ME 335A
Finite Element Analysis
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Problems Set #3

January 24, 2022

Due Friday, January 28, 2022

For this problem set, we are going to play with the inteporlant of a function on a finite element mesh of Lagrange finite elements. Given a domain Ω and a mesh over it with m elements with $k+1$ nodes each, the *local interpolant* of a function f over and element e is a function $\mathcal{I}_e f\colon \Omega^e \to \mathbb{R}$ defined by

$$\mathcal{I}_e f(x) = \sum_{a=1}^{k+1} f(x_a^e) N_a^e(x).$$

where x_a^e is the location of the a-th node in element e, and N_a^e are the basis functions of each element.

The *global interpolant* $\mathcal{I}f\colon \Omega \to \mathbb{R}$ is a function defined by patches over each element, that is,

$$\mathcal{I}f\Big|_{\Omega^e} = \mathcal{I}_e f$$

or

$$\mathcal{I}f(x) = \mathcal{I}_e f(x) \qquad \text{whenever } x \in \Omega^e$$

So, to find the value of the global interpolant at a point x, we need to find the element to which x belongs, and evaluate the local interpolant there.

Your "First" Finite Element Approximation (45)

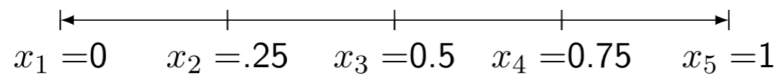
We want to construct a piecewise linear approximation to the weak form: Find $u \in \mathcal{S} = \{u\colon [0,1] \to \mathbb{R} \text{ smooth } \mid u(0) = 0\}$ such that $a(u,w) = \ell(w)$ for all $w \in \mathcal{V} = \mathcal{S}$, where

$$\begin{aligned} a(u,w) &= \int_0^1 (w_{,x}\, u_{,x} + \lambda w u)\, dx + w(1)u(1) \\ \ell(w) &= \int_0^1 w x^2\, dx + w(1), \end{aligned}$$

and $\lambda > 0$.

To this end, we will partition $[0,1]$ into four equal intervals, and build a finite element approximation with continuous functions that are lineal polynomials over each interval.

1. (5) Identify the location of all the vertices in the mesh, and number them. **Solution:** There are m=4 equal intervals, the element size is h= 1/4, where the locations of the vertices are $x_i = (i-1)/m$ for $i = 1...5$. then,



2. (5) Number and sketch all hat functions $\{N_a\}$ over $[0,1]$.

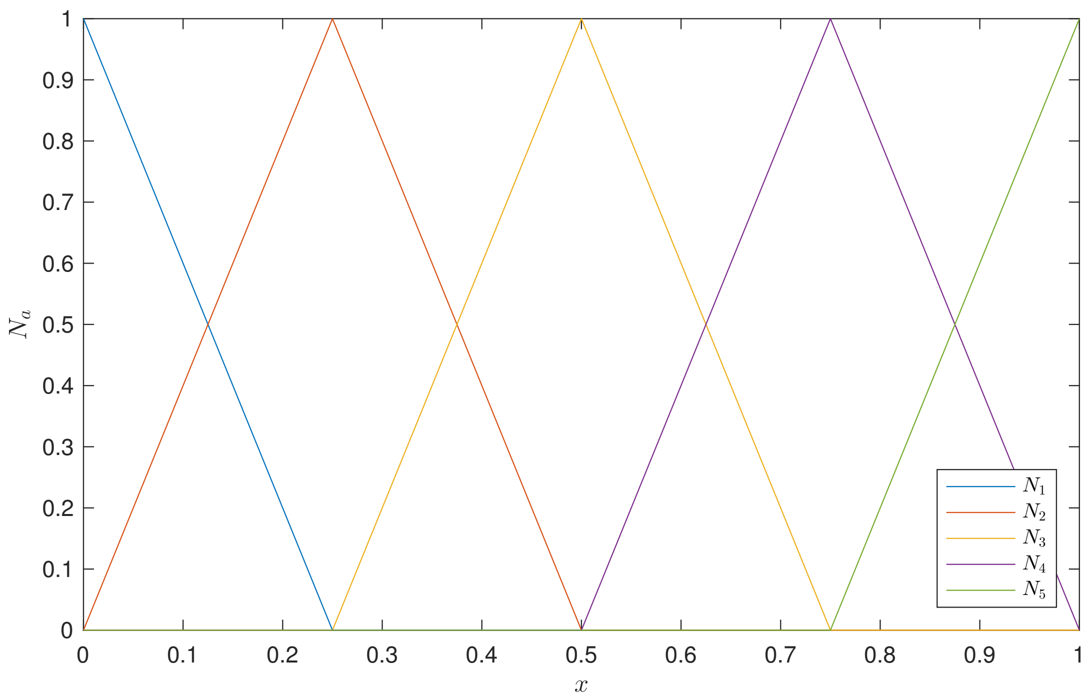
Solution: The hat functions for integer $a \in \{1,2,3,4\}$, $x_a = \frac{a-1}{4}$. For $a = 2,3,4$,

