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SOME SIMPLE R_5 WIGNER COEFFICIENTS AND THEIR APPLICATION

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Abstract: Generalized R_5 Wigner coefficients are calculated in algebraic form for the Kronecker products of an arbitrary irreducible representation with the 4, 5 and 10-dimensional irreducible representations of R_5, the group of rotations in 5-dimensional space. These are expressed in terms of the mathematically natural quantum numbers associated with the group chain $R_5 \supset R_4 \equiv R_3 \times R_3$. The transformation to physically interesting quantum numbers is discussed for two applications. The first involves the seniority force for nuclei with both protons and neutrons (see a companion paper by J. C. Parikh) and the possibility of extracting the N and T dependence of nuclear matrix elements in the seniority scheme. (N and T are the nucleon number and the total isospin). The second application involves the calculation of fractional parentage coefficients for spin-2 phonons in the seniority scheme, for large phonon number N and seniority v. Properties of the generalized Wigner coefficients are used to relate all such fractional parentage coefficients to those of type $\langle N-1 = v-1 | \} N = v \rangle$. A general prescription is given for the calculation of this type of coefficient from the R_5 Wigner coefficients, and numerical tables are given for $v = 5$ and 6.

1. Introduction

The classification of wave functions in nuclear spectroscopy through a chain of symmetry groups can in principle lead to a complete set of commuting operators whose eigenvalues fully specify the states of the system. In practice the group chains of actual physical interest rarely coincide with the mathematically natural chain of subgroups whose invariants would give such a complete specification of states. Besides the physically natural quantum numbers, such as seniority and angular momentum, arbitrary labels are therefore commonly used in place of additional quantum numbers. The recent work of Elliott and others [1-3]) with SU_3 shows that this difficulty may sometimes be overcome. In SU_3 the angular momentum is not a "mathematically natural quantum number" in the sense that the invariants of the group chain $SU_3 \supset R_3 \supset R_2$ do not lead to a complete set of commuting operators. Elliott has shown that the transformation from the mathematically natural group chain $SU_3 \supset SU_2$ to the physically interesting one can be carried out relatively easily by projection techniques. It may therefore be worthwhile to try to exploit in more detail the properties of the other continuous groups of spectroscopic interest.

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In this note the properties of R_5, the group of rotations in 5-dimensional space, are used to derive some simple R_5 Wigner coefficients which have application in nuclear spectroscopy if the transformation from the mathematically natural quantum numbers to the physically interesting ones can be effected. The R_5 Wigner coefficients involving the five-dimensional vector representation are closely connected to the fractional parentage coefficients of spin-2 systems in the seniority scheme. Fractional parentage coefficients for the nuclear d shell have been known for a long time [4]). These have also been used in calculations involving the coupling of quadrupole vibrational excitations (spin-2 phonons) [5]) to other degrees of freedom of the nucleus [6-8]). Jahn's tables give the fractional parentage coefficients for spin-2 phonons (totally symmetric states), only up to a phonon number of four. Although quadrupole vibrational excitations of three or more are probably strongly perturbed by other modes and not directly observable, simplified model calculations have been carried out in which phonon excitations of a high number have been used [9,10]) in an attempt to gain some understanding of the coupling of quadrupole vibrational excitations with other modes, particularly in the transition from the weak to the strong coupling limit. Fractional parentage coefficients of spin-2 phonons of rather high phonon number N may therefore be of interest. In this note properties of the R_5 Wigner coefficients are used to relate all such fractional parentage coefficients to coefficients of the type $\langle (N-1) = (v-1)|\}N = v\rangle$, where v is the seniority quantum number. A prescription is given for the calculation of this type of coefficient which involves the transformation from the mathematically natural to the physically interesting quantum numbers. As an example numerical tables are given for the case of $N = 5$ and 6.

A second application of R_5 arises in the generalization of the work of Kerman [11]), Kerman, Lawson and Macfarlane [12]) on the seniority force to the case of nuclei with both types of nucleons (protons and neutrons) (see the companion paper by Parikh [13])). It has been pointed out by Helmers [14]) that the ten bilinear invariants which commute with the operators of the unitary symplectic group in $(2j+1)$ dimensions, viz.

$$\begin{aligned}
A^+(T_z) &= \frac{1}{\sqrt{2}}\sum_{\tau_1(\tau_2)}\sum_m \frac{(-1)^{j-m}}{\sqrt{(2j+1)}}\, a^+_{m\tau_1} a^+_{-m\tau_2}\langle \tfrac{1}{2}\tau_1 \tfrac{1}{2}\tau_2|1T_z\rangle \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } T_z = 1, 0, -1, \\
A(T_z) &= \frac{1}{\sqrt{2}}\sum_{\tau_1(\tau_2)}\sum_m \frac{(-1)^{j-m}}{\sqrt{(2j+1)}}\, a_{-m\tau_2} a_{m\tau_1}\langle \tfrac{1}{2}\tau_1 \tfrac{1}{2}\tau_2|1T_z\rangle \\
T_\pm &= \sum_m a^+_{m\pm\frac{1}{2}} a_{m\mp\frac{1}{2}}, \qquad T_0 = \tfrac{1}{2}\sum_m [a^+_{m\frac{1}{2}} a_{m\frac{1}{2}} - a^+_{m-\frac{1}{2}} a_{m-\frac{1}{2}}], \\
& \tfrac{1}{2}[N_{\text{op}} - (2j+1)] = \tfrac{1}{2}[\sum_m (a^+_{m\frac{1}{2}} a_{m\frac{1}{2}} + a^+_{m-\frac{1}{2}} a_{m-\frac{1}{2}}) - (2j+1)]
\end{aligned} \tag{1}$$

form the infinitesimal operators which generate a group with the structure of R_5, more precisely the sympletic group in four dimensions Sp_4. (The fact that the groups

R_5 and Sp_4 have Lie algebras of identical structure will be used throughout this paper.) In eq. (1) the $a^+_{m\tau}(a_{m\tau})$ are creation (annihilation) operators for nucleons in a state with space quantum numbers j, m and isospin quantum number $\tau(=\pm\frac{1}{2})$. The four operators $a^+_{m\frac{1}{2}}, a^+_{m-\frac{1}{2}}, (-1)^{j-m}a_{-m\frac{1}{2}}, (-1)^{j-m}a_{-m-\frac{1}{2}}$, with arbitrary m, form a basis for the 4-dimensional irreducible representation of Sp_4 (R_5). From the Kronecker product $\mathbf{4}\times\mathbf{4} = \mathbf{1}+\mathbf{5}+\mathbf{10}$, (for the moment the dimension is used to specify the irreducible representation), it can be seen that operators of the type $a^+_{m\tau}a_{m'\tau'}$ or $a^+_{m\tau}a^+_{m'\tau'}$ will be linear combinations of tensor operators which transform according to the 1, 5 and 10 dimensional irreducible representations of Sp_4 (R_5). (The generalization of the concept of tensor operators to any semi-simple continuous group has been discussed by Stone [15])). Matrix elements of such operators can be split into two factors, an R_5 Wigner coefficient and a reduced matrix element, through the Wigner-Eckart theorem [15]). The R_5 Wigner coefficients will carry all of the dependence on the nucleon number N, and the total isospin T of the nucleus. The detailed applications of these remarks are given in the companion paper by Parikh [13]). It is the purpose of the present note to derive expressions for the R_5 Wigner coefficients involving the product of an arbitrary irreducible representation with the 4, 5 or 10-dimensional irreducible representations, the types of Wigner coefficients which are needed in the applications mentioned here.

In principle the method can also be used to extract the N and T dependence of matrix elements of two-body operators of the type $a^+_{m\tau}a^+_{m'\tau'}a_{m''\tau''}a_{m'''\tau'''}$. The most general such operator for the two-body nucleon-nucleon interaction, when expressed as a linear combination of R_5-irreducible tensor operators, will contain also operators which transform according to a 14-dimensional and a 35-dimensional irreducible representation of R_5. General algebraic expressions for Wigner coefficients involving these irreducible representations would be very cumbersome, but such coefficients might be calculated for specific numerical values of j.

2. Properties of the Group R_5 (Sp_4)

The mathematically natural quantum numbers which completely label the basis vectors of the irreducible representations of R_5 can be related to the group chain

$$R_5 \supset R_4 \supset R_3 \supset R_2, \tag{2a}$$

or alternately to

$$R_5 \supset R_4 \equiv R_3 \times R_3, \tag{2b}$$

where the change from (2a) to (2b) involves ordinary 3-dimensional angular momentum addition. (The Wigner coefficients for R_4 have been discussed by Biedenharn [16])). The fact that R_5 is itself a subgroup of R_6 will also be used. The equivalent group chain involving the symplectic group in four dimensions is

$$SU_4 \supset Sp_4 \supset SU_2 \times SU_2, \tag{3}$$

where the groups SU_4 and R_6, (of order 15), just as the groups Sp_4 and R_5, (of order 10), and the even more familiar pair SU_2 and R_3, (of order 3), have Lie algebras of identical structure. The vector diagram for SU_4 (Cartan's symmetry A_3) is obtained from that of R_6 (D_3) by a simple rotation of the root figure, and the vector diagrams for Sp_4, (C_2) and R_5, (B_2) are similarly related [17]. In the notation natural for R_5 the infinitesimal operators which generate the group can be denoted by $L_{jk} = -i(x_j\partial/\partial x_k - x_k\partial/\partial x_j)$ with $j, k = 1, \ldots 5$. It will often be more convenient to use a notation natural for Sp_4 and the group chain (3). In this case the infinitesimal operators which generate Sp_4 are a subset of those which generate the group SU_4, where the latter are denoted by A_{ij} with $i, j = 1, \ldots 4$, $\sum A_{\alpha\alpha} = 0$, and commutation properties $[A_{ij}, A_{kl}] = A_{il}\delta_{jk} - A_{kj}\delta_{il}$. The infinitesimal operators which generate Sp_4, (R_5), are shown in table 1 in standard form [17]) for Sp_4 in terms

TABLE 1

Infinitesimal operators for $Sp_4(R_5)$

(1)	(2)	(3)	(4)
$H_1(J_0) =$	$\frac{1}{2}(A_{11}-A_{22})$	$\frac{1}{2}(L_{12}+L_{34})$	$T^{(10)}_{10,00}$
$F_{10}(J_+)$	A_{12}	$\frac{1}{2}[(L_{23}+L_{14})+i(L_{31}+L_{24})]$	$-\sqrt{2}T^{(10)}_{10,10}$
$F_{-10}(J_-)$	A_{21}	$\frac{1}{2}[(L_{23}+L_{14})-i(L_{31}+L_{24})]$	$\sqrt{2}T^{(10)}_{10,-10}$
$H_2(\Lambda_0)$	$\frac{1}{2}(A_{33}-A_{44})$	$\frac{1}{2}(L_{12}-L_{34})$	$T^{(10)}_{01,00}$
$F_{01}(\Lambda_+)$	A_{34}	$\frac{1}{2}[(L_{14}-L_{23})+i(L_{24}-L_{31})]$	$-\sqrt{2}T^{(10)}_{01,01}$
$F_{0-1}(\Lambda_-)$	A_{43}	$\frac{1}{2}[(L_{14}-L_{23})-i(L_{24}-L_{31})]$	$\sqrt{2}T^{(10)}_{01,0-1}$
$F_{\frac{1}{2}\frac{1}{2}}$	$\frac{1}{\sqrt{2}}(A_{14}+A_{32})$	$\frac{1}{\sqrt{2}}(L_{52}+iL_{15})$	$-\sqrt{2}T^{(10)}_{\frac{1}{2}\frac{1}{2},\frac{1}{2}\frac{1}{2}}$
$F_{-\frac{1}{2}-\frac{1}{2}}$	$\frac{1}{\sqrt{2}}(A_{41}+A_{23})$	$\frac{1}{\sqrt{2}}(L_{52}-iL_{15})$	$\sqrt{2}T^{(10)}_{\frac{1}{2}\frac{1}{2},-\frac{1}{2}-\frac{1}{2}}$
$F_{\frac{1}{2}-\frac{1}{2}}$	$\frac{1}{\sqrt{2}}(A_{13}-A_{42})$	$\frac{1}{\sqrt{2}}(L_{45}+iL_{53})$	$\sqrt{2}T^{(10)}_{\frac{1}{2}\frac{1}{2},\frac{1}{2}-\frac{1}{2}}$
$F_{-\frac{1}{2}\frac{1}{2}}$	$\frac{1}{\sqrt{2}}(A_{31}-A_{24})$	$\frac{1}{\sqrt{2}}(L_{45}-iL_{53})$	$\sqrt{2}T^{(10)}_{\frac{1}{2}\frac{1}{2},-\frac{1}{2}\frac{1}{2}}$

(1) The infinitesimal operators in standard form for $Sp_4(C_2)$. The infinitesimal operators for the subgroup $SU_2 \times SU_2$ are also labelled in terms of "angular momentum" operators $\boldsymbol{J}$ and $\boldsymbol{\Lambda}$.
(2) The infinitesimal operators for Sp_4 in terms of the infinitesimal operators for SU_4.
(3) The infinitesimal operators in terms of $L_{jk} = -i(x_j\partial/\partial x_k - x_k\partial/\partial x_j)$, with $j, k = 1, \ldots, 5$.
(4) The infinitesimal operators expressed as irreducible tensor operators $T^{(J_m\Lambda_m)}_{J\Lambda, M_J M_\Lambda}$, eq. (4), with proper phase and normalization factors.

of the infinitesimal operators A_{ij} and also in terms of the L_{jk}. The two commuting operators H_1 and H_2 are $\frac{1}{2}(A_{11}-A_{22})$ and $\frac{1}{2}(A_{33}-A_{44})$. The step-up and step-down operators $F_{\alpha_1\alpha_2}$ and $F_{-\alpha_1-\alpha_2}$ are expressed in terms of the root vectors $\boldsymbol{\alpha} = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2$ appropriate to the vector diagram (C_2) of Sp_4. The infinitesimal operators of the subgroup $SU_2 \times SU_2$ are also denoted in terms of the components of two commuting angular momentum operators $\boldsymbol{J}$ and $\boldsymbol{\Lambda}$ where J_0, J_+, J_-, and similarly $\Lambda_0, \Lambda_+, \Lambda_-$ satisfy the usual commutation rules, for example $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_0$. The weights will be labelled by M_J and M_Λ, the eigenvalues of J_0 and Λ_0. (In making this choice the notation is restricted from here on to that natural for Sp_4 (C_2). The notation natural for R_5 (B_2) would have involved the eigenvalues of $L_{12} = J_0+\Lambda_0$ and $L_{34} = J_0-\Lambda_0$ instead.) The irreducible representations will be denoted by $(J_m\Lambda_m)$, where J_m and Λ_m are the values of M_J and M_Λ for the maximum weight state. (In the notation natural for R_5 the irreducible representations would be denoted by $(J_m+\Lambda_m, J_m-\Lambda_m)$. A more common notation for the irreducible representations of Sp_4 would be $(2J_m, 2\Lambda_m)$, but the angular momentum-type notation is preferred in this paper). The basis vectors of the irreducible representations are denoted by $|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle$. They are completely specified by the quantum numbers of the subgroup $SU_2 \times SU_2$. As usual J and Λ give the eigenvalues of $\boldsymbol{J}^2$ and $\boldsymbol{\Lambda}^2$. The irreducible tensor operators $T^{(J_m\Lambda_m)}_{J\Lambda, M_J M_\Lambda}$ satisfy the relations

$$[H_{1(2)}, T^{(J_m\Lambda_m)}_{J\Lambda, M_J M_\Lambda}] = M_J(M_\Lambda) T^{(J_m\Lambda_m)}_{J\Lambda, M_J M_\Lambda},$$

$$\begin{aligned}&[F_{\alpha_1\alpha_2}, T^{(J_m\Lambda_m)}_{J\Lambda, M_J M_\Lambda}] \\ &\quad = \sum_{J'\Lambda'} \langle (J_m\Lambda_m)J'\Lambda'(M_J+\alpha_1)(M_\Lambda+\alpha_2)|F_{\alpha_1\alpha_2}|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle T^{(J_m\Lambda_m)}_{J'\Lambda', (M_J+\alpha_1)(M_\Lambda+\alpha_2)}.\end{aligned} \tag{4}$$

