1 Banach algebras

1.1 Basic properties

Definice 1.1 (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over $\mathbb{K}$, if

- $(A, +, -, 0, \cdot_S)$ is vector space over $\mathbb{K}$;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K}\ \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

Důsledek
1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ $(\forall a \in A)$.

If e_1 is left unit and e_2 is right unit, then $e_1 = e_2$ is unit. $(e_1 = e_1 \cdot e_2 = e_2)$

2) (Algebra) homomorphism $\varphi : A \to B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

Tvrzení 1.1

Let A be algebra over $\mathbb{K}$. Put $A_e = A \times \mathbb{K}$ with operations A_e defined coordinate-wise and multiplication defined by

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \qquad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

Then A_e is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if A is commutative, then A_e is commutative.

Důkaz
We have A_e is vector space (from linear algebra). We easy proof from definition, that A_e is algebra, $(\mathbf{o}, 1)$ is a unit in A_e and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. $\square$

Definice 1.2 (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv A$ is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ $(\forall a, b \in A)$.

Definice 1.3 (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv$ $(A, \|\cdot\|)$ is normed algebra and Banach space.

Například

$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If T is Hausdorff topological space, then

$$\mathcal{C}_b(T) = \{f : T \to \mathbb{K} | f \text{ is continuous and bounded}\} \subseteq l_\infty(T)$$

is closed subalgebra.

If T is locally compact, Hausdorff, not compact. Then

$$\mathcal{C}_0(T) = \{f : T \to \mathbb{K} \text{ continuous } | \forall \varepsilon > 0 : \{t \in T | |f(t)| \geqslant \varepsilon\} \, is \text{ compact}\} \subseteq \mathcal{C}_b(T)$$

is closed subalgebra, which doesn't have unit.

If X is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If X is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

Tvrzení 1.2

If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \to A$ is Lipschitz on bounded sets.

Důkaz

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r) \; \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) : \|ab - cd\| \leqslant$$

$$\leqslant \|a(b-d)\| + \|(a-c) \cdot d\| \leqslant \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leqslant r \cdot (\|b-d\| + \|a-c\|) \leqslant 2r \|(a,b) - (c,d)\|.$$

□

Tvrzení 1.3

Let $(A, \|\cdot\|)$ be a Banach algebra. On A_e we consider the norm

$$\|(a, \alpha)\| := \|a\| + |\alpha|, \qquad (a, \alpha) \in A \times \mathbb{K} = A_e.$$

Then $(A_e, \|\cdot\|)$ is Banach algebra.

Důkaz

It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a,\alpha)\cdot(b,\beta)\| \leqslant \|(a,\alpha)\|\cdot\|(b,\beta)\|,$$

which is easy. □

Poznámka

There is more (natural) ways to define norm on A_e (unlike $\cdot$ on A_e, which is natural).

A has a unit … we may still consider A_e.

If $e \in A\backslash\{\mathbf{o}\}$ is a unit, then $\|e\| \geqslant 1$, because $\|e\| = \|e^2\| \leqslant \|e\|^2$.

Věta 1.4

Let A be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a \cdot x$, $x \in A$. Then $I : A \to \mathcal{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leqslant 1$.

Moreover, if A has a unit e, then I is isomorphism into and $I(e) = \mathrm{id}$.

If $\|x^2\| = \|x\|^2$, $x \in A$, then I is isometry into.

Důkaz

„$L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leqslant 1$“: Linearity is obvious, $\|L_a(x)\| = \|a \cdot x\| \leqslant \|a\| \cdot \|x\|$, so $\|L_a\| \leqslant \|a\|$ and so $\|I\| \leqslant 1$. Since it is easily I preserves multiplication, so we are left to prove the „Moreover“ part.

„A has a unit e“: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geqslant \left\|L_a\left(\frac{e}{\|e\|}\right)\right\| = \frac{|a|}{\|e\|} = \frac{1}{\|e\|}\cdot|a|.$$

So I is bounded from below, so I is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \mathrm{id}\,.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$:

$$\|a\| \geqslant \|I(a)\| = \|L_a\| \geqslant \left\|L_a\left(\frac{a}{\|a\|}\right)\right\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So I is isometry. □

Poznámka
$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies \exists$ equivalent norm $\|\cdot\|$ on A such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

Důkaz
Let $I : A \to \mathcal{L}(A)$ be as before. Put $|\|x\|| := \|I(x)\|$, $x \in A$. Since I is isomorphism, $|\|\cdot\||$ is equivalent norm. Moreover, $|\|x \cdot y\|| = \|I(x \cdot y)\| \leqslant \|I(x)\| \cdot \|I(y)\| = |\|x\|| \cdot |\|y\||$, $x, y \in A$. So $(A, |\|\cdot\||)$ is a Banach algebra. Finally

$$|\|e\|| = \|I(e)\| = \|\operatorname{id}\| = 1.$$

□

1.2 Inverse elements

Definice 1.4

$(M, \cdot, e)$ is monoid ($\cdot$ is associative, e is unit). Then invertible elements form a group ($e^{-1} = e, \exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^\times := \{x \in M \mid \exists x^{-1}\}$

Tvrzení 1.5

If $(A, \cdot, e)$ is monoid and $x_1, \ldots, x_n \in A$ commute, then $x_1 \cdot \ldots \cdot x_n \in A^\times \Leftrightarrow \{x_1, \ldots, x_n\} \subset A^\times$.

Důkaz
It suffices to prove it for $n = 2$ (and use induction). „If x^{-1} and y^{-1} exists, then $(xy)^{-1}$“ is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so z is left inverse to y. Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so $\tilde{z}$ is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. □

Lemma 1.6

Let A be a Banach algebra with a unit.

- $\|x\| < 1 \implies \exists (e - x)^{-1} \wedge (e - x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x + h)^{-1} \wedge \|(x + h)^{-1} - x^{-1}\| \leqslant \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

Důkaz
„First“: We have $\|x^n\| \leqslant \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0}^{\infty} x^n \in A$. Moreover,

$$(e-x) \cdot \left(\sum_{n=0}^{\infty} x^n \right) = \lim_{N \to \infty} (e-x) \cdot (e + x + \ldots + x^N) = \lim_{N \to \infty} e - x^{N+1} = e,$$

because $\lim_{N \to \infty} \|x^{N+1}\| \leqslant \lim_{M \to \infty} \|x\|^M = 0$. And similarly $(\sum x^n) \cdot (e-x) = e$.

„Second item“: $x + h = x \cdot (e + x^{-1}h)$ we have x^{-1} exists and $(e + x^{-1}h)^{-1}$ exists (from first item), so $(x+h)^{-1}$ exists. Moreover

$$(x+h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \overset{1)}{=} \sum_{n=0}^{\infty} \left(-x^{-1}h\right)^n x^{-1},$$

so

$$\|(x+h)^{-1} - x^{-1}\| = \left\| \sum_{n=1}^{\infty} \left(-x^{-1}h\right)^n x^{-1} \right\| \leqslant \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leqslant$$

$$\leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \left(\|x^{-1}\| \cdot \|h\|\right)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}.$$

□

Důsledek
A Banach algebra with a unit $\implies$ $A^{\times} \subset A$ is open and $A^{\times}$ is topological group.

Důkaz
$A^{\times} \subset A$ is open by previous lemma (second item). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^{\times} \ni x_n \to x \in A^{\times} \overset{?}{\implies} x_n^{-1} \to x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \overset{h := x_n - x}{\leqslant} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \to 0.$$

□

1.3 Spectral theory

Definice 1.5 (Resolvent set, spectrum and resolvent)

Let A be a Banach algebra with a unit, $x \in A$. We define resolvent set of x as $\varrho_A(x) := \{\lambda \in \mathbb{K} | \exists (\lambda \cdot e - x)^{-1}\}$. Next we define spectrum of x as $\sigma_A(x) := \mathbb{K} \backslash \varrho_A(x)$. Finally we define resolvent of x as $R_x : \varrho(x) \to A$, $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$.

If A doesn't have a unit, then notions above are defined with respect to A_e.

Tvrzení 1.7

A Banach algebra

a) $\forall x \in A : 0 \in \sigma_{A_e}(x)$ *(in particular, if A has no unit, then* $0 \in \sigma_A(x)$*);*

b) *A has unit* $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

Důkaz (a))

$$\forall (b,\beta) \in A_e : (x,0) \cdot (b,\beta) = (\ldots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x,0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

□

Důkaz (b))
By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in A and second means that $((0,\lambda) - (x,0))^{-1} = (-x,\lambda)^{-1}$ exists in A. We take „$x \to -x$“.

„ $\implies$ “: find $(b,\beta) \in A_e$ such that $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$. So $(x\cdot b+\lambda\cdot b+\beta\cdot x, \lambda\cdot\beta) = (\mathbf{o},1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e+x)^{-1}\cdot x$. Similarly we find left inverse $\left(-\frac{1}{\lambda}x(x+\lambda e)^{-1}, \frac{1}{\lambda}\right)(x,\lambda)$. And next we prove that they are really inverses.

„ $\impliedby$ “: Put $(b,\beta) := (x,\lambda)^{-1}$. Then $(\lambda e+x)^{-1} = b+\beta\cdot e$. We have $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$, so $\lambda\cdot\beta = 1$ and $x\cdot b + \lambda \cdot b + \beta \cdot x = \mathbf{o}$. Then

$$(\lambda e + x)\cdot(b + \beta\cdot e) = \lambda\cdot b + \lambda\cdot\beta\cdot e + x\cdot b + \beta\cdot x = e.$$

Similarly second inverse. □

Věta 1.8

$\{\mathbf{o}\} \neq A$ *complex Banach algebra,* $x \in A$*. Then* $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ *is compact, nonempty.*

Důkaz
After theory. □

Definice 1.6 (Derivative)

Y Banach space, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of f at a.

Tvrzení 1.9 (Fact)

Y Banach, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then $f'(a)$ exists $\implies$ f is continuous at a $\wedge$ $\forall x^* \in Y^*$: $(x^* \circ f)'(a) = x^*(f'(a))$.

Důkaz

Continuity: $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a) = f'(a) \cdot 0 = 0$.

$x^* \in Y^*$ given, then

$$\lim_{x\to a} \frac{x^*(f(x)) - x^*(f(a))}{x-a} = \lim_{x\to a} x^* \left(\frac{f(x)-f(a)}{x-a} \right) = x^*(f'(a)).$$

□

Tvrzení 1.10

A Banach algebra with a unit, $x \in A$. Then

- *$\varrho(x)$ is open set;*
- $\forall \lambda \in \mathbb{K}, |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$*;*
- *(important!) $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$ has derivative at each $\lambda \in \varrho(x)$;*
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$*;*
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$*.*

Důkaz

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left(e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right)^n.$$

Fourth: In general $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$ (proof: $u^{-1}v^{-1} = (vu)^{-1}$). And we apply it for $u = (\mu e - x)$ and $v = (\nu e - x)$.

Fifth: In general $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$ (proof: $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$) so:

$$R_x(\mu) - R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) =$$

$$= R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)R_x(\mu)(R_x(\nu))^{-1} =$$
$$= R_x(\mu)R_x(\nu)\left(R_x(\nu)^{-1} - R_x(\mu)^{-1}\right) = R_x(\mu)R_x(\nu)(\nu - \mu).$$

□

Důkaz
For third we fix $\lambda \in \varrho(x)$ and $t \in (0, \delta)$ for δ small enough ($\lambda + t \in \varrho(x)$ and *). We shall prove that „$R'_x(\lambda) = -R_x(\lambda)^2$“:

$$0 \overset{?}{=} \left\| \frac{R_x(\lambda + t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| =$$

$$= \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leqslant$$

$$\overset{*\text{ for existence of the inverse}}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| (e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t \right\| =$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leqslant$$

$$\overset{\|x^n\| \leqslant \|x\|^n}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \overset{*\text{ for denominator } \leqslant 1/2}{\leqslant} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \to 0.$$

□

Věta 1.11 (Liouville for Banach space valued functions)

Y Banach space over $\mathbb{C}$, $f : \mathbb{C} \to Y$ has derivative at each point, f is bounded ($\equiv \|f\|$ is bounded). Then $f \equiv$ const.

Důkaz
Assume $f \not\equiv$ const, so there are $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$ (by Hahn–Banach theorem) $\exists x^* \in Y^* : x^*(f(x)) \neq x^*(f(x))$. From fact $x^* \circ f : \mathbb{C} \to \mathbb{C}$ has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions. □

Důkaz (Theorem before theory)
First case: „A has a unit“: Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is closed, so $\sigma(x)$ is compact. Assume that $\varrho(x) = \mathbb{C}$. By the previous tvrzeni we have $R_x : \mathbb{C} \to A$ has derivative everywhere, and it is bounded because $\lim_{|\lambda| \to \infty} R_x(\lambda) = \lim_{|\lambda| \to \infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$. From previous theorem $R_x \equiv$ const so $\lim_{|\lambda| \to \infty} R_x(\lambda) = 0 \implies R_x \equiv 0$. In particular $0 = R_x(0) = (-x)^{-1}$. $\lightning$ (If $A \neq \{0\}$ then $x^{-1} \neq 0$ for $x \in A$.)

Second case: „A hasn't a unit“, then $\sigma(x) := \sigma_{A_e}((x, 0))$ so we apply the already proven case. □

Poznámka (Convention)
If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

Věta 1.12 (Gelfand–Mazur)

$\{\mathbf{o}\} \neq A$ Banach algebra with a unit. Assume $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$. Then A is isomorphic to $\mathbb{C}$. If moreover e is a unit in A and $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.

Důkaz

Consider $\psi : \mathbb{C} \to A$ defined as $\psi(\lambda) := \lambda \cdot e$. This is algebraic homomorphism and $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$, so it is isomorphism (and isometry, if $\|e\| = 1$).

It remains „ψ is surjective“: Pick $a \in A$. From previously proved theorem $\exists \lambda \in \sigma(a)$, then $(\lambda e - a) \notin A^\times$. So, $\lambda \cdot e - a = \mathbf{o}$, then $\psi(\lambda) = a$. □

Definice 1.7 (Spectral radius)

A Banach algebra, $x \in A$. Then $r(x) := \sup \{|\lambda|, \lambda \in \sigma(x)\}$ is called spectral radius of x.

Věta 1.13 (Beurling–Gelfand)

A Banach algebra, $x \in A \implies r(x) = \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$.

Lemma 1.14

A Banach algebra with a unit, $x \in A$. For $p(z) = \sum_{j=1}^n \alpha_j z^j \in \mathbb{C}$ a polynom (with complex coefficients) we put $p(x) = \sum_{j=1}^n \alpha_j x^j \in A$. Then $\sigma(p(x)) = p(\sigma(x))$.

Důkaz

Fix $\lambda \in \mathbb{C}$ and write $(\lambda - p)(z) = c \cdot \prod_{i=1}^m (z - z_i)$, where $z_1, \dots, z_m$ are roots of $\lambda - p$. Then $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$ does not exists. $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^m (x - z_i \cdot e)$, so it doesn't exists if and only if $\exists i \in [m]$, such that $(x - z_i \cdot e)^{-1}$ doesn't exists $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow$ $\exists$ root ν of $\lambda - p$ such that $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$. □

Důkaz (Beurling–Gelfand)

WLOG A has a unit. Step 1, „$r(x) \leqslant \inf_n \sqrt[n]{\|x^n\|}$“: fix $\lambda \in \sigma(x)$. By the previous lemma $\forall n : \lambda^n \in \sigma(x^n)$. By theorem 'Before theory' we have $\forall n : |\lambda|^n \leqslant \|x^n\|$.

Step 2, „$r(x) \geqslant \limsup_n \sqrt[n]{\|x^n\|}$“: Pick $r > r(x)$. Claim: „$\frac{x^n}{r^n} \xrightarrow{w} 0$“: Fix $x^* \in A^*$ and put $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$. By fact and tvrzeni after it, f has derivative at each $\lambda \in \varrho(x)$. Moreover for $|\lambda| \geqslant \|x\|$ we have $f(\lambda) = \lambda \cdot x^* \left(\sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}} \right) = \sum_{n=0}^\infty \frac{x^*(x^n)}{\lambda^n}$. Thus $f(\lambda) = \sum_{n=0}^\infty \frac{x^*(x^n)}{\lambda^n}$, $\lambda \in P(0, r(x), \infty)$. From Complex analysis $f \in H(P(0, r, \infty))$ is uniquely given by Laurent series. In particular $f(r) = \sum_{n=0}^\infty \frac{x^*(x^n)}{r^n}$, so $x^* \left(\frac{x^n}{r^n} \right) \to 0$.

From princip of unique boundedness (last semester): $\frac{x^n}{r^n}$ is $\| \cdot \|$-bounded, so $\exists c > 0 :$ $\|x^n\| \leqslant c r^n$, $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{c} \cdot r \to r$. So $\limsup \sqrt[n]{\|x^n\|} \leqslant r$. □

Důsledek
A Banach algebra, $x \in A$ and $|\lambda| > r(x)$. Then $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ is absolutely convergent and $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

Důkaz
Fix q, such that $\frac{r(x)}{|\lambda|} < q < 1$. By the previous theorem, $\exists n_0\ \forall n \geqslant n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$, so $\frac{\|x^n\|}{|\lambda|^n} < q^n$, $n \geqslant n_0$. Thus $\sum \left\| \frac{x^n}{\lambda^n} \right\| \leqslant \infty$, so the sum is absolutely convergent.

Now we easily check that $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$. □

1.4 Subalgebra

Věta 1.15

A Banach algebra with a unit e, $B \subset A$ is closed subalgebra such that $e \in B$. Fix $x \in B$. Then

- *$C \subset \varrho_A(x)$ is component (maximum connected subset) $\implies$ $C \subseteq \sigma_B(x)$ or $C \cap \sigma_B(x) = \varnothing$;*
- *$\partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$;*
- *$\varrho_A(x)$ is connected $\implies$ $\sigma_A(x) = \sigma_B(x)$;*
- *$\operatorname{int} \sigma_B(x) = \varnothing \implies \sigma_A(x) = \sigma_B(x)$.*

Důkaz
„$\sigma_A(x) \subseteq \sigma_B(x)$“: $(\lambda e - x)^{-1}$ exists in B implies it exists (it's same) in A.

„First item“: Let $C \subset \varrho_A(x)$ be component. Pick $\lambda_0 \in C \cap \sigma_B(x)$. Wanted: „$C \backslash \sigma_B(x) = \varnothing$“. Pick $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$ (separate B and $R_x(\lambda) \notin B$). Then $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is holomorphic function on open (because maximum) connected set C. Which is zero[a] on $C \backslash \sigma_B(x)$.

Since $C \backslash \sigma_B(x)$ is open, if it is nonempty it contains a ball, so it has cluster point. Thus $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is such that $\{\lambda \in C | x^*(R_x(\lambda))\} = 0$ has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero. ↯ with $x^*(R_x(\lambda_0)) = 1$.

„Second item“: Pick $\lambda \in \sigma_B(x) \backslash \sigma_A(x)$ and let $C \subset \varrho_A(x)$ be a component containing λ. By first item, $C \subseteq \sigma_B(x)$, C is open, so $\lambda \in C \subseteq \operatorname{int}(\sigma_B(x))$. □

[a] For $\lambda \in C \backslash \sigma_B(x)$, $(\lambda e - x)^{-1}$ exists in B so $R_x(\lambda) \in B$ and therefore, $x^*(R_x(\lambda)) = 0$

Důkaz
„Third item“: If $\varrho_A(x)$ is connected, we can apply first item to $C = \varrho_A(x)$, we have either $\varrho_A(x) \subseteq \sigma_B(x)$ or $\varrho_A(x) \cap \sigma_B(x) = \emptyset$. But first is not possible, because $\varrho_A(x)$ is unbounded and $\sigma_B(x)$ is bounded. Therefore $\sigma_B(x) \subseteq \sigma_A(x)$.

„Fourth item“: If $\operatorname{int}(\sigma_B(x)) = \emptyset$, then (by second item) $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$. □

Důsledek
A Banach algebra, $B \subseteq A$ closed subalgebra, $x \in B$. Then all items from previous theorem hold as well if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.

