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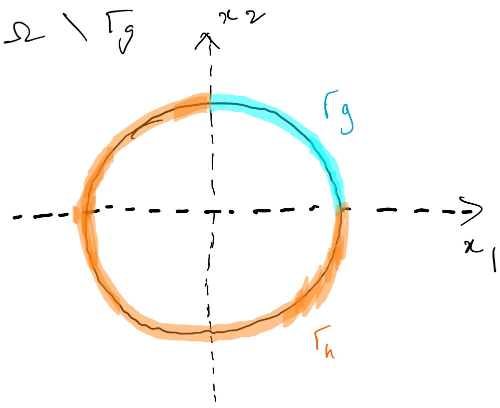
Given domain

$$\Omega = \{(x_1, x_2) \in \mathbb{R} \mid x_1^2 + x_2^2 < R^2\} \quad \text{for } R = 1$$

$$\Gamma_g = \partial\Omega \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0,\ x_2 \geq 0\}$$

$$\Gamma_h = \partial\Omega \setminus \Gamma_g$$



strong form

$$-\frac{1}{2}\Delta u = \frac{2}{R^2} \quad \text{in } \Omega \quad - (1a)$$

$$u = 0 \quad \text{on } \Gamma_g \quad (1b)$$

$$\frac{1}{2}\nabla u \cdot \hat{n} = \frac{-1}{R} \quad \text{on } \Gamma_h \quad (1c)$$

weak form

Multiply (1a) with smooth function $v : \Omega \to \mathbb{R}$ & integrate

$$\int_\Omega -\frac{1}{2}\, \mathrm{div}(\nabla u)\, v \, d\Omega = \int_\Omega \frac{2}{R^2} v \, d\Omega$$

Using integration by parts on LHS term

$$-\frac{1}{2}\left[\underbrace{\int_{\partial\Omega} v \cdot (\nabla u \cdot \hat{n})\, d\Gamma}_{\text{split into Neumann \& dirichlet boundaries}} - \int_\Omega \nabla u \cdot \nabla v \, d\Omega\right] = \int_\Omega \frac{2}{R^2} v \, d\Omega$$

$$\frac{1}{2}\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega \frac{2}{R^2} v \, d\Omega + \frac{1}{2}\left[\int_{\partial\Omega_g} v \cdot (\nabla u \cdot \hat{n})\, d\Gamma_g + \int_{\partial\Omega_h} v \cdot (\nabla u \cdot \hat{n})\, d\Gamma_h\right]$$

$$\frac{1}{2}\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} \frac{2}{R^2} v \, d\Omega \quad \frac{1}{2}\Big[\int_{\partial\Omega_g} \quad \int_{\partial\Omega_h} \Big]$$

we know from strong form

$$\frac{1}{2} \times \nabla u \cdot \hat{n} = -\frac{1}{R} \rightarrow \text{Natural boundary}$$

& we impose $v(x) = 0$ on $\partial\Omega_g$ as our constraint

Essential Boundary $\Rightarrow$ $u = 0$ on $\partial\Omega_g$

$$\Rightarrow \frac{1}{2}\int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} \frac{2}{R^2} v \, d\Omega - \int_{\partial\Omega_h} \frac{v}{R} \, d\Gamma_h$$

with trial space as

$$\mathcal{S} = \{ w : \Omega \rightarrow \mathbb{R}^2 \text{ smooth } \; w(x_1, x_2) = 0 \;\forall\; x \in \partial\Omega_g \}$$

and test space

$$\mathcal{V} = \{ w : \Omega \rightarrow \mathbb{R}^2 \text{ smooth } \; w(x_1, x_2) = 0 \;\forall\; x \in \partial\Omega_g \}$$

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Natural Boundary condition

$$\frac{1}{2} \nabla u \cdot \hat{n} = -\frac{1}{R}$$

acts as our natural B.C as it can be incorporated into our weak form

Essential B.C

$$u = 0$$

impose in trial space as part of constraint indices.

a) Given approximation space

$$W_h = \text{span}\left(\sin(\pi(x_1^2 + x_2^2)),\ \cos\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right),\ 1 \right)$$

Trial Space $\mathcal{S}_h = \mathcal{S} \cap W_h$

at $x_1^2 + x_2^2 = R^2$, $R = 1$ & $x_1 \geq 0$, $x_2 \geq 0$

$$x_1^2 + x_2^2 = 1$$

so the constraint is at $x_1^2 + x_2^2 = 1$ & $x_1 \geq 0$, $x_2 \geq 0$

we need $u(x_1, x_2) = 0$

let $N_1 = \sin(\pi(x_1^2 + x_2^2))$

$N_2 = \cos\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right)$

$N_3 = 1$

$$\mathcal{S}_h = \left\{ c_1 N_1 + c_2 N_2 + c_3 N_3 \mid c_3 = 0 \ \forall\ c_1, c_2 \in \mathbb{R} \right\}$$

$$u_h(x_1, x_2) = c_1 \sin(\pi(x_1^2 + x_2^2)) + c_2 \cos\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right)$$

here if we substitute the constraint $x_1^2 + x_2^2 = 1$ & $x_1 \geq 0$, $x_2 \geq 0$

$$u_h(x_1, x_2) = c_1 \sin\pi + c_2 \cos\left(\frac{\pi}{2}\right) = 0 \ \checkmark$$ satisfied

$$\mathcal{S}_h = \text{span}\left(\sin(\pi(x_1^2 + x_2^2)),\ \cos\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right) \right)$$

