HW 4

Due Wednesday, May 10, 2023

Constructing Some FE Spaces (35)

Consider a mesh of Lagrange P_k-elements (see Example 1.65 in the notes) with $n_{\mathrm{el}} = 3$ elements of equal length in the interval $[0, 3]$. Elements are numbered consecutively from 1 to n_{el} from left to rigth (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

   (a) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

   (b) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

1.

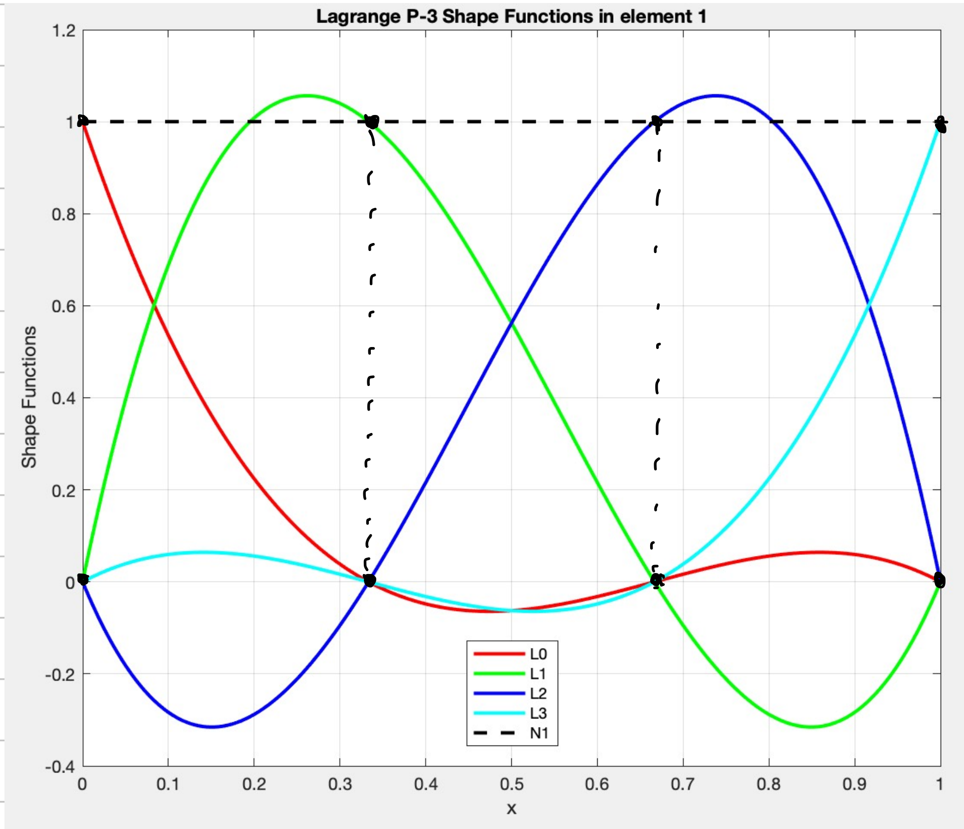
(a) observing LG Map, we know there are 3 global basis functions N_1, N_2, N_3. The dimension of the finite element space is 3.

For each element, there are 4 local degree of freedoms. We can use Lagrange P_3 elements defined as:

$$N_a^e(x) = \frac{\prod_{b=1,\, b\neq a}^{4} (x - x_b^e)}{\prod_{b=1,\, b\neq a}^{4} (x_a^e - x_b^e)}.$$

Consider element 1 for example. All N_a^e only contribute to N^1, we have

$$N^1(x) = N_1^1(x) + N_2^1(x) + N_3^1(x) + N_4^1(x) = 1$$

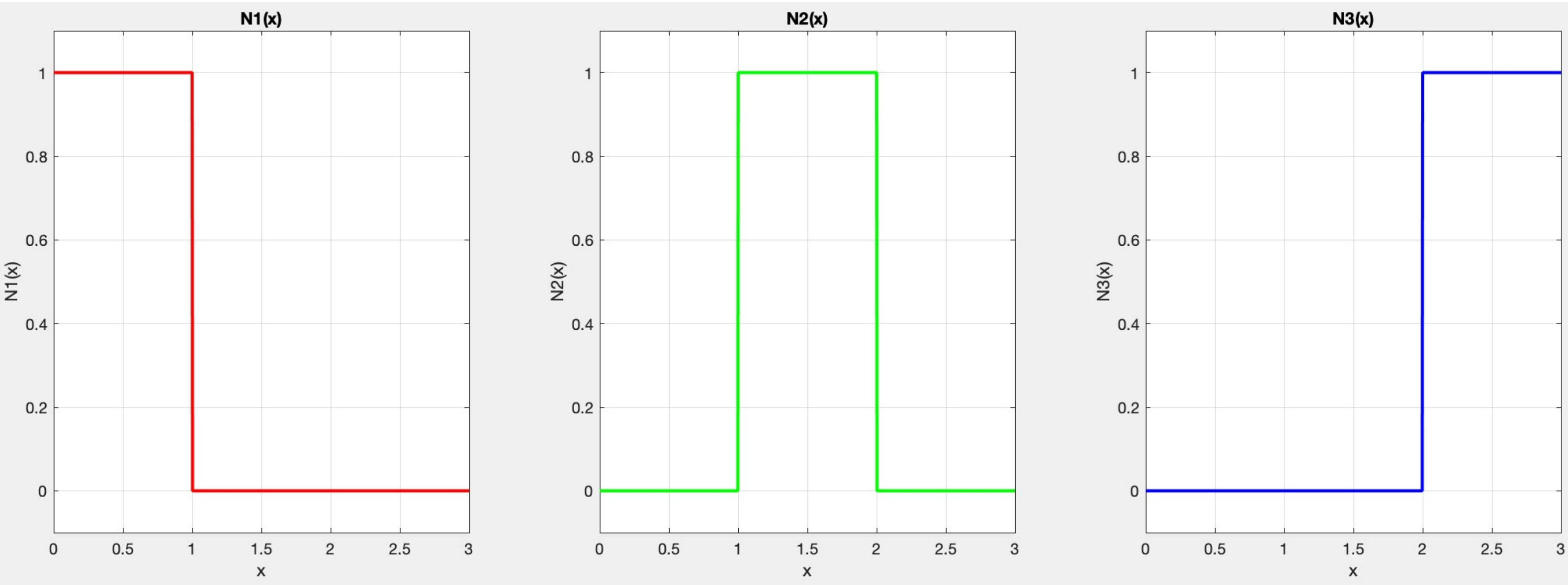


Therefore, $N^1(x) = 1$ (dash line) in element 1.

