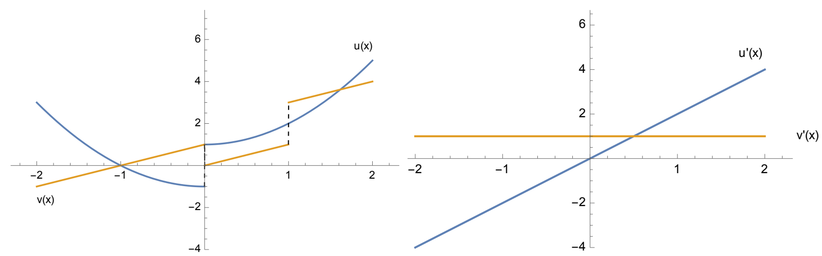
Example 1.24 Consider the functions

$$u(x) = \begin{cases} x^2 - 1 & x < 0, \\ x^2 + 1 & x \geq 0, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} x+1 & x < 0, \\ x & x \in [0,1], \\ x+2 & x > 1. \end{cases}$$

Their derivatives are

$$u'(x) = 2x \text{ for } x \neq 0, \quad \text{and} \quad v'(x) = 1 \text{ for } x \notin \{0,1\}.$$

The two functions and their derivatives are plotted below:



Consider the following expression

$$\int_{-2}^{2} u(x) v'(x)\, dx. \tag{1.46}$$

Its value can be readily computed by direct integration, and it is

$$\int_{-2}^{2} u(x) v'(x)\, dx = \int_{-2}^{0} x^2 - 1\, dx + \int_{-2}^{0} x^2 + 1\, dx = \frac{16}{3}.$$

Let's integrate (1.46) by parts, and verify that returns the same value. To this end, notice that u is smooth in the intervals $(-2,0)$ and $(0,2)$, while v is smooth in the intervals $(-2,0)$, $(0,1)$, and $(1,2)$. The two functions are piecewise smooth in $(-2,2)$ if we select $k=3$ and $c_0 = -2, c_1 = 0, c_2 = 1, c_3 = 2$. Notice that u is smooth in $(0,2)$, and hence it is automatically smooth in the two intervals (c_1, c_2) and (c_2, c_3). From (1.45),

$$\begin{aligned}
\int_{-2}^{2} u(x) v'(x)\, dx &= [\![u(x)v(x)]\!]_{x=-2} + [\![u(x)v(x)]\!]_{x=0} + [\![u(x)v(x)]\!]_{x=1} \\
&\quad + [\![u(x)v(x)]\!]_{x=2} - \int_{-2}^{2} u'(x) v(x)\, dx \\
&= -u(-2)v(-2) + \lim_{x\to 0^-} u(x)v(x) - \lim_{x\to 0^+} u(x)v(x) \\
&\quad + \lim_{x\to 1^-} u(x)v(x) - \lim_{x\to 1^+} u(x)v(x) + u(2)v(2) \\
&\quad - \int_{-2}^{0} 2x(x+1)\, dx - \int_{0}^{1} 2x.x\, dx - \int_{1}^{2} 2x(x+2)\, dx \\
&= -3.(-1) + (-1).1 - 1.0 + 2.1 - 2.3 + 5.4 - \frac{38}{3} \\
&= \frac{16}{3},
\end{aligned}$$

and we verified the identity.

An alternative way to obtain the same result is to split the integral as a sum of integrals over $(-2,0)$, $(0,1)$ and $(1,2)$, and then integrate each one of these integrals by parts. By rearranging the terms, we will arrive to the expression that we used from Thm. 1.2.

1.2 Linear Algebra for Spaces of Functions

The formulation of finite element methods is more easily performed and understood with some basic concepts of linear algebra, in this case, applied to spaces of functions.

1.2.1 Vector Spaces of Functions

The first encounter with finite element methods is for many the first encounter with the use of vector spaces in a context other than one in which vectors represent elementary physics quantities, such as forces or velocities. It is also the first encounter with infinite-dimensional vector spaces. In studying finite element methods, we are interested in vector spaces in which each vector is a function, that is, in *vector spaces of functions*.

For example, consider the set V of all real quadratic polynomials that are zero at zero, namely, functions of the form

$$f(x) = ax^2 + bx$$

for any $a, b \in \mathbb{R}$, such as $f_1(x) = 3x^2$ and $f_2(x) = x$. Notice that $f_1(x) + f_2(x) = 3x^2 + x$ is also a function in V, and $3f_2(x) = 3x$ is another one, so addition of two polynomials in V returns a polynomial in V, and multiplication of a polynomial in V by a scalar (real number) is also a polynomial in V.

