ME 335A
Finite Element Analysis
Instructor: Adrian Lew
Final Review

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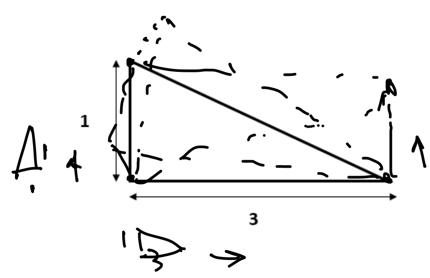
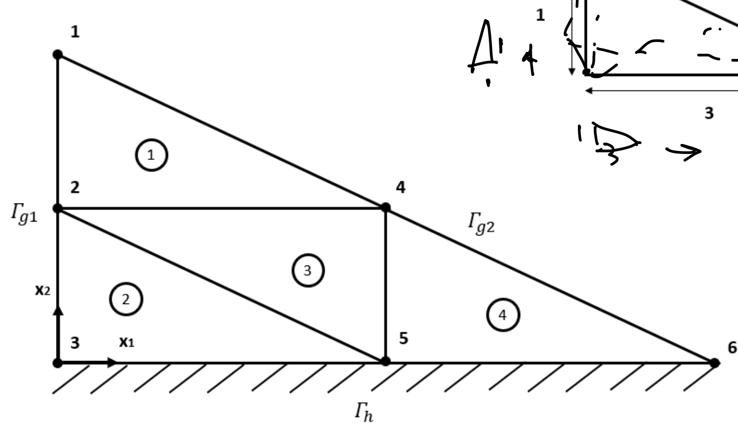
Heat Conduction on a Triangular Bar

The bottom section of a long triangular bar is well insulated, while the sides are maintained at uniform temperatures $T_{\Gamma_{g1}} = 100°C$ and $T_{\Gamma_{g2}} = 50°C$. The domain Ω has a boundary $\partial\Omega$ partitioned as follows $\Gamma_{g1} = \{(x_1, x_2) \in \partial\Omega \mid x_1 = 0\}$, $\Gamma_h = \{(x_1, x_2) \in \partial\Omega \mid x_2 = 0,\ 0 < x_1 < 6\}$ and $\Gamma_{g2} = \partial\Omega \backslash (\Gamma_{g1} \cup \Gamma_h)$. We want to find the temperature $T : \Omega \to \mathbb{R}$ such that

$$
\begin{aligned}
&-\operatorname{div}(K(\mathbf{x})\nabla T) = 0 && \text{on } \Omega \\
&T = 100°C && \text{on } \Gamma_{g1} \\
&T = 50°C && \text{on } \Gamma_{g2} \\
&K(\mathbf{x})\nabla T \cdot \check{n} = 0 && \text{on } \Gamma_h
\end{aligned}
$$

To this end consider the mesh shown in the figure, made of all $\mathcal{P}^1$ elements formed with triangles that have a ratio of 1:3.





$\nabla N_1^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\nabla N_2^2 = \begin{bmatrix} -1 \\ -\frac{1}{3} \end{bmatrix}$

1. Construct a variational equation that T satisfies.

   Solution:

   Applying the divergence theorem (integration by parts)

$$-\int_{\Omega} \operatorname{div}(K\nabla T)v \, d\Omega = \int_{\Omega} (K\nabla T)\cdot\nabla v \, d\Omega - \int_{\Gamma_h} \cancelto{0}{(K\nabla T)\cdot\check{n}} \, v \, d\Gamma_h$$

$$\begin{aligned} a(T,v) &= \int_{\Omega} (K\nabla T)\cdot\nabla v \, d\Omega \\ l(v) &= 0 \end{aligned} \qquad , \forall v \in \mathcal{V}$$

   $\mathcal{V} = \{v : \Omega \to \mathbb{R} \text{ smooth} \mid v = 0, \ \forall \mathbf{x} \in \Gamma_{g1}, \Gamma_{g2}\}$

2. Define $\mathcal{W}_h$, $\mathcal{V}_h$ and $\mathcal{S}_h$ and state the finite element method for this problem using the variational equation obtained in part 1.

