1 Dynamické systémy

Definice 1.1 (Dynamický systém)

(φ, Ω), $\Omega \subset \mathbb{R}^n$ otevřená, $\varphi : \mathbb{R} \times \Omega \to \Omega$ $\varphi(t, x)$.

- $\varphi(0, x) = x$;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- φ je spojité.

Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geqslant 0\}$ je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leqslant 0\}$ je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$ je plný orbit.

Definice 1.3 (Pozitivně, negativně a úplně invariantní)

(φ, Ω) dynamický systém, $M \subset \Omega$.

M je pozitivně invariantní $\equiv \forall x \in M : \gamma^+(x) \subset M$.

M je negativně invariantní $\equiv \forall x \in M : \gamma^-(x) \subset M$.

M je úplně invariantní $\equiv \forall x \in M : \gamma(x) \subset M$.

Poznámka
$\gamma^+(x_0)$ je pozitivně invariantní, $\gamma^-(x_0)$ je negativně invariantní a $\gamma(x_0)$ je úplně invariantní.

Definice 1.4

$$\omega(x_0) = \left\{y \in \Omega | \exists \left\{t_k\right\}_{k=1}^{\infty}, t_k \to \infty : \varphi(t_k, x_0) \to y\right\},$$
$$\alpha(x_0) = \left\{y \in \Omega | \exists \left\{t_k\right\}_{k=1}^{\infty}, t_k \to -\infty : \varphi(t_k, x_0) \to y\right\}.$$

Poznámka (To je ekvivalentní)
$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0\ \forall T > 0\ \exists t \geqslant T : |\varphi(t_r, x_0) - y| < \varepsilon\}$.

Lemma 1.1

$\omega(x_0) = \bigcap_{\tau \geqslant 0} \overline{\gamma^+(\tau, x_0)}$.

Důkaz
„⊆“: $y \in \omega(x_0)$: $\forall \varepsilon > 0\ \forall T\ \exists t \geqslant T : |\varphi(t, x_0) - y| < \varepsilon$. Chceme:

$$\forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists s \geqslant \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

„⊇“: $\forall \tau \geqslant 0\ y \in \overline{\gamma^+(\tau, x_0)} \implies$

$$\implies \forall \varepsilon\ \exists s \geqslant \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

□

Věta 1.2 (Vlastnosti ω-limitní množiny)

Nechť (φ, Ω) je dynamický systém, $x_0 \in \Omega$. Potom

1. *$\omega(x_0)$ je uzavřená, úplně invariantní.*
2. *Pokud $\gamma^+(x_0)$ je relativně kompaktní v $\mathbb{R}^n$, pak $\omega(x_0) \neq \emptyset$, $\omega(x_0)$ je kompaktní, souvislá.*

Důkaz
1. $\omega(x_0)$ je průnik uzavřených množin, tedy uzavřená. $y \in \omega(x_0)\ \exists t_k \nearrow \infty\ \varphi(t_k, x_0) \to y$.

$$\begin{aligned} s_k = t_k + t \qquad & \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0)) \\ t_k \to \infty, \varphi\text{spojitá} \qquad & \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \to \varphi(t, y) \end{aligned}$$

2. $\exists K \subset \mathbb{R}^n$ kompaktní $\gamma^+(x_0) \subset K$. a) pokud $t_n \geqslant 0, t_n \to \infty\ \{\varphi(t_n, x_0)\}_{n=1}^{\infty}$ omezená posloupnost $\implies \exists \{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, podposloupnost, $\exists y \in \Omega \varphi(t_{n_k}, x_0) \to y$. Pak $y \in \omega(x_0)$.

b) $\omega(x_0)$ je tedy úplná a omezená, takže kompaktní. c) ať $\omega(x_0)$ je nesouvislá, tedy $\omega(x_0) \subseteq U \cup V$, U, V otevřené disjunktní neprázdné, $U, V \subseteq K$. Vezměme $y \in \omega(x_0) \cap U$, $z \in \omega(x_0) \cap V$. Nechť t_n je posloupnost taková, že $\varphi(t_{2n} x_0) \to y$, $\varphi(t_{2n+1}, x_0) \to z$, $t_{2n} < t_{2n+1}$, $\varphi(t_{2n}, x_0) \in U$, $\varphi(t_{2n+1}, x_0) \in V$. $F = K \backslash (U \cup V)$ uzavřená, tedy $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$. Tedy $\{\varphi(s_n, x_0)\}$ je omezená posloupnost $\implies \exists$ podposloupnost konvergující k $w \in F$. □

Definice 1.5 (Topologická konjugovanost)

(φ, Ω), ψ, Θ dynamické systémy. $\exists : \Omega \to \Theta$ homeomorfismus (bijekce, spojité, spojitá inverze):

$$\forall x \in \Omega\ \forall t \in \mathbb{R} \qquad h(\varphi(t, x)) = \psi(t, h(x)).$$

Poznámka
Dá se zobecnit ještě zobrazováním časů.

Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0$, (φ, Ω) *příslušný dynamický systém.* $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $y(0) = 0$ *a* (ψ, Θ) *je příslušný dynamický systém. Potom* (φ, Ω), (ψ, Θ) *jsou lokálně topologicky konjugované (*$\exists U$ *okolí* $x_0 \in \Omega$ *a* V *okolí* $\mathbf{o} \in \mathbb{R}^n$ *taková, že* $\exists g : U \to V$ *homeomorfismus* $g(\varphi(t, x)) = \psi(t, g(x))$ $\forall x \in U$, $\forall t : \varphi(t, x) \in U$*).*

Důkaz
BÚNO $f_1(x_0) = \alpha \neq 0$ (první souřadnice funkce f) a $x_0 = \mathbf{o}$. Buď $\tilde{V}$ okolí $\mathbf{o} \in \mathbb{R}^n$ $G : \tilde{V} \to \mathbb{R}^n$, $G(y_1, \ldots, y_n) = \varphi(y, (0, y_2, \ldots, y_n))$. Chceme ukázat, že G je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \ldots, y_n)}{\partial y_1}|_{(0,\ldots,0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \ldots, y_n))|_{y_1=0,\ldots,y_n=0} = f(\varphi(y_1(0, y_2, \ldots, y_n)))|_{y_1=0,\ldots,y_n=0} = f(\varphi$$

$$\frac{\partial G(y_1, \ldots, y_n)}{\partial y_j}|_{(0,\ldots,0)} = \lim_{h\to 0} \frac{G(0, \ldots, h, \ldots, 0) - G(0, \ldots, 0)}{h} = \lim_{h\to 0} \frac{(0, \ldots, h, \ldots, 0)^T - (0, \ldots, 0)^T}{h} = (0$$

Tedy $\nabla G(0, \ldots, 0)$ je „jednotková matice, až na to, že a_{11} je α“, tudíž podle věty o inverzi funkce $\exists V \subseteq \tilde{V}$ okolí 0, $\exists U$ okolí bodu x_0 tak, že $G : V \to U$ je homeomorfismus. Položme $g = G^{-1}$.

Nyní už stačí $g(\varphi(t, x_0)) = \psi(t, g(x_0))$ $\forall x_0 \in U$ $\forall t : \varphi(t, x_0) \in U$. $\varphi(t, x_0) = G(\psi(t, g(x_0)))$

3. $x \in U = G(V)$ $\exists y \in V$ $x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))$$

□

Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \to \Omega, floc.Lip.$$

$$\exists V : \Omega \to \mathbb{R}, \text{ bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leqslant l\} - -bounded$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leqslant 0 \qquad \forall x \in \Omega_l.$$

$$R = \left\{ x \in \Omega_l | \dot{V}_f(x) = 0 \right\}, \quad M = \left\{ y \in R | \gamma^+(y) \subset R \right\}.$$

Then $\forall x \in \Omega_l : \omega(x) \subset M$.

Důkaz

Let $x \in \Omega_l$. $\forall y \in \omega(x)\ \exists t_k \nearrow \infty : x(t_k) \to y$. $\varphi(t, x_0) = x(t)$.

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leqslant 0.$$

$V(x(t)) \searrow$ and $\exists C : \forall x \in \Omega : V(x) > -C$ so $\exists \lim_{t\to\infty} V(x(t)) = c$.

So $\exists c\ \forall y \in \omega(x_0) V(y) = c$. $V(x(t_k)) \to V(y) = c$.

$$\gamma^+(y) \subset \omega(x_0)\ V(\varphi(t, y)) = c\ \forall t \geqslant 0 \implies$$

$$\implies \frac{d}{dt} V(\varphi(t, y)) = 0.$$

$\gamma^+(y) \subset R$ in particular, $y \in R$. Hence $y \in M$. □

2 Poincaré-Bendixson theory

Věta 2.1 (Poincaré-Bendixson)

Let $p \in \Omega$, Ω *open connected.* $\omega(p)$ *doesn't contain stat points and* $\gamma^+(p)$ *is relatively compact* ($\overline{\gamma^+(p)}$ *is compact). Then* $\omega(p) = \Gamma$*-periodic orbit.*

Věta 2.2 (Bendixson-Dulas)

Ω*-simply connected (*$\forall$ *closed Jordan curve* γ *in* Ω, $\operatorname{int}(\gamma) \subset \Omega$*).* $\exists B : \Omega \to \mathbb{R} : (\operatorname{div} Bf)(x) = \frac{\partial Bf_1}{\partial x_1}(x_1, x_2) + \frac{\partial Bf_2}{\partial x_2}(x_1, x_2) > 0$ *for almost every* $x \in \Omega$. *Then* $x' = f(x)$ *doesn't have nontrivial periodic solutions.*

Definice 2.1 (Transverzála)

Σ segment on a line such that $\forall p \in \Sigma : \Sigma \nparallel f(p)$.

