Poznámka
At least 1 from (3-)4 homework (flexible deadlines – last lecture).

Poznámka
In this lecture, there was also the revision of topology. (Topological space, topology, basis of topology, continuous map, quotient space, product topology, Hausdorff spaces).

Poznámka
World Homotopy comes from homós (= same, simiar) and topos (place).

Definice 0.1 (Homotopic functions)

Given two topological spaces X and Y and two continuous functions $f, g : X \to Y$, we say that f is homotopic to g ($f \sim g$) if there is a 1-parametric family $f_t : X \to Y$: $f_0 = f$, $f_1 = g$ and the map $F : [0,1] \times X \to Y$ defined by $(t, x) \mapsto f_t(x)$ is continuous.

Definice 0.2 (Homotopy equivalent spaces)

Given two topological spaces X and Y we say that X and Y are homotopy equivalent if there is a pair of continuous maps (f, g) such that $f : X \to Y$ and $g : Y \to X$ and $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$, $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$.

Příklad
Given $\mathbb{R}$, $\mathbb{R}^2$ with the standard Euclidean topology and two maps $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto f(x) = (x, x^3)$, $g : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto g(x) = (x, e^x)$.

Are f and g homotopic? (Show that by constructing homotopy.)

Řešení

$$F(t, x) = (1 - t)(x, x^3) + t(x, e^x) = (x, (1 - t)x^3 + te^x).$$

Příklad
Given three topological spaces $(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$ and two pairs of continuous maps $f_1, g_1 : (X, \tau_X) \to (Y, \tau_Y)$ and $f_2, g_2 : (Y, \tau_Y) \to (Z, \tau_Z)$. Assume that f_1 is homotopic to g_1 and f_2 is homotopic to g_2. Show that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$.

Řešení

$$F(t, x) = F_2(t, F_1(t, x)).$$

Příklad
Take $B^n := \left\{x, \ldots, x_n | \sqrt{x_1^2 + \ldots + x_n^2} \leqslant 1\right\} \subseteq \mathbb{R}^n$. And take a map $f : B^n \to B^n$: $f(x) = (0, \ldots, 0) \in B^n$ for all $x \in B^n$. Shows that there is a homotopy from id to f.

Řešení

$$F : [0,1] \times B^n \to B^n, \qquad (t,x) \mapsto (1-t)x.$$

Příklad
Take a 2-ball B^2. B^2 is homotopy equivalent to its center by previous problem, but it is not homeomorphic to $(0,0)$.

Definice 0.3 (Deformation retraction)

A deformation retraction of a topological space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in [0,1]$: $f_0 = \mathrm{id}_X$, $f_1(X) = A$ and $f_t|_A = \mathrm{id}_A$. And family f_t is continuous in the following sense:

$$F : [0,1] \times X \to X, (t,x) \to f_t(x), \text{ is continuous.}$$

Tvrzení 0.1

Given a deformation retraction $f_t : X \to X$, there is a pair $(f,g) : X \xrightarrow{f} A \xrightarrow{g} X : g \circ f \sim \mathrm{id}_X, f \circ g \sim \mathrm{id}_A$.

Poznámka (Suggestion)
$f = f_1$, $g = f_i \circ i_A$ ($A \overset{i_A}{\hookrightarrow} X$), tj. $f \circ g : A \overset{i_A}{\hookrightarrow} X \xrightarrow{f_1} X \xrightarrow{f_1} X$, $a \mapsto a \mapsto a \mapsto a$ (or A) $\implies f \circ g = \mathrm{id}_A$. $g \circ f : X \xrightarrow{f_i} A \xrightarrow{i_A} X \implies f_1(x) \sim \mathrm{id}_X$.

Definice 0.4

Given two topological spaces X and Y and a continuous map $f : X \to Y$, the mapping cylinder M_f is defined to be the quotient space of $X \times [0,1] \coprod Y$ and $\sim$: $(x,1) \sim f(x)$. $M_f = X \times [0,1] \coprod Y / \sim$.

