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Noise generation by aeroengines

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Abstract

We formulate equations in terms of distributions which describe the motion of fluid in a linear shear flow with an arbitrary body present. We combine these equations of motion into a single equation involving the biharmonic operator, and then we calculate both the free space and infinite strip Green's function for the the biharmonic equation. We make use of the Papkovich-Fadle eigenfunctions to get a series expansion for the infinite strip Green's function which satisfies both Dirichlet and Neumann boundary conditions. This then allows us to explicitly calculate the expression for the pressure field in an infinite strip with a body present. We find both volume and surface source terms, which are then analysed numerically in a specific case to determine which have the largest contribution to the pressure field. The techniques we use throughout the report are not new, but applying them to this relatively simple example has not been done before and allows more detailed analysis because we have less complexity.

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1 Introduction

Although modern aircraft are now significantly quieter than early aircraft, there is still room for improvement. With demand for aircraft travel every increasing, regulations controlling the effective perceived noise level are becoming tighter. Aircraft have many sources of noise, and the noise heard depends on both the position of the observer and whether the aircraft is landing, taking off or flying overhead. The main contributions to the total noise of the aircraft come from from the engine (fan, compressor, combustor, turbine and jet) and the airframe itself. We will be primarily concerned with the noise from the fan, with the other sources of noise discussed in [Rolls-Royce, 2005, §1.4]. Noise comes from the aerodynamic interaction between the fan blades and outlet guide vanes (also known as stators) as air flows between them, and is influenced by many factors such as position and number of blades. The fan noise is made up of two types of noise, broadband noise and tone noise.

Broadband noise is caused by turbulent air in the boundary layer near the blades and also in the wake behind the blades and vanes [Rolls-Royce, 2005], and is made up of many frequencies. The more aerodynamically efficient the blades and vanes are, the less noise produced, but there is a trade-off between performance of the fan and aerodynamic efficiency. Tonal noise is only concentrated at one frequency, the blade passing frequency. It occurs since the pressure wave in front of each fan blade produces a sound pulse as the blade passes. The mathematical progress in understanding these noise sources for both turbofan and open-rotor engines is documented in [Peake and Parry, 2012, §3].

The main theme of this report is to understand the sources of noise in the simple case of an incompressible linear shear flow, as considered in [Rienstra et al., 2013], by using the same method as considered in [Posson and Peake, 2013] in the study of an acoustic analogy in an annular duct. Using this method allows the source terms to arise naturally in the derivation of the conservation equations, rather than the unphysical approach of just putting a source term on the right hand side of the mass equation, as in [Rienstra et al., 2013]. We follow a similar outline to [Posson and Peake, 2013], which is as follows.

In Section 2 we briefly review distributions and their derivatives, which we use in Section 3 to derive the general mass, momentum and vorticity equations for distributions with an arbitrary body present. We also linearise these equations, which are considered for a base flow with small perturbations.

In Section 4 we apply these equations to the specific case of shear flow and derive a single differential equation for the pressure field. In Section 5 we go on to find the free space Green's function and also the Green's function in an infinite strip for the partial differential operator of the pressure field. In Section 6 we formally calculate the expression for the pressure field, which involves a volume source term and several surface source terms.

Finally, in Section 7 we use the work of the previous sections to numerically compute approximations to the volume source term and surface source terms. We analyse each source term separately and also consider the effect of shear on the pressure field.

2 Generalised functions

We begin by reviewing generalised functions (or distributions), with most of the content based on [Crighton et al., 1992] and [Farassat, 1996]. We consider distributions because we want to differentiate functions which are are not differentiable in the classical sense. Given a function $f \in L^1_{loc}(\mathbb{R})$, and $\phi \in C^\infty_c(\mathbb{R})$, a test function, we define the generalised function

$$F[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\mathrm{d}x.$$

We introduce the notions of the Dirac delta function $\delta(x)$ and Heaviside function $h(x)$ through the definitions

$$\delta[\phi] = \int_{-\infty}^{\infty} \delta(x)\phi(x)\mathrm{d}x = \phi(0) \text{ and } H[\phi] = \int_{-\infty}^{\infty} h(x)\phi(x)\mathrm{d}x = \int_{0}^{\infty} \phi(x)\mathrm{d}x.$$

We define the derivative of F to be the generalised function F' that satisfies

$$F'[\phi] = -F[\phi']. \tag{2.1}$$

If there is a locally integrable function g such that $F'[\phi] = \int_{-\infty}^{\infty} g(x)\phi(x)\mathrm{d}x$ for all test functions $\phi \in C^\infty_c(\mathbb{R})$, then we say that f has generalised derivative g, which we write as $\overline{f'}(x) = g(x)$. In this sense we have $\overline{h'}(x) = \delta(x)$. We now consider a function φ which is piecewise smooth apart from a discontinuity at x_0. If we let $[\varphi]_{x_0-}^{x_0+} = \varphi(x_0+) - \varphi(x_0-)$ be the jump that φ makes at the discontinuity, then it is simple to show that

$$\overline{\varphi'} = \varphi'(x) + [\varphi]_{x_0-}^{x_0+}\delta(x - x_0).$$

If we now extend this result to three dimensions, where the discontinuity is at a surface described by $f(x) = 0$, then we can show that

$$\frac{\overline{\partial}\varphi}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + [\varphi]_{f=0-}^{f=0+}\frac{\partial f}{\partial x_i}\delta(f).$$

We can further extend this result to the generalised gradient $\overline{\boldsymbol{\nabla}}\varphi$, generalised divergence $\overline{\boldsymbol{\nabla}} \cdot \boldsymbol{\varphi}$ and generalised curl $\overline{\boldsymbol{\nabla}} \times \boldsymbol{\varphi}$ (see [Farassat, 1996]).

We now wish to consider a surface $\Sigma(t)$ defined by $f(\boldsymbol{x}, t) = 0$ which divides a fluid into a region $\mathcal{V}(t)$ where the fluid is moving and a region $\mathcal{W}(t)$ where it is at rest. For example, we could choose the surface to be the edge of the blades of a rotor, and then we want the fluid to be at rest inside this surface (since it is metal), and moving outside the surface. For convention we choose f such that $f < 0$ where the fluid is at rest, $f > 0$ where the fluid is moving and such that $|\boldsymbol{\nabla} f| = 1$. This is the approach as considered in [Posson and Peake, 2013]. Let

$$\boldsymbol{\nabla} f(\boldsymbol{x}, t) = \boldsymbol{n} \text{ and } \frac{\partial f}{\partial t} = -\boldsymbol{v}^s \cdot \boldsymbol{n},$$

where $\boldsymbol{v}^s$ is surface speed and $\boldsymbol{n} = (n_1, n_2, n_3)$. Now, given functions $\varphi(\boldsymbol{x}, t)$ and $\boldsymbol{w}(\boldsymbol{x}, t)$, we can define the generalised functions

$$\widetilde{\varphi}(\boldsymbol{x}, t) := h(f)\varphi(\boldsymbol{x}, t) \text{ and } \widetilde{\boldsymbol{w}}(\boldsymbol{x}, t) := h(f)\boldsymbol{w}(\boldsymbol{x}, t),$$

so the tilde functions are zero where the fluid is at rest and equal to the original function otherwise. Then the following relations are easy to show

$$\frac{\overline{\partial}\widetilde{\varphi}}{\partial t} = h(f)\frac{\partial \varphi}{\partial t} - [(\boldsymbol{v}^s \cdot \boldsymbol{n})\varphi]\,\delta(f), \tag{2.2}$$

$$\frac{\overline{\partial}\widetilde{\varphi}}{\partial x_i} = h(f)\frac{\partial \varphi}{\partial x_i} + n_i\varphi\delta(f), \tag{2.3}$$

$$\overline{\boldsymbol{\nabla}}\widetilde{\varphi} = h(f)\boldsymbol{\nabla}\varphi + [\varphi\boldsymbol{n}]\,\delta(f), \tag{2.4}$$

where we have omitted the argument from each expression for clarity. We can see from these definitions that rules such as the following hold:

$$\frac{\overline{\partial}}{\partial t}(\widetilde{\varphi} + \widetilde{\psi}) = \frac{\overline{\partial}\widetilde{\varphi}}{\partial t} + \frac{\overline{\partial}\widetilde{\psi}}{\partial t} \text{ and } \frac{\overline{\partial}}{\partial t}(\widetilde{\phi\psi}) = \psi\frac{\overline{\partial}\widetilde{\varphi}}{\partial t} \text{ if } \psi \text{ is independent of } t.$$

We also have a general product rule for differentiating the product of a smooth function g and a generalised function φ, (see for example [Grubb, 2009] for a proof) which is

$$\frac{\overline{\partial}}{\partial x_i}(g\varphi) = g\frac{\overline{\partial}\varphi}{\partial x_i} + \varphi\frac{\partial \varphi}{\partial x_i}, \tag{2.5}$$

while it is clear from (2.2) that

$$\frac{\overline{\partial}}{\partial x_i}(\widetilde{\phi\psi}) = \phi\frac{\overline{\partial}}{\partial x_i}\widetilde{\psi} + \psi\frac{\overline{\partial}}{\partial x_i}\widetilde{\phi} - \phi\psi n_i\delta(f). \tag{2.6}$$

Finally, in §4, we will need to be able to exchange the order of derivatives. This follows at once from (2.1) and the smoothness of test functions, so

$$\frac{\overline{\partial}}{\partial x_i}\left[\frac{\overline{\partial}}{\partial x_j}\widetilde{\psi}\right] = \frac{\overline{\partial}}{\partial x_j}\left[\frac{\overline{\partial}}{\partial x_i}\widetilde{\psi}\right]. \tag{2.7}$$

Additionally, we have that the following statement holds, which follows from applying (2.2) and (2.5) twice:

$$\frac{\overline{\partial}}{\partial x_j}\left[\phi\frac{\overline{\partial}}{\partial x_i}\widetilde{\psi}\right] = \phi\frac{\overline{\partial}}{\partial x_j}\left[\frac{\overline{\partial}}{\partial x_i}\widetilde{\psi}\right] + \frac{\partial\phi}{\partial x_j}\frac{\overline{\partial}}{\partial x_i}\widetilde{\psi}. \tag{2.8}$$

3 Conservation Equations

The mass and momentum equations for an **inviscid** flow, with density ρ, pressure p and velocity $\boldsymbol{u}$ are given by

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{3.1}$$

$$\rho\frac{\partial\boldsymbol{u}}{\partial t} + \rho\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) + \boldsymbol{\nabla}p = 0. \tag{3.2}$$

We can assume the fluid is **inviscid** if we are outside the boundary layer. We also have the vorticity equation for a **inviscid**, **barotropic** fluid

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{3.3}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity. In the vorticity equation the first term is the unsteadiness of the flow, the second term is advection while the third term the stretching vorticity due to the velocity gradient, and the fourth term is vorticity stretching due to the compressibility of the fluid.

We consider the conservation equations above with total variables $\boldsymbol{u}_{to}, \rho_{to}, p_{to}, \boldsymbol{\omega}_{to}$, with

$$\boldsymbol{u}_{to} = \boldsymbol{U} + \boldsymbol{u}, \quad \rho_{to} = \rho_0 + \rho, \quad p_{to} = p_0 + p, \quad \boldsymbol{\omega}_{to} = \boldsymbol{\Omega} + \boldsymbol{\omega},$$

where $\boldsymbol{U}$, ρ_0, p_0, and $\boldsymbol{\Omega}$ are bases flow and $\boldsymbol{u}$, ρ, p and $\boldsymbol{\omega}$ are perturbations. We denote $\boldsymbol{U} = (U, V, W)$ and $\boldsymbol{u} = (u, v, w)$, and note that $\boldsymbol{\Omega} = \boldsymbol{\nabla}\times\boldsymbol{U}$ and $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$. We consider the conservation equations for generalised functions and obtain similar results for mass and momentum as in [Ffowcs Williams and Hawkings, 1969] if we had a surface which didn't move in time ($\boldsymbol{v}^s = \boldsymbol{0}$), while we get a new result for vorticity.

3.1 Mass

We firstly consider the mass equation with the generalised functions $\widetilde{\rho}$, $\widetilde{\boldsymbol{u}}$ along with their base flows and perturbations. We begin by rewriting the mass equation (3.1) so that the left hand side is a linear operator acting on perturbations and the right hand side contains the non-linear and base flow effects.

$$\begin{aligned}\frac{\partial \rho_{to}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho_{to}\boldsymbol{u}_{to}) =& \frac{\partial}{\partial t}(\rho_0+\rho) + \frac{\partial}{\partial x_j}\left((\rho_0+\rho)(U_j+u_j)\right)\\ =& \frac{\partial \rho_0}{\partial t} + \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) + \frac{\partial}{\partial x_j}(\rho_0 u_j) + \frac{\partial}{\partial x_j}(\rho_0 U_j) + \frac{\partial}{\partial x_j}(\rho u_j) = 0.\end{aligned}$$

Rearranging this, we get the form that we want:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) + \frac{\partial}{\partial x_j}(\rho_0 u_j) = -\frac{\partial \rho_0}{\partial t} - \frac{\partial}{\partial x_j}(\rho_0 U_j) - \frac{\partial}{\partial x_j}(\rho u_j). \tag{3.4}$$

We then multiply this expression by the Heaviside function and then use (2.2) and (2.3) to get

$$\begin{aligned}&\frac{\bar{\partial}\widetilde{\rho}}{\partial t} + (\boldsymbol{v}^s\cdot\boldsymbol{n})\rho\delta(f) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho U_j} + \widetilde{\rho_0 u_j}) - \left[\rho\boldsymbol{U}\cdot\boldsymbol{n} + \rho_0\boldsymbol{u}\cdot\boldsymbol{n}\right]\delta(f) =\\ &\qquad -\frac{\bar{\partial}\widetilde{\rho_0}}{\partial t} - (\boldsymbol{v}^s\cdot\boldsymbol{n})\rho_0\delta(f) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_j} + \widetilde{\rho u_j}) + \left[\rho_0\boldsymbol{U}\cdot\boldsymbol{n} + \rho\boldsymbol{u}\cdot\boldsymbol{n}\right]\delta(f).\end{aligned}$$

Finally, collecting all the $\delta(f)$ terms gives

$$\boxed{\frac{\bar{\partial}\widetilde{\rho}}{\partial t} + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho U_j} + \widetilde{\rho_0 u_j}) = -\frac{\bar{\partial}\widetilde{\rho_0}}{\partial t} - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_j} + \widetilde{\rho u_j}) + \delta(f)\left[\rho_{to}(\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot\boldsymbol{n}\right].} \tag{3.5}$$

In the case where $\boldsymbol{u}_{to} = \boldsymbol{v}^s$, so the fluid moves at the same speed as the surface, we have no $\delta(f)$ terms in (3.5) as we would expect and there is no source term for mass. An example of when $\boldsymbol{u}_{to} = \boldsymbol{v}^s$ would be near the boundary for an **impermeable** surface.

3.1.1 Linearised mass equation

Since the perturbations are small, we can ignore quadratic terms in (3.4) so the last term disappears and it becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho U_j) + \frac{\partial}{\partial x_j}(\rho_0 u_j) = -\frac{\partial \rho_0}{\partial t} - \frac{\partial}{\partial x_j}(\rho_0 U_j).$$

Performing similar analysis to before we then get the generalised, linearised mass equation:

$$\frac{\bar{\partial}\widetilde{\rho}}{\partial t} + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho U_j} + \widetilde{\rho_0 u_j}) = -\frac{\bar{\partial}\widetilde{\rho_0}}{\partial t} - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_j}) + \delta(f)\left[\rho_{to}(\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot\boldsymbol{n} - \rho\boldsymbol{u}\cdot\boldsymbol{n}\right]. \tag{3.6}$$

3.2 Momentum

In components the momentum equation (3.2) reads

$$\rho_{to}\frac{\partial u_{to,i}}{\partial t} + \rho_{to}\frac{\partial u_{to,i}}{\partial x_j}u_{to,j} + \frac{\partial p_{to}}{\partial x_i} = \frac{\partial}{\partial t}(\rho_{to}u_{to,i}) + \frac{\partial}{\partial x_j}(\rho_{to}u_{to,i}u_{to,j}) + \frac{\partial p_{to}}{\partial x_i} = 0,$$

with the equality coming from conservation of mass. As before, we rewrite in terms of perturbations and the base flow so the left hand side is a linear operator acting on perturbations:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho U_i + \rho_0 u_i) + &\frac{\partial}{\partial x_j}(\rho_0 u_i U_j + \rho_0 u_j U_i + \rho U_i U_j) + \frac{\partial p}{\partial x_i} \\ &= -\frac{\partial}{\partial t}(\rho_0 U_i + \rho u_i) - \frac{\partial}{\partial x_j}(\rho_0 U_i U_j + \rho_{to} u_j u_i + \rho u_i U_j + \rho U_i u_j) - \frac{\partial p_0}{\partial x_i}.\end{aligned} \tag{3.7}$$

We then multiply this expression by the Heaviside function and use (2.2) and (2.3) to get

$$\begin{aligned}\frac{\bar{\partial}}{\partial t}(\widetilde{\rho U_i} + \widetilde{\rho_0 u_i}) + &(\boldsymbol{v}^s \cdot \boldsymbol{n})\left[\rho U_i + \rho_0 u_i\right]\delta(f) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 u_i U_j} + \widetilde{\rho_0 u_j U_i} + \widetilde{\rho U_i U_j}) \\ &- n_j\left[\rho_0 u_i U_j + \rho_0 u_j U_i + \rho U_i U_j\right]\delta(f) + \frac{\bar{\partial}\widetilde{p}}{\partial x_i} - n_i p \delta(f) \\ = -\frac{\bar{\partial}}{\partial t}(\widetilde{\rho_0 U_i} + \widetilde{\rho u_i}) - &(\boldsymbol{v}^s \cdot \boldsymbol{n})\left[\rho_0 U_i + \rho u_i\right]\delta(f) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_i U_j} + \widetilde{\rho_{to} u_j u_i} + \widetilde{\rho u_i U_j} + \widetilde{\rho U_i u_j}) \\ &+ n_j\left[\rho_0 U_i U_j + \rho_{to} u_j u_i + \rho u_i U_j + \rho U_i u_j\right]\delta(f) - \frac{\bar{\partial}\widetilde{p_0}}{\partial x_i} - n_i p_0 \delta(f).\end{aligned}$$

Finally, rearranging and combining all the $\delta(f)$ functions and writing in vector form gives

$$\boxed{\begin{aligned}\frac{\bar{\partial}}{\partial t}(\widetilde{\rho \boldsymbol{U}} + \widetilde{\rho_0 \boldsymbol{u}}) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 u_i U_j} + \widetilde{\rho_0 u_j U_i} + \widetilde{\rho U_i U_j})\boldsymbol{e}_i + \overline{\boldsymbol{\nabla}}\widetilde{p} = -\frac{\bar{\partial}}{\partial t}(\widetilde{\rho_0 \boldsymbol{U}} + \widetilde{\rho \boldsymbol{u}}) \\ - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_i U_j} + \widetilde{\rho_{to} u_j u_i} + \widetilde{\rho u_i U_j} + \widetilde{\rho U_i u_j})\boldsymbol{e}_i - \overline{\boldsymbol{\nabla}}\widetilde{p_0} + \delta(f)\left[\rho_{to}\boldsymbol{u}_{to}((\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot \boldsymbol{n}) + p_{to}\boldsymbol{n}\right].\end{aligned}} \tag{3.8}$$

In the case $\boldsymbol{u}_{to} = \boldsymbol{v}^s$ the source term for momentum reduces to $\delta(f) p_{to} \boldsymbol{n}$. We now consider some specific forms of the momentum equation which will be useful later.

