Two-point functions in defect versions of $\mathcal{N} = 4$ super Yang Mills theory

Master thesis in theoretical physics

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Abstract

We study various two-point functions in certain defect versions of $\mathcal{N} = 4$ super Yang Mills theory. These defect theories are obtained by insertion of a D7 probe-brane, with either $AdS_4 \times S^2 \times S^2$ or $AdS_4 \times S^4$ geometry, into the standard D3 brane configuration of AdS / CFT. The $\mathcal{N} = 4$ SYM theories, arising from the decoupling limit of these brane configurations, have non-zero vacuum expectation values (*vevs*) for the scalar fields ϕ_i. These non-zero vevs breaks super symmetry completely and conformal symmetry partially, thus presenting us with an interesting opportunity to make non-trivial tests of the AdS / CFT duality.
We focus first on two-point functions with $SO(3) \times SO(3)$ symmetric vevs, between chiral primary operators of the forms $\operatorname{tr} Z^L$, $\operatorname{tr} \bar{Z}^L$, $\operatorname{tr} X^L$, where $X = \phi_1 + i\phi_4$, $Y = \phi_2 + i\phi_5$ and $Z = \phi_3 + i\phi_6$. By use of pertubative methods, we were able to reduce the connected tree-level contributions to these two-point functions, down to expressions involving complicated infinite sums. These infinite sums unfortunately seem unevaluable in general. However, for specific values of L and parameters associated to the stabilization of the brane configurations, we were able to evaluate the sums explicitly.
We also study two-point functions, first with $SO(3) \times SO(3)$ symmetric vevs, between short scalar operators $\mathcal{O}_{W_1 W_2} = \operatorname{tr}[W_1 W_2]$ with scalars $W_1, W_2 = X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}$, and Bethe state operators $\mathcal{O}_L = \Psi_M^{i_1 \ldots i_L} \operatorname{tr}[V_{i_1} \cdots V_{i_l}]$, with $V_i = X, Z$ and Ψ_M being a Bethe wavefunction with M excitations. By use of integrability techniques, we find that certain choices of W_1, W_2 allows for the tree-level contribution to these two-point functions to be expressed in terms of the tree-level value of $\langle \mathcal{O}_L \rangle$. The computations of these various types of two-point functions provide the first step towards a very non-trival check of the AdS / CFT duality. We hope that future work will enable us to complete this endeavor, by studying the corresponding objects on the gravity side of the duality.

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1 Introduction

It is hardly a controversial statement, that one of the most challenging problems of modern high energy physics, is that of finding a model of gravity which is also consistent with the laws of quantum mechanics. Altough no complete theory of quantum gravity currently exists, some incomplete candidate theories do exist, of which the most well known and most extensively studied is probably *string theory*. Although we do not yet have a complete description of quantum gravity, we have nevertheless been able to obtain valuable insights, particularly by studying *black holes* in the settings of classical and semi-classical general relativity. For example, it was proposed by Bekenstein and later confirmed by Hawking, that black holes have entropy proportional to their event horizon area, and not their volume which we would naively expect. This lead a number of people, including 't Hooft and Suskind, to suggest that black holes, and by extension quantum gravity in any region of spactime, might be described by a theory on the boundary of the region in question [1]. This idea is now known as *holographic duality*. In 1998, Jaun Maldacena found the first explicit realization of holographic duality [2], by considering N coincident $D3$-branes in so called *type IIb super string theory*. Using this setup, Maldacena was able to show that a theory of closed type IIb strings on an $AdS_5 \times S^5$ background, is equivalent to a gauge theory with degree $\mathcal{N} = 4$ super symmetry and gauge-group $U(N)$ on a standard M_4 background. The gauge theory in question is the so called $\mathcal{N} = 4$ *super Yang Mills theory* (*SYM theory for short*). This particular realization of holographic duality is known as the *AdS / CFT correspondence*.

The discovery of the AdS / CFT correspondence has been the catalyst of a great deal of research, the result of which has lead to big advances in many areas of physics, such as: high energy particle physics, black hole physics, condensed matter physics and more. Even though the discovery of the AdS / CFT correspondence has been hughly impactful, there are several features of the origial setup which are considered undesirable for different reasons. One of these features, which will serve as part of the motivation for this thesis, is the supersymmetric nature of the theories involved. Super symmetry (*which is a symmetry that relates bosonic and fermionic degrees of freedom*) has, at the time of writing this thesis not been oberved in nature to any degree. The search for super symmetry in the standard model has in fact already been caried out to very high energies [4], which means that if supersymmetry is indeed a symmetry of nature, it would have to be a badly broken one. Knowing this, it would be very interesting if we could somehow study a less super symmetric version of the AdS / CFT correspondence. It turns out that this can indeed be done, and in a number of different ways. The aim of this thesis will be to study a certain subset of these less supersymmetric setups from the gauge theory side of the correspondence.

Before describing in greater detail what exact field theoretic quantities will be investigated in this thesis, we first provide the exact AdS / CFT framework that we will be working in. The core idea is to modify the original setup by introducing a so called *probe bane*, or in other words, a brane whose interactions are not strong enough to affect the resulting $AdS_5 \times S^5$ background. The purpose of this probe brane will primarily be to provide a place for the $D3$-branes to terminate. In general, we will now be able to have a different number of $D3$-branes on either side of the probe brane, which result in a corresponding field theory dual with two different gauge groups on either side of a domain wall. To be more specific, if we have N coincident $D3$-branes on one side of the probe brane and $N - d$ coincident $D3$-branes on the other, we end up with dual $\mathcal{N} = 4$ SYM field theories with gauge groups $U(N)$ and $U(N - d)$ on either side of a domain wall. These domain walls are examples of what is known as defects, and the field theories to which they belong are naturally classified as types of defect field theories. Because $\mathcal{N} = 4$ SYM theory is invariant under 4D conformal symmetry, we will refer to the field theory duals of the probe brane setups as *defect conformal field theories* (*dCFTs for short*). A sketch of the D-brane setup and dual field theory setup can be seen in figure 1. As is also indicated in the aforementioned figure, we choose coordinates such that the co-dimension one defect is located at $x_3 = 0$, and such that the gauge group is $U(N)$ for $x_3 > 0$.

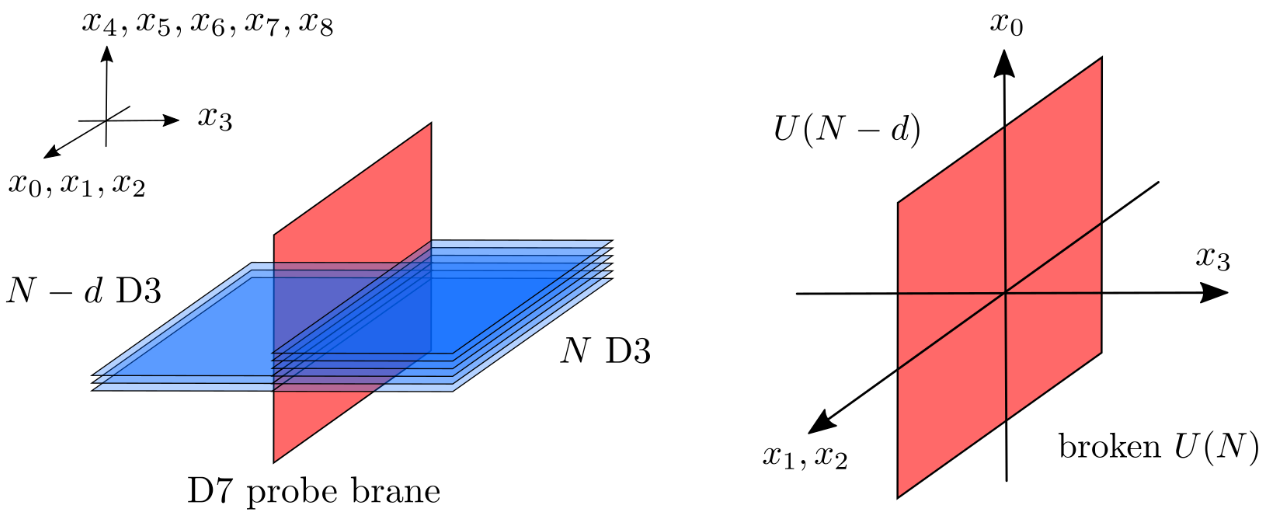

Figure 1: The brane configuration in string theory on the left vs. the dual field theory configuration, with different gauge groups on each side of the defect at $x_3 = 0$, on the right. This figure has been recreated from [11].

By varying the dimensionality and geometry of the probe brane, we obtain different supersymmetry breaking AdS / CFT setups. The setup obtained by introducing a D5 brane with $AdS_4 \times S^2$ geometry has already been extensively studied. See for example [9, 10, 16, 18, 19]. What will be the focus of this thesis, are the setups obtained by introducing a D7 brane, with geometry given by either $AdS_4 \times S^2 \times S^2$ or $AdS_4 \times S^4$. In order for these setups to be stable, one has to add either external gauge field fluxes k_1 and k_2 on $S^2 \times S^2$ [25], or a non-trivial instanton bundle on S^4 [26]. In the dual field theories, some or all of the scalar fields have to acquire non-zero vacuum expectation values (*vevs for short*) on the $x_3 > 0$ side of the defect. At tree-level, these vevs are given by the following classical solutions to the $\mathcal{N} = 4$ SYM equations of motion (2.1.3) (*more on these classical scalar field solutions in section 2*).

$$\mathfrak{so}(3) \times \mathfrak{so}(3): \quad \Phi_i(x) = -\frac{1}{x_3} \begin{cases} t_i^{k_1} \otimes \mathbb{1}_{k_2} \oplus 0_{N-k_1k_2} & \text{for } i = 1,2,3 \\ \mathbb{1}_{k_1} \otimes t_{i-3}^{k_2} \oplus 0_{N-k_1k_2} & \text{for } i = 4,5,6 \end{cases} \tag{1.0.1}$$

$$\mathfrak{so}(5): \quad \Phi_i(x) = \frac{1}{\sqrt{2}x_3} \begin{cases} G_{i6}^{d_n} \oplus 0_{N-d_n} & \text{for } i = 1,2,3,4,5 \\ 0_N & \text{for } i = 6 \end{cases} \tag{1.0.2}$$

Where $t_i^{k_s}$, with $s = 1,2$, constitute k_s dimensional representations of the $\mathfrak{so}(3)$ Lie algebra generators, and $G_{i6}^{d_n}$ constitute a subset of the $d_n = d_6\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$[1] dimensional representations of the $\mathfrak{so}(6)$ Lie algebra generators. The labels $\mathfrak{so}(3) \times \mathfrak{so}(3)$ and $\mathfrak{so}(5)$ in (1.0.1) and (1.0.2) denotes the remaining unbroken subgroups of the full $SO(6)$ R-symmetry, corresponding to the respective classical solutions. They also correspond to the isometries of the compact parts of the D7 probe brane geometries, $S^2 \times S^2$ and S^4 respectively.

1.1 Symmetries of the dCFTs

Let us at this point briefly discuss what symmetries of the $\mathcal{N} = 4$ SYM field theories survive the introduction of a defect in spacetime. First of all, the presence of the defect quite obviously break

[1] For more information, see appendix D on representation theory for $\mathfrak{so}(5)$ and $\mathfrak{so}(6)$.

translation invariance in the direction perpendicular to itself. Also, the local $U(N - k_1 k_2)$ gauge symmetry in the $x_3 < 0$ region is trivially intact, since we have only zero vevs on this side of the defect. More interesting is the non-zero vevs in the $x_3 > 0$ region, which result from the classical scalar field solutions (1.0.1) and (1.0.2). These vevs partially break both the local $U(N)$ gauge group and the global $PSU(2, 2|4)$ super conformal symmetry group of $4D$ $\mathcal{N} = 4$ SYM theory. From the forms of the scalar field solutions, we see that the global $PSU(2, 2|4)$ symmetry reduces to the following.

$$PSU(2,2|4) \to SO(3,2) \times \begin{cases} SO(3) \times SO(3) \\ SO(5) \end{cases} \tag{1.1.1}$$

Where clearly, the upper case corresponds to (1.0.1) vevs, and the lower case corresponds to (1.0.2) vevs. We see that super symmetry is completely broken in these setups, which will become apparent when we look at the mass spectrum of the fields in subsection 2.5. The remaining spacetime symmetry is constituted by the Poincaré transformations parallel to the defect: $SO(2, 1)$, and dilatations of spacetime: $SO(1, 1)$, together with special conformal transformations that complete $SO(3, 2)$. As already mentioned, the $U(N)$ gauge group is also partially broken by the non zero vevs, and is reduced to the following.

$$U(N) \to U(N - k_1 k_2) \times \begin{cases} U(1) \times U(1) \\ U(1) \end{cases} \tag{1.1.2}$$

Where the upper and lower cases again correspond to $\mathfrak{so}(3) \times \mathfrak{so}(3)$ and $\mathfrak{so}(5)$ symmetric vevs respectively. This is easily seen, as only multiples of $\mathbb{1}_{k_1}$, $\mathbb{1}_{k_2}$ commutes with $t_i^{k_1}$, $t_i^{k_2}$ respectively. Similarly, only multiples of $\mathbb{1}_{d_n}$ commutes with $G_{i6}^{d_n}$. Thus, the scalar field solutions Φ_i commutes with matrices of the form: $U = e^{i\theta_1}\mathbb{1}_{k_1} \otimes e^{i\theta_2}\mathbb{1}_{k_2} \oplus U_{N-k_1k_2}$, in the $\mathfrak{so}(3) \times \mathfrak{so}(3)$ case, and matrices of the form: $U = e^{i\theta}\mathbb{1}_{d_n} \oplus U_{N-d_n}$, in the $\mathfrak{so}(5)$ case. However, because the gauge group in the $x_3 < 0$ region is given by either $U(N - k_1k_2)$ or $U(N - d_n)$, any gauge transformation in the $x_3 > 0$ region has to reduce to either $U(N - k_1k_2)$ or $U(N - d_n)$ transformations at the boundary. This is not possible for unitary transformations of the forms above, which means that the gauge group in both cases is further reduced, so that the gauge groups on both sides of the defect agrees.

$$U(N) \to U(N - k_1 k_2) \tag{1.1.3}$$

1.2 The aim of the thesis

The primary focus of this thesis will be to study different kinds of bulk-to-bulk two-point functions in $\mathcal{N} = 4$ defect conformal field theories, with vevs given by either (1.0.1) or (1.0.2). More precisely, we will be looking at two-point functions of various local scalar single-trace operators $\mathcal{O}_a(x)$ and $\mathcal{O}_b(y)$, where the indicies a, b label operators of the form.

$$\mathcal{O}_{a,b}(x) = \Psi_{a,b}^{i_1 \ldots i_L} \operatorname{tr}\left[\boldsymbol{\phi}_{i_1}(x) \ldots \boldsymbol{\phi}_{i_L}(x)\right] \quad , \quad x_3, y_3 > 0 \tag{1.2.1}$$

One-point functions between certain operators of the above types, have already been studied in [11] for the case of $\mathfrak{so}(3) \times \mathfrak{so}(3)$ symmetric vevs, and in [12] for the case of $\mathfrak{so}(5)$ symmetric vevs. The techniques developed in [11, 12] will also be intrumental in computing two-point functions, and so will the ideas presented in [19]. The structure of this thesis will be as follow. First, in section 2, we will go through the necessary steps to bring the theory to a form suitable for making perturbatively calculations, of which the biggest challenge is to diagonalize all the new quadratic terms in the Lagrangian. We then proceed, in section 3, to pertubativly compute the leading order contributions to two-point functions of different scalar single-trace chiral primary operators. Lastly, in section 4, we attempt to compute the leading order contribution to the two-point fuctions of a certain type of non-protected operators with definite conformal dimension at 1-loop level, and single-trace scalar operators of length two.

2 Defect conformal field theory setup

In this section, we go through the ground work necessary for subsequent pertubative calculations in the dCFT setups under consideration. This is essentially a matter of finding the propagators of all fields in the theory, such that they can be used in computing Feynmann diagrams. However, this seemingly straight forward task is vastly complicated by the appearance of a completely scrambled mass-matrix (*this mass-matrix is non diagonal in both color and flavor indices, as we will see later on*). We first look at the particular non trivial solutions of the scalar equations of motion (*EOM's*) in $\mathcal{N}=4$ SYM theory (*SYM*), that will server as the vacua for our defect conformal field theory setups. The fields of the theory will then acquire masses in the usual Higgs-like manner. After gauge-fixing the $\mathcal{N}=4$ action and reducing the 10D Majorana-Weyl fermions, we expand around the classical scalar field solutions and find the form of the mass-matrix. Seeing as the procedure of diagonalizing the mass-matrix is a rather long and complicated one, we will not go through all the details in this thesis, but instead refer the reader to the original paper in which this was done [11]. We conclude this section by finding the propagators of the diagonal fields, which is slightly complicated by the fact that the mass eigenvalues are spacetime dependent.

2.1 Non zero classical scalar field solutions

Our starting point for the entire proceeding analysis is the $\mathcal{N}=4$ SYM action, which looks as follow.

$$S_{\mathcal{N}=4} = \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x \,\mathrm{tr}\left[-\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}(D_\mu\phi_i)^2 + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{1}{2}\bar{\Psi}\tilde{\Gamma}^i[\phi_i,\Psi] + \frac{1}{4}[\phi_i,\phi_j]^2\right] \tag{2.1.1}$$

The field content of the theory is: one $U(N)$ gauge field A_μ, six Lorentz scalars ϕ_i and one 10D Majorana-Weyl spinor Ψ (*in section 2.2 we explain how to reduce this 10D spinor to four 4D spinors*). The set of matrices $\{\Gamma^\mu, \tilde{\Gamma}^i\}$ constitute 10D gamma matrices. All fields transform in the same representation of $U(N)$ (*as is necessary for the theory to be super symmetric*), namely the adjoint representation[2]. We use the following conventions for the field strength $F_{\mu\nu}$ and the adjoint covariant derivative D_μ.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad , \quad D_\mu\chi = \partial_\mu\chi - i[A_\mu,\chi] \quad , \quad \chi \in \{F_{\mu\nu}, \Psi, \phi_i\} \tag{2.1.2}$$

We now look for solutions to the EOM's of this action, such that the scalar fields ϕ_i, are non-zero while all other fields vanish. We can now vary the action with respect to the scalars, and assuming all other fields vanish we find that.

$$\frac{\delta S_{\mathcal{N}=4}}{\delta\phi_i} = 0 \quad \Rightarrow \quad \partial_\mu\partial^\mu\phi_i = [\phi_j,[\phi_j,\phi_i]] \tag{2.1.3}$$

A non-zero solution to these equations was presented in [11]. The solution has the following form.

$$\mathfrak{so}(3)\times\mathfrak{so}(3): \quad \Phi_i(x) = -\frac{1}{x_3}\begin{cases} t_i^{k_1}\otimes \mathbb{1}_{k_2}\oplus 0_{N-k_1k_2} & \text{for } i=1,2,3 \\ \mathbb{1}_{k_1}\otimes t_{i-3}^{k_2}\oplus 0_{N-k_1k_2} & \text{for } i=4,5,6 \end{cases} \tag{2.1.4}$$

Here, $t_i^{k_1}$ and $t_i^{k_2}$ are k_1 and k_2 dimensional representations of the $\mathfrak{so}(3)$ generators respectively, and $k_1k_2 \le N$. It is easy to verify that this is indeed a solution of equ. (2.1.3) using the $\mathfrak{so}(3)$ commutation relations: $[t_i,t_j] = i\varepsilon_{ijk}t_k$, and the identity: $\varepsilon_{jkl}\varepsilon_{jik} = -2\delta_{li}$.

$$[t_j,[t_j,t_i]] = i\varepsilon_{jik}[t_j,t_k] = -\varepsilon_{jik}\varepsilon_{jkl}t_l = 2t_i \quad , \quad i,j,k,l = 1,2,3 \tag{2.1.5}$$

[2] Of course A_μ does not transform exactly as an adjoint field, but rather the field strength $F_{\mu\nu}$ does.

Using the result in (2.1.5) and the fact that $t_i^{k_1} \otimes \mathbb{1}_{k_2}$ and $\mathbb{1}_{k_1} \otimes t_j^{k_2}$ commute, it should be clear that (2.1.4) is indeed a solution to (2.1.3). It should be noted at this point, that a certain limit of this solution ($k_1 = 1$ *or* $k_2 = 1$) is dual to the so called fuzzy-funnel solution of the probe D5-D3 brane setup with $AdS_4 \times S^2$ geometry [5].
In addition to the above classical field solution, another non-zero solution also exists [12], and it is given by the following similar looking expression.

$$\mathfrak{so}(5): \quad \Phi_i(x) = \frac{1}{\sqrt{2}x_3} \begin{cases} G_{i6}^{d_n} \oplus 0_{N-d_n} & \text{for } i = 1,2,3,4,5 \\ 0_N & \text{for } i = 6 \end{cases} \tag{2.1.6}$$

Where $G_{i6}^{d_n}$ are a subset of the $d_n = d_6\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}\right)$ dimensional representations of $\mathfrak{so}(6)$ generators. It is again easy to verify that the above is indeed a solution of (2.1.3), this time using the commutation relations of $\mathfrak{so}(6)$ and the fact that $G_{ij} = -G_{ji}$.

$$[G_{ij}, G_{kl}] = i\left(\delta_{ik}G_{jl} + \delta_{jl}G_{ik} - \delta_{il}G_{jk} - \delta_{jk}G_{il}\right) \quad , \quad i,j,k,l = 1,\ldots 6 \tag{2.1.7}$$

$$[G_{j6}, G_{i6}] = iG_{ji} \quad , \quad i,j,k,l = 1,\ldots 5 \tag{2.1.8}$$

$$[G_{j6}, [G_{j6}, G_{i6}]] = i[G_{j6}, G_{ji}] = -\left(\delta_{jj}G_{6i} - \delta_{ji}G_{6j}\right) = 4G_{i6} \quad , \quad i,j,k,l = 1,\ldots 5 \tag{2.1.9}$$

Using the result (2.1.9), it should again be clear that (2.1.6) is indeed also a solution to equ. (2.1.3). It should again be noted, that this solution is also dual to a fuzzy-funnel solution, this time of the probe D7-D3 brane setup with $AdS_4 \times S^4$ geometry [6].

2.2 Reducing the 10D Majorana-Weyl fermion

When we presented the fields of $\mathcal{N}=4$ SYM back in section 2.1, one field might have seemed a bit out of place; namely the $10D$ Majorana-Weyl fermion. This $10D$ fermion is in fact a remnant of the $10D$ $\mathcal{N}=1$ SYM action, from which we can obtain the $4D$ $\mathcal{N}=4$ action by dimensional reduction. For the sake of completness, we present here the $\mathcal{N}=1$ SYM action.

$$S_{\mathcal{N}=1} = \frac{2}{\tilde{g}^2} \int_{\mathbb{R}^{10}} \mathrm{d}^{10}x \,\mathrm{tr}\left[-\frac{1}{4}(F_{MN})^2 + \frac{i}{2}\bar{\Psi}\Gamma^M D_M \Psi\right] \quad , \quad M,N = 0,...,9 \tag{2.2.1}$$

Where $\bar{\Psi} \equiv \Psi^\dagger \Gamma^0$. We now move on to the task of decomposing the $10D$ Majorana-Weyl fermion into a set of $4D$ fermions. To do this, we need to take into account both the Majorana and Weyl constraints for the $10D$ Majorana-Weyl fermion.

$$\Psi = \Psi^C \equiv \mathcal{C}_{10}\Gamma^0\Psi^* \quad , \quad \Gamma^{11}\Psi = -\Psi \tag{2.2.2}$$

Where Γ^M, Γ^{11} are $10D$ gamma matrices, which have to obey the Clifford anti-commutator algebra.

$$\{\Gamma^M, \Gamma^N\} = -2\eta^{MN} \quad , \quad \Gamma^{11} = i\,\Gamma^0 \cdots \Gamma^9 \tag{2.2.3}$$

In the above, C_{10} is the $10D$ charge conjugation matrix, which implicitly defined by the following relation: $-\left(\Gamma^M\right)^* = \Gamma^0\, C_{10}^{-1}\,\Gamma^M\, C_{10}\,\Gamma^0$. In what follows, we will employ the representation of $10D$ gamma matrices and the $10D$ charge conjugation matrix given below.

$$\Gamma^\mu = \gamma^\mu \otimes \mathbb{1}_4 \otimes \mathbb{1}_2 \quad , \quad \mu = 0,1,2,3 \tag{2.2.4}$$

$$\tilde{\Gamma}^i \equiv \Gamma^{i+3} = \begin{cases} -i\,\gamma^5 \otimes G^i \otimes \sigma_2 & \text{for } i = 1,2,3 \\ \gamma^5 \otimes G^i \otimes \sigma_1 & \text{for } i = 4,5,6 \end{cases} \tag{2.2.5}$$

$$\Gamma^{11} = -\gamma^5 \otimes \mathbb{1}_4 \otimes \sigma_3 \quad , \quad C_{10} = C_4 \otimes \mathbb{1}_4 \otimes \sigma_1 \quad , \quad C_4 = i\,\sigma_2 \otimes \sigma_3 \tag{2.2.6}$$

Where the 4×4 matrices G^i are given by the following expressions.

$$\begin{aligned} G^1 = \sigma_3 \otimes \sigma_2 \quad , \quad G^2 = -\sigma_2 \otimes \sigma_2 \quad , \quad G^3 = \sigma_2 \otimes \mathbb{1}_2 \\ G^4 = -i\,\sigma_2 \otimes \sigma_1 \quad , \quad G^5 = -i\,\mathbb{1}_2 \otimes \sigma_2 \quad , \quad G^6 = i\,\sigma_2 \otimes \sigma_3 \end{aligned} \tag{2.2.7}$$

And the 4×4 matrices γ^μ, γ^5 are given by the following expressions.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad , \quad \gamma^5 = i\,\gamma^0 \cdots \gamma^3 \quad , \quad \sigma^\mu = (\mathbb{1}_2, \sigma^i) \quad , \quad \bar{\sigma}^\mu = (\mathbb{1}_2, -\sigma^i) \tag{2.2.8}$$

We start now with an unconstrained $10D$ Dirac fermion. The way we have expressed the $10D$ gamma matrices suggest a decomposition into two blocks of four $4D$ fermions. We therefore write out the $10D$ fermion in the following way.

$$\Psi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_4 \\ \chi_5 \\ \vdots \\ \chi_8 \end{pmatrix} \tag{2.2.9}$$

Where the χ's are unconstrained $4D$ Dirac fermions. The Weyl condition in equ. (2.2.2) amounts to the following when applied to the $10D$ Dirac spinor (2.2.9).

$$\begin{pmatrix} \chi_1 \\ \vdots \\ \chi_4 \\ \chi_5 \\ \vdots \\ \chi_8 \end{pmatrix} = \begin{pmatrix} +\gamma_5\chi_1 \\ \vdots \\ +\gamma_5\chi_4 \\ -\gamma_5\chi_5 \\ \vdots \\ -\gamma_5\chi_8 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_4 \\ \chi_5 \\ \vdots \\ \chi_8 \end{pmatrix} = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_5 \\ \vdots \\ R\psi_8 \end{pmatrix} \tag{2.2.10}$$

Where $L = \frac{\mathbb{1}_4+\gamma_5}{2}$ and $R = \frac{\mathbb{1}_4-\gamma^5}{2}$ are standard projection operators, and the ψ's are new unconstrained $4D$ Dirac fermions. The Majorana condition in equ. (2.2.2) amounts to the following when applied to the result in (2.2.10).

$$\begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_5 \\ \vdots \\ R\psi_8 \end{pmatrix} = \begin{pmatrix} C_4\gamma^0 R\psi_5^* \\ \vdots \\ C_4\gamma^0 R\psi_8^* \\ C_4\gamma^0 L\psi_1^* \\ \vdots \\ C_4\gamma^0 L\psi_4^* \end{pmatrix} = \begin{pmatrix} R\psi_5^C \\ \vdots \\ R\psi_8^C \\ L\psi_1^C \\ \vdots \\ L\psi_4^C \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_5 \\ \vdots \\ R\psi_8 \end{pmatrix} = \begin{pmatrix} L\psi_1 \\ \vdots \\ L\psi_4 \\ R\psi_1 \\ \vdots \\ R\psi_4 \end{pmatrix} \tag{2.2.11}$$

Where we have used that $C_4\,\gamma^0\,\gamma^5\,\psi_i^* = -C_4\,\gamma^0\,\gamma^5\,\gamma^0\,C_4\,\psi_i^C = -\gamma^5\,\psi_i^C$, and concluded that the $4D$ spinors ψ_i with $a = 1,2,3,4$, must be Majorana spinors: $\psi_i = \psi_i^C \equiv \mathcal{C}_4\,\gamma^0\,\psi_i^*$.
Now that we know how the $10D$ Majorana-Weyl fermion splits into four left-chiral and four right-chiral $4D$ Majorana fermions, we can use this information to figure out how the terms in the $\mathcal{N}=4$ action (2.1.1), involving the Majorana-Weyl fermion, reduce. The terms in question are the following.

$$\frac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi \xrightarrow{\text{Dim. Red.}} \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + \frac{1}{2}\bar{\Psi}\tilde{\Gamma}_i[\boldsymbol{\phi}_i, \Psi] \tag{2.2.12}$$

Because of the simple sctructure of the Γ^μ matrices, given in equ. (2.2.4), we find that the kinetic term of the Majorana-Weyl spinor reduce in the following simple way.

$$\frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi = \frac{i}{2}(L\psi_a)^\dagger\gamma^0\gamma^\mu D_\mu L\psi^a + \frac{i}{2}(R\psi_a)^\dagger\gamma^0\gamma^\mu D_\mu R\psi^a = \frac{i}{2}\bar{\psi}_a\gamma^\mu D_\mu\psi^a \tag{2.2.13}$$

Where we have used that L, R are Hermitian, $\{\gamma^5, \gamma^\mu\} = 0$ and the simple property: $L\psi + R\psi = \psi$. We have also made use of the definition: $\bar{\psi} \equiv \psi^\dagger \gamma_0$. In reducing the fermion-scalar interaction term, the simple struture of the gamma matrices, this time $\tilde{\Gamma}^i$ (2.2.5), again simplify the procedure.

$$\frac{1}{2}\bar{\Psi}\tilde{\Gamma}_i[\boldsymbol{\phi}_i, \Psi] = \sum_{i=1}^{3}\frac{1}{2}(R\psi_a)^\dagger\gamma^0\gamma^5(G_i)^a{}_b[\boldsymbol{\phi}_i, L\psi^b] - \sum_{i=1}^{3}\frac{1}{2}(L\psi_a)^\dagger\gamma^0\gamma^5(G_i)^a{}_b[\boldsymbol{\phi}_i, R\psi^b]$$

$$+\sum_{i=4}^{6}\frac{1}{2}(R\psi_a)^\dagger\gamma^0\gamma^5(G_i)^a{}_b[\boldsymbol{\phi}_i, L\psi^b] + \sum_{i=4}^{6}\frac{1}{2}(L\psi_a)^\dagger\gamma^0\gamma^5(G_i)^a{}_b[\boldsymbol{\phi}_i, R\psi^b] \tag{2.2.14}$$

$$= \sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\boldsymbol{\phi}_i, \psi^b] + \sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\boldsymbol{\phi}_i, \gamma^5\psi^b] \tag{2.2.15}$$

Where we have used the same properties to simplify as we did for the kinetic term, and additionally $L\gamma^5 = L$ and $R\gamma^5 = -R$. With both the kinetic and the interaction term reduced, we finally have the spinor terms in a form appropriate for extracting propagators and vertex rules for the fields in our dCFT setups. As a last aside for this section, we note that the $(G_i)^a{}_b$ matrices are related to (*Euclidian*) $6D$ gamma matrices. In our conventions, these gamma matrices would take the forms:

$$\gamma_i^{(6)} = \begin{cases} -i\, G_i \otimes \sigma_2 & \text{for } i = 1,2,3 \\ G_i \otimes \sigma_1 & \text{for } i = 4,5,6 \end{cases} \tag{2.2.16}$$

2.3 Gauge fixing and ghost fields

In order to simplify the diagonilazation of the mass matrix of the spontaneously broken theory (*more on that in section 2.5*), we want to get rid of the terms in the expanded action (2.4.9) quadritic in the fields and containing a derivative. We will see in section 2.4 that one such term appears, and has the form.

$$\mathrm{tr}[i[A_\mu, \Phi_i]\partial^\mu\phi_i] \quad \text{with} \quad \boldsymbol{\phi}_i = \Phi_i + \phi_i \tag{2.3.1}$$

It turns out we can get rid of this term by gauge-fixing in a clever way [9]; effectively killing two birds with one stone. To be more precise, we choose to gauge-fix the action (2.1.1) using the following gauge-fixing function.

$$G[A_\mu, \boldsymbol{\phi}_i] = \partial_\mu A^\mu + i[\boldsymbol{\phi}_i, \Phi_i] \quad , \quad S_{\text{gf}} = \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}\left[-\frac{1}{2\xi}G[A_\mu, \boldsymbol{\phi}_i]^2\right] \tag{2.3.2}$$

Where S_{gf} is an extra term, which appears in the action after performing the *Faddev-Popov gauge-fixing procedure*. The slightly unusual thing about the gauge-fixing function above, is the fact that it depends also on the scalar fields $\boldsymbol{\phi}_i$ in addition to the gauge fields A_μ. By looking at the form of (2.3.1), it should however be fairly obvious that we need the $\boldsymbol{\phi}_i$ dependence in the gauge-fixing function, if there is to be any hope of canceling this unwanted term.
In what follows, we will always work in the gauge for which $\xi = 1$. We can now insert our gauge-fixing function into S_{gf}. The result is the following.

$$S_{\text{gf}} = \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}\left[-\frac{1}{2}(\partial_\mu A^\mu)^2 - [\boldsymbol{\phi}_i, \Phi_i]^2 + 2i[\boldsymbol{\phi}_i, \Phi_i]\partial_\mu A^\mu\right]$$

$$= \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{tr}\left[-\frac{1}{2}(\partial_\mu A^\mu)^2 + \frac{1}{2}[\phi_i, \Phi_i]^2 + i[\partial_\mu\phi_i, \Phi_i]A^\mu + i[\phi_i, \partial_\mu\Phi_i]A^\mu\right] \tag{2.3.3}$$

Where we have used integration by parts to get the second equality. It can easily be seen (*using the cylic property of the trace operation*) that the third term in the above gauge-fixing action exactly cancels the unwanted term (2.3.1). As usual, the first term in the gauge-fixing action cancels the problematic part of the kinetic term for the A_μ fields, leaving an invertible and diagonal term.
Now we turn our attention to the ghost part of the action. Under a infinitesimal gauge tranformation, A_μ and $\boldsymbol{\phi}_i$ transform in the following ways.

$$\delta A_\mu = D_\mu \varepsilon = \partial_\mu \varepsilon - i[A_\mu, \varepsilon] \quad , \quad \delta \boldsymbol{\phi}_i = -i[\boldsymbol{\phi}_i, \varepsilon] \tag{2.3.4}$$

The ghost part of the action S_{gh} can then be extracted from the following functional determinant.

$$\det\left(\frac{\delta G[A_\mu + D_\mu \varepsilon, \boldsymbol{\phi}_i - i[\boldsymbol{\phi}_i, \varepsilon]]}{\delta \varepsilon}\right) = \det\left(\partial^\mu D_\mu[\,\cdot\,] - [\Phi_i, [\Phi_i + \phi_i, \,\cdot\,]]\right) \tag{2.3.5}$$

Where the $[\,\cdot\,]$ denote an unfilled argument of an operator. We can now make use of the fact that the functional determinant of any operator $\mathcal{O}$ can be written as a Grassman path integral as follow.

$$\det(\mathcal{O}) = \int \mathcal{D}\bar{c}\,\mathcal{D}c \exp\left(i\,\frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x \,\mathrm{tr}\,[\bar{c}\,\mathcal{O}\,c]\right) \tag{2.3.6}$$

$$\Rightarrow S_{\text{gh}} = \frac{2}{g^2}\int_{\mathbb{R}^4} \mathrm{d}^4x \,\mathrm{tr}\,[\bar{c}\,\partial^\mu D_\mu c - \bar{c}\,[\Phi_i, [\Phi_i + \phi_i, c]]] \tag{2.3.7}$$

The prefactor of $2/g^2$ is of course purely conventional. We now see the price we have to pay to get rid of the unwanted term (2.3.1); namely the appearance of massive ghosts which couple directly to the scalar fields. Nevertheless, we can now write the full gauge-fixed actions with Faddeev-Popov ghosts.

$$S = S_{\mathcal{N}=4} + S_{\text{gf}} + S_{\text{gh}} \tag{2.3.8}$$

Where the $\mathcal{N}=4$ super Yang Mills action $S_{\mathcal{N}=4}$ is given in equ. (2.1.1) in the previous section 2.1.

2.4 Fluctuations around the classical scalar solutions

Now that we have managed to both gauge fix the action (*in a way which will simplify the mass matrix diagonalization*) and reduce the $10D$ Majorana-Weyl fermion, we are now ready to expand the six scalar fields $\boldsymbol{\phi}_i$ around the classical solutions (2.1.4), (2.1.6) and find the effective form of the action (2.1.1). We first write the scalar fields as follow.

$$\boldsymbol{\phi}_i = \Phi_i + \phi_i \tag{2.4.1}$$

Where ϕ_i are perturbations around the classical solutions. Before we get further into the process of expanding the $\mathcal{N}=4$ action, we note the following two things. Firstly, the field independent part of the expanded action.

$$\frac{1}{2}\Phi_i\,\partial^\mu\partial_\mu\Phi_i - \frac{1}{4}\Phi_i[\Phi_j, [\Phi_j, \Phi_i]] \tag{2.4.2}$$

Will be ignored from now on, as it does not affect any perturbative calculations. Secondly, the part of the expanded action linear in ϕ_i will vanish due to the EOM's for the scalar fields.

$$\phi_i\,\partial^\mu\partial_\mu\Phi_i - \phi_i\,[\Phi_j, [\Phi_j, \Phi_i]] = 0 \tag{2.4.3}$$

We can thus ignore these two terms when expanding the action. Let us now start by expanding the kinetic term for the ϕ_i fields. We write here the covariant derivate of Φ_i and ϕ_i for convenience.

$$D_\mu\Phi_i = \partial_\mu\Phi_i - i[A_\mu, \Phi_i] \quad , \quad D_\mu\phi_i = \partial_\mu\phi_i - i[A_\mu, \phi_i] \tag{2.4.4}$$

It is now straight forward to expand the kinetic term for the ϕ_i fields. The result is following.

$$\operatorname{tr}\left[-\frac{1}{2}D_\mu\boldsymbol{\phi}_i D^\mu\boldsymbol{\phi}_i\right]=\operatorname{tr}\left[-\frac{1}{2}D_\mu\Phi_i D^\mu\Phi_i-D_\mu\phi_i D^\mu\Phi_i-\frac{1}{2}D_\mu\phi_i D^\mu\phi_i\right] \tag{2.4.5}$$

$$\begin{aligned}=\operatorname{tr}\Bigg[&i[A_\mu,\Phi_i]\,\partial^\mu\Phi_i+\frac{1}{2}[A_\mu,\Phi_i][A^\mu,\Phi_i]\\&+i[A_\mu,\Phi_i]\,\partial^\mu\phi_i+i[A_\mu,\phi_i]\,\partial^\mu\Phi_i+[A_\mu,\phi_i][A^\mu,\Phi_i]\\&-\frac{1}{2}\partial_\mu\phi_i\,\partial^\mu\phi_i+i[A_\mu,\phi_i]\,\partial^\mu\phi_i+\frac{1}{2}[A_\mu,\phi_i][A^\mu,\phi_i]\Bigg]\end{aligned} \tag{2.4.6}$$

Notice that the problematic term mentioned back in section 2.3 appears in the above expansion. Notice also that the term linear in A_μ will only ever be relevant in the computation of one-point function of said field, and we will therefore ignore it from this point on. Next up, we expand the self interaction term for the ϕ_i fields, and the ϕ_i, ψ interaction terms.

$$\begin{aligned}\operatorname{tr}\left[\frac{1}{4}[\boldsymbol{\phi}_i,\boldsymbol{\phi}_j][\boldsymbol{\phi}_i,\boldsymbol{\phi}_j]\right]=\operatorname{tr}\Bigg[&\frac{1}{4}[\Phi_i,\Phi_j][\Phi_i,\Phi_j]+[\phi_i,\Phi_j][\Phi_i,\Phi_j]+\frac{1}{2}[\phi_i,\phi_j][\Phi_i,\Phi_j]\\&+\frac{1}{2}[\phi_i,\Phi_j][\phi_i,\Phi_j]+\frac{1}{2}[\phi_i,\Phi_j][\Phi_i,\phi_j]+[\phi_i,\phi_j][\phi_i,\Phi_j]+\frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi^j]\Bigg]\end{aligned} \tag{2.4.7}$$

$$\begin{aligned}\operatorname{tr}\left[\sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\boldsymbol{\phi}_i,\psi^b]+\sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\boldsymbol{\phi}_i,\gamma^5\psi^b]\right]=\operatorname{tr}\Bigg[&\sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\Phi_i,\psi^b]\\+\sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\Phi_i,\gamma^5\psi^b]+\sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\phi_i,\psi^b]+&\sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\phi_i,\gamma^5\psi^b]\Bigg]\end{aligned} \tag{2.4.8}$$

Now that we have expanded the terms in the $\mathcal{N}=4$ action involving the ϕ_i fields, we can finally write down the entire expanded action in a well organized form.

$$S_{\mathcal{N}=4}=S_{\text{kinetic}}+S_{\text{m,b}}+S_{\text{m,f}}+S_{\text{qubic}}+S_{\text{quadratic}} \tag{2.4.9}$$

$$S_{\text{kinetic}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\operatorname{tr}\left[\frac{1}{2}A_\mu\partial_\nu\partial^\nu A^\mu+\frac{1}{2}\phi_i\partial_\nu\partial^\nu\phi_i+\frac{i}{2}\bar{\psi}_a\gamma^\mu\partial_\mu\psi^a+\bar{c}\,\partial_\nu\partial^\nu c\right] \tag{2.4.10}$$

$$\begin{aligned}S_{\text{m,b}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\operatorname{tr}\Bigg[&\frac{1}{2}[A_\mu,\Phi_i][A^\mu,\Phi_i]+2i[A_\mu,\phi_i]\,\partial^\mu\Phi_i\\+\frac{1}{2}[\phi_i,\phi_j][\Phi_i,\Phi_j]+\frac{1}{2}[\phi_i,\Phi_j][\phi_i,\Phi_j]+&\frac{1}{2}[\phi_i,\Phi_j][\Phi_i,\phi_j]+\frac{1}{2}[\phi_i,\Phi_i][\phi_j,\Phi_j]\Bigg]\end{aligned} \tag{2.4.11}$$

$$S_{\text{m,f}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\operatorname{tr}\left[\sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\Phi_i,\psi^b]+\sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\Phi_i,\gamma^5\psi^b]-\bar{c}\,[\Phi_i,[\Phi_i,c]]\right] \tag{2.4.12}$$

$$\begin{aligned}S_{\text{cubic}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\operatorname{tr}\,\Big[&i[A_\mu,A_\nu]\,\partial^\mu A_\nu+[A_\mu,\phi_i][A^\mu,\Phi_i]+i[A_\mu,\phi_i]\,\partial^\mu\phi_i+[\phi_i,\phi_j][\phi_i,\Phi_j]\\&+\frac{1}{2}\bar{\psi}_a\gamma^\mu[A_\mu,\psi^a]+\sum_{i=1}^{3}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\phi_i,\psi^b]+\sum_{i=4}^{6}\frac{1}{2}\bar{\psi}_a(G_i)^a{}_b[\phi_i,\gamma^5\psi^b]\\&+i\,(\partial^\mu\bar{c})[A_\mu,c]-\bar{c}\,[\Phi_i,[\phi_i,c]]\Big]\end{aligned} \tag{2.4.13}$$

$$S_{\text{quadratic}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\operatorname{tr}\left[\frac{1}{4}[A_\mu,A_\nu][A^\mu,A^\nu]+\frac{1}{2}[A_\mu,\phi_i][A^\mu,\phi_i]+\frac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\right] \tag{2.4.14}$$

Where in the above, we have also included the terms from the gauge-fixing and ghost parts of the action (*see section 2.3*), as well as the expanded forms of the kinetic terms for both the A_μ and ψ^a fields (*see equ. (2.1.1) for the unexpanded forms*).

