HW 4

Due Wednesday, May 10, 2023

Constructing Some FE Spaces (35)

Consider a mesh of Lagrange P_k-elements (see Example 1.65 in the notes) with $n_{\text{el}} = 3$ elements of equal length in the interval $[0, 3]$. Elements are numbered consecutively from 1 to n_{el} from left to rigth (from 0 to 3).

1. Let $k = 3$. For the following local-to-global maps, state the dimension of the finite element space, and plot each one of the basis functions.

   (a) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

   (b) (5)

$$\mathsf{LG} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

1.

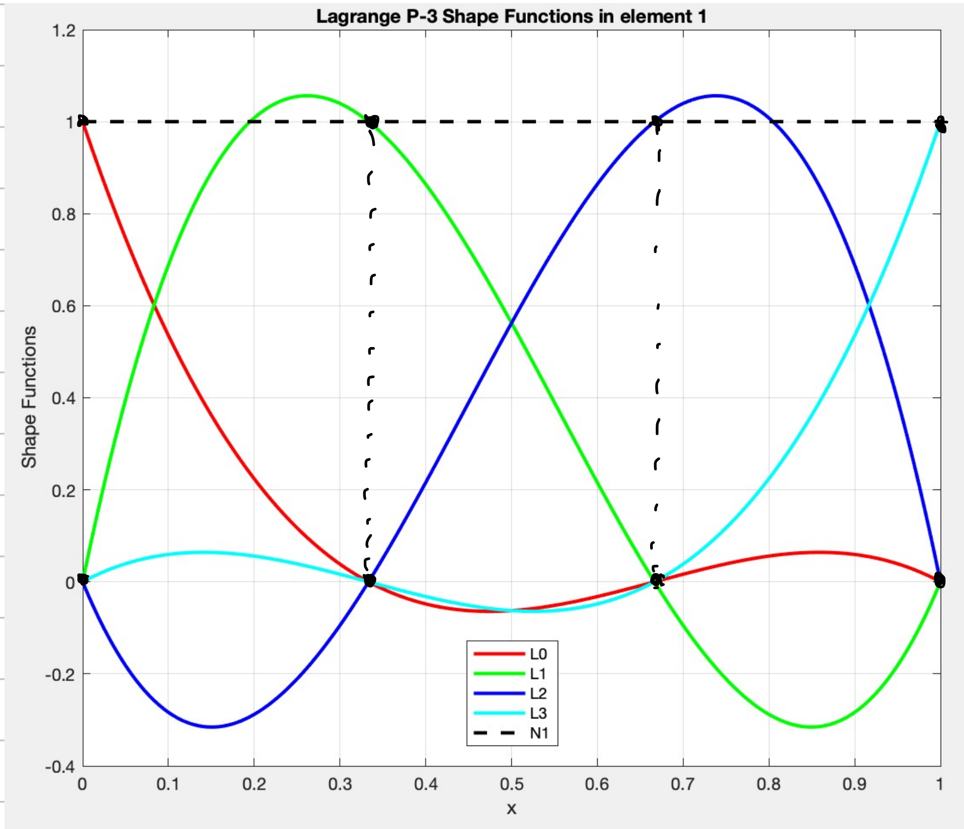
(a) observing LG Map, we know there are 3 global basis functions N_1, N_2, N_3. The dimension of the finite element space is 3.

For each element, there are 4 local degree of freedoms. We can use Lagrange P_3 elements defined as:

$$N_a^e(x) = \frac{\prod_{b=1,\, b\neq a}^{4} (x - x_b^e)}{\prod_{b=1,\, b\neq a}^{4} (x_a^e - x_b^e)}.$$