3.2.1 Constant base flow

Using the conservation of mass equation we can rewrite (3.7) as

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_0 u_i) + &\frac{\partial}{\partial x_j}(\rho_0 u_i U_j) + \frac{\partial p}{\partial x_i} \\ &= -\rho_{to}\left(\frac{\partial U_i}{\partial t} + \frac{\partial U_i}{\partial x_j}(U_j + u_j)\right) - \frac{\partial}{\partial t}(\rho u_i) - \frac{\partial}{\partial x_j}(\rho_{to} u_j u_i + \rho u_i U_j) - \frac{\partial p_0}{\partial x_i},\end{aligned} \tag{3.9}$$

so if the base flow is constant then it reduces to

$$\frac{\partial}{\partial t}(\rho_0 u_i) + \frac{\partial}{\partial x_j}(\rho_0 u_i U_j) + \frac{\partial p}{\partial x_i} = -\frac{\partial}{\partial t}(\rho u_i) - \frac{\partial}{\partial x_j}(\rho_{to} u_j u_i + \rho u_i U_j) - \frac{\partial p_0}{\partial x_i}.$$

Then the generalised momentum equation for constant base flow is

$$\begin{aligned}\frac{\bar{\partial}}{\partial t}(\widetilde{\rho_0 \boldsymbol{u}}) + &\frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 u_i U_j})\boldsymbol{e}_i + \overline{\boldsymbol{\nabla}}\widetilde{p} \\ &= -\frac{\bar{\partial}}{\partial t}(\widetilde{\rho \boldsymbol{u}}) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_{to} u_j u_i} + \widetilde{\rho u_i U_j})\boldsymbol{e}_i - \overline{\boldsymbol{\nabla}}\widetilde{p_0} + \delta(f)\left[\rho_{to}\boldsymbol{u}((\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot \boldsymbol{n}) + p_{to}\boldsymbol{n}\right].\end{aligned}$$

3.2.2 Incompressible perturbations, constant density ρ_{to} and $\frac{D}{Dt}\boldsymbol{U} = 0$

By incompressible perturbations we mean that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ and by $\frac{D}{Dt}\boldsymbol{U} = 0$ we mean zero material derivative with respect to the base flow itself, so

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{U} = 0.$$

In this case (3.9) becomes

$$\frac{\partial}{\partial t}(\rho_0 u_i) + \frac{\partial}{\partial x_j}(\rho_0 u_i U_j) + \frac{\partial p}{\partial x_i} = -\frac{\partial}{\partial x_j}(\rho_{to} u_j U_i) - \frac{\partial}{\partial t}(\rho u_i) - \frac{\partial}{\partial x_j}(\rho_{to} u_j u_i + \rho u_i U_j) - \frac{\partial p_0}{\partial x_i},$$

and thus the generalised momentum equation for this case is

$$\begin{aligned}\frac{\bar{\partial}}{\partial t}(\widetilde{\rho_0 \boldsymbol{u}}) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 u_i U_j})\boldsymbol{e}_i + \overline{\boldsymbol{\nabla}}\widetilde{p} = &-\frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_{to} u_j U_i})\boldsymbol{e}_i - \frac{\bar{\partial}}{\partial t}(\widetilde{\rho \boldsymbol{u}}) \\ &- \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_{to} u_j u_i} + \widetilde{\rho u_i U_j})\boldsymbol{e}_i - \overline{\boldsymbol{\nabla}}\widetilde{p_0} + \delta(f)\left[\rho_{to}\boldsymbol{U}(\boldsymbol{u} \cdot \boldsymbol{n}) + \rho_{to}\boldsymbol{u}((\boldsymbol{u}_{to} - \boldsymbol{v}^s) \cdot \boldsymbol{n}) + p_{to}\boldsymbol{n}\right].\end{aligned}$$

3.2.3 Linearised momentum equation

Since the perturbations are small, we can ignore quadratic (and higher order) terms in (3.7) so it becomes

$$\frac{\partial}{\partial t}(\rho U_i + \rho_0 u_i) + \frac{\partial}{\partial x_j}(\rho_0 u_i U_j + \rho_0 u_j U_i + \rho U_i U_j) + \frac{\partial p}{\partial x_i} = -\frac{\partial}{\partial t}(\rho_0 U_i) - \frac{\partial}{\partial x_j}(\rho_0 U_i U_j) - \frac{\partial p_0}{\partial x_i}.$$

Performing similar analysis to before we then get the generalised, linearised momentum equation:

$$\begin{aligned}\frac{\bar{\partial}}{\partial t}(\widetilde{\rho \boldsymbol{U}} + \widetilde{\rho_0 \boldsymbol{u}}) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 u_i U_j} + \widetilde{\rho_0 u_j U_i} + \widetilde{\rho U_i U_j})\boldsymbol{e}_i + \overline{\boldsymbol{\nabla}}\widetilde{p} = -\frac{\bar{\partial}}{\partial t}(\widetilde{\rho_0 \boldsymbol{U}}) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho_0 U_i U_j})\boldsymbol{e}_i - \overline{\boldsymbol{\nabla}}\widetilde{p_0} \\ + \delta(f)\left[\rho_0 \boldsymbol{u}_{to}((\boldsymbol{U} - \boldsymbol{v}^s) \cdot \boldsymbol{n}) + \rho(\boldsymbol{U}(\boldsymbol{U} - \boldsymbol{v}^s) \cdot \boldsymbol{n}) + \rho_0 \boldsymbol{U}(\boldsymbol{u} \cdot \boldsymbol{n}) + p_{to}\boldsymbol{n}\right]. \quad (3.10)\end{aligned}$$

3.3 Vorticity

The vorticity equation (3.3) can be written in components as

$$\begin{aligned}0 = \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} - \omega_j \frac{\partial u_i}{\partial x_j} + \omega_i \frac{\partial u_j}{\partial x_j} &= \frac{\partial \omega_i}{\partial t} + \frac{\partial}{\partial x_j}(\omega_i u_j) - \frac{\partial}{\partial x_j}(u_i \omega_j) + u_i \frac{\partial \omega_j}{\partial x_j} \\ &= \frac{\partial \omega_i}{\partial t} + \frac{\partial}{\partial x_j}(\omega_i u_j) - \frac{\partial}{\partial x_j}(u_i \omega_j),\end{aligned}$$

with the last line following from the fact that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0$. As before, we rewrite in terms of perturbations and the base flow so the left hand side is a linear operator acting on perturbations:

$$\frac{\partial \Omega_i}{\partial t} + \frac{\partial}{\partial x_j}(\Omega_i u_j + \omega_i U_j) - \frac{\partial}{\partial x_j}(u_i \Omega_j + U_i \omega_j) = -\frac{\partial \omega_i}{\partial t} - \frac{\partial}{\partial x_j}(\omega_i u_j + \Omega_i U_j) + \frac{\partial}{\partial x_j}(u_i \omega_j + U_i \Omega_j). \quad (3.11)$$

We now multiply by the Heaviside function and use (2.2) and (2.3), which gives

$$\frac{\bar{\partial}\widetilde{\Omega}_i}{\partial t} + \left[(\boldsymbol{v}^s\cdot\boldsymbol{n})\omega_i\right]\delta(f) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega_i u_j} + \widetilde{\omega_i U_j}) - n_j\left[\Omega_i u_j + \omega_i U_j\right]\delta(f) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{u_i\Omega_j} + \widetilde{U_i\omega_j})$$
$$+n_j\left[u_i\Omega_j + U_i\omega_j\right]\delta(f) = -\frac{\bar{\partial}\widetilde{\omega}_i}{\partial t} - \left[(\boldsymbol{v}^s\cdot\boldsymbol{n})\Omega_i\right]\delta(f) - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\omega_i u_j} + \widetilde{\Omega_i U_j})$$
$$+ n_j\left[\omega_i u_j + \Omega_i U_j\right]\delta(f) + \frac{\bar{\partial}}{\partial x_j}(\widetilde{u_i\omega_j} + \widetilde{U_i\Omega_j}) - n_j\left[u_i\omega_j + U_i\Omega_j\right]\delta(f).$$

Finally, collecting all the $\delta(f)$ terms we get the generalised vorticity equation in vector form

$$\boxed{\begin{aligned}\frac{\bar{\partial}\widetilde{\boldsymbol{\Omega}}}{\partial t} + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega_i u_j} + \widetilde{\omega_i U_j})\boldsymbol{e}_i - \frac{\bar{\partial}}{\partial x_j}(\widetilde{u_i\Omega_j} + \widetilde{U_i\omega_j})\boldsymbol{e}_i = -\frac{\bar{\partial}\widetilde{\boldsymbol{\omega}}}{\partial t} - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\omega_i u_j} + \widetilde{\Omega_i U_j})\boldsymbol{e}_i \\ +\frac{\bar{\partial}}{\partial x_j}(\widetilde{u_i\omega_j} + \widetilde{U_i\Omega_j})\boldsymbol{e}_i + \delta(f)\left[\boldsymbol{\omega}_{to}((\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot\boldsymbol{n}) - \boldsymbol{u}_{to}(\boldsymbol{\omega}_{to}\cdot\boldsymbol{n})\right].\end{aligned}} \tag{3.12}$$

In the case $\boldsymbol{u}_{to} = \boldsymbol{v}^s$ the source term for vorticity reduces to $-\delta(f)\left[\boldsymbol{u}_{to}(\boldsymbol{\omega}_{to}\cdot\boldsymbol{n})\right]$ and thus would vanish if $\boldsymbol{\omega}_{to}$ is also orthogonal to the surface normal.

3.3.1 Linearised vorticity equation

Since the perturbations are small, we can ignore quadratic terms in (3.11) so it becomes

$$\frac{\partial\Omega_i}{\partial t} + \frac{\partial}{\partial x_j}(\Omega_i u_j + \omega_i U_j) - \frac{\partial}{\partial x_j}(u_i\Omega_j + U_i\omega_j) = -\frac{\partial\omega_i}{\partial t} - \frac{\partial}{\partial x_j}(\Omega_i U_j) + \frac{\partial}{\partial x_j}(U_i\Omega_j)$$

Performing similar analysis to before we then get the generalised, linearised vorticity equation:

$$\frac{\bar{\partial}\widetilde{\boldsymbol{\Omega}}}{\partial t} + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega_i u_j} + \widetilde{\omega_i U_j})\boldsymbol{e}_i - \frac{\bar{\partial}}{\partial x_j}(\widetilde{u_i\Omega_j} + \widetilde{U_i\omega_j})\boldsymbol{e}_i = -\frac{\bar{\partial}\widetilde{\boldsymbol{\omega}}}{\partial t} - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega_i U_j})\boldsymbol{e}_i$$
$$+\frac{\bar{\partial}}{\partial x_j}(\widetilde{U_i\Omega_j})\boldsymbol{e}_i + \delta(f)\left[\boldsymbol{\Omega}((\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot\boldsymbol{n}) + \boldsymbol{\omega}((\boldsymbol{U} - \boldsymbol{v}^s)\cdot\boldsymbol{n}) - \boldsymbol{U}(\boldsymbol{\omega}\cdot\boldsymbol{n}) - \boldsymbol{u}_{to}(\boldsymbol{\Omega}\cdot\boldsymbol{n})\right].$$

For a two dimensional flow, with $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega$ and $\Omega_1 = \Omega_2 = 0$, $\Omega_3 = \Omega$ the linearised vorticity equation reduces to

$$\frac{\bar{\partial}\widetilde{\Omega}}{\partial t} + \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega u_j} + \widetilde{\omega U_j}) = -\frac{\bar{\partial}\widetilde{\omega}}{\partial t} - \frac{\bar{\partial}}{\partial x_j}(\widetilde{\Omega U_j}) + \delta(f)\left[\Omega((\boldsymbol{u}_{to} - \boldsymbol{v}^s)\cdot\boldsymbol{n}) + \omega((\boldsymbol{U} - \boldsymbol{v}^s)\cdot\boldsymbol{n})\right]. \tag{3.13}$$

4 Shear flow

We now consider a specific example, and firstly formulate the generalised conservation equations and then proceed to solve them. We consider a two-dimensional incompressible **inviscid** model, with time dependent perturbations of a linearly sheared mean flow. Thus we have

$$\boldsymbol{U} = (U(y), 0, 0), \quad \boldsymbol{u} = (u(x,y)e^{i\omega t}, v(x,y)e^{i\omega t}, 0), \quad \rho_0 = \text{constant}, \quad \rho = 0, \quad p_0 = 0.$$

We also have by definition of vorticity for a two dimensional flow

$$\Omega = -U'(y), \quad \omega = \chi e^{i\omega t} \text{ where } \chi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

We derive similar results to [Rienstra et al., 2013, §2.1] but now in terms of generalised functions.

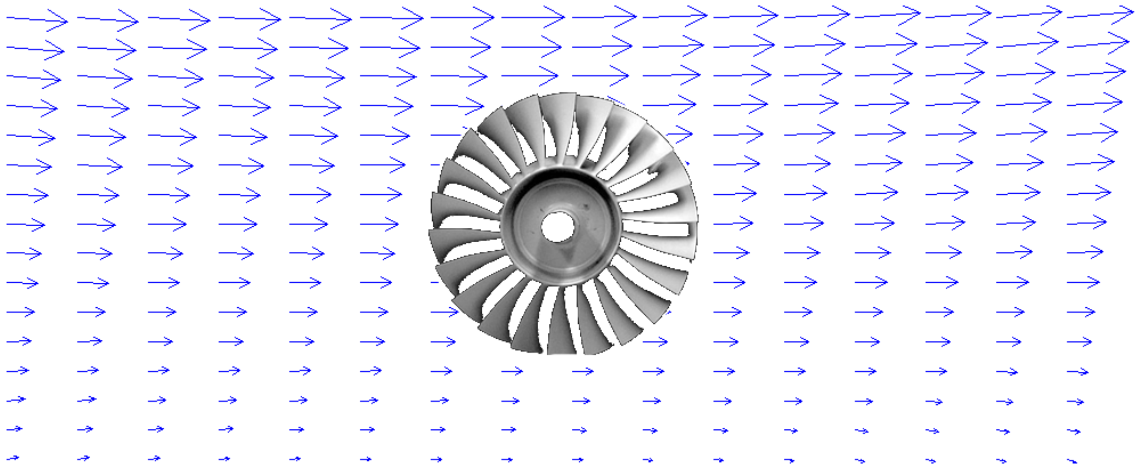


Figure 1: Shear flow with small perturbations (in this case of the form $u, v \propto xy$) and constant density around a jet engine with occupies the space $f < 0$. Picture of jet engine from `http://www.grc.nasa.gov/WWW/RT/2005/RX/RX41S-min.html`.

