Racah's method for general subalgebra chains: Coupling coefficients of SO(5) in canonical and physical bases

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It is shown that the method of infinitesimal generators ("Racah's method") can be broadly and systematically formulated as a method applicable to the calculation of reduced coupling coefficients for a generic subalgebra chain $G \supset H$, provided the reduced matrix elements of the generators of G and the recoupling coefficients of G are known. The calculation of G and G reduced coupling coefficients is considered as an example, and a procedure for transformation of reduced coupling coefficients between canonical and physical subalgebra chains is presented. The problem of calculating coupling coefficients for generic irreps of G reduced with respect to any of its subalgebra chains, is completely resolved by this approach.

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I. INTRODUCTION

Continuous symmetries and their associated Lie algebras facilitate the description of many-body systems both directly and indirectly. When a symmetry occurs as a dynamical symmetry of the system, the corresponding algebra immediately gives the spectroscopic properties of the system. However, even when a symmetry is strongly broken, the algebraic structure nonetheless provides a calculational tool, classifying the basis states used in a full computational treatment of the many-body problem and greatly simplifying the underlying calculational machinery. Lie algebras have a long history of application, in both these capacities, to nuclear spectroscopy and related problems.^{1–3} The fundamental quantities underlying calculations within a Lie algebraic framework are the coupling coefficients of the algebra, also known as generalized Clebsch-Gordan coefficients or Wigner coefficients. These are needed in order to couple states (or operators) of good symmetry to yield new states (or operators) of good symmetry, and they are required for the calculation of matrix elements through the generalized Wigner-Eckart theorem of the algebra.

The Lie algebra SO(5), isomorphic to Sp(4), has several distinct applications in nuclear theory, involving different physical realizations of the operators, and in which different subalgebra chains are relevant to the symmetry properties. The natural construction of SO(5) in terms of generators of rotation in five-dimensional space gives rise to a canonical $SO(4) \sim SO(3) \otimes SO(3)$ subalgebra.⁴ However, application as the proton-neutron pairing quasispin algebra^{5–9} requires reduction with respect to the $U(1) \otimes SO(3)$ algebra of isospin and occupation number operators. For the dynamics of spin-2 bosons (as in the interacting boson model^{10,11}) and for the Bohr collective model, ^{12–16} the appropriate reduction is instead with respect to a physical angular momentum SO(3) subalgebra.

In this article, it is shown that the method of infinitesimal generators ("Racah's method") can be broadly and systematically formulated as a method applicable to the calculation of reduced coupling coefficients for a generic subalgebra chain $G \supset H$, provided the reduced matrix elements of the generators of G and the recoupling coefficients of H are known (Sec. II). More specifically, the problem of calculating coupling coefficients for generic irreps of SO(5), reduced with respect to any of the subalgebra chains, is completely resolved by this approach. The calculation of reduced coupling coefficients for the SO(5) \supset SO(4) canonical chain is considered in detail (Sec. III). Coupling coefficients reduced with respect to the noncanonical subalgebra chains of SO(5) may be obtained by a similar application of Racah's method, or they can be deduced from the canonical chain coupling coefficients by unitary transformation. The general formulation in the presence of outer multiplicities for H, numerical examples for SO(5), and a detailed account of the transformation procedure between subalgebra chains are given in the appendices.

II. METHOD

A. Background and definitions

Consider a Lie algebra G and subalgebra H. States which reduce this subalgebra chain may be identified by the irrep labels Γ of G, the irrep labels Λ of H, and a label λ (typically the Cartan weights) to distinguish basis states within Λ , as $|\Gamma \Lambda \lambda\rangle$. The coupling coefficients, or generalized Clebsch-Gordan coefficients, for G relate the uncoupled product states of two irreps of G to the coupled states, as

$$\begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma \\ \lambda \\ \lambda \end{vmatrix} = \sum_{\substack{\Lambda_1 \Lambda_2 \\ \lambda_1 \lambda_2}} \begin{pmatrix} \Gamma_1 & \Gamma_2 & \Gamma \\ \Lambda_1 & \Lambda_2 & \Lambda \\ \lambda_1 & \lambda_2 & \lambda \end{pmatrix} \begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \Lambda_1 & \Lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix}$$
 (1)

In general, additional labels will be required to resolve multiplicities. There may be "outer" multiplicities in the Clebsch-Gordan series for the outer product of G (that is, $\Gamma_1 \otimes \Gamma_2$ may contain the irrep Γ more than once), and there may be "branching" multiplicites under the restriction of G to H (that is, the given irrep Γ of G may contain an irrep Λ of H more than once). The coupling relation (1) generalizes, with multiplicities, to

$$\begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \rho \Gamma \\ a \Lambda \\ \lambda \end{vmatrix} = \sum_{\substack{a_1 \Lambda_1 a_2 \Lambda_2 \\ \lambda_1 \lambda_2}} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} \begin{vmatrix} \rho \Gamma \\ a \Lambda \\ \lambda \end{pmatrix} \begin{vmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix},$$
 (2)

where ρ is the outer multiplicity index for $G \otimes G \to G$, and the a indices resolve the branching multiplicities for $G \to H$. Furthermore, H may be subject to outer multiplicites $(H \otimes H \to H)$. In the following discussion, we shall for simplicity take the subalgebra H to be multiplicity free. Such is the case for the commonly encountered situation in which the physically relevant subalgebra H is SO(3), as well as for the subalgebra SO(4) considered in Sec. III. However, the necessary generalizations in the presence of outer multiplicities on H are given in Appendix A, as would be needed for consideration of, e.g., chains involving SU(3) as a subalgebra.

Racah's factorization lemma¹⁷ allows the coupling coefficient appearing in (2) to be decomposed as the product

$$\begin{pmatrix}
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\rho \Gamma \\
\alpha \Gamma$$

of a coupling coefficient of H, embodying all the dependence upon weights λ , with a reduced coupling coefficient (or isoscalar factor) for $G \supset H$. The reduced coupling coefficient is nonzero only if Γ is contained in the outer product of Γ_1 and Γ_2 (i.e., $\Gamma_1 \otimes \Gamma_2 \to \Gamma$), each irrep of H is contained in the corresponding irrep of G (i.e., $\Gamma_1 \to \Lambda_1$, $\Gamma_2 \to \Lambda_2$, and $\Gamma \to \Lambda$), and Λ is contained in the outer product of Λ_1 and Λ_2 (i.e., $\Lambda_1 \otimes \Lambda_2 \to \Lambda$). The reduced coupling coefficients satisfy the orthonormality conditions¹⁸

$$\sum_{a_1\Lambda_1 a_2\Lambda_2} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \rho'\Gamma' \\ a'\Lambda \end{pmatrix} = \delta_{(\rho\Gamma)(\rho'\Gamma')}\delta_{aa'} \tag{4}$$

and

$$\sum_{a\Gamma a} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix} \begin{pmatrix} \rho \Gamma \\ a_1' \Lambda_1' & a_2' \Lambda_2' \\ a_1' \Lambda_1' & a_2' \Lambda_2' \end{pmatrix} \begin{pmatrix} \rho \Gamma \\ a_1 \Lambda_1 & a_2' \Lambda_2' \\ a_1 \Lambda_2 & a_2' \Lambda_2' \end{pmatrix} = \delta_{(a_1 \Lambda_1)(a_1' \Lambda_1')} \delta_{(a_2 \Lambda_2)(a_2' \Lambda_2')}, \tag{5}$$

for any irrep Λ such that $\Gamma \to \Lambda$.

If T^{Λ_T} is an irreducible tensor operator with respect to H, the Wigner-Eckart theorem for H permits the expression of a general matrix element of $T^{\Lambda_T}_{\lambda_T}$ as 18

$$\left\langle \begin{array}{c} \Gamma' \\ a'\Lambda' \\ \lambda' \end{array} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{array}{c} \Gamma \\ a\Lambda \\ \lambda \end{array} \right\rangle = \left(\begin{array}{c} \Lambda & \Lambda_T \\ \lambda & \lambda_T \end{array} \middle| \begin{array}{c} \Lambda' \\ \lambda' \end{array} \right) \left\langle \begin{array}{c} \Gamma' \\ a'\Lambda' \end{array} \middle| T^{\Lambda_T} \middle| \begin{array}{c} \Gamma \\ a\Lambda \end{array} \right\rangle,$$
 (6)

in terms of a coupling coefficient for H and a reduced matrix element with respect to H. A Wigner-Eckart theorem of this form [or its generalization (A4)] may be obtained whenever H is a compact, semi-simple Lie algebra.

Several methods may be considered, in general, for constructing the reduced coupling coefficients of Lie algebras:

- (1) Recurrence relations among coupling coefficients may be obtained by considering the action of an infinitesimal generator $G_i = G_i^{(1)} + G_i^{(2)}$ on uncoupled and coupled states. This approach, used in the present construction, is broadly termed "Racah's method" (see Ref. 18) and generalizes the classic recurrence method for evaluating SU(2) \sim SO(3) Clebsch-Gordan coefficients.¹⁹
- (2) Recurrence relations and seed values may be obtained by considering the action of a "shift tensor", lying outside the algebra, which connects different irreps of the algebra.²⁰
- (3) Consistency relations among coupling and recoupling coefficients serve as the basis for a "building up" process, ^{9,21} in which unknown coupling coefficients can be deduced from a few known coefficients.

(4) Explicit realizations of an algebra can be obtained in terms of bosonic or fermionic creation and annihilation operators. Relations among coupling coefficients follow from considering the matrix elements of tensor operators acting on bosonic or fermionic states (e.g., Ref. 22). This approach is generally restricted to symmetric irreps, antisymmetric irreps, or irreps which can be obtained as simple combinations thereof.

Indeed, all of these approaches have been applied or suggested, in various forms, for the calculation of specific classes of SO(5) coupling coefficients.^{1,9,18,22–29} For the *symmetric* irreps of SO(5), one may also work with an explicit realization in terms of five-dimensional spherical harmonics as functions on the four-sphere. Their triple overlap integrals are then proportional to SO(5) coupling coefficients.^{14,30}

B. Racah relations among reduced coupling coefficients

Let us now consider how the first approach, *i.e.*, Racah's method^{1,17} based on the action of infinitesimal generators, can be generally and systematically formulated as a method applicable to the calculation of reduced coupling coefficients involving generic irreps of an arbitrary subalgebra chain. Consider the action of an infinitesimal generator G_i of G on the coupled product state of (2). The generator on the product space is of the form $G_i = G_i^{(1)} + G_i^{(2)}$, where $G_i^{(1)}$ acts only on the space carrying the irrep Γ_1 and $G_i^{(2)}$ acts only on the space carrying the irrep Γ_2 . The equivalence of the action of G_i on the two sides of (2) imposes conditions on the coupling coefficients connecting the different basis states used on the two sides. For effective application of Racah's method, it is most convenient to recast these relations among coupling coefficients so that they involve only (1) reduced coupling coefficients of G with respect to H, (2) reduced matrix elements of the generators of G, and (3) recoupling coefficients of H, as obtained in this section.

Racah's approach requires that the action of the generators on the basis states of an irrep be known explicitly. In general, if coupling coefficients are to be determined for states which reduce $G \supset H$, it is necessary to consider the action of the generators which are in G but not in H, since only these generators can connect different irreps Λ of H. Note that Racah's method is essentially an extension of the classic scheme¹⁹ for calculating the ordinary SO(3) Clebsch-Gordan coefficients, via recurrence relations obtained by considering the known matrix elements of $J_{\pm} = J_{\pm}^{(1)} + J_{\pm}^{(2)}$, between the uncoupled product states $\left| \begin{smallmatrix} J_1 & J_2 \\ M_1 & M_2 \end{smallmatrix} \right\rangle$ and coupled states $\left| \begin{smallmatrix} J_1 & J_2 \\ M & M_2 \end{smallmatrix} \right\rangle$ (see also Sec. II C).

If $T_{\lambda_T}^{\Lambda_T}$ is a generator of G, expressed as an irreducible tensor operator with respect to H, we begin by considering the matrix element of $T_{\lambda_T}^{\Lambda_T} = T_{\lambda_T}^{\Lambda_T(1)} + T_{\lambda_T}^{\Lambda_T(2)}$, between uncoupled and coupled product states,

$$\left\langle \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{array} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ \rho \Gamma \\ a_{\Lambda} \end{array} \right\rangle = \left\langle \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{array} \middle| T_{\lambda_T}^{\Lambda_T (1)} \middle| \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ \rho \Gamma \\ a_{\Lambda} \end{array} \right\rangle + \left\langle \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{array} \middle| T_{\lambda_T}^{\Lambda_T (2)} \middle| \begin{array}{cc|c} \Gamma_1 & \Gamma_2 \\ \rho \Gamma \\ a_{\Lambda} \end{array} \right\rangle. \tag{7}$$

The coupling relation (2) may be used to express each ket on the right hand side of (7) entirely in terms of uncoupled states, and its inverse, obtained by orthonormality of coupling coefficients, may be used to express the bra on the left hand side entirely in terms of coupled states. Since T^{Λ_T} , as a generator of G, does not connect different irreps of G, and since the matrix elements of T^{Λ_T} between states within an irrep of G depends only upon the irrep

labels, the result simplifies to

$$\sum_{\substack{a',\Lambda'\\(\lambda')}} \left\langle \begin{matrix} \Gamma\\ a'\Lambda'\\ \lambda' \end{matrix} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{matrix} \Gamma\\ a\Lambda\\ \lambda \end{matrix} \right\rangle \left(\begin{matrix} \Gamma_1 & \Gamma_2\\ a_1\Lambda_1 & a_2\Lambda_2\\ \lambda_1 & \lambda_2 \end{matrix} \middle| \begin{matrix} \rho\Gamma\\ a'\Lambda'\\ \lambda' \end{matrix} \right) = \sum_{\substack{a'_1\Lambda'_1\\(\lambda'_1)}} \left\langle \begin{matrix} \Gamma_1\\ a_1\Lambda_1\\ \lambda_1 \end{matrix} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{matrix} \Gamma_1\\ a'_1\Lambda'_1\\ \lambda'_1 \end{matrix} \right\rangle \left(\begin{matrix} \Gamma_1 & \Gamma_2\\ a'_1\Lambda'_1 & a_2\Lambda_2\\ \lambda'_1 & \lambda_2 \end{matrix} \middle| \begin{matrix} \rho\Gamma\\ a\Lambda\\ \lambda' \end{matrix} \right) \\
+ \sum_{\substack{a'_2\Lambda'_2\\(\lambda'_2)}} \left\langle \begin{matrix} \Gamma_2\\ a_2\Lambda_2\\ \lambda_2 \end{matrix} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{matrix} \Gamma_2\\ a'_2\Lambda'_2\\ \lambda'_2 \end{matrix} \right\rangle \left(\begin{matrix} \Gamma_1 & \Gamma_2\\ a_1\Lambda_1 & a'_2\Lambda'_2\\ \lambda_1 & \lambda'_2 \end{matrix} \middle| \begin{matrix} \rho\Gamma\\ a\Lambda\\ \lambda \end{matrix} \right). \quad (8)$$