2.5 Diagonalizing the mass matrix

Now that we finally have the $\mathcal{N}=4$ SYM action in the fully operational form given above, we are ready to tackle the problem of diagonalizing the mass terms of said action. Looking at both the bosonic (2.4.11) and fermionic (2.4.12) mass terms of the action (2.4.9), we see that all of these are either non-diagonal with regards to the $U(N)$ matrix structure (*color mixing*), non-diagonal with regards to the species of fields (*flavor mixing*) or both. The techniques necessary to solve this diagonalization problem was first presented in [11] for the case of $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric vevs, and in [12] for the case of $\mathfrak{so}(5)$ symmetric vevs. The following subsection will constitute a review of the work contained in those aforementioned articles.

2.5.1 The boson mass matrix for $\mathrm{SO}(3)\times\mathrm{SO}(3)$ symmetric vevs

In order to more clearly see the structure which makes the diagonalization of the boson mass matrix possible, we have to work a bit witht the form of $S_{\text{m,b}}$. It turns out that we can rewrite the boson mass part of the total action (2.4.11) to the following.

$$S_{\text{m,b}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\,\mathrm{tr}\left[-\frac{1}{2}A_\mu[\Phi_i,[\Phi_i,A^\mu]]-2iA_\mu[\partial^\mu\Phi_i,\phi_i]\right.$$
$$\left.-\frac{1}{2}\phi_i[\Phi_j,[\Phi_j,\phi_i]]-\phi_i[[\Phi_i,\Phi_j],\phi_j]\right] \tag{2.5.1}$$

Where we have used cyclicity of the trace to produce the nested commutators, and the Jacobi identity: $[A,[B,C]]+[C,[A,B]]+[B,[C,A]]=0$, to combine some terms in the action. We now define the following two operators, to make simplify the form of the action further.

$$L_i^{(1)}=\mathrm{ad}(t_i^{(1)})\equiv[t_i^{(1)},\,\cdot\,]\quad,\quad t_i^{(1)}\equiv t_i^{k_1}\otimes\mathbb{1}_{k_2}\oplus 0_{N-k_1k_2} \tag{2.5.2}$$

$$L_i^{(2)}=\mathrm{ad}(t_i^{(2)})\equiv[t_i^{(2)},\,\cdot\,]\quad,\quad t_i^{(2)}\equiv \mathbb{1}_{k_1}\otimes t_i^{k_2}\oplus 0_{N-k_1k_2} \tag{2.5.3}$$

Recall now that the $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric Φ_i solutions are constructed from the $\mathfrak{so}(3)$ generators in such a way that their commutators are given by the following.

$$[\Phi_i,\Phi_j]=\frac{i}{x_3^2}\,\varepsilon_{ijk}\,t_k^{(1)}\quad,\quad\text{for } i,j,k=1,2,3 \tag{2.5.4}$$

$$[\Phi_{i+3},\Phi_{j+3}]=\frac{i}{x_3^2}\,\varepsilon_{ijk}\,t_k^{(2)}\quad,\quad\text{for } i,j,k=1,2,3 \tag{2.5.5}$$

Using the above commutators, we can now further rewrite the boson mass part of the action (2.5.1).

$$S_{\text{m,b}}=\frac{2}{g^2}\int_{\mathbb{R}^4}\mathrm{d}^4x\frac{1}{x_3^2}\,\mathrm{tr}\left[-\frac{1}{2}A_\mu\left(L_{(1)}^2+L_{(2)}^2\right)A^\mu-\sum_{i=1}^{6}\frac{1}{2}\phi_i\left(L_{(1)}^2+L_{(2)}^2\right)\phi_i\right.$$
$$+i\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\phi_iL_j^{(1)}\phi_k+i\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\phi_{i+3}L_j^{(2)}\phi_{k+3}$$
$$\left.+i\sum_{i=1}^{3}\left(\phi_iL_i^{(1)}A_3-A_3L_i^{(1)}\phi_i\right)+i\sum_{i=1}^{3}\left(\phi_{i+3}L_i^{(2)}A_3-A_3L_i^{(2)}\phi_{i+3}\right)\right] \tag{2.5.6}$$

Where $L^2_{(s)} \equiv L^{(s)}_i L^{(s)}_i$ for $s \in \{1, 2\}$ are $\mathfrak{so}(3)$ Casimir operators. We can now group together the various fields in (2.5.6) according to wether or not their mass terms are flavor diagonal. We call the fields which are flavor diagonal *easy fields*, and denote them collectively by E. The fields which are not flavor diagonal we call *complicated fields*, and denote collectively by $\tilde{C}$.

$$E = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} \quad , \quad \tilde{C} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_6 \\ A_3 \end{pmatrix} \tag{2.5.7}$$

In order to write the terms in (2.5.1) involving the complicated fields $\tilde{C}$ in a more suggestive way, we define the following two 7×7 matrices to act on the space of the 7 complicated fields in $\tilde{C}$.

$$\tilde{S}^{(1)}_i = \begin{pmatrix} \tilde{T}_i & 0 & \tilde{R}_i \\ 0 & 0 & 0 \\ \tilde{R}^\dagger_i & 0 & 0 \end{pmatrix} \quad , \quad \tilde{S}^{(2)}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{T}_i & \tilde{R}_i \\ 0 & \tilde{R}^\dagger_i & 0 \end{pmatrix} \tag{2.5.8}$$

$$\tilde{T}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad \tilde{T}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad , \quad \tilde{T}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.5.9}$$

Where in the above, the $\tilde{R}_j$ matrices are 3×1, with only an i in the j-th row, and zeros in all other rows: $[\tilde{R}_j]_k = i\delta_{jk}$. Notice also that the set of matrices $\{\tilde{T}_i\}$ constitute the 3-dimensional irreducible representation of $\mathfrak{so}(3)$. Using the matrices $\tilde{S}^{(1)}_i$ and $\tilde{S}^{(2)}_i$, we can now rewrite the bosonic mass part of the action for a final time.

$$S_{\mathrm{m,b}} = -\frac{1}{g^2} \int_{\mathbb{R}^4} \mathrm{d}^4x \frac{1}{x_3^2} \operatorname{tr} \left[\sum_{s \in \{1,2\}} E^\dagger \, L^2_{(s)} \, E + \tilde{C}^\dagger \left(L^2_{(s)} - 2\tilde{S}_{(s)} \cdot L_{(s)} \right) \tilde{C} \right] \tag{2.5.10}$$

We can now see that the problem of diagonalizing the easy fields in (2.5.10) is structurally very reminiscent to the problem of finding eigenvectors of the total angular momentum in standard quantum mechanics. Similarly, the problem of diagonalizing the complicated fields is structurally very reminiscent to the problem of finding eigenvectors of the total angular momentum with spin-orbit coupling. As might be suspected from these apparent similarities, introducing a kind of spherical harmonics and making use of the machinery of angular momentum addition (*or equivalently decomposition of reducible* $\mathfrak{su}(2)$ *representations*) will be crucial in solving the mass diagonalization problem at hand.

Before we proceed further with the task of finding the fields which diagonalize $S_{\mathrm{m,b}}$, it is useful to make the following decomposition of the matrix-valued fields in both E and $\tilde{C}$.

$$\Psi = \left[\begin{array}{c|c} \Psi_{n,n'} E^n{}_{n'} & \Psi_{n,a} E^n{}_a \\ \hline \Psi_{a,n} E^a{}_n & \Psi_{a,a'} E^a{}_{a'} \end{array} \right] \tag{2.5.11}$$

Here, Ψ is a stand-in for any field contained in either E or $\tilde{C}$. The basis matrices $E^n{}_{n'}$ are defined such that: $[E^n{}_{n'}]^m{}_{m'} = \delta^{nm}\delta_{n'm'}$. In other words, $E^n{}_{n'}$ are the matrices which have 1 at entry (n, n') and 0 at every other entry. The indices n n' runs over the values: $n, n' = 1, \ldots, k_1k_2$, while the indices a, a' runs over the values: $a, a' = k_1k_2 + 1, \ldots, N$.

2.5.1.1 The Easy Fields Let us first focus on diagonalizing the part of $S_{\text{m,b}}$ containing the easy fields E. Firstly, the fields spanned by the $E^a{}_{a'}$ matrices in the $(N - k_1k_2) \times (N - k_1k_2)$ block are all anihilated by the angular momentum operators.

$$L_i^{(1)} E^a{}_{a'} = [t_i^{k_1} \otimes \mathbb{1}_{k_2} \oplus 0_{N-k_1k_2}, E^a{}_{a'}] = 0 \tag{2.5.12}$$

$$L_i^{(2)} E^a{}_{a'} = [\mathbb{1}_{k_1} \otimes t_i^{k_2} \oplus 0_{N-k_1k_2}, E^a{}_{a'}] = 0 \tag{2.5.13}$$

Plugging the lower diagonal fields into the easy part of $S_{\text{m,b}}$, we conclude that all fields in the lower diagonal block have the same mass, which is given by.

$$m^2_{\text{lower diag.}} = 0 \quad , \quad \textit{multiplicity: } (N - k_1k_2)^2 \tag{2.5.14}$$

Next, we look at the fields spanned by the $E^n{}_a$ matrices in the $(N - k_1k_2) \times k_1k_2$ block, and the fields spanned by the $E^a{}_n$ matrices in the $k_1k_2 \times (N - k_1k_2)$ block. The result of applying the angular momentum operators $L_i^{(1)}$, $L_i^{(2)}$ to these fields are as follow.

$$L_i^{(1)} E^n{}_a = [t_i^{k_1} \otimes \mathbb{1}_{k_2}]_{n,n'} E^{n'}{}_a \quad , \quad L_i^{(2)} E^n{}_a = [\mathbb{1}_{k_1} \otimes t_i^{k_2}]_{n,n'} E^{n'}{}_a \tag{2.5.15}$$

$$L_i^{(1)} E^a{}_n = -[t_i^{k_1} \otimes 2_{k_2}]_{n,n'} E^a{}_{n'} \quad , \quad L_i^{(2)} E^a{}_n = -[\mathbb{1}_{k_1} \otimes t_i^{k_2}]_{n,n'} E^a{}_{n'} \tag{2.5.16}$$

Becuase the $\{t_i^{k_1}\}$ and $\{t_i^{k_2}\}$ matrices are generators of the k_1-dimensional and the k_2-dimensional irreducible representations of $\mathfrak{su}(2)$ respectively, we know that $t^2_{k_s} = \ell_s(\ell_s+1)\mathbb{1}_{k_s}$ with $k_s = 2\ell_s + 1$, are the Casimir operators of the k_s-dimensional irreps. of $\mathfrak{su}(2)$. Therefore, we obtain the following results by applying the angular momentum operators for a second time.

$$L^2_{(1)} E^n{}_a = \frac{k_1^2 - 1}{4} E^n{}_a \quad , \quad L^2_{(2)} E^n{}_a = \frac{k_2^2 - 1}{4} E^n{}_a \tag{2.5.17}$$

$$L^2_{(1)} E^a{}_n = \frac{k_1^2 - 1}{4} E^a{}_n \quad , \quad L^2_{(2)} E^a{}_n = \frac{k_2^2 - 1}{4} E^a{}_n \tag{2.5.18}$$

Plugging the off diagonal fields into the easy part of $S_{\text{m,b}}$, and using the orthogonality relations for $E^n{}_a$, we conclude that all fields in the two off-diagonal blocks have the same mass, which is given by.

$$\operatorname{tr}\Big[(E^n{}_a)^\dagger E^{n'}{}_{a'}\Big] = \delta_{a,a'}\delta_{n,n'} \quad , \quad (E^n{}_a)^\dagger = E^a{}_n \quad , \quad (\Psi_{n,a})^\dagger = \Psi_{a,n} \tag{2.5.19}$$

$$m^2_{\text{off diag.}} = \frac{k_1^2 - 1}{4} + \frac{k_2^2 - 1}{4} \quad , \quad \textit{multiplicity: } 2\,k_1k_2\,(N - k_1k_2) \tag{2.5.20}$$

Lastly, we will discuss the fields spanned by the $E^n{}_{n'}$ matrices in the $k_1k_2 \times k_1k_2$ block. We observed for the off-diagonal fields, that the matrices $E^n{}_a$, $E^a{}_n$ transform in the $\left(\frac{k_1-1}{2}, \frac{k_2-1}{2}\right)$ irreducible representation of $\mathfrak{su}(2) \times \mathfrak{su}(2)$. To be more precise, only the n-indices transform under $\mathfrak{su}(2) \times \mathfrak{su}(2)$, while the a-indeices did not transform what so ever. By this line of reasoning, we see that the matrices $E^n{}_{n'}$ transform under the following product representation of $\mathfrak{su}(2) \times \mathfrak{su}(2)$ algebra.

$$\left(\frac{k_1 - 1}{2}, \frac{k_2 - 1}{2}\right) \otimes \left(\frac{k_1 - 1}{2}, \frac{k_2 - 1}{2}\right) = \bigoplus_{\ell_1=0}^{k_1-1} \bigoplus_{\ell_2=0}^{k_2-1} (\ell_1, \ell_2) \tag{2.5.21}$$

Mass eigenstates	Mass m^2	Multiplicity
$\Psi^{a}{}_{a'}$	$m^2_{\text{lower diag.}}=0$	$(N-k_2k_2)^2$
$\Psi^{n}{}_{a}$	$m^2_{\text{off diag.}}=\frac{k_1^2-1}{4}+\frac{k_2^2-1}{4}$	$2k_1k_2(N-k_2k_2)$
$\Psi^{n}{}_{n'}$	$m^2_{\text{upper diag.}}=\ell_1(\ell_1+1)+\ell_2(\ell_2+1)$	$(2\ell_2+1)(2\ell_1+1)$

Table 1: Masses and eigenstates of the easy bosons: $\Psi=\{A_0,A_1,A_2\}$, with respect to the block decomposition in (2.5.11), for $SO(3)\times SO(3)$ symmetric vevs. In the above, $\ell_1=0,\ldots,k_1-1$ and $\ell_2=0,\ldots,k_2-1$.

Where we have decomposed the reducible $\mathfrak{su}(2)\times\mathfrak{su}(2)$ representation on the LHS into the irreducible representations (ℓ_1,ℓ_2) on the RHS. If we now want to find fields which have definite masses, we need to decompose $E^{n}{}_{n'}$ into matrices which transform under the (ℓ_1,ℓ_2) irreps. of $\mathfrak{su}(2)\times\mathfrak{su}(2)$. In practise, we do this by choosing a different basis of matrices for the $k_1k_2\times k_1k_2$ block.

$$\Psi_{n,n'}E^{n}{}_{n'}=\sum_{\ell_1=0}^{k_1-1}\sum_{\ell_2=0}^{k_2-1}\sum_{m_1=-\ell_1}^{\ell_1}\sum_{m_2=-\ell_2}^{\ell_2}\Psi_{\ell_1,m_1;\ell_2,m_2}\,\hat{Y}^{m_1}_{\ell_1}\otimes\hat{Y}^{m_2}_{\ell_2} \tag{2.5.22}$$

The matrices $\hat{Y}^m_\ell$ are so called *fuzzy spherical harmonics*[3], and they are the matrix analogs of the well known spherical harmonic functions $Y^m_\ell(\vec{r})$ over $\mathbb{R}^3$. An explicit contruction of $\hat{Y}^m_\ell$ can be found in [10]. In what follows, we will not need thier explicit form, only the knowledge that they exists and that they can be defined implicitly as solutions to the equations.

$$L_3^{(s)}\hat{Y}^m_\ell=[t_3^{(s)},\hat{Y}^m_\ell]=m\hat{Y}^m_\ell \quad , \quad L^2_{(s)}\hat{Y}^m_\ell=[t^2_{(s)},\hat{Y}^m_\ell]=\ell_s(\ell_s+1)\hat{Y}^m_\ell \tag{2.5.23}$$

Plugging the expansion (2.5.22) into the easy part of $S_{\text{m,b}}$, and using the orthogonality realtions for $\hat{Y}^m_\ell$, we see that the fields in the $k_1k_2\times k_1k_2$ block all have the samme mass, which is given by the following expression.

$$\operatorname{tr}\left[(\hat{Y}^m_\ell)^\dagger\hat{Y}^{m'}_{\ell'}\right]=\delta_{m,m'}\delta_{\ell,\ell'} \quad , \quad (\hat{Y}^m_\ell)^\dagger=(-1)^m\hat{Y}^{-m}_\ell \tag{2.5.24}$$

$$m^2_{\text{upper diag.}}=\ell_1(\ell_1+1)+\ell_2(\ell_2+1) \quad , \quad \textit{multiplicity:}\ (2\ell_1+1)(2\ell_2+1) \tag{2.5.25}$$

This concludes the diagonalization of the term in $S_{\text{m,b}}$ containing the easy fields E. The masses and corresponding eigen-fields of the easy sector are summarized in table 1 for convenience. We now move on to the diagonalization of the part of $S_{\text{m,b}}$ containing the complicated fields.

2.5.1.2 The Complicated Fields Let us now begin the diagonalization procedure of the part of $S_{\text{m,b}}$ containing the complicated fields $\tilde{C}$. First, we perform a unitary transformation U to bring the $\tilde{T}_i$ matrices to the usual form of the $\mathfrak{su}(2)$ *spin*-1 irrep. To do this in practise, we form a 7×7 unitary matrix V from the 3×3 unitary matrix U in the following way.

$$V=\frac{1}{\sqrt{2}}\begin{pmatrix}U&0&0\\0&U&0\\0&0&1\end{pmatrix} \quad , \quad U=\frac{1}{\sqrt{2}}\begin{pmatrix}-1&0&1\\-i&0&-i\\0&\sqrt{2}&0\end{pmatrix} \tag{2.5.26}$$

[3] More information about these matrices can be found in appendix B.

After performing the unitary basis transformation U, the *spin*-1 generators given by: $T_i = U^\dagger \tilde{T}_i U$, end up with taking the well know forms, in which the T_3 generator specifically is diagonal.

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad T_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.5.27}$$

Under the unitary transformation V, the flavor-mixing matrices $S_i^{(1)}$ and $S_i^{(2)}$ transform such that: $S_i^{(s)} = V^\dagger \tilde{S}_i^{(s)} V$, and the 3×1 matrices $\tilde{R}_i$ transform such that: $R_i = U^\dagger \tilde{R}_i$. The total spin-orbit coupling operator in (2.5.10), is then given by the following after we perform the transfomation V.

$$S \cdot L \equiv \sum_{s \in \{1,2\}} S_{(s)} \cdot L_{(s)} = \begin{pmatrix} T_i L_i^{(1)} & 0 & R_i L_i^{(1)} \\ 0 & T_i L_i^{(2)} & R_i L_i^{(2)} \\ R_i^\dagger L_i^{(1)} & R_i^\dagger L_i^{(2)} & 0 \end{pmatrix} \tag{2.5.28}$$

$$R_i^\dagger L_i^{(s)} = i \left(\frac{1}{\sqrt{2}} L_+^{(s)}, -L_3^{(s)}, -\frac{1}{\sqrt{2}} L_-^{(s)} \right) \tag{2.5.29}$$

Where the operators: $L_\pm^{(s)} = L_1^{(s)} \pm i L_-^{(s)}$, are the usual ladder operators of $\mathfrak{su}(2)$. The result of acting with one of these ladder operators on a fuzzy spherical harmonic is the following.

$$L_\pm \hat{Y}_\ell^m = \sqrt{\ell(\ell+1) - m(m \pm 1)}\, \hat{Y}_\ell^{m \pm 1} \tag{2.5.30}$$

The complicated fields will also transforms under the unitary transformation V, in the following way.

$$C = V^\dagger \tilde{C} = \begin{pmatrix} C^{(1)} \\ C^{(2)} \\ A_3 \end{pmatrix} \tag{2.5.31}$$

The entries of the 3×1 column vectors $C^{(1)}$ and $C^{(2)}$ are given as linear combinations of the scalars.

$$C^{(1)} \equiv \begin{pmatrix} C_+^{(1)} \\ C_0^{(1)} \\ C_-^{(1)} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}(-\phi_1 + i\phi_2) \\ \phi_3 \\ \frac{1}{\sqrt{2}}(+\phi_1 + i\phi_2) \end{pmatrix} \tag{2.5.32}$$

$$C^{(2)} \equiv \begin{pmatrix} C_+^{(2)} \\ C_0^{(2)} \\ C_-^{(2)} \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{\sqrt{2}}(-\phi_4 + i\phi_5) \\ \phi_6 \\ \frac{1}{\sqrt{2}}(+\phi_4 + i\phi_5) \end{pmatrix} \tag{2.5.33}$$

The subscripts $+, 0, -$ of the fields above, denote their respective eigenvalues $1, 0, -1$ with respect to the generator T_3. Now that we have fleshed out what happens to all the objects connected with the complicated fields $\tilde{C}$ under the transformatio V, we are ready to make contact with ideas concerning addition of angular momentum. First, we define the total angular momentum operators $J_i^{(s)}$ as follow.

$$J_i^{(s)} = L_i^{(s)} + T_i \quad \Rightarrow \quad T_i L_i^{(s)} = \frac{1}{2} \left(J_{(s)}^2 - L_{(s)}^2 - T^2 \right) \tag{2.5.34}$$

Looking at the spin-orbit coupling operator (2.5.28), we see that if we can find eigenvectors of $T_i L_i^{(s)}$ which are also annihilated by $R_i^\dagger L_i^{(s)}$, we can obtain eigenvectors of the entire 7×7 matrix $S \cdot L$ by padding the 3-dimensional eigenvectors of the (1) and (2) sectors with 0's as shown below.

$$T_i L_i^{(s)}\, X^{(s)} = \lambda^{(s)}\, X^{(s)} \quad , \quad R_i^\dagger L_i^{(s)}\, X^{(s)} = 0$$

$$\Rightarrow \quad S \cdot L \begin{pmatrix} X^{(1)} \\ 0 \\ 0 \end{pmatrix} = \lambda^{(1)} \begin{pmatrix} X^{(1)} \\ 0 \\ 0 \end{pmatrix} \quad , \quad S \cdot L \begin{pmatrix} 0 \\ X^{(2)} \\ 0 \end{pmatrix} = \lambda^{(2)} \begin{pmatrix} 0 \\ X^{(2)} \\ 0 \end{pmatrix} \tag{2.5.35}$$

Not all eigenvectors of $S \cdot L$ will be of the above type, but it turns out the we can easily find the rest by diagonalizing a simple 3×3 matrix, as we will see shortly. First, we need to find eigenvectors of $T_i L_i^{(1)}$ and $T_i L_i^{(2)}$. At this point, it should be noted that we can focus on finding eigenvectors of the $k_1 k_2 \times k_1 k_2$ block of the decomposition (2.5.11). The reasons for this are as follow.

1. The fields in the lower diagonal $(N-k_1k_2) \times (N-k_1k_2)$ block are annihilated by both $S \cdot L$ and $L^2_{(s)}$, and so these fields always have zero mass.

2. The fields in the off-diagonal $(N-k_1k_2) \times k_1k_2$ and $k_1k_2 \times (N-k_1k_2)$ blocks are actually covered by the analysis of the $k_1k_2 \times k_1k_2$ block. This is because the $E^n{}_a$ and $E^a{}_n$ matrices transform in the $(\frac{k_1-1}{2}, \frac{k_2-1}{2})$ of $\mathfrak{su}(2) \times \mathfrak{su}(2)$, which is covered in the decomposition (2.5.21). We can thus map results from the $k_1k_2 \times k_1k_2$ block to the off-diagonal blocks by making the following substitutions.

$$\ell_1 \to \frac{k_1-1}{2} \quad , \quad \ell_2 \to \frac{k_2-1}{2} \quad , \quad \hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2} \to E^n{}_a, E^a{}_n \tag{2.5.36}$$

The multiplicities of the masses in the off-diagonal blocks can also be obtained from those in the $k_1k_2 \times k_1k_2$ block, simply by multiplying by $(N-k_1k_2)$, which is the number of possible values the a, a' indices can take.

Since we are now going to couple a *spin*-1 representation (*spanned by the generators* $\{T_i\}$) and a *spin*-ℓ_s representation (*spanned by the generators* $\{L_i^{(s)}\}$), we can express the eigenvectors of the $\{J^2_{(s)}, J^{(s)}_3, L^2_{(s)}, T^2\}$ operators as linear combinations of the eigenvectors of $\{L^2_{(s)}, L^{(s)}_3, T^2, T_3\}$. The expansion coefficients are given by the well known $\mathfrak{su}(2)$ *Clebsch Gordan coefficients*. Let us now write out this expansion explicitly.

$$(\mathcal{C}^{(1)})_{j_1,n_1,\ell_1;\ell_2,m_2} = \sum_{m_T=-1}^{+1} \sum_{m_1=-\ell_1}^{\ell_1} \langle \ell_1, m_1; 1, m_T | j_1, n_1 \rangle \, (C^{(1)}_{m_T})_{\ell_1,m_1;\ell_2,m_2} \tag{2.5.37}$$

$$(\mathcal{C}^{(2)})_{\ell_1,m_1;j_2,n_2,\ell_2} = \sum_{m_T=-1}^{+1} \sum_{m_2=-\ell_2}^{\ell_2} \langle \ell_2, m_2; 1, m_T | j_2, n_2 \rangle \, (C^{(2)}_{m_T})_{\ell_1,m_1;\ell_2,m_2} \tag{2.5.38}$$

Here, the $(C^{(s)}_{m_T})_{\ell_1,m_1;\ell_2,m_2}$ fields are the coefficient of the matrix valued fields $C^{(s)}$ (2.5.32, 2.5.33), when expanded in terms of the basis vectors $\hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2} \otimes \hat{e}_{m_T}$. The fields $(\mathcal{C}^{(s)})_{j_1,n_1,\ell_1;\ell_2,m_2}$ are also coefficients of the matrix valued fields $C^{(s)}$, but with respect to an expansion in terms of the following modified fuzzy spherical harmonics.

$$(\hat{Y}^{(1)})^{n_1,m_2}_{j_1,\ell_1,\ell_2} \equiv \hat{Y}^{n_1}_{j_1,\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2} \quad , \quad (\hat{Y}^{(2)})^{m_1,n_2}_{\ell_1,j_2,\ell_2} \equiv \hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{n_2}_{j_2,\ell_2} \tag{2.5.39}$$

$$\hat{Y}^n_{j,\ell} = \sum_{m_T=-1}^{+1} \sum_{m=-\ell}^{\ell} \langle \ell, m; 1, m_T | j, n \rangle \, \hat{Y}^m_\ell \otimes \hat{e}_{m_T} = \sum_{m=-\ell}^{\ell} \begin{pmatrix} \langle \ell, m; 1, +1 | j, n \rangle \, \hat{Y}^m_\ell \\ \langle \ell, m; 1, 0 | j, n \rangle \, \hat{Y}^m_\ell \\ \langle \ell, m; 1, -1 | j, n \rangle \, \hat{Y}^m_\ell \end{pmatrix} \tag{2.5.40}$$

Where the vectors $\hat{e}_{m_T}$ are eigenvectors of T_3 with eigenvalues m_T. In what follows, it will be most convenient to know of the following simplifying properties of the $\mathfrak{su}(2)$ Clebsch Gordan coefficients.

1. For the Clebsch Gordan coefficients to be non-vanishing, the following equality must hold true: $m_s + m_T = n_s$.

2. For the Clebsch Gordan coefficients to be non-vanishing, the following inequality must hold: $|\ell - \ell_T| \le j \le \ell + \ell_T$, where $\ell_T = 1$. This means that $j = \ell-1, \ell, \ell+1$, except for the case $\ell = 0$ for which $j = 1$.

Using the above information, we can simplify the sums appearing in (2.5.37), (2.5.38) and (2.5.40), and also slightly simplify the notation, using that $j=\ell+\alpha$ with $\alpha=-1,0,1$. The result of these simplifications are written out below.

$$(\mathcal{C}^{(1)}_{\alpha_1})_{\ell_1,n_1;\ell_2,m_2}=\sum_{m_T=-1}^{+1}\langle\ell_1,n_1-m_T;1,m_T|\ell_1+\alpha_1,n_1\rangle\,(C^{(1)}_{m_T})_{\ell_1,n_1-m_T;\ell_2,m_2} \tag{2.5.41}$$

$$(\mathcal{C}^{(2)}_{\alpha_2})_{\ell_1,m_1;\ell_2,n_2}=\sum_{m_T=-1}^{+1}\langle\ell_2,n_2-m_T;1,m_T|\ell_2+\alpha_2,n_2\rangle\,(C^{(2)}_{m_T})_{\ell_1,m_1;\ell_2,n_2-m_T} \tag{2.5.42}$$

$$(\hat{Y}^{(1)}_{\alpha_1})^{n_1,m_2}_{\ell_1,\ell_2}\equiv(\hat{Y}_{\alpha_1})^{n_1}_{\ell_1}\otimes\hat{Y}^{m_2}_{\ell_2}\quad,\quad(\hat{Y}^{(2)}_{\alpha_2})^{m_1,n_2}_{\ell_1,\ell_2}\equiv\hat{Y}^{m_1}_{\ell_1}\otimes(\hat{Y}_{\alpha_2})^{n_2}_{\ell_2} \tag{2.5.43}$$

$$(\hat{Y}_\alpha)^n_\ell=\sum_{m_T=-1}^{+1}\langle\ell,n-m_T;1,m_T|\ell+\alpha,n\rangle\,\hat{Y}^{n-m_T}_\ell\otimes\hat{e}_{m_T} \tag{2.5.44}$$

Using the fact that the $(\hat{Y}_\alpha)^n_\ell$ matrices are constructed to be eigenvectors of $\{J^2_{(s)},J^{(s)}_3,L^2_{(s)},T^2\}$, we can easily use (2.5.34) to find their eigenvalues with respect to the T_iL_i operators appearing in $S\cdot L$.

$$T_iL_i(\hat{Y}_\alpha)^n_\ell=\frac{1}{2}\left[(\ell+\alpha)(\ell+\alpha+1)-\ell(\ell+1)-2\right](\hat{Y}_\alpha)^n_\ell=\mu_\alpha(\hat{Y}_\alpha)^n_\ell \tag{2.5.45}$$

$$\mu_\alpha=\frac{1}{2}\left[\alpha(2\ell+\alpha+1)-2\right]=\begin{cases}\ell & \text{for } \alpha=+1\\ -1 & \text{for } \alpha=0\\ -(\ell+1) & \text{for } \alpha=-1\end{cases} \tag{2.5.46}$$

We need to find out which, if any, of the $(\hat{Y}_\alpha)^n_\ell$ matrices are annihilated by $R^\dagger_iL_i$. When we evaluate the action of $R^\dagger_iL_i$ on $(\hat{Y}_\alpha)^n_\ell$, it is convenient to make use of following Clebsch Gordan coefficients.

$$\langle\ell,n;1,0|\ell,n\rangle=\frac{n}{\sqrt{\ell(\ell+1)}} \tag{2.5.47}$$

$$\langle\ell,n\mp1;1,\pm1|\ell,n\rangle=\mp\frac{\sqrt{\ell(\ell+1)-n(n\pm1)}}{\sqrt{2\ell(\ell+1)}} \tag{2.5.48}$$

Using the above Clebsch Gordan coefficients together with (2.5.30) and (2.5.44), we can now evaluate the action of $R^\dagger_iL_i$ on $(\hat{Y}_\alpha)^n_\ell$. The result is the following.

$$\begin{aligned}R^\dagger_iL_i(\hat{Y}_\alpha)^n_\ell&=-i\sqrt{\ell(\ell+1)}\sum_{m_T=-1}^{+1}\sum_{m=-\ell}^{\ell}\langle\ell,n|\ell,m;1,m_T\rangle\,\langle\ell,m;1,m_T|\ell+\alpha,n\rangle\,\hat{Y}^n_\ell\\&=-i\sqrt{\ell(\ell+1)}\sum_{m_T=-1}^{+1}\langle\ell,n-m_T;1,m_T|\ell,n\rangle\,\langle\ell,n-m_T;1,m_T|\ell+\alpha,n\rangle\,\hat{Y}^n_\ell\\&=-i\delta_{\ell,\ell+\alpha}\sqrt{\ell(\ell+1)}\,\hat{Y}^n_\ell\end{aligned} \tag{2.5.49}$$

Where we have used that the Clebsch Gordan coefficients can be taken to be real, and that the set of states $\{|\ell,m;1,m_T\rangle\}$ form a complete basis. Alternatively to using the completeness of the Clebsch Gordan coefficients, the above sum can be explicitly evaluated using the *Wigner 3-j symbols* and the following relation.

$$\langle\ell_1,m_1;\ell_2,m_2|\ell,m\rangle=(-1)^{j_1-j_2+m}\begin{pmatrix}\ell_1 & \ell_2 & \ell\\ m_1 & m_2 & -m\end{pmatrix} \tag{2.5.50}$$

From (2.5.49), we can now contruct four eigenvectors of $S\cdot L$, using the prescription given in (2.5.35).

$$\begin{pmatrix} (\hat{Y}^{(1)}_{\alpha_1})^{n_1,m_2}_{\ell_1,\ell_2} \\ 0 \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 \\ (\hat{Y}^{(2)}_{\alpha_2})^{m_1,n_2}_{\ell_1,\ell_2} \\ 0 \end{pmatrix} \quad , \quad \alpha_1,\alpha_2=-1,+1 \tag{2.5.51}$$

Taking linear combinations of the above eigenvectors with the coefficient fields $(\mathcal{C}^{(1)}_0)_{\ell_1,n_1;\ell_2,m_2}$ and $(\mathcal{C}^{(1)}_0)_{\ell_1,n_1;\ell_2,m_2}$, plugging them into the complicated part of $S_{\text{m,b}}$ and using (2.5.46, 2.5.23), we find the following masses for the fields.

$$m^2_{(1),+}=\ell_1(\ell_1-1)+\ell_2(\ell_2+1) \quad , \quad \textit{miltiplicity: } (2\ell_1+3)(2\ell_2+1) \tag{2.5.52}$$
$$m^2_{(1),-}=(\ell_1+1)(\ell_1+2)+\ell_2(\ell_2+1) \quad , \quad \textit{miltiplicity: } (2\ell_1-1)(2\ell_2+1) \tag{2.5.53}$$
$$m^2_{(2),+}=\ell_2(\ell_2-1)+\ell_1(\ell_1+1) \quad , \quad \textit{miltiplicity: } (2\ell_2+3)(2\ell_1+1) \tag{2.5.54}$$
$$m^2_{(2),-}=(\ell_2+1)(\ell_2+2)+\ell_1(\ell_1+1) \quad , \quad \textit{miltiplicity: } (2\ell_2-1)(2\ell_1+1) \tag{2.5.55}$$

What remains now, is to find the last three eigenvectors of the 7×7 coupling matrix $S\cdot L$. In order to do this, we first write down the most general complicated field vector possible, which does not contain any of the four known eigenvectors.

$$\mathcal{C}=\begin{pmatrix} (\hat{Y}^{(1)}_0)^{n_1,m_2}_{\ell_1,\ell_2}(\mathcal{C}^{(1)}_0)_{\ell_1,n_1;\ell_2,m_2} \\ (\hat{Y}^{(2)}_0)^{m_1,n_2}_{\ell_1,\ell_2}(\mathcal{C}^{(2)}_0)_{\ell_1,m_1;\ell_2,n_2} \\ \hat{Y}^{m_1}_{\ell_1}\otimes\hat{Y}^{m_2}_{\ell_2}(A_3)_{\ell_1,m_1;\ell_2,m_2} \end{pmatrix} \tag{2.5.56}$$

We now want to take the complicated field vector above, and insert it into the term in $S_{\text{m,b}}$ containing the coupling operator $S\cdot L$. We will need the following matrix elements in order to simplify what we obtain after inserting $\mathcal{C}$.

$$\operatorname{tr}\Big[(\hat{Y}_{\alpha'})^\dagger_{\ell',n'}T_iL_i(\hat{Y}_\alpha)_{\ell,n}\Big]=\mu_\alpha\delta_{n,n'}\delta_{\ell,\ell'}\delta_{\alpha,\alpha'} \tag{2.5.57}$$

$$\operatorname{tr}\Big[(\hat{Y}^{m'}_{\ell'})^\dagger R^\dagger_iL_i(\hat{Y}_\alpha)_{\ell,n}\Big]=-i\delta_{n,m'}\delta_{\ell,\ell'}\delta_{\ell,\ell+\alpha}\sqrt{\ell(\ell+1)} \tag{2.5.58}$$

$$\operatorname{tr}\Big[(\hat{Y}_{\alpha'})^\dagger_{\ell',n'}R_iL_i\hat{Y}^m_\ell\Big]=+i\delta_{m,n'}\delta_{\ell,\ell'}\delta_{\ell,\ell+\alpha'}\sqrt{\ell(\ell+1)} \tag{2.5.59}$$

These results are easily obtained from (2.5.46) and (2.5.49) and the orthogonality relation for fuzzy spherical harmonics. We have also used the fact that $L^\dagger_i=L_i$, to obtain the last result. We now insert (2.5.56) into the part of $S_{\text{m,b}}$ containing $S\cdot L$, and simplify using the above traces. The result looks as follow.

$$\operatorname{tr}\left[\mathcal{C}^\dagger\, S\cdot L\,\mathcal{C}\right]=$$
$$\Big((\mathcal{C}^{(1)}_0)^\dagger\ (\mathcal{C}^{(2)}_0)^\dagger\ (A_3)^\dagger\Big)\begin{pmatrix} -1 & 0 & -i\sqrt{\ell_1(\ell_1+1)} \\ 0 & -1 & -i\sqrt{\ell_2(\ell_2+1)} \\ i\sqrt{\ell_1(\ell_1+1)} & i\sqrt{\ell_2(\ell_2+1)} & 0 \end{pmatrix}\begin{pmatrix} \mathcal{C}^{(1)}_0 \\ \mathcal{C}^{(2)}_0 \\ A_3 \end{pmatrix} \tag{2.5.60}$$

Where we have dropped all but the α-indices on the coefficient fields in the above, in order to make the result more easily readable. We can now easily transform the above matrix into diagonal form.

$$\Big((\mathfrak{C}_0)^\dagger\ (\mathfrak{C}_+)^\dagger\ (\mathfrak{C}_-)^\dagger\Big)\begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_+ & 0 \\ 0 & 0 & \lambda_- \end{pmatrix}\begin{pmatrix} \mathfrak{C}_0 \\ \mathfrak{C}_+ \\ \mathfrak{C}_- \end{pmatrix} \tag{2.5.61}$$

Where the eigenvalues λ_0, $\lambda_\pm$ and the corresponding eigen-fields $\mathfrak{C}_0$, $\mathfrak{C}_\pm$ are given by the following.