Důkaz
Without proof. (Basically same that previous; we add unit to A and B, so this unit is same $((\mathbf{o}, 1))$, etc.) □

1.5 Holomorphic calculus

Definice 1.8

X Banach, $\gamma : [a, b] \to \mathbb{C}$ path (continuous, piecewise smooth (C^1)), $f : \langle\gamma\rangle \to X$ continuous. Then

$$\int_\gamma f := \int_{[a,b]} \gamma'(t) f(\gamma(t))dt. \qquad \text{(As Bochner integral.)}$$

If $\Gamma = \gamma_1 + \ldots + \gamma_n$ is chain in $\mathbb{C}$, $f : \langle\Gamma\rangle \to X$ continuous, then

$$\int_\Gamma f := \sum_{i=1}^{n} \int_{\gamma_i} f.$$

Lemma 1.16

Γ chain in $\mathbb{C}$, X Banach, $f : \langle\Gamma\rangle \to X$, $x \in X$. Then

$$\int_\Gamma f = x \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_\Gamma x^* \circ f.$$

Důkaz
„ $\Longleftarrow$ “ by Hahn–Banach theorem. „ $\Longrightarrow$ “: (by previous semester x^* and $\int$ ”commutes”)

$$x^*\left(\int_\Gamma f\right) = \sum_{i=1}^{n} x^*\left(\int_{\gamma_i} f\right) = \sum_{i=1}^{n} \int_{[a_i,b_i]} \gamma_i'(t) x^*(f(\gamma_i(t)))dt = \int_\Gamma x^* \circ f.$$

□

Poznámka (Recall)
If $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact. Then there is a cycle Γ such that $\langle \Gamma \rangle \subset \Omega \backslash K$ and $\operatorname{ind}_\Gamma z = 1$ if $z \in K$ and 0 if $z \notin \Omega$.

Then we say that Γ circulates K in Ω.

Definice 1.9

Let A be a Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open and $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $f(x) := \frac{1}{2\pi i} \int_\Gamma f \cdot R_x$, where Γ is any cycle which circulates $\sigma(x)$ in Ω.

Poznámka
$f(x)$ exists ($f \cdot R_x$ is continuous on $\langle \Gamma \rangle$), $f(x)$ does not depend on the choice of Γ (Pick $x^* \in X^*$, then $(x^* \circ f \cdot R_x)(\lambda) = f(\lambda) \cdot x^*(R_x(\lambda))$ is holomorphic. Pick Γ_1, Γ_2 cycles circulating $\sigma(x)$ in Ω, then $\int_{\Gamma_1 - \Gamma_2} x^* \circ f \cdot R_x = 0$ from Cauchy).

Věta 1.17 (Holomorphic calculus)

A Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open such that $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $\Phi : \mathcal{H}(\Omega) \to A$ defined as $\Phi(f) = f(x)$ (from definition above) has the following properties:

- *Φ is algebra homomorphism, $\Phi(1) = e$, $\Phi(\mathrm{id}) = x$;*
- *$f_n \overset{loc.}{\rightrightarrows} f$ in $\mathcal{H}(\Omega)$, then $f_n(x) \to f(x)$;*
- *$f(x)^{-1}$ exists $\Leftrightarrow f \neq 0$ on $\sigma(x)$, in this case $f(x)^{-1} = \frac{1}{f}(x)$;*
- *$\sigma(f(x)) = f(\sigma(x))$;*
- *if Ω_1 is open and $f(\sigma(x)) \subseteq \Omega_1$, $g \in \mathcal{H}(\Omega_1)$, then $(g \circ f)(x) = g(f(x))$;*
- *if $y \in A$ commutes with x, then y commutes with $f(x)$.*

Moreover, if $\psi : \mathcal{H}(\Omega) \to A$ satisfy first two item, then $\psi = \Phi$.

Lemma 1.18

(Ω, μ) complete measurable space, A Banach algebra, $f \in L_1(\mu, A)$. Let $x \in A$ and $E \subset \Omega$ is measurable. Then

$$x \cdot \left(\int_E f(t) d\mu(t) \right) = \int_E x \cdot f(t) d\mu(t), \qquad \left(\int_E f(t) d\mu(t) \right) \cdot x = \int_E f(t) \cdot x d\mu(t).$$

Důkaz
Easy (by commutation of integral and linear operator from last semester), skipped. □

Důkaz (Holomorphic calculus)
„1st item“: „Φ is linear“ is easy, „Φ is multipicative“: Pick $f, g \in \mathcal{H}(\Omega)$, open set U such that $\sigma(x) \subset U \subset \overline{U} \subset \Omega$. Let Γ cycle circulating $\sigma(x)$ in U, Λ cycle circulating $\overline{U}$ in Ω. Then

$$f(x) \cdot g(x) = \left(\frac{1}{2\pi i} \int_\Gamma f \cdot R_x \right) \cdot g(x) \overset{\text{lemma}}{=}$$

$$= \frac{1}{2\pi i} \int_\Gamma f(t) R_x(t) g(x) dt = \frac{1}{2\pi i} \int_\Gamma f(t) \cdot R_x(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(s) ds dt \overset{\text{lemma}}{=}$$

$$= \frac{1}{2\pi i} \int_\Gamma f(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(t) \cdot R_x(s) ds dt =$$

because $\langle \Lambda \rangle \cap \langle \Gamma \rangle = \varnothing$, we can use theorem after definition of R_x:

$$= \frac{1}{(2\pi i)^2} \int_\Gamma \int_\Lambda f(t) \cdot g(s) \cdot \frac{R_x(t) - R_x(s)}{s - t} ds dt \overset{\text{Fubini to } x^*(\dots) \text{ and lemma}}{=}$$

$$= \frac{1}{(2\pi i)^2} \int_\Gamma f(t) \left(\int_\Lambda \frac{g(s)}{s-t} ds \right) R_x(t) dt - \frac{1}{(2\pi i)^2} \int_\Lambda g(s) \left(\int_\Gamma \frac{f(t)}{s-t} \right) R_x(s) ds =$$

(Now we use Cauchy theorem ($f(z) \operatorname{ind}_\Gamma z = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw$). $\forall s \in \langle \Lambda \rangle : (t \mapsto \frac{f(t)}{s-t}) \in \mathcal{H}(U) \wedge \operatorname{ind}_\Gamma z = 0, z \notin U$, so $\int_\Gamma \frac{f(t)}{s-t} dt = 0$. $\forall t \in \langle \Gamma \rangle : \operatorname{ind}_\Lambda t = 1 \wedge (s \mapsto g(s)) \in \mathcal{H}(\Omega) \implies g(t) = \frac{1}{2\pi i} \int_\Lambda \frac{g(s)}{s-t} ds$.)

$$= \frac{1}{2\pi i} \int_\Gamma f(t) g(t) R_x(t) dt - 0.$$

It remains that „if $f(z) = z^k$, $k \in \mathbb{N} \cup \{0\}$ then $f(x) = x^k$“ (we want it for $k = 0$ and $k = 1$). Put $\Gamma(t) = r \cdot e^{it}$, $t \in [0, 2\pi]$, where $r > \|x\|$ arbitrary. By some theorem:

$$R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}, \qquad |\lambda| > \|x\|.$$

Thus (we switch integral and sum, because later we realize that sum of integral of absolute value is finite)

$$\forall x^* \in A^* : x^*(f(x)) = \frac{1}{2\pi i} \int_\Gamma \lambda^k x^* \left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} \right) d\lambda = \frac{1}{2\pi i} \int_\Gamma \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_\Gamma \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^{\infty} x^*(x^n) \int_\Gamma \frac{1}{\lambda^{n-k+1}} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} x^*(x^n) \int_0^{2\pi} i \frac{1}{\Gamma(t)^{n-k}} dt = x^*(x^k) + \sum 0,$$

because Γ (is 2π periodic).

„2nd item“: For $\Gamma = \gamma_1 + \ldots + \gamma_N$:

$$\|f_n(x) - f(x)\| = \frac{1}{2\pi i}\left\|\int_\Gamma (f_n(\lambda) - f(\lambda))R_x(\lambda)d\lambda\right\| \leqslant \frac{1}{2\pi}\int_\Gamma |f_n(\lambda) - f(\lambda)| \cdot \|R_x(\lambda)\|d\lambda \leqslant$$

$$\leqslant \frac{1}{2\pi}\sum_{i=1}^N \int_{a_i}^{b_i} |\gamma_i'(t)| \sup_{z\in\langle\Gamma\rangle} |f_n(z) - f(z)| \cdot \|R_x(\gamma_i(t))\|dt =$$

$$= \sup_{z\in\langle\Gamma\rangle} |f_n(z) - f(z)| \cdot \frac{1}{2\pi}\sum_{i=1}^N \int_{a_i}^{b_i} \|R_x(\gamma_i(t))\| \cdot |\gamma_i'(t)|dt \to 0.$$

„Moreover part“: By Runge theorem (and second item) it is enough prove it for rational functions. If R was polynom, then $\Phi(R) = \Psi(R)$ by second item. So it suffices „$\forall p$ polynom: $\frac{1}{p} \in \mathcal{H}(\Omega) \implies \Phi(\frac{1}{p}) = \psi(\frac{1}{p})$“. Pick p polynom. Then $e = \psi(1) = \psi(p \cdot \frac{1}{p}) = \psi(p) \cdot \psi(\frac{1}{p}) = \Phi(p) \cdot \psi(\frac{1}{p})$ (similarly for $\frac{1}{p} \cdot p$). So $\psi(\frac{1}{p}) = \Phi(p)^{-1} = \Phi(\frac{1}{p})$.

„3rd item“: „ $\implies$ “ Let $f(z) = 0$ for some $z \in \sigma(x)$. Then exists $g \in H(\Omega) : f(u) = (z-u)g(z)$. By item one, we have $(ze-x)g(x) = f(x) = g(x)(ze-x)$. But $(ze-x)^{-1}$ does not exist, so $f(x)^{-1}$ does not exists.

„ $\impliedby$ “ Suppose $f \neq 0$ on $\sigma(x)$ by compactness. $\exists \Omega_1 \subset \Omega$ open: $\sigma(x) \subset \Omega_1$ and $f \neq 0$ on Ω_1. Then $\frac{1}{f} \in H(\Omega_1)$ and by first item we have $e = (f \cdot \frac{1}{f})(x) = f(x)\frac{1}{f}(x) = \ldots = \frac{1}{f}(x) \cdot f(x) \implies f(x)^{-1} = \frac{1}{f}(x)$. □

Poznámka
$f = g$ on a neighbourhood of $\sigma(x) \implies f(x) = g(x)$ (from definition), other implication doesn't hold!

1.6 Multiplicative functionals

Definice 1.10 (Multiplicative functional)

Let A be a Banach algebra. We say $\varphi : A \to \mathbb{C}$ is multiplicative linear functional $\equiv \varphi$ preserves $+, \cdot, \cdot_S$.

$$\Delta(A) := \{\varphi : A \to \mathbb{C} | \varphi \text{ multiplicative linear functional }, \varphi \not\equiv 0\}.$$

Tvrzení 1.19

A Banach algebra, $\varphi \in \Delta(A) \cup \{0\}$. Then

- $\exists! \tilde{\varphi} \in \Delta(A_e) : \tilde{\varphi}((x, 0)) = \varphi(x), \forall x \in A$. *It is given by* $\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda$. *Moreover,* $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$.
- $\forall x \in A : \varphi(x) \in \sigma(x)$ *whenever* $\varphi \not\equiv 0$.
- $\Delta(A) \subseteq B_{A^*}$.
- *A has a unit,* $\varphi \not\equiv 0 \implies \|\varphi\| \geqslant \frac{1}{\|e\|}$. *In particular if* $\|e\| = 1$, *then* $\|\varphi\| = 1$.

Důkaz

„1. uniqueness“: For $\tilde{\varphi} \in \Delta(A_e)$ such that $\tilde{\varphi}((x,0)) = \varphi(x)$, $x \in A$:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda\tilde{\varphi}((\mathbf{o},1)) = \varphi(x) + \lambda,$$

second equality by $\varphi \in \Delta(A) \implies \varphi(e) = \varphi(e^2) = \varphi^2(e)$. „1. existence“ is proven by check that defined $\tilde{\varphi}$ is multiplicative linear functional (and it is nonzero, but $\tilde{\varphi}((0,1)) = 1 \neq 0$). This is easy (omitted).

„$\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$“: „$\subseteq$“: $\varphi \in LHS$, put $\varphi(x) := \psi((x,0))$. Then $\varphi \in \Delta(A) \cup \{0\}$ and $\tilde{\varphi} = \psi$ became:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda = \psi((x,0)) + \lambda = \psi((x,\lambda)).$$

„$\supseteq$“: We know already that $\tilde{\varphi} \in \Delta(A_e)$.

„2. with A has unit e“: $\varphi \neq 0$, $\varphi \in \Delta(A)$: If $\lambda \in \varrho(x)$, then $\varphi(\lambda e - x) \neq 0$ ($\varphi(x) \neq 0$ if x^{-1} exists). $0 \neq \varphi(\lambda e - x) = \lambda - \varphi(x) \implies \lambda \neq \varphi(x)$. Thus $\varphi(x) \notin \varrho(x)$, so $\varphi(x) \in \sigma(x)$. „2. with A hasn't unit“, then $\varphi(x) = \tilde{\varphi}((x,0)) \in \sigma_{A_e}((x,0)) = \sigma_A(x)$.

„3.“: $\varphi \in \Delta(A)$. Then $\forall x \in A : \varphi(x) \in \sigma(x) \subseteq B(\mathbf{o}, \|x\|)$, so $|\varphi(x)| \leqslant \|x\|$.

„4.“: A has a unit e, then $\|\varphi\| \geqslant \left|\varphi\left(\frac{e}{\|e\|}\right)\right| = \frac{1}{\|e\|}$. □

Věta 1.20

A Banach algebra, $M := \Delta(A) \cup \{0\}$. Then $M \subset (B_{A^*}, w^*)$ is compact, $\Delta(A)$ is locally compact and if A has u unit, then $\Delta(A)$ is compact. The mapping $\Phi : M \to \Delta(A_e)$, $\Phi(\varphi) = \tilde{\varphi}$ is w^*–w^* homeomorphism.

Důkaz

By the previous proposition, $M \subset (B_{A^*}, w^*)$ ((B_{A^*}, w^*) is compact by previous semester). So, it suffices to check that M is w^*-closed.

$$M = \bigcap_{x,y \in A} \{\varphi \in A^* | \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}.$$

Sets from RHS is closed by previous semester, so, M is closed. Thus M is compact.

$\Delta \subset M$ is open, so $\Delta(A)$ is locally compact (and M is 1-point compactification of $\Delta(A)$). If Δ has a unit, then $\Delta(A) = \{\varphi \in M | \varphi(e) = 1\}$ is w^*-closed, so $\Delta(A)$ is compact (and 0 is isolated in M).

Finally, by previous proposition, Φ is bijection. Φ is w^*-continuous:

$$\varphi_i \xrightarrow{w^*} \varphi \implies \forall (x,\lambda) : \tilde{\varphi}_i((x,\lambda)) = \varphi_i(x) + \lambda \to \varphi(x) + \lambda = \tilde{\varphi}((x,\lambda)) \implies \tilde{\varphi}_i \xrightarrow{w^*} \tilde{\varphi}$$

So, Φ is homeomorphism (continuous bijection on compact, last semester?). □

Například
$\Delta(\mathcal{C}(K)) = \{\delta_x | x \in K\}$. ($f \mapsto f(x)$ is multiplicative. Suppose $\varphi \in \Delta(\mathcal{C}(K))$, $\varphi \notin \{\delta_x | x \in K\}$. So for $x \in K$ there is $g_x \in C(x) : \varphi(g_x) \neq g_x(x)$. Consider $f_x = g_x - \varphi(g_x)$. Then $\varphi(f_x) = 0$, $f_x(x) \neq 0$. So there is U_x open neighbourhood of x such that $f_x \neq 0$ on U_x. Compactness implies $\exists x_1, \dots, x_n \in K : K \subset \bigcup_{i=1}^n U_{x_i}$. Consider $h := \sum_{i=1}^n |f_{x_i}|^2$. Then $h > 0$ on K, so h^{-1} exists and therefore $\varphi(h) \neq 0$. But $\varphi(h) = \sum_{i=1}^n \varphi(f_{x_i})\overline{\varphi_{x_i}} = 0$.)

$\Delta\{M_n\} = \emptyset$, $n \geqslant 2$, where M_n is (non-commutative) algebra of $n \times n$ matrices. ($M_n = \text{LO}\{E^{i,j}\}$. $E^{ij} \cdot E^{kl} = E^{il}$ if $j = k$, else 0. So $\varphi(E^{ij}) \cdot \varphi(E^{ij}) = \varphi(E^{ij} \cdot E^{ij}) = 0$ if $i \neq j$. $\varphi(E^{ii}) = \varphi(E^{in}E^{ni}) = \varphi(E^{in})\varphi(E^{in}) = 0$. $\varphi(E^{nn}) = \varphi(E^{n1}E^{1n}) = 0$.)

Definice 1.11 (Ideal, maximal ideal)

A Banach algebra. Ideal in A is a subspace $I \subset A$ if $\forall x \in I\ \forall y \in A : x \cdot y \in I \wedge y \cdot x \in I$.

Maximal ideal $\equiv$ proper ($I \neq A$) ideal and it is maximal proper ideal with respect to inclusion.

Například (2021, Johnson-Schetman, Acta mathematica)
$\mathcal{L}(L_p)$ has 2^{2^ω} non-isomorphic closed ideals.

Tvrzení 1.21

A Banach algebra with a unit. Then:

- *Any proper ideal is contained in a maximum ideal. (From Zorn's lemma. And $I \subset A$ ideal is proper $\Leftrightarrow e \notin I$.)*
- *$I \subset A$ proper ideal $\implies \overline{I} \subset A$ is proper ideal. In particular, maximal ideals are closed. (Easy: $\overline{I}$ is ideal. Moreover, $I \cap A^\times = \emptyset$ (if $x \in I$ was invertible thus $e = x \cdot x^{-1} \in I$, but $e \notin I$). So ($A^\times$ is open) $\overline{I} \cap A^\times = \emptyset$ and therefore $e \notin \overline{I}$.)*

Tvrzení 1.22

A Banach algebra, $I \subseteq A$ closed ideal $\implies A/I$ is Banach algebra ($[x] \cdot [y] := [x \cdot y]$).

Důkaz
Straightforward from definition. (Omitted.) □

Poznámka
From now on, A will be commutative.

Věta 1.23

A commutative Banach algebra with a unit. Then $\Phi : \Delta(A) \to \{$*maximal ideals in* $A\}$, $\Phi(\varphi) := \operatorname{Ker}\varphi$, *is bijection.*

Důkaz

Pick $\varphi \in \Delta(A)$. Then „$\operatorname{Ker}\varphi$ is maximal ideal“: ideal: $y \in \operatorname{Ker}\varphi, x \in A : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \ldots \cdot 0 = 0$, proper: $\varphi \not\equiv 0$, maximal: $\operatorname{codim}\operatorname{Ker}\varphi = 1$: pick $x_0 : \varphi(x_0) \neq 0$, $a = a - \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} + \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} \in \operatorname{Ker}\varphi \oplus \mathbb{R}$.

„Φ is one-to-one“: Pick $\varphi, \psi \in \Delta(A)$: $\operatorname{Ker}\varphi = \operatorname{Ker}\psi$. Then (by lemma from previous semester) $\varphi = c \cdot \psi$ for some $c \in \mathbb{K}$. But $\varphi(e) = 1 = \psi(I)$ so $\varphi = \psi$.

„Φ is surjective“: Let $I \subset A$ be maximal ideal ($\implies$ closed). Step 1 „Any nonzero element in A/I is invertible“: For contradiction assume $\exists q(x) \in A/I$ ($q(x) = [x]$), $q(x) \neq 0 \wedge q(x)^{-1}$ does not exist. By next lemma $q(x)(A/I)$ is proper ideal. Then $q^{-1}\left(q(x)(A/I)\right)$ is an ideal in A which is proper and $I \subsetneq q^{-1}\left(q(x)(A/I)\right)$, which contradicts maximality of I. It follows from: ideal: follows from the fact that q is algebra homomorphism; proper: $q(e) = [e] \notin q(x)A/I$; $I \subseteq q^{-1}(\ldots)$: $0 \in q(x)A/I$; $I \neq q^{-1}(\ldots)$: $q(x) \neq 0 \implies x \notin I$, but $q(x) = q(x)q(e) \in q(x)(A/I)$, so $x \in q^{-1}(\ldots)$.