We can think of each polynomial in V as the vector in $\mathbb{R}^2$ that starts at the origin and ends at the point with coordinates (a, b), see Fig. 1.3. The sum of two functions in V corresponds to adding the two vectors, and similarly with the multiplication by a scalar (real number). You may be wondering about why to call each function a vector, or simply why to talk about vector spaces of functions? With this identification we can define the *dimension* and a basis for V, and by using a basis, we will be able to build any function (vector) in the space.

We begin by reviewing the definition of vector spaces.

Definition 1.1 (Vector Space)**.** *A Vector Space V is a set for which two operations $+$ and $\cdot$ are defined, called* **vector addition** *and* **multiplication by a scalar***, such that for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$ they satisfy:*

1. **Closure***: $u + v \in V$, and $\alpha \cdot u \in V$.*
2. **Commutativity:** *$u + v = v + u$.*

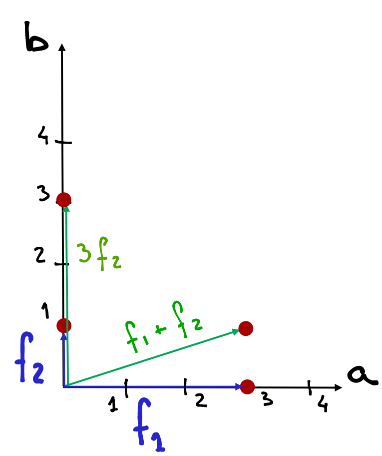


Figure 1.3 Identification of polynomials with vectors in $\mathbb{R}^2$.

3. **Associativity:** $u + (v + w) = (u + v) + w$, *and* $\alpha \cdot (\beta \cdot u) = (\alpha\beta) \cdot u$.

4. **Identity:** *There exists an element* $0 \in V$, *called "zero," such that* $u + 0 = u$, *and* $1 \cdot u = u$.

5. **Additive Inverse:** *For any* $u \in V$, *there exists* $v \in V$ *such that* $v + u = 0$.

6. **Distributivity:** $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$ *and* $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.

The elements of V are called vectors.

In the following, we will drop the symbol $\cdot$ to indicate multiplication by a scalar, unless there is ambiguity. So, for example, for $\alpha \cdot u$ will write $\alpha\, u$.

You are by now very familiar with $\mathbb{R}^n$ as a vector space, $n \in \mathbb{N}$. Each point of $\mathbb{R}^n$ defines a vector under the standard definition of vector addition and multiplication by a scalar in $\mathbb{R}^n$. As aforementioned, what might be new for you is that sets of functions can also be vector spaces. In this case, each function in the set is a "vector."

To complete the depiction of functions as elements of a vector space, we need to specify the **vector-addition** and **multiplication-by-a-scalar** operations. Fortunately, they are defined exactly as you would expect: Let V be a set of functions over a domain $\Omega \subseteq \mathbb{R}^n$, and let $f, g \in V$ and $\alpha \in \mathbb{R}$, then

- **Vector addition:** the function defined as $h(x) = f(x) + g(x)$ for all $x \in \Omega$.

- **Multiplication by a scalar:** the function defined as $w(x) = \alpha f(x)$ for all $x \in \Omega$.

To illustrate this definition, we will consider sets of *smooth functions*, which as in previous sections, are functions in which all derivatives exist and are continuous.

Examples:

1.25 The set $V_1 = \{f : [a, b] \to \mathbb{R} \text{ smooth}\}$ is a vector space. We can check each one of the non-trivial properties:

   i. Closure: If $u, v \in V_1$ and $\alpha \in \mathbb{R}$, then $u + v \in V_1$ and $\alpha u \in V_1$, since the sum of smooth functions is another smooth function, and multiplication of a smooth function by a scalar is anoter smooth function.

   ii. Identity: the function $z(x) = 0$ for all $x \in [a, b]$ is the "zero" of the space.

   The rest of the properties are easy to check.

1.26 The set $V_2 = \{f : [a, b] \to \mathbb{R} \text{ smooth} \mid w(a) = w(b) = 0\}$ is a vector space. Since $V_2 \subset V_1$ in Example 1.25, V_2 inherits commutativity, associativity, and distributivity in Def. 1.1 from V_1. We need to only check closure, identity and additive inverse. Closure follows because if $u, v \in V_2$, and $w = u + v$, then $w(a) = u(a) + v(a) = 0$ and $w(b) = u(b) + v(b) = 0$, and hence $w \in V_2$. Since the zero function is in V_2, identity is satisfied. Finally, if $u \in V_2$, then $-u \in V_2$ because $u(a) = -u(a) = 0$ and $u(b) = -u(b) = 0$, and hence additive inverse is also satisfied.

