CP-1.

Strong Form:

Given $f(x) = -2$; $g = 1$ and $h = -1$; find $u : [0,1] \to \mathbb{R}$ such that

$$-u_{,xx} + \lambda u = f \qquad x \in [0,1)$$
$$u(1) = g$$
$$-u_{,x}(0) = h$$

Weak Form:

Find $u \in S = \{u : [0,1) \to \mathbb{R} \text{ smooth} \mid u(1) = g\}$.

$v \in \mathcal{V} = \{v : [0,1) \to \mathbb{R} \text{ smooth} \mid v(1) = 0\}$.

such that

$a(u, v) = l(v)$ for all $v \in \mathcal{V}$, where

$$a(u,v) = \int_0^1 v_{,x} u_{,x} + \lambda \int_0^1 v u$$
$$l(v) = \int_0^1 f v - u_{,x}(0) v(0)$$
$$= \int_0^1 f v - v(0).$$

Discretized Form:

Let $W_h = \text{span}(\{N_1, \cdots, N_{el+1}\})$.

$$S_h = \{u_h \in W_h \mid u_h(1) = g\}.$$
$$\mathcal{V}_h = \{v_h \in W_h \mid v_h(1) = 0\}.$$
$$S_h = \{\sum_{i=1}^{el} c_i N_i + N_{el+1} g\}.$$
$$\mathcal{V}_h = \{\sum_{i=1}^{el} c_i N_i\}.$$

Solve:

$$\sum_{i=1}^{el+1} c_i\, a(N_i, N_j) = \ell(N_j) \quad \forall j = 1, \cdots, el$$

$$c_i = g \qquad j = el+1.$$

Question 6 Once your code is working, vary the number of elements used to compute the solution (the value of `nel`). Generate figures of the FE solution and its derivative for `nel =1, 10 and 100`, for each one of the three cases $\lambda = -10, 0$ and 10.

