HW-6.

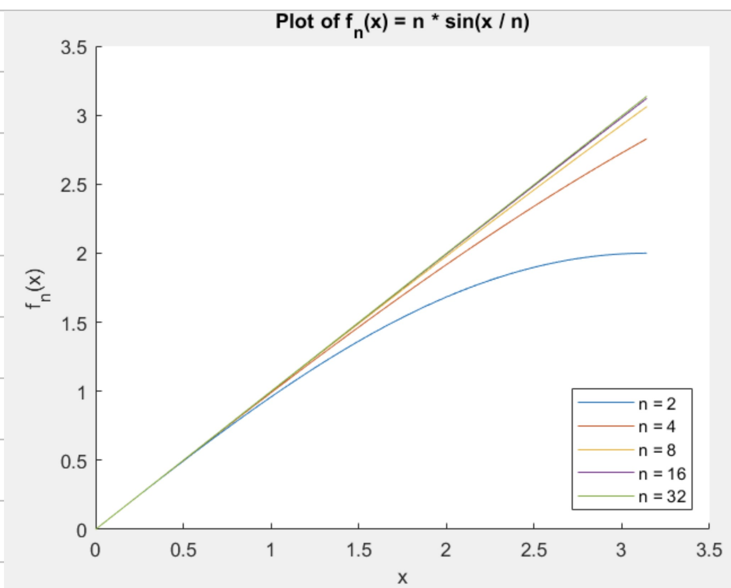
In this problem we would like to play with the convergence, norms, and membership of sequence of functions in different spaces. To this end, let $I = (0, \pi)$, and recall (see Appendix A in the notes) that a function $f\colon I \to \mathbb{R}$ is a member of the following spaces if

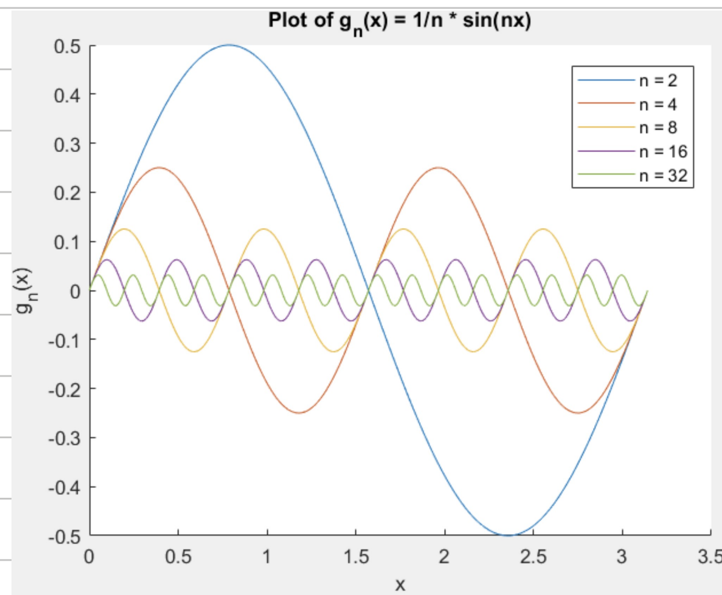
$$f \in L^2(I) \Leftrightarrow \|f\|_{0,2} = \left(\int_0^\pi f^2\,dx\right)^{1/2} < \infty$$
$$f \in L^\infty(I) \Leftrightarrow \|f\|_{0,\infty} = \max_{x\in I}|f(x)| < \infty$$
$$f \in H^1(I) \Leftrightarrow \|f\|_{1,2} = \left(\|f\|_{0,2}^2 + \|f'\|_{0,2}^2\right)^{1/2} < \infty.$$

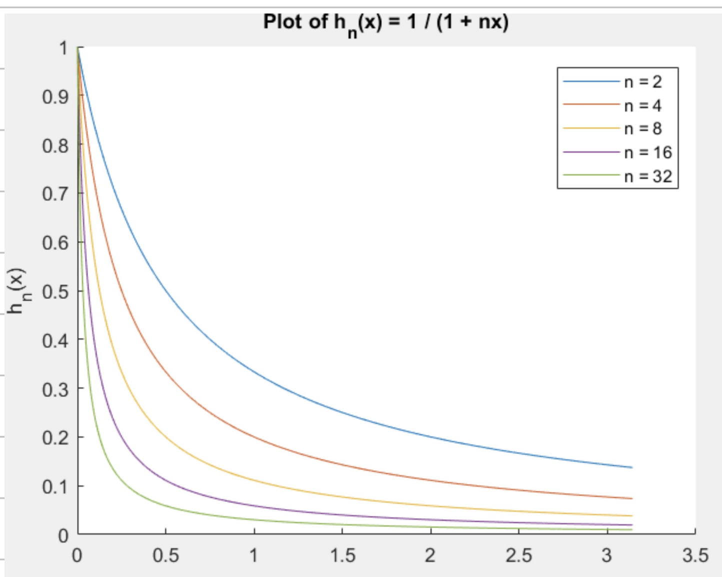
Consider the sequences of functions for $n = 1, 2 \ldots$:

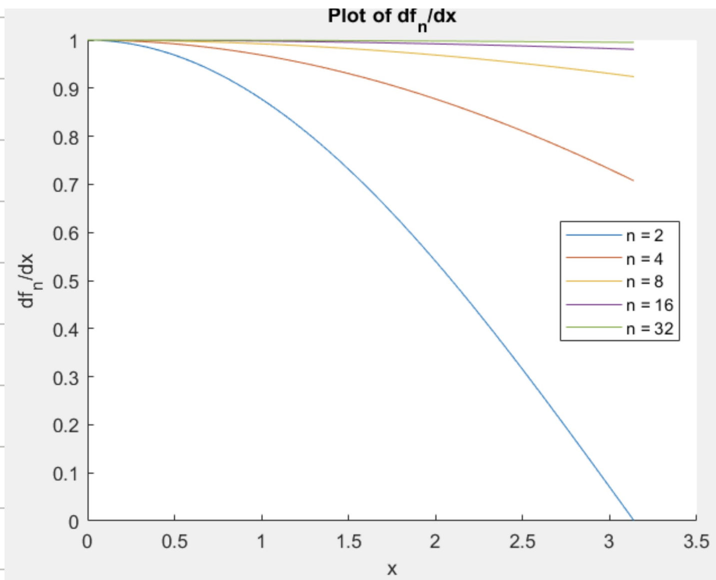
$$f_n(x) = n\sin\left(\frac{x}{n}\right), \quad f_\infty(x) = x,$$
$$g_n(x) = \frac{1}{n}\sin(nx), \quad g_\infty(x) = 0,$$
$$h_n(x) = \frac{1}{1+nx}, \quad h_\infty(x) = 0.$$

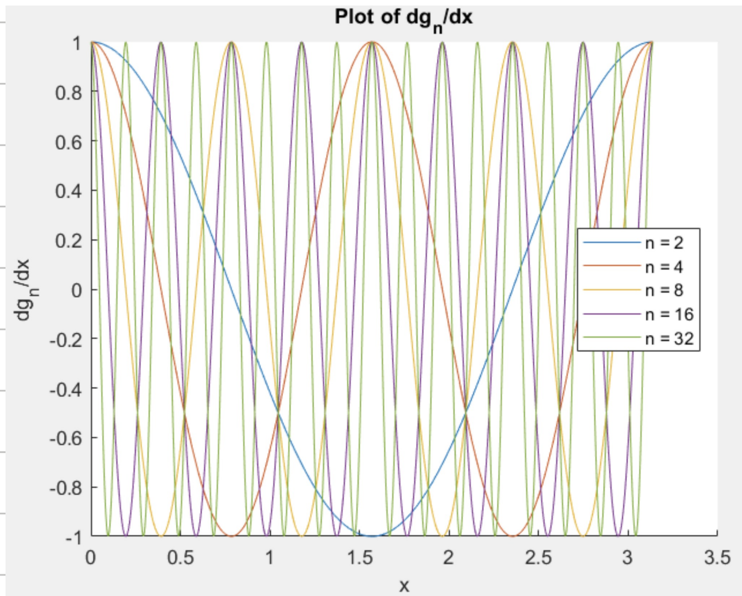
1.

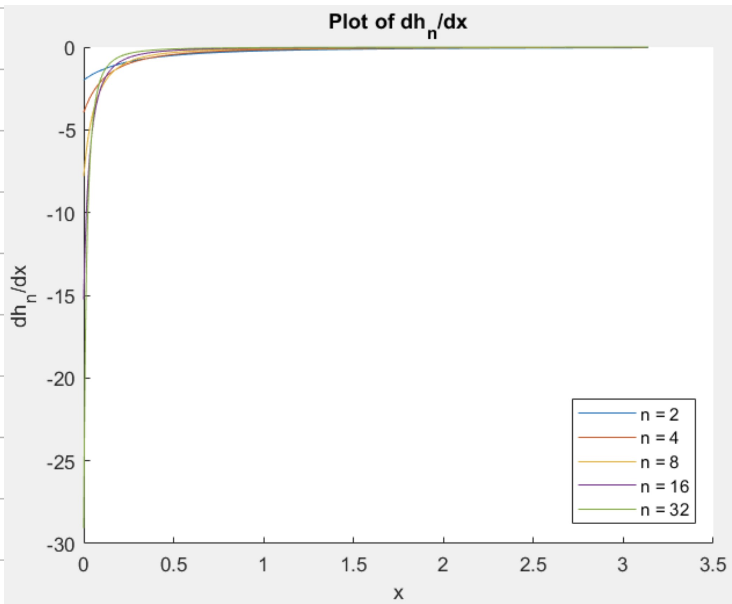












2 ① Given $f_\infty(x) = x$, $\|f_\infty\|_{0,2} = \left(\int_0^\pi x^2\,dx\right)^{\frac{1}{2}}$

$$= \left(\frac{\pi^3}{3}\right)^{\frac{1}{2}} < \infty.$$

Hence, $f_\infty \in L^2(I)$.

② Given $f_\infty(x) = x$, $\|f_\infty\|_{0,\infty} = \max_{x\in I} |f(x)|$

$$= \pi < \infty$$

$f_\infty(x) \in \mathcal{L}^\infty(I)$.

③. Given $f_\infty(x) = x$, $\|f_\infty\|_{1,2} = (\|f_\infty\|_{0,2}^2 + \|f_\infty'\|_{0,2}^2)^{1/2}$

$f_\infty'(x) = \lim_{n\to\infty} \cos(\frac{x}{n}) = 1$; $\|f_\infty'\|_{0,2}^2 = 1$.

$\|f_\infty\|_{0,2}^2 = \frac{\pi^3}{3}$.

Hence, $\|f_\infty\|_{1,2} = (\frac{\pi^3}{3}+1)^{\frac{1}{2}} < \infty$

$f_\infty \in H^1(I)$.

3.

$\mathcal{L}^2(I)$ for f_n is as follows:

$$\|f\|_{0,2} = \left(\int_0^\pi (n\sin(\frac{x}{n}))^2\,dx\right)^{\frac{1}{2}}$$

$$= \left(\int_0^\pi n^2 \sin^2(\frac{x}{n})\,dx\right)^{\frac{1}{2}}$$

$$= \left[\frac{\pi n^2}{2} - \frac{n^3 \sin(\frac{2\pi}{n})}{4}\right]^{\frac{1}{2}}$$

$H^1(I)$ for f_n is similiar

$$\|f\|_{1,2} = (\|f\|_{0,2}^2 + \|f'\|_{0,2}^2)^{1/2}$$

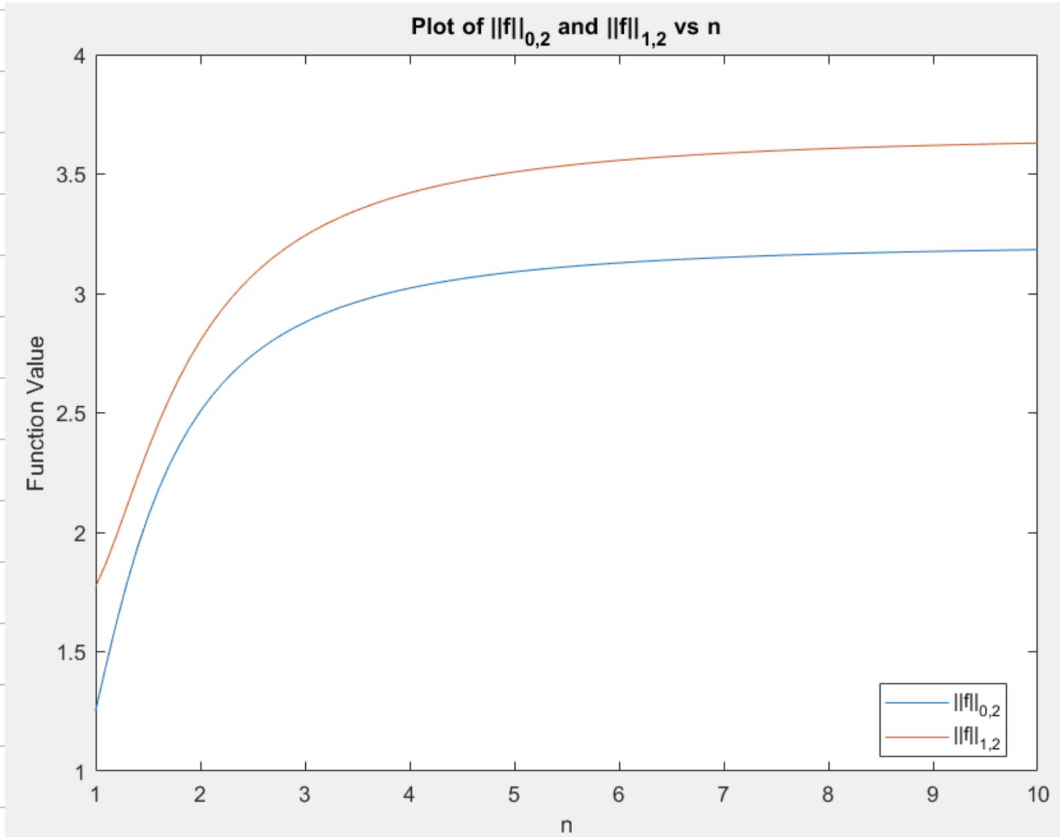
Calculating the second part in the parenthesis

$$\|f'\|_{0,2} = \left[\int_0^\pi (\cos(\frac{x}{n}))^2\,dx\right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi}{2} + \frac{n \sin\left(\frac{2\pi}{n}\right)}{4} \right]^{\frac{1}{2}}$$

$$\|f\|_{1,2} = \left(\frac{\pi n^2}{2} - \frac{n^3 \sin\left(\frac{2\pi}{n}\right)}{4} + \frac{\pi}{2} + \frac{n \sin\left(\frac{2\pi}{n}\right)}{4} \right)^{\frac{1}{2}}$$

The $\|f\|_{0,2}$ and $\|f\|_{1,2}$ vs n are as follows:



4.

①. $f_n(x) = n \sin\left(\frac{x}{n}\right)$; $f_\infty(x) = x$.

$$\|f_n(x) - f_\infty(x)\|_{0,2} = \left(\int_0^{\pi} \left(n \sin\left(\frac{x}{n}\right) - x\right)^2 \right)^{\frac{1}{2}}$$

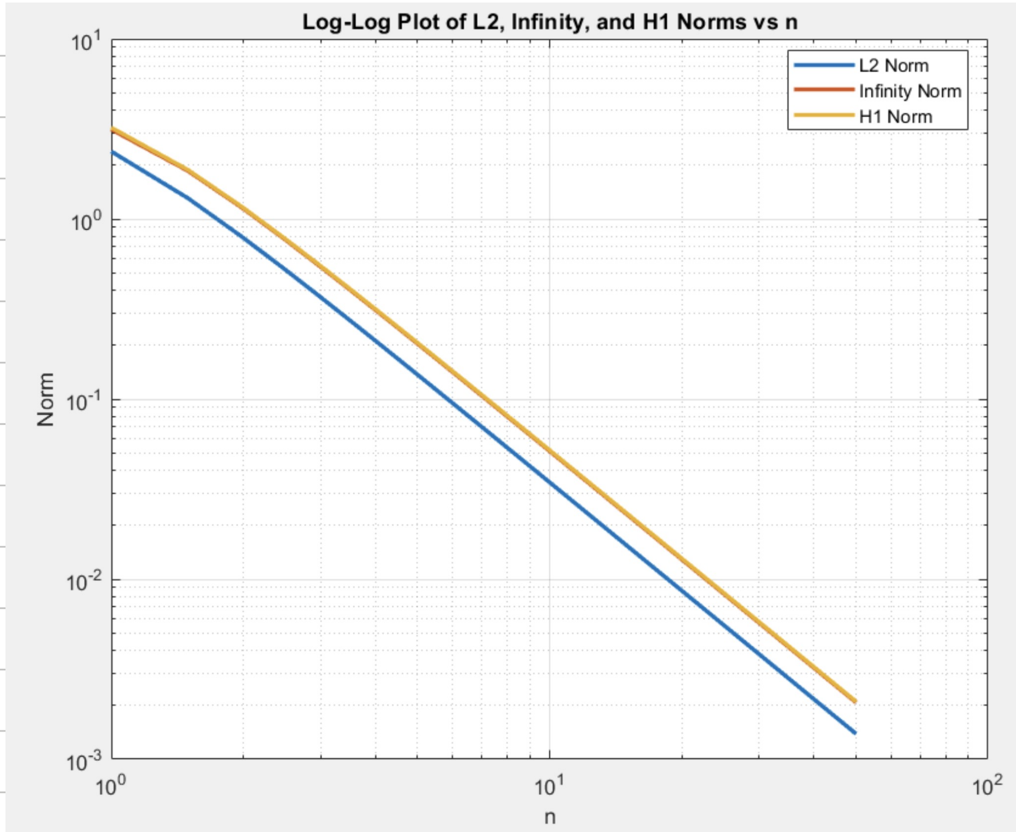
$$= \left[\int_0^{\pi} \left(n^2 \sin^2\left(\frac{x}{n}\right) - 2n \sin\left(\frac{x}{n}\right) x + x^2\right) dx \right]^{\frac{1}{2}}$$

$$= \left[\frac{\pi n^2}{2} - 2n^3 \sin\left(\frac{\pi}{n}\right) + \frac{\pi^3}{3} + 2n^2 \pi \cos(\pi/n) - \frac{n^3 \cos\left(\frac{\pi}{n}\right) \sin\left(\frac{\pi}{n}\right)}{2} \right]^{\frac{1}{2}}$$

$$\| f_n(x) - f_\infty(x) \|_{0,\infty} = \max_{x \in I} \left| n \sin\left(\frac{x}{n}\right) - x \right|$$

$$\| f_n(x) - f_\infty(x) \|_{1,2} = \left(\| f \|_{0,2}^2 + \| f' \|_{0,2}^2 \right)^{\frac{1}{2}}$$

Using Matlab Symbolic calculation, we have:



The figure shows that $f_n(x) \rightarrow f_\infty$ in $L^2(I)$

$f_n(x) \rightarrow f_\infty$ in $H^1(I)$.

$f_n(x) \rightarrow f_\infty$ in $L^\infty(I)$.

This also reflects the observations in part 1, as larger n leads to a better approximation for $f(x)$ and $f'(x)$.

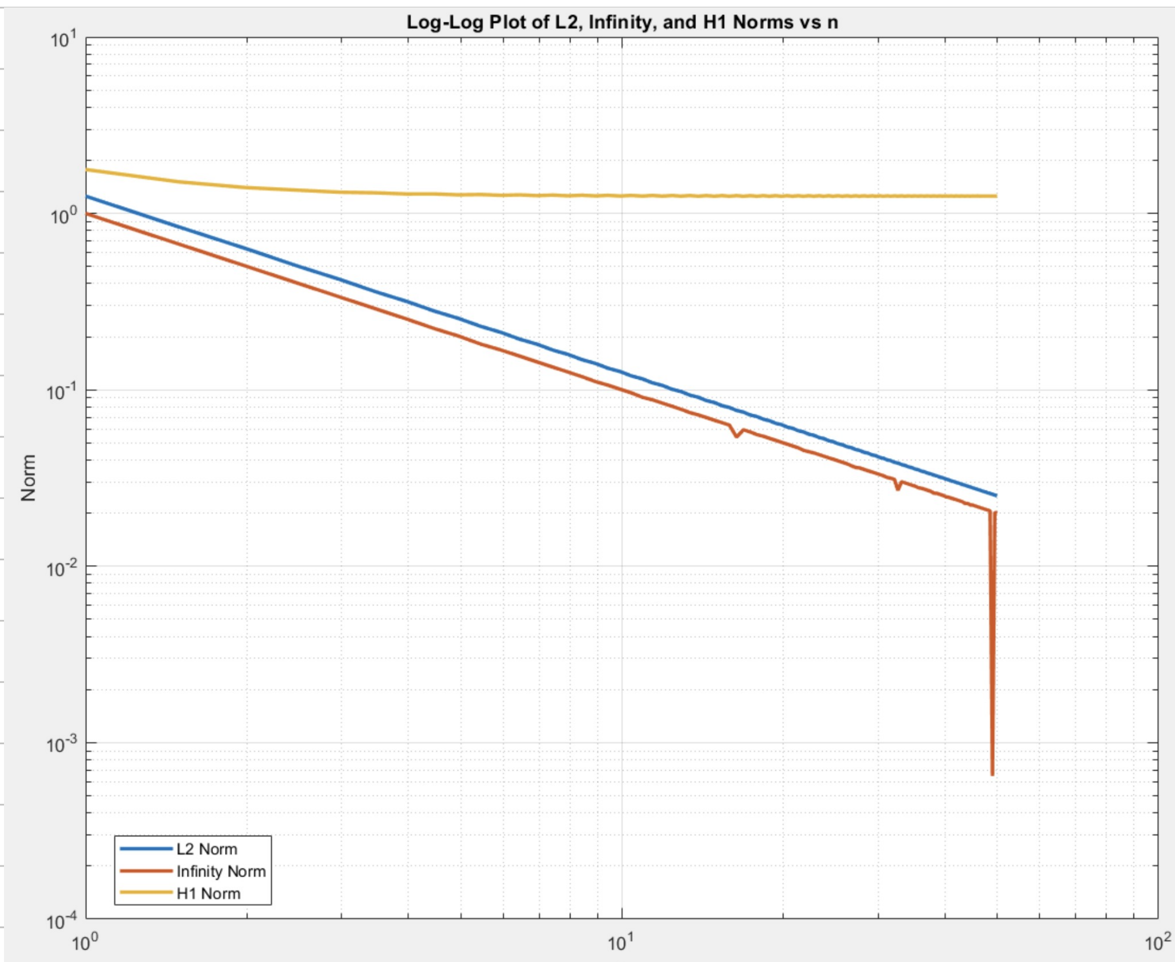
② $g_n = \frac{1}{n}\sin(nx)$, $g_\infty(x) = 0$.

$$\|g_n(x) - g_\infty(x)\|_{0,2} = \left[\int_0^{\pi} \left(\frac{1}{n}\sin(nx) - 0\right)^2 dx\right]^{\frac{1}{2}}$$

$$\|g_n(x) - g_\infty(x)\|_{0,\infty} = \max_{x\in I} \left|\frac{1}{n}\sin(nx) - 0\right|$$

$$\|g_n(x) - g_\infty(x)\|_{1,\infty} = \left(\|f\|_{0,\infty}^2 + \|f'\|_{0,2}^2\right)^{\frac{1}{2}}$$

Following the integreation procedure above, we can plot norms vs n ~~of~~ g_n as follows:



We have observed that

$$f_n(x) \to f_\infty \text{ in } L^2(I)$$

$$f_n(x) \to f_\infty \text{ in } L^\infty(I).$$

But, $f_n(x)$ does not converge in $H^1(I)$.

This is because the analytical expression of $H^1(I)$ is as follows:

$$\left[\frac{\pi}{2} + \frac{\left[\frac{\pi n}{2} - \frac{\sin(2n\pi)}{4} + \frac{n^2\sin(2n\pi)}{4}\right]}{n^3}\right]^{\frac{1}{2}}$$

$$\to \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \approx 1.2533.$$

No matter how large the n is, f_n never converge in H_1 Norm space. This phenomenon can be explained from the plot of $f'(x)$. As the $f'(x)$ is sinodial by nature and $f'_\infty(x)$ is 1, $f'_n(x)$ can not approximate $f'_\infty(x)$.

③ $h_n(x) = \frac{1}{1+nx}$, $h_\infty(x) = 0$.

$$\| h_n(x) - h_\infty(x) \|_{0,2} = \left[\int_0^\pi \left(\frac{1}{1+nx} \right)^2 dx \right]^{\frac{1}{2}}$$

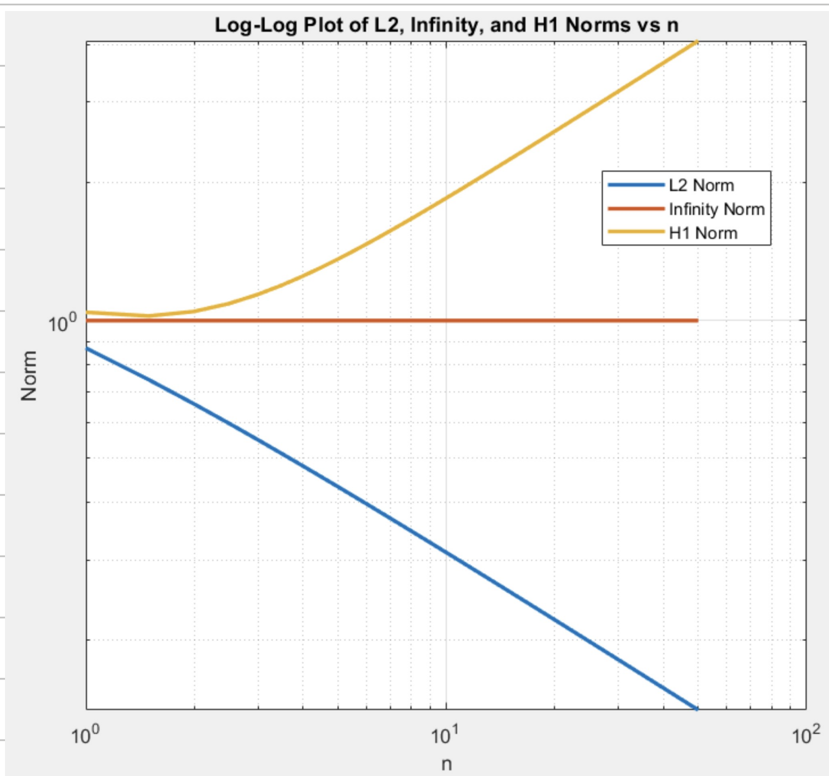
$$= \left[\frac{\pi}{n\pi+1} \right]^{\frac{1}{2}}$$

$$\| h_n(x) - h_\infty(x) \|_{0,\infty} = \max_{x \in I} \left| \frac{1}{1+nx} \right|$$

$$= 1$$

$$\| h_n(x) - h_\infty(x) \|_{1,2} = \left(\| f \|_{0,2}^2 + \| f' \|_{0,2}^2 \right)^{1/2}$$

$$= \left[\frac{\pi}{n\pi+1} + \frac{n}{3} - \frac{n}{(3(n\pi+1)^3)} \right]^{\frac{1}{2}}$$



We observe that $h_n \to h_\infty$ in L_2 Norm. But it doesn't converge with infinity norm and H_1 Norm. The $f_n(x)$ even diverges in the H_1 Norm.

On Interpolation Errors (70)

Consider the interval $\Omega = [-1, 1]$, and a mesh of $n_{\text{el}} \in \mathbb{N}$ equally long P_k-elements on it, for $k = 1, 2, 4$. for P_k-elements is shown in Example 3.24 in the notes, while The Lagrange finite element interpolant $\mathcal{I}u$ is constructed through (3.24) in the notes.

For $\omega \in \mathbb{R}$, consider the functions

$$v_\omega(x) = \cos(\omega x)$$

$$w_\omega(x) = \begin{cases} 0 & x < 0 \\ x^\omega & x \geq 0. \end{cases}$$