4.1 Derivation of conservation equations

Mass

We firstly consider the linearised mass equation (3.6) (although since $\rho = 0$ the two forms (3.5) and (3.6) are identical), which gives

$$\rho_0 \frac{\overline{\partial}}{\partial x_j}(\widetilde{u_j}) = -\frac{\overline{\partial}\widetilde{\rho_0}}{\partial t} - \rho_0 \frac{\overline{\partial}}{\partial x_j}(\widetilde{U_j}) + \delta(f)\left[\rho_0 \boldsymbol{u}\cdot\boldsymbol{n} + \rho_0 \boldsymbol{U}\cdot\boldsymbol{n} - \rho_0 \boldsymbol{v}^s\cdot\boldsymbol{n}\right].$$

Upon using $\boldsymbol{\nabla}\cdot\boldsymbol{U} = 0$ and the constancy of ρ_0 in time we get

$$\rho_0 e^{i\omega t}\left[\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v}\right] = \delta(f)\left[\rho_0 \boldsymbol{u}\cdot\boldsymbol{n}\right],$$

and thus we have

$$\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v} = \left[u(x,y)n_1 + v(x,y)n_2\right]\delta(f). \tag{4.1}$$

Vorticity

We have a two dimensional flow, so we can use (3.13) to get

$$\frac{\overline{\partial}}{\partial x}(-\widetilde{U'ue^{i\omega t}} + \widetilde{U\chi e^{i\omega t}}) + \frac{\overline{\partial}}{\partial y}(-\widetilde{U'v e^{i\omega t}}) = -\frac{\overline{\partial}}{\partial t}\widetilde{\chi e^{i\omega t}} + \delta(f)\left[-U'(\boldsymbol{u}\cdot\boldsymbol{n}) + \chi e^{i\omega t}((\boldsymbol{U}-\boldsymbol{v}^s)\cdot\boldsymbol{n})\right].$$

Upon simplifying some of the derivatives, rearranging and using the product rule (2.6) we get

$$U e^{i\omega t}\frac{\overline{\partial}}{\partial x}\widetilde{\chi} + i\omega\widetilde{\chi}e^{i\omega t} - U'e^{i\omega t}\left(\frac{\overline{\partial}}{\partial x}\widetilde{u}+\frac{\overline{\partial}}{\partial y}\widetilde{v}\right) - ve^{i\omega t}U'' = \delta(f)\left[-U'(\boldsymbol{u}\cdot\boldsymbol{n}) + \chi e^{i\omega t}(\boldsymbol{U}\cdot\boldsymbol{n})\right].$$

If $U''(y) = 0$ (as it will be in §4.2) or is of a similar magnitude to the perturbations then we can ignore that term since it is quadratic. Finally, using conservation of mass (4.1) we simplify the vorticity equation to

$$\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{\chi} = \chi U n_1 \delta(f). \tag{4.2}$$

Momentum

Next, the linearised momentum equation (3.10) becomes

$$\rho_0 \frac{\overline{\partial}}{\partial t}\widetilde{\boldsymbol{u}} + \rho_0 \frac{\overline{\partial}}{\partial x_j}(\widetilde{u_i U_j} + \widetilde{u_j U_i})\boldsymbol{e}_i + \overline{\boldsymbol{\nabla}}\widetilde{p} = -\rho_0 \frac{\overline{\partial}}{\partial t}(\widetilde{\boldsymbol{U}}) - \rho_0 \frac{\overline{\partial}}{\partial x_j}(\widetilde{U_i U_j})\boldsymbol{e}_i$$
$$+ \delta(f)\left[\rho_0 \boldsymbol{u}((\boldsymbol{U} - \boldsymbol{v}^s)\cdot \boldsymbol{n}) + \rho_0 \boldsymbol{U}((\boldsymbol{U} - \boldsymbol{v}^s)\cdot \boldsymbol{n}) + \rho_0 \boldsymbol{U}(\boldsymbol{u}\cdot\boldsymbol{n}) + p\boldsymbol{n}\right], \tag{4.3}$$

and after simplifying some of the derivatives the first component is given by

$$\rho_0 u \frac{\overline{\partial}}{\partial t}\widetilde{e^{i\omega t}} + 2\rho_0 U e^{i\omega t}\frac{\overline{\partial}}{\partial x}\widetilde{u} + \rho_0 e^{i\omega t}\frac{\overline{\partial}}{\partial y}\widetilde{vU} + \frac{\overline{\partial}}{\partial x}\widetilde{p} = \rho_0 U(\boldsymbol{v}^s\cdot\boldsymbol{n})\delta(f) - \rho_0 U^2 n_1 \delta(f)$$
$$+ \delta(f)\left[\rho_0 u e^{i\omega t}(Un_1 - \boldsymbol{v}^s\cdot\boldsymbol{n}) + \rho_0 U(Un_1 - \boldsymbol{v}^s\cdot\boldsymbol{n}) + \rho_0 U\left[un_1 + vn_2\right]e^{i\omega t} + pn_1\right].$$

Upon using the product rule this equation becomes

$$\rho_0 u \frac{\overline{\partial}}{\partial t}\widetilde{e^{i\omega t}} + \rho_0 U e^{i\omega t}\frac{\overline{\partial}}{\partial x}\widetilde{u} + \rho_0 U e^{i\omega t}\left[\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v}\right] + \rho_0 e^{i\omega t} v \frac{\overline{\partial}}{\partial y}\widetilde{U} - \rho_0 e^{i\omega t} vUn_2\delta(f) + \frac{\overline{\partial}}{\partial x}\widetilde{p}$$
$$= \delta(f)\left[\rho_0 u e^{i\omega t}(Un_1 - \boldsymbol{v}^s\cdot\boldsymbol{n}) + \rho_0 U\left[un_1 + vn_2\right]e^{i\omega t} + pn_1\right]. \tag{4.4}$$

Finally, using the result that

$$\rho_0 u \frac{\overline{\partial}}{\partial t}\widetilde{e^{i\omega t}} = \rho_0 i\omega e^{i\omega t}\widetilde{u} - \rho_0 u e^{i\omega t}(\boldsymbol{v}^s\cdot\boldsymbol{n})\delta(f), \tag{4.5}$$

and conservation of mass equation (4.1) we reduce (4.4) to

$$\rho_0 e^{i\omega t}\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{u} + \rho_0 e^{i\omega t} v\frac{\overline{\partial}}{\partial y}\widetilde{U} + \frac{\overline{\partial}}{\partial x}\widetilde{p} = \delta(f)\left[\rho_0 U(un_1 + vn_2)e^{i\omega t} + pn_1\right]. \tag{4.6}$$

The second component of (4.3) is

$$\rho_0 v \frac{\overline{\partial}}{\partial t}\widetilde{e^{i\omega t}} + \rho_0 U e^{i\omega t}\frac{\overline{\partial}}{\partial x}\widetilde{v} + \frac{\overline{\partial}}{\partial y}\widetilde{p} = \delta(f)\left[\rho_0 v e^{i\omega t}(Un_1 - \boldsymbol{v}^s\cdot\boldsymbol{n}) + pn_2\right].$$

Upon using a result similar to (4.5), we get the final equation to be solved is

$$\rho_0 e^{i\omega t}\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{v} + \frac{\overline{\partial}}{\partial y}\widetilde{p} = \delta(f)\left[\rho_0 vUn_1 e^{i\omega t} + pn_2\right]. \tag{4.7}$$

Finally, we replace p by $pe^{i\omega t}$ for the rest of this section for ease of notation, and will add it back in again at the start of Section 6. This then gives the system of equations to be solved as

$$\boxed{\begin{aligned}
&\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v} = \left[un_1 + vn_2\right]\delta(f). && \text{(Mass)}\\
&\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{\chi} = \chi U n_1 \delta(f). && \text{(Vorticity)}\\
&\rho_0\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{u} + \rho_0 v\frac{\overline{\partial}}{\partial y}\widetilde{U} + \frac{\overline{\partial}}{\partial x}\widetilde{p} = \delta(f)\left[\rho_0 U(un_1 + vn_2) + pn_1\right]. && \text{(Momentum 1)}\\
&\rho_0\left(i\omega + U\frac{\overline{\partial}}{\partial x}\right)\widetilde{v} + \frac{\overline{\partial}}{\partial y}\widetilde{p} = \delta(f)\left[\rho_0 U v n_1 + pn_2\right]. && \text{(Momentum 2)}
\end{aligned}}$$

These are the same equations that were derived in [Rienstra et al., 2013], if we replaced the generalised functions by normal functions and set the right hand side of the mass equation to be $2\pi S\delta(x)\delta(y)$ and the right hand side of the other equations to be 0.

4.2 Partial differential equation for the pressure field

Throughout this section we assume that $U''(y) = 0$, so U is a linear shear flow of the form $U(y) = U_0 + \sigma y$ for some U_0 and σ. The first equation we solve is the vorticity equation, since it is the only equation to contain only one unknown. A simple calculation shows that

$$\widetilde{\chi} = Se^{-i\omega x/U(y)}h(f), \tag{4.8}$$

solves the vorticity equation, with S some constant determining the strength of the source term, which is

$$\left[SU(y)e^{-i\omega x/U(y)}\frac{\partial f}{\partial x}\right]\delta(f).$$

This compares to the vorticity source term in [Rienstra et al., 2013] of $2\pi S\delta(x)\delta(y)$, and thus generalises the result to arbitrary surfaces. Next, we solve for $\widetilde{v}$. An easy calculation shows that

$$\widetilde{\chi} = \frac{\overline{\partial}\widetilde{v}}{\partial x} - \frac{\overline{\partial}\widetilde{u}}{\partial y} + (un_2 - vn_1)\delta(f), \tag{4.9}$$

and then differentiating this with respect to x and exchanging the order of derivatives in the term involving $\widetilde{u}$ using (2.7) gives

$$\frac{\overline{\partial}}{\partial x}\widetilde{\chi} = \frac{\overline{\partial}}{\partial x}\frac{\overline{\partial}\widetilde{v}}{\partial x} - \frac{\overline{\partial}}{\partial y}\frac{\overline{\partial}\widetilde{u}}{\partial x} + \frac{\overline{\partial}}{\partial x}\left[(un_2 - vn_1)\delta(f)\right],$$

and upon using the Mass equation we get

$$\frac{\overline{\partial}}{\partial x}\widetilde{\chi} = \frac{\overline{\partial}^2\widetilde{v}}{\partial x^2} + \frac{\overline{\partial}^2\widetilde{v}}{\partial y^2} + \frac{\overline{\partial}}{\partial x}\left[(un_2 - vn_1)\delta(f)\right] - \frac{\overline{\partial}}{\partial y}\left[(un_1 + vn_2)\delta(f)\right].$$

Thus we have, on calculating the derivative of $\widetilde{\chi}$ with respect to x (or using the Vorticity equation),

$$\boxed{\overline{\Delta}\widetilde{v} = -i\omega\frac{\widetilde{\chi}}{U} + \mathbb{S}_1,} \tag{4.10}$$

where $\overline{\Delta}$ is the two dimensional Laplacian (and will be throughout the rest of the report) and

$$\mathbb{S}_1 = \chi n_1\delta(f) + \frac{\overline{\partial}}{\partial y}\left[(un_1 + vn_2)\delta(f)\right] - \frac{\overline{\partial}}{\partial x}\left[(un_2 - vn_1)\delta(f)\right].$$

Having solved for $\widetilde{v}$ we now follow a similar approach to determine $\widetilde{u}$. Differentiating (4.9) by y, exchanging the order of derivatives and using the mass equation gives

$$\frac{\overline{\partial}}{\partial y}\widetilde{\chi} = -\frac{\overline{\partial}^2\widetilde{u}}{\partial x^2} - \frac{\overline{\partial}^2\widetilde{u}}{\partial y^2} + \frac{\overline{\partial}}{\partial y}\left[(un_2 - vn_1)\delta(f)\right] + \frac{\overline{\partial}}{\partial x}\left[(un_1 + vn_2)\delta(f)\right],$$

and upon calculating the y derivative of χ we get

$$\boxed{\overline{\Delta}\widetilde{u} = -i\omega\sigma x\frac{\widetilde{\chi}}{U^2} + \mathbb{S}_2,} \tag{4.11}$$

where

$$\mathbb{S}_2 = \frac{\overline{\partial}}{\partial x}\left[(un_1 + vn_2)\delta(f)\right] + \frac{\overline{\partial}}{\partial y}\left[(un_2 - vn_1)\delta(f)\right] - \chi n_2 \delta(f).$$

Instead of using the vorticity equation to calculate the expressions for $\widetilde{u}$ and $\widetilde{v}$, we could have used the two momentum equations; taking the y derivative of the first and the x derivative of the second, and then subtracting to eliminate pressure.

Finally, we need to calculate an expression for the pressure $\widetilde{p}$ which is what we actually want, but we needed to determine expressions for $\widetilde{u}$ and $\widetilde{v}$ first. We differentiate the first momentum equation with respect to x and differentiate the second momentum equation with respect to y, which gives us

$$\rho_0 i\omega \frac{\overline{\partial}}{\partial x}\widetilde{u} + \rho_0 \frac{\overline{\partial}}{\partial x}\left(U\frac{\overline{\partial}}{\partial x}\widetilde{u}\right) + \rho_0 \frac{\overline{\partial}}{\partial x}\left(v\frac{\overline{\partial}}{\partial y}\widetilde{U}\right) + \frac{\overline{\partial}^2}{\partial x^2}\widetilde{p} = S_1,$$

and

$$\rho_0 i\omega \frac{\overline{\partial}}{\partial y}\widetilde{v} + \rho_0 \frac{\overline{\partial}}{\partial y}\left(U\frac{\overline{\partial}}{\partial x}\widetilde{v}\right) + \frac{\overline{\partial}^2}{\partial y^2}\widetilde{p} = S_2,$$

$$S_1 := \frac{\overline{\partial}}{\partial x}\left[\left(\rho_0 U(un_1 + vn_2) + pn_1\right)\delta(f)\right] \text{ and } S_2 := \frac{\overline{\partial}}{\partial y}\left[\left(\rho_0 U v n_1 + pn_2\right)\delta(f)\right].$$

Upon using (2.8) and the definition of U the two equations become

$$\rho_0 i\omega \frac{\overline{\partial}}{\partial x}\widetilde{u} + \rho_0 U\frac{\overline{\partial}}{\partial x}\left(\frac{\overline{\partial}}{\partial x}\widetilde{u}\right) + \rho_0 v \frac{\overline{\partial}}{\partial x}\left(\frac{\overline{\partial}}{\partial y}\widetilde{U}\right) + \rho_0 \frac{\partial v}{\partial x}\frac{\overline{\partial}}{\partial y}\widetilde{U} + \frac{\overline{\partial}^2}{\partial x^2}\widetilde{p} = S_1,$$

and

$$\rho_0 i\omega \frac{\overline{\partial}}{\partial y}\widetilde{v} + \rho_0 U\frac{\overline{\partial}}{\partial y}\left(\frac{\overline{\partial}}{\partial x}\widetilde{v}\right) + \rho_0 U'(y)\frac{\overline{\partial}}{\partial x}\widetilde{v} + \frac{\overline{\partial}^2}{\partial y^2}\widetilde{p} = S_2,$$

respectively. Upon adding together the above equations, calculating the mixed derivative of U and changing the term involving derivatives of v to generalised derivatives of v we have (where $\sigma = U'(y)$)

$$\rho_0 i\omega\left(\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v}\right) + \rho_0 U\frac{\overline{\partial}}{\partial x}\left(\frac{\overline{\partial}}{\partial x}\widetilde{u} + \frac{\overline{\partial}}{\partial y}\widetilde{v}\right) + 2\rho_0\sigma\frac{\overline{\partial}}{\partial x}\widetilde{v} + \overline{\Delta}\widetilde{p} = S_1 + S_2 + S_3 - S_4,$$

with

$$S_3 := \rho_0\left[n_1\sigma v - n_2\frac{\partial v}{\partial x}U\right]\delta(f) \text{ and } S_4 := \rho_0 v\left[\frac{\overline{\partial}}{\partial y}(n_1 U\delta(f))\right].$$

Upon using the mass equation we then get

$$\overline{\Delta}\widetilde{p} + 2\rho_0\sigma\frac{\overline{\partial}\widetilde{v}}{\partial x} = S_1 + S_2 + S_3 - S_4 - S_5 - S_6 := \mathbb{T}, \tag{4.12}$$

with

$$S_5 := \rho_0 i\omega\left[un_1 + vn_2\right]\delta(f), \quad S_6 := \rho_0 U\frac{\overline{\partial}}{\partial x}\left[(un_1 + vn_2)\delta(f)\right].$$

We now take the Laplacian of (4.12), and since σ is constant we thus have

$$\overline{\Delta}^2\widetilde{p} + 2\rho_0\sigma\frac{\overline{\partial}}{\partial x}\overline{\Delta}\widetilde{v} = \overline{\Delta}\mathbb{T},$$

and then by using (4.10) we get

$$\overline{\Delta}^2\widetilde{p} = \overline{\Delta}\mathbb{T} - 2\rho_0\sigma\frac{\overline{\partial}}{\partial x}\mathbb{S}_1 + 2\rho_0\sigma\frac{\overline{\partial}}{\partial x}\left[i\omega\frac{\widetilde{\chi}}{U}\right] = \overline{\Delta}\mathbb{T} + 2\rho_0\sigma\left(-\frac{\overline{\partial}}{\partial x}\mathbb{S}_1 + \frac{i\omega}{U}\chi n_1\delta(f) + \frac{\omega^2\widetilde{\chi}}{U^2}\right).$$

Thus

$$\boxed{\overline{\Delta}^2\widetilde{p} = \mathbb{S}_s + \mathbb{S}_v,} \tag{4.13}$$

where

$$\mathbb{S}_s := \overline{\Delta}\mathbb{T} - 2\rho_0\sigma\frac{\overline{\partial}}{\partial x}\mathbb{S}_1 + 2\rho_0\sigma\frac{i\omega}{U}\chi n_1\delta(f) \text{ and } \mathbb{S}_v := 2\rho_0\sigma\frac{\omega^2\widetilde{\chi}}{U^2}.$$

Thus we have show that the pressure field satisfies a fourth order partial differential equation, the biharmonic operator with a right hand side consisting of surface source terms $\mathbb{S}_s$ and volume source terms $\mathbb{S}_v$. Our next step is to find the Green's functions for the biharmonic operator.

5 Green's function for biharmonic operator

5.1 Free space Green's function

We are looking to find the free space Green's function $g(x, y|x_0, y_0)$ for the two dimensional biharmonic operator; that is find g such that

$$\Delta^2 g = \frac{\partial^4 g}{\partial x^4} + 2\frac{\partial^4 g}{\partial x^2\partial y^2} + \frac{\partial^4 g}{\partial y^4} = \delta(x-x_0)\delta(y-y_0). \tag{5.1}$$

We follow the approach by [Duffy, 2001], introducing the Fourier Transform of g,

$$G(k, l|x_0, y_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y|x_0, y_0)e^{-ily}e^{-ikx}\mathrm{d}y\mathrm{d}x,$$

and then transforming (5.1) into

$$k^4G + 2k^2l^2G + l^4G = e^{-ily_0}e^{-ikx_0}.$$

Thus, the Green's function can be computed by

$$g(x, y|x_0, y_0) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{e^{il(y-y_0)}e^{ik(x-x_0)}}{(k^2+l^2)^2}\mathrm{d}k\mathrm{d}l. \tag{5.2}$$

We now differentiate underneath the integral sign, leaving the justification as an exercise to any concerned readers (it follows from taking a limit of integral and using the dominated convergence theorem). The reason for differentiating is that we then get a contour integral which converges. We compute that

$$\frac{\partial^2}{\partial x\partial y}g(x, y|x_0, y_0) = \frac{1}{2\pi^2 i}\int_0^{\infty}\left(\int_{\mathbb{R}}\frac{ke^{ik(x-x_0)}}{(k^2+l^2)^2}\mathrm{d}k\right)l\sin\left(l(y-y_0)\right)\mathrm{d}l.$$

We now evaluate the inner integral using a contour integral, and after some elementary complex analysis we get that

$$\int_{\mathbb{R}}\frac{ke^{ik(x-x_0)}}{(k^2+l^2)^2}\mathrm{d}k = \frac{\pi i}{2l}(x-x_0)e^{-l|x-x_0|},$$

and hence

$$\frac{\partial^2}{\partial x \partial y} g(x, y|x_0, y_0) = \frac{x - x_0}{4\pi} \int_0^\infty \sin\left(l(y - y_0)\right) e^{-l|x - x_0|} \mathrm{d}l. \tag{5.3}$$

Carrying on, we can evaluate the integral in (5.3) using standard integrals, and this gives us

$$\begin{aligned}
\frac{\partial^2}{\partial x \partial y} g(x, y|x_0, y_0) &= \frac{x - x_0}{4\pi} \left[-\frac{(y - y_0)\cos((y - y_0)l) - |x - x_0| \sin((y - y_0)l)}{((x - x_0)^2 + (y - y_0)^2)\, e^{|x - x_0|l}} \right]_0^\infty \\
&= \frac{1}{4\pi} \frac{(x - x_0)(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}.
\end{aligned}$$

Upon integrating once with respect to x and once with respect to y we get that

$$g(x, y|x_0, y_0) = \frac{(x - x_0)^2 + (y - y_0)^2}{8\pi} \left(\log \sqrt{(x - x_0)^2 + (y - y_0)^2} - \frac{1}{2} \right) + h(y - y_0) + k(x - x_0),$$

where h and k are arbitrary functions. We now choose the simplest k and h such that:

(1) For $x \neq x_0$ and $y \neq y_0$, $\Delta^2 g(x, y|x_0, y_0) = 0$.

