

CP-2.

Strong form:

Find $u \in C^2(\Omega)$ such that:

$$-k \Delta u = f \text{ in } \Omega$$

$$u = g_1 \text{ on } \Gamma_{g_1}$$

$$u = g_2 \text{ on } \Gamma_{g_2}$$

$$k \nabla u \cdot \vec{n} = h_1 \text{ on } \Gamma_{h_1}$$

$$k \nabla u \cdot \vec{n} = h_2 \text{ on } \Gamma_{h_2}$$

$$k \nabla u \cdot \vec{n} = h_3 \text{ on } \Gamma_{h_3}$$

$$k \nabla u \cdot \vec{n} = h_4 \text{ on } \Gamma_{h_4}$$

Define

$S = \{u : \Omega \rightarrow \mathbb{R} \text{ smooth such that}$

$u(x) = g_1 \text{ for } x \in \Gamma_{g_1}; u(x) = g_2 \text{ for } x \in \Gamma_{g_2}\}$.

$V = \{v : \Omega \rightarrow \mathbb{R} \text{ smooth} \mid v(x) = 0 \text{ for } x \in \Gamma_{g_1} \cup \Gamma_{g_2}\}$.

Find $u \in S$ such that:

$$a(u, v) = f(v) \text{ for all } v \in V$$

where $a(u, v) = k \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega$

$$f(v) = \int_{\Omega} fv \, d\Omega + \int_{\partial\Omega_N} Hv \, d\Gamma$$

where $\partial\Omega_N = \{\Gamma_{h_1} \cup \Gamma_{h_2} \cup \Gamma_{h_3} \cup \Gamma_{h_4}\}$.

$$H = \begin{cases} h_1 & x \in \Gamma_{h_1} \\ h_2 & x \in \Gamma_{h_2} \\ h_3 & x \in \Gamma_{h_3} \\ h_4 & x \in \Gamma_{h_4} \end{cases}$$

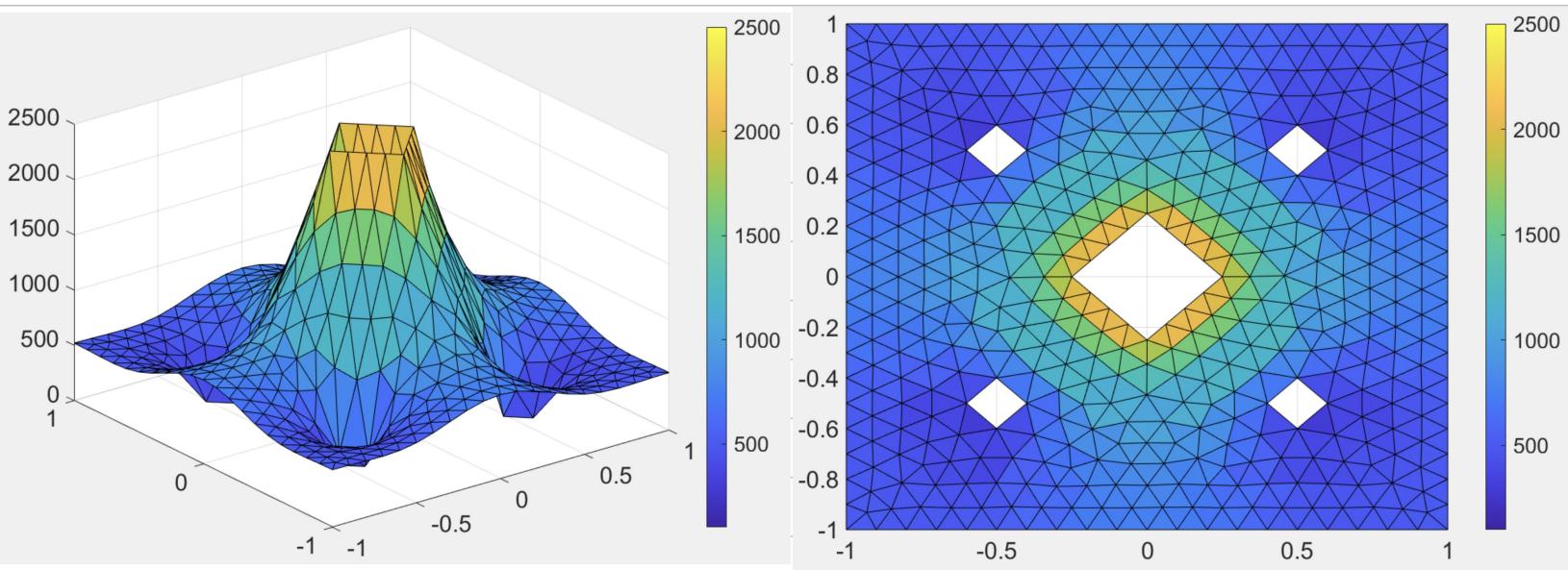
Discretized Form:

Let $\mathcal{W}_h = \text{span}(\{N_1, \dots, N_{e+1}\})$.

$$S_h = \{u_h \in \mathcal{W}_h \mid u_h(x_a^e) = q(x_a^e) \quad \forall x_a^e \in J_g\}$$

$$\mathcal{V}_h = \{v_h \in \mathcal{W}_h \mid v_h(x_a^e) = 0 \quad \forall x_a^e \in J_g\}$$

1.



2. & 3

See implementations of function. uValue.

function P1 Function.

4.

At $(0, 0.9)$,

$$u_h = 769.4303 \text{ } ^\circ\text{C}$$

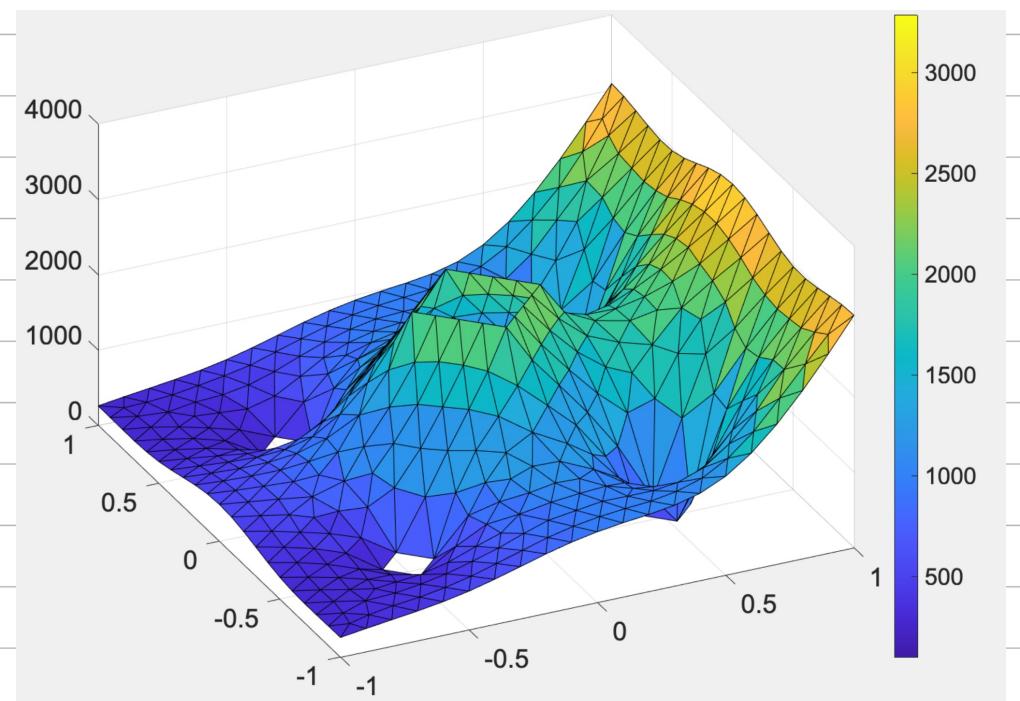
$$\frac{\partial u_h}{\partial x_1} = -0.5502 \text{ } ^\circ\text{C/ft}; \frac{\partial u_h}{\partial x_2} = -337.7738 \text{ } ^\circ\text{C/ft}$$