Mass eigenstates	Mass m^2	Multiplicity
$\mathcal{C}_+^{(1)}$	$m^2_{(1),+} = \ell_1(\ell_1-1)+\ell_2(\ell_2+1)$	$(2\ell_1+3)(2\ell_2+1)$
$\mathcal{C}_-^{(1)}$	$m^2_{(1),-} = (\ell_1+1)(\ell_1+2)+\ell_2(\ell_2+1)$	$(2\ell_1-1)(2\ell_2+1)$
$\mathcal{C}_+^{(2)}$	$m^2_{(2),+} = \ell_2(\ell_2-1)+\ell_1(\ell_1+1)$	$(2\ell_2+3)(2\ell_1+1)$
$\mathcal{C}_-^{(2)}$	$m^2_{(2),-} = (\ell_2+1)(\ell_2+2)+\ell_1(\ell_1+1)$	$(2\ell_2-1)(2\ell_1+1)$
$\mathfrak{C}_0$	$m_0^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)+2$	$(2\ell_1+1)(2\ell_2+1)$
$\mathfrak{C}_+$	$m_+^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)-2\lambda_+$	$(2\ell_1+1)(2\ell_2+1)$
$\mathfrak{C}_-$	$m_-^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)-2\lambda_-$	$(2\ell_1+1)(2\ell_2+1)$

Table 2: Masses and eigenstates of the complicated bosons in the $k_1k_2 \times k_1k_2$ block for the case of $SO(3)\times SO(3)$ symmetric vevs. In the above, $\ell_1 = 1,\dots,k_1-1$ and $\ell_2 = 1,\dots,k_2-1$. The masses and multiplicities for the fields in the off-diagonal blocks can be obtained from those above by using (2.5.36) and the following steps.

$$\lambda_0 = -1 \quad , \quad \lambda_\pm = -\frac{1}{2} \pm \sqrt{\ell_1(\ell_1+1)+\ell_2(\ell_2+1)+\frac{1}{4}} \tag{2.5.62}$$

$$\mathfrak{C}_0 = \frac{1}{\sqrt{N_0}}\left(-\sqrt{\ell_2(\ell_2+1)}\,\mathcal{C}_0^{(1)} + \sqrt{\ell_1(\ell_1+1)}\,\mathcal{C}_0^{(2)}\right) \tag{2.5.63}$$

$$\mathfrak{C}_\pm = \frac{1}{\sqrt{N_\pm}}\left(i\sqrt{\ell_1(\ell_1+1)}\,\mathcal{C}_0^{(1)} + i\sqrt{\ell_2(\ell_2+1)}\,\mathcal{C}_0^{(2)} + \lambda_\mp A_3\right) \tag{2.5.64}$$

$$N_0 = -\lambda_+\lambda_- \quad , \quad N_\pm = \lambda_\mp(\lambda_\mp - \lambda_\pm) \tag{2.5.65}$$

Using the above eigenvalues with respect to $S\cdot L$ together with the $L^2_{(s)}$ eigenvalues given in (2.5.23), we obtain that following masses for the fields $\mathfrak{C}_0$ and $\mathfrak{C}_\pm$ from the form of $S_{\mathrm{m,b}}$ given in (2.5.10).

$$m_0^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)-2\lambda_0 \quad , \quad \textit{miltiplicity: } (2\ell_1+1)(2\ell_2+1) \tag{2.5.66}$$
$$m_+^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)-2\lambda_+ \quad , \quad \textit{miltiplicity: } (2\ell_1+1)(2\ell_2+1) \tag{2.5.67}$$
$$m_-^2 = \ell_1(\ell_1+1)+\ell_2(\ell_2+1)-2\lambda_- \quad , \quad \textit{miltiplicity: } (2\ell_1+1)(2\ell_2+1) \tag{2.5.68}$$

This concludes the diagonalization of the complicated fields. The masses and corresponding eigenfields of the complicated sector are summarized in table 2 for convenience. We note here that for the case of $\ell_s = 0$, the fields $\mathcal{C}_-^{(s)}$, $\mathcal{C}_0^{(s)}$ do not exist, as we must have $j=1$. Furthermore, the fields $\mathfrak{C}_0$ do not exist either, since the matrix in (2.5.61) is effectively reduced to a 2×2 matrix by the absence of $\mathcal{C}_0^{(s)}$ [10]. We will now move on to discuss the diagonalization procedure for the case of $SO(5)$ symmetric vevs.

2.5.2 The boson mass matrix for SO(5) symmetric vevs

Work in progress..

2.6 The field propagators

Now that we know the masses of the fields which diagonalize the mass terms $S_{\text{m,b}}$ and $S_{\text{m,f}}$, we have almost done all the ground work needed to begin perturbative calculations in our dCFT setups. The last unusual thing we need to deal with, is the spacetime dependence in $S_{\text{m,b}}$ and $S_{\text{m,f}}$, which effectively change the standard Minkowski-space propagator equation to the following.

$$\left(-\partial_\mu \partial^\mu + \frac{m^2}{x_3^2}\right) K(x,y) = \frac{g^2}{2}\delta(x-y) \tag{2.6.1}$$

The trick to solving the above equation [9], is to define the following modified propagator $\tilde{K}(x,y)$.

$$K(x,y) = \frac{g^2}{2}\frac{\tilde{K}(x,y)}{x_3 y_3} \tag{2.6.2}$$

We can now insert the above definition into the propagator equation (2.6.1). The result is as follow.

$$\left(-x_3^2\partial_\mu\partial^\mu + 2x_3\partial_3 + m^2 - 2\right)\tilde{K}(x,y) = x_3^4\delta(x-y) \tag{2.6.3}$$

The above form of the propagator equation is indetical to the propagator equation of AdS_4 space. To see this, we first write the general curved space version of the Klein Gordon propagator equation.

$$\left(-\nabla_\mu\nabla^\mu + \tilde{m}^2\right) K_{AdS}(x,y) = \frac{\delta(x-y)}{\sqrt{g}} \tag{2.6.4}$$

Given that we use Poincaré coordinates on AdS_4, the metric components take the following form.

$$g_{\mu\nu} = \frac{1}{x_3^2}\delta_{\mu\nu} \quad , \quad g^{\mu\nu} = x_3^2\delta^{\mu\nu} \quad , \quad \sqrt{g} = \frac{1}{x_3^4} \tag{2.6.5}$$

$$-\nabla_\mu\nabla^\mu = -\frac{1}{\sqrt{g}}\partial_\mu\left(g^{\mu\nu}\sqrt{g}\,\partial_\nu\right) = -x_3^2\partial_\mu\partial^\mu + 2x_3\partial_3 \tag{2.6.6}$$

Inserting the above into equation (2.6.4), we find it takes exactly the same form as (2.6.3), given that we make the identification: $\tilde{m}^2 = m^2 - 2$. The solutions to equation (2.6.3) as propagators on AdS_4 are well known in the literature, and can be written in the following way using *hypergeometric functions* [10]. The solutions to equation (2.6.1) can then be written as follow.

$$K^{m^2}(x,y) = \frac{g^2}{16\pi^2 x_3 y_3}\binom{2\nu+1}{\nu+\frac{1}{2}}^{-1}\frac{{}_2F_1\left(\nu-\frac{1}{2},\nu+\frac{1}{2};2\nu+1;-\xi^{-1}\right)}{(1+\xi)\xi^{\nu+\frac{1}{2}}} \tag{2.6.7}$$

$$\xi = \frac{|x-y|^2}{4x_3y_3} \quad , \quad \nu = \sqrt{m^2+\frac{1}{4}} \tag{2.6.8}$$

We note here, that the specific combination ξ of x_3 and y_3, appearing in the above propagator, is actually connected to the $SO(3,2)$ symmetry of the dCFT setups. The same goes for the combination $(x_3y_3)^{-1}$. More on this in section 3. For a treatment of the fermion propagators, see for example [9].

3 Two-point functions of chiral primary operators

In the following section we concern ourselves with the computations of two-point functions between a certain type of local single-trace scalar operators in $\mathcal{N}=4$ SYM; namely oprators of the following simple form.

$$\mathcal{O}_{\boldsymbol{Z}} = \operatorname{tr} \boldsymbol{Z}^L \quad , \quad \mathcal{O}_{\boldsymbol{X}} = \operatorname{tr} \boldsymbol{X}^L \quad , \quad \boldsymbol{Z} = \boldsymbol{\phi}_3 + i\,\boldsymbol{\phi}_6 \quad , \quad \boldsymbol{X} = \boldsymbol{\phi}_1 + i\,\boldsymbol{\phi}_4 \tag{3.0.1}$$

Where we have suppressed the spacetime dependence of all scalar fields in the above definitions, to avoid notational clutter. We will attempt to compute two-point functions between the above operators using the standard field theoretic perturbative technique of Feynman diagrams, which is only possible due to the ground work layed out in section 2 (*gauge fixing, mass matrix diagonalization etc.*). The reason we focus on these operators in particular is very pratical; the local operators $\mathcal{O}_{\boldsymbol{Z}}$ and $\mathcal{O}_{\boldsymbol{X}}$ are perhaps some of the simplest examples of *chiral primary operators*. In order to understand what makes chiral primary operators comparatively easier to study, we first have to know what they are exactly.

3.1 Chiral primary operators in SCFTs

In the context of super conformal field theories, a chiral primary operator is first of all a *super conformal primary* operator. By definition, super conformal primary operators have to satisfy the conditions.

$$[D, \mathcal{O}(0)] = -i\Delta\,\mathcal{O}(0) \quad , \quad [S^a_\alpha, \mathcal{O}(0)] = 0 \quad , \quad [\bar{S}_{a\dot{\alpha}}, \mathcal{O}(0)] = 0 \tag{3.1.1}$$

Where S^a_α are super-partners to the generators of special conformal transformations K_μ, and D is the generator of dilatations (*scale transformations*). Furthermore, a chiral primary operator needs to be anihilated by at least one of the super-partners Q^a_α to the generators of spacetime translations P_μ.

$$\exists a \in \{1, \ldots, \mathcal{N}\}\, \exists \alpha \in \{1,2\} : \; [Q^a_\alpha, \mathcal{O}(0)] = 0 \tag{3.1.2}$$

The conditions (3.1.1) and (3.1.2) together leads to a very powerful restriction on the conformal dimensions Δ of chiral primaries; namely that they can not depend on the coupling constant g. To show that this is indeed the case, we need the following anti-commutation relation from the super conformal algebra $\mathfrak{psu}(2,2|4)$ [3].

$$\{Q^a_\alpha, S_{b\beta}\} = -\frac{i}{2}\,\varepsilon_{\alpha\beta}(\sigma^{IJ})^a{}_b\, R_{IJ} - \frac{1}{2}\varepsilon_{\alpha\beta}\delta^a{}_b D + (\sigma^{\mu\nu})_{\alpha\beta}\delta^a{}_b\, M_{\mu\nu} \tag{3.1.3}$$

Where $(\sigma^{IJ})^a{}_b$ are the sigma matrices of the $SU(4)$ fundamental representation, R_{IJ} are the generators of the $SO(6)$ R-symmetry, and $M_{\mu\nu}$ are the generators of $SO(3,1)$. If we now assume that $\mathcal{O}$ is a scalar operator, such that $[M_{\mu\nu}, \mathcal{O}(0)] = 0$, we can use the above anti-commutation relations of the super conformal algebra, the graded Jacobi identity and the aforementioned two conditions (3.1.1) and (3.1.2) to obtain.

$$\begin{aligned}
[\{Q^a_\alpha, S_{b\beta}\}, \mathcal{O}(0)] &= [-i\,\varepsilon_{\alpha\beta}(\sigma^{IJ})^a{}_b\, R_{IJ} - \varepsilon_{\alpha\beta}\delta^a{}_b D, \mathcal{O}(0)] = 0 \\
\Leftrightarrow \quad (\sigma^{IJ})^a{}_b\, [R_{IJ}, \mathcal{O}(0)] &= \Delta\,\delta^a{}_b\, \mathcal{O}(0)
\end{aligned} \tag{3.1.4}$$

We have now shown that any chiral primary $\mathcal{O}$ has to satisfy (3.1.4). It is also true that any super conformal primary $\mathcal{O}$ satisfying (3.1.4) has to be a chiral primary. This is because.

$$[[\mathcal{O}(0), Q^a_\alpha], S_{b\beta}] = 0 \quad \Rightarrow \quad [\mathcal{O}(0), Q^a_\alpha] = 0 \tag{3.1.5}$$

Since Q^a_α would otherwise increase the conformal dimension of $\mathcal{O}$ by $1/2$, making $S_{b\beta}$ unable to anihilate $\mathcal{O}$. Now, because the eigenvalues of the $SO(6)$ generators (*also know as the R-charges*) are discrete and unchanged when interactions are included, the conformal dimensions Δ of the chiral primaries will also remain unchanged because of (3.1.4). Thus, the conformal dimensions of chiral primaries can not receive any quantum corrections (*also known as anomolus dimensions*) when interactions are included. Comming back to equation (3.1.4), it can easily be checked using the explicit form of the $(\sigma^{IJ})^a{}_b$ matrices.

$$\sigma^{14} = \text{diag}(1,1,-1,-1) \quad , \quad \sigma^{25} = \text{diag}(1,-1,1,-1) \quad , \quad \sigma^{36} = \text{diag}(1,-1,-1,1) \tag{3.1.6}$$

That a solution to this equation is given by operators with R-charges $(J_1,0,0)$ and conformal dimension $\Delta = J_1$, for $a = 1,2$. The R-charges correspond to eigenvalues of R_{14}, R_{25} and R_{36}, such that.

$$[R_{14}, \mathcal{O}(0)] = J_1\,\mathcal{O}(0) \quad , \quad [R_{25}, \mathcal{O}(0)] = [R_{36}, \mathcal{O}(0)] = 0 \tag{3.1.7}$$

It turns out that conformal primary operators with R-charges $(J_1,0,0)$ and $\Delta = J_1$, are also anihilated by the super-charges $\bar{Q}_{3,\dot{\alpha}}$ and $\bar{Q}_{4,\dot{\alpha}}$. This follows from yet another anti-commutation relation from the super conformal algebra $\mathfrak{psu}(2,2|4)$, together with (3.1.6).

$$\{\bar{Q}_{a\dot{\alpha}}, \bar{S}^b_{\dot{\beta}}\} = \frac{i}{2}\,\varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^{IJ})_a{}^b\,R_{IJ} - \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\delta_a{}^b D + (\sigma^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\delta_a{}^b\,M_{\mu\nu} \tag{3.1.8}$$

$$\Leftrightarrow \quad (\sigma^{IJ})_a{}^b\,[R_{IJ}, \mathcal{O}(0)] = -\Delta\,\delta_a{}^b\,\mathcal{O}(0) \tag{3.1.9}$$

Thus, the operators with R-charges $(J_1,0,0)$ and $\Delta = J_1$ are anihilated by half of the translational super charges. Such operators are referred to as $1/2$-*BPS operators*. Of cause there is nothing particularly special about the R-charges $(J_1,0,0)$, compared to the $(0,J_2,0)$ or the $(0,0,J_3)$ R-charges. Therefore, operators which have these collections of R-charges also solve (3.1.4) for half the values of a and $\dot{\alpha}$, given that Δ is chosen analogously. Now that we know that these types of operators constitute chiral primaries, we can quickly verify that $\mathcal{O}_{\boldsymbol{Z}}$ and $\mathcal{O}_{\boldsymbol{X}}$ are indeed such operators. With the definitions of $\boldsymbol{X}$ and $\boldsymbol{Z}$ given in (3.0.1), it is easy to verify that.

$$[R^{(0)}_{14}, \mathcal{O}_{\boldsymbol{X}}(0)] = L\,\mathcal{O}_{\boldsymbol{X}}(0) \quad , \quad [R^{(0)}_{36}, \mathcal{O}_{\boldsymbol{Z}}(0)] = L\,\mathcal{O}_{\boldsymbol{Z}}(0) \tag{3.1.10}$$

$$[D^{(0)}, \mathcal{O}_{\boldsymbol{X}}(0)] = -i\,L\,\mathcal{O}_{\boldsymbol{X}}(0) \quad , \quad [D^{(0)}, \mathcal{O}_{\boldsymbol{Z}}(0)] = -i\,L\,\mathcal{O}_{\boldsymbol{Z}}(0) \tag{3.1.11}$$

Where $D^{(0)}$ and $R^{(0)}_{IJ}$ denotes the dilatation operator and $SO(6)$ generators respectively, at zero coupling. Thus, we see that for the free theory, these operators have the R-charges $(L,0,0)$ and $(0,0,L)$, and conformal dimensions $\Delta = J_1 = L$ and $\Delta = J_3 = L$ respectively. Furthermore, since $\mathcal{O}_{\boldsymbol{Z}}$, $\mathcal{O}_{\boldsymbol{X}}$ are constructed out of scalar fields only, and the scalar fields have the lowest conformal dimension of all $\mathcal{N}=4$ fields, they must indeed be super conformal primaries. Because $\mathcal{O}_{\boldsymbol{X}}$ and $\mathcal{O}_{\boldsymbol{Z}}$ are super conformal primaries, and also solve the equation (3.1.4) at zero coupling, they must also commute with half the translational supercharges at zero coupling. Because the number of operators in a given $\mathfrak{psu}(2,2|4)$ representation does not depend on the coupling, we conclude that the operators $\mathcal{O}_{\boldsymbol{X}}$ and $\mathcal{O}_{\boldsymbol{Z}}$ have R-charges $(L,0,0)$ and $(0,0,L)$, and conformal dimensions $\Delta = L$, for all values of the coupling constant g.

Instead of thinking of chiral primaries as operators with a given set of R-charges, it is often useful to think of them as elements of some tensor representation of $SU(4) \simeq SO(6)$. The operators $\mathcal{O}_{\boldsymbol{X}}$ and $\mathcal{O}_{\boldsymbol{Z}}$ in particular turn out to be highest weight operators of the L-fold symmetric-traceless representation of $SO(6)$.

$$\mathcal{O}_{\boldsymbol{X}} = \Phi^{i_1 \ldots i_L}_{\boldsymbol{X}}\,\mathrm{tr}\big[\boldsymbol{\phi}_{i_1}\cdots\boldsymbol{\phi}_{i_L}\big] \quad , \quad \mathcal{O}_{\boldsymbol{Z}} = \Phi^{i_1 \ldots i_L}_{\boldsymbol{Z}}\,\mathrm{tr}\big[\boldsymbol{\phi}_{i_1}\cdots\boldsymbol{\phi}_{i_L}\big] \tag{3.1.12}$$

Where the wavefunctions $\Phi^{i_1 \ldots i_L}_{\boldsymbol{X}}$ and $\Phi^{i_1 \ldots i_L}_{\boldsymbol{Z}}$, are symmetric and traceless in any pair of indices. To see that this is indeed the case, let us look at $\mathcal{O}_{\boldsymbol{Z}}$ as an example. Expanding $\boldsymbol{Z}$-field number ℓ and $\ell+1$ in the trace, we find that.

$$\cdots \boldsymbol{Z}(\phi_3 + i\phi_6)(\phi_3 + i\phi_6)\boldsymbol{Z}\cdots = \cdots \boldsymbol{Z}(\phi_3\phi_3 - \phi_6\phi_6 + i\phi_3\phi_6 + i\phi_6\phi_3)\boldsymbol{Z}\cdots \tag{3.1.13}$$

$$\Phi_{\boldsymbol{Z}}^{...36...} = \Phi_{\boldsymbol{Z}}^{...63...} \quad , \quad \Phi_{\boldsymbol{Z}}^{...33...} + \Phi_{\boldsymbol{Z}}^{...66...} = 0 \quad , \quad \Phi_{\boldsymbol{Z}}^{...i_\ell i_{\ell+1}...} = 0 \quad \text{for} \quad i_\ell, i_{\ell+1} \neq 3,6 \tag{3.1.14}$$

Using the information contained in (3.1.14), we can now conclude that the wavefunction $\Phi_{\boldsymbol{Z}}^{i_1...i_L}$ obeys.

$$\Phi_{\boldsymbol{Z}}^{i_1...i_\ell\, i_{\ell+1}...i_L} = \Phi_{\boldsymbol{Z}}^{i_1...i_{\ell+1} i_\ell...i_L} \quad , \quad \sum_{i_\ell=1}^{6} \Phi_{\boldsymbol{Z}}^{i_1...i_\ell\, i_\ell...i_L} = 0 \tag{3.1.15}$$

It is clear that by the same arguments, the wavefunction $\Phi_{\boldsymbol{X}}^{i_1...i_L}$ also has exactly the above properties.

3.2 Symmetries and the form of two-point functions in dCFTs

Before we make any attempts to compute the leading order corrections to any two-point functions, let us briefly discuss how the remaining unbroken symmetries in our dCFT setups restrict the forms of scalar two-point functions. Firstly, the unbroken $SO(2,1)$ Poincaré invariance parallel to the defect, dictates that any scalar two-point function $\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle$ must be a function of the combination $|\vec{x}-\vec{y}|$. Here, x, y are any points on the $x_3, y_3 > 0$ side of the defect, and $|\vec{x}-\vec{y}|$ is the standard Euclidian norm. We additionally use the notation.

$$x = (\vec{x}, x_3) \quad , \quad \vec{x} = (x_0, x_1, x_2) \quad , \quad |x-y|^2 = |\vec{x}-\vec{y}|^2 + |x_3 - y_3|^2 \tag{3.2.1}$$

We can further restrict the form of $\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle$ by taking advantage of the fact that the discrete symmetry known as *inversion*, also remains unbroken by the non-zero vevs.

$$x \to \tilde{x} = \frac{x}{|x|^2} \quad \Rightarrow \quad |x| \to |\tilde{x}| = \frac{|x|}{|x|^2} = \frac{1}{|x|} \quad \Rightarrow \quad \tilde{x} \to \frac{\tilde{x}}{|\tilde{x}|^2} = x \tag{3.2.2}$$

The invariance under inversion and $SO(2,1)$ Poincaré symmetries greatly restricts the possible forms of $\langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle$, which stems from the fact that only one so called *conformal ratio* ξ, can be constructed such as to be invaraint under the aforementioned symmetries. Using (3.2.2) we find that.

$$|\tilde{x}-\tilde{y}|^2 = \frac{|x-y|^2}{x^2y^2} \quad , \quad \tilde{x}_3\tilde{y}_3 = \frac{x_3y_3}{x^2y^2} \quad , \quad \xi = \frac{|x-y|^2}{4x_3y_3} \quad \Rightarrow \quad \tilde{\xi} = \xi \tag{3.2.3}$$

Finally, to ensure the correct scaling under the last unbroken symmetry: $SO(1,1)$ *scale transformations*, the two-point functions are forced to be of the following form.

$$x \to \tilde{x} = \lambda x \quad , \quad \mathcal{O}_a(x) \to \tilde{\mathcal{O}}_a(x) = \lambda^{-\Delta_a}\mathcal{O}_a(\lambda^{-1}x) \tag{3.2.4}$$

$$\Rightarrow \quad \langle \mathcal{O}_a(x)\mathcal{O}_b(y)\rangle = \frac{f_{ab}(\xi)}{(2x_3)^{\Delta_a}(2y_3)^{\Delta_b}} \tag{3.2.5}$$

We now see that the only freedom left on the form of scalar two-point functions are encoded in the functions $f_{ab}(\xi)$, which depend only on the scalar operators in question and the conformal ratio ξ. Thus, for chiral primary operators, which have their conformal dimension Δ protected, the information that we want to later extract from our perturbative computations are exactly these conformal functions $f_{ab}(\xi)$.

Another way to extract information about two-point functions based on symmetry is to consider the limit of points far away from the defect. We know that the scalar vevs vanish when we take the distance from the defect to ∞, which means that the two-point function should reduce to the standard form dictated by the full $SO(4,2)$ symmetry.

$$\lim_{z_3\to\infty} \big\langle \mathcal{O}_a(x+z)\bar{\mathcal{O}}_b(y+z)\big\rangle = \frac{M_{ab}}{|x-y|^{\Delta_a+\Delta_b}} \tag{3.2.6}$$

From this consideration we learn two things. Firstly, we see that conformal dimensions of operators remain unchanged in this limit. Therefore, if we want to find corrections to the conformal dimensions of operators caused by interactions, it will be easier to do the analysis in the standard $\mathcal{N} = 4$ theory and subsequently carry over the results to the defect theory. More on how to do this in section 4. Secondly, to ensure the correct limiting behavior of the conformal functions $f_{ab}(\xi)$ when going far from the defect, they must be of the following form.

$$f_{ab}(\xi) = \xi^{-\frac{\Delta_a+\Delta_b}{2}} \left[M_{ab} + \sum_{n=1}^{\infty} c_{ab,n}\, \xi^n \right] \tag{3.2.7}$$

Thus, we can not have terms with arbitrarily high negative powers of ξ appearing in the conformal functions $f_{ab}(\xi)$. This fact can provide a nice check of the results obtained by perturbative methods.

3.3 Two-point functions at tree level (*disconnected*)

In this subsection, we finally begin the computation of two-point functions between the chiral primary operators $\mathcal{O}_{\boldsymbol{X}}(x)$ and $\mathcal{O}_{\boldsymbol{Z}}(x)$. There are three different kinds of two-point functions we can make from these operators.

$$\langle \mathcal{O}_{\boldsymbol{Z}}(x)\, \mathcal{O}_{\boldsymbol{Z}}(y) \rangle \quad , \quad \langle \mathcal{O}_{\boldsymbol{Z}}(x)\, \mathcal{O}_{\bar{\boldsymbol{Z}}}(y) \rangle \quad , \quad \langle \mathcal{O}_{\boldsymbol{X}}(x)\, \mathcal{O}_{\boldsymbol{Z}}(y) \rangle \tag{3.3.1}$$

To start of, we want to find the disconnected tree-level contribution to these two-point functions, obtained by inserting the classical values of the $\boldsymbol{Z}$ and $\boldsymbol{X}$ fields into the operators $\mathcal{O}_{\boldsymbol{X}}(x)$ and $\mathcal{O}_{\boldsymbol{Z}}(x)$.

$$\begin{aligned} &\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,dc.}} = \operatorname{tr} \mathcal{Z}^{L_1} \operatorname{tr} \mathcal{Z}^{L_2} \quad , \quad \left\langle \operatorname{tr} \boldsymbol{Z}^{L_1} \operatorname{tr} \bar{\boldsymbol{Z}}^{L_2} \right\rangle_{\text{tree,dc.}} = \operatorname{tr} \mathcal{Z}^{L_1} \operatorname{tr} \bar{\mathcal{Z}}^{L_2} \\ &\left\langle \operatorname{tr} \boldsymbol{X}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,dc.}} = \operatorname{tr} \mathcal{X}^{L_1} \operatorname{tr} \mathcal{Z}^{L_2} \quad , \quad \boldsymbol{Z} = \mathcal{Z} + Z \quad , \quad \boldsymbol{X} = \mathcal{X} + X \end{aligned} \tag{3.3.2}$$

Where $\bar{\boldsymbol{Z}}$ is just the adjoint of $\boldsymbol{Z}$. Also, the spacetime dependence of the $\boldsymbol{X}$ and $\boldsymbol{Z}$ fields in the above have been suppressed to avoid notational clutter. We will continue to do so throughout this section.

3.3.1 SO(3) $\times$ SO(3) symmetric vevs

The classical parts of the $\boldsymbol{X}$ and $\boldsymbol{Z}$ fields in the $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric setup, can easily be expressed in terms of the classical parts of the scalar fields they are constructed from (3.0.1). The results are as follow.

$$\mathcal{Z} = \Phi_3 + i\,\Phi_6 = -\frac{1}{x^3}\left(t_3^{k_1} \otimes \mathbb{1}_{k_2} \oplus 0_{N-k_1k_2} + i\,\mathbb{1}_{k_1} \otimes t_3^{k_2} \oplus 0_{N-k_1k_2}\right) \tag{3.3.3}$$

$$\mathcal{X} = \Phi_1 + i\,\Phi_4 = -\frac{1}{x^3}\left(t_1^{k_1} \otimes \mathbb{1}_{k_2} \oplus 0_{N-k_1k_2} + i\,\mathbb{1}_{k_1} \otimes t_1^{k_2} \oplus 0_{N-k_1k_2}\right) \tag{3.3.4}$$

To find the tree-level contributions to (3.3.1), we need to evaluate $\operatorname{tr}\mathcal{X}^L$ and $\operatorname{tr}\mathcal{Z}^L$, as can be seen in (3.3.2). As one might intuitively expect, there exists a unitary transformation U, which takes $t_1 \to t_3$ and $t_3 \to t_1$, which means that the two kinds of traces are equal.

$$U t_1 U^\dagger = t_3 \quad , \quad U t_2 U^\dagger = -t_2 \quad , \quad U t_3 U^\dagger = t_1 \quad , \quad U = e^{i\pi t_3} e^{i\pi t_2/2} \tag{3.3.5}$$

$$\operatorname{tr}[\mathcal{X}\cdots\mathcal{X}] = \operatorname{tr}\big[V\mathcal{X}V^\dagger \cdots V\mathcal{X}V^\dagger\big] = \operatorname{tr}[\mathcal{Z}\cdots\mathcal{Z}] \quad , \quad V = U^{k_1} \otimes U^{k_2} \oplus 0_{N-k_1k_2} \tag{3.3.6}$$

Thus, we need only evaluate $\operatorname{tr}\mathcal{Z}^L$. To do this, we need to know an expression for $\operatorname{tr}\left(t_3^k\right)^L$. It has been shown in [16], that this particular trace can in fact be written in terms of *Bernoulli polynomials* $B_n(x)$, and is given by the following.

$$\operatorname{tr}\left(t_3^k\right)^L = \begin{cases} (-1)^{L+1} \frac{2}{L+1} B_{L+1}\left(\frac{1-k}{2}\right) \simeq \frac{k^{L+1}}{2^L(L+1)} & \text{for } L \text{ even} \\ 0 & \text{for } L \text{ odd} \end{cases} \tag{3.3.7}$$

Where $\simeq$ in this context means equal to highest order in k. Using the above expression for $\operatorname{tr}\left(t_3^k\right)^L$, we can now write down an expression for $\operatorname{tr}\mathcal{Z}^L$, given in terms of the following binomial expansion.

$$\operatorname{tr}\mathcal{Z}^L = \sum_{n=0}^{L}\binom{L}{n} i^n \operatorname{tr}\left[(t_3^{k_1})^{L-n}\otimes(t_3^{k_2})^n \oplus 0_{N-k_1k_2}\right]$$

$$= \sum_{n=0}^{L}\binom{L}{n} i^n \operatorname{tr}\left[(t_3^{k_1})^{L-n}\right]\operatorname{tr}\left[(t_3^{k_2})^n\right] \simeq \frac{(-1)^L}{x_3^L}\sum_{n=0}^{L}\binom{L}{n}\frac{i^n k_1^{L-n+1}k_2^{n+1}}{2^L(n+1)(L-n+1)} \tag{3.3.8}$$

The above sum has been evaluated in [11], in the limit of large k_1, k_2. The final result looks as follow.

$$\operatorname{tr}\mathcal{X}^L = \operatorname{tr}\mathcal{Z}^L = \operatorname{tr}\bar{\mathcal{Z}}^L \simeq \frac{(-i)^L(k_1^2+k_2^2)^{\frac{L}{2}+1}\sin[(L+2)\phi_0]}{2^L x_3^L(L+1)(L+2)} \tag{3.3.9}$$

Where $\tan(\phi_0)\equiv k_1/k_2$. Thus, we find that in the $\mathfrak{so}(3)\times\mathfrak{so}(3)$ setup, the disconnected part of the tree-level contribution to the two-point functions (3.3.1) are given by the following expression.

$$\boxed{\begin{aligned} &\mathfrak{so}(3)\times\mathfrak{so}(3):\quad \left\langle \operatorname{tr}Z^{L_1}\operatorname{tr}Z^{L_2}\right\rangle_{\text{tree,dc.}} \simeq \frac{f_{\text{tree,dc.}}}{(2x_3)^{L_1}(2y_3)^{L_2}} && (3.3.10a)\\ &f_{\text{tree,dc.}} = (-1)^{\frac{L_1+L_2}{2}}\frac{(k_1^2+k_2^2)^{\frac{L_1+L_2}{2}+2}\sin[(L_1+2)\phi_0]\sin[(L_2+2)\phi_0]}{(L_1+1)(L_1+2)(L_2+1)(L_2+2)} && (3.3.10b)\end{aligned}}$$

Where the above result hold for the other two-point functions in (3.3.1) as well. We see that the above result is consistent with general form of scalar two-point functions, derived in the previous subsection. Not surprisingly, we have found the part of $f(\xi)$ independent of ξ.

3.3.2 SO(5) symmetric vevs

Also in the $\mathfrak{so}(5)$ symmetric setup, the classical parts of the X and Z fields can easily be expressed in terms of the classical parts of the scalar fields they are constructed from. The results are as follow.

$$\mathcal{Z} = \Phi_5 + i\,\Phi_6 = \frac{1}{\sqrt{2}x^3}G_{56}^{d_n}\oplus 0_{N-d_n} \tag{3.3.11}$$

$$\mathcal{X} = \Phi_1 + i\,\Phi_2 = \frac{1}{\sqrt{2}x^3}\left(G_{16}^{d_n}\oplus 0_{N-d_n} + i\,G_{26}^{d_n}\oplus 0_{N-d_n}\right) \tag{3.3.12}$$

In this case, we also only have to consider $\operatorname{tr}\mathcal{Z}^L$, but for a different reason. It was shown in [17], that any trace containing an non-equal number of $\mathcal{X}$'s and $\bar{\mathcal{X}}$'s must vanish, since there exists a similarity transformation U_a for all $a\in\mathbb{C}$, such that.

$$U_a(G_{16}\pm i\,G_{26})U_a^{-1} = a^{\pm1}(G_{16}\pm i\,G_{26}) \tag{3.3.13}$$

Similarly to the $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric case, it has been shown in [12] that $\operatorname{tr}\mathcal{Z}^L$, in the $\mathfrak{so}(5)$ symmetric setup, can also be expressed in terms of Bernoulli polynomials, and is given by the following.

$$\operatorname{tr}\left(G_{d_n}^{56}\right)^L = \begin{cases} \frac{2}{L+3}B_{L+3}\left(-\frac{n}{2}\right) - \frac{(n+2)^2}{2(L+1)}B_{L+1}\left(-\frac{n}{2}\right) \simeq \frac{(-1)^L n^{L+3}}{2^{L+1}(L+3)(L+1)} & \text{for } L \text{ even}\\ 0 & \text{for } L \text{ odd}\end{cases} \tag{3.3.14}$$

Where $\simeq$ now means equal to highest order in n. Using the above expression for $\operatorname{tr}\left(G_{56}^{d_n}\right)^L$, we can now write down an explicit expression for the traces: $\operatorname{tr}\mathcal{X}^L$, $\operatorname{tr}\mathcal{Z}^L$ and $\operatorname{tr}\bar{\mathcal{Z}}^L$.

$$\operatorname{tr}\mathcal{X}^L = 0 \quad , \quad \operatorname{tr}\mathcal{Z}^L = \operatorname{tr}\bar{\mathcal{Z}}^L \simeq \frac{(-1)^L}{2^{\frac{L}{2}}x_3^L}\frac{n^{L+3}}{2^{L+1}(L+3)(L+1)} \tag{3.3.15}$$

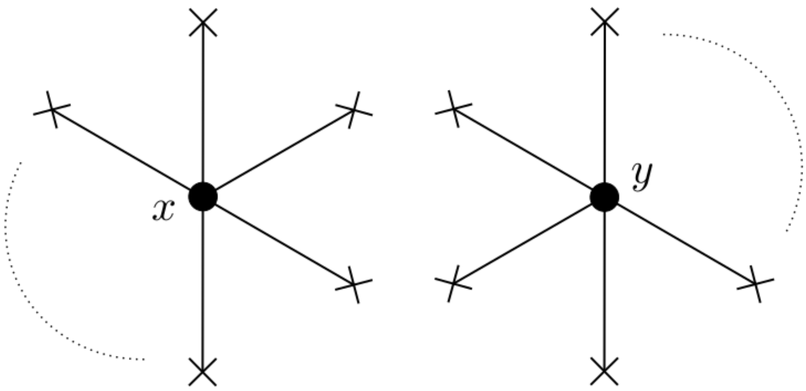

Figure 2: Disconnected tree-level contribution to the chiral primary two-point functions given in (3.3.1). In the diagram above, the lines with a cross at the end represents insertions of the classical parts, $\mathcal{X}$, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$, of the $\boldsymbol{X}$, $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ fields respectively.

Thus, we find that in the $\mathfrak{so}(5)$ symmetric setup, the disconnected part of the tree-level contributions to the two-point functions in (3.3.1) are given by the following expression.

$$\boxed{\begin{aligned} &\mathfrak{so}(5): \quad \left\langle \operatorname{tr} Z^{L_1} \operatorname{tr} Z^{L_2} \right\rangle_{\text{tree,dc.}} \simeq \frac{f_{\text{tree,dc.}}}{(2x_3)^{L_1}(2y_3)^{L_2}} \qquad (3.3.16\text{a}) \\ &f_{\text{tree,dc.}} = (-1)^{L_1+L_2} \frac{n^{L_1+L_2+6}}{2^{\frac{L_1+L_2}{2}+2}(L_1+3)(L_1+1)(L_2+3)(L_2+1)} \qquad (3.3.16\text{b}) \end{aligned}}$$

Where the above result hold for all two-point functions in (3.3.1), expect $\left\langle \operatorname{tr} \boldsymbol{X}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle$ which vanish at tree-level. Also in this case, the above result is consistent with general form of scalar two-point functions, and idependent of ξ.

3.4 Two-point functions at tree level (*connected*)

At tree level, the two-point functions $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle$, $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}} \rangle$ and $\langle \mathcal{O}_{\boldsymbol{X}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle$ all have only one connected contribution; namely the contribution which arise from Wick-contracting one complex scalar field from the first operator with one complex scalar field from the second.

$$\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,c.}} = L_1 L_2 \operatorname{tr}\left[\mathcal{Z}^{L_1-1} Z\right] \operatorname{tr}\left[Z \mathcal{Z}^{L_2-1}\right] \tag{3.4.1}$$

$$\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1} \operatorname{tr} \bar{\boldsymbol{Z}}^{L_2} \right\rangle_{\text{tree,c.}} = L_1 L_2 \operatorname{tr}\left[\mathcal{Z}^{L_1-1} Z\right] \operatorname{tr}\left[\bar{Z} \bar{\mathcal{Z}}^{L_2-1}\right] \tag{3.4.2}$$

$$\left\langle \operatorname{tr} \boldsymbol{X}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,c.}} = L_1 L_2 \operatorname{tr}\left[\mathcal{X}^{L_1-1} X\right] \operatorname{tr}\left[Z \mathcal{Z}^{L_2-1}\right] \tag{3.4.3}$$

If any more contractions are made in any way, we obtain loops due to the local nature of the operators. Note that the off-diagonal fields (*w.r.t the decompositions in section* (2.5)) do not contribute becuase $\mathcal{Z}$, $\bar{\mathcal{Z}}$ and $\mathcal{X}$ all have no off-diagonal elements. Thus, we only have to consider the upper diagonal parts of Z, $\bar{Z}$ and X when doing the contractions.

3.4.1 $\mathrm{SO}(3) \times \mathrm{SO}(3)$ symmetric vevs

To evaluate the Wick-contractions in the above expressions, we expand the upper diagonal components of Z, $\bar{Z}$ and X in a product basis of $\mathfrak{so}(3)$ fussy spherical harmonics, exactly as when we diagonalized the $\mathfrak{so}(3) \times \mathfrak{so}(3)$ symmetric mass matrix back in section 2.5.

$$Z = Z_{\ell_1,m_1;\ell_2,m_2}\hat{Y}^{m_1}_{\ell_1}\otimes\hat{Y}^{m_2}_{\ell_2} \tag{3.4.4}$$
$$\bar{Z} = Z^{\dagger}_{\ell_1,m_1;\ell_2,m_2}(\hat{Y}^{m_1}_{\ell_1})^{\dagger}\otimes(\hat{Y}^{m_2}_{\ell_2})^{\dagger} \tag{3.4.5}$$
$$X = X_{\ell_1,m_1;\ell_2,m_2}\hat{Y}^{m_1}_{\ell_1}\otimes\hat{Y}^{m_2}_{\ell_2} \tag{3.4.6}$$

The propargators between the coefficient fields above, can now be expressed using the propagators between the scalar coefficient fields $[\phi_i]_{\ell_1,m_1;\ell_2,m_2}$. Since the scalar fields ϕ_i are all *complicated*, one must reverse the transformations performed in the process of diagonalizing the complicated mass matrix, in order to obtained these propagators. The details of this reversal can be found in [11].

