ME 335A
Finite Element Analysis
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Problems Set #5–Solutions

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Due Wednesday, May 17, 2023

A Variational Method with a$_{\text{n almost}}$ Spectral Basis (53)

Let $\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < R^2\}$ for $R = 1$, $\partial\Omega_D = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. Consider the problem: Find $u \in \Omega \to \mathbb{R}$ such that

$$-\frac{1}{2}\Delta u = \frac{2}{R^2} \qquad \text{in } \Omega \tag{1a}$$

$$u = 0 \qquad \text{on } \partial\Omega_D \tag{1b}$$

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{1}{R} \qquad \text{on } \partial\Omega_N. \tag{1c}$$

1. (10) Construct a variational equation that u satisfies, following the standard recipe.

   Solution: In the following, we use indicial notation, in which $u_{,i}$ indicates $\partial u/\partial x_i$, and an index repeated twice implies sum over that index.

   Setting $R = 1$, the PDE is

$$-u_{,ii} = 4 \qquad \text{in } \Omega \tag{2a}$$

$$u = 0 \qquad \text{on } \partial\Omega_D \tag{2b}$$

$$u_{,i}\,\check{n}_i = -2 \qquad \text{on } \partial\Omega_N. \tag{2c}$$

   where $\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < 1\}$ for $R > 0$, $\partial\Omega_D = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. Consider v smooth enough. We proceed with the procedure to obtain the variational equation we have seen in class:

$$\int_\Omega (-u_{,ii} v)\, d\Omega = \int_\Omega 4v\, d\Omega \tag{3}$$

$$\int_\Omega u_{,i}\, v_{,i}\; d\Omega = \int_\Omega 4v\, d\Omega + \int_{\partial\Omega} v u_{,i}\, \check{n}_i\, d\Gamma \tag{4}$$

$$= \int_\Omega 4v\, d\Omega + \int_{\partial\Omega_N} v u_{,i}\, \check{n}_i\, d\Gamma \qquad \text{request } v = 0 \text{ on } \partial\Omega_D \tag{5}$$

$$= \int_\Omega 4v\, d\Omega - 2\int_{\partial\Omega_N} v\, d\Gamma \tag{6}$$

The variational equation that u satisfies is:

$$a(u,v) = l(v) \qquad \forall v \in \mathcal{V} \tag{7}$$

$$a(u,v) = \int_\Omega u,_i v,_i \; d\Omega \tag{8}$$

$$l(v) = \int_\Omega 4v d\Omega - 2\int_{\partial\Omega_N} v \; d\Gamma. \tag{9}$$

where $\mathcal{V} = \{u \colon \Omega \to \mathbb{R} \mid \; u = 0 \text{ on } \partial\Omega_D\}$.

2. (3) Identify essential and natural boundary conditions.

   Solution: Equation (1c) was incorporated in the variational equation, and hence it is a natural boundary condition. Therefore,

$$\text{Essential B.C.} \qquad u = 0 \text{ on } \partial\Omega_D \tag{10}$$

$$\text{Natural B.C.} \qquad u,_i \check{n}_i = -2 \text{ on } \partial\Omega_N. \tag{11}$$

3. Consider the approximation space

$$\mathcal{W}_h = \operatorname{span}\left(\sin\left(\pi\left(x_1^2+x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right), 1\right).$$

   (a) (10) Identify test and trial spaces, and active and constrained indices, naming the basis functions with indices in the order they appear above.

   Solution: We see that the essential boundary condition is satisfied by $\sin\left(\pi\left(x_1^2+x_2^2\right)\right)$ and $\cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right)$ but not by 1. The test and trial spaces therefore are

$$\mathcal{S}_h = \mathcal{V}_h = \operatorname{span}\left\{\sin\left(\pi\left(x_1^2+x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right)\right\}$$

   The active and constrained indices are $\eta_a = \{1,2\}$, and $\eta_g = \{3\}$.

   (b) (5) In this problem, there is a possibility of selecting a smaller space $\mathcal{W}_h$ without changing the results. What is this smaller space $\mathcal{W}_h$? Identify active and constrained indices in this new space.