To introduce reduced coupling coefficients and reduced matrix elements, we apply Racah's factorization lemma (3) and the Wigner-Eckart theorem (6), yielding

$$\sum_{\substack{a'\Lambda'\\(\lambda')}} \begin{pmatrix} \Lambda & \Lambda_T \\ \lambda & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma \\ \lambda' \end{pmatrix} \begin{pmatrix} \Gamma \\ a'\Lambda' \end{pmatrix} \begin{pmatrix} \Gamma \\ a\Lambda \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Lambda' \\ \lambda_1 & \Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ a'_1\Lambda' \end{pmatrix} \\
= \sum_{\substack{a'_1\Lambda'_1\\(\lambda'_1)}} \begin{pmatrix} \Lambda'_1 & \Lambda_T \\ \lambda'_1 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ a_1\Lambda_1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ a_1\Lambda_1 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ a'_1\Lambda'_1 \end{pmatrix} \begin{pmatrix} \Lambda'_1 & \Lambda_2 \\ \lambda'_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_2 \\ \lambda_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_2 \\ a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a'_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda'_2 \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \alpha_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \alpha_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \alpha_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \alpha_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda'_2 & \lambda_T \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \alpha_2\Lambda'_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda_1 & \lambda'_2 \end{pmatrix} \begin{pmatrix} \rho\Gamma \\ \lambda$$

It now remains to eliminate the coupling coefficients of H, and thus all reference to weights. The orthogonality relations allow these coefficients to be moved to the right hand side, resulting in sums of quadruple products of coupling coefficients. These sums are recognized as recoupling coefficients of H, specifically, the "unitary 6- Λ symbols", or transformation brackets between basis states in the coupling schemes $[(\Lambda_1\Lambda_2)^{\Lambda_{12}}\Lambda_3]^{\Lambda}$ and $[\Lambda_1(\Lambda_2\Lambda_3)^{\Lambda_{23}}]^{\Lambda}$, which are given by

$$\begin{bmatrix}
\Lambda_1 & \Lambda_2 & \Lambda_{12} \\
\Lambda_3 & \Lambda & \Lambda_{23}
\end{bmatrix} = \sum_{\substack{\lambda_1 \lambda_2 \lambda_3 \\ \lambda_{12} \lambda_{13}}} \begin{pmatrix}
\Lambda_1 & \Lambda_2 & | \Lambda_{12} \\
\lambda_1 & \lambda_2 & | \lambda_{12}
\end{pmatrix} \begin{pmatrix}
\Lambda_{12} & \Lambda_3 & | \Lambda \\
\lambda_{12} & \lambda_3 & | \lambda
\end{pmatrix} \begin{pmatrix}
\Lambda_2 & \Lambda_3 & | \Lambda_{23} \\
\lambda_2 & \lambda_3 & | \lambda_{23}
\end{pmatrix} \begin{pmatrix}
\Lambda_1 & \Lambda_{23} & | \Lambda \\
\lambda_1 & \lambda_{23} & | \lambda
\end{pmatrix}.$$
(10)

The 6- Λ symbol is nonvanishing only if the Clebsch-Gordan series relations $\Lambda_1 \otimes \Lambda_2 \to \Lambda_{12}$, $\Lambda_{12} \otimes \Lambda_3 \to \Lambda$, $\Lambda_2 \otimes \Lambda_3 \to \Lambda_{23}$, and $\Lambda_1 \otimes \Lambda_{23} \to \Lambda$ are satisfied. Let $\Phi(\Lambda_1 \Lambda_2; \Lambda)$ denote the phase factor incurred by interchange of the first and second irreps in a coupling coefficient of H, i.e., $\begin{pmatrix} \Lambda_2 & \Lambda_1 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} \Lambda \\ \lambda \end{pmatrix} = \Phi(\Lambda_2 \Lambda_1; \Lambda) \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Lambda \\ \lambda \end{pmatrix}$. Then the condition (9) becomes, with labels renamed for simplicity,

$$\sum_{a} \left\langle \begin{array}{c} \Gamma \\ a\Lambda \end{array} \right| T^{\Lambda_{T}} \left\| \begin{array}{c} \Gamma \\ a'\Lambda' \end{array} \right\rangle \left(\begin{array}{ccc} \Gamma_{1} & \Gamma_{2} \\ a_{1}\Lambda_{1} & a_{2}\Lambda_{2} \end{array} \right| \rho \Gamma \right) \\
= \sum_{a'_{1}\Lambda'_{1}} \Phi(\Lambda_{1}\Lambda_{2}; \Lambda) \Phi(\Lambda'_{1}\Lambda_{2}; \Lambda') \left[\begin{array}{ccc} \Lambda_{2} & \Lambda'_{1} & \Lambda' \\ \Lambda_{T} & \Lambda & \Lambda_{1} \end{array} \right] \left\langle \begin{array}{ccc} \Gamma_{1} \\ a_{1}\Lambda_{1} \end{array} \right| T^{\Lambda_{T}} \left\| \begin{array}{ccc} \Gamma_{1} & \Gamma_{2} \\ a'_{1}\Lambda'_{1} & a_{2}\Lambda_{2} \end{array} \right| \rho \Gamma \right) \\
+ \sum_{a'_{2}\Lambda'_{2}} \left[\begin{array}{ccc} \Lambda_{1} & \Lambda'_{2} & \Lambda' \\ \Lambda_{T} & \Lambda & \Lambda_{2} \end{array} \right] \left\langle \begin{array}{ccc} \Gamma_{2} \\ a_{2}\Lambda_{2} \end{array} \right| T^{\Lambda_{T}} \left\| \begin{array}{ccc} \Gamma_{2} \\ a'_{2}\Lambda'_{2} \end{array} \right\rangle \left(\begin{array}{ccc} \Gamma_{1} & \Gamma_{2} \\ a_{1}\Lambda_{1} & a'_{2}\Lambda'_{2} \end{array} \right| \rho \Gamma \right), \quad (11)$$

expressed entirely in terms of the reduced coupling coefficients to be calculated, reduced matrix elements, and recoupling coefficients of the lower algebra H.

For the important special case in which H is the angular momentum algebra $SO(3) \sim SU(2)$, the relation (11) becomes

$$\sum_{a} \left\langle \prod_{aJ} \left\| T^{(J_{T})} \right\| \prod_{a'J'} \right\rangle_{SO(3)} \left(\prod_{a_{1}J_{1}} \prod_{a_{2}J_{2}} \rho \Gamma \right) \\
= \sum_{a'_{1}J'_{1}} (-)^{J_{2}+J_{T}+J'} \hat{J} \hat{J}' \left\{ \prod_{J_{1}} J' \prod_{J_{2}} J' \prod_{J_{1}} J' \right\} \left\langle \prod_{a_{1}J_{1}} \left\| T^{(J_{T})} \right\| \prod_{a'_{1}J'_{1}} \right\rangle_{SO(3)} \left(\prod_{a'_{1}J'_{1}} \prod_{a_{2}J_{2}} \rho \Gamma \right) \\
+ \sum_{a'_{2}J'_{2}} (-)^{J_{1}+J'_{2}+J} \hat{J} \hat{J}' \left\{ \prod_{J_{1}} J' \prod_{J_{2}} J' \prod_{J_{2}} \left\{ \prod_{a_{2}J_{2}} \left\| T^{(J_{T})} \right\| \prod_{a'_{2}J'_{2}} \right\} \left\langle \prod_{a_{1}J_{1}} \prod_{a'_{2}J'_{2}} \rho \Gamma \right\rangle, \quad (12)$$

where $\hat{J} \equiv (2J+1)^{1/2}$. Note that the customary form¹⁹ of the Wigner-Eckhart theorem for SO(3) is defined in terms of a 3-J symbol rather than a Clebsch-Gordan coefficient. This results in a reduced matrix element which differs in normalization and phase from the definition implied by the generic statement of the Wigner-Eckhart theorem in (6). The reduced matrix elements under these two conventions are related by $\langle J_3 \parallel T^{(J_2)} \parallel J_1 \rangle_{\text{SO(3)}} = (-)^{2J_2} \hat{J}_3 \langle J_3 \parallel T^{(J_2)} \parallel J_1 \rangle$.

C. Solution of the homogeneous system

The condition (11) yields a different relation among specific reduced coupling coefficients for each choice of values for the four irrep labels $a_1\Lambda_1$, $a_2\Lambda_2$, Λ , and $a'\Lambda'$ (the multiplicity index a is summed over). If there are N coupling coefficients for the coupling $\Gamma_1 \otimes \Gamma_2 \to \rho \Gamma$, then the relations obtained from (11) constitute a linear, homogeneous system of equations in N unknowns for these coupling coefficients.

Note that the most familiar and traditional approach to extracting coupling coefficients, after obtaining some set of relations among them, is to proceed by recurrence (e.g., in the familiar case of $SO(3)^{19}$ and in the "building-up process", ¹⁸ as well as in prior applications of Racah's method to higher algebras^{23,28}). That is, a seed value is given for one coupling coefficient, and further coefficients are deduced inductively (one by one) from those already obtained.

A recurrence approach is indeed natural in the case of SO(3). The relations obtained by considering the actions of J_{+} are

$$K_{\pm}(JM)\left(\begin{smallmatrix} J_1 & J_2 \\ M_1 & M_2 \end{smallmatrix}\right|_{M\pm 1}^{J}\right) = K_{\pm}(J_1M_1 \mp 1)\left(\begin{smallmatrix} J_1 & J_2 \\ M_1 \mp 1 & M_2 \end{smallmatrix}\right|_{M}^{J}\right) + K_{\pm}(J_2M_2 \mp 1)\left(\begin{smallmatrix} J_1 & J_2 \\ M_1 & M_2 \mp 1 \end{smallmatrix}\right|_{M}^{J}\right), (13)$$

where $M = M_1 + M_2 \mp 1$, and $K_{\pm}(JM) \equiv \langle JM \pm 1 \mid J_{\pm} \mid JM \rangle = [(J \mp M)(J \pm M + 1)]^{1/2}$ is the generator matrix element. These relations connect at most three coupling coefficients, and a natural order for traversing the coefficients can easily be chosen, such that only one unknown arises at each step, as illustrated in Fig. 1(a). [This is accomplished by involving certain known-zero or "forbidden" Clebsch-Gordan coefficients, represented by the triangle vertices without dots in Fig. 1(a), in the relations.] Since classic treatments (e.g., Ref. 31) apply orthonormality relations interspersed with the recurrence relations at intermediate stages of the calculation, we stress that recourse to orthonormality conditions is not actually necessary.

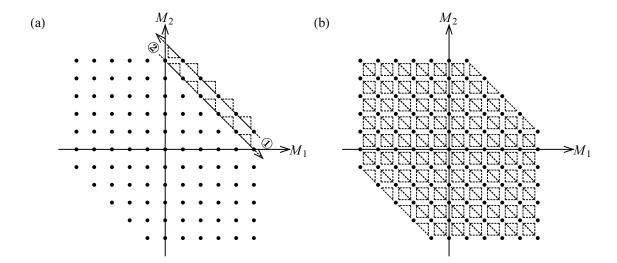


FIG. 1: The classic problem of constructing the SO(3) Clebsch-Gordan coefficients $\begin{pmatrix} J_1 & J_2 \\ M_1 & M_2 \end{pmatrix} \begin{vmatrix} J \\ M_1+M_2 \end{pmatrix}$ by use of the relations (13). Dots indicate allowed non-zero coefficients. Coefficients at the vertices of a dashed triangle are connected by (13). (a) The conventional recurrence approach, in which coefficients are calculated inductively from a seed coefficient, making use of relations which in some cases invoke known-zero Clebsch-Gordan coefficients. (b) A full set of relations among allowed Clebsch-Gordan coefficients, yielding a linear, homogeneous system of equations in the Clebsch-Gordan coefficients.

It may be seen from the figure that all coefficients are accessible by the relations (13). In anticipation of the treatment of higher algebras, we also observe that all allowed coefficients may be connected by the relations directly, without involving any forbidden coefficients, as in Fig. 1(b). This yields a system of equations which fully determines the coefficients, to within an overall phase and normalization, although in this case the system is not amenable to solution by recursive calculation of successive coefficients from a single seed coefficient.

