in a similar manner to establish (1) for arbitrary nonnegative integers a and d. The details are left to the reader.

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THE TEACHING OF MATHEMATICS

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A NOTE ON COMPLEX ITERATION

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The aim of this note is to describe how a recent result about the sequence

$$a, a^a, a^{(a^a)}, \ldots, \qquad a \in \mathbb{C}, \tag{1}$$

might be included in a course on complex analysis (one that is based for instance on [**1**]). The study of the sequence (1) goes back to Euler [**5**] who showed that, for $a > 0$, it is convergent if and only if

$$e^{-e} \leqslant a \leqslant e^{1/e}.$$

To discuss the general case it is convenient to let c be some fixed determination of $\log a$ and to put

$$f(z) = e^{cz}, \qquad z \in \mathbb{C}, \tag{2}$$

so that, if $f^{n+1} = f \circ f^n$, $n = 1, 2, \ldots$, with $f^1 = f$, then

$$f^{n+1}(0), \qquad n = 1, 2, \ldots, \tag{3}$$

is a well-defined version of (1).

It was shown by Carlsson [**4**] that if $f^n(0) \to w$ as $n \to \infty$ and $f^n(0) \neq w$, $n = 1, 2, \ldots$, then c must lie in the closure of a certain cardioid

$$D = \{ te^{-t} : |t| < 1 \},$$

which is illustrated in Fig. 1.

To prove Carlsson's result, note that $\lim_{n \to \infty} f^n(0) = w$ implies that $w = f(w) = e^{cw}$. Thus if $t = cw$, then $w = e^t$ and so $c = te^{-t}$. To prove that $|t| \leqslant 1$ we use the fact that $f'(w) = ce^{cw} = t$.

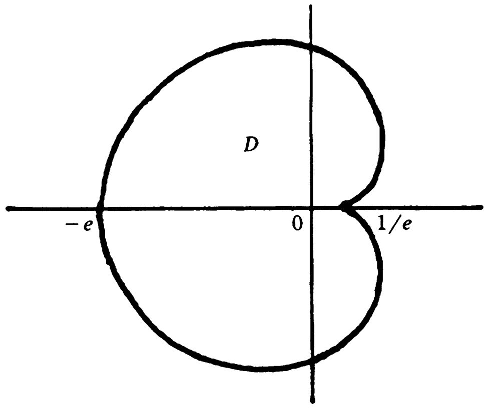

FIG. 1

Since $f^n(0) \neq w$, $n = 1, 2, \ldots$, we have

$$(4) \qquad \lim_{n\to\infty}\left(\frac{f^{n+1}(0)-w}{f^n(0)-w}\right) = \lim_{n\to\infty}\left(\frac{f(f^n(0))-f(w)}{f^n(0)-w}\right) = f'(w) = t.$$

This implies that $|t| \leqslant 1$, as required. Also, if t is not real, then, according to (4), the convergence of $f^n(0)$ to w is eventually spiral-like. Some illustrations of the kinds of convergence which occur are given in [**8**].

We shall now give a sufficient condition for the convergence of (3). Previous results in this direction are surveyed in [7].

THEOREM. *The sequence* (3) *is convergent whenever c lies in D.*

This result was proved in [**2**] using the theory of iteration due to Fatou and Julia (see, for instance, [**6**]). We prove it below using only those parts of this general theory which seem to be essential. Note that the case $c = 0$ can be disposed of immediately.

If $c = te^{-t}$ with $0 < |t| < 1$, then $w = e^t$ is a fixed point of f with $f'(w) = t$. Thus

$$\lim_{z\to w}\left(\frac{f(z)-w}{z-w}\right) = \lim_{z\to w}\left(\frac{f(z)-f(w)}{z-w}\right) = f'(w) = t,$$

and since $|t| < 1$ there exist $r > 0$ and $\lambda < 1$ such that

$$\left|\frac{f(z)-w}{z-w}\right| \leqslant \lambda, \qquad 0 < |z-w| < r.$$

We deduce that $f^n(z) \to w$ uniformly as $n \to \infty$ for $|z - w| < r$.