Tvrzení 0.2

Given X, Y and f, M_f deformation retracts to Y.

Důkaz (/ Idea of proof)
The way to construct $f_t = F(\cdot, t) : M_f \to M_f$ is to slide each point (x, t) along the segment $\{x\} \times [0, 1]$ to $f(x)$:

$$F : (x, t) \mapsto f_t(x), \qquad \forall y \in Y : y = F(y, t) \mapsto \{f_1 = \operatorname{id} Y \to Y\}$$

In your HW you will check that $F(x, t)$ is continuous. □

Poznámka
Cell complex (CW complex) is a topological space with a nice decomposition into small pieces.

1. Start with a discrete set X^0, whose points are called 0-cells.

2. We form the n-skeleton X^n from X^{n-1} by attaching cells $e^n_\alpha = I^n = [0, 1]^n$. By the attachment we mean $(e^n_a = B^n_\alpha, \partial e^n_a = S^n_\alpha)$ $\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$. Hence we can view $X^n = X^{n-1} \coprod \coprod B^n_\alpha / \sim$, where $x \sim \varphi_\alpha(x)$ for $x \in \partial\partial B^n_\alpha$.

3. We can either stop this inductive process at a certain finite steps or take an infinite number of steps. In the first case $X = X^n$ for some n, in the second one $X = \bigcup_{n \in \mathbb{N}_0} X^n$ with the weak topology ($A \subset X$ is open $\leftrightarrow$ $A \cap X^n$ is open for all n).

Například
Example of 1-skeleton is graph.

Definice 0.6

Given a cell complex X. Each cell e^n_α has a characteristic map $\Phi_\alpha : e^n_\alpha = B^n_\alpha \to X$ which extends the attaching map $\varphi_\alpha : \partial B^n_\alpha \to X^n$, it is homeomorphism from the interior of B^n_α onto e^n_α. Namely

$$B^n_\alpha \hookrightarrow X^{n-1} \coprod \coprod_\beta B^n_\beta \overset{quotient}{\to} X^n \to X, \qquad B^n_\alpha \to X$$

Definice 0.7

A subcomplex of CW complex is a closed subspace $A \subset X$ that is a union of cells with the corresponding attachments.

Příklad
Construct two different CW structures on S^2.

Řešení
$S^2 = e^0 \cup e^2$, $S^2 = e^0 \cup e^1 \cup \{e^2_1, e^2_2\}$. (See practicals.)

Příklad
We define $\mathbb{R}P^n$ to be the quotient of $S^n/\sim$, where $V \sim$ the antipodal point to V. TODO?

Definice 0.8

Consider a pair (X, A) where X is a CW complex and A is subcomplex. Then we define the quotient complex X/A to be the CW complex with the structure: There are all the cells of $X\backslash A$ with the corresponding attaching maps, and there is a extra 0-cell which is A in $X\backslash A$. For a cell e^n_α of $X\backslash A$ attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map in the corresponding cell in $X\backslash A$ is the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

Příklad
Show that $S^n = e^0 \cup e^n$ is $B^n/S^{n-1} = TODO/e^0 \cup e^{n-1}$.

TODO!!!

Tvrzení 0.3

There is an isomorphism $\Pi_1(X, x_1) \to \Pi_1(X, x_0)$ for x_0 and x_1 in the same path connected component.

Důkaz
Since x_0, x_1 are in one path connected component $\tilde{X}$, $\exists$ path $h : [0, 1] \to X$: h is in $\tilde{X}$ and $h(0) = x_0, h(1) = x_1$. $\overline{h}(s) := h^{-1}(s) := h(1 - s)$, $s \in [0, 1]$.

