Notes

1. expression of solution set

① general solution: $\begin{cases} x_1 = f(\text{free variable}) \\ x_2 = \uparrow \\ \vdots \\ x_n = \uparrow \end{cases}$

② parametric vector form

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \langle \text{free variabe} \rangle \begin{bmatrix} \\ \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix}$

③ parametric form

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} f(\text{free variable}) \\ \vdots \end{pmatrix}$

2. $\vec{x} \in \mathbb{R}^n$: $\{\vec{x}\}$ is only linear dependent when $\vec{x} = \vec{0}$

Any set of vectors containing $\vec{0}$ is a linear dependent set

3. image of $\vec{x}$ is $T(\vec{x})$

$\begin{cases} \text{dilation}: T(\vec{x}) = C_1\vec{x} \text{ where } C_1 > 1 \ (C_1 \text{ is a scalar}) \\ \text{contraction}: T(\vec{x}) = C_2\vec{x} \text{ where } 0 \le C_2 \le 1 \end{cases}$ ← C_2 too

4. 假設先对 $\vec{v} \in \mathbb{R}^n$ 做 f 操作，再做 g 操作

那么 image 就是 $g(f(\vec{x}))$

体现在 linear trastormation 上就是 $\underset{\text{standard matrix}}{G \cdot F} \cdot \vec{x}$

但是不能说B和C row equivalent，因为让A有一排0 那么C也会有一排0 (即C的那一排还是B的每排的linear combination，但coefficient都是0)

5. $AB = C = [Ab_1 \cdots Ab_n]$ ($b_1 \sim b_n$ 为B的n列) ☆☆☆

$\| \begin{bmatrix} a_1B \\ \vdots \\ a_mB \end{bmatrix}$ ($a_1 \sim a_m$ 为A的m行)

$\boxed{A \cdot B = C \ (\text{C是A的 colomn combination} \ \ \text{是B的 row combination})}$

when A.B.C invertible

$C \begin{cases} \text{与A column equivalent} \\ \text{与B row equivalent} \end{cases}$

$$\begin{array}{cc} l_1 & l_2 \\ \end{array} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{array}{c} r_1 \\ r_2 \end{array} = \begin{bmatrix} a_1b_1 + a_2b_3 & a_1b_2 + a_2b_4 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{bmatrix} \begin{array}{l} \Rightarrow a_1r_1 + a_2r_2 \\ \Rightarrow a_3r_1 + a_4r_2 \end{array}$$

$$\Downarrow b_1l_1 + b_3l_2 \qquad \Downarrow b_2l_1 + b_4l_2$$

$$A^{-1}[A \ \ I] = [I \ A^{-1}]$$

6. power of matrix

A is a square matrix. $A^0 = I_n$

Transpose: $\begin{cases} (A^T)^T = A \\ (A+B)^T = A^T + B^T \\ (AB)^T = B^TA^T \end{cases}$

注意 对 $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $\begin{cases} v \cdot v^T = \begin{bmatrix} 25 & 15 \\ 15 & 9 \end{bmatrix} \\ v^T \cdot v = 34 \end{cases}$

7. ① Inverse

invertible ⇔ $\exists C$, $AC = I_n$ & $CA = I_n$

C是唯一的

properties: $\begin{cases} (A^{-1})^{-1} = A \\ (AB)^{-1} = B^{-1} \cdot A^{-1} \\ (A^T)^{-1} = (A^{-1})^T \end{cases}$

② Elementary matrices

* If an elementary row operation is performed on an mxn matrix A, the resulting matrix can be written as EA, where the mxm matrix E is created by performing the same row operation on Im
* Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

③ find A^{-1}

$RREF([A \ I_n]) = [I_n \ A^{-1}]$

④ Determinants

(1) 按行/列展开

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \cdot (-1)^{1+1} \underbrace{\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}_{A_{11}} + a_{12} \cdot (-1)^{1+2} \cdot \underbrace{\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}}_{A_{12}} + a_{13} \cdot (-1)^{1+3} \cdot \underbrace{\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}_{A_{13}}$$

(2) triangular matrix:

$\det A = \prod \langle \text{diagonal-entry} \rangle$ (主对角线矩阵) $[\ddots]$

$\det A = (-1)^{\frac{n(n-1)}{2}} \prod \langle \text{diagonal-entry} \rangle$ (副对角线矩阵) $[\iddots]$

(3)