Let us now begin to construct the propagators we need to compute the tree-level contribution to the three different two-point functions. Firstly, the relevant ϕ_i propagators for computing the contribution (3.4.1) to $\langle\mathcal{O}_Z\,\mathcal{O}_Z\rangle$, are the following ones.

$$\begin{aligned}\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_3]_{\ell_1,m_1;\ell_2,m_2}[\phi_3]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2+m_2',0}\left[\delta_{m_1+m_1',0}K^{\phi,(1)}_{\text{sing}} - [t_3^{(\ell_1)}t_3^{(\ell_1)}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{sym}}\right]\end{aligned} \tag{3.4.7}$$

$$\begin{aligned}\langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_6]_{\ell_1,m_1;\ell_2,m_2}[\phi_6]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\left[\delta_{m_2+m_2',0}K^{\phi,(2)}_{\text{sing}} - [t_3^{(\ell_2)}t_3^{(\ell_2)}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{sym}}\right]\end{aligned} \tag{3.4.8}$$

$$\begin{aligned}\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_3]_{\ell_1,m_1;\ell_2,m_2}[\phi_6]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}[t_3^{(\ell_1)}]_{m_1,-m_1'}[t_3^{(\ell_2)}]_{m_2,-m_2'}K^{\phi}_{\text{opp}}\end{aligned} \tag{3.4.9}$$

Where we have introduced the notation $\boldsymbol{\ell}=(\ell_1,m_1;\ell_2,m_2)$, in an attempt to unclutter the notation. Next, the relevant propagators for computing the contribution (3.4.2) to $\langle\mathcal{O}_Z\,\mathcal{O}_{\bar{Z}}\rangle$, are the following.

$$\left\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_3]^{\dagger}_{\boldsymbol{\ell}'}\right\rangle = \delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2,m_2'}\left[\delta_{m_1,m_1'}K^{\phi,(1)}_{\text{sing}} - [t_3^{(\ell_1)}t_3^{(\ell_1)}]_{m_1,m_1'}K^{\phi,(1)}_{\text{sym}}\right] \tag{3.4.10}$$

$$\left\langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_6]^{\dagger}_{\boldsymbol{\ell}'}\right\rangle = \delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1,m_1'}\left[\delta_{m_2,m_2'}K^{\phi,(2)}_{\text{sing}} - [t_3^{(\ell_2)}t_3^{(\ell_2)}]_{m_2,m_2'}K^{\phi,(2)}_{\text{sym}}\right] \tag{3.4.11}$$

$$\left\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_6]^{\dagger}_{\boldsymbol{\ell}'}\right\rangle = \delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}[t_3^{(\ell_1)}]_{m_1,m_1'}[t_3^{(\ell_2)}]_{m_2,m_2'}K^{\phi}_{\text{opp}} \tag{3.4.12}$$

Lastly, the relevant propagators for computing the contribution (3.4.3) to $\langle\mathcal{O}_X\,\mathcal{O}_Z\rangle$, are the following.

$$\begin{aligned}\langle[\phi_1]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_1]_{\ell_1,m_1;\ell_2,m_2}[\phi_3]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2+m_2',0}\left[i\,[t_2^{(\ell_1)}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{anti}} - [t_1^{(\ell_1)}t_3^{(\ell_1)}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{sym}}\right]\end{aligned} \tag{3.4.13}$$

$$\begin{aligned}\langle[\phi_4]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_4]_{\ell_1,m_1;\ell_2,m_2}[\phi_6]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\left[i\,[t_2^{(\ell_2)}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{anti}} - [t_1^{(\ell_2)}t_3^{(\ell_2)}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{sym}}\right]\end{aligned} \tag{3.4.14}$$

$$\begin{aligned}\langle[\phi_1]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_1]_{\ell_1,m_1;\ell_2,m_2}[\phi_6]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}[t_1^{(\ell_1)}]_{m_1,-m_1'}[t_3^{(\ell_2)}]_{m_2,-m_2'}K^{\phi}_{\text{opp}}\end{aligned} \tag{3.4.15}$$

$$\begin{aligned}\langle[\phi_4]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\left\langle[\phi_4]_{\ell_1,m_1;\ell_2,m_2}[\phi_3]^{\dagger}_{\ell_1',-m_1';\ell_2',-m_2'}\right\rangle\\ &= (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}[t_1^{(\ell_2)}]_{m_2,-m_2'}[t_3^{(\ell_1)}]_{m_1,-m_1'}K^{\phi}_{\text{opp}}\end{aligned} \tag{3.4.16}$$

Where in writing down expressions for the above scalar propagators, we have made repeated use of the following property of fuzzy spherical harmonics.

$$(\hat{Y}_\ell^m)^\dagger = (-1)^m \hat{Y}_\ell^{-m} \quad \Rightarrow \quad [\phi_i]^\dagger_{\ell_1,m_1;\ell_2,m_2} = (-1)^{m_1'+m_2'} [\phi_i]_{\ell_1,-m_1;\ell_2,-m_2} \tag{3.4.17}$$

The explicit expressions for the propagators $K^{\phi,(s)}_{\text{sym}}$, $K^{\phi,(s)}_{\text{anti}}$ and K^{ϕ}_{opp} can be found in appendix A, and the matrix elements of the $\mathfrak{su}(2)$ generators $[t_i^{(\ell)}]_{m,m'} \equiv \langle \ell, m| \, t_i^{(\ell)} \, |\ell, m'\rangle$, are given by the following standard expressions.

$$[t_1^{(\ell)}]_{m,m'} = \frac{1}{2}\left(C^\ell_{m,m'}\delta_{m',m-1} + C^\ell_{m',m}\delta_{m',m+1}\right) \tag{3.4.18}$$

$$[t_2^{(\ell)}]_{m,m'} = \frac{1}{2i}\left(C^\ell_{m,m'}\delta_{m',m-1} - C^\ell_{m',m}\delta_{m',m+1}\right) \tag{3.4.19}$$

$$[t_3^{(\ell)}]_{m,m'} = m\,\delta_{m',m} \quad , \quad C^\ell_{m,m'} = \sqrt{(\ell+m)(\ell-m')} \tag{3.4.20}$$

Using all the scalar propagators written above, we can easily find the propergators: $\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, $\langle Z_{\boldsymbol{\ell}}\, Z^\dagger_{\boldsymbol{\ell}'}\rangle$ and $\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, by simply using definition (3.0.1) and expanding the products appearing in the propagators. The results are given below.

$$\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle = \langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle - \langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle + i\left(\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle + \langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle\right) \tag{3.4.21}$$

$$\langle Z_{\boldsymbol{\ell}}\, Z^\dagger_{\boldsymbol{\ell}'}\rangle = \left\langle[\phi_3]_{\boldsymbol{\ell}}\,[\phi_3]^\dagger_{\boldsymbol{\ell}'}\right\rangle + \left\langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_6]^\dagger_{\boldsymbol{\ell}'}\right\rangle + i\left(\left\langle[\phi_6]_{\boldsymbol{\ell}}\,[\phi_3]^\dagger_{\boldsymbol{\ell}'}\right\rangle - \left\langle[\phi_3]_{\boldsymbol{\ell}}[\phi_6]^\dagger_{\boldsymbol{\ell}'}\right\rangle\right) \tag{3.4.22}$$

$$\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle = \langle[\phi_1]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle - \langle[\phi_4]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle + i\left(\langle[\phi_4]_{\boldsymbol{\ell}}\,[\phi_3]_{\boldsymbol{\ell}'}\rangle + \langle[\phi_1]_{\boldsymbol{\ell}}\,[\phi_6]_{\boldsymbol{\ell}'}\rangle\right) \tag{3.4.23}$$

If we now insert the explicit forms of the ϕ_i field propagators into the above expressions and make use of (3.4.18), (3.4.19) and (3.4.20), we find that the complex scalar propagators simplify quite a lot. For $\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, we find that.

$$\begin{aligned}\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle &= (-1)^{m_1'+m_2'}\,\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0} \\ &\times \left.\left[K^{\phi,(1)}_{\text{sing}} - K^{\phi,(2)}_{\text{sing}} - m_1^2 K^{\phi,(1)}_{\text{sym}} + m_2^2 K^{\phi,(2)}_{\text{sym}} + 2i m_1 m_2 K^{\phi}_{\text{opp}}\right]\right\}\, K^{ZZ}\end{aligned} \tag{3.4.24}$$

For $\langle Z_{\boldsymbol{\ell}}\, Z^\dagger_{\boldsymbol{\ell}'}\rangle$, we find that.

$$\begin{aligned}\langle Z_{\boldsymbol{\ell}}\, Z^\dagger_{\boldsymbol{\ell}'}\rangle &= \delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1m_1'}\delta_{m_2m_2'} \\ &\times \left.\left[K^{\phi,(1)}_{\text{sing}} + K^{\phi,(2)}_{\text{sing}} - m_1^2 K^{\phi,(1)}_{\text{sym}} - m_2^2 K^{\phi,(2)}_{\text{sym}}\right]\right\}\, K^{Z\bar{Z}}\end{aligned} \tag{3.4.25}$$

For $\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, we find that.

$$\begin{aligned}\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle &= \frac{1}{2}(-1)^{m_1'+m_2'}\,\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'} \\ &\times\delta_{m_2+m_2',0}\delta_{m_1+m_1',1}C^{\ell_1}_{m_1,-m_1'}\,\left.\left[K^{\phi,(1)}_{\text{anti}} - m_1 K^{\phi,(1)}_{\text{sym}} + i\,m_2 K^{\phi}_{\text{opp}}\right]\right\}\, K^{XZ}_{(1),+} \\ &\times\delta_{m_2+m_2',0}\delta_{m_1+m_1',-1}C^{\ell_1}_{-m_1',m_1}\,\left.\left[-K^{\phi,(1)}_{\text{anti}} - m_1 K^{\phi,(1)}_{\text{sym}} + i\,m_2 K^{\phi}_{\text{opp}}\right]\right\}\, K^{XZ}_{(1),-} \\ &\times\delta_{m_1+m_1',0}\delta_{m_2+m_2',1}C^{\ell_2}_{m_2,-m_2'}\,\left.\left[-K^{\phi,(2)}_{\text{anti}} + m_2 K^{\phi,(2)}_{\text{sym}} + i\,m_1 K^{\phi}_{\text{opp}}\right]\right\}\, K^{XZ}_{(2),+} \\ &\times\,\delta_{m_1+m_1',0}\delta_{m_2+m_2',-1}C^{\ell_2}_{-m_2',m_2}\,\left.\left[K^{\phi,(2)}_{\text{anti}} + m_2 K^{\phi,(2)}_{\text{sym}} + i\,m_1 K^{\phi}_{\text{opp}}\right]\right\}\, K^{XZ}_{(2),-}\end{aligned} \tag{3.4.26}$$

Now that we have expressions for all the relevant propagators between the complex scalar coefficient fields, we can attempt to compute the leading order connected contribution to the two-point functions $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}}\rangle$, $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}}\rangle$ and $\langle \mathcal{O}_{\boldsymbol{X}}\, \mathcal{O}_{\boldsymbol{Z}}\rangle$. We start out by considering the constribution (3.4.1) to $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}}\rangle$.

$$\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1+1} \operatorname{tr} \boldsymbol{Z}^{L_2+1} \right\rangle_{\text{tree,c.}} = \frac{(-1)^{L_1+L_2}(L_1+1)(L_2+1)}{x_3^{L_1} y_3^{L_2}} \langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'} \rangle$$
$$\operatorname{tr}\Big[(t_3^{k_1} \otimes \mathbb{1}_{k_2} + i\, \mathbb{1}_{k_1} \otimes t_3^{k_2})^{L_1}\, \hat{Y}_{\ell_1}^{m_1} \otimes \hat{Y}_{\ell_2}^{m_2}\Big] \operatorname{tr}\Big[(t_3^{k_1} \otimes \mathbb{1}_{k_2} + i\, \mathbb{1}_{k_1} \otimes t_3^{k_2})^{L_2}\, \hat{Y}_{\ell_1'}^{m_1'} \otimes \hat{Y}_{\ell_2'}^{m_2'}\Big] \tag{3.4.27}$$

To proceed further, we now rewrite the products appearing in the traces above as binomial expansions.

$$\operatorname{tr}\Big[(t_3^{k_1} \otimes \mathbb{1}_{k_2} + i\, \mathbb{1}_{k_1} \otimes t_3^{k_2})^{L}\, \hat{Y}_{\ell_1}^{m_1} \otimes \hat{Y}_{\ell_2}^{m_2}\Big] = \sum_{p=0}^{L} \binom{L}{p} i^p \operatorname{tr}\left[(t_3^{k_1})^{L-p}\, \hat{Y}_{\ell_1}^{m_1} \otimes (t_3^{k_2})^{p}\, \hat{Y}_{\ell_2}^{m_2}\right] \tag{3.4.28}$$

The traces of the $\mathfrak{so}(3)$ generators $t_3^{k_s}$ with the fuzzy spherical harmonics $\hat{Y}_\ell^m$ now have to be evaluated somehow. This was first done in [10], by taking advantage of certain properties of the fuzzy spherical harmonics. The result of the whole procedure is the following.

$$\operatorname{tr}\left[(t_3^k)^{p_1}\, \hat{Y}_{\ell_1}^{m_1} \otimes (t_3^k)^{p_2}\, \hat{Y}_{\ell_2}^{m_2}\right] = \operatorname{tr}\left[(t_3^k)^{p_1}\, \hat{Y}_{\ell_1}^{m_1}\right] \operatorname{tr}\left[(t_3^k)^{p_2}\, \hat{Y}_{\ell_2}^{m_2}\right] = \alpha_{\ell_1}^{p_1} \alpha_{\ell_2}^{p_2} \delta^{m_1,0} \delta^{m_2,0} \tag{3.4.29}$$

$$\alpha_m^L \simeq (-1)^k \left(\frac{k}{2}\right)^{L+\frac{1}{2}} i^m \sqrt{m+\frac{1}{2}} \begin{cases} \frac{1}{L+1} \frac{\left(\frac{2-L}{2}\right)_{\frac{m-2}{2}}}{\left(\frac{L+3}{2}\right)_{\frac{m}{2}}} & \text{for } L, m \text{ even} \\ i \frac{\left(\frac{1-L}{2}\right)_{\frac{m-1}{2}}}{\left(\frac{L+2}{2}\right)_{\frac{m+1}{2}}} & \text{for } L, m \text{ odd} \end{cases} \tag{3.4.30}$$

Where in this context, $\simeq$ means equal to highest order in k, Furthermore, α_m^L will evaluate to zero if certain conditions are not met. These conditions can be found below.

1. Both L and m was to be either even or odd.
2. The inequality $L > m$ has to be satisfied.

It turns out that it is possible to bring (3.4.30) to a form which is much more symmetric between the L, m even and L, m odd cases. This more symmetric form looks as follow.

$$\alpha_m^L \simeq (-1)^{k+m} \left(\frac{k}{2}\right)^{L+\frac{1}{2}} \frac{\sqrt{\pi\left(m+\frac{1}{2}\right)}}{2^L} \frac{\Gamma(L+1)}{\Gamma\left(\frac{L+m+3}{2}\right)\Gamma\left(\frac{L-m+2}{2}\right)} \begin{cases} -1 & \text{for } L, m \text{ even} \\ 1 & \text{for } L, m \text{ odd} \end{cases} \tag{3.4.31}$$

We now need to evaluate the sum (3.4.28). It turns out that we can actually obtain an explicit result, using for example *Mathematica*. This explicitly result looks as follow.

$$\sum_{p=0}^{L} i^p \binom{L}{p} \alpha_{\ell_1}^{L-p} \alpha_{\ell_2}^{p} = \Gamma(L+1) \left[\frac{{}_3\tilde{F}_2\left(1, -\frac{1+\ell_2+L}{2}, \frac{\ell_2-L}{2}; \frac{2-\ell_1}{2}, \frac{3+\ell_1}{2}; -1\right)}{\Gamma(\frac{2+L-\ell_2}{2})\Gamma(\frac{3+L+\ell_2}{2})} \right.$$
$$\left. + i\, \frac{{}_3\tilde{F}_2\left(1, -\frac{L+\ell_2}{2}, \frac{1-L+\ell_2}{2}; \frac{3-\ell_1}{2}, \frac{4+\ell_1}{2}; -1\right)}{\Gamma(\frac{1+L-\ell_2}{2})\Gamma(\frac{2+L+\ell_2}{2})} \right] \tag{3.4.32}$$

$$\text{where} \quad {}_p\tilde{F}_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z) \equiv \frac{{}_pF_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z)}{\Gamma(b_1)\cdots\Gamma(b_q)} \tag{3.4.33}$$

In the above, ${}_p\tilde{F}_q(a_1,\ldots,a_p; b_1,\ldots,b_q; z)$ represents *generalized hypergeometric functions*. It turns out that the complicated expression (3.4.32) simplifies significantly for small values of ℓ_1 and ℓ_2. For example, in the case of $\ell_1 = 0$ and $\ell_2 = 0$ the sum (3.4.32) evaluates to following simple expression.

$$\sum_{p=0}^{L} i^p \binom{L}{p} \alpha_{\ell_1=0}^{L-p} \alpha_{\ell_2=0}^{p} = \frac{2^{2+L}\left(i + 2^{1+\frac{L}{2}}\, e^{\frac{i\pi L}{4}}\right)}{(1+L)(2+L)\pi} \tag{3.4.34}$$

Now that we have a better understanding of the traces which appear in (3.4.27), we can continue the evaluation of this connected contribution to $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$. By inserting (3.4.24), (3.4.28) and (3.4.29) into (3.4.27), we find the following expression for the connected tree-level contribution to $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$.

$$\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1+1} \operatorname{tr} \boldsymbol{Z}^{L_2+1}\right\rangle_{\text{tree,c.}} \simeq \frac{(L_1+1)(L_2+1)}{x_3^{L_1} y_3^{L_2}}$$
$$\times \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\left[K^{\phi,(1)}_{\text{sing}} - K^{\phi,(2)}_{\text{sing}}\right]\sum_{p_1=0}^{L_1} i^{p_1}\binom{L_1}{p_1}\alpha_{\ell_1}^{L_1-p_1}\alpha_{\ell_2}^{p_1}\sum_{p_2=0}^{L_2} i^{p_2}\binom{L_2}{p_2}\alpha_{\ell_1}^{L_2-p_2}\alpha_{\ell_2}^{p_2} \tag{3.4.35}$$

Note that the factor of $(-1)^{L_1+L_2}$ present in (3.4.27) drops out due to the properties of α^L_m. In principle, all we now need to do is insert the values (3.4.32) for the binomial sums and subsequently evaluate the sums over ℓ_1 and ℓ_2. Unfortunately, at the moment of writing this , we do not know whether or not this is possible for general L_1, L_2 and k_2, k_1. The sums can however be evaluated for some particularly simple values of the parameters L_1, L_2 and k_2, k_1, which we will explore further shortly. Before we do so however, let us write down reduced expressions for the connected contributions to $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\bar{\boldsymbol{Z}}}\rangle$ and $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$. For $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\bar{\boldsymbol{Z}}}\rangle$, we find something very similar to (3.4.35).

$$\left\langle \operatorname{tr} \boldsymbol{Z}^{L_1+1} \operatorname{tr} \bar{\boldsymbol{Z}}^{L_2+1}\right\rangle_{\text{tree,c.}} \simeq \frac{(L_1+1)(L_2+1)}{x_3^{L_1} y_3^{L_2}}$$
$$\times \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\left[K^{\phi,(1)}_{\text{sing}} + K^{\phi,(2)}_{\text{sing}}\right]\sum_{p_1=0}^{L_1} i^{p_1}\binom{L_1}{p_1}\alpha_{\ell_1}^{L_1-p_1}\alpha_{\ell_2}^{p_1}\sum_{p_2=0}^{L_2} (-i)^{p_2}\binom{L_2}{p_2}\alpha_{\ell_1}^{L_2-p_2}\alpha_{\ell_2}^{p_2} \tag{3.4.36}$$

In contrast to the the sum (3.4.28) which contains the factor i^p, we were not able to find an explicit expression for the same sum when changing $i^p \to (-i)^p$. Therefore, the above expression is even more challenging to evaluate than (3.4.35). Lastly, we write down the reduced expressions for the connected contributions to $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$. To do this, we will need following relation between the traces $\operatorname{tr}\left[(t_1^k)^p\,\hat{Y}_\ell^m\right]$ and $\operatorname{tr}\left[(t_3^k)^p\,\hat{Y}_\ell^0\right]$, which was worked out in [10].

$$\operatorname{tr}\left[(t_1^k)^p\,\hat{Y}_\ell^m\right] = \frac{i^{\ell+m}}{2^\ell}\sqrt{\frac{\Gamma(\ell+m+1)}{\Gamma(\ell-m+1)}}\frac{\Gamma(\ell-m+1)}{\Gamma\left(\frac{\ell-m+2}{2}\right)\Gamma\left(\frac{\ell+m+2}{2}\right)}\operatorname{tr}\left[(t_3^k)^p\,\hat{Y}_\ell^0\right] \tag{3.4.37}$$

The relation above is valid for L,ℓ,m all even, or L,ℓ,m all odd, otherwise the RHS should be set to zero. Because of the structure of $\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell'}}\rangle$, the cases of $m=\pm1$ are of particuar interest, and we write these explicitly below.

$$\operatorname{tr}\left[(t_1^k)^p\,\hat{Y}_\ell^{\pm1}\right] = \frac{i^{\ell\pm1}}{2^\ell}\sqrt{\frac{\ell+1}{\ell}}\binom{\ell}{\frac{\ell-1}{2}}\operatorname{tr}\left[(t_3^k)^p\,\hat{Y}_\ell^0\right] \tag{3.4.38}$$

We can now use the above trace relations together with (3.4.26) and (3.4.29), to write down the tree-level connected contribution to the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\bar{\boldsymbol{Z}}}\rangle$. When doing this, we notice the following simplifying observation about the traces.

$$C^\ell_{0,-1}\operatorname{tr}\left[(t_1^k)^p\,\hat{Y}_\ell^{+1}\right] = -C^\ell_{1,0}\operatorname{tr}\left[(t_1^k)^p\,\hat{Y}_\ell^{-1}\right] = \frac{i^{\ell+1}}{2^\ell}(\ell+1)\binom{\ell}{\frac{\ell-1}{2}}\operatorname{tr}\left[(t_3^k)^p\,\hat{Y}_\ell^0\right] \tag{3.4.39}$$

Using the above, we find the following expression for the connected tree-level contribution to $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$.

$$\left\langle \operatorname{tr} \boldsymbol{X}^{L_1+1} \operatorname{tr} \boldsymbol{Z}^{L_2+1}\right\rangle_{\text{tree,c.}} \simeq -\frac{(L_1+1)(L_2+1)}{x_3^{L_1} y_3^{L_2}}\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\left[\frac{i^{\ell_1+1}(\ell_1+1)!}{2^{\ell_1}\left(\frac{\ell_1-1}{2}\right)!\left(\frac{\ell_1+1}{2}\right)!}K^{\phi,(1)}_{\text{anti}}\right.$$
$$\left.-\frac{i^{\ell_2+1}(\ell_2+1)!}{2^{\ell_2}\left(\frac{\ell_2-1}{2}\right)!\left(\frac{\ell_2+1}{2}\right)!}K^{\phi,(2)}_{\text{anti}}\right]\sum_{p_1=0}^{L_1} i^{p_1}\binom{L_1}{p_1}\alpha_{\ell_1}^{L_1-p_1}\alpha_{\ell_2}^{p_1}\sum_{p_2=0}^{L_2} i^{p_2}\binom{L_2}{p_2}\alpha_{\ell_1}^{L_2-p_2}\alpha_{\ell_2}^{p_2} \tag{3.4.40}$$

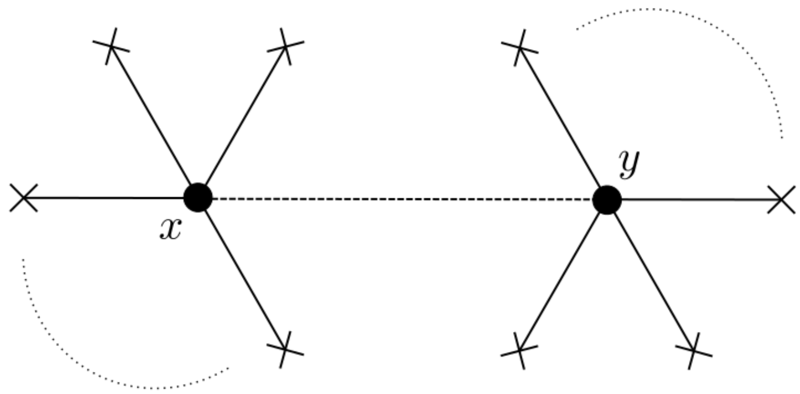

Figure 3: Connected tree-level contribution to the chiral primary two-point functions given in (3.3.1). In the diargram above, the lines with a cross at the end represents insertions of the classical parts, $\mathcal{X}$, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$, of the $\boldsymbol{X}$, $\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}$ fields respectively. The doted line represents either a $\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, $\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$ or $\langle Z_{\boldsymbol{\ell}}\, Z^{\dagger}_{\boldsymbol{\ell}'}\rangle$ propagator.

As was also the case for the other tree-level connected contributions, we do not know whether or not it is possible to explicitly evaluate the sums over ℓ_1 and ℓ_2, appearing in the above expression. We suspect that at least part of the difficulty lies with the fact that the masses of the fields are complicated. More specifically, we find that for general ℓ_1 and ℓ_2.

$$\nu = \sqrt{m^2 + \frac{1}{4}} \notin \mathbb{N}_0 \tag{3.4.41}$$

Where m^2 refers to the masses in table 2. That ν is typically not an integer means that we have square roots appearing in the sums over ℓ_1 and ℓ_2, which complicates the evaluation. This hindrance, together with that of the complicated binomial sums, makes it seem unlikely that more explicit expressions for the connected tree-level contributions can be obtained (*at least using the approach of this thesis*).

3.4.1.1 The case of $\mathbf{L_1 = 2}$ and $\mathbf{L_2 = 2}$: One of the special cases in which the connected tree-level contributions to the two-point functions (3.3.1) can be evaluated explicitly, is whenever we have $L_1 = L_2 = 2$ and k_1, k_2 arbitrary. In this case, the expression (3.4.35) for the connected contribution to $\langle \mathcal{O}_{\boldsymbol{Z}} \mathcal{O}_{\boldsymbol{Z}} \rangle$ reduces to the following.

$$\left\langle \operatorname{tr} \boldsymbol{Z}^2 \operatorname{tr} \boldsymbol{Z}^2 \right\rangle_{\text{tree,c.}} = \frac{4}{x_3\, y_3} \langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'} \rangle$$
$$\times \sum_{p_1=0}^{1} i^{p_1} \operatorname{tr}\left[(t_3^{k_1})^{1-p_1}\, \hat{Y}^{m_1}_{\ell_1} \otimes (t_3^{k_2})^{p_1}\, \hat{Y}^{m_2}_{\ell_2} \right] \sum_{p_2=0}^{1} i^{p_2} \operatorname{tr}\left[(t_3^{k_1})^{1-p_2}\, \hat{Y}^{m_1'}_{\ell_1'} \otimes (t_3^{k_2})^{p_2}\, \hat{Y}^{m_2'}_{\ell_2'} \right] \tag{3.4.42}$$

The reason why this case is particularly simple is because the only traces appearing in the expression above are of the following form: $\operatorname{tr}\left[\hat{Y}^m_\ell\right]$ and $\operatorname{tr}\left[t_3^k\, \hat{Y}^m_\ell\right]$, which are both simple to evaluate. All we need to evaluate these types of traces are three pieces of information. *First piece of information*: the fuzzy harmonic $\hat{Y}^0_0$ is proportional to the identity matrix. This is because.

$$L_3\, \mathbb{1}_k = [t_3, \mathbb{1}_k] = 0 \quad , \quad L^2\, \mathbb{1}_k = [t_i\, t^i, \mathbb{1}_k] = 0 \quad \Rightarrow \quad \hat{Y}^0_0 \sim \mathbb{1}_k \tag{3.4.43}$$

Second piece of information: fuzzy spherical harmonics satisfy the following orthogonality realtion.

$$\operatorname{tr}\left[\hat{Y}^m_\ell\, \hat{Y}^{m'}_{\ell'}\right] = (-1)^{m'}\, \delta_{\ell,\ell'}\, \delta_{m+m',0} \tag{3.4.44}$$

Using the above orthogonality condition, and the fact that $\hat{Y}_0^0$ must be proportional to the identity matrix, we find that: $(-1)^{k+1}\sqrt{k}\,\hat{Y}_0^0 = \mathbb{1}_k$, where the factor of $(-1)^{k+1}$ is purely conventional. Using this expression for the fuzzy harmonic $\hat{Y}_0^0$, we can evaluate the first type of traces we encounter.

$$\mathrm{tr}\left[\hat{Y}_\ell^m\,\hat{Y}_0^0\right] = \delta_{m,0}\,\delta_{\ell,0} \quad\Rightarrow\quad \mathrm{tr}\left[\hat{Y}_\ell^m\right] = (-1)^{k+1}\sqrt{k}\,\delta_{m,0}\,\delta_{\ell,0} \tag{3.4.45}$$

In order to evaluate the other type of trace, we need to know how the generator t_3 is related to the fuzzy spherical harmonics. *Third piece of information*: the fuzzy harmonic $\hat{Y}_0^1$ is proportional to the generator t_3. This is because.

$$L_3\,t_3 = [t_3, t_3] = 0 \quad , \quad L^2\,t_3 = [t_i\,t^i, t_3] = 2\,t_3 \quad\Rightarrow\quad \hat{Y}_1^0 \sim t_3 \tag{3.4.46}$$

Using the above together with the matrix elements of the t_3^k generator: $\langle \ell, m|\, t_3^k\,|\ell, m'\rangle = m\,\delta_{m,m'}$, we conclude that t_3^k must take the following form.

$$\mathrm{tr}\big[t_3^k\,t_3^k\big] = \sum_{m=-\frac{k-1}{2}}^{\frac{k-1}{2}} m^2 = \frac{k(k^2-1)}{12} \quad\Rightarrow\quad t_3^k = \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{3}}\,\hat{Y}_1^0 \tag{3.4.47}$$

Where again, the factor of $(-1)^{k+1}$ is purely conventional. We now have all the information we need to evaluate the second kind of traces appearing in the connected tree-level contribution (3.4.1.1).

$$\mathrm{tr}\left[t_3^k\,\hat{Y}_\ell^m\right] = \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{3}}\,\mathrm{tr}\left[\hat{Y}_1^0\,\hat{Y}_\ell^m\right] = \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{3}}\,\delta_{m,0}\,\delta_{\ell,1} \tag{3.4.48}$$

Using equ. (3.4.45) and (3.4.48), we can now begin to evaluate the two-point function $\langle\mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle_{L_1=L_2=1}$.

$$\begin{aligned}
\big\langle \mathrm{tr}\,\boldsymbol{Z}^2\,\mathrm{tr}\,\boldsymbol{Z}^2\big\rangle_{\text{tree,c.}} &= \frac{4}{x_3\,y_3}\,\langle Z_{\boldsymbol{\ell}}\,Z_{\boldsymbol{\ell}'}\rangle \\
\times \sum_{p_1=0}^{1} i^{p_1}\,\mathrm{tr}\left[(t_3^{k_1})^{1-p_1}\,\hat{Y}_{\ell_1}^{m_1}\otimes(t_3^{k_2})^{p_2}\,\hat{Y}_{\ell_2}^{m_2}\right] &\sum_{p_2=0}^{1} i^{p_2}\,\mathrm{tr}\left[(t_3^{k_1})^{1-p_2}\,\hat{Y}_{\ell_1'}^{m_1'}\otimes(t_3^{k_2})^{p_2}\,\hat{Y}_{\ell_2'}^{m_2'}\right] \\
&\simeq \frac{4}{x_3\,y_3}\,\frac{k_1^2\,k_2^2}{12}\left(K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)}\right)\delta_{\ell_1\ell_1'}\,\delta_{\ell_2\ell_2'}\,\delta_{m_1+m_1',0}\,\delta_{m_2+m_2',0} \\
\times\left(\delta_{m_1,0}\,\delta_{\ell_1,1}\,\delta_{m_2,0}\,\delta_{\ell_2,0} + i\,\delta_{m_1,0}\,\delta_{\ell_1,0}\,\delta_{m_2,0}\,\delta_{\ell_2,1}\right)&\left(\delta_{m_1',0}\,\delta_{\ell_1',1}\,\delta_{m_2',0}\,\delta_{\ell_2',0} + i\,\delta_{m_1',0}\,\delta_{\ell_1',0}\,\delta_{m_2',0}\,\delta_{\ell_2',1}\right) \\
&= \frac{k_1^2\,k_2^2}{3\,x_3\,y_3}\left(K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)}\right)\left(\delta_{\ell_1,1}\,\delta_{\ell_2,0} - \delta_{\ell_1,0}\,\delta_{\ell_2,1}\right)
\end{aligned} \tag{3.4.49}$$

Where $\simeq$ in the context means equal to highest order in k_1, k_2. In order to further simplify the above expression, we need the explicit forms of the scalar propagators $K_{\text{sing}}^{\phi,(1)}$ and $K_{\text{sing}}^{\phi,(2)}$. For the cases of $\ell_1 = 1$, $\ell_2 = 0$ and $\ell_1 = 0$, $\ell_2 = 1$, the two propagators are given as follow[4].

$$\ell_1 = 1\ ,\ \ell_2 = 0:\quad K_{\text{sing}}^{\phi,(1)} = \frac{2}{3}\,K^{m^2=0} + \frac{1}{3}\,K^{m^2=6} \quad , \quad K_{\text{sing}}^{\phi,(2)} = K^{m^2=2} \tag{3.4.50}$$

$$\ell_1 = 0\ ,\ \ell_2 = 1:\quad K_{\text{sing}}^{\phi,(1)} = K^{m^2=2} \quad , \quad K_{\text{sing}}^{\phi,(2)} = \frac{2}{3}\,K^{m^2=0} + \frac{1}{3}\,K^{m^2=6} \tag{3.4.51}$$

[4]See appendix A for more information about these propagators

Using the above propagators, we obtain the final expression for the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle_{L_1=L_2=1}$.

$$
\begin{aligned}
\left\langle \operatorname{tr} \boldsymbol{Z}^2 \operatorname{tr} \boldsymbol{Z}^2 \right\rangle_{\text{tree,c.}} &\simeq \frac{1}{x_3\, y_3} \frac{k_1^2\, k_2^2}{3} \left(\frac{4}{3} K^{m^2=0} + \frac{2}{3} K^{m=6} - 2\, K^{m^2=2} \right) \\
&= \frac{g^2}{(2x_3)^2\, (2y_3)^2} \frac{16\, k_1^2\, k_2^2}{3} \left(\frac{2}{3} K^{\nu=1/2}_{AdS} + \frac{1}{3} K^{\nu=5/2}_{AdS} - K^{\nu=3/2}_{AdS} \right)
\end{aligned} \tag{3.4.52}
$$

Where in the above, we have used that: $K^{m^2} = \frac{g^2}{2\, x_3\, y_3} K^{\nu}_{AdS}$, with $\nu = \sqrt{m^2 + \frac{1}{4}}$, in order to obtain the g^2 dependency[5]. We see that the two-point function has exactly the form we would expect, considering the general symmetry arguments of subsection 3.2.

$$
\mathfrak{so}(3) \times \mathfrak{so}(3) : \quad \left\langle \operatorname{tr} \boldsymbol{Z}^2 \operatorname{tr} \boldsymbol{Z}^2 \right\rangle_{\text{tree,c}} \simeq \frac{f_{\text{tree,c}}(\xi)}{(2x_3)^2 (2y_3)^2} \tag{3.4.53a}
$$

$$
f_{\text{tree,c.}}(\xi) = g^2\, \frac{16\, k_1^2\, k_2^2}{3} \left(\frac{2}{3} K^{\nu=1/2}_{AdS} + \frac{1}{3} K^{\nu=5/2}_{AdS} - K^{\nu=3/2}_{AdS} \right) \tag{3.4.53b}
$$

Where all the ξ dependence reside in the auxiliary AdS_4 propagators. We can likewise obtain similar expressions for the two-point functions $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}} \rangle_{L_1=L_2=1}$ and $\langle \mathcal{O}_{\boldsymbol{X}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle_{L_1=L_2=1}$. In the case of the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}} \rangle_{L_1=L_2=1}$, the treatment is almost exactly identical to that of the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle_{L_1=L_2=1}$, and the result is the following.

$$
\mathfrak{so}(3) \times \mathfrak{so}(3) : \quad \left\langle \operatorname{tr} \boldsymbol{Z}^2 \operatorname{tr} \bar{\boldsymbol{Z}}^2 \right\rangle_{\text{tree,c}} \simeq \frac{f_{\text{tree,c}}(\xi)}{(2x_3)^2 (2y_3)^2} \tag{3.4.54a}
$$

$$
f_{\text{tree,c.}}(\xi) = g^2\, \frac{16\, k_1^2\, k_2^2}{3} \left(\frac{2}{3} K^{\nu=1/2}_{AdS} + \frac{1}{3} K^{\nu=5/2}_{AdS} + K^{\nu=3/2}_{AdS} \right) \tag{3.4.54b}
$$

In the case of the two-point function $\langle \mathcal{O}_{\boldsymbol{X}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle_{L_1=L_2=1}$, we also need an expression for the traces between fuzzy spherical harmonics and the generator t_1^k. To find such an expression, we need some additional information. *Fourth piece of information*: the fuzzy harmonics $\hat{Y}_1^{\pm 1}$ are proportional to the generators $t_\pm$ respectively. This is because.

$$
L_3\, t_\pm = [t_3, t_\pm] = \pm t_\pm \quad , \quad L^2\, t_\pm = [t_i\, t^i, t_\pm] = 2\, t_\pm \quad \Rightarrow \quad \hat{Y}_1^{\pm 1} \sim t_\pm \tag{3.4.55}
$$

Just as we did with the trace types, we can now use the matrix elements of the $t_\pm$ generators: $\langle \ell, m|\, t_\pm^k\, |\ell, m' \rangle = \sqrt{(\ell \pm m)(\ell \mp m')}\, \delta_{m,m'\pm 1}$, together with the orthogonality relation (3.4.44), to obtain the proportinallity constant.

$$
\operatorname{tr}\left[t_\pm^k\, t_\mp^k \right] = \sum_{m=-\frac{k-1}{2}}^{\frac{k-1}{2}} \left(\frac{k-1}{2} + m \right) \left(\frac{k+1}{2} - m \right) = \frac{k(k^2-1)}{6} \tag{3.4.56}
$$

$$
\Rightarrow \quad t_\pm^k = \mp (-1)^{k+1} \sqrt{\frac{k(k^2-1)}{6}}\, \hat{Y}_1^{\pm 1} \quad , \quad t_1^k = \frac{t_+^k + t_-^k}{2} \tag{3.4.57}
$$

Using the above together with (3.4.44), we find the following expression for the third kind of traces.