(2) That $\Delta g = \hat{g}$ where $\hat{g}$ is the Green's function for the Laplacian, so $\hat{g} = \frac{1}{2\pi} \log r$.

After some calculations, this gives us that k and h have to solve the equations

$$h''''(y - y_0) + k''''(x - x_0) = 0 \text{ and } \frac{1}{4\pi} + h''(y - y_0) + k''(x - x_0) = 0,$$

of which the simplest solution is

$$h(y - y_0) + k(x - x_0) = -\frac{1}{8\pi} r^2.$$

Thus, we get the widely accepted result that

$$\boxed{g(x, y|x_0, y_0) = \frac{r^2}{8\pi} (\log r - 1) \text{ where } r = \sqrt{(x - x_0)^2 + (y - y_0)^2}.} \tag{5.4}$$

5.2 Green's function in an infinite strip

We notice that in the source terms $\mathbb{S}_v$ and $\mathbb{S}_s$ we are dividing by $U(y)$. If we consider the whole region then at $y = -U_0/\sigma$, $U(y) = 0$, so instead we focus on a strip with $0 \leq y \leq L$ and $-\infty < x < \infty$ where $U \neq 0$, for some arbitrary L. We consider several different boundary conditions for the Green's function.

5.2.1 Neumann boundary conditions

We assume firstly in the strip the boundary condition from [Posson and Peake, 2013]; that the normal velocity of the Green's function is zero on the boundary of the strip (as if there was a wall at the boundary). In this case we require $\partial g/\partial y = 0$ on the boundary at $y = 0$ and $y = L$. We assume that our Green's function is of the form

$$g(x, y|x_0, y_0) = \sum_{n=-\infty}^{\infty} G_n(x|x_0) \cos \frac{n\pi y_0}{L} \cos \frac{n\pi y}{L}, \tag{5.5}$$

with G_n to be determined. It is clear that this expression satisfies the boundary conditions $g_y(x,0|x_0,y_0) = g_y(x,L|x_0,y_0) = 0$. It can also be easily shown that

$$\delta(y-y_0) = \frac{1}{L}\sum_{n=-\infty}^{\infty} \cos\frac{n\pi y_0}{L}\cos\frac{n\pi y}{L}. \tag{5.6}$$

Thus, upon inserting (5.5) and (5.6) into $\Delta^2 g = \delta(x-x_0)\delta(y-y_0)$ we require that for each n, G_n satisfies the differential equation

$$\frac{d^4G_n}{dx^4} - 2\frac{n^2\pi^2}{L^2}\frac{d^2G_n}{dx^2} + \frac{n^4\pi^4}{L^4}G_n = \frac{1}{L}\delta(x-x_0). \tag{5.7}$$

For ease of notation we let $k = n\pi/L$. We note that the homogeneous solution to this equation is given by (for $k \neq 0$)

$$c_1e^{kx} + c_2e^{-kx} + c_3xe^{kx} + c_4e^{-kx},$$

where c_i are constants to be determined. We let $f_1(x)$ be the homogeneous solution valid for $-\infty < x < x_0$ and $f_2(x)$ be the homogeneous solution valid for $x_0 < x < \infty$. Assuming that the Green's function is zero at $\pm\infty$, we then have (for $k > 0$)

$$f_1(x) = c_1e^{kx} + c_3xe^{kx} \text{ and } f_2(x) = c_2e^{-kx} + c_4xe^{-kx}.$$

We choose the constants c_i such that f_i satisfy the continuity and jump conditions:

$$f_1(x_0) - f_2(x_0) = 0,\ \ f_1'(x_0) - f_2'(x_0) = 0,\ \ f_1''(x_0) - f_2''(x_0) = 0,\ \ f_1'''(x_0) - f_2'''(x_0) = -\frac{1}{L}. \tag{5.8}$$

Solving these system of equations gives

$$c_1 = \frac{1+kx_0}{4Lk^3}e^{-kx_0},\ c_2 = \frac{1-kx_0}{4Lk^3}e^{kx_0},\ c_3 = -\frac{1}{4Lk^2}e^{-kx_0} \text{ and } c_4 = \frac{1}{4Lk^2}e^{kx_0},$$

and thus for $k \neq 0$

$$G_k(x|x_0) = \begin{cases} \dfrac{1}{4Lk^3}e^{k(x-x_0)} - \dfrac{x-x_0}{4Lk^2}e^{k(x-x_0)} & \text{when } x \leq x_0 \\ \dfrac{1}{4Lk^3}e^{k(x_0-x)} - \dfrac{x_0-x}{4Lk^2}e^{k(x_0-x)} & \text{when } x > x_0. \end{cases} \tag{5.9}$$

When $k = 0$ the homogeneous solution to (5.7) is

$$Ax^3 + Bx^2 + Cx + D,$$

where we need to determine the constants. Assuming f_1 and f_2 as before, we let $f_1 = Ax^3 + Bx^2 + Cx + D$ and $f_2(x) = -Ax^3 - Bx^2 - Cx - D$, which ensures the solution will have some symmetry. Using (5.8) we then get that

$$A = -\frac{1}{12L},\ B = \frac{1}{4L}x_0,\ C = -\frac{1}{4L}x_0^2 \text{ and } D = \frac{1}{12L}x_0^3,$$

and hence that

$$G_0(x|x_0) = \begin{cases} \dfrac{1}{12L}(x_0-x)^3 & \text{when } x \leq x_0 \\ \dfrac{1}{12L}(x-x_0)^3 & \text{when } x > x_0. \end{cases} \tag{5.10}$$

Although we would like a solution that is zero as $|x| \to \infty$, any solution to $G_0^{(4)}(x|x_0) = \frac{1}{L}\delta(x-x_0)$ is a cubic polynomial with a jump, but the only polynomial which is zero as $|x| \to \infty$ is the polynomial identical to zero. Instead, we have cubic growth as $x \to \infty$, which is not entirely surprising since the Green's function for the whole plane had $x^2 \log x$ growth. Our Green's function can be written as the infinite series

$$\boxed{g(x,y|x_0,y_0) = \frac{1}{12L}|x-x_0|^3 + \sum_{k=1}^{\infty}\left[\frac{1}{2Lk^3}e^{-k|x-x_0|} + \frac{|x-x_0|}{2Lk^2}e^{-k|x-x_0|}\right]\cos(ky_0)\cos(ky).} \tag{5.11}$$

5.2.2 Dirichlet and Neumann boundary conditions

For our purposes, having polynomial growth is unsatisfactory. We try to find another Green's function which avoids this, and it turns out the idea is to impose both Dirichlet and Neumann boundary conditions. We consider the symmetric strip $-1 \leq y \leq 1$, $-\infty < x < \infty$ instead and follow the approach by [Gregory, 1979]. We start of by Fourier transforming in x, so we let

$$G(k,y|x_0,y_0) = \int_{-\infty}^{\infty} g(x,y|x_0,y_0)e^{-ikx}\mathrm{d}x.$$

We then consider the Fourier Transform of $\Delta^2 g = \delta(x-x_0)\delta(y-y_0)$, which gives

$$\frac{d^4G}{dy^4} - 2k^2\frac{d^2G}{dy^2} + k^4G = e^{-ikx_0}\delta(y-y_0).$$

To solve this we look for two homogeneous solutions to this equation, $f_1(y)$ which is valid for $-1 \leq y \leq y_0$ and $f_2(y)$ which is valid for $y_0 \leq y \leq 1$, together with the boundary conditions that

$$\begin{cases} f_1(-1) = \dfrac{df_1}{dy}(-1) = 0 \\ f_2(1) = \dfrac{df_2}{dy}(1) = 0 \end{cases}.$$

We then use (5.8) (at y_0, with the last condition equal to $-e^{ikx_0}$) to determine the matching conditions between the two solutions. We then 'patch' our solutions together to find G, and after some rather long and tedious algebra, we get [Gregory, 1979, Theorem 1] by taking the inverse Fourier transform

$$g(x,y|x_0,y_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(k,y|x_0,y_0)e^{-ikx}\mathrm{d}k.$$

Gregory also proves in the theorem that the solution makes sense and converges to zero as $|x| \to \infty$. He then goes on to prove that the Green's function can be expanded in terms of special eigenfunctions, the Papkovich-Fadle eigenfunctions of the infinite strip, in [Gregory, 1979, Theorem 2], with

$$\boxed{g(x,y|x_0,y_0) = \sum_{\lambda\in L}\frac{2\pi\tan^2\left(\frac{1}{2}\lambda\right)}{\lambda^3}E_\lambda(y)E_\lambda(y_0)e^{-\frac{1}{2}\lambda|x-x_0|} + \sum_{\mu\in M}\frac{2\pi\cot^2\left(\frac{1}{2}\mu\right)}{\mu^3}F_\mu(y)F_\mu(y_0)e^{-\frac{1}{2}\mu|x-x_0|}.} \tag{5.12}$$

In this expression

$$L = \{\lambda; \sin\lambda + \lambda = 0, \Re(\lambda) > 0\} = \{4.21239 \pm 2.25073i, 10.71254 \pm 3.10315i, 17.07336 \pm 3.55109i, ..\}$$

and

$$M = \{\mu; \sin\mu - \mu = 0, \Re(\mu) > 0\} = \{7.49768 \pm 2.76868i, 13.89996 \pm 3.35221i, 20.23852 \pm 3.71677i, ..\}$$

are the complex non-zero eigenvalues of $-\sin z$ and $\sin z$ respectively, while

$$E_\lambda(y) = (y-1)\sin\left(\frac{1}{2}\lambda(y+1)\right) + (y+1)\sin\left(\frac{1}{2}\lambda(y-1)\right),$$

and

$$F_\mu(y) = (y-1)\sin\left(\frac{1}{2}\mu(y+1)\right) - (y+1)\sin\left(\frac{1}{2}\mu(y-1)\right),$$

are the Papkovich-Fadle eigenfunctions which satisfy both the Dirichlet and Neumann boundary conditions. It is clear that the Green's function is real, and for ease of calculations later, we let

$$G_\lambda(x, y, x_0, y_0) = 4\pi \sum_{\lambda \in L} \Re\left(\frac{\tan^2\left(\frac{1}{2}\lambda\right)}{\lambda^3} E_\lambda(y) E_\lambda(y_0) e^{-\frac{1}{2}\lambda|x-x_0|}\right) \tag{5.13}$$

and

$$G_\mu(x, y, x_0, y_0) = 4\pi \sum_{\mu \in M} \Re\left(\frac{\cot^2\left(\frac{1}{2}\mu\right)}{\mu^3} F_\mu(y) F_\mu(y_0) e^{-\frac{1}{2}\mu|x-x_0|}\right). \tag{5.14}$$

An approximation to the Green's function is plotted in Figure 2.

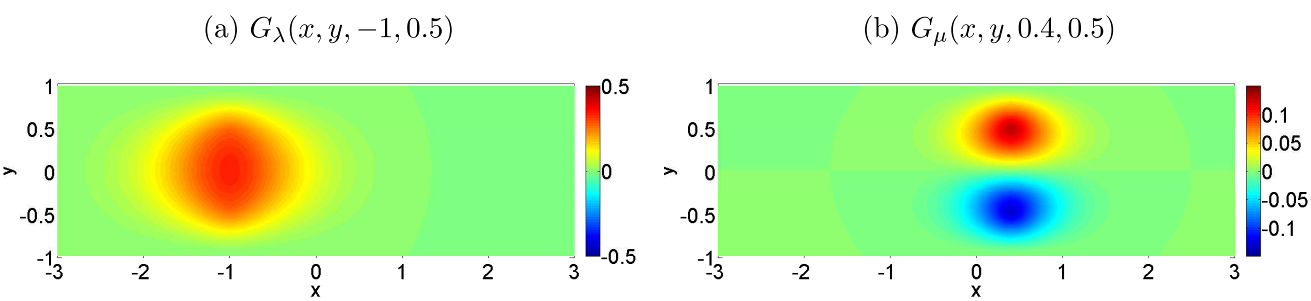


Figure 2: Approximations to the Green's function using the first seven terms of L on the left and the first seven terms of M on the right.

6 Pressure field solution

Having found the Green's function in the last section, we now let $\widetilde{g}(x, y|x_0, y_0)$ be a generalised Green's function, which is equal to $g(x, y|x_0, y_0)$ in $\Sigma(t) \cup \mathcal{V}(t)$ (where $f \geq 0$) and any value in $\mathcal{W}(t)$ (where $f < 0$). For convenience in later evaluating derivatives of $\widetilde{g}$ we let $\widetilde{g} = g$. We also add back the $e^{i\omega t}$ we omitted early in Section 4. From (4.13) we have that for $\boldsymbol{x} \notin \mathcal{W}(t)$ at time t

$$p(\boldsymbol{x}, t) = e^{-i\omega t} \iint_{\mathbb{R}^2} g(x, y|x_0, y_0) \left(\mathbb{S}_s(x_0, y_0) + \mathbb{S}_v(x_0, y_0)\right) \mathrm{d}x_0 \mathrm{d}y_0 := \mathcal{P}_s(x, t) e^{-i\omega t} + \mathcal{P}_v(x, t) e^{-i\omega t}, \tag{6.1}$$

where

$$\mathcal{P}_s(\boldsymbol{x},t) = \iint\limits_{\mathbb{R}^2} g(x,y|x_0,y_0)\mathbb{S}_s(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0$$

$$= \iint\limits_{\mathbb{R}^2} g(x,y|x_0,y_0)\left(\overline{\Delta}_0\mathbb{T} - 2\rho_0\sigma\frac{\overline{\partial}}{\partial x_0}\mathbb{S}_1 + 2\rho_0\sigma\frac{i\omega}{U}\chi n_1\delta(f)\right)(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0, \tag{6.2}$$

and

$$\mathcal{P}_v(\boldsymbol{x},t) = \iint\limits_{\mathbb{R}^2} g(x,y|x_0,y_0)\mathbb{S}_v(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0 = 2\rho_0\omega^2\sigma\iint\limits_{\mathbb{R}^2} g(x,y|x_0,y_0)\frac{\widetilde{\chi}}{U^2}(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0, \tag{6.3}$$

with the subscript denoting differentiation with respect to the source variables x_0 and y_0. On inserting the formula for the vorticity into (6.3) we get

$$\boxed{\mathcal{P}_v(\boldsymbol{x},t) = 2\rho_0\omega^2 S\sigma\iint\limits_{\mathcal{V}(t)} g(x,y|x_0,y_0)\frac{e^{-i\omega x_0/(U_0+\sigma y_0)}}{(U_0+\sigma y_0)^2}\mathrm{d}x_0\mathrm{d}y_0.} \tag{6.4}$$

This volume source arises due to the shear flow, if there is no shear ($\sigma = 0$) then the term is identically zero. We now calculate the surface source terms, and to do this we need to move the derivatives from the delta functions to the Green's function by using integration by parts. Assuming that $\Sigma(t)$ is finite, then we get no boundary terms so (6.2) becomes (omitting the argument of g for conciseness)

$$\mathcal{P}_s(\boldsymbol{x},t) = \mathcal{P}_s^i(\boldsymbol{x},t) + \mathcal{P}_s^{ii}(\boldsymbol{x},t) + \mathcal{P}_s^{iii}(\boldsymbol{x},t), \tag{6.5}$$

where [1]

$$\mathcal{P}_s^i(\boldsymbol{x},t) := \iint\limits_{\mathbb{R}^2}\Delta_0 g\mathbb{T}(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0, \quad \mathcal{P}_s^{ii}(\boldsymbol{x},t) := 2\rho_0\sigma\iint\limits_{\mathbb{R}^2}\frac{\partial g}{\partial x_0}\mathbb{S}_1(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0,$$

and

$$\mathcal{P}_s^{iii}(\boldsymbol{x},t) := 2\rho_0\sigma i\omega\iint\limits_{\mathbb{R}^2} g\frac{\chi n_1}{U}(x_0,y_0)\delta(f)\mathrm{d}x_0\mathrm{d}y_0.$$

We deal with each of these terms in turn. We have from the definition of $\mathbb{T}$ that

$$\mathcal{P}_s^i(\boldsymbol{x},t) = \iint\limits_{\mathbb{R}^2}\Delta_0 g\left(S_1+S_2+S_3-S_4-S_5-S_6\right)(x_0,y_0)\mathrm{d}x_0\mathrm{d}y_0.$$

After some calculations this then gives (by using integration by parts and cancelling some terms)

$$\mathcal{P}_s^i(\boldsymbol{x},t) = -\int\limits_{\Sigma(t)}\boldsymbol{\nabla}_0(\Delta_0 g)\cdot p\boldsymbol{n}(x_0,y_0)\mathrm{d}\Sigma(t) + \rho_0\int\limits_{\Sigma(t)}\Delta_0 g\Big[n_1\sigma v + U\boldsymbol{n}^\perp\cdot\boldsymbol{\nabla}v\Big](x_0,y_0)\mathrm{d}\Sigma(t)$$
$$- i\rho_0\omega e^{-i\omega t}\int\limits_{\Sigma(t)}\Delta_0 g\boldsymbol{u}\cdot\boldsymbol{n}(x_0,y_0)\mathrm{d}\Sigma(t), \tag{6.6}$$

[1]We reserve $\mathcal{P}_s^1, \mathcal{P}_s^2, \ldots$ for each individual term in the pressure surface source term.