1.27 The set of polynomials of degree less or equal than $k \in \mathbb{N} \cup \{0\}$ over an interval $I \subset \mathbb{R}$, $\mathbb{P}_k(I)$, is a vector space. This is because the sum of polynomials in $\mathbb{P}_k$ and the product by a scalar is still a polynomial of degree less or equal than k (closure), and because the function $0 \in \mathbb{P}_k$ (identity). The rest of the properties are easy to check.

In addition to vector spaces, we will use a closely related concept, that of an affine subspace, which we define next.

Definition 1.2 (Vector Subspace)**.** *Let W be a vector space. A vector space $V \subset W$ is vector subspace of W.*

Definition 1.3 (Affine Subspace)**.** *Let W be a vector space. An affine subspace of W is a set $S \subset W$ such that for any $s_1 \in S$ the set*

$$V = \{s_2 - s_1 \mid s_2 \in S\}$$

is a vector subspace of W. The vector space V is called the **direction** *of S, or the associated vector space to S.*

The direction V is independent of the choice of s_1.

The direction V is independent of the choice of s_1.

To see this, let $s_a, s_b \in S$ and

$$V_a = \{s - s_a \mid s \in S\}, \qquad V_b = \{s - s_b \mid s \in S\}$$

be the associated vector spaces to S. We will prove that $V_a = V_b$, and hence that the direction is independent of s_1. We will use the fact that $\Delta s = s_a - s_b$ belongs to both V_a and V_b, by definition. So, if $v \in V_a$, then there exists $\bar{s} \in S$ such that

$$\bar{s} = v + s_a = v + \Delta s + s_b,$$

and hence $w = v + \Delta s = \bar{s} - s_b \in V_b$. But $w = v + \Delta s \in V_a$, since $\Delta s \in V_a$, from where $v = w - \Delta s \in V_b$ as well, since $\Delta s \in V_b$. We conclude then that if $v \in V_a$, $v \in V_b$, or $V_a \subseteq V_b$. A similar argument leads to $V_a \supseteq V_b$, and hence to $V_a = V_b$.

Of course, using the notation from the definition, a vector space $V \subset W$ is an affine subspace of W. Elements of an affine subspace are called points and not vectors, since it is not a vector space.

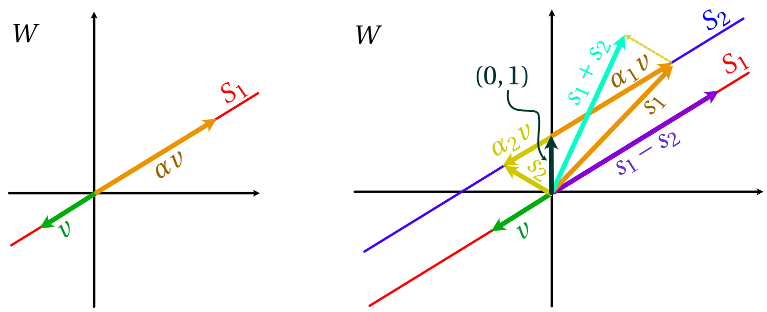


Figure 1.4 Sketch of sets S_1 and S_2 in Example 1.28. The former is a vector space, while the latter is not; it is an affine subspace of W instead.

Examples:

1.28 Let $v = (-1,-1) \in W = \mathbb{R}^2$. The set $S_1 = \{\alpha v \mid \alpha \in \mathbb{R}\}$ is a vector space. Instead, the set $S_2 = \{\alpha v + (0,1) \mid \alpha \in \mathbb{R}\}$ is *not* a vector space. This is because if $s_1 = \alpha_1 v + (0,1)$ and $s_2 = \alpha_2 v + (0,1)$, then $h = s_1 + s_2 = \alpha_1 v + (0,1) + \alpha_2 v + (0,1) = (\alpha_1 + \alpha_2) v + (0,2)$, and hence $h \not\in S_2$.

Instead, the set S_2 is an affine subspace of W, since $s_1 - s_2 = (\alpha_1 - \alpha_2) v$, and hence $s_1 - s_2$ can be any element of the vector space S_1. Please see Fig. 1.4 for a sketch.