$$N_a = \begin{cases} \frac{x-x_{a-1}}{x_a-x_{a-1}}, & x_{a-1} \le x < x_a \\ \frac{x_{a+1}-x}{x_{a+1}-x_a}, & x_a \le x \le x_{a+1} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$N_1 = \begin{cases} \frac{x_2-x}{x_2-x_1}, & x_1 \le x \le x_2 \\ 0, & x_2 < x \end{cases}, \qquad N_5 = \begin{cases} 0, & x < x_4 \\ \frac{x-x_4}{x_5-x_4}, & x_4 \le x \le x_5 \end{cases}$$

The hat functions are sketched in Fig. 2



3. (5) Let $\mathcal{W}_h = \text{span}(N_1, \ldots, N_5)$. What are the trial space $\mathcal{S}_h$ and test space $\mathcal{V}_h$ for Galerkin Method, and the constrained and active index sets? Select $\overline{u}_h \in \mathcal{S}_h$. **Solution:** For this problem, we require $u(0) = 0$ and $w(0) = 0$. Therefore, we need

$$\mathcal{V}_h = \mathcal{S}_h = \{v_h \in \mathcal{W}_h \mid v_h(0) = 0\}. \tag{1}$$

A basis for $\mathcal{V}_h$ is obtained by setting $\eta_a = \{2, 3, 4, 5\}$ and $\eta_g = \{1\}$. So,

$$\mathcal{V}_h = \mathcal{S}_h = \text{span}(N_2, N_3, N_4, N_5).$$

Also, if $\overline{u}_h \in \mathcal{S}_h$, then $\overline{u}_1 = 0$. We can select $\overline{u}_h = 0$, for example.

4. (5) State Galerkin Method for this problem.

Solution: The Galerkin Method is: Find $u_h \in \mathcal{S}_h$ such that

$$a(u_h, w_h) = l(w_h) \text{ for all } w_h \in \mathcal{V}_h. \tag{2}$$

where

$$a(u, w) = \int_0^1 (w_{,x} u_{,x} + \lambda w u)\, dx + w(1)u(1) \tag{3}$$

$$\ell(w) = \int_0^1 w x^2\, dx + w(1) \tag{4}$$

5. (10) Compute the stiffness matrix, assuming that $\lambda = 2$. **Solution:**

The entries of the stiffness matrix for $a \in \eta_a$ and for any $b \in \eta$ are

$$K_{ba} = a(N_a, N_b) = \int_0^1 (N_{a,x} N_{b,x} + 2 N_a N_b)\, dx + N_a(1) N_b(1), \tag{5}$$

while for $a \in \eta_g$ and any $b \in \eta$ we have

$$K_{ab} = \delta_{ab}.$$

Computing,

$$K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 & 0 \\ 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} & 0 \\ 0 & 0 & -\frac{47}{12} & \frac{25}{3} & -\frac{47}{12} \\ 0 & 0 & 0 & -\frac{47}{12} & \frac{31}{6} \end{bmatrix}$$

6. (10) Compute the load vector.

Solution:
For $a \in \eta_a$,

$$f_a = \ell(N_a) = \int_0^1 N_a x^2\, dx + N_a(1) \tag{6}$$

and

$$f_1 = \overline{u}_1 = 0.$$

Then,

$$f = \begin{bmatrix} 0 \\ \frac{7}{384} \\ \frac{25}{384} \\ \frac{55}{384} \\ \frac{283}{256} \end{bmatrix}$$

7. (5) Find the approximate solution u_h.