The matrix elements of a component of an irreducible tensor operator are given through the generalized Wigner-Eckart theorem [15]) in terms of Sp_4 (R_5) Wigner coefficients and reduced (double-barred) matrix elements

$$\begin{aligned}&\langle (J'_m\Lambda'_m)J'\Lambda'(M_J+m_j)(M_\Lambda+m_\lambda)|T^{(j_m\lambda_m)}_{j\lambda, m_j m_\lambda}|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle \\ &\quad = \sum_\rho \langle (J_m\Lambda_m)J\Lambda, M_J M_\Lambda; (j_m\lambda_m)j\lambda m_j m_\lambda|(J'_m\Lambda'_m)J'\Lambda'(M_J+m_j)(M_\Lambda+m_\lambda)\rangle_\rho \\ &\qquad\qquad \times \langle (J'_m\Lambda'_m)||\boldsymbol{T}^{(j_m\lambda_m)}||(J_m\Lambda_m)\rangle_\rho,\end{aligned} \tag{5}$$

where the index ρ is needed to distinguish the several coupled states in those cases in which the Kronecker product $(J_m\Lambda_m) \times (j_m\lambda_m)$ contains the irreducible representation $(J'_m\Lambda'_m)$ more than once. The Wigner coefficients can be split into a factor independent of M_J and M_Λ, to be designated double-barred Sp_4 (R_5) Wigner coefficient and two ordinary SU_2 (R_3) Wigner coefficients

$$\begin{aligned}&\langle (J_m\Lambda_m)J\Lambda M_J M_\Lambda; (j_m\lambda_m)j\lambda m_j m_\lambda|(J'_m\Lambda'_m)J'\Lambda' M'_J M'_\Lambda\rangle_\rho \\ &\quad = \langle (J_m\Lambda_m)J\Lambda; (j_m\lambda_m)j\lambda||(J'_m\Lambda'_m)J'\Lambda'\rangle_\rho \langle JM_J; jm_j|J'M'_J\rangle\langle \Lambda M_\Lambda; \lambda m_\lambda|\Lambda' M'_\Lambda\rangle.\end{aligned} \tag{6}$$

The products of particular interest in this investigation involve the 4-dimensional irreducible representation $(j_m \lambda_m) = (\frac{1}{2}0)$ with $j\lambda = \frac{1}{2}0$ and $0\frac{1}{2}$; the 5-dimensional irreducible representation $(j_m \lambda_m) = (\frac{1}{2}\,\frac{1}{2})$ with $j\lambda = \frac{1}{2}\,\frac{1}{2}$ and $0\,0$; and the 10-dimensional irreducible representation $(j_m \lambda_m) = (10)$ with $j\lambda = 10, \frac{1}{2}\,\frac{1}{2}$, and 01. The Kronecker products are

$$(J_m \Lambda_m) \times (\tfrac{1}{2}0) = (J_m + \tfrac{1}{2}, \Lambda_m) + (J_m - \tfrac{1}{2}, \Lambda_m) + (J_m, \Lambda_m + \tfrac{1}{2}) + (J_m, \Lambda_m - \tfrac{1}{2}), \tag{7a}$$

$$\begin{aligned}(J_m \Lambda_m) \times (\tfrac{1}{2}\tfrac{1}{2}) = {}& (J_m + \tfrac{1}{2}, \Lambda_m + \tfrac{1}{2}) + (J_m - \tfrac{1}{2}, \Lambda_m - \tfrac{1}{2}) \\ & + (J_m + \tfrac{1}{2}, \Lambda_m - \tfrac{1}{2}) + (J_m - \tfrac{1}{2}, \Lambda_m + \tfrac{1}{2}) + (J_m \Lambda_m),\end{aligned} \tag{7b}$$

$$\begin{aligned}(J_m \Lambda_m) \times (10) = {}& (J_m + 1, \Lambda_m) + (J_m - 1, \Lambda_m) + (J_m, \Lambda_m + 1) + (J_m, \Lambda_m - 1) \\ & + 2(J_m \Lambda_m) + (J_m + \tfrac{1}{2}, \Lambda_m + \tfrac{1}{2}) + (J_m - \tfrac{1}{2}, \Lambda_m - \tfrac{1}{2}) + (J_m + \tfrac{1}{2}, \Lambda_m - \tfrac{1}{2}) + (J_m - \tfrac{1}{2}, \Lambda_m + \tfrac{1}{2}).\end{aligned} \tag{7c}$$

The index ρ is therefore needed only in the case $(j_m \lambda_m) = (10)$, $(J'_m \Lambda'_m) = (J_m \Lambda_m)$, the only time $(J'_m \Lambda'_m)$ occurs more than once in the Kronecker products of interest here. The ten infinitesimal operators of the group themselves transform according to the irreducible representation (10) (see table 1). A logical choice for the two independent coupled states of type $|[(J_m \Lambda_m) \times (10)](J_m \Lambda_m)\rho, J\Lambda M_J M_\Lambda\rangle$ with $\rho = 1$ or 2, would therefore be the following. States of type $\rho = 1$ are constructed to transform according to $F|(J_m \Lambda_m) J \Lambda M_J M_\Lambda\rangle$ where F stands for the ten infinitesimal operators of the group, while states with $\rho = 2$ are constructed orthogonal to these.

For values of J_m and Λ_m such that some of the irreducible representations in eqs. (7) are not admissible, modification rules must be used [18]). The Kronecker product $(10) \times (10)$, for example, contains only six, rather than ten, irreducible representations

$$(10) \times (10) = (20) + (\tfrac{3}{2}\tfrac{1}{2}) + (11) + (10) + (\tfrac{1}{2}\tfrac{1}{2}) + (00). \tag{7d}$$

The explicit expressions for the Wigner coefficients which are derived here are identically zero for values of $J_m \Lambda_m$ and $J'_m \Lambda'_m$ which would lead to inadmissible irreducible representations.

3. Explicit Construction of the States $|(J_m \Lambda_m) J \Lambda M_J M_\Lambda\rangle$

The states $|(J_m \Lambda_m) J \Lambda M_J M_\Lambda\rangle$ can be generated from the maximum weight state $|(J_m \Lambda_m) J_m \Lambda_m J_m \Lambda_m\rangle$ by the techniques of Racah [17]) through step-down operations. It is convenient to do this in two steps. The set of states with $M_J = J$, $M_\Lambda = \Lambda$ are constructed first. The well known step-down operations for M_J and M_Λ can then be used to construct the whole set of states. From the commutation properties of the infinitesimal operators it can be seen that the operator

$$O_{-+} = A_{21}(A_{14} + A_{32}) + (A_{31} - A_{24})(A_{11} - A_{22} + 1) \tag{8a}$$

acts as J step-down, Λ step-up operator, in $\frac{1}{2}$-integral steps, when acting on a state with $M_J = J$ and $M_\Lambda = \Lambda$, while

$$O_{--} = A_{43}O_{-+} + [A_{21}(A_{13}-A_{42}) - (A_{41}+A_{23})(A_{11}-A_{22}+1)](A_{33}-A_{44}+1) \tag{8b}$$

acts as J step-down, Λ step-down operator, again in $\frac{1}{2}$-integral steps, when acting on a state with $M_J = J$ and $M_\Lambda = \Lambda$. This follows, for example, from the commutation properties

$$\begin{aligned}
[J_0, O_{-+}]|(J_m\Lambda_m)J\Lambda J\Lambda\rangle &= -\tfrac{1}{2}O_{-+}|(J_m\Lambda_m)J\Lambda J\Lambda\rangle,\\
[\Lambda_0, O_{-+}]|(J_m\Lambda_m)J\Lambda J\Lambda\rangle &= +\tfrac{1}{2}O_{-+}|(J_m\Lambda_m)J\Lambda J\Lambda\rangle,\\
[J^2, O_{-+}]|(J_m\Lambda_m)J\Lambda J\Lambda\rangle &= -O_{-+}(J_0+\tfrac{1}{4})|(J_m\Lambda_m)J\Lambda J\Lambda\rangle,\\
[\Lambda^2, O_{-+}]|(J_m\Lambda_m)J\Lambda J\Lambda\rangle &= +O_{-+}(\Lambda_0+\tfrac{3}{4})|(J_m\Lambda_m)J\Lambda J\Lambda\rangle,
\end{aligned} \tag{9}$$

so that

$$O_{-+}|(J_m\Lambda_m)J\Lambda J\Lambda\rangle = c|(J_m\Lambda_m)(J-\tfrac{1}{2})(\Lambda+\tfrac{1}{2})(J-\tfrac{1}{2})(\Lambda+\tfrac{1}{2})\rangle, \tag{10a}$$

and similarly

$$O_{--}|(J_m\Lambda_m)J\Lambda J\Lambda\rangle = c'|(J_m\Lambda_m)(J-\tfrac{1}{2})(\Lambda-\tfrac{1}{2})(J-\tfrac{1}{2})(\Lambda-\tfrac{1}{2})\rangle, \tag{10b}$$

where c and c' are constants. The operators O_{-+}, O_{--} commute with each other so that the order of these step-down operations is arbitrary. The general state $|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle$ can be written

$$|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle = N(J_m\Lambda_m, n, m, x, y)J_-^x \Lambda_-^y O_{--}^m O_{-+}^n|(J_m\Lambda_m)J_m\Lambda_m J_m\Lambda_m\rangle, \tag{11a}$$

with

$$\begin{aligned}
J &= J_m - \tfrac{1}{2}n - \tfrac{1}{2}m, & 0 &\leqq n \leqq 2(J_m-\Lambda_m),\\
\Lambda &= \Lambda_m + \tfrac{1}{2}n - \tfrac{1}{2}m, & 0 &\leqq m \leqq 2\Lambda_m,\\
M_J &= J - x, & 0 &\leqq x \leqq 2J,\\
M_\Lambda &= \Lambda - y, & 0 &\leqq y \leqq 2\Lambda,
\end{aligned} \tag{11b}$$

where the normalization constant $N(J_m\Lambda_m, n, m, x, y) = N(n, m)N(x)N(y)$ is calculated in appendix 1:

$$N(n, m) = \left[\frac{(2J_m+1-n)!(2J_m+1-m)!(2J_m-2\Lambda_m-n)!(2\Lambda_m-m)! \times (2J_m+2\Lambda_m+2-m)!(2J_m+1-n-m)!(2\Lambda_m+1+n-m)!}{n!m![(2J_m+1)!]^3(2J_m-2\Lambda_m)!(2\Lambda_m)!(2J_m+2\Lambda_m+2)!(2\Lambda_m+1+n)!}\right]^{\frac{1}{2}}, \tag{11c}$$

$$N(x) = \left[\frac{(2J-x)!}{(2J)!x!}\right]^{\frac{1}{2}}, \qquad N(y) = \left[\frac{(2\Lambda-y)!}{(2\Lambda)!y!}\right]^{\frac{1}{2}}.$$

The range of the integers n and m ensures that J and Λ remain positive. As a check, the full set of states must give the dimension of the irreducible representations $(J_m \Lambda_m)$

$$\dim (J_m \Lambda_m) = \sum_{J,\Lambda} (2J+1)(2\Lambda+1) = \sum_{n=0}^{2(J_m-\Lambda_m)} \sum_{m=0}^{2\Lambda_m} (2J_m - n - m + 1)(2\Lambda_m + n - m + 1)$$
$$= \tfrac{1}{6}(2J_m + 2\Lambda_m + 3)(2J_m - 2\Lambda_m + 1)(2\Lambda_m + 1)(2J_m + 2). \tag{12}$$

4. Matrix Elements of the Infinitesimal Operators

The matrix elements of the ten infinitesimal operators follow from the explicit expression for the state $|(J_m \Lambda_m) J \Lambda J M_J M_\Lambda\rangle$. Matrix elements of $J_0, J_\pm, \Lambda_0$ and $\Lambda_\pm$ are the well known angular momentum matrix elements. Matrix elements of the remaining four infinitesimal operators can be obtained directly by operating with any one on the state $|(J_m \Lambda_m) J \Lambda M_J M_\Lambda\rangle \equiv |x, y, n, m\rangle$ of eq. (11) and using the commutation properties of the infinitesimal operators. For example

$$F_{-\frac{1}{2}-\frac{1}{2}}|x, y, n, m\rangle = \frac{1}{\sqrt{2}} (A_{41} + A_{23})|x, y, n, m\rangle$$
$$= \frac{1}{\sqrt{2}} N(J_m \Lambda_m, n, m, x, y) \left\{ \frac{-J_-^x \Lambda_-^y O_{--}^{m+1} O_{-+}^n + J_-^x \Lambda_-^{y+1} O_{--}^m O_{-+}^{n+1}}{(2J_m - n - m + 1)(2\Lambda_m + n - m + 1)} \right.$$
$$+ \frac{n(2J_m + 2 - n)(2J_m - 2\Lambda_m + 1 - n)(2\Lambda_m + 1 + n)}{(2J_m - n - m + 1)(2\Lambda_m + n - m + 1)} J_-^{x+1} \Lambda_-^y O_{--}^m O_{-+}^{n-1}$$
$$\left. + \frac{m(2J_m + 2 - m)(2J_m + 2\Lambda_m + 3 - m)(2\Lambda_m + 1 - m)}{(2J_m - n - m + 1)(2\Lambda_m + n - m + 1)} J_-^{x+1} \Lambda_-^{y+1} O_{--}^{m-1} O_{-+}^n \right\}$$
$$\times |x = 0, y = 0, n = 0, m = 0\rangle$$
$$= - \left[\frac{(m+1)(2J_m + 1 - m)(2\Lambda_m - m)(2J_m + 2\Lambda_m + 2 - m)(2J_m - n - m - x) \times (2\Lambda_m + n - m - y)}{2(2J_m + 1 - n - m)(2J_m - n - m)(2\Lambda_m + 1 + n - m)(2\Lambda_m + n - m)} \right]^{\frac{1}{2}}$$
$$\times |x, y, n, m+1\rangle$$
$$+ \left[\frac{(n+1)(y+1)(2\Lambda_m + 2 + n)(2J_m + 1 - n)(2J_m - 2\Lambda_m - n)(2J_m - n - m - x)}{2(2J_m + 1 - n - m)(2J_m - n - m)(2\Lambda_m + 2 + n - m)(2\Lambda_m + 1 + n - m)} \right]^{\frac{1}{2}}$$
$$\times |x, y+1, n+1, m\rangle$$
$$+ \left[\frac{n(x+1)(2J_m + 2 - n)(2\Lambda_m + 1 + n)(2J_m - 2\Lambda_m + 1 - n)(2\Lambda_m + n - m - y)}{2(2J_m + 2 - n - m)(2J_m + 1 - n - m)(2\Lambda_m + 1 + n - m)(2\Lambda_m + n - m)} \right]^{\frac{1}{2}}$$
$$\times |x+1, y, n-1, m\rangle$$
$$+ \left[\frac{m(x+1)(y+1)(2J_m + 2 - m)(2J_m + 2\Lambda_m + 3 - m)(2\Lambda_m + 1 - m)}{2(2J_m + 2 - n - m)(2J_m + 1 - n - m)(2\Lambda_m + 2 + n - m)(2\Lambda_m + 1 + n - m)} \right]^{\frac{1}{2}}$$
$$\times |x+1, y+1, n, m-1\rangle, \tag{13}$$

where the states $|x', y', n', m'\rangle$ are normalized. When expressed in terms of J, Λ, M_J, M_Λ rather than x, y, n, m, eq. (13) gives the matrix elements of the infinitesimal operator $F_{-\frac{1}{2}-\frac{1}{2}}$. (Note that this is equal to $\sqrt{2}T^{(10)}_{\frac{1}{2}\frac{1}{2},-\frac{1}{2}-\frac{1}{2}}$ when expressed as an irreducible tensor component of the 10-dimensional type, table 1.) Such matrix elements are best expressed in terms of the Wigner-Eckart theorem, eqs. (5) and (6), with

$$\begin{aligned}\langle(J_m\Lambda_m)||T^{(10)}||(J_m\Lambda_m)\rangle_1 &= [J_m(J_m+2)+\Lambda_m(\Lambda_m+1)]^{\frac{1}{2}},\\ \langle(J_m\Lambda_m)||T^{(10)}||(J_m\Lambda_m)\rangle_2 &\equiv 0,\end{aligned} \tag{14}$$

and double-barred Sp_4 (R_5) Wigner coefficients given by

$$\begin{aligned}\langle(J_m\Lambda_m)J'\Lambda';(10)10||(J_m\Lambda_m)J\Lambda\rangle_1 &= \delta_{JJ'}\delta_{\Lambda\Lambda'}\left[\frac{J(J+1)}{J_m(J_m+2)+\Lambda_m(\Lambda_m+1)}\right]^{\frac{1}{2}},\\ \langle(J_m\Lambda_m)J'\Lambda';(10)01||(J_m\Lambda_m)J\Lambda\rangle_1 &= \delta_{JJ'}\delta_{\Lambda\Lambda'}\left[\frac{\Lambda(\Lambda+1)}{J_m(J_m+2)+\Lambda_m(\Lambda_m+1)}\right]^{\frac{1}{2}},\end{aligned} \tag{15}$$

$$\langle(J_m\Lambda_m)(J-\tfrac{1}{2})(\Lambda+\tfrac{1}{2});(10)\tfrac{1}{2}\tfrac{1}{2}||(J_m\Lambda_m)J\Lambda\rangle_1$$
$$=\tfrac{1}{2}\left[\frac{(J_m-\Lambda_m-J+\Lambda+1)(J_m+\Lambda_m-J+\Lambda+2)(J_m+\Lambda_m+J-\Lambda+1)(J_m-\Lambda_m+J-\Lambda)}{(2J+1)(2\Lambda+1)[J_m(J_m+2)+\Lambda_m(\Lambda_m+1)]}\right]^{\frac{1}{2}},$$

$$\langle(J_m\Lambda_m)(J-\tfrac{1}{2})(\Lambda-\tfrac{1}{2});(10)\tfrac{1}{2}\tfrac{1}{2}||(J_m\Lambda_m)J\Lambda\rangle_1$$
$$=\tfrac{1}{2}\left[\frac{(J_m+\Lambda_m-J-\Lambda+1)(J_m+\Lambda_m+J+\Lambda+2)(J_m-\Lambda_m+J+\Lambda+1)(\Lambda_m-J_m+J+\Lambda)}{(2J+1)(2\Lambda+1)[J_m(J_m+2)+\Lambda_m(\Lambda_m+1)]}\right]^{\frac{1}{2}},$$

$$\langle(J_m\Lambda_m)(J+\tfrac{1}{2})(\Lambda-\tfrac{1}{2});(10)\tfrac{1}{2}\tfrac{1}{2}||(J_m\Lambda_m)J\Lambda\rangle_1$$
$$=\tfrac{1}{2}\left[\frac{(J_m-\Lambda_m-J+\Lambda)(J_m+\Lambda_m-J+\Lambda+1)(J_m+\Lambda_m+J-\Lambda+2)(J_m-\Lambda_m+J-\Lambda+1)}{(2J+1)(2\Lambda+1)[J_m(J_m+2)+\Lambda_m(\Lambda_m+1)]}\right]^{\frac{1}{2}},$$

$$\langle(J_m\Lambda_m)(J+\tfrac{1}{2})(\Lambda+\tfrac{1}{2});(10)\tfrac{1}{2}\tfrac{1}{2}||(J_m\Lambda_m)J\Lambda\rangle_1$$
$$=-\tfrac{1}{2}\left[\frac{(J_m+\Lambda_m-J-\Lambda)(J_m-\Lambda_m+J+\Lambda+2)(\Lambda_m-J_m+J+\Lambda+1)(J_m+\Lambda_m+J+\Lambda+3)}{(2J+1)(2\Lambda+1)[J_m(J_m+2)+\Lambda_m(\Lambda_m+1)]}\right]^{\frac{1}{2}}.$$

