ME 335A
Finite Element Analysis
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Problems Set #2

January 19, 2022

Due Friday, January 21, 2022

A Simple Galerkin Method Example (33)

1. (15) Consider the problem: Find $u\colon [0,1] \to \mathbb{R}$ continuous such that

$$\begin{aligned} (1+x^2)u_{,xx} + xu_{,x} + x^2 u &= 0 \\ u_{,x}(1) - 3u(1) &= 0 \\ u(0) &= 1 \end{aligned}$$

   Find the weak form of the problem, using

$$\begin{aligned} \mathcal{S} &= \{w\colon [0,1] \to \mathbb{R} \quad \text{smooth} \ \mid w(0) = 1\} \\ \mathcal{V} &= \{w\colon [0,1] \to \mathbb{R} \quad \text{smooth} \ \mid w(0) = 0\}. \end{aligned}$$

   Identify essential and natural boundary conditions.

2. (2) Identify the bilinear form and the linear functional of the problem so that the variational equation can be written as $a(u,v) = \ell(v)$. Is a symmetric?

3. Consider a subspace of functions $\mathcal{W}_h = \text{span}\{1, x, x^2, x^3\}$. We want to formulate Galerkin Method for this problem and find its solution.

   (a) (5) What are the spaces $\mathcal{S}_h$ and $\mathcal{V}_h$?

   (b) (7) Find the stiffness matrix and load vector.

   (c) (3) Find the solution to Galerkin Method, and plot it.

   Hint: Recall that the stiffness matrix $K_{ab} = a(N_b, N_a)$; the order is important for non-symmetric bilinear forms.

4. (1) Is the natural boundary condition satisfied exactly by the solution of Galerkin Method?

On Convergence (50)

As mentioned in class, approximating the solution of a boundary value problem involves the construction of a method to obtain numerical solutions (approximate solutions) that are as close as we want to the exact solution of the problem. Let's illustrate here two cases that illuminate the wonders and the perils that we may find in walking this path.

In this problem we will take advantage of Legendre polynomials over the interval $[0,1]$. For $n = 0, 1, \ldots$, the Legendre polynomial of order n can be written as

$$P_n(x) = \sqrt{2n+1}(-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-x)^k. \tag{1}$$

Here

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{2}$$

We can also define their integrals

$$iP_n(x) = \int_0^x P_n(y)\, dy = \sqrt{2n+1}(-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{(-x)^{k+1}}{k+1}. \tag{3}$$

Legendre Polynomials satisfy the following "orthogonality" condition:

$$\int_0^1 P_n(x) P_m(x)\, dx = \delta_{mn}. \tag{4}$$

Here δ_{mn} is the Kronecker delta, whose values are

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases} \tag{5}$$

For this problem, we strongly suggest you use Matlab, Mathematica, or Maple to perform all integrals and calculations, as well as to plot the results.

In this exercise, we consider the following problem in its weak from:

Problem (Weak Form)**.** *Given* $f\colon [0,1] \to \mathbb{R}$*, find* $u \in \mathcal{V} = \{w\colon [0,1] \to \mathbb{R} \text{ smooth } \mid w(0) = 0\}$ *such that*

$$\int_0^1 u'(x) v'(x)\, dx = \int_0^1 f(x) v(x)\, dx \tag{W}$$

for any $v \in \mathcal{V}$*.*

We set $f(x) = \cos(4\pi x)$, and will examine the use of Galerkin Method to approximate the solution to this problem.

1. (10) Is $u(x) = -\frac{\sin(2\pi x)^2}{8\pi^2}$ the exact solution to this problem ? Justify.

   Hint: It is convenient to replace in the weak form and integrate by parts to verify it is satisfied for any $v \in \mathcal{V}$.

2. Consider the set of functions $\mathcal{W}_h = \{w = \sum_{k=0}^{n} c_k iP_k(x) \mid (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}\}$. We will denote with ${}^n u_h$ the solution of Galerkin Method for a given n.

(a) (15) Compute ${}^1u_h$, ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^1u_h$, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

(b) (5) Is ${}^nu_h$ "visually converging" to u as n grows?

3. Consider next the set of functions $\mathcal{W}_h = \{w = \sum_{k=2}^{n} c_k iP_k(x) \mid (c_0, \dots, c_n) \in \mathbb{R}^{n+1}\}$ (notice that k begins at 2 instead of at 0).

   (a) (15) Compute ${}^3u_h$, ${}^5u_h$, ${}^7u_h$. In the same plot, plot u, ${}^3u_h$, ${}^5u_h$ and ${}^7u_h$.

   (b) (5) Is ${}^nu_h$ "visually converging" to u as n grows? Do you think that your answer to this question will change if we keep increasing n beyond 7?