Consider element 1 for example. All N_a^e only contribute to N^1, we have

$$N^1(x) = N_1^1(x) + N_2^1(x) + N_3^1(x) + N_4^1(x) = 1$$

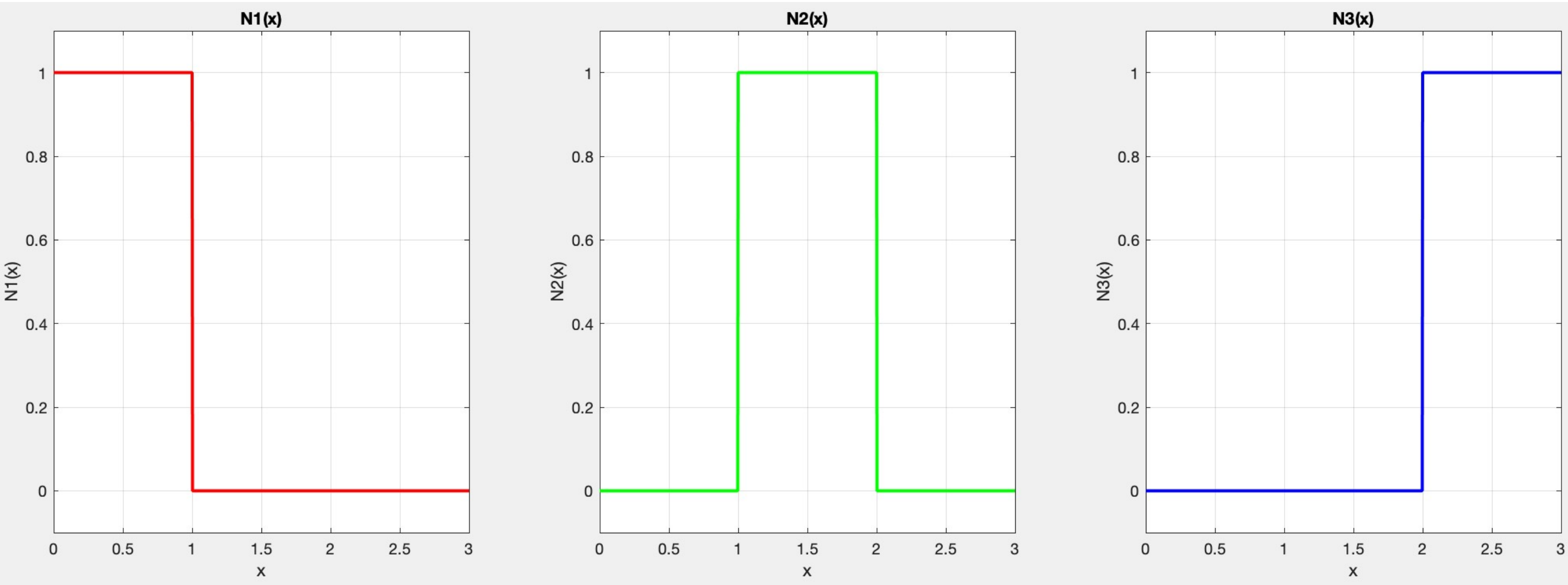


Therefore, $N^1(x) = 1$ (dash line) in element 1.

Similiarly, we have $N^2(x) = N_1^2(x) + N_2^2(x) + N_3^2(x) + N_4^2(x) = 1$.

$N^3(x) = N_1^3(x) + N_2^3(x) + N_3^3(x) + N_4^3(x) = 1$

So, the global basis functions N_1, N_2, N_3 are



(b). We have $LG = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 5 & 6 & 6 \end{bmatrix}$. We know there are 6 global basis functions. The dimension of FE space is 6.