For higher algebras, many irreps of H may be connected by the generator $T^{(\Lambda_T)}$, and therefore each relation obtained from (11) may involve many unknown coupling coefficients. A simple recurrence pattern, as in Fig. 1(a), may be impractical to devise. A more generally applicable and straightforward approach is to directly solve the linear, homogeneous system of equations for the unknown coupling coefficients, by standard linear algebraic methods, e.g., Euler row reduction.³²

Let us therefore summarize the system of equations which must be constructed and solved. The N unknown coupling coefficients for $\Gamma_1 \otimes \Gamma_2 \to \rho \Gamma$ may be labeled with a single counting index as $C_i \equiv \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix} \begin{pmatrix} \rho \Gamma \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix}$ with $i = 1, \ldots, N$. Each value of the index i therefore designates a specific combination $(a_1 \Lambda_1 a_2 \Lambda_2 a \Lambda)$. The numerical coefficients of the unknown quantities C_i in the relations (11) do not depend upon the outer multiplicity index ρ , to be discussed further below. For each generator $T^{(\Lambda_T)}$ in G but not in H, and for each quadruplet of irrep labels $(a_1 \Lambda_1 a_2 \Lambda_2 \Lambda a' \Lambda')$, the relation (11) yields an equation (which we label by a counting index k) of the form $\sum_{i=1}^{N} a_{ki} C_i = 0$, i.e., linear and homogeneous in the C_i . The equation is nonnull (i.e., some of the coefficients a_{ki} are nonvanishing) only if the Clebsch-Gordan series conditions $\Lambda_1 \otimes \Lambda_2 \to \Lambda$ and $\Lambda' \otimes \Lambda_T \to \Lambda$ are met.³³

The resulting equations must be aggregated to yield the full system, which may be ex-

pressed in matrix form as

Normally, it suffices to consider the conditions obtained with $(\Lambda_1\Lambda_2\Lambda\Lambda')$ such that $\Gamma_1 \to \Lambda_1$, $\Gamma_2 \to \Lambda_2$, $\Gamma \to \Lambda$, and $\Gamma \to \Lambda'$, that is, relations involving only "allowed" coupling coefficients, as in Fig. 1(b). However, in certain exceptional cases,³⁴ additional conditions involving known-zero coupling coefficients may be necessary, analogous to Fig. 1(a). These may be obtained, e.g., by considering some Λ' with $\Gamma \to \Lambda'$.

The problem of solving this linear homogeneous system of equations (14) is equivalent to finding the null vector (or vectors) of the matrix A appearing on the left hand side of (14). In general, there may be many more rows (equations) than columns (unknown coupling coefficients). However, these rows are not linearly independent. In the case where $\Gamma_1 \otimes \Gamma_2 \to \Gamma$ is free of outer multiplicity, the matrix A can be expected to be of rank N-1. The null vector is then uniquely determined, to within normalization and phase, and its entries are the coupling coefficients $[C_1 C_2 \cdots C_N]$. The proper normalization, yielding coefficients satisfying the condition (4), is obtained by evaluating

$$\mathcal{N}^2 \equiv \sum_{\substack{i \text{ (same } a\Lambda)}} C_i^2. \tag{15}$$

That is, the summation runs over the subset of entries C_i sharing the same value for $a\Lambda$. An identical result for \mathcal{N} must be obtained, regardless of the choice of $a\Lambda$, provided $\Gamma \to a\Lambda$. (In fact, the requirement of equality may be used as an internal consistency check on the calculation.) The normalized coupling coefficients are then obtained by dividing the null vector by \mathcal{N} .

An overall sign remains to be chosen for the entire set of coupling coefficients for $\Gamma_1 \otimes \Gamma_2 \to \Gamma$. For instance, Refs. 23,24 suggest a "generalized Condon-Shortley phase convention", such that $\binom{\Gamma_1 \quad \Gamma_2}{\Lambda_{1m} \quad a_2 \tilde{\Lambda}_2} \binom{\Gamma}{\Lambda_m} > 0$, that is, a positive value is adopted for the coupling coefficient involving the highest weight irreps of H contained in Γ_1 and Γ and the highest weight irrep $\tilde{\Lambda}_2$ consistent with these.

More generally, when the coupling $\Gamma_1 \otimes \Gamma_2 \to \Gamma$ has outer multiplicity D ($\rho = 1, \ldots, D$), the matrix A may be expected to be of rank N-D. That is, the system of equations given by (14) yields D linearly independent null vectors (or its null space has dimension D). The null vectors obtained by Euler row reduction must be orthonormalized, 35 e.g., by the Gram-Schmidt procedure, to yield a set of coupling coefficients satisfying the orthonormality relation (4). Note that the appropriate inner product for this orthonormalization is not the standard vector dot product on \mathbb{R}^N . Rather, if we label the entries of each null vector \mathbf{C}_{ρ} as $[C_{\rho 1} C_{\rho 2} \cdots C_{\rho N}]$, then the inner product to be used for orthonormalization is

$$\mathcal{M}_{\rho'\rho} \equiv \sum_{\substack{i \text{(same } a\Lambda)}} C_{\rho'i} C_{\rho i}. \tag{16}$$

The same value of $\mathcal{M}_{\rho'\rho}$ is obtained regardless of the choice of $a\Lambda$ used in evaluating the sum. (Again, requirement of this equality provides an internal consistency check on the calculation.) The orthonormal coupling coefficients are then simply the entries of the orthonormalized null vectors.

When an outer multiplicity is present, it should be noted that the coupling coefficients are defined only to within a unitary transformation, arising from the arbitrariness inherent in defining the resolution of the outer multiplicity, *i.e.*, in choosing the basis states $|\rho\Gamma\cdots\rangle$ ($\rho=1,\ldots,D$) spanning the *D*-dimensional space of irreps of type Γ . In the present calculational procedure, the freedom in resolution of the multiplicity is manifested in the freedom to choose different sets of orthogonal basis vectors for the null space of A.

III. COUPLING COEFFICIENTS FOR SO(5) IN THE CANONICAL BASIS

A. Overview

For a concrete example of the application of Racah's method in terms of reduced coupling coefficients, as developed in Sec. II, we consider the calculation of coupling coefficients for SO(5), reduced with respect to the canonical subalgebra SO(4). That is, we have $SO(5) \supset SO(4)$ as the algebras $G \supset H$. Both essential criteria for application of the method are met: (1) the coupling and recoupling coefficients (Wigner calculus) for SO(4) are known,³⁶ and (2) the reduced matrix elements of the SO(5) generators, considered as tensor operators with respect to SO(4), are also known.^{23,37}

The algebra SO(5) contains several subalgebra chains, involving distinct $SO(3) \sim SU(2)$ subalgebras,

$$SO(5) \supset SO(4) \supset SO_{J}(3) \supset SO_{J}(2)$$

$$\supset SO(4) \sim SO_{X}(3) \otimes SO_{Y}(3) \supset SO_{X}(2) \otimes SO_{Y}(2)$$

$$\supset U_{N}(1) \otimes SO_{T}(3) \supset SO_{T}(2)$$

$$\simeq SO_{L}(3) \supset SO_{L}(2)$$

$$\simeq SO_{L}(3) \supset SO$$

where the irrep label has been noted beneath each subalgebra. Branching multiplicity labels are indicated by κ and α in the last two chains. Chain (I) is the standard canonical chain, while in (I') the canonical SO(4) subalgebra is reexpressed using the isomorphism SO(4) \sim SO(3) \otimes SO(3). [As far as definition of reduced coupling coefficients is concerned, the two chains (I) and (I') are equivalent, but branching rules, coupling coefficients, etc., are simpler when expressed with respect to the latter chain (I').] The prerequisite definitions and algebraic results are summarized in Sec. III B, and the calculation of reduced coupling coefficients for the canonical chain is discussed in Sec. III C.

Physical applications require the coupling coefficients of SO(5) reduced with respect to the noncanonical subalgebras of chains (II) and (III). The isospin algebra $SO_T(3)$ of chain (II) is the relevant subalgebra for the description of proton-neutron pairing.^{5-9,38-40} In this context, the SO(5) generators arise as quasispin operators for pairing of protons and neutrons occupying the same j-shell. On the other hand, the "physical" or "geometric" angular momentum subalgebra $SO_L(3)$ of chain (III) is the relevant subalgebra for application to

systems of spin-2 bosons^{10,11} or the nuclear collective model.^{12–14,16} Explicit constructions of these subalgebras and further algebraic properties for the noncanonical chains are detailed in Appendix C, where the transformation between canonical and noncanonical bases is considered.

B. Definitions and algebraic properties

Let us begin with a concise but comprehensive summary of the construction of SO(5) and the algebraic properties needed for the application of Racah's method. Such a review is particularly necessary since notations and conventions for nearly all aspects of the treatment of SO(5) vary widely (e.g., Refs. 18,23,37,41), and phases and normalizations play an essential role in the calculation of coupling coefficients.

The basic construction proceeds from the generators of rotation,

$$L_{rs} \equiv -i(x_r \partial_s - x_s \partial_r). \tag{18}$$

These operators are Hermitian $(L_{rs}^{\dagger} = L_{rs})$, are antisymmetric in the indices, and have commutators

$$[L_{pq}, L_{rs}] = -i(\delta_{qr}L_{ps} + \delta_{ps}L_{qr} + \delta_{sq}L_{rp} + \delta_{rp}L_{sq}).$$
(19)

First, for SO(4), let $J_r \equiv \frac{1}{2}\varepsilon_{rst}L_{st}$ and $N_r \equiv L_{r4}$ (1 \le r, s, t \le 3), i.e.,

$$J_1 = L_{23}$$
 $J_2 = L_{31}$ $J_3 = L_{12}$
 $N_1 = L_{14}$ $N_2 = L_{24}$ $N_3 = L_{34}$. (20)

Then the J_r span the usual three-dimensional angular momentum algebra, which we denote by $SO_J(3)$. The J_r and N_r together span SO(4), with commutators $[J_r, J_s] = i\varepsilon_{rst}J_t$, $[N_r, N_s] = i\varepsilon_{rst}J_t$, and $[J_r, N_s] = i\varepsilon_{rst}N_t$. The standard Cartan weight operators for SO(4) are J_3 and N_3 . An SO(4) irrep is labeled by the highest weight defined by these operators, which is of the form [pq], with $p \geq |q|$, both integer or both odd half integer.

The isomorphism $SO(4) \sim SO_X(3) \otimes SO_Y(3)$ is realized by taking

$$X_k \equiv \frac{1}{2}(J_k + N_k) \quad Y_k \equiv \frac{1}{2}(J_k - N_k),$$
 (21)

so $[X_r,X_s]=i\varepsilon_{rst}X_t, [Y_r,Y_s]=i\varepsilon_{rst}Y_t$, and $[X_r,Y_s]=0$. The ladder operators for each SO(3) algebra are thus $X_\pm\equiv X_1\pm iX_2$ and $Y_\pm\equiv Y_1\pm iY_2$. The natural Cartan weight operators in this scheme are then the SO(3) angular momentum projections $X_0\equiv X_3$ and $Y_0\equiv Y_3$, defining weight labels M_X and M_Y . An SO(4) irrep is then labeled by the highest weight (XY), *i.e.*, the angular momenta associated with the SO_X(3) and SO_Y(3) subalgebras. The SO_X(3) \otimes SO_Y(3) irrep labels are related to the standard SO(4) labels by $X=\frac{1}{2}(p+q)$ and $Y=\frac{1}{2}(p-q)$ or, conversely, [p,q]=[X+Y,X-Y]. Note that the canonical SO_J(3) is obtained as the sum angular momentum algebra of SO_X(3) and SO_Y(3), since $J_k=X_k+Y_k$, and thus the basis states reducing chains (I) and (I') are related to each other by ordinary angular momentum coupling.

The Clebsch-Gordan series and coupling and recoupling coefficients for SO(4) follow immediately from the SO(3) \otimes SO(3) structure,³⁶ most transparently with the (XY) labeling scheme for the irreps. The weights contained within (XY) are $M_X = -X, \ldots, X-1, X$ and $M_Y = -Y, \ldots, Y-1, Y$. The Clebsch-Gordan series is given by application of the triangle

inequality separately to each of the SO(3) algebras, that is, for $(X_1Y_1) \otimes (X_2Y_2) \to (XY)$, $X = |X_1 - X_2|, |X_1 - X_2| + 1, \dots, X_1 + X_2$ and $Y = |Y_1 - Y_2|, |Y_1 - Y_2| + 1, \dots, Y_1 + Y_2$. Hence, no inner or outer multiplicities are obtained for SO(4). Coupling coefficients factorize into products of ordinary SO(3) Clebsch-Gordan coefficients, as

$$\begin{pmatrix} (X_1Y_1) & (X_2Y_2) \\ M_{X1}M_{Y1} & M_{X2}M_{Y2} \end{pmatrix} \begin{pmatrix} (XY) \\ M_{X}M_{Y} \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ M_{X1} & M_{X2} \end{pmatrix} \begin{pmatrix} X_1 & Y_2 \\ M_{Y1} & M_{Y2} \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ M_{Y1} & M_{Y2} \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ M_{Y1} & M_{Y2} \end{pmatrix}.$$
(22)

By inspection of (10), it is immediately apparent that the recoupling coefficients factorize as well, as

$$\begin{bmatrix}
(X_1Y_1) & (X_2Y_2) & (X_{12}Y_{12}) \\
(X_3Y_3) & (XY) & (X_{23}Y_{23})
\end{bmatrix} = \begin{bmatrix}
X_1 & X_2 & X_{12} \\
X_3 & X & X_{23}
\end{bmatrix} \begin{bmatrix}
Y_1 & Y_2 & Y_{12} \\
Y_3 & Y & Y_{23}
\end{bmatrix}
= (-)^{X_1 + X_2 + X_3 + X} (-)^{Y_1 + Y_2 + Y_3 + Y} \hat{X}_{12} \hat{X}_{23} \hat{Y}_{12} \hat{Y}_{23} \begin{cases}
X_1 & X_2 & X_{12} \\
X_3 & X & X_{23}
\end{cases} \begin{cases}
Y_1 & Y_2 & Y_{12} \\
Y_3 & Y & Y_{23}
\end{cases}. (23)$$

An equivalent result is given with standard SO(4) labels in Ref. 42. However, note that the result is considerably more cumbersome to derive if one uses the standard canonical chain (I).⁴³

The algebra SO(5) includes the additional four generators L_{r5} (r = 1, ..., 4). A tensor operator with respect to SO(4) is simply a simultaneous spherical tensor with respect to both the SO_X(3) and SO_Y(3) algebras, *i.e.*, a spherical "bitensor". For the SO(5) generators, we have bitensor expressions⁴⁴

$$X_{\pm 10}^{(10)} \equiv X_{\pm 1} = \mp \frac{1}{2\sqrt{2}} [(L_{23} + L_{14}) \pm i(L_{31} + L_{24})] \qquad X_{00}^{(10)} \equiv X_0 = \frac{1}{2} (L_{12} + L_{34})$$

$$Y_{0\pm 1}^{(01)} \equiv Y_{\pm 1} = \mp \frac{1}{2\sqrt{2}} [(L_{23} - L_{14}) \pm i(L_{31} - L_{24})] \qquad Y_{00}^{(01)} \equiv Y_0 = \frac{1}{2} (L_{12} - L_{34})$$

$$T_{\pm \frac{1}{2} + \frac{1}{2}}^{(\frac{1}{2} \frac{1}{2})} \equiv T_{++} = -\frac{1}{2} (L_{15} + iL_{25}) \qquad T_{\pm \frac{1}{2} - \frac{1}{2}}^{(\frac{1}{2} \frac{1}{2})} \equiv T_{+-} = \frac{1}{2} (L_{35} + iL_{45})$$

$$T_{-\frac{1}{2} + \frac{1}{2}}^{(\frac{1}{2} \frac{1}{2})} \equiv T_{--} = \frac{1}{2} (L_{15} - iL_{25}).$$

$$T_{-\frac{1}{2} - \frac{1}{2}}^{(\frac{1}{2} \frac{1}{2})} \equiv T_{--} = \frac{1}{2} (L_{15} - iL_{25}).$$

The phases are chosen so that these operators obey $A_{M_XM_Y}^{(XY)\dagger}=(-)^{M_X+M_Y}A_{-M_X-M_Y}^{(XY)}$, a generalization of the usual condition for a self-adjoint spherical tensor. ¹⁹ All commutators involving X_{μ} or Y_{μ} have the values implied by the spherical bitensor notation of (24), e.g., $[X_{\pm 1}, A_{\mu\mu'}^{(\lambda\lambda')}] = \mp [\frac{1}{2}(\lambda \mp \mu)(\lambda \pm \mu + 1)]^{1/2}A_{(\mu\pm 1)\mu'}^{(\lambda\lambda')}$. The commutators between components of $T^{(1/2\,1/2)}$ are given explicitly in Table I.