Similarly, we have $N^2(x) = N_1^2(x) + N_2^2(x) + N_3^2(x) + N_4^2(x) = 1$.

$N^3(x) = N_1^3(x) + N_2^3(x) + N_3^3(x) + N_4^3(x) = 1$

So, the global basis functions N_1, N_2, N_3 are



(b). We have $LG = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$. We know there are 6 global basis functions. The dimension of FE space is 6.

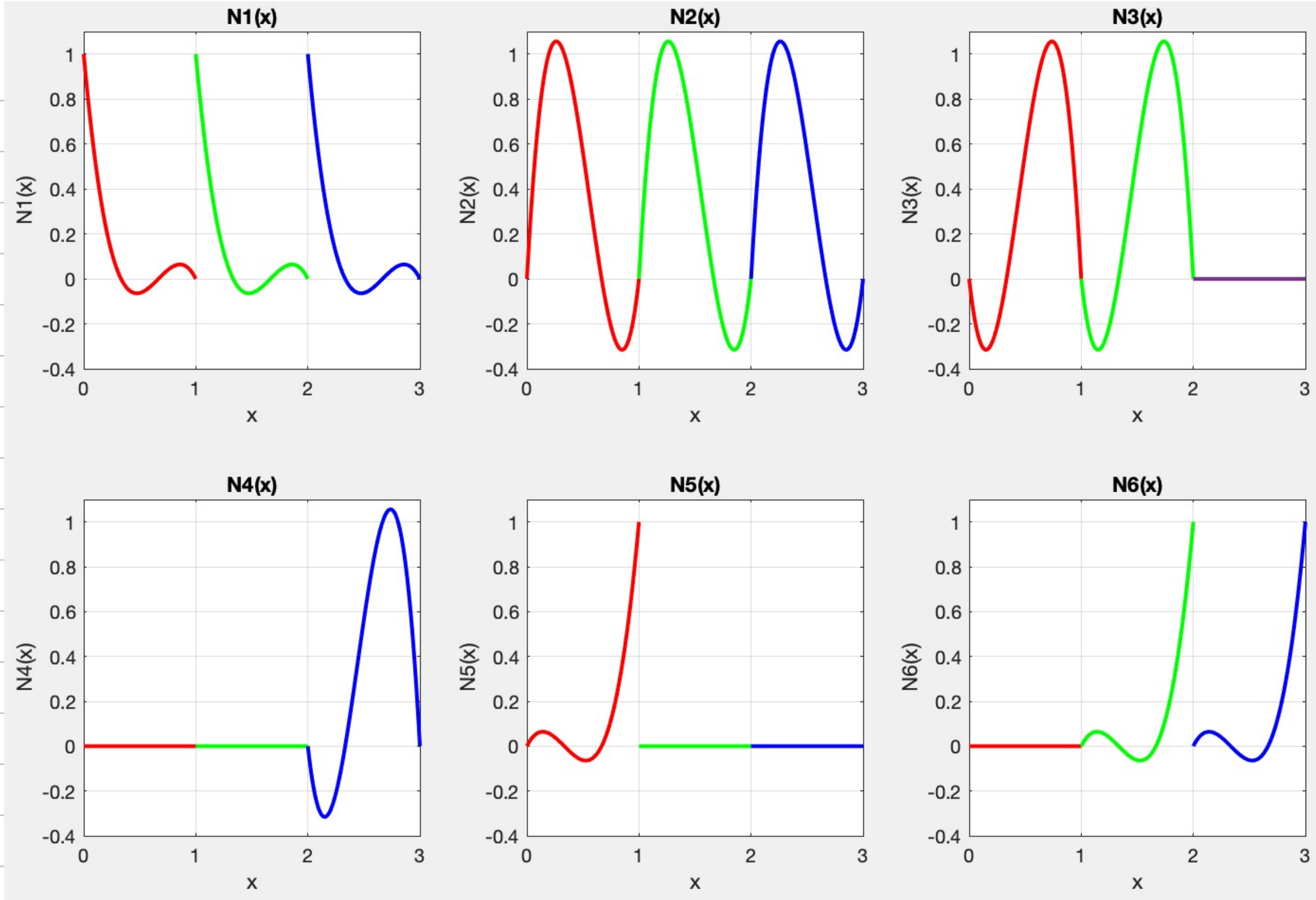
Observing first row, we know N_1^1, N_1^2, N_1^3 contribute to N_1. Here, shape functions $N_a^e(x)$ are defined as P_3 lagrange functions.

Second row tells N_2^1, N_2^2, N_2^3 contribute to N_2

N_3^1, N_3^2, contribute to N_3. and N_3^3 contribute to N_4

N_4^1 contribute to N_5; N_4^2 and N_4^3 contribute to N_6

Therefore, the plots of all basis functions are:



2. The following local-to-global map renders the basis functions to be continuous and have minimal support.

(a). For P_3 - elements.

Each element have 4 shape functions.

