ME 335A
Finite Element Analysis
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Problems Set #5

May 11, 2023

Due Wednesday, May 17, 2023

A Variational Method with an almost Spectral Basis (53)

Let $\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < R^2\}$ for $R = 1$, $\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the problem: Find $u \in \Omega \to \mathbb{R}$ such that

$$-\frac{1}{2}\Delta u = \frac{2}{R^2} \qquad \text{in } \Omega \tag{1a}$$

$$u = 0 \qquad \text{on } \Gamma_g \tag{1b}$$

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{1}{R} \qquad \text{on } \Gamma_h. \tag{1c}$$

1. (10) Construct a variational equation that u satisfies, following the standard recipe.
2. (3) Identify essential and natural boundary conditions.
3. Consider the approximation space

$$\mathcal{W}_h = \text{span}\left(\sin\left(\pi\left(x_1^2 + x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2 + x_2^2\right)\right), 1\right).$$

   (a) (10) Identify test and trial spaces, and active and constrained indices, naming the basis functions with indices in the order they appear above.
   (b) (5) In this problem, there is a possibility of selecting a smaller space $\mathcal{W}_h$ without changing the results. What is this smaller space $\mathcal{W}_h$? Identify active and constrained indices in this new space.
   (c) (15) Using the smaller space $\mathcal{W}_h$, compute the stiffness matrix and load vector.
   (d) (5) Find the numerical approximation. Plot it together with the the exact solution.
   (e) (5) Do you think the numerical approximation would change if we change the boundary condition on the Neumann boundary to

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{2}{R}?$$

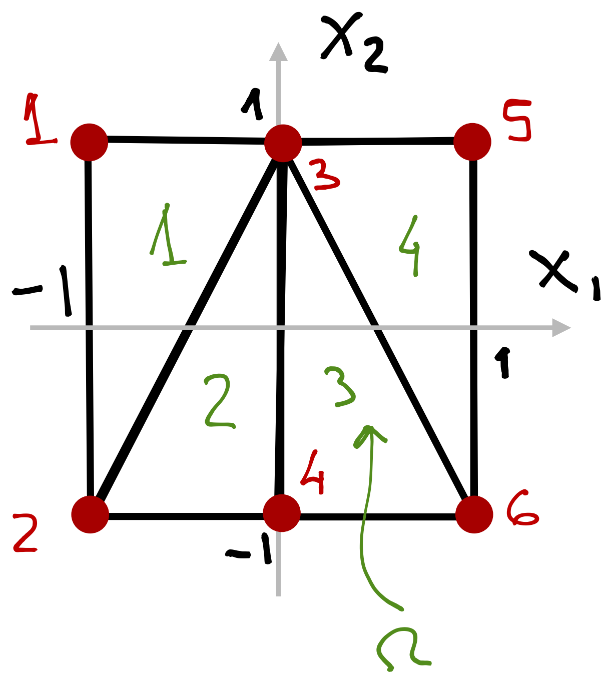
Manual Assembly, Once More (60)

Let $\Omega = [-1,1] \times [-1,1]$, $\Gamma_g = \{1\} \times [-1,1]$, and $\Gamma_h = \partial\Omega \setminus \Gamma_g$. Consider the variational equation that $u\colon \Omega \to \mathbb{R}$ satisfies:

$$\int_\Omega \nabla u \cdot \nabla v + uv \, d\Omega = \int_\Omega (x_1 + x_2) v \, d\Omega + \int_{\Gamma_h} (x_2^2 - 1) v \, d\Gamma$$

for all $v \in \mathcal{V} = \{v\colon \Omega \to \mathbb{R} \mid v = 0 \text{ on } \Gamma_g\}$, where (x_1, x_2) are the Cartesian coordinates in Ω. The function u satisfies the essential boundary condition $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in \Gamma_g$.

Consider then the mesh shown in the figure, made of all P_1 elements:



1. (5) What is the local-to-global map for the mesh?
2. (5) Identify $\mathcal{S}_h$ and $\mathcal{V}_h$ by providing the general expression for their functions in terms of the P_1 basis functions of the mesh. Identify active and constrained indices.
3. (10) Evaluate the shape function N_3^2 and its derivative ∇N_3^2 at $(x_1, x_2) = (-0.5, -0.1)$.
4. (20) Compute the element stiffness matrix and load vector for each element.
5. (5) Denote Γ_g as line 1, and Γ_h as line 2. Construct the array of boundary edges BE and compute the load vector associated to the natural boundary condition.
6. (10) Build the stiffness matrix and load vector.
7. (5) Compute the finite element approximation, express it as a linear combination of basis functions, and plot it over the square.