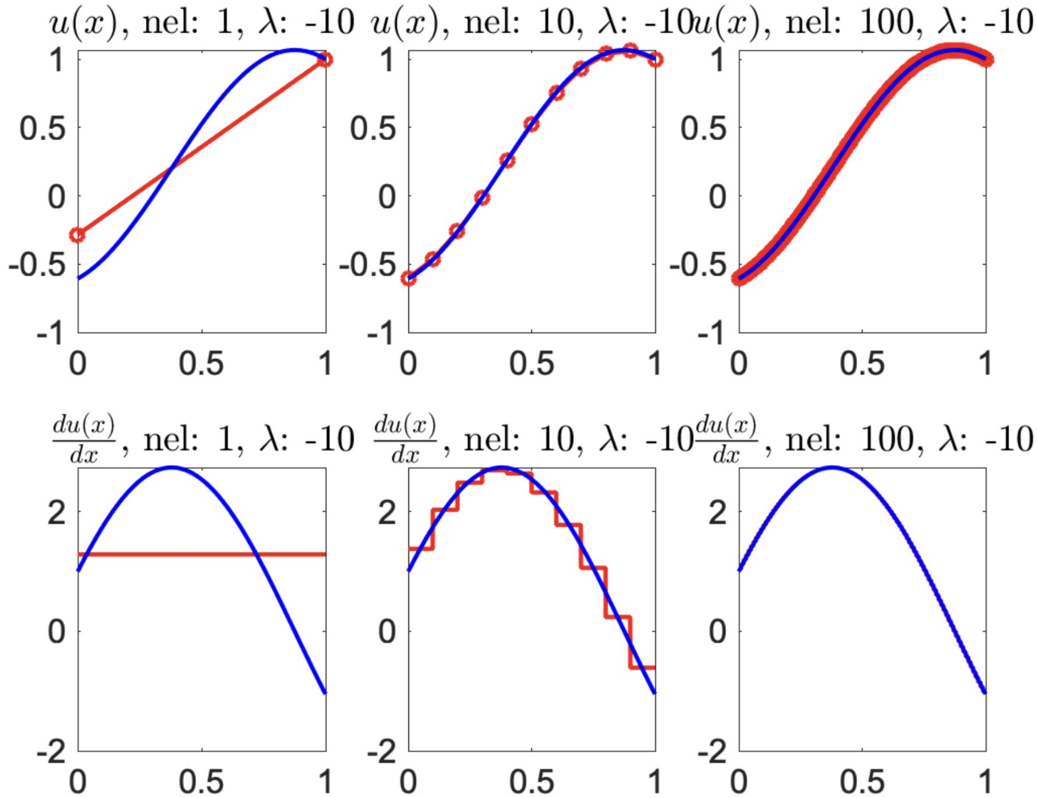


Figure above shows the effect of mesh size, represented by nel, on the solutions of $u(x)$ and $\frac{du(x)}{dx}$ with λ being 10. In the first row, red line with dots denote numerical solutions while blue curve denote exact solution. In the second row, numerical and exact solutions are shown in red and blue, respectively.

Clearly, refining mesh size results in a better approximation for both $u(x)$ and $\frac{du(x)}{dx}$. Particularly, we use a first-order piece-wise polynomial function as basis functions, which results in a constant $\frac{du(x)}{dx}$ with each element block.

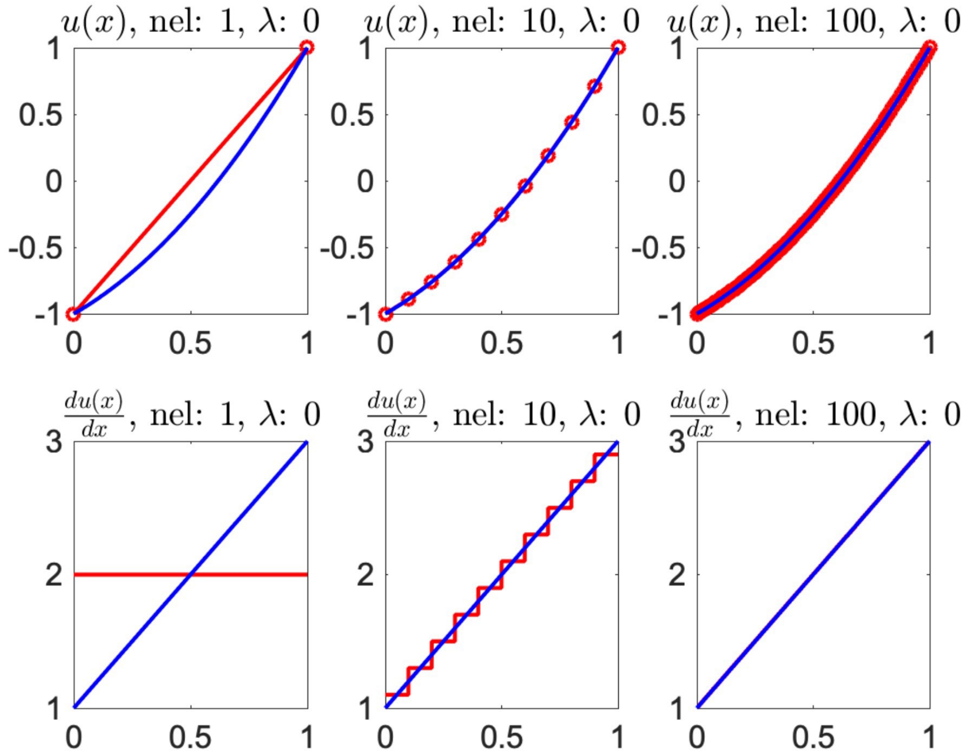


Figure above shows the effect of mesh size, represented by nel, on the solutions of $u(x)$ and $\frac{du(x)}{dx}$ with λ being 0. The effect of mesh size on solutions are observed to be similiar to the ones found above.