The root vector diagram of SO(5) is shown for reference in Fig. 2(a), with the generators (24) placed according to their $SO_X(3) \otimes SO_Y(3)$ weights $(M_X M_Y)$. The canonical subalgebra is highlighted in Fig. 2(b), and the construction of the physical subalgebras, of chains (I) and (III), is indicated in Fig. 2(c,d) (see Appendix C).

The standard Cartan highest weight labels for an SO(5) irrep are defined with respect to weight operators J_3 and N_3 and have the form $[l_1l_2]$, with $l_1 \geq l_2$, both integer or both odd half integer. It is more convenient in the present context to label SO(5) irreps by the highest weight defined by the $SO_X(3) \otimes SO_Y(3)$ weight operators X_0 and Y_0 (following Hecht²³). The resulting label has the form (RS), with $R \geq S$, each independently either integer or odd half integer. This label may also be considered as representing the angular momenta $(X_m Y_m)$

TABLE I: Commutation relations between the components of $T^{(1/2 \, 1/2)}$, for the SO(5) generator normalization and phase conventions defined in (24).

	T_{++}	T_{+-}	T_{-+}	$T_{}$
T_{++}	0	$-\frac{1}{\sqrt{2}}X_{+1}$	$-\frac{1}{\sqrt{2}}Y_{+1}$	$-\frac{1}{2}(X_0+Y_0)$
T_{+-}	$+\frac{1}{\sqrt{2}}X_{+1}$	0	$+\frac{1}{2}(\dot{X}_0-Y_0)$	$-\frac{1}{\sqrt{2}}Y_{-1}$
T_{-+}	$+\frac{1}{\sqrt{2}}Y_{+1}$	$-\frac{1}{2}(X_0-Y_0)$	0	$-\frac{1}{\sqrt{2}}X_{-1}$
$T_{}$	$+\frac{1}{2}(X_0 + Y_0)$	$+\frac{1}{\sqrt{2}}Y_{-1}$	$+\frac{1}{\sqrt{2}}X_{-1}$	0

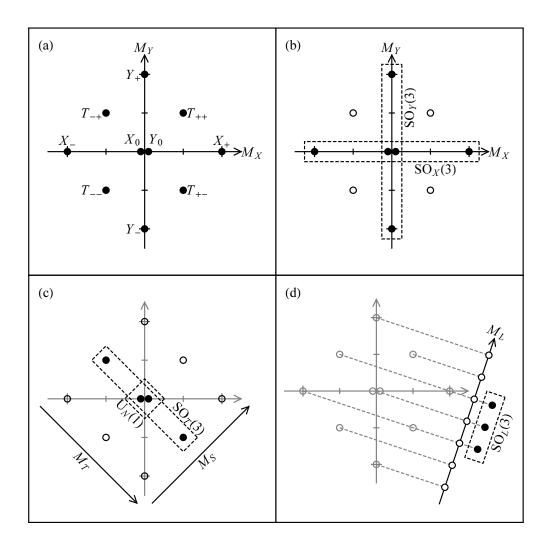


FIG. 2: Root vector diagram for SO(5) and its subalgebras. (a) The generators of SO(5), labeled by their Cartan weights M_X and M_Y . (b) The canonical subalgebra SO(4) \sim SO_X(3) \otimes SO_Y(3), which begins chains (I) and (I'). (c) The U_N(1) \otimes SO_T(3) subalgebra of chain (II). (d) The physical angular momentum SO_L(3) subalgebra of chain (III). These generators are obtained as linear combinations of the canonical generators, as indicated by the dashed lines, such that all generators have good SO_L(3) \supset SO_L(2) tensorial character.

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Labels	Range	Relations	Description	Refs.
$[l_1l_2]$	$l_1 = l_2, l_2 + 1, \dots$	_	SO(5) Cartan highest weight	6,37
	$l_2 = 0, \frac{1}{2}, \dots$			
(a_1a_2)	$a_1=0,\bar{1},\ldots$	$a_1 = l_1 - l_2$	SO(5) Dynkin	46
	$a_2 = 0, 1, \dots$	$a_2 = 2l_2$		
(vf)	$v=0,1,\ldots$	$v = l_1 - l_2$	SO(5) Dynkin (modified)	47
	$f = 0, \frac{1}{2}, \dots$	$f = l_2$		
$\langle l_1' l_2' \rangle$	$l_1' = l_2', \overline{l_2'} + 1, \dots$	$l_1' = l_1 + l_2$	Sp(4) Cartan highest weight	$5,\!45$
	$l_2' = 0, 1, \dots$	$l_2' = l_1 - l_2$		
$(a_{1}^{\prime}a_{2}^{\prime})$	$a_1' = 0, 1, \dots$	$a_1'=2l_2$	Sp(4) Dynkin	8,41
	$a_2' = 0, 1, \dots$	$a_2' = l_1 - l_2$		
(RS)	$R = S, S + \frac{1}{2}, \dots$	$R = \frac{1}{2}(l_1 + l_2)$	$SO(3) \otimes SO(3)$ highest weight	23,48
	$S=0,\frac{1}{2},$	$S = \frac{1}{2}(l_1 - l_2)$		

TABLE II: Labeling schemes for irreps of SO(5) in use in the physics literature, with relations for interconversion.

of the highest weight $SO_X(3) \otimes SO_Y(3)$ irrep contained in the SO(5) irrep. The relation to the standard labels is $R = \frac{1}{2}(l_1 + l_2)$ and $S = \frac{1}{2}(l_1 - l_2)$. A plethora of labeling schemes for SO(5) irreps are in use in the physics literature, interrelated as summarized in Table II (even more schemes arise if we consider the translation to physical labels, such as reduced isospin⁴⁵).

The branching rule for SO(5) to SO(4), i.e., $(RS) \rightarrow (XY)$, is given by^{23,37}

$$X = R - \frac{1}{2}n - \frac{1}{2}m$$

$$Y = S + \frac{1}{2}n - \frac{1}{2}m,$$
(25)

with $0 \le n \le 2(R-S)$ and $0 \le m \le 2S$, m and n integers. Graphically, the SO(4) irreps form a lattice bounded by a tilted rectangle, as illustrated by the dotted line in Fig. 3(c). The rectangle's "right" corner is at the highest weight $(RS) = (X_m Y_m)$, its "bottom" corner lies on the M_X axis, and the remaining two corners are specified by symmetry about the line $M_X = M_Y$. The branching rule is shown for example irreps of SO(5) in Fig. 3: symmetric [Fig. 3(a)], antisymmetric [Fig. 3(b)], and generic [Fig. 3(c)].

The Clebsch-Gordan series for SO(5) may be obtained by a relatively efficient elementary approach based on the method of weights, but "reduced" with respect to SO(4). Since the SO(5) \rightarrow SO(4) branching rules (25) are known, as is the SO(4) Clebsch-Gordan series, the tabulation of weights can be replaced by tabulation of SO(4) irrep labels, which then imply all the weights contained within these irreps. To decompose the SO(5) outer product $(R_1S_1) \otimes (R_2S_2)$, first the SO(4) irreps in the branchings $(R_1S_1) \rightarrow (X_1Y_1)$ and $(R_2S_2) \rightarrow (X_2Y_2)$ are enumerated. Then, for each pair of SO(4) irreps (X_1Y_1) and (X_2Y_2) , the product irreps $(X_1Y_1) \otimes (X_2Y_2) \rightarrow (XY)$ are enumerated. The aggregate set of these product irreps represents the SO(4) content of $(R_1S_1) \otimes (R_2S_2)$. The SO(5) content can now be extracted. Namely, the highest weight SO(4) label in the set gives the highest weight SO(5) irrep contained in $(R_1S_1) \otimes (R_2S_2)$ [which will, incidentally, simply be the sum of (R_1S_1) and (R_2S_2) as weights]. The SO(4) content of this SO(5) irrep can now be deleted from the set, after which the next highest remaining SO(4) label gives the next highest weight SO(5) irrep in $(R_1S_1) \otimes (R_2S_2)$, etc. The process is repeated until the set of SO(4) irreps has been

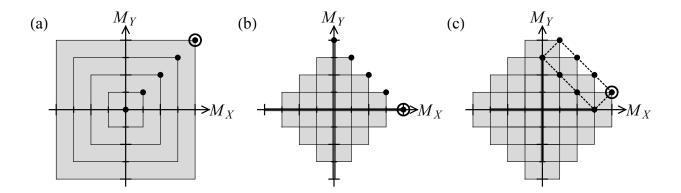


FIG. 3: Branching diagrams for $SO(5) \supset SO(4)$, shown for (a) the symmetric irrep (22), (b) the antisymmetric irrep (20), and (c) a representative generic irrep $(\frac{3}{2}\frac{1}{2})$. The open circle indicates the highest weight for the SO(5) irrep, and the solid dots indicate SO(4) highest weights, according to branching rule (25). The shaded rectangles are the boundaries of the weight sets for these SO(4) irreps. The dashed rectangle in panel (c) is the boundary of the SO(4) highest weight set, as discussed in the text.

exhausted. The Clebsch-Gordan series for SO(5) may also be obtained by group character methods (see Refs. 18,49).

The remaining ingredients needed for application of Racah's method are the SO(4)-reduced matrix elements of the "additional" generators of SO(5) not contained in SO(4), i.e., $T^{(1/2\,1/2)}$. These were obtained in closed form, by solving certain recurrence relations obtained from the commutators of the algebra, by Hecht²³ and by Kemmer, Pursey, and Williams,³⁷ as

$$\left\langle \binom{(RS)}{(X+\frac{1}{2}Y+\frac{1}{2})} \right\| T^{(\frac{1}{2}\frac{1}{2})} \right\| \binom{(RS)}{(XY)} \right\rangle = \frac{\left[(R+S-X-Y)(R+S+X+Y+3) + (R+S+X+Y+2) \right]^{1/2}}{\frac{2X+\frac{1}{2}Y+\frac{1}{2}}{2}} \left[(R+S-X+Y+1)(R+S+X-Y+2) + (R+S+X+Y+1)(R+S+X-Y+2) + (R+S+X+Y+1)(R+S+X-Y+1) \right]^{1/2}}{\frac{2X+\frac{1}{2}Y-\frac{1}{2}}{2X+\frac{1}{2}Y-\frac{1}{2}}}.$$
(26)

These expressions are appropriate to the Wigner-Eckart theorem normalization defined in (6) and the normalization of $T^{(1/2\,1/2)}$ defined by (24). The remaining matrix elements, connecting (XY) with $(X-\frac{1}{2}\,Y+\frac{1}{2})$ or $(X-\frac{1}{2}\,Y-\frac{1}{2})$, follow from these by the self-adjoint property of the generators,³⁷ as

$$\left\langle \binom{(RS)}{(XY)} \middle\| T^{(\frac{1}{2}\frac{1}{2})} \middle\| \binom{(RS)}{(X'Y')} \right\rangle = \frac{\hat{X}'\hat{Y}'}{\hat{X}\hat{Y}} (-)^{X-X'+Y-Y'} \left\langle \binom{(RS)}{(X'Y')} \middle\| T^{(\frac{1}{2}\frac{1}{2})} \middle\| \binom{(RS)}{(XY)} \right\rangle. \tag{27}$$