   Solution: Because the third basis function, $N_3(x) = 1$, is not a basis function for either $\mathcal{S}_h$ or $\mathcal{V}_h$, we can eliminate it from $\mathcal{W}_h$. The smaller space is

$$\mathcal{W}_h = \operatorname{span}\left(\sin\left(\pi\left(x_1^2+x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right)\right).$$

   In this space, $\eta_a = \{1,2\}$, and $\eta_g = \emptyset$.

   (c) (15) Using the smaller space $\mathcal{W}_h$, compute the stiffness matrix and load vector.

   Solution:

$$u^h = u_1 \sin\left(\pi\left(x_1^2+x_2^2\right)\right) + u_2 \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right) = u_1 N_1 + u_2 N_2 \tag{12}$$

$$v^h = v_1 \sin\left(\pi\left(x_1^2+x_2^2\right)\right) + v_2 \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right) = v_1 N_1 + v_2 N_2 \tag{13}$$

   The solution procedure for a variational method implies that for any $i = 1, 2$,

$$\sum_{j=1}^{2} a(N_j, N_i) u_j = l(N_i) \tag{14}$$

The stiffness matrix has entries:

$$K_{11} = \int_\Omega (N_{1,1}^2 + N_{1,2}^2) d\Omega = \int_\Omega \nabla N_1 \cdot \nabla N_1 d\Omega \tag{15}$$

$$K_{12} = K_{21} = \int_\Omega N_{1,1}N_{2,1} + N_{1,2}N_{2,2} d\Omega = \int_\Omega \nabla N_1 \cdot \nabla N_2 d\Omega \tag{16}$$

$$K_{22} = \int_\Omega (N_{2,1}^2 + N_{2,2}^2) d\Omega = \int_\Omega \nabla N_2 \cdot \nabla N_2 d\Omega \tag{17}$$

We can reduce the complexity of integration by introducing the change of variables to polar coordinates. For a generic function f,

$$\nabla f = \frac{\partial f}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{u}_\theta \tag{18}$$

$$N_1 = \sin(\pi r^2) \tag{19}$$

$$N_2 = \cos\left(\frac{\pi}{2} r^2\right) \tag{20}$$

$$\nabla N_1 = 2\pi r \cos(\pi r^2)\hat{u}_r \tag{21}$$

$$\nabla N_2 = -\pi r \sin(\frac{\pi}{2} r^2)\hat{u}_r \tag{22}$$

$$K_{11} = \int_0^{2\pi} d\theta \int_0^1 \nabla N_1 \cdot \nabla N_1 r dr = \pi^3 \tag{23}$$

$$K_{12} = \int_0^{2\pi} d\theta \int_0^1 \nabla N_1 \cdot \nabla N_2 r dr = \frac{40\pi}{9} \tag{24}$$

$$K_{22} = \int_0^{2\pi} d\theta \int_0^1 \nabla N_2 \cdot \nabla N_2 r dr = \frac{\pi(\pi^2+4)}{4} \tag{25}$$

$$F_1 = \int_\Omega 4N_1 d\Omega - 2\int_{\partial\Omega_N} N_1 d\Gamma = \int_0^{2\pi} d\theta \int_0^1 4N_1 r dr - 2\int_{\partial\Omega_N} 0 d\Gamma = 8 \tag{26}$$

$$F_2 = \int_\Omega 4N_2 d\Omega - 2\int_{\partial\Omega_N} N_2 d\Gamma = \int_0^{2\pi} d\theta \int_0^1 4N_2 r dr - 2\int_{\partial\Omega_N} 0 d\Gamma = 8 \tag{27}$$

$$K = \begin{bmatrix} \pi^3 & \frac{40\pi}{9} \\ \frac{40\pi}{9} & \frac{\pi(\pi^2+4)}{4} \end{bmatrix}$$

$$F = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

(d) (5) Find the numerical approximation. Plot it together with the the exact solution.
Solution: We solve $Kd = F$,

$$d = \begin{bmatrix} \frac{72(9\pi^2-124)}{\pi(324\pi^2+81\pi^4-6400)} \\ \frac{288(9\pi^2-40)}{\pi(324\pi^2+81\pi^4-6400)} \end{bmatrix} \approx \begin{bmatrix} -0.1720 \\ 0.9548 \end{bmatrix}$$

The plot is in Fig.1.