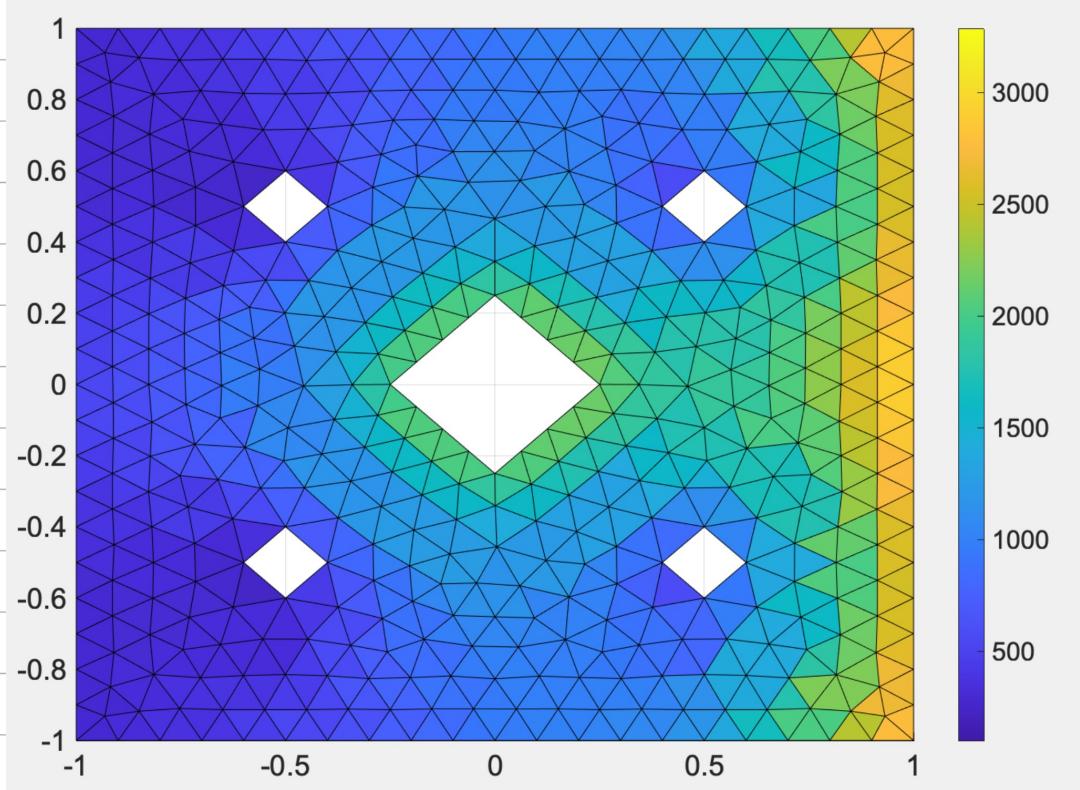
At $(0.25, 0.25)$,

$$u_h = 1.5811 \text{ e}3 \text{ } ^\circ\text{C}$$

$$\frac{\partial u_h}{\partial x_1} = -3.4063 \text{ e}3 \text{ } ^\circ\text{C/ft}; \frac{\partial u_h}{\partial x_2} = -3.4060 \text{ e}3 \text{ } ^\circ\text{C/ft}$$