C. Calculation of $SO(5) \supset SO(4)$ coupling coefficients

Consider calculation of the set of SO(5) \supset SO(4) reduced coupling coefficients for $(R_1S_1)\otimes$ $(R_2S_2) \rightarrow (RS)$. It is now straightforward to construct the terms appearing in the Racah

condition (11), by use of the chain (I') branching rules, SO(4) Clebsch-Gordan series, SO(4) Wigner calculus, and SO(4)-reduced matrix elements of $T^{(1/2 \, 1/2)}$, compiled in the preceding section. For SO(5) \supset SO(4), the relation (11) may be written

$$\left\langle \begin{pmatrix} (RS) \\ (XY) \end{pmatrix} \middle| T^{(\frac{1}{2}\frac{1}{2})} \middle| \begin{pmatrix} (RS) \\ (X'Y') \end{pmatrix} \right\rangle \left(\begin{pmatrix} (R_1S_1) & (R_2S_2) \\ (X_1Y_1) & (X_2Y_2) \end{pmatrix} (RS) \right)
= \sum_{(X_1'Y_1')} \Phi[(X_1Y_1)(X_2Y_2); (XY)] \Phi[(X_1'Y_1')(X_2Y_2); (X'Y')]
\times \left[\begin{pmatrix} (X_2Y_2) & (X_1'Y_1') & (X'Y') \\ (\frac{1}{2}\frac{1}{2}) & (XY) & (X_1Y_1) \end{pmatrix} \right] \left\langle \begin{pmatrix} (R_1S_1) \\ (X_1Y_1) \end{pmatrix} \middle| T^{(\frac{1}{2}\frac{1}{2})} \middle| \begin{pmatrix} (R_1S_1) \\ (X_1'Y_1') \end{pmatrix} \left\langle \begin{pmatrix} (R_1S_1) & (R_2S_2) \\ (X_1'Y_1') & (X_2Y_2) \end{pmatrix} \middle| (X'Y') \right)
+ \sum_{(X_2'Y_2')} \left[\begin{pmatrix} (X_1Y_1) & (X_2'Y_2') & (X'Y') \\ (\frac{1}{2}\frac{1}{2}) & (XY) & (X_2Y_2) \end{pmatrix} \middle| T^{(\frac{1}{2}\frac{1}{2})} \middle| \begin{pmatrix} (R_2S_2) \\ (X_2'Y_2') \end{pmatrix} \middle| \begin{pmatrix} (R_1S_1) & (R_2S_2) \\ (X_1Y_1) & (X_2'Y_2') \end{pmatrix} \middle| (X'Y') \right). \tag{28}$$

The system of equations for the coupling coefficients is assembled following the approach of Sec. II C. A different condition is obtained from (28) for each quadruplet of SO(4) irreps (X_1Y_1) , (X_2Y_2) , (XY), and (X'Y'), chosen from the branchings $(R_1S_1) \to (X_1Y_1)$, $(R_2S_2) \to (X_2Y_2)$, $(RS) \to (XY)$, and $(RS) \to (X'Y')$ [for couplings involving the identity irrep (00), see endnote 34]. A nonnull condition on the coupling coefficients is obtained only if $(X_1Y_1) \otimes (X_2Y_2) \to (XY)$ and $(X'Y') \otimes (\frac{1}{2}\frac{1}{2}) \to (XY)$. Since all quantities involved in the conditions (28) are known exactly, in the form of square roots of rational numbers, and since Euler row reduction can be carried out in exact (*i.e.*, symbolic) arithmetic, all coupling coefficients can be obtained exactly through the present process, again as (signed) square roots of rational numbers.

Two concrete numerical examples are provided as illustrations of the method in Appendix B. A simple low-dimensional example is provided by the coupling $(\frac{1}{2}\frac{1}{2})\otimes(\frac{1}{2}0)\to(\frac{1}{2}0)$, which involves only a 4×4 coefficient matrix, and an example involving an outer multiplicity is provided by the coupling $(10)\otimes(1\frac{1}{2})\to(1\frac{1}{2})$, for which the coefficient matrix has dimensions 36×18 . (In the canonical labeling scheme, these examples are $[10]\otimes[\frac{1}{2}\frac{1}{2}]\to[\frac{1}{2}\frac{1}{2}]$ and $[11]\otimes[\frac{3}{2}\frac{1}{2}]\to[\frac{3}{2}\frac{1}{2}]$, respectively.)

D. Noncanonical chains

In the previous section, it was seen how the coupling coefficients for SO(5) reduced with respect to the *canonical* chain may be evaluated by Racah's method, as formulated in Sec. II. The necessary ingredients take on a particularly simple form for the canonical chain, in that the reduced matrix elements of the generators are given by closed form expressions (26). However, the matrix elements of the generators reduced with respect to the noncanonical isospin subalgebra [chain (II)] and physical angular momentum algebra [chain (III)] are also known. Recurrence relations for the reduced matrix elements have been obtained from vector coherent state realizations, ^{47,50} or an elementary construction has also been demonstrated for chain (II). Hence, Racah's method may be applied directly to the calculation of chain (II) and chain (III) reduced coupling coefficients.

Alternatively, once coupling coefficients for the canonical chain have been obtained, the coupling coefficients for the noncanonical chains can readily be deduced by a unitary transformation. For instance, the coupling coefficients of SU(3) reduced with respect to its

physical angular momentum subalgebra are conventionally obtained from the canonical $SU(3) \supset U(1) \otimes SU(2)$ coupling coefficients^{51,52} through such a process. The transformation brackets between basis states reducing the canonical and noncanonical chains are only known in closed form for a few special cases.⁹ However, they can be obtained in a straightforward fashion, either (1) by diagonalizing the appropriate Casimir operator, *i.e.*, \mathbf{T}^2 or \mathbf{L}^2 , in the canonical basis, as in Ref. 15, or (2) by a combination of laddering and orthogonalization operations. The procedure for transformation of coupling coefficients to either of the noncanonical chains is discussed in Appendix C.

IV. CONCLUSION

It has been shown that Racah's method of infinitesimal generators can be systematically generalized to the calculation of reduced coupling coefficients for an arbitrary subalgebra chain, provided the matrix elements of the generators (reduced with respect to the lower algebra) and the recoupling coefficients of the lower algebra are known. For the algebra SO(5), the problem of calculating coupling coefficients for generic irreps, reduced with respect to the canonical or noncanonical chains, is thereby completely resolved.

The specific example of SO(5) coupling coefficients may be considered as a prototype for the systematic calculation of coupling coefficients for other higher algebras. For instance, the computational machinery for SU(3) is well established^{51,52} and may therefore be used as the starting point for calculation of Sp(6) \supset U(3) reduced coupling coefficients, for the fermion dynamical symmetry model,⁵³ or Sp(6, \mathbb{R}) \supset U(3) reduced coupling coefficients, for the symplectic shell model.⁵⁴ The requisite generator matrix elements for Sp(6) and Sp(6, \mathbb{R}) may be calculated from vector coherent state realizations.^{55,56} The Sp(6, \mathbb{R}) \supset U(3) coupling coefficients are required, for instance, if large-scale calculations are to be carried out in the ab initio symplectic scheme of Dytrych et al.^{57,58}

Acknowledgments

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Appendix A: General relations in the presence of outer multiplicities for H

The derivation of Racah's method in terms of reduced quantities, as given in Sec. II, can readily be generalized to the case in which the subalgebra H has outer multiplicities, *i.e.*, its Kronecker product is not simply reducible. In this appendix, the necessary generalizations of the algebraic relations (e.g., Ref. 18) entering into the derivation of Sec. II are summarized, and the fundamental relation (11) for Racah's method is extended to incorporate outer multiplicities of H.

The general form of Racah's factorization lemma for $G \supset H$ is

$$\begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} = \sum_{\sigma} \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \sigma \Lambda \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \\ a \end{pmatrix} \begin{pmatrix} \rho \Gamma \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix} \sigma \Lambda, \tag{A1}$$

where σ is the multiplicity index for the coupling $\Lambda_1 \otimes \Lambda_2 \to \Lambda$. The reduced coupling coefficients satisfy orthonormality relations

$$\sum_{\substack{a_1\Lambda_1 a_2\Lambda_2\\\sigma}} \begin{pmatrix} \Gamma_1 & \Gamma_2\\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 & \Gamma_2\\ a_1\Lambda_1 & a_2\Lambda_2 \end{pmatrix} \begin{pmatrix} \rho'\Gamma'\\ a'\Lambda \end{pmatrix}_{\sigma} = \delta_{(\rho\Gamma)(\rho'\Gamma')}\delta_{aa'} \tag{A2}$$

and

$$\sum_{\rho\Gamma a} \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix} \rho \Gamma \\ a_1 \Lambda_1 & a_2 \Lambda_2 \end{pmatrix} \rho \Gamma \\ a_1 \Lambda_1' & a_2' \Lambda_1' & a_2' \Lambda_2' \\ a_1 \Lambda_1' & a_2' \Lambda_2' \\ a_1 \Lambda_2' \\ a_2 \Lambda_2' \\ a_1 \Lambda_2' \\ a_2 \Lambda_2'$$

for any irrep Λ such that $\Gamma \to \Lambda$ and, in the second relation, any values of the multiplicity indices σ and σ' , which resolve $\Lambda_1 \otimes \Lambda_2 \to \Lambda$ and $\Lambda'_1 \otimes \Lambda'_2 \to \Lambda$, respectively.

For the subalgebra H, the Wigner-Eckart theorem is of the form

$$\left\langle \begin{array}{c} \Gamma' \\ a'\Lambda' \\ \lambda' \end{array} \middle| T_{\lambda_T}^{\Lambda_T} \middle| \begin{array}{c} \Gamma \\ a\Lambda \\ \lambda \end{array} \right\rangle = \sum_{\sigma} \left(\begin{array}{c} \Lambda & \Lambda_T \\ \lambda & \lambda_T \end{array} \middle| \begin{array}{c} \sigma\Lambda' \\ \lambda' \end{array} \right) \left\langle \begin{array}{c} \Gamma' \\ a'\Lambda' \end{array} \middle\| T^{\Lambda_T} \middle\| \begin{array}{c} \Gamma \\ a\Lambda \end{array} \right\rangle_{\sigma}. \tag{A4}$$

The recoupling coefficients (unitary 6- Λ symbols) of H are

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & \sigma_{12}\Lambda_{12} \\ \Lambda_3 & \Lambda & \sigma_{23}\Lambda_{23} \end{bmatrix}_{\sigma\sigma'} = \sum_{\substack{\lambda_1\lambda_2\lambda_3 \\ \lambda_{12}\lambda_{13}}} \begin{pmatrix} \Lambda_1 & \Lambda_2 & \sigma_{12}\Lambda_{12} \\ \lambda_1 & \lambda_2 & \lambda_{12} \end{pmatrix} \begin{pmatrix} \Lambda_{12} & \Lambda_3 & \sigma_{\Lambda} \\ \lambda_{12} & \lambda_3 & \lambda_3 & \lambda_2 \end{pmatrix} \begin{pmatrix} \Lambda_2 & \Lambda_3 & \sigma_{23}\Lambda_{23} \\ \lambda_2 & \lambda_3 & \lambda_{23} & \lambda_{23} \end{pmatrix} \begin{pmatrix} \Lambda_1 & \Lambda_{23} & \sigma'_{\Lambda} \\ \lambda_1 & \lambda_{23} & \lambda_3 & \lambda_2 \end{pmatrix}.$$
(A5)

This represents the transformation bracket between basis states in the coupling schemes $[(\Lambda_1\Lambda_2)^{\sigma_{12}\Lambda_{12}}\Lambda_3]^{\sigma\Lambda}$ and $[\Lambda_1(\Lambda_2\Lambda_3)^{\sigma_{23}\Lambda_{23}}]^{\sigma'\Lambda}$. Under interchange of the first and second irreps, the coupling coefficients may be expected to satisfy a symmetry relation of the form [e.g., for SU(3), see Refs. 51,59]

$$\begin{pmatrix} \Lambda_2 & \Lambda_1 \\ \lambda_2 & \lambda_1 \end{pmatrix} \sigma \Lambda = \sum_{\sigma'} \Phi_{\sigma\sigma'}(\Lambda_2 \Lambda_1; \Lambda) \begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \sigma' \Lambda . \tag{A6}$$

Following the same arguments as in Sec. II, the reduced coupling coefficients for $G \supset H$ are found to satisfy a homogeneous system of equations of the form

$$\sum_{a} \left\langle \prod_{\alpha'} \prod_{\alpha$$

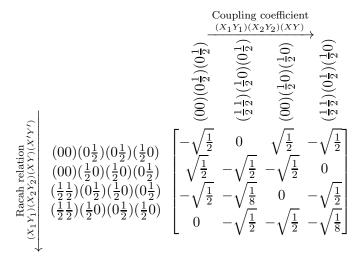


FIG. 4: Coefficient matrix for the linear, homogenous system of equations determining the SO(5) \supset SO(4) coupling coefficients for $(\frac{1}{2}\frac{1}{2}) \otimes (\frac{1}{2}0) \rightarrow (\frac{1}{2}0)$.

A different equation is obtained for each choice of $(a_1\Lambda_1a_2\Lambda_2\Lambda a'\Lambda')$ and indices σ and τ , where σ resolves $\Lambda_1 \otimes \Lambda_2 \to \Lambda$ and τ resolves $\Lambda' \otimes \Lambda_T \to \Lambda$.

Appendix B: Numerical examples for the calculation of $SO(5) \supset SO(4)$ reduced coupling coefficients

For a simple numerical example of the calculation of SO(5) \supset SO(4) reduced coupling coefficients, according to the methods of Sec. III C, consider the coupling $(\frac{1}{2}\frac{1}{2})\otimes(\frac{1}{2}0)\to(\frac{1}{2}0)$. The coupling coefficients to be obtained are $\binom{(1/21/2)}{(00)}\binom{(1/20)}{(01/2)}\binom{(1/20)}{(01/2)}$, $\binom{(1/21/2)}{(1/20)}\binom{(1/20)}{(1/20)}\binom{(1/21/2)}{(1/20)}\binom{(1/20)}{(1/20)}$, and $\binom{(1/21/2)}{(1/21)}\binom{(1/20)}{(1/20)}\binom{(1/20)}{(1/20)}$. The system of four equations in four unknowns obtained from (28) has the coefficient matrix shown in Fig. 4. This matrix is of rank 3, admitting the null vector

$$\left[-\frac{1}{2} -1 \mid \frac{1}{2} \mid 1 \right],$$

shown as a row vector for easier comparison with the coupling coefficient labels across the top of Fig. 4. It remains to obtain the proper normalization (and phase, if a phase convention is to be enforced). The vertical bar demarcates coupling coefficients involving the same product irrep (XY), $(0\frac{1}{2})$ for the first two coefficients and $(\frac{1}{2}0)$ for the remaining two coefficients, that is, the coefficients which appear together in the normalization sum (15). The squared norm, calculated using either pair of coefficients, is $\mathcal{N}^2 = \frac{5}{4}$. Hence, the normalized coupling coefficients are given by

$$\left[-\sqrt{\frac{1}{5}} - \sqrt{\frac{4}{5}} \mid \sqrt{\frac{1}{5}} \sqrt{\frac{4}{5}}\right],$$

which incidentally also conforms to the generalized Condon-Shortly phase convention.