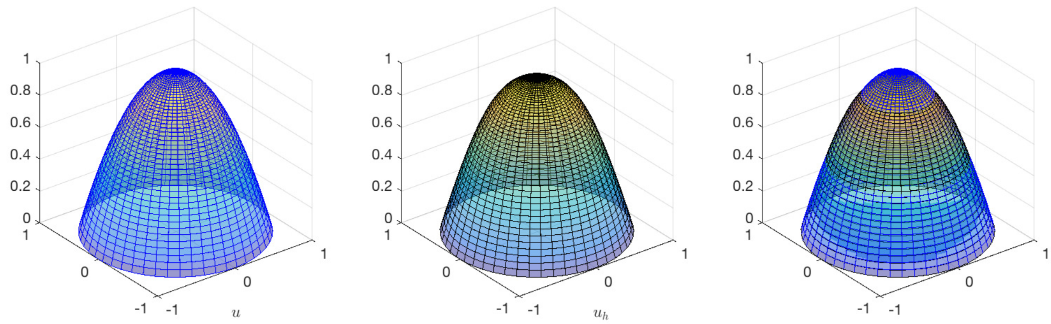


Figure 1: Plot of the exact solution u in blue grids, numerical solution u_h in black grids and two plots superposed together. We can see that the numerical solution underestimates the value at $(0, 0)$

(e) (5) Do you think the numerical approximation would change if we change the boundary condition on the Neumann boundary to

$$\frac{1}{2}\nabla u \cdot \check{n} = -\frac{2}{R}?$$

Solution: Because the two basis functions in $\mathcal{V}_h$ are zero on the entire boundary, the Galerkin approximation with this choice of space is not sensitive to the Neumann boundary conditions. A richer approximation space $\mathcal{W}_h$ is needed, in particular, one that is not made of only axi-symmetric functions (functions that depend only on r).

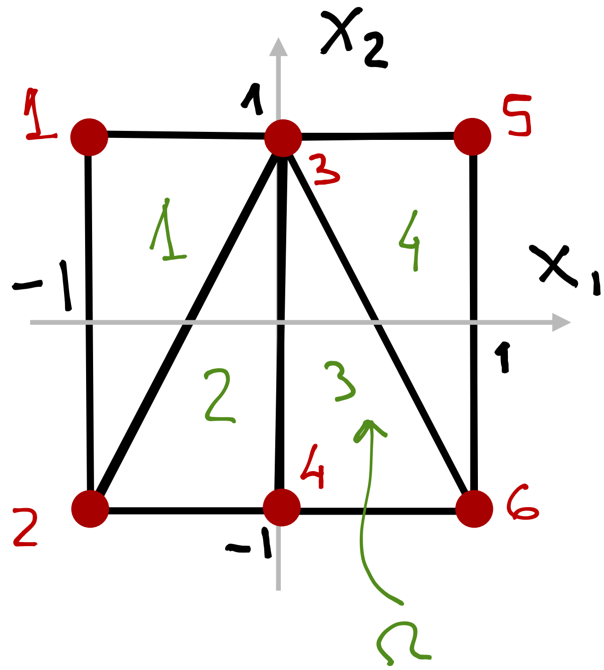
Manual Assembly, Once More (60)

Let $\Omega = [-1, 1] \times [-1, 1]$, $\partial\Omega_D = \{1\} \times [-1, 1]$, and $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$. Consider the variational equation that $u\colon \Omega \to \mathbb{R}$ satisfies:

$$\int_\Omega \nabla u \cdot \nabla v + uv \; d\Omega = \int_\Omega (x_1 + x_2)v \; d\Omega + \int_{\partial\Omega_N} (x_2^2 - 1)v \; d\Gamma$$

for all $v \in \mathcal{V} = \{v\colon \Omega \to \mathbb{R} \mid v = 0 \text{ on } \partial\Omega_D\}$, where (x_1, x_2) are the Cartesian coordinates in Ω. The function u satisfies the essential boundary condition $u(x_1, x_2) = x_2$ for $(x_1, x_2) \in \partial\Omega_D$.