1° 把一行的倍数加到另一行

det 值不变

2° 两行交换

det 值变相反数

3° 一整行 ×k

det 值也 ×k

(4) 一个 square matrix A is invertible if and only if $\det(A) \ne 0$

⇒ $\det(A) = 0$ ⇔ rows in A are linear dependent

(5) determinant 性质

1° $\det(A^T) = \det(A)$

column operations：对column操作对det值的影响和对row操作是类似的

2° $\det(AB) = \det(A) \cdot \det(B)$

3° $\det(A+B) \ne \det(A) + \det(B)$

4° $\det(A) \cdot \det(A^{-1}) = \det(I_n) = 1$

(if A is invertible)

5° determinant function

$\det(\vec{a_1}, \vec{a_2}, \vec{a_3}, \cdots, \vec{a_n}) = \det([\vec{a_1} \vec{a_2} \cdots \vec{a_n}])$

$\Rightarrow \begin{cases} \det(\vec{a_1}, \cdots, c\vec{a_i}, \cdots, \vec{a_n}) = c \cdot \det(\vec{a_1}, \cdots, \vec{a_n}) \\ \det(\vec{a_1}, \cdots, \vec{a_n} + \vec{b}, \cdots, \vec{a_n}) = \det(\vec{a_1}, \cdots, \vec{a_n}) + \det(\vec{a_1}, \cdots, \vec{b}, \cdots, \vec{a_n}) \end{cases}$

⑤ Cramer's rule, Volume, Linear Transformations

(1) Cramer's rule

A is a n*n invertible matrix

for $\forall \vec{b} \in \mathbb{R}^n$, the unique solution $\vec{x}$ for $Ax = b$

is $x_i = \dfrac{\det A_i(B)}{\det A}$ $(i = 1, 2, 3, \cdots, n)$

$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $A_i(B) = [\vec{a_1}, \cdots, \vec{a_{i-1}}, \vec{b}, \vec{a_{i+1}}, \cdots, \vec{a_n}]$

(2) a formula for A^{-1}

A is a invertible nxn matrix

$A^{-1} = \dfrac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & & \ddots & \vdots \\ C_{m1} & - & - & C_{mn} \end{bmatrix}$ --- adj A

$C_{ij} = (-1)^{i+j} \det A_{ij}$

⑥ area & volume

(1) $A = [a_1 \ a_2]$, $a_i \in \mathbb{R}^2$

⇒ a_1, a_2 形成的平行四边形 $S = |\det(A)|$

如果C是$\mathbb{R}^2$中的一个平行四边形

<area of A·C> = <area of C> · $|\det(A)|$

(2) $A = [a_1 \ a_2 \ a_3]$, $a_i \in \mathbb{R}^3$

⇒ a_1, a_2, a_3 形成的平行六面体 $V = |\det(A)|$

如果C是$\mathbb{R}^3$中的一个平行六面体

<volumn of A·C> = <volumn of C> · $|\det(A)|$

其实是 <Area/volume T(C)> = <Area/Volume C> · $|\det(T)|$

只要C是封闭图形都ok

8.

① 注意题干表述: maps ··· onto ···

说明standard matrix 是onto的

② if $v_1 \sim v_n$ is linear (in)dependent 的

那么 $T(v_1) \sim T(v_n)$ 还是 linear (in)dependent的

③ A, B 均 n*n

⇒ $AB = I_n \Leftrightarrow A = B^{-1}$

④ $\vec{0}$ 平行与 ∀ 向量

⑤ ☆ 一组vector linear dependent 不意味着里面每条 vector 都可以被剩下的表示(里面可能有 independent 小圈)

9. vector space & subspace

subspace V 定义: { 含 $\vec{0}$ (满足 $\vec{u}+\vec{0}=\vec{u}$)
$\vec{u},\vec{v}\in V \Rightarrow (\vec{u}+\vec{v})\in V$
$\vec{u}\in V \Rightarrow c\cdot\vec{u}\in V$

Equivalent: 只要 $\vec{u},\vec{v}\in H$, 那么 $a\vec{u}+b\vec{v}\in H$
那么 H 就是subspace

(zero subspace: $\{\vec{0}\}$)

Show H= ⋯ is a subspace { show H is a matrix A´s colomn space/ null space
or
proof that $\vec{0}\in H$ and for $\forall \vec{u},\vec{v}\in H$, $(c_1\cdot\vec{u}+c_2\cdot\vec{v})\in H$

