02471 Machine Learning for Signal Processing

Solution

Exercise 11: State-space models – Kalman filtering

11.1 Derivation of the Kalman filter

Exercise 11.1.3

We have

$$P_{n|n-1} = \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big]$$

But the error can be rewritten as

$$\begin{aligned}
\mathbf{e}_{n|n-1} &= \mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1} \\
&= F_n\mathbf{x}_{n-1} + \boldsymbol{\eta}_n - F_n\hat{\mathbf{x}}_{n-1|n-1} \\
&= F_n(\mathbf{x}_{n-1} - \hat{\mathbf{x}}_{n-1|n-1}) + \boldsymbol{\eta}_n \\
&= F_n\mathbf{e}_{n-1|n-1} + \boldsymbol{\eta}_n
\end{aligned}$$

Substituting we get

$$\begin{aligned}
P_{n|n-1} &= \mathbb{E}\big[(F_n\mathbf{e}_{n-1|n-1} + \boldsymbol{\eta}_n)(F_n\mathbf{e}_{n-1|n-1} + \boldsymbol{\eta}_n)^T\big] \\
&= \mathbb{E}\big[F_n\mathbf{e}_{n-1|n-1}\mathbf{e}_{n-1|n-1}^T F_n^T + \boldsymbol{\eta}_n\boldsymbol{\eta}_n^T\big] \\
&= F_n\mathbb{E}\big[\mathbf{e}_{n-1|n-1}\mathbf{e}_{n-1|n-1}^T\big] F_n^T + \mathbb{E}\big[\boldsymbol{\eta}_n\boldsymbol{\eta}_n^T\big] \\
&= F_n P_{n-1|n-1} F_n^T + Q_n
\end{aligned}$$

Where we have used that $\mathbb{E}\big[F_n\mathbf{e}_{n-1|n-1}\boldsymbol{\eta}_n\big] = F_n\mathbb{E}\big[\mathbf{e}_{n-1|n-1}\big]\mathbb{E}[\boldsymbol{\eta}_n] = 0$, since $\mathbb{E}[\boldsymbol{\eta}_n] = 0$.

Exercise 11.1.4

With these definitions we can now carry out the following rewrites (using $\hat{\mathbf{x}}_{n|n} = \hat{\mathbf{x}}_{n|n-1} - K_n\mathbf{e}_n$)

$$\begin{aligned}
P_{n|n} &= \mathbb{E}\big[\mathbf{e}_{n|n}\mathbf{e}_{n|n}^T\big] \\
&= \mathbb{E}\big[(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n})(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n})^T\big] \\
&= \mathbb{E}\big[(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1} - K_n\mathbf{e}_n)(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1} - K_n\mathbf{e}_n)^T\big] \\
&= \mathbb{E}\big[(\mathbf{e}_{n|n-1} - K_n\mathbf{e}_n)(\mathbf{e}_{n|n-1} - K_n\mathbf{e}_n)^T\big] \\
&= \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big] - \mathbb{E}\big[K_n\mathbf{e}_n\mathbf{e}_{n|n-1}^T\big] - \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_n^T K_n^T\big] + \mathbb{E}\big[K_n\mathbf{e}_n\mathbf{e}_n^T K_n^T\big] \\
&= \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big] - K_n\mathbb{E}\big[\mathbf{e}_n\mathbf{e}_{n|n-1}^T\big] - \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_n^T\big] K_n^T + K_n\mathbb{E}\big[\mathbf{e}_n\mathbf{e}_n^T\big] K_n^T
\end{aligned}$$

Let us inspect the expectations term by term

$$\begin{aligned}
\mathbb{E}\big[\mathbf{e}_n\mathbf{e}_{n|n-1}^T\big] &= \mathbb{E}\big[(\mathbf{y}_n - \hat{\mathbf{y}}_n)\mathbf{e}_{n|n-1}^T\big] \\
&= \mathbb{E}\big[(H_n\mathbf{x}_n + \mathbf{v}_n - H_n\hat{\mathbf{x}}_{n|n-1})\mathbf{e}_{n|n-1}^T\big] \\
&= \mathbb{E}\big[H_n(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})\mathbf{e}_{n|n-1}^T\big] + \mathbb{E}\big[\mathbf{v}_n\mathbf{e}_{n|n-1}^T\big] \\
&= H_n\mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big] + \mathbb{E}\big[\mathbf{v}_n\mathbf{e}_{n|n-1}^T\big] \\
&= H_nP_{n|n-1} + \mathbb{E}\big[\mathbf{v}_n(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T\big] \\
&= H_nP_{n|n-1}
\end{aligned}$$

where the last term vanishes because we assumed $\mathbf{v}_n$ is uncorrelated with $\mathbf{x}_n$ and $\hat{\mathbf{x}}_{n|n-1}$. Using identical derivations, we get

$$\begin{aligned}
\mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_n^T\big] &= \mathbb{E}\big[\mathbf{e}_{n|n-1}(\mathbf{y}_n - \hat{\mathbf{y}}_n)^T\big] \\
&= \mathbb{E}\big[\mathbf{e}_{n|n-1}(H_n\mathbf{x}_n + \mathbf{v}_n - H_n\hat{\mathbf{x}}_{n|n-1})^T\big] \\
&= \mathbb{E}\big[\mathbf{e}_{n|n-1}(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1})^T H_n^T\big] + \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{v}_n^T\big] \\
&= \mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big]H_n^T \\
&= P_{n|n-1}H_n^T
\end{aligned}$$