where $\boldsymbol{n}^{\perp} = (-n_2, n_1, 0)$ is the tangent to the surface described by $f(\boldsymbol{x}, t) = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = (un_1 + vn_2)e^{i\omega t}$. We can also calculate that

$$\mathcal{P}_s^{ii}(\boldsymbol{x}, t) = -2\rho_0\sigma \int\limits_{\Sigma(t)} \boldsymbol{\nabla}_0 \left(\frac{\partial g}{\partial x_0}\right) \cdot \left(u\boldsymbol{n}^{\perp} + v\boldsymbol{n}\right)(x_0, y_0)\mathrm{d}\Sigma(t) + 2\rho_0\sigma \int\limits_{\Sigma(t)} \frac{\partial g}{\partial x_0}\chi n_1(x_0, y_0)\mathrm{d}\Sigma(t), \quad (6.7)$$

and

$$\mathcal{P}_s^{iii}(\boldsymbol{x}, t) = 2\rho_0\sigma i\omega \int\limits_{\Sigma(t)} g\frac{\chi n_1}{U}(x_0, y_0)\mathrm{d}\Sigma(t). \quad (6.8)$$

Thus, by combining (6.5),(6.6),(6.7) and (6.8) we have the complete surface source term:

$$\boxed{\begin{aligned} \mathcal{P}_s(\boldsymbol{x}, t) = &-\int\limits_{\Sigma(t)} \boldsymbol{\nabla}_0(\Delta_0 g) \cdot p\boldsymbol{n}(x_0, y_0)\mathrm{d}\Sigma(t) + \rho_0 \int\limits_{\Sigma(t)} \Delta_0 g \left[U\boldsymbol{n}^{\perp} \cdot \boldsymbol{\nabla} v\right](x_0, y_0)\mathrm{d}\Sigma(t) \quad (6.9) \\ &-i\rho_0\omega e^{-i\omega t} \int\limits_{\Sigma(t)} (\Delta_0 g)\boldsymbol{u} \cdot \boldsymbol{n}(x_0, y_0)\mathrm{d}\Sigma(t) - 2\rho_0\sigma \int\limits_{\Sigma(t)} \boldsymbol{\nabla}_0 \left(\frac{\partial g}{\partial x_0}\right) \cdot \left(u\boldsymbol{n}^{\perp} + v\boldsymbol{n}\right)(x_0, y_0)\mathrm{d}\Sigma(t) \\ &+2\rho_0\sigma i\omega \int\limits_{\Sigma(t)} g\frac{\chi n_1}{U}(x_0, y_0)\mathrm{d}\Sigma(t) + 2\rho_0\sigma \int\limits_{\Sigma(t)} \frac{\partial g}{\partial x_0}\chi n_1(x_0, y_0)\mathrm{d}\Sigma(t) + \rho_0\sigma \int\limits_{\Sigma(t)} (\Delta_0 g)n_1 v(x_0, y_0)\mathrm{d}\Sigma(t). \end{aligned}}$$

The first term is a force term which relates the source term to the pressure itself, while the second term accounts for the pressure source directly from the base flow and gradient of the velocity and disappears with zero base flow. The third term is slightly concerning, since it is imaginary, but since u and v have imaginary parts as well it will cancel out. It accounts for the source term arising from the perturbations in the normal direction. The last four terms are the surface source terms due to the shear flow, and disappear is the flow is constant ($\sigma = 0$). The first and last of these account for the effect of the velocity on the pressure field, while the second and third terms account for the effect of the vorticity on the pressure field.

To solve for the pressure field, we also need to have similar expressions for u and v. We let

$$\hat{g}(x, y|x_0, y_0) = \frac{1}{2\pi}\log r,$$

be the Green's function for the Laplacian and then we calculate from (4.10) and (4.11) that

$$u(x, y) = -i\omega\sigma \iint\limits_{\mathcal{V}(t)} \hat{g}x_0\frac{\chi}{U^2}(x_0, y_0)\mathrm{d}x_0\mathrm{d}y_0 - \int\limits_{\Sigma(t)} \left[\hat{g}\chi n_2(x_0,y_0) + \boldsymbol{\nabla}_0\hat{g} \cdot \left(u\boldsymbol{n} - v\boldsymbol{n}^{\perp}\right)(x_0, y_0)\right]\mathrm{d}\Sigma(t), \quad (6.10)$$

and

$$v(x, y) = -i\omega \iint\limits_{\mathcal{V}(t)} \hat{g}\frac{\chi}{U}(x_0, y_0)\mathrm{d}x_0\mathrm{d}y_0 + \int\limits_{\Sigma(t)} \left[\hat{g}\chi n_1(x_0, y_0) - \boldsymbol{\nabla}_0\hat{g} \cdot \left(u\boldsymbol{n}^{\perp} + v\boldsymbol{n}\right)(x_0, y_0)\right]\mathrm{d}\Sigma(t). \quad (6.11)$$

7 Computation of source terms for a pulsating 2-sphere

We now wish to calculate the source terms for a specific example, a pulsating 2-sphere in the infinite strip. We define the pulsating 2-sphere to have a radius of $1/2 + \cos(\pi t)/4$. We choose this example as it is just about the simplest case with time dependence.

7.1 Volume term

Recall that we need to integrate over the region $\mathcal{V}(t)$, which is now the infinite strip minus a 2-sphere of varying radius. If we use the first Green's function (5.11) then it is clear that due to the polynomial growth that the volume term is infinite unless there is zero shear, in which case it is zero. Thus, we use the more interesting Green's function (5.12) which gives us a finite volume term. We need to calculate

$$\mathcal{P}_v(\boldsymbol{x}, t) = 2\rho_0\omega^2 S\sigma \left[\iint\limits_{\mathcal{V}(t)} I_1(x, y, x_0, y_0)\mathrm{d}x_0\mathrm{d}y_0 + \iint\limits_{\mathcal{V}(t)} I_2(x, y, x_0, y_0)\mathrm{d}x_0\mathrm{d}y_0 \right], \tag{7.1}$$

where

$$I_1(x, y, x_0, y_0) = G_\lambda(x, y, x_0, y_0)\frac{e^{-i\omega x_0/(U_0+\sigma y_0)}}{(U_0 + \sigma y_0)^2} \text{ and } I_2(x, y, x_0, y_0) = G_\mu(x, y, x_0, y_0)\frac{e^{-i\omega x_0/(U_0+\sigma y_0)}}{(U_0 + \sigma y_0)^2},$$

with G_λ and G_μ defined in (5.13) and (5.14). In Figures 3 and 4 we plot approximations to the real and imaginary parts of I_1 and I_2 for several values of x and y. We fix $\omega = \pi$ and $U_0 = 2$ for the whole of this section, but allow the shear to vary.

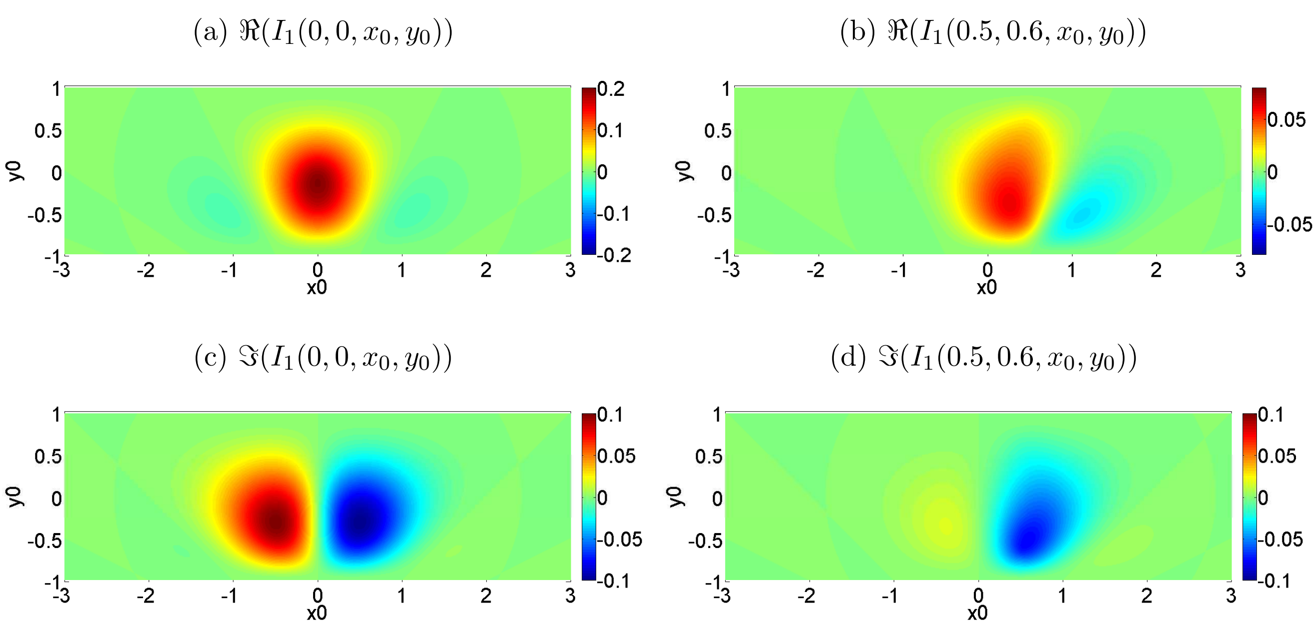


Figure 3: Approximations to the real and imaginary parts of I_1 at $(x, y) = (0, 0)$ and $(0.5, 0.6)$, using the first seven terms of L. We have set $\omega = \pi$, $U_0 = 2$ and $\sigma = 1$.

While we can do the integration with respect to x_0 without too much trouble (we have to split it up in a couple of cases depending on the value of x), the y integration is not possible using

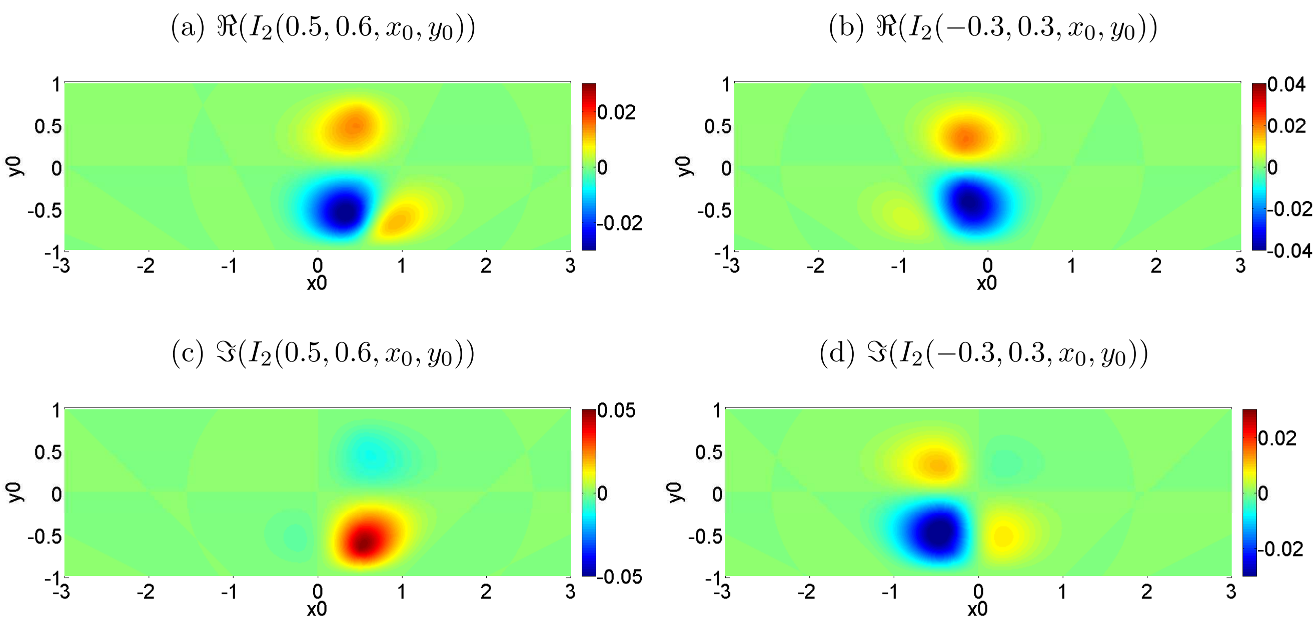


Figure 4: Approximations to the real and imaginary parts of I_2 at $(x, y) = (0.5, 0.6)$ and $(-0.3, 0.3)$, using the first seven terms of M. We have set $\omega = \pi$, $U_0 = 2$ and $\sigma = 1$. We do not plot at $(x, y) = (0, 0)$ since $I_2(0, 0, x_0, y_0) \equiv 0$.

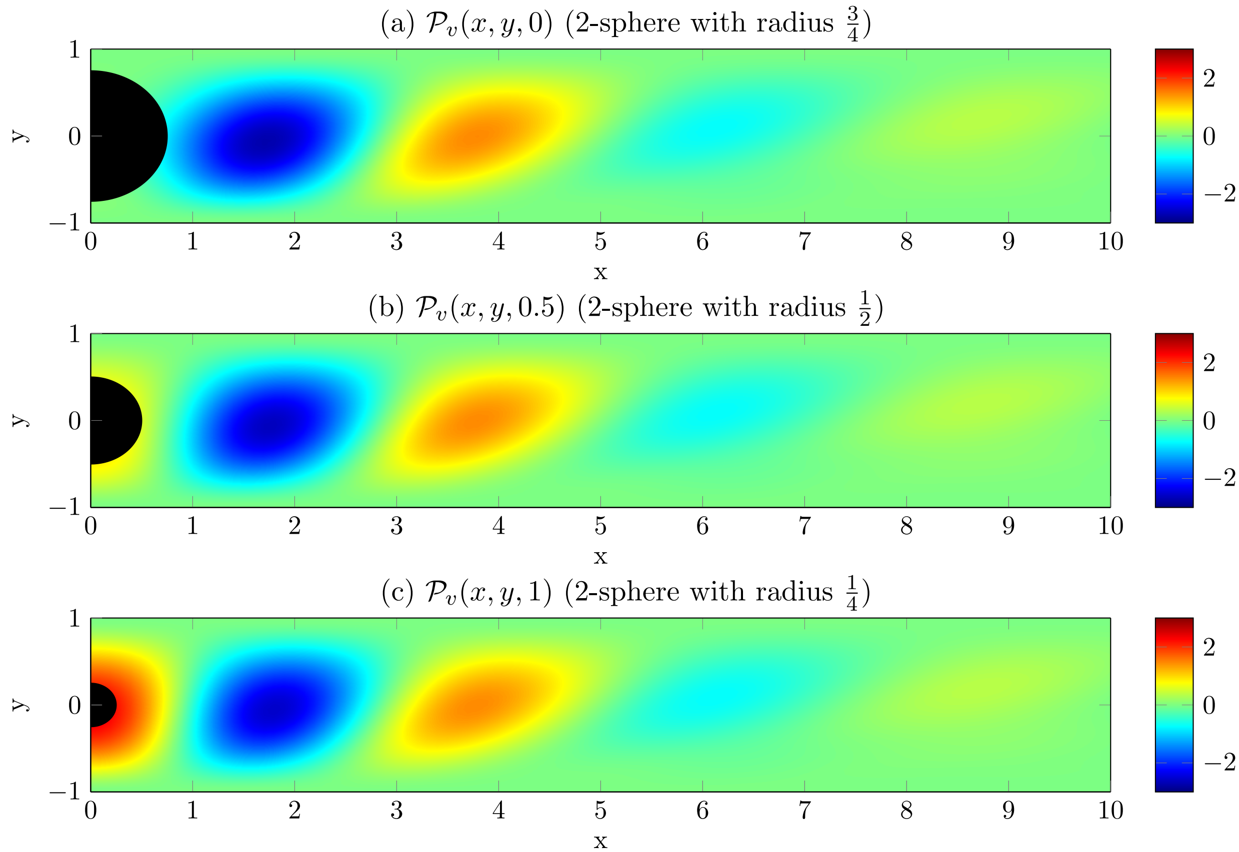


Figure 5: Medium shear. $\mathcal{P}_v(x, y, t)$ (up to a constant of $2\rho_0\omega^2 S$) when $\sigma = 1$ and $U_0 = 2$. For conciseness and since the pressure field is symmetric in x we only include the postive region of x.

conventional functions and instead we have to turn to numerical integration. However, by treating the x_0 integral as an integral across the strip minus the integral over the pulsating 2-sphere, we see that the pulsating 2-sphere integral is proportional to $e^{-0.5\lambda x}$ and thus for large x the effect of the pulsating 2-sphere is negligible. Thus, far enough downstream of the 2-sphere or arbitrary body (since we can enclose any body in some 2-sphere) the volume term does not depend on the shape of the body.

We use a two dimensional numerical method based on quadrature ('quad2d' on Matlab) to evaluate the integrals I_1 and I_2 at each x and y value. The real part of $\mathcal{P}_v$ is plotted up to a constant of $\rho_0 S$ in Figures 5, 6 and 7 for several different times and different shears, which includes the maximum and minimum size of the pulsating 2-sphere. From Figure 5, it is clear that the size of the 2-sphere only affects the region nearby, with the tail of the pressure field unaffected and decaying to zero as we would expect. The central peak and the first trough both clearly change shape as the 2-sphere pulsates, with the first trough getting more elongated as the 2-sphere grows in size, while the central peak reduces in radius and become less elongated as the 2-sphere grows. The first peak out from the centre is pretty much unaffected as the 2-sphere changes size. The shear flow has caused the distortion and antisymmetry in the y direction, and the case of no shear is considered in Figure 6.