The quantity $J_m(J_m+2)+\Lambda_m(\Lambda_m+1)$ is the eigenvalue of the Casimir operator

$$G = \tfrac{1}{2}\sum_{\alpha+}(F_\alpha F_{-\alpha}+F_{-\alpha}F_\alpha)+H_1^2+H_2^2. \tag{16}$$

Matrix elements of the infinitesimal operators of the rotation groups have previously been derived by Gel'fand and Tseitlin [19]) in a slightly different basis corresponding to the group chain (2a) rather than (2b) or its equivalent (3). For R_5, Gel'fand and Tseitlin's basis corresponds to $|(J_m\Lambda_m)J\Lambda\Omega M_\Omega\rangle$ where $\boldsymbol{\Omega} = \boldsymbol{J}+\boldsymbol{\Lambda}$, so that it is related to the basis used in this investigation by ordinary angular momentum coupling techniques. In the $|JA\Omega M_\Omega\rangle$ basis the R_5 Wigner coefficient can again be factored into the product of a double-barred R_5 Wigner coefficient and now a single R_3 Wigner coefficient:

$$\begin{aligned}&\langle (J_m A_m)JA\Omega M_\Omega;\ (j_m \lambda_m)j\lambda\omega m_\omega|(J'_m A'_m)J'A'\Omega'(M_\Omega+m_\omega)\rangle\\ &\quad = \langle (J_m A_m)JA\Omega;\ (j_m \lambda_m)j\lambda\omega||(J'_m A'_m)J'A'\Omega'\rangle\langle \Omega M_\Omega;\ \omega m_\omega|\Omega'(M_\Omega+m_\omega)\rangle, \qquad (17)\end{aligned}$$

where the two types of double-barred R_5 coefficients, those of eqs. (17) and (6), can be related through ordinary angular momentum coupling techniques by

$$\begin{aligned}&\langle (J_m A_m)JA\Omega;\ (j_m \lambda_m)j\lambda\omega||(J'_m A'_m)J'A'\Omega'\rangle\\ &\qquad = [(2\Omega+1)(2\omega+1)(2J'+1)(2A'+1)]^{\frac{1}{2}}\begin{Bmatrix} J & A & \Omega\\ j & \lambda & \omega\\ J' & A' & \Omega'\end{Bmatrix}\\ &\qquad \times\langle (J_m A_m)JA;\ (j_m\lambda_m)j\lambda||(J'_m A'_m)J'A'\rangle, \qquad (18)\end{aligned}$$

involving the usual 9 j-symbol.

5. Calculation of Wigner Coefficients

Simple Wigner coefficients of the type needed in this investigation can be calculated through recursion techniques. By operating with an operator

$$F_{\alpha_1\alpha_2} = F_{\alpha_1\alpha_2}(1)+F_{\alpha_1\alpha_2}(2) \qquad \text{with } \alpha_1, \alpha_2 = \pm\tfrac{1}{2}$$

on a wave function of a coupled system, built from systems 1 and 2

$$\begin{aligned}&\sum_{J_1A_1J_2A_2}\ \sum_{M_{J_1}(M_{J_2})M_{A_1}(M_{A_2})} |1;(J_{m_1}A_{m_1})J_1A_1M_{J_1}M_{A_1}\rangle|2;(J_{m_2}A_{m_2})J_2A_2M_{J_2}M_{A_2}\rangle\\ &\qquad\times\langle (J_{m_1}A_{m_1})J_1A_1M_{J_1}M_{A_1};(J_{m_2}A_{m_2})J_2A_2M_{J_2}M_{A_2}|(J_mA_m)JAM_JM_A\rangle, \qquad (19)\end{aligned}$$

a recursion relation for the Wigner coefficients can be obtained in the usual way. This recursion relation has just four times the complexity of the analogous relation for the rotation group R_3:

$$\begin{aligned}&\sum_{a_1,a_2=-\frac{1}{2}}^{\frac{1}{2}}\langle (J_mA_m)(J+a_1)(A+a_2)(M_J+\alpha_1)(M_A+\alpha_2)|F_{\alpha_1\alpha_2}|(J_mA_m)JAM_JM_A\rangle\\ &\quad\times\langle (J_{m_1}A_{m_1})J_1A_1M_{J_1}M_{A_1};(J_{m_2}A_{m_2})J_2A_2M_{J_2}M_{A_2}|(J_mA_m)(J+a_1)(A+a_2)\\ &\qquad\times(M_J+\alpha_1)(M_A+\alpha_2)\rangle \qquad (20)\\ &= \sum_{a_1,a_2=-\frac{1}{2}}^{\frac{1}{2}}\langle (J_{m_1}A_{m_1})J_1A_1M_{J_1}M_{A_1}|F_{\alpha_1\alpha_2}(1)|(J_{m_1}A_{m_1})(J_1-a_1)(A_1-a_2)(M_{J_1}-\alpha_1)\\ &\quad(M_{A_1}-\alpha_2)\rangle\langle (J_{m_1}A_{m_1})(J_1-a_1)(A_1-a_2)(M_{J_1}-\alpha_1)(M_{A_1}-\alpha_2);\\ &\quad(J_{m_2}A_{m_2})J_2A_2M_{J_2}M_{A_2}|(J_mA_m)JAM_JM_A\rangle\\ &+\sum_{a_1,a_2=-\frac{1}{2}}^{\frac{1}{2}}\langle (J_{m_2}A_{m_2})J_2A_2M_{J_2}M_{A_2}|F_{\alpha_1\alpha_2}(2)|(J_{m_2}A_{m_2})(J_2-a_1)(A_2-a_2)(M_{J_2}-\alpha_1)\\ &\quad(M_{A_2}-\alpha_2)\rangle\langle (J_{m_1}A_{m_1})J_1A_1M_{J_1}M_{A_1};(J_{m_2}A_{m_2})(J_2-a_1)(A_2-a_2)(M_{J_2}-\alpha_1)\\ &\quad(M_{A_2}-\alpha_2)|(J_mA_m)JAM_JM_A\rangle.\end{aligned}$$

The matrix elements of $F_{\alpha_1\alpha_2}$ can be read off from eqs. (14) and (15). At first glance this 12-term recursion formula appears formidable. By proper choice of α_1, α_2 and M_J, M_Λ, however, it can be made to collapse to simple two or three-term recursion formulae in all the cases of interest here, so that the Wigner coefficients can be calculated by successive application of the recursion formula and finite difference equation techniques.

In the case of the Kronecker products involving the 5-dimensional irreducible representation $(\frac{1}{2}\frac{1}{2})$ a simpler method can be used. The matrix elements of the infinitesimal operators of R_6 are known [19]). The 15 infinitesimal operators of R_6 (SU_4) contain the 10 infinitesimal operators of R_5 (Sp_4) together with a set of 5 infinitesimal operators which form a 5-dimensional vector operator [20]) under R_5. The matrix elements of these five infinitesimal operators of R_6 contain the Wigner coefficients for the Kronecker product $(J_m \Lambda_m) \times (\frac{1}{2}\frac{1}{2})$. Irreducible tensor operators under R_6, $T^{(PP'P'')}$, have components which can be specified by the irreducible representations of the subgroups $R_5 \supset R_4 \supset R_3 \supset R_2$ into which the irreducible representation $(PP'P'')$ of R_6 decomposes. In the notation natural for the rotation groups such tensor operators would be designated by $T^{(PP'P'')}_{(J_m+\Lambda_m,\, J_m-\Lambda_m)J+\Lambda,\, J-\Lambda;\, \Omega M_\Omega}$ with the restrictions [19]) $P \geqq J_m+\Lambda_m \geqq P' \geqq J_m-\Lambda_m \geqq P''$. In the notation natural for the group chain $SU_4 \supset Sp_4 \supset SU_2 \times SU_2$ which is preferred here such tensor operators would be designated by $T^{[f]}_{(J_m\Lambda_m)J\Lambda M_J M_\Lambda}$ where $[f] = [f_1 f_2 f_3 f_4]$ is related to $(PP'P'')$ by

$$P = \tfrac{1}{2}(f_1+f_2-f_3-f_4), \quad P' = \tfrac{1}{2}(f_1-f_2+f_3-f_4), \quad P'' = \tfrac{1}{2}(f_1-f_2-f_3+f_4).$$

The infinitesimal operators of SU_4 (R_6) transform according to the 15-dimensional irreducible representation $[f] = [211]$, or $(PP'P'') = (110)$, with $(J_m\Lambda_m) = (10)$ and $(\frac{1}{2}\frac{1}{2})$. A matrix element of an infinitesimal operator of SU_4 (R_6) can again be factored through the Wigner-Eckart theorem into a Wigner coefficient and a reduced matrix element

$$\begin{aligned}\langle [f](J'_m\Lambda'_m)J'\Lambda'M'_J M'_\Lambda | T^{[211]}_{(\frac{1}{2}\frac{1}{2})j\lambda m_j m_\lambda} | [f](J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle \\ = \langle [f] \| T^{[211]} \| [f]\rangle \langle [f](J_m\Lambda_m)J\Lambda M_J M_\Lambda;\, [211](\tfrac{1}{2}\tfrac{1}{2})j\lambda m_j m_\lambda \\ | [f](J'_m\Lambda'_m)J'\Lambda'M'_J M'_\Lambda\rangle, \quad (21)\end{aligned}$$

where the R_6 Wigner coefficient can be factored into two pieces, a double-barred SU_4 (R_6) Wigner coefficient which is a function of the quantum numbers of type $[f]$ and $(J_m\Lambda_m)$ only, and the Sp_4 (R_5) Wigner coefficient which is of interest here

$$\begin{aligned}\langle [f](J_m\Lambda_m);\, [211](\tfrac{1}{2}\tfrac{1}{2}) \| [f](J'_m\Lambda'_m)\rangle \\ \times \langle (J_m\Lambda_m)J\Lambda M_J M_\Lambda;\, (\tfrac{1}{2}\tfrac{1}{2})j\lambda m_j m_\gamma | (J'_m\Lambda'_m)J'\Lambda'M'_J M'_\Lambda\rangle.\end{aligned}$$

By splitting off the J, Λ, M_J, M_Λ-dependent piece, the Sp_4 (R_5) Wigner coefficient involving the 5-dimensional irreducible representation $(\frac{1}{2}\frac{1}{2})$ can therefore be read off. The normalization follows from the unitary condition which holds for both the double-barred as well as the full Wigner coefficients. The choice of phases is as always somewhat arbitrary. The following generalized Condon and Shortley phase convention is made. The Sp_4 (R_5) Wigner coefficients are chosen real, and coefficients of the type $\langle (J_{m_1}\Lambda_{m_1})J_{m_1}\Lambda_{m_1}; (J_{m_2}\Lambda_{m_2})J_2\Lambda_2 || (J_m\Lambda_m)J_m\Lambda_m\rangle$ are chosen positive. (For Kronecker products for which $(J_m\Lambda_m)$ occurs only once there is only one coefficient of this type, and this prescription is sufficient to fix the phases. For Kronecker products for which $(J_m\Lambda_m)$ occurs more than once, as in the case of $(J_{m_2}\Lambda_{m_2}) = (10)$ $(J_m\Lambda_m) = (J_{m_1}\Lambda_{m_1})$, for which the index ρ is needed, there is more than one coefficient of this type, and their relative phases are not necessarily positive. In this case an arbitrary overall choice of phase has been made).

The 4×4 matrix for the Kronecker product $(J_m\Lambda_m)\times(\frac{1}{2}0)$ and the 5×5 matrix for the product $(J_m\Lambda_m)\times(\frac{1}{2}\frac{1}{2})$ are tabulated in tables 2 and 3. The number of independent coefficients in these tables can be reduced considerably through symmetry properties of the Wigner coefficients (see appendix 2). In the Kronecker product of $(J_m\Lambda_m)$ with the 10-dimensional irreducible representation, (10), coupled functions with $(J'_m\Lambda'_m) = (J_m\Lambda_m)$ are of greatest interest in the applications mentioned in sect. 1. Wigner coefficients of the type $\langle (J_m\Lambda_m)J_1\Lambda_1; (10)J_2\Lambda_2 || (J'_m\Lambda'_m)J\Lambda\rangle_\rho$ with $(J'_m\Lambda'_m) = (J_m\Lambda_m)$ and $\rho = 1$ are given by eqs. (15). Those with $\rho = 2$, orthogonal to the former, are shown in table 4(a). In the special case, with $(J_m\Lambda_m) = (10)$, the Wigner coefficients with $\rho = 2$ are identically zero. The Kronecker product $(10)\times(10)$ contains the irreducible representation (10) only once, (compare with eq. (7d)), corresponding to an antisymmetric coupling of the two (10) states. With $(J'_m\Lambda'_m) \neq (J_m\Lambda_m)$ the index ρ is not needed. Wigner coefficients with $(J'_m\Lambda'_m) = (J_m+1, \Lambda_m)$, (J_m, Λ_m+1), $(J_m+\frac{1}{2}, \Lambda_m+\frac{1}{2})$, and $(J_m+\frac{1}{2}, \Lambda_m-\frac{1}{2})$ are tabulated in tables 4(b)–(e), respectively. Wigner coefficients with $(J'_m\Lambda'_m) = (J_m-1, \Lambda_m)$, (J_m, Λ_m-1), $(J_m-\frac{1}{2}, \Lambda_m-\frac{1}{2})$, and $(J_m-\frac{1}{2}, \Lambda_m+\frac{1}{2})$ are not explicitly tabulated. They can be obtained from those of tables 4(b)–(e) through the symmetry relation, eq. (II.9), derived in appendix 2.

6. Transformations to the Physically Interesting Quantum Numbers. The $|(T_p)TNT_z\rangle$ Scheme

Wigner coefficients of the type given in tables 2–4 cannot be used directly in the application to physical problems since the mathematically natural quantum numbers such as J, Λ, M_J, M_Λ or J, Λ, Ω, M_Ω are not the eigenvalues of operators of actual physical interest.