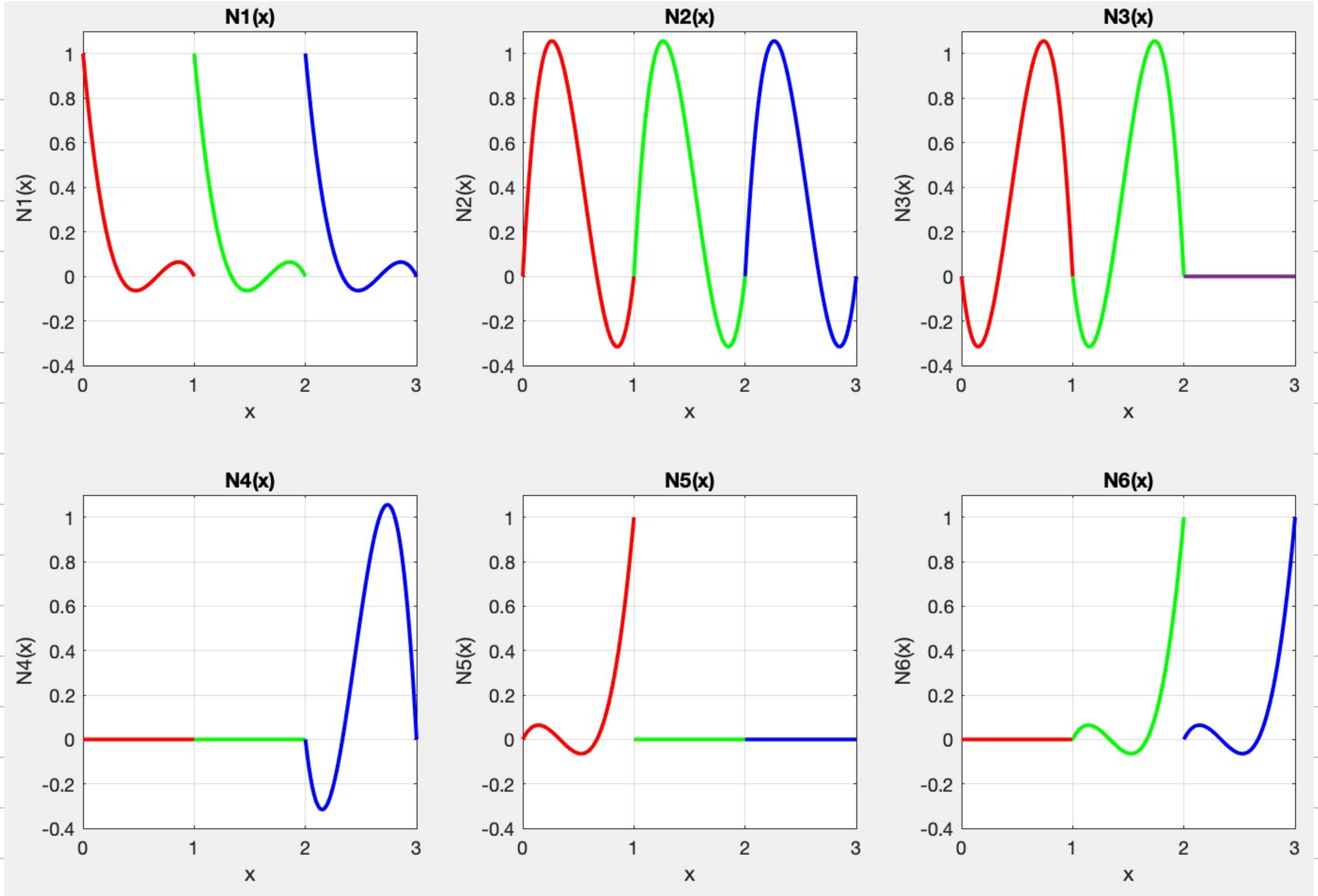
Observing first row, we know N_1^1, N_1^2, N_1^3 contribute to N_1. Here, shape functions $N_a^e(x)$ are defined as P_3 lagrange functions.

Second row tells N_2^1, N_2^2, N_2^3 contribute to N_2

$N_3^1, N_3^2,$ contribute to N_3. and N_3^3 contribute to N_4

N_4^1 contribute to N_5 ; N_4^2 and N_4^3 contribute to N_6

Therefore, the plots of all basis functions are:



2. The following local-to-global map renders the basis functions to be continuous and have minimal support.

(a). For P_3 - elements.

Each element have 4 shape functions.