   Solution:

$$\mathcal{W}_h = \operatorname{span}(N_1, N_2, N_3, N_4, N_5, N_6)$$
$$\mathcal{V}_h = cN_5, \quad c \in \mathbb{R}$$
$$\mathcal{S}_h = T_{\Gamma_{g1}}(N_1 + N_2 + N_3) + T_{\Gamma_{g2}}(N_4 + N_6) + cN_5$$

   Find $T_h \in \mathcal{S}_h$ such that:

$$a(T_h, v_h) = l(v_h), \qquad \forall v_h \in \mathcal{V}_h$$

$$\begin{aligned} a(T_h, v_h) &= \int_{\Omega} (K\nabla T_h)\cdot\nabla v_h \, d\Omega \\ l(v_h) &= 0 \end{aligned}$$

3. Find LV for the given mesh.

   Solution:

$$LV = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 3 & 5 & 5 \\ 4 & 5 & 4 & 6 \end{bmatrix}$$

4. Since the bar has anisotropic properties let us assume that the thermal conductivity is constant for each element (i.e. $K(\mathbf{x}) \approx k^e\mathbf{I}, \ \mathbf{x} \in \Omega^e, \ k^e \in \mathbb{R}$) and have the following values (assume that the units of k^e are consistent with the problem). The expressions for N_a^e and A are also provided to you in case you need them.

$$N_1^e = \frac{1}{2A}[-(X_2^3 - X_2^2)(x_1 - X_1^2) + (X_1^3 - X_1^2)(x_2 - X_2^2)]$$
$$N_2^e = \frac{1}{2A}[-(X_2^1 - X_2^3)(x_1 - X_1^3) + (X_1^1 - X_1^3)(x_2 - X_2^3)]$$
$$N_3^e = \frac{1}{2A}[-(X_2^2 - X_2^1)(x_1 - X_1^1) + (X_1^2 - X_1^1)(x_2 - X_2^1)]$$
$$A = \frac{1}{2}(X_1^2 - X_1^1)(X_2^3 - X_2^1) - (X_2^2 - X_2^1)(X_1^3 - X_1^1)$$

k^e	**Value**
k^1	14
k^2	27
k^3	45
k^4	27

With this information find the stiffness matrix and load vector. Provide the finite element approximation T_h as a linear combination of the basis functions.

Solution:

Since $\eta_g = \{1,2,3,4,6\}$ and $l(v_h) = 0$, the stiffness matrix and load vector are

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} T_{\Gamma_{g1}} \\ T_{\Gamma_{g1}} \\ T_{\Gamma_{g1}} \\ T_{\Gamma_{g2}} \\ 0 \\ T_{\Gamma_{g2}} \end{bmatrix}$$

From the LV = LG matrix we know that $K_{ab}^e \rightarrow K_{LG(a,e)LG(b,e)}$, therefore

$$\begin{aligned} K_{51} &= 0 \\ K_{52} &= K_{31}^2 + K_{21}^3 \\ K_{53} &= K_{32}^2 \\ K_{54} &= K_{23}^3 + K_{21}^4 \\ K_{55} &= K_{33}^2 + K_{22}^3 + K_{22}^4 \\ K_{56} &= K_{23}^4 \end{aligned}$$

Because all the elements have the same area A and we assume that K is constant for each element:

$$K_{ab}^e = \int_{\Omega^e} k^e \nabla N_b^e \cdot \nabla N_a^e \, d\Omega^e = k^e \nabla N_b^e \cdot \nabla N_a^e \int_{\Omega^e} d\Omega^e = \boxed{k^e A \nabla N_b^e \cdot \nabla N_a^e}$$

$$\nabla N_1^2 = [0,1]^T, \quad \nabla N_2^2 = [-1/3,-1]^T, \quad \nabla N_3^2 = [1/3,0]^T$$
$$\nabla N_1^3 = [-1/3,0]^T, \quad \nabla N_2^3 = [0,-1]^T, \quad \nabla N_3^3 = [1/3,1]^T$$
$$\nabla N_1^4 = [0,1]^T, \quad \nabla N_2^4 = [-1/3,-1]^T, \quad \nabla N_3^4 = [1/3,0]^T$$

$$\begin{aligned}
K_{51} &= 0 \\
K_{52} &= K_{31}^2 + K_{21}^3 = 0 \\
K_{53} &= K_{32}^2 = -3A \\
K_{54} &= K_{23}^3 + K_{21}^4 = -72A \\
K_{55} &= K_{33}^2 + K_{22}^3 + K_{22}^4 = 78A \\
K_{56} &= K_{23}^4 = -3A
\end{aligned}$$

$$\begin{aligned}
&0 = T_{\Gamma_{g1}}(K_{51}+K_{52}+K_{53}) + T_{\Gamma_{g2}}(K_{54}+K_{56}) + u_5 K_{55} \\
&0 = 100(0+0-3A) + 50(-72A-3A) + u_5 78A \\
&u_5 = 675/13
\end{aligned}$$

