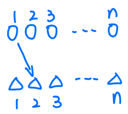
cheatsheet 内容 ✓

1. derangement 公式

$$D_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right)$$

$$\lim_{n\to\infty}\frac{D_n}{n!}=\frac{1}{e}$$



假设如1选了2（这步有(n-1)种选择）
然后假设如2选了1，那么剩下的就是a_{n-2}
2不选1，那么相当于a_{n-1}

$$a_n=(n-1)(a_{n-1}+a_{n-2})$$

2. logic 公式：什么时候predicate的quantifier可以分离

① $\forall x\,(A\wedge P(x)) \equiv A\wedge \forall x\,P(x)$

$\forall$换成$\exists$，$\wedge$换成$\vee$，四种组合均可

② $\exists x\,P(x)\to\forall y\,Q(y)$

$\equiv \neg\exists x\,P(x)\vee\forall y\,Q(y)$

$\equiv \forall x\neg P(x)\vee\forall y\,Q(y)$

$\equiv \forall x\,\forall y\,(\neg P(x)\vee\forall y\,Q(y))$

$\equiv \forall x\,\forall y\,(P(x)\to\forall y\,Q(y))$

③ $p\vee(q\wedge r)\equiv(p\vee q)\wedge(p\vee r)$

$p\vee(p\wedge q)\equiv p$

$p\wedge(p\vee q)\equiv p$

要注意的点：

1. 如果用了contradiction/contrapositive一定要说明用了
2. predicate 专题

(1) 在写predicate的时候注意括号的使用

因为quantifier是默认和离它最近的东西绑定的，所以如果后面跟了一大串东西那还是打一个大括号吧

还有：多个quantifier的顺序是不能被随意改变的

(2) 写predicate proposition的题：假如说，A，怎么怎么样 那我就直接写 P(A) … 而不是 $\exists x\,P(x)$了

有说是俩不同的人的要写 $x\neq y$

3. 让prove false 的举反例就行
4. 存在 + imply 是个奇怪的组合 遇到小心点，

$\exists x\,\exists y\; p(x)\to q(x)$，只要找到 $P(x)=F$就行

5. countable 题

注意：(1) 2^N量级的都算 uncountable

(2) 注意可能的对"无穷"的暗示 比如如果说某小数只能有偶数位 那就是暗示一定不是无穷位

6. $p\to q$

(1) proof by contradiction:

suppose p is true and q is false

then sth. goes wrong

(e.g.: p is false, which caused contradiction)

(2) proof by contraposition

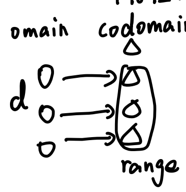
直接证明 $\neg q\to\neg p$ 即可

7. 证明存在 bijection

$\Leftrightarrow$ domain 和 codomain 元素数量一样

证明存在 injection $\Leftrightarrow$ |domain| $\leq$ |codomain|

存在 surjection $\Leftrightarrow$ |domain| $\geq$ |codomain|



domain = range : injective

domain < range: X

domain > range: not injective

domain ≥ codomain: 说明不了啥

domain < codomain: not surjective

range = codomain: surjective

domain = range = codomain: bijective

codomain 可以取不到 但 range 必须要取到

8. 过程：

Problem 4: [8 pts] Consider the following argument:

Premise 1: $\exists x\; P(x)$.
Premise 2: $\forall x\; (P(x)\to Q(x))$.
Premise 3: $\forall x\; (P(x)\to R(x))$.
Conclusion: $\exists x\; (Q(x)\wedge R(x))$.

Show that the argument is valid. Number your steps and refer to those numbers in the reason you give for each step, but it is not necessary to give the name of the inference rule.

