Problem categorizing

General form of ODE: $a_n(t)\cdot y^{(n)}+a_{n-1}(t)\cdot y^{(n-1)}+\cdots+a_1(t)\cdot y=g(t)$ (order n)

Linear: $y^{(n)}$ 的次数都是1次的

1. first order ODE

(1) constant coefficients

$(y'=ay+b)$

直接把 x, y 分离两边分别求积分

(2) variable coefficients

$(y'+p(t)y=g(t))$

两边 $\times\, \mu(t)$ 凑形式, 使得 $\mu'(t)=p(t)\mu(t)$

$\mu(t)$ 算法: $p(t)=\frac{1}{\mu(t)}\cdot\frac{d\mu(t)}{dt}=\frac{d\ln\mu(t)}{dt}$

$\therefore \ln\mu(t)=\int p(t)dt+C$

(3) nonlinear type

$(y'=-\frac{2}{ty}+4t)$

1° seperable

change the form into $M(y)dy=N(t)dt$

然后两边分别积分

2° not seperable

没讲 应该不考

2. second order ODE

General form: $P(t)\cdot y''+Q(t)\cdot y'+R(t)\cdot y=G(t)$

$(P(t)\neq 0)$: $y''+q(t)y'+r(t)y=g(t)$

Homogeneous: $G(t)=0$ / $g(t)=0$

只学了coefficient都是constant的情况, variable coefficient 只能 given 一个, 用 order of reduction 求另一个

(1) homogeneous equation $(ay''+by'+cy=0)$

substitude y to e^{rt}

characteristic equation: $ar^2+br+c=0$

1° $b^2-4ac>0$

$y=C_1\cdot e^{r_1t}+C_2\cdot e^{r_2t}$

($W\neq 0$, 不用验证)

2° $b^2-4ac<0$

$$\begin{cases} e^{r_1t}=e^{\lambda t}(\cos\mu t+i\sin\mu t)\\ e^{r_2t}=e^{\lambda t}(\cos\mu t-i\sin\mu t)\end{cases}$$

$$\Rightarrow\begin{cases} y_1=e^{\lambda t}\cos\mu t\\ y_2=e^{\lambda t}\sin\mu t\end{cases}$$

$$\Rightarrow\begin{cases} y_1'=e^{\lambda t}\cos\mu t\\ y_2'=e^{\lambda t}\sin\mu t\end{cases}$$

$\therefore y=e^{\lambda t}(C_1\cos\mu t+C_2\cdot\sin\mu t)$

($W\neq 0$, 不用验)

3° $b^2=4ac$

$y_1=e^{-\frac{b}{2a}t}$

$\Rightarrow$ Method of reduction of order

假设 $y_2=u(t)y_1$ 是另一个解

把 y_2 代入 $ay''+by'+c=0$ 里

$\Rightarrow$ 发现 $u(t)=t$ 就work

$\Rightarrow y_2=ty_1=te^{-\frac{b}{2a}t}$

($W\neq 0$, 不用验)

(2) Non-homogeneous equation $(y''+p(t)y'+q(t)y=g(t)\neq 0)$

把对应的homogeneous general solution 和 non-homogeneous solution 加一起即可

前面已经讲过如何求homogeneous solution, 即只需求non-homogeneous即可

猜形式 + 待定系数

3. Method of Reduction of Order

已知一个根 y_1, 求另一个根 y_2 的办法

拿 second order 举例: 对 $\forall\, ay''+by'+c=0$. 已知 y_1 是一个 solution

设 $y_2=u(t)\cdot y_1$

然后寻找合适的 $u(t)$, 使得 $ay_2''+by_2'+c=0$

算完要用 Wronskian determinant 来检查

4. Wronskian 的应用场景

用完reduction of order 或题目强调要用

5. 其它题型

Theorem: Solution uniqueness

Consider $$\begin{cases} y''+p(t)y'+q(t)y=g(t)\\ y(t_0)=y_0\\ y'(t_0)=y_0'\end{cases}$$

where p, q and g are continuous on (a_0, b_0) that contains t_0

then there exists a unique solution $y=\phi(t)$ on (a_0, b_0)

- Consider the second order linear initial value problem
  $$y''+p(t)y'+q(t)y=0,\ y(0)=0,\ y'(0)=0$$
  where p, q are continuous on an open interval I containing t_0.
- In light of the initial conditions, note that $y=0$ is a solution to this homogeneous initial value problem.
- Since the hypotheses of Theorem 3.2.1 are satisfied, it follows that $y=0$ is the only solution of this problem.