1.29 The set $V_3 = \{w\colon [a,b] \to \mathbb{R} \text{ smooth } \mid w(a) = 1 = w(b) = 1\}$ is *not* a vector space. This is because if $u, v \in V_3$, and $w = u + v$, then $w(a) = u(a) + v(a) = 2$, and similarly for $w(b)$, and hence $w \not\in V_3$.

Instead, V_3 is an affine subspace of the vector space $W = \{w\colon [a,b] \to \mathbb{R} \text{ smooth}\}$. To see this, first notice first that for any $v_1, v_2 \in V_3$, $v_2 - v_1 \in V_2$ of Example 1.26, a vector space, since $v_1(a) - v_2(a) = 0 = v_1(b) - v_2(b)$. This implies that for any $v_1 \in V_3$,

$$V = \{v_2 - v_1 \mid v_2 \in V_3\} \subseteq V_2.$$

Second, notice that if $v \in V_2$, then $v_1 + v = v_2 \in V_3$, so $v_2 - v_1 = v$, from where

$$V \supseteq V_2.$$

Therefore, $V = V_2$, and hence V_3 is an affine subspace of W with V_2 as its direction.

Vector spaces that are subspaces of $\mathbb{R}^n$ can be identified with (hyper-)planes that contain the origin. Affine subspaces of $\mathbb{R}^n$ are (hyper-)planes that may not contain the origin, and hence, will be parallel to a vector subspace, their direction, as illustrated by S_1 and S_2 in Fig. 1.4. Also in the figure, notice that the role of the vector $(0,1)$ is to "transport" the vector space S_1 parallel to itself to become the affine subspace S_2. More generally, if s_1 is any element of an affine subspace S, then any other element $s \in S$ can be written as $s = s_1 + w$ for $w \in V$, since by the definition of an affine subspace, $s - s_1 = w \in V$. In Fig. 1.4, when $s = s_2$, then $w = \alpha_2 v$.

More importantly, if $s_1 \in S$, then we can reach all elements in S by adding an element in its direction V, namely,

$$S = \{s_1 + w \mid w \in V\}. \tag{1.47}$$

$S = \{s_1 + w \mid w \in V\}$ **for any** $s_1 \in S$**.**

To see this, let

$$U = \{s_1 + w \mid w \in V\}.$$

We want to show that $\mathscr{U} = S$. For any $s_2 \in S$, $w = s_2 - s_1 \in V$, by definition, so $s_2 = w + s_1 \in U$. Therefore $U \supseteq S$.

To see that $U \subseteq S$, notice that for any $w \in V$, there exists $s_2 \in S$ such that $s_2 - s_1 = w$, since all elements of V are generated by such differences. This implies that $s_2 = s_1 + w$, and hence that $w + s_1 \in S$, or $U \subseteq S$.

1.2.1.1 Bases in a vector space of functions

Next, we review the definition of a basis for a vector space, and see examples of bases in vector spaces of functions.

Definition 1.4 (Linear combinations, or span)**.** *Let V be a vector space and $U \subset V$ be a set of vectors in V. The* **span** *of U, $span(U)$, is the set*

$$span(U) = \left\{ \sum_{i=1}^{n} c_i e_i \mid n \in \mathbb{N}, e_i \in U, c_i \in \mathbb{R} \right\}. \tag{1.48}$$

The set $span(U)$ contains all **linear combinations** *of vectors in the set U.*

In the following, we denote vectors in $\mathbb{R}^n$ by the Cartesian coordinates of their end points.

Examples:

1.30 $U_1 = \{e_1, e_2\} \subset \mathbb{R}^3$, where $e_1 = (1,0,0)$ and $e_2 = (1,0,1)$. Then

$$\text{span}(U_1) = c_1 e_1 + c_2 e_2 = (c_1 + c_2, 0, c_2)$$

is the plane that contains 0 and vectors e_1 and e_2, or whose normal is in the direction $e_1 \times e_2$, where $'\times'$ is the vector cross-product. Notice that vectors in U do not need to be unit vectors.

1.31 $U_2 = \{1, x, x^2\}$. Then span(U_2) is the set of all quadratic polynomials. The vector with components $(3,4,5)$, or $c_1 = 3$, $c_2 = 4$ and $c_3 = 5$ is the polynomial

$$p(x) = 3 + 4x + 5x^2,$$

and the quadratic polynomial $q(x) = 5 - 2x - 6x^2$ has components $(5, -2, -6)$ in this basis.