Solution:
After obtaining K and f, we can solve the linear system $Kd = f$

$$d \approx \begin{bmatrix} 0 \\ 0.1044 \\ 0.2175 \\ 0.3417 \\ 0.4730 \end{bmatrix}$$

The approximate solution u_h is

$$u_h = 0.1044N_2 + 0.2175N_3 + 0.3417N_4 + 0.4730N_5 \tag{7}$$

Constructing Some FE Spaces (35)

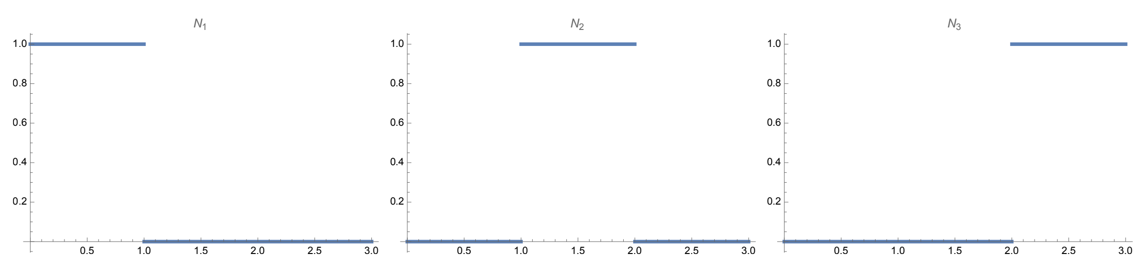
Consider a mesh of Lagrange P_k-elements (see Example 1.57 in the notes) with $n_{\text{el}} = 3$ elements of equal length in the interval $[0, 3]$. Elements are numbered consecutively from 1 to n_{el} from left to rigth (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

   (a) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

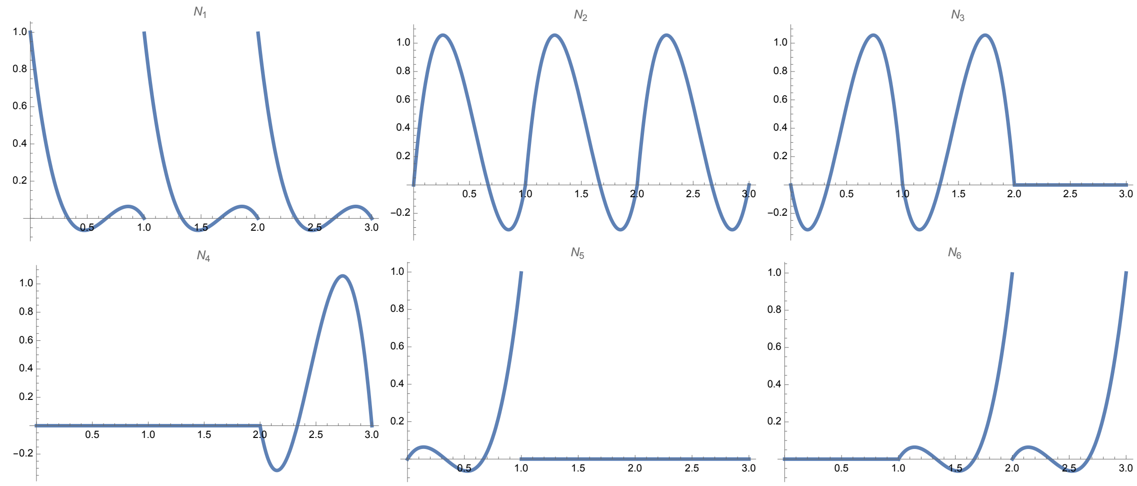
Solution: In this case, all shape functions of the same element need to be added up. Their sum is equal to 1 in each element. There is a total of 3 basis functions, and hence the dimension of the space is 3. Therefore, the basis functions are



   (b) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

Solution: In this case, there are 6 basis functions, and hence the dimension of the space is 6. The basis functions are



2. For each one of the following finite elements, write the local-to-global map so that the basis functions are continuous and have minimal support (each basis function should be zero in the minimum number of elements).