10. colomn space & null space

row space ⊥ null space → $A=[a_1 \cdots a_n]$, $a_i\in R^m$
say $v\in (\mathrm{Col}A)^{\perp}$ $(v\in R^m)$
$\Rightarrow v\cdot\vec{a_i}=0$
$\therefore A^T\cdot v=0$
$\therefore v\in \mathrm{Nul}\,A^T$
$\therefore (\mathrm{Col}\,A)^{\perp}=\mathrm{Nul}\,A^T$
$\quad\quad \| $
$(\mathrm{Row}A^T)^{\perp}$
$\therefore \mathrm{Row}A \perp \mathrm{Nul}\,A$

Null space of matrix

$\mathrm{Nul}\,A=\{\vec{x}\mid A\vec{x}=0,\ \vec{x}\in R^m\}$

Basis of null space of matrix A:
calculate RREF([A 0])
and find it´s general solution
↓
$\dim(\mathrm{Nul}(A))$ = num of free variables in [A 0]

Colomn space of matrix $A=[a_1\cdots a_n]$

$\mathrm{Col}\,A=\{\vec{x}\mid \sum_{i=1}^{n} c_ia_i=\vec{x},\ c_i\in R,\ a_i\in R^m\}$

Basis of colomn space of matrix A
↓
A´s basis

$\dim(\mathrm{Col}(A))$ = num of basic variables in A

$\therefore \dim(\mathrm{Col}A)+\dim(\mathrm{Nul}A)=n$

11. linear transformation

a transformation that satisfies $T(c\vec{v}+b\vec{u})=aT(\vec{v})+bT(\vec{u})$
$(T: R^m\to R^n,\ n\times m)$

kernel of T: (null space)

$\{\vec{u}\in R^m \mid T(\vec{u})=\vec{0}\}$

range of T: (colomn space)

$\{\vec{u}\in R^n \mid \vec{u}=T(\vec{v}),\ \vec{v}\in R^m\}$

row operation 不影响 colomn的 dependence 关系 (∵做 row operation就是乘一个elemantary matrix
新colomn 还是原colomn的linear combination)
→ Col A 的 basis 可以选 RREF后的pivot colomn, 也可以选本身的pivot colomn

12. coordinate system

Unique Representation Theorem

for $\forall\vec{v}\in H$, say $B=\{\vec{b_1},\cdots,\vec{b_k}\}$ is basis of H

$\exists$ unique $c_1\sim c_k\in R^m$, s.t. $\vec{v}=\sum_{i=1}^{k}c_i\vec{b_i}$

Denote $[\vec{v}]_B=\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix}$

coordinate mapping: $\vec{x}\mapsto[\vec{x}]_B$
(1-to-1 mapping. from the unique representable thm)
↳ is linear: $\begin{cases}c[\vec{v}]_B=[c\vec{v}]_B\\ a[\vec{v}]_B+b[\vec{u}]_B=[a\vec{v}+b\vec{u}]_B\end{cases}$

Change-of-coordinates matrix $P_B=[b_1\cdots b_n]$ → invertible
(where $b_1\sim b_n$ are basis of R^n)

$\therefore \forall\vec{v}\in R^n$, $\vec{v}=P_B\cdot[\vec{v}]_B$

$\Rightarrow [\vec{v}]_B=P_B^{-1}\cdot\vec{v}$

Isomorphism: 1-to-1 linear transformation from vector space V→W

13. Dimension

$\dim\{\vec{0}\}=0$

Ex: The subspaces of R^3 can be classified by dimension.
0-dimensional subspaces: Only the zero subspace.
1-dimensional subspaces: Any subspace spanned by a single nonzero vector.
2-dimensional subspaces: Any subspace spanned by two linearly independent vectors. Such subspaces are planes through the origin.
3-dimensional subspaces: Only R^3 itself. Any three linearly independent vectors in R^3 span all of R^3.

Method to show $\dim A\le\dim B$ (if $A\subset B$)
let basis of A is $A_B=\{a_1,\cdots,a_m\}$ ⇒ $\mathrm{span}(A_B)$ is a subset of B
$\because A\subset B$ ∴ A_B could be expanded to a basis of B

14. row space

$A=\begin{bmatrix}r_1\\\vdots\\r_m\end{bmatrix}$ $(m\times n)$

row space $=\{\vec{v}\mid\vec{v}\in R^n,\ \vec{v}\in\mathrm{span}(r_1,\cdots,r_m)\}$

Basis for Row A → ① transpose, then find Basis(Col(A^T))

row operation will change the linear dependence of rows

② pivot rows (row operation is reversible)
$\therefore \dim(\mathrm{Row}\,A)=\dim(\mathrm{Col}\,A)$