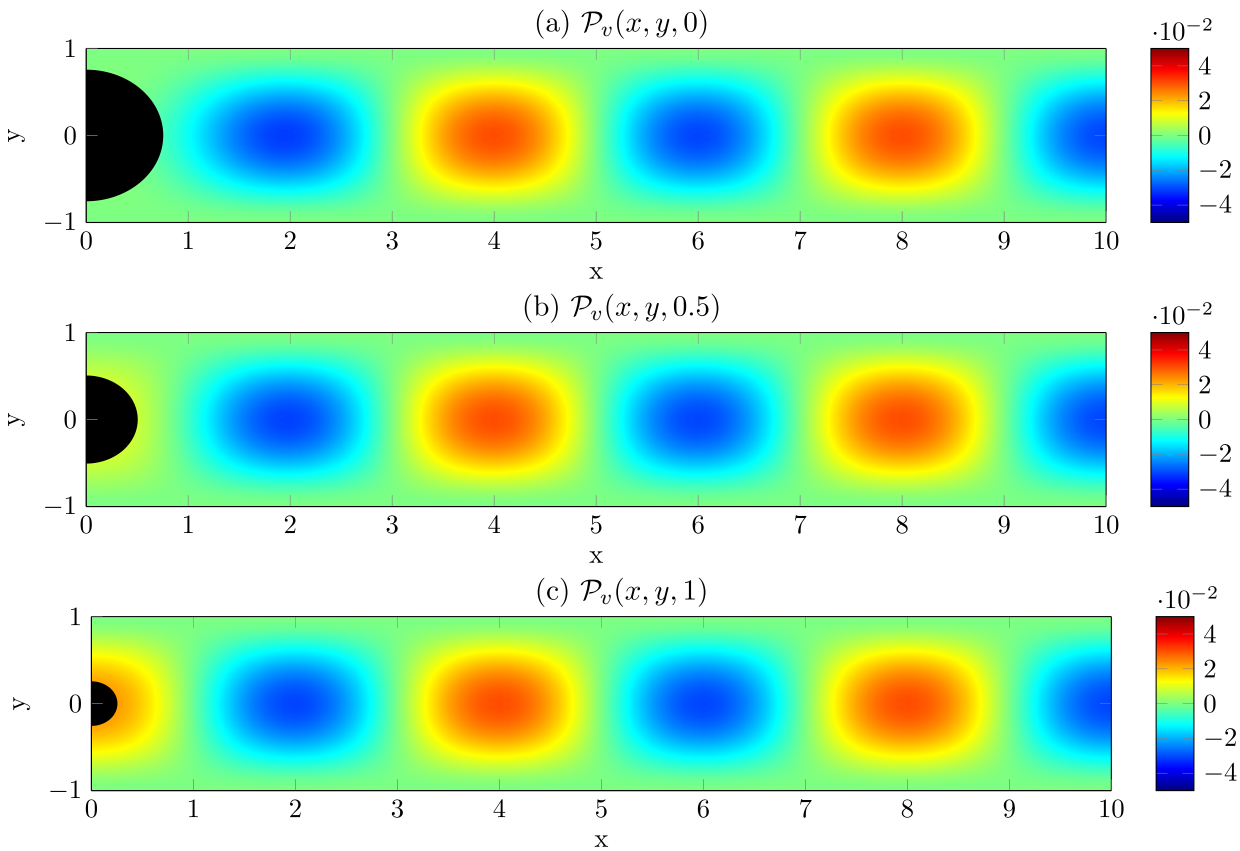


Figure 6: Zero shear. $\mathcal{P}_v(x, y, t)$ (up to a constant of $\rho_0 S$) when $\sigma = 0.01$ and $U_0 = 2$.

With no shear, the pressure volume term is zero, so instead we consider the case when $\sigma = 0.01$ (from now on when we say zero shear we mean $\sigma = 0.01$). As before, we leave out the unknown constant $\rho_0 S$ when we plot $\mathcal{P}_v$, although we have scaled the plot by $\sigma = 0.01$ to compare to Figure 5. The first thing to notice is that the relative decay is a lot slower, and that it is not at all noticeable, because all the terms are small. In fact, if we were to plot on the same axis before we would not notice the troughs or peaks, and it would appear flat. The pattern of the peaks and troughs are the same as in the case with $\sigma = 1$, with the central peak and first trough affected by the pulsating 2-sphere and changing their elongation, although less so. In addition, We do now have symmetry in both the x and y directions.

With a high shear ($\sigma = 1.5$), we get Figure 7, where we have scaled by σ make it directly comparable with the previous figures and is the pressure volume term up to the constant of $2\rho_0\omega^2 S$. One immediate thing we notice is that we have less troughs and peaks as in the two figures before, with each peak and trough more elongated and having a greater 'diameter'. We also notice that the peaks and troughs decay quicker than in Figure 5; the central values of $\mathcal{P}_v$ are larger, while the tail behaviour is very similar and is of a similar absolute size to the tail in Figures 5. We get similar behaviour to before; as the 2-sphere pulses outwards the central peak elongates inwards, and the first trough elongates outwards. It is also more noticeable than before that the first peak also elongates inwards as the 2-sphere grows in size.

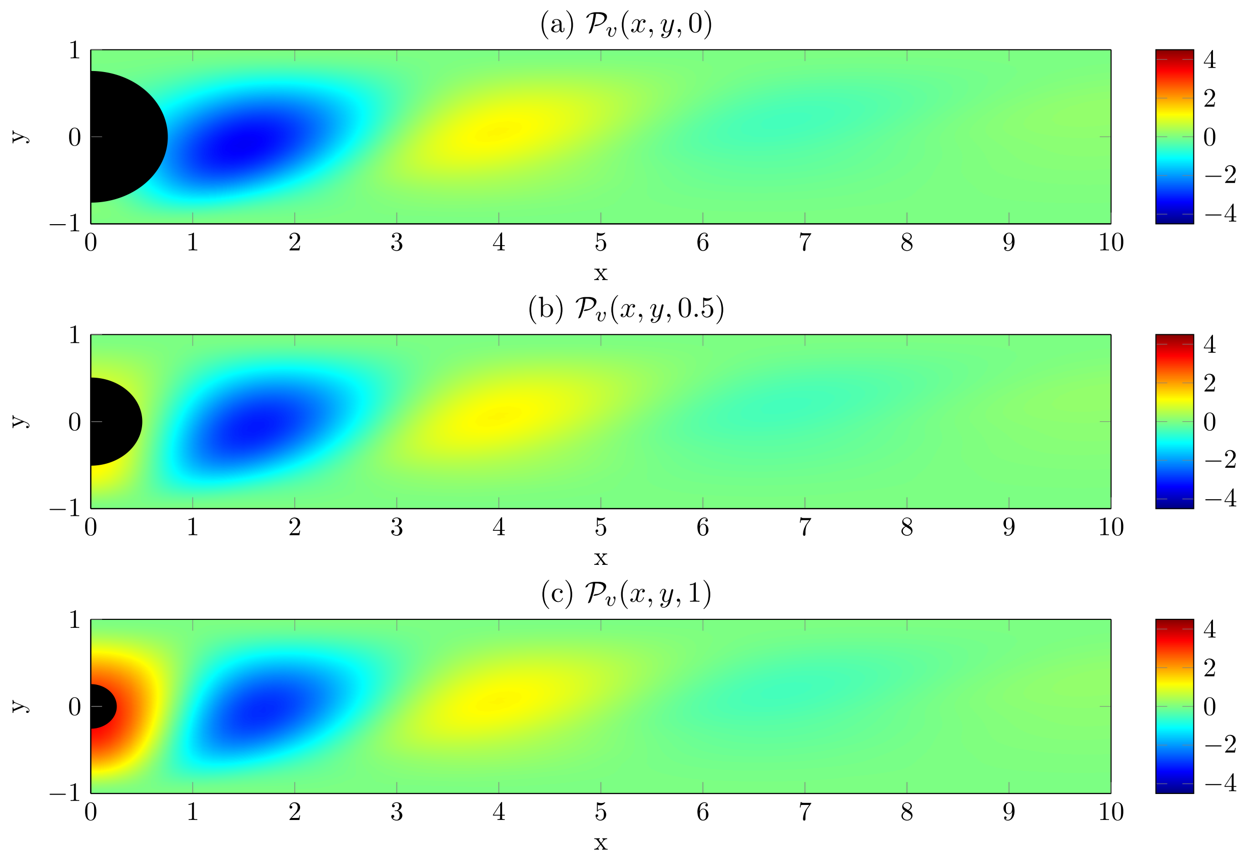


Figure 7: High shear. $\mathcal{P}_v(x, y, t)$ (up to a constant of $\rho_0 S$) when $\sigma = 1.5$ and $U_0 = 2$.

Overall, the effect of the shear is quite apparent. Increasing the shear increases the "diameter" of the peaks and troughs, and reduces the number. It also increases the speed of the relative decay, with hardly any decay when $\sigma = 0.01$ and significant decay at $\sigma = 1.5$. It also significantly elongates the shapes of the peaks and troughs, with nearly circular shapes when $\sigma = 0.01$ to very elongated ellipses at high shear. The numerical computations were performed on a 40×100 grid, and if we were to consider the effect of an arbitrary body rather than a pulsating 2-sphere we would need to increase the number of points to account for the non-uniform shape. We only choose $\sigma = 1.5$ for high shear since when we increase σ any more the integrals we need to calculate take too long to compute at present. By multiplying $\mathcal{P}_v$ by $e^{i\omega t}$ to get the actual pressure volume term, we just get a phase shift of the the above plots, so we omit the details.

7.2 Surface terms with no feedback

Next, we want to compute the surface terms calculated in (6.9) for the pulsating 2-sphere. However, many of these terms involve already knowing what the pressure p or the velocities u and v are over the surface $\Sigma(t)$. We leave the computation of these terms for later, and for now focus on the two vorticity terms that can be computed without already knowing the pressure field or velocity, which are

$$\mathcal{P}_s^3(\boldsymbol{x}, t) := 2\rho_0\sigma i\omega \int\limits_{\Sigma(t)} g\frac{\chi n_1}{U}(x_0, y_0)\mathrm{d}\Sigma(t) \text{ and } \mathcal{P}_s^6(\boldsymbol{x}, t) := 2\rho_0\sigma \int\limits_{\Sigma(t)} \frac{\partial g}{\partial x_0}\chi n_1(x_0, y_0)\mathrm{d}\Sigma(t). \quad (7.2)$$

Surface term 5

We deal with the former of these first, and after parametrising the line integral we need to compute

$$\mathcal{P}_s^5(\boldsymbol{x}, t) = 2\rho_0\omega i\sigma RS \int_0^{2\pi} g(x, y, R\cos\theta, R\sin\theta)\frac{e^{-i\omega R\cos\theta/(U_0+\sigma R\sin\theta)}}{U_0 + \sigma R\sin\theta}\cos\theta\mathrm{d}\theta,$$

where $R(t) = 0.5 + 0.25\cos\pi t$, the radius of the pulsating 2-sphere and $g = G_\lambda + G_\mu$ as before.

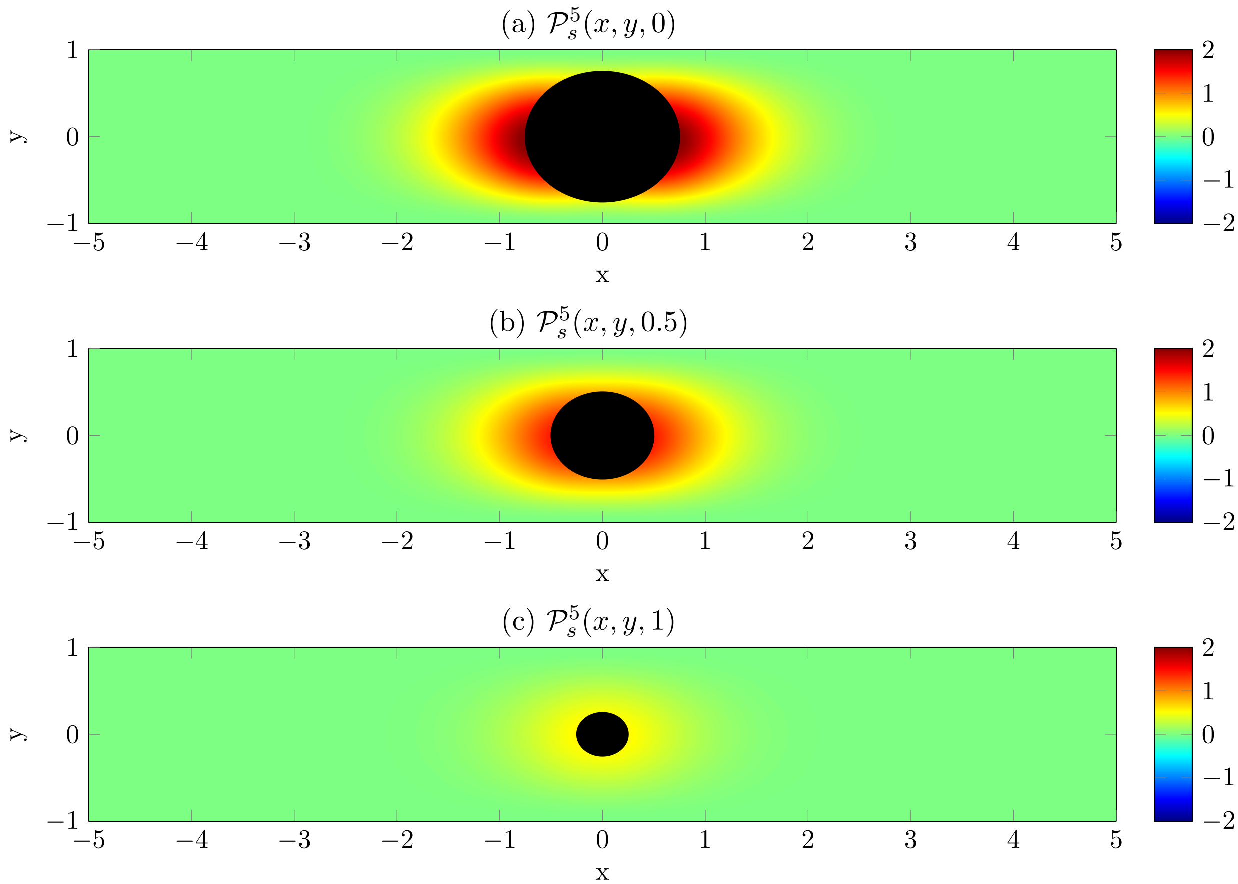


Figure 8: $\mathcal{P}_s^5(x, y, t)$ (up to a constant of $\rho_0 S$) when $\sigma = 1$ and $U_0 = 2$.

Again, we turn to numerical integration, using the routine 'quad' on Matlab. Since we have an i in the front of the integral, we calculate the integral of the imaginary part of the integrand so that we plot the real part of $\mathcal{P}_s^5(\boldsymbol{x}, t)$ (up to a constant of $\rho_0 S$) in Figure 8.

In Figure 8 we can clearly see the effect of the pulsating 2-sphere, as it causes pressure waves to propagate horizontally outwards. We see that the pressure on the boundary of the 2-sphere increases at it pulses outwards, which is what we would expect. It it also noticeable how much the pressure drops as the 2-sphere reduces in size, and this comes from the R term in front of the integral. Although it is not obvious, the figure is not quite symmetric in the y coordinate and this is caused by the shear of the initial flow. By considering high and low shear we can see this more clearly. When we have low shear the flow is symmetrical in the y direction, while at high shear the flow is clearly shifted downwards from the symmetry line. Also, the intensity of the flow scales with σ, and this is the main effect of the shear on this term.

Surface term 6

We now consider the other integral in (7.2), which is

$$\mathcal{P}_s^6(\boldsymbol{x}, t) = 2\rho_0\sigma \int\limits_{\Sigma(t)} \frac{\partial g}{\partial x_0}\chi n_1(x_0, y_0)\mathrm{d}\Sigma(t).$$

We can calculate that for a given λ (or μ) that

$$\frac{\partial}{\partial x_0}e^{-\frac{1}{2}\lambda|x-x_0|} = \begin{cases} \dfrac{1}{2}\lambda e^{-\frac{1}{2}\lambda(x-x_0)} & \text{when } x > x_0 \\ -\dfrac{1}{2}\lambda e^{-\frac{1}{2}\lambda(x_0-x)} & \text{when } x < x_0. \end{cases}.$$

Since we are integrating this expression we do not need to worry about the value at $x = x_0$, but for simplicity we set it to be zero. Thus, it easy to compute $\frac{\partial g}{\partial x_0}$, since we just multiply each λ (or μ) term in the sum by $\frac{1}{2}\lambda\,\mathrm{sgn}(x - x_0)$. We then parametrise the pulsating 2-sphere as before, and thus we need to compute the real part of

$$\mathcal{P}_s^6(\boldsymbol{x}, t) = 2\rho_0\sigma RS \int\limits_0^{2\pi} \frac{\partial g}{\partial x_0}(x, y, R\cos\theta, R\sin\theta)e^{-i\omega R\cos\theta/(U_0+\sigma R\sin\theta)}\cos\theta\mathrm{d}\theta.$$

This is plotted in Figure 9 for a medium shear value, and we see that the size of the 2-sphere has quite an impact on the pressure field. As the 2-sphere increases in size from $1/4$ (Figure 9c) to $1/2$ (Figure 9b) we can see that both the peaks and the troughs of the pressure field expand in size and intensity, with the boundary of the 2-sphere 'catching up' to the boundary of the central trough of pressure. Up to the size of $1/2$, it appears that the pressure field is very close to being symmetric in y, suggesting that the shear doesn't affect this source of noise for small objects. This is further confirmed in Figure 10 which shows that at size $1/2$ (and for smaller sizes) the shear has little effect on the size of the peaks and troughs. We have scaled the pressure field axis by σ in the Figure, and it is very apparent that the main effect of the shear on a small 2-sphere is to change the intensity at a rate proportional to σ.

However, as the 2-sphere expands to a size of $3/4$ (Figure 9a) the shear flow causes it to lose the symmetry as the peaks slowly rotate, without changing size or intensity. We also notice that this pressure surface term rotates in the opposite direction to the pressure volume term, and in addition doesn't stretch the peaks. As the 2-sphere expands, it also fully catches up to the central trough, leaving just a small negative region around the boundary of the 2-sphere. Increasing the shear increases the amount of rotation, while zero shear causes no rotation. We see this in Figure 11. We also see that at high intensity ($\sigma = 1.5$) the intensity at the centre of the peaks has decreased (relatively) and that there is now a larger negative areas of pressure above the 2-sphere than below it.