Now consider the set Ω of those complex numbers in some neighbourhood of which the sequence $f^n \to w$ uniformly as $n \to \infty$. To prove the theorem we show that $0 \in \Omega$. It is clear that Ω is open, that $w \in \Omega$ and that Ω is completely invariant under f, by which we mean that

$$z \in \Omega \Leftrightarrow f(z) \in \Omega.$$

Also, by the Heine-Borel theorem, the sequence $f^n \to w$ as $n \to \infty$ uniformly on any compact subset of Ω. It follows from this that each component of Ω is simply connected. Indeed if Γ is any Jordan curve in Ω, then $f^n - w \to 0$ uniformly on Γ and so $f^n - w \to 0$ uniformly in the interior of Γ, by the maximum principle. This implies that Ω has no "holes".

Let Ω_w be the component of Ω which contains w and suppose that $0 \notin \Omega_w$. We shall show that this leads to a contradiction. Since Ω_w is simply connected, we can choose in Ω_w [**1**, p. 143] a single-valued analytic branch g of

$$f^{-1}(z) = (\log z)/c,$$

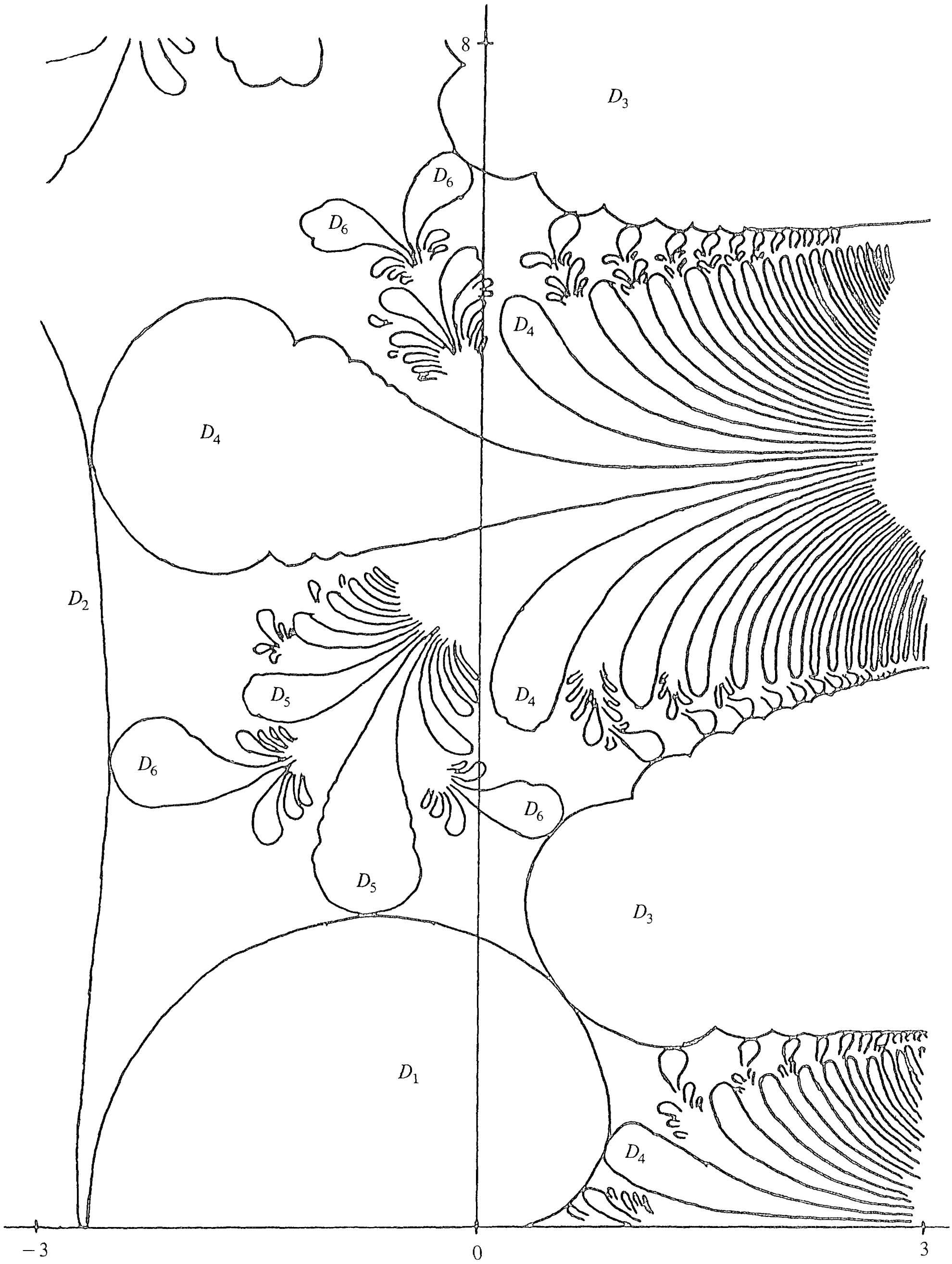

FIG. 2

such that $g(w) = w$. The complete invariance of Ω and the continuity of g then imply that $g(\Omega_w) \subseteq \Omega_w$ and we deduce, from Schwarz's lemma, that

$$|g'(w)| \leqslant 1.$$

Thus $|f'(w)| = |g'(w)|^{-1} \geq 1$ and this contradiction completes the proof.

The form of Schwarz's lemma used here is as follows.

LEMMA. *Let g be an analytic function which maps a simply connected domain $\Omega_w \neq \mathbb{C}$ into itself with fixed point $w \in \Omega_w$. Then*

$$|g'(w)| \leqslant 1.$$

This can be proved by using the Riemann mapping theorem [**1**, p. 222] to choose a conformal mapping ϕ from Ω_w onto the unit disc U, such that $\phi(w) = 0$. The result follows by applying the usual form of Schwarz's lemma to $\phi \circ g \circ \phi^{-1}$, which maps U to U and fixes 0.

If a more elementary argument is required, then we can choose, as in the first stage of the proof of the Riemann mapping theorem [**1**, p. 222], an analytic function ϕ from Ω_w into U such that $\phi'(w) \neq 0$. The sequence of functions $\phi_n = \phi \circ g^n$, $n = 1, 2, \ldots$, is then uniformly bounded in Ω_w and so, by [**1**, p. 122], the sequence ϕ_n' is uniformly bounded in some neighbourhood of w. But

$$\phi_n'(w) = \phi'(w)(g'(w))^n,$$

since $g(w) = w$ and, in view of the fact that $\phi'(w) \neq 0$, we deduce that $|g'(w)| \leqslant 1$.

REMARKS. In [**2**] we also describe the behaviour of the sequence (3) for most boundary points of D. It is convergent if $c = te^{-t}$, where t is an nth root of unity, but divergent almost everywhere on ∂D. The behaviour of (3) for values of c outside $\overline{D}$ is described in a forthcoming paper [**3**]. Briefly, in the c-plane there are infinitely many disjoint domains, in each of which the sequence (3) forms a number of convergent subsequences. In those domains illustrated in Fig. 2 the label D_k indicates that, for the corresponding numbers c, the sequence (3) has k convergent subsequences $\{f^{nk+j}(0)\}_{n=1}^{\infty}$, $j = 1, 2, \ldots, k$, each with a distinct limit.

The domain D_1 is the cardioid D discussed above. Each of the domains D_k, $k \geqslant 2$, is simply connected and unbounded, there is a single D_2, but for $k \geqslant 3$ there appear to be infinitely many D_k. The relative positions of these domains are described in more detail in [**3**], but there remain a number of open questions. For example, we do not know whether the union of all these domains D_k is dense in $\mathbb{C}$.

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COMPACTNESS AND CLOSEDNESS IN LOCALLY COMPACT HAUSDORFF SPACES

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The purpose of this note is to illuminate a reasoning error in point set topology that is seductively easy for experts as well as students to make—namely, the belief that

> *If Y is a locally compact subset of a locally compact Hausdorff space X, then $Y \cap K$ is compact for every compact set $K \subset X$.*

(A simple counterexample is the open segment $(0,1)$, which is a locally compact subset of the real line, although its intersection with the compact interval $[0,1]$ is not compact.) Since this unjustified assumption has, in fact, led to circular reasoning in a number of sources in proving the