Finally we get that

$$T_h = 100(N_1+N_2+N_3) + 50(N_4+N_6) + \frac{675}{13}N_5$$

5. Now that you know T_h, find the value of the temperature at the centroid of the elements $\bar{\mathbf{x}}^e$.

$\bar{\mathbf{x}}^e$
(1, 4/3)
(1, 1/3)
(2, 2/3)
(4, 1/3)

Solution:

The barycentric coordinates of the centroid of the elements correspond to (1/3, 1/3, 1/3)

$$\begin{aligned}
T_h(\bar{\mathbf{x}}^1) &= \frac{1}{3}(100+100+50) \\
T_h(\bar{\mathbf{x}}^2) &= \frac{1}{3}\left(100+100+\frac{675}{13}\right) \\
T_h(\bar{\mathbf{x}}^3) &= \frac{1}{3}\left(100+50+\frac{675}{13}\right) \\
T_h(\bar{\mathbf{x}}^4) &= \frac{1}{3}\left(50+50+\frac{675}{13}\right)
\end{aligned}$$

6. With this finite element approximation, assuming that you are in the asymptotic region of convergence, what convergence rate r_1 would you expect to have for $||T - T_h||_{0,2,\Omega}$ and for $||T - T_h||_{1,2,\Omega}$?

Solution:

The expected order for convergence for $||T - T_h||_{0,2,\Omega}$ is $r_1 = k + 1 = 2$ (Aubin-Nitsche trick) and for $||T - T_h||_{0,2,\Omega}$ is $r_1 = k = 1$

7. You are not satisfied with the approximation of the temperature that you get with $\mathcal{P}^1$ elements and for this reason you will use $\mathcal{P}^2$ elements. In this case, assume that you have access to a thermocouple that allows you to get measurements of the temperature T_{meas} at any point $\mathbf{x}_{meas}$. You can incorporate them in the problem as

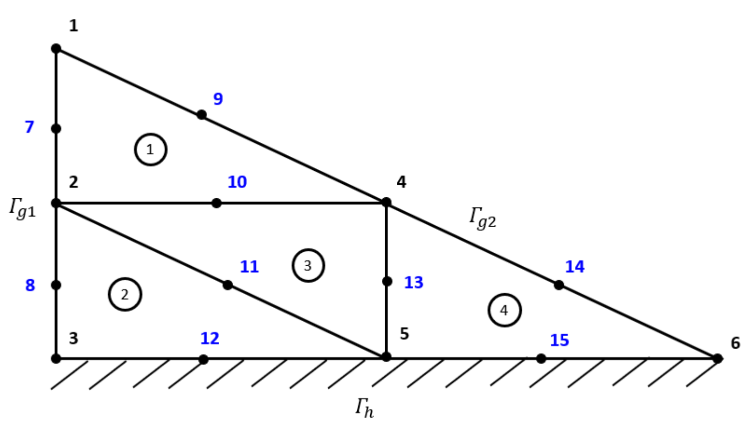
$$
\begin{aligned}
&-\operatorname{div}(K(\mathbf{x})\nabla T) = 0 && \text{on } \Omega \\
&T = 100°C && \text{on } \Gamma_{g1} \\
&T = 50°C && \text{on } \Gamma_{g2} \\
&K(\mathbf{x})\nabla T \cdot \check{n} = 0 && \text{on } \Gamma_h \\
&T(\mathbf{x}) = T_{meas}(\mathbf{x}_{meas})
\end{aligned}
$$

Specify the locations $\mathbf{x}_{meas}$ at which you would measure the temperature. Number the nodes of the new mesh that you get using $\mathcal{P}^2$ elements, find the stiffness matrix and load vector and provide the finite element approximation T_h as a linear combination of the basis functions that is consistent with the mesh that you provide.

