Q1

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Given Strong form

$f : (0,1) \to \mathbb{R}$ continuous and constant $\lambda > 0$

$$-u''(x) + \lambda u'(x) = f(x) \quad , x \in (0,1) \longrightarrow (A)$$

$$u(0) = 0.1$$

$$u'(1) = 1$$

Introduce trial and test space

$\mathcal{S} = \{ u : [0,1] \to \mathbb{R} \text{ smooth} \mid u(0) = 0.1 \}$ → look for solution

$\mathcal{V} = \{ v : [0,1] \to \mathbb{R} \text{ smooth} \mid v(0) = 0 \}$ → test the solution

Multiply (A) with $v(x)$ → test function & integrate over Ω

$$\int_0^1 v(x) \left[u''(x) - \lambda u'(x) + f(x) \right] dx = 0 \qquad \forall x \in \Omega, \ \forall v \in \mathcal{V}$$

by parts

Integrate by parts on 1st Term →

$$\Rightarrow u'(x) v(x) \Big|_{0+}^{1-} - \int_0^1 u'(x) v'(x) dx - \lambda \int_0^1 u'(x) v(x) dx + \int_0^1 f(x) v(x) dx = 0$$

$u'(1) v(1) = v(1)$ as $u'(1) = 1$

$u'(0) v(0) = 0$ as $v(0) = 0$

$$\Rightarrow \int_0^1 u'(x) v'(x) dx + \lambda \int_0^1 u'(x) v(x) dx - v(1) = \int_0^1 f(x) v(x) dx \quad \forall v \in \mathcal{V} - (B)$$

This is the weak form of the problem which gives weak solution

* So the weak form states ⇒ Find $u \in \mathcal{S}$ such that equation (B) holds for any $v \in \mathcal{V}$

with $u(0) = 0.1$ being an essential B.C

$v(0) = 0$ & $u'(1) = 1$ are to occur naturally

Q2 a

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Given equation (functions)

$$a(u, w) = \int_0^1 (w_x u_x + \lambda w u)\, dx + w(1)\, u(1)$$

bilinear form needs to be $\rightarrow$

1. both u, w need to be in vector space $\mathcal{V}$
2. As $u, w \in \mathcal{V}$ $a(u, w)$ can be computed

and

$$a(u + \alpha v, w) = \int_0^1 w_x (u + \alpha v)' + \lambda w (u + \alpha v)\, dx + w(1)\left[u(1) + \alpha v(1)\right]$$

$$= \int_0^1 \left[w_x u_x + \alpha w_x v_x + \lambda w u + \alpha \lambda w v\right] dx + w(1)\, u(1) + \alpha\, w(1)\, v(1)$$

$$= \int_0^1 (w_x u_x + \lambda w u)\, dx + w(1)\, u(1) + \alpha \left[\int_0^1 (w_x v_x + \lambda w v)\, dx + w(1)\, v(1)\right]$$

$$= a(u, w) + \alpha \left[a(v, w)\right]$$

similarly

$$a(u, w + \alpha v) = \int_0^1 (w + \alpha v)' u_x + \lambda u (w + \alpha v)\, dx + u(1)\left[w(1) + \alpha v(1)\right]$$

$$= \int_0^1 \left[w_x u_x + \alpha v_x u_x + \lambda u w + \lambda \alpha u v\right] dx + u(1)\, w(1) + \alpha\, u(1)\, v(1)$$

$$= \int_0^1 (w_x u_x + \lambda u w)\, dx + u(1)\, w(1) + \alpha \left[\int_0^1 (v_x u_x + \lambda u v)\, dx + u(1) v(1) \right]$$

$$= a(u, w) + \alpha\, a(u, v)$$

$\therefore$ Bilinearity proved as function $a(u,w)$ is linear for both input functions u, w

Q2 b

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$$a(u,v) = \int_0^1 (v_x u_x + \lambda u v)\,dx + u(1)\,v(1)$$

$$a(v,u) = \int_0^1 (u_x v_x + \lambda v u)\,dx + v(1)\,u(1)$$

$$= a(u,v)$$

$\therefore$ the function is symmetric as changing or replacing functions u, v with v, u results in same form for $a(\cdot\ )$