FE should be continuous with and across element boundaries. One possible LG map is:

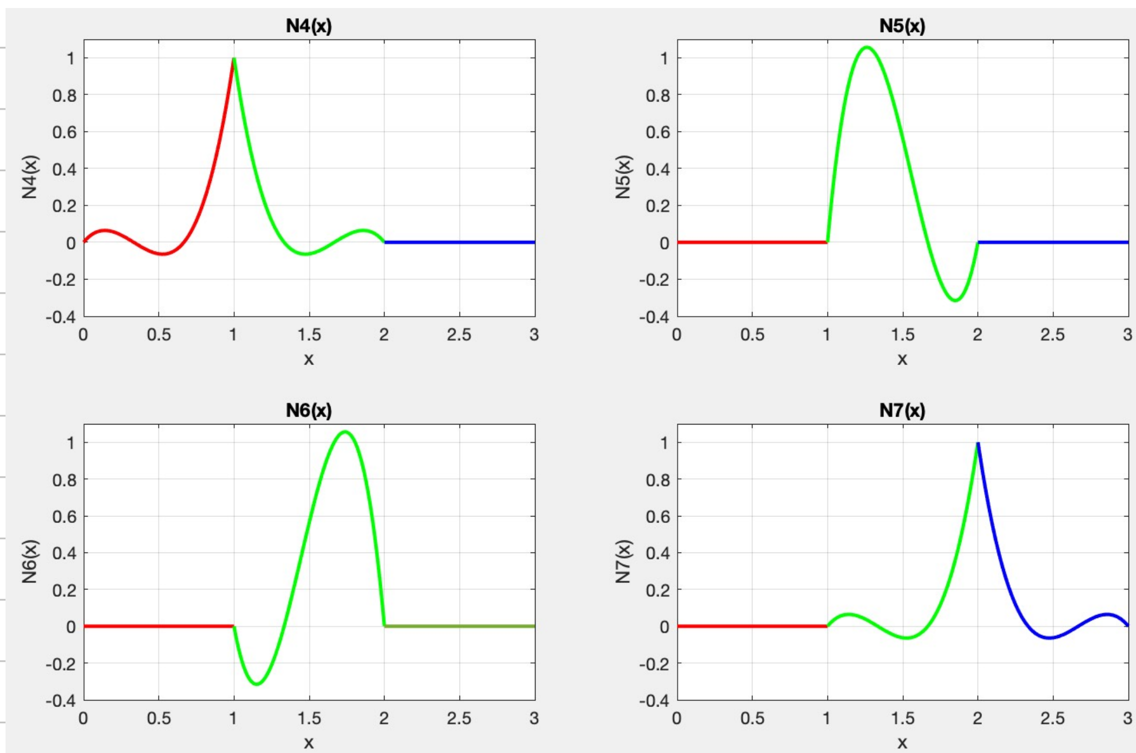
$$LG = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix}$$

Therefore, there 10 global basi functions.