From Gelfand–Mazur theorem $\exists$ surjective isomorphism $j : A/I \to \mathbb{C}$. Then $\varphi := j \circ q \in \Delta(A)$. It remains „$I = \operatorname{Ker}\varphi$“: $x \in \operatorname{Ker}\varphi \Leftrightarrow j(q(x)) = 0 \Leftrightarrow q(x) = 0 \Leftrightarrow x \in I$. □

Lemma 1.24

A commutative *Banach algebra with a unit,* $x \in A$ *does not have inverse* $\implies$ xA *is proper ideal.*

Důkaz

xA is ideal, because A is commutative. Then xA is proper ($e \notin xA$). □

Důsledek (Hahn–Banach like theorem)
A is commutative Banach algebra with a unit, $I \subset A$ proper ideal. Then $\exists\varphi \in \Delta(A) : \varphi|_I \equiv 0$.

Důkaz

Let $\tilde{I} \supseteq I$ be maximal ideal. By the previous theorem there is $\varphi \in \Delta(A) : \tilde{I} = \operatorname{Ker}\varphi$. □

Tvrzení 1.25

A, B Banach algebras, $\Phi : A \to B$ *algebraic isomorphism. Then* $\Phi^{\#} : \Delta(B) \to \Delta(A)$ *defined as* $\Phi^{\#}(\varphi) := \varphi \circ \Phi$ *is homeomorphism.*

Důkaz

„$\Phi^\#(\varphi) \in \Delta(A)$“: $\Phi^\#(\varphi) = \varphi \circ \Phi \in \Delta(A) \cup \{0\}$ and $\varphi \not\equiv 0 \wedge \Phi$ is onto $\implies \varphi \circ \Phi \neq 0$.

„$\Phi^\#$ is w^*-w^* continuous“: $\varphi_i \xrightarrow{w^*} \varphi \implies \varphi_i \circ \Phi \xrightarrow{w^*} \varphi \circ \Phi$.

Apply the proven part to Φ^{-1}, obtain that $(\Phi^{-1})^\# : \Delta(A) \to \Delta(B)$ is w^*-w^* continuous. Moreover we have $\Phi^\# \circ (\Phi^{-1})^\# = \mathrm{id} \wedge (\Phi^{-1})^\# \circ \Phi^\#$. □

Tvrzení 1.26

L locally compact T_2. Then $\delta : L \to \Delta(C_0(L)), x \mapsto \delta_x$ is homeomorphism onto.

Důkaz

„Case 1: L is compact“: By example δ is onto. Of course, δ is one-to-one, continuous. So δ is homeomorphism.

„Case 2: L is not compact“: Then there is $K = L \cup \{\infty\}$, one-point compactification, and $\{f \in \mathcal{C}(K) | f(\infty) = 0\} \ni f \mapsto f|_L \in C_0(L)$ is isometric isomorphism. Moreover $\Phi : \mathcal{C}_0(L)_e \to \mathcal{C}(K)$, $\Phi(f, \lambda) := f + \lambda$, is algebraic isomorphism.

So, we have $K \xrightarrow{\eta} \Delta(C(K)) \xrightarrow{\Phi^\#} \Delta(C_0(L)_e) \xrightarrow{\psi} \Delta(C_0(L)) \cup \{0\}$, where η is homeomorphism from case 1 and $\psi(\varphi) := \varphi|_{C_0(L)}$.

Thus $\delta := \psi \circ \Phi^\# \circ \eta$ is homeomorphism between $L \cup \{\infty\}$ and $\Delta(C_0(L)) \cup \{0\}$. Finally, for $x \in K$ and $f \in \mathcal{C}_0(L)$:

$$\Phi^\# \circ \eta(x)(f) = (\eta(x) \circ \Phi)(f) = f(x),$$

so $\psi \circ \Phi^\# \circ \eta(x) = \Phi^\# \circ \eta(x)|_{C_0(L)} = \delta_x|_{C_0(L)}$. □

Věta 1.27

K, L locally compact T_2. Then following is ekvivalent

- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as Banach algebra;*
- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as algebras;*
- *$K \approx L$ as topological spaces.*

Důkaz

„1 $\implies$ 2“ trivial. „2 $\implies$ 3“: $K \approx \Delta(\mathcal{C}_0(K)) \approx \Delta(\mathcal{C}_0(L)) \approx L$ from previous two tvrzeni. „3 $\implies$ 1“: Given $h : K \to L$ homeomorphism, $f \mapsto f \circ h$ is isometry between Banach algebras. □

Definice 1.12 (Semi-simple Banach algebra)

A commutative Banach algebra. It is semi-simple $\equiv$ $\Delta(A)$ separates points of A. ($\Leftrightarrow$ $\bigcap\{\operatorname{Ker}\varphi | \varphi \in \Delta(A)\} = \{\mathbf{o}\}$.)

Poznámka
Semi-simple $\implies$ commutative. (Semi-simple and $x \cdot y \neq y \cdot x \implies \exists \varphi \in \Delta(A)$: $\varphi(x) \cdot \varphi(y) = \varphi(x \cdot y) \neq \varphi(y \cdot x) = \varphi(y) \cdot \varphi(x)$ ↯.)

Věta 1.28

A, B Banach algebras, B is semi-simple, then every (algebra) homomorphism $\Phi : A \to B$ is continuous.

Důkaz
Use Closed graph theorem. Pick $x_n \to x$, $\varphi(x_n) \to y$. Wanted „$\Phi(x) = y$“ ($\Leftrightarrow \forall \varphi \in \Delta(B) : \varphi(\Phi(x)) = \varphi(y)$). For $\varphi \in \Delta(B)$ we have $\varphi(y) = \lim_n \varphi(\Phi(x_n)) \overset{\varphi\circ\Phi\in\Delta(A)\subseteq A^*}{=} \varphi \circ \Phi(\lim_n x_n) = \varphi(\Phi(x))$. □

Důsledek
$(A, \|\cdot\|)$ semi-simple Banach algebra and $(A, |||\cdot|||)$ is Banach algebra (with other norm), then $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Důkaz
We have that $\mathrm{id} : (A, |||\cdot|||) \to (A, \|\cdot\|)$ is algebra homomorphism, so continuous by previous theorem. Similarly inverse is continuous (semi-simplicity doesn't depend on norm). So, id is isomorphism. □

2 Gelfand transformation

Definice 2.1 (Gelfand transformation)

A Banach algebra. For $x \in A$ we define $\hat{x} : \Delta(A) \to \mathbb{C}$, $\hat{x}(\varphi) := \varphi(x)$. We say that $\hat{x}$ is Gelfand transformation of x.

Poznámka
$\hat{x} \in \mathcal{C}_0(\Delta(A))$.

$A = \mathcal{C}_0(L) \implies \Delta(A) = \{\delta_x | x \in L\} \implies \forall f \in A : \hat{f}(\delta_x) = f(x), x \in L$. So, $\hat{f} = f$.

$A = L_1(\mathbb{R}^d) \implies \Delta(A) = \{e^{it\cdot x} | x \in \mathbb{R}\} \subseteq L_\infty(\mathbb{R}^d) = A^*$ and $\hat{f}$ is Fourier transformation.

Věta 2.1

A commutative Banach algebra, $x \in A$. Then

- *A has a unit $\implies$ $\sigma(x) = \operatorname{Rng} \hat{x}$;*
- *A doesn't have a unit $\implies$ $\sigma(x) = \operatorname{Rng} \hat{x} \cup \{0\}$;*
- $\|\hat{x}\|_\infty = r(x) = \sup\{|\lambda| | \lambda \in \sigma(x)\}$.

Důkaz

„a) $\subseteq$“: $\lambda \in \sigma(x) \Leftrightarrow (\lambda \cdot e - x)^{-1}$ does not exists $\implies$ (Lemma above) $(\lambda e - x)A$ is proper ideal $\implies$ $\exists \varphi \in \Delta(A) : \varphi|_{(\lambda e - x)A} \equiv 0$ $\implies$ $\exists \varphi \in \Delta(A) : 0 = \varphi(\lambda e - x) = \lambda - \varphi(x) = \lambda - \hat{x}(\varphi)$ $\implies$ $\lambda \in \operatorname{Rng} \hat{x}$.

„$\supseteq$“ follows from the Tvrzeni above, $\varphi(x) \in \sigma(x)$ for $\varphi \in \Delta(A)$.

„b)“ For $x \in A$:

$$\sigma(x) = \sigma_{A_e}((x,0)) \overset{\text{a)}}{=} \operatorname{Rng} \widehat{(x,0)} = (\{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}) =$$

$$= \{\varphi(x) | \varphi \in \Delta(A) \cup \{0\}\} = \operatorname{Rng} \hat{x} \cup \{0\}\,.$$

„c)“ $\|\hat{x}\|_\infty = \sup\{|\lambda| | \lambda \in \operatorname{Rng} \hat{x}\} = \sup\{|\lambda| | \lambda \in \operatorname{Rng} \hat{x} \cup \{0\}\} = \sup\{|\lambda| | \lambda \in \sigma(x)\} = r(x)$. □

Definice 2.2 (Gelfand transformation of algebra)

A Banach algebra, then $\Gamma : A \to \mathcal{C}_0(\Delta(A))$, $\Gamma(x) := \hat{x}$ is the Gelfand transformation of A.

Věta 2.2

A commutative Banach algebra, Γ Gelfand transformation. Then

- *Γ is algebra transformation, continuous, $\|\Gamma\| \leqslant 1$;*
- *$\Gamma(A)$ separates the points of $\Delta(A)$;*
- *Γ is one-to-one $\Leftrightarrow$ A is semi-simple;*
- *Γ is an isomorphism into $\Leftrightarrow$ $\exists K > 0 : \|x^2\| \geqslant K \cdot \|x\|^2$, $x \in A$; ($\Leftrightarrow$ Γ is one-to-one and $\Gamma(A)$ is closed;)*
- *Γ is an isometry into $\Leftrightarrow$ $\|x^2\| = \|x\|^2$, $x \in A$.*

Důkaz
„a)“: Γ is linear (obvious), Γ preserves multiplication (obvious). Finally, $\|\Gamma(x)\|_\infty = \|\hat{x}\|_\infty = r(x) \leqslant \|x\|$. So $\|\Gamma\| \leqslant 1$.

„b)“: Let $\varphi \neq \psi \in \Delta(A)$ and $x \in A : \hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$.

„c)“: $\Gamma(x) = 0 \Leftrightarrow \hat{x}(\varphi) = 0, \varphi \in \Delta(A) \Leftrightarrow \varphi(x) = 0, \varphi \in \Delta(A)$. So, Γ is one-to-one $\Leftrightarrow \forall x \neq 0\ \exists \varphi \in \Delta(A) : \varphi(x) \neq 0 \Leftrightarrow A$ is semi-simple.

„d) second“: Γ is isomorphism into $\Leftrightarrow$ Γ is bijection between A and $\Gamma(A) \wedge \Gamma(A)$ is closed. ($\Gamma(A)$ is closed, then we use Open mapping theorem; if Γ is isomorphism, $\Gamma(A)$ is a Banach space.).

„d) + e), $\implies$ “: Suppose $\exists c > 0$: $\|\Gamma(x)\| \geqslant c \cdot \|x\|$, $x \in A$. Then $\forall x \in A : \|x^2\| \overset{\text{a)}}{\geqslant} \|\Gamma(x^2)\| = \|\Gamma(x)\|^2 \geqslant c^2 \cdot \|x\|^2$.

„d) + e), $\impliedby$ “: Let d) hold with K (this holds in every algebra). Then (proven by induction)

$$\forall x \in A : \|x^{2^n}\| \geqslant K^{2^{n-1}} \|x\|^{2^n}, \qquad n \in \mathbb{N}.$$

$$\implies \sqrt[2^n]{\|x^{2^n}\|} \geqslant K^{1-2^{-n}} \|x\|,$$

where left side converges (by Beurling) to $r(x)$ and right side converges to $\|x\|$. So $r(x) \geqslant K \cdot \|x\|$ and from previous theorem $r(x) \geqslant \|\hat{x}\|_\infty = \|\Gamma(x)\|$. ▫

2.1 C^*-algebras

Definice 2.3 (Involution)

A is a Banach algebra. Involution is a mapping $* : A \to A$ such that

$$\forall x, y \in A\ \forall \lambda \in \mathbb{C} :$$

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x^*, \qquad (xy)^* = y^* \cdot x^*, \qquad (x^*)^* = x.$$

Definice 2.4 (C^*-algebra)

Banach algebra with involution $*$ is a C^*-algebra, if

$$\forall x \in A : \|x \cdot x^*\| = \|x\|^2, x \in A.$$

Definice 2.5 (Self-adjoint element, normal element)

For A with involution $*$ and $x \in A$ we say that x is self-adjoint $\equiv x = x^*$, and x is normal $\equiv x \cdot x^* = x^* \cdot x$.

Tvrzení 2.3 (Properties)

A Banach algebra with involution, $x \in A$. Then

- *e is left/right unit $\implies$ e is unit and $e = e^*$. (e is left unit $\Leftrightarrow$ e^* is right unit. So there is unit.)*
- *A is C^*-algebra $\Leftrightarrow$ $\|x \cdot x^*\| \geqslant \|x\|^2$, $x \in A$. Then $\|x^*\| = \|x\|$, $x \in A$. („ $\implies$ “: clear, „ $\impliedby$ “: Then $\forall x \in A : \|x\|^2 \leqslant \|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\|$, so $\|x\| \leqslant \|x^*\|$, and applying to x^* we get $\|x^*\| \leqslant \|x\|$. But then we have $\|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\| = \|x\|^2$.)*
- *Let A has a unit, then x^{-1} exists $\Leftrightarrow$ $(x^*)^{-1}$ exists. Then $(x^*)^{-1} = (x^{-1})^*$. („ $\implies$ “: $x^* \cdot (x^{-1})^* = (x^{-1}x)^* = e^* = e$, analogically $(x^{-1})^* x^* = e$. „ $\impliedby$ “: Apply the proven part to x^*.)*
- *$\lambda \in \sigma(x) \Leftrightarrow \overline{\lambda} \in \sigma(x^*)$. ($A$ has a unit: $\lambda \notin \sigma(x) \Leftrightarrow \exists (\lambda e - x)^{-1} \Leftrightarrow \exists \left((\lambda e - x)^*\right)^{-1} \Leftrightarrow \overline{\lambda} \notin \sigma(x^*)$. If A has not a unit, then we use previous sentence and next theorem?)*
- *$x + x^*$, $x^* \cdot x$, $x \cdot x^*$, $i \cdot (x - x^*)$ are self-adjoint. (Easy, omitted.)*
- *$\exists! u, v \in A$ self-adjoint: $x = u + i \cdot v$. Then $x^* = u - i \cdot v$, and x is normal $\Leftrightarrow$ $uv = vu$. („Existence“: $u := \frac{1}{2}(x + x^*), v := \frac{1}{2i}(x - x^*)$. Then $x = u + iv$. „Formulas“: $(u + i \cdot v)^* = u^* + \bar{i} v^*$. „Uniqueness“: Pick $a, b \in A_{sa} : x = a + i \cdot b$. Then $a + i \cdot b = x = u + i \cdot v$, $a - i \cdot b = x^* = u - i \cdot v$. By subtracting or summing equation we get $a = u$ and $b = v$. „Normality“: x normal $\Leftrightarrow (u + i \cdot v)(u - i \cdot v) = (u - i \cdot v)(u + i \cdot v)$ $\Leftrightarrow -i \cdot u \cdot v + i \cdot v \cdot u = i \cdot u \cdot v - i \cdot v \cdot u \Leftrightarrow u \cdot v = v \cdot u$.)*

Věta 2.4

A is C^*-algebra, $x \in A$ is normal. Then $r(x) = \|x\|$.

Důkaz

„Step 1: $\|x^2\| = \|x\|^2$“:

$$\|x\|^4 = \|x^* x\|^2 = \|(x^* x)^* (x^* x)\| = \|x^* x x^* x\| = \|x^* x^* x x\| = \|(xx)^* xx\| = \|xx\|^2 = \|x^2\|^2.$$

Thus inductively, we obtain $\|x^{2^k}\| = \|x\|^{2^k}$, $k \in \mathbb{N}$. Thus, Beurling gives

$$r(x) = \lim_k \sqrt[2^k]{\|x^{2^k}\|} = \|x\|.$$

$\square$

Důsledek

A (Banach) algebra with involution. Then there is at most one norm $\| \cdot \|$ on A, such that $(A, \| \cdot \|)$ is C^*-algebra.

Důkaz
If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on A such that $(A, \|\cdot\|)$ is C^*-algebra, then by previous theorem

$$\forall x \in A : \|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \|x^*x\|_2 = \|x\|_2^2.$$

□

Věta 2.5

$(A, \|\cdot\|)$ Banach algebra.

- *$(a, \lambda)^* = (a^*, \overline{\lambda})$, $(a, \lambda) \in A_e$ defines an involution on A_e. (Trivial.)*
- *If A is C^*-algebra, then on A_e there exists a norm $|||\cdot|||$ (equivalent to the norm from $A \oplus_1 \mathbb{K}$) such that $(A_e, |||\cdot|||)$ is C^*-algebra and $|||(a,0)||| = \|a\|$, $a \in A$.*

Věta 2.6

A is C^*-algebra, $x \in A$. Then

- *$x = x^* \implies \sigma(x) \subseteq \mathbb{R}$;*
- *A has a unit and $x^* = x^{-1}$ (that is, x is unitary) $\implies \sigma(x) \subseteq \{\lambda | |\lambda| = 1\}$.*

Důkaz
By the previous theorem, WLOG A has a unit.

„a)“: Let $\alpha + i\beta \in \sigma(x)$, $\alpha, \beta \in \mathbb{R}$. We want $\beta = 0$. Trick: $x_t := x + i \cdot t \cdot e$, $t \in \mathbb{R}$. Then

$$\alpha + i \cdot (\beta + t) \in \sigma(x_t) (\impliedby (\alpha + i(\beta+t))e - x_t = (\alpha + i \cdot \beta)e - x),$$

$$\alpha^2 + (\beta+t)^2 = |\alpha + i(\beta+t)|^2 \leqslant \|x_t\|^2 = \|x_t^* x_t\| = \|(x - i\cdot t\cdot e)\cdot(x + i \cdot t \cdot e)\| = \|x^2 + (t\cdot e)^2\| \leqslant \leqslant \|x^2\| + t^2.$$

So, $\alpha^2 + (\beta+t)^2 - t^2 \leqslant \|x^2\|$, $t \in \mathbb{R} \implies \beta = 0$ (Otherwise $LHS \to +\infty$ for $t \to \pm\infty$.)

„b)“: ($\|e\| = \|e^2\| = \|e\|^2$.) $1 = \|e\| = \|x^*x\| = \|x\|^2$, so $\|x\| = 1$. Then, for $\lambda \in \sigma(x)$, we have $|\lambda| \leqslant \|x\| = 1$. On the other hand $\frac{1}{\lambda} \in \sigma(x^{-1})$ (because if not, then $\frac{1}{\lambda}e - x^{-1}$ ha inverse $\implies \lambda e - x = (\lambda e - x)x^{-1}x = (\lambda x^{-1} - e)x = -\lambda(\frac{1}{\lambda}e - x^{-1})x \implies \lambda e - x$ has inverse.) So

$$\left|\frac{1}{\lambda}\right| \leqslant \|x^{-1}\| = \|x^*\| = \|x\| = 1.$$

□

Definice 2.6

A, B are C^*-algebras, then $\Phi : A \to B$ is $*$-homomorphism if Φ is homomorphism preserving $*$ (that is, $\Phi(x^*) = (\Phi(x))^*$).

Důsledek
Let A be a C^*-algebra and $\Phi \in \Delta(A)$. Then Φ is $*$-homomorphism.

Důkaz
„If $x = x^*$“, then $\Phi(x) \in \sigma(x) \subseteq \mathbb{R}$, so $\Phi(x^*) = \Phi(x) = \overline{\Phi(x)}$.

„In general“, if $x = u + i \cdot v$ ($u = u^*$, $v = v^*$), then $\Phi(x^*) = \Phi(u - i \cdot v) = \Phi(u) - i \cdot \Phi(v) = \overline{\Phi(u) + i \cdot \Phi(v)} = \overline{\Phi(x)}$. □

Tvrzení 2.7 (Automatical continuous)

Let A, B be C^*-algebras, $\Phi : A \to B$ is $*$-homomorphism. Then Φ is continuous and $\|\Phi\| \leqslant 1$.

Důkaz

$\forall x \in A : \|\Phi(x)\|^2 = \|\Phi(x)^* \cdot \Phi(x)\| = r(\Phi(x^*) \cdot \Phi(x)) = r(\Phi(x^* x)) \overset{\circledast}{=} r(x^* x) = \|x^* x\| = \|x\|^2$.