   (a) (5) For P_3-elements.
   Solution:

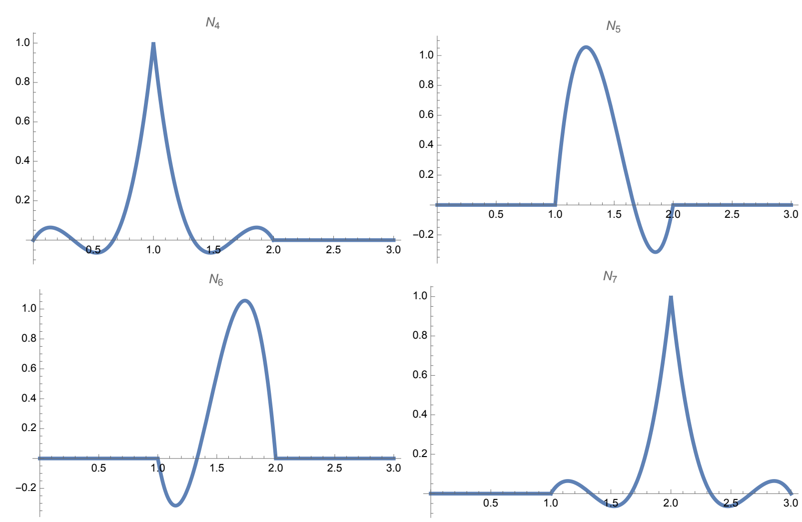
$$\mathsf{LG} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

   (b) (5) For P_4-elements.
   Solution:

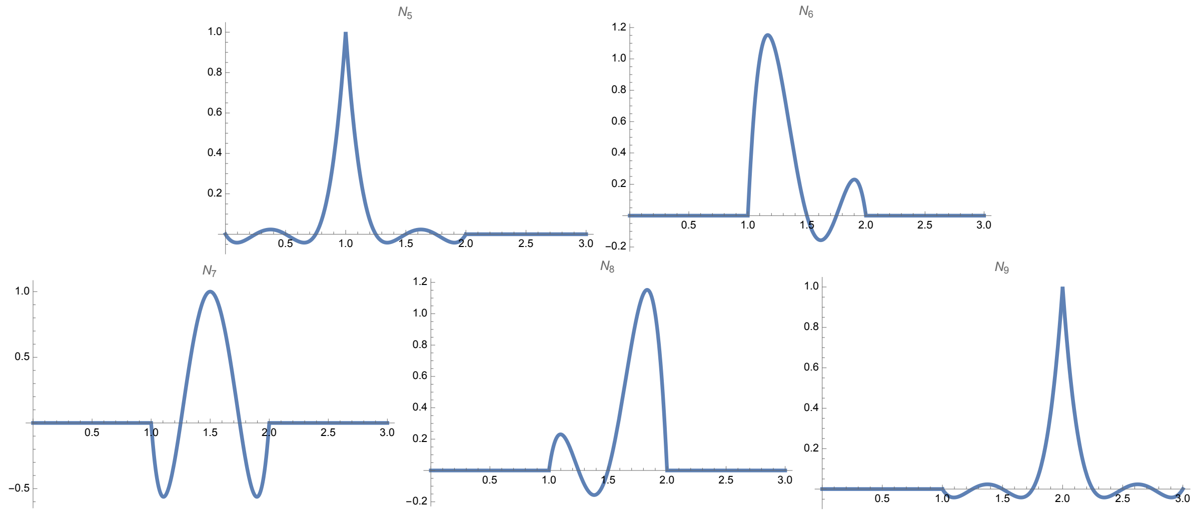
$$\mathsf{LG} = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{bmatrix}$$

3. (5) Plot the basis functions that are non-zero in the second element for each of the cases in part 2 of this problem.

   Solution: For P_3,



For P_4,



4. (5) For a problem with periodic boundary conditions of the form $u(0) = u(3)$, it is convenient to build a finite element space in which all functions in the space satisfy this periodicity constraint. If $k = 1$, write the required local-to-global map. What is the dimension of the finite element space?

   Solution: The space has dimension 3, and the local-to-global map is

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}.$$

5. (5) Let $f(x) = \sin(nx)$. Plot f, $\mathcal{I}f$ and $f - \mathcal{I}f$ when all elements in the mesh are P_3-elements, for $n = 2, 4, 6$.

   Solution: