ME 335A
Finite Element Analysis
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Final Review

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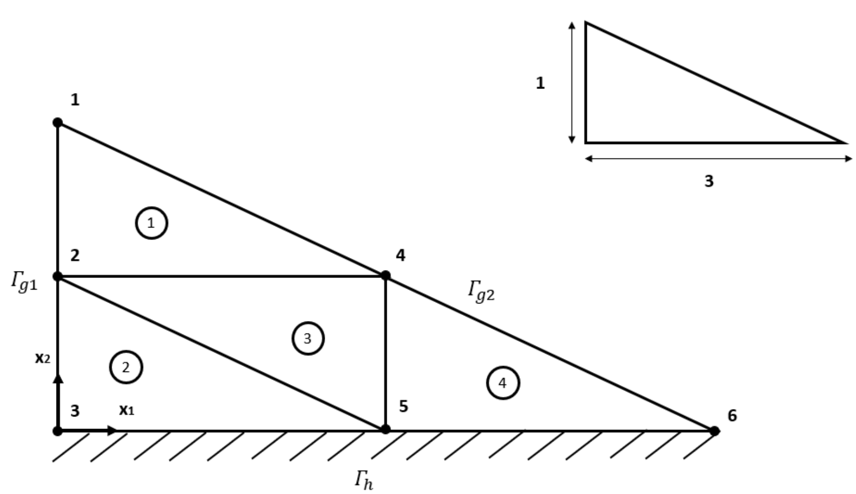
Heat Conduction on a Triangular Bar

The bottom section of a long triangular bar is well insulated, while the sides are maintained at uniform temperatures $T_{\Gamma_{g1}} = 100°C$ and $T_{\Gamma_{g2}} = 50°C$. The domain Ω has a boundary $\partial\Omega$ partitioned as follows $\Gamma_{g1} = \{(x_1, x_2) \in \partial\Omega \mid x_1 = 0\}$, $\Gamma_h = \{(x_1, x_2) \in \partial\Omega \mid x_2 = 0,\ x_1 \neq 0\}$ and $\Gamma_{g2} = \partial\Omega \backslash (\Gamma_{g1} \cup \Gamma_h)$. We want to find the temperature $T : \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
&-\operatorname{div}(K(\mathbf{x})\nabla T) = 0 && \text{on } \Omega \\
&T = 100°C && \text{on } \Gamma_{g1} \\
&T = 50°C && \text{on } \Gamma_{g2} \\
&K(\mathbf{x})\nabla T \cdot \check{n} = 0 && \text{on } \Gamma_h
\end{aligned}$$

To this end consider the mesh shown in the figure, made of all $\mathcal{P}^1$ elements formed with triangles that have a ratio of 1:3.



1. Construct a variational equation that T satisfies.

2. Define $\mathcal{W}_h$, $\mathcal{V}_h$ and $\mathcal{S}_h$ and state the finite element method for this problem using the variational equation obtained in part 1.

3. Find LV for the given mesh.

4. Since the bar has anisotropic properties let us assume that the thermal conductivity is constant for each element (i.e. $K(\mathbf{x}) \approx k^e \mathbf{I},\ \mathbf{x} \in \Omega^e,\ k^e \in \mathbb{R}$) and have the following values (assume that the units of k^e are consistent with the problem). The expressions for N_a^e and A are also provided to you in case you need them.

$$
\begin{aligned}
N_1^e &= \frac{1}{2A}[-(X_2^3 - X_2^2)(x_1 - X_1^2) + (X_1^3 - X_1^2)(x_2 - X_2^2)] \\
N_2^e &= \frac{1}{2A}[-(X_2^1 - X_2^3)(x_1 - X_1^3) + (X_1^1 - X_1^3)(x_2 - X_2^3)] \\
N_3^e &= \frac{1}{2A}[-(X_2^2 - X_2^1)(x_1 - X_1^1) + (X_1^2 - X_1^1)(x_2 - X_2^1)] \\
A &= \frac{1}{2}(X_1^2 - X_1^1)(X_2^3 - X_2^1) - (X_2^2 - X_2^1)(X_1^3 - X_1^1)
\end{aligned}
$$

k^e	**Value**
k^1	14
k^2	27
k^3	45
k^4	27

With this information find the stiffness matrix and load vector. Provide the finite element approximation T_h as a linear combination of the basis functions.