As $\mathcal{S}$ & $\mathcal{V}$ are similar i.e

$w(x_1, x_2) = 0$ on $\partial\Omega_g$

we will have

we will have

$$\mathcal{V}_h = \text{span}\left(\sin\left(\pi\left(x_1^2 + x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2 + x_2^2\right)\right)\right)$$

So the active indices are 1, 2 and $\eta_a = \{1, 2\}$

constrained index is $3 \rightarrow \{1\}$ function $\eta_g = \{3\}$

b) Smaller space $\mathcal{W}_h$

As N_3 is not in use for $\mathcal{U}_h$ or $\mathcal{V}_h$

we can select $\mathcal{W}_h$ as

$$\mathcal{W}_h = \text{span}\left(\sin\left(\pi\left(x_1^2 + x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2 + x_2^2\right)\right)\right)$$

in this case both $\mathcal{U}_h$ & $\mathcal{V}_h$ will be

same as $\mathcal{W}_h$ i-e

$$\mathcal{U}_h = \mathcal{V}_h = \mathcal{W}_h = \text{span}\left(\sin\left(\pi\left(x_1^2 + x_2^2\right)\right), \cos\left(\frac{\pi}{2}\left(x_1^2 + x_2^2\right)\right)\right)$$

where active indices are 1, 2

and there are No constrained indices

$$\eta_a = \{1, 2\}$$

$$\eta_g = \{\phi\}$$

c) we have

$$a(u_h, v_h) = \frac{1}{2} \int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega$$

$$l(v_h) = \frac{2}{R^2} \int_{\Omega} v_h \, d\Omega - \int_{\partial \Omega_h} \frac{v_h}{R} \, d\Gamma_h$$

and use N_1 , N_2 $\longrightarrow$ $r\theta$ transform

$$\nabla N_1 = \begin{bmatrix} 2\pi x_1 \cos(\pi(x_1^2 + x_2^2)) \\ 2\pi x_2 \cos(\pi(x_1^2 + x_2^2)) \end{bmatrix}$$

$$\nabla N_2 = \begin{bmatrix} -\pi x_1 \sin\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right) \\ -\pi x_2 \sin\left(\frac{\pi}{2}(x_1^2 + x_2^2)\right) \end{bmatrix}$$

$$k_{11} = \frac{1}{2} \int_{-1}^{1}\int_{-1}^{1} \nabla N_1 \cdot \nabla N_1 \, dx_1 \, dx_2$$

$$k_{12} = k_{21} = \frac{1}{2} \int_{-1}^{1}\int_{-1}^{1} \nabla N_1 \cdot \nabla N_2 \, dx_1 \, dx_2$$

$$k_{22} = \frac{1}{2} \int_{-1}^{1}\int_{-1}^{1} \nabla N_2 \cdot \nabla N_2 \, dx_1 \, dx_2$$

Convert to r, θ basis

$$k_{12} = \frac{1}{2} \int_{0}^{1}\int_{0}^{2\pi} \nabla N_1 \cdot \nabla N_2 \, r \, d\theta \, dr$$

$N_1 = \sin(\pi r^2)$, $N_2 = \cos \frac{\pi}{2} r^2$

Stiffness Matrix and load vector are computed in Matlab $\longrightarrow$ solution below.

$K =$

	N_1 (1)	N_2 (2)
1	15.5031	6.9813
2	6.9813	5.4466

$F =$

	1
1	4.0000
2	4.0000

d) Exact solution :

$$u(x,y) = g - \frac{f}{\ } (x_1^2 + x_2^2 - R^2)$$

/

$$u(x,y) = g - \frac{f}{4K}\left(x_1^2 + x_2^2 - R^2\right)$$

$K = 1/2$, $f = 2$, $R = 1$

$g = 0$

$$= -\frac{2}{4\times\left(\frac{1}{2}\right)}\left(x_1^2 + x_2^2 - 1\right) = 1 - x_1^2 - x_2^2$$

Galerkin :

$$-0.1720 \times \sin\left(\pi\left(x_1^2+x_2^2\right)\right) + 0.9548 \times \cos\left(\frac{\pi}{2}\left(x_1^2+x_2^2\right)\right)$$

U1	-0.1720
U2	0.9548

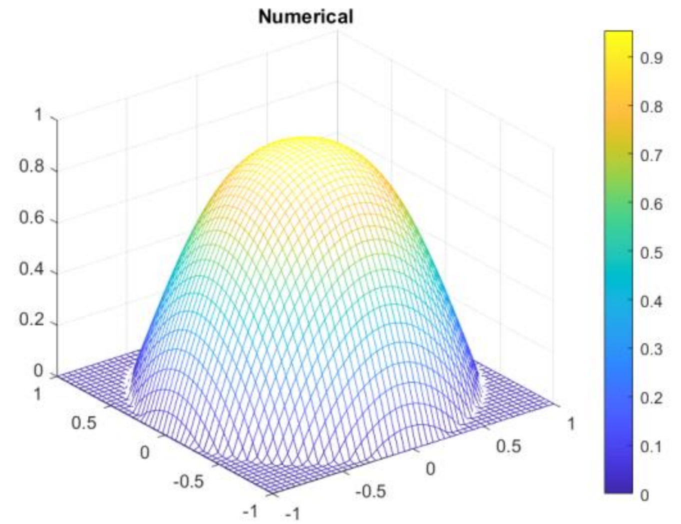
e) No the approximation will not change as the value of both basis functions is zero along the whole boundary.