In the applications involving the seniority force and nuclei with both protons and neutrons, the physically interesting commuting operators are the number operator, more precisely $\frac{1}{2}[N_{\text{op}}-(2j+1)]$, and the total isospin operators T_z and $\boldsymbol{T}^2$. These are most naturally associated with the infinitesimal operators of R_5 in the following way [13]):

$$\tfrac{1}{2}[N_{\text{op}}-(2j+1)] \equiv L_{12}, \qquad T_z \equiv L_{34}, \qquad \boldsymbol{T}^2 = L_{34}^2+L_{35}^2+L_{45}^2 .$$

(Compare with eq. (1) and table 1.) States of definite number N and T_z are therefore states of definite M_J and M_Λ; $M_J+M_\Lambda = \frac{1}{2}[N-(2j+1)]$, $M_J-M_\Lambda = T_z$; but states of definite T will in general be linear combinations of states involving all pairs $J\Lambda$. (If T were identified with J, Λ or Ω, the number operator would not be diagonal in such a representation.) Also, there seems to be no natural fourth operator which commutes with the commuting set L_{12}, L_{34} and $L_{34}^2+L_{35}^2+L_{45}^2$. The operator $\sum_{ij}\sum_{\alpha\beta} L_{\alpha i}L_{\beta j}L_{\alpha j}L_{\beta i}$, where the sums are over $i, j = 1, 2$, and $\alpha, \beta = 3, 4, 5$ only, would be a possible operator of this type, but this operator, quartic in the infinitesimal operators of R_5, seems to be a very unnatural operator from the mathematical point of view and seems to have no ready physical interpretation. In many of the cases of actual interest [13]) the operators N, T_z and T^2, are sufficient to label the states of the irreducible representations of R_5. In other cases, where there is more than one state of given N, T, T_z in a given irreducible representation of R_5, an additional label must be used. One possible natural label [21]) would be the resultant isospin T_p, of the p pairs of nucleons, each coupled to angular momentum 0 and isospin 1, with the possible values $T_p = p, (p-2), \ldots, 1$ (or 0). For states with seniority (v), zero or one, T_p is uniquely defined by N and T and is therefore a redundant label. For $v = 0$, $T_p \equiv T$, and for $v = 1$, $T_p = (T-\frac{1}{2})$ for $(\frac{1}{2}N-T)$ even, and $T_p = (T+\frac{1}{2})$ for $(\frac{1}{2}N-T)$ odd. The transformation of the Wigner coefficients from the mathematically natural basis $|J\Lambda M_J M_\Lambda\rangle$ to the physically interesting one $|(T_p)TNT_z\rangle$ involves a unitary transformation of each of the coupled functions

$$|(T_p)TNT_z\rangle = \sum_{J\Lambda} |J\Lambda M_J M_\Lambda\rangle\langle J\Lambda M_J M_\Lambda|(T_p)TNT_z\rangle,$$

with $M_J+M_\Lambda = \frac{1}{2}[N-(2j+1)]$ and $M_J-M_\Lambda = T_z$. Since the quantum numbers J_m, Λ_m which characterize the irreducible representations are identified with the values of M_J and M_Λ for the maximum weight state, their physical significance follows from these relations. In states with seniority $v = 0$ the maximum possible number of nucleons is $N_{\max} = 4j+2$. In states with arbitrary seniority v the maximum possible number of nucleons is $N_{\max} = 4j+2-v$. In states with this maximum number of nucleons the isospin is unique and has the value t, the so-called reduced isospin introduced by Flowers [22]). Thus $T_z = t$ in the maximum weight state, and the quantum numbers J_m, Λ_m have the values $J_m+\Lambda_m = j+\frac{1}{2}-\frac{1}{2}v$, $J_m-\Lambda_m = t$. (For a full discussion see the work of Parikh [13])).

The transformation coefficients $\langle J\Lambda M_J M_\Lambda|(T_p)TNT_z\rangle$ can be calculated if the state $|(T_p)TNT_z)$ with arbitrary quantum numbers is generated from the state of maximum weight through successive application of step-down operators built from the infinitesimal operators. Since the matrix elements of the infinitesimal operators are known it is possible to evaluate the coefficients of the states $|J\Lambda M_J M_\Lambda\rangle$ generated by these step-down operators. Details for states of seniority $v = 1$ are given by Parikh [13]). Alternately, if the transformation coefficients from the $|(T_p)TNT_z\rangle$ scheme to the $|J\Lambda M_J M_\Lambda\rangle$ scheme are known for states with seniority $v = 0$, that is $J_m = \Lambda_m$, (and $T \equiv T_p$), the transformation coefficients for all other irreducible representations can be calculated by operating on these states with operators $a^+_{m\tau}$ (or $a_{m\tau}$) coupled to appropriate values of t. The operators $a^+_{m\frac{1}{2}}$, $a^+_{m-\frac{1}{2}}$, $(-1)^{j-m}a_{-m+\frac{1}{2}}$, $(-1)^{j-m}a_{-m-\frac{1}{2}}$ transform according to the four-dimensional irreducible representation $(J_m\Lambda_m) = (\frac{1}{2}0)$ with $M_J M_\Lambda$ values equal to $\frac{1}{2}0$, $0\frac{1}{2}$, $-\frac{1}{2}0$, $0-\frac{1}{2}$, respectively. Since the Wigner coefficients for the product $(J_m J_m)\times(\frac{1}{2}0)$ in the $|J\Lambda M_J M_\Lambda\rangle$ scheme are known (table 2), the transformation coefficients $\langle J\Lambda M_J M_\Lambda|(T_p)TNT_z\rangle$ for states with $v = 1$, $t = \frac{1}{2}$ follow from those for states with $v = 0$. The method can be generalized to states with $t > \frac{1}{2}$. States with $v = 2$, $t = 1$, for example, can be generated by operating on states with $v = 0$, $T \equiv T_p$ with operators of the type $\sum_\tau\langle\frac{1}{2}\tau_1\frac{1}{2}\tau_2|1M_T\rangle a^+_{m\tau_1}a^+_{m'\tau_2}$, (or the corresponding annihilation operators). From the Wigner coefficients of table 2 it can be seen that such operators transform according to the 10-dimensional irreducible representation (10) with $J\Lambda M_J M_\Lambda$ values of 1010, 0101, and $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$ for the linear combinations with $M_T = 1$, -1 and 0, respectively. From the Wigner coefficients for the product $(J_m J_m)\times(10)$ in the $|J\Lambda M_J M_\Lambda\rangle$ scheme (table 4), the transformation coefficients $\langle J\Lambda M_J M_\Lambda|(T_p)TNT_z\rangle$ for states with $v = 2$, $t = 1$ again follow from those for $v = 0$.

The starting point of the calculation involves the transformation coefficients for states with $v = 0$. Some of the details are given in appendix 3. States with $v = 0$, arbitrary N and T, but with $T = T_z$, can be generated from the maximum weight state with $N = 4j+2$, $T = 0$ through successive operation with two commuting step-down operators (see table 1)

$$|NT = T_z\rangle = \mathcal{N}(a, b)(F^2_{-\frac{1}{2}-\frac{1}{2}}-2J_-\Lambda_-)^b\Lambda^a_-|\max\rangle, \tag{22a}$$

with resultant $N = 4j+2-2a-4b$, and $T = T_z = a$. The operator Λ_- which annihilates a pair of nucleons coupled to $T = 1$, $T_z = -1$, when acting on a state with $T = T_z$, steps up T (and T_z) by one unit[13]), while the operator $(F^2_{-\frac{1}{2}-\frac{1}{2}}-2J_-\Lambda_-)$ which annihilates two $T = 1$ pairs of nucleons, coupled to resultant isospin zero, leaves T invariant while stepping down N by 4 units. The normalization constant (appendix 3) has the value

$$\mathcal{N}(a, b) = 2^b\left[\frac{(4J_m+1-2b)!(2J_m-a-b)!(2a+1)!(a+b)!}{(4J_m+1)!(2J_m-b)!(a!)^2 b!(2a+2b+1)!}\right]^{\frac{1}{2}}. \tag{22b}$$

Since $\Lambda_m = J_m$, $(= \frac{1}{2}j+\frac{1}{4})$, in states with $v = 0$, the quantum numbers J and Λ are related by $J = \Lambda$, (see eqs. (11b)). As a result the transformation coefficients from the $|J\Lambda M_J M_\Lambda\rangle$ to the $|TNT_z\rangle$ scheme are characterized by a single index μ and can thus be evaluated by simple recursion techniques. For states with $T = T_z = a$, $N = 4j+2-4b-2a$,

$$\langle J\Lambda M_J M_\Lambda|NTT_z\rangle = \langle J_m-\mu, J_m-\mu, J_m-b, J_m-a-b|NTT_z\rangle \equiv c_\mu,$$

with

$$|(v = 0)NT = T_z\rangle = \sum_{\mu=0}^{b} c_\mu|(J_m J_m)J_m-\mu, J_m-\mu, J_m-b, J_m-a-b\rangle. \tag{23a}$$

An algebraic expression for the transformation coefficient $c_\mu(T = T_z)$ is derived in appendix 3:

$$c_\mu(T = T_z) = \frac{(-1)^{b-\mu}}{a!\mu!(2J_m+1-\mu)!} \times \left[\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu)!(4J_m+2-2\mu)!(2J_m-a-b)! \times(2J_m+1-b)!(2J_m-\mu-b)!(2J_m+1-2\mu)}{(2a+2b+1)!(b-\mu)!(4J_m+2-2b)!(2J_m-\mu-a-b)!}\right]^{\frac{1}{2}}. \tag{23b}$$

Transformation coefficient for states with $T_z < T$ follow from these by operating on equation (23b) with T_-. Matrix elements of T_- in the $|JAM_J M_A\rangle$ scheme can be read off from eqs. (15), with the defining relation $T_- = L_{45}-iL_{53} = \sqrt{2}F_{-\frac{1}{2}\frac{1}{2}}$. The transformation coefficient with $T_z = T-1$, for example, is given by

$$c_\mu(T_z = T-1) = \frac{(-1)^{b-\mu}}{\mu!(2J_m-\mu)!} \times \left[\frac{2(2a+1)!(a+b)!b!(2\mu+1)!(a+b-\mu-1)!(4J_m+1-2\mu)!(2J_m-a-b)! \times(2J_m+1-b)!(2J_m-\mu-b-1)!(2J_m-2\mu)}{a!(a-1)!(2a+2b+1)!(b-\mu)!(4J_m+2-2b)!(2J_m-\mu-b-a)!}\right]^{\frac{1}{2}} \tag{23c}$$

The annihilation operator $a_{-m-\frac{1}{2}}$ when operating on a state with $v = 0$, N_p and $T_z = T \equiv T_p$ produces a state of $N = N_p-1$ nucleons with $v = 1$, $t = \frac{1}{2}$ and $T = T_z = T_p+\frac{1}{2}$. (An annihilation operator is preferred over a creation operator since all states have been expressed in terms of step-down operators starting from the maximum weight or closed shell state.) Since the operator $a_{-m,\,-\frac{1}{2}}$ has R_5-irreducible tensor character $T^{(\frac{1}{2}0)}_{0\frac{1}{2};\,0\,-\frac{1}{2}}$ its matrix elements in the $|JAM_J M_A\rangle$ scheme, except for a trivial reduced matrix element, can be read off from table 2. Thus

$$\begin{aligned}
|(v = 1, t = \tfrac{1}{2})N-1, T_z = T = T_p+\tfrac{1}{2}\rangle &= \mathcal{N} a_{-m-\frac{1}{2}}|(v = 0)NT_p = T_z\rangle \\
&= \sum_\mu \mathcal{N} c_\mu(T = T_z) T^{(\frac{1}{2}0)}_{0\frac{1}{2}\,0\,-\frac{1}{2}}|(J_m J_m)J_m-\mu, J_m-\mu, J_m-b, J_m-a-b\rangle \\
&= \sum_\mu a_\mu|(J_m J_m-\tfrac{1}{2})J_m-\mu, J_m-\mu+\tfrac{1}{2}, J_m-b, J_m-a-b-\tfrac{1}{2}\rangle \\
&\quad + \sum_\mu b_\mu|(J_m J_m-\tfrac{1}{2})J_m-\mu, J_m-\mu-\tfrac{1}{2}, J_m-b, J_m-a-b-\tfrac{1}{2}\rangle,
\end{aligned} \tag{24a}$$

where a_μ and b_μ are given by

$$\left.\begin{matrix} a_\mu \\ b_\mu \end{matrix}\right\} = \mathcal{N} c_\mu\langle(J_m J_m)J_m-\mu, J_m-\mu; (\tfrac{1}{2}0)0\tfrac{1}{2}||(J_m J_m-\tfrac{1}{2})J_m-\mu, J_m-\mu\pm\tfrac{1}{2}\rangle \times\langle J_m-\mu, J_m-a-b; \tfrac{1}{2}-\tfrac{1}{2}|J_m-\mu\pm\tfrac{1}{2}, J_m-a-b-\tfrac{1}{2}\rangle. \tag{24b}$$

The reduced matrix element has the value 1, and the new normalization factor $\mathcal{N}$ is equal to $[(4J_m+3)/(4J_m-2a-2b)]^{\frac{1}{2}}$.

In a similar way states with $v = 1$, $t = \frac{1}{2}$ and $T = T_p-\frac{1}{2}$ can be built from $v = 0$ states with $T \equiv T_p$ by transforming the operator equation

$$\begin{aligned} &|(v = 1, t = \tfrac{1}{2})N-1, T_z = T = T_p-\tfrac{1}{2}\rangle \\ &\qquad = \mathcal{N}'\{a_{-m\frac{1}{2}}|(v = 0)NT_p, T_z = T_p\rangle\langle T_pT_p\tfrac{1}{2}\, -\tfrac{1}{2}|T_p-\tfrac{1}{2}, T_p-\tfrac{1}{2}\rangle \\ &\qquad -a_{-m,-\frac{1}{2}}|(v = 0)NT_p, T_z = T_p-1\rangle\langle T_p(T_p-1)\tfrac{1}{2}\tfrac{1}{2}|T_p-\tfrac{1}{2}, T_p-\tfrac{1}{2}\rangle\} \end{aligned} \tag{25}$$

to the $|JAM_J M_A\rangle$ scheme, using the expansion coefficient of both eqs. (23b) and (c) and the fact that $a_{-m\frac{1}{2}}$ and $a_{-m,-\frac{1}{2}}$ have R_5-tensor character $T^{(\frac{1}{2}0)}_{\frac{1}{2}0;-\frac{1}{2}0}$ and $T^{(\frac{1}{2}0)}_{0\frac{1}{2};0-\frac{1}{2}}$.

In summary, the transformation coefficients $\langle JAM_JM_A|(T_p)NT = T_z\rangle$ for states with $v = 1$, $t = \frac{1}{2}$, that is $(J_mA_m) = (J_mJ_m-\frac{1}{2})$ with $J_m = \frac{1}{2}j+\frac{1}{4}$, have the following expressions:

<u>Case 1: $T = T_p+\frac{1}{2}$</u>

$$\begin{aligned} &\langle J_m-\mu, J_m-\mu+\tfrac{1}{2}, J_m-b, J_m-a-b-\tfrac{1}{2}|(T_p)NT = T_z = T_p+\tfrac{1}{2}\rangle \\ &\quad = \frac{(-1)^{1+b-\mu}}{a!(2J_m+1-\mu)!} \\ &\quad \times \left[\frac{(2a+1)!(a+b)!b!(a+b-\mu+1)!(2\mu)!(4J_m+2-2\mu)!(2J_m-\mu-b)!\times(2J_m-a-b-1)!(2J_m+1-b)!}{(2a+2b+1)!(b-\mu)!\mu!(\mu-1)!(4J_m+2-2b)!(2J_m-\mu-a-b)!}\right]^{\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} &\langle J_m-\mu, J_m-\mu-\tfrac{1}{2}, J_m-b, J_m-a-b-\tfrac{1}{2}|(T_p)NT = T_z = T_p+\tfrac{1}{2}\rangle \\ &\quad = \frac{(-1)^{b-\mu}}{a!\mu!} \\ &\quad \times \left[\frac{(2a+1)!(a+b)!b!(a+b-\mu)!(2\mu)!(4J_m+2-2\mu)!(2J_m-\mu-b)!\times(2J_m-a-b-1)!(2J_m+1-b)!}{(2a+2b+1)!(b-\mu)!(4J_m+2-2b)!(2J_m+1-\mu)!(2J_m-\mu)!\times(2J_m-\mu-a-b-1)!}\right]^{\frac{1}{2}}; \end{aligned} \tag{26a}$$

<u>Case 2: $T = T_p-\frac{1}{2}$</u>

$$\begin{aligned} &\langle J_m-\mu+\tfrac{1}{2}, J_m-\mu, J_m-b-\tfrac{1}{2}, J_m-a-b|(T_p)NT = T_z = T_p-\tfrac{1}{2}\rangle \\ &\quad = \frac{(-1)^{1+b-\mu}}{(\mu-1)!(2J_m+1-\mu)!} \\ &\quad \times \left[\frac{2(2a)!(a+b)!b!(a+b-\mu)!(2\mu-1)!(4J_m+2-2\mu)!(2J_m-\mu-b)!\times(2J_m-b-a)!(2J_m+1-b)!(4J_m+1-2b)}{a!(a-1)!(2a+2b+1)!(b+1-\mu)!(4J_m+2-2b)!(2J_m-\mu-a-b)!}\right]^{\frac{1}{2}}, \end{aligned}$$

$$\langle J_m-\mu-\tfrac{1}{2}, J_m-\mu, J_m-b-\tfrac{1}{2}, J_m-a-b|(T_p)NT = T_z = T_p-\tfrac{1}{2}\rangle$$
$$= \frac{(-1)^{1+b-\mu}}{\mu!(2J_m-\mu)!} \tag{26b}$$
$$\times \left[\frac{2(2a)!(a+b)!b!(a+b-\mu)!(2\mu)!(4J_m+1-2\mu)!(2J_m-\mu-b-1)! \times(2J_m-b-a)!(2J_m+1-b)!(4J_m+1-2b)}{a!(a-1)!(2a+2b+1)!(b-\mu)!(4J_m+2-2b)!(2J_m-\mu-a-b)!}\right]^{\frac{1}{2}}$$

both with $T_p = a,\quad N = 4j+1-2a-4b,\quad \mu = 0, 1, \dots b,\quad N = N_p-1.$

For certain states with $t > \frac{1}{2}$ the concept of a fractional parent T_p may have to be introduced to insure the orthogonality of states † with the same NTT_z but different T_p. The transformation of the R_5 Wigner coefficients to the $|(T_p)TNT_z\rangle$ scheme through transformation coefficients such as those of eqs. (26) involves a summation over the index μ of complicated functions of μ. No techniques have been discovered to perform these in general, but they can be easily performed for specific values of the integer b. From these the dependence on the quantum numbers N and T can be discovered. Specific results are given in ref. [13]).