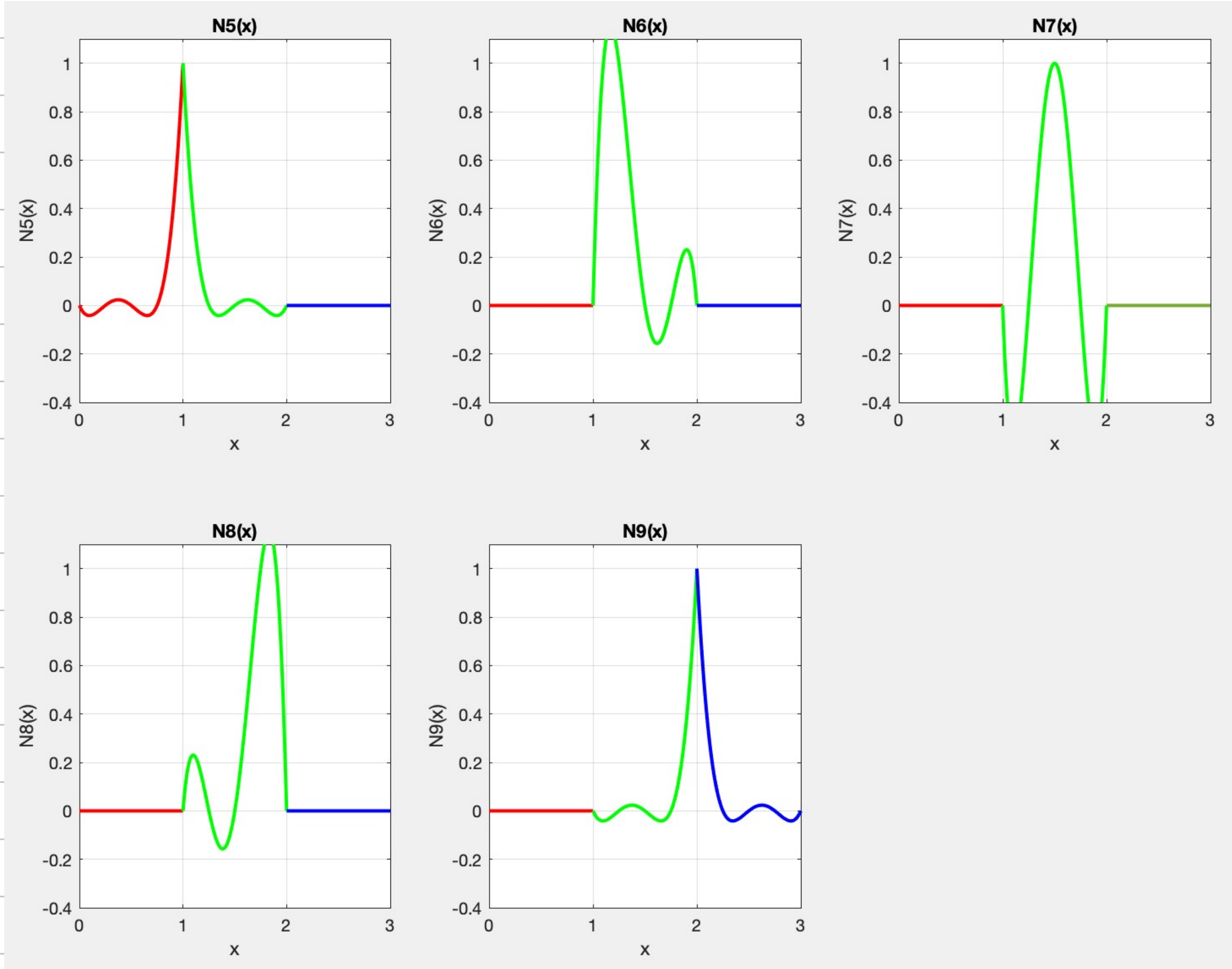
(b). Similiarly, for P_4-elemene, we have 5 local basis functions. The LG can be written as:

$$LG = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \\ 5 & 9 & 13 \end{bmatrix}$$

