02471 Machine Learning for Signal Processing

Solution

Exercise 3: Stochastic Processes and Linear Filtering

3.1 Correlation functions

Exercise 3.1.1

Now we consider the following signal:

$$\mathrm{u}_n = c_1 \mathrm{x}_n + c_2 \mathrm{y}_{n-d}$$

Using the expression for auto-correlation we obtain

$$\begin{aligned}
r_u(k) &= \mathbb{E}[\mathrm{u}_n \mathrm{u}_{n-k}] \\
&= \mathbb{E}[(c_1 \mathrm{x}_n + c_2 \mathrm{y}_{n-d})(c_1 \mathrm{x}_{n-k} + c_2 \mathrm{y}_{n-d-k})] \\
&= c_1^2 \mathbb{E}[\mathrm{x}_n \mathrm{x}_{n-k}] + c_1 c_2 \mathbb{E}\left[\mathrm{x}_n \mathrm{y}_{n-(d+k)}\right] + \\
&\qquad\qquad c_1 c_2 \mathbb{E}[\mathrm{y}_{n-d} \mathrm{x}_{n-k}] + c_2^2 \mathbb{E}[\mathrm{y}_{n-d} \mathrm{y}_{n-d-k}]
\end{aligned}$$

Where we used the linear properties of expectation. Using the fact that x_n and y_n are WSS (we are free to shift) we obtain

$$\begin{aligned}
r_u(k) &= c_1^2 \mathbb{E}[\mathrm{x}_n \mathrm{x}_{n-k}] + c_1 c_2 \mathbb{E}\left[\mathrm{x}_n \mathrm{y}_{n-(d+k)}\right] + \\
&\qquad\qquad c_1 c_2 \mathbb{E}\left[\mathrm{y}_n \mathrm{x}_{n-(k-d)}\right] + c_2^2 \mathbb{E}[\mathrm{y}_n \mathrm{y}_{n-k}] \\
&= c_1^2 r_x(k) + c_1 c_2 r_{xy}(d+k) + c_1 c_2 r_{xy}(d-k) + c_2^2 r_y(k)
\end{aligned}$$

Exercise 3.1.4

A 1st order AR process is represented as: $\mathrm{u}_n = a\mathrm{u}_{n-1} + \eta_n$, where $a \in \mathbb{R}$ and η_n is a white noise sequence, which has the properties $\mathbb{E}[\eta_n] = 0$ and $r_\eta(k) = \sigma_\eta^2 \delta(k)$.

Insert into the expression for auto-correlation:

$$\begin{aligned}
r_u(k) &= \mathbb{E}[\mathrm{u}_n \mathrm{u}_{n-k}] \\
&= \mathbb{E}[(a\mathrm{u}_{n-1} + \eta_n)\mathrm{u}_{n-k}] \\
&= \mathbb{E}[a\mathrm{u}_{n-1}\mathrm{u}_{n-k} + \eta_n \mathrm{u}_{n-k}] \\
&= a\mathbb{E}\left[\mathrm{u}_n \mathrm{u}_{n-(k-1)}\right] + \mathbb{E}[\eta_n \mathrm{u}_{n-k}]
\end{aligned}$$

The latter term can easily be reduced if η_n and u_{n-k} is uncorrelated. If $k > 0$ this will be the case since u_{n-k} cannot be correlated with the future η_n noise sequence. Then we get $\mathbb{E}[\eta_n \mathrm{u}_{n-k}] = \mathbb{E}[\eta_n]\mathbb{E}[\mathrm{u}_{n-k}] = 0$, since $\mathbb{E}[\eta_n] = 0$.

So we get

$$\begin{aligned}
r_u(k) &= a\mathbb{E}\left[\mathrm{u}_n \mathrm{u}_{n-(k-1)}\right] \\
&= a r_u(k-1)
\end{aligned}$$

So, for $k > 0$ we have recursion.

Let us consider $k = 0$.

$$\begin{aligned} r_u(0) &= \mathbb{E}\left[\mathrm{u}_n^2\right] \\ &= \mathbb{E}\left[(a\mathrm{u}_{n-1} + \eta_n)^2\right] \\ &= \mathbb{E}\left[a^2\mathrm{u}_{n-1}\mathrm{u}_{n-1}\right] + \mathbb{E}\left[\eta_n^2\right] + 2a\mathbb{E}[\mathrm{u}_{n-1}\eta_n] \\ &= \mathbb{E}\left[a^2\mathrm{u}_{n-1}\mathrm{u}_{n-1}\right] + \sigma_\eta^2 \end{aligned}$$

where the latter term vanish since u_{n-1} and η_n are uncorrelated. Using the fact that u_n WSS, we shift the signal to obtain

$$\begin{aligned} r_u(0) &= \mathbb{E}\left[a^2\mathrm{u}_{n-1}\mathrm{u}_{n-1}\right] + \sigma_\eta^2 \\ &= a^2\mathbb{E}[\mathrm{u}_n\mathrm{u}_n] + \sigma_\eta^2 \\ &= a^2 r_u(0) + \sigma_\eta^2 \end{aligned}$$