and in our weak form the as the Neumann boundary is incorporated like

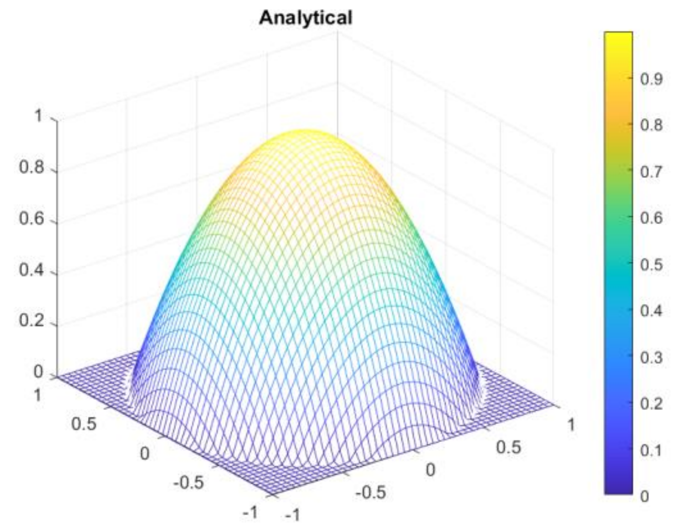
$$-\int \frac{v}{R}\, d\Gamma_h \Rightarrow -\int \frac{2}{R} v\, d\Gamma_h$$

but as v is always zero the solution doesnot change

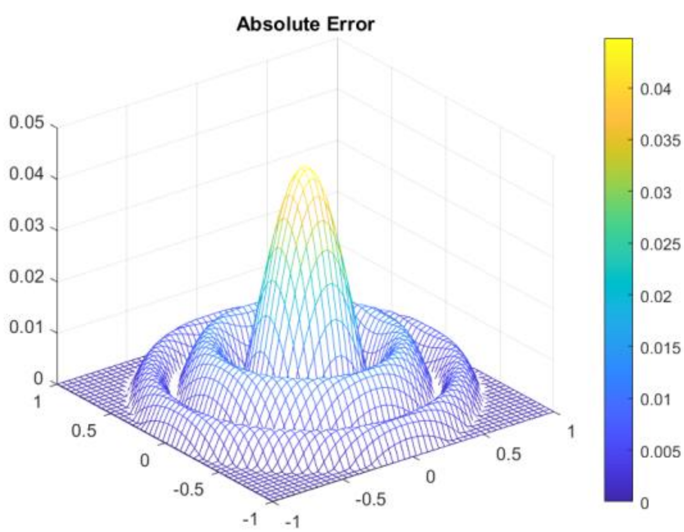
d) Galerkin :

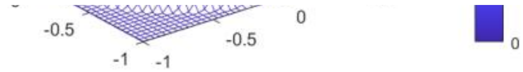


Analytical solution :



Error (|Num - Anal|)