15. Rank

$\mathrm{Rank}(A)=\dim(\mathrm{Col}(A))=\dim(\mathrm{Row}(A))$
definition

$\mathrm{rank}(A)+\dim(\mathrm{Nul}A)=n$

16. eigenvector & eigenvalue

$A\vec{x}=\lambda\cdot\vec{x}$ $\quad(A-\lambda I)\cdot\vec{x}=0$ (applicable only to square matrix)

∴ eigenvector ∈ the Nul space of $(A-\lambda I)$
↳ basis of eigenspace: finding basis of $\mathrm{Nul}(A-\lambda I)$

Eigenvalue for triangle matrix: entries on main diagonal
(upper or lower ✓)

All basis of eigenspace的交集of the same matrix must be linearly independent
(First proof when evalue⇔evector, all l.i.
Then proof if some evalue has multiple evector
its eigenspace l.i. with others)

Calculate eigenvalue: λ, s.t. $\det(A-\lambda I)=0$
∴ A is invertible ⇔ 0 is not a eigenvalue of A
⇔ $\det(A)\ne0$

multiplicity: $f(\lambda)=\det(A-\lambda I)$
$=(\lambda-c_1)^{k_1}(\lambda-c_2)^{k_2}\cdots(\lambda-c_n)^{k_n}$
k_i

Calculate A^n: → 要 $[\vec{v_1}\ \vec{v_2}]$ invertible
∴ 得 diagonalizable
才能这样搞

$\vec{x_k}=A^n\vec{x_0}$ → 要算 A^n
$A^n\vec{v_1}=\lambda_1^n\vec{v_1}$
$A^n\vec{v_2}=\lambda_2^n\vec{v_2}$
$A^n[\vec{v_1}\ \vec{v_2}]=[\lambda_1^n\vec{v_1}\ \ \lambda_2^n\vec{v_2}]$
$A^n=[\lambda_1^n\vec{v_1}\ \ \lambda_2^n\vec{v_2}][\vec{v_1}\ \vec{v_2}]^{-1}$

17. Similarity

→ square matrix
A is similar with B: if $\exists P$, s.t. $A=PBP^{-1}$ && $B=PAP^{-1}$
$A\to B=P^{-1}AP$: similarity transformation

A similar with B ⟹ $\det(A-\lambda I)=\det(B-\lambda I)$

similar ⇎ same eigenvalue | similar $\cancel{\Leftarrow}$ ✓⇒ same characteristic polynomial
↳ same characteristic function
↳ $\det A=\det B$

A, A^T has same characteristic equation

A must be similar to A^T

18. Diagonalization (diagonal: 只有对角线上有非0值)

If $A=PDP^{-1}$, P invertible, D diagonal
then $A^2=PD^2P^{-1}$, $A^k=PD^kP^{-1}$

Diagonalizable: A is diagonalizable
⇔ A is similar to a diagonal matrix
⇔ A has n linearly independent eigenvectors → 即 $\sum\dim(\text{eigenspace})=n$
(⇔ A的 eigenspace的basis的并集的元素个数为n)
⇔ { P´s colomns are A´s eigenvectors
D´s diagonal entries are A´s eigenvalues
有一一对应关系, D_{11}对应P_1, D_{22}对应P_2, …

Diagonalizing matrices: 把diagonalizable 的matrix A 写成 PDP^{-1}的形式

1. compute eigenvalue
2. compute basises of eigenspace
(n l.i. eigenvector)
3. construct D & P (对于同一个eigenvalue, eigenvector的顺序是无所谓的)

判断是否 diagonalizable: 计算是否有n个l.i.的eigenvector
假设 A有n个不同的 eigenvalue, 那么 A一定 diagonalizable
→ $\sum\dim(\text{eigenspace}(\lambda_i))=n$
→ 对于$\forall\lambda_k$, $\mathrm{multi}(\lambda_k)=\dim(\text{eigenspace}(\lambda_k))$
→ $f(\lambda)=\det(A-\lambda I)$ 全部可拆成一次式之积

对于$\forall\lambda_i$ = eigenvalue
λ_i´s multiplicity $\ge\dim(\text{eigenspace}(\lambda_i))$

A diagonalizable ⇒ { A^{-1} diagonalizable (if A is invertible)
A^T diagonalizable