Lemma 2.3

Σ *transverzála,* $p \in \Sigma \subset \Omega$. *Then* $\exists \tilde{\subset} U$ *neighborhood of* p. $\exists \Delta > 0$ *such that*

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U\ \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

Důkaz
Use Th. of rect. □

Lemma 2.4

Let $p \in \Omega$ and assume that $|\gamma^+(p) \cap \Sigma| \geqslant 3$, i. e. $\exists t_1 < t_2 < t_3$ $\varphi(t_j, p) \in \Sigma$, $j = 1, 2, 3$. Then $\varphi(t_2, p)$ lie between $\varphi(t, p)$ and $\varphi(t_3, p)$.

TODO!!!

TODO!!!

2.1 Controllability

Definice 2.2 (Control theory)

$$x' = f(x, u), f : \Omega \times U, \Omega \subset \mathbb{R}^n, U \subset \mathbb{R}^n,$$

$$u \in \mathcal{U} := \{u : [0, T] \to \mathbb{R}^n | \text{measurable}, ||u||_\infty < \infty\} = L^\infty(0, T, \mathbb{R}^n).$$

($\mathcal{U}$ is admissible functions).

Definice 2.3 (Linear task)

$x' = Ax + Bu$, $A, B \in \mathbb{R}^{n \times m}$, $m < n$.

Definice 2.4

$x_0 \underset{u(0)}{\overset{t}{\to}} 0$ iff $x(0) = x_0$, $x(t) = 0$.

Definice 2.5 (Area of controlability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in L^\infty(0, t, \mathbb{R}^n) : x_0 \underset{u(0)}{\overset{t}{\to}} 0 \right\}$$

Definice 2.6 (Kalman matrix)

$$\mathcal{K}(A, B) := (B|AB|A^2B| \dots |A^{n-1}B)$$

Věta 2.5

For linear problem $\mathcal{R}(t) = \mathrm{LO}(g_1, g_2, \dots, g_{n \cdot m})$, where $\mathcal{K}(A, B) = (g_1|g_2| \dots |g_{n \cdot m})$

Tvrzení 2.6 (Observation)

$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)Bu(s)ds}$.

$$x_0 \xrightarrow[u(0)]{t} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = -\int_0^t e^{-As}Bu(s)ds \qquad (KO)$$

Lemma 2.7 (1)

$$A^k \in \mathrm{LO}(I, A, A^2, \dots, A^{n-1}), k \in \mathbb{N}_0$$

Důkaz

Cayley-Hamilton. □

Důkaz

1) $\mathcal{R}(t)$ is vector subspace of $\mathbb{R}^n$ from definition $x_0 + x_1 \xrightarrow[(u_1+u_2)(0)]{t} 0$, $\alpha x \xrightarrow[\alpha u(0)]{t} 0$.

2) We want $\mathcal{R}(t)^\perp = (\mathrm{LO}(g_1, \dots, g_n))^\perp$. „$\supseteq$“: $p \in (\mathrm{LO}(g_1, \dots, g_n))^\perp$. $x_0 \in \mathcal{R}(t)$ arbitrary. From KO:

$$0 \stackrel{?}{=} p^T x_0 = -\int_0^t p^T e^{-As}Bu(s)ds = -\int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k Bu(s)ds$$

We know $(p, g_j) = 0$, $p^T g_j = 0$, $p^T \mathcal{K}(A, B) = 0$, $p^T A^k B = 0$, $k \in [n-1]$. And from lemma 1 $k \in \mathbb{N}$. „$\subseteq$“: $p \in \mathbb{R}^n$, $p \in \mathcal{R}(t)^\perp$. We want to prove $p \perp B, AB, A^2B, \dots, A^{n-1}B$. $B = (b_1|\dots|b_m)$. $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$. $\varphi \in L^\infty(0, T, \mathbb{R})$, $u(t) = \varphi(t) \cdot \mathbf{e}_j$, where $x_0 = -\int_0^t e^{-As}Bu(s)ds$. We have $x_0 \xrightarrow[u(0)]{t} 0$, hence $x_0 \in \mathcal{R}(t)$.

$$0 = p^T x_0 = -p^T \int_0^t e^{-As}Bu(s)ds = -\int_0^t p^T e^{-As} b_j \varphi(s)ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have $p^T e^{-As} b_j \equiv 0$, we derivate it, $p^T A^n e^{-As} b_j \equiv 0$, and set $s = 0$. □

Důsledek

$\mathcal{R}(t)$ doesn't depend on time.

Definice 2.7 (Locally and globally controllable)

Linear problem is called locally controllable, iff $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 | \, |x_0| < \delta\} \subset \mathcal{R}(t)$. And globally if $\mathbb{R}^n = \mathcal{R}(t)$.

Důsledek
Linear problem is controllable $\Leftrightarrow$ rank $K(A,B)=n$.

2.2 Observability

Definice 2.8 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n y = g(x), g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, m < n.$$

Definice 2.9

We say that system $x' = f(x)$ is observable through $g(\cdot)$ on $[0,t]$, iff $\forall x_1(\cdot), x_2(\cdot) : [0,T] \to \mathbb{R}^n : g(x_1(t)) = g(x_2(t))\ \forall t \in [0,T] \implies x_1(0) = x_2(0)$.