-0.5
-1
-1
-0.5
0
0

Code

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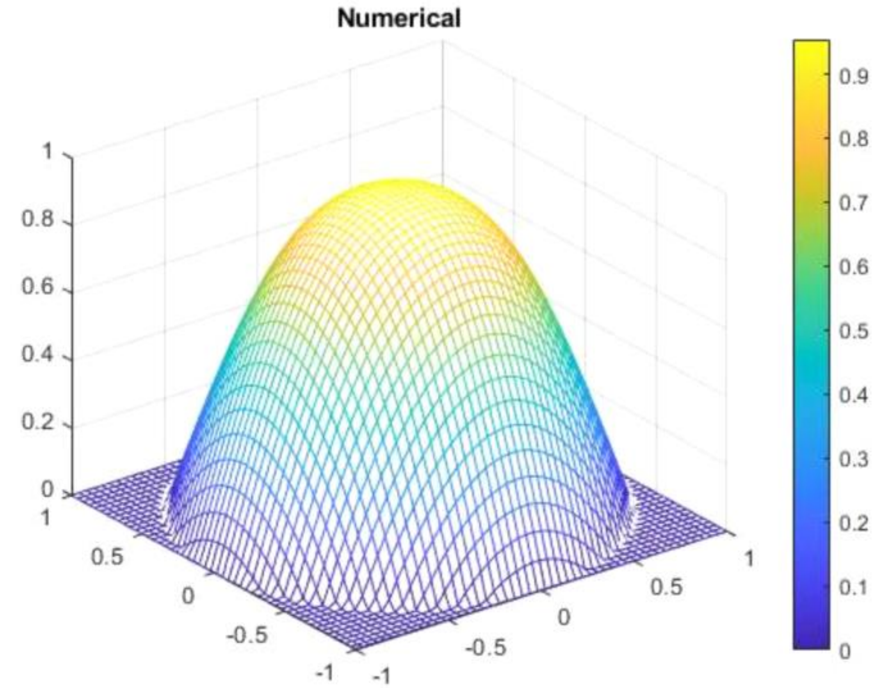
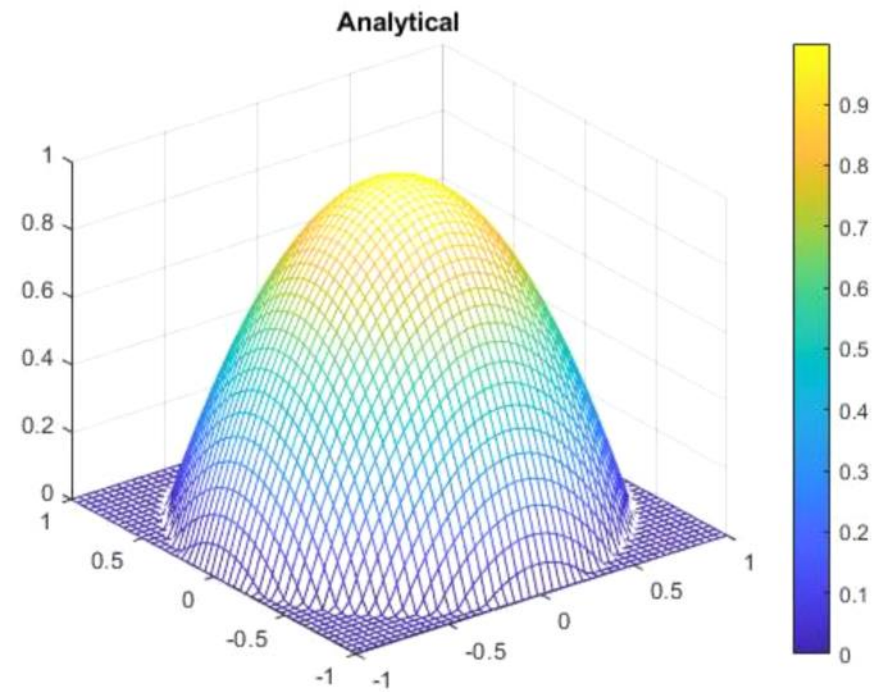
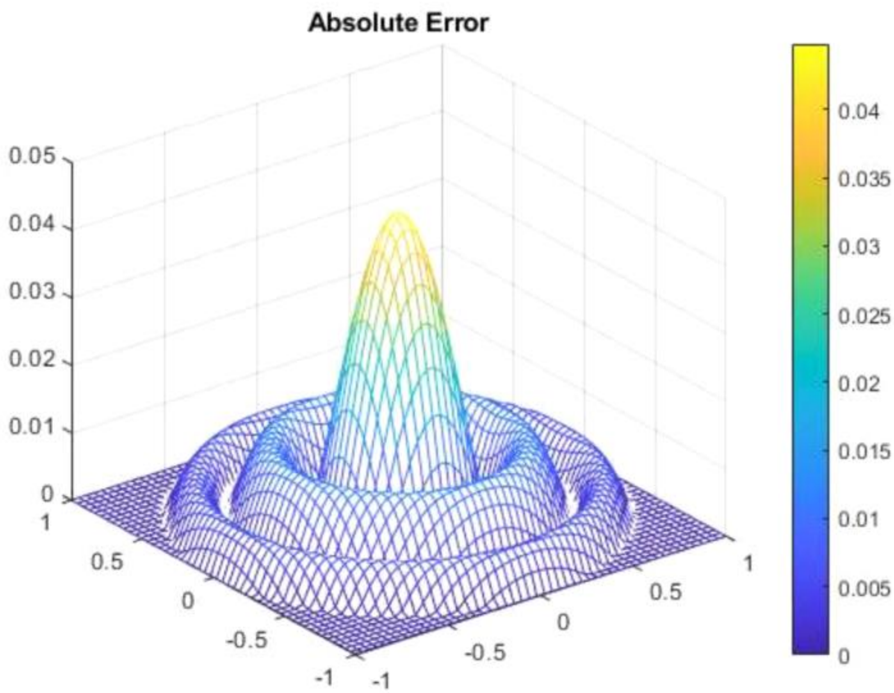
```
N11 = @(r,Th) 2*pi*r.*cos(pi*(r.^2));
N21 = @(r,Th) -pi*r.*sin((pi/2)*(r.^2));

K12_N = integralNab(N11, N21);
K11_N = integralNab(N11, N11);
K22_N = integralNab(N21, N21);

N1 = @(r,Th) r.*sin(pi*(r.^2));
N2 = @(r,Th) r.*cos((pi/2)*(r.^2));

F1 = integral2(N1, 0, 1, 0, 2*pi);
F2 = integral2(N2, 0, 1, 0, 2*pi);

F = 2*[F1; F2];
K = [K11_N, K12_N; K12_N, K22_N];
U = K\F;

N1_n = @(x,y) sin(pi*(x.^2 + y.^2));
N2_n = @(x,y) cos((pi/2)*(x.^2 + y.^2));

x = linspace(-1,1, 50);
y = linspace(-1,1, 50);
[X,Y] = meshgrid(x,y);
Z = (U(1)*N1_n(X,Y) + U(2)*N2_n(X,Y));
Z(Z<0)=0;
figure(1)
mesh(X,Y,Z)
title('Numerical')
colorbar
Z_an = (1 - X.^2 - Y.^2);
Z_an(Z_an<0)=0;
figure(2)

mesh(X,Y,Z_an)
title('Analytical')
colorbar

figure(3)

mesh(X,Y,abs(Z - Z_an))
title('Absolute Error')
colorbar

r_1 = linspace(0, 1, 200);
Area = 0;
for ii=1:200
    if ii>1
        deltr = r_1(ii) - r_1(ii-1);
        Area = Area + deltr * N1(r_1(ii),20);
    end
end

function [Ke]=integralNab(N11, N21)
  fun_3 = @(r,Th) r.*N11(r,Th).*N21(r,Th) ;
  Ke = 0.5*integral2(fun_3, 0, 1, 0, 2*pi);
  % <<-------------------------->>
end
```