To each loop f based at x_1 we associate a loop $h \cdot f \cdot h^{-1}$. $h \cdot f \cdot h^{-1}$ is based at x_0. $\beta_h : \Pi_1(x, x_1) \to \Pi_1(x, x_0), [f] \mapsto [h \cdot f \cdot h^{-1}]$. We claim, that β_h is an isomorphism. „β_h is homomorhism“:

$$\beta_h([f \cdot h]) = [hfgh^{-1}] = [hfh^{-1}hgh^{-1}] = [hfh^{-1}] \cdot [hgh^{-1}] = \beta_h([f]) \cdot \beta_h([g]).$$

„β_h is isomorphism“: „the inverse of β_h is $\beta_{h^{-1}}$“ (which is homomorphism too by the argument we used for β_h):

$$\beta_{h^{-1}}(\beta_h([f])) = \beta_{h^{-1}}([hfh^{-1}]) = [h^{-1}hfh^{-1}h] = [f].$$

□

Věta 0.4 (Fundamental group of S^1)

S^1 is path connected, thus $\Pi_1(S^1, x_0) = \Pi_1(S^1)$.

$\Pi_1(S^1) \simeq \mathbb{Z}$.

Důkaz
We claim that $\Pi_1(S^1) \simeq \langle[\omega]\rangle$, where $\omega : [0,1] \to S^1$, $s \mapsto (\cos(2\pi s), \sin(2\pi s)) \in \mathbb{R}^2$, $s \in [0,1]$. $\omega_n(s) := (\cos(2\pi ns), \sin(2\pi ns)) \sim \omega^n$, so $[\omega]^n = [\omega_n]$.

Now our theorem is equivalent to the statement that every loop in S^1 based at $(1,0)$ is homotopic to the unique ω_n. We use the following two facts:

Fact 1: For every path $f : I \to X$ starting at $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at x_0.

Fact 2: For each homotopy $f_x : I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists$ unique lifted homotopy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

p that we need: $p : \mathbb{R} \to S^1$; $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If we define $\tilde{\omega}_n(s) = n \cdot s$. We will apply Facts 1 and 2 to $p : \mathbb{R} \to S^1$, $\tilde{\omega}_n$: Given $f : [0,1] \to S^1$ based at $(0,1)$ representing some element of $\Pi_1(S^1)$. We take $\tilde{f}$. Since $p\tilde{f}(1) = f(1) = (1,0)$ (and $p^{-1}(1) \in \mathbb{Z}$), we can argument that if $\tilde{f}$ ends at u (i.e. $\tilde{f}(1) = f$), it is homotopoc to $\tilde{\omega}_n$ by the homotopy $\tilde{F} = (1-t)\tilde{f} + t\tilde{\omega}_n$.

From fact 1 exists $\tilde{f}$ starting at 0 and ending at $p^{-1}(1) \in \mathbb{Z}$.

Theorem: Exists homotopy $\tilde{F}$ from $\tilde{\omega}_k$ to $\tilde{f}$ denoted by $(*)$.

So we define homotopy F from ω_n to f by $F = p \circ \tilde{F}$, homotopy from ω_n to f. Since $[\omega_n] = n \cdot [\omega]$, $\Pi_1(S^1) \simeq \mathbb{Z}$.

Now we would like to show that $[f]$ is uniformly determined. Assume that $f \sim \omega_n$ and $f \sim \omega_m$, then using Facts 1 and 2 we have $[\omega_n] = [\omega_m]$ which lends to contradiction since they have different endpoints on $\mathbb{R}$. □

Definice 0.9

Given a topological space X, a covering space of X consists of a topological space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ satisfying that $\forall x \in X$ $\exists$ open neighbourhood U of x in X such that $p^{-1}(U)$ is a disjoint union of open subsets U_α each of which is homeomorphically mapped to U.

Definice 0.10

Given a map $[0,1] \xrightarrow{f} X$ and $p : \tilde{X} \to X$ we say that $\tilde{f} : [0,1] \to \tilde{X}$ is a lift of f if $p \circ \tilde{f} = f$.

The same construction can be defined for homotopy.

Tvrzení 0.5 (*)

Given a map $F : Y \times [0,1] \to X$ and a map $\tilde{F} : Y \times \{\mathbf{o}\} \to \tilde{X}$, where $p : \tilde{X} \to X$ is a covering space, and $\tilde{F}$ lifts $F|_{Y \times \{\mathbf{o}\}}$; there restricting to $\tilde{F}$ on $Y \times \{\mathbf{o}\}$.

