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An Elementary Treatment of the Lambert-W Relation

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In terms of early-college mathematics, one of the most common problems that is studied is solving equations of various types; including linear, quadratic, exponential, logarithmic, trigonometric, and so on. The Lambert-W relation, which has seen several applications in topics such as enzyme reactions [4], black body radiation [7], fracture dynamics [6], economics [3], and the repair of irradiated calls [1] (to name a few), is a relation that allows us to solve a class of equations that involve polynomial-exponential products and tetrations (to name a couple). We present a very basic approach to presenting the relation that can be incorporated into late high-school and early college mathematical discussions. Not only does this approach introduce another strategy in solving equations, but it also introduces students to concepts such as multi-functions and branches.

The Lambert-W Relation and Basic Properties

We will begin by discussing what the Lambert-W relation is and some of its basic properties. Consider the function:

$$f(x) := xe^x.$$

From its definition, it is easy to see that the graph of f passes through the origin and grows without bound as x gets arbitrary large. Rewriting this expression as $\frac{x}{e^{-x}}$, we can see that as x tends to negative infinity, the quotient gets arbitrarily close to 0; hence, $y = 0$ will serve as a horizontal asymptote of f. Since both relations $y_1 = x$ and $y_2 = e^x$ are continuous for all $x \in \mathbb{R}$, together with the comments above, the function f will be bounded below. Using basic differential calculus, one can determine the exact location of this minimum to be:

$$\min f = -\frac{1}{e} \quad \text{when } x = -1.$$

Using curve-sketching strategies or computer software, one can construct the graph of f to be as shown as the solid curve in Figure 1. Since f has a minimum at $\left(-1, -\frac{1}{e}\right)$ with $f(x \pm \epsilon) > -1$ for $\epsilon > 0$, we have that f defined on $\mathbb{R}$ is not one-to-one, hence

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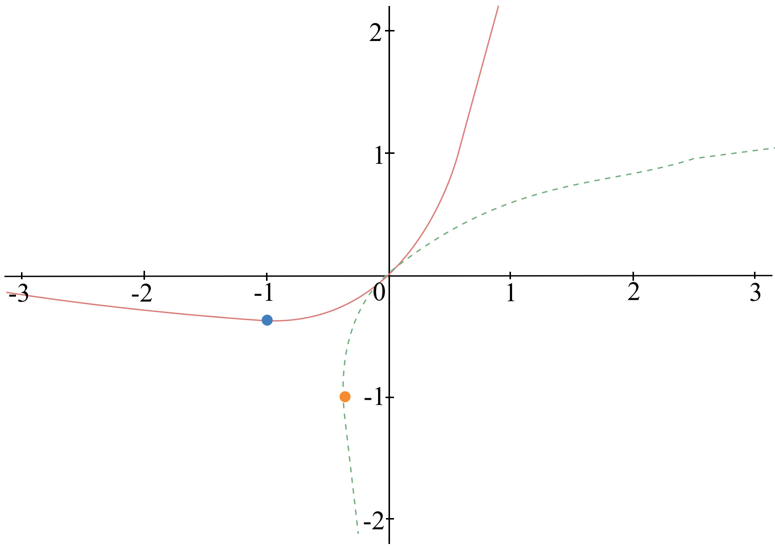

Figure 1. Lambert-W and its inverse.

its correspond inverse image f^{-1} will not be a function of x. Nonetheless, we define a relation W to be:

$$W := \{(x, y) \mid x = ye^y\},$$

By properties of f, we can find that:

$$\operatorname{dom}(W) = \left[-\frac{1}{e}, \infty\right) \quad \text{and} \quad \operatorname{ran}(W) = \mathbb{R},$$

and using the fact that W is simply the mirror reflection of f about $y = x$, we can obtain the graph of W as shown in green in Figure A. We will refer to W as "the Lambert-W multi-function." Two trivial points that lie on the graph of W are:

$$W(0) = 0 \quad \text{and} \quad W\left(-\frac{1}{e}, -1\right).$$

From the graph of W, it is easy to see that if we choose any $x \geq 0$, there exists a unique y such that $W(x) = y$. On the other hand, if we choose any $t \in \left(-\frac{1}{e}, 0\right)$, there exists distinct numbers $y_2 < -1 < y_1 < 0$ such that $W(t) = y_1$ and $W(t) = y_2$. For a reasonable partition of our multi-function, we will create a break-point of W when $y = -1$ by defining the following two relations:

$$W_0 := \{(x, y) \mid x = ye^y, y \geq -1\},$$
$$W_{-1} := \{(x, y) \mid x = ye^y, y < -1\}.$$

We will refer to the relations W_0 and W_{-1} as the principle and secondary branch of W, respectively. It is easy to see that both branches of W are single-valued functions of x, with:

$$\operatorname{dom}(W_0) = \left[-\frac{1}{e}, \infty\right), \qquad \operatorname{ran}(W_0) = [-1, \infty),$$
$$\operatorname{dom}(W_{-1}) = \left(-\frac{1}{e}, 0\right), \qquad \operatorname{ran}(W_{-1}) = (-\infty, -1).$$

To re-emphasize the point made above, if $x \geq 0$, then there exists $y \in W_0$ such that $W_0(x) = y$, and if $t \in \left(-\frac{1}{e}, 0\right)$, then there exists $y_1 \in W_0$ and $y_2 \in W_{-1}$ such that $W_0(t) = y_1$ and $W_{-1}(t) = y_2$.

We now present the most important property of W that makes it useful for solving a new class of equations. Let W the Lambert-W multi-function and f is inverse relation (as defined previously). Since f and W are inverse relations of one another, then for some x for which f and W are both defined, we have:

$$(f \circ W)(x) = (W \circ f)(x) = x.$$

Expanding this statement, we have:

$$(W \circ f)(x) = W(f(x)) = W\left(xe^x\right) = x.$$

Since the value of x is arbitrary, this relation applies to both branches of W.

Solving Polynomial-Exponential Product Equations

Before proceeding into a set of examples, we need to mention that for most of the examples at the level for which one can introduce W, the equations will have answers such as $x = W_0(7)$ or $z = W_{-1}\left(\frac{1}{7}\right)$. This is no different than us accepting answers like $x = \sqrt{2}$ for quadratic equations or $z = \ln(\pi)$ for exponential equations. Currently, most calculators that are used in secondary schools do not have the capability to generate numerical answers of W, hence their "closed-form" representations are the best we can do, unless the students have access to computer software that have already implemented W or if they are familiar with numerical schemes such as the Newton-Raphson method (which we will mention later).

Example 1. Solve $-\frac{1}{3}xe^{2x} = 1$ for x.

Solution. Rearranging the equation yields:

$$(2x)e^{2x} = -6.$$

Note that we introduced a coefficient of 2 to our linear-exponential term in order to have our leading monomial and exponent to be the same. Since $W(ue^u) = u$ for any u, we have:

$$x = \frac{1}{2}W(-3).$$

Since $-3 \notin \operatorname{dom}(W)$ (since $-3 < \frac{1}{e}$), we have that $W(-3)$ does not belong to either W_0 nor W_{-1}. Therefore, there are no real solutions to the equation. ■

Using this example as a starting guide, one should be able to obtain the following generalization:

$$(ax+b)e^{cx}=d \quad \to \quad x=\frac{1}{c}\left[W\left(\frac{cd}{a}e^{\frac{bc}{a}}\right)-\frac{bc}{a}\right].$$

To further demonstrate how these problems can be technical for the more mature students, consider the following examples that contain polynomial leading terms, and different exponential bases.

Example 2. Solve $3x^2e^{5x}=7$ for x.

Solution. We begin by applying the power of $\frac{1}{2}$ (equivalently the principle square root) to both sides of the equation.

$$|x|e^{\frac{5}{2}x}=\sqrt{\frac{7}{3}}.$$

Decomposing the equation as we normally do with absolute-value equations, we have:

$$\frac{5}{2}xe^{\frac{5}{2}x}=\frac{5}{2}\sqrt{\frac{7}{3}} \quad \text{or} \quad \frac{5}{2}xe^{\frac{5}{2}x}=-\frac{5}{2}\sqrt{\frac{7}{3}}$$

At this stage, it is possible that we have one or two solutions, depending on the size of the constant on the right-hand side of the equations. One can find that the negative of that constant is outside of the domain of W. Therefore, the solution of the equation is:

$$x=\frac{2}{5}W_0\left(5\sqrt{\frac{7}{12}}\right).$$

■

A similar process can be applied to solve any equation with a monomial as the leading function and in the exponent:

$$x^me^{x^n}=a \quad \to \quad x=\sqrt[n]{\left(\frac{m}{n}\right)W\left(\frac{n}{m}\sqrt[m]{a}\right)}.$$

Example 3. Solve $(x^2-2x+1)e^{6x+1}=4$ for x.

Solution. We begin by noticing that the leading quadratic function is simply the square of a linear-binomial:

$$(x-1)^2e^{6x+1}=4.$$

Performing the substitution $x:=u+1$, we obtain:

$$u^2e^{6u+7}=4.$$

If one were to encounter a case where the leading quadratic is not factorable, you can always complete the square. This new form of our equation now is similar to that of Example 2 Performing the necessary manipulations gives us:

$$3ue^{3u}=6e^{-\frac{7}{2}} \quad \text{or} \quad 3ue^{3u}=-6e^{-\frac{7}{2}}.$$

For the first equation, we see that $6e^{-\frac{7}{2}} > 0$. For the second equation, one can verify that $-6e^{-\frac{7}{2}} \in \left[-\frac{1}{e}, 0\right)$. Hence, we have three solutions: one pertaining to the first and two pertaining to the second. Switching back in terms of x, the solutions are given by:

$$\begin{aligned} x_1 &= 1 + \frac{1}{3} W_0\left(6e^{-\frac{7}{2}}\right), \\ x_2 &= 1 + \frac{1}{3} W_0\left(-6e^{-\frac{7}{2}}\right), \\ x_3 &= 1 + \frac{1}{3} W_{-1}\left(-6e^{-\frac{7}{2}}\right). \end{aligned}$$

■

Example 4. Let $a > 1$. Solve $xa^x = b$ for x, where $b \in \mathbb{R}$.

Solution. We begin by defining $x := \log_a(e^u) = \frac{u}{\ln a}$, which gives us $a^x = e^u$. Replacing our substitution and rearranging gives us:

$$x = \frac{1}{\ln a} W(b \ln a).$$

■

In general, the Lambert-W function can be applied to any equation of the form $p(x)e^{q(x)} = a$ where p, q are polynomials and $a \in \mathbb{R}$. We will not go into the journey of generalizing results to this extent here , but the examples above lay down the foundations for those that are curious.

Numerical Approximation to $W(a)$ for $a \geq -\frac{1}{e}$

Define the function $\varphi(x) := xe^x - a$ where $a \geq -\frac{1}{e}$. It is easy to verify that $W(a)$ is the solution to the equation $\varphi(x) = 0$. Therefore, we can use the Newton-Raphson method for numerical convergence. We know that $W_0\left(-\frac{1}{e}\right) = -1$ and $W_0(0) = 0$, hence these two values can be hardcoded into any system, in addition to $W(a) \notin \mathbb{R}$ for $a < -\frac{1}{e}$. If you are familiar with the Newton-Raphson iteration method, recall that we need an initial guess x_0 in order to use the iteration scheme:

$$x_{n+1} = x_n - \frac{\varphi(x_n)}{\varphi'(x_n)}, \; n = 0, 1, 2, 3, \ldots$$

For values of $a > 0$, there will be only one value associated to $W(a)$, namely $W_0(a)$. One can show that the sequence $\{x_n\}_{n=1}^{\infty}$ will converge to $W_0(a)$ provided $x_0 > -1$ (the location of the minimum of φ). Hence, we can define $x_0 := 0$ if $a > 0$. Otherwise, if $a \in \left(-\frac{1}{e}, 0\right)$, then we have two values associated to $W(a)$, namely $W_0(a)$ and $W_{-1}(a)$. Again, the iteration scheme will converge to $W_0(a)$ if $x_0 > -1$ and will converge to $W_{-1}(a)$ if $x_0 < -1$. We can write this as:

$$x_0(a) = \begin{cases} 0, & a > 0 \\ -2, & a \in \left(-\frac{1}{e}, 0\right) \end{cases}$$

For the implementation in MatLab or other languages, one can use the central difference approximation of φ' (as shown below), or an alternative or built-in derivative approximation:

$$\varphi'(x_n) \approx \frac{1}{2h}(\varphi(x_n + h) - \varphi(x_n - h)),\ h \approx 0.$$

If one is to use a software or pre-built function, its important to test to see if it will give you all solutions that you desire. For example, if you want only real values of $W(-0.1)$, then it should return the approximate values -0.11 and -3.58.

Solving Basic Tetration Equations

Now that we have practice with W in regards to polynomial-exponential product equations and are able to calculate the numerical representations of W_0 and W_{-1}, we now turn to another class of equations that involve tetrations. We consider a basic form of this class as an example.

Example 5. Solve $x^x = \frac{8}{11}$ for x.

Solution. As one will do (or have done) in calculus, we begin by taking the (natural) logarithm of both sides of our equation to turn it into an equation that we are more comfortable with.

$$x \ln x = \ln\left(\frac{8}{11}\right).$$

Defining $x := e^u$ will give us $\ln x = u$, which gives us an equation for which are already familiar with:

$$ue^u = \ln\left(\frac{8}{11}\right).$$

Applying our normal approach, we have:

$$x_1 = e^{W_0\left(\ln\left(\frac{8}{11}\right)\right)} \approx 0.5744,$$

$$x_2 = e^{W_{-1}\left(\ln\left(\frac{8}{11}\right)\right)} \approx 0.1945.$$

■

A common notation for x^x is ${}^2x$. Similarly, $x^{x^x} = {}^3x$, and so on. We leave as an exercise to prove the following more general result:

$${}^2x = a \quad \to \quad x = e^{W(\ln a)}.$$

The Lambert-W Relation on $\left(-\infty, -\frac{1}{e}\right)$

We have already determined that if $x \in \left(-\infty, -\frac{1}{e}\right)$, then $W(x) \notin \mathbb{R}$. The question now becomes "Is $W(x) \in \mathbb{C}$, and if so, what is the structure of its real and imaginary parts?". Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ and let $t \in \mathbb{C}$. Our goal is now to solve the equation:

$$ze^z = t.$$

Replacing z with $x + iy$ and implementing the Euler-Cotes theorem, once we rearrange, we have:

$$(x \cos y - y \sin y) + i(x \sin y + y \cos y) = t.$$

Since t will be the input of W, we can assume that its real and imaginary parts are known, and denote them by t_r and t_i, respectively. Equating real and imaginary parts of the equation above, gives us the system:

$$x \cos y - y \sin y = t_r e^{-x} \qquad \text{and} \qquad x \sin y + y \cos y = t_i e^{-x}.$$

Note here that y lies on the interior of both sine and cosine, hence if we were to attempt to solve this system algebraically, use of arc-sine and arc-cosine would be needed. As we may already know, these two relations are also multi-functions, which have principle branches $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $[0, \pi)$, respectively. Hence we should only consider values $y \in [0, \pi/2]$. One can then apply numerical methods to approximate the value(s) of (x, y) that satisfy these criteria; we will denote this value as $W_0(t)$. In terms of details on numerical schemes on solving nonlinear systems, there are several methods depending on their structure, one of the general methods being a modification of the Newton and secant methods, as described here [5].

Example 6. Approximate the value of $W_0(i)$.

Solution. Taking $t_r = 0$ and $t_i = 1$, we have the system:

$$x \cos y - y \sin y = 0 \qquad \text{and} \qquad x \sin y + y \cos y = e^{-x}.$$

This is one special case for which we solve the system using substitution by isolating x in the first equation and substituting into the second equation for an implicit equation in y. Finding the solution y that lies in $\left[0, \frac{\pi}{2}\right]$ via Newton-Raphson, gives us $\mathrm{Im}(W_0(i)) := y^\star$. This then gives $\mathrm{Re}(W_0(i)) = y^\star \tan y^\star$, whose approximate values are given to be:

$$W_0(i) \approx 0.375 + 0.576i.$$

A graph of the system and the location of its principle solution are shown in Figure 2. ∎

Application of the Lambert-W Relation

As mentioned in the introduction, there have been a wide variety of applications and problems for which the Lambert-W relation naturally arises. In [2], they discuss some relationships that describe the flow of fluids passing through pipes that could be either smooth (the type that a lot of people are familiar with) or rough (which may sometimes occur in nature). One of the constants that can be used to approximate the properties of a fluid as it undergoes movement is known as the Reynolds number, for which we will denote by R_e. Two other constants that are used in relation to flow of fluids in a pipe