Thus for $\circledast$ it suffices to show that (by following lemma)

$$\sigma(\Phi(x^* x)) \subseteq \sigma(x^* x) \cup \{0\}\,.$$

□

Lemma 2.8

Let A, B be Banach algebras, $\Phi : A \to B$ algebra homomorphism. Then $\forall x \in A : \sigma_B(\Phi(x)) \subseteq \sigma_A(x) \cup \{0\}$.

Důkaz
Consider $\tilde{\Phi} : A_e \to B_e$ defined as $\tilde{\Phi}(a, \lambda) := (\Phi(a), \lambda)$. Then $\tilde{\Phi}$ is algebra homomorphism preserving unit. Moreover $\sigma_B(\Phi(x)) \subseteq \sigma_{B_e}((\Phi(x), 0)) \cup \{0\}$ and $\sigma_{A_e}((x, 0)) \subseteq \sigma_A(x) \cup \{0\}$. Thus, WLOG A, B have units and $\Phi(e_A) = e_B$.

But then for $\lambda \neq 0$ and $x \in A : \lambda e - x$ has inverse in A, then $\Phi(\lambda e - x) = \lambda \Phi(e) - \Phi(x)$ has inverse in B. So, $\lambda \notin \sigma_A(x) \cup \{0\} \implies \lambda \notin \sigma_B(\Phi(x))$. □

Věta 2.9 (Gelfand–Naimark)

A commutative C^*-algebra. Then the Gelfand transformation $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ is isometric $*$-isomorphism onto.

Důkaz
By proposition above, Γ is algebra homomorphism, $\|\Gamma\| \leqslant 1$ and from theorem above $\|\Gamma(x)\|_\infty = r(x)$, $x \in A$. „Γ is $*$-homomorphism“:

$$\forall a \in A\ \forall \varphi \in \Delta(A) : \Gamma(a^*)(\varphi) = \varphi(a^*) = \overline{(\varphi(a))} = \overline{\Gamma(a)}(\varphi).$$

„Γ is isometry“:

$$\forall x \in A : \|\Gamma(x)\|^2 = \|\overline{\Gamma(x)} \cdot \Gamma(x)\| = \|\Gamma(x^* x)\| = r(x^* x) = \|x^* x\| = \|x\|^2.$$

„Γ is onto“: $\Gamma(A)$ is Banach space so $\Gamma(A) \subseteq \mathcal{C}_0(\Delta(A))$ is closed and $*$-subalgebra. And $\Gamma(A)$ separates points of $\Delta(A)$. So from Stone–Weierstrass theorem ($A \subset \mathcal{C}_0(K)$ is $*$-subalgebra separating the points, then $\overline{A}^{\|\cdot\|} = \mathcal{C}_0(K)$) $\Gamma(A) = \mathcal{C}_0(\Delta(A))$. □

Důsledek
A, B commutative C^*-algebras. Then the following items are equivalent:

- A and B are isometrically $*$-isomorphic;
- A and B are algebraically isomorphic;
- $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

Důkaz
„2. $\Leftrightarrow$ 3.“ follows from theorem above (where it is proven for $\mathcal{C}_0(K)$-spaces). „1. $\implies$ 2.“: trivial.

„3. $\implies$ 1.“: easy for $\mathcal{C}_0(K)$-spaces, because if $h : K \to L$ is homeomorphism, then $f \mapsto f \circ h$ is isometrical $*$-isomorphism. □

Definice 2.7

A Banach algebra, $M \subset A$. Then $alg(M) = \bigcap \{B \supseteq M | B \text{ is subalgebra of } A\}$.

Poznámka (Easy)

$$= \left\{ \sum_{i=1}^{n} \alpha_i \prod_{j=1}^{m} x_{ij} | n, m \in \mathbb{N}, \alpha_i \in \mathbb{C}, x_{ij} \in M \right\}.$$

Moreover $\overline{alg} M = \bigcap \{B \supseteq M | B \text{ is closed subalgebra of } A\}$.

Poznámka (Easy)

$$= \overline{alg M}^{\|\cdot\|}.$$

Tvrzení 2.10 (Fact)

A is C^*-algebra, $M \subset A$ is commutative and closed under $*$, then $\overline{alg} M$ is commutative C^*-subalgebra of A.

Věta 2.11

A, B are C^*-algebras, $h : A \to B$ is $*$-homomorphism, one-to-one. Then h is isometry.

Důkaz

WLOG A, B have units and $h(e)$ is a unit ($(a, \lambda) \mapsto (h(a), \lambda)$ is 1-to-1 $*$-homomorphism). Suffices: $\forall x \in A$ self-adjoint: $\|x\| = \|h(x)\|$ ($\forall y \in A : \|h(y)\|^2 = \|h(y^*y)\| = \|y^*y\| = \|y\|^2$). Let $x \in A$ be self-adjoint. Put $A_0 := \overline{alg}\,\{e, x\} = \overline{\mathrm{LO}}\,\{e, x, x^2, x^3, \dots\}$ is commutative and C^*-subalgebra.

$$B_y := \overline{alg}\,\{e, h(x)\} = \overline{\mathrm{LO}}\,\left\{e, h(x), h(x^2), \dots\right\}$$

is commutative and C^*-subalgebra. So, we have $A_0 \xrightarrow{h} B_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(B_0))$, $A_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(A_0))$. So there is $\tilde{h} : \mathcal{C}(\Delta(A_0)) \to \mathcal{C}(\Delta(B_0))$ one-to-one $*$-homeomorphism, $\tilde{h}(1) = 1$. So, it suffices to prove the following lemma. $\square$

Lemma 2.12

Let K, L be T_2 compact spaces, $\varphi : \mathcal{C}(K) \to \mathcal{C}(L)$ $*$-homomorphism, $\varphi(1) = 1$. Then $\exists \alpha : L \to K$ continuous mapping such that $\varphi(f) := f \circ \alpha$, $f \in \mathcal{C}(K)$.

Moreover, if φ is one-to-one, then α is onto and so φ is isometry.

Důkaz

By proposition above $\|\varphi\| \leqslant 1$ and φ is continuous. Consider $\varphi^* : \mathcal{M}(L) \to \mathcal{M}(K)$. Then „$\varphi^*(\Delta(\mathcal{C}(L)) \subseteq \Delta(\mathcal{C}(K))$“:

$$\forall h \in \Delta(\mathcal{C}(L))\ \forall f, g : \varphi^* h(fg) = h(\varphi(fg)) = h(\varphi(f))h(\varphi(g)) = \varphi^* h(f)\varphi^* h(g).$$

So, we have: $L \xrightarrow{\delta} \Delta(\mathcal{C}(L)) \xrightarrow{\varphi^*} \Delta(\mathcal{C}(K)) \xrightarrow{\delta^{-1}} K$. So, $\alpha(x) := \delta^{-1}(\varphi^*(\delta(x)))$, $x \in L$ is continuous from L to K.

For this α we have:

$$\forall x \in L\ \forall f \in \mathcal{C}(K) : \varphi(f)(x) = \delta_x(\varphi(f)) = (\varphi^* \circ \delta_x)(f) = f(\delta^{-1}\varphi^*\delta_x) = f(\alpha(x)).$$

Moreover, „if φ is one-to-one, then α is onto“: Suppose $\alpha(L) \subsetneq K \implies \exists f \in C(K) \setminus \{0\} : f|_{\alpha(L)} \equiv 0$. But then $\varphi(f) \equiv 0$, but $f \neq 0$. ↯(φ should be one-to-one.) Thus φ is isometry. $\square$

Poznámka (GNS construction)
A is C^*-algebra $\implies \exists H$ Hilbert $\exists \varphi : A \to B(H)$ $*$-isomorphism into.

Důkaz (Sketch)
$f \geqslant 0$ $(\sigma(f) \geqslant 0)$ on $A|_{\{a|f(a*a)=0\}}$ constructs inner product $\langle [x],[y]\rangle := f(y^*x)$. Put $H := \overline{A|_{\{a|f(a*a)=0\}}}$. Then $\varphi(a)([x]) = [ax]$. ▫

3 Continuous calculus for formal elements of C^*-algebras

Poznámka
Idea: $\varphi(\sigma(x)) \ni f \mapsto f(x) \in A$.

For $A = C(K)$:

$$g \in \mathcal{C}(K), \varphi(\sigma(x)) \ni f \implies g \circ f \in C(K).$$

Let A be C^*-algebra with a unit, $x \in A$ normal. Consider

$$B = \overline{alg}\,\{e, x, x^*\} \in A \implies \Gamma_B : B \to \mathcal{C}(\Delta(B)) \wedge f(x) := \Gamma_B^{-1}(f \circ \Gamma_B(x)), f \in \mathcal{C}(\sigma_A(x)).$$

Problem is when $\Gamma_B(x) \subseteq \sigma_A(x)$.

Lemma 3.1

A is C^*-algebra, $B \subset A$ is C^*-algebra. Then

- *If A and B have the same unit $\implies \forall x \in B : \exists x^{-1} \in B \Leftrightarrow \exists x^{-1} \in A$;*
- *$\forall x \in B$: B has a unit, which is not a unit in $A \implies \sigma_A(x) = \sigma_B(x) \cup \{0\}$, otherwise $\sigma_A(x) = \sigma_B(x)$;*
- *(In any case $\sigma_B(x) \subseteq \sigma_A(x)$).*

Důkaz
„1.“: Pick $x \in B$. „ $\implies$ “: easy. „ $\impliedby$ “: If x^{-1} exists in A, then $(x^*x)^{-1}$ exists in A. So $0 \notin \sigma_A(x^*x) = \sigma_B(x^*x) \implies (x^*x)^{-1}$ exists in B. $x^{-1} = x^{-1}(x^*)^{-1}x^* = x^{-1}(x^*x)^{-1}x^*$.

„2.“: If A and B have the same unit, we have $\sigma_A(x) = \sigma_B(x)$. WLOG A has a unit $e \notin B$ (Because $B \in A_e$ and $\sigma_{A_e}(x) = \sigma_A(x)$ if A has not unit). Then $\sigma_A(x) = \sigma_{B+\mathrm{LO}(e)}(x) \overset{*}{=} \sigma_{B_e}((x,0)) = \sigma_B(x)$ if B has no unit and $\sigma_B(x) \cup \{0\}$ if B has a unit.

$*$: $\varphi : B + \mathrm{LO}(e) \to B_e, b + \lambda e \mapsto (b, \lambda)$ is algebra homomorphism. ▫

Věta 3.2

Let A be a C^*-algebra with a unit, $x \in A$ normal, $f \in \mathcal{C}(\sigma(x))$. Then the mapping

$$\Phi : \mathcal{C}(\sigma(x)) \to A, \qquad \Phi(g) := g(x) := \Gamma_B^{-1}(g \circ \Gamma_B(x))$$

has the following properties:

1. Φ *is isometric* $*$*-isomorphism onto* $B = \overline{alg}\,\{e, x, x^*\}$, $\Phi(1) = e$ *and* $\Phi(\mathrm{id}) = x$.
2. *If* $\psi : \mathcal{C}(\sigma(x)) \to A$ *is* $*$*-homomorphism,* $\psi(1) = e$, $\psi(\mathrm{id}) = x$, *then* $\psi = \Phi$.
3. *If* $g \in \mathcal{H}(\Omega)$, *where* $\Omega \subset \mathbb{C}$ *open,* $\sigma(x) \subset \Omega$, *then* $\Phi(g|_{\sigma(x)}) = \psi(g)$, *where* ψ *is from holomorphic calculus.*
4. $f(x)^{-1}$ *exists in* $A \Leftrightarrow f \neq 0$ *on* $\sigma(x)$. *In this case* $f(x)^{-1} = \left(\frac{1}{f}\right)(x)$.
5. $\sigma(f(x)) = f(\sigma(x))$.
6. $\forall g \in \mathcal{C}(f(\sigma(x))) : (g \circ f)(x) = g(f(x))$.
7. $\forall y \in A : yx = xy : yf(x) = f(x)y$.

Důkaz
„1.“: Recall theorem above $\Gamma_B(x) : \Delta(B) \to \mathbb{C}$ continuous and onto $\sigma_B(x)$. And it is „one-to-one“:

$$\forall \varphi_1, \varphi_2 \in \Delta(B) : \varphi_1(x) = \varphi_2(x) \implies \varphi_1 = \varphi_2 \text{ on } B.$$

So $\Gamma_B(x) : \Delta(B) \to \sigma(x)$ is homeomorphism, then $\mathcal{C}(\sigma(x)) \ni g \mapsto g \circ \Gamma_B(x) \in \mathcal{C}(\Delta(A))$ is isometric $*$-isomorphism onto. Thus $\mathcal{C}(\sigma(x)) \ni g \mapsto \Gamma_B^{-1}(g \circ \Gamma_B(x)) \in B$ is isometric $*$-isomorphism onto.

Moreover, $\Phi(1) = \Gamma_B^{-1}(1) = e$ $(\forall \varphi \in \Delta(B) : \varphi(e) = 1)$. $\Phi(\mathrm{id}) = \Gamma_B^{-1}(\Gamma_B(x)) = x$.

„2.“: By theorem above, ψ is continuous (because it is $*$-isomorphism), moreover $\psi = \Phi$ on complex polynomials. Since complex polynomials are dense in $\mathcal{C}(\sigma(x))$ by (S-W), by continuity $\Phi = \psi$ everywhere.

„3.“: Omitted (on polynomials, on inverse, on rationals, rationals are dense in $\mathcal{H}$).

„4.“: Since $f(x) \in B$, we have $f(x)^-1$ exists in $B \Leftrightarrow f(x)^{-1}$ exists in $A \overset{\Phi \text{ is ?}}{\Leftrightarrow} f^{-1}$ exists in $\mathcal{C}(\sigma(x)) \Leftrightarrow f \neq 0$ on $\sigma(x)$. And if $f \neq 0$ on $\sigma(x)$, then $f(x)^{-1} = \Phi(f^{-1}) = \Phi\left(\frac{1}{f}\right) = \left(\frac{1}{f}\right)(x)$.

„5.“: $f(x) \in B$, so $\sigma_A(f(x)) \overset{\text{Lemma}}{=} \sigma_B(f(x)) = \sigma_B(\Phi(f)) \overset{\Phi\text{is isomorphism}}{=} \sigma_{\mathcal{C}(\sigma(x))} = \operatorname{Rng} f = f(\sigma(x))$.

„6.“: Omitted.

„7.“: TODO!!! □

Věta 3.3 (Bent Fuglede (1950), Calvin R. Putnam (1951))

Let A be complex C^*-algebra, $x \in A$ and $a, b \in A$ be normal such that $ax = xb$. Then $a^*x = xb^*$.

Důkaz
Omitted. □

4 Operators on Hilbert spaces

Definice 4.1 (Sesquilinear map, sesquilinear form)

Let X, Y be vector spaces over $\mathbb{C}$. Map $S : X \times X \to Y$ is called sesquilinear, if it is linear in the first variable and conjugate-linear in the second one. If $Y = \mathbb{C}$, S is a sesquilinear form.

Tvrzení 4.1 (Polarization identity)

X, Y vector spaces over $\mathbb{C}$ and $S : X \times X \to Y$ is a sesquilinear map. Then for all $x, y \in X$, it holds that

$$S(x,y) = \frac{1}{4}(S(x+y,x+y) - S(x-y,x-y) + iS(x+iy,x+iy) - iS(x-iy,x-iy)).$$

Důkaz (TODO!!!)
$(RHS = \frac{1}{4}(4 \cdot S(x,y) + 0 \cdot S(y,x) + 0 \cdot S(x,x) + 0 \cdot S(y,y)).)$ □

Důsledek
$\{\mathbf{o}\} \neq H$ Hilbert space, $T, S \in \mathcal{L}(H)$. Then $T = S$ iff $\forall x \in H : \langle Tx, x\rangle = \langle Sx, x\rangle$.

Důkaz (TODO!!!)
$(\langle Tx, x\rangle = \langle Sx, x\rangle \implies \langle Tx, y\rangle = \langle Sx, y\rangle \implies S = T)$ □

Věta 4.2

$\{\mathbf{o}\} \neq H$ Hilbert space and $T \in \mathcal{L}(H)$. Then

- *T is self-adjoint iff $\forall x \in H : \langle Tx, x\rangle \in \mathbb{R}$;*
- *T is normal iff $\forall x \in H : \|Tx\| = \|T^*x\|$;*
- *$\forall x \in H : \langle Tx, x\rangle \geqslant 0$ iff T is self-adjoint and $\sigma(T) \subseteq [0, \infty)$.*

Důkaz (TODO!!!)
$(\langle Tx, x\rangle \overset{\Rightarrow}{=} \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} \overset{\Leftarrow}{=} \langle Tx, x\rangle.)$ $(\langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle.)$ („ $\implies$ “: $\sigma(T) \subseteq \overline{N_T}$, „ $\impliedby$ “: $\inf N_T \in \sigma(T)$.) □

Definice 4.2 (Non-negative)

A C^*-algebra and $x \in A$. We say that x is non-negative ($x \geqslant 0$), if x is self-adjoint and $\sigma(x) \subseteq [0, +\infty)$.

Věta 4.3

H Hilbert space and $T \in \mathcal{L}(H)$ normal. Then

- *$\operatorname{Ker} T = \operatorname{Ker} T^*$ and $\operatorname{Ker} T = (\operatorname{Rng} T)^\perp$;*
- *$\operatorname{Rng} T$ is dense in H iff T is one-to-one;*
- *$\lambda \in \sigma_P(T)$ iff $\overline{\lambda} \in \sigma_P(T^*)$, eigenspace of T associated with λ is equal to eigenspace of T^* associated with $\overline{\lambda}$;*
- *if $\lambda_1 \neq \lambda_2$ are eigenvalues of T, then $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$.*

Důkaz
TODO!!! See Funkcionalka (OM3). □

Věta 4.4 (Hilbert–Schmidt)

H Hilbert space and $T \in \mathcal{K}(H)$ nonzero normal. Then exists orthonormal basis B of space H consisting of eigenvectors of T. The set of vectors from B associated with nonzero eigenvalues of T is at most countable and we can arrange them to sequence $\{e_n\}_{n=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, then $\{e_n\}$ is orthonormal basis of $\overline{\operatorname{Rng} T}$ and for every $x \in H$:

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n \rangle e_n,$$

where λ_n is eigenvalue associated with the eigenvector e_n.

Důkaz
Omitted. „`OM3/Funkcionalka.pdf`“ □

Věta 4.5 (Schmidt)

H Hilbert space and $T \in \mathcal{L}(H)$ nonzero compact. Then exists $N \in \mathbb{N}_0 \cup \{\infty\}$, sequence of positive numbers $\{\lambda_n\}_{n=1}^N$ and orthonormal systems $\{u_n\}_{n=1}^N$ and $\{v_n\}_{n=1}^N$ such that for every $x \in H$:

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n.$$

Důkaz
TODO!!! Hilbert–Schmidt on TT^*. □

Věta 4.6

H Hilbert space and $P \in \mathcal{L}(H)$ projection. Then following are equivalent: P is orthogonal ($\operatorname{Rng} P \perp \operatorname{Ker} P$); $P \geqslant 0$; P is self-adjoint; P is normal.

Moreover, if $P, Q \in \mathcal{L}(H)$ are orthogonal projections, then $\operatorname{Rng}(P) \perp \operatorname{Rng}(Q)$ iff $PQ = 0$.

Důkaz (TODO!!!)
(4 $\implies$ 1 and 2 $\implies$ 3 was in the theorem above, 3 $\implies$ 4 trivial, 1 $\implies$ 2 obvious.) □

Definice 4.3 (Unitary operator)

H, K Hilbert spaces. Operator $T \in \mathcal{L}(H, K)$ is called unitary, if $T^{-1} = T^*$, i.e., $T^* \circ T = I_H$ and $T \circ T^* = I_K$.

Tvrzení 4.7

H, K Hilbert spaces and $T \in \mathcal{L}(H, K)$. Then T is unitary $\Leftrightarrow$ T is isometry onto $\implies$ T is isometry $\Leftrightarrow$ $\langle Tx, Ty\rangle = \langle x, y\rangle$ for every $x, y \in H$. Moreover if T is onto, then all propositions are equivalent.

Důkaz (TODO!!!) □

Definice 4.4 (Partial isometry, initial subspace)

H Hilbert space. Operator $U \in \mathcal{L}(H)$ is called partial isometry, if there is closed subspace $K \subset H$ (initial subspace of U) such that $U|_K$ is isometry and $U|_{K^\perp} \equiv \mathbf{o}$.

Věta 4.8 (Polar decomposition)

H Hilbert space, $T \in \mathcal{L}(H)$.