3. ① For P_3. elemene



③. For P_φ element



4. To enforce $u(0) = u(3)$, we could make

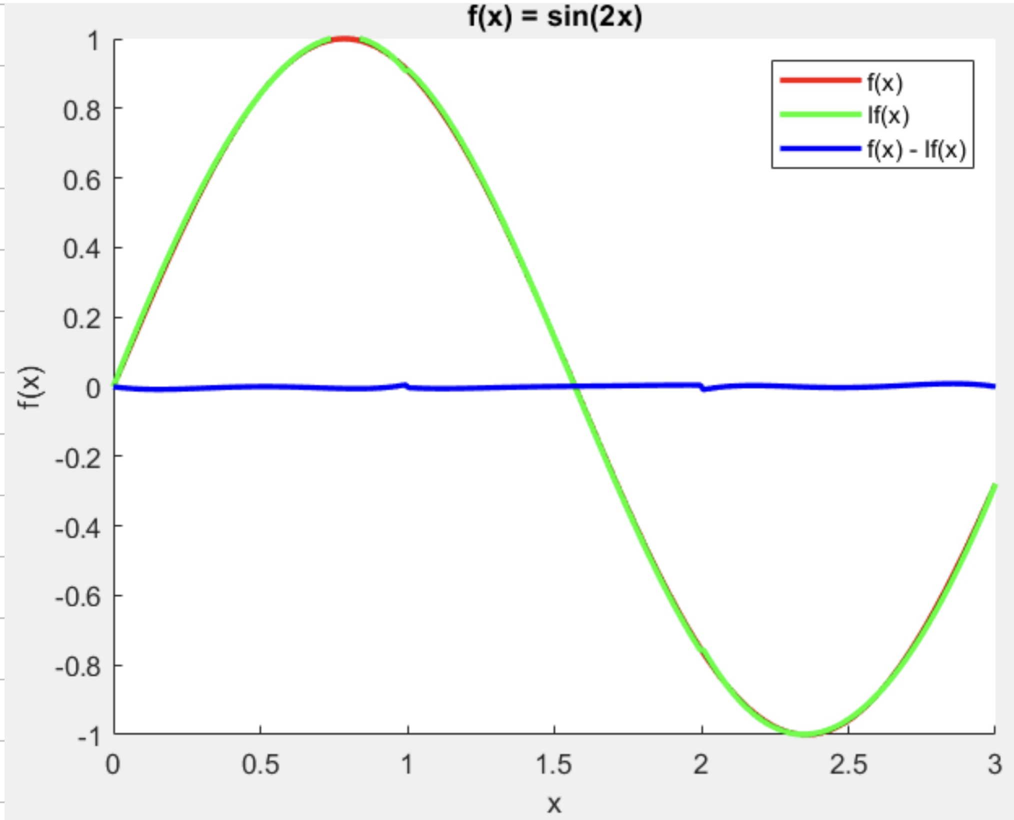
$$\sum_{i=1}^{\alpha} N_i(x=0) = \sum_{i=1}^{\alpha} N_i(x=3).$$

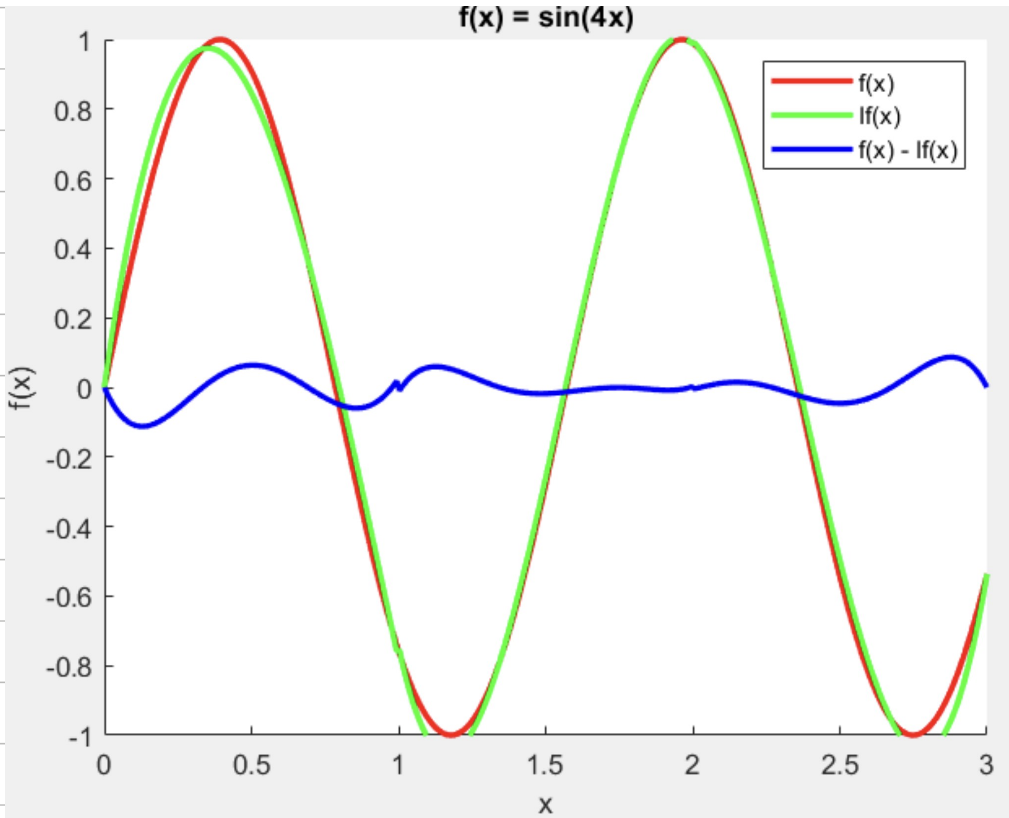
If we assume using P_1 element and continuous everywhere; we can write LG map as:

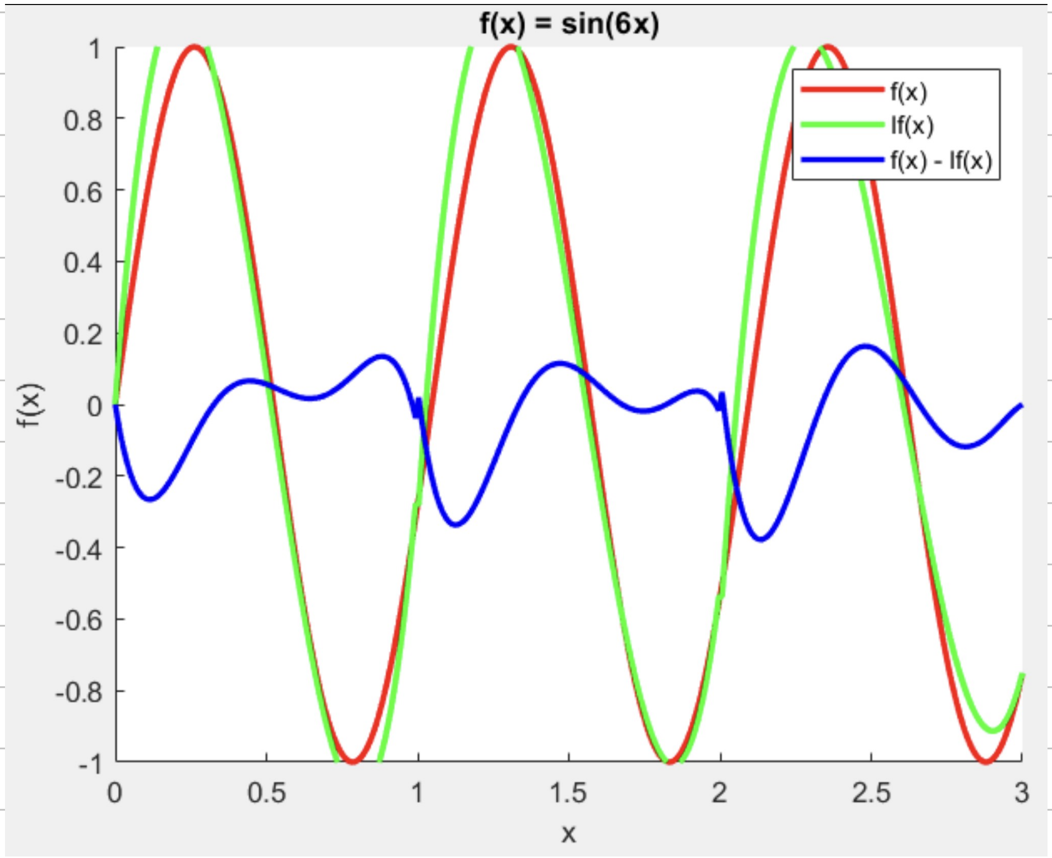
$$LG = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

The dimension of FE space is 3.

5.


f(x) = sin(2x)
f(x)
If(x)
f(x) - If(x)
f(x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3


f(x) = sin(4x)
f(x)
If(x)
f(x) - If(x)
f(x)
x
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3


f(x) = sin(6x)
f(x)
lf(x)
f(x) - lf(x)
x
f(x)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
0.5
1
1.5
2
2.5
3