One of the simplest cases in which an outer multiplicity occurs is in the coupling $(10) \otimes (1\frac{1}{2}) \to (1\frac{1}{2})$, which has multiplicity D=2. There are 18 coupling coefficients for these irreps, related by a system of 36 equations. The first few rows of the coefficient matrix are shown in Fig. 5(a). The matrix is of rank 16, admitting two null vectors, given in Fig. 5(b). Again, these are shown as row vectors, and groups of coefficients sharing the same (XY)

(a)
$$\begin{array}{c} \text{Coupling coefficient} \\ \text{Coupling coefficient} \\ (x_1 Y_1)(x_2 Y_2)(xY) \\ (x_1 Y_1)(x_1 Y_2)(x_1 Y_1) \\ (x_1 Y_1)(x_1 Y_1)(x_1 Y_1) \\ (x_1 Y_1)(x_1 Y_1)(x_1$$

FIG. 5: (a) Coefficient matrix for the linear, homogenous system of equations determining the SO(5) \supset SO(4) coupling coefficients for (10) \otimes (1 $\frac{1}{2}$) \rightarrow (1 $\frac{1}{2}$). The matrix has dimension 36 \times 18. Only the first few rows are shown. (b) Basis vectors for the null space (D=2). (c) Orthonormal coupling coefficient vectors, consisting of the SO(5) \supset SO(4) reduced coupling coefficients for $\rho=1$ (upper row) and $\rho=2$ (lower row).

are delimited by vertical bars. The inner product matrix (16), which may be obtained using any of these four groups of coefficients, is

$$\mathcal{M} = \begin{bmatrix} \frac{15}{8} & \sqrt{\frac{45}{32}} \\ \sqrt{\frac{45}{32}} & \frac{31}{4} \end{bmatrix}.$$

The orthonormal null vectors obtained by the Gram-Schmidt procedure with respect to \mathcal{M} are then given in Fig. 5(c). The entries of the upper and lower rows may be taken as the $SO(5) \supset SO(4)$ reduced coupling coefficients for $\rho = 1$ and 2, respectively.

Appendix C: Transformation of coupling coefficients

In this appendix, a general algorithm is outlined for determination of the transformation brackets between canonical and $U_N(1) \otimes SO_T(3)$ [chain (II)] bases of an SO(5) irrep. These are the necessary ingredients for deducing the chain (II) reduced coupling coefficients from the canonical reduced coupling coefficients. The analogous procedure for transformation to the $SO_L(3)$ basis [chain (III)] is also briefly considered.

First, let us review the relevant properties of the proton-neutron quasispin realization of SO(5) and the chain (II) basis and branching rules. For protons and neutrons occupying a single level of angular momentum j (degeneracy 2j + 1), consider the proton pair quasispin operators $(X_+, X_0, \text{ and } X_-)$, neutron pair quasispin operators $(Y_+, Y_0, \text{ and } Y_-)$, proton-neutron pair quasispin operators $(S_+, S_0, \text{ and } S_-)$, and isospin operators $(T_+, T_0, \text{ and } T_-)$. Each set spans an angular momentum algebra, which we denote by $SO_X(3)$, $SO_Y(3)$, $SO_S(3)$, or $SO_T(3)$, respectively. Explicitly,

$$X_{+} = \frac{1}{2}(a_{p}^{\dagger} \cdot a_{p}^{\dagger}) \qquad X_{0} = -\frac{1}{4}(a_{p}^{\dagger} \cdot \tilde{a}_{p} + \tilde{a}_{p} \cdot a_{p}^{\dagger}) \qquad X_{-} = -\frac{1}{2}(\tilde{a}_{p} \cdot \tilde{a}_{p})$$

$$Y_{+} = \frac{1}{2}(a_{n}^{\dagger} \cdot a_{n}^{\dagger}) \qquad Y_{0} = -\frac{1}{4}(a_{n}^{\dagger} \cdot \tilde{a}_{n} + \tilde{a}_{n} \cdot a_{n}^{\dagger}) \qquad Y_{-} = -\frac{1}{2}(\tilde{a}_{n} \cdot \tilde{a}_{n})$$

$$S_{+} = \frac{1}{2}(a_{p}^{\dagger} \cdot a_{n}^{\dagger} + a_{n}^{\dagger} \cdot a_{p}^{\dagger}) \qquad S_{0} = -\frac{1}{2}(a_{p}^{\dagger} \cdot \tilde{a}_{p} + \tilde{a}_{n} \cdot a_{n}^{\dagger}) \qquad S_{-} = -\frac{1}{2}(\tilde{a}_{n} \cdot \tilde{a}_{p} + \tilde{a}_{p} \cdot \tilde{a}_{n})$$

$$T_{+} = -(a_{p}^{\dagger} \cdot \tilde{a}_{n}) \qquad T_{0} = -\frac{1}{2}(a_{p}^{\dagger} \cdot \tilde{a}_{p} - a_{n}^{\dagger} \cdot \tilde{a}_{n}) \qquad T_{-} = -(a_{n}^{\dagger} \cdot \tilde{a}_{p}),$$
(C1)

where $\tilde{a}_{m}^{(j)} \equiv (-)^{j-m} a_{-m}^{(j)}$, and we define $A^{(j)} \cdot B^{(j)} \equiv \hat{\jmath}(A \times B)_{0}^{(0)} = \sum_{m} A_{m} \tilde{B}_{m}$ for half-integer j. Thus, for instance, $X_{+} = \frac{1}{2} \sum_{m} (-)^{j-m} a_{p,m}^{\dagger} a_{p,-m}^{\dagger}$, $X_{0} = \frac{1}{4} \sum_{m} (a_{p,m}^{\dagger} a_{p,m} - a_{p,m} a_{p,m}^{\dagger})$, and $X_{-} = \frac{1}{2} \sum_{m} (-)^{j-m} a_{p,-m} a_{p,m}$. There are only ten independent operators, since $S_{0} = X_{0} + Y_{0}$ and $T_{0} = X_{0} - Y_{0}$. The operators defined in (C1) obey the commutation relations of the SO(5) generators of Sec. III B, with the identifications

$$S_{+} = -2T_{++}$$
 $S_{-} = 2T_{--}$ $T_{+} = 2T_{+-}$ $T_{-} = 2T_{-+}$, (C2)

and thus span an SO(5) algebra.

The generators T_+ , T_0 , T_- , and S_0 span the $U_N(1) \otimes SO_T(3)$ subalgebra of SO(5), as shown in Fig. 2(c). Here $U_N(1)$ is the one-dimensional algebra of S_0 , so denoted in recognition of the relation of this operator to the total proton-neutron number operator $N = N_p + N_n$. The natural weights for chain (II), M_S and M_T , are related to the chain (I') weights by $M_S = M_X + M_Y$ and $M_T = M_X - M_Y$.⁶¹ For the proton-neutron quasispin realization of SO(5) defined in (C1), the weights are simply related to proton and neutron occupation numbers

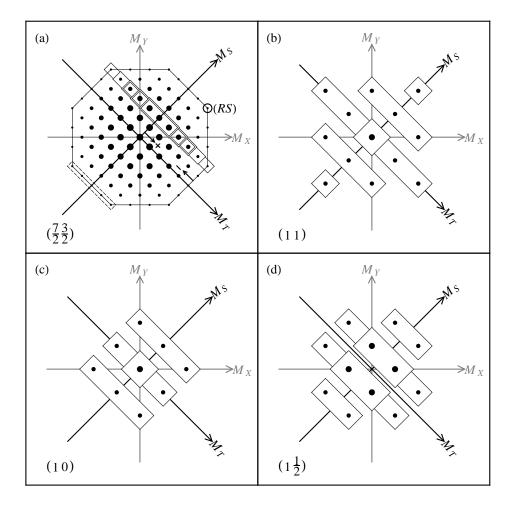


FIG. 6: Weight diagrams for SO(5) irreps, illustrating the decomposition into $U_N(1) \otimes SO_T(3)$ irreps, *i.e.*, the chain (II) branching. The area of each dot indicates the multiplicity of the weight point, and the rectangles indicate grouping of weight points into isospin multiplets. (a) The general characteristics, in the presence of branching multiplicites, are illustrated with the irrep $(\frac{7}{2}\frac{3}{2})$. The branching is obtained by decomposing the weights at a given value of M_S into multiplets, *e.g.*, for $M_S = +2$, in this example, $T = 1^2$, 2^2 , 3^2 , 4, and 5, where the exponents indicate branching multiplicities. The mathematical labeling schemes for the SO(5) irrep are based on the highest weight point (circle), but the (v,t) labeling scheme for pairing applications is based on the quantum numbers of the "pair vacuum" $U_N(1) \otimes SO_T(3)$ irrep (dashed box). The arrows indicate the action of the isospin laddering operators T_{\pm} considered in the algorithms presented in the text. (b) Decomposition of the low-dimensional symmetric irrep (11). (c) Decomposition for the low-dimensional antisymmetric irrep (10), which is the adjoint (or generator) irrep. (d) Decomposition of the low-dimensional generic irrep $(1\frac{1}{2})$, illustrating isospin multiplets of half-integer isospin.

by $M_X = \frac{1}{2}(N_p - \Omega)$ and $M_Y = \frac{1}{2}(N_n - \Omega)$, or $M_S = \frac{1}{2}(N - 2\Omega)$ and $M_T = \frac{1}{2}(N_p - N_n)$, where $\Omega = \frac{1}{2}(2j + 1)$ is the half-degeneracy.

The SO(5) irrep (RS) may be decomposed into $U_N(1) \otimes SO_T(3)$ irreps labeled by M_S and T, as indicated in (17). By inspection of the weight diagram [Fig. 6(a)] and considering the relation between canonical weights (horizontal and vertical axes) and chain (II) weights

(diagonal axes), it can be seen that the labels are taken from the possible values $M_S = -(R+S), \ldots, R+S-1, R+S$ and T=0 or $\frac{1}{2}, \ldots, R+S-1, R+S$. For a given value of M_S , the highest weight M_T and thus highest isospin is given by $T_{\max}(RS; M_S) = R+S$ for $|M_S| \leq R-S$, and $T_{\max}(RS; M_S) = 2R-|M_S|$ for $|M_S| > R-S$. Note that one of the $U_N(1) \otimes SO_T(3)$ irreps, at the bottom left of the weight diagram, is a "pair vacuum", annihilated by X_-, Y_- , and S_- [dashed box in Fig. 6(a)]. In applications to proton-neutron pairing, SO(5) irreps are conventionally labeled not by the usual mathematical labels (Table II), but rather by the seniority v and reduced isospin t. These are the occupation number and isospin of the pair vacuum, $i.e., M_{S \text{vac}} \equiv \frac{1}{2}(v-2\Omega) = -(R+S)$ and $T_{\text{vac}} \equiv t = R-S$.

The isospin content may be seen by decomposing the weights M_T for each given value of M_S , *i.e.*, occurring on single diagonal of the weight diagram, into isospin multiplets [solid boxes in Fig. 6(a)]. A given pair of $U_N(1) \otimes SO_T(3)$ labels may occur more than once within an SO(5) irrep and is therefore labeled by a multiplicity index $\kappa = 1, 2, ..., \text{mult}(RS; M_ST)$. The algorithm for constructing the branching $SO(5) \to U_N(1) \otimes SO_T(3)$ is derived in, *e.g.*, Ref. 8. Simple counting arguments then give a closed form multiplicity formula

$$\operatorname{mult}(RS; M_S T) = \begin{cases} f(2S, T; M_S) & T \le R - S \\ f(R + S - T, R - S; M_S) & T > R - S, \end{cases}$$
 (C3)

where $f(u, v; w) = \lfloor \min[u, \frac{1}{2}(u + v + w)] \rfloor - \lceil \max[0, \frac{1}{2}(u - v + w)] \rceil + 1.^{62}$ This relation fully defines the branching rule for chain (II). The branching into $U_N(1) \otimes SO_T(3)$ irreps is depicted in Fig. 6, for example irreps of SO(5): symmetric [Fig. 6(b)], antisymmetric [Fig. 6(c)], and generic [Fig. 6(d)].

Basis states for chain (II) involve only a linear combination of canonical chain (I') basis states at the *same* point in weight space. This point may be labeled interchangably either by M_X and M_Y or by M_S and M_T . Thus,

$$\begin{vmatrix}
(RS) \\
M_S \kappa T \\
M_T
\end{vmatrix} = \sum_{(XY)} \left\langle \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix} \begin{pmatrix} (RS) \\ M_S \kappa T \\ M_T \end{pmatrix} \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix},$$
(C4)

where it is to be understood that $M_X = \frac{1}{2}(M_S + M_T)$ and $M_Y = \frac{1}{2}(M_S - M_T)$. The SO(4) irreps (XY) of the canonical basis states contributing to a given chain (II) basis state are constrained by the branching condition $(RS) \to (XY)$ and by the usual angular momentum projection rules $(|M_X| \le X \text{ and } |M_Y| \le Y)$.

The transformation brackets in (C4) are only known in closed form for a restricted set of cases involving low branching multiplicites.⁹ However, the transformation brackets can be systematically calculated for any SO(5) irrep, regardless of multiplicity, through two possible procedures, outlined here, involving isospin laddering operations and either matrix diagonalization or orthogonalization.

The first method relies upon the construction of chain (II) basis states as eigenstates of \mathbf{T}^2 . Suppose \mathbf{T}^2 is written as its matrix realization in the (I') basis. The transformation brackets appearing in (C4) are the coefficients for decomposition of the eigenstate with respect to the chain (I') basis. Therefore, they must constitute the entries of an eigenvector of the \mathbf{T}^2 matrix, of eigenvalue T(T+1), and may be computed by diagonalization of this matrix. Since each chain (II) basis vector involves only a single weight point, the diagonalization can be carried out separately on subspaces corresponding to different weight points, in which case the eigenvalue T(T+1) occurs with degeneracy given by $\text{mult}(RS; M_ST)$ from (C3).