Answer:

Step	Reason
1. $\exists x\; P(x)$	Hypothesis
2. $P(c)$	(1), Existential instantiation
3. $\forall x\; (P(x)\to Q(x))$	Hypothesis
4. $P(c)\to Q(c)$	(3), Universal instantiation
5. $\forall x\; (P(x)\to R(x))$	Hypothesis
6. $P(c)\to R(c)$	(5), Universal instantiation
7. $Q(c)$	(2) and (4), Modus ponens
8. $R(c)$	(2) and (6), Modus ponens
9. $Q(c)\wedge R(c)$	(7) and (8), Conjunction
10. $\exists x\; (Q(x)\wedge R(w))$	(9), Existential generalization

9. conjunction: $\wedge$ disjunction: $\vee$

$\oplus$: XOR，p,q一样为0，不一样为1

converse: $q\to p$ inverse: $\neg p\to\neg q$

biconditional: $p\leftrightarrow q$

paradox 悖论 postulate = premise = 前提

p if q: $q\to p$

p only if q (p iff q): $p\to q$

PNF: 把所有quantifier扔到最前面

10. Roster method：枚举set中的element

singleton set: 只有一个元素的set

proper subset: 真包含的 subset

cardinality: unique elements 的数量

power set: 所有子集的集合

ordered n-tuple: 有序n元集合，或者看成 $v\in R^n$ 也行

cartesian product: A{a,b} B{c,d} ⟶ A*B {(a,c),(a,d),(b,c),(b,d)}

Relation: a relation from A to B (A*B的一个子集)

disjoint: A and B is disjoint ($A\cap B=\emptyset$)

difference: $A-B=\{1,3,5\}-\{1,2,3\}=\{5\}$

(也叫complement) Universal compliment: 关于全集的补集

11. countable, uncountable 例子

$\mathbb{N}^k$: countable

$\mathbb{Q}$: countable (有理数) 理解为 $\frac{q}{p}$，即 N^2

$\mathbb{R}$: uncountable (实数)

Problem 4: [8 pts] Let $\mathbb{N}$ be the set of all natural numbers.

(a) Show that $\{S \mid S\subseteq\mathbb{N}, |S| \text{ is finite}\}$ is countable.

(b) Is $\{S \mid S\subseteq\mathbb{N}, |S| \text{ is infinite}\}$ countable or uncountable? Justification is not necessary.

Solution : (a) Here is one enumeration method for $\{S \mid S\subseteq\mathbb{N}, |S| \text{ is finite}\}$. We first enumerate $\emptyset$. Then for $k=0,1,\ldots,$ we enumerate all $S\subseteq\mathbb{N}$ such that $\max S = k$. For each k, there are finitely many such S, and we can enumerate them in any order.

(b) Uncountable.

Problem 5: [8 pts] For each of the following sets, determine if it is countable or uncountable. No justification needed.

(a) $\{x\in\mathbb{R} \mid x^2\in\mathbb{Z}\}$

(b) $\{x\in\mathbb{R} \mid$ there is a decimal expansion of x with only even digits$\}$

(c) The union of countably many countable sets.

(d) The set of circles in the plane.

Solution: (a) Countable (c) Countable
(b) Uncountable (d) Uncountable

(b) [3 pts] Without the need for providing any explanation, determine if the following sets are infinitely countable (C) or infinitely uncountable (U):

(i) [1 pts] The union of countably many countable sets

(ii) [1 pts] The set of all functions from $\{1,2\}\to\mathbb{N}$

(iii) [1 pts] The set of all functions from $\mathbb{N}\to\{1,2\}$

(b) C, C, U

12. combination: 组合，无序

permutation: 排列，有序

k-combination: 从原来的set中无序地选k个distinct element

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}.$$

$$\binom{n+1}{r+1}=\sum_{j=r}^{n}\binom{j}{r}.$$

Solution 1: The left-hand side counts the number of bit strings of length $n+1$ containing $(r+1)$ 1's.

The right-hand side counts the same objects in an alternative way by considering the possible locations of the last 1 in a string with $(r+1)$ 1's: the last 1 must occur at location $r+1, r+2, \ldots,$ or $n+1$.