Poznámka (Corollary: Fact 1 and Fact 2 from the previous proof)
Fact 1 is free, it comes when $Y = \{\text{point}\}$, Fact 2 also follows.

Příklad
We say that a topological (path-connected) space is simply connected $\Leftrightarrow \Pi_1(X) = \{e\}$. Examples of simply connected topological spaces: $\mathbb{R}, \mathbb{R}^2, \ldots$. S^1 is not simply connected.

Příklad
Given X, Y path-connected and $x_0 \in X$, $y_0 \in Y$. Show that $\Pi_1(X \times Y, (x_0, y_0)) \simeq \Pi_1(X, x_0) \times \Pi_1(Y, y_0)$.

Řešení
Product topology is defined to be such that a map $f : Z \to X \times Y$ is continuous $\Leftrightarrow$ $(p_x : X \times Y \to X,\ p_y : X \times Y \to Y)$ $p_x \circ f$ and $p_y \circ f$ are continuous.

A loop $\gamma : [0,1] \to X \times Y$ based at (x_0, y_0) splits at two loops $\gamma_1 : [0,1] \to X$, $\gamma_2 : [0,1] \to Y$. The same holds for homotopy, i.e. F from γ to $\tilde{\gamma}$ splits into (F_1, F_2), where F_1 is a homotopy on X from γ_1 to $\tilde{\gamma}_1$ and F_2 is a homotopy on Y from γ_2 to $\tilde{\gamma}_2$.

Důsledek
$\Pi_1(T^n) := \Pi_1(S^1 \times S^1 \times \ldots \times S^1) = \mathbb{Z}^n$.

Příklad
Show that TODO!!! is a covering space for $S^1 V S^1$.

TODO!!!

Důkaz (Proposition *)
To prove our proposition we need to construct $\tilde{F} : N \times I \to \tilde{X}$, where N is an open neighbourhood in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighbourhood of $F(y_0, t)$.

By compactness of $\{y_0\} \times I$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times I$. This implies that we can choose a single neighbourhood N of y_0, and a partition of $[0,1]$ $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ such that $F|_{N \times [t_i, t_{i+1}]}$ is contained in an evenly covered neighbourhood U_i.

Assume inductively that $\tilde{F}$ has been constructed for $N \times [0, t_i]$ starting at a given $\tilde{F}$ on $N \times \{0\}$. We have that $F(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, there is an open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically to U_i via p and $\tilde{F}((y_0, t_i)) \in \tilde{U}_i$. After replacing N by a smaller neighbourhood of y_0. (We replace $N \times \{t_i\}$ with the intersection with ?) we may assume that $\tilde{F}(N \times \{t_0\}) \in \tilde{U}_i$. Now we define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F

with $p^{-1} : U_i \to \tilde{U}_i$. After a finite number of steps, we eventually get a lift $\tilde{F} : N \times I \to \tilde{X}$ for N (some neighbourhood of y_0).

Next we show the uniqueness for $Y = \{\text{point}\}$. In this case we ? Suppose there are two lifts $\tilde{F} : I \to \tilde{X}$, $\tilde{F}' : I \to \tilde{X}$. As before we choose a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ so that $\forall i : F([t_i, t_{i+1}])$ is contained in some evenly covered neighbourhood U_i. Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, co is $\tilde{F}([t_i, t_{i+1}])$, which must therefore lie in one of the disjoint open sets $\tilde{U}_i$ projecting homeomorphically to U_i. By the same token, $\tilde{F}'([t_i, t_{i+1}])$ lies in a single $\tilde{U}_i$, in fact in the same containing $\tilde{F}([t_i, t_{i+1}])$ (by the assumption of induction). Since p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$ and the induction step follows.