Note that the matrix realization of \mathbf{T}^2 at weight point $(M_X M_Y)$ is readily obtained from the known matrix elements (26) of the SO(5) generators. From (C2), $\mathbf{T}^2 = (X_0 - Y_0)^2 + 2(T_{+-}T_{-+} + T_{-+}T_{+-})$. This operator can be reexpressed in terms of spherical bitensor coupled products of the generator $T^{(1/21/2)}$ as $\mathbf{T}^2 = (X_0 - Y_0)^2 + 2[(T \times T)_{00}^{(11)} - (T \times T)_{00}^{(00)}]$. The requisite matrix elements are therefore

$$\begin{pmatrix}
(RS) \\
(X'Y') \\
M_X M_Y
\end{pmatrix} \mathbf{T}^2 \begin{vmatrix}
(RS) \\
(XY) \\
M_X M_Y
\end{pmatrix} = (M_X - M_Y)^2 + 2 \begin{pmatrix}
(XY) \\
M_X M_Y
\end{vmatrix} \begin{pmatrix}
(11) \\
M_X M_Y
\end{pmatrix} \begin{pmatrix}
(X'Y') \\
M_X M_Y
\end{pmatrix}$$

$$\times \left\langle \begin{pmatrix}
(RS) \\
(X'Y')
\end{pmatrix} \| (T \times T)^{(11)} \| \begin{pmatrix}
(RS) \\
(XY)
\end{pmatrix} - 2 \left\langle \begin{pmatrix}
(RS) \\
(X'Y')
\end{pmatrix} \| (T \times T)^{(00)} \| \begin{pmatrix}
(RS) \\
(XY)
\end{pmatrix}. (C5)$$

These may be evaluated using Racah's reduction formula, 19 as naturally extended to spherical bitensors, *i.e.*, SO(4) tensors, giving

and

$$\left\langle \binom{(RS)}{(X'Y')} \middle\| (T \times T)^{(00)} \middle\| \binom{(RS)}{(XY)} \right\rangle = -\frac{1}{2} \delta_{(XY)(X'Y')} \sum_{(X''Y'')} \left\langle \binom{(RS)}{(XY)} \middle\| T \middle\| \binom{(RS)}{(X''Y'')} \right\rangle^2. \tag{C7}$$

These expressions involve only SO(4) coupling coefficients (22), SO(4) recoupling coefficients (23), and $SO(5) \supset SO(4)$ reduced matrix elements (26), all of which are readily calculated.

However, it is important to note that the transformation brackets must be consistent among weight points within an isospin multiplet, since these are connected by the isospin laddering operators T_{\pm} [Fig. 6(a)]. The chain (II) basis states of a given M_S , κ , and T but different M_T must be related under laddering by T_{\pm} with the correct (positive) phase. Furthermore, in the presence of branching multiplicities, the choice of basis vectors (resolution of the multiplicity) must be consistent between weight points, *i.e.*, laddering should not connect different κ values at adjacent weight points. The relation among transformation brackets at adjacent weight points along a diagonal of given M_S , obtained by comparing the action of T_{\pm} on the chain (I') and chain (II) states, is

$$[(T \mp M_T)(T \pm M_T + 1)]^{1/2} \begin{pmatrix} (RS) \\ (X'Y') \\ M'_X M'_Y \end{pmatrix} \begin{pmatrix} (RS) \\ M_S \kappa T \\ M'_T \end{pmatrix}$$

$$= 2 \sum_{(XY)} \begin{pmatrix} (RS) \\ (X'Y') \\ M'_X M'_Y \end{pmatrix} \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix} \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix} \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix} \begin{pmatrix} (RS) \\ M_S \kappa T \\ M_T \end{pmatrix}, \quad (C8)$$

where $M'_X = M_X \pm \frac{1}{2}$, $M'_Y = M_Y \mp \frac{1}{2}$, and $M'_T = M_T \pm 1$, and where it is understood that $M_X = \frac{1}{2}(M_S + M_T)$, $M_Y = \frac{1}{2}(M_S - M_T)$, $M'_X = \frac{1}{2}(M_S + M'_T)$, and $M'_Y = \frac{1}{2}(M_S - M'_T)$.

At most four SO(4) irreps (XY) contribute to the sum, under the triangularity condition $(XY) \otimes (\frac{1}{2}\frac{1}{2}) \to (X'Y')$. The matrix element of $T_{\pm \mp}$ can be calculated from (6) and (26).

If \mathbf{T}^2 were diagonalized independently at each weight point, this would not guarantee consistency between weight points under laddering. As usual in diagonalization problems: (1) Eigenvectors, even in the absence of degenerate eigenvalues, are only defined to within an arbitrary phase (sign). (2) In the presence of degenerate eigenvalues (here, multiple occurrence of the same isospin at given M_S), the choice of orthogonal basis vectors for the degenerate eigenspace is arbitrary. Diagonalization of \mathbf{T}^2 must therefore be augmented by further conditions.

A consistent prescription for the transformation brackets is provided by diagonalizing \mathbf{T}^2 only once, for each M_S , at the most central weight point on the diagonal [Fig. 6(a)], either $M_T=0$ for integer isospin [Fig. 6(b,c)] or $M_T=+\frac{1}{2}$ for half-integer isospin [Fig. 6(d)]. The remaining transformation brackets are obtained by laddering outward, to more peripheral weight points of larger $|M_T|$ [the "+" arrow in Fig. 6(a)]. If the transformation brackets (C4) are considered as entries of numerical eigenvectors, then the laddering operation (C8) consists of multiplication by the (generally nonsquare and sparse) matrix realization of $T_{\pm \mp}$ between weight points. For a vector of isospin T, laddering past the weight point $M_T=T$ gives a null result. The process terminates at $M_T=T_{\max}(RS;M_S)$, or $M_T=-T_{\max}(RS;M_S)$ for laddering towards negative M_T . Although the laddering procedure provides a consistent set of coupling coefficients, the resolution of multiplicities arising from the diagonalization (at $M_T=0$ or $+\frac{1}{2}$) remains arbitrary and thus still does not provide a unique and reproducible prescription. Uniqueness (to within phase) can be obtained by further diagonalizing a "second" operator, as detailed in Ref. 9.⁶³

The second, alternative approach to constructing the transformation brackets does not involve explicitly diagonalizing \mathbf{T}^2 . Rather, it is based on laddering inward along a diagonal of constant M_S , from the most peripheral weight point [the "-" arrow in Fig. 6(a)] in conjunction with orthogonalization at each weight point. The weight point $M_T = T_{\text{max}}(RS; M_S)$ contains only a single basis state, which therefore is also the $T = T_{\text{max}}$ basis vector for chain (II), to within sign. We are free to choose this sign e.g., as always positive. Laddering inward to $M_T = T_{\rm max} - 1$ yields the $T = T_{\rm max}$ basis vector at this weight point. If the number of basis states (i.e., the dimension of the subspace) at this point is larger than one, then the remaining orthogonal vector (or vectors) needed to span the space must be the chain (II) basis vector (or vectors) of isospin $T = T_{\text{max}} - 1$. The degeneracy will be $\operatorname{mult}(RS; M_S, T_{\max} - 1)$. The orthogonal basis of good isospin may therefore be found by Gram-Schmidt orthogonalization of a complete but nonorthogonal basis, starting from the known $T = T_{\text{max}}$ basis vector, which is supplemented by $\text{mult}(RS; M_S, T_{\text{max}} - 1)$ further linearly independent vectors as needed to span the subspace. A unique set of transformation brackets, both in terms of phases and resolution of multiplicities, is obtained if a well-defined prescription is used for specifying these further independent vectors. For instance, one might use basis states taken from chain (I'), in order of increasing weight for the SO(4) label (XY), starting with the lowest-weight SO(4) label available at the weight point. (Alternatively, uniqueness can be enforced by diagonalizing a "second" operator within the $T = T_{\text{max}} - 1$ space, supplemented by a phase convention.) The process of laddering followed by orthonormalization must then be repeated for $M_T = T_{\text{max}} - 2$, $T_{\text{max}} - 3$, etc., until $M_T = 0$ or $+\frac{1}{2}$ is reached.

For large-scale computations, the choice between the two methods, (1) diagonalization of \mathbf{T}^2 followed by outward laddering or (2) inward laddering alternating with orthogonalization,

will be dictated by considerations of numerical efficiency and accuracy. These will depend upon the numerical linear algebra algorithms being used. The latter method provides the simplest route to a unique, reproducible set of phases and resolution of the branching multiplicity.

Once the transformation brackets between bases reducing chains (I') and (II) have been obtained, the transformation of reduced coupling coefficients follows immediately. The full (unreduced) coupling coefficient may be interpreted as the inner product of a coupled state with an uncoupled product of two states, each described by (C4). Then, for the SO(5) \supset U_N(1) \otimes SO_T(3) reduced coupling coefficient it follows from the factorization lemma and orthonormality that

$$\underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{2}S_{2}) & \rho(RS) \\ M_{S1}\kappa_{1}T_{1} & M_{S2}\kappa_{2}T_{2} & M_{S}\kappa T \end{pmatrix}}_{SO(5)\supset U_{N}(1)\otimes SO_{T}(3)} = \sum_{\substack{(X_{1}Y_{1})(X_{2}Y_{2})(XY) \\ M_{T1}(M_{T2})}} \underbrace{\begin{pmatrix} T_{1} & T_{2} & T \\ M_{T1} & M_{T2} & M_{T} \end{pmatrix}}_{SO(3)} \underbrace{\begin{pmatrix} (X_{1}Y_{1}) & (X_{2}Y_{2}) & (XY) \\ M_{X1}M_{Y1} & M_{X2}M_{Y2} & M_{X}M_{Y} \end{pmatrix}}_{SO(4)} + \underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{1}S_{1}) \\ M_{S1}\kappa_{1}T_{1} & (X_{1}Y_{1}) \\ M_{T1} & M_{X1}M_{Y1} \end{pmatrix} \begin{pmatrix} (R_{2}S_{2}) & (R_{2}S_{2}) \\ M_{S2}\kappa_{2}T_{2} & (X_{2}Y_{2}) \\ M_{X2}M_{Y2} \end{pmatrix} \begin{pmatrix} (RS) & (RS) \\ M_{S}\kappa T & (XY) \\ M_{T} & M_{X}M_{Y} \end{pmatrix}}_{SO(5)\supset SO(4)} + \underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{2}S_{2}) & \rho(RS) \\ (X_{1}Y_{1}) & (X_{2}Y_{2}) & (XY) \end{pmatrix}}_{SO(5)\supset SO(4)}, \quad (C9)$$

for any value of M_T allowed given isospin T, where again it is to be understood that $M_X = \frac{1}{2}(M_S + M_T)$, $M_Y = \frac{1}{2}(M_S - M_T)$, and similarly for M_{X1} , M_{Y1} , M_{X2} , and M_{Y2} . The sum over SO(4) irrep labels is subject to the usual branching and triangularity constraints on the canonical reduced coupling coefficients. For the resulting chain (II) reduced coupling coefficient to be nonvanishing, it must obey the SO(5) \rightarrow U_N(1) \otimes SO_T(3) branching rules (C3), the U_N(1) additivity condition $M_{S1} + M_{S2} = M_S$, and the SO_T(3) triangularity condition $T_1 \otimes T_2 \rightarrow T$.

In Appendix B, the canonical reduced coupling coefficients for the SO(5) coupling $(10) \otimes (1\frac{1}{2}) \to (1\frac{1}{2})$ were calculated [Fig. 5(c)], as a relatively simple numerical example involving an outer multiplicity (D=2). For a concrete illustration of the transformation procedure just described, let us consider the transformation of these coefficients into chain (II) reduced coupling coefficients. The decomposition of the irreps (10) and $(1\frac{1}{2})$ into $U_N(1) \otimes SO_T(3)$ irreps is shown in Fig. 6(c,d). For the transformation brackets, we follow the second method above. For instance, consider the $M_S = +\frac{1}{2}$ diagonal of the weight diagram for the $(1\frac{1}{2})$ irrep of SO(5) [Fig. 6(d)]. The $T = \frac{3}{2}$ seed state at $M_T = +\frac{3}{2}$ is $|+\frac{1}{2}\frac{3}{2}+\frac{3}{2}\rangle = +|(1\frac{1}{2})+1-\frac{1}{2}\rangle$, where we abbreviate chain (I') basis states as $|(XY)M_XM_Y\rangle$ and chain (II) basis states as $|M_STM_T\rangle$. Laddering inward to $M_T = +\frac{1}{2}$ yields $T = \frac{3}{2}$ state

$$|+\frac{1}{2}\frac{3}{2}+\frac{1}{2}\rangle = +\sqrt{\frac{5}{6}}|(\frac{1}{2}0)+\frac{1}{2}0\rangle + \sqrt{\frac{1}{6}}|(\frac{1}{2}1)+\frac{1}{2}0\rangle.$$

Orthogonalization, using $|(\frac{1}{2}0)+\frac{1}{2}0\rangle$ as the independent Gram-Schmidt basis vector, gives

TABLE III: Chain (II) reduced coupling coefficients for the SO(5) coupling $(10) \otimes (1\frac{1}{2}) \to (1\frac{1}{2})$, obtained from the canonical coupling coefficients of Fig. 5(c) according to the transformation conventions described in the text. For each $U_N(1) \otimes SO_T(3)$ coupling $(M_{S1}T_1) \otimes (M_{S2}T_2) \to (M_ST)$, the coefficients are given for SO(5) outer multiplicity index values $\rho = 1$ and 2, respectively.