Solution:

$\mathcal{P}^2$ elements have additional degrees of at the midpoints of the edges of the elements, therefore we can measure the temperature at the midpoints for which we don't have information (nodes 10, 11, 12, 13 and 15) and at node 5 and we can get a better approximation T_h. The measurements enter as constrained values and therefore

$$K = \mathbf{I}, \quad F = U$$



$$T_h = 100(N_1 + N_2 + N_3 + N_7 + N_8) + 50(N_4 + N_6 + N_9 + N_{14}) + \sum_{j \in \mathcal{J}} T^j_{meas} N_j$$

$$\mathcal{J} = \{5, 10, 11, 12, 13, 15\}$$

Euler-Lagrange Equations and Assembly in 1D

Consider the weak form: For $g,\ \alpha \in \mathbb{R}$ find $y \in \mathcal{S} = \{s : \Omega = (1,3) \to \mathbb{R} \text{ smooth} \mid s(3) = g\}$ such that $a(u,v) = l(v),\ \forall v \in \mathcal{V} = \{v : \Omega = (1,3) \to \mathbb{R} \text{ smooth} \mid v(3) = 0\}$

$$a(u,v) = \int_1^3 -x^2 v' y' - xvy' + (x^2 - \alpha^2) yv \; dx$$
$$l(v) = v(1)$$

1. Obtain the Euler-Lagrange equations. Identify essential and natural boundary conditions.

   Solution:

   Applying integration by parts

   $$\begin{aligned} \int_1^3 -x^2 v' y' \; dx &= \left[-x^2 v y'\right]_1^3 + \int_1^3 2xvy' + x^2 v y'' \; dx \\ &= -9\cancelto{0}{v(3)} y'(3) + v(1)y'(1) + \int_1^3 2xvy' + x^2 v y'' \; dx \end{aligned}$$

   Then

   $$\begin{aligned} v(1)y'(1) - v(1) + \int_1^3 2xvy' + x^2 v y'' - xvy' + (x^2 - \alpha^2) yv \; dx = 0 \\ [y'(1) - 1]v(1) + \int_1^3 [xy' + x^2 y'' + (x^2 - \alpha^2) y] v \; dx = 0 \end{aligned}$$

   Testing with all v such that $v(1) = 0$ we get that

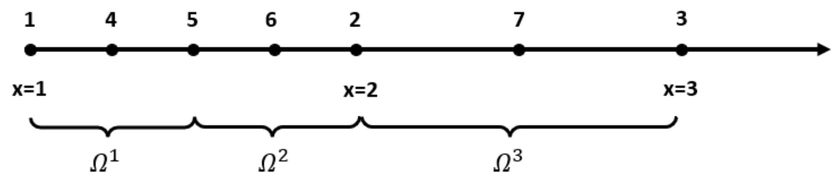
   $$\int_1^3 [xy' + x^2 y'' + (x^2 - \alpha^2) y] v \; dx = 0 \implies x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$$

   If $x^2 y'' + xy' + (x^2 - \alpha^2) y = 0$ then $y'(1) - 1 = 0$. The Euler-Lagrange equations are:

   $$\begin{aligned} &x^2 y'' + xy' + (x^2 - \alpha^2) y = 0, \qquad \forall x \in (1,3) \\ &y'(1) - 1 = 0 \end{aligned}$$

   $y(3) = g$ is the essential boundary condition and $y'(1) = 1$ is the natural boundary condition. You don't need to know this to solve this problem but if $\alpha \in \mathbb{C}$ the differential equation is called the Bessel differential equation. This equation has application in electromagnetic waves, quantum mechanics and to model the vibration of membranes.

2. Consider the nodes 1 to 7 with positions $\{1, 2, 3, 1.25, 1.5, 1.75, 2.5\}$, respectively as it is shown in the figure. These nodes form $\mathcal{P}^2$ elements 1, 2, and 3, whose domains are $\Omega^1 = [1,\ 1.5]$, $\Omega^2 = [1.5,\ 2]$ and $\Omega^3 = [2,\ 3]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.

