Poznámka (Note of Me – autor of notes)
Bad English in this text is my fault, not lecturer's one.

Úvod

Poznámka
3 part exam: theorem -> proof; scientific paper -> understand + explain; therms + concepts -> explain

credits: homework (time demanding)

Microsoft teams

0.1 Matrix analysis / linear algebra

Poznámka
Scalar product: $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$: $\mathbf{u} \cdot \mathbf{v}$, cross product: $\mathbf{u} \times \mathbf{v}$, and more: $\mathbf{u} = u^i \mathbf{e}_i$ $\mathbf{u} \cdot \mathbf{v} = \delta_{ij}(u^i v^j)$, $(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k$ (where ε_{ijk}, Levi-Civita symbol, does expecting thing).

Definice 0.1 (Tensor product)

$$\mathbf{u} \otimes \mathbf{v} \qquad (\mathbf{u} \otimes \mathbf{v})\mathbf{w} := \mathbf{u}(\mathbf{v} \cdot \mathbf{w})$$

Tvrzení 0.1 (Identities for Levi-Civita symbol)

$$\varepsilon_{ijk}\varepsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}$$

$$\varepsilon_{ijk} \cdot \delta_{lm} = \varepsilon_{jkm} \cdot \delta_{il} + \varepsilon_{klm} \cdot \delta_{jl} + \varepsilon_{ijm} \cdot \delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{km} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{ijm}\varepsilon_{ijn} = 2\delta_{mn}$$

Definice 0.2 (Transpose matrix)

$\mathbb{A} \in \mathbb{R}^{3\times3}$, $\mathbb{A}^T$ is defined as $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 : \mathbb{A}^T\mathbf{u} \cdot \mathbf{v} := \mathbf{u} \cdot \mathbb{A}\mathbf{v}$.

Definice 0.3 (Trace of matrix)

$\mathbb{A} \in \mathbb{R}^{3\times3}$, $\operatorname{tr} \mathbb{A}$ is defined as $\operatorname{tr}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

Poznámka
Matrix, tensor and linear operator is the same.

$$\mathbb{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbb{A}\mathbf{v} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(v_m\mathbf{e}_m) = A_{ij}v_m\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_m) = (A_{ij}v_j)\mathbf{e}_i.$$

Definice 0.4 (Axial vector)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$, $\mathbb{A}$ is stew-symetric ($-\mathbb{A} = \mathbb{A}^T$). Then we can prove that $\forall \mathbf{w} \in \mathbb{R}^3 : \mathbb{A}\mathbf{w} = \mathbf{v}_{\mathbb{A}} \times \mathbf{w}$. We call $\mathbf{v}$ the axial vector.

Poznámka
$\mathbf{v}_{\mathbb{A}} = (A_{23}, A_{13}, A_{12})^T$.

Tvrzení 0.2

$\mathbb{A}\mathbf{v}_{\mathbb{A}} = \mathbf{o}$ *and* $(\mathbf{u} \otimes \mathbf{v})^T = (\mathbf{v} \otimes \mathbf{u})$.

Definice 0.5 (Determinant in 3D)

$\det \mathbb{A} := \frac{\mathbb{A}\mathbf{u}\cdot(\mathbb{A}\mathbf{v}\times\mathbb{A}\mathbf{w})}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}$ for three arbitrary vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

Poznámka (Nanson formula)

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\det \mathbb{A})^{-1}\mathbb{A}\mathbf{w} \cdot (\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) = \mathbf{w} \cdot (\det \mathbb{A})^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) \implies$$
$$\implies \mathbf{u} \times \mathbf{v} = (\det A)^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v})$$
$$\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v} = (\det \mathbb{A})\mathbb{A}^{-T}(\mathbf{u} \times \mathbf{v})$$

Definice 0.6 (Cofactor)

$$\operatorname{cof} \mathbb{A} := (\det \mathbb{A})\mathbb{A}^{-T}.$$

(Change of surface area under linear mapping $\mathbb{A}$.)

Definice 0.7 (Eigenvalues, eigenvectors)

$\mathbb{A}\mathbf{v} = \lambda\mathbf{v}$.

Characteristic polynomial: $\det(\mathbb{A} - \mu\mathbb{I}) = -\mu^3 + c_1\mu^2 - c_2\mu + c_3$.

Věta 0.3 (Cayley-Hamilton)

$$-\mathbb{A}^3 + c_1\mathbb{A}^2 - c_2\mathbb{A} + c_3\mathbb{I} = \mathbb{O}$$

Tvrzení 0.4

$$c_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det \mathbb{A}$$
$$c_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \operatorname{tr}\operatorname{cof}\mathbb{A} = \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}(\mathbb{A}^2))$$
$$c_1 = \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}\mathbb{A}$$

Důkaz
With definition of characteristic polynomial, Cayley-Hamilton and Schur decomposition. Schur decomposition: $\mathbb{A} \in \mathbb{R}^{3\times 3}$. There exists an invertible matrix $\mathbb{U}$ and upper triangular matrix $\mathbb{T}$ such that

$$\mathbb{A} = \mathbb{U}^{-1}\mathbb{T}\mathbb{U}, \qquad \mathbb{T} = \begin{pmatrix} \lambda_1 & T_{12} & T_{13} \\ 0 & \lambda_2 & T_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

□

Tvrzení 0.5 (Useful identity from CH)

$$\mathbb{A}^{-1} = \frac{1}{c_3}\mathbb{A}^2 - \frac{c_1}{c_3}\mathbb{A} + \frac{c_2}{c_3}\mathbb{I} = \frac{1}{\det\mathbb{A}}\mathbb{A}^2 - \frac{\operatorname{tr}\mathbb{A}}{\det\mathbb{A}}\mathbb{A} + \frac{\operatorname{tr}\operatorname{cof}\mathbb{A}}{\det\mathbb{A}}\mathbb{I}$$

Poznámka (Functions of matrices)
$\exp\mathbb{A}$, $\ln\mathbb{A}$, $\sin\mathbb{A}$, ...

There are several ways of define it: Analytics calculus = Taylor series, Borel calculus: $\mathbb{A} = \sum \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$ () $\implies f(\mathbb{A}) := \sum f(\lambda_i)\mathbf{v}_i\otimes\mathbf{v}_i$, Holomorphic calculus ($f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)}d\zeta$) $f(\mathbb{A}) = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta\mathbb{I} - \mathbb{A})^{-1}d\zeta$ (where curve ζ envelops eigenvalues of $\mathbb{A}$)

Tvrzení 0.6 (Useful identities for functions)

$$\det(\exp\mathbb{A}) = \exp(\operatorname{tr}\mathbb{A})$$
$$\exp\mathbb{A} = \lim_{n\to\infty}\left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n$$