Numerical
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0


Analytical
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0



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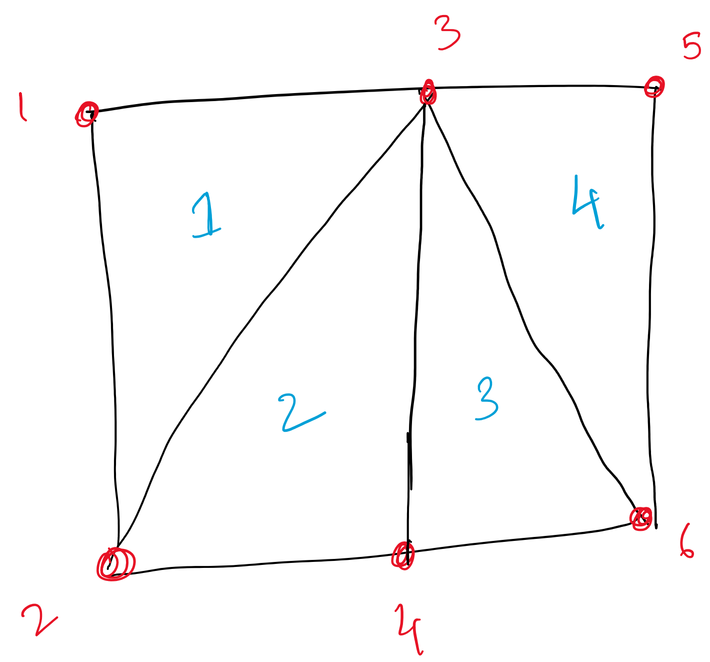
LG Map for the Mesh



here we have 4 conforming elements & 6 nodes

LG for P_1 elements will be

$$
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \rightarrow \text{elements} \\
= \begin{bmatrix} 1 & 3 & 4 & 3 \\ 2 & 2 & 6 & 6 \\ 3 & 4 & 3 & 5 \end{bmatrix}
\end{array}
$$

We take our vertices in Anti clockwise direction

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Here we have P_1 basis functions

So a general form of these basis function is

$$N_1^e = \frac{1}{2A^e}\left[-(x_2^3 - x_2^2)(x_1 - X_1^2) + (x_1^3 - x_1^2)(x_2 - X_2^2)\right]^e$$

for the full node

$N_1 = N_1^{e_1} + N_2^{e_2} + \cdots$ where $e_1, e_2 \ldots$ are the elements which share the vertex / node 1

x_1^i, x_2^i are x_1, x_2 coordinates for 3 vertices $i = 1, 2, 3$ in the triangular element.

so as we have 6 vertices we will have 6 basis functions

$$\mathcal{S}_h = \text{span}(N_1, N_2, N_3, N_4, N_5, N_6)$$

where dirichlet condition is imposed on node 5, 6 as $v = 0$ on Γ_g

$$\mathcal{V}_h = \text{span}(N_1, N_2, N_3, N_4)$$

Active indices are $\eta_a = \{1, 2, 3, 4\}$

constrained indices are $n_g = \{5, 6\}$

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shape function N_3^2

As per our LV   element 2 has vertices

$$\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \begin{matrix} \to (0,1) \\ \to (-1,-1) \\ \to (0,-1) \end{matrix}$$

$N_3^2 \to$ corresponds to vertex 4

$2A = 2$

$$N_3^2 = \frac{1}{2}\left[(1-(-1))(x-0) + (-1-0)(y-1)\right]$$

at $-0.5, -0.1$

$$N_3^2 = 0.05$$

$$\nabla N_3^2 = \begin{bmatrix} 1, & -0.5 \end{bmatrix}$$

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Element stiffness Matrix & load vector

Before we clear out our constrained indices for Global stiffness, element stiffness is computed assuming No dirichlet

dirichlet is directly reflected in global

Element 1

$$K_e = \begin{matrix} 1.4167 & -0.1667 & -0.9167 \\ -0.1667 & 0.4167 & 0.0833 \\ -0.9167 & 0.0833 & 1.1667 \end{matrix}$$

$$F_e = \begin{matrix} -0.0833 \\ -0.2500 \\ 0.0000 \end{matrix}$$

Element 2

$$K_e = \begin{matrix} 0.4167 & 0.0833 & -0.1667 \\ 0.0833 & 1.1667 & -0.9167 \\ -0.1667 & -0.9167 & 1.4167 \end{matrix}$$

$$F_e = \begin{matrix} -0.0833 \\ -0.3333 \\ -0.2500 \end{matrix}$$

Element 3

$$F_e = \begin{matrix} -0.0833 \\ -0.0000 \end{matrix}$$

$$f_e = \begin{matrix} -0.0833 \\ -0.0000 \\ 0.0833 \end{matrix}$$

$$k_e = \begin{matrix} 1.4167 & -0.9167 & -0.1667 \\ -0.9167 & 1.1667 & 0.0833 \\ -0.1667 & 0.0833 & 0.4167 \end{matrix}$$

Element 4

$$F_e = \begin{matrix} 0.3333 \\ 0.2500 \\ 0.4167 \end{matrix}$$

$$k_e = \begin{matrix} 1.1667 & 0.0833 & -0.9167 \\ 0.0833 & 0.4167 & -0.1667 \\ -0.9167 & -0.1667 & 1.4167 \end{matrix}$$

Load vector with natural boundary conditions

Over Natural B.C is $F_{\Gamma_h} = \int_{\Gamma_h} (x_2^2 - 1)\, v \, d\Gamma$

the boundaries $[-1, 1] \times \{1\}$ & $[-1, 1] \times \{-1\}$

have $F_{\Gamma_h} = 0$ as $x_2^2 - 1 = 0$

$\therefore$ only 1 boundary remains $\{-1\} \times [-1, 1]$

this boundary has 2 nodes/vertices $\{1, 2\}$

Integrate from -1 to 1

$$q_1 = \int_{-1}^{1} (x_2^2 - 1) \; N_1^1 \; dx_2 \quad , \; q_2 = \int_{-1}^{1} (x_2^2 - 1) \; N_2^1 \; dx_2$$

we get

$$q_1 = -0.6667 \quad , \quad q_2 = -0.6667$$

$$\therefore \quad F_{\Gamma_h} = [-0.6667, -0.6667, 0, 0, 0, 0]$$

$\downarrow$

Contributes to nodes 1 & 2

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Total Stiffness Matrix and load vector