Isolate for $r_u(0)$ to obtain

$$r_u(0) = \frac{\sigma_\eta^2}{1 - a^2}$$

Let's have a look on the term we derived a few steps before: $r_u(k) = ar_u(k-1)$. We see the recursive pattern:

$$\begin{aligned} r_u(k) &= ar_u(k-1) \\ r_u(1) &= ar_u(0) \\ r_u(2) &= ar_u(1) = a^2 \cdot r_u(0) \end{aligned}$$

So, we see for $k > 0$, we have $r_u(k) = ar_u(k-1)$. From properties of the auto-correlation sequence, equation 2.113 from the book, we know that $r_u(k) = r_u(-k)$, so for all k we have:

$$r_u(k) = a^{|k|} \cdot r_u(0) = \frac{a^{|k|}}{1 - a^2}\sigma_\eta^2$$

3.2 Wiener filter

Exercise 3.2.1

From the description we have $\mathrm{u}_n = \mathrm{s}_n + \epsilon_n$ where s_n is an AR(1) process and ϵ_n is a white noise sequence. Using the expression for auto-correlation we obtian

$$\begin{aligned} r_u(k) &= \mathbb{E}[\mathrm{u}_n\mathrm{u}_{n-k}] \\ &= \mathbb{E}[(\mathrm{s}_n + \epsilon_n)(\mathrm{s}_{n-k} + \epsilon_{n-k})] \\ &= \mathbb{E}[\mathrm{s}_n\mathrm{s}_{n-k}] + \mathbb{E}[\mathrm{s}_n\epsilon_{n-k}] + \mathbb{E}[\epsilon_n\mathrm{s}_{n-k}] + \mathbb{E}[\epsilon_n\epsilon_{n-k}] \end{aligned}$$

Since signal s_n and ϵ_n are uncorrelated and ϵ_n is a white noise sequence the cross-terms $\mathbb{E}[\mathrm{s}_n\epsilon_{n-k}]$ and $\mathbb{E}[\epsilon_n\mathrm{s}_{n-k}]$ vanish and we obtain

$$\begin{aligned} r_u(k) &= \mathbb{E}[\mathrm{s}_n\mathrm{s}_{n-k}] + \mathbb{E}[\epsilon_n\epsilon_{n-k}] \\ &= r_s(k) + r_\epsilon(k) \end{aligned}$$

From 3.1.4 we know that

$$r_s(k) = \frac{a^{|k|}}{1-a^2}\sigma_\eta^2$$

For $r_\epsilon(k)$ we have (since ϵ_n is a white noise sequence) $r_\epsilon(k) = \delta(k)\sigma_w^2$. So, we obtain by substitution:

$$r_u(k) = \frac{a^{|k|}}{1-a^2}\sigma_v^2 + \delta(k)\sigma_\epsilon^2$$

Exercise 3.2.2

The setup specifies that $\mathrm{d}_n = \mathrm{s}_n$ so using the expression for cross-correlation we obtain

$$\begin{aligned} r_{du}(k) &= \mathbb{E}[\mathrm{d}_n \mathrm{u}_{n-k}] \\ &= \mathbb{E}[\mathrm{s}_n \mathrm{s}_{n-k} + \epsilon_{n-k})] \\ &= \mathbb{E}[\mathrm{s}_n \mathrm{s}_{n-k}] + \mathbb{E}[\mathrm{s}_n \epsilon_{n-k}] \\ &= r_s(k) \end{aligned}$$

Where, in the last line, we used ϵ_n is a white noise sequence.

Exercise 3.2.3

Using equation (4.43) in the book, we get

$$\left(\begin{bmatrix} r_s(0) & r_s(1) & r_s(2) \\ r_s(1) & r_s(0) & r_s(1) \\ r_s(2) & r_s(1) & r_s(0) \end{bmatrix} + \begin{bmatrix} \sigma_\epsilon^2 & 0 & 0 \\ 0 & \sigma_\epsilon^2 & 0 \\ 0 & 0 & \sigma_\epsilon^2 \end{bmatrix} \right) \boldsymbol{w} = \begin{bmatrix} r_s(0) \\ r_s(1) \\ r_s(2) \end{bmatrix}$$

Exercise 3.2.4

For $\sigma_\epsilon^2 \to 0$:

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Thus filtering by $\boldsymbol{w}$ does not alter the signal (try and write the convolution for the filter, to see we get $\hat{\mathrm{d}}_n = \mathrm{u}_n$). This is a reasonable solution, since there is no noise to filter out.

Exercise 3.2.5

For $\sigma_\epsilon^2 \gg \sigma_\eta^2$:

$$\boldsymbol{w} = \frac{1}{\sigma_\epsilon^2} \begin{bmatrix} r_s(0) \\ r_s(1) \\ r_s(2) \end{bmatrix}$$

$\boldsymbol{w}$ tends to 0. This is a sensible solution, since there is only (unpredictable) noise in the signal. That means the energy output of the filter will be smaller as the noise dominates.