1.32 $U_3 = \{e_1, \dots, e_k\} \subset V$. Then

$$\operatorname{span}(U_3) = \sum_{i=1}^{k} c_i e_k = c_1 e_1 + \ldots + c_k e_k.$$

This examples illustrates the construction of the span of a set made of a finite number of vectors in V.

1.33 Let $W = \{(0, y, z) \mid (y, z) \in \mathbb{R}^2\}$ and $U_4 = \{(1,0,0)\} \cup W$. Then, span$(U_4) = \mathbb{R}^3$. For example, if $w = (2,3,4) \in \mathbb{R}^3$, we can write it as $w = 2 \times (1,0,0) + 1 \times (0,3,4)$. In Def. 1.4, this follows after setting $n = 2$, $c_1 = 2$, $c_2 = 1$, $e_1 = (1,0,0)$ and $e_2 = (0,3,4)$. Alternatively, we could have written $w = 2 \times (1,0,0) + 2 \times (0,3/2) + 8 \times (0,0,1/2)$, in which we set $n = 3$, $c_2 = 2$, $c_3 = 8$, $e_2 = (0,3/2,2)$ and $e_3 = (0,0,1/2)$ instead.

When U has an infinite number of vectors, elements of span(U) are formed through linear combinations of any finite number of vectors in U.

Definition 1.5 (Linearly Independent and Linearly Dependent Set of Vectors)**.** *Let V be a vector space and let $e_i \in V$ for $i = 1, \dots, n$. The set of vectors $U = \{e_1, \dots, e_n\}$ is linearly independent whenever*

$$\sum_{i=1}^{n} c_i e_i = 0 \Longleftrightarrow c_i = 0 \text{ for } i = 1, \dots, n. \tag{1.49}$$

Otherwise, the set of vectors U is linearly dependent.

Examples:

1.34 $U_1 = \{e_1, e_2\} \subset \mathbb{R}^3$, where $e_1 = (1,0,0)$ and $e_2 = (1,0,1)$. The set of vectors in U_1 is linearly independent:

$$c_1(1,0,0) + c_2(1,0,1) = (0,0,0) \Longleftrightarrow \begin{cases} c_1 + c_2 & = 0 \\ \quad c_2 = & 0 \end{cases} \Longleftrightarrow c_1 = c_2 = 0.$$

1.35 $U_2 = \{1, x, x^2\}$. The set of vectors U_2 is linearly independent:

$$p(x) = c_1 + c_2 x + c_3 x^2 = 0 \quad x \in [0,1] \Longleftrightarrow c_1 = c_2 = c_3 = 0.$$

To see this, it is enough to evaluate $p(x)$ at three different locations, for example. Say, $p(0) = 0$, $p(1/2) = 0$, $p(1) = 0$. The resulting system of equations has $c_1 = c_2 = c_3 = 0$ as the only solution.

1.36 $U_3 = \{1, x, 2+3x\}$. This is a linearly dependent set of vectors, since if we let $e_1 = 1$, $e_2 = x$, and $e_3 = 2+3x$, then $2e_1 + 3e_2 - e_3 = 0$.

1.37 Consider the set $U_4 = \{\min\{0, x\}, x\}$ of functions with domain $[a, b]$. If $[a, b] = [-1, 1]$, U_4 is a set of linearly independent functions, since for

$$f(x) = c_1 \min\{0, x\} + c_2 x,$$

we have that

$$0 = f(-1) = -c_2, \qquad 0 = f(1) = c_1 + c_2 \Longrightarrow c_1 = c_2 = 0.$$

Instead, if $[a, b] = [0, 1]$, this is a linearly dependent set. To see this, notice that for $x \in [0, 1]$, $\min\{0, x\} = x$, so the two functions are precisely the same function over this interval.

Definition 1.6 (Basis and Dimension of a Vector Space)**.** *Let V be a vector space and $e_i \in V$ for $i = 1, \ldots, n$, $n \in \mathbb{N}$. The set $U = \{e_1, \ldots, e_n\}$ is a basis of V if U is linearly independent and span$(U) = V$. The number of vectors in a basis is the dimension of V.*

A vector space that does not have a basis is an infinite-dimensional space.

Given a basis $U = \{e_1, \ldots, e_n\}$ in a vector space V, and a vector $v \in V$, there there exists a *unique* set of numbers $(c_1, \ldots, c_n) \in \mathbb{R}^n$ such that

$$v = c_1 e_1 + \ldots + c_n e_n. \tag{1.50}$$

The numbers $c_1, \ldots, c_n$ are called the **components** of v in basis U.