Stiffness Matrix 6 nodes

	1	2	3	4	5	6
1	1.4167	-0.1667	-0.9167	0	0	0
2	-0.1667	1.5833	0.1667	-0.9167	0	0
3	-0.9167	0.1667	3.1667	-0.3333	-0.9167	0.1667
4	0	-0.9167	-0.3333	2.8333	0	-0.9167
5	0	0	0	0	1	0
6	0	0	0	0	0	1

Load Vector

1	
-0.7500	1 → vertex
-1.2500	2
0.3333	3
-0.3333	4
1	5
-1	6

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Approximate Solution is

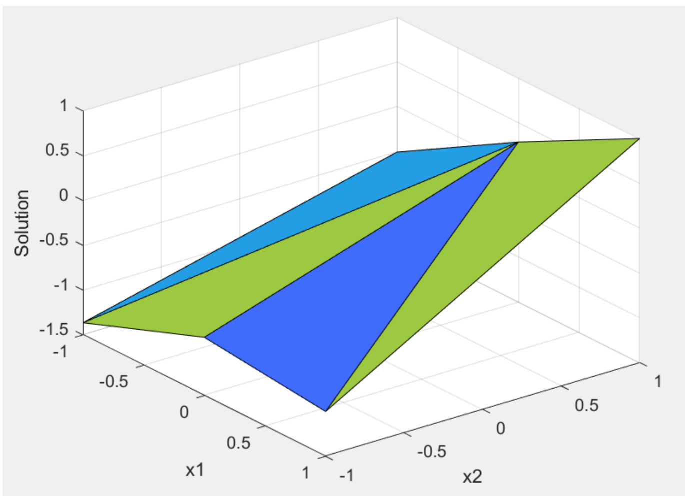
$$u = -0.5067\,N_1 - 1.3643\,N_2 + 0.2831\,N_3 - 0.8493\,N_4 + N_5 - N_6$$

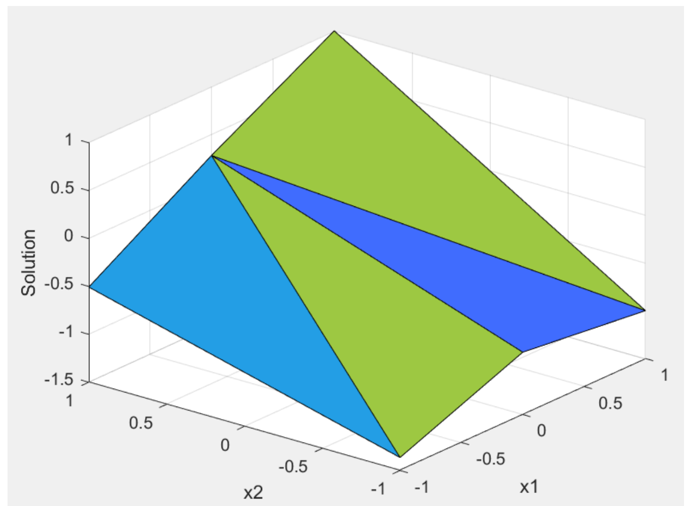
$N_1, N_2, N_3, N_4, N_5, N_6 \rightarrow$ basis functions