[5] For more information about the auxiliary AdS_4 propagators, refer to subsection 2.6

$$\begin{aligned}\operatorname{tr}\left[t_1^k\,\hat{Y}_\ell^m\right] &= \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}}\,\operatorname{tr}\left[\hat{Y}_1^{-1}\,\hat{Y}_\ell^m - \hat{Y}_1^{+1}\,\hat{Y}_\ell^m\right]\\ &= \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}}\,(\delta_{m+1,0}\,\delta_{\ell,1} - \delta_{m-1,0}\,\delta_{\ell,1})\end{aligned} \tag{3.4.58}$$

Using equ. (3.4.45) and (3.4.58), we can now begin to evaluate the two-point function $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle_{L_1=L_2=1}$.

$$\begin{aligned}\left\langle \operatorname{tr}\boldsymbol{X}^2\operatorname{tr}\boldsymbol{Z}^2\right\rangle_{\text{tree,c.}} &= \frac{4}{x_3\,y_3}\,\langle X_{\boldsymbol{\ell}}\,Z_{\boldsymbol{\ell}'}\rangle\\ &\times\sum_{p_1=0}^{1} i^{p_1}\operatorname{tr}\left[(t_3^{k_1})^{1-p_1}\,\hat{Y}_{\ell_1}^{m_1}\otimes(t_3^{k_2})^{p_2}\,\hat{Y}_{\ell_2}^{m_2}\right]\sum_{p_2=0}^{1} i^{p_2}\operatorname{tr}\left[(t_1^{k_1})^{1-p_2}\,\hat{Y}_{\ell_1'}^{m_1'}\otimes(t_1^{k_2})^{p_2}\,\hat{Y}_{\ell_2'}^{m_2'}\right]\\ &\simeq -\frac{2}{x_3\,y_3}\,\frac{k_1^2\,k_2^2}{12\sqrt{2}}\left(\sqrt{\ell_1(\ell_1+1)}\,K_{\text{anti}}^{\phi,(1)}\,\delta_{m_2+m_2',0}\,(\delta_{m_1+m_1',1}-\delta_{m_1+m_1',-1})-\right.\\ &\left.\sqrt{\ell_2(\ell_2+1)}\,K_{\text{anti}}^{\phi,(2)}\,\delta_{m_1+m_1',0}\,(\delta_{m_2+m_2',1}-\delta_{m_2+m_2',-1})\right)\,\delta_{\ell_1\ell_1'}\,\delta_{\ell_2\ell_2'}\\ &\times(\delta_{m_1,0}\,\delta_{\ell_1,1}\,\delta_{m_2,0}\,\delta_{\ell_2,0}+i\,\delta_{m_1,0}\,\delta_{\ell_1,0}\,\delta_{m_2,0}\,\delta_{\ell_2,1})\\ &\times\left(\delta_{m_2',0}\,\delta_{\ell_2',0}\,(\delta_{m_1'+1,0}\,\delta_{\ell_1',1}-\delta_{m_1'-1,0}\,\delta_{\ell_1',1})+i\,\delta_{m_1',0}\,\delta_{\ell_1',0}\,(\delta_{m_2'+1,0}\,\delta_{\ell_2',1}-\delta_{m_2'-1,0}\,\delta_{\ell_2',1})\right)\\ &= \frac{k_1^2\,k_2^2}{3\sqrt{2}\,x_3\,y_3}\left(\sqrt{\ell_1(\ell_1+1)}\,K_{\text{anti}}^{\phi,(1)}-\sqrt{\ell_2(\ell_2+1)}\,K_{\text{anti}}^{\phi,(2)}\right)(\delta_{\ell_1,1}\,\delta_{\ell_2,0}-\delta_{\ell_1,0}\,\delta_{\ell_2,1})\end{aligned} \tag{3.4.59}$$

In order to further simplify the above expression, we need the explicit forms of the scalar propagators $K_{\text{anti}}^{\phi,(1)}$ and $K_{\text{anti}}^{\phi,(2)}$. For the cases of $\ell_1 = 1$, $\ell_2 = 0$ and $\ell_1 = 0$, $\ell_2 = 1$, the two propagators are given as follow[6].

$$\ell_1 = 1\,,\ \ell_2 = 0: \quad K_{\text{anti}}^{\phi,(1)} = \frac{1}{3}\,K^{m^2=0} - \frac{1}{3}\,K^{m^2=6} \quad , \quad K_{\text{anti}}^{\phi,(2)} = K^{m^2=2} - K^{m^2=4} \tag{3.4.60}$$

$$\ell_1 = 0\,,\ \ell_2 = 1: \quad K_{\text{anti}}^{\phi,(1)} = K^{m^2=2} - K^{m^2=4} \quad , \quad K_{\text{sing}}^{\phi,(2)} = \frac{1}{3}\,K^{m^2=0} - \frac{1}{3}\,K^{m^2=6} \tag{3.4.61}$$

Using the above propagators, we obtain the final expression for the two-point function $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle_{L_1=L_2=1}$.

$$\mathfrak{so}(3)\times\mathfrak{so}(3): \quad \left\langle \operatorname{tr}\boldsymbol{X}^2\operatorname{tr}\boldsymbol{Z}^2\right\rangle_{\text{tree,c}} \simeq \frac{f_{\text{tree,c}}(\xi)}{(2x_3)^2(2y_3)^2} \tag{3.4.62a}$$

$$f_{\text{tree,c.}}(\xi) = g^2\,\frac{16\,k_1^2\,k_2^2}{3}\left(\frac{1}{3}K_{AdS}^{\nu=1/2} - \frac{1}{3}K_{AdS}^{\nu=5/2}\right) \tag{3.4.62b}$$

The above result now completes the evaluations of the connected tree-level contributions to the two-point functions (3.3.1), in the case of $L_1 = L_2 = 1$. We now move on to discuss the special case of $k_1 = 1$ or $k_2 = 1$.

3.4.1.2 The case of $\mathbf{k_1 = 1}$ or $\mathbf{k_2 = 1}$: In the case where either $k_1 = 1$ or $k_2 = 1$, the connected tree-level contributions to the two-point functions (3.3.1) also simplify dramatically, this time because the 1-dimensional representation of $\mathfrak{so}(3)$ is trival, i.e. $t_i^{k=1} = 0$. Thus, only the term in the binomial sums with $p_1 = p_2 = 0$ survives. For concreteness, we focus here on the case of $k_2 = 1$. The treatment for $k_1 = 1$ is completely analogous. We find that the connected tree-level contribution to the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$ looks as follow.

[6] See appendix A for more information about these propagators

$$\begin{aligned}\left\langle \mathrm{tr}\, \boldsymbol{Z}^{L_1}\, \mathrm{tr}\, \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,c.}} &= \frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}} \left\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'} \right\rangle \\ &\times \mathrm{tr}\left[(t_3^{k_1})^{L_1-1}\, \hat{Y}_{\ell_1}^{m_1} \otimes \hat{Y}_{\ell_2}^{m_2}\right] \mathrm{tr}\left[(t_3^{k_1})^{L_2-1}\, \hat{Y}_{\ell_1'}^{m_1'} \otimes \hat{Y}_{\ell_2'}^{m_2'}\right] \\ &= \frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}}\, \mathrm{tr}\left[(t_3^{k_1})^{L_1-1}\, \hat{Y}_{\ell_1}^{m_1}\right] \mathrm{tr}\left[(t_3^{k_1})^{L_2-1}\, \hat{Y}_{\ell_1'}^{m_1'}\right] \left\langle Z_{\ell_1,m_1;0,0}\, Z_{\ell_1',m_1';0,0} \right\rangle \\ &= \frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}} \sum_{\ell_1=0}^{\infty} \alpha_{\ell_1}^{L_1-1} \alpha_{\ell_1}^{L_2-1} \left[K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)}\right]_{\ell_1,\ell_2=0}\end{aligned} \tag{3.4.63}$$

Where we have used (3.4.45) and (3.4.29) together with (3.4.24) in order to simplify the above. Now, in the case where $\ell_2 = 0$, the propagators take the forms.

$$K_{\text{sing}}^{\phi,(1)} = \frac{\ell_1+1}{2\ell_1+1} K^{m^2=\ell_1(\ell_1-1)} + \frac{\ell_1}{2\ell_1+1} K^{m^2=(\ell_1+1)(\ell_1+2)} \quad , \quad K_{\text{sing}}^{\phi,(2)} = K^{m^2=\ell_1(\ell_1+1)} \tag{3.4.64}$$

As expected, the propargators reduce to those of a dCFT setup with $\mathfrak{so}(3)$ symmetric vevs [9]. It can be shown using several hypergeometric identities [10], that the combination of propargators in the two-point function (3.4.63) can be rewritten in the following way.

$$\left[K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)}\right]_{\ell_1,\ell_2=0} = \frac{g^2}{16\pi^2} \frac{1}{x_3 y_3} \frac{{}_2F_1(\ell_1, \ell_1+1; 2\ell_1+2; -\xi^{-1})}{\binom{2\ell_1+1}{\ell_1+1} \xi^{\ell_1+1}} \frac{\xi}{\xi+1} \tag{3.4.65}$$

Using the above expression for the propagators, the sum in (3.4.63) can actually be evaluated in the limit of $L_1, L_2 \gg 1$ [7]. Using *Mathematica*, one can first expand the summand around $L_1, L_2 = \infty$, and then subsequently evaluate the sum for a finite number of terms. If the result is now expanded in powers of ξ^{-1} around $\xi^{-1} = 0$, we see a clear pattern.

$$\begin{aligned} L_1 L_2 \sum_{\ell_1=0}^{n} \alpha_{\ell_1}^{L_1-1} \alpha_{\ell_1}^{L_2-1} \frac{{}_2F_1(\ell_1, \ell_1+1; 2\ell_1+2; -\xi^{-1})}{\binom{2\ell_1+1}{\ell_1+1} \xi^{\ell_1+1}} &= \left(\frac{k_1}{2}\right)^{L_1+L_2-1} \\ \times \begin{cases} 2\,\xi^{-2} - \xi^{-3} + \xi^{-4} - \ldots - \xi^{-(2n-1)} + \xi^{-2n} + \mathcal{O}(\xi^{-(2n+1)}) & \text{for } L_1, L_2 \text{ even} \\ 2\,\xi^{-1} + \xi^{-3} - \xi^{-4} + \ldots + \xi^{-(2n-1)} - \xi^{-2n} + \mathcal{O}(\xi^{-(2n+1)}) & \text{for } L_1, L_2 \text{ odd} \end{cases} \\ = \left(\frac{k_1}{2}\right)^{L_1+L_2-1} \frac{1}{(\xi+1)^2\,\xi} \begin{cases} 2\,\xi + 1 & \text{for } L_1, L_2 \text{ even} \\ 2\,\xi\,(\xi+1)+1 & \text{for } L_1, L_2 \text{ odd} \end{cases} \quad , \quad L_1, L_2 \gg 1 \end{aligned} \tag{3.4.66}$$

The last equality is obtained by assuming that the pattern holds for $n \to \infty$, and subsequently summing the geometric series. We are ultimately interested in taking the following double-scaling limit.

$$\text{Large } N \text{ expansion}: \quad N \to \infty \quad , \quad \lambda = g^2 N \quad \text{finite} \tag{3.4.67}$$

$$\text{Large } k_1 \text{ expansion}: \quad k_1 \to \infty \quad , \quad \mu = \frac{\lambda}{k_1^2} \quad \text{finite} \tag{3.4.68}$$

When evaluated in the above double-scaling limit, the two-point function takes the following form.

$$\boxed{\begin{aligned} \mathfrak{so}(3) \times \mathfrak{so}(3): \quad \left\langle \mathrm{tr}\, \boldsymbol{Z}^{L_1}\, \mathrm{tr}\, \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,c.}}^{k_2=1} &= \frac{\mu}{8\pi^2} \frac{1}{N} \frac{k_1^{L_1+L_2+1}}{(2x_3)^{L_1}(2y_3)^{L_2}} && \text{(3.4.69a)} \\ \times \frac{1}{(\xi+1)^2\,\xi} \begin{cases} 2\,\xi+1 & \text{for } L_1, L_2 \text{ even} \\ 2\,\xi\,(\xi+1)+1 & \text{for } L_1, L_2 \text{ odd} \end{cases} &\quad , \quad L_1, L_2 \gg 1 && \text{(3.4.69b)} \end{aligned}}$$

[7] One does not actually need the simplification (3.4.65) in order to evaluate the sum. The sum can also be evaluated using just (3.4.64), but one has to be careful about the propargators with $\nu = -1/2$ [10].

For the two-point function $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}} \rangle$, the treatment is again very similar to that of $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle$. Effectively, all that changes is the sign of the propagator $K^{\phi,(2)}_{\text{sing}}$ in (3.4.63). It can again be shown using several hypergeometric identities [10], that this new combination of propargators can be rewritten in a more compact form.

$$\left[K^{\phi,(1)}_{\text{sing}} + K^{\phi,(2)}_{\text{sing}}\right]_{\ell_1,\ell_2=0} = \frac{g^2}{16\pi^2}\frac{1}{x_3 y_3}\frac{{}_2F_1(\ell_1,\ell_1+1;2\ell_1+2;-\xi^{-1})}{\binom{2\ell_1+1}{\ell_1+1}\xi^{\ell_1+1}} \tag{3.4.70}$$

Comparing the above to (3.4.65), we see that they only differ by a factor of $\frac{\xi}{\xi+1}$. Thus, the connected tree-level contribution to $\langle \mathcal{O}_{\boldsymbol{Z}}\, \mathcal{O}_{\bar{\boldsymbol{Z}}} \rangle$ takes the following form in the double-scaling limit.

$$\boxed{\begin{aligned} &\mathfrak{so}(3)\times\mathfrak{so}(3):\quad \left\langle \operatorname{tr} \boldsymbol{Z}^{L_1} \operatorname{tr} \bar{\boldsymbol{Z}}^{L_2} \right\rangle^{k_2=1}_{\text{tree,c.}} = \frac{\mu}{8\pi^2}\frac{1}{N}\frac{k_1^{L_1+L_2+1}}{(2x_3)^{L_1}(2y_3)^{L_2}} &&\text{(3.4.71a)}\\ &\times\frac{1}{(\xi+1)\,\xi^2}\begin{cases} 2\,\xi+1 & \text{for } L_1, L_2 \text{ even}\\ 2\,\xi\,(\xi+1)+1 & \text{for } L_1, L_2 \text{ odd}\end{cases}\quad,\quad L_1,L_2\gg 1 &&\text{(3.4.71b)}\end{aligned}}$$

Lastly, for the two-point function $\langle \mathcal{O}_{\boldsymbol{X}}\, \mathcal{O}_{\boldsymbol{Z}} \rangle$, the connected tree-level contribution looks as follow.

$$\begin{aligned} \left\langle \operatorname{tr} \boldsymbol{X}^{L_1} \operatorname{tr} \boldsymbol{Z}^{L_2} \right\rangle_{\text{tree,c.}} &= \frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}} \left\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'} \right\rangle \\ &\times \operatorname{tr}\left[(t_3^{k_1})^{L_1-1}\, \hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2}\right] \operatorname{tr}\left[(t_1^{k_1})^{L_2-1}\, \hat{Y}^{m_1'}_{\ell_1'} \otimes \hat{Y}^{m_2'}_{\ell_2'}\right] \\ &= \frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}} \operatorname{tr}\left[(t_3^{k_1})^{L_1-1}\, \hat{Y}^{m_1}_{\ell_1}\right] \operatorname{tr}\left[(t_1^{k_1})^{L_2-1}\, \hat{Y}^{m_1'}_{\ell_1'}\right] \left\langle X_{\ell_1,m_1;0,0}\, Z_{\ell_1',m_1';0,0} \right\rangle \\ &= -\frac{L_1 L_2}{x_3^{L_1-1} y_3^{L_2-1}} \sum_{\ell_1=1}^{\infty} \alpha_{\ell_1}^{L_1-1} \alpha_{\ell_1}^{L_2-1} \frac{i^{\ell_1+1}(\ell_1+1)!}{2^{\ell_1}\left(\frac{\ell_1-1}{2}\right)!\left(\frac{\ell_1+1}{2}\right)!} K^{\phi,(1)}_{\text{anti}}\Big|_{\ell_1,\ell_2=0} \end{aligned} \tag{3.4.72}$$

Where we have used (3.4.45) and (3.4.29) together with (3.4.26) and (3.4.37), in order to obtain the above expression. In the case where $\ell_2=0$, the propagator take the form.

$$K^{\phi,(1)}_{\text{anti}} = \frac{1}{2\ell_1+1} K^{m^2=\ell_1(\ell_1-1)} - \frac{1}{2\ell_1+1} K^{m^2=(\ell_1+1)(\ell_1+2)} \tag{3.4.73}$$

Once again, it can again be shown using several hypergeometric identities [10], that this propergator can be rewritten in a simple form. The result of this procedure is the following.

$$K^{\phi,(1)}_{\text{anti}}\Big|_{\ell_1,\ell_2=0} = \frac{g^2}{16\pi^2}\frac{1}{x_3 y_3}\frac{{}_2F_1(\ell_1+1,\ell_1+1;2\ell_1+2;-\xi^{-1})}{(\ell_1+1)\binom{2\ell_1+1}{\ell_1+1}} \tag{3.4.74}$$

Using the above expression for the propagators, the sum in (3.4.72) can now be evaluated in the limit of $L_1, L_2 \gg 1$. Using *Mathematica*, one can again expand the summand around $L_1, L_2 = \infty$, and then subsequently evaluate the sum for a finite number of terms. Expanding the result in powers of ξ^{-1} around $\xi^{-1}=0$, we again see a clear pattern.

$$\begin{aligned} &-L_1 L_2 \sum_{\ell_1=1}^{n} \alpha_{\ell_1}^{L_1-1}\alpha_{\ell_1}^{L_2-1} \frac{i^{\ell_1+1}\ell_1!}{2^{\ell_1}\left(\frac{\ell_1-1}{2}\right)!\left(\frac{\ell_1+1}{2}\right)!} \frac{{}_2F_1(\ell_1+1,\ell_1+1;2\ell_1+2;-\xi^{-1})}{\binom{2\ell_1+1}{\ell_1+1}\xi^{\ell_1+1}} = \left(\frac{k_1}{2}\right)^{L_1+L_2-1} \\ &\times\begin{cases} \xi^{-2}+\xi^{-3}+\frac{3}{4}\xi^{-4}-\ldots+\frac{2n-2}{2^{2n-3}}\xi^{-(2n-1)}-\frac{2n-1}{2^{2n-2}}\xi^{-2n}+\mathcal{O}(\xi^{-(2n+1)}) & \text{for } L_1, L_2 \text{ even}\\ 0 & \text{for } L_1, L_2 \text{ odd}\end{cases} \\ &= \left(\frac{k_1}{2}\right)^{L_1+L_2-1}\frac{1}{(2\,\xi+1)^2}\begin{cases} 4 & \text{for } L_1, L_2 \text{ even}\\ 0 & \text{for } L_1, L_2 \text{ odd}\end{cases}\quad,\quad L_1,L_2\gg 1 \end{aligned} \tag{3.4.75}$$

The last equality is obtained by assuming that the pattern (*starting from the third term in the sum*) holds for $n\to\infty$, and subsequently summing the series. The reason why above sum vanishes for L_1, L_2 odd can be tracked back to (3.4.72), and the fact that $\mathrm{tr}\left[(t_1^k)^L \hat{Y}_\ell^m\right]$ vanishes if we do not have either L,ℓ,m all even or L,ℓ,m all odd [10]. Thus, the connected tree-level contribution to $\langle \mathcal{O}_{\boldsymbol{X}}\,\mathcal{O}_{\boldsymbol{Z}}\rangle$ takes the following form in the double-scaling limit.

$$\boxed{\begin{aligned}&\mathfrak{so}(3)\times\mathfrak{so}(3):\quad \left\langle \mathrm{tr}\,\boldsymbol{X}^{L_1}\,\mathrm{tr}\,\boldsymbol{Z}^{L_2}\right\rangle^{k_2=1}_{\text{tree,c.}} = \frac{\mu}{8\pi^2}\frac{1}{N}\frac{k_1^{L_1+L_2+1}}{(2x_3)^{L_1}(2y_3)^{L_2}} && (3.4.76a)\\ &\qquad\times\frac{1}{(2\,\xi+1)^2}\begin{cases}4 & \text{for } L_1,L_2 \text{ even}\\ 0 & \text{for } L_1,L_2 \text{ odd}\end{cases},\quad L_1,L_2\gg 1 && (3.4.76b)\end{aligned}}$$

This conlcudes our discussion of the connected tree-level contribution to the two-point functions (3.3.1), for the different cases with $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric vevs. We now proceed to discuss what happens for $\mathfrak{so}(5)$ symmetric vevs.

3.4.2 SO(5) symmetric vevs

Work in progress..

3.5 Two-point functions at 1-loop (*disconnected*)

For the sake of completeness, we now discuss how to compute the other first order in λ contributions to the chiral primary two-point functions; namely the disconnected 1-loop contributions. These disconnected 1-loop contributions are given simply in terms of 1-loop contributions to one-point functions of the chiral primary operators. For example, the disconnected 1-loop contribution to $\left\langle \mathrm{tr}\,\boldsymbol{Z}^{L_1}\,\mathrm{tr}\,\boldsymbol{Z}^{L_2}\right\rangle$ is given by the following.

$$\left\langle \mathrm{tr}\,\boldsymbol{Z}^{L_1}\,\mathrm{tr}\,\boldsymbol{Z}^{L_2}\right\rangle_{1-\text{loop,dc.}} = \left\langle \mathrm{tr}\,\boldsymbol{Z}^{L_1}\right\rangle_{1-\text{loop}}\mathrm{tr}\,\mathcal{Z}^{L_2} + \mathrm{tr}\,\mathcal{Z}^{L_1}\left\langle \mathrm{tr}\,\boldsymbol{Z}^{L_2}\right\rangle_{1-\text{loop}} \tag{3.5.1}$$

The 1-loop contribution to the one-point function $\langle \mathrm{tr}\,\boldsymbol{Z}^L\rangle$ specifically have already been computed in previous works [11, 12]. Also, the 1-loop contributions to the one-point functions $\langle \mathrm{tr}\,\bar{\boldsymbol{Z}}^L\rangle$ and $\langle \mathrm{tr}\,\boldsymbol{X}^L\rangle$ can easily be obtained from the result for $\langle \mathrm{tr}\,\boldsymbol{Z}^L\rangle$, as we shall explain below.

3.5.1 SO(3) × SO(3) symmetric vevs

The one-loop contributions to the one-point function $\left\langle \mathrm{tr}\,\boldsymbol{Z}^L\right\rangle$ in the $SO(3)\times SO(3)$ symmetric setup, was computed in [11]. The final result of these computations is the following expression for the one-loop contribution.

$$\begin{aligned}\left\langle \mathrm{tr}\,\boldsymbol{Z}^L\right\rangle_{1-\text{loop}} = \Bigg[&\frac{\mu}{4\pi^2(L+1)(k_1^2+k_1^2)^2}\left(4(k_1k_2)^2+(L^2+3L-2)(k_1^4+k_2^4)\right.\\ &\left.+2(L-1)(L+2)k_1k_2(k_1^2-k_2^2)\cot[(L+2)\psi_0]\right)\Bigg]\mathrm{tr}\,\mathcal{Z}^L+\mathcal{O}(\mu^2)\end{aligned} \tag{3.5.2}$$

Now, if one writes out the raw unsimplified expression for $\langle \mathrm{tr}\,\bar{\boldsymbol{Z}}^L\rangle_{1-\text{loop}}$ using the perturbative techniques of [11], one finds that.

$$\langle \mathrm{tr}\,\bar{\boldsymbol{Z}}^L\rangle_{1-\text{loop}} = \left\langle \mathrm{tr}\,\boldsymbol{Z}^L\right\rangle^*_{1-\text{loop}} \quad\Rightarrow\quad \langle \mathrm{tr}\,\bar{\boldsymbol{Z}}^L\rangle_{1-\text{loop}} = \left\langle \mathrm{tr}\,\boldsymbol{Z}^L\right\rangle_{1-\text{loop}} \tag{3.5.3}$$

Similarly if one writes out the unsimplified expression for $\langle \operatorname{tr} \boldsymbol{X}^L \rangle_{1-\text{loop}}$, it turns out that.

$$\left\langle \operatorname{tr} \boldsymbol{X}^L \right\rangle_{1-\text{loop}} = V \left\langle \operatorname{tr} \boldsymbol{Z}^L \right\rangle_{1-\text{loop}} V^\dagger \tag{3.5.4}$$

Where V is the similarity transformation previously employed in this section, which takes $t_1 \to t_3$, $t_2 \to -t_2$ and $t_3 \to t_1$. Combining the above observation with the result $\operatorname{tr} \mathcal{X}^L = \operatorname{tr} \mathcal{Z}^L$ from earlier, we see that.

$$\left\langle \operatorname{tr} \boldsymbol{X}^L \right\rangle_{1-\text{loop}} = \left\langle \operatorname{tr} \boldsymbol{Z}^L \right\rangle_{1-\text{loop}} \tag{3.5.5}$$

With the results (3.5.2), (3.5.3) and (3.5.5), together with the tree-level results $\operatorname{tr} \mathcal{Z}^L$, $\operatorname{tr} \bar{\mathcal{Z}}^L$ and $\operatorname{tr} \mathcal{X}^L$, we now effectively know the disconnected one-loop contributions to the chiral primary two-point functions in the $SO(3) \times SO(3)$ symmetric defect setup.

3.5.2 SO(5) symmetric vevs

The one-loop contributions to the one-point function $\left\langle \operatorname{tr} \boldsymbol{Z}^L \right\rangle$ in the $SO(3) \times SO(3)$ symmetric setup, was computed in [12]. The final result of these computations is the following expression for the one-loop contribution.

$$\left\langle \operatorname{tr} \boldsymbol{Z}^L \right\rangle_{1-\text{loop}} = \left[\frac{\mu L(L+3)}{4(L-1)} \right] \operatorname{tr} \mathcal{Z}^L + \mathcal{O}(\mu^2) \tag{3.5.6}$$

If again one writes out the raw unsimplified expression for $\langle \operatorname{tr} \bar{\boldsymbol{Z}}^L \rangle_{1-\text{loop}}$, this time using the perturbative techniques of [12], one directly sees that.

$$\langle \operatorname{tr} \bar{\boldsymbol{Z}}^L \rangle_{1-\text{loop}} = \left\langle \operatorname{tr} \boldsymbol{Z}^L \right\rangle_{1-\text{loop}} \tag{3.5.7}$$

If one writes out the unsimplified expression for $\langle \operatorname{tr} \boldsymbol{X}^L \rangle_{1-\text{loop}}$, it now turns out that.

$$\left\langle \operatorname{tr} \boldsymbol{X}^L \right\rangle_{1-\text{loop}} = A \operatorname{tr} \mathcal{X}^{L-2} + B \operatorname{tr} \mathcal{X}^L \tag{3.5.8}$$

The precise forms of A and B are not important here since we know from an earlier computation in this section, that $\operatorname{tr} \mathcal{X}^L = 0$, in the $SO(5)$ symmetric setup for any L. Thus, we find that.

$$\left\langle \operatorname{tr} \boldsymbol{X}^L \right\rangle_{1-\text{loop}} = 0 \tag{3.5.9}$$

With the results (3.5.6), (3.5.7) and (3.5.9), together with the tree-level results $\operatorname{tr} \mathcal{Z}^L$, $\operatorname{tr} \bar{\mathcal{Z}}^L$ and $\operatorname{tr} \mathcal{X}^L$, we now effectively know the disconnected one-loop contributions to the chiral primary two-point functions in the $SO(5)$ symmetric defect setup.

4 Two-point functions between long and short operators

The aim of this following section is to compute two-point functions of the form: $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W}_1\boldsymbol{W}_2} \rangle$. The operator $\mathcal{O}_{\boldsymbol{L}}$ is a non-protected scalar operator of length L corresponding to a Bethe state in the spin-chain picture: $\mathcal{O}_{\boldsymbol{L}} = \Psi^{i_1 \dots i_L}\, \mathrm{tr}[\boldsymbol{V}_{i_1} \cdots \boldsymbol{V}_{i_L}]$, where $\boldsymbol{V}_{i_\ell} \in \{\boldsymbol{X}, \boldsymbol{Z}\}$ and $\Psi^{i_1 \dots i_L}$ is a Bethe wave function. The operator $\mathcal{O}_{\boldsymbol{W}_1\boldsymbol{W}_2}$ is a scalar operator of length two, which does not have to correspond to a Bethe state: $\mathcal{O}_{\boldsymbol{W}_1\boldsymbol{W}_2} = \mathrm{tr}[\boldsymbol{W}_1\boldsymbol{W}_2]$, where $\boldsymbol{W}_1, \boldsymbol{W}_2 \in \{\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \bar{\boldsymbol{X}}, \bar{\boldsymbol{Y}}, \bar{\boldsymbol{Z}}\}$. The complex scalars and their conjugates are defined as follow.

$$\boldsymbol{X} = \boldsymbol{\phi}_1 + i\,\boldsymbol{\phi}_4 \quad , \quad \boldsymbol{Y} = \boldsymbol{\phi}_2 + i\,\boldsymbol{\phi}_5 \quad , \quad \boldsymbol{Z} = \boldsymbol{\phi}_3 + i\,\boldsymbol{\phi}_6 \tag{4.0.1}$$

$$\bar{\boldsymbol{X}} = \boldsymbol{\phi}_1 - i\,\boldsymbol{\phi}_4 \quad , \quad \bar{\boldsymbol{Y}} = \boldsymbol{\phi}_2 - i\,\boldsymbol{\phi}_5 \quad , \quad \bar{\boldsymbol{Z}} = \boldsymbol{\phi}_3 - i\,\boldsymbol{\phi}_6 \tag{4.0.2}$$

Before going into the details of how to compute these two-point functions, we will first decribe the connection between gauge invariant single trace operators in $\mathcal{N}=4$ SYM, and states of certain spin-chain systems. We will focus on a subset of single trace scalar operators, namely the operators $\mathcal{O}_{\boldsymbol{L}}$, corresponding to states of a Heisenberg spin-chain (*all spins are spin-$\frac{1}{2}$*). We will also discuss how to account for the non-zero expectation values of the scalar fields by introducing the so-called *Matrix Product State* of the spin-chain, before finally addressing the computation of $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W}_1\boldsymbol{W}_2} \rangle$.

4.1 Two-point functions of non-protected operators

As we have already discussed in section 3.2, the form of the two-point functions of scalar operators in our dCFT setups are partially fixed by the remaining $SO(3,2)$ symmetry. In particular, we saw that the conformal dimensions Δ_a of the operators in question appeared in these partially fixed forms. Thus, if we want to know the two-point functions of non-protected operators, we need to know how to find loop corrections to the conformal dimensions of these operators. To this end, it is convenient to study the limit of two-point functions far away from the defect, where the vevs vanish and the full $SO(4,2)$ symmetry is restored. Recall, again from section 3.2, that in this limit the two-point function of any two scalar operators are completely fixed by the symmetry, and takes the form.

$$\langle \mathcal{O}_a(x)\, \bar{\mathcal{O}}_b(y) \rangle = \frac{M_{ab}}{|x-y|^{\Delta_a + \Delta_b}} \tag{4.1.1}$$

One can now find an orthonormal set of operators by diagonalizing the matrix of two-point functions: $M_{ab} = \langle \mathcal{O}_a(x)\, \bar{\mathcal{O}}_b(y) \rangle |_{|x-y|=1}$. Upon subsequently rescaling the orthogonal operators, we find that $M_{ab} \to \delta_{ab}$. Now, to find the loop corrections to the conformal dimensions of the theory, we first split the conformal dimensions into two pieces: $\Delta = \Delta_0 + \gamma_{\mathcal{O}}$. We call Δ_0 the the *bare dimension*, and it is the conformal dimension at zero coupling. We call $\gamma_{\mathcal{O}}$ the *anomalous dimension* of the operator, and it encodes the loop corrections to the total conformal dimension. For small coupling $g \ll 1$ we assume that also $\gamma_{\mathcal{O}} \ll 1$, and we can write.

$$\big\langle \mathcal{O}_a(x)\, \bar{\mathcal{O}}_b(y) \big\rangle = \frac{\delta_{ab}}{|x-y|^{2\Delta}} = \frac{\delta_{ab}}{|x-y|^{2\Delta_0 + 2\gamma_{\mathcal{O}}}} \approx \frac{\delta_{ab}}{|x-y|^{2\Delta_0}} \left[1 - \gamma_{\mathcal{O}} \log\!\big(\mu^2 |x-y|^2\big)\right] \tag{4.1.2}$$

Where we have introduced a constant μ with mass dimension 1, in order for the argument of the $\log$ to be dimensionless. We can now perturbatively compute the loop corrections to the two-point function (4.1.2). In what follows, we will only be concerned with the 1-loop corrections to the scalar single-trace two-point functions, for which the explicit calculation has been carried out in [3]. The result is the following.

$$\big\langle \mathcal{O}_a(x)\, \bar{\mathcal{O}}_a(y) \big\rangle = \frac{1}{|x-y|^{2L}} (\bar{\Psi}_a)_{j_1 \dots j_L} \left[1 - (\Gamma)^{j_1 \dots j_L}_{i_1 \dots i_L} \log\!\big(\mu^2 |x-y|^2\big)\right] (\Psi_a)^{i_1 \dots i_L} \tag{4.1.3}$$

Where $\mathcal{O}_a(x) = (\Psi_a)^{i_1\ldots i_L}\,\mathrm{tr}\big[\boldsymbol{\phi}_{i_1}(x)\cdots\boldsymbol{\phi}_{i_L}(x)\big]$. The matrix Γ, which contains the information about the anomalous dimensions of the scalar single trace operators, is given by the following expression.

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{\ell=1}^{L}\left(2 - 2\,P_{\ell,\ell+1} + K_{\ell,\ell+1}\right) \tag{4.1.4}$$

Where $\lambda = g^2 N$ is the 't Hooft coupling, and the operators $P_{\ell,\ell+1}$ and $K_{\ell,\ell+1}$ are the so-called permutation and trace operators respectively. Their action on any wavefunction is given by the following, where any indices which are not written explicitly are unaffected by both operators.

$$(P_{\ell,\ell+1})^{j_\ell\,j_{\ell+1}}_{i_\ell\,i_{\ell+1}}\,(\Psi_a)^{i_\ell\,i_{\ell+1}} = (\Psi_a)^{j_{\ell+1}\,j_\ell} \tag{4.1.5}$$

$$(K_{\ell,\ell+1})^{j_\ell\,j_{\ell+1}}_{i_\ell\,i_{\ell+1}}\,(\Psi_a)^{i_\ell\,i_{\ell+1}} = \delta^{j_\ell\,j_{\ell+1}}\,\delta_{i_\ell\,i_{\ell+1}}\,(\Psi_a)^{i_\ell\,i_{\ell+1}} \tag{4.1.6}$$

Despite what we might have expected, we see that Γ is not diagonal in the $\mathrm{tr}\big[\boldsymbol{\phi}_{i_1}(x)\cdots\boldsymbol{\phi}_{i_L}(x)\big]$ basis. This is just a reflection of the fact that the operators which where eigenstates of the Dilatation operator at tree level, fail to remain eigenstates at 1-loop level. Before going on to discuss what the eigenstates of Γ actually are and how to find them, let us first make some quick comments on renormalization. Firstly, the dimensionful constant μ that we introduced in (4.1.2) actually appears naturally as a renormalization scale in the derivation of (4.1.3). Secondly, the anomalous dimension $\gamma_{\mathcal{O}}$ as we have defined it above is indeed the same as in the context of the *Renormalization Group Equations*. In particular, the *Callan-Symanzik equation* for the renormalized two-point function $\boldsymbol{\Delta}_{ab}(x) = \big\langle \mathcal{O}_a(x)\,\bar{\mathcal{O}}_b(0)\big\rangle$, reads as follow.

$$\left[\frac{\partial}{\partial\log(\mu)} + \beta(g)\frac{\partial}{\partial g} + m\gamma_m(g)\frac{\partial}{\partial m} + 2\gamma_{\mathcal{O}}(g)\right]\boldsymbol{\Delta}_{ab}(x) = 0 \tag{4.1.7}$$

For $\mathcal{N}=4$ SYM, the beta-function is believed to vanish for all orders in the coupling g, so that $\beta(g)=0$. Furthermore, we are only considering massless operators $\mathcal{O}_a$, so that $m=0$. The Callan-Symanzik equation then reduces to.

$$\left[\frac{\partial}{\partial\log(\mu)} + 2\gamma_{\mathcal{O}}(g)\right]\boldsymbol{\Delta}_{ab}(x) = 0 \quad\Rightarrow\quad \boldsymbol{\Delta}_{ab}(x) = \frac{M_{ab}(g)}{|x|^{2\Delta_0}}\left(\mu^2|x|^2\right)^{-\gamma_{\mathcal{O}}(g)} \tag{4.1.8}$$

Where we have used that the mass dimensions of $\mathcal{O}_a(x)$ and $\mathcal{O}_b(x)$ are Δ_0, in order to determine the $|x|$ dependence of the two-point function. Again, under the assumption that $\gamma_{\mathcal{O}} \ll 1$ when $g \ll 1$, we find that (4.1.8) reproduces (4.1.2).

4.2 Spin-chains and integrability in $\mathcal{N}=4$ SYM

In order to determine the possible anomalous dimensions at 1-loop level, we need to find a way to diagonalize the anomalous dimensions operator Γ (4.1.4). To this end, we recognize that Γ can be thought of as a linear operator on the space: $\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_\ell\otimes\cdots\otimes\mathcal{H}_L$, with $\mathcal{H}_\ell = \mathbb{C}^6$. In other words, we can think Γ as acting on the Hilbert space of an $SO(6)$ spin-chain with length L, and the basis of single trace operators as equivalent to a standard basis on the spin-chain.

$$\mathrm{tr}\big[\boldsymbol{\phi}_{i_1}(x)\cdots\boldsymbol{\phi}_{i_L}(x)\big] \to |s_1\cdots s_L\rangle \tag{4.2.1}$$

The actions of $P_{\ell,\ell+1}$, $K_{\ell,\ell+1}$ have the following straightforward translations to the spin-chain picture.

$$P_{\ell,\ell+1}\,|s_1\cdots s_\ell\,s_{\ell+1}\cdots s_L\rangle = |s_1\cdots s_{\ell+1}\,s_\ell\cdots s_L\rangle \tag{4.2.2}$$

$$K_{\ell,\ell+1}\,|s_1\cdots s_\ell\,s_{\ell+1}\cdots s_L\rangle = \delta_{s_\ell\,s_{\ell+1}}\,\delta^{s'_\ell\,s'_{\ell+1}}\,\big|s_1\cdots s'_\ell\,s'_{\ell+1}\cdots s_L\big\rangle \tag{4.2.3}$$

Because the opertaors $\mathcal{O}_a(x)$ are traced, we need to make sure that we only consider states of the spin chain which are invariant under uniform shifts.

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_L \to \mathcal{H}_L \otimes \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{L-1} \tag{4.2.4}$$

Furthermore, it is easily shown that Γ is in fact a Hermitian operator in the spin-chain Hilbert space.

$$\bar{\Phi}_{kl}\, P^{kl}_{ij}\, \Psi^{ij} = \bar{\Phi}_{kl}\, \Psi^{lk} = \overline{\left[\bar{\Psi}_{kl}\, \Phi^{lk}\right]} = \overline{\left[\bar{\Psi}_{kl}\, P^{kl}_{ij}\, \Phi^{ij}\right]} \tag{4.2.5}$$

$$\bar{\Phi}_{kl}\, K^{kl}_{ij}\, \Psi^{ij} = \bar{\Phi}_{kl}\, \delta^{kl}\, \delta_{ij}\, \Psi^{ij} = \overline{\left[\bar{\Psi}_{ij}\, \delta^{ij}\, \delta_{kl}\, \Phi^{kl}\right]} = \overline{\left[\bar{\Psi}_{ij}\, K^{ij}_{kl}\, \Phi^{kl}\right]} \tag{4.2.6}$$

Where we have droped the $\ell, \ell+1$ specifier on the operators, and used i, k and j, l for indices on site ℓ and site $\ell+1$ respectively. This means that may think of Γ as the Hamiltonian of the $SO(6)$ spin-chain Hilbert space.

4.2.1 The $SU(2)$ subsector

If we now choose to restrict our attention to a subset of the spin-chain states spanned by the following basis of operators / states.

$$\mathrm{tr}[\boldsymbol{V}_{i_1} \cdots \boldsymbol{V}_{i_L}] \to |\sigma_1 \cdots \sigma_L\rangle \quad , \quad \boldsymbol{V}_{i_\ell} \in \{\boldsymbol{X}, \boldsymbol{Z}\} \quad , \quad \sigma_\ell \in \{\downarrow, \uparrow\} \tag{4.2.7}$$

We observe that all states in this restricted Hilbert space are annihilated by the trace operator $K_{\ell,\ell+1}$, simply because any piece $\boldsymbol{XZ} = \phi_1\phi_3 - \phi_4\phi_6 + i\,\phi_1\phi_6 + i\,\phi_4\phi_3$, of a basis element has no diagonal components, and any piece $\boldsymbol{XX} = \phi_1\phi_1 - \phi_4\phi_4 + i\,\phi_1\phi_4 + i\,\phi_4\phi_1$, has canceling diagnoal components. Similar arguments can of course be made for parts $\boldsymbol{ZX}$ and $\boldsymbol{ZZ}$ respectively. This means that the spin-chain Hamiltonian (4.1.4) reduces to the following.