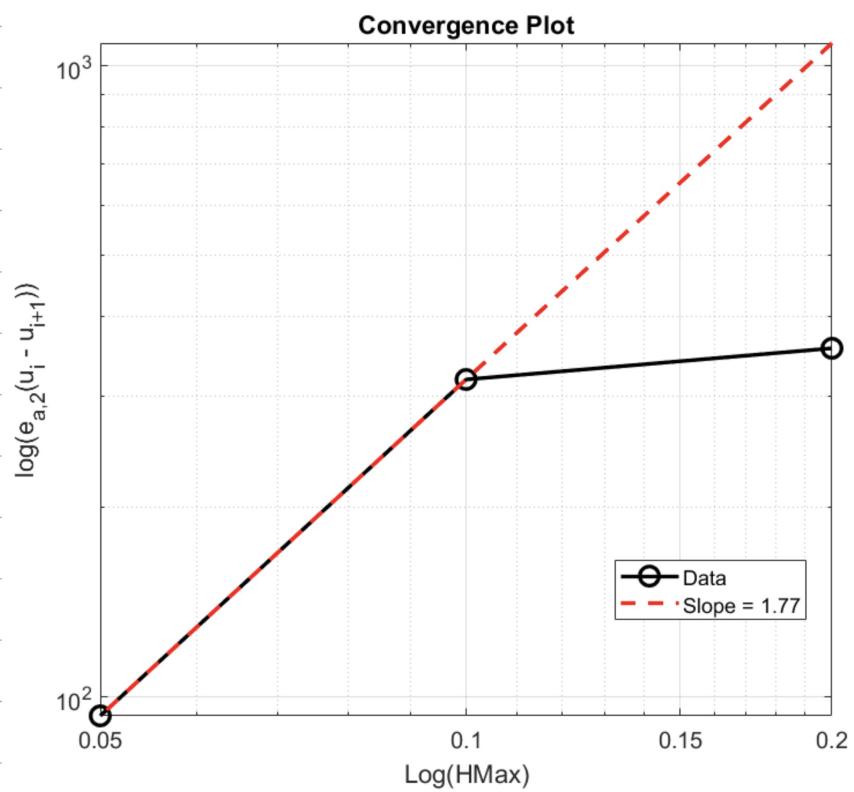
5.





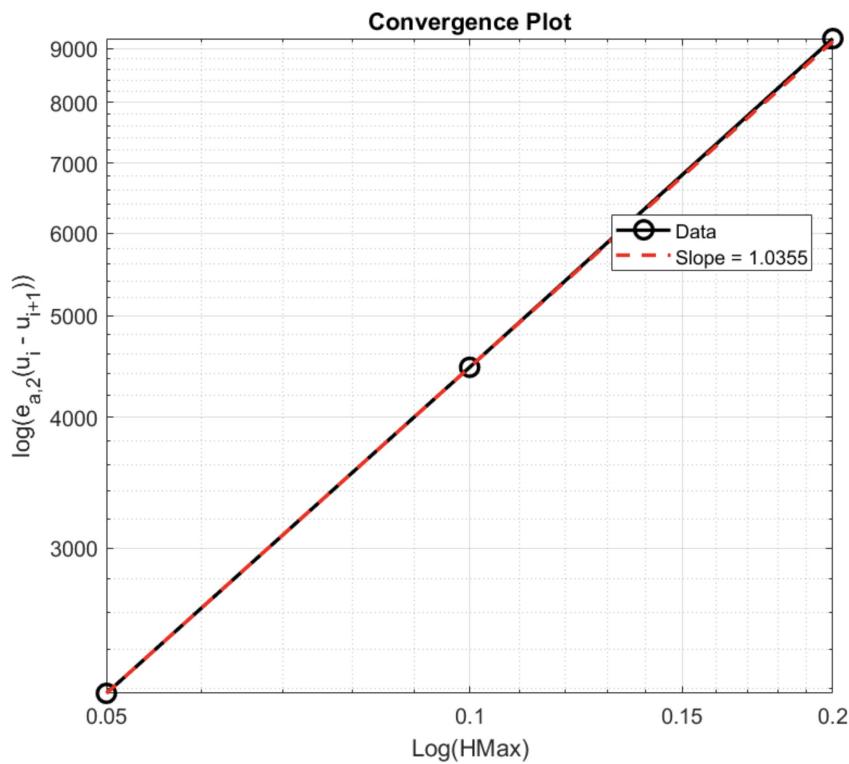
b. (a).

$$e_{0,2}(u) = \left[\sum_{i,j=0}^N u(X_{ij}) \right]^{\frac{1}{2}}$$



The figure above shows the log log plot of $e_{0,2}(u_i - u_{i+1})$ vs HMax. We can clearly observe that the convergence order is around 2.0 (1.77) for this problem with P1 element.