U =

1	-0.5067
2	-1.3643
3	0.2831
4	-0.8493
5	1
6	-1

FE solution plotted on x1, x2 axis





Code

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Contents


```
X = [-1, -1, 0, 0, 1, 1;...
    1, -1, 1, -1, 1, -1];
LV = [1, 3, 4, 3;...
    2, 2, 6, 6;...
    3, 4, 3, 5];
BE = [1, 2, 4, 6, 5, 3;...
    2, 4, 6, 5, 3, 1;...
    1, 2, 3, 4, 4, 1];
HH = [1, 0, 0, 0, 0, 0];
nh = size(BE, 2);
EtaG = [5, 6];
nel = 4;
nod = 6;
ke = 1;
ng=length(EtaG); II=eye(nod);
centrd = zeros([2, 4]);
K=zeros(nod,nod); F=zeros(nod,1);
for ii=1:nel
    xe = X(:,LV(:,ii));
    centrd(:,ii) = mean(xe,2);
    lge=LV(:,ii);
    dN=[xe(2,2)-xe(2,3),xe(2,3)-xe(2,1),xe(2,1)-xe(2,2);...
    xe(1,3)-xe(1,2),xe(1,1)-xe(1,3),xe(1,2)-xe(1,1)];
    Ae2=dN(2,3)*dN(1,2)-dN(1,3)*dN(2,2);
    N1 = @(x, y) (1/Ae2)*(dN(1,1).*(x - xe(1,2))+ dN(2,1).*(y - xe(2,2)));
    N2 = @(x, y) (1/Ae2)*(dN(1,2).*(x - xe(1,3))+ dN(2,2).*(y - xe(2,3)));
    N3 = @(x, y) (1/Ae2)*(dN(1,3).*(x - xe(1,1))+ dN(2,3).*(y - xe(2,1)));
    if ii==2
        N_test = N3;
        gradN_test = (1/Ae2)*[dN(1,3), dN(2,3)];
        N_test(-0.5, -0.1)
    end
%     Nall = @(x, y) [N1(x,y), N2(x,y), N3(x,y)];
%     N_mat = @(x, y) Nall(x, y)'*Nall(x, y);
%     B = N_mat(1, 1);
%     B(1, 1)
    N11 = @(x, y) N1(x,y).*N1(x,y); N12 = @(x, y) N1(x,y).*N2(x,y); N13 = @(x, y) N1(x,y).*N3(x,y);
    N21 = @(x, y) N2(x,y).*N1(x,y); N22 = @(x, y) N2(x,y).*N2(x,y); N23 = @(x, y) N2(x,y).*N3(x,y);
    N31 = @(x, y) N3(x,y).*N1(x,y); N32 = @(x, y) N3(x,y).*N2(x,y); N33 = @(x, y) N3(x,y).*N3(x,y);
    N_mat = [TriIntegral(N11,xe(1,:),xe(2,:)), TriIntegral(N12,xe(1,:),xe(2,:)), TriIntegral(N13,xe(1,:),xe(2,:));...
        TriIntegral(N21,xe(1,:),xe(2,:)), TriIntegral(N22,xe(1,:),xe(2,:)), TriIntegral(N23,xe(1,:),xe(2,:));...
        TriIntegral(N31,xe(1,:),xe(2,:)), TriIntegral(N32,xe(1,:),xe(2,:)), TriIntegral(N33,xe(1,:),xe(2,:))];

    N_p = @(x, y) N1(x,y).*N1(x,y);
%     TriIntegral(N_p,xe(1,:),xe(2,:))

    lp1 = @(x, y) (x +y).*N1(x,y);
    lp2 = @(x, y) (x +y).*N2(x,y);
    lp3 = @(x, y) (x +y).*N3(x,y);
    Fe = [TriIntegral(lp1,xe(1,:),xe(2,:)); TriIntegral(lp2,xe(1,:),xe(2,:)); TriIntegral(lp3,xe(1,:),xe(2,:))];
    dN=dN/Ae2;
    Ke=Ae2/2*ke*dN'*dN + N_mat;
    K(lge,lge) = K(lge,lge) + Ke;
    F(lge) = F(lge) + Fe;
    Ke, Fe
end

for ig=1:ng
    K(EtaG(ig),:)=II(EtaG(ig),:);
```

```
    F(EtaG(ig))=X(2,EtaG(ig));
end

for ied=1:nh
    if HH(1,ied)~=0
        lged=BE(1:2,ied);
        xed(:,1:2)=X(:,lged);
        Led=norm(xed(:,1)-xed(:,2));
        pts = LV(:,BE(3,ied));
        xe = X(:,pts);
        dN=[xe(2,2)-xe(2,3),xe(2,3)-xe(2,1),xe(2,1)-xe(2,2);...
        xe(1,3)-xe(1,2),xe(1,1)-xe(1,3),xe(1,2)-xe(1,1)];
        Ae2=dN(2,3)*dN(1,2)-dN(1,3)*dN(2,2);
        N1 = @(y) (1/Ae2)*(dN(2,1).*(y - xe(2,2))).*(y.^2 - 1);
        N2 = @(y) (1/Ae2)*(dN(2,2).*(y - xe(2,3))).*(y.^2 - 1);
        q1 = integral(N1, -1, 1);
        q2 = integral(N2, -1, 1);
        q1, q2
        F(lged)=F(lged)+[q1;q2];
    end
end
```