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1}) \quad , \quad \mathcal{H} = \bigotimes_{\ell=1}^{L} \mathcal{H}_\ell \quad , \quad \mathcal{H}_\ell = \mathbb{C}^2 \tag{4.2.8}$$

It turns out that the permutation operator when acting on this $SU(2)$ spin-chain can be written in the following simple form [13].

$$P_{\ell,\ell+1} = \frac{1}{2}\left(\mathbb{1}_\ell \otimes \mathbb{1}_{\ell+1} + \sum_\alpha \sigma^\alpha_\ell \otimes \sigma^\alpha_{\ell+1}\right) \tag{4.2.9}$$

Where σ^α are the standard Pauli matrices. Using the above form of the permutation operator, we can now rewrite the spin-chain Hamiltonian as follow.

$$\Gamma_{SU(2)} \equiv \frac{\lambda}{4\pi^2} H \quad , \quad H = \sum_{\ell=1}^{L} \left(\frac{1}{4} - \vec{S}_\ell \cdot \vec{S}_{\ell+1}\right) \tag{4.2.10}$$

Thus, we see that upon rescaling, the spin-chain Hamiltonian above is exactly that of a ferromagnetic Heisenberg spin-chain. At this point, our problem of finding the possible anomalous dimension of single trace scalar operators, has therefore been reduced to the problem of finding eigenvalues and egienvectors of the ferromagnetic Heisenberg spin-chain. It turns out that this spin-chain system is actually integrable (*there exists $L-1$ independent operators which commute with H*), and the eigenstates and eigenvalues can be found analytically by use of the so-called *Algebraic Bethe Ansatz* [13], or *ABA* for short. In what follows, we will briefly summarize the main results and ideas behind this approach. The first step in the procedure is to introduce two auxiliary spaces $V_1, V_2 = \mathbb{C}^2$, with which we extend the Hilbert space to $\mathcal{H}\otimes V_1\otimes V_2$. Next, we can now define a central object called the *Lax operator*, which is a linear operator on the Hilbert space $\mathcal{H}_\ell\otimes V$, with either $V=V_1,V_2$.

$$\mathcal{L}_{\ell,a}(u):\mathcal{H}_\ell\otimes V\mapsto\mathcal{H}_\ell\otimes V\quad,\quad\mathcal{L}_{\ell,a}(u)=u\,\mathbb{1}_\ell\otimes\mathbb{1}_a+i\sum_\alpha S^\alpha_\ell\otimes\sigma^\alpha_a \tag{4.2.11}$$

Where we have used the index a to denote the operators which act on the space V. The Lax operator can also be written in terms of the raising and lowering operators on each site of the spin-chain, or alternatively in terms of the permutation operator $P_{\ell,a}$ on the space $\mathcal{H}_\ell\otimes V$.

$$\mathcal{L}_{\ell,a}(u)=\left(u-\frac{i}{2}\right)\mathbb{1}_\ell\otimes\mathbb{1}_a+i\,P_{\ell,a}=\begin{pmatrix}u+iS^3_\ell & iS^-_\ell\\ iS^+_\ell & u-iS^3_\ell\end{pmatrix} \tag{4.2.12}$$

We now define another important object in the ABA, namely the so called *R-matrix*, which is a linear operator acting on the product of the two auxiliary spaces V_1 and V_2.

$$\mathcal{R}_{a_1,a_2}(u):V_1\otimes V_2\mapsto V_1\otimes V_2\quad,\quad\mathcal{R}_{a_1,a_2}(u)=u\,\mathbb{1}_{a_1}\otimes\mathbb{1}_{a_2}+iP_{a_1,a_2} \tag{4.2.13}$$

The Lax operator and the R-matrix together satisfy a Yang-Baxter type relation, which serves as the foundation of the entire ABA approach. This relation looks as follow.

$$\mathcal{R}_{a_1,a_2}(u-v)\,\mathcal{L}_{\ell,a_1}(u)\,\mathcal{L}_{\ell,a_2}(v)=\mathcal{L}_{\ell,a_2}(v)\,\mathcal{L}_{\ell,a_1}(u)\,\mathcal{R}_{a_1,a_2}(u-v) \tag{4.2.14}$$

Where the above is to be understood as a relation on the space $\mathcal{H}_\ell\otimes V_1\otimes V_2$, with the operators trivially extended using appropriate identity matrices. The relation (4.2.14) implies a more practically useful Yang-Baxter type relation, which is given as.

$$\mathcal{R}_{a_1,a_2}(u-v)\,T_{a_1}(u)\,T_{a_2}(v)=T_{a_2}(v)\,T_{a_1}(u)\,\mathcal{R}_{a_1,a_2}(u-v) \tag{4.2.15}$$

Where the operators $T_a(u)$ act on the space $\mathcal{H}\otimes V$, and can be though of as monodromy operators on this modified Hilbert space. The monodromy operators $T_a(u)$ are defined as follow.

$$T_a(u)=\mathcal{L}_{L,a}(u)\cdots\mathcal{L}_{1,a}(u)=\begin{pmatrix}A(u) & B(u)\\ C(u) & D(u)\end{pmatrix} \tag{4.2.16}$$

Where the operators $A(u)$, $B(u)$, $C(u)$, $D(u)$, are operators only on the Hilbert space $\mathcal{H}$. seeing as it will become important momentarily, we note that using definition (4.2.11), the momodromy operator $T_a(u)$ can be written as a polynomial in u of order L.

$$T_a(u)=u^L+i\,u^{L-1}\sum_\alpha S^\alpha_\ell\otimes\sigma^\alpha_a+\sum_{n=0}^{L-2}\tilde{Q}_{(n),a}\,u^n \tag{4.2.17}$$

If we now trace over the auxiliary spaces V_1 and V_2 in the relation (4.2.15), it turns out that we end up with a family $\{Q_{(n)}\}$ of commuting operators. This is because the operators $F(u)$ defined below commutes for different input values. We find that.

$$F(u)=\mathrm{tr}[T_a(u)]=A(u)+D(u)=2\,u^L+\sum_{n=0}^{L-2}Q_{(n)}\,u^n \tag{4.2.18}$$

$$[F(u),F(v)]=0\quad\Rightarrow\quad[Q_{(n)},Q_{(m)}]=0\quad,\quad n,m=0,\dots,L-2 \tag{4.2.19}$$

It can be shown that the spin-chain Hamiltonian H can be generated from $F(u)$ in the following way.

$$H=\frac{L}{2}-\frac{i}{2}\frac{d}{du}\log F(u)\bigg|_{u=i/2} \tag{4.2.20}$$

Thus, we can take H to be some linear combination of the operators $\{Q_{(n)}\}$. By letting $v \to \infty$ in relation (4.2.15), it can also be shown that the total spin operator $\vec{S}$ commutes with the operator $F(u)$.

$$\left[T_a(u), S^\alpha + \frac{1}{2}\sigma^\alpha\right] = 0 \quad \Rightarrow \quad [F(u), \vec{S}] = 0 \quad , \quad \vec{S} = \sum_{\ell=1}^{L} \vec{S}_\ell \tag{4.2.21}$$

We can then extend the set of commuting operators by adding for example S^3. The set $\{Q_{(n)}, S^3\}$ then constitutes a set of L commuting operators, from which H in particular can be contructed. With all the necessary machinery laid out, we can now begin to describe the process of obtaining the eiegnvectors and eigenvalues of the spin-chain. Instead of looking for the eigenvectors and eigenvalues of H directly, it turns out to be more convenient to look for eigenvectors and eigenvalues of $F(u)$, which will be equivalent according to (4.2.20). We now first look for a lowest weight state, which in this context will be a state that is annihilated by $C(u)$. Using (4.2.12), it can easily be shown that the lowset weight state is given by.

$$C(u)\left|0\right\rangle = 0 \quad , \quad \left|0\right\rangle = \left|\uparrow \ldots \uparrow\right\rangle \tag{4.2.22}$$

Again, using (4.2.12), we also find that $|0\rangle$ is an eigenvector of $A(u)$ and $D(u)$, with the eigenvalues.

$$A(u)\left|0\right\rangle = (u+i/2)^L\left|0\right\rangle \quad , \quad D(u)\left|0\right\rangle = (u-i/2)^L\left|0\right\rangle \tag{4.2.23}$$

Thus, $|0\rangle$ is an eigenvector of $F(u)$. We now try to look for new eigenvectors of the following form.

$$\left|\Psi_M\right\rangle = B(u_1)\cdots B(u_M)\left|0\right\rangle \quad , \quad [B(u), B(v)] = 0 \tag{4.2.24}$$

Where the commutativity of the $B(u)$ operators follows from the Yang-Baxter relation (4.2.15). It is also possible to obtain commutation relations between $A(u)$, $B(v)$ and $D(u)$, $B(v)$ using (4.2.15). The explicit relations can be found in [13]. Using these commutation relations, one finds that states $|\Psi_M\rangle$ of the form (4.2.24) are eigenvectors of $F(u)$ with eigenvalues.

$$A(u)B(u_1)\cdots B(u_M)\left|0\right\rangle = (u+i/2)^L \prod_{j=1}^{M}\left(\frac{u-u_j-i}{u-u_j}\right) B(u_1)\cdots B(u_M)\left|0\right\rangle \tag{4.2.25}$$

$$D(u)B(u_1)\cdots B(u_M)\left|0\right\rangle = (u-i/2)^L \prod_{j=1}^{M}\left(\frac{u+u_j-i}{u-u_j}\right) B(u_1)\cdots B(u_M)\left|0\right\rangle \tag{4.2.26}$$

Only if the parameter set $\{u\}_M \equiv \{u_1, \ldots, u_M\}$ satisfy the *Bethe Ansatz Equations* (*BAE for short*).

$$\left(\frac{u_j+i/2}{u_j-i/2}\right)^L = \prod_{k\neq j}^{M} \frac{u_j-u_k+i}{u_j-u_k-i} \tag{4.2.27}$$

Another important observation is that the eigenstates $|\Psi_M\rangle$ are all simultaneously eigenvectors of S^3. This can be shown using the following information.

$$S^3\left|0\right\rangle = \frac{L}{2} \quad , \quad [S^3, B(u)] = -B(u) \quad \Rightarrow \quad S^3\left|\Psi_M\right\rangle = \left(\frac{L}{2} - M\right)\left|\Psi_M\right\rangle \tag{4.2.28}$$

This means that we can disregard all the contributions to $B(u)$ which contain any S_ℓ^+ operators, when constructing the eigenvectors $|\Psi_M\rangle$. Using (4.2.12), we can now show that the operator $B(u)$ can effectively be written in the following way.

$$B(u) = i\sum_{\ell=1}^{L}\left[\prod_{k=\ell+1}^{L}\left(u+iS_k^3\right)\right] S_\ell^- \left[\prod_{k=1}^{\ell-1}\left(u-iS_k^3\right)\right] \tag{4.2.29}$$

Now that we know all the eigenvalues of $F(u)$ and how to explicitly construct the associated eigenvectors, we can now use (4.2.20) to find the eigenvalues of H. The result is given as follow.

$$H\,|\Psi_M\rangle = \sum_{j=1}^{M} \varepsilon(u_j)\,|\Psi_M\rangle \quad , \quad \varepsilon(u) = \frac{1}{2}\frac{1}{u^2+1/4} \tag{4.2.30}$$

We have now obtained all the eigenvalues and eigenvectors of the Heisenberg spin-chain using the Algebraic Bethe Ansatz. All we now need to to is enforce the uniform shift invariance (4.2.4) condition on the states $|\Psi_M\rangle$. To this end, we define the shift operator U on the spin-chain. Given that X_ℓ is an operator which acts only on the ℓ'th site of the chain, it can be shown that.

$$U = i^{-L}\,\mathrm{tr}[T_a(i/2)] = P_{1,2}\cdots P_{L-1,L} \quad , \quad U^\dagger X_\ell\, U = X_{\ell+1} \tag{4.2.31}$$

The uniform shift invariance condition can then be expressed using U in the following simple way.

$$U\,|\Psi_M\rangle = |\Psi_M\rangle \tag{4.2.32}$$

As the notation suggests, the shift operator U is a unitary operator on the Hilbert space $\mathcal{H}$, which can easily be seen, using the following properties of the permutation operators.

$$P^\dagger_{\ell,m} = P_{\ell,m} \quad , \quad P^2_{\ell,m} = \mathbb{1}_{\ell,m} \quad \Rightarrow \quad U\,U^\dagger = \mathbb{1} \quad \Rightarrow \quad U = e^{i\,P} \tag{4.2.33}$$

Where the operator P is Hermitian, and can be interpreted as the momentum operator of the spin-chain. We can now re-express (4.2.32) in terms of P, by requiring that the eigenvectors $|\Psi_M\rangle$ be zero momentum states. Using (4.2.31) and the eigenvalues of $F(u)$, we explicitly write the zero momentum condition in the following way.

$$P\,|\Psi_M\rangle = \sum_{j=1}^{M} p(u_j)\,|\Psi_M\rangle \quad , \quad p(u) = \frac{1}{i}\log\left(\frac{u+i/2}{u-i/2}\right) \quad , \quad \sum_{j=1}^{M} p(u_j) = 0 \tag{4.2.34}$$

This concludes our discussion of how to obtain eigenvalues and eigenvectors of the Heisenberg spin-chain. By the correspondece layed out earlier, we now also know how to obtain anomalous dimensions, and their corresponding operators, at 1-loop level. We now move on to discuss how to find expressions for these 1-loop definite operators in the presence of a defect, by introducing one very particularly chosen state on the spin-chain.

4.3 Defects represented as spin-chain states

In what follows, we will explain more precisely what exactly is ment by introducing the defect as a spin-chain state in the $SU(2)$ subsector. We will subsequently show how to use this formalism to find a closed form expression for the tree-level contribution to single-trace Bethe state operators. We begin by defining the so-called *matrix product state* (*MPS* for short).

$$SO(3): \quad \langle\mathrm{MPS}| = \mathrm{tr}\left[\langle\uparrow|\,t_3^k + \langle\downarrow|\,t_1^k\right]^{\otimes L} \tag{4.3.1}$$

$$SO(3)\times SO(3): \quad \langle\mathrm{MPS}| = \mathrm{tr}\left[\langle\uparrow|\,T_3^{k_1,k_2} + \langle\downarrow|\,T_1^{k_1,k_2}\right]^{\otimes L} \tag{4.3.2}$$

$$T_i^{k_1,k_2} = t_i^{k_1}\otimes\mathbb{1}_{k_2} + i\,\mathbb{1}_{k_1}\otimes t_i^{k_2} \tag{4.3.3}$$

$$SO(5): \quad \langle\mathrm{MPS}| = \mathrm{tr}\left[\langle\uparrow|\,G_{56}^{d_n} + \langle\downarrow|\,G_{16}^{d_n}\right]^{\otimes L} \tag{4.3.4}$$

The three different matrix product states listed above, corresponds to different dCFT setups, distinguished by the geometries of the probe-branes in the dual string theory setups.

$$SO(3) \leftrightarrow \text{D5 probe-brane setup with brane geometry: } AdS_4 \times S^2$$
$$SO(3)\times SO(3) \leftrightarrow \text{D7 probe-brane setup with brane geometry: } AdS_4 \times S^2 \times S^2$$
$$SO(5) \leftrightarrow \text{D7 probe-brane setup with brane geometry: } AdS_4 \times S^4$$

The traces in the MPS are with respect to the $k_1k_2 \times k_1k_2$ generator matrices. The MPS is constructed in this very particular way, such that it can be used to express the tree-level contribution to any Bethe state operator as an inner product.

$$\mathcal{O}_{\boldsymbol{L}} = \Psi_m^{i_1\ldots i_L}\,\mathrm{tr}[\boldsymbol{V}_{i_1}\cdots\boldsymbol{V}_{i_L}] \quad \leftrightarrow \quad |\Psi_m\rangle = \Psi_m^{i_1\ldots i_L}\,|\sigma_{i_1}\ldots\sigma_{i_L}\rangle \tag{4.3.5}$$

$$\Rightarrow \quad \langle \text{MPS}|\Psi_m\rangle = \Psi_m^{i_1\ldots i_L}\,\mathrm{tr}[\mathcal{V}_{i_1}\cdots\mathcal{V}_{i_L}] = \langle\mathcal{O}_{\boldsymbol{L}}\rangle_{\text{tree}} \tag{4.3.6}$$

To properly understand the motivation for re-expressing these tree-level contributions as overlaps of spin-chain states, we first need to discuss a key features of the MPS. It turns out that in the cases with $SO(3)$ and $SO(5)$ symmetry, the MPS is annihilated by one of the conserved charges generated by the $F(\lambda)$ operator discussed in the previous section. We will denote this conserved charge by Q_3. Its definition is given as follow.

$$Q_3 = \sum_{\ell=1}^{L} Q_{\ell-1,\ell,\ell+1} \quad , \quad Q_{\ell-1,\ell,\ell+1} = [H_{\ell-1,\ell}, H_{\ell,\ell+1}] \tag{4.3.7}$$

$$H_{\ell m} = 2 - 2\,P_{\ell m} + K_{\ell m} \quad , \quad (P_{\ell m})_{ij}^{ks} = \delta_i^s\delta_j^k \quad , \quad (K_{\ell m})_{ij}^{ks} = \delta^{ks}\delta_{ij} \tag{4.3.8}$$

We now proceed to prove the claim that the MPS is annihilated by the conserved charge Q_3 in the $SO(3)$ and $SO(5)$ symmetric setups, but not the $SO(3)\times SO(3)$ symmetric setup.

4.3.1 The action of Q_3 on the MPS

In the various dCFT setups listed above, the action of Q_3 on the MPS, can be obtained by summing the actions of all $Q_{\ell-1,\ell,\ell+1}$ on the MPS, as can be seen from (4.3.7). We see also from (4.3.7) and (4.3.8), that $Q_{\ell-1,\ell,\ell+1}$ is given by the following commutator.

$$[H_{\ell-1,\ell}, H_{\ell,\ell+1}] = [2\,P_{\ell-1,\ell}, 2\,P_{\ell,\ell+1}] + [K_{\ell-1,\ell}, K_{\ell,\ell+1}] - [2\,P_{\ell-1,\ell}, K_{\ell,\ell+1}] - [K_{\ell-1,\ell}, 2\,P_{\ell,\ell+1}] \tag{4.3.9}$$

We can now use the definitions of the trace and permutation operators P and K, given in (4.3.8) to evaluate the action of each piece of $Q_{\ell-1,\ell,\ell+1}$ on the MPS.

$$\begin{aligned}([2\,P_{\ell-1,\ell}, 2\,P_{\ell,\ell+1}]\cdot\text{MPS})_{ijk} &= 4\,P_{ij}^{ru}P_{uk}^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 4\,P_{jk}^{ut}P_{iu}^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t \\ = 4\,\delta_j^r\,\delta_i^u\,\delta_k^s\,\delta_u^t\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 4\,\delta_k^u\,\delta_j^t\,\delta_u^r\,\delta_i^s\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t &= 4\,\mathcal{V}_j\mathcal{V}_k\mathcal{V}_i - 4\,\mathcal{V}_k\mathcal{V}_i\mathcal{V}_j\end{aligned} \tag{4.3.10}$$

$$\begin{aligned}([K_{\ell-1,\ell}, K_{\ell,\ell+1}]\cdot\text{MPS})_{ijk} &= K_{ij}^{ru}K_{uk}^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - K_{jk}^{ut}K_{iu}^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t \\ = \delta_{ij}\,\delta^{ru}\,\delta_{uk}\,\delta^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - \delta_{jk}\,\delta^{ut}\,\delta_{iu}\,\delta^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t &= \delta_{ij}\,\mathcal{V}_k\mathcal{V}^t\mathcal{V}_t - \delta_{jk}\,\mathcal{V}^s\mathcal{V}_s\mathcal{V}_i\end{aligned} \tag{4.3.11}$$

$$\begin{aligned}([2\,P_{\ell-1,\ell}, K_{\ell,\ell+1}]\cdot\text{MPS})_{ijk} &= 2\,P_{ij}^{ru}K_{uk}^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 2\,K_{jk}^{ut}P_{iu}^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t \\ = 2\,\delta_j^r\,\delta_i^u\,\delta_{uk}\,\delta^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 2\,\delta_{jk}\,\delta^{ut}\,\delta_u^r\,\delta_i^s\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t &= 2\,\delta_{ik}\mathcal{V}_j\mathcal{V}^t\mathcal{V}_t - 2\,\delta_{jk}\,\mathcal{V}^t\mathcal{V}_i\mathcal{V}_t\end{aligned} \tag{4.3.12}$$

$$\begin{aligned}([K_{\ell-1,\ell}, 2\,P_{\ell,\ell+1}]\cdot\text{MPS})_{ijk} &= 2\,K_{ij}^{ru}P_{uk}^{st}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 2\,P_{jk}^{ut}K_{iu}^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t \\ = 2\,\delta_{ij}\,\delta^{ru}\,\delta_k^s\,\delta_u^t\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t - 2\,\delta_k^u\,\delta_j^t\,\delta_{iu}\,\delta^{rs}\,\mathcal{V}_r\mathcal{V}_s\mathcal{V}_t &= 2\,\delta_{ij}\mathcal{V}^t\mathcal{V}_k\mathcal{V}_t - 2\,\delta_{ki}\,\mathcal{V}^s\mathcal{V}_s\mathcal{V}_j\end{aligned} \tag{4.3.13}$$

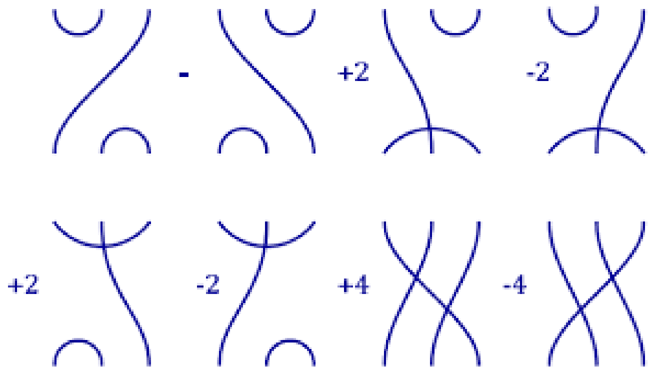

Figure 4: Graphical representation of the action Q_{ijk} on $|\text{MPS}\rangle$, For a $SO(3)\times SO(3)$ symmetric $|\text{MPS}\rangle$. The three indices i, j, k are all adjacent in $\text{MPS}^{i_1\ldots i_\ell \ldots i_L}$. As can be seen from the picture, the different contributions do not in general cancel when we sum over ℓ. This figure was taken from [17].

In the above expressions, $i\equiv i_{\ell-1}$, $j\equiv i_\ell$, $k\equiv i_{\ell+1}$ are the wave-function indices of three adjacent sites on the spin-chain. We have also suppressed the site labels on the permutation and trace operators in some cases to avoid notational clutter. Altogether, we find that the action of $Q_{\ell-1,\ell,\ell+1}$ on the MPS is given by the following expression.

$$\begin{aligned}(Q_{\ell-1,\ell,\ell+1}\cdot \text{MPS})_{ijk} &= 4\,\mathcal{V}_j\mathcal{V}_k\mathcal{V}_i - 4\,\mathcal{V}_k\mathcal{V}_i\mathcal{V}_j + \delta_{ij}\,\mathcal{V}_k\mathcal{V}^t\mathcal{V}_t - \delta_{jk}\,\mathcal{V}^s\mathcal{V}_s\mathcal{V}_i \\ &+ 2\,\delta_{jk}\,\mathcal{V}^t\mathcal{V}_i\mathcal{V}_t - 2\,\delta_{ik}\mathcal{V}_j\mathcal{V}^t\mathcal{V}_t + 2\,\delta_{ki}\,\mathcal{V}^s\mathcal{V}_s\mathcal{V}_j - 2\,\delta_{ij}\mathcal{V}^t\mathcal{V}_k\mathcal{V}_t \end{aligned} \tag{4.3.14}$$

The action of $Q_{\ell-1,\ell,\ell+1}$ on the MPS can also be represented graphically, which can be seen in figure 4.

4.3.1.1 The SO(3) symmetric setup for this particular setup, the matrices $\mathcal{V}_i$ are just $SO(3)$ generators, and thus satisfy the commutation relations: $[\mathcal{V}_i,\mathcal{V}_j]=i\,\varepsilon_{ijk}\mathcal{V}^k$. We can use these commutation relations, and the fact that $\mathcal{V}_s\mathcal{V}^s=(k^2-1)/4$, to further simplify the action of $Q_{\ell-1,\ell,\ell+1}$ on the MPS.

$$\begin{aligned}(Q_{\ell-1,\ell,\ell+1}\cdot \text{MPS})_{ijk} &= \frac{1}{4}\left(k^2-1\right)\,\mathcal{V}_i\,\delta_{jk} + 2\,\mathcal{V}_i\,\delta_{jk} + 4\,i\,\mathcal{V}_i\,\varepsilon_{jks}\,\mathcal{V}^s \\ &- \frac{1}{4}\left(k^2-1\right)\,\delta_{ij}\,\mathcal{V}_k - 2\,\delta_{ij}\,\mathcal{V}_k - 4\,i\,\varepsilon_{ijs}\,\mathcal{V}^s\mathcal{V}_k \end{aligned} \tag{4.3.15}$$

Upon summing over all ℓ and taking the trace over the $\mathcal{V}_{i_\ell}$ matrices, we find that the above expression vanishes. For example, if we focus on the terms $2\,\mathcal{V}_i\,\delta_{jk}$ and $-2\,\delta_{ij}\,\mathcal{V}_k$ in the above expression, and define $p\equiv i_{\ell+2}$, we see that.

$$\text{tr}[\cdots(2\,\mathcal{V}_i\,\delta_{jk})\,\mathcal{V}_p\cdots] + \text{tr}[\cdots\mathcal{V}_i\,(-2\,\delta_{jk}\,\mathcal{V}_p)\cdots] = 0 \tag{4.3.16}$$

The remaining four terms can analogously be paired up, and one finds that they will cancel each other in the same manner as in the example above. This cancelation can also be expressed graphically, as seen in figure 5.

4.3.1.2 The SO(5) symmetric setup in this dCFT setup, the matrices $\mathcal{V}_i$ are now given by the subset $G^{d_n}_{i6}$, with $i=1,2,3,4,5$, of the d_n dimensional representation of $SO(6)$ generators. The commutation relation between $\mathcal{V}_i$ matrices in this setup therefore takes the following form.

$$[\mathcal{V}_i,\mathcal{V}_j] = i\,G^{d_n}_{ij}\equiv i\,\mathcal{V}_{ij} \quad , \quad \mathcal{V}_s\mathcal{V}^s = C_6\left(\frac{n}{2},\frac{n}{2},\frac{n}{2}\right) - C_5\left(\frac{n}{2},0\right) = \frac{n(n+1)}{4} \tag{4.3.17}$$

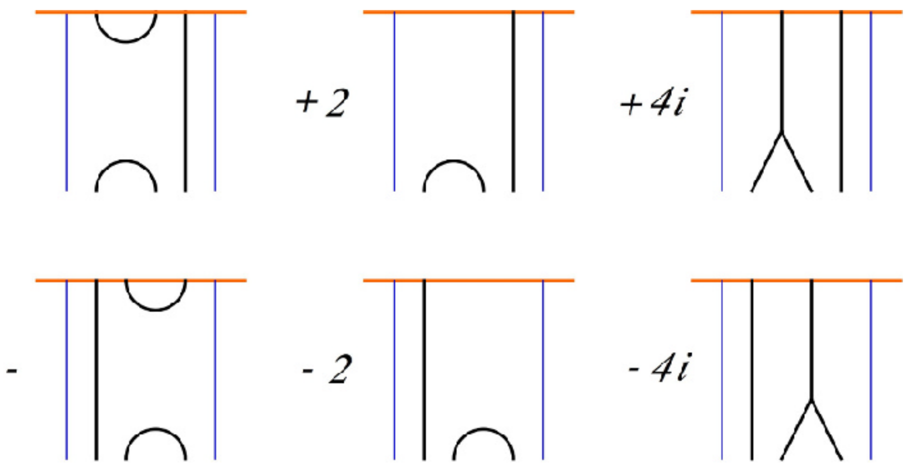

Figure 5: Graphical representation of the action Q_{ijk} on $|\text{MPS}\rangle$, For a $SO(3)$ symmetric $|\text{MPS}\rangle$. The three indices i, j, k are all adjacent in $\text{MPS}^{i_1 \ldots i_\ell \ldots i_L}$. As can be seen from the picture, the different contributions can in this case be unravelled, an are seen to cancel when we sum over ℓ. This figure was taken from [16].

Where the matrices $G_{ij}^{d_n}$ represents the last $5 \cdot 4/2 = 10$ generators of $SO(6)$. For more information about the $SO(5)$ and $SO(6)$ Casimir operators, refer to appendix D. Using the above commutation relations together with the Casimir operator, we find that.

$$(Q_{\ell-1,\ell,\ell+1} \cdot \text{MPS})_{ijk} = \frac{1}{4} n(n+4)\,(\delta_{ij}\,\mathcal{V}_k - \mathcal{V}_i\,\delta_{jk}) + 4\,i\,(\mathcal{V}_{ij}\mathcal{V}_k - \mathcal{V}_i\mathcal{V}_{jk}) \tag{4.3.18}$$

Upon summing over all ℓ and taking the trace over the generator matrices, we again find that the MPS is annihilated by Q_3. The reasoning is exactly the same as in the $SO(3)$ symmetric case.

4.3.1.3 The $\mathbf{SO(3) \times SO(3)}$ symmetric setup for this case, the $\mathcal{V}_i$ matrices do not constitute a set of Lie algebra generators, and so do not satisfy any commutation relation which would help reduce (4.3.14). Thus, one would not expect the MPS to be annihilated by Q_3 in this particular dCFT setup, and this is indeed the case on spin-chains with $L \geq 12$ [22].

4.3.2 The tree-level constribution to Bethe operator one-point functions

In the dCFT setups in which the MPS is annihilated by Q_3, we can now make use of the spin-chain formalism in order to obtain the tree-level contributions to Bethe state operators $\mathcal{O}_L$. To do this, we need to first introduce the parity operator $\mathcal{P}$ of the spin-chain. In what follows, we label the sites of the spin-chain by $\ell = -\ell_L, -\ell_L + 1, \ldots, \ell_L - 1, \ell_L$, where $\ell_L = \lfloor \frac{L}{2} \rfloor$ and $\ell = 0$ is excluded for L even. We define the parity operator $\mathcal{P}$ by its action on the basis states as follow.

$$\mathcal{P}\,|\sigma_{-\ell_L}\sigma_{-\ell_L+1}\ldots\sigma_{\ell_L-1}\sigma_{\ell_L}\rangle = |\sigma_{\ell_L}\sigma_{\ell_L-1}\ldots\sigma_{-\ell_L+1}\sigma_{-\ell_L}\rangle \tag{4.3.19}$$

Using the above definition, we can now easily show that the conserved charge Q_3 is odd under parity.

$$\mathcal{P}\,Q_{\ell-1,\ell,\ell+1}\,\mathcal{P}^{-1} = Q_{-\ell+1,-\ell,-\ell-1} = [\mathcal{H}_{-\ell+1,-\ell}, \mathcal{H}_{-\ell,-\ell-1}] = -[\mathcal{H}_{-\ell-1,-\ell}, \mathcal{H}_{-\ell,-\ell+1}] \tag{4.3.20}$$

$$\Rightarrow \quad \mathcal{P}\,Q_3\,\mathcal{P}^{-1} = \sum_\ell \mathcal{P}\,Q_{\ell-1,\ell,\ell+1}\,\mathcal{P}^{-1} = -\sum_\ell Q_{-\ell-1,-\ell,-\ell+1} = -Q_3 \tag{4.3.21}$$

Until this point we have been referring to the eigenstates of the spin-chain as $|\Psi_M\rangle$. In what follows, it will however be convenient to be more precise and label the eigenstates by the rapidity parameters: $|\{u_j\}\rangle$. Using (4.2.34) and the fact that the momentum operator is odd under parity, we find that.

$$\mathcal{P}\,P\,\mathcal{P}^{-1} = -P \quad \Rightarrow \quad \mathcal{P}\,|\{u_j\}\rangle \sim |\{-u_j\}\rangle \tag{4.3.22}$$

From the above, we see that generally, an eigenstate of the spin-chain gets mapped to a different eigenstates with opposite momentum. For the special case of so-called *unpaired states*, where the rapidity parameters come in opposite sign pairs, the parity operator only multiplies by a factor. In other words, the states of the form $|\{u_j, -u_j\}\rangle$ are eigenstates of the parity operator. This implies.

$$Q_3\,|\{u_j,-u_j\}\rangle = q_3\,|\{u_j,-u_j\}\rangle \quad \Rightarrow \quad Q_3\,|\{u_j,-u_j\}\rangle = -q_3\,|\{u_j,-u_j\}\rangle \tag{4.3.23}$$

Thus we conclude that $q_3 = 0$ for unpaired states. Moreover, it can be shown that all spin-chain eigenstates which are not unpaired, will have $q_3 \neq 0$ [16]. This means that all Bethe state operators $\mathcal{O}_{\boldsymbol{L}}(x)$ which do not correspond to unpaired states will vanish at tree-level, since.

$$0 = \langle \text{MPS}|\,Q_3\,|\{u_j\}\rangle = q_3\,\langle \text{MPS}|\{u_j\}\rangle \quad \Rightarrow \quad \langle \text{MPS}|\{u_j\}\rangle = 0 \tag{4.3.24}$$

Where we have used that the MPS is annihilated by Q_3. Thus, we can focus on overlaps between the MPS and unpaired Bethe states. Furthermore, it can be shown that the overlap between the MPS and a given Bethe state $|\Psi_M\rangle$, vanishes unless both L and M are even [16].
Using the spin-chain picture, and all the insight which can be extracted from this approach, it is possible to find tree-level expressions for one-point functions of the Bethe state type. For example, in the case of the $SO(3)$ symmetric defect setup, the tree-level contribution to $\langle \mathcal{O}_{\boldsymbol{L}}(x)\rangle$ can be written as follow [16, 18].

$$SO(3): \quad \langle \mathcal{O}_{\boldsymbol{L}}(x)\rangle_{\text{tree}} = \frac{C_k(\{u_j\})}{x_3^L} \tag{4.3.25}$$

Where $C_k(\{u_j\})$ is given by the following expression.

$$C_k(\{u_j\}) = 2^{L-1} C_2(\{u_j\}) \sum_{j=\frac{1-k}{2}}^{\frac{k-1}{2}} j^L \prod_{i=1}^{\frac{M}{2}} \frac{u_i^2\left(u_i^2 + \frac{k^2}{4}\right)}{\left[u_i^2 + \left(j-\frac{1}{2}\right)^2\right]\left[u_i^2 + \left(j+\frac{1}{2}\right)^2\right]} \tag{4.3.26}$$

And $C_2(\{u_j\})$ is given by the expression.

$$C_2(\{u_j\}) = 2\left[\left(\frac{2\pi^2}{\lambda}\right)^L \frac{1}{L}\prod_{j=1}^{M} \frac{u_j^2 + \frac{1}{4}}{u_j^2}\frac{\det G^+}{\det G^-}\right]^{1/2} \tag{4.3.27}$$

And lastly, the $\frac{M}{2} \times \frac{M}{2}$ matrices $G^\pm$ are given by the expression.

$$G_{jk}^{\pm} = \left(\frac{L}{u_j^2 + \frac{1}{4}} - \sum_{n=1}^{\frac{M}{2}} K_{jn}^{+}\right)\delta_{jk} + K_{jk}^{\pm} \quad , \quad K_{jk}^{\pm} = \frac{2}{1+(u_j-u_k)^2} \pm \frac{2}{1+(u_j+u_k)^2} \tag{4.3.28}$$

In the cases of the $SO(3) \times SO(3)$ and $SO(5)$ symmetric defect setups, general expressions for the tree-level contribution to Bethe state one-point functions are not yet known, at the time of writing this thesis. However, for special values of parameters, such as L and M, results are known both for the $SO(3) \times SO(3)$ symmetric setup [22] and the $SO(5)$ symmetric setup [17].
This concludes our discussion of how to include defects into the spin-chain picture using the MPS, and how these states in some cases can be used to find explicit expressions for tree-level contributions to Bethe state one-point functions. In the next subsection, we will discuss how we can use these results to find tree-level contributions to the long / short two-point functions $\langle \mathcal{O}_L \mathcal{O}_{W_1 W_2}\rangle$ described at the beginning of this entire section.

4.4 Computing long / short two-point functions

In this subsection, we finally begin to tackle the central problem layed out at the start of this section; namely how to compute the leading order connected contribution to two-point functions of the form.

$$\langle \mathcal{O}_{\boldsymbol{L}}(x)\mathcal{O}_{\boldsymbol{W}_1\boldsymbol{W}_2}(y)\rangle = \sum_{\ell=1}^{L} \Psi_M^{i_1\cdots i_L}\,\mathrm{tr}\left[\mathcal{V}_{i_1}\cdots \overline{V_{i_\ell}\cdots \mathcal{V}_{i_L}\right]\mathrm{tr}\left[W_1}\mathcal{W}_2\right] + (W_1 \leftrightarrow W_2) \tag{4.4.1}$$

In the above expression, $V_{i_\ell} \in \{X, Z\}$ and $W_1, W_2 \in \{X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}\}$. We will postpone the discussion of the cases $W_1 \in \{Y, \bar{Y}\}$ or $W_2 \in \{Y, \bar{Y}\}$ untill the end of this subsection, as these require a slightly different treatment, for reason which will hopefully become clear later on. It should be noted at this point, that our method for computing the two-point functions 4.4.1 has been greatly inspired by the work presented in [19], in which two-point functions of the form 4.4.1 are evaluated in the $SO(3)$ symmetric D5-D3 probe brane setup.

4.4.1 $\mathrm{SO}(3)\times\mathrm{SO}(3)$ symmetric vevs

As an intermediate step in the process of computing these two-point funtions, we will be interested in all operators of the form.

$$T_{V_{i_\ell}W_1W_2} \equiv \overline{V_{i_\ell}\,\mathrm{tr}\left[W_1}\mathcal{W}_2\right] = \hat{Y}_{\ell_1}^{m_1}\otimes \hat{Y}_{\ell_2}^{m_2}\,\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes \hat{Y}_{\ell_2'}^{m_2'}\,\mathcal{W}_2\right]\langle [V_{i_\ell}]_{\boldsymbol{\ell}}\,[W_1]_{\boldsymbol{\ell}'}\rangle \tag{4.4.2}$$

Where we have used the $\boldsymbol{\ell} = (\ell_1, m_1; \ell_2, m_2)$ introduced in earlier sections. Looking at the form of (4.4.1), the motivation for why one would be interested in the T operators should be fairly obvious. A priori, there seem to be a large number of different T operators, but fortunately it turns out that the forms of all T operators can be infered from just a few specific cases, as we shall now see.

4.4.1.1 Computing the T operators: ideas and examples Before we start the process of computing the T operators, let us first make some simplifying redefinitions of some of the objects in (4.4.2). These redefinitions looks as follow.

$$\langle [V_{i_\ell}]_{\boldsymbol{\ell}}\,[W_1]_{\boldsymbol{\ell}'}\rangle \to x_3 y_3\,\langle [V_{i_\ell}]_{\boldsymbol{\ell}}\,[W_1]_{\boldsymbol{\ell}'}\rangle \quad , \quad \mathcal{V}_{i_\ell} \to x_3\,\mathcal{V}_{i_\ell} \quad , \quad \mathcal{W}_{1,2} \to y_3\,\mathcal{W}_{1,2} \tag{4.4.3}$$

These rescalings exactly cancel the spacetime dependence of the objects being rescaled, and thus the T operators all become spacetime independent after these rescalings. If need be, we can always recover the spacetime dependence of any T operator by undoing 4.4.3.

The operator $\mathbf{T_{ZZZ}}$: Let us, somewhat arbitrarily, take this specific T operator to be the starting point in our process of computing all T operators. According to (4.4.2), T_{ZZZ} is given as follow.

$$T_{ZZZ} = \hat{Y}_{\ell_1}^{m_1}\otimes \hat{Y}_{\ell_2}^{m_2}\,\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes \hat{Y}_{\ell_2'}^{m_2'}\,\mathcal{Z}\right]\left\langle Z_{\ell_1,m_1;\ell_2,m_2}\,Z_{\ell_1',m_1';\ell_2',m_2'}\right\rangle \tag{4.4.4}$$

In order to compute the trace, we take advantage of the orthogonality of the fuzzy spherical harmonics, together with relation between t_3^k, $\mathbb{1}_k$ and the fuzzy spherical harmonics $\hat{Y}_\ell^m$. We restate these relations below for convenience.

$$\mathrm{tr}\left[\hat{Y}_\ell^m\,\hat{Y}_{\ell'}^{m'}\right] = (-1)^{m'}\delta_{\ell,\ell'}\,\delta_{m+m',0} \tag{4.4.5}$$

$$\mathbb{1} = d_0^k\,\hat{Y}_0^0 \quad , \quad t_3 = \sqrt{2}\,d_1^k\,\hat{Y}_1^0 \quad ; \quad d_0^k = (-1)^{k+1}\sqrt{k} \quad , \quad d_1^k = \frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}} \tag{4.4.6}$$

Using the above information, we can evaluate the trace appearing in T_{ZZZ}, which yields the result.

$$\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes \hat{Y}_{\ell_2'}^{m_2'}\,\mathcal{Z}\right] = \mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\,t_3^{k_1}\otimes \hat{Y}_{\ell_2'}^{m_2'}\right] + i\,\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes \hat{Y}_{\ell_2'}^{m_2'}\,t_3^{k_2}\right]$$

$$= \sqrt{2}\, d_1^{k_1}\, \delta_{\ell_1',1}\, \delta_{m_1',0}\, d_0^{k_2}\, \delta_{\ell_2',0}\, \delta_{m_2',0} + i\, d_0^{k_1}\, \delta_{\ell_1',0}\, \delta_{m_1',0}\, \sqrt{2}\, d_1^{k_2}\, \delta_{\ell_2',1}\, \delta_{m_2',0} \tag{4.4.7}$$

Now, let us turn our attention to propagator appearing in T_{ZZZ}. The propagator is between Z-fields, which makes it particularly simple. We restate the form of the ZZ-propagator below for convenience.

$$\left\langle Z_{\ell_1,m_1;\ell_2,m_2} Z_{\ell_1',m_1';\ell_2',m_2'} \right\rangle = (-1)^{m_1'+m_2'} \delta_{\ell_1\ell_1'} \delta_{\ell_2\ell_2'} \delta_{m_1+m_1',0} \delta_{m_2+m_2',0} K^{ZZ} \tag{4.4.8}$$

$$K^{ZZ} = \left[K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)} - m_1^2\, K_{\text{sym}}^{\phi,(1)} + m_2^2\, K_{\text{sym}}^{\phi,(2)} + 2i\, m_1 m_2\, K_{\text{opp}}^{\phi} \right] \tag{4.4.9}$$

The exact expressions for the different K's can be found in appendix A, and also in [11]. Now that we have computed the trace and re-familiarized ourselves with the form of the ZZ-propagator, we are ready to simplify the form of the operator T_{ZZZ}. All terms in K^{ZZ} proportional to m_1 or m_2 drop out due to the δ-symbols, and we are left with the following expression.

$$\begin{aligned} T_{ZZZ} &= \sqrt{2}\, d_1^{k_1}\, d_0^{k_2}\, \hat{Y}_1^0 \otimes \hat{Y}_0^0 \left[K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)} \right]_{\ell_1=1,\ell_2=0} \\ &\quad + i\,\sqrt{2}\, d_0^{k_1}\, d_1^{k_2}\, \hat{Y}_0^0 \otimes \hat{Y}_1^0 \left[K_{\text{sing}}^{\phi,(1)} - K_{\text{sing}}^{\phi,(2)} \right]_{\ell_1=0,\ell_2=1} \\ &= \frac{1}{3} \left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6} \right) \bar{\mathcal{Z}} \end{aligned} \tag{4.4.10}$$

Where we have used the following information.

$$\ell_1 = 1\,,\ \ell_2 = 0: \quad K_{\text{sing}}^{\phi,(1)} = \frac{2}{3}\, K^{m^2=0} + \frac{1}{3}\, K^{m^2=6} \quad , \quad K_{\text{sing}}^{\phi,(2)} = K^{m^2=2} \tag{4.4.11}$$

$$\ell_1 = 0\,,\ \ell_2 = 1: \quad K_{\text{sing}}^{\phi,(1)} = K^{m^2=2} \quad , \quad K_{\text{sing}}^{\phi,(2)} = \frac{2}{3}\, K^{m^2=0} + \frac{1}{3}\, K^{m^2=6} \tag{4.4.12}$$

In order to obtain the last equality. Note that $T_{ZZZ} \sim \bar{\mathcal{Z}}$. This turns out to be an important observation.