$\Rightarrow$ 所以说有时候当条件满足有唯一解的时候, 可以通过猜根&有唯一解来做题

$\Rightarrow$ 另外一个 take away 是, 题也可以问你使得有唯一解的最长区间(且包括initial value)是多少, 这时可通过把原equation化为standard form 然后看coefficient在哪段上是 continuous 的

6. second order ODE with variable coefficient

Power series 复习: $$\begin{cases}\text{converge at point } x: \lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_0)^n \text{ exists}\\ \text{converge absolutely at point } x: \lim_{N\to\infty}\sum_{n=0}^{N} |a_n(x-x_0)^n| \text{ exists}\end{cases}$$

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right|=|x-x_0|\cdot\left|\frac{a_{n+1}}{a_n}\right|$ radio of convergence $\rho=\left|\frac{a_n}{a_{n+1}}\right|$

Index shifting: 转换下标(对齐下标)

关于常见power series: $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$

泰勒展开: $f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$

$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots \quad R=1$

$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots \quad R=\infty$

$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \quad R=\infty$

$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \quad R=\infty$

$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \quad R=\infty$

$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots \quad R=1$

$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\cdots \quad R=1$

analytic: $y=y(x)$ is **analytic** at $x=x_0$ if y has a **Taylor series expansion** about x_0:

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n=y(x_0)+y'(x_0)(x-x_0)+\frac{y''(x_0)}{2}(x-x_0)^2+\cdots,$$

(1) polynomial coefficient

$P(x)\cdot y''+Q(x)\cdot y'+R(x)\cdot y=0$

(规定 $P(x), Q(x), R(x)$ 不能再约分了)

$$\begin{cases}\text{ordinary point: 使 } P(x)\neq 0 \text{ 的 } x\\ \text{singular point: 使 } P(x)=0 \text{ 的 } x\end{cases}$$

$$\begin{cases}\text{regular singular point: 满足 } \lim_{x\to x_0}x\cdot\frac{Q(x)}{P(x)}=\alpha,\ \lim_{x\to x_0}x^2\frac{R(x)}{P(x)}=\beta \text{ 的 } x_0\ (\alpha,\beta\text{有限})\\ \text{irregular singular point: ?}\end{cases}$$

Initial point 是 ordinary point

步骤: ① 验证 Initial point x_0 是ordinary point

② 令 $y=\sum_{i=0}^{\infty}a_i(x-x_0)^i$

③ 算出 a_n 之间的递推关系

④ 用 a_0, a_1 表示其他 a_i, 算出 y_1, y_2

⑤ 用Wronskian验证

Initial point 是 regular singular point (Special case)

步骤: ① 原方程是 $(x-x_0)^2y''+\alpha(x-x_0)y'+\beta\cdot y=0$ 的形式 (Euler equation)

② 换元 $y=|x-x_0|^r$ $\longrightarrow$ 注意在general solution的地方判断范围并脱去绝对值

$\Rightarrow$ 变为 $r^2+(\alpha-1)r+\beta=0$ (characteristic equation)

1° 相异实根 r_1, r_2

$y=C_1\cdot|x-x_0|^{r_1}+C_2\cdot|x-x_0|^{r_2}$

2° 相异虚根 $r=\lambda\pm\mu i$

$$\begin{cases} y_1=|x-x_0|^{\lambda+\mu i}\\ y_2=|x-x_0|^{\lambda-\mu i}\end{cases}\Rightarrow y=\ln|x-x_0|\cdot|x-x_0|^{\lambda}\cdot(C_1\cos\mu+C_2\sin\mu)$$