M_{S1}	M_{S2}	M_S	T_1	T_2	T	Coeffi	cients	M_{S1}	M_{S2}	M_S	T_1	T_2	T	Coeffi	icients
1	$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{7}}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{1}{30}}$	$\sqrt{\frac{9}{70}}$
			1	$\frac{3}{2}$	$\frac{1}{2}$	$\sqrt{\frac{4}{15}}$	$\sqrt{\frac{16}{35}}$				0	$\frac{3}{2}$	$\frac{3}{2}$	$\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{9}{70}}$
1	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{45}}$	$-\sqrt{\frac{12}{35}}$				1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{210}}$
			1	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{2}{9}}$	0				1	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\sqrt{\frac{4}{21}}$
			1	$\frac{3}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{9}}$	0				1	$\frac{3}{2}$	$\frac{1}{2}$	0	$\sqrt{\frac{8}{21}}$
			1	$\frac{3}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{3}{7}}$				1	$\frac{3}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{42}}$
1	$-\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{7}}$	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{1}{70}}$
			1	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{2}{15}}$	$\sqrt{\frac{8}{35}}$				1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{27}{70}}$
0	$\frac{3}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{1}{70}}$	-1	$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\frac{1}{3}}$	$-\sqrt{\frac{1}{7}}$
			1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{27}{70}}$				1	$\frac{1}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{2}{15}}$	$-\sqrt{\frac{8}{35}}$
0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{30}}$	$-\sqrt{\frac{9}{70}}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{45}}$	$-\sqrt{\frac{12}{35}}$
			0	$\frac{3}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{1}{30}}$	$\sqrt{\frac{9}{70}}$				1	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{2}{9}}$	0
			1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{10}}$	$\sqrt{\frac{1}{210}}$				1	$\frac{3}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{9}}$	0
			1	$\frac{1}{2}$	$\frac{3}{2}$	0	$-\sqrt{\frac{4}{21}}$				1	$\frac{3}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{1}{9}}$	$\sqrt{\frac{3}{7}}$
			1	$\frac{3}{2}$	$\frac{1}{2}$	0	$\sqrt{\frac{8}{21}}$	-1	$-\frac{1}{2}$	$-\frac{3}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{7}}$
			1	$\frac{3}{2}$	$\frac{3}{2}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{42}}$				1	$\frac{3}{2}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{15}}$	$-\sqrt{\frac{16}{35}}$

 $T = \frac{1}{2}$ state

$$|+\frac{1}{2}\frac{1}{2}+\frac{1}{2}\rangle = +\sqrt{\frac{1}{6}}|(\frac{1}{2}0)+\frac{1}{2}0\rangle - \sqrt{\frac{5}{6}}|(\frac{1}{2}1)+\frac{1}{2}0\rangle.$$

Continued laddering to negative M_T gives the remaining transformation brackets along the diagonal. Straightforward application of (C9), using the full set of transformation brackets derived in this fashion, then yields the chain (II) reduced coupling coefficients in Table III. Note that the coefficients are either symmetric or antisymmetric under the $U_N(1)$ particle-hole conjugation operation $M_S \to -M_S$ (see Ref. 9).

The $SO_L(3)$ subalgebra of chain (III) is the maximal SO(3) subalgebra, *i.e.*, one which is not contained within any larger proper subalgebra of SO(5). This subalgebra is obtained by letting^{64–66}

$$L_{+1}^{(1)} = -\sqrt{2}X_{+} - \sqrt{6}T_{-+} \quad L_{0}^{(1)} = X_{0} + 3Y_{0} \quad L_{-1}^{(1)} = \sqrt{2}X_{-} + \sqrt{6}T_{+-}$$

$$O_{+3}^{(3)} = -\sqrt{5}Y_{+} \quad O_{+2}^{(3)} = \sqrt{10}T_{++} \quad O_{+1}^{(3)} = -\sqrt{3}X_{+} + 2T_{-+} \quad O_{0}^{(3)} = 3X_{0} - Y_{0} \quad (C10)$$

$$O_{-1}^{(3)} = \sqrt{3}X_{-} - 2T_{+-} \quad O_{-2}^{(3)} = -\sqrt{10}T_{--} \quad O_{-3}^{(3)} = \sqrt{5}Y_{-}.$$

Then $SO_L(3)$ has generators $L_{M_L}^{(1)}$, where, as usual, $L_{\pm 1}^{(1)} = \mp \frac{1}{\sqrt{2}} L_{\pm}$. The remaining generators $O_{M_L}^{(3)}$ constitute an octupole tensor with respect to $SO_L(3)$. The commutation relations of the generators are given (most compactly in spherical tensor coupled form⁶⁷) by

$$[L, L]^{(1)} = -\sqrt{2}L \qquad [L, O]^{(3)} = -2\sqrt{3}O$$

$$[O, O]^{(1)} = -2\sqrt{7}L \qquad [O, O]^{(3)} = \sqrt{6}O.$$
(C11)

The $SO_L(3)$ weight M_L is related to the canonical weights by $M_L = M_X + 3M_Y$, according to (C10), and thus defines an oblique axis in the weight space. The SO(5) generators are shown classified according to this weight in Fig. 2(d). The dashed lines connect the canonical generators of Sec. III B, which have good M_L but not L, to the linear combinations $L_{M_L}^{(1)}$ and $O_{M_L}^{(3)}$, which have both good M_L and good L.

The labels for basis states reducing the $SO_L(3)$ subalgebra are indicated in (17). Basis states for chain (III) involve a linear combination of chain (I') basis states of the same M_L . The transformation between bases therefore involves a sum not only over (XY), as in (C4), but also over distinct weight points $(M_X M_Y)$,

$$\begin{vmatrix} (RS) \\ \alpha L \\ M_L \end{vmatrix} = \sum_{\substack{(XY) \\ M_X(M_Y)}} \left\langle \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix} \begin{vmatrix} (RS) \\ \alpha L \\ M_L \end{pmatrix} \middle| \begin{pmatrix} (RS) \\ (XY) \\ M_X M_Y \end{pmatrix},$$
 (C12)

subject to the constraint $M_L = M_X + 3M_Y$. The transformation brackets may be systematically evaluated using an adaptation of either of the methods proposed above for Chain (II): (1) diagonalization of \mathbf{L}^2 for the $M_L = 0$ or $+\frac{1}{2}$ subspace of the SO(5) irrep, followed by laddering outward to larger- M_L spaces, or (2) laddering inward from the maximal- M_L space, in conjunction with orthogonalization within each successive lower- M_L space. For a unique resolution of the SO(5) \supset SO_L(3) branching multiplicity, it has been suggested that chain (III) basis states be chosen in which the octupole tensor is diagonal, equivalent to diagonalizing the Hermitian "second" operator $(L \times L \times O \times L)^{(0)} + (L \times O \times L \times L)^{(0)}$. Once transformation brackets have been obtained, the transformation of reduced coupling coefficients is given by

$$\underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{2}S_{2}) & \rho(RS) \\ \alpha_{1}L_{1} & \alpha_{2}L_{2} & \alpha L \end{pmatrix}}_{SO(5)\supset SO_{L}(3)} = \underbrace{\sum_{\substack{(X_{1}Y_{1})(X_{2}Y_{2})(XY) \\ M_{X1}(M_{Y1})M_{X2}(M_{Y2})M_{X}(M_{Y}) \\ M_{L1}(M_{L2})}}_{SO(3)} \underbrace{\begin{pmatrix} L_{1} & L_{2} & L \\ M_{L1} & M_{L2} & M_{L} \end{pmatrix}}_{SO(3)} \underbrace{\begin{pmatrix} (X_{1}Y_{1}) & (X_{2}Y_{2}) & (XY) \\ M_{X1}M_{Y1} & M_{X2}M_{Y2} & M_{X}M_{Y} \end{pmatrix}}_{SO(4)} \\
\times \underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{1}S_{1}) \\ \alpha_{1}L_{1} & (X_{1}Y_{1}) \\ M_{L1} & M_{X1}M_{Y1} \end{pmatrix}}_{SO(5)\supset[SO_{L}(3)\leftrightarrow SO(4)]} \begin{pmatrix} (R_{2}S_{2}) & (R_{2}S_{2}) & (R_{2}S_{2}) \\ M_{L2} & M_{X2}M_{Y2} \end{pmatrix}}_{SO(5)\supset SO(4)} \underbrace{\begin{pmatrix} (R_{1}S_{1}) & (R_{2}S_{2}) \\ (X_{1}Y_{1}) & (X_{2}Y_{2}) & (XY) \\ (X_{1}Y_{1}) & (X_{2}Y_{2}) & (XY) \end{pmatrix}}_{SO(5)\supset SO(4)}, \quad (C13)$$

for any value of M_L allowed given angular momentum L, where the summations over M_{X1} , M_{Y1} , M_{X2} , M_{Y2} , M_{X3} , and M_{Y3} are subject to the constraints $M_{L1} = M_{X1} + 3M_{Y1}$, $M_{L2} = M_{X2} + 3M_{Y2}$, and $M_{L} = M_{X1} + 3M_{Y2}$.

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⁴ Following common notation in the relevant literature, an algebra will be represented by its associated group under the exponential map, e.g., the direct sum algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ by the corresponding direct product group $SO(3) \otimes SO(3)$.

- Euler row reduction permits solution for the null vectors in exact symbolic arithmetic, as illustrated in the examples of Appendix B. However, for large-scale numerical calculation of coupling coefficients, QR decomposition or singular value decomposition algorithms would be more suitable.
- The first term on the right hand side of (11) only contributes when, moreover, $\Lambda'_1 \otimes \Lambda_2 \to \Lambda$ and $\Lambda'_1 \otimes \Lambda_T \to \Lambda_1$ for some Λ'_1 satisfying $\Gamma_1 \to \Lambda_1$. Likewise, the second term on the right hand side only contributes when $\Lambda_1 \otimes \Lambda'_2 \to \Lambda$ and $\Lambda'_2 \otimes \Lambda_T \to \Lambda_2$ for some Λ'_2 satisfying $\Gamma_2 \to \Lambda_2$.
- In particular, for coupling of a representations to its conjugate to give the identity representation, $\Gamma_1 \otimes \Gamma_2 \to (0)$ (see Sec. 19.7 of Ref. 18), if we restrict consideration to conditions (11) involving only nonnull coefficients, we are left without any viable $(\Lambda_1 \Lambda_2 \Lambda \Lambda')$ values. [Observe that $\Gamma \to \Lambda$ implies $\Lambda = (0)$, and $\Gamma \to \Lambda'$ implies $\Lambda' = (0)$, but it is then impossible to satisfy $\Lambda \otimes \Lambda_T \to \Lambda'$, given $\Lambda_T \neq (0)$.] This is already seen in the case of SO(3). For the coupling $J \otimes J \to 0$, all nonvanishing Clebsch-Gordan coefficients have $(M_1 M_2)$ values lying on a single diagonal $(M_1 = -M_2)$. Therefore, only a recurrence pattern of the type shown in Fig. 1(a), *i.e.*, involving known-zero Clebsch-Gordan coefficients on the adjacent diagonal, can relate the $J \otimes J \to 0$ coefficients.
- Null vectors obtained numerically by QR decomposition or singular value decomposition (see endnote 32) will already constitute an orthonormal set with respect to the \mathbb{R}^N dot product (i.e., $\mathbf{C}_{\rho'} \cdot \mathbf{C}_{\rho} = \delta_{\rho'\rho}$). Although this is not the relevant inner product $\mathcal{M}_{\rho'\rho}$, for obtaining orthonormal coupling coefficients, note that $\mathbf{C}_{\rho'} \cdot \mathbf{C}_{\rho} = n \mathcal{M}_{\rho'\rho}$, where n is the number of distinct $a\Lambda$ values arising in the set of coupling coefficients. Thus, vectors which are orthonormal with respect to the \mathbb{R}^N dot product are also already orthogonal with respect to (16) and need only be renormalized by a factor \sqrt{n} .
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- The SO(4) generators as defined here are exactly as given by Kemmer, Pursey, and Williams³⁷ (with $X \to p$ and $Y \to q$). However, the generators of Hecht²³ (with $X \to J$ and $Y \to \Lambda$) differ by the change of phase $Y_{\pm}^{\text{Hecht}} = -Y_{\pm}$ in the definition of $\text{SO}_Y(3)$. The additional SO(5) generators $T^{(1/2\,1/2)}$ as defined here follow the phases of Kemmer, Pursey, and Williams and are multiplied by a factor of $1/\sqrt{2}$ relative to Kemmer, Pursey, and Williams to match the "SO(5) tensor" normalization of Hecht. [However, note an apparent misprint in (5) of Ref. 37, interchanging the definitions of T_{+-} and T_{-+} .] The relation between the generators of (24) and Hecht's realization is $T_{++}^{\text{Hecht}} = iT_{++}$, $T_{+-}^{\text{Hecht}} = -iT_{+-}$, $T_{-+}^{\text{Hecht}} = iT_{-+}$, and $T_{--}^{\text{Hecht}} = -iT_{--}$. This mapping preserves the self-adjoint property on $T^{(1/2\,1/2)}$, and the commutators for the generators are also preserved if taken in conjunction with the change of phase on Y_{\pm} . Note also that the generators as defined here give commutation relations exactly matching those of Ref. 66.
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- Here we adopt a dot product notation for the scalar product of two spherical tensors of half-integer rank j, in the same spirit as the usual dot product for tensors of integer rank. However, for integer rank, $A^{(L)} \cdot B^{(L)} \equiv (-)^L \hat{L}(A \times B)_0^{(0)}$ involves an additional phase factor $(-)^L$.
- In relation to the Cartan weight operators of chain (I), observe that $S_0 = J_3$ and $T_0 = N_3$. Thus, the natural weight labels of chain (I) and chain (II) are identical, with $[\lambda_1 \lambda_2] = [M_S M_T]$. However, in the context of (20), N_3 was not the "M" weight operator of any SO(3) algebra.
- The function f(u, v; w) counts the number of lattice points (x, y), with $x = -u, -u + 2, \ldots, +u$ and $y = -v, -v + 1, \ldots, +v$, such that x + y = w.
- A convention may then be chosen for the signs of the eigenvectors. In particular, it may be desirable to impose symmetry or antisymmetry conditions between the transformation brackets related by particle-hole conjugation $(M_S \leftrightarrow -M_S)$. These relations will determine the conjugation symmetry of the coupling coefficients, discussed in Ref. 9.
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