The operator $\mathbf{T_{XZZ}}$: this T operator is very similar in form to the T_{ZZZ} operator. The trace which appears in the operator is the same as for T_{ZZZ}, but the propagator is now of XZ-type, as seen below.

$$T_{XZZ} = \hat{Y}_{\ell_1}^{m_1} \otimes \hat{Y}_{\ell_2}^{m_2}\, \text{tr}\left[\hat{Y}_{\ell_1'}^{m_1'} \otimes \hat{Y}_{\ell_2'}^{m_2'}\, \mathcal{Z} \right] \left\langle X_{\ell_1,m_1;\ell_2,m_2}\, Z_{\ell_1',m_1';\ell_2',m_2'} \right\rangle \tag{4.4.13}$$

Seeing as we just evaluated the trace appearing in T_{XZZ}, all we need is the form of the XZ-propagator, before we can begin to simplify. We restate the form of the XZ-propagator below for convenience.

$$\begin{aligned} &\left\langle X_{\ell_1,m_1;\ell_2,m_2} Z_{\ell_1',m_1';\ell_2',m_2'} \right\rangle = \\ &(-1)^{m_1'+m_2'} \delta_{\ell_1\ell_1'} \delta_{\ell_2\ell_2'} \delta_{m_2+m_2',0} \left(i[t_2^{k_1}]_{m_1,-m_1'}\, K_{\text{anti}}^{\phi,(1)} - [t_1^{k_1} t_3^{k_1}]_{m_1,-m_1'}\, K_{\text{sym}}^{\phi,(1)} \right) \\ &-(-1)^{m_1'+m_2'} \delta_{\ell_1\ell_1'} \delta_{\ell_2\ell_2'} \delta_{m_1+m_1',0} \left(i[t_2^{k_2}]_{m_2,-m_2'}\, K_{\text{anti}}^{\phi,(2)} - [t_1^{k_2} t_3^{k_2}]_{m_2,-m_2'}\, K_{\text{sym}}^{\phi,(2)} \right) \\ &+ i\,(-1)^{m_1'+m_2'} \delta_{\ell_1\ell_1'} \delta_{\ell_2\ell_2'} \left([t_1^{k_2}]_{m_2,-m_2'} [t_3^{k_1}]_{m_1,-m_1'} + [t_1^{k_1}]_{m_1,-m_1'} [t_3^{k_2}]_{m_2,-m_2'} \right) K_{\text{opp}}^{\phi} \end{aligned} \tag{4.4.14}$$

Where in the above, the matrix representations of the different $\mathfrak{su}(2)$ generators are given as follow.

$$[t_1]_{m,m'} = \frac{1}{2} \left(\sqrt{(\ell+m)(\ell-m')}\, \delta_{m',m-1} + \sqrt{(\ell+m')(\ell-m)}\, \delta_{m',m+1} \right) \tag{4.4.15}$$

$$[t_2]_{m,m'} = \frac{1}{2\,i} \left(\sqrt{(\ell+m)(\ell-m')}\, \delta_{m',m-1} - \sqrt{(\ell+m')(\ell-m)}\, \delta_{m',m+1} \right) \tag{4.4.16}$$

$$[t_3]_{m,m'} = m'\, \delta_{m,m'} \tag{4.4.17}$$

$$[t_1 t_3]_{m,m'} = [t_1]_{m,m''} [t_3]_{m'',m'} = m'\, [t_1]_{m,m'} \tag{4.4.18}$$

Now that we know the form of the XZ-propagator, we are ready to simplify the T_{XZZ} operator. Using the trace result (4.4.7) and propagator form (4.4.71), we can make some immediate simplifications.

1. Since the $k=1$ representation of $\mathfrak{su}(2)$ is trivial, meaning that $t_i^{k=1}=0$, all terms containing either $\delta_{\ell_1,0}\, t_i^{k_1}$ or $\delta_{\ell_2,0}\, t_i^{k_2}$ will vanish.
2. It turns out that the propagators $K_{\text{sym}}^{\phi,(1)}$ and $K_{\text{sym}}^{\phi,(2)}$, vanish for the cases $\ell_1=1$, $\ell_2=0$ and $\ell_1=0$, $\ell_2=1$ respectively. The explicit forms of $K_{\text{sym}}^{\phi,(1)}$ and $K_{\text{sym}}^{\phi,(2)}$ can be found in appendix A and in [11].

Using the above information, we see that the 2×6 terms constituting T_{XZZ} collapses down to only 2.

$$
\begin{aligned}
T_{XZZ} &= -\frac{1}{2}\sqrt{2}\, d_1^{k_1}\, d_0^{k_2}\left[\hat{Y}_1^{-1}\otimes\hat{Y}_0^0-\hat{Y}_1^1\otimes\hat{Y}_0^0\right] K_{\text{anti}}^{\phi,(1)}\Big|_{\ell_1=1,\ell_2=0} \\
&\quad +i\,\frac{1}{2}\sqrt{2}\, d_0^{k_1}\, d_1^{k_2}\left[\hat{Y}_0^0\otimes\hat{Y}_1^{-1}-\hat{Y}_0^0\otimes\hat{Y}_1^1\right] K_{\text{anti}}^{\phi,(2)}\Big|_{\ell_1=0,\ell_2=1} \\
&= -\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}
\end{aligned}
\tag{4.4.19}
$$

Where we have used the following information.

$$
\ell_1=1\ ,\ \ell_2=0: \quad K_{\text{anti}}^{\phi,(1)}=\frac{1}{3}\,K^{m^2=0}-\frac{1}{3}\,K^{m^2=6} \tag{4.4.20}
$$

$$
\ell_1=0\ ,\ \ell_2=1: \quad K_{\text{anti}}^{\phi,(2)}=\frac{1}{3}\,K^{m^2=0}-\frac{1}{3}\,K^{m^2=6} \tag{4.4.21}
$$

In order to obtain the last equality. Once again, we see that the result after simplification is proportional to a single complex classical scalar. This time we find $T_{XZZ}\sim\bar{\mathcal{X}}$.

The operator $\mathbf{T_{ZXZ}}$: we now encounter our first example that the form of one T operator can be infered from another. The unsimplified form of T_{ZXZ} is given by the following expression.

$$
T_{ZXZ}=\hat{Y}_{\ell_1}^{m_1}\otimes\hat{Y}_{\ell_2}^{m_2}\,\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes\hat{Y}_{\ell_2'}^{m_2'}\,\mathcal{Z}\right]\left\langle Z_{\ell_1,m_1;\ell_2,m_2}\,X_{\ell_1',m_1';\ell_2',m_2'}\right\rangle \tag{4.4.22}
$$

As can be seen from the above expression, T_{ZXZ} is almost exactly identical to T_{XZZ}. The only difference is that X and Z are switched in the propagator. It turns out that the ZX-propagator can be obtained from the XZ-propagator by making the substitutions $t_1\to t_3$, $t_3\to t_1$ and $t_2\to -t_2$. This can be seen from the generic expression for the ϕ_i propagators, which can be found in either appendix A or [11]. Thus, we find that the operator T_{ZXZ} can be written as.

$$
T_{ZXZ}=\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}} \tag{4.4.23}
$$

Since all the terms containing t_1 and t_3 vanish for the same reasons as in the T_{XZZ} operator case.

The operator $\mathbf{T_{ZZX}}$: this operator is again one which cannot be infered from any T operator we have alreday computed. More work is once again needed. The explicit un-simplified expression for T_{ZZX} looks as follow.

$$
T_{ZZX}=\hat{Y}_{\ell_1}^{m_1}\otimes\hat{Y}_{\ell_2}^{m_2}\,\mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'}\otimes\hat{Y}_{\ell_2'}^{m_2'}\,\mathcal{X}\right]\left\langle Z_{\ell_1,m_1;\ell_2,m_2}\,Z_{\ell_1',m_1';\ell_2',m_2'}\right\rangle \tag{4.4.24}
$$

To compute the trace, we take advantage of the orthogonality of the fuzzy spherical harmonics (4.4.5), together with the expasion of t_1 in terms of the fuzzy spherical harmonics.

$$
t_1=d_1^k\left(\hat{Y}_1^{-1}-\hat{Y}_1^1\right)\quad,\quad d_1^k=\frac{(-1)^{k+1}}{2}\sqrt{\frac{k(k^2-1)}{6}} \tag{4.4.25}
$$

Using the above information, we can now evaluate the trace appearing in T_{ZZX}. The result is this.

$$\mathrm{tr}\Big[\hat{Y}^{m_1'}_{\ell_1'} \otimes \hat{Y}^{m_2'}_{\ell_2'} \mathcal{X}\Big] = \mathrm{tr}\Big[\hat{Y}^{m_1'}_{\ell_1'} t_1^{k_1} \otimes \hat{Y}^{m_2'}_{\ell_2'}\Big] + i\, \mathrm{tr}\Big[\hat{Y}^{m_1'}_{\ell_1'} \otimes \hat{Y}^{m_2'}_{\ell_2'} t_1^{k_2}\Big]$$

$$= d_1^{k_1}\, \delta_{\ell_1',1}\, (\delta_{m_1',-1} - \delta_{m_1',1})\, d_0^{k_2}\, \delta_{\ell_2',0}\, \delta_{m_2',0} + i\, d_0^{k_1}\, \delta_{\ell_1',0}\, \delta_{m_1',0}\, d_1^{k_2}\, \delta_{\ell_2',1}\, (\delta_{m_2',-1} - \delta_{m_2',1}) \tag{4.4.26}$$

Now that we have computed the trace, we are ready to simplify the form of the operator T_{ZZX}. The terms in K^{ZZ} proportional to m_2 drops out when multiplied by the first term of the above trace, and the terms proportional to m_1 drops out when multiplied by the second term of the trace. Furthermore, we found earlier on that $K^{\phi,(1)}_{\mathrm{sym}}|_{\ell_1=1,\ell_2=0} = K^{\phi,(2)}_{\mathrm{sym}}|_{\ell_1=0,\ell_2=1} = 0$. Using this information, we obtain the following simplified expression for T_{ZZX}.

$$T_{ZZX} = d_1^{k_1}\, d_0^{k_2} \left(\hat{Y}^{-1}_1 \otimes \hat{Y}^0_0 - \hat{Y}^1_1 \otimes \hat{Y}^0_0\right) \left[K^{\phi,(1)}_{\mathrm{sing}} - K^{\phi,(2)}_{\mathrm{sing}}\right]_{\ell_1=1,\ell_2=0}$$

$$+ i\, d_0^{k_1}\, d_1^{k_2} \left(\hat{Y}^0_0 \otimes \hat{Y}^{-1}_1 - \hat{Y}^0_0 \otimes \hat{Y}^1_1\right) \left[K^{\phi,(1)}_{\mathrm{sing}} - K^{\phi,(2)}_{\mathrm{sing}}\right]_{\ell_1=0,\ell_2=1}$$

$$= \frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right)\bar{\mathcal{X}} \tag{4.4.27}$$

It turns out that the T operator above was the last one we need to compute explicitly. The rest of the T operators can be inferred from the ones we already know, as we will now proceed to show.

The operator $\mathbf{T_{XXZ}}$: this particular operator can be obtained from T_{ZXX} by use of a similarity transformation. This transformation is an fact the exact same one we made use of back in section 3. The transformation sends $t_1 \to t_3$, $t_2 \to -t_2$ and $t_3 \to t_1$, and is defined as follow.

$$V = U^{k_1} \otimes U^{k_2} \quad , \quad U^{k_s} = e^{i\pi t_3^{k_s}}\, e^{i\pi t_2^{k_s}/2} \tag{4.4.28}$$

Using the following transformation properties of the fuzzy spherical harmonics and $\mathfrak{su}(2)$ generators.

$$U\, \hat{Y}^m_\ell\, U^\dagger = U_{n,m}\, \hat{Y}^n_\ell \quad , \quad U\, (\hat{Y}^m_\ell)^\dagger\, U^\dagger = \bar{U}_{n,m}\, \hat{Y}^n_\ell \quad , \quad U_{m,m'} = \langle m|\, U\, |m'\rangle \tag{4.4.29}$$

$$U_{m,n}\, \bar{U}_{m',n'}\, [t_3]_{n,n'} = [t_1]_{m,m'} \quad , \quad U_{m,n}\, \bar{U}_{m',n'}\, [t_1]_{n,n'} = [t_3]_{m,m'} \tag{4.4.30}$$

We can now prove that T_{XXZ} can be obtained from T_{ZZX}. Transforming T_{ZZX} using V, we find that.

$$V\, T_{ZZX}\, V^\dagger = V\, \hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2}\, V^\dagger\, \mathrm{tr}\Big[V\, \hat{Y}^{m_1'}_{\ell_1'} \otimes \hat{Y}^{m_2'}_{\ell_2'} V^\dagger\, V\, \mathcal{X}\, V^\dagger\Big]\, \langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle\, K^{\phi,(1)}_{\mathrm{anti}}$$

$$= \hat{Y}^{n_1}_{\ell_1} \otimes \hat{Y}^{n_2}_{\ell_2}\, \mathrm{tr}\Big[\hat{Y}^{n_1'}_{\ell_1'} \otimes \hat{Y}^{n_2'}_{\ell_2'}\, \mathcal{Z}\Big]\, U^{k_1}_{n_1,m_1}\, U^{k_2}_{n_2,m_2}\, \bar{U}^{k_1}_{-n_1',-m_1'}\, \bar{U}^{k_2}_{-n_2',-m_2'}\, (-1)^{m_1'+m_2'}(-1)^{n_1'+n_2'}\, \langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$$

$$= \hat{Y}^{n_1}_{\ell_1} \otimes \hat{Y}^{n_2}_{\ell_2}\, \mathrm{tr}\Big[\hat{Y}^{n_1'}_{\ell_1'} \otimes \hat{Y}^{n_2'}_{\ell_2'}\, \mathcal{Z}\Big]\, \langle X_{\boldsymbol{\ell}}\, X_{\boldsymbol{\ell}'}\rangle = T_{XXZ} \tag{4.4.31}$$

We can now obtain the simplified form of T_{XXZ}, simply by transforming the result we already found for T_{ZZX}, using the V. The result is the following.

$$T_{XXZ} = \frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right)\bar{\mathcal{Z}} \tag{4.4.32}$$

The operator $\mathbf{T_{XZX}}$: this operator can by infered from the operator T_{ZXZ}, again by use of the similarity transformation V. The explicit unsimplified form of T_{XZX} looks as follow.

$$T_{XZX} = \hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2}\, \mathrm{tr}\Big[\hat{Y}^{m_1'}_{\ell_1'} \otimes \hat{Y}^{m_2'}_{\ell_2'}\, \mathcal{X}\Big]\, \Big\langle X_{\ell_1,m_1;\ell_2,m_2}\, Z_{\ell_1',m_1';\ell_2',m_2'}\Big\rangle \tag{4.4.33}$$

Transforming T_{ZXZ} using V, we find that T_{XZX} is given by the following simple expression.

$$T_{XZX} = \frac{1}{3}\left(K^{m^2=0} - K^{m^2=6}\right)\bar{\mathcal{Z}} \tag{4.4.34}$$

4.4.1.2 Computing the T operators: obtaining the rest It is now time to use the insight we have gained from computing a few of the T operators, to infer the forms of the rest. First, we look at how to extend the results we found for T_{ZZZ} and T_{ZZX}. Given the expression for the $Z\bar{Z}$-propagator.

$$\left\langle Z_{\ell_1,m_1;\ell_2,m_2}\bar{Z}_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \delta_{\ell_1,\ell_1'}\,\delta_{\ell_2,\ell_2'}\,\delta_{m_1,m_1'}\,\delta_{m_2,m_2'}\,K^{Z\bar{Z}} \tag{4.4.35}$$

$$K^{Z\bar{Z}} = \left[K^{\phi,(1)}_{\text{sing}} + K^{\phi,(2)}_{\text{sing}} - m_1^2\,K^{\phi,(1)}_{\text{sym}} - m_2^2\,K^{\phi,(2)}_{\text{sym}}\right] \tag{4.4.36}$$

It is relatively easy to obtain the following results, by extend the results found for T_{ZZZ} and T_{ZZX}.

$$T_{Z\bar{Z}Z} = \frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right)\mathcal{Z} \tag{4.4.37}$$

$$T_{Z\bar{Z}X} = \frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right)\mathcal{X} \tag{4.4.38}$$

$$T_{Z\bar{Z}\bar{Z}} = \frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right)\bar{\mathcal{Z}}K^{\phi,(1)}_{\text{anti}} \tag{4.4.39}$$

$$T_{Z\bar{Z}\bar{X}} = \frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right)\bar{\mathcal{X}} \tag{4.4.40}$$

$$T_{ZZ\bar{Z}} = \frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right)\mathcal{Z} \tag{4.4.41}$$

$$T_{ZZ\bar{X}} = \frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right)\mathcal{X} \tag{4.4.42}$$

If we look back at the computations of T_{ZZZ} and T_{ZZX}, we see that only the $K^{\phi,(1)}_{\text{sing}}$ and $K^{\phi,(2)}_{\text{sing}}$ terms in K^{ZZ} contributed. This will also be the case with $K^{Z\bar{Z}}$. The change in the relative sign between $K^{\phi,(1)}_{\text{sing}}$ and $K^{\phi,(2)}_{\text{sing}}$ when going from K^{ZZ} to $K^{Z\bar{Z}}$, produce the sign change on the $K^{m^2=2}$ terms, in the $T_{Z\bar{Z}W_2}$ operators. The same relative sign is also responsible for the fact that operators of the form $T_{Z\bar{Z}W_2} \sim \mathcal{W}_2$, and not $\bar{\mathcal{W}}_2$.
Another important thing to note, is that the complex conjagated fields $\bar{Z}$ and $\bar{X}$ are expanded in terms of complex conjagated fuzzy harmonics $(\hat{Y}^m_\ell)^\dagger$, in the following way.

$$\bar{Z} = Z^\dagger_{\ell_1,m_1;\ell_2,m_2}\,(\hat{Y}^{m_1}_{\ell_1})^\dagger \otimes (\hat{Y}^{m_2}_{\ell_2})^\dagger \quad , \quad \bar{X} = X^\dagger_{\ell_1,m_1;\ell_2,m_2}\,(\hat{Y}^{m_1}_{\ell_1})^\dagger \otimes (\hat{Y}^{m_2}_{\ell_2})^\dagger \tag{4.4.43}$$

For operators of the form $T_{V_{i_\ell}\bar{Z}W_2}$ and $T_{V_{i_\ell}\bar{X}W_2}$, we would therefore have to evaluate traces of $\bar{Z}$ and $\bar{X}$ with conjugated fuzzy harmonics. However, it can readily be verified that.

$$\text{tr}\left[(\hat{Y}^{m_1}_{\ell_1})^\dagger \otimes (\hat{Y}^{m_2}_{\ell_2})^\dagger\,\mathcal{Z}\right] = \text{tr}\left[\hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2}\,\mathcal{Z}\right] \tag{4.4.44}$$

$$\text{tr}\left[(\hat{Y}^{m_1}_{\ell_1})^\dagger \otimes (\hat{Y}^{m_2}_{\ell_2})^\dagger\,\mathcal{X}\right] = \text{tr}\left[\hat{Y}^{m_1}_{\ell_1} \otimes \hat{Y}^{m_2}_{\ell_2}\,\mathcal{X}\right] \tag{4.4.45}$$

Next, we look at how to extend the results we found for T_{XZZ} and T_{XZX}. Given the expression for the $X\bar{Z}$-propagator.

$$\begin{aligned}
&\left\langle X_{\ell_1,m_1;\ell_2,m_2}\bar{Z}_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2,m_2'}\left(i[t_2^{k_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{anti}} - [t_1^{k_1}t_3^{k_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{sym}}\right) \\
&+\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1,m_1'}\left(i[t_2^{k_2}]_{m_2,m_2'}K^{\phi,(2)}_{\text{anti}} - [t_1^{k_2}t_3^{k_2}]_{m_2,m_2'}K^{\phi,(2)}_{\text{sym}}\right) \\
&+ i\,\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\left([t_1^{k_2}]_{m_2,m_2'}[t_3^{k_1}]_{m_1,m_1'} - [t_1^{k_1}]_{m_1,m_1'}[t_3^{k_2}]_{m_2,m_2'}\right)K^{\phi}_{\text{opp}}
\end{aligned} \tag{4.4.46}$$

T_{ZZZ}	$\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Z}}$		
T_{XZZ}	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$		
T_{ZXZ}	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$	T_{ZZX}	$\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{X}}$
T_{XXZ}	$\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Z}}$	T_{XZX}	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Z}}$

Table 3: The T operators necessary for constructing the Q_{ZZ} and Q_{XZ} operators. Because these operators are all proportional to conjugated classical scalar fields, they can not be interpreted as $\mathfrak{su}(2)$ spin-chain operators.

$T_{Z\bar{Z}\bar{Z}}$	$\frac{1}{3}\left(2K^{m^2=0}+3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Z}}$		
$T_{X\bar{Z}\bar{Z}}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$		
$T_{Z\bar{X}\bar{Z}}$	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$	$T_{Z\bar{Z}\bar{X}}$	$\frac{1}{3}\left(2K^{m^2=0}+3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{X}}$
$T_{X\bar{X}\bar{Z}}$	$\frac{1}{3}\left(2K^{m^2=0}+3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Z}}$	$T_{X\bar{Z}\bar{X}}$	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Z}}$

Table 4: The T operators necessary for constructing the $Q_{\bar{Z}\bar{Z}}$ and $Q_{\bar{X}\bar{Z}}$ operators. Because these operators are all proportional to conjugated classical scalar fields, they can not be interpreted as $\mathfrak{su}(2)$ spin-chain operators.

It is relatively easy to obtain the following results, by extend the results found for T_{XZZ} and T_{XZX}.

$$T_{X\bar{Z}Z}=-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{X} \tag{4.4.47}$$

$$T_{X\bar{Z}X}=\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Z} \tag{4.4.48}$$

$$T_{X\bar{Z}\bar{Z}}=-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}} \tag{4.4.49}$$

$$T_{X\bar{Z}\bar{X}}=\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Z}} \tag{4.4.50}$$

$$T_{XZ\bar{Z}}=-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{X} \tag{4.4.51}$$

$$T_{XZ\bar{X}}=\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Z} \tag{4.4.52}$$

If we look back at the computations of T_{XZZ} and T_{XZX}, we see that only the $K^{\phi,(1)}_{\text{anti}}$ and $K^{\phi,(2)}_{\text{anti}}$ terms in the XZ-propagator contributed. This will also be the case with the $X\bar{Z}$-propagator. The sign change on the $K^{\phi,(2)}_{\text{anti}}$ term, is responisble for the fact that operators of the form $T_{X\bar{Z}W_2}$ are proportional to $\mathcal{W}_2$, while operators of the form T_{XZW_2} are proportional to $\bar{\mathcal{W}}_2$.
By using the similarity transformation V, we can obtain all the T operators which are not explicitly written. The results for a representative selection of T operators can be found in figs. 3, 4 and 5.

$T_{ZZ\bar{Z}}$	$\frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right) Z$	$T_{Z\bar{Z}Z}$	$\frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right) \mathcal{Z}$
$T_{XZ\bar{Z}}$	$-\frac{1}{3}\left(K^{m^2=0} - K^{m^2=6}\right) \mathcal{X}$	$T_{X\bar{Z}Z}$	$-\frac{1}{3}\left(K^{m^2=0} - K^{m^2=6}\right) \mathcal{X}$
$T_{ZX\bar{Z}}$	$\frac{1}{3}\left(K^{m^2=0} - K^{m^2=6}\right) \mathcal{X}$	$T_{Z\bar{Z}X}$	$\frac{1}{3}\left(2K^{m^2=0} + 3K^{m^2=2} + K^{m^2=6}\right) \mathcal{X}$
$T_{XX\bar{Z}}$	$\frac{1}{3}\left(2K^{m^2=0} - 3K^{m^2=2} + K^{m^2=6}\right) \mathcal{Z}$	$T_{X\bar{Z}X}$	$\frac{1}{3}\left(K^{m^2=0} - K^{m^2=6}\right) \mathcal{Z}$

Table 5: The T operators necessary for constructing the $Q_{Z\bar{Z}}$ and $Q_{X\bar{Z}}$ operators. Because these operators are all proportional to non-conjugated classical scalar fields, they can in fact bthey can note interpreted as $\mathfrak{su}(2)$ spin-chain operators.

4.4.1.3 The relation between the T operators and spin-chain operators From the results in figs. 3, 4 and 5, we see that for certain choices of W_1, W_2 the operators $T_{ZW_1W_2}$, $T_{ZW_2W_1}$, $T_{XW_1W_2}$, $T_{XW_2W_1}$ will be proportional to either $\mathcal{Z}$ or $\mathcal{X}$. For this to work, we have to chooce W_1, W_2 such that one is conjugated while the other is not. The T operators presented in fig. 5 are examples of such operators. For any set of four operators $T_{ZW_1W_2}$, $T_{ZW_2W_1}$, $T_{XW_1W_2}, T_{XW_2W_1}$, which are all proportional to either $\mathcal{Z}$ or $\mathcal{X}$, it is possible to interpret their combined contribution to $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1W_2}} \rangle$ through the introduction of a certain spin-chain operator. This can more easily be seen by first writing out the unexpanded forms of such a set of T operators.

$$Z \operatorname{tr}\left[W_1 \mathcal{W}_2\right] + Z \operatorname{tr}\left[W_2 \mathcal{W}_1\right] = T_{ZW_1W_2} + T_{ZW_2W_1} = c^{\uparrow}\mathcal{Z} + c^{-}\mathcal{X} \tag{4.4.53}$$

$$X \operatorname{tr}\left[W_1 \mathcal{W}_2\right] + X \operatorname{tr}\left[W_2 \mathcal{W}_1\right] = T_{XW_1W_2} + T_{XW_2W_1} = c^{\downarrow}\mathcal{X} + c^{+}\mathcal{Z} \tag{4.4.54}$$

Given the mapping from the complex scalars $\boldsymbol{Z}$ and $\boldsymbol{X}$, to Heisenberg spin-chain states.

$$\operatorname{tr}[\boldsymbol{V}_{i_1} \cdots \boldsymbol{V}_{i_L}] \to |\sigma_1 \cdots \sigma_L\rangle \quad , \quad \boldsymbol{V}_{i_\ell} \in \{\boldsymbol{X}, \boldsymbol{Z}\} \quad , \quad \sigma_\ell \in \{\downarrow, \uparrow\} \tag{4.4.55}$$

It is useful to think of individual scalars as mapping to individual spins as: $\boldsymbol{Z} \to |\uparrow\rangle$ and $\boldsymbol{X} \to |\downarrow\rangle$. We can now define a new operator $Q^{\ell}_{W_1,W_2}$ by its action on a single spin-states at site ℓ of the chain.

$$Q^{\ell}_{W_1,W_2} |\uparrow\rangle = c^{\uparrow} |\uparrow\rangle + c^{-} |\downarrow\rangle \quad , \quad Q^{\ell}_{W_1,W_2} |\downarrow\rangle = c^{\downarrow} |\downarrow\rangle + c^{+} |\uparrow\rangle \tag{4.4.56}$$

The spin-chain operators $Q^{\ell}_{W_1,W_2}$ can now be used to rewrite the entire connected tree-level contribution to the long / short two-point functions $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1W_2}} \rangle$, as a certain inner product on the spin-chain.

$$\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1W_2}} \rangle_{\text{tree,c.}} = \sum_{\ell=1}^{L} \Psi_M^{i_1 \cdots i_L} \operatorname{tr}\left[\mathcal{V}_{i_1} \cdots V_{i_\ell} \cdots \mathcal{V}_{i_L}\right] \operatorname{tr}\left[W_1 \mathcal{W}_2\right] + (W_1 \leftrightarrow W_2)$$

$$= \sum_{\ell=1}^{L} \Psi_M^{i_1 \cdots i_L} \operatorname{tr}\left[\mathcal{V}_{i_1} \cdots Q^{\ell}_{W_1,W_2} \mathcal{V}_{i_\ell} \cdots \mathcal{V}_{i_L}\right] \quad \to \quad \langle \text{MPS}| \, Q_{W_1,W_2} \, |\Psi_M\rangle \tag{4.4.57}$$

Where the operator Q_{W_1,W_2} acts on the complete spin-chain Hilbert space, and is defined as follow.

$$Q_{W_1,W_2} = \sum_{\ell=1}^{L} \left(\mathbb{1} \otimes \cdots \mathbb{1} \otimes Q^{\ell}_{W_1,W_2} \otimes \mathbb{1} \cdots \otimes \mathbb{1}\right) \tag{4.4.58}$$

By the introduction of the Q_{W_1,W_2} operators, we have now revealed that the connected tree-level structure of $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1 W_2}} \rangle$, is in fact very similar to the tree-level structure of a Bethe state one-point function $\langle \mathcal{O}_{\boldsymbol{L}} \rangle$. We write down the two slightly different overlaps below for easy comparison.

$$\langle \mathcal{O}_{\boldsymbol{L}}(x) \rangle_{\text{tree}} \sim \frac{\langle \text{MPS} | \, |\Psi_M \rangle}{x_3^L} \quad , \quad \langle \mathcal{O}_{\boldsymbol{L}}(x) \mathcal{O}_{\boldsymbol{W_1 W_2}}(y) \rangle_{\text{tree,c.}} \sim \frac{\langle \text{MPS} | \, Q_{W_1,W_2} \, |\Psi_M \rangle}{(x_3 y_3)^L} \tag{4.4.59}$$

It turns out that the connection between $\langle \mathcal{O}_{\boldsymbol{L}} \rangle_{\text{tree}}$ and $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1 W_2}} \rangle_{\text{tree,c.}}$ runs deeper than just that surface level structural similarity. In fact, for specific choices of Q_{W_1,W_2}, we find that $\langle \mathcal{O}_{\boldsymbol{L}} \mathcal{O}_{\boldsymbol{W_1 W_2}} \rangle_{\text{tree,c.}}$ is proportional to $\langle \mathcal{O}_{\boldsymbol{L}} \rangle_{\text{tree}}$. In order to show this result, it is convenient to organize the $Q^{\ell}_{W_1,W_2}$ operators into two distinct classes; diagonal operators $Q^{\ell}_{=}$, and off-diagonal operators $Q^{\ell}_{\neq}$, depending on whether $W_1 = \bar{W}_2$ or $W_1 \neq \bar{W}_2$.

$$[Q^{\ell}_{=}]_{\sigma,\sigma'} = \begin{pmatrix} c^{\uparrow} & 0 \\ 0 & c^{\downarrow} \end{pmatrix} \quad , \quad [Q^{\ell}_{\neq}]_{\sigma,\sigma'} = \begin{pmatrix} 0 & c^{+} \\ c^{-} & 0 \end{pmatrix} \quad , \quad \sigma, \sigma' \in \{\uparrow, \downarrow\} \tag{4.4.60}$$

Where the matrix entries $c^{\uparrow}$, $c^{\downarrow}$, c^{+} and c^{-} can all be extracted from T operators in the following way.

$$c^{\uparrow} = \frac{1}{2} \operatorname{tr} \left[T_{ZW_1W_2} \bar{\mathcal{Z}} \right] + (W_1 \leftrightarrow W_2) \quad , \quad c^{\downarrow} = \frac{1}{2} \operatorname{tr} \left[T_{XW_1W_2} \bar{\mathcal{X}} \right] + (W_1 \leftrightarrow W_2) \tag{4.4.61}$$

$$c^{+} = \frac{1}{2} \operatorname{tr} \left[T_{XW_1W_2} \bar{\mathcal{Z}} \right] + (W_1 \leftrightarrow W_2) \quad , \quad c^{-} = \frac{1}{2} \operatorname{tr} \left[T_{ZW_1W_2} \bar{\mathcal{X}} \right] + (W_1 \leftrightarrow W_2) \tag{4.4.62}$$

In the case of a $Q_{=}$ type spin-chain operator, the overlap $\langle \text{MPS} | \, Q_{W_1,W_2} \, |\Psi_M \rangle$ is now easily evaluated.

$$Q_{=} \, |\Psi_M \rangle = [c^{\uparrow}(L-M) + c^{\downarrow} M] \, |\Psi_M \rangle \;\Rightarrow\; \boxed{\langle \mathcal{O}_L \mathcal{O}_{=} \rangle_{\text{tree,c.}} = [c^{\uparrow}(L-M) + c^{\downarrow} M] \, \langle \mathcal{O}_L \rangle_{\text{tree}}} \tag{4.4.63}$$

For the case of $Q_{\neq}$ type spin-chain operators, the overlap is not as straightforwardly evalauted. Firstly, $Q_{\neq}$ type operators change the number of excitations of a Bethe state $|\Psi_M\rangle$, which is easy to see if written in terms of the spin-raising and spin-lowering operators S^{+} and S^{-}.

$$Q_{\neq} = c^{+} S^{+} + c^{-} S^{-} \tag{4.4.64}$$

Since Bethe states are states of highest weight, meaning that $S^{+} \, |\Psi_M \rangle = 0$ [13], the overlap (4.4.57) can be written in the following way for the case of $Q_{\neq}$ type operators.

$$\langle \text{MPS} | \, Q_{\neq} \, |\Psi_M \rangle = c^{-} \, \langle \text{MPS} | \, S^{-} \, |\Psi_M \rangle \tag{4.4.65}$$

It turns out that the action of S^{-} on a Bethe state $|\Psi_M\rangle$ produces another Bethe state. This Bethe state has one additional excitation with zero associated rapidity. This can be seen from the form of the magnon creation operators $B(u)$, find in (4.2.29).

$$\lim_{u \to 0} B(u) \sim S^{-} \quad \Rightarrow \quad S^{-} \, |\Psi_M \rangle = \lim_{u_{M+1} \to 0} |\Psi_{M+1} \rangle \tag{4.4.66}$$

The Bethe state obtained by acting with S^{-} on a Bethe state $|\Psi_M\rangle$ are called *Bethe descendants*, and we can now write the overlap (4.4.57) in terms of this type of spin-chain eigenstate.

$$\langle \text{MPS} | \, Q_{\neq} \, |\Psi_M \rangle = \lim_{u_{M+1} \to 0} c^{-} \, \langle \text{MPS} | \Psi_{M+1} \rangle \tag{4.4.67}$$

When working in the $SO(3) \times SO(3)$ symmetric defect setup, a general expression for overlaps of the form $\langle \text{MPS} | \Psi_{M+1} \rangle$ has, to the best of our knowledge, not been found. As previously mention, the value of the overlap is known for some specific values of L, M, k_1, k_2, and these values can for example be found in [22]. For the $SO(3)$ symmetric defect setup in contrast, a general expression for the overlap $\langle \text{MPS} | \Psi_{M+1} \rangle$ can be found, due to the fact that the MPS is annihilated by the conserved charge Q_3. More details on this can be found in [19].

T_{ZYY}	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Z}}$		
T_{XYY}	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$		
$T_{Z\bar{Y}\bar{Y}}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Z}}$		
$T_{X\bar{Y}\bar{Y}}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{X}}$		
$T_{ZY\bar{Y}}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Z}$	$T_{Z\bar{Y}Y}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Z}$
$T_{XY\bar{Y}}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{X}$	$T_{X\bar{Y}Y}$	$-\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{X}$

Table 6: The T operators necessary for constructing the Q_{YY}, $Q_{\bar{Y}\bar{Y}}$ and $Q_{Y\bar{Y}}$ operators. Because only the T operators making up $Q_{Y\bar{Y}}$ are proportional to non-conjugated classical scalar fields, only this can be interpreted as a $\mathfrak{su}(2)$ spin-chain operator.

4.4.1.4 The T operators containing Y and $\bar{\mathbf{Y}}$ scalars As promised at the begining of this subsection, we will now discuss the computation of the $T_{V_{i_\ell}W_1W_2}$ operators which contain $W_1 = Y, \bar{Y}$ OR $W_2 = Y, \bar{Y}$. We will also discuss the difficulties with associating some of these T operators to spin-chain operators. As we shall see soon, T operators for which $W_1 = Y, \bar{Y}$, $W_2 = X, Z, \bar{X}, \bar{Z}$ will be proportional to $\mathcal{Y}$. The same be will true for T operators with $W_1 \leftrightarrow W_2$. Thus, the Q operators constructed from these kind of T operators can not be regarded as proper spin-chain operators. This turns out not to be a big problem however, as we shall soon see.

The T operators with $\mathbf{W_1, W_2 = Y, \bar{Y}}$ In order to obtain the forms of all T operators of this type, we can again employ the trick of using similarity transfomations to relate the unknown T operators to known ones. Using the transfomation V_1, which takes $t_1 \to t_3$, $t_2 \to t_1$ and $t_3 \to t_2$.

$$V_1 = U_1^{k_1} \otimes U_1^{k_2} \quad , \quad U_1^{k_s} = e^{-i\frac{\pi}{2}t_3^{k_s}}\, e^{-i\frac{\pi}{2}t_2^{k_s}} \tag{4.4.68}$$

Together with the transfomation V_2, which takes $t_1 \to t_2$, $t_2 \to t_3$ and $t_3 \to t_1$.

$$V_2 = U_2^{k_1} \otimes U_2^{k_2} \quad , \quad U_2^{k_s} = \left(U_1^{k_s}\right)^\dagger = e^{i\frac{\pi}{2}t_2^{k_s}}\, e^{i\frac{\pi}{2}t_3^{k_s}} \tag{4.4.69}$$

We can obtain all T operators of type $W_1, W_2 = Y, \bar{Y}$ from those presented in fig. 3, fig. 4 and fig. 5[8]. For example, the operator T_{ZYY} can be obtained from T_{XZZ} by application of V_1. The results for all $W_1, W_2 = Y, \bar{Y}$ type T operators are presented in fig. 6.

The vanishing T operators It turns out that for the T operators of type $W_1 = Y, \bar{Y}$, $W_2 = X, Z, \bar{X}, \bar{Z}$ and $W_1 \leftrightarrow W_2$, a large subset vanish as we shall now demonstrate. We start out by evaluating the operator T_{XYZ}. This operator has the following unsimplified form.

$$T_{XYZ} = \hat{Y}_{\ell_1}^{m_1} \otimes \hat{Y}_{\ell_2}^{m_2}\, \mathrm{tr}\left[\hat{Y}_{\ell_1'}^{m_1'} \otimes \hat{Y}_{\ell_2'}^{m_2'}\, \mathcal{Z}\right] \left\langle X_{\ell_1,m_1;\ell_2,m_2}\, Y_{\ell_1',m_1';\ell_2',m_2'}\right\rangle \tag{4.4.70}$$

The X, Y-propagator can as usual be obtained by writting out the complex scalars in terms of ϕ_i-fields, and using the ϕ_i, ϕ_j-propagators [11]. The result of this procedure is the following expression.

[8] One might have to also make use of the transformation V, which takes $t_1 \to t_3$, $t_2 \to -t_2$ and $t_3 \to t_1$, presented earlier in this subsection 4.4.28.