7. Fractional Parentage Coefficients for Spin-2 Phonons

In the applications involving spin-2 systems the states of physical interest are related to the well known group chain $SU_5 \supset R_5 \supset R_3$. In particular, the states of interest in the case of spin-2 phonons are the totally symmetric states of N spin-2 systems in the seniority scheme. For such states the irreducible representations $[N]$ of SU_5 decompose into irreducible representations $(v0) = (J_m+\Lambda_m, J_m-\Lambda_m)$ of R_5, that is, into states with $J_m = \Lambda_m = \frac{1}{2}v$, where $v = N, (N-2), (N-4), \dots, 0$ (or 1). If b_μ^+ (b_μ) denote the spin-2 phonon creation (annihilation) operators, the infinitesimal operators of R_5 are the odd rank spherical tensor operators

$$T_q^k = \sum_\mu \langle 2\mu kq|2(\mu+q)\rangle b_{\mu+q}^+ b_\mu .$$

The three components of the tensor operator with $k = 1$ are the components of the angular momentum operator $\boldsymbol{I}$. These are related to the standard form of the infinitesimal operators of R_5 (table 1) in the following way:

$$\begin{aligned} I_0 &= 3J_0+\Lambda_0, \\ I_\pm &= 2\Lambda_\pm+\sqrt{6}F_{\pm\frac{1}{2}\mp\frac{1}{2}}. \end{aligned} \tag{27}$$

The operators I_0 and $\boldsymbol{I}^2$ do not form a complete set of commuting operators, and two additional operators are needed to fully specify the states corresponding to a

† Since the label T_p is not the eigenvalue of an operator which commutes with T^2, T_z and N_{op}, it is not a true quantum number and in this sense very similar in its properties to the label K used in SU_3 calculations, ref. [1]).

given irreducible representation of R_5. These operators must be scalars in the space of R_3. Scalar operators, cubic in the T^k_q, with k odd, collapse to operators quadratic in the T^k_q through the commutation relations for odd rank tensor operators, and the quadratic scalars are linear combinations of I^2 and the Casimir invariant for R_5. Scalar operators whose eigenvalues could be used to distinguish states of a given v and I must therefore be operators such as $\sum_q(-1)^{k-q}[T^3\times T^3]^k_q\cdot[T^3\times T^3]^k_{-q}$ with k even. Since these are cumbersome operators with no ready physical significance, and since, for $v \leqq 10$, there are relatively few states for which additional quantum numbers are needed, it may be best to construct linearly independent (orthogonal) states in an arbitrary way in those few cases where the quantum numbers v, I and M_I are insufficient to label the states. (A complete table of v and I values for $v \leqq 18$ has recently been given by LeTourneux [10]).

The fractional parentage coefficients for spin-2 systems are closely related to the Wigner coefficients of SU_5 for the Kronecker product $[N]\times[1]$. (See the discussion given by Moshinsky [2]). For the totally symmetric states $[N]$, with $J_m = \Lambda_m = \frac{1}{2}v$, the basis vectors for SU_5 can be denoted by $|[N](\frac{1}{2}v\,\frac{1}{2}v)J\Lambda M_J M_\Lambda\rangle$, with $J=\Lambda$ in this case, see eq. (11b); or alternately in terms of the physically interesting quantum numbers by $|[N]v\alpha IM_I\rangle$ where α labels the independent (orthogonal) states in some manner in those cases in which there is more than one state I for given v. The fractional parentage coefficients $\langle|\}\rangle$ and the SU_5 Wigner coefficients are related through an ordinary (SU_2) Wigner coefficient

$$\langle[N]v\alpha IM_I;\,[1]1\_2(M_I'-M_I)|[N+1]v'\alpha'I'M_I'\rangle$$
$$= \langle Nv\alpha I;\,11\_2|\}N+1,\,v'\alpha'I'\rangle\langle IM_I 2(M_I'-M_I)|I'M_I'\rangle, \quad (28)$$

where the SU_5 Wigner coefficient can again be factored into an SU_5 double-barred Wigner coefficient and an R_5 Wigner coefficient

$$\langle[N]v\alpha IM_I;\,[1]1\_2(M_I'-M_I)|[N+1]v'\alpha'I'M_I'\rangle$$
$$= \langle[N]v;[1]1\|[N+1]v'\rangle\langle v\alpha IM_I;\,1\_2(M_I'-M_I)|v'\alpha'I'M_I'\rangle. \quad (29)$$

The SU_5 double-barred coefficient is a function only of the quantum numbers which label the irreducible representations of SU_5 and R_5. The R_5 Wigner coefficient is here expressed in the physically interesting basis $|v\alpha IM_I\rangle$. It is related to the R_5 Wigner coefficients of table 3 through a unitary transformation

$$\langle v\alpha IM_I;\,1\_2(M_I'-M_I)v'\alpha'I'M_I'\rangle$$
$$= \sum_{J\Lambda M_J M_\Lambda}\ \sum_{J'\Lambda'M'_J M'_\Lambda}\langle\alpha IM_I|J\Lambda M_J M_\Lambda\rangle\langle(\tfrac{1}{2}v\tfrac{1}{2}v)J\Lambda M_J M_\Lambda;\,(\tfrac{1}{2}\tfrac{1}{2})j\lambda m_j m_\lambda$$
$$|(\tfrac{1}{2}v'\tfrac{1}{2}v')J'\Lambda'M_J'M_\Lambda'\rangle\langle J'\Lambda'M_J'M_\Lambda'|\alpha'I'M_I'\rangle \quad (30)$$

where

$$|v\alpha IM_I\rangle = \sum_{J\Lambda M_J}|(\tfrac{1}{2}v\tfrac{1}{2}v)J\Lambda M_J M_\Lambda\rangle\langle J\Lambda M_J M_\Lambda|\alpha IM_I\rangle,$$

with $J = \Lambda$, and $M_I = 3M_J + M_\Lambda$. States with $M_I = I$ are sufficient to determine the fractional parentage coefficients, and the transformation coefficients $\langle J\Lambda M_J M_\Lambda|\alpha II\rangle$ can be calculated for each v by simple projection techniques. The requirement

$$I_+|v\alpha II\rangle = 0 \tag{31a}$$

leads through

$$\begin{aligned}(2\Lambda_+ + \sqrt{6}F_{\frac{1}{2}-\frac{1}{2}}) &\sum_{JM_J(M_\Lambda)} |(\tfrac{1}{2}v\tfrac{1}{2}v)JJM_JM_\Lambda\rangle\langle JJM_JM_\Lambda|\alpha II\rangle \\ = \sum_{JM_J(M_\Lambda)} \Big\{ & 2[(\Lambda - M_\Lambda)(\Lambda + M_\Lambda + 1)]^{\frac{1}{2}}|(\tfrac{1}{2}v\tfrac{1}{2}v)JJM_J(M_\Lambda+1)\rangle \\ & + \left[\frac{3(v-2J+1)(v+2J+2)(J-M_J)(J+M_\Lambda)}{2J(2J+1)}\right]^{\frac{1}{2}} \\ & \qquad \times |(\tfrac{1}{2}v\tfrac{1}{2}v)(J-\tfrac{1}{2})(J-\tfrac{1}{2})(M_J+\tfrac{1}{2})(M_\Lambda-\tfrac{1}{2})\rangle \\ & + \left[\frac{3(v-2J)(v+2J+3)(J+M_J+1)(J-M_\Lambda+1)}{(2J+1)(2J+2)}\right]^{\frac{1}{2}} \\ & \qquad \times |(\tfrac{1}{2}v\tfrac{1}{2}v)(J+\tfrac{1}{2})(J+\tfrac{1}{2})(M_J+\tfrac{1}{2})(M_\Lambda-\tfrac{1}{2})\rangle\Big\}\langle JJM_JM_\Lambda|\alpha II\rangle\end{aligned} \tag{31b}$$

to a simple system of linear equations in the $\langle JJM_JM_\Lambda|\alpha II\rangle$. Where the label α is required the independent solutions are best chosen arbitrarily, but they must be chosen orthogonal to each other to preserve the unitary character of the transformation.

The SU_5 double-barred coefficients of eq. (29) could be calculated by recursion techniques similar to those outlined for the R_5 coefficients. For the totally symmetric states $[N]$ of interest here, however, they follow, except for a trivial normalization factor, directly from the radial matrix elements for the 5-dimensional harmonic oscillator [23])†. The only non-zero double-barred coefficients involving totally symmetric states $[N]$ are given by

$$\begin{aligned}\langle[N]v;[1]1||[N+1]v+1\rangle &= \left[\frac{(N+v+5)(v+1)}{(2v+5)(N+1)}\right]^{\frac{1}{2}}, \\ \langle[N]v;[1]1||[N+1]v-1\rangle &= \left[\frac{(N-v+2)(v+2)}{(2v+1)(N+1)}\right]^{\frac{1}{2}}.\end{aligned} \tag{32}$$

These carry the whole N-dependence of the fractional parentage coefficients, so that only coefficients with $N = v$ need be explicitly calculated. In addition, fractional parentage coefficients of the type $\langle N = v, v\alpha I; 11\_2|\}v+1, v-1, \alpha'I'\rangle$ can be related

† Matrix elements of the infinitesimal operators L_{jk} and of the 5-dimensional vector operators between states $(J_m\Lambda_m)$ with the special restriction $J_m = \Lambda_m$ also follow from the work of Louck [23]) on the n-dimensional harmonic oscillator.

to those of type $\langle N = v, v\alpha I;\, 11\_2|\}v+1, v+1, \alpha' I'\rangle$ through a symmetry property of the R_5 Wigner coefficients[25]). In the mathematically natural basis, coefficients in which the role of the irreducible representations $(J_{m_1} \Lambda_{m_1})$ and $(J_{m_3} \Lambda_{m_3})$ are interchanged, are related by (see appendix 2)

$$\begin{aligned}
&\langle (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 M_{J_1} M_{\Lambda_1};\, (\tfrac{1}{2}\tfrac{1}{2}) J_2 \Lambda_2 M_{J_2} M_{\Lambda_2} | (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 M_{J_3} M_{\Lambda_3}\rangle \\
&\quad = (-1)^{J_{m_1} - J_{m_3} + \Lambda_{m_1} - \Lambda_{m_3} - \tilde{J}_2 - \tilde{\Lambda}_2 + J_2 + \Lambda_2 + M_{J_2} + M_{\Lambda_2}} \left[\frac{\dim (J_{m_3} \Lambda_{m_3})}{\dim (J_{m_1} \Lambda_{m_1})}\right]^{\frac{1}{2}} \\
&\quad \times \langle (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 M_{J_3} M_{\Lambda_3};\, (\tfrac{1}{2}\tfrac{1}{2}) J_2 \Lambda_2 - M_{J_2} - M_{\Lambda_2} | (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 M_{J_1} M_{\Lambda_1}\rangle, \quad (33)
\end{aligned}$$

where the dimension of the irreducible representation of R_5, eq. (12), for the case $J_m = \Lambda_m = \frac{1}{2}v$ becomes, $\dim(v) = \frac{1}{6}(v+1)(v+2)(2v+3)$. The symbols $\tilde{J}_2, \tilde{\Lambda}_2$ are defined in appendix 2. For both cases $v' = v \pm 1$, $(J_{m_3} - J_{m_1} = \Lambda_{m_3} - \Lambda_{m_1} = \pm\frac{1}{2})$, they have the specific values $\tilde{J}_2 = \tilde{\Lambda}_2 = \frac{1}{2}$. The phase factor for both $v' = v \pm 1$ thus reduces to $(-1)^{J_2 + \Lambda_2 + M_{J_2} + M_{\Lambda_2}}$ and (through $J_2 = \Lambda_2$, $M_{I_2} = 3M_{J_2} + M_{\Lambda_2}$) to $(-1)^{2J_2 - 2M_{J_2} + M_{I_2}} = (-1)^{M_{I_2}}$, a phase independent of the quantum numbers $J, \Lambda, M_J, M_\Lambda$. The R_5 Wigner coefficient in the $|v\alpha I M_I\rangle$ basis therefore has a similar symmetry property

$$\begin{aligned}
&\langle v\alpha I M_I;\, 1_2 M_{I_2} | v'\alpha' I' M_I'\rangle \\
&\quad = (-1)^{+M_{I_2}} \left[\frac{\dim (v')}{\dim (v)}\right]^{\frac{1}{2}} \langle v'\alpha' I' M_I';\, 1_2, -M_{I_2} | v\alpha I M_I\rangle. \quad (34)
\end{aligned}$$

Eqs. (33) and (34) are the generalization for R_5 of the well-known symmetry property of the ordinary (SU_2) Wigner coefficient

$$\langle I_1 M_1, I_2 M_2 | I_3 M_3\rangle = (-1)^{I_3 - I_1 + M_{I_2}} \left[\frac{2I_3+1}{2I_1+1}\right]^{\frac{1}{2}} \langle I_3 M_3, I_2 - M_{I_2} | I_1 M_1\rangle. \quad (35)$$

The symmetry properties, eqs. (34) and (35), when substituted into eqs. (28) and (29), relate the coefficients in which the role of v and v' are interchanged. In summary

$$\begin{aligned}
&\langle N v \alpha_v I_v;\, 11_2|\}(N+1)(v-1)\alpha_{v-1} I_{v-1}\rangle \\
&\quad = (-1)^{I_{v-1} - I_v} \left[\frac{(N-v+2)(v+2) \dim (v-1)(2I_v+1)}{(2v+1)(N+1) \dim (v)(2I_{v-1}+1)}\right]^{\frac{1}{2}} \\
&\qquad \times \langle v-1, v-1, \alpha_{v-1} I_{v-1};\, 11_2|\} v v \alpha_v I_v\rangle \quad (36)
\end{aligned}$$

$$\begin{aligned}
&\langle N v \alpha_v I_v;\, 11-2|\}(N+1)(v+1)\alpha_{v+1} I_{v+1}\rangle \\
&\quad = \left[\frac{(N+v+5)(v+1)}{(2v+5)(N+1)}\right]^{\frac{1}{2}} \langle v v \alpha_v I_v;\, 11_2|\} v+1, v+1, \alpha_{v+1} I_{v+1}\rangle.
\end{aligned}$$

Thus all fractional parentage coefficients have been related to those of type $\langle vv\alpha_v I_v;\, 11\_2|\}v+1,\, v+1,\, \alpha_{v+1} I_{v+1}\rangle$, and these are the only ones which remain to be calculated. Numerical values of such coefficients are given in table 5 for the case $v+1 = 5$ and 6, for which the transformation from the mathematically natural to the physically interesting quantum numbers has been carried out through explicit use of eq. (31).