Exists unique operators $P, U \in \mathcal{L}(H)$ such that $P \geqslant 0$, U is partial isometry with initial subspace $\overline{\operatorname{Rng} P}$ and $T = UP$. Moreover $P = \sqrt{T^*T} = U^*T$.

If T is invertible, then exists unique $P, U \in \mathcal{L}(H)$ such that $P \geqslant 0$ is invertible, U is unitary and $T = UP$.

Důkaz (TODO!!!) □

5 Borel measurable calculus

Lemma 5.1 (Lax–Milgram)

H Hilbert, $S : H \times H \to C$ sesquilinear, $\|S\| := \sup_{x,y \in S_H} |S(x, y)| < \infty$. Then $\exists! T \in \mathcal{L}(H)$: $\|T\| = \|S\| \wedge \langle Tx, y\rangle = S(x, y)$.

Důkaz

Fix $y \in H$. Then $H \ni x \mapsto S(x, y)$ is a point in $H^* \implies \exists! U(y) \in H : S(x, y) = \langle x, U(y)\rangle$, $x \in H$. Then $U \in \mathcal{L}(H)$, $\|U\| = \|S\|$.

„Linearity“: Easy: $\forall y, z \in H, \alpha \in \mathbb{K} \implies$

$$\forall x \in H : \langle x, U(\alpha y + z)\rangle = S(x, \alpha y + z) = \overline{\alpha} S(x, y) + S(x, z) = \overline{\alpha} \langle x, Uy\rangle + \langle x, Uz\rangle .$$

„$\|U\| \leqslant \|S\|$“:

$$\forall y \in H : \|Uy\|^2 = \langle Uy, Uy\rangle = S(Uy, y) \leqslant \|S\| \cdot \|Uy\| \cdot \|y\| \implies \|Uy\| \leqslant \|S\| \cdot \|y\|.$$

„$\|U\| \geqslant \|S\|$“:

$$\forall x, y \in S_H : \|S(x, y)| = |\langle x, Uy\rangle| \leqslant \|x\| \cdot \|U\| \cdot \|y\| = \|U\|.$$

„Uniqueness“: Bounded operator is given by values of $\langle Tx, y\rangle$. □

Definice 5.1

H Hilbert, $T \in \mathcal{L}(H)$ normal, $\Phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(H)$ continuous from "Continuous calculus".

- $\forall x, y \in H$: $\mu_{x,y} \in M(\sigma(T))$ is the unique measure satisfying
$$\int_{\sigma(T)} f d\mu_{x,y} = \langle \Phi(f)x, y \rangle, \qquad f \in \mathcal{C}(\sigma(T)).$$
- $\forall f \in Bor_b(\sigma(T))$ (bounded, Borel) we define $\Phi(f) \in \mathcal{L}(H)$ as the unique operator such that
$$\langle \Phi(f)x, y \rangle = \int_{\sigma(T)} f d\mu_{x,y}, \qquad x, y \in H$$

Důkaz

„1.“: $f \mapsto \langle \Phi(f)x, y \rangle$ is linear and $|\langle \Phi(f)x, y \rangle| \leqslant \|\Phi(f)\| \cdot \|x\| \cdot \|y\|$. So $f \mapsto \langle \Phi(f)x, y \rangle \in \mathcal{C}(\sigma(T))^* = M(\sigma(T)) \implies \mu$ exists by Riesz representation theorem.

„2.“: $\forall x, x_2, y \in H \ \forall \alpha \in \mathbb{K} \ \forall f \in \mathcal{C}(\sigma(T))$:
$$\langle \Phi(f)(\alpha x_1 + x_2), y \rangle = \alpha \langle \Phi(f)x_1, y \rangle + \langle \Phi(f)x_2, y \rangle = \alpha \mu_{x,y}(f) + \mu_{x_2,y}(f).$$

Thus $\cdot \mapsto \mu_{\cdot,y}$ is linear (for each y). Analogously $\cdot \mapsto \mu_{x,\cdot}$ is conjugate-linear.

Thus, $(x, y) \mapsto \mu_{x,y}(f) \in \mathbb{C}$ is sesquilinear form.
$$\forall x, y \in S_H : |\mu_{x,y}(f)| \leqslant \int |f| d|\mu_{x,y}| \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\| = \|f\|_\infty.$$

And from Lax–Milgram:
$$\exists! \Phi(f) \in \mathcal{L}(H) : \langle \Phi(f)x, y \rangle = \mu_{x,y}.$$

Moreover $\|\Phi(f)\| \leqslant \|f\|_\infty$. □

Poznámka

H Hilbert, $T \in \mathcal{L}(H)$ normal:

- Mapping $H \times H \ni (x, y) \mapsto \mu_{x,y}$ is sesquilinear, so
$$\mu_{x,y} = \frac{1}{4} \left(\mu_{x+y,x+y} - \mu_{x-y,x-y} + i\mu_{x+iy,x+iy} - i\mu_{x-iy,x-iy} \right).$$
- $\forall x \in H : \mu_{x,x} \geqslant 0$. (Proof: „$f \geqslant 0 \implies \mu_{x,x}(f) \geqslant 0, f \in \mathcal{C}(\sigma(T))$“: $f \geqslant 0 \implies \Phi(f) \geqslant 0$ $(\sigma(\Phi(f)) = f(\sigma(T)) \subseteq [0, \infty) \implies \Phi(f) \geqslant 0.)$ So $\int_{\sigma(T)} f d\mu_{x,x} = \Phi(f)x, x \geqslant 0$.)

- $Bor_b(\sigma(T)) \subseteq l_\infty(\sigma(x)) \mapsto \mathcal{L}(H)$ is C^*-subalgebra.
- The mapping $\Phi : Bor_b(\sigma(x)) \to \mathcal{L}(H)$ from previous definition, is extension of continuous calculus from theorem above.

Věta 5.2

Let P be a metric space, Φ be the smallest system of functions such that $\mathcal{C}_b(P) \subset \Phi$ and Φ is closed under point-wise bounded convergent sequences. Then $\Phi = Bor_b(P)$.

Důkaz (Sketch)
Suffices: „$\forall A \subset P$ Borel: $\chi_A \in \Phi$.“

$$\mathcal{F} := \{A \subset P \text{ Borel } | \chi_A \in \Phi\}$$

is σ-algebra containing closed sets $\implies \mathcal{F} = Bor(P)$. □

Definice 5.2

Let X, Y be normed linear spaces. On $\mathcal{L}(X, Y)$ we define the following two Hausdorff locally convex topologies:

- τ_{SOT} generated by pseudonorms $\{P_x(T) = \|Tx\| | x \in X\}$ (so, $T_i \overset{\text{SOT}}{\to} T \Leftrightarrow \forall x \in X : T_i x \overset{\|\cdot\|}{\to} Tx$);
- τ_{WOT} generated by pseudonorms $\{P_{x,y^*}(T) = y^*(Tx) | x \in X \wedge y^* \in Y^*\}$ (so, $T_i \overset{\text{WOT}}{\to} T \Leftrightarrow \forall x \in X : T_i x \overset{w}{\to} Tx$) (in $X = Y = H$ Hilbert: $\Leftrightarrow \forall x, y \in H : \langle T_i x, y\rangle \to \langle Tx, y\rangle$).

Poznámka

$$T_i \overset{\|\cdot\|}{\to} T \implies T_i \overset{\text{SOT}}{\to} T \implies T_i \overset{\text{WOT}}{\to} T.$$

Například
$R_n x := (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$, $x \in l_2$. Then $R_n \in \mathcal{L}(l_2)$, $n \in \mathbb{N}$, and $R_n \overset{\|\cdot\|}{\nrightarrow} 0$, because $\|R_n(e_{n+1})\| = 1$, $n \in \mathbb{N}$. But $R_n \overset{\text{SOT}}{\to} 0$, because $\forall x \in l_2 : \|R_n x\|_2^2 = \sum_{i=n+1}^\infty |x_i|^2 \to 0$.

$S_n x := (0, 0, \ldots, 0, x_1, x_2, \ldots)$, $x \in l_2$. Then $S_n \in \mathcal{L}(l_2)$ is isometry $\forall n \in \mathbb{N}$. But $S_n \overset{\text{SOT}}{\nrightarrow} 0$, because $\|S_n(e_1)\| = 1 \nrightarrow 0$. But $S_n \overset{\text{WOT}}{\to} 0$, because $\forall x, y \in l_2$:

$$|\langle S_n x, y\rangle| = \left|\sum_{i=1}^\infty x_i y_{n+i}\right| \leqslant \|x\|_2 \sqrt{\sum_{i=n+1}^\infty |y_i|^2} \to 0.$$

Věta 5.3

H Hilbert, $T \in \mathcal{L}(H)$ normal, $f \in Bor_b(\sigma(T))$, $\Phi : Bor_b(\sigma(T)) \to \mathcal{L}(H)$ as in definition above. Then

Φ *is continuous $*$-homomorphism and $\|\Phi\| = 1$;*

Důkaz

Φ is linear (easy from definition). $\|\Phi\| \leqslant 1$ follows from the second point of the previous theorem, and $\|\Phi(1)\| = \|\operatorname{id}\| = 1$, so $\|\Phi\| = 1$.

„Φ is multiplicative“: Step 1: „$\mathcal{F} \coloneqq \{g \in Bor_b(\sigma(T)) | \forall f \in \mathcal{C}(\sigma(t)) \colon \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$, then $\mathcal{F} = Bor_b(\sigma(T))$“: $\mathcal{F} \subseteq \mathcal{C}(\sigma(T))$ follows from continuous calculus, „$\mathcal{F}$ closed under point-wise limits of bounded sequences“: Let $\mathcal{F} \ni g_n \to g$ and $f \in \mathcal{C}(\sigma(T))$. Then $g_n f \to gf$ point-wise. So, for $x, y \in H$:

$$\langle \Phi(g, f)x, y \rangle = \int_{\sigma(T)} gf d\mu_{x,y} = \lim \int g_n f d\mu_{x,y} = \lim \langle \Phi(g_n)x, y \rangle =$$

$$= \lim \langle \Phi(g_n)(\Phi(f)x), y \rangle = \lim \int g_n d\mu_{\Phi(f)x,y} = \langle \Phi(g)(\Phi(f)x), y \rangle .$$

$\implies \mathcal{F} = Bor_b(\sigma(T))$.

Step 2: „$\mathcal{H} := \{g \in Bor_b(\sigma(T)) | \forall f \in Bor_b(\sigma(t)) : \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$, then $\mathcal{H} = Bor_b(\sigma(T))$“: „$\mathcal{H}$ is closed under point-wise limits of bounded sequences“: $\mathcal{H} \ni f_n \overset{\tau_p}{f}$, f_n bounded, then $\forall x, y \in H$ $\forall g \in Bor_b(\sigma(T))$:

$$\langle \Phi(gf)x, y \rangle \overset{\text{Lebesgue}}{=} \lim_n \langle \varphi(gf_n)x, y \rangle = \lim_n \langle \Phi(g)\Phi(f_n)x, y \rangle = \lim_n \langle \Phi(f_n)x, \Phi(g)^* y \rangle =$$

$$= \lim_n \int f_n d\mu_{x,\Phi(g)^* y} \overset{\text{Lebesgue}}{=} \int f d\mu_{x,\Phi(g)^* y} = \langle \Phi(f)x, \Phi(g)^* y \rangle = \langle \Phi(g)\Phi(f)x, y \rangle .$$

Thus $\Phi(gf) = \Phi(g)\Phi(f)$.

„Φ preserves $*$“: $\mathcal{F} := \{g \in Bor_b(\sigma(T)) | \Phi(g)^* = \Phi(\overline{g})\}$. Then $\mathcal{F} \subseteq \mathcal{C}(\sigma)$ by continuous calculus and $\mathcal{F}$ is ”closed under taking limits” analogously as above. $\implies \mathcal{F} = Bor_b(\sigma(T))$.

□

$(f_n) \in Bor_b(\sigma(T))^{\mathbb{N}}$ bounded and $f_n \overset{\tau_p}{\to} f$, then $\Phi(f_n) \overset{SOT}{\to} \Phi(f)$.

Důkaz
Step 1: „$\Phi(f_n) \overset{\text{WOT}}{\to} \Phi(f)$“:

$$\forall x, y \in H : \langle \Phi(f_n)x, y \rangle \overset{\text{Lebesgue}}{\to} \langle \Phi(f)x, uy \rangle .$$

Step 2: „$\|\Phi(f_n)x\| \to \|\Phi(f)x\|, x \in H$“:

$$\|\Phi(f_n)x\|^2 = \left\langle \Phi(\overline{f_n})\Phi(f_n)x, x \right\rangle = \left\langle \Phi(\overline{f_n} f_n)x, x \right\rangle \overset{\text{Lebesgue}}{\to} \left\langle \Phi(\overline{f} f)x, x \right\rangle = \|\Phi(f)x\|^2.$$

Step 3: From steps 1 and 2:

$$\|\Phi(f_n)x - \Phi(f)x\|^2 \overset{\text{Cos. věta}}{=} \|\Phi(f_n)x\|^2 + \|\Phi(f)x\|^2 - 2\Re\langle \Phi(f_n)x, \Phi(f)x \rangle \to 0.$$

□

If $K \subset \mathbb{C}$ is compact, $K \supseteq \sigma(T)$, $\psi : Bor_b(K) \to \mathcal{L}(H)$ is continuous $*$-homomorphism, $\psi(1) = \mathrm{id}$, $\psi(\mathrm{id}) = T$ and $f_n \overset{\tau_p}{\to} f \implies \psi(f_n) \overset{WOT}{\to} \psi(f)$. Then $\psi(g) = \Phi(g|_{\sigma(T)})$, $g \in Bor_b(K)$.

Důkaz
Skipped. Using characterization of Bor_b. □

$\Phi(f)$ is normal, $\Phi(f)$ is self-adjoint $\Leftrightarrow$ f is real.

Důkaz
Skipped. Easy from first part of theorem. □

$\sigma(\Phi(f)) \subseteq \overline{f(\sigma(T))}$.

$g \in Bor_b(\overline{\operatorname{Rng} f}) \implies (g \circ f)(T) = g(f(T))$.

$\forall S \in \mathcal{L}(H), ST = TS : Sf(T) = f(T)S$.

6 Spectral decomposition of normal operator

Definice 6.1 (Spectral measure)

H Hilbert space, $(X, \mathcal{A})$ measurable space. Then $E : \mathcal{A} \mapsto \mathcal{L}(H)$ is spectral measure for $(X, \mathcal{A}, H)$ if

- $\forall A \in \mathcal{A} : E(A)$ is orthogonal projection;
- $E(X) = \mathrm{id}$, $E(\emptyset) = \mathbf{o}$;
- if $\{A_n, n \in \mathbb{N}\} \subset \mathcal{A}$ is point-wise disjoint, then

$$E\left(\bigcup A_n\right) x = \sum_{n=1}^{\infty} E(A_n)x, x \in H.$$

Tvrzení 6.1 (Properties of spectral measure)

H Hilbert, $(X, \mathcal{A})$ measurable space, E is spectral measure for $(X, \mathcal{A}, H)$. Then

1. *$\forall A, B \in \mathcal{A}, A \subset B : E(A) \leqslant E(B)$ (that's $E(B) - E(A) \geqslant 0$);*
2. *$\forall A, B \in \mathcal{A} : E(A \cap B) = E(A) \cdot E(B)$, in particular, if $A \cap B = \emptyset$, then $E(A) \cdot E(B) = \emptyset$.*
3. *$\forall x, y \in H : \mathcal{A} \ni A \mapsto \langle E(A)x, y\rangle$ is a complex measure (denoted by $E_{x,y}$), with total variation $\|E_{x,y}\| \leqslant \|x\| \cdot \|y\|$.*
4. *$(x, y) \mapsto E_{x,y}$ is sesquilinear mapping.*
5. *$\forall x, y \in H\ \forall A \in \mathcal{A}$:*
$$|E_{x,y}(A)| \leqslant \frac{1}{2}\left(E_{x,x}(A) + E_{y,y}(A)\right).$$
6. *$\forall x, y \in H$:*
$$E_{x+y,x+y} \leqslant 2\left(E_{x,x} + E_{y,y}\right).$$

Důkaz
„1.“: $E(A) + E(B\backslash A) = E(B)$, so $E(B) - E(A) = E(B\backslash A) \geqslant 0$.

„2.“: „Step 1: $A \cap B = \varnothing$“:

$$\mathrm{id} = E(X) = E(A) + E(A^c) \geqslant E(A) + E(B),$$

so $E(B) \leqslant \mathrm{id} - E(A)$, which is orthogonal projection onto $(\operatorname{Rng} E(A))^\perp$. Thus („$P, Q \in \mathcal{L}(A)$ orthogonal projections, $Q - P \geqslant 0$, then $\operatorname{Rng} P \subset (\operatorname{Rng} Q)^\perp$“: $\|Px\|^2 =$

$$= \|QPx\|^2 + \|(\mathrm{id} - Q)Px\|^2 = \langle QPx, Px\rangle + \|(\mathrm{id} - Q)Px\|^2 \geqslant \underbrace{\langle PPx, Px\rangle}_{\|Px\|^2} + \|(\mathrm{id} - Q)Px\|^2,$$

thus, $(\mathrm{id} - Q)Px = 0$, so $\operatorname{Rng} P \subseteq \operatorname{Ker}(\mathrm{id} - Q) = \operatorname{Rng} Q$.) $\operatorname{Rng} E(B) \subseteq (\operatorname{Rng} E(A))^\perp$. Thus $\operatorname{Rng} E(A) \perp \operatorname{Rng} E(B)$, so $E(A) \cdot E(B) = 0$.

„Step 2: In general“:

$$E(A) = E(A \cap B) + E(A\backslash B), \qquad E(B) = E(A \cap B) + E(B\backslash) \implies$$

$$E(A) \cdot E(B) = (E(A\backslash B) + E(A\backslash B)) \cdot (E(A \cap B) + E(B\backslash A)) = E^2(A \cap B) + 3 \cdot 0 = E(A \cap B).$$

„3.“: „$E_{x,y}$ is countably additive“ is easy. By this it is a complex measure. „Calculation of $\|E_{x,y}\|$“: Fix $A_1, \ldots, A_n \in \mathcal{A}$ disjoint such that $\bigcup_{i=1}^n A_i = X$. For $i \in [n]$ pick $\alpha_i \in S_{\mathbb{C}}$: $\alpha_i \langle E(A_i)x, y\rangle = |\langle E(A_i)x, y\rangle|$. Then

$$\sum_{i=1}^n |E_{x,y}(A_i)| = \sum_{i=1}^n \alpha_i \langle E(A_i)x, y\rangle \overset{\text{Cauchy–Schwartz}}{\leqslant} \|\sum_{i=1}^n \alpha_i E(A_i)x\| \cdot \|y\|.$$

$$\left\|\sum_{i=1}^n E(A_i)(\alpha_i x)\right\|^2 \overset{\text{Pythagoras}}{=} \sum_{i=1}^n \|E(A_i)(\alpha_i x)\| = \sum_{i=1}^n \|E(A_i)(x)\| = \sum_{i=1}^n \langle E(A_i)x, x\rangle =$$
$$= \left\langle E\left(\bigcup A_i\right) x, x\right\rangle = \langle x, x\rangle = \|x\|^2.$$

„4.“: Easy, using definition. „5.“:

$$|E_{x,y}(A)| = |\langle E(A)x, y\rangle| = |\langle E(A)x, E(A)y\rangle| \overset{\text{Cauchy-Schwartz}}{\leqslant} \|E(A)x\| \cdot \|E(A)y\| =$$
$$= \sqrt{E_{x,x}(A)} \cdot \sqrt{E_{y,y}(A)} \overset{\text{A–G}}{\leqslant} \frac{1}{2}\left(E_{x,x}(A) + E_{y,y}(A)\right).$$

„6.“:

$$E_{x+y,x+y}(A) = E_{x,x}(A) + E_{y,x}(A) + E_{x,y}(A) + E_{y,y}(A) \leqslant E_{x,x}(A) + 2\Re E_{y,x}(A) + E_{y,y}(A) \leqslant$$
$$\leqslant E_{x,x}(A) + 2 \cdot \frac{1}{2}\left(E_{x,x}(A) + E_{y,y}(A)\right) + E_{y,y}(A) = 2\left(E_{x,x}(A) + E_{y,y}(A)\right).$$

□

Poznámka
From 4. we get $E_{x,y}(A) = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle E(A)(x+i^k y), x+iky \right\rangle$. Thus 3. is equivalent to $\forall x \in H : E_{x,x} \geqslant 0$ is measure.