The last expectation gives

$$\begin{aligned}
\mathbb{E}\big[\mathbf{e}_n\mathbf{e}_n^T\big] &= \mathbb{E}\big[(\mathbf{y}_n - \hat{\mathbf{y}}_n)(\mathbf{y}_n - \hat{\mathbf{y}}_n)^T\big] \\
&= \mathbb{E}\big[(H_n\mathbf{x}_n + \mathbf{v}_n - H_n\hat{\mathbf{x}}_{n|n-1})(H_n\mathbf{x}_n + \mathbf{v}_n - H_n\hat{\mathbf{x}}_{n|n-1})^T\big] \\
&= \mathbb{E}\big[(H_n(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{v}_n)(H_n(\mathbf{x}_n - \hat{\mathbf{x}}_{n|n-1}) + \mathbf{v}_n)^T\big] \\
&= \mathbb{E}\big[(H_n\mathbf{e}_{n|n-1} + \mathbf{v}_n)(H_n\mathbf{e}_{n|n-1} + \mathbf{v}_n)^T\big] \\
&= H_n\mathbb{E}\big[\mathbf{e}_{n|n-1}\mathbf{e}_{n|n-1}^T\big]H_n^T + \mathbb{E}\big[\mathbf{v}_n\mathbf{v}_n^T\big] \\
&= H_nP_{n|n-1}H_n^T + R_n
\end{aligned}$$

Let $S = H_nP_{n|n-1}H_n^T + R_n$, so that $\mathbb{E}\big[\mathbf{e}_n\mathbf{e}_n^T\big] = S$, then put it all together to get

$$P_{n|n} = P_{n|n-1} - K_nH_nP_{n|n-1} - P_{n|n-1}H_n^TK_n^T + K_nSK_n^T$$

Exercise 11.1.5

We have to minimize trace($P_{n|n}$) with respect to K_n, so we take the derivate w.r.t. K_n and set to zero. We will use that the trace is a linear operator i.e. $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$, and $\text{trace}(A) = \text{trace}(A^T)$, and the following two differentiation rules

$$\begin{aligned}
\frac{\partial \text{trace}(AB)}{\partial A} &= B^T \\
\frac{\partial \text{trace}(ACA^T)}{\partial A} &= 2AC
\end{aligned}$$

First we rewrite a bit

$$\begin{aligned}
\text{trace}(P_{n|n-1}H_n^TK_n^T) &= \text{trace}((P_{n|n-1}H_n^TK_n^T)^T) \\
&= \text{trace}(K_nH_nP_{n|n-1}^T)
\end{aligned}$$

Since $P_{n|n-1}^T = P_{n|n-1}$ we have $\text{trace}(P_{n|n-1}H_n^T K_n^T) = \text{trace}(K_n H_n P_{n|n-1})$ an we get

$$\text{trace}(P_{n|n}) = \text{trace}(P_{n|n-1}) - 2\text{trace}(K_n H_n P_{n|n-1}) + \text{trace}(K_n S K_n^T)$$

Using the rules of differentation specified earlier, we get

$$\begin{aligned}
\frac{\partial \text{trace}(P_{n|n})}{\partial K_n} &= \frac{\partial}{\partial K_n}\text{trace}(P_{n|n-1}) - 2\frac{\partial}{\partial K_n}\text{trace}(K_n H_n P_{n|n-1}) + \frac{\partial}{\partial K_n}\text{trace}(K_n S K_n^T) \\
&= -2(H_n P_{n|n-1})^T + 2K_n S
\end{aligned}$$

Setting the derivative to zero gives

$$\begin{aligned}
2K_n S &= 2(H_n P_{n|n-1})^T \quad \Leftrightarrow \\
K_n &= (H_n P_{n|n-1})^T S^{-1} \\
&= P_{n|n-1}^T H_n^T S^{-1} \\
&= P_{n|n-1} H_n^T S^{-1}
\end{aligned}$$

Exercise 11.1.6

We can now go back and finalize the recursion for $P_{n|n}$

$$P_{n|n} = P_{n|n-1} - K_n H_n P_{n|n-1} - P_{n|n-1} H_n^T K_n^T + K_n S K_n^T$$

The last term can be rewritten

$$\begin{aligned}
K_n S K_n^T &= P_{n|n-1} H_n^T S^{-1} S K_n^T \\
&= P_{n|n-1} H_n^T K_n^T
\end{aligned}$$

Using substitution we get

$$\begin{aligned}
P_{n|n} &= P_{n|n-1} - K_n H_n P_{n|n-1} - P_{n|n-1} H_n^T K_n^T + P_{n|n-1} H_n^T K_n^T \\
&= P_{n|n-1} - K_n H_n P_{n|n-1}
\end{aligned}$$