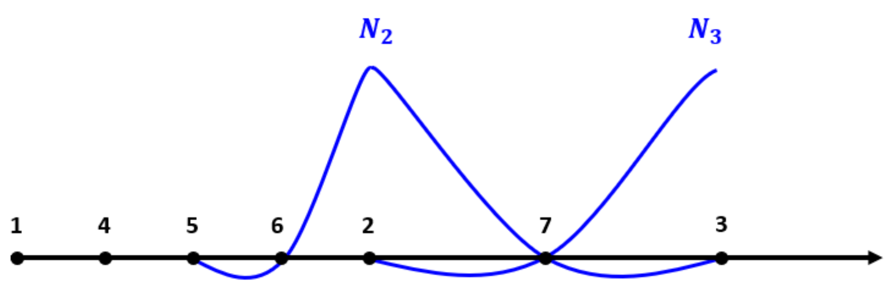


Solution:

$$LG = \begin{bmatrix} 1 & 5 & 2 \\ 4 & 6 & 7 \\ 5 & 2 & 3 \end{bmatrix}$$

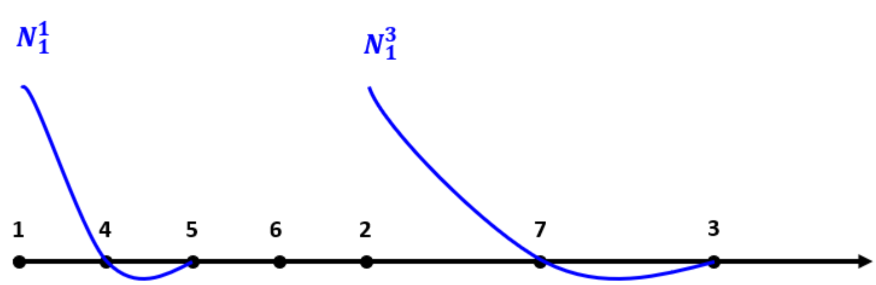
3. Sketch the basis functions N_2, N_3

Solution:



4. Sketch the shape functions N_1^3, N_1^1.

Solution:



5. State the finite element method for this problem using the given variational equation. Identify $\mathcal{V}_h$, $\mathcal{S}_h$, η_a, η_g and $\bar{u}_h \in \mathcal{S}_h$.

   Solution:

$$
\begin{aligned}
\mathcal{W}_h &= \mathrm{span}(N_1, N_2, N_3, N_4, N_5, N_6, N_7)\\
\mathcal{V}_h &= \mathrm{span}(N_1, N_2, N_4, N_5, N_6, N_7)\\
\mathcal{S}_h &= \{gN_3 + v \mid v \in \mathcal{V}_h\}, \quad \bar{u}_h = gN_3\\
\eta_a &= \{1, 2, 3, 4, 5, 6\}, \quad \eta_g = \{3\}
\end{aligned}
$$

   Find $y_h \in \mathcal{S}_h$ such that:

$$
a(y_h, v_h) = l(v_h), \qquad \forall v_h \in \mathcal{V}_h
$$

$$
\begin{aligned}
a(y_h, v_h) &= \int_1^3 -x^2 v_h' y_h' - x v_h y_h' + (x^2 - \alpha^2) y_h v_h \, dx\\
l(v_h) &= v(1)
\end{aligned}
$$

6. Provide expressions to compute K_{23}^1, K_{32}^1 and K_{11}^3 in terms of the appropriate shape functions. Do not compute the integrals.

   Solution:

$$
\begin{aligned}
K_{23}^1 = a(N_3^1, N_2^1) &= \int_1^{1.5} -x^2 N_2^{1\prime} N_3^{1\prime} - x N_2^1 N_3^{1\prime} + (x^2 - \alpha^2) N_3^1 N_2^1 \, dx\\
K_{32}^1 = a(N_2^1, N_3^1) &= \int_1^{1.5} -x^2 N_3^{1\prime} N_2^{1\prime} - x N_3^1 N_2^{1\prime} + (x^2 - \alpha^2) N_2^1 N_3^1 \, dx\\
K_{12}^3 = a(N_2^3, N_1^3) &= \int_2^{3} -x^2 N_1^{3\prime} N_2^{3\prime} - x N_1^3 N_2^{3\prime} + (x^2 - \alpha^2) N_2^3 N_1^3 \, dx
\end{aligned}
$$

7. State where each entry in the second row of K^2 is assembled in the stiffness matrix.

   Solution:

   From the LG matrix we know that $K_{ab}^e \to K_{LG(a,e)LG(b,e)}$, therefore

$$
\begin{aligned}
K_{21}^2 &\to K_{65}\\
K_{22}^2 &\to K_{66}\\
K_{23}^2 &\to K_{62}
\end{aligned}
$$

8. Provide the numerical values for the load vector.

   Solution:

$$
F = \begin{bmatrix} 1 & 0 & g & 0 & 0 & 0 & 0 \end{bmatrix}^T
$$