Definice 6.2 (Integral)

H Hilbert space, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$. $f : X \to \mathbb{C}$ bounded $\mathcal{A}$-measurable function. Then integral of f with respect to E is the operator $T \in \mathcal{L}(H)$ such that

$$\langle Tx, y\rangle = \int_X f dE_{x,y}, \qquad x, y \in H.$$

Notation: Then $\int f dE := T$.

Poznámka
It always exists due to Lax–Milgram: $(x,y) \mapsto \int f dE_{x,y}$ is sesquilinear and $\left|\int f dE_{x,y}\right| \leqslant \|f\|_\infty \cdot \|E_{x,y}\| \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\|$. So T exists and $\|T\| \leqslant \|f\|_\infty$.

Tvrzení 6.2

H Hilbert, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$, $f : X \to \mathbb{C}$ bounded $\mathcal{A}$-measurable. Then for $\varepsilon > 0$ pick $A_1, \ldots, A_m \in \mathcal{A}$ disjoint partition of X such that $\operatorname{diam} f(A_i) < \varepsilon$ *and for $x_i \in A_i$, $i \in [n]$*

$$\left\| \int f dE - \sum_{i=1}^{n} f(x_i) E(A_i) \right\| < \varepsilon.$$

Důkaz
Denote $T = \int f dE$. For $x, y \in H : |\langle Tx, y\rangle - \langle \sum f(x_i) E(A_i) x, y\rangle| =$

$$= \left| \sum_{i=1}^{n} \int_{A_i} (f(t) - f(x_i))\, dE_{x,y} \right| \leqslant \sum_{i=1}^{n} \int_{A_i} |f(t) - f(x_i)| d|E_{x,y}| \leqslant \varepsilon \cdot \int_X d|E_{x,y}| \leqslant \varepsilon \cdot \|x\| \cdot \|y\|.$$

This finishes the proof. ($|\langle Sx, y\rangle| \leqslant \varepsilon \cdot \|x\| \cdot \|y\| \implies \|S\| < \varepsilon$.) □

Definice 6.3 (Notation)

$(X, \mathcal{A})$ measurable space, $B(X, \mathcal{A}) \subset l_\infty(X)$ C^*-algebra consisting of bounded $f : X \to \mathcal{C}$ $\mathcal{A}$-measurable functions.

Tvrzení 6.3

H Hilbert, $(X, \mathcal{A})$ measurable space, E spectral measure for $(X, \mathcal{A}, H)$. Consider $\varrho : B(X, \mathcal{A}) \to \mathcal{L}(H)$, $\varrho(f) = \int f dE$. Then

1. *ϱ is continuous $*$-homomorphism, $\|\varrho\| = 1$, $\varrho(1) = \mathrm{id}$.*

2. *$\forall f \in B(X, \mathcal{A})$: $\varrho(f)$ is normal. f is real $\implies$ $\varrho(f)$ is self-adjoint, $f \geqslant 0$ $\implies$ $\varrho(f) \geqslant 0$.*

3. *$f_n \in B(X, \mathcal{A})^n$ bounded, $f_n \to f$ point-wise $\implies$ $\varrho(f_n) \overset{WOT}{\to} \varrho(f)$.*

4. *$\forall f \in B(X, \mathcal{A})$ $\forall x \in H$: $\|\varrho(f)x\| = \sqrt{\int |f|^2 dE_{x,x}}.i$*

5. *$\int f dE$ is the unique $T \in \mathcal{L}(H)$: $\langle Tx, y\rangle = \int f dE_{x,y}$, $x, y \in H$.*

Důkaz
1.) „ϱ is linear“: easy. „$\|\varrho\| \leqslant 1$“: easy as well. „ϱ preserves $*$“:

$$\forall x \in H : \langle \varrho(f)^* x, x\rangle = \langle x, \varrho(f)x\rangle = \overline{\langle \varrho(f)x, x\rangle} = \overline{\int f dE_{x,x}} = \int \overline{f} dE_{x,x} = \left\langle \varrho(\overline{f})x, x\right\rangle.$$

„ϱ is multiplicative“: For $f, g \in B(X, \mathcal{A})$, $\varepsilon > 0$. Find disjoint partition $A_1, \ldots, A_n \in \mathcal{A}$ of X such that for $\omega \in \{f, g, f \cdot g\}$ we have $\operatorname{diam} \omega(A_i) < \varepsilon$ for $i \in [n]$. Pick $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$. Thus using previous proposition we have

$$\left\| \int fg dE - \left(\int f dE\right)\left(\int g dE\right)\right\| \leqslant \varepsilon +$$

$$+ \left\| \sum_{i=1}^{n} (f \cdot g)(x_i) E(A_i) - \left(\sum f(x_i) E(A_i)\right)\left(\sum g(x_i) E(A_i)\right)\right\| +$$

$$+ \left\| \left(\sum f(x_i) E(A_i)\right)\left(\sum g(x_i) E(A_i)\right) - \left(\int f dE\right)\left(\int g dE\right)\right\| \leqslant \varepsilon + 0 +$$

$$+ \left\| \left(\sum f(x_i) E(A_i)\right)\left(\sum g(x_i) E(A_i) - \int g dE\right)\right\| + \|(\sum f(x_i) E(A_i)) TODO <$$

$$< \|f\|_\infty \cdot \varepsilon + \varepsilon \cdot \|g\|_\infty .$$

„$\|\varrho\| = 1$“: TODO!!!

„$\varrho(1) = \mathrm{id}$“: $\forall x \in H : \langle \varrho(1)x, x\rangle = \int_X 1 dE_{x,x} = \langle E(X)x, x\rangle = \langle x, x\rangle = \langle \mathrm{id}\, x, x\rangle$.

2.)
$$\varrho(f)^* \varrho(f) = \varrho(\overline{f} f) = \varrho(f \overline{f}) = \varrho(f)\varrho(f)^* \implies \varrho(f) \text{ is normal.}$$
$$f \text{ is real} \implies f = \overline{f} \implies \varrho(f) = \varrho(f)^*.$$
$$f \geqslant 0 \implies \forall x \in H : \langle \varrho(f)x, x\rangle = \int f dE_{x,x} \geqslant 0 \implies \varrho(f) \geqslant 0.$$

3.)
$$\forall x, y \in H : \langle \varrho(f_n)x, y\rangle = \int f_n dE_{x,y} \overset{\text{Lebesgue}}{\to} \int f dE_{x,y} = \langle \varrho(f)x, y\rangle.$$

4.)
$$\|\varrho(f)x\|^2 = \langle \varrho(f)x, \varrho(f)x\rangle = \left\langle \varrho(\overline{f} f)x, x\right\rangle = \int \overline{f} f dE_{x,x} = \int |f|^2 dE_{x,x}.$$

□

Důsledek (Spectral decomposition of normal operator)
H Hilbert, $T \in \mathcal{L}(H)$ normal $\implies \exists!$ spectral measure E for $(\sigma(T), Bor(\sigma(T)), H)$: $T = \int \mathrm{id}\, dE$. Moreover $E(A) = \Phi(\chi_A)$ for any $A \in Bor(\sigma(T))$, where $\Phi : Bor_b(\sigma(T)) \to \mathcal{L}(H)$ is borel calculus from definition above.

Důkaz
Whenever E is spectral measure for $(\sigma(T), Bor(\sigma(T)), H)$ satisfying $T = \int \mathrm{id}\, dE$, then $\int f dE = \Phi(f)$, $f \in \mathcal{B}(\sigma(T), Bor(\sigma(T)))$. This proves uniqueness.

„Existence“: Put $E(A) := \Phi(A)$, $A \subset \sigma(T)$ borel. Then E is spectral measure: $E(A)$ is orthogonal projection ($\chi_A^2 = \chi_A$, χ_A is real), $E(\sigma(T)) = \mathrm{id}$, $E(\varnothing) = 0$ ($\chi_{\sigma(T)} = 1$ and $\Phi(1) = \mathrm{id}$, $\chi_\varnothing = 0$), $A_i \in Borel(\sigma(T))$ disjoint, $x \in H$, then

$$\left\| E\left(\bigcup A_n\right) x - \sum E(A_i)x \right\| = \left\langle E\left(\bigcup A_i\right) x, E\left(\bigcup A_i\right) x \right\rangle = \left\langle E\left(\bigcup A_i\right) x, x \right\rangle =$$

$$= \int \chi_{\bigcup A_i} d\mu_{x,x} = \sum_{N+1}^{\infty} \mu_{x,x}(A_i) \to 0.$$

„$T = \int \mathrm{id}\, dE$“: $E_{x,y} = \mu_{x,y}$ ($E_{x,y}(A) = \langle E(A)x, y\rangle = \int \chi_A d\mu_{x,y} = \mu_{x,y}(A)$). Thus

$$\left\langle \int \mathrm{id}\, dEx, y \right\rangle = \int \mathrm{id}\, dE_{x,y} = \int \mathrm{id}\, d\mu_{x,y} = \langle \Phi(\mathrm{id})x, y\rangle = \langle Tx, y\rangle.$$

□

7 Unbounded operators

Definice 7.1

X, Y Banach spaces. Operator from X to Y is a linear mapping defined on a linear space $D(T) \subset X$ with values in $R(T) \subset Y$. If $X = Y$, we say T is operator on X. Then graph of T is $G(T) = \{(x, Tx) | x \in D(T)\} \subseteq X \times Y$.

We say that T is densely defined $\equiv \overline{D(T)} = X$. We say that T is closed $\equiv G(T) \subset X \times Y$ is closed.

Definice 7.2 (Notations)

X, Y Banach spaces. If T, S is operator from X to Y, then $S + T$ is operator from X to Y defined as $(S + T)(x) = Sx + T(x)$ for $x \in D(S + T) = D(S) \cap D(T)$.

If T is operator from X to Y and S is operator from Y to a Banach space Z, then ST is operator with $D(ST) = \{x \in D(T) | Tx \in D(S)\}$ defined as $(ST)x = S(Tx)$ for $x \in D(ST)$.

Operator S from X to Y is extension of T, if $G(S) \supset G(T)$ (and we write $T \subset S$).

Například
$D(T) = c_{00} \subset l_2 = X$, $Tx = (\sum_{n=1}^{\infty} x_n, 0, 0, 0, 0, \ldots)$. Then T is densely defined, but it doesn't have closed extension.

Důkaz
Consider $x^n = \left(\frac{1}{2^n}, \ldots, \frac{1}{2^n}, 0, \ldots\right)$ then $(x_n, Tx_n) \to (\mathbf{o}, e_1)$, so if there is extension, then $(\mathbf{o}, e_1) \in G(S)$, but $S\mathbf{o} = \mathbf{o}$, because of linearity. □

Poznámka
It is easy to check:

$$(S+T)+V = S+(T+V),$$
$$(ST)V = S(TV),$$
$$(S+T)V = SV+TV.$$

Pozor

$$V(S+T) \supseteq VS+VT.$$

Lemma 7.1

X, Y Banach and $L \subseteq X \times Y$. Then $\exists$ operator T from X to Y such that $L = G(T) \Leftrightarrow L$ is a subspace and $\{(x,y) \in L | x = 0\} = \{(0,0)\}$.

Důkaz
„ $\implies$ “: Easy.

„ $\impliedby$ “: Put $D(T) = \{x \in X | \exists y \in Y : (x,y) \in L\}$. Then $\forall x \in D(T)$ $\exists! y \in Y$: $(x,y) \in L$. $((x,y_1),(x,y_2) \in L \implies (0, y_1 - y_2) \in L.)$ So, we put $Tx := y$, where $y \in L$ is such that $(x,y) \in L$. Then T is linear and $G(T) = L$ □

Tvrzení 7.2

X, Y Banach spaces, T operator from X to Y.

- *$D(T) = X \wedge T$ is closed $\implies T \in \mathcal{L}(X,Y)$.*
- *Equivalence:*
  1. *T has closed extension;*
  2. *$(x_n, Tx_n) \to (0,y)$ in $D(T) \times Y \implies y = 0$;*
  3. *$\overline{G(T)} \subset X \times Y$ is graph of an operator from X to Y.*
- *T is one-to-one and closed $\implies T^{-1}$ is closed.*

Důkaz
First point follows immediately from closed graph theorem.

„1.) $\implies$ 2.)“: Let $S \supset T$ be closed. If $(x_n, Tx_n) \to (\mathbf{o}, y)$, then $(\mathbf{o}, y) \in G(S)$, so $\mathbf{o} = S\mathbf{o} = y$.

„2.) $\implies$ 3.)“ We will show, using the previous lemma, that $\overline{G(T)}$ is graph of an operator: $\overline{G(T)}$ is linear, because $G(T)$ is linear. If $(\mathbf{o}, y) \in \overline{G(T)}$, then $\exists (x_n) \in D(T)^{\mathbb{N}}$: $(x_n, Tx_n) \to (\mathbf{o}, y)$, so $y = \mathbf{o}$ from 2)..

„3.) $\implies$ 1.)“: Clear.

Third point $\Phi : X \times Y \to Y \times X$ defined as $(x, y) \mapsto (y, x)$ is homeomorphism, so, $G(T)$ is closed $\Leftrightarrow \Phi(G(T)) = G(T^{-1})$ is closed. □

Definice 7.3 (Closure of operator)

X, Y Banach spaces, T operator from X to Y, T has closed extension. Then $\overline{T}$ is operator satisfying $\overline{T} \supset T$ and $G(\overline{T}) = \overline{G(T)}$.

Tvrzení 7.3

X, Y, Z Banach spaces, T operator from X to Y, which is closed.

- *If $S \in \mathcal{L}(X, Y)$, then $S + T$ is closed and $D(S + T) = D(T)$.*
- *If $S \in \mathcal{L}(Y, Z)$, then $D(ST) = D(T)$ and if S is isomorphism into, then ST is closed.*
- *If $S = \mathcal{L}(Z, X)$, then TS is closed.*

Důkaz
Of course $D(S + T) = D(S) \cap D(T) = D(T)$. If $(x_n, (S + T)x_n) \to (x, y)$, then $Tx_n = (S+T)x_n - Sx_n \to y - Sx$. So $(x_n, Tx_n) \to (x, y - Sx) \in G(T)$, so $Tx = y - Sx \implies y = (T + S)x$.

$$D(ST) = \{x \in D(T) | Tx \in D(S) = Y\} = D(T).$$

Suppose S is isomorphism into, $(x_n, STx_n) \to (x, z)$, then $Tx_n = S^{-1}STx_n \to S^{-1}z$. So $(x_n, Tx_n) \to (x, S^{-1}z) \in G(T)$, so $Tx = S^{-1}z$, then $STx = z$.

$(z_n, TSz_n) \to (x, y)$, then $Sz_n \to Sx$, so $(Sz_n, TSz_n) \to (Sx, y) \in G(T)$, thus $TSx = y$. □

TODO example?

Tvrzení 7.4

X, Y Banach, T one-to-one closed operator from X to Y. Then following statements are equivalent:

$$\operatorname{Rng} T = Y \wedge T^{-1} \in \mathcal{L}(Y, X); \quad \overline{\operatorname{Rng} T} = Y; \quad \operatorname{Rng} T \textit{ is dense and } T^{-1} \in \mathcal{L}(\operatorname{Rng} T, X).$$

Důkaz

„1) $\implies$ 2)“: trivial. „2) $\implies$ 3)“: $\operatorname{Rng} T$ is dense and $T^{-1}(\operatorname{Rng} T, X)$ due to previous proposition (by which T^{-1} is closed).

„3) $\implies$ 1)“: Let $S \in \mathcal{L}(Y, X)$ be continuous extension of T^{-1}. Pick $y \in Y$. Since $\overline{\operatorname{Rng} T} = Y$, there is $(x_n) \in X^{\mathbb{N}}$ such that $Tx_n \to y$. Then $STx_n = T^{-1}Tx_n = x_n \to Sy$. So $(x_n, Tx_n) \to (Sy, y) \in G(T)$, thus $TSy = y \in \operatorname{Rng} T$. □

Definice 7.4 (Resolvent set, resolvent function, spectrum of operator)

X Banach, T linear operator on X. Then resolvent set is

$$\varrho(T) := \{\lambda \in \mathbb{K} | \lambda I - T \text{ has inverse which belongs to } \mathcal{L}(X)\};$$

resolvent function is $R_T(\lambda) := (\lambda I - T)^{-1}$, $\lambda \in \varrho(T)$; spectrum of T is $\sigma(T) := \mathbb{K} \backslash \varrho(T)$.

Věta 7.5

X Banach, T linear operator on X. Then $\varrho(T)$ is open, $\sigma(T)$ is closed and R_T has derivative at each point of $\varrho(T)$. (So, if X is complex, then R_T is holomorphic on $\varrho(T)$).

Důkaz

„$\varrho(T)$ is open“: Pick $\lambda \in \varrho(T)$ and $h \in \mathbb{K}$ small ($|\cdot|$) enough: $|h| < \frac{1}{\|(\lambda I - T)^{-1}\|}$. Then $h(\lambda I - T)^{-1} =: S \in \mathcal{L}(X)$, $\|S\| < 1$. Thus, $(I + S)^{-1}$ exists, so $(\lambda + h)I - T = (I + S) \cdot (\lambda I - T)$ has inverse $(\lambda I - T)^{-1} \circ (I + S)^{-1} \in \mathcal{L}(X)$. $(\lambda I - T)^{-1} \circ (I + S)^{-1} \in \mathcal{L}(X)$. So $U(\lambda, \frac{1}{\|(\lambda I - T^{-1})\|}) \subset \varrho(T)$.

„R_T has derivative at each $\lambda \in \varrho(T)$“: $R_T'(\lambda) = -R_T(\lambda)^2$:

$$\forall h \text{ small enough}: \left\| \frac{R_T(\lambda + h) - R_T(\lambda)}{h} + R_T(\lambda)^2 \right\| =$$

$$\frac{1}{h} \|R_T(\lambda + h) - R_T(\lambda) + R_T(\lambda) h R_T(\lambda)\| = \frac{\|R_T(\lambda)\|}{\|h\|} \cdot \|(I + S)^{-1} - I + hR_T(\lambda)\| =$$

$$\left((I + S)^{-1} = \sum_{n=0}^{\infty} (-S)^n = I - S + \sum_{n=2}^{\infty} (-S)^n = I - hR_T(\lambda) + \sum_{n=2}^{\infty} (-hR_T(\lambda))^n \right)$$

$$= \frac{\|R_T(\lambda)\|}{|h|} \cdot \left\| \sum_{n=2}^{\infty} (-hR_T(\lambda))^n \right\| \leqslant \frac{\|R_T(\lambda)\|}{|h|} \sum_{n=2}^{\infty} \|hR_T(\lambda)\|^n = \frac{\|R_T(\lambda)\|}{|h|} \cdot \frac{\|hR_T(\lambda)\|^2}{1 - \|hR_T(\lambda)\|} \leqslant$$

$$\leqslant \frac{\|R_T(\lambda)\|}{|h|} \cdot \frac{|h|^2 \|R_T(\lambda)\|^2}{1/2} = 2|h| \cdot \|R_T(\lambda)\|^3 \to 0.$$

□

Lemma 7.6

X Banach space, T operator in X, $0 \notin \sigma(T)$. Then $\forall \lambda \neq 0 : \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$.

Důkaz

Since $0 \in \varrho(T)$, so $T^{-1} \in \mathcal{L}(X)$. Moreover, $T = (T^{-1})^{-1}$ is closed (by proposition above). In the same time, since T is closed, we have $\lambda \in \varrho(T) \Leftrightarrow \lambda I - T$ is bijection („ $\Longrightarrow$ “: trivial, „ $\Longleftarrow$ “: $\lambda I - T$ is bijection and closed operator, so by previous proposition $(\lambda I - T)^{-1} \in \mathcal{L}(X)$).

So, it suffices: „$\forall \lambda \neq 0$: $\lambda I - T$ bijection $\Leftrightarrow \frac{1}{\lambda} I - T^{-1}$ bijection“:

$$\frac{1}{\lambda} I - T^{-1} = -\frac{1}{\lambda}(\lambda I - T)T^{-1} \qquad \left(\text{so } (\lambda I - T)^{-1} \text{ exists} \implies (\frac{1}{\lambda} I - T^{-1})^{-1} \text{ exists}\right)$$

$$\lambda I - T = -\lambda(\frac{1}{\lambda} I - T^{-1})T \qquad \left(\text{so } (\frac{1}{\lambda} I - T^{-1})^{-1} \text{ exists} \implies (\lambda I - T)^{-1} \text{ exists}\right).$$

□

Důsledek

X complex Banach, T operator on X, $\sigma(T) = \emptyset$. Then $T^{-1} \in \mathcal{L}(X)$ and $\sigma(T^{-1}) = \{0\}$.