3° 重根 r_1 $\Rightarrow y=|x-x_0|^{r_1}$

reduction of order: $y'=|x-x_0|^{r_1}\cdot \ln|x-x_0|$

$y=C_1|x-x_0|^{r_1}+C_2\cdot|x-x_0|^{r_1}\ln|x-x_0|$

(2) Laplace transform (一种方法, 而非题型)

$f(t)\longrightarrow F(s)=\int_0^{\infty}e^{-st}f(t)dt=\lim_{A\to\infty}\int_0^{A}e^{-st}f(t)dt$

$\Rightarrow F(s)=\mathcal{L}(f(t))$, $\mathcal{L}^{-1}(F(s))=f(t)$

① $\mathcal{L}(\ )$ 是linear的: $\mathcal{L}(ax+by)=a\mathcal{L}(x)+b\mathcal{L}(y)$

② $\mathcal{L}(f'(t))=\lim_{A\to\infty}\int_0^A e^{-st}\cdot f'(t)dt$

$=\lim_{A\to\infty}\int_0^A e^{-st}df(t)$

$=\lim_{A\to\infty}\left(e^{-st}f(t)\Big|_0^A-\int_0^A f(t)\,de^{-st}\right)$

$=\lim_{A\to\infty}\left(-f(0)+\int_0^A s\,f(t)e^{-st}dt\right)$

$=-f(0)+s\mathcal{L}(f(t))$

$\Rightarrow \mathcal{L}(f''(t))=s^2\mathcal{L}(f(t))-sf(0)-f'(0)$

$\mathcal{L}(f^{n}(t))=s^n\mathcal{L}(f(t))-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

方法: ① 两边取 Laplace

② LHS化为关于 $\mathcal{L}(y)$ 的形式

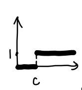
③ 从②里得到 $\mathcal{L}(y)=f(t)$

把 $f(t)$ 拆成 Laplace表里有的形式之和

然后取 $\mathcal{L}^{-1}(\ )$ 反推 y

特殊函数: ① step function

$u_c(t)=\begin{cases}0 & (t<c)\\ 1 & (t\geq c)\end{cases}$



可表示所有分段polynomial

$u_i(t)-u_j(t)=\begin{cases}0 & \text{otherwise}\\ 1 & (j\leq x<i)\end{cases}$ (如果 $i>j$)

$t(u_i(t)-u_j(t))=\begin{cases}0 & \text{otherwise}\\ t & (j\leq x<i)\end{cases}$

② Unit Impulse function

$$\delta(t)=0,\quad \text{for all}\quad t\neq 0,\qquad \text{and}\qquad \int_{-\infty}^{\infty}\delta(t)\,dt=1.$$

By the definition, $\delta(t)$ is a function which is zero almost everywhere and its nonzero part "concentrates" at 0. Similarly, $\delta(t-c)$ is an impulse function which is zero almost everywhere and the nonzero part concentrating at $t=c$.

☆ 矩阵求导

假设算出来eigenvalue是 $\lambda_1\sim\lambda_m$, eigenvector是 $v_1\sim v_n$ (一一对应)

general solution $=C_1\cdot v_1\cdot e^{\lambda_1 t}+\cdots+C_n\cdot v_n\cdot e^{\lambda_m t}$

initial value 就把 $t=t_0$ 带进去解 $C_1\sim C_n$

Concepts

1. equilibrium solution 当所有导数为0时，使式子满足的解. 如 $y' = y(1-y) \Rightarrow y_{e_1} = 0,\ y_{e_2} = 1$

2. general solution $y = c_1 \cdot \_\_ + c_2 \cdot \_\_ + \cdots$

   particular solution (initial value solution) $y = \_\_ + \_\_ + \cdots$

3. fundamental set of solutions

   $\{\_\_, \_\_\}$ → a component of general solution