Consider then the mesh shown in the figure, made of all P_1 elements:



1. (5) What is the local-to-global map for the mesh?

   Solution:

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 6 & 6 \\ 3 & 3 & 3 & 5 \end{bmatrix}.$$

2. (5) Identify $\mathcal{S}_h$ and $\mathcal{V}_h$ by providing the general expression for their functions in terms of the P_1 basis functions of the mesh. Identify active and constrained indices.

   Solution:
   Functions in $\mathcal{S}_h$ need to be equal to x_2 on $\partial\Omega_D$, and functions in $\mathcal{V}_h$ need to be equal to zero on $\partial\Omega_D$. Fortunately, the function x_2 is in the P_1 finite element space over this mesh, so we can impose this restriction. To this end, we need to set $u_h(1,1) = u_5 = 1$ and $u_h(1,-1) = u_6 = -1$, so that $u_h(1,x_2) = u_5 N_5(1,x_2) + u_6 N_6(1,x2) = x_2$.

   For $\mathcal{V}_h$, we need to set the components of basis functions N_5 and N_6 to zero. Therefore,

$$\mathcal{S}_h = \{u_h = u_1 N_1 + u_2 N_2 + u_3 N_3 + u_4 N_4 + N_5 - N_6 \mid (u_1,u_2,u_3,u_4) \in \mathbb{R}^4\},$$
$$\mathcal{V}_h = \{v_h = v_1 N_1 + v_2 N_2 + v_3 N_3 + v_4 N_4 \mid (v_1,v_2,v_3,v_4) \in \mathbb{R}^4\}.$$

3. (10) Evaluate the shape function N_3^2 and its derivative ∇N_3^2 at $(x_1,x_2) = (-0.5,-0.1)$.

   Solution:

   The shape function N_3^2 is the shape function that is equal to 1 at node $\mathsf{LG}(3,2) = 3$ and zero at nodes $\mathsf{LG}(1,2) = 2$ and $\mathsf{LG}(2,2) = 4$. Hence, $N_3^2 = \lambda_3$ on element 2.

   To solve this problem, we can use the formulas from the notes that give $\lambda_3(x_1,x_2)$. In this case, however, it is simple to build it by inspection. Since N_3^2 is a linear polynomial that is zero when

$x_2 = -1$ and is 1 when $x_2 = 1$, $N_3^2(x_1, x_2) = (1 + x_2)/2$. Hence,

$$N_3^2(-0.5, -0.1) = 0.45.$$

Its gradient is then

$$\nabla N_3^2(x_1, x_2) = (0, 1/2),$$

which is constant in the element.

4. (20) Compute the element stiffness matrix and load vector for each element. **Solution:** We can write out the expression of every element shape function.

$$N_1^1 = \frac{1}{2}(x_2 - 2x_1 - 1) \tag{28}$$
$$N_2^1 = \frac{1}{2}(-x_2 + 1) \tag{29}$$
$$N_3^1 = x_1 + 1 \tag{30}$$
$$N_1^2 = -x_1 \tag{31}$$
$$N_2^2 = \frac{1}{2}(-x_2 + 2x_1 + 1) \tag{32}$$
$$N_3^2 = \frac{1}{2}(x_2 + 1) \tag{33}$$
$$N_1^3 = \frac{1}{2}(1 - 2x_1 - x_2) \tag{34}$$
$$N_2^3 = x_1 \tag{35}$$
$$N_3^3 = \frac{1}{2}(x_2 + 1) \tag{36}$$
$$N_1^4 = \frac{1}{2}(1 - x_1) \tag{37}$$
$$N_2^4 = \frac{1}{2}(-x_2 + 1) \tag{38}$$
$$N_3^4 = \frac{1}{2}(x_2 + 2x_1 - 1) \tag{39}$$

Then we proceed to assemble element stiffness matrix and load vectors. Because the essential boundary condition is 0, the load vectors associated with the essential boundary condition is $\mathbf{0}$.