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Appendix 1

CALCULATION OF THE NORMALIZATION CONSTANT

Since the matrix elements associated with the M-step down operators, J_- and Λ_-, are well-known, it is sufficient to calculate the normalization constant associated with the J and Λ step operators, the constant $N(n, m)$ of eq. (11). Its value follows from the condition

$$\langle \max|(O^\dagger_{-+})^n(O^\dagger_{--})^m O^m_{--}\, O^n_{-+}|\max\rangle |N(n, m)|^2 = 1 \tag{A.1}$$

and it is best evaluated by recursion techniques. In eq. (A.1) the state of maximum weight $|(J_m\Lambda_m)J_m\Lambda_m J_m\Lambda_m\rangle$ is denoted by $|\max\rangle$, and the hermitian conjugate of the operator O_{--}, for example, is given by

$$\begin{aligned} O^\dagger_{--} = {} & (A_{41}+A_{23})A_{34}A_{12}+(A_{11}-A_{22}+1)(A_{13}-A_{42})A_{34} \\ & +(A_{33}-A_{44}+1)(A_{31}-A_{24})A_{12}-(A_{33}-A_{44}+1)(A_{11}-A_{22}+1)(A_{14}+A_{32}). \end{aligned} \tag{A.2}$$

Since the operators $A_{12}(=J_+)$ and $A_{34}(=\Lambda_+)$ give zero when operating on a state with $M_J = J$ and $M_\Lambda = \Lambda$, substitution of (A.2) for one of the operators $O^\dagger_{--}$ in eq. (4.1) leads to

$$\begin{aligned} & -(2\Lambda_m+n-m+2)(2J_m-n-m+2) \\ & \qquad \langle \max|(O^\dagger_{-+})^n(O^\dagger_{--})^{m-1}(A_{14}+A_{32})O^m_{--}O^n_{-+}|\max\rangle|N(n, m)|^2 = 1. \end{aligned} \tag{A.3}$$

The commutator of $(A_{14}+A_{32})$ with O_{--} is relatively complicated but when operating on a state with $M_J = J$ and $M_\Lambda = \Lambda$ it can be expressed simply in terms of the operators J_0, Λ_0 and G, the Casimir operator (eq. (16))

$$\begin{aligned} & [(A_{14}+A_{32}),\, O_{--}]O^{m'}_{--}O^n_{-+}|\max\rangle \\ & \quad = 4\{G-(J_0+\Lambda_0)(J_0+\Lambda_0+2)\}(J_0+\Lambda_0+1)O^{m'}_{--}O^n_{-+}|\max\rangle \qquad \text{(A.4)} \\ & \quad = -4\{(2J_m+1)\Lambda_m-2m'(J_m+\Lambda_m+1)+m'^2\}(J_m+\Lambda_m+1-m')O^{m'}_{--}O^n_{-+}|\max\rangle. \end{aligned}$$

By repeated application of this commutator the operator $(A_{14}+A_{32})$ can be "worked to the right" in eq. (A.3) to give

$$(A_{14}+A_{32})O^{m}_{--}O^{n}_{-+}|\max\rangle = -4\sum_{m'=0}^{m-1}\{(2J_m+1)\Lambda_m-2m'(J_m+\Lambda_m+1)+m'^2\}$$
$$\times(J_m+\Lambda_m+1-m')O^{m-1}_{--}O^{n}_{-+}|\max\rangle$$
$$= -m(2J_m+2\Lambda_m+3-m)(2J_m+2-m)(2\Lambda_m+1-m)O^{m-1}_{--}O^{n}_{-+}|\max\rangle, \quad (A.5)$$

in which the relation $(A_{14}+A_{32})O^{n}_{-+}|\max\rangle = 0$ has been used. This follows again from repeated application of the commutator of $(A_{14}+A_{32})$ now with O_{-+} and the condition $(A_{14}+A_{32})|\max\rangle = 0$. Eqs. (A.5) and (A.3) together give

$$(2\Lambda_m+n-m+2)(2J_m-n-m+2)m(2J_m+2\Lambda_m+3-m)(2J_m+2-m)(2\Lambda_m+1-m)$$
$$\times\langle\max|(O^{\dagger}_{-+})^{n}(O^{\dagger}_{--})^{m-1}(O_{--})^{m-1}O^{n}_{-+}|\max\rangle|N(n,m)|^2 = 1, \quad (A.6)$$

which leads to the desired recursion relation

$$\frac{|N(n,m)|^2}{|N(n,m-1)|^2}$$
$$= \frac{1}{(2\Lambda_m+n-m+2)(2J_m-n-m+2)m(2J_m+2\Lambda_m+3-m)(2J_m+2-m)(2\Lambda_m+1-m)}. \quad (A.7)$$

Repeated application of this recursion relation gives

$$\frac{|N(n,m)|^2}{|N(n,0)|^2}$$
$$= \frac{(2\Lambda_m+1+n-m)!(2J_m+1-n-m)!(2J_m+2\Lambda_m+2-m)!(2J_m+1-m)!(2\Lambda_m-m)!}{(2\Lambda_m+1+n)!(2J_m+1-n)!m!(2J_m+2\Lambda_m+2)!(2J_m+1)!(2\Lambda_m)!}. \quad (A.8)$$

The constant $N(n,0)$ follows from the condition

$$\langle\max|(O^{\dagger}_{-+})^{n}O^{n}_{-+}|\max\rangle|N(n,0)|^2 = 1 \quad (A.9)$$

and is evaluated by similar techniques. Substitution of $O^{\dagger}_{-+} = (A_{41}+A_{23})A_{12} +(A_{11}-A_{22}+1)(A_{13}-A_{42})$ for one of the factors in this relation leads to

$$(2J_m+2-n)\langle\max|(O^{\dagger}_{-+})^{n-1}(A_{13}-A_{42})O^{n}_{-+}|\max\rangle|N(n,0)|^2 = 1. \quad (A.10)$$

Repeated application of the commutation property

$$[(A_{13}-A_{42}),O_{-+}]O^{n'}_{-+}|\max\rangle$$
$$= 2\{3J_0^2+3J_0+\Lambda_0^2-2J_0\Lambda_0-G\}O^{n'}_{-+}|\max\rangle$$
$$= 2\{(2J_m+1)(J_m-\Lambda_m)+n'(2\Lambda_m-4J_m-\tfrac{3}{2})+\tfrac{3}{2}n'^2\}O^{n'}_{-+}|\max\rangle \quad (A.11)$$

leads with eq. (A.10) to the desired recursion relation

$$(2J_m+2-n)n(2J_m+2-n)(2J_m-2\Lambda_m+1-n)$$
$$\times\langle \max|(O^\dagger_{-+})^{n-1}O^{n-1}_{-+}|\max\rangle|N(n,0)|^2 = 1. \quad (A.12)$$

Repeated application of the new recursion relations gives

$$|N(n,0)|^2 = \frac{[(2J_m+1-n)!]^2(2J_m-2\Lambda_m-n)!}{n![(2J_m+1)!]^2(2J_m-2\Lambda_m)!}. \quad (A.13)$$

Together with the phase convention, by which $N(n, m)$ is chosen real and positive, eqs. (A.13) and (A.8) give the normalization constant.

Appendix 2

SYMMETRY PROPERTY OF THE WIGNER COEFFICIENTS

The symmetry property expressed by eq. (33) follows from the behaviour under complex conjugation of the basis vectors of the irreducible representations. Except for a possible overall phase factor which can be a function only of the quantum numbers J_m and Λ_m the behaviour under complex conjugation is given by the properties of the basis vectors associated with the subgroup $SU_2\times SU_2$

$$|(J_m\Lambda_m)J\Lambda M_J M_\Lambda\rangle^* = C(J_m\Lambda_m)(-1)^{J+M_J+\Lambda+M_\Lambda}|(J_m\Lambda_m)J\Lambda, -M_J, -M_\Lambda\rangle, \quad (A.14)$$

where the phase factor $C(J_m, \Lambda_m)$ is independent of $J, \Lambda, M_J, M_\Lambda$, with $|C| = 1$. This expression can be used to determine the behaviour under complex conjugation of the matrix element of a finite 5-dimensional rotation, the D-function for R_5. In what follows it will be convenient to use a shorthand notation. Let (j) stand for the quantum numbers $(J_m\Lambda_m)$ which characterize the irreducible representation, and let m stand for the set of quantum numbers $J, \Lambda, M_J, M_\Lambda$, with $-m \equiv J, \Lambda, -M_J, -M_\Lambda$. In addition, use $\sigma(m)$ for the m-dependent function in the phase factor of eq. (A.14), $\sigma = J+M_J+\Lambda+M_\Lambda$. If the basis vector $|(j)m\rangle$ is transformed into $|(j)m'\rangle'$ under a finite 5-dimensional rotation, denoted by θ, which stands collectively for the ten parameters of the group, the D-functions are defined by

$$|(j)m\rangle = \sum_{m'} D^{(j)}_{mm'}(\theta)|(j)m'\rangle'. \quad (A.15)$$

From eqs. (A.14) and (A.15) the complex conjugate of the D-function for R_5 is given by the relation

$$D^{(j)}_{mm'}(\theta)^* = (-1)^{\sigma'-\sigma}D^{(j)}_{-m\,-m'}(\theta). \quad (A.16)$$

If the ten-parameter volume element for the group is normalized such that $\int d\Omega(\theta) = 1$, integrals involving three D-functions can be expressed in terms of R_5

Wigner coefficients by

$$I = I^* = \int D^{(j_3)}_{m_3 m'_3}(\theta)^* D^{(j_1)}_{m_1 m'_1}(\theta) D^{(j_2)}_{m_2 m'_2}(\theta) \mathrm{d}\Omega(\theta)$$

$$= \frac{1}{\dim(j_3)} \sum_\rho \langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle_\rho \langle j_1 m'_1; j_2 m'_2 | j_3 m'_3 \rangle_\rho, \tag{A.17}$$

(see, e.g., ref. [24])). Using eq. (A.16) for the D-function associated with (j_2) in the expression for I^*, the integral can also be expressed by

$$I^* = \frac{1}{\dim(j_1)} \sum_\rho (-1)^{\sigma(m_2)} \langle j_3 m_3; j_2 - m_2 | j_1 m_1 \rangle (-1)^{\sigma(m'_2)} \langle j_3 m'_3; j_2 - m'_2 | j_1 m'_1 \rangle. \tag{A.18}$$

Since the two expressions for I hold for all arbitrary values of m and m', the coefficients $(-1)^{\sigma(m_2)} \langle j_3 m_3; j_2 - m_2 | j_1 m_1 \rangle$ and $\langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle$ are related through an m-independent proportionality factor, at least in those cases in which the index ρ has only one value and is thus a redundant quantum number. In particular

$$\langle j_1 m_1; j_2 m_2 | j_3 m_3 \rangle = \left[\frac{\dim(j_3)}{\dim(j_1)}\right]^{\frac{1}{2}} c(j)(-1)^{\sigma(m_2)} \langle j_3 m_3; j_2 - m_2 | j_1 m_1 \rangle, \tag{A.19}$$

where the phase factor $c(j)$ is independent of the quantum numbers m. Reverting to the full notation for the quantum numbers this becomes

$$\begin{aligned} &\langle (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 M_{J_1} M_{\Lambda_1}; (J_{m_2} \Lambda_{m_2}) J_2 \Lambda_2 M_{J_2} M_{\Lambda_2} | (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 M_{J_3} M_{\Lambda_3} \rangle \\ &\quad = \left[\frac{\dim(J_{m_3} \Lambda_{m_3})}{\dim(J_{m_1} \Lambda_{m_1})}\right]^{\frac{1}{2}} c(J_m \Lambda_m)(-1)^{J_2 + M_{J_2} + \Lambda_2 + M_{\Lambda_2}} \\ &\quad \times \langle (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 M_{J_3} M_{\Lambda_3}; (J_{m_2} \Lambda_{m_2}) J_2 \Lambda_2 - M_{J_2} - M_{\Lambda_2} | (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 M_{J_1} M_{\Lambda_1} \rangle \end{aligned} \tag{A.20a}$$

where $|c(J_m \Lambda_m)| = 1$, and arg c follows from the phase convention for the Wigner coefficients with $J_1 = M_{J_1} = J_{m_1}$, $\Lambda_1 = M_{\Lambda_1} = \Lambda_{m_1}$, $J_3 = M_{J_3} = J_{m_3}$, $\Lambda_3 = M_{\Lambda_3} = \Lambda_{m_3}$. For these values of the quantum numbers both Wigner coefficients in eq. (A.20a) are positive, the M quantum numbers are related by $M_{J_2} = J_{m_3} - J_{m_1}$, $M_{\Lambda_2} = \Lambda_{m_3} - \Lambda_{m_1}$, while the values of J_2 and Λ_2 are uniquely determined by the quantum numbers J_{m_1}, Λ_{m_1}, J_{m_3}, Λ_{m_3} in those cases in which the index ρ is not needed. If these special values of J_2 and Λ_2 are denoted by $\tilde{J}_2$ and $\tilde{\Lambda}_2$, the phase of c is given by

$$\arg c = (-1)^{J_{m_1} - J_{m_3} + \Lambda_{m_1} - \Lambda_{m_3} - \tilde{J}_2 - \tilde{\Lambda}_2}. \tag{A.20b}$$

Using the analogous symmetry property of the ordinary (SU_2) Wigner coefficient, for example

$$\langle J_1 M_1\, J_2 M_2 | J_3 M_3\rangle = \left[\frac{2J_3+1}{2J_1+1}\right]^{\frac{1}{2}} (-1)^{J_1-J_3+M_2} \langle J_3 M_3\, J_2 -M_2 | J_1 M_1\rangle. \quad \text{(A.21)}$$

The symmetry relation (A.20) can be written in terms of the double-barred R_5 Wigner coefficients

$$\begin{aligned}\langle (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 ; (J_{m_2} \Lambda_{m_2}) J_2 \Lambda_2 || (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 \rangle & \\ = \left[\frac{\dim (J_{m_3} \Lambda_{m_3})}{\dim (J_{m_1} \Lambda_{m_1})} \frac{(2J_1+1)}{(2J_3+1)} \frac{(2\Lambda_1+1)}{(2\Lambda_3+1)}\right]^{\frac{1}{2}} & \\ \times (-1)^{J_{m_1}-J_{m_3}+\Lambda_{m_1}-\Lambda_{m_3}-\tilde{J}_2-\tilde{\Lambda}_2+J_2+J_3-J_1+\Lambda_2+\Lambda_3-\Lambda_1} & \\ \times \langle (J_{m_3} \Lambda_{m_3}) J_3 \Lambda_3 ; (J_{m_2} \Lambda_{m_2}) J_2 \Lambda_2 || (J_{m_1} \Lambda_{m_1}) J_1 \Lambda_1 \rangle. & \quad \text{(A.22)}\end{aligned}$$

For those Kronecker products for which the index ρ has more than one value, there will be more than one set of values for $\tilde{J}_2 \tilde{\Lambda}_2$. If the various coupled states $\rho = 1, 2, \ldots$ can be chosen such that $(-1)^{\tilde{J}_2+\tilde{\Lambda}_2}$ has a unique value for each ρ, the symmetry relations (A.20) and (A.22) can be made valid in this case also. In the Kronecker product $(J_m \Lambda_m) \times (10)$ with $(J'_m \Lambda'_m) = (J_m \Lambda_m)$ of interest in this investigation the values $\tilde{J}_2 \tilde{\Lambda}_2$ satisfy the restriction $\tilde{J}_2 + \tilde{\Lambda}_2 = 1$. The Wigner coefficients $\langle (J_m \Lambda_m) J_1 \Lambda_1 ; (10) J_2 \Lambda_2 || (J_m \Lambda_m) J_3 \Lambda_3 \rangle_\rho$ with $\rho = 1$ and 2, eqs. (15) and table 4a, thus satisfy the symmetry relation (A.22). The symmetry relation (A.22) considerably reduces the number of independent Wigner coefficient in tables 2–4.