FE should be continuous with and across element boundaries. One possible LG map is:

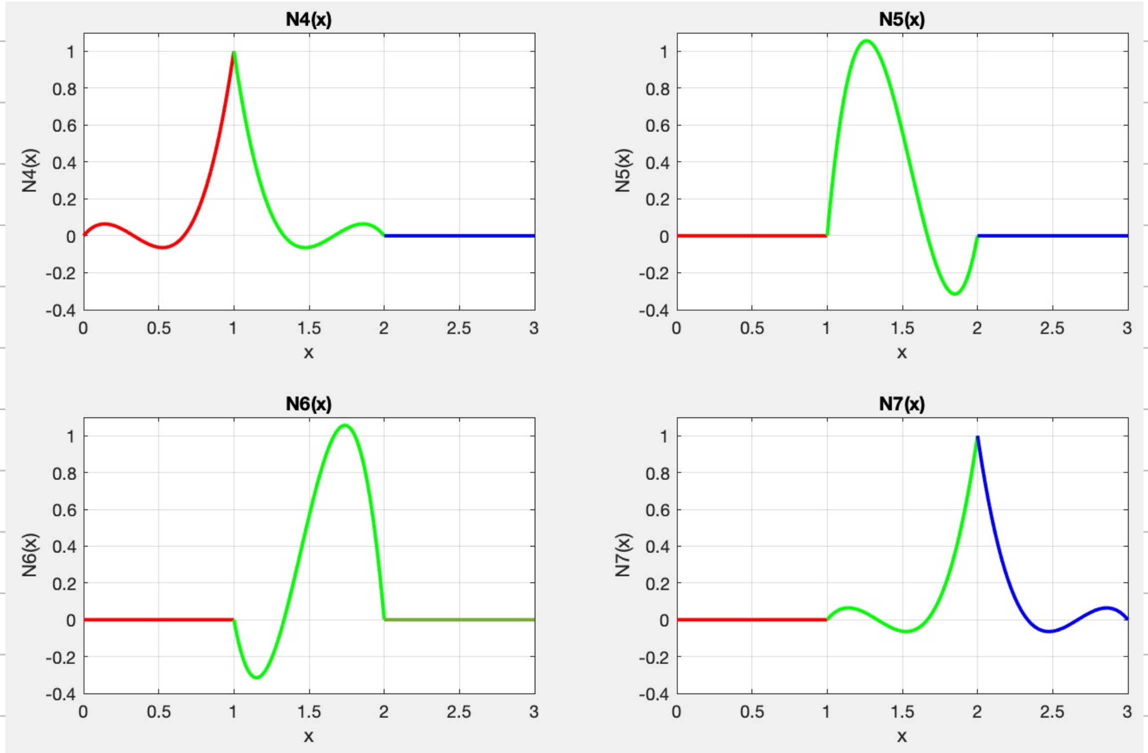
$$LG = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

Therefore, there 10 global bas functions.

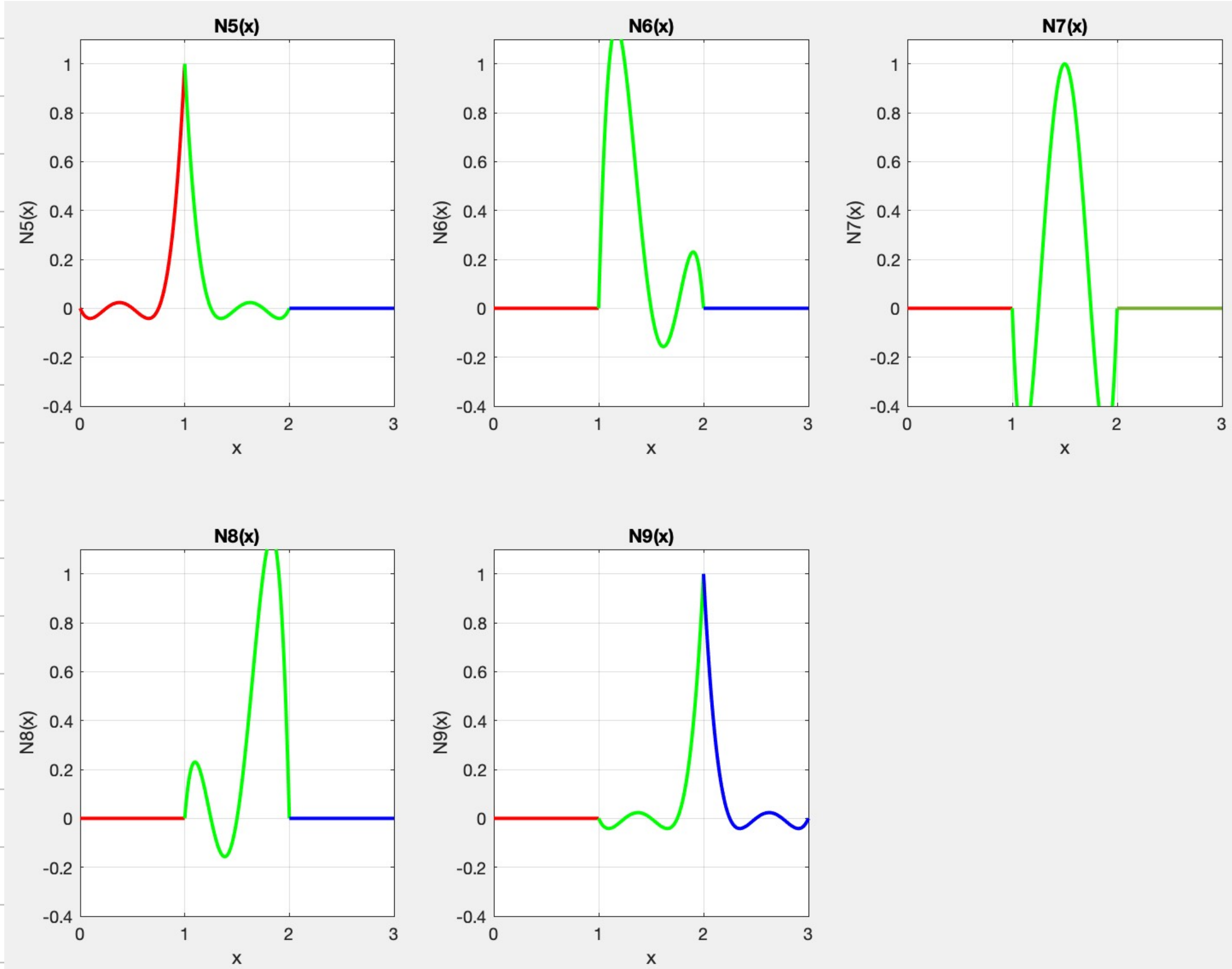
(b). Similiarly, for P_4-element, we have 5 local basis functions. The LG can be written as:

$$LG = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{bmatrix}$$

3. ① For P_3. element



3. For P_4 element



4. To enforce $u(0) = u(3)$, we could make

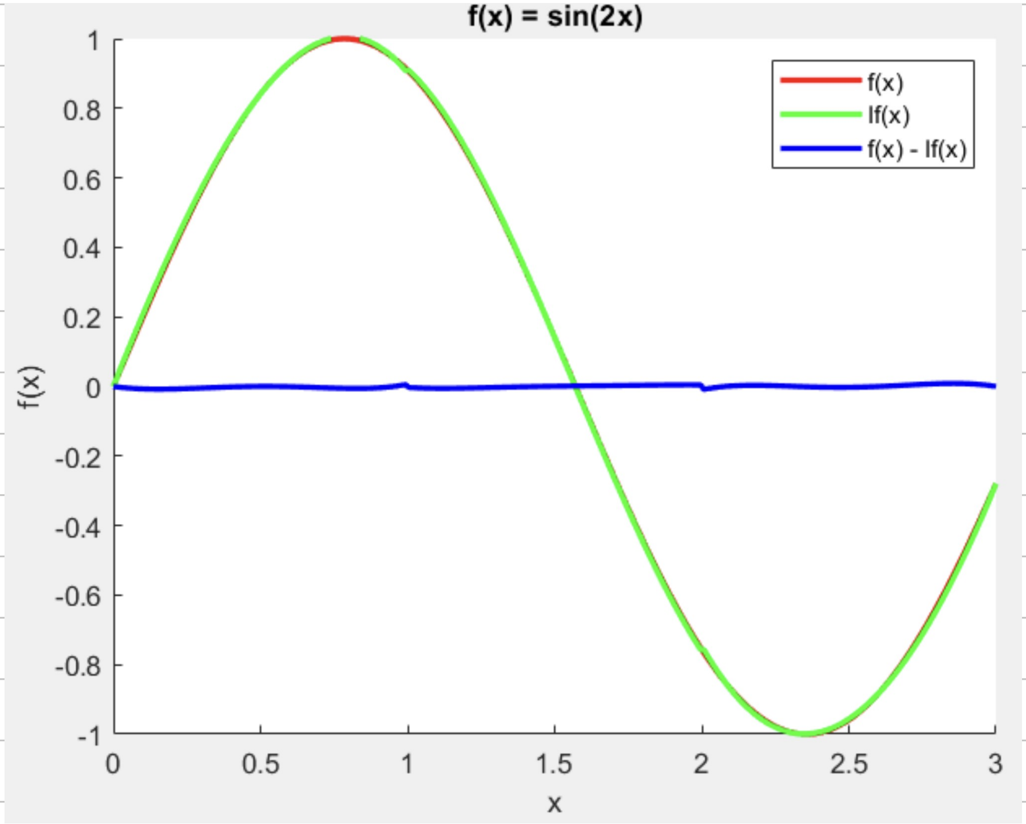
$$\sum_{i=1}^{a} N_i(x=0) = \sum_{i=1}^{a} N_i(x=3).$$

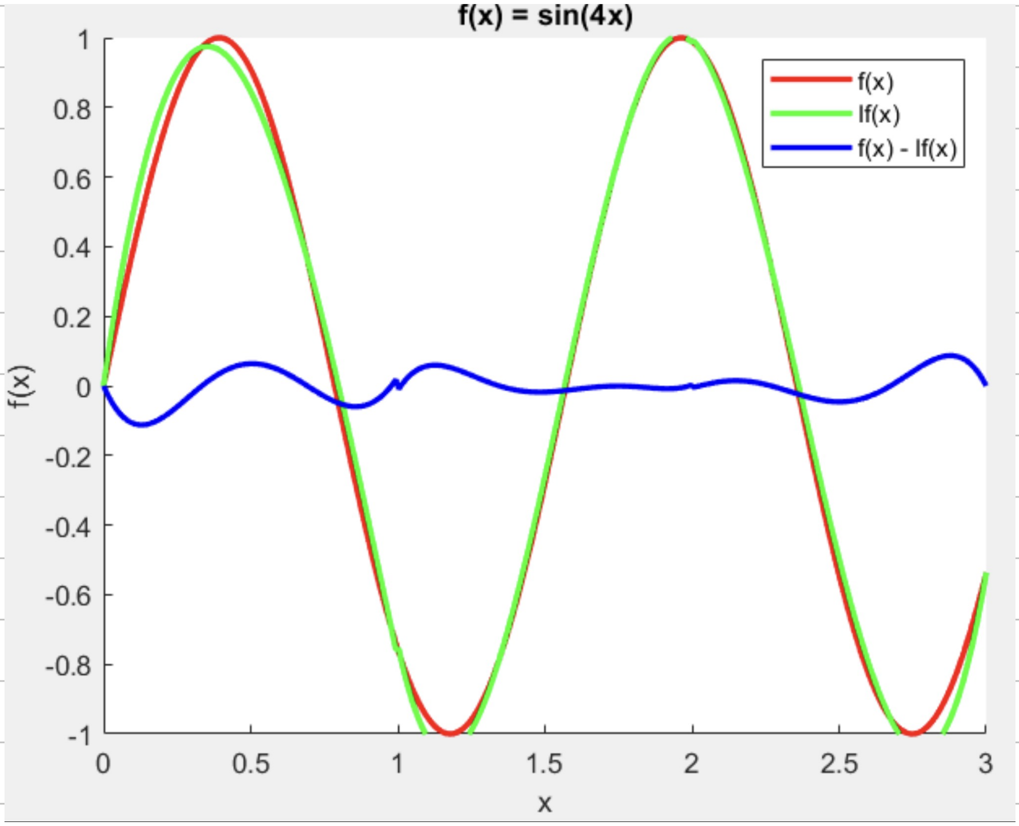
If we assume using P_1 element and continuous every where; we can write LG map as:

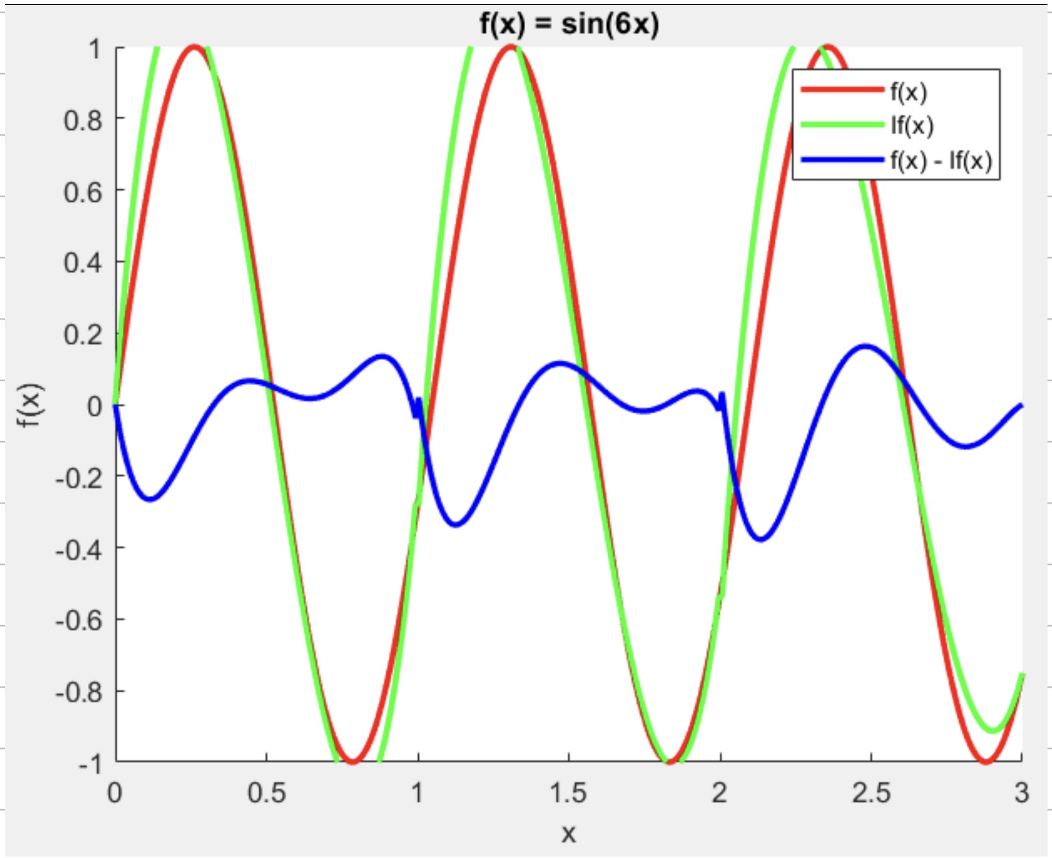
$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

The dimension of FE space is 3.

5.







f(x) = sin(6x)
f(x)
lf(x)
f(x) - lf(x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3

(Manual) Assembly (80)

Consider the domain $\Omega = [1, 7]$, and the convection-diffusion problem: Find $u : \Omega \to \mathbb{R}$ such that

$$-\varepsilon u_{,xx} + c u_{,x} = f \qquad x \in \mathring{\Omega}, \tag{1a}$$

$$-\varepsilon u_{,x}(7) = h, \tag{1b}$$

$$u(1) = g, \tag{1c}$$

where $\mathring{\Omega} = (1, 7)$ indicates the *interior* of the set Ω, $c \in \mathbb{R}$ is the convection velocity, $\varepsilon > 0$ is the diffusion coefficient, $h, g \in \mathbb{R}$ are boundary conditions, and $f : \Omega \to \mathbb{R}$ is a source. We would like to construct a finite element approximation of the solution of this problem.

1. (10) Find a variational equation according to the recipe in the notes, and identify natural and essential boundary conditions.

1. Define solution space and test space as follows:

$$S = \{u : [1, 7] \to \mathbb{R} \text{ Smooth} \mid u(1) = g\}$$
$$V = \{v : [1, 7] \to \mathbb{R} \text{ Smooth} \mid v(1) = 0\}$$

The variational form of the strong form can be derived as:

$$\int_1^7 (-\varepsilon u_{,xx} + c u_{,x}) v \, dx = \int_1^7 f v \, dx.$$

$$\int_1^7 \varepsilon v_{,x} u_{,x} - \int_1^7 \varepsilon (v u_{,x})' + \int_1^7 c u_{,x} v = \int f v$$

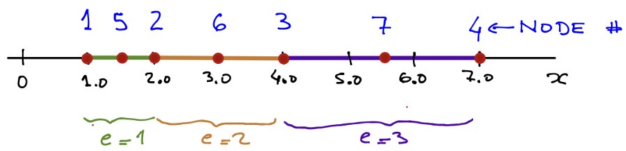
$$-\varepsilon v(7) u_{,x}(7) + \cancel{\varepsilon v(1) u_{,x}(1)} + \int_1^7 \varepsilon v_{,x} u_{,x} + c u_{,x} v = \int_1^7 f v$$

$$\int_1^7 \varepsilon v_{,x} u_{,x} + c u_{,x} v \, dx = \int_1^7 f v \, dx - v(7) h$$

Neutural B.C.: $-\varepsilon u_{,x}(7) = h$

Essential B.C.: $u(1) = g$

2. (5) Consider the nodes 1 to 7 with positions $\{1, 2, 4, 7, 1.5, 3, 5.5\}$, respectively; see figure. These nodes form P_2 elements 1,2, and 3, whose domains are $\Omega^1 = [1, 2]$, $\Omega^2 = [2, 4]$, $\Omega^3 = [4, 7]$. Using the node number as the index of global degree of freedom, write down the local-to-global map LG to build a space of continuous basis functions.