Conversely, when the components $(c_1, \ldots, c_n)$ span all points in $\mathbb{R}^n$, the vector v in (1.50) spans the space V. Because all possible vectors in V are obtained by evaluating all possible values of $(c_1, \ldots, c_n)$, the variables $c_1, \ldots, c_n$ are called **degrees of freedom** of V.

Examples:

1.38 Consider the set $U_1 = \{e_1, e_2\} \subset \mathbb{R}^3$, where $e_1 = (1, 0, 0)$ and $e_2 = (1, 0, 1)$. The set of vectors in U_1 is not a basis for $\mathbb{R}^3$, since the vector $(0, 1, 0) \not\in$ span(U_1).

1.39 Consider the set $U_4 = \{e_1, e_2, e_3\} \subset \mathbb{R}^3$, where $e_1 = (1, 0, 0)$, $e_2 = (1, 0, 1)$, and $e_3 = (0, 1, 0)$. The set U_4 is a basis for $\mathbb{R}^3$, since it can be seen to be linearly independent, and any vector in $\mathbb{R}^3$ is a linear combination of the basis: If $(x, y, z) \in \mathbb{R}^3$, then $(x, y, z) = (x - y)e_1 + ye_2 + ze_3$. The dimension of $\mathbb{R}^3$ is then 3.

1.40 The set $U_2 = \{1, x, x^2\} \subset V_1 = \{f : (0, 1) \to \mathbb{R} \text{ smooth}\}$. The set of vectors U_2 is not a basis for V_1. In fact, there is no basis for V_1, and hence it is an **infinite dimensional space**.

The set U_2 is a basis for the vector space $\mathbb{P}_2$ formed by all quadratic polynomials, whose dimension is 3.

1.41 The set $U_5 = \{1+x, x-x^2, x^2-1\}$ is another basis for $\mathbb{P}_2$. To see this, notice that given a polynomial $p(x) = a+bx+cx^2 \in \mathbb{P}_2$ for $a,b,c \in \mathbb{R}$, we can write it as

$$p(x) = \frac{a+b+c}{2}(1+x) + \frac{b-c-a}{2}(x-x^2) + \frac{b+c-a}{2}(x^2-1).$$

Instead, the set $U_5' = \{1+x, x-x^2, x^2+1\}$ is not a basis, since the three functions are not linearly independent: $(x^2+1)+(x-x^2) = 1+x$.

1.42 The set $U_6 = \{\sin(x), \sin(2x), \sin(3x), \sin(4x)\}$ is a basis for $\mathrm{span}(U_6)$ over the interval $(0,2\pi)$, since the 4 functions are linearly independent. One way to see the linear independence is as follows: If we have

$$\sum_{i=0}^{n} c_i \sin(ix) = 0$$

with $n=4$ here, we need to show that this implies that $c_i = 0$ for any i. This follows by multiplying the last equation by $\sin(jx)$ for any $j \in \{1,\dots,n\}$ and integrating over the interval $(0,2\pi)$. In this case we get that

$$\sum_{i=0}^{n} c_i \int_0^{2\pi} \sin(ix)\sin(jx)\, dx = 0. \tag{1.51}$$

We then notice that

$$\int_0^{2\pi} \sin(ix)\sin(jx)\, dx = \begin{cases} \pi & i=j, \\ 0 & i \neq j. \end{cases}$$

Using this in (1.51) allows us to conclude that

$$0 = \sum_{i=0}^{n} c_i \underbrace{\int_0^{2\pi} \sin(ix)\sin(jx)\, dx}_{\neq 0 \text{ only if } i=j} \Longrightarrow c_j = 0,$$

and since this is true for any j, we can conclude that U_6 is a linearly independent set, and hence it is a basis for $\mathrm{span}(U_6)$.

1.2.1.2 Linear functional and bilinear form

We conclude this section by introducing two more definitions, which will allow us to talk about variational equations in an abstract way.