Appendix 3

CALCULATION OF THE TRANSFORMATION COEFFICIENTS $\langle J\Lambda M_J M_\Lambda | (T_v) T N T_z \rangle$ FOR STATES WITH $v = 0$

States with $v = 0$ and arbitrary nucleon number N and isospin T, but with $T = T_z$, can be generated from the maximum weight state, ($N = 4j+2$, $T = 0$), through successive application of the two commuting step-down operators Λ_- and $(F^2_{-\frac{1}{2}-\frac{1}{2}} - 2J_- \Lambda_-)$, (see eq. (22a)),

$$\begin{aligned}|(v = 0) N = 4j+2-2a-4b,\, T = T_z = a\rangle & \\ = \mathcal{N}(a, b) \Lambda_-^a (F^2_{-\frac{1}{2}-\frac{1}{2}} - 2J_- \Lambda_-)^b | (J_m J_m) J_m J_m J_m J_m \rangle. & \quad \text{(A.23)}\end{aligned}$$

Since the matrix elements of the angular momentum-type operator Λ_- are well-known it is convenient to consider first the case $a = 0$. Since $T = 0$ in this case, the calculations are simple. The normalization constant is again evaluated from the condition

$$\langle \max | (O^\dagger)^b O^b | \max \rangle | \mathcal{N}(0, b) |^2 = 1,$$

through recursion techniques. The short-hand notation $O=(F^2_{-\frac{1}{2}-\frac{1}{2}}-2J_-\Lambda_-)$ and $O^\dagger=(F^2_{\frac{1}{2}\frac{1}{2}}-2\Lambda_+J_+)$ has been used. In general the commutator of $O^\dagger$ with O is complicated, but when operating on a state with $T=0$ it can again be expressed as a function of the operators J_0, Λ_0 and the R_5 Casimir operator G:

$$
\begin{aligned}
&[O^\dagger, O]O^\sigma|(J_mJ_m)J_mJ_mJ_mJ_m\rangle \\
&\qquad =\{2(J_0+\Lambda_0)[2G-(J_0+\Lambda_0)^2]+3(J_0+\Lambda_0)\}O^\sigma|(J_mJ_m)J_mJ_mJ_mJ_m\rangle \qquad \text{(A.24)}\\
&\qquad = F(\sigma, J_m)O^\sigma|\max\rangle = 2(\sigma-J_m)[8\sigma^2-16J_m\sigma-(12J_m+3)]O^\sigma|\max\rangle .
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\mathcal{N}(0,b)|^2\langle\max|(O^\dagger)^bO^b|\max\rangle &= \sum_{\sigma=0}^{b-1}F(\sigma,J_m)\langle\max|(O^\dagger)^{b-1}O^{b-1}|\max\rangle|\mathcal{N}(0,b)|^2\\
&= \frac{|\mathcal{N}(0,b)|^2}{|\mathcal{N}(0,b-1)|^2}\,b(2b+1)(4J_m+3-2b)(2J_m+1-b) = 1.
\end{aligned}
$$

This recursion formula leads to

$$
\mathcal{N}(0,b) = 2^b\left[\frac{(4J_m+1-2b)!}{(4J_m+1)!(2b+1)!}\right]^{\frac{1}{2}}. \qquad \text{(A.25)}
$$

From the matrix elements of the infinitesimal operators it can be seen that the operator O shifts J in integral steps only when acting on a state $|(J_mJ_m)JJMM\rangle$:

$$
\begin{aligned}
O|(J_mJ_m)JJMM\rangle &= f_-(J_mJ,M)|(J_mJ_m)J-1,J-1,M-1,M-1\rangle\\
&\quad +f_0(J_m,J,M)|(J_mJ_m)JJM-1,M-1\rangle\\
&\quad +f_+(J_m,J,M)|(J_mJ_m)J+1,J+1,M-1,M-1\rangle, \qquad \text{(A.26)}
\end{aligned}
$$

where $f_-(J_m,J,M)$, for example, has the value

$$
\begin{aligned}
&f_-(J_m,J_m-\mu,M=J_m-\nu)=f_-(\mu,\nu)\\
&\quad =\tfrac{1}{2}\left[\frac{(2\mu+1)(2\mu+2)(4J_m+1-2\mu)(4J_m+2-2\mu)}{(2J_m+1-2\mu)(2J_m-1-2\mu)}\right]^{\frac{1}{2}}\\
&\qquad\qquad\times\frac{(2J_m-\mu-\nu)(2J_m-\mu-\nu-1)}{(2J_m-2\mu)}. \qquad \text{(A.27)}
\end{aligned}
$$

The expansion coefficients of the function $\mathcal{N}O^b|(J_mJ_m)J_mJ_mJ_mJ_m\rangle$ can thus be characterized by a single integer μ:

$$
\mathcal{N}O^b|(J_mJ_m)J_mJ_mJ_mJ_m\rangle = \sum_{\mu=0}^{b}c_\mu(a=0,b)|(J_mJ_m)J_m-\mu,J_m-\mu,J_m-b,J_m-b\rangle .
$$

In particular

$$c_b(0,b) = \mathcal{N}(0,b)\prod_{\mu=0}^{b-1} f_-(\mu,\mu) = \left[\frac{2(2J_m+1-2b)}{(2b+1)(4J_m+2-2b)}\right]^{\frac{1}{2}}. \tag{A.28}$$

The remaining coefficients c_μ are best evaluated through recursion techniques. Since the state $O^b|\max\rangle$ has $T = 0$

$$T_\mp O^b|\max\rangle = 0$$

or

$$F_{\mp\frac{1}{2},\pm\frac{1}{2}} \sum_{\mu=0}^{b} c_\mu |(J_m J_m)(J_m-\mu), (J_m-\mu), J_m-b, J_m-b\rangle = 0,$$

which leads to the recursion formula

$$\frac{c_{\mu-1}}{c_\mu} = -\left[\frac{(2\mu-1)(4J_m+4-2\mu)(2J_m+1-2\mu)}{2\mu(4J_m+3-2\mu)(2J_m+3-2\mu)}\right]^{\frac{1}{2}}. \tag{A.29}$$

Together with eq. (A.28) this gives the expansion coefficient

$$c_\mu(0,b) = (-1)^{b-\mu}\frac{b!}{\mu!}\frac{(2J_m+1-b)!}{(2J_m+1-\mu)!}\left[\frac{2(2\mu)!(4J_m+2-2\mu)!(2J_m+1-2\mu)}{(2b+1)!(4J_m+2-2b)!(4J_m+2-2b)}\right]^{\frac{1}{2}}. \tag{A.30}$$

The expansion coefficients $c_\mu(a,b)$ for the general function $\mathcal{N}\Lambda^a O^b|\max\rangle$ follow from those with $a = 0$ from the usual angular momentum type matrix elements of Λ_-:

$$\Lambda_-^a|(J_m J_m)J_m-\mu, J_m-\mu, J_m-b, J_m-b\rangle$$
$$= \left[\frac{(2J_m-\mu-b)!(b-\mu+a)!}{(2J_m-\mu-b-a)!(b-\mu)!}\right]^{\frac{1}{2}}|(J_m J_m)J_m-\mu, J_m-\mu, J_m-b, J_m-b-a\rangle, \tag{A.31}$$

so that

$$c_\mu(a,b) = \frac{\mathcal{N}(a,b)}{\mathcal{N}(0,b)}c_\mu(0,b)\left[\frac{(2J_m-\mu-b)!(b-\mu+a)!}{(2J_m-\mu-b-a)!(b-\mu)!}\right]^{\frac{1}{2}}. \tag{A.32}$$

The new normalization constant follows from the condition $\sum_\mu |c_\mu(a,b)|^2 = 1$. No elegant techniques have been discovered to perform this sum. Straightforward but tedious calculation shows that

$$\frac{\mathcal{N}(a,b)}{\mathcal{N}(0,b)} = \left[\frac{(2J_m-b-a)!(2b+1)!(2a+1)!(a+b)!}{(2J_m-b)!(a!)^2 b!(2a+2b+1)!}\right]^{\frac{1}{2}} \tag{A.33}$$

for all values of b which might be of interest in nucleon configurations $(j)^N$. Eqs. (A.30), (A.32) and (A.33) give the full transformation coefficient $\langle J\Lambda M_J M_\Lambda|NT=T_z\rangle$ for states with $v = 0$, eq. (23b) of the text.

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TABLE 2

$\langle (J_m\Lambda_m)J_1\Lambda_1;\ (\tfrac{1}{2}0)J_2\Lambda_2 \| (J'_m\Lambda'_m)J\Lambda\rangle$
with $J = J_m+\tfrac{1}{2}-\tfrac{1}{2}n-\tfrac{1}{2}m,\quad \Lambda = \Lambda_m+\tfrac{1}{2}n-\tfrac{1}{2}m$

$(J'_m\Lambda'_m)$	$J_1 = J+\tfrac{1}{2}, \Lambda_1 = \Lambda \quad J_2\Lambda_2 = \tfrac{1}{2}0$	$J_1 = J-\tfrac{1}{2}, \Lambda_1 = \Lambda \quad J_2\Lambda_2 = \tfrac{1}{2}0$
$(J_m+\tfrac{1}{2}, \Lambda_m)$	$-\left[\dfrac{nm(2\Lambda_m+1+n)(2\Lambda_m+1-m)}{(2J+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(2J_m+2-n)(2J_m+2-m)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m)$	$\left[\dfrac{(2J_m+2-n)(2J_m+2-m)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{nm(2\Lambda_m+1+n)(2\Lambda_m+1-m)}{(2J+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m, \Lambda_m+\tfrac{1}{2})$	$-\left[\dfrac{m(2J_m+2-m)(2\Lambda_m+1+n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$-\left[\dfrac{n(2J_m+2-n)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m, \Lambda_m-\tfrac{1}{2})$	$-\left[\dfrac{n(2J_m+2-n)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{m(2J_m+2-m)(2\Lambda_m+1+n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$

$(J'_m\Lambda'_m)$	$J_1 = J, \Lambda_1 = \Lambda+\tfrac{1}{2} \qquad J_2\Lambda_2 = 0\tfrac{1}{2}$	$J_1 = J, \Lambda_1 = \Lambda-\tfrac{1}{2} \qquad J_2\Lambda_2 = 0\tfrac{1}{2}$
$(J_m+\tfrac{1}{2}, \Lambda_m)$	$-\left[\dfrac{m(2\Lambda_m+1-m)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)}{(2\Lambda+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{n(2\Lambda_m+1+n)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)}{(2\Lambda+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m)$	$\left[\dfrac{n(2\Lambda_m+1+n)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)}{(2\Lambda+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{m(2\Lambda_m+1-m)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)}{(2\Lambda+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m, \Lambda_m+\tfrac{1}{2})$	$\left[\dfrac{nm(2J_m+2-n)(2J_m+2-m)}{(2\Lambda+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$\left[\dfrac{(2\Lambda_m+1+n)(2\Lambda_m+1-m)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+3-m)}{(2\Lambda+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m, \Lambda_m-\tfrac{1}{2})$	$\left[\dfrac{(2\Lambda_m+1+n)(2\Lambda_m+1-m)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+3-m)}{(2\Lambda+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$	$-\left[\dfrac{nm(2J_m+2-n)(2J_m+2-m)}{(2\Lambda+1)(2\Lambda_m+1)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$

TABLE 3

$\langle (J_m \Lambda_m) J_1 \Lambda_1;\ (\tfrac{1}{2}\tfrac{1}{2}) J_2 \Lambda_2 \| (J'_m \Lambda'_m) J \Lambda \rangle$

with $J = J_m - \tfrac{1}{2}n - \tfrac{1}{2}m, \quad \Lambda = \Lambda_m + \tfrac{1}{2}n - \tfrac{1}{2}m$

$(J'_m \Lambda'_m)$	$J_1 = J+\tfrac{1}{2}, \Lambda_1 = \Lambda+\tfrac{1}{2} \qquad J_2 = \Lambda_2 = \tfrac{1}{2}$
$(J_m+\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$-\left[\dfrac{m(m+1)(2J_m+2-n)(2\Lambda_m+2+n)(2\Lambda_m+1-m)(2J_m+2-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+2)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$\left[\dfrac{(2J_m+1-n)(2\Lambda_m+1+n)(2J_m+2-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
(J_m, Λ_m)	$-\dfrac{(J_m-\Lambda_m-n)[m(2\Lambda_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)]^{\frac{1}{2}}}{[2(2J+1)(2\Lambda+1)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$(J_m+\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$-\left[\dfrac{m(n+1)(2J_m-2\Lambda_m+1-n)(2\Lambda_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m+1)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$\left[\dfrac{nm(2J_m-2\Lambda_m-n)(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$

$(J'_m \Lambda'_m)$	$J_1 = J-\tfrac{1}{2}, \Lambda_1 = \Lambda-\tfrac{1}{2} \qquad J_2 = \Lambda_2 = \tfrac{1}{2}$
$(J_m+\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$\left[\dfrac{(2J_m+2-n)(2\Lambda_m+2+n)(2J_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+2)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$-\left[\dfrac{m(m+1)(2J_m+1-n)(2\Lambda_m+1+n)(2J_m+1-m)(2\Lambda_m-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
(J_m, Λ_m)	$\dfrac{-(J_m-\Lambda_m-n)[(m+1)(2\Lambda_m-m)(2J_m+1-m)(2J_m+2\Lambda_m+2-m)]^{\frac{1}{2}}}{[2(2J+1)(2\Lambda+1)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$(J_m+\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$-\left[\dfrac{(n+1)(m+1)(2J_m-2\Lambda_m+1-n)(2J_m+2-m)(2J_m+1-m)(2J_m+2\Lambda_m+2-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m+1)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$\left[\dfrac{n(m+1)(2J_m-2\Lambda_m-n)(2\Lambda_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m)(2J-2\Lambda_m+1)}\right]^{\frac{1}{2}}$

$(J'_m \Lambda'_m)$	$J_1 = J, \Lambda_1 = \Lambda \qquad J_2 = \Lambda_2 = 0$
$(J_m+\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$\left[\dfrac{(m+1)(2J_m+2-n)(2\Lambda_m+2+n)(2J_m+2\Lambda_m+3-m)}{(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+2)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$\left[\dfrac{m(2J_m+1-n)(2\Lambda_m+1+n)(2J_m+2\Lambda_m+2-m)}{(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
(J_m, Λ_m)	$\dfrac{(J_m-\Lambda_m-n)(J_m+\Lambda_m+1-m)}{[(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$(J_m+\tfrac{1}{2}, \Lambda_m-\tfrac{1}{2})$	$\left[\dfrac{(n+1)(2J_m-2\Lambda_m+1-n)(2\Lambda_m-m)(2J_m+2-m)}{(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m+1)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$
$(J_m-\tfrac{1}{2}, \Lambda_m+\tfrac{1}{2})$	$-\left[\dfrac{n(2J_m-2\Lambda_m-n)(2\Lambda_m+1-m)(2J_m+1-m)}{(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$

TABLE 3 (continued)

$\langle (J_m \Lambda_m) J_1 \Lambda_1; (\frac{1}{2}\frac{1}{2}) J_2 \Lambda_2 || (J'_m \Lambda'_m) J \Lambda \rangle$

with $J = J_m - \frac{1}{2}n - \frac{1}{2}m, \quad \Lambda = \Lambda_m + \frac{1}{2}n - \frac{1}{2}m$

$(J'_m \Lambda'_m)$	$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda - \frac{1}{2} \qquad J_2 = \Lambda_2 = \frac{1}{2}$
$(J_m+\frac{1}{2}, \Lambda_m+\frac{1}{2})$	$\left[\frac{n(m+1)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+2)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\frac{1}{2}, \Lambda_m-\frac{1}{2})$	$-\left[\frac{nm(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)(2J_m+2\Lambda_m+2-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m \Lambda_m)$	$\frac{(J_m+\Lambda_m+1-m)[n(2\Lambda_m+1+n)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)]^{\frac{1}{2}}}{[2(2J+1)(2\Lambda+1)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$(J_m+\frac{1}{2}, \Lambda_m-\frac{1}{2})$	$\left[\frac{n(n+1)(2\Lambda_m+1+n)(2J_m+2-m)(2J_m+2-n)(2\Lambda_m-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m+1)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$
$(J_m-\frac{1}{2}, \Lambda_m+\frac{1}{2})$	$\left[\frac{(2\Lambda_m+1+n)(2J_m+2-n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)(2J_m+1-m)(2\Lambda_m+1-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$

$(J'_m \Lambda'_m)$	$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda + \frac{1}{2} \qquad J_2 = \Lambda_2 = \frac{1}{2}$
$(J_m+\frac{1}{2}, \Lambda_m+\frac{1}{2})$	$\left[\frac{(n+1)(m+1)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)(2J_m+2\Lambda_m+3-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+2)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m-\frac{1}{2}, \Lambda_m-\frac{1}{2})$	$-\left[\frac{(n+1)m(2J_m-2\Lambda_m-n)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m+2\Lambda_m+2-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m+\Lambda_m+1)(2J_m+2\Lambda_m+3)}\right]^{\frac{1}{2}}$
$(J_m \Lambda_m)$	$-\frac{(J_m+\Lambda_m+1-m)[(n+1)(2\Lambda_m+2+n)(2J_m+1-n)(2J_m-2\Lambda_m-n)]^{\frac{1}{2}}}{[2(2J+1)(2\Lambda+1)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$(J_m+\frac{1}{2}, \Lambda_m-\frac{1}{2})$	$\left[\frac{(2\Lambda_m+2+n)(2J_m+1-n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)(2J_m+2-m)(2\Lambda_m-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m+1)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$
$(J_m-\frac{1}{2}, \Lambda_m+\frac{1}{2})$	$\left[\frac{n(n+1)(2\Lambda_m+2+n)(2J_m+1-n)(2J_m+1-m)(2\Lambda_m+1-m)}{2(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(J_m-\Lambda_m)(2J_m-2\Lambda_m+1)}\right]^{\frac{1}{2}}$

Table 4a

$\langle (J_m\Lambda_m)J_1\Lambda_1;\ (10)J_2\Lambda_2 \| (J_m\Lambda_m)J\Lambda\rangle_2$

with $J = J_m-\frac{1}{2}n-\frac{1}{2}m,\quad \Lambda = \Lambda_m+\frac{1}{2}n-\frac{1}{2}m,\quad G_m = J_m(J_m+2)+\Lambda_m(\Lambda_m+1)$