T_{ZZY}	$\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Y}}$	T_{ZYZ}	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Y}}$
$T_{ZZ\bar{Y}}$	$\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\mathcal{Y}$	$T_{Z\bar{Y}Z}$	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Y}$
$T_{Z\bar{Z}Y}$	$\frac{1}{3}\left(2K^{m^2=0}+3K^{m^2=2}+K^{m^2=6}\right)\mathcal{Y}$	$T_{ZY\bar{Z}}$	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\mathcal{Y}$
$T_{Z\bar{Z}\bar{Y}}$	$\frac{1}{3}\left(2K^{m^2=0}+3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Y}}$	$T_{Z\bar{Y}\bar{Z}}$	$\frac{1}{3}\left(K^{m^2=0}-K^{m^2=6}\right)\bar{\mathcal{Y}}$

Table 7: The non zero T operators necessary for constructing the Q_{ZY}, $Q_{Z\bar{Y}}$, $Q_{\bar{Z}Y}$ and $Q_{\bar{Z}\bar{Y}}$ operators. Because these operators are all proportional to $\mathcal{Y}$, they can not be interpreted as proper $\mathfrak{su}(2)$ spin-chain operators.

$$\begin{aligned}\left\langle X_{\ell_1,m_1;\ell_2,m_2}Y_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\ (-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2+m_2',0}\left(-i[t_3^{k_1}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{anti}}-[t_1^{k_1}t_2^{k_1}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{sym}}\right) \\ -(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\left(-i[t_3^{k_2}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{anti}}-[t_1^{k_2}t_2^{k_2}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{sym}}\right) \\ +\,i\,(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\left([t_1^{k_2}]_{m_2,-m_2'}[t_2^{k_1}]_{m_1,-m_1'}+[t_2^{k_1}]_{m_1,-m_1'}[t_1^{k_2}]_{m_2,-m_2'}\right)K^{\phi}_{\text{opp}}\end{aligned} \tag{4.4.71}$$

The trace appearing in (4.4.70) was evaluated earlier in this subsection. We write out the result below for convenience.

$$\operatorname{tr}\left[\hat{Y}^{m_1'}_{\ell_1'}\otimes\hat{Y}^{m_2'}_{\ell_2'}\,\mathcal{Z}\right]=\sqrt{2}\,d_1^{k_1}\,\delta_{\ell_1',1}\,\delta_{m_1',0}\,d_0^{k_2}\,\delta_{\ell_2',0}\,\delta_{m_2',0}+i\,d_0^{k_1}\,\delta_{\ell_1',0}\,\delta_{m_1',0}\sqrt{2}\,d_1^{k_2}\,\delta_{\ell_2',1}\,\delta_{m_2',0} \tag{4.4.72}$$

The above expression for the trace, together with following four previously discussed observations.

$$[t_i^{k=1}]_{m,m'}=0 \quad , \quad [t_3^k]_{m,m'}\delta_{m',0}=0 \quad , \quad K^{\phi,(1)}_{\text{sym}}\Big|_{\ell_1=1,\ell_2=0}=0 \quad , \quad K^{\phi,(2)}_{\text{sym}}\Big|_{\ell_1=0,\ell_2=1}=0 \tag{4.4.73}$$

Leads us to the conclusion that $T_{XYZ}=0$. In fact, this result can easily be extended to a large number of similar T operators by use of the similarity transformations V_1, V_2 and the transformation V (4.4.28), which takes $t_1\to t_3$, $t_2\to -t_2$ and $t_3\to t_1$.

$$T_{XYZ}=T_{XZY}=T_{ZXY}=T_{ZYX}=0 \tag{4.4.74}$$

$$T_{XY\bar{Z}}=T_{X\bar{Z}Y}=T_{Z\bar{X}Y}=T_{ZY\bar{X}}=0 \tag{4.4.75}$$

$$T_{X\bar{Y}Z}=T_{XZ\bar{Y}}=T_{ZX\bar{Y}}=T_{Z\bar{Y}X}=0 \tag{4.4.76}$$

$$T_{X\bar{Y}\bar{Z}}=T_{X\bar{Z}\bar{Y}}=T_{Z\bar{X}\bar{Y}}=T_{Z\bar{Y}\bar{X}}=0 \tag{4.4.77}$$

Thus, we see that half of the T operators needed to construct the operators Q_{YZ}, Q_{YX}, $Q_{\bar{Y}Z}$, $Q_{\bar{Y}X}$, $Q_{Y\bar{Z}}$, $Q_{Y\bar{X}}$, $Q_{\bar{Y}\bar{Z}}$ and $Q_{\bar{Y}\bar{X}}$ vanish. We now turn to the computation of the remaining non zero T operators needed to construct aforementioned Q operators.

The T operators proportional to $\mathcal{Y}$ As we shall now see, all the non zero T operators of type $W_1=Y,\bar{Y}$, $W_2=X,Z,\bar{X},\bar{Z}$ and $W_1\leftrightarrow W_2$, turn out to all be proportional to the classical scalar $\mathcal{Y}$. We again start out by evaluating an example of such a T operator; this time T_{ZZY}. For this particular example, we can transform the known form of T_{XXZ}, found in fig. 3, using V_1 to obtain the desired operator T_{ZZY}. The result is then.

$$T_{ZZY}=\frac{1}{3}\left(2K^{m^2=0}-3K^{m^2=2}+K^{m^2=6}\right)\bar{\mathcal{Y}} \tag{4.4.78}$$

All the remaining non zero T operators of type $W_1 = Y, \bar{Y}$, $W_2 = X, Z, \bar{X}, \bar{Z}$ and $W_1 \leftrightarrow W_2$, can similarly be obtained through appropriate applications of V, V_1 and V_2. More examples of these T operators can be found in fig. 7, although their exact forms are actually irrelevant for our purposes. We only need to know that they all share the property of being proportional to $\mathcal{Y}$, and we shall now explain why. From the form of the long / short two-point functions (4.4.1), we see that the T operators always appears as part of a trace of length L. If $T \sim \mathcal{Y}$, we thus find that.

$$\mathrm{tr}[\mathcal{V}_{i_1} \cdots T \cdots \mathcal{V}_{i_L}] \sim \mathrm{tr}[\mathcal{V}_{i_1} \cdots \mathcal{Y} \cdots \mathcal{V}_{i_L}] \quad , \quad \mathcal{V}_{i_\ell} \in \{\mathcal{X}, \mathcal{Z}\} \tag{4.4.79}$$

Now, by introducing the anti-unitary time reversal operator $\mathcal{T}^k$ associated to the k-dimensional $\mathfrak{su}(2)$ representation, we can construct a transformation which takes $t_1 \to t_1$, $t_2 \to -t_2$ and $t_3 \to t_3$. This transformation looks as follow.

$$V_3 = U_3^{k_1} \otimes U_3^{k_2} \quad , \quad U_3^{k_s} = \mathcal{T}^{k_s} \, e^{-i\pi t_1^{k_s}} \, e^{-i\pi t_3^{k_s}} \tag{4.4.80}$$

Thus, by applying the transformation V_3 to the expression (4.4.79), we find that it simply vanishes.

$$\mathrm{tr}[\mathcal{V}_{i_1} \cdots \mathcal{Y} \cdots \mathcal{V}_{i_L}] = -\,\mathrm{tr}[\mathcal{V}_{i_1} \cdots \mathcal{Y} \cdots \mathcal{V}_{i_L}] \quad \Rightarrow \quad \mathrm{tr}[\mathcal{V}_{i_1} \cdots \mathcal{Y} \cdots \mathcal{V}_{i_L}] = 0 \tag{4.4.81}$$

Using the above result, together with the results (4.4.74), (4.4.75), (4.4.76) and (4.4.77), we find that the following Q operator overlaps vanish.

$$\langle \mathrm{MPS}| \, Q_{ZY} \, |\Psi_M\rangle = 0 \quad , \quad \langle \mathrm{MPS}| \, Q_{XY} \, |\Psi_M\rangle = 0 \tag{4.4.82}$$

$$\langle \mathrm{MPS}| \, Q_{Z\bar{Y}} \, |\Psi_M\rangle = 0 \quad , \quad \langle \mathrm{MPS}| \, Q_{X\bar{Y}} \, |\Psi_M\rangle = 0 \tag{4.4.83}$$

$$\langle \mathrm{MPS}| \, Q_{\bar{Z}Y} \, |\Psi_M\rangle = 0 \quad , \quad \langle \mathrm{MPS}| \, Q_{\bar{X}Y} \, |\Psi_M\rangle = 0 \tag{4.4.84}$$

$$\langle \mathrm{MPS}| \, Q_{\bar{Z}\bar{Y}} \, |\Psi_M\rangle = 0 \quad , \quad \langle \mathrm{MPS}| \, Q_{\bar{X}\bar{Y}} \, |\Psi_M\rangle = 0 \tag{4.4.85}$$

In conclusion, we have found that only the Q operators with $W_1, W_2 = Y, \bar{Y}$ can produce non zero overlaps of the type $\langle \mathrm{MPS}| \, Q \, |\Psi_M\rangle$. Furthermore, of the $W_1, W_2 = Y, \bar{Y}$ type Q operators, only $Q_{Y\bar{Y}}$ is a proper spin-chain operator. $Q_{Y\bar{Y}}$ happens to also of diagonal type, which means that its associated overlap can be expressed in terms of $\langle \mathcal{O}_{\boldsymbol{L}} \rangle_{\text{tree}}$.
With that, we conclude our discussion on evaluating long / short two-point functions of form (4.4.1).

5 Conclusion and outlook

As explained in the introductory part of the thesis, the ultimate aim of our work was twofold. Firstly, we wanted to find explicit expressions for the two-point functions between the scalar chiral primary operators $\mathcal{O}_Z = \operatorname{tr} Z^L$, $\mathcal{O}_{\bar{Z}} = \operatorname{tr} \bar{Z}^L$ and $\mathcal{O}_X = \operatorname{tr} X^L$, in the dCFT setups arising from D3-D7 probe-brane configurations. Both the disconnected tree-level and the disconnected 1-loop contributions to these two-point functions in both the $SO(3)\times SO(3)$ and the $SO(5)$ symmetric setups, we were easily able to obtain by direct extension of the works [11, 12]. For the connected tree-level constribution however, in contrast to the $SO(3)$ symmetric D5-D3 probe brane setup studied in [10], the infinite sums obtained in the $SO(3) \times SO(3)$ symmetric setup are seemingly unevaluable. Thus, we could not in general find a closed form expression for the connected tree-level contributions to the chiral primary two-point functions. We were however able to evaluate the infinite sums for very short chiral primary two-point functions; specifically for $L_1 = 2$, $L_2 = 2$. We were also able to reproduce and correct[9] the results found in [10], for the cases of $k_1 = 1$ and $k_2 = 1$ units of external gauge-field flux through the $S^2 \times S^2$. In conclusion, we were able to find explicit expressions to first order in λ, for the chiral primary two-point functions in the $SO(3) \times SO(3)$ symmetric setup, to the extend of seeming possibility.
Our second ultimate aim was to extend the work of [19] from the D5-D3 probe-brane setup to the D3-D7 probe-brane setups. In other words, we wanted to also find explicit expressions for two-point functions between short scalar operators of the form $\mathcal{O}_{W_1W_2} = \operatorname{tr}[W_1W_2]$, and Bethe state operators $\mathcal{O}_L = \Psi_M^{i_1\ldots i_L} \operatorname{tr}[V_{i_1}\cdots V_{i_L}]$, in the $SO(3) \times SO(3)$ and $SO(5)$ symmetric dCFT setups. The insight presented in [19] was that the connected tree-level contribution to these two-point functions could be re-expressed as an overlap of the form $\langle \text{MPS}| Q_{W_1W_2} |\Psi_M\rangle$. This overlap can then be further reduced down to the overlap $\langle \text{MPS}|\Psi_M\rangle$, which in the $SO(3)$ symmetric D5-D3 setup can be expressed in a compact determinant form [18]. In the case of the $SO(3) \times SO(3)$ symmetric setup, we find that it is also possible to re-express the connected tree-level contribution to these long / short two-point functions in terms of spin-chain operators $Q_{W_1W_2}$, but only when $W_2 = \bar{W}_1$. If we want to insist on a spin-chain picture for $W_2 \neq \bar{W}_1$, we can think of the associated $Q_{W_1W_2}$ as mapping the $\mathfrak{su}(2)$ spin-chain states out of the $\mathfrak{su}(2)$ subsector. For the $SO(3)\times SO(3)$ symmetric setup, we furthermore have the problem that the third conserved charge Q_3 of the $\mathfrak{su}(2)$ subsector, does not annihilate the MPS. The consequence of this is first of all that only a subset of the $\langle \text{MPS}| Q_{W_1W_2} |\Psi_M\rangle$ overlaps with $W_2 = \bar{W}_1$ can be written in terms of the simpler $\langle \text{MPS}|\Psi_M\rangle$ overlap. Even more problematic, is the fact that $\langle \text{MPS}|\Psi_M\rangle$ is not even known for general values of k_1, k_2 and M. For certain specific parameter values, $\langle \text{MPS}|\Psi_M\rangle$ has however been computed in the $SO(3) \times SO(3)$ symmetric setup, and the results can be found in [22]. In further conclusion, we were indeed able to find explicit expressions for the long / short two-point functions in the $SO(3) \times SO(3)$ symmetric setup, but only for very specific choices of W_1, W_2, due in part to the more complicated nature of the MPS.
With the contributions to first order in λ to the these different two-point functions now at hand, we have made some significant progress towards a very non-trivial check of the AdS / CFT duality. Looking forward, it would be very interesting to first precisely identify and subsequently compute the objects on the gravity side, dual to the two-point functions. One might somewhat intuitively expect these dual objects to somehow be related to string configurations, with two string-ends connected to the AdS_5 boundary, and one string-end connected to the D7 brane. To the best of our knowledge however, there currently exists no concrete attempts to compute the area of such string configurations, nor any other suggestions to what the gravity dual objects might be. Needless to say, there is certainly more work to be done pertaining to an AdS / CFT check through the various dCFT two-point functions presented in this thesis.

[9] With our analysis, we were able to reproduce the connected tree-level contributions find in [10], only for L_1, L_2 even. For L_1, L_2 odd, we very clearly obtain different, although similar results. We belive this to be an oversight on the part of the authors.

A Propagators of the complex scalar fields

In the evaluation of two-point functions between chiral primary operators from section 3, and likewise in the evaluation of two-point functions between Bethe state operators and scalar operators of length two from section 4, we made repeated use of the propagators between the complex scalar operators.

$$X = \phi_1 + i\phi_4 \quad , \quad Y = \phi_2 + i\phi_5 \quad , \quad Z = \phi_3 + i\phi_6 \tag{A.0.1}$$

In this appendix, we will discuss how to obtain all of these propagators, both for the case of $SO(3)\times SO(3)$ symmetric vevs and for the case of $SO(5)$ symmetric vevs. Just as when we derived the $\langle Z_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$, $\langle Z_{\boldsymbol{\ell}}\, Z^{\dagger}_{\boldsymbol{\ell}'}\rangle$ and $\langle X_{\boldsymbol{\ell}}\, Z_{\boldsymbol{\ell}'}\rangle$ propargators back in section 3, we write out the remaining complex scalars propergators in terms of the $\langle[\phi_i]\,[\phi_j]^{\dagger}\rangle$ propagators. As also mentioned in section 3, the $\langle[\phi_i]\,[\phi_j]^{\dagger}\rangle$ propagators can be obtained by writting out the ϕ_i fields in terms of the mass matrix eigenfields 2 for the defect setup in question. For the $SO(3)\times SO(3)$ and $SO(5)$ symmetric setups, we respectively refer to [11] and [12] for further details.

A.1 Propagators for $\mathfrak{so}(3)\times\mathfrak{so}(3)$ symmetric vevs

For the case of $SO(3)\times SO(3)$ symmetric vevs, the $\langle[\phi_i]\,[\phi_j]^{\dagger}\rangle$ propagators takes one of two forms, depending on wether the ϕ_i fields in question are in the same sector. Sector (1) contains the fields ϕ_1,ϕ_2,ϕ_3, and the (2) sector contains the fields ϕ_4,ϕ_5,ϕ_6. We write out the propagators using the notation $\phi_i^{(1)}\equiv\phi_i$ and $\phi_i^{(2)}\equiv\phi_{i+3}$, with $i=1,2,3$. The two kinds of propagators looks as follow.

$$\langle[\phi_i^{(1)}]_{\ell_1,m_1;\ell_2,m_2}[\phi_j^{(2)}]^{\dagger}_{\ell_1',m_1';\ell_2',m_2'}\rangle = \delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}[t_i^{(\ell_1)}]_{m_1,m_1'}[t_j^{(\ell_2)}]_{m_2,m_2'}K^{\phi}_{\text{opp}} \tag{A.1.1}$$

$$\begin{aligned}\langle[\phi_i^{(1)}]_{\ell_1,m_1;\ell_2,m_2}[\phi_j^{(1)}]^{\dagger}_{\ell_1',m_1';\ell_2',m_2'}\rangle &= \delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}\delta_{m_2,m_2'}\\ \times\Big[\delta_{ij}\delta_{m_1,m_1'}K^{\phi,(1)}_{\text{sing}} - i\varepsilon_{ijk}[t_k^{\ell_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{anti}} &- [t_i^{\ell_1}t_i^{\ell_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{sym}}\Big]\end{aligned} \tag{A.1.2}$$

$$\langle[\phi_i^{(2)}]_{\ell_1,m_1;\ell_2,m_2}[\phi_j^{(1)}]^{\dagger}_{\ell_1',m_1';\ell_2',m_2'}\rangle = \langle[\phi_i^{(1)}]_{\ell_2,m_2;\ell_1,m_1}[\phi_j^{(2)}]^{\dagger}_{\ell_2',m_2';\ell_1',m_1'}\rangle \tag{A.1.3}$$

$$\langle[\phi_i^{(2)}]_{\ell_1,m_1;\ell_2,m_2}[\phi_j^{(2)}]^{\dagger}_{\ell_1',m_1';\ell_2',m_2'}\rangle = \langle[\phi_i^{(1)}]_{\ell_2,m_2;\ell_1,m_1}[\phi_j^{(1)}]^{\dagger}_{\ell_2',m_2';\ell_1',m_1'}\rangle \tag{A.1.4}$$

The propagators K^{ϕ}_{opp}, $K^{\phi,(1)}_{\text{sing}}$, $K^{\phi,(1)}_{\text{anti}}$ and $K^{\phi,(1)}_{\text{sym}}$ in the above expressions, are given by the following.

$$K^{\phi}_{\text{opp}} = \frac{K^{m_-^2}}{N_-}+\frac{K^{m_+^2}}{N_+}-\frac{K^{m_0^2}}{N_0} \quad , \quad K^{\phi,(1)}_{\text{anti}} = \frac{K^{m^2_{(1),+}}}{2\ell_1+1}-\frac{K^{m^2_{(1),-}}}{2\ell_1+1} \tag{A.1.5}$$

$$K^{\phi,(1)}_{\text{sing}} = \frac{\ell_1+1}{2\ell_1+1}K^{m^2_{(1),+}}+\frac{\ell_1}{2\ell_1+1}K^{m^2_{(1),-}} \tag{A.1.6}$$

$$K^{\phi,(1)}_{\text{sym}} = \frac{K^{m^2_{(1),+}}}{(2\ell_1+1)(\ell_1+1)}+\frac{K^{m^2_{(1),-}}}{(2\ell_1+1)\ell_1}-\frac{\ell_2(\ell_2+1)}{\ell_1(\ell_1+1)}\frac{K^{m_0^2}}{N_0}-\frac{K^{m_-^2}}{N_-}-\frac{K^{m_+^2}}{N_+} \tag{A.1.7}$$

$$N_0 = -\lambda_+\lambda_- \quad , \quad N_\pm = \lambda_\mp(\lambda_\mp-\lambda_\pm) \tag{A.1.8}$$

$$\lambda_0 = -1 \quad , \quad \lambda_\pm = -\frac{1}{2}\pm\sqrt{\ell_1(\ell_1+1)+\ell_2(\ell_2+1)+\frac{1}{4}} \tag{A.1.9}$$

$$\langle[\phi_i]^{\dagger}_{\ell_1,m_1;\ell_2,m_2}\rangle = (-1)^{m_1+m_2}\,\langle[\phi_i]_{\ell_1,-m_1;\ell_2,-m_2}\rangle \tag{A.1.10}$$

$$[t_3^{(\ell)}]_{m,m'} = m\,\delta_{m',m} \quad , \quad C^{\ell}_{m,m'} = \sqrt{(\ell+m)(\ell-m')} \tag{A.1.11}$$

$$[t_2^{(\ell)}]_{m,m'} = \frac{1}{2i}\left(C^{\ell}_{m,m'}\delta_{m',m-1} - C^{\ell}_{m',m}\delta_{m',m+1}\right) \tag{A.1.12}$$

$$[t_1^{(\ell)}]_{m,m'} = \frac{1}{2}\left(C^{\ell}_{m,m'}\delta_{m',m-1} + C^{\ell}_{m',m}\delta_{m',m+1}\right) \tag{A.1.13}$$

$$\begin{aligned}
&\left\langle Z_{\ell_1,m_1;\ell_2,m_2} Z_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0}\left(K^{\phi,(1)}_{\text{sing}} - m_1^2\, K^{\phi,(1)}_{\text{sym}}\right) \\
&-(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0}\left(K^{\phi,(2)}_{\text{sing}} - m_2^2\, K^{\phi,(2)}_{\text{sym}}\right) \\
&2i\,(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0}\, m_1 m_2\, K^{\phi}_{\text{opp}}
\end{aligned} \tag{A.1.14}$$

$$\begin{aligned}
&\left\langle Z_{\ell_1,m_1;\ell_2,m_2} \bar{Z}_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1,m_1'}\delta_{m_2,m_2'}\left(K^{\phi,(1)}_{\text{sing}} - m_1^2\, K^{\phi,(1)}_{\text{sym}}\right) \\
&+\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1,m_1'}\delta_{m_2,m_2'}\left(K^{\phi,(2)}_{\text{sing}} - m_2^2\, K^{\phi,(2)}_{\text{sym}}\right)
\end{aligned} \tag{A.1.15}$$

$$\begin{aligned}
&\left\langle X_{\ell_1,m_1;\ell_2,m_2} Z_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2+m_2',0}\left(i[t_2^{\ell_1}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{anti}} - [t_1^{\ell_1}t_3^{\ell_1}]_{m_1,-m_1'}K^{\phi,(1)}_{\text{sym}}\right) \\
&-(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1+m_1',0}\left(i[t_2^{\ell_2}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{anti}} - [t_1^{\ell_2}t_3^{\ell_2}]_{m_2,-m_2'}K^{\phi,(2)}_{\text{sym}}\right) \\
&+i\,(-1)^{m_1'+m_2'}\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\left([t_1^{\ell_2}]_{m_2,-m_2'}[t_3^{\ell_1}]_{m_1,-m_1'} + [t_1^{\ell_1}]_{m_1,-m_1'}[t_3^{\ell_2}]_{m_2,-m_2'}\right)K^{\phi}_{\text{opp}}
\end{aligned} \tag{A.1.16}$$

$$\begin{aligned}
&\left\langle X_{\ell_1,m_1;\ell_2,m_2} \bar{Z}_{\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_2,m_2',0}\left(i[t_2^{\ell_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{anti}} - [t_1^{\ell_1}t_3^{\ell_1}]_{m_1,m_1'}K^{\phi,(1)}_{\text{sym}}\right) \\
&\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{m_1,m_1',0}\left(i[t_2^{\ell_2}]_{m_2,m_2'}K^{\phi,(2)}_{\text{anti}} - [t_1^{\ell_2}t_3^{\ell_2}]_{m_2,m_2'}K^{\phi,(2)}_{\text{sym}}\right) \\
&+i\,\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\left([t_1^{\ell_2}]_{m_2,m_2'}[t_3^{\ell_1}]_{m_1,m_1'} - [t_1^{\ell_1}]_{m_1,m_1'}[t_3^{\ell_2}]_{m_2,m_2'}\right)K^{\phi}_{\text{opp}}
\end{aligned} \tag{A.1.17}$$

Work in progress...

A.2 Propagators for $\mathfrak{so}(5)$ symmetric vevs

$$X = \phi_3 + i\phi_6 \quad , \quad Z = \phi_5 + i\phi_6 \tag{A.2.1}$$

$$\langle \mathbf{L}|\, L_{ij}\, |\mathbf{L}'\rangle = \operatorname{tr}\Big[\hat{Y}^{\dagger}_{\mathbf{L}} L_{ij} \hat{Y}_{\mathbf{L}'}\Big] = \operatorname{tr}\Big[\hat{Y}^{\dagger}_{\mathbf{L}}\,[G_{ij}, \hat{Y}_{\mathbf{L}'}]\Big] \tag{A.2.2}$$

$$\langle \phi^{\dagger}_{L_1,L_2;\ell_1,m_1;\ell_2,m_2}\rangle = -(-1)^{L_1-L_2+\ell_1+\ell_2+m_1+m_2}\,\langle \phi_{L_1,L_2;\ell_1,-m_1;\ell_2,-m_2}\rangle \tag{A.2.3}$$

$$\begin{aligned}
&\left\langle Z_{L_1,L_2;\ell_1,m_1;\ell_2,m_2} Z_{L_1',L_2';\ell_1',m_1';\ell_2',m_2'}\right\rangle = \\
&-(-1)^{L_1-L_2+\ell_1+\ell_2+m_1+m_2}\,\delta_{L_1,L_1'}\delta_{L_2,L_2'}\delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0}\,K^{\hat{m}^2_{\text{easy}}} \\
&-(-1)^{L_1-L_2+\ell_1+\ell_2+m_1+m_2}\,\delta_{L_1,L_1'}\delta_{L_2,L_2'}\delta_{\ell_1,\ell_1'}\delta_{\ell_2,\ell_2'}\delta_{m_1+m_1',0}\delta_{m_2+m_2',0}\,\hat{f}^{\text{sing}} \\
&-(-1)^{L_1-L_2+\ell_1+\ell_2+m_1+m_2}\,\langle \mathbf{L}|\,\{L_{6k}, L_{6l}\}L_{kl}\,|\mathbf{L}'\rangle\;\hat{f}^{\text{cubic}} \\
&-(-1)^{L_1-L_2+\ell_1+\ell_2+m_1+m_2}\,\langle \mathbf{L}|\,\{L_{6k}, L_{k6}\}\,|\mathbf{L}'\rangle\;\hat{f}^{\text{sym}}_5
\end{aligned} \tag{A.2.4}$$

Work in progress...

B Fuzzy spherical harmonics

B.1 Fuzzy spherical harmonics on $\mathfrak{so}(3)$

Using the *Wigner-Eckart theorem*, we can find the matrix elements of the fuzzy spherical harmonics.

$$\langle n| \hat{Y}_\ell^m |n'\rangle = [\hat{Y}_\ell^m]_{n,n'} = (-1)^{k-n}\sqrt{2\ell+1} \begin{pmatrix} \frac{k-1}{2} & \ell & \frac{k-1}{2} \\ n-\frac{k+1}{2} & m & \frac{k+1}{2}-n' \end{pmatrix} \tag{B.1.1}$$

Work in progress...

B.2 Fuzzy spherical harmonics on $\mathfrak{so}(5)$

$$\langle \mathbf{S}| \hat{Y}_{\mathbf{L}} |\mathbf{S}'\rangle = \delta_{S_1,S_1'}\delta_{S_2,S_2'}\sqrt{S_1(S_1+2)+S_2(S_2+1)}\,\langle \mathbf{S};\mathbf{L}|\mathbf{S}'\rangle \tag{B.2.1}$$

$$\mathbf{S} = \left(\frac{n}{2}, 0\right), \ell_1, \ell_2, m_1, m_2 \tag{B.2.2}$$

Work in progress...

C Trace relations for Lie algebra generators

C.1 $\mathfrak{so}(3)$ trace relations

In order to compute the two-point functions, we need to know the following trace:

$$\operatorname{tr}\left[t_3^L \hat{Y}_\ell^m\right] \tag{C.1.1}$$

A defining feature of the $\hat{Y}_\ell^m$ matrices is that they obey:

$$L_3\hat{Y}_\ell^m = [t_3, \hat{Y}_\ell^m] = m\hat{Y}_\ell^m \tag{C.1.2}$$

Let us now attempt to compute the following trace:

$$\operatorname{tr}\left[t_3^L L_3 \hat{Y}_\ell^m\right] = \operatorname{tr}\left[t_3^L [t_3, \hat{Y}_\ell^m]\right] = \operatorname{tr}\left[[t_3^L, t_3]\hat{Y}_\ell^m\right] = 0 \tag{C.1.3}$$

But we also know that:

$$\operatorname{tr}\left[t_3^L L_3 \hat{Y}_\ell^m\right] = m \operatorname{tr}\left[t_3^L \hat{Y}_\ell^m\right] \tag{C.1.4}$$

Thus, we conclude that:

$$\operatorname{tr}\left[t_3^L \hat{Y}_\ell^m\right] = 0 \quad \text{for} \quad m \neq 0 \quad \Rightarrow \quad t_3^L = \sum_{\ell=0}^{\infty} \alpha_\ell^L \, \hat{Y}_\ell^0 \tag{C.1.5}$$

Another defining feature of the $\hat{Y}_\ell^m$ matrices is that they obey:

$$L^2\hat{Y}_l^m = [t_i, [t^i, \hat{Y}_\ell^m]] = \ell(\ell+1)\hat{Y}_\ell^m \tag{C.1.6}$$

It will be useful to split up L^2 in the following standard way:

$$L^2 = L_- L_+ + L_3^2 + L_3 \tag{C.1.7}$$

Let us now attempt to compute the following trace:

$$\operatorname{tr}\left[t_3^L L^2 \hat{Y}_\ell^m\right] \tag{C.1.8}$$

We do this in steps using the split up:

$$\operatorname{tr}\left[t_3^L L_3^2 \hat{Y}_\ell^m\right] = \operatorname{tr}\left[t_3^L [t_3, [t_3, \hat{Y}_\ell^m]]\right] = \operatorname{tr}\left[[t_3^L, t_3][t_3, \hat{Y}_\ell^m]\right] = 0 \tag{C.1.9}$$

Now, the non trivial contribution:

$$\operatorname{tr}\left[t_3^L L_- L_+ \hat{Y}_\ell^m\right] = \operatorname{tr}\left[t_3^L [t_-, [t_+, \hat{Y}_\ell^m]]\right] = \operatorname{tr}\left[[t_3^L, t_-][t_+, \hat{Y}_\ell^m]\right] = \operatorname{tr}\left[[[t_3^L, t_-], t_+]\hat{Y}_\ell^m\right] \tag{C.1.10}$$

$$\hat{Y}_\ell^m = \sum_{m_1=-\ell_1}^{\ell_1} \sum_{m_2=-\ell_2}^{\ell_2} C^{\ell,m}_{\ell_1,m_1;\ell_2,m_2} \hat{Y}_{\ell_1}^{m_1} \hat{Y}_{\ell_2}^{m_2} \tag{C.1.11}$$

$$\hat{Y}_{\ell_1}^{m_1} \hat{Y}_{\ell_2}^{m_2} = \sum_{\ell=0}^{k-1} \sum_{m=-\ell}^{\ell} F^{\ell,m}_{\ell_1,m_1;\ell_2,m_2} \hat{Y}_\ell^m \tag{C.1.12}$$

$$m_1 + m_2 = m \quad , \quad |\ell_2 - \ell_1| \leq \ell \leq \ell_1 + \ell_2 \tag{C.1.13}$$

$$\begin{pmatrix} \hat{Y}_2^0 \\ \hat{Y}_1^0 \\ \hat{Y}_0^0 \end{pmatrix} = \begin{pmatrix} C^{2,0}_{1,1;1,-1} & C^{2,0}_{1,0;1,0} & C^{2,0}_{1,-1;1,1} \\ C^{1,0}_{1,1;1,-1} & C^{1,0}_{1,0;1,0} & C^{1,0}_{1,-1;1,1} \\ C^{0,0}_{1,1;1,-1} & C^{0,0}_{1,0;1,0} & C^{0,0}_{1,-1;1,1} \end{pmatrix} \begin{pmatrix} \hat{Y}_1^1 \hat{Y}_1^{-1} \\ \hat{Y}_1^0 \hat{Y}_1^0 \\ \hat{Y}_1^{-1} \hat{Y}_1^1 \end{pmatrix} \tag{C.1.14}$$

$$\hat{Y}_1^0 \hat{Y}_1^0 = F^{2,0}_{1,0;1,0} \hat{Y}_2^0 + F^{1,0}_{1,0;1,0} \hat{Y}_1^0 + F^{0,0}_{1,0;1,0} \hat{Y}_0^0 \tag{C.1.15}$$

Work in progress...

C.2 $\mathfrak{so}(5)$ trace relations

Work in progress...

D Representation theory for $\mathfrak{so}(5)$ and $\mathfrak{so}(6)$

The commutation relations between generators of $\mathfrak{so}(n)$.

$$[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}) \quad , \quad i,j,k,l = 1,\dots,n \tag{D.0.1}$$

D.1 $\mathfrak{so}(5)$ representation theory

The simple root vectors of $\mathfrak{so}(5)$.

$$\alpha_1 = (0,1) \quad , \quad \alpha_2 = \left(\frac{1}{2}, -\frac{1}{2}\right) \tag{D.1.1}$$

The Cartan generators which also serve as parts of the two $\mathfrak{su}(2)$ sub-algebra.

$$H_1 = J_0 = \frac{1}{2}(L_{12} + L_{34}) \quad , \quad H_2 = \Lambda_0 = \frac{1}{2}(L_{12} - L_{34}) \tag{D.1.2}$$

The raising and lowering operators for each $\mathfrak{su}(2)$ sub-algebra.

$$F_{\pm 1,0} = J_\pm = \frac{1}{2}[(L_{14} + L_{23}) \pm i(L_{24} + L_{31})] \tag{D.1.3}$$

$$F_{0,\pm 1} = \Lambda_\pm = \frac{1}{2}[(L_{14} - L_{23}) \pm i(L_{24} - L_{31})] \tag{D.1.4}$$

The remaining two raising and lowering operators of the $\mathfrak{so}(5)$ algebra.

$$F_{\pm\frac{1}{2},\pm\frac{1}{2}} = \frac{1}{\sqrt{2}}(L_{52} \pm iL_{15}) \quad , \quad F_{\pm\frac{1}{2},\mp\frac{1}{2}} = \frac{1}{\sqrt{2}}(L_{45} \pm iL_{53}) \tag{D.1.5}$$

Important commutation relations.

$$[J_0, F_{\pm\frac{1}{2},\pm\frac{1}{2}}] = \pm\frac{1}{2}F_{\pm\frac{1}{2},\pm\frac{1}{2}} \quad , \quad [\Lambda_0, F_{\pm\frac{1}{2},\pm\frac{1}{2}}] = \pm\frac{1}{2}F_{\pm\frac{1}{2},\pm\frac{1}{2}} \tag{D.1.6}$$

$$[J_0, F_{\pm\frac{1}{2},\mp\frac{1}{2}}] = \pm\frac{1}{2}F_{\pm\frac{1}{2},\mp\frac{1}{2}} \quad , \quad [\Lambda_0, F_{\pm\frac{1}{2},\mp\frac{1}{2}}] = \mp\frac{1}{2}F_{\pm\frac{1}{2},\mp\frac{1}{2}} \tag{D.1.7}$$

Ranges of the different quantum numbers, assuming $J_m \geq \Lambda_m$.

$$J = J_m - \frac{1}{2}m - \frac{1}{2}n \quad , \quad \Lambda = \Lambda_m - \frac{1}{2}m + \frac{1}{2}n \tag{D.1.8}$$

$$0 \leq m \leq 2\Lambda_m \quad , \quad 0 \leq n \leq 2(J_m - \Lambda_m) \tag{D.1.9}$$

$$m_J = -J, -J+1, \dots, J-1, J \quad , \quad m_\Lambda = -\Lambda, -\Lambda+1, \dots, \Lambda-1, \Lambda \tag{D.1.10}$$

The total number of states in a given irreduceable representation of $\mathfrak{so}(5)$ is then given by.

$$d_5(J_m, \Lambda_m) = \sum_{n=0}^{2(J_m-\Lambda_m)} \sum_{m=0}^{2\Lambda_m} [2J_m - m - n + 1][2\Lambda_m - m + n + 1] \tag{D.1.11}$$

$$= \frac{1}{3}(2J_m + 2\Lambda_m + 3)(2J_m - 2\Lambda_m + 1)(J_m + 1)(2\Lambda_m + 1) \tag{D.1.12}$$

Work in progress...

D.2 $\mathfrak{so}(6)$ representation theory

The simple root vectors of $\mathfrak{so}(6)$.

$$\alpha_1 = (0,1,0) \quad , \quad \alpha_2 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right) \quad , \quad \alpha_3 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}\right) \tag{D.2.1}$$

$$P'' \leq J_m - \Lambda_m \leq P' \leq J_m + \Lambda_m \leq P \tag{D.2.2}$$

Work in progress...

References

[1] L. Suskind, ”The World as a Hologram”, *J. Math. Phys.* **36** (1995) 6377-6396, arXiv:hep-th/9409089

[2] J. Maldacena, ”The Large N Limit of Superconformal field theories and supergravity”, *Int. J. Theor. Phys.* **38** (1999) 1113-1133, *Adv. Theor. Math. Phys.* **2** (1998) 231-252, arXiv:hep-th/9711200

[3] J. A. Minahan, ”Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N = 4 Super Yang-Mills”, arXiv:1012.3983 [hep-th]

[4] V. A. Mitsou, ”Experimental status of particle and astroparticle searches for supersymmetry”, 2013 *J. Phys.: Conf. Sereffectively.* **447** 012019, arXiv:1304.1414 [hep-ph]

[5] N. R. Constable, R. C. Myers, and O. Tafjord, ”The Noncommutative bion core”, *Phys. Rev.* **D61** (2000) 106009, arXiv:hep-th/9911136 [hep-th]

[6] R. Bhattacharyya and R. de Mello Koch, ”Fluctuating Fuzzy Funnels”, *JHEP* **0510** (2005) 036 arXiv:hep-th/0508131

[7] D. Gaiotto and E. Witten, ”Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory”, *J. Statist. Phys.* **135** (2009) 789-855, arXiv:0804.2902 [hep-th]

[8] P. Liendo, L. Rastelli and B. C. van Rees, ”The Bootstrap Program for Boundary CFT$_d$”, *JHEP* **1307** (2013) 113, arXiv:1210.4258 [hep-th] YITP-SB-12-37

[9] I. Buhl-Mortensen, M. de Leeuw, A. C. Ipsen, C. Kristjansen and M. Wilhelm, ”A Quantum Check of AdS/dCFT”, *JHEP* **1701** (2017) 098, arXiv:1611.04603 [hep-th]

[10] M. de Leeuw, A. C. Ipsen, C. Kristjansen, K. E. Vardinghus and M. Wilhelm, ”Two-point functions in AdS/dCFT and the boundary conformal bootstrap equations”, *JHEP* **1708** (2017) 020, arXiv:1705.03898 [hep-th]

[11] A. G. Grau, C. Kristjansen, M. Volk and M. Wilhelm, ”A Quantum Check of Non-Supersymmetric AdS/dCFT”, *JHEP* **1901** (2019) 007, arXiv:1810.11463 [hep-th]

[12] A. G. Grau, C. Kristjansen, M. Volk and M. Wilhelm, ”A Quantum Framework for AdS/dCFT through Fuzzy Spherical Harmonics on S^4”, DESY-19-220, arXiv:1912.02468 [hep-th]

[13] L. D. Faddeev, ”How Algebraic Bethe Ansatz works for integrable model”, arXiv:hep-th/9605187

[14] M. Staudacher, ”The Factorized S-Matrix of CFT/AdS”, *JHEP* **0505** (2005) 054, arXiv:hep-th/0412188

[15] J. A. Minahan and K. Zarembo, ”The Bethe-Ansatz for $\mathcal{N} = 4$ Super Yang-Mills”, *JHEP* **0303** (2003) 013, arXiv:hep-th/0212208

[16] M. de Leeuw, C. Kristjansen and K. Zarembo, ”One-point Functions in Defect CFT and Integrability”, *JHEP* **1508** (2015) 098, arXiv:1506.06958 [hep-th], NORDITA-2015-72, UUITP-12-15

[17] M. de Leeuw, C. Kristjansen and G. Linardopoulos, ”One-Point Functions of Non-protected Operators in the $SO(5)$ Symmetric D3-D7 dCFT”, *Journal of Physics A Mathematical and Theoretical* **50(25)**, arXiv:1612.06236 [hep-th]

[18] I. Buhl-Mortensen, M. de Leeuw, C. Kristjansen and K. Zarembo, ”One-point Functions in AdS/dCFT from Matrix Product States”, *JHEP* **1602** (2016) 052, arXiv:1512.02532 [hep-th], NORDITA-2015-132, UUITP-26-15

[19] E. Widen, ”Two-point functions of SU(2)-subsector and length-two operators in dCFT”, *Phys. Lett.* **B773** (2017) 435-439, arXiv:1705.08679 [hep-th]

[20] J. Hoppe, ”Quantum theory of a massless relativistic surface and a two-dimensional bound state problem”, PhD thesis, MIT 1982.

[21] S. Kawamoto and T. Kuroki, ”Existence of new nonlocal field theory on noncommutative space and spiral flow in renormalization group analysis of matrix models”, *JHEP* **06** (2015) 062, arXiv:1503.08411 [hep-th]

[22] M. de Leeuw, C. Kristjansen and K. E. Vardinghus, ”A non-integrable quench from AdS/dCFT”, *Phys.Lett.* **B798** (2019) 134940, arXiv:1906.10714 [hep-th]

[23] K. Hecht, ”Some simple R5 Wigner Coefficients and their application”, *Nuclear Physics* **63** (1965) 177.

[24] M. Volk and A. G. Grau, ”One-Loop One-Point Functions in Non-Supersymmetric AdS/dCFT”, *Master thesis 2017 / 2018*, at the *Niels Bohr Institute*

[25] O. Bergman, N. Jokela, G. Lifschytz, and M. Lippert, ”Quantum Hall Effect in a Holographic Model”, *JHEP* **10** (2010) 063, arXiv:1003.4965 [hep-th]

[26] R. C. Myers and M. C. Wapler, ”Transport Properties of Holographic Defects”, *JHEP* **12** (2008) 115, arXiv:0811.0480 [hep-th]