Definition 1.7 (Linear Functional)**.** *Let V be a vector space. A linear functional is a function $\ell\colon V \to \mathbb{R}$ such that for any $u, v \in V$ and $\alpha \in \mathbb{R}$*

$$\ell(u+\alpha v) = \ell(u) + \alpha \ell(v). \tag{1.52}$$

Examples:

1.43 Let $V_1 = \{f\colon [0,1] \to \mathbb{R} \text{ smooth }\}$, then

$$\ell(v) = \int_0^1 x^2 v(x)\, dx$$

is a linear functional in V_1. This is because:

- The value of $\ell(v)$ can be computed for any function $v \in V_1$, so it is defined for *any* function in V_1.
- It is simple to see that (1.52) is true, to wit, for $u, v \in V_1$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned}\ell(u+\alpha v) &= \int_0^1 x^2(u(x)+\alpha v(x))\, dx \\ &= \int_0^1 x^2 u(x)\, dx + \alpha \int_0^1 x^2 v(x)\, dx \\ &= \ell(u) + \alpha \ell(v).\end{aligned}$$

Let's compute the value of the linear functional for a couple of functions:

- $\ell(\cos(x)) = \int_0^1 x^2 \cos(x)\, dx = 2\cos(1) - \sin(1)$.
- $\ell(x^4) = \int_0^1 x^2 x^4\, dx = \frac{x^7}{7}\Big|_{x=1} = \frac{1}{7}$.

This is an example of linear functionals of the form

$$\ell(v) = \int_a^b f(x) v(x)\, dx$$

for some function f, which we will encounter often in later sections.

1.44 Let $V = \mathbb{R}^2$, and $f = (f_1, f_2) \in \mathbb{R}^2$. For $v = (v_1, v_2) \in V$,

$$\ell(v) = v_1 f_1 + v_2 f_2 \tag{1.53}$$

is a linear functional.

1.45 Let V be the set of continuous functions over $\mathbb{R}$. For $v \in V$, let

$$\ell(v) = v(0) \tag{1.54}$$

This is a linear functional. You may have encountered this functional written in a different way:

$$\ell(v) = \int_{\mathbb{R}} \delta(x) v(x)\, dx,$$

namely, using the *Dirac delta function.* A problem with the denomination of $\delta(x)$ as a function is that $\delta(x)$ is not a function over the real line, since it does not assign a value to points in the real line. Instead, it is a linear functional, since it assigns a scalar value to each function over the real line.

A linear functional is also called a **one-form**.

Definition 1.8 (Bilinear Form)**.** *Let V be a vector space. A bilinear form is a function $a\colon V \times V \to \mathbb{R}$ that is linear in each argument. More precisely, for any $u, v, w \in V$ and $\alpha \in \mathbb{R}$*

$$\begin{aligned} a(u+\alpha v, w) &= a(u,w) + \alpha a(v,w) \\ a(w, u+\alpha v) &= a(w,u) + \alpha a(w,v). \end{aligned} \tag{1.55}$$

If, additionally, for all $u, v \in V$

$$a(u,v) = a(v,u), \tag{1.56}$$

then "a" is a symmetric bilinear form.

Examples:

1.46 Let $V_1 = \{f\colon [0,1] \to \mathbb{R} \text{ smooth}\}$, then

$$a(u,v) = \int_0^1 u'(x)v'(x)\,dx$$

is a bilinear form, since $a(u,v)$ can be computed for any functions $u, v \in V_1$, and it is simple to see that (1.55) is true. To wit,

$$\begin{aligned} a(u+\alpha v, w) &= \int_0^1 (u' + \alpha v')w'\,dx \\ &= \int_0^1 u'w'\,dx + \alpha \int_0^1 v'w'\,dx \\ &= a(u,w) + \alpha a(v,w), \end{aligned}$$

and similarly with the other slot.
This bilinear form is symmetric.
Let's compute the value of the bilinear form for a few functions:

- $a(\sin(x), x^2) = \int_0^1 \cos(x)2x\,dx = 2(\sin(1) - \cos(1))$.
- $a((x-1)^2, x^3) = \int_0^1 2(x-1)3x^2\,dx = -1/2$.

1.47 Let $V = \mathbb{R}^2$, and

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are vectors in $\mathbb{R}^2$, written in a column matrix form. Then, we can define

$$\begin{aligned} a(u,v) &= u^{\mathsf{T}} M v \\ &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= 2u_1v_1 + 3u_2v_1 + u_1v_2 + 2u_2v_2. \end{aligned} \tag{1.57}$$

This is a bilinear form in V. It is *not* a symmetric bilinear form. For example, $a((0,1),(2,0)) = 6$, and $a((2,0),(0,1)) = 2$.

Now, if

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.58}$$

then $a(u, v) = u \cdot v$, or the dot product between vectors u and v. So, the dot product *is* a bilinear form.