Důkaz

$0 \in \varrho(T) \implies T^{-1} \in \mathcal{L}(x)$. By the previous lemma, $\forall \lambda \neq 0 : \frac{1}{\lambda} \notin \sigma(T^{-1})$. So $\sigma(T^{-1}) \subset \{0\}$. Since $\sigma(T^{-1}) \neq \emptyset$, we have $\sigma(T^{-1}) = \{0\}$. □

7.1 Unbounded operators in Hilbert spaces

Definice 7.5 (Convention)

From now, all Banach spaces are over $\mathbb{K} = \mathbb{C}$ (if not said otherwise).

Definice 7.6 (Hilbert adjoint of operator)

H Hilbert, T densely defined operator on H. Hilbert adjoint of T, denoted as T^*, is defined on $D(T^*) := \{y \in H | x \mapsto \langle Tx, y\rangle \text{ is continuous linear on } D(T)\}$. For $y \in D(T^*)$, T^*y is the unique point from H satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $x \in D(T)$.

Důkaz

„T^*y exists“: any $\varphi \in D(T)^*$ can be extended to $H^* = H$. □

Tvrzení 7.7

H Hilbert, S and T densely defined in H.

- $S \subset T \implies T^* \subset S^*$.

  Důkaz
  $D(T^*) = \{y | x \mapsto \langle Tx, y\rangle = \langle Sx, y\rangle$ is continuous on $D(T) \supset D(S)\} \subset D(S^*)$. And for $y \in D(T^*)$:

  $$\forall x \in D(S) : \langle x, T^*y\rangle = \langle Tx, y\rangle = \langle Sx, y\rangle = \langle x, S^*y\rangle \implies T^*y = S^*y.$$

  □

- *$S + T$ is densely defined $\implies$ $S^* + T^* \subset (S+T)^*$ and if $S \in \mathcal{L}(H)$, then there is equality.*

  Důkaz
  For $y \in D(S^* + T^*) = D(S^*) \cap D(T^*)$ and $x \in D(S+T)$:

  $$\langle (S+T)x, y\rangle = \langle x, S^*y\rangle + \langle x, T^*y\rangle = \langle x, (S^* + T^*)y\rangle .$$

  So, $y \in D((S+T)^*)$ and $(S+T)^*y = (S^* + T^*)(y)$. This proves the inclusion.

  „If $S \in \mathcal{L}(H)$" For $y \in D((S+T)^*)$ and for $x \in D(S+T) = D(T)$:

  $$D(T) \ni x \mapsto \langle Tx, y\rangle = \langle (S+T)x, y\rangle - \langle Sx, y\rangle$$

  is constant on $D(T)$. So, $y \in D(T^*) = D(T^*) \cap D(S^*) = D(S^* + T^*)$. Thus, $D(S^* + T^*) = D((S+T)^*) \wedge S^* + T^* \subset (S+T)^*$, so $S^* + T^* = (S+T)^*$. □

- *ST is densely defined $\implies$ $T^*S^* \subset (ST)^*$ and if $S \in \mathcal{L}(H)$ then there is equality.*

  Důkaz
  Pick $y \in D(T^*S^*)$. Then for $x \in D(ST)$:

  $$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle .$$

  So, $y \in D((ST)^*)$ and $(ST)^*y = T^*S^*y$.

  „If $S \in \mathcal{L}(H)$": Then $D(ST) = D(T)$ and for $y \in D((ST)^*)$ we want „$S^*y \in D(T^*)$" (then $y \in D(T^*S^*)$ and we are done):

  $$D(T) \ni\mapsto \langle Tx, S^*y\rangle = \langle STx, y\rangle = \langle x, (ST)^*y\rangle .$$

  So, $x \mapsto \langle Tx, S^*y\rangle$ is continuous on $D(T)$. □

Tvrzení 7.8

H Hilbert, T densely defined on H.

- *T^* is closed operator on H;*
- *T has closed extension $\Leftrightarrow$ T^* is densely defined. Then $(T^*)^* = \overline{T}$.*
- *T is closed $\Leftrightarrow$ T^* is densely defined and $T = (T^*)^*$.*

Lemma 7.9

H Hilbert, T densely defined on H. Consider $V \in \mathcal{L}(H \oplus H)$ such that $V(x, y) := (-y, x)$. Then V is unitary and $G(T^*) = V(G(T))^\perp$.

Důkaz
„V is unitary:" obvious (V is isometry onto).

„$G(T^*) \subseteq V(G(T))^\perp$": Pick $y \in D(T^*)$ and $x \in D(T)$. Then

$$\langle (y, T^*y), V(x, Tx) \rangle = \langle (y, T^*y), (-Tx, x) \rangle = \langle y, -Tx \rangle + \langle T^*y, x \rangle = 0.$$

„$V(G(T))^\perp \subseteq G(T^*)$": Pick $(x, y) \in V(G(T))^\perp$. Then for $z \in D(T)$:

$$0 = \langle (x, y), (-Tz, z) \rangle = -\langle x, Tz \rangle + \langle y, z \rangle,$$

so $\langle x, Tz \rangle = \langle y, z \rangle$, so $D(T) \ni z \mapsto \langle Tz, x \rangle$ $(= \langle z, y \rangle)$ is continuous. So $x \in D(T^*)$ and $T^*x = y$, co $(x, y) \in G(T^*)$. □

Poznámka
$U \in \mathcal{L}(H)$ unitary, $A \subset H$. Then $U(A^\perp) = U(A)^\perp$.

Důkaz
$x \in U(A)^\perp \Leftrightarrow \forall a \in A : 0 = \langle x, Ua \rangle = \langle U^*x, a \rangle \Leftrightarrow U^*x \in A^\perp \Leftrightarrow x \in U(A^\perp)$. □

Důkaz (Of the previous proposition)
First point follows from the previous lemma.

„Second point, $\Longrightarrow$ ": Pick $y_0 \in D(T^*)^\perp$. Wanted: $y_0 = 0$. We $\underline{\text{have}}$ $(y_0, 0) \in G(T^*)^\perp$ $(\forall z \in D(T^*) : \langle (z, T^*z), (y_0, 0) \rangle = 0)$. $G(T^*)^\perp = V(G(T))^{\perp\perp} = \overline{V(G(T))} = V(\overline{G(T)})$. So $(0, -y_0) = V^*(y_0, 0) \in V^*V(\overline{G(T)}) = \overline{G(T)}$. Thus $y_0 = 0$ (because T is closed).

„Second point, $\Longleftarrow$ ": T^* is densely defined. Then $(T^*)^*$ is defined and, by first point, it is closed. Moreover, „$T \subset (T^*)^*$": Pick $x \in D(T)$. Then $D(T^*) \ni y \mapsto \langle T^*y, x \rangle = \langle y, Tx \rangle$, so $x \in D((T^*)^*)$ and $(T^*)^*x = Tx$.

„Second point, then part": $T \subseteq (T^*)^*$ is done, „$(T^*)^* \subseteq \overline{T}$": it suffices to prove

„$G((T^*)^*) = \overline{G(T)}$“: By the previous lemma, $G((T^*)^*) = V(G(T^*))^\perp = V^*(G(T^*))^\perp = V^*(V(G(T))^\perp)^\perp = V^*V(G(T)^{\perp\perp}) = \overline{G(T)}$.

„Third point“: „ $\implies$ “ follows directly from second point, „ $\impliedby$ “ by second point, T has closed extension and $\overline{T} = (T^*)^* = T$, so ti is closed. □

Tvrzení 7.10

H Hilbert, T densely defined on H. Then

- $\operatorname{Rng}(T)^\perp = \operatorname{Ker} T^*$*;*

  Důkaz
  $y \in \operatorname{Ker} T^* \Leftrightarrow T^*y = 0 \Leftrightarrow \forall x \in D(T) : \langle Tx, y\rangle = 0 \Leftrightarrow y \in \operatorname{Rng} T^\perp$. □

- *If T is moreover closed, then* $\operatorname{Ker} T = (\operatorname{Rng} T^*)^\perp$.

  Důkaz
  By the previous proposition T^* is densely defined and $T^{**} = T$. By the previous point, $\operatorname{Ker} T = \operatorname{Ker} T^{**} = (\operatorname{Rng} T^*)^\perp$. □

Tvrzení 7.11

H Hilbert, T is one-to-one densely defined on H, $\overline{\operatorname{Rng} T} = H$. Then T^* is one-to-one and $(T^*)^{-1} = (T^{-1})$.

Důkaz
Proof omitted (using the previous proposition and lemma). □

Definice 7.7 (Self-adjoint operator, symmetric operator, maximally symmetric operator)

H Hilbert, T operator on H. T is self-adjoint $\equiv T = T^*$. T is symmetric $\equiv \forall x, y \in D(T) : \langle Tx, y\rangle = \langle x, Ty\rangle$. T is maximally symmetric $\equiv$ T is symmetric, and there is no $S \supsetneq T$ symmetric.

Poznámka
T is self-adjoint $\implies$ T is densely defined. T is densely defined, then it is symmetric $\Leftrightarrow$ $T \subseteq T^*$. If T is densely defined, then T is self-adjoint $\implies$ symmetric. (And the other implication doesn't hold.)

Tvrzení 7.12

H Hilbert, T densely defined and symmetric.

- *T has closed extension and $\overline{T}$ is symmetric;*
- *$R(T)$ is dense $\implies$ T is one-to-one;*
- *$D(T) = H \implies T = T^*$ and $T \in \mathcal{L}(H)$;*
- *$R(T) = H \implies T$ is one-to-one, self-adjoint and $T^{-1} \in \mathcal{L}(H)$;*
- *T is self-adjoint $\implies$ T is maximally symmetric.*

Důkaz
Omitted. □

Věta 7.13

H Hilbert space, $H \neq \{0\}$, T is self-adjoint operator on H. Then $\emptyset \neq \sigma(T) \subseteq \mathbb{R}$.

Důkaz
BÚNO $0 \not\equiv T = T^*$. Kdyby $\sigma(T) = \emptyset$, pak $T^{-1} \in \mathcal{L}(H)$. Pak T^{-1} je samoadjungovaný ($(T^{-1})^* = (T^*)^{-1} = T^{-1}$.).)

„$\sigma(T) \subseteq \mathbb{R}$“: Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. Then

$$\overline{\mathrm{Rng}(\lambda I - T)} = \mathrm{Ker}((\lambda I - T)^*)^\perp = \mathrm{Ker}(\overline{\lambda} I - T^*)^\perp = \{\mathbf{o}\}^\perp = H.$$

By next lemma, $\lambda I - T$ is onto. (Because T is closed because T is self-adjoint) and $(\lambda I - T)^{-1}$ is continuous. Thus $\lambda \notin \sigma(T)$.

□

Lemma 7.14

T is symmetric on Hilbert H, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then $(\lambda I - T)$ is one-to-one, $(\lambda I - T)^{-1}$ is continuous on $R(\lambda I - T)$, and moreover T is closed $\Leftrightarrow$ $R(\lambda I - T)$ is closed.

Důkaz
$\lambda = \alpha + i \cdot \beta$, $\beta \neq 0$, $\alpha, \beta \in \mathbb{R}$. Then $\alpha I - T$ is symmetric, so $\forall x \in D(T)$:

$$\|(\lambda I - T)x\|^2 = \|(\alpha I - T)x + i \cdot \beta x\|^2 = \|i \cdot \beta \cdot x\|^2 + \|(\alpha I - T)x\|^2 + 2\Re\langle i \cdot \beta \cdot x, (\alpha I - T)x\rangle =$$

$$= |\beta|^2 \cdot \|x\|^2 + \|(\alpha I - T)x\|^2 + 0 \geqslant |\beta|^2 \cdot \|x\|^2,$$

cause S is symmetric, then $\langle Sx, x\rangle \in \mathbb{R}$, $x \in D(S)$. So, $\|(\lambda I - T)x\| \geqslant |\beta| \cdot \|x\|$, $x \in D(T)$, thus $(\lambda I - T)$ is one-to-one. And $(\lambda I - T)^{-1}$ is bounded on its domain, so continuous on its domain.

It suffices: For $S := \lambda I - T$: S is closed $\Leftrightarrow$ $R(S)$ is closed. And proof of this is omitted.

„Moreover“: Denote $S := \lambda I - T$ (S closed $\Leftrightarrow$ T closed). „ $\implies$ “: Let S be closed, then „$\operatorname{Rng} S$“ is closed: $\operatorname{Rng} S \ni y_n \to y \implies (S^{-1}(y_n))$ is Cauchy, so there is $x \in D(S)$: $S^{-1}y_n \to x$. Then $(S^{-1}y_n, y_n) \to (x, y)$, so $Sx = y$.

„ $\impliedby$ “: Let $\operatorname{Rng} S$ be closed. Then „$G(S)$ is closed“: $(x_n, Sx_n) \to (x, y) \implies x_n = S^{-1}Sx_n \to S^{-1}y$. So $S^{-1}y = x$. □

Důsledek (Of the previous theorem)
H Hilbert, T operator on H. Then next propositions are equivalent

- T is self-adjoint;
- T is densely defined, symmetric and $\sigma(T) \subseteq \mathbb{R}$;
- T is densely defined, symmetric and there is $\lambda \in \mathbb{C}\backslash\mathbb{R} : \lambda, \overline{\lambda} \in \varrho(T)$.

Důkaz
„1. $\implies$ 2.“ use the previous theorem. „2. $\implies$ 3.“ easy. „3. $\implies$ 1.“: $T \subset T^*$ by third point. Wanted: „$D(T^*) \subset D(T)$“: Pick $x \in D(T^*)$. Put

$$y := (\lambda I - T)^{-1}\left((\lambda I - T^*)x\right) \in \operatorname{Rng}((\lambda I - T)^{-1}) = D(\lambda I - T).$$

Then

$$(\lambda I - T^*)x = (\lambda I - T)y = \lambda y - Ty = \lambda y - T^*y = (\lambda I - T^*)y.$$

$\lambda I - T^*$ is one-to-one ($\operatorname{Ker}(\lambda I - T^*) = \operatorname{Ker}((\overline{\lambda} I - T)^*) = \operatorname{Rng}(\overline{\lambda} I - T)^{\perp} = H^{\perp} = \{\mathbf{o}\}$). So, $x = y \in D(T)$. □

8 Cayley transform

Poznámka (Motivation)
T self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$ and $M(z) = \frac{z-i}{z+i}$, $z \in \mathbb{R}$ is bijection between $\mathbb{R}$ and $\mathbb{D} \setminus \{1\}$.

Definice 8.1 (Cayley transform of operator)

H Hilbert, T symmetric operator on H. Then Cayley transform of T is the operator $\mathcal{C}(T) := (T - iI) \cdot (T + i \cdot I)^{-1}$.

Poznámka
$\mathcal{C}(T)$ is well defined: $T + iI$ is one-to-one, $\mathrm{Rng}(T + iI)^{-1} = D(T + iI) = D(T - iI)$.

$$Tx + ix \overset{\mathcal{C}(T)}{\to} Tx - ix.$$

Věta 8.1

H Hilbert, T symmetric operator on H, $\mathcal{C}(T)$ Cauchy transform. Then

- *$\mathcal{C}(T)$ is linear isometry $D(\mathcal{C}(T)) = R(T+iI)$ onto $R(\mathcal{C}(T)) = R(T-iI)$;*

  Důkaz
  $D(\mathcal{C}(T)) = R(T + iI)$ by definition. $R(\mathcal{C}(T)) = R(T - iI)$ by definition too.

  For $y = Tx + ix \in D(\mathcal{C}(T))$ we have

  $$\|\mathcal{C}(T)y\|^2 = \|Tx + ix\|^2 \overset{\text{COS}}{=} \|Tx\|^2 + \|x\|^2 + 2\Re\langle Tx, -ix\rangle = \|Tx\|^2 + \|x\|^2$$

  $$\|y\|^2 = \|Tx + ix\|^2 = \ldots = \|Tx\|^2 + \|x\|^2.$$

  So, $\mathcal{C}(T)$ is isometry. □

- *$I - \mathcal{C}(T) = 2i(T + iI)^{-1}$, and so $I - \mathcal{C}(T)$ is one-to-one and $R(I - \mathcal{C}(T)) = D(T)$;*

  Důkaz
  Let $y = Tx + ix \in D(\mathcal{C}(T))$, then

  $$(I - \mathcal{C}(T))y = y - \mathcal{C}(T)y = Tx + ix - (Tx - ix) = 2ix = (T + iI)^{-1}y$$

  $\implies$ formula holds.

  Since $(T + iI)^{-1}$ is one-to-one, $I - \mathcal{C}(T)$ is one-to-one. Moreover, $R(I - \mathcal{C}(T)) = R((T + iI)^{-1}) = D(T + iI) = D(T)$. □

- $T = i\,(I + \mathcal{C}(T)) \cdot (I - \mathcal{C}(T))^{-1}$.

Důkaz
We know $D(T) = R(I - \mathcal{C}(T))$ and $R\left((I - \mathcal{C}(T))^{-1}\right) = D(I - \mathcal{C}(T)) = D(I + \mathcal{C}(T))$. So operator on RHS is well-defined and LHS have same domain as RHS.

Pick $y \in D(T)$ and $x \in D(\mathcal{C}(T))$ such that $(I - \mathcal{C}(T))x = y$. Then

$$y - (I - \mathcal{C}(T))x = 2i(T + iI)^{-1}x,$$

so $i(I + \mathcal{C}(T)) \cdot (I - \mathcal{C}(T))\, y = i(I + \mathcal{C}(T))x = i\,(x + (T - iI)(T + iI)^{-1}x) =$

$$= i\,(x + (T - iI) \cdot (y/2i)) = \frac{i}{2i}\,(2ix + (T - iI)y) = \frac{1}{2}((T + iI)y + (T - iI)y) = Ty.$$

□

- *T closed $\Leftrightarrow$ $\mathcal{C}(T)$ closed $\Leftrightarrow$ $D(\mathcal{C}(T))$ closed $\Leftrightarrow$ $R(\mathcal{C}(T))$ closed.*

Důkaz (Omitted.) □

Věta 8.2

Let H be a Hilbert space and U isometry form $D(U)$ onto $R(U)$. Let $I - U$ be one-to-one. Then $T := i(I + U)(I - U)^{-1}$ is symmetric and $\mathcal{C}(T) = U$. Moreover T is densely defined if and only if $R(I - U)$ is dense.

Důkaz
T is well-defined: $R((I - U)^{-1}) = D(I - U) = D(I + U)$. $D(T) = R(I - U)$, so T is densely defined iff $R(I - U)$ is dense.

„T is symmetric“: Let $x = (I - U)x' \in D(T)$, $y = (I - U)y' \in D(T)$.

$$\langle Tx, y\rangle = \langle i(I + U)x', y\rangle = i\,\langle x' + Ux', y' - Uy'\rangle \overset{U \text{ isometry}}{=} i\,(-\langle x'Uy'\rangle + \langle Ux', y'\rangle)\,,$$

$$\langle x, Ty\rangle = \ldots = \langle x, i(I + u)y'\rangle = -i\,\langle x' - Ux'\rangle = -i\,(\langle x', Uy'\rangle - \langle Ux', y'\rangle)\,.$$

„$\mathcal{C}(T) = U$“: Let $x = (I - U)x' \in D(T)$:

$$(T - iI)x = i(I + U)x' - ix = i(x' + Ux') - i(x' - Ux') = 2iUx',$$

$$(T + iI)x = \ldots + ix = \ldots + \ldots = 2ix'.$$

So, $x' \in R(T + iI) = D(\mathcal{C}(T))$ and $D(U) \subseteq D(\mathcal{C}(T))$ and $D(\mathcal{C}(T)) = R(T + iI) \subseteq D(U)$. Thus, $D(U) = D(\mathcal{C}(T))$. Finally, for $x \in D(T)$:

$$U(Tx + ix) = U(2ix') = 2iUx' = (T - iI)x = Tx - ix.$$

□

Věta 8.3

H Hilbert:

a) *Let T be a symmetric operator on H. Then T is self-adjoint $\Leftrightarrow \mathcal{C}(T)$ is unitary (i.e. $D(\mathbb{C}(T)) = H = R(\mathcal{C}(T))$).*

b) *$U \in \mathcal{U}(H)$ such that $I - U$ is one-to-one, then*

$$T := i(I + U)(I - U)^{-1}$$

is self-adjoint and $\mathcal{C}(T) = U$.i

Důkaz

„a) $\implies$“: Since $\sigma(T) \subseteq \mathbb{R}$, we have $\pm i \in \varrho(T)$, so $T \pm iI$ are onto, so $D(\mathcal{C}(T)) = H = R(\mathcal{C}(T))$ by the theorem above.