Definice 2.10 (Linear observability)

$x' = Ax$, $y = Bx$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$.

Věta 2.8

$x' = Ax$ is observable on $[0,T]$ through $y = Bx \Leftrightarrow x' = A^T x + B^T u$ is controllable.

Důkaz
(We will prove equivalence with rank $\mathcal{K}(A^T, B^T) = n$.) „ $\Longleftarrow$ “: For contradiction

$$\exists x_1(t), x_2(t), [0,T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At}x_0, Bx(t) = B^{At}x_0 \equiv 0 \qquad \forall t \in [0,T].$$

We differentiate it, set $t = 0$ and get $Bx_0 = 0$, $BAx_0 = 0$, ..., $BA^{n-1}x_0 = 0$. So $x_0^T B^T = 0$, ..., $x_0^T \left(A^T\right)^{n-1} B^T = 0$. $x_0^T \mathcal{K}(A^T, B^T) = 0$, $x_0 \perp \mathcal{K}(A^T, B^T)$, ↯.

„ $\Longrightarrow$ “: For contradiction rank$(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T \mathcal{K}(A^T, B^T) = 0$. $x_0^T \left(A^T\right)^k B^T = 0\ \forall k \in [n-1]$ and from lemma 1 $\forall k \in \mathbb{N}$. $BA^T x_0 = 0$. $Be^{At}x_0 = 0$ $\forall t \in [0,T]$. ↯. □

Věta 2.9

$V \subset \mathbb{R}^n$ neighbourhood of 0, $U \subset \mathbb{R}^n$ neighbourhood of 0, $f : V \times U \to \mathbb{R}^n$ C^1 smooth, $f(0,0) = 0, \mathcal{U} = \{u : [0,T] \to U$ measurable$\}$, $A = \nabla_x f(0,0)$, $B = \nabla_u f(0,0)$, rank $\mathcal{K}(A,B) = n$. Then

$$x' = f(x,u), x(0) = x_0 \text{ is locally controllable } \forall t \in (0,T].$$

Důkaz

Fix $t > 0$, consider $x' = Ax + Bu$. Since $\operatorname{rank}(A, B) = n$, the linear problem is globally controllable. Take initial condition $y_1, \dots, y_n$ linearly independent.

$$\exists u_i \in L^\infty(0, t, \mathbb{R}^n) : y_j \to^t_{u_\lambda(0)} 0$$

$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ denote by $u_{\lambda(t)} = \sum_{j=1}^n \lambda_j u_j(t)$. We know $\sum_{j=1}^n \lambda_j y_j \to^t_{u_\lambda(0)} 0$.

Step 2:

$$x'_\lambda = f(x_\lambda, u_\lambda), \qquad x_\lambda(t) = 0$$

If $\lambda = 0$, then $u_\lambda \pm 0$, then $x_\lambda \equiv 0$.

$$\psi(\lambda) := x_\lambda(0), \psi : U_\lambda(0) \subset \mathbb{R}^n \to \mathbb{R}^n.$$

We want to prove $\psi(U_\lambda(0)) \supseteq \tilde{V}$, for some $\tilde{V} \subset \mathbb{R}^n$ open, $0 \in \tilde{V}$. We will prove that ψ is C^1 smooth, and that $\nabla\varphi(0)$ is regular (if this is proved, than ψ is local diffeomorphism).

Step 3:

$$x_\lambda(s) = x_\lambda(t) + \int_t^s f(x_\lambda(s), u_\lambda(s)) ds.$$

Formally differentiate:

$$\frac{\partial x_\lambda(s)}{\partial \lambda_j} = \int_t^s (\nabla_x f(x_lambda(s), u_\lambda(s)) \cdot \frac{\partial x_\lambda(s)}{\partial \lambda_j} + \nabla_u f(x_\lambda(s), u_j(s))) ds.$$

Denote $y_{\lambda,j}(s) = \frac{\partial x_\lambda(s)}{\partial \lambda_j}$.

$$y'_{\lambda,j}(s) = \nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot y_{\lambda,j}(s) + \nabla_u f(x_\lambda(s), u_\lambda(s)) \cdot u_j(s).$$

$$y_{\lambda,j}(t) = 0.$$

Consider $(LPy) \to y_{\lambda,j}(\cdot)$.

$$x_{\lambda+\Delta\lambda}(s) - x_\lambda(s) - \Delta\lambda \cdot y_{\lambda,j}(s) = 0$$

(as in Thn? of differentiability w. r. t. initial condition)

$$\frac{\partial \psi}{\partial \lambda_j}(\lambda = 0) = \frac{\partial x_\lambda(s = 0)}{\partial \lambda_j}|_{\lambda=0} = y_{\lambda,j}(s = 0)|_{\lambda=0} = y_{\lambda,j}(s = 0)|_{\lambda=0} = y_j.$$

If $\lambda = 0$, then (LPy): $y'_{0,j}(s) = Ay_{0,t}(s) + Bu_j(s)$, $y_{0,j}(t) = 0$. From uniq.: $y_{0,j}(0) = y_{j,0}$.