1.48 Let V_1 be that of Example 1.46, and set

$$a(u, v) = \int_0^1 \sin(x) u(x) v'(x)\, dx + u'(1/2) v(1/2).$$

This is a bilinear form, since it is defined for any pair of smooth functions, and it is linear in each argument. It is not symmetric, as it can be inferred from the different roles u and v play in each term. An alternative way to see this is by choosing two functions u and v and evaluating $a(u, v)$; there is a high chance that $a(u, v)$ will be different than $a(v, u)$ if a is not symmetric, and single pair of functions for which $a(u, v) \neq a(v, u)$ is enough to show that it is not symmetric. Set $u(x) = x^2$ and $v(x) = x^3$, then

$$a(x^2, x^3) = \int_0^1 \sin(x) x^2 3x^2\, dx + 2(1/2)(1/2)^3 = 577/8 - 39\cos(1) - 60\sin(1),$$
$$a(x^3, x^2) = \int_0^1 \sin(x) x^3 2x\, dx + 3(1/2)^2(1/2)^2 = 771/16 - 26\cos(1) - 40\sin(1).$$

so $a(x^2, x^3) \neq a(x^3, x^2)$, and this proves that a is not symmetric.

1.2.2 Linear Variational Equations

Having defined linear functionals and bilinear forms, we can now write the variational equations we have seen so far in a simple, abstract way. In paarticular, we are now ready to give a proper definition of a variational equation.

Definition 1.9 (Variational Equation)**.** *Let $\mathscr{W}, \mathscr{V}$ be vector spaces and $F\colon \mathscr{W} \times \mathscr{V} \to \mathbb{R}$ be linear in the second argument, that is,*

$$F(u, v + \alpha w) = F(u, v) + \alpha F(u, w)$$

for any $u \in \mathscr{W}$, $v, w \in \mathscr{V}$ and $\alpha \in \mathbb{R}$.

A variational equation is an equation of the form

$$F(u, v) = 0 \qquad \forall v \in \mathscr{V}, \tag{1.59}$$

The space $\mathscr{V}$ is called the test space.

Definition 1.10 (Linear Variational Equation)**.** *A linear variational equation is a variational equation that is linear in the first argument.*

For completeness, F from (1.59) is linear in the first argument if

$$F(u+\alpha w, v) = F(u,v)+\alpha F(w,v)$$

for any $u, w \in \mathscr{W}$, $v \in V$ and $\alpha \in \mathbb{R}$.

The variational equations we have seen in §1.1.2 are all linear variational equations that have the abstract form

$$0 = F(u,v) = a(u,v) - \ell(v) \qquad \forall v \in \mathcal{V}, \tag{1.60}$$

or

$$\boxed{a(u,v) = \ell(v) \qquad \forall v \in \mathcal{V}} \tag{1.61}$$

where $a\colon \mathscr{W} \times \mathscr{W} \to \mathbb{R}$ is a bilinear form, and $\ell\colon \mathscr{W} \to \mathbb{R}$ is a linear functional.

Examples:

1.49 Let's identify the bilinear form and linear functional for the model problem. Consider the variational equation (1.9a) that the solution u of Problem 1.1 satisfies, namely,

$$\int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\,dx \\ - k(L)d_L v(L) = \int_\Omega f(x)v(x)\,dx. \tag{1.62}$$

for any $v \in \mathcal{V} = \{w\colon \Omega \to \mathbb{R} \text{ smooth} \mid w(0) = 0\}$, with $\Omega = (0, L)$. Let's identify

$$a(u,v) = \int_\Omega \left[k(x)u'(x)v'(x) + b(x)u(x)'v(x) + c(x)u(x)v(x)\right]\,dx, \tag{1.63a}$$

$$\ell(v) = \int_\Omega f(x)v(x)\,dx - k(L)d_L v(L), \tag{1.63b}$$

as a bilinear form and a linear functional, respectively.

1.50 Consider the differential equation

$$u(x)u'(x) + u''(x) = 0$$

for all $x \in (0, L)$. This is a nonlinear equation because of the first term. If u satisfies it, then it also satisfies the variational equation

$$F(u,v) = \int_0^L u(x)u'(x)v(x) - u'(x)v'(x)\,dx = 0 \tag{1.64}$$

for any $v \in \mathcal{V} = \{w\colon [0,L] \text{ smooth} \mid v(0) = v(L) = 0\}$. This equation follows after multiplying the differential equation by v, integrating over $(0, L)$, and then integrating by parts the second term.
Equation (1.64) is a variational equation, but because F is non-linear in the first argument, it is not a linear variational equation.

The specification of the space $\mathscr{W}$ to build the domains of a and ℓ is not important, at least at this stage, so we will skip it.