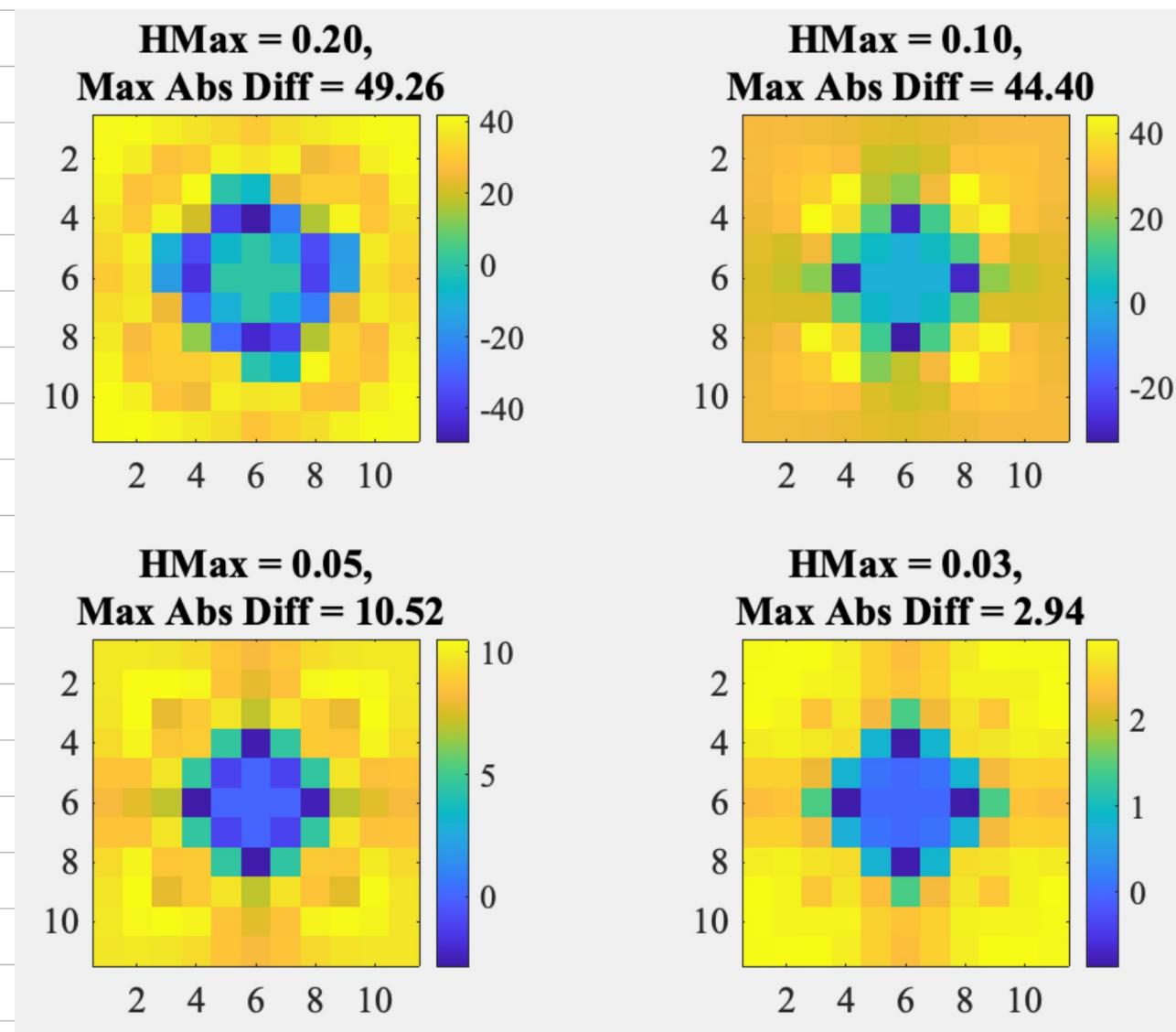
$$e_{1,2}(u) = \left[\sum_{i,j=0}^N \| \nabla u(x_{ij}) \|^2 \right]^{1/2}$$



The figure above shows the log log plot of $e_{1,2}(u_i - u_{i+1})$ vs HMax. For the gradient of $u(x)$, we observed a convergence rate of 1.035.

(b).

$$U_j(X) - U_{j+1}(X)$$



For $U(X)$, the mesh of $Hmax = 0.05$ will result in a max absolute difference between two consecutive solutions to be less than $30^\circ C$.

As the difference between steps will become smaller with mesh refinement, the maximum mesh meeting the requirement is 0.05.

$$\nabla \mathcal{U}_j(\mathbf{x}) - \nabla \mathcal{U}_{j+1}(\mathbf{x})$$