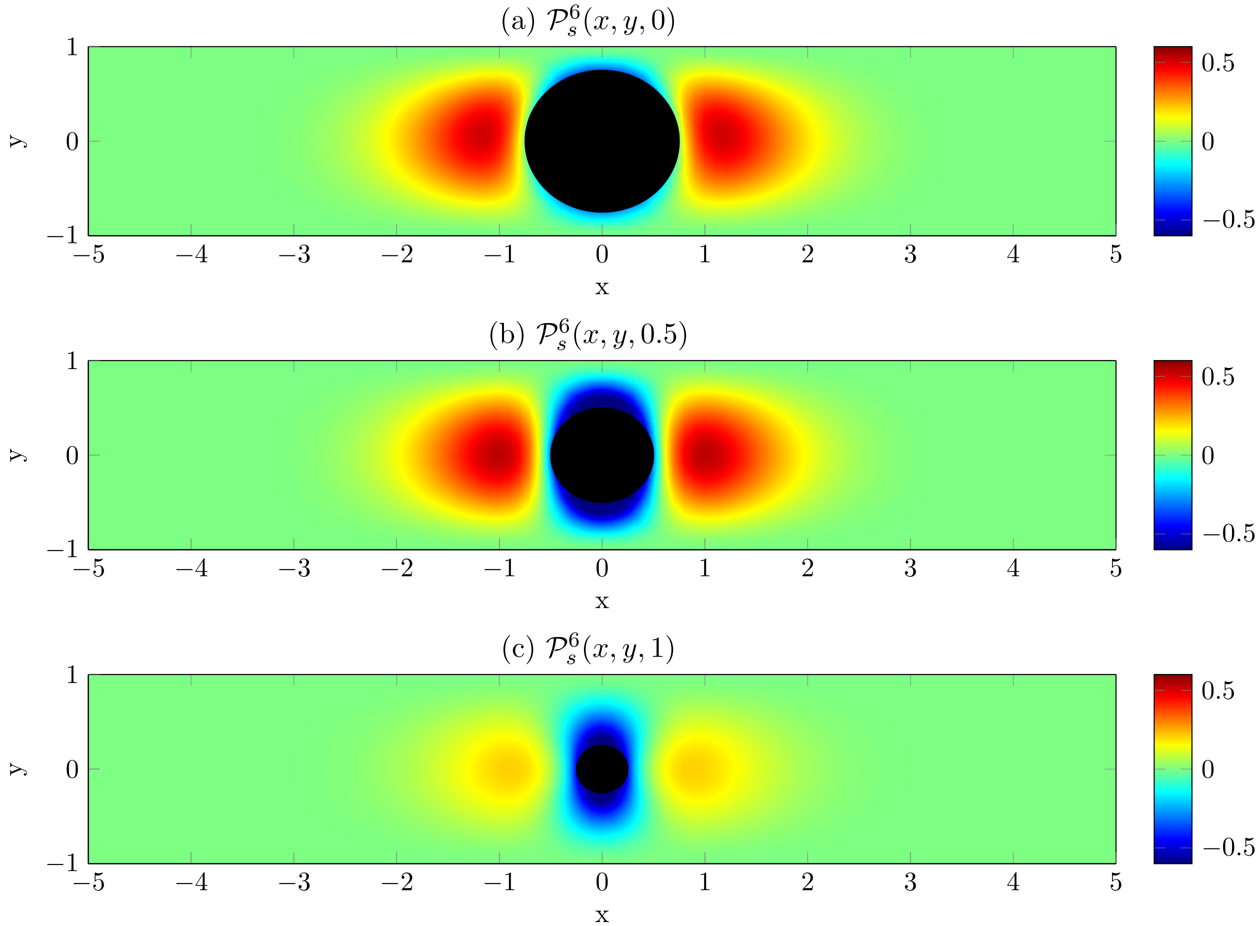


Figure 9: Medium Shear. $\mathcal{P}_s^6(x,y,t)$ (up to a constant of $\rho_0 S$) when $\sigma = 1$ and $U_0 = 2$.

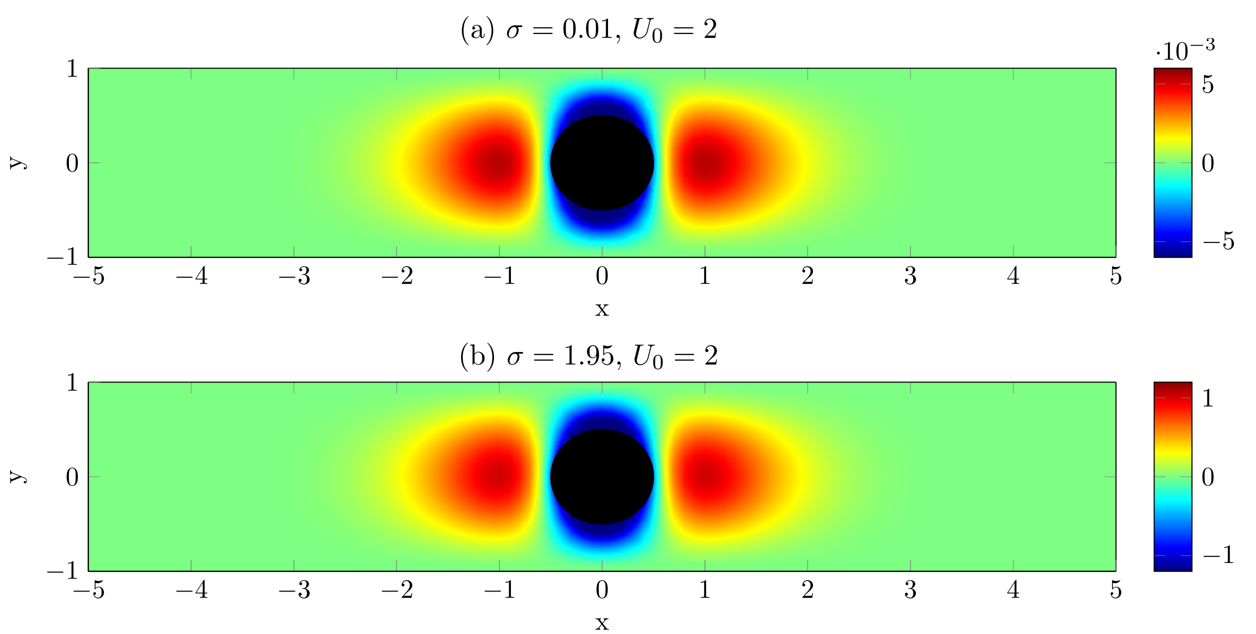


Figure 10: $\mathcal{P}_s^6(x,y,0.5)$ (up to a constant of $\rho_0 S$) for low shear (10a) and high shear (10b).

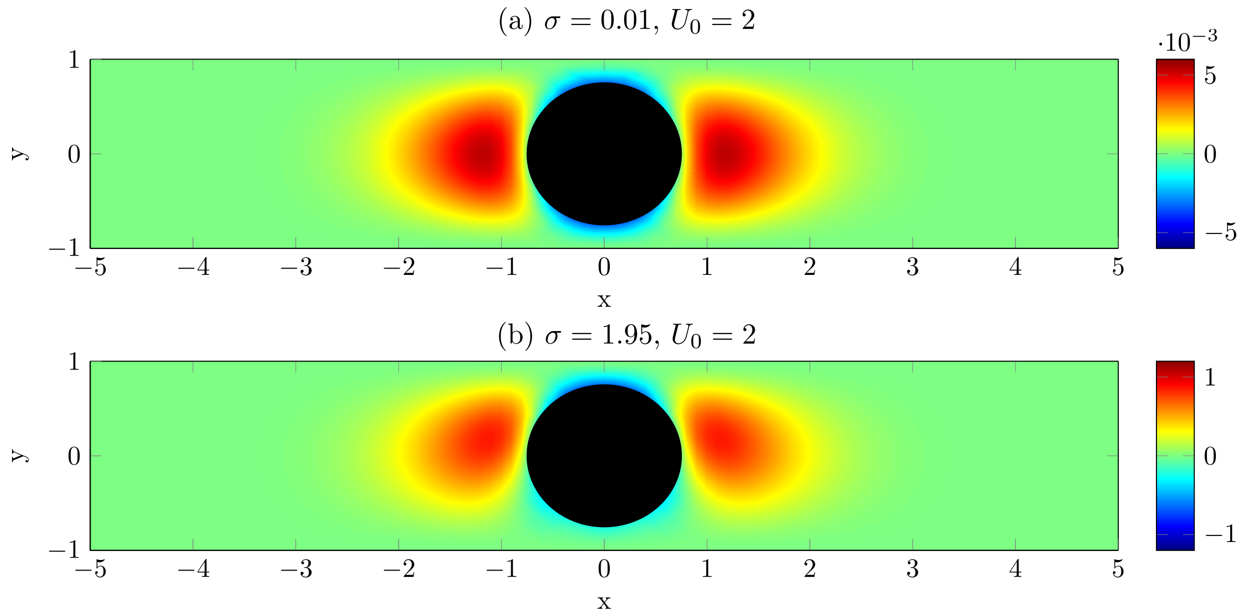


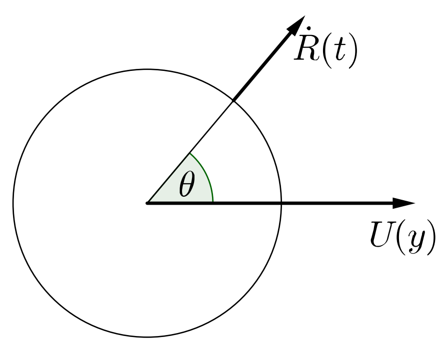
Figure 11: $\mathcal{P}_s^6(x, y, 0)$ (up to a constant of $\rho_0 S$) for low shear (11a) and high shear (11b).

7.3 Surface terms with feedback

Having computing the surface source terms which do not involve feedback (so they do not involve already knowing p, u and v), we turn to the terms involving feedback. The way we will calculate these terms is to 'guess' at an initial solution for p, u and v and then iterate using (6.9), (6.10) and (6.11) until we get convergence to the actual solution. The question of whether we get convergence and at what rate and what sense is to be considered at a later date. We firstly consider the initial guess to a solution.

7.3.1 Initial guess

We assume that close to the boundary of the 2-sphere the fluid will be practically incompressible (as in [Brambley, 2013]) and irrotational, so $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ for some potential $\phi(r, \theta)$. Thus, we need to solve $\Delta\phi = 0$ subject to boundary conditions. Letting $R(t) = 0.5 + 0.25\cos\pi t$ be the boundary of the 2-sphere, then the first boundary condition is



$$\frac{\partial\phi}{\partial r}(R(t), \theta) = \dot{R}(t) + U(y)\cos\theta = \dot{R}(t) + U_0\cos\theta + \sigma R(t)\sin\theta\cos\theta. \tag{7.3}$$

We also impose the boundary condition that ϕ has no more than logarithm growth as $r \to \infty$. Ideally, we would have ϕ decaying to zero but there is no solution to $\Delta\phi = 0$ and the first boundary condition which decays to zero. After letting $\phi(r, \theta) = f(r)g(\theta)$ we need to solve

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta) = 0,$$

and thus we can compute that the solution is

$$\phi(r,\theta,t) = \dot{R}R\log r - \frac{U_0 R^2\cos\theta}{r} - \frac{\sigma R^4\sin\theta\cos\theta}{2r^2}.$$

From this we can calculate that

$$u(r,\theta,t) = \frac{\partial\phi}{\partial r}\cos\theta - \frac{\partial\phi}{\partial\theta}\frac{\sin\theta}{r}, \tag{7.4}$$

and

$$v(r,\theta,t) = \frac{\partial\phi}{\partial r}\sin\theta + \frac{\partial\phi}{\partial\theta}\frac{\cos\theta}{r}, \tag{7.5}$$

while using Bernoulli's equation we get that

$$p(r,\theta,t) = f(t) - \rho_0\frac{\partial\phi}{\partial t} - \frac{\rho_0}{2}|\boldsymbol{\nabla}\phi|^2 = f(t) - \rho_0\frac{\partial\phi}{\partial t} - \frac{\rho_0}{2}\left(\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right), \tag{7.6}$$

where f is some function of time. We also calculate that the θ and t derivatives at $r = R$ (since we will perform a line integral around the circle of radius R) are given by

$$\frac{\partial\phi}{\partial\theta}(R,\theta) = U_0 R\sin\theta - \frac{\sigma R^2\cos 2\theta}{2},$$

and

$$\frac{\partial\phi}{\partial t}(R,\theta) = (\dot{R}^2 + R\ddot{R})\log R - 2\dot{R}U_0\cos\theta - 2\sigma R\dot{R}\sin\theta\cos\theta.$$

7.3.2 Derivatives of the Green's function

Next, we need to compute some more derivatives of the Green's functions for the surface terms. Firstly, we can compute that $\Delta_0 g = \Delta_0 G_\lambda + \Delta_0 G_\mu$, where for $x \neq x_0$

$$\Delta_0 G_\lambda(x,y,x_0,y_0) = 4\pi\sum_{\lambda\in L}\Re\left(\frac{\tan^2\left(\frac{1}{2}\lambda\right)}{\lambda^3}E_\lambda(y)H_\lambda(y_0)e^{-\frac{1}{2}\lambda|x-x_0|}\right), \tag{7.7}$$

and

$$\Delta_0 G_\mu(x,y,x_0,y_0) = 4\pi\sum_{\mu\in M}\Re\left(\frac{\cot^2\left(\frac{1}{2}\mu\right)}{\mu^3}F_\mu(y)K_\mu(y_0)e^{-\frac{1}{2}\mu|x-x_0|}\right), \tag{7.8}$$

with

$$H_\lambda(y_0) = \lambda\left[\cos\left(\frac{\lambda}{2}(y_0+1)\right) + \cos\left(\frac{\lambda}{2}(y_0-1)\right)\right],$$

and

$$K_\mu(y_0) = \mu\left[\cos\left(\frac{\mu}{2}(y_0+1)\right) - \cos\left(\frac{\mu}{2}(y_0-1)\right)\right].$$

One thing to notice is that the Laplacian of the Green's functions is zero on the boundary, so $\Delta_0 g(x,\pm1,x_0,y_0) = 0$. We compute the derivatives $\boldsymbol{\nabla}_0(\Delta_0 g)$ and $\boldsymbol{\nabla}_0(\frac{\partial g}{\partial x_0})$ in a similar way, and it is clear when we calculate these that the derivatives are also zero on the boundary $y = \pm1$. We now calculate the first approximation to the remaining pressure terms, using (7.4), (7.5) and (7.6).

7.3.3 Calculation of first approximation to remaining pressure terms

Surface term 1

The first feedback surface term we deal with is the one that involves the pressure, which is

$$\mathcal{P}_s^1(\boldsymbol{x}, t) := -\int_{\Sigma(t)} \boldsymbol{\nabla}_0(\Delta_0 g) \cdot p\boldsymbol{n}(x_0, y_0)\mathrm{d}\Sigma(t).$$

We can calculate the gradient of the Laplacian of the Green's function using (7.7) and (7.8), and then we can parametrise $\mathcal{P}_s^1(\boldsymbol{x}, t)$. As in all of the surface terms we compute it to a constant, which is ρ_0, and this comes from (7.6). Compared to the other source terms we also have the additional choice of the function f in (7.6). We choose this f such that $\mathcal{P}_s^1(\boldsymbol{x}, t)$ is not too big, and has some symmetry. In Figure 12 we can see this source term plotted for different values of time and $f(t) = 1.825 + 0.675\cos(\pi t)$, which was experimentally chosen. The choice of $f(t)$ is definitely something to be investigated, and the choice may influence the convergence of the iterative scheme for finding the final value of the pressure surface source term.

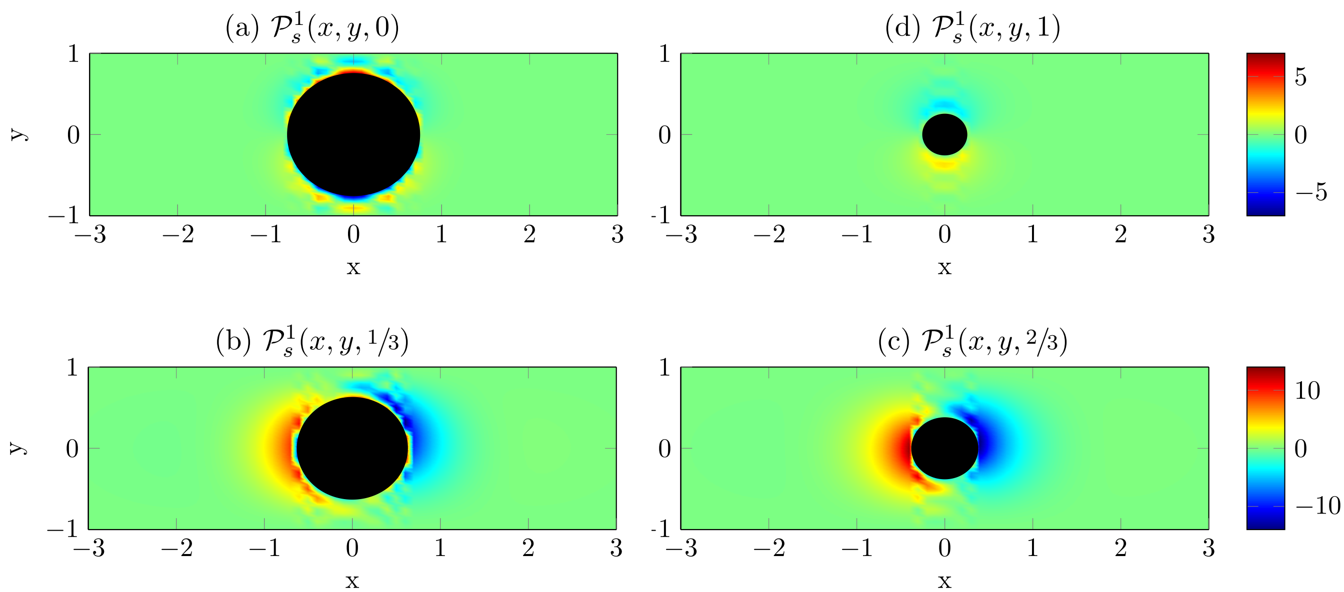


Figure 12: First iteration for $\mathcal{P}_s^1(x, y, t)$ (up to a constant of ρ_0) for $\sigma = 1$ (medium shear) and $U_0 = 2$. At times $t = 0$ and $t = 1$ we have the intensity in the interval $[-7, 7]$ while at times $t = 1/3$ and $t = 2/3$ we have the intensity in the interval $[-14, 14]$.

We can see that initially we have areas of high pressure on the boundary above the 2-sphere and areas of low pressure on the boundary below the 2-sphere. Slightly further away from the boundary, we have small bubbles of pressure, with low pressure bubbles near the high pressure boundary and vice versa. These bubbles are also of a lower intensity than the pressure at the boundary of the 2-sphere. As the 2-sphere contracts the areas of high and low pressure on the boundary rotate clockwise by $\pi/2$ and stay on the boundary. The bubbles of high and low pressure also rotate clockwise by $\pi/2$, but they all join together and increase in intensity significantly.