$$\nabla\psi(0) = \left(\frac{\partial \psi}{\partial \lambda_1}(0) \dots \frac{\partial \psi}{\partial \lambda_n}(0) \right) = (y_1, \dots, y_n)$$

regular matrix. □

Poznámka

$$x' = Ax + Bu, u \in \mathcal{U} = \{u : [0,T] \to [-1,1] \text{ measurable}\}, x(0) = x_0.$$

Definice 2.11

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U} \wedge x_0 \to^t_{u(0)} 0 \right\}.$$

Definice 2.12

$u_n \in \mathcal{U}_0$: $u_n \rightharpoonup^* u \in \mathcal{U} \equiv \forall f \in L(0,T,\mathbb{R}^n) : \int_0^T f(s) u_n(s) ds \to \int_0^T f(s) u^*(s) ds.$

Věta 2.10 (Alaoglu)

$\mathcal{U}$ is weak- sequentially compact (i. e. $\forall \{u_n\}_{n=1}^\infty \in \mathcal{U}\ \exists \{u_{n_k}\}$ weekly-* convergent).*

Věta 2.11

$\mathcal{R}(t)$ convex, symmetric, closed

$$0 < t_1 < t_2 \implies \mathcal{R}(t_1) \subset \mathcal{R}(t_2).$$

Důkaz

Convex: $x_{01}, x_{02} \in \mathcal{H}(t)$, $\alpha \in [0,1] \implies \alpha x_{01} + (1-\alpha) x_{02} \in \mathcal{R}(t)$.

$$x(t) = e^{At} x_0 + \int_0^t e^{As} Bu(s) ds. x_{01} \to^t_u 0 \wedge x_{02} \to^t_u 0 \Leftrightarrow x_1 = -\int_0^t e^{(s-t)A} Bu_1(s) ds.$$

Symmetry: $x_0 \in \mathcal{R}(t) \implies -x_0 \in \mathcal{R}(t)$, $x_0 \to^t_u 0 \implies -x_0 \to^t_u 0$.

Closedness: $x_{0n} \in \mathbb{R}(t), x_{0n} \to x_0$. $x_0 \in \mathcal{R}(t)$? $\exists u_n(0) \in \mathcal{U}$, $x_{0n} = -\int_0^t e^{(s-t)A} Bu_n(s) ds \to -\int_0^t e^{(s-t)A} Bu(s) ds$. WLOG $u_n \rightharpoonup^* u \in \mathcal{U}$. Then $x_0 \to^t_u 0$.

$$\mathcal{R}(t_1) \subset \mathcal{R}(t_2) 0 < t_1 < t_2 < T x_0$$

$$\exists u_1 \in \mathcal{U} \qquad x_0 = -\int_0^t e^{(s-t)A} Bu_1(s) ds.$$

Define $u_2(s) = u_1(s)$ if $0 \leqslant s \leqslant t$, else 0. □

Definice 2.13 (Area of controllability)

$$\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t).$$

Věta 2.12

$$\operatorname{rank}\mathcal{K}(A,B) - n \Leftrightarrow \forall t > 0 \mathcal{R}(t) \supseteq U(0),$$

where $U(0) \subset \mathbb{R}^n$ is some neighbourhood of 0.

Důkaz

„ $\Longleftarrow$ “: If $\exists t > 0$ $\mathcal{R}(t) \supset U(0)$ open, $0 \in U(0)$. $\tilde{\mathcal{R}}$: $u \in L^\infty, \mathcal{R}$: $||u||_\infty \leqslant 1$, then $\tilde{\mathcal{R}}(t) \supset \mathcal{R}(t) \supset U(0) \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n$. $\implies \operatorname{rank}\mathcal{K}(A,B) = n$.

„ $\implies$ “: $\operatorname{rank}(A,B) = n \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n$. TODO? □

Věta 2.13 (Minimal time)

$$x' = Ax + Bu$$
$$\forall x_0 \in \mathcal{R} = \bigcup_{s>0} \mathcal{R}(s)$$
$$\exists t > 0 \ \exists u(0) \in \mathcal{U} : x_0 \to_u^t 0$$
$$t = \inf\{s > 0 | x_0 \in \mathcal{R}(s)\}.$$

Důkaz

$$t > 0, \exists t_n \searrow t, t_n \in (0,T], \exists u_k \in U, x_0 = -\int_0^{t_n} e^{(t_n-s)A} V u_n(s).$$

Alaoglu: WLOG $u_n \xrightharpoonup{*} u \in U$.

$$x_0 = -\int_0^{t_n} e^{(t-s)A} B u_n(s) ds - \int_0^{t_n} \left[e^{(t-s)A} - e^{(t_n-s)A}\right] B u_n(s) ds$$

$$x_0 = -\underbrace{\int_0^t e^{(t-s)A} B u_n(s) ds}_{\xrightharpoonup{*} \int_0^t e^{(t-s)A} B u(s) ds} - \underbrace{\int_t^{t_n} e^{(t-s)A} B u_n(s) ds}_{\to 0} - \underbrace{\int_0^T \left[e^{(t-s)A} - e^{(t_n-s)A}\right] B u_n(s) ds}_{\to 0 (DLCT)}.$$

□

Definice 2.14 (Bang-bang)

We say that a regulation $u \in U(0)$ is of type bang-bang, if for almost every $t \in [0,T]$: $u(t) = \pm 1$.

Věta 2.14 (Bang-bang)

If $x_0 \in \mathcal{R}(t) \implies \exists \tilde{u}(0)$ of type bang-bang $x_0 \to_{\tilde{u}}^t 0$.

Definice 2.15 (Extremal point)

X vector space, $K \subset X$. $x \in K$ is called an extremal point, if it cannot be written as $x = \frac{y+z}{2}$, $y, z \in K$, $y \neq z$. We denote $ex(K)$ the set of extremal points.

Tvrzení 2.15 (Krein-Milman theorem)

X locally convex vector space, $K \subset X$: $K \neq \emptyset$, K convex and compact. Then $ex(K) \cap K \neq \emptyset$.

Důkaz (Bang-bang)

$$K = \left\{u \in \mathcal{U} | x_0 \to^t_{u(0)} 0\right\}, \qquad X = L^\infty(0, T, \mathbb{R}^n).$$

$K \neq \emptyset$ ($u \in \mathcal{R}(t)$), K convex, K is compact (sequential compactness: Alaoglu theorem? $L'(0, T, \mathbb{R}^n)$ separable $\implies$ $L^\infty(0, T, \mathbb{R}^n)$ with locale $*$ topology is metrizable $\implies$ sequential compactness $\implies$ compactness.

It remains to check that $\tilde{u}_j(s) = \pm 1$, $\forall j \in [n]$ for almost every $s \in (0, t)$. For contradiction: for some $j \in [n]$ $\exists E \subset (0, t)$, $\lambda(E) > 0$ $\forall s \in E$ $|\tilde{u}_j(s)| < 1$. WLOG

$$\exists \varepsilon > 0 \ \forall s \in E |\tilde{u}_j(s)| < 1 - \varepsilon \cdot \left[E = \bigcup_{n \in \mathbb{N}} \left\{s \in (0, t) | |\tilde{u}_j(s)| \leqslant 1 - \frac{1}{n}\right\}\right].$$

$$x_0 = -\int_0^t e^{-sA} B \tilde{u}(s) ds$$

We want to find $\varphi \in L^\infty(0, T, \mathbb{R})$ such that:

1. $\operatorname{supp} \varphi \subset E$;
2. $\int_E e^{-sA} B(0, \ldots, 0, \varphi(s), 0, \ldots, 0)^T ds = 0$;
3. $\forall s \in E |\varphi(s)| < \varepsilon$.

Define $u_1(s) = \tilde{u}(s) + (0, \ldots, 0, \varphi(s), 0, \ldots, 0)^T$ and $u_2(s) = \tilde{u}(s) - (0, \ldots, 0, \varphi(s), 0, \ldots, 0)^T$. Then $x_0 \to^t_{u_{1,2}(0)} 0$, and $u_1, u_2 \in \mathcal{K}$. □

Věta 2.16 (Global controlability)

We have (LTP) $x' = Ax + Bu$, $x(0) = x_0$, $u \in \mathcal{U}$.

1. $\operatorname{rank} \mathcal{K}(A, B) = n \implies$ *(LTP) is locally controllable.*

2. $\operatorname{rank} \mathcal{K}(A, B) = n$ *and* $\Re\lambda \leqslant 0$ $\forall \lambda$*-eigenvalues of A. Then (LTP) is globally controllable* $\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t) = \mathbb{R}^n$.

Důkaz
1) folows from „In theorem of local controllability for the problem $x' = f(x,u)$ we could take $u \in \mathcal{U}$."

2a) If $\forall\lambda$ eigenvalue of A we have $\Re\lambda < 0 \implies$ theorem follows from text above: first, set $u = 0$. Then we arrive at a neighbourhood of zero.

2b) For contradiction $x_0 \in \mathbb{R}^n \backslash \mathcal{R}$. $\mathcal{R}$ convex $\exists z_0 \in \partial\mathcal{R}$, n normal vector. $\forall x_1 \in \mathcal{R}$: $n^T(x_1 - x_0) \leqslant 0$, $n^T x_1 \leqslant n^T x_0 =: M$.

$$x_1 = -\int_0^t e^{-sA} Bu(s) ds$$

$$n^T x_1 = -\int_0^t \underbrace{n^T e^{-sA} B}_{v(s)} u(s) ds$$

$$\tilde{u}(s) := \begin{cases} 0, & v(s) = 0, \\ \frac{-v(s)}{||v(s)||_2}, & v(s) \neq 0. \end{cases}$$

If $v(s) \equiv -$, then apply $\frac{d^p}{(ds)^p}$, $n^T A^p e^{-sA} B \equiv 0$, then $n^T \mathcal{K}(A,B) = 0$. ↯

$$\int_0^\infty ||v(s)||_2 ds = \infty.$$

If this is true, then $t_k \nearrow \infty$, $u_k = \tilde{u}|_{[0,t_k]}$, $x_{1,k} = -\int_0^{t_k} e^{-sA} Bu_k(s) ds$.

$$n^T x_{1,k} = -\int_0^{t_k} v^t(s) \cdot \tilde{u}(s) ds = \int_0^{t_k} ||v(s)||_2 ds \to \infty. ↯$$

$v(s)$ is linear combination of $s^j e^{-s\lambda_p}, \Re\lambda_p \leqslant 0$. Then $\int_0^\infty |v(s)| ds = \infty$. □

Věta 2.17 (Pontrjagin maximum)

$$x' = Ax + Bu, ||u||_\infty \leqslant 1, x(0) = x_0.$$

Let $x_0 \to^{t^*}_{u^*(0)} 0$, t^* *is the minimal. Then* $\exists h \in \mathbb{R}^n \backslash \{\mathbf{o}\} : h^T \cdot e^{-sA} Bu^*(s) = \max_{\eta \in [-1,1]^m h^t e^{-sA} B\eta}$ *for almost every* $s \in (0, t^*)$.

Důkaz
$x_0 \in \partial\mathcal{R}(t^*)$.

Step 2 – contradiction: $\exists E \subset (0,t^*), \lambda(E) > 0$, $\forall s \in E$ $\exists \eta_s \in [-1,1]^m$ $h^T e^{-sA} B u^*(s) < h^T e^{-sA} B \eta_s$. But $x_j(\delta) \in \mathcal{R}(t^* - \delta)$, hence $x_0 \in \mathcal{R}(t^* - \delta)$ and t^* is not minimal.

Step 1: $x_0 \in \partial\mathcal{R}(t^*)$. For contradiction $x_0 \in \operatorname{int}\mathcal{R}(t^*)$.

$$\exists x_1, \dots, x_{n+1} \in \mathcal{R}(t^*), x_0 \in CO(x_1, \dots, x_{n+1}).$$

$$\exists u_1, \dots, u_{n+1} \in U, x_j \to^{t^*}_{u_j(\cdot)} 0 \ \forall j \in [n+1].$$

Let $\tilde{u}_j(t)$ are the corresponding solutions

TODO!!! □

Věta 2.18 (Pontrjagin)

$x'(f,u)$, $x(0) = x_0$, $u \in \mathcal{U} = \{u : (0,T) \to U \subset \mathbb{R}^n\}$, T fixed,

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s))ds \to \textit{maximum}.$$

$f, g, r, \nabla_x f, \nabla_x g, \nabla_x r$ are continuous.

Let u is a local maximum of this problem (it maximizes P), then for p solving ($H(x,p,u) := p^T f(x,u) + r(x,u)$):

$$p' = -\nabla_x H(x,p,u),$$
$$p(T) = \nabla_x g(x(T)),$$

we have

$$H(x,p,u) = \max_{\eta \in U} H(x,p,\eta) \textit{ for almost every } t \in (0,T).$$

Důkaz
Step one „WLOG $r = 0$": We set

$$x' = f(x,u), \qquad x'_{n+1} = r(x,u), x_{n+1}(0) = 0, P[u(\cdot)] = \hat{g}(\hat{x}(T)) = g(x(T)) + x_{n+1}(T).$$

Step 2: Fix $\tau \in (0,T)$, $\eta \in U$, $u_\varepsilon(T) = \begin{cases} \eta, & t \in (\tau - \varepsilon, \tau), \\ u(t), & \text{elsewhere,} \end{cases}$ and corresponding $x_\varepsilon(t)$.

$$u \text{ "best"} \implies P[u_\varepsilon(0)] \leqslant P[u(0)] \implies g(x_\varepsilon(T)) \leqslant g(x(t)).$$

$$D = \frac{d}{d\varepsilon}|_{\varepsilon=0^+} Dg(x_\varepsilon(T))|_{\varepsilon=0^+} \leqslant 0$$
$$\nabla_x g(x(T)) \cdot Dx_\varepsilon(T)|_{\varepsilon=0^+} \leqslant 0.$$

Step 2.2: $x_\varepsilon(t) = x_0 + \int_0^t f(x_\varepsilon(s), u_\varepsilon(s))ds$. If $t < \tau$, then $u_\varepsilon \equiv u$, $x_\varepsilon \equiv x$, $Dx_\varepsilon(t) \equiv 0$ on $[0,t]$. If $t > \tau$, then $x_\varepsilon(t) =: y(t)$, $y'(t) = f(y(t), u(t))$, $u(\tau) = x_\varepsilon(\tau)$,

$$Dx_\varepsilon(t) \equiv z(t) : z' = \nabla_x f(y(t), u(t))z, z(\tau) = Dx_\varepsilon(\tau), \qquad \text{variational equation.}$$

Statement: $z' = A(t)z, p' = -A^T(t)p \implies p^T z = const$. Proof: $(p^T z)' = (p^T)'z + p^T z' = -p^T Az + p^T Az = 0$.

Step 2.3: $p' = -(\nabla_x f(y(t), u(t)))^T p$, $p(T) = (\nabla_x g(x(T)))^T$. Then $p^T(t)z(t)$ constant on (τ, T), $p^T(\tau)z(\tau) \leqslant 0$.

Step 2.4: $Dx_\varepsilon(\tau)|_{\varepsilon=0^+} \stackrel{?}{=} f(x(\tau), \eta) - f(x(\tau), u(\tau))$. Then

$$p^T(\tau)\left(f(x(\tau), \eta) - f(x(\tau), u(\tau))\right) \leqslant 0$$

$$\frac{1}{\varepsilon}(x_\varepsilon(\tau) - x(\tau)) = \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), u(s))]\, ds =$$

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), \eta)]\, ds + \int_{\tau-\varepsilon}^{\tau} [f(x(s), \eta) - f(x(s), u(s))]\, ds.$$

Fist converge to zero from Lebesgue theorem about average value. Second to $f(x(\tau), \eta) - f(x(\tau), u(\tau)) \to 0$. □

Věta 2.19 (Potrjagin for fixed point („fixed finish"))

Same as previous, but T is not fixed, $x(T)$ is fixed $\implies$ $g \equiv 0$ (we don't "rate" final point, because it's the same for all u).

3 Bifurcation

Definice 3.1

$x' = \mu - x^2$ is saddle-node bifurcation, $x' = \mu x - x^2 = x(\mu - x)$ is transcritical bifurcation, $x' = \mu x - x^3 = x(\mu - x^2)$ is fork bifurcation, in $x' = x - \sin\mu$ there is no bifurcation.

Pozorování
$f(x_0, \mu_0) \neq 0 \implies$ no bifurcation. (From lemma of rect.) (Bifurcation $\implies f = 0$.)

Pozorování

$$f(x_0, \mu_0) = 0, \sigma(\nabla_x f(x_0, \mu_0)) = \{\lambda_j | \Re\lambda_j \neq 0\}.$$

Definice 3.2

Point from previous observation is called hyperbolic stationary point.

Věta 3.1

$f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be C^1, (x_0, μ_0) is a hyperbolic stationary point. Then $\exists \Delta > 0$ $\exists \delta > 0$ $\forall \mu \in U_\delta(\mu_0)$ $\exists x = x(\mu) \in U_\Delta(x_0)$, stationary point $x(\mu)$ is a hyperbolic stationary point of $\mu \mapsto x(\mu)$, which is C^1.

Důkaz
IFT:

$$f(x_0, \mu_0) = 0 \wedge \nabla_x f(x_0, \mu_0) \text{regular?} \wedge f \in C^1 \implies x = x(\mu), f(x(\mu), \mu) = 0.$$

Hyperbolic? Eigenvalues of $A = \nabla_x f(x(\mu), \mu)$, $\det(\lambda I - A(\mu))$ – polynomial in λ, deg $= n$. □

Věta 3.2

$f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 on neighborhood $(0,0) \in \mathbb{R}^2$.

$$f(0,0) = 0, \qquad f_\mu(0,0) \neq 0, \qquad f_x(0,0) = 0, \qquad f_{xx}(0,0) \neq 0.$$

Then f has bifurcation at $(0,0)$ of the type saddle-node.

Důkaz
Without proof. □

Věta 3.3

$f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 on neighborhood $(0,0) \in \mathbb{R}^2$.

$$f(0,0) = 0, \qquad f_x(0,0) = 0, \qquad f(0,\mu) = 0 \ \forall \mu \in U_\delta(0), \qquad f_{xx}(0,0) \neq 0, \qquad f_{x\mu}(0,0) \neq 0.$$

Then f has bifurcation at $(0,0)$ of the type transcritical.

Důkaz
Without proof. ◻

Věta 3.4

$f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 on neighborhood $(0,0) \in \mathbb{R}^2$.

$$f(0,0) = 0, \qquad f_x(0,0) \neq 0, \qquad f_{xx}(0,0) = 0, \qquad f_{xxx}(0,0) \neq 0,$$

$$f(0,\mu) = 0 \ \forall \mu \in U_\delta(0), f_{x\mu}(0,0) \neq 0.$$

Then f has bifurcation at $(0,0)$ of the type fork.

Důkaz
Without proof. ◻

Lemma 3.5 (About division)

$h : U(0,0) \to \mathbb{R}$ be C^k for some $k \in \mathbb{N}$. $h(0,\lambda) = 0 \ \forall \lambda \in U_\delta(0)$. Then

$$h(x,\lambda) = xH(x,\lambda), H \in C^{k-1}(U(0,0),\mathbb{R}).$$

$$H(0,0) = h_x(0,0), \qquad H_x(0,0) = \frac{1}{2}h_{xx}(0,0), \qquad H_\lambda(0,0) = h_{x\lambda}(0,0),$$

$$H_{xx}(0,0) = \frac{1}{3}h_{xxx}(0,0),$$

if k is sufficiently large.

Důkaz

$$H(x,\lambda) := \int_0^1 \partial_x h(\sigma x, \lambda) d\sigma.$$

◻