The last step of the prove is to observe that since $\tilde{F}$, $\tilde{F}'$ are constructed on the sets of form $N \times I$ and are unique when we restrict to each segment $\{y\} \times I$, they must agree whenever two such sets $N \times I$ overlap, so we get in fact a well-defined lift $\tilde{F}$ on $Y \times I$. This $\tilde{F}$ is continuous since it is continuous on each segment $\{y\} \times I$. □

Poznámka

We would like to see π, as a functor $\pi_1 : Top \to Grp$. In order for π_1 to be a functor, we want for $\varphi : (X, x_0) \overset{\text{cont.}}{\to} (Y, y_0)$ associate $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. How to get φ_*? Given a loop γ on X based at x_0. We have a loop $\varphi \circ \gamma$ on Y based at y_0. $\varphi_*([\gamma]) := [\varphi \circ \gamma]$. Is φ_* a homomorphism? $\varphi_*([\gamma; \gamma_2]) = [\varphi \circ (\gamma_1 \cdot \gamma_2)] = [\varphi \circ \gamma_1 \cdot \varphi \cdot \gamma_2] = [\varphi \circ \gamma_1] \cdot [\varphi \circ \gamma_2]$. Hence φ_* is a homomorphism.

$$\forall (X, x_0) \in Fb(Top) \exists \varphi = 1 \in Hom((X, x_0), (X, x_0)).$$

We want $\pi_1(1) = \mathrm{id}_{\pi_1(X, x_0)}$. This because of the definition of $\pi_1(1)$, which maps $[\gamma] \to [\gamma]$ $\implies \pi_1(1) : [\gamma] \mapsto [\gamma]$, so it is $\mathrm{id}_{\pi_1(X, x_0)}$.

In order for π_1 to be a functor we also need: given $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$ then $i(\psi \circ \varphi)_* = \pi_1(\psi \circ \varphi) = \pi_1(\psi) \circ \pi_1(\varphi) = (\psi)_* \cdot (\varphi)_*$. This is because

$$\pi_1(\psi \circ \varphi)([\gamma]) = (\psi \circ \varphi)_*([\gamma]) = [\psi \circ \varphi \circ \gamma] = \psi_*([\varphi \circ \gamma]) = \pi_1(\psi)(\varphi_*([\gamma])) = \pi_1(\psi) \circ \pi_1(\varphi)([\gamma]).$$

Hence it holds and π_1 is functor $TOP \to GRP$.

Tvrzení 0.6

Given a topological space X. If $X = \bigcup A_\alpha$, where each of A_α is a path connected subspace of X and $A_\alpha \cap A_\beta$ is path connected $\forall \alpha, \beta$, then each loop in X based at x_0 can be decomposed as a product of loops each of which is in some A_α.

Důkaz
Given $f : I \to X$ with the basepoint x_0, we claim that there is a partition $0 = s_0 < s_1 < \ldots < s_m = 1$ of I such that $[s_{i-1}, s_i]$ is mapped by f to a single A_α (that by call A_i). Since f is continuous, each $s \in I$ has an open neighbourhood V_i in I mapped by f to some A_α. We may in fact take V_s to be an interval whose closure is mapped to a single A_α. By compactness of I, we see that a finite number of such intervals cover I. The endpoints of these intervals form a partition of I: $0 = s_0 < s_1 < \ldots < s_m = 1$. Again A_i we call $A_\alpha : f([s_{i-1}, s_i]) \subset A_\alpha$. Let f_i be denoted as $f|_{[s_{i-1}, s_i]}$. Then $f = f_i, \ldots, f_m$, where f_i is a path in A_i.

Since $f([s_{i-1}, s_i]) \subset A_i \wedge f([s_i, s_{i+1}]) \subset A_{i+1} \implies f(s_i) \in A_i \cap A_{i-1}$. Since $A_i \cap A_{i+1}$ is path connected, we can choose path $g_i \subset A_i \cap A_{i+1}$, g_i starts at x_0 and ends at $f(s_i)$. Hence $[f][f_1 g_1^{-1}][g_1 f_2 g_2^{-1}] \ldots [g_{n-1} f_n g_n^{-1}]$, loops in $A_1, A_2, \ldots, A_n$. $\square$

TODO?