2. Given we are using P_2 lagrange element, the shape functions are defined as:

$$N_1^e(x) = \frac{(x - x_2^e)(x - x_3^e)}{(x_1^e - x_2^e)(x_1^e - x_3^e)}$$

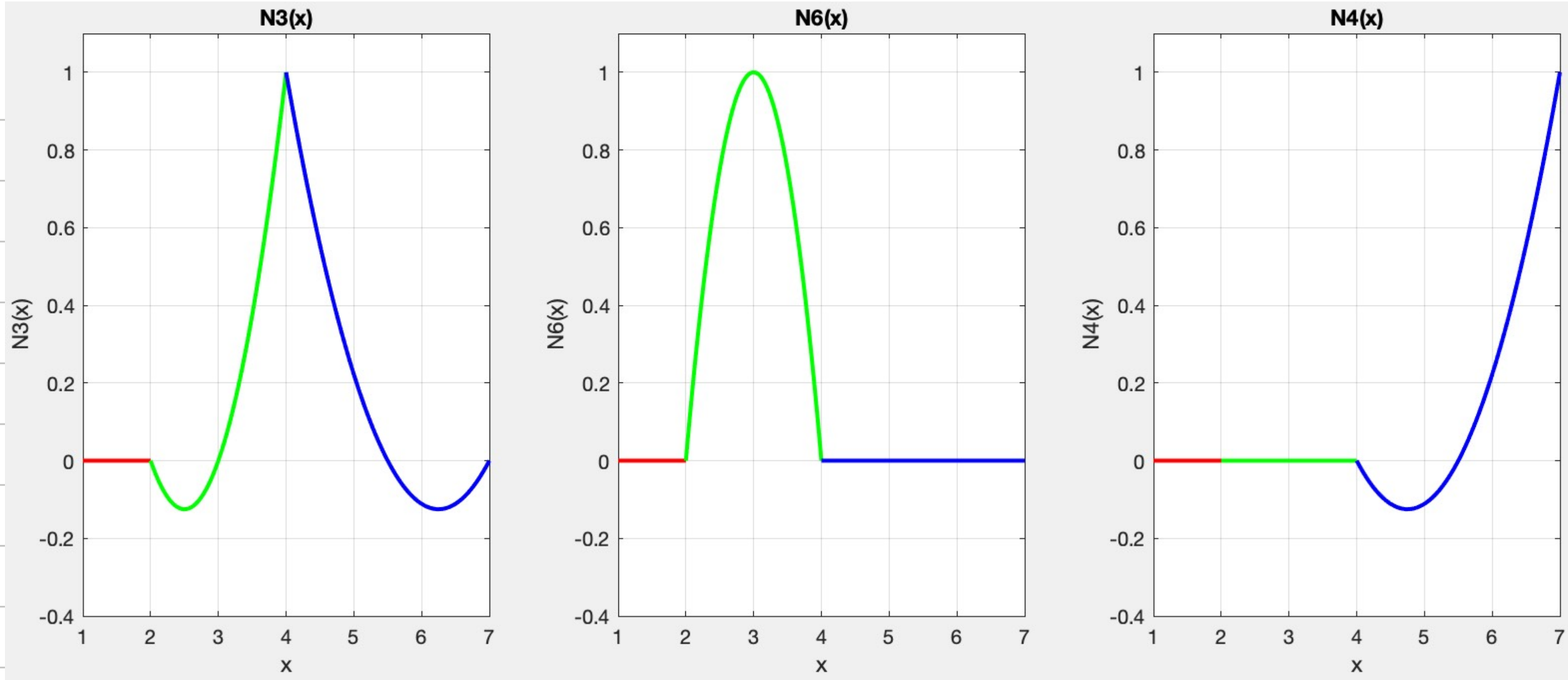
$$N_2^e(x) = \frac{(x - x_1^e)(x - x_3^e)}{(x_2^e - x_1^e)(x_2^e - x_3^e)}$$

$$N_3^e(x) = \frac{(x - x_1^e)(x - x_2^e)}{(x_3^e - x_1^e)(x_3^e - x_2^e)}.$$

According the global DOFs defined in the figure and continuous function requirement, we can write LG as:

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

3. Plot the global Basis functions N_3, N_6, and N_4.



4.

So $v_h \in \mathbb{P}^2$ can be written as:

$$v_h = 1.5\, N_1^2(x) + (-1.5)\, N_2^2(x) + 3\, N_3^2(x)$$

We know $N_1^2(x) = \dfrac{(x-3)(x-4)}{(-1)\cdot(-2)} = \dfrac{(x-3)(x-4)}{2}$

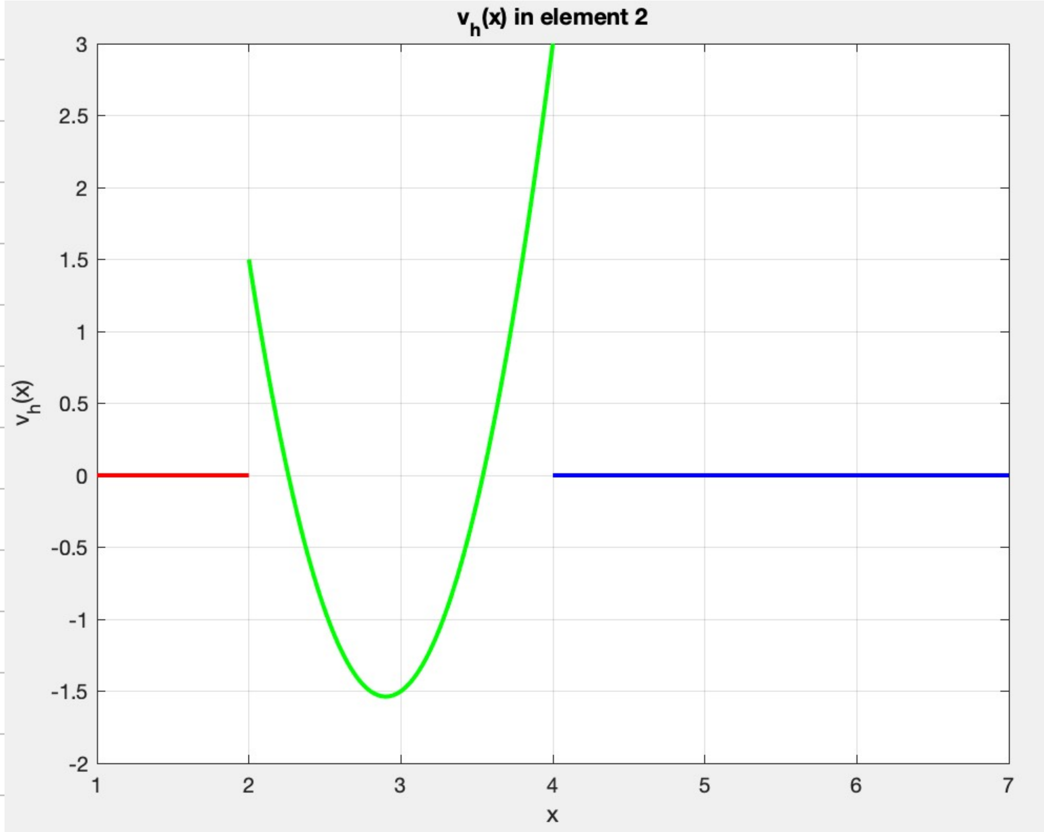
$$N_2^2(x) = \frac{(x-2)(x-4)}{(1)(-1)} = -\frac{(x-2)(x-4)}{1}$$

$$N_3^2(x) = \frac{(x-2)\cdot(x-3)}{(2)\cdot(1)} = \frac{(x-2)(x-3)}{2}$$

$$v_h = \frac{3}{4}(x-3)(x-4) + \frac{3}{2}(x-2)(x-4) + \frac{3}{2}(x-2)(x-3)$$

$$= \frac{15}{4}x^2 - \frac{93}{4}x + 33$$

The plot of v_h looks like:

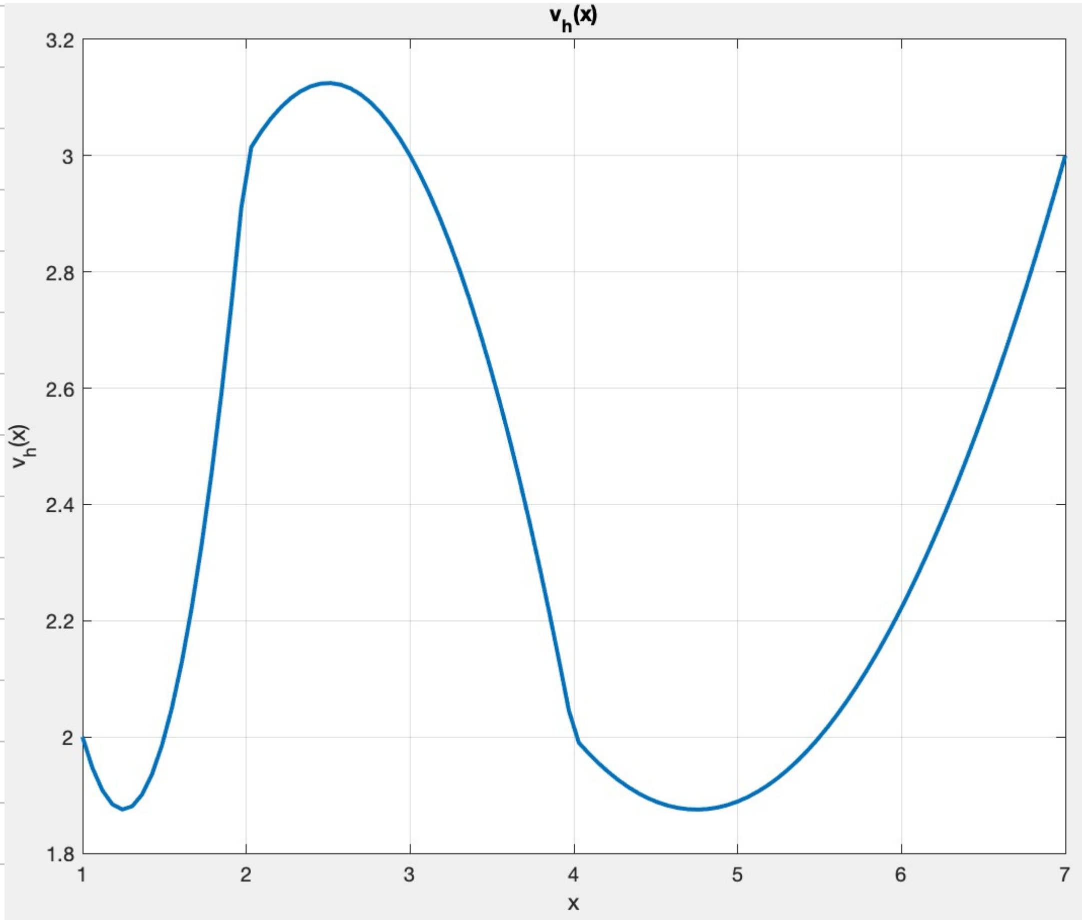


5. We know the LG map, which defines the relationship between local shape functions and global basis functions.

$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 2 & 3 & 4 \end{bmatrix}$$

Given $\mathcal{W}_h$ ~~be the~~ FE space of continuous functions, the function $v_h \in \mathcal{W}_h$ that is equal to 2 on odd nodes, to 3 on even nodes, can be written as:

$$v_h = 2N_1 + 2N_5 + 3N_2 + 3N_6 + 2N_3 + 2N_7 + 3N_4$$


v_h(x)
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1
2
3
4
5
6
7
x

6.

Let $\mathcal{W}_h = \text{span}(\{N_1, \dots, N_7\})$

$\mathcal{V}_h = \{v_h \in \mathcal{W}_h \mid v_h(1) = 0\}$.

$\mathcal{S}_h = \{\mathcal{V}_h + g\} = \{u_h \in \mathcal{W}_h \mid u_h(1) = g\}$.

A basis for $\mathcal{V}_h$ is obtained by setting active indice set $\eta_\alpha = \{2, 3, 4, 5, 6, 7\}$, and $\eta_g = \{1\}$. So,

$$\mathcal{V}_h = \text{span}(N_2, N_3, N_4, M_5, N_6, N_7)$$

$$\mathcal{S}_h = \mathcal{V}_h + \bar{u}_h \text{ , where } \bar{u}_n = g.$$

Find $u_h \in \mathcal{S}_h$ such that

$$a(u_h, v_h) = f(v_h) \quad \text{for all } v_h \in \mathcal{V}_h$$

where

$$a(u, v) = \int_1^7 e v_{,x} u_{,x} + c u_{,x} v \, dx$$

$$f(v) = \int_1^7 f v \, dx - v(7) h$$

7. Assume that $f(x) = x$, $h = -20$, $\epsilon = 1$, $c = 1$, $g = 2$

Plug these parameters into the weak form above, we have:

Find $u_h \in S_h$ such that

$$a(u_h, v_h) = f(v_h) \quad \text{for all } v_h \in V_h$$

where

$$a(u, v) = \int_1^7 (v_{,x} u_{,x} + u_{,x} v)\, dx$$

$$f(v) = \int_1^7 x v \, dx + 20 v(7)$$

If we assume a nature ordering (namely from 1 to 7) for dofs, the local stiffness matrix follows as:

For element 1, we have: K^e_{ab} where e is element #

a is the index of basis functions ~~for~~ test space,
b is the index of local basis functions in solution space.

$$K_{11} = 1, \quad K_{12} = K_{15} = 0.$$

$$K_{21} = \int_1^2 N^1_{3,x} N^1_{1,x} + N^1_{1,x} N^1_3 \, dx$$

$$K_{22} = \int_1^2 N^1_{3,x} N^1_{3,x} + N^1_{3,x} N^1_3 \, dx.$$

$$K_{25} = \int_1^2 N^1_{3,x} N^1_{2,x} + N^1_{2,x} N^1_3 \, dx$$

$$K_{31} = \int_1^2 N'_{2,x} \; N'_{1,x} \quad + \quad N'_{1,x} N'_{3} \; dx$$

$$K_{32} = \int_1^2 N'_{2,x} \; N'_{3,x} \quad + \quad N'_{3,x} N'_{2} \; dx$$

$$K_{33} = \int_1^2 N'_{2,x} \; N'_{2,x} \quad + \quad N'_{2,x} N'_{2} \; dx$$

Ke =

1.0000	0	0
-3.3333	5.3333	-2.0000
0.5000	-3.3333	2.8333

$$f_1 = \int_1^2 f N'_1 \, dx = \int_1^2 x N'_1 \, dx$$

$$f_2 = \int_1^2 x N'_3 \, dx$$

$$f_3 = \int_1^2 x N'_2 \, dx$$

Fe =

2.0000
1.0000
0.3333

For element 2.

$$K_{22} = \int_2^4 N^2_{1,x} N^2_{1,x} + N^2_{1,x} N^2_1 \, dx$$

$$K_{26} = \int_2^4 N^2_{1,x} N^2_{2,x} + N^2_{2,x} N^2_1 \, dx$$

$$K_{23} = \int_2^4 N^2_{1,x} N^2_{3,x} + N^2_{3,x} N^2_1 \, dx$$

$$K_{62} = \int_2^4 N^2_{2,x} N^2_{1,x} + N^2_{1,x} N^2_2 \, dx$$

$$K_{66} = \int_2^4 N^2_{2,x} N^2_{2,x} + N^2_{2,x} N^2_2 \, dx$$

$$K_{63} = \int_2^4 N^2_{2,x} N^2_{3,x} + N^2_{3,x} N^2_2 \, dx$$

$$K_{32} = \int_2^4 N^2_{3,x} N^2_{1,x} + N^2_{1,x} N^2_3 \, dx$$

$$K_{36} = \int_2^4 N^2_{3,x} N^2_{2,x} + N^2_{2,x} N^2_3 \, dx$$

$$K_{33} = \int_2^4 N^2_{3,x} N^2_{3,x} + N^2_{3,x} N^2_3 \, dx$$

$$f_2 = \int_2^4 x N^2_1 \, dx$$

$$f_6 = \int_2^4 x N^2_2 \, dx$$

$$f_3 = \int_2^4 x N^2_3 \, dx$$

```
Ke =

    0.6667   -0.6667         0
   -2.0000    2.6667   -0.6667
    0.3333   -2.0000    1.6667


Fe =

    0.6667
    4.0000
    1.3333
```

For element 3.

$$K_{33} = \int_{4}^{7} N^3_{1,x} N^3_{1,x} + N^3_{1,x} N^3_1 \, dx$$

$$K_{37} = \int_{4}^{7} N^3_{1,x} N^3_{2,x} + N^3_{2,x} N^3_1 \, dx$$

$$K_{34} = \int_{4}^{7} N^3_{1,x} N^3_{3,x} + N^3_{3,x} N^3_1 \, dx$$

$$K_{73} = \int_{4}^{7} N^3_{2,x} N^3_{1,x} + N^3_{1,x} N^3_2 \, dx$$

$$K_{77} = \int_{4}^{7} N^3_{2,x} N^3_{2,x} + N^3_{2,x} N^3_2 \, dx$$

$$K_{74} = \int_{4}^{7} N^3_{2,x} N^3_{3,x} + N^3_{3,x} N^3_2 \, dx$$

$$K_{43} = \int_{4}^{7} N^3_{3,x} N^3_{1,x} + N^3_{1,x} N^3_3 \, dx$$

$$K_{47} = \int_{4}^{7} N^3_{3,x} N^3_{2,x} + N^3_{2,x} N^3_3 \, dx$$

$$K_{44} = \int_{4}^{7} N^3_{3,x} N^3_{3,x} + N^3_{3,x} N^3_3 \, dx$$

$$f_3 = \int_{4}^{7} x N^3_1 \, dx$$

$$f_7 = \int_{4}^{7} x N^3_2 \, dx$$

$$f_4 = \int_{4}^{7} x N^3_3 \, dx + 20 N^3_3 (7)$$

```
Ke =

    0.2778   -0.2222   -0.0556
   -1.5556    1.7778   -0.2222
    0.2778   -1.5556    1.2778


Fe =

    2.0000
   11.0000
   23.5000
```

(c) Assembled matrix:

	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	0.5000	3.5000	0	0	-3.3333	-0.6667	0
3	0	0.3333	1.9444	-0.0556	0	-2	-0.2222
4	0	0	0.2778	1.2778	0	0	-1.5556
5	-3.3333	-2	0	0	5.3333	0	0
6	0	-2	-0.6667	0	0	2.6667	0
7	0	0	-1.5556	-0.2222	0	0	1.7778

Assembled Load Vector

	1
1	2
2	1
3	3.3333
4	23.5000
5	1
6	4
7	11

(d). $U = K^{-1}F$

U can be calculated using direct solver. The solution follows as:

	1
1	2.0000
2	4.5833
3	13.3745
4	43.9514
5	3.1562
6	8.2811
7	23.3841

$$U(x) = 2N_1(x) + 4.5833N_2(x) + 13.3745N_3(x) + 43.9514N_4(x) + 3.1562N_5(x) + 8.28811N_6(x) + 23.3841N_7(x).$$