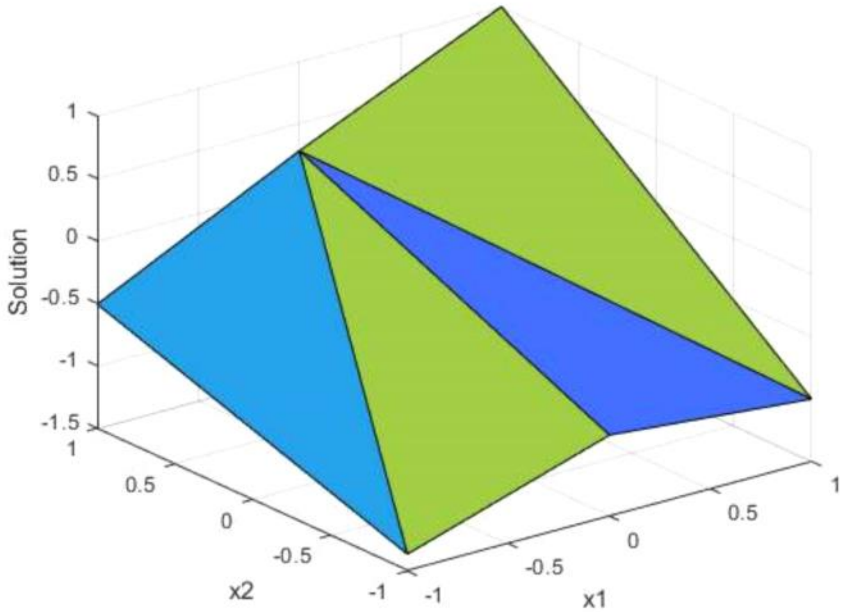
solve algebraic system

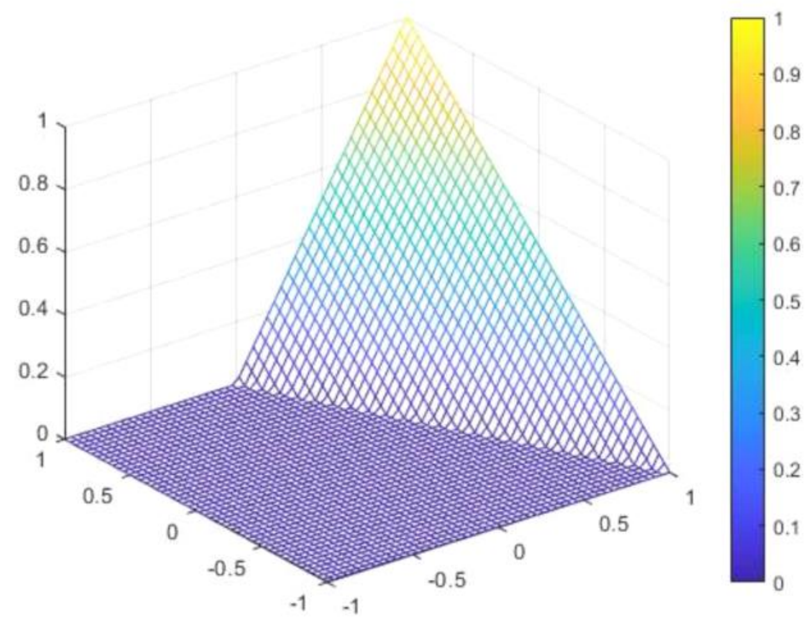
```
U=K\F;
```

plot

```
trisurf(LV',X(1,:),X(2,:),U)
xlabel('x1'), ylabel('x2'), zlabel('Solution')
x = linspace(-1,1, 50);
y = linspace(-1,1, 50);
[X1,Y1] = meshgrid(x,y);
Z = X1+Y1;

for ii=1:50
    for jj=1:50
        Z(ii,jj) = pllt(N3,X1(ii,jj),Y1(ii,jj),X(1,LV(:,4)),X(2,LV(:,4)));
    end
end

figure(2)
mesh(X1,Y1,Z)
colorbar

% for ii=1:nel
%     xe = X(:,LV(:,ii));
%     centrd(:,ii) = mean(xe,2);
%     lge=LV(:,ii);
%     dN=[xe(2,2)-xe(2,3),xe(2,3)-xe(2,1),xe(2,1)-xe(2,2);...
%     xe(1,3)-xe(1,2),xe(1,1)-xe(1,3),xe(1,2)-xe(1,1)];
%     Ae2=dN(2,3)*dN(1,2)-dN(1,3)*dN(2,2);
%     N1 = @(x, y) (1/Ae2)*(dN(1,1).*(x - xe(1,2))+ dN(2,1).*(y - xe(2,2)));
%     N2 = @(x, y) (1/Ae2)*(dN(1,2).*(x - xe(1,3))+ dN(2,2).*(y - xe(2,3)));
%     N3 = @(x, y) (1/Ae2)*(dN(1,3).*(x - xe(1,1))+ dN(2,3).*(y - xe(2,1)));
%
%     for j=1:50
%         for k=1:50
%             Z(j,k) = Z(j,k)+ computeSol(xe(1,:), xe(2,:), U(lge), N1, N2, N3,X1(j,k),Y1(j,k));
%         end
%     end
% end
%
```

```
% figure(3)
% mesh(X1,Y1,Z)
% colorbar
```





Reference for TriIntegral: https://www.mathworks.com/matlabcentral/answers/430420-integrate-a-function-over-a-triangle-area

```
function I = TriIntegral(f, Tx, Ty)
% I = TriIntegral(f, Tx, Ty)
% 2D integration of f on a triangle
```

```
% INPUTS:
%   - f is the vectorized function handle that when calling f(x,y) returns
%       function value at (x,y), x and y are column vectors
%   - Tx,Ty are is two vectors of length 3, coordinates of the triangle
% OUTPUT
%   I: integral of f in T
T = [Tx(:), Ty(:)];
I = integral2(@(s,t) fw(f,s,t,T),0,1,0,1);
A = det(T(2:3,:)-T(1,:));
I = I*abs(A);
end % TriIntegral
```

```
function y = fw(f, s, t, T)
sz = size(s);
w1 = (1-s); % Bug fix
w2 = s.*t;
w3 = 1-w1-w2;
P = [w1(:),w2(:),w3(:)] * T;
y = feval(f,P(:,1),P(:,2));
y = s(:).*y(:);
y = reshape(y,sz);
end

function z = pllt(N,xq,yq,xv,yv)
    if inpolygon(xq,yq,xv,yv)==1
        z = N(xq, yq);
    else
        z = 0;
    end
end

function z = computeSol(xv, yv, Unod, N1, N2, N3,xq,yq)
    if inpolygon(xq,yq,xv,yv)==1
        z = Unod(1)*pllt(N1,xq,yq,xv,yv) + Unod(2)*pllt(N2,xq,yq,xv,yv) + Unod(3)*pllt(N3,xq,yq,xv,yv);
    else
        z = 0;
    end
end
```

```
Ke =

    1.4167   -0.1667   -0.9167
   -0.1667    0.4167    0.0833
   -0.9167    0.0833    1.1667


Fe =

   -0.0833
   -0.2500
    0.0000


ans =

    0.0500


Ke =

    0.4167    0.0833   -0.1667
    0.0833    1.1667   -0.9167
   -0.1667   -0.9167    1.4167
```

```
Fe =

   -0.0833
   -0.3333
   -0.2500


Ke =

    1.4167   -0.9167   -0.1667
   -0.9167    1.1667    0.0833
   -0.1667    0.0833    0.4167


Fe =

   -0.0833
   -0.0000
    0.0833


Ke =

    1.1667    0.0833   -0.9167
    0.0833    0.4167   -0.1667
   -0.9167   -0.1667    1.4167


Fe =

    0.3333
    0.2500
    0.4167


q1 =

   -0.6667


q2 =

   -0.6667
```

Published with MATLAB® R2021b