As we reach time $2/3$, the bubbles of high and low pressure continue to increase in intensity and rotate a little more, although they reduce in size. Finally, as the 2-sphere contracts to its smallest size, the bubbles of pressure and the pressure on the boundary rotate and join up, with high pressure below the 2-sphere and low pressure above it. Ideally, we would perform these calculations with more grid points to get more accurate images, but we leave this task for a later date and more powerful computer. Due to the complicated nature of this term, which depends heavily on $f(t)$, we do not investigate the effect of high and low shear since changing the shear changes $f(t)$.

Surface term 2

The term we deal with next involves the base flow U and the gradient of the velocity, and is given by

$$\mathcal{P}_s^2(\boldsymbol{x}, t) := \rho_0 \int\limits_{\Sigma(t)} \Delta_0 g \left[U \boldsymbol{n}^\perp \cdot \boldsymbol{\nabla} v\right](x_0, y_0) \mathrm{d}\Sigma(t).$$

We calculate that $\boldsymbol{\nabla} v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$, where

$$\frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial r^2}\frac{\sin 2\theta}{2} + \frac{\partial^2 \phi}{\partial \theta \partial r}\frac{\cos 2\theta}{r} - \frac{\partial^2 \phi}{\partial \theta^2}\frac{\sin 2\theta}{2r^2} - \frac{\partial \phi}{\partial r}\frac{\sin 2\theta}{2r} - \frac{\partial \phi}{\partial \theta}\frac{\cos 2\theta}{r^2},$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial r^2}\sin^2\theta + \frac{\partial^2 \phi}{\partial \theta \partial r}\frac{\sin 2\theta}{r} + \frac{\partial^2 \phi}{\partial \theta^2}\frac{\cos^2\theta}{r^2} + \frac{\partial \phi}{\partial r}\frac{\cos^2\theta}{r} - \frac{\partial \phi}{\partial \theta}\frac{\sin 2\theta}{r^2}.$$

In future iterations, we would need to calculate this derivative numerically since we will no longer have an explicit formula for v. After parametrising this integral and then computing it numerically

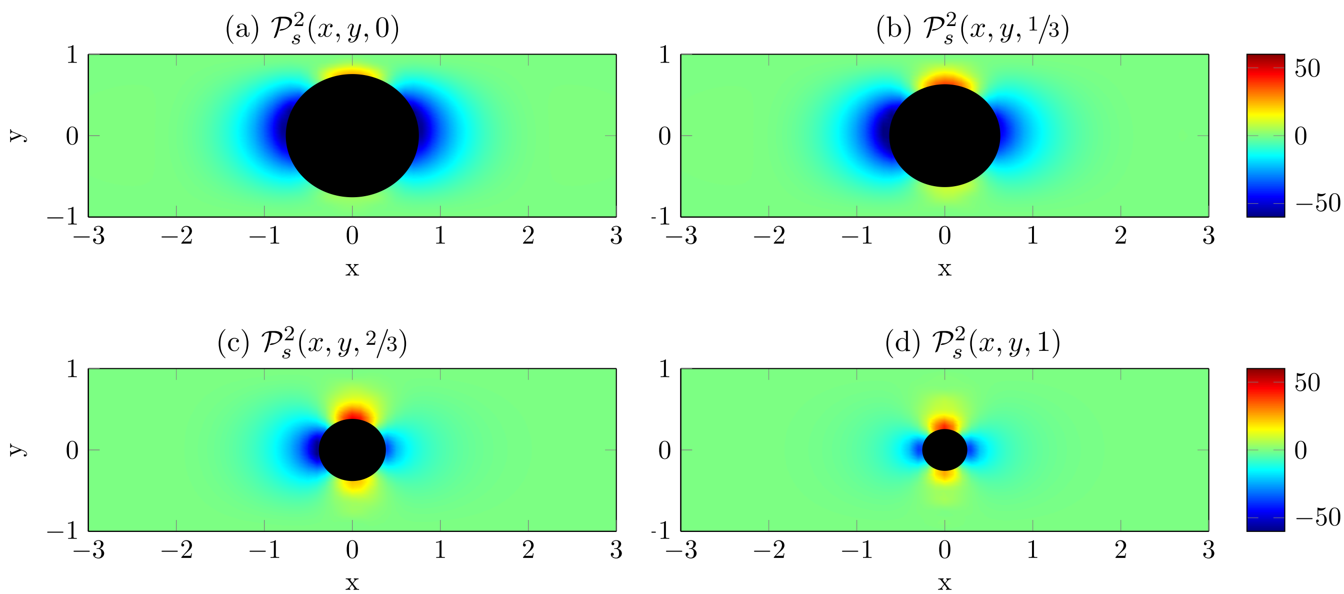


Figure 13: First iteration for $\mathcal{P}_s^2(x, y, t)$ (up to a constant of ρ_0) for $\sigma = 1$ (medium Shear) and $U_0 = 2$.

we get Figure 13. We firstly note that that the intensity of the pressure field is significantly higher than the other pressure terms we have encountered so far, and we consider the relative size of each term in 7.4. We start of with a small patch of high pressure above the 2-sphere, and very high intensity areas of low pressure to either side. As the 2-sphere contracts, the right area of low pressure reduces in size, while the areas of high pressure increase, with the pressure at the bottom of the 2-sphere now visible. As we reach time 2/3 both areas of low pressure have shrunk in size, although the right area of pressure is still smaller, while the areas of high pressure continue to grow. When the 2-sphere is at its smallest the areas of high and low pressure are roughly the same size, with the two areas of lower pressure the same size while the top area of high pressure is larger than the bottom one.

Unlike before, this pressure source term depends not only on the radius of the 2-sphere, but also whether it is expanding or contracting. Thus, the pressure field when it is expanding will be different to the pressure field when it is contracting, as shown in Figure 15. It turns out that when we expand rather than contract, we just reverse the symmetry in the x direction, so the right area of low pressure would be bigger than the left area at times $1^1/_3$ and $1^2/_3$. The shear, as usual, affects the symmetry in the y direction, and with zero shear the pressure field would be symmetric in the y direction, while high shear causes less symmetry, with the top area of high pressure significantly larger than the bottom area.

Surface term 3

The next term we deal with is

$$\mathcal{P}_s^3(\boldsymbol{x},t) := -i\rho_0\omega e^{-i\omega t}\int\limits_{\Sigma(t)}(\Delta_0 g)\boldsymbol{u}\cdot\boldsymbol{n}(x_0,y_0)\mathrm{d}\Sigma(t).$$

However, our first approximations to u and v are real valued, as is $\Delta_0 g$, so $\Re(\mathcal{P}_s^3)=0$. However, it is clear from the expressions of u (6.10) and v (6.11) that future iterations will involve a real and imaginary part, and thus for future iterations we would need to calculate the real part of $\mathcal{P}_s^3(\boldsymbol{x},t)$, which would be given by

$$\Re\left(\mathcal{P}_s^3(\boldsymbol{x},t)\right) = \rho_0\omega R\int\limits_0^{2\pi}(\Delta_0 g)(x,y,R\cos\theta,R\sin\theta)\Im(u\sin\theta+v\cos\theta)\mathrm{d}\theta.$$

Surface term 4

The penultimate term we deal one is

$$\mathcal{P}_s^4(\boldsymbol{x},t) := -2\rho_0\sigma\int\limits_{\Sigma(t)}\boldsymbol{\nabla}_0\left(\frac{\partial g}{\partial x_0}\right)\cdot\left(u\boldsymbol{n}^\perp+v\boldsymbol{n}\right)(x_0,y_0)\mathrm{d}\Sigma(t).$$

We can compute the necessary derivatives of the Green's function in a similar manner to before, and once we have these derivatives we can parametrise the integral. In Figure 14 we see the effect of this source term. Initially, we start of with areas of high and low pressure on the boundary, as well as two bubbles of high pressure and two of low pressure. We notice again that we have antisymmetry in the y direction, which is caused by the shear. This is the main effect of the shear, apart from the intensity scaling with the shear, and we do not plot any figures for this term illustrating the relatively minor role of the shear.

As the 2-sphere contracts the left hand bubbles increase slowly in intensity until time $^2/_3$, and then they decrease in intensity. The right hand bubbles just decrease in intensity as the 2-sphere contracts, and by the time the 2-sphere is at its smallest the left and right hand bubbles are the same intensity, and have also both rotated. The more interesting part of the pressure field is the vertical wave caused by the 2-sphere contacting, which comes from the areas of high and

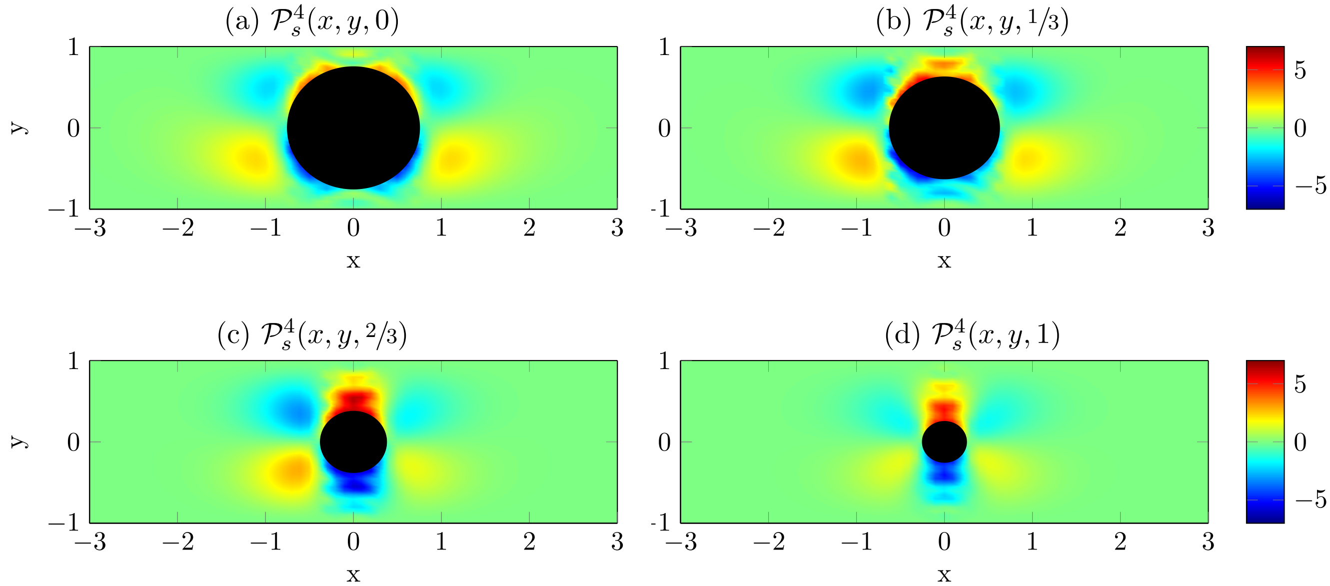


Figure 14: First iteration for $\mathcal{P}_s^4(x, y, t)$ (up to a constant of ρ_0) for $\sigma = 1$ (medium Shear) and $U_0 = 2$.

low pressure on the boundary. This is the first surface term causing a wave like motion in the vertical direction as opposed to the horizontal direction. When the 2-sphere expands rather than contracts, we reverse the symmetry in the x direction, so as before we avoid plotting the figure here.

Surface term 7

The last term that we calculate is the last term involving $\Delta_0 g$, and the last one that directly scales with shear, and is given by

$$\mathcal{P}_s^7(\boldsymbol{x}, t) := \rho_0 \sigma \int\limits_{\Sigma(t)} (\Delta_0 g) n_1 v(x_0, y_0) \mathrm{d}\Sigma(t).$$

After parametrising this integral we see the effect of this term on the pressure in Figure 15. Initially, when the 2-sphere is largest at time 0, we can see two bubbles of high pressure below the 2-sphere and two bubbles of low pressure above, which are opposite in intensity. We also notice that that are not the same size, and this antisymmetry in the y direction is present at all times for this surface term.

As the 2-sphere contracts the left areas of high and low pressure initially stay the same size and intensity, but the right areas reduce in size and intensity, causing the pressure field to lose the symmetry in the x direction. By time $2/3$, all the areas have reduced in size and intensity, although the left hand areas are still larger in size. Also, the bubbles of high pressure have joined into a single bubble of high pressure, and the same for the low pressure. As the 2-sphere reaches its smallest size, we can see that we have one small bubble of high pressure, and one small bubble of low pressure, and they are approximately equal in size, and we nearly have symmetry in the x direction. As usual, when the sphere expands rather than contract, we just reverse the symmetry in the x direction, so we do not plot it here.

We also investigate the effect of the shear, and it turns out that the effect of shear is pretty minimal on this pressure term, apart from the intensity scaling with σ. However, we do notice

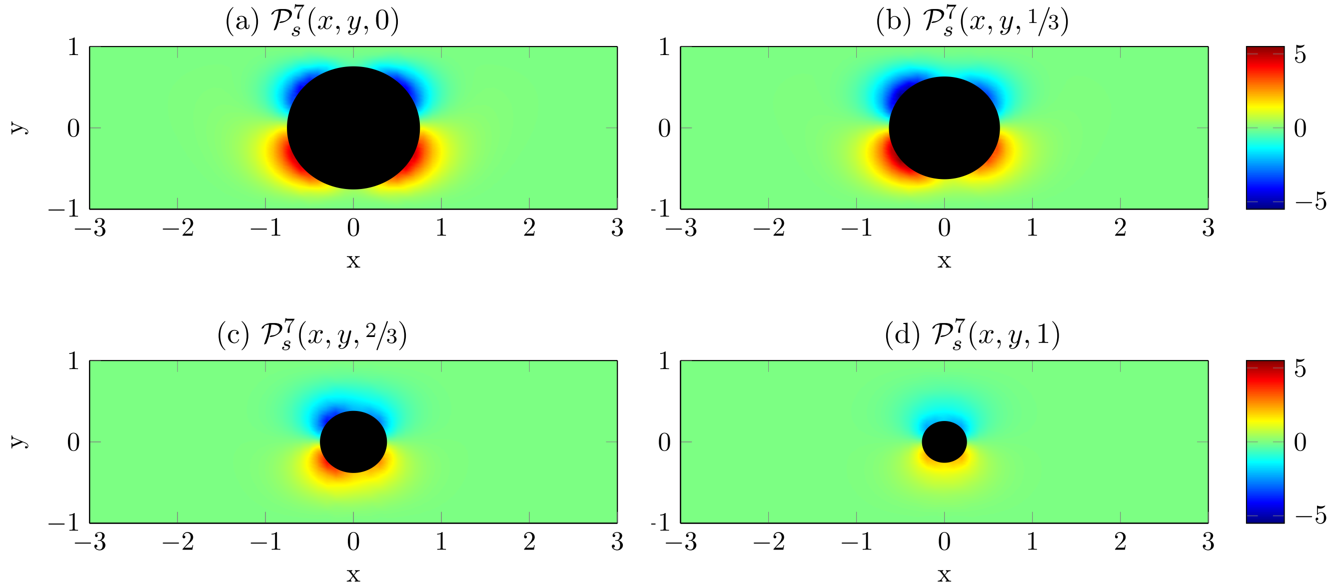


Figure 15: First iteration for $\mathcal{P}_s^7(x, y, t)$ (up to a constant of ρ_0) for $\sigma = 1$ (medium shear) and $U_0 = 2$.

that at high shear values the pressure field is no longer antisymmetric in the y direction (it is antisymmetric when there is no shear), with the positive pressure field higher than it should be for antisymmetry.

7.4 Summary

Finally, all the work in Sections 7.2 and 7.3 has been to calculate the value of $\mathcal{P}_s(\boldsymbol{x}, t)$. To calculate the effect of $\mathcal{P}_s(\boldsymbol{x}, t)$ on the pressure, we need to multiply by $e^{-i\omega t}$ which is just a phase shift. As we explained earlier, for the terms involving u, v and p we have found the first approximation $p^{(1)}$ to the pressure field term (which we get by summing up the seven surface terms). We would also calculate the first iteration of the velocity terms $u^{(1)}$ and $v^{(1)}$ by feeding in our initial guess into (6.10) and (6.11). We would then feed in $p^{(1)}$, $u^{(1)}$ and $v^{(1)}$ into (6.9) and the equations for u and v to get $p^{(2)}$, $u^{(2)}$ and $v^{(2)}$. We would carry on this process until we (hopefully) get convergence. This task was too computationally intensive to be performed now, and could be done at a later date. For now we compare the first approximations to the pressure field.

The constants ρ_0 and S prevent us from comparing the magnitude of the terms directly, but before multiplying by constants, surface term 2 is significantly bigger than all the other terms. Fixing for example $\sigma = 1$ and $S = 20$ would make the volume term of a similar size to surface term 2 and surface term 5 would not be too far behind in size. Surface term 7 is approximately ten times smaller than surface term 2 at these constants, so is fairly insignificant. Surface term 6 is also not particularly significant, since it is approximately five times smaller than surface term 2, while surface term 4 is of a similar magnitude to surface term 7. The next step would be to fix the constants ρ_0 and S and plot the sum of all the pressure volume and surface terms.

Further Work

Further work in this area could be achieved in a number of ways. Firstly, instead of the pulsating 2-sphere we could look at a more aeronautical relevant shape such as a two dimensional fan. This would increase the complexity because parametrising the boundary of the fan to compute the pressure surface terms would require piecewise parametrisations. Also, finding the initial approximation to the velocities u, v and pressure p would be a lot harder.

Next, instead of an infinite strip we could look at a tapered strip or a circular region, or possibly even curved ducts. This would place considerable emphasis on finding the Green's function in the region, since we have new boundary conditions, and from then it would be similar analysis, although clever numerical schemes might need to be implemented.

Another possible direction would be to consider quadratic shear or similar, which increases the complexity because the Vorticity equation in Section 4.1 is no longer trivial to solve. This would then give a different differential operator which the pressure field solves, and then we would need to find the Green's function of another operator. We could also consider other two dimensional examples which are not shear flows, which again changes the differential operator which the pressure field satisfies. Finally, We could incorporate temperature and entropy into the model by adding the Energy equation to the mass, momentum and vorticity equations.

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