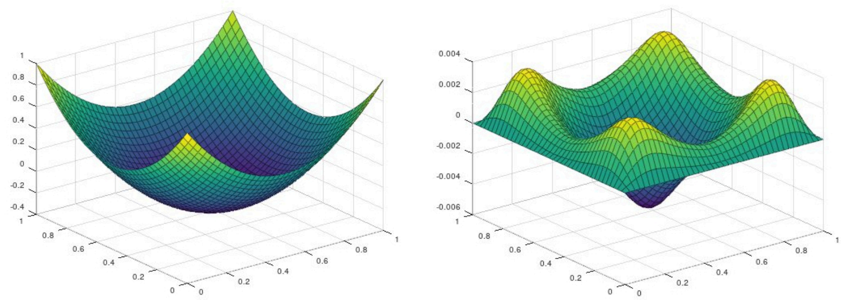

Figure 2.7 The residual function $r_h(x_1, x_2)$ (left) and the error function $e_h(x_1, x_2)$ (right).

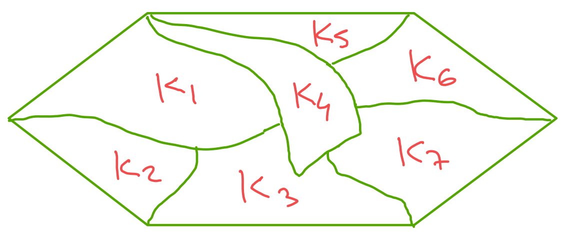

Figure 2.8 Example of a mesh with $n_{\text{el}} = 7$

2.4 Finite Element Spaces in Two Dimensions

Over the years many finite element spaces have been introduced with ever-increasing sophistication for different specific applications. For diffusion problems the classical and still most popular ones consist of **piecewise polynomial functions that are continuous in** Ω. These are the ones that we discuss next.

2.4.1 The Simplest C^0 Finite Element Space in Two Dimensions

How to define and build a space of piecewise polynomials that only consist of continuous functions? Consider the domain subdivided into element subdomains, over which the functions of the space need to be polynomials of degree r in two variables. If r is zero, the function is continuous if and only if its value is the same in all element subdomains, which makes the *piecewise* constant functions to be *globally* constant. So, $r = 0$ does not produce a space of continuous functions that can be used to approximate anything.

But what about piecewise *linear* polynomials ($r = 1$)? Or polynomials of higher degree? To consider the simplest case, lets answer this question for $r = 1$. The answer will introduce us to the **continuous** P_1 **finite element space**, also known simply as P_1 space. To introduce this space, let's first define what a mesh is.

In the following, an **element domain** is a set in $\mathbb{R}^2$ (or $\mathbb{R}^3$ in 3D) that can be obtained by a bijective deformation of a disc (or a sphere in 3D).