„a) $\impliedby$“: We have $D(T)^{\perp} = R(I - \mathcal{C}(T))^{\perp} = \mathrm{Ker}(I - \mathcal{C}(T))^* = \mathrm{Ker}(I - \mathcal{C}(T)) = \{\mathbf{o}\}$, co T is densely defined. Moreover, $T \pm iI$ is onto, so $Ii \in \varrho(T)$. Thus, from the corollary above, T is self-adjoint.

„b)“: $\mathcal{C}(T) = U$ by the previous theorem. Moreover $D(T)^{\perp} = R(I - U)^{\perp} = \ldots = \{\mathbf{o}\}$, so T is densely-defined. It remains „$T \pm iI$ is onto“: Fix $y \in H$, put $zi = (I - U)y$, then:

$$(T + iI)z = Tz + iz = i(I + U)y + i(I - U)y = 2iy,$$

$$(T - iI)z = Tz - iz = i(I + U)y - i(I - U)y = 2iUy.$$

So, (Since $D(U) = H = R(U)$), we have $T \pm iI$ is onto. □

Definice 8.2 (n_+ and n_i (deficiency indices))

Let T be a symmetric closed operator in a Hilbert space H. Then

$$n_+(T) = \dim(\mathrm{Rng}(T + iI))^{\perp} = \dim D(\mathcal{C}(T))^{\perp},$$

$$n_-(T) = \dim(\mathrm{Rng}(T - iI))^{\perp} = \dim \mathrm{Rng}(\mathcal{C}(T))^{\perp}$$

are called deficiency indices of the operator T.

Věta 8.4

T symmetric, densely defined, closed operator on separable (we prove it only for separable) H. Then

a) *T is self-adjoint $\Leftrightarrow n_+(T) = n_-(T) = 0$;*

b) *(T is maximal symmetric $\Leftrightarrow \min(n_+(T), n_-(T)) = 0$;)*

c) *T has self-adjoint extension $\Leftrightarrow n_+(T) = n_-(T)$.*

Důkaz
„a)“: T self-adjoint $\Leftrightarrow \mathcal{C}(T)$ is unitary $\Leftrightarrow D(\mathcal{C}(T)) = R(\mathcal{C}(T)) = H \overset{*}{\Leftrightarrow} n_+(T) = 0 = n_-(T)$.
*) T is closed, so $D(\mathcal{C}(T)) \neq H \Leftrightarrow n_+(T) > 0$ and $R(\mathcal{C}(T)) \neq 0 \Leftrightarrow n_-(T) > 0$ (from item d) from the theorem above).

„b)“ omitted.

„c) $\implies$ “: Let $S \supseteq T$ be self-adjoint. Then $\mathcal{C}(S) \supseteq \mathcal{C}(T)$ and $\mathcal{C}(S)$ is unitary and $\mathcal{C}(S)(D(\mathcal{C}(T))) = R(\mathcal{C}(T))$, $\mathcal{C}(S)(\ldots^\perp) = R(\mathcal{C}(T))^\perp$ (U unitary, $U(A) = B \overset{\text{easy}}{\implies} U(A^\perp) = B^\perp$). So,

$$n_+(T) = \dim D(\mathcal{C}(T))^\perp = \dim R(\mathcal{C}(T))^\perp = n_-(T)$$

Since H is separable, we have $n_+(T) = n_-(T) \Leftrightarrow \exists$ isometry between $D(\mathcal{C}(T))^\perp$ and $R(\mathcal{C}(T))^\perp$ (because Hilbert spaces are isometric to right l_2). Let $V \supseteq \mathcal{C}(T)$ is unitary operator such that $V(R(\mathcal{C}(T))^\perp) = R(\mathcal{C}(T))^\perp$.

Then „$R(I - V)$ is dense and $I - V$ is one-to one.“:

$$R(I - V) \supseteq R(I - \mathcal{C}(T)) = D(T),$$

so $R(I - V)$ is dense. Fix $x \in \operatorname{Ker}(I - V)$ and $y \in D(V)$. Then

$$\langle x, (I - V)y\rangle = \langle x, y\rangle - \langle x, Vy\rangle = \langle Vx, Vy\rangle - \langle x, Vy\rangle = \langle Vx - x, Vy\rangle = \langle \mathbf{o}, Vy\rangle = 0.$$

Thus, $x \in R(I - V)^\perp = \{\mathbf{o}\}$.

$\implies$ $\exists S$ symmetric and densely defined such that $\mathcal{C}(S) = V \supseteq \mathcal{C}(T)$, so $S \supseteq T$ ($S = i(I + V)(I - V)^{-1} \supseteq i(I + \mathcal{C}(T))(I - \mathcal{C}(T))^{-1} = T$). □

9 Integral of unbounded function with respect to a spectral measure

Definice 9.1

H Hilbert, $(X, \mathcal{A})$ is measurable space, E spectral measure for $(X, \mathcal{A}, H)$, E spectral measure for $(X, \mathcal{A}, H)$, $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable. Then $\int f dE$ is the operator on H such that

$$D\left(\int f dE\right) := \left\{x \in H \middle| \int |f|^2 dE_{x,x} < \infty\right\}, \qquad \langle Tx, y\rangle := \int_X f dE_{x,y}, \quad x, y \in D(T).$$

Věta 9.1

H Hilbert, $(X, \mathcal{A})$ is measurable space, E spectral measure for $(X, \mathcal{A}, H)$, E spectral measure for $(X, \mathcal{A}, H)$, $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable. Then $D := \left\{x \in H \middle| \int_X |f|^2 dE_{x,x} < \infty\right\}$ is dense subspace of H, $\int f dE$ exists (and it is unique).

Moreover, $\|Tx\|^2 = \int_X |f(\lambda)| dE_{x,x}$*,* $x \in D(\int f dE)$*.*

Důkaz
„D is subspace“: From proposition (basic properties of spectral measure) sixth item (addition) and fourth point (multiplication).

„For $A_n := f^{-1}(B(\mathbf{o}, n))$ we have $\operatorname{Rng} E(A_n) \subseteq D(\int f dE)$, $n \in \mathbb{N}$“: $\forall x \in \operatorname{Rng} E(A_n)$:

$$E_{x,x}(A_n) = \langle E(A_n)x, x\rangle = \langle x, x\rangle = \langle E(X)x, x\rangle = E_{x,x}(X).$$

So, $E_{x,x}(X \backslash A_n) = 0$, so $|f| \leqslant n$ $E_{x,x}$-almost everywhere, so

$$\int_X |f|^2 dE_{x,x} \leqslant n^2 \int_X 1 \cdot E_{x,x} < \infty.$$

„D is dense“: Pick $y \in H$, then $D \ni E(A_n)y \to y$ ($\|E(a_n)y - y\|^2 = \|E(X \backslash A_n)y\|^2 = E_{y,y}(X \backslash A_n) \to 0$.)

„$\forall x, y \in D : \int f dE_{x,y} \in \mathbb{C}$“: $(x, y) \mapsto E_{x,y}$ is sesquilinear, so it suffices to check it for $x = y$. But $f \in L^2(E_{x,x}) \subseteq L^1(E_{x,x})$, so $\int f dE_{x,x} \in \mathbb{C}$.

„Definition of T“: For $x \in D$ put $Tx := \lim_{n \to \infty} \left(\int_X f \chi_{A_n} dE\right) x$. „$T$ well defined“: limit exists, because the sequence is cauchy:

$$\forall m < n : \| \int f \chi_{A_n} dEx - \int f \chi_{A_m} dEx\|^2 = \| \int f \chi_{A_n \backslash A_m} dEx\|^2 = \int_{A_n \backslash A_m} |f|^2 dE_{x,x} \to 0.$$

„T linear“: easy (VAL + Linearity of the integral). „For T equation holds“: By sesquilinearity, suffices to check for $x = y \in D$:

$$\langle Tx, x\rangle = \lim \left\langle \int_X f \chi_{A_n} dEx, x \right\rangle = \lim \int f \chi_{a_n} dE_{x,x} \overset{\text{Lebesgue}}{=} \int \lim f \chi_{a_n} dE_{x,x} = \int f dE_{x,x}.$$

„$\|Tx\| = \sqrt{\ldots}$“:

$$\|Tx\|^2 = \lim \left\langle \int f \chi_{A_n} dEx, \int f \chi_{A_n} dEx \right\rangle = \lim \int |f \chi_{A_n}|^2 dE_{x,x} \overset{\text{Lebesgue}}{=} \int |f|^2 dE_{x,x}.$$

„Uniqueness“: $\langle Tx, y\rangle = \langle z, y\rangle$, $y \in D \implies Tx = z$ on H, because D is dense. □

Věta 9.2

Let H *Hilbert space,* $(X, \mathcal{A})$ *measurable space,* E *spectral measure for* $(X, \mathcal{A}, H)$ *and* $f, g : X \to \mathbb{C}$ *be* $\mathcal{A}$*-measurable functions. Then the following assertions hold:*

$\int f dE + \int g dE \subset \int f + g dE$*;*

Důkaz (Omitted. (From definition.)) □

$\left(\int fdE\right)\left(\int gdE\right) \subset \int fgdE$ *and* $D\left(\left(\int fdE\right)\left(\int gdE\right)\right) = D\left(\int gdE\right) \cap D\left(\int fgdE\right)$*;*

Důkaz (Omitted. (Technical, difficult, from definition of bounded version.)) □

$(\int fdE)^* = \int \overline{f}dE$ *and* $\int fdE\left(\int fdE\right)^* = \int |f|^2 dE = (\int fdE)^* \int fdE$*, that is,* $\int fdE$ *is normal;*

Důkaz (Omitted.) □

$\int fdE$ *is closed;*

Důkaz
From the previous item: $\int fdE = \int \overline{\overline{f}}dE = \left(\int \overline{f}dE\right)^* \implies$ (by the proposition above) $\int fdE$ is closed. □

$\int fdE \in \mathcal{L}(H) \Leftrightarrow \exists A \in \mathcal{A}$*:* $E(X \backslash A) = \mathbf{o} \wedge f$ *is bounded on* A*.*

Důkaz
„ $\Longleftarrow$ “: „$D(\int fdE) = H$“: $\forall x \in H : \int_X |f|^2 dE_{x,x} = \int_A |f|^2 dE_{x,x} < \infty$. „$\forall x \in H : \|\int fdEx\|^2 \leqslant C \cdot \|x\|^2$“: from the previous theorem:

$$\|\int fdEx\|^2 = \int_X |f|^2 dE_{x,x} = \int_A |f|^2 dE_{x,x} \leqslant \|f|_A\|_\infty \cdot E_{x,x}(X) \leqslant \|f|_A\|_\infty \cdot \|x\|^2.$$

„ $\Longrightarrow$ “: Put $K := \|\int |f|dE\| < \infty$, $A := \{t | |f(t)| \leqslant K+1\}$. Then „$E(X \backslash A) = 0$“: If not, $\exists x \in S_H \cap \operatorname{Rng} E(X \backslash A)$ and then

$$K+1 = \int (K+1) dE_{x,x} \leqslant \int_{A^c} |f| dE_{x,x} = \int |f| \chi_{A^c} dE_{x,x} = \left\langle \int \chi_{A^c} dE \int |f| dEx, x \right\rangle =$$

$$= \left\langle E(A^c) \cdot \int |f| dEx, x \right\rangle = \left\langle \int |f| dE_{x,x}, E(A^c)x \right\rangle = \left\langle \int |f| dEx, x \right\rangle \leqslant$$

$$\leqslant \left\| \int |f| dEx \right\| \cdot 1 \leqslant \left\| \int |f| dE \right\| \cdot 1 \cdot 1 = K.$$

□

Věta 9.3

Let H *be a Hilbert space,* $(X, \mathcal{A})$ *measurable space,* E *spectral measure for* $(X, \mathcal{A}, H)$ *and* $f : X \to \mathbb{C}$ *be* $\mathcal{A}$*-measurable function. Then*

$$\sigma\left(\int fdE\right) = ess \operatorname{Rng} f := \left\{\lambda \in \mathcal{C} | \forall r > 0 : E(f^{-1}(U(\lambda, r))) \neq 0\right\}.$$

Moreover, for $\lambda \in \mathbb{C}$ *we have* $\operatorname{Ker}(\lambda I - \int fdE) = \operatorname{Rng}(E(f^{-1}(\{\lambda\})))$*. Thus* $\lambda \in \sigma_P(\int TODO)$ *TODO.*

Lemma 9.4

H Hilbert, $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ measurable spaces, E spectral measure for $(X, \mathcal{A}, H)$, $\varphi : X \to Y$ measurable. Then $\varphi(E) : \mathcal{B} \to \mathcal{L}(H)$ defined by $\varphi(E)(B) := E(\varphi^{-1}(B))$, $B \in \mathcal{B}$ is spectral measure for $(X, \mathcal{B}, H)$ such that $\int g d\varphi(E) = \int g \circ \varphi dE$, $g : Y \to \mathbb{C}$ measurable. In particular if $Y \subseteq \mathbb{C}$, $\int \varphi dE = \int \mathrm{id}\, d\varphi(E)$.

Důkaz

„$\varphi(E)$ spectral measure“: Easy from definition.

Fix $g : Y \to \mathbb{C}$ measurable. Then „$D(\ldots) = D(\ldots)$“:

$$\forall x \in H : \int |g|^2 d\varphi(E)_{x,x} \overset{\triangle}{=} \int |g|^2 d\varphi(E_{x,y}) = \int |g \circ \varphi|^2 dE_{x,y}.$$

(„$\triangle$“: $\varphi(E)_{x,y} = \varphi(E_{x,y})$, because $\forall A$ measurable: $\varphi(E)_{x,y}(A) = \langle \varphi(E)(A)x, y\rangle = \langle E(\varphi^{-1}(A))x, y\rangle = E_{x,y}(\varphi^{-1}(A)) = \varphi(E_{x,y})(A)$.)

„$\int g d\varphi(E) = \int g \circ \varphi dE$“:

$$\forall x, y \in D(\int g d\varphi(E)) : \left\langle \int g d\varphi(E) x, y \right\rangle = \int g d\varphi(E)_{x,y} \overset{\triangle}{=} \int g d\varphi(E_{x,y}) =$$

$$= \int g \circ \varphi dE_{x,y} = \left\langle \int g \circ \varphi dE x, y \right\rangle.$$

„In particular“: We set $g = \mathrm{id}$. □

Věta 9.5

T is self-adjoint operator on $H \neq \{\mathbf{o}\}$ Hilbert. Then $\exists! E$ spectral measure for $(\mathbb{C}, Borel(\mathbb{C}), H)$ such that $T = \int \mathrm{id}\, dE$.

Moreover $E(\mathbb{C} \backslash \sigma(T)) = 0$.

Důkaz (Existence)
Assume $U := \mathcal{C}(T)$. Then $U \in \mathcal{L}(H)$ is unitary onto, $I-U$ is one-to-one, $T = i(I+U)(I-U)^{-1}$. Let F is spectral measure for $(\sigma(U), Borel(\sigma(U)), H)$ such that $U = \int \mathrm{id}\, dF$. Then $F(\{1\}) = 0$ (moreover part of the previous theorem). Let $F' = F|_{Borel(\sigma(U)\setminus\{1\})}$, then „$U = \int \mathrm{id}\, dF'$“:

$$\forall x, y \in H : \left\langle \int \mathrm{id}'\, dF' x, y \right\rangle = \int \mathrm{id}\, dF'_{x,y} = \int \mathrm{id}\, dF_{x,y} = \langle Ux, y \rangle .$$

Assume $\varphi : \sigma(U)\setminus\{1\} \to \mathbb{C}$, $\sigma(z) := i\frac{1+z}{1-z}$. Then $\operatorname{Rng}\varphi \subseteq \mathbb{R}$ ($\varphi(z) = i\frac{1+z}{1-z} \cdot \frac{1-\overline{z}}{1-\overline{z}} = i \cdot \frac{1-|z|^2+z-\overline{z}}{|1-z|^2} \overset{\sigma(U)\subseteq\{z,|z|=1\}}{=} -\frac{2\operatorname{Im} z}{|1-z|^2} \in \mathbb{R}$).

Put $E := \varphi(F')$, then $\int \mathrm{id}\, dE = \int \varphi dF'$. We want: $\int \varphi dF' = T$. Denote $S := \int \varphi dF'$. Then S is self-adjoint. Then, $\varphi(z)(1-z) = i(1+z)$, so

$$\left(\int \varphi dF'\right)\left(\int (1-z) dF'\right) = S(I-U)$$

$$LHS = \int i(1+z) dF' = i(I+U).$$

$$\left(\text{because } D((\int \varphi)(\int 1-z)) = D(1-z) \cap D(\int i(1+z)) = D(U) \cap D(U) = D(\int i(1+z))\right)$$

Thus $(D(S(I-U)) = D(I+U) = D(I-U))$ $D(S) \subseteq \operatorname{Rng}(I-U) = D(T)$. And $T = i(I+U)(I-U)^{-1} = S(I-U)(I-U)^{-1} = S|_{D(T)}$ $(R(I-U) = D(T))$. So, $T \subseteq S$. And because S and T are self-adjoint, so $S = S^* \subseteq T^* = T$. Thus $T = S$. □

Důkaz (Moreover)
„In general, $E'((ess \operatorname{Rng} id)^C) = 0$ (whenever E' spectral measure such that $\int \mathrm{id}\, dE' = T$)“: Choose $\lambda \notin ess \operatorname{Rng} \mathrm{id}$, then $\exists r > 0$: $E(U(\lambda, r)) = 0$. Then, from Lindelöf's property, exist λ_n, r_n such that $E(U(\lambda_n, r_n)) = 0$ and $\bigcup U(\lambda_n, r_n) \supseteq (ess \operatorname{Rng} \mathrm{id})^C$. Then $E((ess \operatorname{Rng} \mathrm{id})^C) = 0$ (countable union of zero sets).

So, from the theorem above, $E'(\sigma(T)^C) = E((ess \operatorname{Rng} \mathrm{id})^C) = 0$. □

Důkaz (Uniqueness)
Let E' be such that $T = \mathrm{id}\, dE'$. We know $E'(\sigma(T)^C) = 0$. Let $\psi := \frac{z-i}{z+i}$, $z \neq -i$, and $\psi := 0$, $z = i$, and put $U := \int \psi dE'$. Then „$U = \mathcal{C}(T)$“: We have $E'((ess\,\mathrm{Rng}\,\mathrm{id})^C) = 0$ and $|\psi(z)| = 1$ on $ess\,\mathrm{Rng}\,\mathrm{id} = \sigma(T) \subseteq \mathbb{R}$. Thus $U \in \mathcal{L}(H)$. Next $\psi(z)(z+i) = z - i \implies U(T + iI) = \left(\int \psi dE'\right)\left(\int (z+i)dE'\right) = \int (z-i)dE' = T - iI$. So $U = \mathcal{C}(T)$.

Next step: Choose $\tilde{\psi} : \mathbb{C} \to \sigma(\mathcal{C}(T))$ measurable function such that $\tilde{\psi} = \psi$ on $\psi^{-1}(\sigma(\mathcal{C}(T)))$. Then whenever E', E'' are spectral measures from ?, then $\tilde{\psi}(E') = \tilde{\varphi}(E'')$. Then it suffices „$\int \mathrm{id}\, d\tilde{\psi}(E') = \mathcal{C}(T) = U$“: We have $U = \int \psi dE' = \int \mathrm{id}\, d\psi(E')$, so $E'(\psi^{-1}(\sigma(U)^C))\psi(E')(\sigma(U)^C) = 0$.

Then $E'(A) = ((\varphi \circ \psi)(E))(A) = E'(\psi^{-1}\varphi^{-1}(A)) = E'(\tilde{\psi}^{-1}\tilde{\varphi}^{-1}(A) \cap \sigma(T)) = = E''(\tilde{\psi}^{-1}\tilde{\varphi}^{-1}(A) \cap \sigma(T)) = \ldots = E''(A)$. □

Důsledek
Let T be self-adjoint operator on a Hilbert space. Then T is continuous iff $\sigma(T)$ is bounded.

Důkaz
„ $\implies$ “: We already know $\sigma(T) \subset B(0, \|T\|)$. „ $\impliedby$ “: We have $T = \int \mathrm{id}\, dE$ for E spectral measure for $(\mathbb{C}, Borel(\mathbb{C}), H)$ and $E(\sigma(T)^C) = 0$. Thus id is ”E-almost everywhere” bounded. □