19. orthogonal

a set of vectors $\{\vec{v_1},\cdots,\vec{v_n}\mid\vec{v_i}\in R^n\}$ is a orthogonal set
iff $\vec{v_i}\cdot\vec{v_j}=0$ $(i\ne j)$

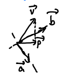
if $S_n=\{\vec{v_1},\cdots,\vec{v_n}\mid\vec{v_i}\in R^n\}$ is orthogonal & $\vec{v_i}\ne\vec{0}$
then S_n is the basis of R^n

orthogonal basis: an orthogonal set of vectors that is also a basis of some vector space V

let $U=\{\vec{u_1},\cdots,\vec{u_p}\}$ is a orthogonal basis of V
then $\forall\vec{v}\in V$, $\vec{v}=\sum_{i=1}^{p}c_i\cdot\vec{u_i}$, where $c_i=\dfrac{\vec{v}\cdot\vec{u_i}}{|\vec{u_i}|^2}$
($\because \vec{v}\cdot\vec{u_i}=(\sum_{i=1}^{p}c_i\cdot\vec{u_i})\cdot\vec{u_i}=c_i\cdot|\vec{u_i}|^2\Rightarrow c_i=\dfrac{\vec{v}\cdot\vec{u_i}}{|\vec{u_i}|^2}$)

orthogonal projection:
找vector到 vector/vector space的投影

vector:
$\vec{v}\mapsto\vec{p}$ (orthogonal projection), s.t. $(\vec{v}-\vec{p})\cdot\vec{u}=0$ and $\vec{p}$和$\vec{u}$共线
(找投影)
$\vec{p}=T(\vec{v})=\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\cdot\vec{u}$
(intuition: $(\vec{v}-\vec{p})\cdot\vec{u}=0$
$\Leftrightarrow \vec{v}\cdot\vec{u}=\vec{p}\cdot\vec{u}$
when $\vec{p}=\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\cdot\vec{u}$,
$\vec{p}\cdot\vec{u}=\dfrac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\cdot\vec{u}\cdot\vec{u}$
$=\vec{v}\cdot\vec{u}$)

vector space:
类似高中找到平面的投影  只需上于basis即可

$\therefore \vec{p}=T(\vec{v})$
$=\sum_{i=1}^{k}\dfrac{v\cdot u_i}{|u_i|^2}\cdot u_i$ (u_i is a orthogonal basis)

If $\vec{u}\in V$. V´s basis is $U=\{\vec{u_1},\cdots,\vec{u_k}\}$
then $T(\vec{u})=\vec{u}$ $(\vec{u}-T(\vec{u})=\vec{0},\ \vec{0}\perp V)$

matrix形式: 求到一个subspace 的 orthogonal projection的 standard matrix
$\vec{p}=\sum_{i=1}^{k}\dfrac{\vec{v}\cdot\vec{u_i}}{|\vec{u_i}|^2}\cdot\vec{u_i}$
$=[\vec{u_1}\cdots\vec{u_k}]\cdot\begin{bmatrix}\frac{\vec{v}\cdot\vec{u_1}}{|\vec{u_1}|^2}\\\vdots\\\frac{\vec{v}\cdot\vec{u_k}}{|\vec{u_k}|^2}\end{bmatrix}$
$=U\cdot(U\cdot U^T)^{-1}\cdot U^T\cdot\vec{v}$
$(U\cdot U^T)^{-1}$是 normalize
$U^T\cdot\vec{v}$是 $\begin{bmatrix}\vec{v}\cdot\vec{u_1}\\\vdots\\\vec{v}\cdot\vec{u_k}\end{bmatrix}$

orthonormal sets:
$\{\vec{u_1},\cdots,\vec{u_k}\}$ that $\begin{cases}\vec{u_i}\cdot\vec{u_j}=0 & \text{(orthogonal set)}\\ |\vec{u_i}|=1\end{cases}$

a matrix A has orthonormal colomn ⇔ $A^T\cdot A=I$
→ orthogonal ⇓ square
→ if $T(v)=Av$
then $\begin{cases}T(\vec{v})\cdot T(\vec{u})=\vec{v}\cdot\vec{u}\\ |T(\vec{v})|=|\vec{v}|\end{cases}$ (保长度&角度)
→ $\det(A)=\pm1$

orthonormal basis: a set of basis which is orthonormal

Best Approximation Theorem:  1≤2
→ projection, 不是垂直向量

distance from $\vec{v}\in R^n$ to a subspace W
is defined by $[\vec{v}-\mathrm{Proj}_W(\vec{v})]$
least square: $d=|\vec{b}-A\vec{x_0}|$