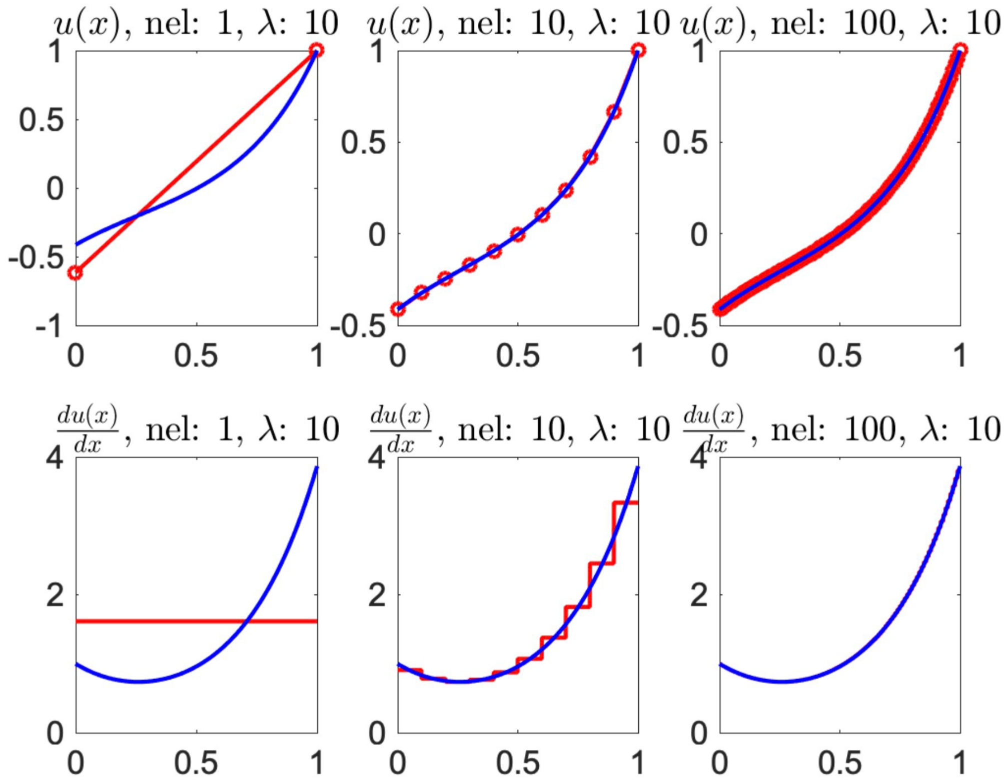


Figure above shows the effect of mesh size, represented by nel, on the solutions of $u(x)$ and $\frac{du(x)}{dx}$ with λ being 10. The effect of mesh size on solutions are observed to be similiar to the ones found above.

Question 7 In line 89 the code to plot the results begins. Arrays `up` and `dup` contain the values of u_h and u'_h, respectively, at 1000 equispaced points, while `uep` and `duep` should have the values of u and u', respectively, at the same points.

For each value of $\lambda = -10, 10$ and each `nel`$=10, 100, 500$, compute the following error measures:

$$\text{error}_u = \sqrt{\sum_{i=1}^{1000} |u(x_i) - u_h(x_i)|^2}, \qquad \text{error}_{u'} = \sqrt{\sum_{i=1}^{1000} |u'(x_i) - u'_h(x_i)|^2}.$$

Report the errors for each case in a table.

For each λ, they should behave approximately as $\sim h^k$, where $h = 1/\text{nel}$ is the mesh size and k is some real number.

Find and report what k is for each λ, for both the function and its derivative. To this end, it is convenient to plot log(`nel`) – log(error) and fit a straight line through them. The slope is $-k$. Please show these plots, and the fitted line.

Comment on your results, paying close attention to the accuracy of the solution, its derivative, and how that accuracy changes with the number of elements.

[1] You can use the `dsolve` command by Matlab to obtain the analytical solution of the given equation.