$J_1 = J,\ \Lambda_1 = \Lambda+1$ $J_2\Lambda_2 = 01$	$\left[\frac{m(n+1)(2\Lambda_m+2+n)(2J_m+1-n)(2J_m-2\Lambda_m-n)(2\Lambda_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)G_m}{2(2\Lambda+1)(2\Lambda+2)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda-1$ $J_2\Lambda_2 = 01$	$-\left[\frac{n(m+1)(2\Lambda_m+1+n)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)(2\Lambda_m-m)(2J_m+1-m)(2J_m+2\Lambda_m+2-m)G_m}{2(2\Lambda)(2\Lambda+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 01$	$\frac{2\Lambda\{G_m m(n+1)(2J_m+1-n)(2J_m-2\Lambda_m-n)\}+(2\Lambda+2)\{(2J_m+1)(J_m+\Lambda_m+1)[2(J_m+2)(J_m-\Lambda_m)(2\Lambda_m-m)-2\Lambda_m n(J_m+\Lambda_m+3)] -G_m nm[n^2-2n(2J_m-\Lambda_m+1)+(4J_m^2-4J_m\Lambda_m+2J_m-4\Lambda_m-1]\}}{[4G_m(2\Lambda)(2\Lambda+2)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$J_1 = J+1,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\frac{nm(2\Lambda_m+1+n)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)(2J_m+2-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)G_m}{2(2J+1)(2J+2)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)}\right]^{\frac{1}{2}}$
$J_1 = J-1,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\left[\frac{(m+1)(n+1)(2\Lambda_m+2+n)(2J_m+1-n)(2J_m-2\Lambda_m-n)(2J_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)G_m}{2(2J)(2J+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\frac{2J\{nmG_m(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)\}+(2J+2)\{(2J_m+1)[2\Lambda_m(J_m+\Lambda_m+1)[2(\Lambda_m+1)(J_m-\Lambda_m)-n(J_m+\Lambda_m+3)] +2(\Lambda_m+1)(J_m-\Lambda_m)(J_m-\Lambda_m+2)m]-G_m nm[m^2-2m(J_m+2\Lambda_m+1)+4\Lambda_m^2+4J_m\Lambda_m+4\Lambda_m-2J_m-1]\}}{[4G_m(2J)(2J+2)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2},\ \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\frac{[m(2J_m+2\Lambda_m+3-m)(2J_m+2-m)(2\Lambda_m+1-m)]^{\frac{1}{2}}\{(2J_m+1)(J_m-\Lambda_m)(\Lambda_m+1)(J_m-\Lambda_m+2)+2nG_m(J_m-\Lambda_m)-n^2G_m\}}{[(2J+1)(2\Lambda+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)G_m]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2},\ \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\frac{[(m+1)(2J_m+2\Lambda_m+2-m)(2J_m+1-m)(2\Lambda_m-m)]^{\frac{1}{2}}\{(2J_m+1)(J_m-\Lambda_m)(\Lambda_m+1)(J_m-\Lambda_m+2)+2nG_m(J_m-\Lambda_m)-n^2G_m\}}{[(2J+1)(2\Lambda+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)G_m]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2},\ \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\frac{[n(2\Lambda_m+1+n)(2J_m+2-n)(2J_m-2\Lambda_m+1-n)]^{\frac{1}{2}}\{(2J_m+1)\Lambda_m(J_m+\Lambda_m+1)(J_m+\Lambda_m+3)-2mG_m(J_m+\Lambda_m+1)+m^2G_m\}}{[(2J+1)(2\Lambda+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2},\ \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\frac{[(n+1)(2\Lambda_m+2+n)(2J_m+1-n)(2J_m-2\Lambda_m-n)]^{\frac{1}{2}}\{(2J_m+1)\Lambda_m(J_m+\Lambda_m+1)(J_m+\Lambda_m+3)-2mG_m(J_m+\Lambda_m+1)+m^2G_m\}}{[(2J+1)(2\Lambda+1)(2\Lambda_m)(2\Lambda_m+2)(2J_m+1)(2J_m+3)(J_m-\Lambda_m)(J_m-\Lambda_m+1)(J_m+\Lambda_m+1)(J_m+\Lambda_m+2)]^{\frac{1}{2}}}$

TABLE 4b

$\langle (J_m\Lambda_m)J_1\Lambda_1;\ (10)J_2\Lambda_2 \| (J_m+1,\Lambda_m)J\Lambda\rangle$

with $J = J_m-\frac{1}{2}n-\frac{1}{2}m,\quad \Lambda = \Lambda_m+\frac{1}{2}n-\frac{1}{2}m$

$J_1 = J, \Lambda_1 = \Lambda+1$ $J_2\Lambda_2 = 01$	$\left[\frac{m(m+1)(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{(2\Lambda+1)(2\Lambda+2)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda-1$ $J_2\Lambda_2 = 01$	$\left[\frac{n(n+1)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2\Lambda(2\Lambda+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 01$	$-\left[\frac{2(m+1)(2\Lambda_m-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)(n+1)(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)}{2\Lambda(2\Lambda+2)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\frac{n(n+1)m(m+1)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2\Lambda_m-m)(2\Lambda_m+1-m)}{(2J+1)(2J+2)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\frac{(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{2J(2J+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\left[\frac{2(m+1)(2\Lambda_m-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)(n+1)(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)}{2J(2J+2)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\left[\frac{2m(m+1)(2\Lambda_m-m)(2\Lambda_m+1-m)(n+1)(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\left[\frac{2(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)(n+1)(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\frac{2n(n+1)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(m+1)(2\Lambda_m-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2\Lambda+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\frac{2(m+1)(2\Lambda_m-m)(2J_m+2-m)(2J_m+2\Lambda_m+3-m)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda+1)(2J_m+2)(2J_m+3)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$

TABLE 4c

$\langle (J_m \Lambda_m) J_1 \Lambda_1;\quad (10) J_2 \Lambda_2 \| (J_m, \Lambda_m + 1) J \Lambda \rangle$

with $J = J_m - \frac{1}{2}n - \frac{1}{2}m,\quad \Lambda = \Lambda_m + \frac{1}{2}n - \frac{1}{2}m$

$J_1 = J,\ \Lambda_1 = \Lambda+1$ $J_2\Lambda_2 = 01$	$\left[\dfrac{n(n+1)(2J_m+1-n)(2J_m+2-n)m(m+1)(2J_m+1-m)(2J_m+2-m)}{(2\Lambda+1)(2\Lambda+2)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda-1$ $J_2\Lambda_2 = 01$	$\left[\dfrac{(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2\Lambda(2\Lambda+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 01$	$\left[\dfrac{2n(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(m+1)(2J_m+1-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)}{2\Lambda(2\Lambda+2)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+1,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{m(m+1)(2J_m+1-m)(2J_m+2-m)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2J+2)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-1,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{n(n+1)(2J_m+1-n)(2J_m+2-n)(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2J(2J+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J,\ \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{2n(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(m+1)(2J_m+1-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)}{2J(2J+2)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+\frac{1}{2},\ \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\dfrac{2n(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)m(m+1)(2J_m+1-m)(2J_m+2-m)}{(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-\frac{1}{2},\ \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\dfrac{2n(2J_m+2-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J+\frac{1}{2},\ \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\dfrac{2(m+1)(2J_m+1-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-\frac{1}{2},\ \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\left[\dfrac{2(m+1)(2J_m+1-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)n(n+1)(2J_m+1-n)(2J_m+2-n)}{(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2\Lambda_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+1)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$

TABLE 4d

$\langle (J_m\Lambda_m)J_1\Lambda_1;\ (10)J_2\Lambda_2 \| (J_m+\frac{1}{2}, \Lambda_m+\frac{1}{2})J\Lambda \rangle$

with $J = J_m-\frac{1}{2}n-\frac{1}{2}m, \quad \Lambda = \Lambda_m+\frac{1}{2}n-\frac{1}{2}m$

$J_1 = J, \Lambda_1 = \Lambda+1$ $J_2\Lambda_2 = 01$	$-\left[\dfrac{2m(m+1)(2J_m+2-m)(2\Lambda_m+1-m)(n+1)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)}{(2\Lambda+1)(2\Lambda+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda-1$ $J_2\Lambda_2 = 01$	$\left[\dfrac{2n(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)(2J_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2\Lambda(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 01$	$-\dfrac{\{(2J_m-2\Lambda_m)(2\Lambda_m-m)-2n(J_m+\Lambda_m+1-m)\}[(m+1)(2J_m+2\Lambda_m+3-m)(2J_m+2-n)(2\Lambda_m+2+n)]^{\frac{1}{2}}}{[2\Lambda(2\Lambda+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{2m(m+1)(2J_m+2-m)(2\Lambda_m+1-m)n(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)}{(2J+1)(2J+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\left[\dfrac{2(n+1)(2J_m-2\Lambda_m-n)(2J_m+1-n)(2J_m+2-n)(2J_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)}{2J(2J+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\dfrac{\{(2J_m-2\Lambda_m)(2J_m+2-m)-2n(J_m+\Lambda_m+1-m)\}[(m+1)(2J_m+2\Lambda_m+3-m)(2J_m+2-n)(2\Lambda_m+2+n)]^{\frac{1}{2}}}{[2J(2J+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\dfrac{(2J_m-2\Lambda_m-2n)[m(m+1)(2J_m+2-m)(2\Lambda_m+1-m)(2J_m+2-n)(2\Lambda_m+2+n)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\dfrac{(2J_m-2\Lambda_m-2n)[(2J_m+1-m)(2\Lambda_m-m)(2J_m+2\Lambda_m+2-m)(2J_m+2\Lambda_m+3-m)(2J_m+2-n)(2\Lambda_m+2+n)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\dfrac{(2J_m+2\Lambda_m+2-2m)[(m+1)(2J_m+2\Lambda_m+3-m)n(2\Lambda_m+1+n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m+1-n)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\dfrac{(2J_m+2\Lambda_m+2-2m)[(m+1)(2J_m+2\Lambda_m+3-m)(n+1)(2J_m+1-n)(2J_m+2-n)(2J_m-2\Lambda_m-n)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+3)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$

TABLE 4e

$\langle (J_m \Lambda_m) J_1 \Lambda_1;\ (10) J_2 \Lambda_2 \| J_m + \tfrac{1}{2},\ \Lambda_m - \tfrac{1}{2}) J \Lambda \rangle$

with $J = J_m - \tfrac{1}{2}n - \tfrac{1}{2}m, \quad \Lambda = \Lambda_m + \tfrac{1}{2}n - \tfrac{1}{2}m$

$J_1 = J, \Lambda_1 = \Lambda+1$ $J_2\Lambda_2 = 01$	$-\left[\dfrac{2m(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)(2J_m+1-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{(2\Lambda+1)(2\Lambda+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda-1$ $J_2\Lambda_2 = 01$	$-\left[\dfrac{2n(n+1)(2J_m+2-n)(2\Lambda_m+1+n)(m+1)(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)}{2\Lambda(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 01$	$\dfrac{\{(2J_m+2\Lambda_m+2)(2\Lambda_m+2+n)+m(2J_m-2\Lambda_m-2n)\}[(n+1)(2\Lambda_m-m)(2J_m-2\Lambda_m+1-n)(2J_m+2-m)]^{\frac{1}{2}}}{[2\Lambda(2\Lambda+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{2n(n+1)(2J_m+2-n)(2\Lambda_m+1+n)m(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)}{(2J+1)(2J+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J-1, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$\left[\dfrac{2(m+1)(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)(2J_m+1-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)}{2J(2J+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)}\right]^{\frac{1}{2}}$
$J_1 = J, \Lambda_1 = \Lambda$ $J_2\Lambda_2 = 10$	$-\dfrac{\{(2J_m+2\Lambda_m+2)(2J_m+2-n)-m(2J_m-2\Lambda_m-2n)\}[(n+1)(2\Lambda_m-m)(2J_m-2\Lambda_m+1-n)(2J_m+2-m)]^{\frac{1}{2}}}{[2J(2J+2)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\dfrac{(2J_m-2\Lambda_m-2n)[(n+1)(2J_m-2\Lambda_m+1-n)m(2\Lambda_m-m)(2\Lambda_m+1-m)(2J_m+2\Lambda_m+3-m)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\dfrac{(2J_m-2\Lambda_m-2n)[(n+1)(2J_m-2\Lambda_m+1-n)(m+1)(2J_m+1-m)(2J_m+2-m)(2J_m+2\Lambda_m+2-m)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J+\frac{1}{2}, \Lambda_1 = \Lambda-\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$-\dfrac{(2J_m+2\Lambda_m+2-2m)[n(n+1)(2\Lambda_m+1+n)(2J_m+2-n)(2\Lambda_m-m)(2J_m+2-m)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m+2\Lambda_m+4)]^{\frac{1}{2}}}$
$J_1 = J-\frac{1}{2}, \Lambda_1 = \Lambda+\frac{1}{2}$ $J_2\Lambda_2 = \frac{1}{2}\frac{1}{2}$	$\dfrac{(2J_m+2\Lambda_m+2-2m)[(2J_m+1-n)(2\Lambda_m+2+n)(2J_m-2\Lambda_m-n)(2J_m-2\Lambda_m+1-n)(2\Lambda_m-m)(2J_m+2-m)]^{\frac{1}{2}}}{[(2J+1)(2\Lambda+1)(2\Lambda_m+1)(2J_m+2)(2J_m-2\Lambda_m+1)(2J_m-2\Lambda_m+2)(2J_m+2\Lambda_m+2)(2J_m-2\Lambda_m+4)]^{\frac{1}{2}}}$

TABLE 5

$\langle N=v,\ v(\alpha_v)I_v;\ 112|\}N=v+1,\ v+1(\alpha_{v+1})I_{v+1}\rangle$

$[N=v=4)$

	I_{v+1}	$I_v=2$	4	5	6	8
	2	$\frac{-\sqrt{2\cdot 13}}{\sqrt{5\cdot 7}}$	$\frac{3}{\sqrt{5\cdot 7}}$			
$N+1=5$, $v+1=5$	4	$\frac{-\sqrt{3\cdot 13}}{\sqrt{7\cdot 11}}$	$\frac{-24\sqrt{3}}{55\sqrt{7}}$	$\frac{-\sqrt{7\cdot 13}}{5\sqrt{11}}$	$\frac{-7}{11\sqrt{5}}$	
	5		$\frac{-3\sqrt{3\cdot 7}}{5\sqrt{11}}$	$\frac{\sqrt{6}}{5}$	$\frac{-2}{\sqrt{5\cdot 11}}$	
	6		$\frac{-3\sqrt{3\cdot 7\cdot 17}}{11\sqrt{5\cdot 13}}$	$\frac{-2\sqrt{17}}{\sqrt{5\cdot 7\cdot 11}}$	$\frac{6\sqrt{2\cdot 17}}{5\cdot 11}$	$\frac{-16}{5\sqrt{7\cdot 11\cdot 13}}$
	7			$\frac{2\sqrt{6}}{\sqrt{5\cdot 7}}$	$\frac{\sqrt{3}}{5}$	$\frac{-\sqrt{2\cdot 17}}{5\sqrt{7}}$
	8				$\frac{\sqrt{19}}{5}$	$\frac{\sqrt{6}}{5}$
	10					1

$[N=v=5]$

	I_{v+1}	$I_v=2$	4	5	6	7	8	10
	0	1						
	3	$\frac{\sqrt{3\cdot 5}}{2\sqrt{7}}$	$\frac{-\sqrt{11}}{\sqrt{5\cdot 7}}$	$\frac{-\sqrt{3}}{2\sqrt{5}}$				
	4	$\frac{\sqrt{17}}{2\sqrt{13}}$	$\frac{\sqrt{17}}{\sqrt{3\cdot 13}}$	$\frac{-\sqrt{17}}{2\sqrt{3\cdot 13}}$	$\frac{\sqrt{5}}{\sqrt{3\cdot 13}}$			
$N+1=6$, $v+1=6$	6_1		$\frac{-\sqrt{2\cdot 251}}{13\sqrt{5}}$	$\frac{-56}{13\sqrt{2\cdot 5\cdot 251}}$	$\frac{-4\sqrt{17}}{\sqrt{13\cdot 251}}$	$\frac{135\sqrt{5}}{2\cdot 13\sqrt{2\cdot 251}}$	$\frac{7\sqrt{17\cdot 19}}{2\cdot 13\sqrt{2\cdot 251}}$	
	6_2		0	$\frac{-5\sqrt{2\cdot 17\cdot 19}}{\sqrt{7\cdot 13\cdot 251}}$	$\frac{11\sqrt{19}}{\sqrt{5\cdot 7\cdot 251}}$	$\frac{\sqrt{2\cdot 17\cdot 19}}{\sqrt{7\cdot 13\cdot 251}}$	$\frac{-\sqrt{2\cdot 13}}{\sqrt{5\cdot 7\cdot 251}}$	
	7			$\frac{\sqrt{2\cdot 17}}{\sqrt{7\cdot 13}}$	$\frac{\sqrt{11}}{\sqrt{5\cdot 7}}$	$\frac{11}{\sqrt{2\cdot 3\cdot 7\cdot 13}}$	$\frac{-\sqrt{19}}{\sqrt{2\cdot 3\cdot 5\cdot 7}}$	
	8				$\frac{\sqrt{3\cdot 11\cdot 19}}{\sqrt{5\cdot 13\cdot 17}}$	$\frac{-\sqrt{19}}{3\sqrt{2\cdot 13}}$	$\frac{11\sqrt{7}}{\sqrt{2\cdot 5\cdot 13\cdot 19}}$	$\frac{-8\sqrt{5}}{3\sqrt{13\cdot 17\cdot 19}}$
	9					$\frac{-\sqrt{5\cdot 11}}{6\sqrt{2}}$	$\frac{-\sqrt{11}}{2\sqrt{2\cdot 19}}$	$\frac{2\sqrt{7}}{3\sqrt{19}}$
	10						$\frac{-\sqrt{2\cdot 23}}{\sqrt{3\cdot 19}}$	$\frac{-\sqrt{11}}{\sqrt{3\cdot 19}}$
	12							1