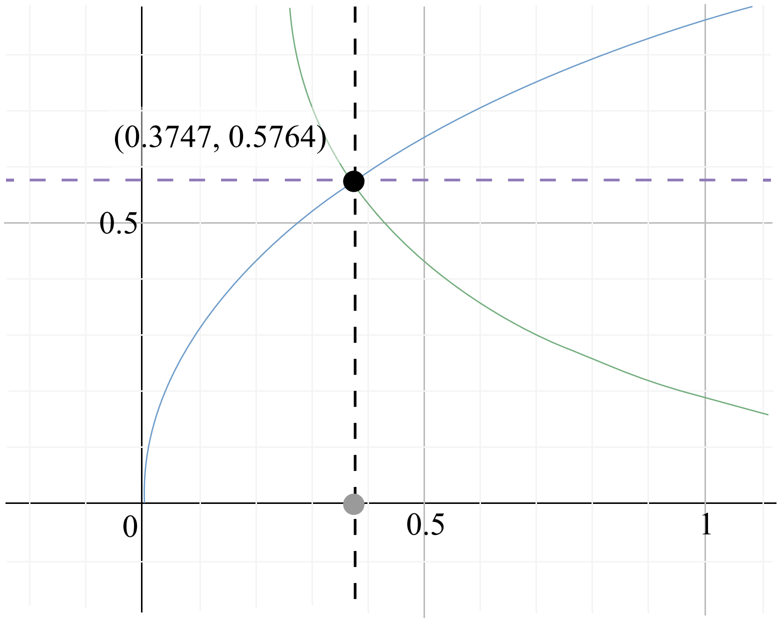

Figure 2. Solution to the nonlinear system for W(i).

are the pipes diameter, D, and its roughness height, ε, where ε being close to 0 length units indicates a "perfectly smooth" pipe, and larger values (such as about 3 mm for some concrete pipes) indicate more "rougher" pipes. In [2], they state a relationship between these constants and the (Darcy) friction factor f for which can describe how much effect the friction that is generated from the roughness has on the fluid and the flow of the fluid in that pipe. For some constants $a, b, c \in \mathbb{R}_{\geq 0}$ that are determined experimentally, the proposed model was given by

$$\frac{1}{\sqrt{f}} = -a \ln\left(\frac{b\,\varepsilon}{D} + \frac{c}{R_e\sqrt{f}}\right)$$

Note that in order for this equation to make mathematical sense, the argument of the logarithm should be between 0 and 1. Typically, the constants ε, D, and R_e are known to us, so friction factor f is a common unknown variable of this relation. To solve this equation for f, we can perform a few substitutions upfront to make the manipulations more tolerable for those learning algebra. To start, we can define $x := \frac{1}{\sqrt{f}}$, $K := \frac{b\,\varepsilon}{D}$, and $L = \frac{c}{R_e}$. After rearranging, this gives us the equation

$$(Lx + K)e^{\frac{1}{a}x} = 1.$$

Then we can directly use one of the previously-derived relations to obtain

$$x = aW\left(\frac{1}{aL}e^{\frac{K}{aL}}\right) - \frac{K}{L},$$

from which we can use our temporary solutions to revert to the original physical constants for which the problem was formulated under. Note also that this friction factor will result in a single number (from the principal branch of W) since the physics of the problem mandate that a, c, and R_e are all positive and finite; it may give rise to an interesting discussion in a fluid mechanics environment for which these restrictions are lifted, and we then have two friction factors, or possibly complex-valued friction factors.

Summary. In this discussion, we introduced the Lambert-W relation, how to solve equations that are solvable via this relation and not solvable otherwise, discussed how to numerically calculate values, how to generate the real and imaginary parts of W for which have sufficiently negative or complex arguments. We then gave a brief insight into one of the physical applications of the Lambert-W relation that is used in fluid mechanics. Although one could introduce several new functions into the algebraic toolbox for those exploring mathematics, we have demonstrated that the Lambert-W relation is one that can easily be adapted into the toolbox of elementary-level mathematics students.

References

[1] Belkić, D. (2018). The Euler T and Lambert W functions in mechanistic radiobiological models with chemical kinetics for repair of irradiated cells. *J. Math. Chem.* 56: 2133–2193.

[2] Colebrook, C. F., White, C. M. Experiments with fluid friction in roughened pipes. *Proc. Royal Soc. A*. 161(906): 367–381.

[3] Disney, S., Warburton, R. (2012). On the Lambert W function: Economic order quantity applications and pedagogical considerations. *Int. J. Prod. Econ.* 140: 756–764.

[4] Golicnik, M. (2012). On the Lambert-W function and its utility in biochemical kinetics. *Biochem. Eng. J.* 63: 116–123.

[5] Mahwash, K. N., Gyang, G. D. (2018). Numerical solution of nonlinear systems of algebraic equations. *Proc. Royal Soc. London. Series A Math. Phys. Sci.* 161 (1937) *Int. J. Data Sci. Anal.* 4: 1–23.

[6] Roberts, K. (2014). Lambert W function applications and methods. Available at: https://www.researchgate.net/publication/273486861_Lambert_W_Function_Applications_and_Methods

[7] Valluri, S. R., Corless, R. M., Jeffrey, D. J. (2000). Some applications of the Lambert W function to physics. *Can. J. Phys.* 78(9): 823–831.