$$K_{ab}^e = \int_\Omega \nabla N_a^e \cdot \nabla N_b^e + N_a^e N_b^e \, d\Omega \tag{40}$$
$$F_a^e = \int_\Omega (x_1 + x_2) N_a^e \, d\Omega \tag{41}$$

$$K^1 = \begin{bmatrix} 17/12 & -1/6 & -11/12 \\ -1/6 & 5/12 & 1/12 \\ -11/12 & 1/12 & 7/6 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} 7/6 & -11/12 & 1/12 \\ -11/12 & 17/12 & -1/6 \\ 1/12 & -1/6 & 5/12 \end{bmatrix}$$

$$K^3 = \begin{bmatrix} 17/12 & -11/12 & -1/6 \\ -11/12 & 7/6 & 1/12 \\ -1/6 & 1/12 & 5/12 \end{bmatrix}$$

$$K^4 = \begin{bmatrix} 7/6 & 1/12 & -11/12 \\ 1/12 & 5/12 & -1/6 \\ -11/12 & -1/6 & 17/12 \end{bmatrix}$$

$$F^1 = \begin{bmatrix} -1/12 \\ -1/4 \\ 0 \end{bmatrix}$$

$$F^2 = \begin{bmatrix} -1/3 \\ -1/4 \\ -1/12 \end{bmatrix}$$

$$F^3 = \begin{bmatrix} -1/12 \\ 0 \\ 1/12 \end{bmatrix}$$

$$F^4 = \begin{bmatrix} 1/3 \\ 1/4 \\ 5/12 \end{bmatrix}$$

5. (5) Denote $\partial\Omega_D$ as line 1, and $\partial\Omega_N$ as line 2. Construct the array of boundary edges BE and compute the load vector associated to the natural boundary condition.

   Solution: The array of boundary edges is

$$\mathsf{BE} = \begin{bmatrix} 5 & 1 & 2 & 6 \\ 1 & 2 & 6 & 5 \\ 2 & 2 & 2 & 1 \end{bmatrix}.$$

   For the load vector associated with natural boundary condition. We can see that we only need to compute the vector with the edge with nodes 1,2, because $x_2^2 - 1 = 0$ for other edges with natural boundary condition.

$$F^{12} = \begin{bmatrix} -2/3 \\ -2/3 \end{bmatrix}$$

6. (10) Build the stiffness matrix and load vector.

   Solution: After the assembly,

$$K = \begin{bmatrix} 17/12 & -1/6 & -11/12 & 0 & 0 & 0 \\ -1/6 & 19/12 & 1/6 & -11/12 & 0 & 0 \\ -11/12 & 1/6 & 19/6 & -1/3 & -11/12 & 1/6 \\ 0 & -11/12 & -1/3 & 17/6 & 0 & -11/12 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -3/4 \\ -5/4 \\ 1/3 \\ -1/3 \\ 1 \\ -1 \end{bmatrix}$$

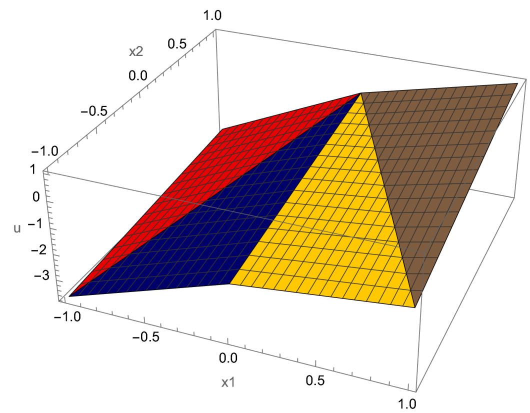


Figure 2: Plot of the numerical solution u_h

7. (5) Compute the finite element approximation, express it as a linear combination of basis functions, and plot it over the square.

Solution:

$$U = \begin{bmatrix} -135345/267089 \\ -364387/267089 \\ 75610/267089 \\ -226828/267089 \\ 1 \\ -1 \end{bmatrix} \approx \begin{bmatrix} -0.506741 \\ -1.36429 \\ 0.283089 \\ -0.84926 \\ 1. \\ -1 \end{bmatrix}$$