The Gram-Schmidt Process

用来produce orthogonal basis的

1. 初始化: 令 $u_1=v_1$。
2. 迭代步骤: 对于每个 $i=2,3,\ldots,n$, 执行以下操作:
 1. 计算 v_i 在前面所有正交向量 $\{u_1,u_2,\ldots,u_{i-1}\}$ 上的投影, 并从 v_i 中减去这些投影, 得到新的向量 u_i。
 $$u_i=v_i-\mathrm{proj}_{u_1}v_i-\mathrm{proj}_{u_2}v_i-\cdots-\mathrm{proj}_{u_{i-1}}v_i$$
 其中, 投影公式为:
 $$\mathrm{proj}_{u_j}v_i=\frac{v_i\cdot u_j}{u_j\cdot u_j}u_j$$
 2. 如果需要正交归一化, 可以将 u_i 归一化:
 $$e_i=\frac{u_i}{\|u_i\|}$$
 这里, e_i 就是归一化后的单位向量。

QR Factorization

QR 分解 (QR Factorization) 是一种将一个矩阵分解为两个矩阵乘积的过程。通常用于解线性方程组、最小二乘问题和计算矩阵的特征值等。QR 分解将一个矩阵 A 分解为一个正交矩阵 Q 和一个上三角矩阵 R, 即:
$$A=QR$$
其中:
- A 是一个 $m\times n$ 的矩阵。
- Q 是一个 $m\times m$ 的正交矩阵 (即 $Q^TQ=I$, 其中 I 是单位矩阵), 如果需要, 也可以是正交归一矩阵。
- R 是一个 $m\times n$ 的上三角矩阵。

QR 分解的几种方法
1. Gram-Schmidt 过程:
 - 使用 Gram-Schmidt 过程可以通过一系列向量投影来生成正交矩阵 Q, 然后通过这些正交向量计算上三角矩阵 R。
 具体来说, 给定矩阵 A 中的列向量 $a_1,a_2,\ldots,a_n$, 通过 Gram-Schmidt 过程得到:
 $$q_1=a_1,\quad q_2=a_2-\mathrm{proj}_{q_1}a_2,\quad\ldots,\quad q_n=a_n-\sum_{j=1}^{n-1}\mathrm{proj}_{q_j}a_n$$
 然后可以计算矩阵 R, 即:
 $$R=\begin{bmatrix}q_1\cdot a_1 & q_1\cdot a_2 & \cdots & q_1\cdot a_n\\ 0 & q_2\cdot a_2 & \cdots & q_2\cdot a_n\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & q_n\cdot a_n\end{bmatrix}$$
 这种方法虽然直观, 但计算效率较低, 尤其是在处理大规模矩阵时。

$A=QR$
$\because Q$ orthogonal
$\therefore Q^T\cdot Q=I$
$\therefore Q^T\cdot A=R$

20. Least square problems

当 $A\vec{x}=\vec{b}$ inconsistent 时, 求近似解

① 用解 $A^TA\vec{x}=A^T\vec{b}$ 的解来逼近代替 (only applicable when $\mathrm{Col}\,A=R^n$)

② 假如 A linear independent ? 哪用了
还可以用 $\vec{x}=R^{-1}Q^T\vec{b}$ 来表示 $A\vec{x}=\vec{b}$ 的 least square solution
$\because A\vec{x}=\vec{b}\Leftrightarrow QR\vec{x}=\vec{b}\Leftrightarrow QR\cdot R^{-1}Q^T\vec{b}=\vec{b}\Leftrightarrow QQ^T\vec{b}=\vec{b}$
orthogonal projection($\vec{b}$) → colA

? 解的uniqueness是∵ colomns of A are l.i.

? 当 $\vec{b}$ 与 ColA orthogonal 时, $A\vec{x}=\vec{b}$ 的 least square 解 $\vec{x_0}$
满足 $A\vec{x_0}=\vec{0}$

Equivalent { $A\vec{x}=\vec{b}$ has a unique least-square solution for $\forall b\in R^m$
colomns of A is linear independent
A^TA is invertible

21. symmetric matrix

A symmetric ⇒ ∀ 2 eigenvectors from different eigenspaces are orthogonal

orthogonally diagonalizable ⇔ $A=PDP^{-1}=PDP^T$ (P orthogonal & $P^T=P^{-1}$, D diagonal)
⇕
symmetric ($\because A^T=(PDP^T)^T=PD^TP^T=PDP^T=A$)