Q2 c

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linear functional of a function $l : V \to \mathbb{R}$ such that for any $u, v \in V$ & $\alpha \in \mathbb{R}$

$$l(u + \alpha v) = l(u) + \alpha\, l(v)$$

we have $l(w) = \int_0^1 w\, x^2\, dx + w(1)$

$$l(u + \alpha v) = \int_0^1 (u + \alpha v)\, x^2\, dx + u(1) + \alpha\, v(1)$$

$$= \int_0^1 u\, x^2\, dx + u(1) + \alpha \left[\int_0^1 v\, x^2\, dx + v(1) \right]$$

$$= l(u) + \alpha\, l(v)$$

$\therefore$ proved the function $l(w)$ is linear functional

$\rightarrow$ value of function $l(w)$ can be computed for any function $w \in V$ $\rightarrow$ which is given

Q2 d

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Given u is smooth & the variational equations is

$$\int_0^1 (w' u' + \lambda w u)\, dx + w(1)\, u(1) = \int_0^1 w x^2\, dx + w(1)$$

where w is the test function, $w \in \mathcal{V}$

Use integrate by parts on 1st Term

$$\Rightarrow \int_0^1 (w' u' + \lambda w u - w x^2)\, dx + w(1)u(1) - w(1) = 0$$

$$\Rightarrow \int_0^1 (-(w u'') + \lambda w u - w x^2)\, dx + w(1)\, u'(1) - w(0)\, u'(0) + w(1)\, u(1) - w(1) = 0$$

As $w \in \mathcal{V} \Rightarrow w(0) = 0$

$$\Rightarrow \int_0^1 (-u'' + \lambda u - x^2)\, w\, dx + w(1)\left[u'(1) + u(1) - 1 \right] = 0$$

This resulting equation should be valid for any $w \in \mathcal{V}$, so every Term that multiplies with test function should be $= 0$ to satisfy the above residual (i.e multiply with $w(x)$ & $w(1)$)

$$\Rightarrow \quad -u''(x) + \lambda u(x) - x^2 = 0 \qquad x \in (0,1)$$

$$u'(1) + u(1) - 1 = 0$$

$\therefore$ the strong form of our problem is

$$u''(x) - \lambda u(x) = x^2 \qquad \forall\, x \in (0,1) \quad \Omega$$

$$\left.\begin{array}{l} u'(1) = 1 - u(1) \\ u(0) = 0 \end{array}\right\} \partial\Omega \quad \text{boundary conditions}$$

Q2 e

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In part 'd' the natural boundary condition is the equation which is obtained when we construct the strong form from weak form

So, $u'(1) + u(1) = 1$ is natural boundary condition

Whereas essential B.C is required by us to actually construct our equation & vector space

So, $u(0) = 0$ is an essential B.C

Q1

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to find if sets are vector spaces of functions,
we need to check closure, identity and additive inverse

given vector space $\rightarrow$ $W = \{u : [-1,1] \rightarrow R \text{ smooth}\}$

(a) $V_1 = \{ u = W \mid u(0) = 0 \}$

if $u, v \in V_1$

$u(0) = 0$, $v(0) = 0$

then we can take $w = u + v$

$\Rightarrow w(0) = u(0) + v(0) = 0$

$\therefore w \in V_1$ & zero function is also satisfied in V_1

also $\alpha u \in V_2$ as $\alpha u(0) = 0$

$-u(0) = 0 = u(0) \rightarrow$ satisfied

$-u \in V_1$

(b) $V_2 = \{ u \in W \mid u''(0) = 0 \}$

$u, v \in V_2$

$u''(0) = v''(0) = 0$

$w = u + v$

$w'' = (u+v)'' = u'' + v''$

$\Rightarrow w''(0) = u''(0) + v''(0) = 0$

& zero function in V_2

$\alpha u \in V_2$ as $(\alpha u)'' = \alpha u''(0) = 0$

$-u''(0) = 0 = u''(0)$ $\therefore -u \in V_2$

$\therefore V_2$ is vector spaces of functions

(c) $V_3 = \{ u \in W \mid u(x) \neq 0 \ \forall x \in [-1,1] \}$

(c) $V_3 = \{ u \in W \mid u(x) \neq 0 \quad \forall x \in [-1, 1] \}$

let us take $u(x) \mid u(x) \neq 0 \ \forall x \in [-1, 1]$

& $v(x) = -u(x)$

here $u \in V_3$ & $v \in V_3$ as $-u(x) \neq 0 \ \forall x \in [-1, 1]$

Now consider function

$w = v + u$

$w(x) = v(x) + u(x) = u(x) - u(x) = 0$

w contradicts condition in V_3 as $w(x) = 0 \ \forall x \in [-1, 1]$

closure not satisfied

So V_3 set is not a vector space of functions

(d) $V_4 = \{ u \in W \mid \int_{-1}^{1} u''(x)\, dx = 0 \}$

let $u, v \in V_4 \Rightarrow \int_{-1}^{1} u''(x)\, dx = 0, \ \int_{-1}^{1} v''(x)\, dx = 0$

if $w = u + v \Rightarrow \int_{-1}^{1} w''(x)\, dx = \int_{-1}^{1} (u+v)''\, dx$

$= \int_{-1}^{1} u''(x)\, dx + \int_{-1}^{1} v''(x)\, dx = 0$

$\alpha u \Rightarrow \int_{-1}^{1} \alpha u''(x)\, dx = \alpha \int_{-1}^{1} u''(x)\, dx = 0$

$\therefore$ V_4 satisfies closure & zero function is in V_4

also $-u(x) \in V_4$ $\therefore$ V_4 is sets of vectorspaces of functions.

(e) $V_5 = \{ u \in W \mid \int_{-1}^{1} x^2 u(x)\, dx = 1 \}$

let $u, v \in V_5 \Rightarrow \int^{1} x^2 u(x)\, dx = 1; \ \int^{1} x^2 v(x)\, dx = 1.$

(5) $V_5 = \{ \ldots$

let $u, v \in V_5 \Rightarrow \int_{-1}^{1} x^2 u(x)\,dx = 1 ; \quad \int_{-1}^{1} x^2 v(x)\,dx = 1.$

if $w = u + v \in V_5 \Rightarrow$

$$\int_{-1}^{1} x^2 w(x)\,dx = \int_{-1}^{1} x^2 (u+v)\,dx$$

$$= \int_{-1}^{1} x^2 u(x)\,dx + \int_{-1}^{1} x^2 v(x)\,dx$$

$$= 1 + 1 = 2$$

$\therefore$ closure not satisfied

$\therefore$ V_5 is not a set of vector spaces of functions.

(6) $V_6 = \{ u \in W \mid u(0) = -5 \}$

let $u, v \in V_6 \Rightarrow u(0) = v(0) = -5$

if $w = u + v \in V_6 \Rightarrow w(0) = u(0) + v(0)$

$$= -5 - 5 = -10 \neq -5$$

$\therefore$ closure not satisfied

$\therefore$ V_6 is not a set of vector spaces of functions

Q2

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We have

$V_6 = \{ u \in W \mid u(0) = -5 \}$

V_6 is an affine subspace of W & define

V_1 from part (a)

$V_1 = \{ u \in W \mid u(0) = 0 \}$

if $u, v \in V_6$ then $w = v - u \in V_1$

as $v(0) - u(0) = -5 + 5 = 0$

$w(0) = 0 \in V_1$

so for any $u \in V_6$

$V = \{ v - u \mid v \in V_6 \} \subseteq V_1$

also $V \supseteq V_1$ as $v_1 \in V_1$ $v_1 + v = u \in V_6$

so $\cancel{v} - u \in V_1$

$V = V_1$

$\therefore$ V_1 is the direction of V_6

& V_6 is affine subspace of W

Q3

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Given $l : V_1 \to \mathbb{R}$

$$l(u) = \int_{-1}^{1} u''(x)\, dx$$

$\Rightarrow$ the value of $l(u)$ can be computed for any function $u \in V_1 \longrightarrow$ u should be continuous & derivative exists

$\Rightarrow$ $l(u + \alpha v)$ for $u, v \in V_1$ & $\alpha \in \mathbb{R}$,

$$= \int_{-1}^{1} (u + \alpha v)''\, dx = \int_{-1}^{1} u''\, dx + \alpha \int_{-1}^{1} v''\, dx$$

$$= l(u) + \alpha\, l(v)$$

$\therefore$ This proves $l(u)$ is linear functional as W is smooth

Q1

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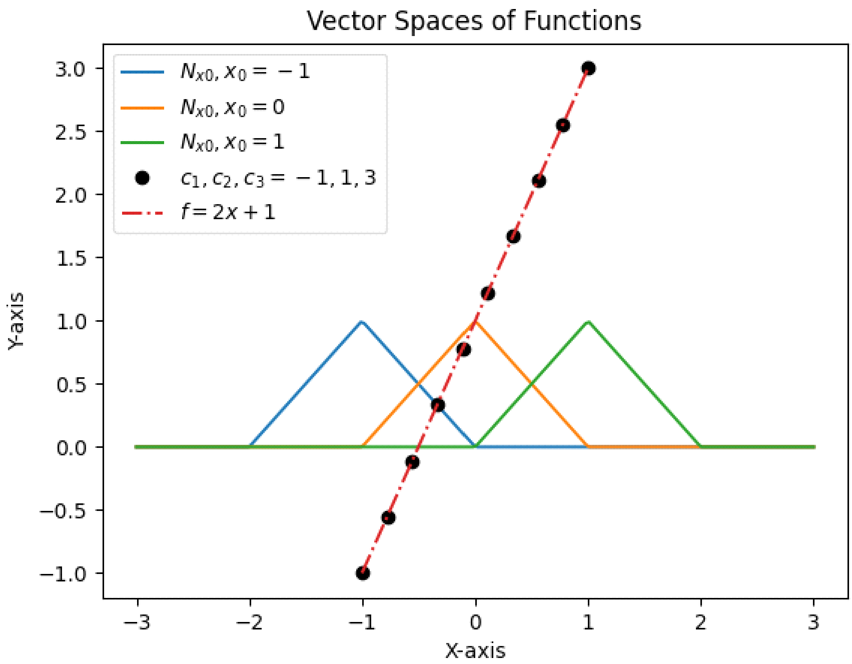

All 3 functions from Q1 and two functions from part 4 are plotted. Plot is also in the interval (-3, 3).

Q2

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given 4 functions $\{N_{-1}, N_0, N_1, g\}$ in interval $(-3, 3)$, $g(x) = 1$

$$p(x) = c_1(1) + c_2 N_{-1}(x) + c_3 N_0(x) + c_4 N_1(x) = 0 \quad x \in (-3, 3)$$

from functions N_{-1}, N_0, N_1 plots at $x = 2.5$ $N_{-1} = N_0 = N_1 = 0$

$p(2.5) = c_1 + c_2(0) + c_3(0) + c_4(0) = 0$

at $x = 0.5$

$p(0.5) = c_1 + c_2(0) + c_3(0.5) + c_4(0.5) = 0$

at $x = -0.5$

$p(0.5) = c_1 + c_2(0.5) + c_3(0.5) + c_4(0) = 0$

at $x = 1$

$p(-2.5) = c_1 + c_2(0) + c_3(0) + c_4(1) = 0$

$c_1 = 0$

$c_4 = c_1 = 0$

$c_3 = -c_4 = 0$

$c_2 = -c_3 = 0$

$\therefore$ only solution of $p(x)$ is $c_1 = c_2 = c_3 = c_4 = 0$ $\forall\, x \in (-3, 3)$

$\therefore$ they are linearly Independent

Q3

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In the interval $x \in (-1, 1)$ the functions (N_0, N_{-1}, N_1, g) will be linearly dependent

Just by observation or taking

$$p(x) = c_1 + c_2 N_{-1}(x) + c_3 N_0(x) + c_4 N_{+1}(x) = 0$$

$x = 0 \Rightarrow c_1 + c_3 = 0$

$x = 1 \Rightarrow c_1 + c_4 = 0$

$x = -1 \Rightarrow c_1 + c_2 = 0$

$x = 0.5 \Rightarrow c_1 + \frac{c_3}{2} + \frac{c_4}{2} = 0$

These equations are lin dependent

$\underline{c_3 = c_4 = c_2 = -c_1}$

1st 3 eqns

$c_1 - \frac{c_1}{2} - \frac{c_1}{2} = 0$

$\therefore c_1 \in \mathbb{R}$

just take $c_1 = 1$

$$p(x) = 1 - N_{-1} - N_0 - N_{+1} = 0 \text{ always}$$

$$\forall x \in (-1, 1)$$

$\therefore \{g, N_{-1}, N_0, N_1\}$ are linearly dependent in domain $(-1, 1)$

Q4

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given $f(x) = 2x+1$

check if $f \in \text{span}(N_{-1}, N_0, N_1)$

$f = c_1 N_{-1} + c_2 N_0 + c_3 N_1$

at $x = -1, 0, 1$

$c_1(1) + c_2(0) + c_3(0) = -1$

$c_1 = -1$

$c_1(0) + c_2(1) + c_3(0) = 1$

$c_2 = 1$

$c_1(0) + c_2(0) + c_3(1) = 3$

$c_3 = 3$

$\therefore f = -N_{-1} + N_0 + 3N_1$

This is also plotted to verify if f & $-N_{-1} + N_0 + 3N_1$ are same