5. Now that you know T_h, find the value of the temperature at the centroid of the elements $\bar{\mathbf{x}}^e$.

$\bar{\mathbf{x}}^e$
(1, 4/3)
(1/3, 1/3)
(2, 2/3)
(4, 1/3)

6. With this finite element approximation, assuming that you are in the asymptotic region of convergence, what convergence rate r_1 would you expect to have for $||T - T_h||_{0,2,\Omega}$ and for $||T - T_h||_{1,2,\Omega}$?

7. You are not satisfied with the approximation of the temperature that you get with $\mathcal{P}^1$ elements and for this reason you will use $\mathcal{P}^2$. In this case, assume that you have access to a thermocouple that allows you to get measurements of the temperature T_{meas} at any point $\mathbf{x}_{meas}$. You can incorporate them in the problem as

$$
\begin{aligned}
&-\operatorname{div}(K(\mathbf{x})\nabla T) = 0 && \text{on } \Omega\\
&T = 100°C && \text{on } \Gamma_{g1}\\
&T = 50°C && \text{on } \Gamma_{g2}\\
&K(\mathbf{x})\nabla T\cdot\check{n} = 0 && \text{on } \Gamma_h\\
&T(\mathbf{x}) = T_{meas}(\mathbf{x}_{meas})
\end{aligned}
$$

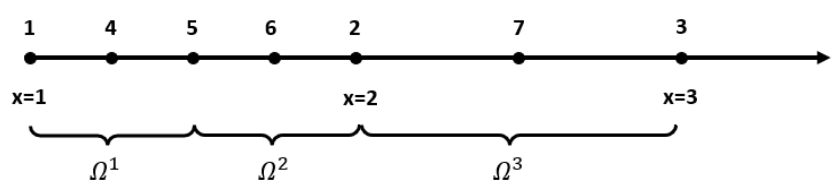
Specify the locations $\mathbf{x}_{meas}$ at which you would measure the temperature. Number the nodes of the new mesh that you get using $\mathcal{P}^2$ elements, find the stiffness matrix and load vector and provide the finite element approximation T_h as a linear combination of the basis functions that is consistent with the mesh that you provide.

Euler-Lagrange Equations and Assembly in 1D

Consider the weak form: For $g,\ \alpha \in \mathbb{R}$ find $y \in \mathcal{S} = \{s : \Omega = (1,3) \to \mathbb{R} \text{ smooth} \mid s(3) = g\}$ such that $a(u,v) = l(v),\ \forall v \in \mathcal{V} = \{v : \Omega = (1,3) \to \mathbb{R} \text{ smooth} \mid v(3) = 0\}$

$$
\begin{aligned}
a(u,v) = \int_1^3 -x^2 v' y' - xvy' + (x^2 - \alpha^2) yv\, dx\\
l(v) = v(1)
\end{aligned}
$$

1. Obtain the Euler-Lagrange equations. Identify essential and natural boundary conditions.
2. Consider the nodes 1 to 7 with positions $\{1, 2, 3, 1.25, 1.5, 1.75, 2.5\}$, respectively as it is shown in the figure. These nodes form $\mathcal{P}^2$ elements 1, 2, and 3, whose domains are $\Omega^1 = [1,\ 1.5]$, $\Omega^2 = [1.5,\ 2]$ and $\Omega^3 = [2,\ 3]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.



3. Sketch the basis functions N_2, N_3
4. Sketch the shape functions N_1^3, N_1^1.
5. State the finite element method for this problem using the given variational equation. Identify $\mathcal{V}_h$, $\mathcal{S}_h$, η_a, η_g and $\bar{u}_h \in \mathcal{S}_h$.
6. Provide expressions to compute K_{23}^1, K_{32}^1 and K_{11}^3 in terms of the appropriate shape functions. Do not compute the integrals.
7. State where each entry in the second row of K^2 is assembled in the stiffness matrix.
8. Provide the numerical values for the load vector.