Table: Error

Lambda	nel	h	||u-u_h||	||u'-u_h'||
-10	10	0.1	0.27975	5.6371
-10	100	0.01	0.0028427	0.56708
-10	500	0.002	0.00010616	0.13746

Table: Error

Lambda	nel	h	||u-u_h||	||u'-u_h'||
0	10	0.1	0.057735	1.8287
0	100	0.01	0.00057732	0.18466
0	500	0.002	2.2361e-05	0.044766

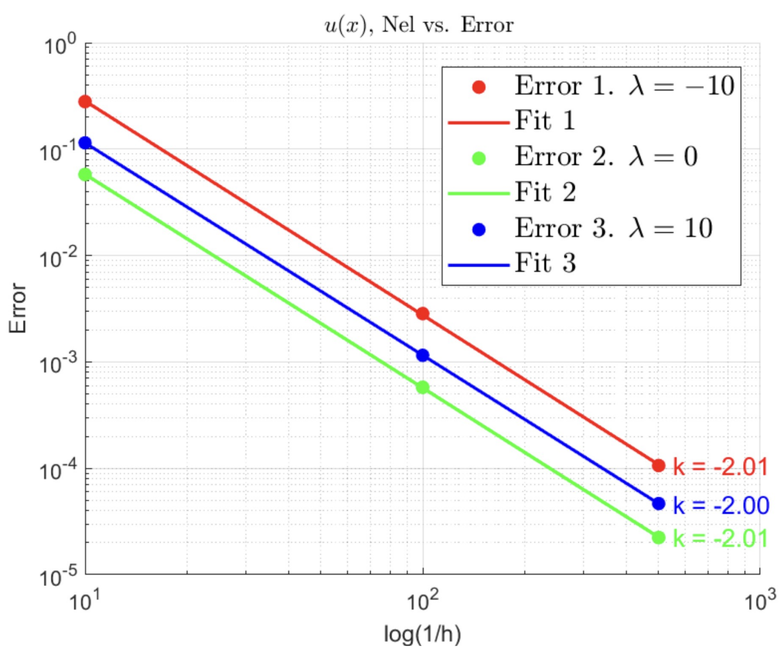
Table: Error

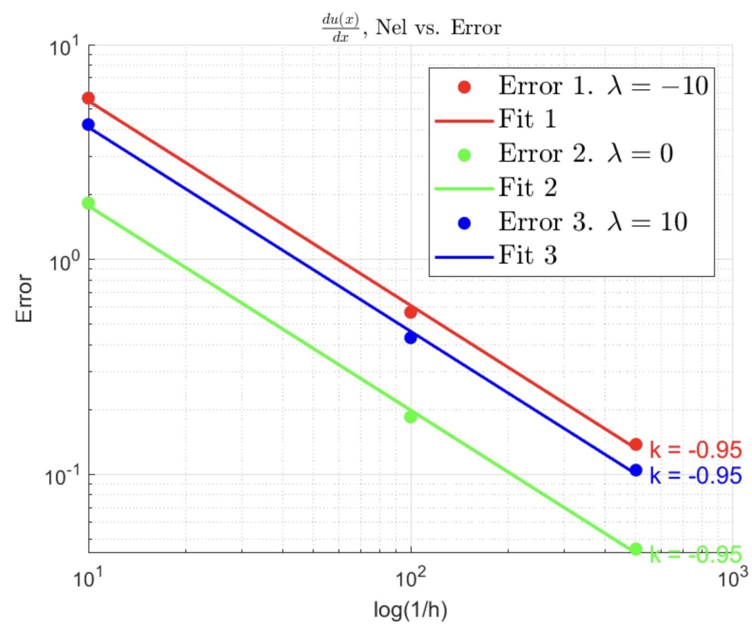
Lambda	nel	h	||u-u_h||	||u'-u_h'||
10	10	0.1	0.11456	4.2431
10	100	0.01	0.0011538	0.43127
10	500	0.002	4.6608e-05	0.10436

The table above lists the effect of mesh size (nel) on errors.
Note that $\|u - u_h\|$ denotes error_u; $\|u' - u'_h\|$ is $\text{error}_{u'}$.

Consistent with findings above, refining mesh leads to better approximations.

The FEM approximated solutions match the exact solutions best when λ is 0.





The figures above give the convergence rates of $u(x)$ and $u'(x)$ with respect to mesh refinement. Clearly, error decreases exponentially with decreasing mesh size. Particularly, the slopes of fitted convergence lines are given in the plots as well.

For $u(x)$, the convergence rate, represented by $-k$, is around 2.00.

For $u'(x)$, the convergence rate is around 0.95.