

Úvod

Poznámka (Organizační úvod)

K ukončení předmětu je třeba pouze udělat zkoušku: 2 příklady na definice ($2 \cdot 10$), 2 věta-důkaz ($15 + 20, 15 + 30$). (Hranice 85, 70, 55.)

Literatura:

- L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.
- W. Rudin, Analýza v reálném a komplexním oboru, Academia, 2003.

1 Differentiation of measures

1.1 Covering theorems

Definition 1.1 (Vitali cover)

Let $A \subset \mathbb{R}^n$ we say that a system \mathcal{V} consisting of closed balls from \mathbb{R}^n forms Vitali cover of A , if

$$\forall x \in A \forall \varepsilon > 0 \exists B \in \mathcal{V} : x \in B \wedge \text{diam } B < \varepsilon.$$

Definition 1.2 (Notation)

λ_n is Lebesgue measure on \mathbb{R}^n . λ_n^* is outer Lebesgue measure on \mathbb{R}^n . If $B \subset \mathbb{R}^n$ is a ball and $\alpha > 0$, then $\alpha \cdot B$ stands for the ball, which is concentric with B and with α -times greater radius than B .

Věta 1.1 (Vitali)

Let $A \subset \mathbb{R}^n$ and \mathcal{V} be a system of closed balls forming a Vitali cover of A . Then there exists a countable disjoint subsystem $\mathcal{A} \subseteq \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.

┌ *Důkaz*

First assume that A is bounded. Take an open bounded set $G \subset \mathbb{R}^n$ with $A \subset G$. We set

$$\mathcal{V}^* = \{B \in \mathcal{V} \mid B \subset G\}.$$

Then \mathcal{V}^* is a Vitali cover of A . If there exists a finite disjoint subsystem of \mathcal{V}^* covering A , we are done. So Assume that there is no such subsystem. Mathematical induction:

First step: We set $s_1 = \sup \{\text{diam } B \mid B \in \mathcal{V}^*\}$. We choose a ball $B_1 \in \mathcal{V}^*$ such that $B_1 > \frac{1}{2}s_1$.

k -th step: Suppose that we have already constructed balls B_1, B_2, \dots, B_{k-1} . We set

$$s_k = \sup \left\{ \text{diam } B \mid B \in \mathcal{V}^* \wedge B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset \right\}.$$

We find $B_k \in \mathcal{V}^*$ such that $\text{diam } B_k > \frac{1}{2}s_k > 0$, $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$.

Let $\mathcal{A} = \{B_k \mid k \in \mathbb{N}\}$. It is disjoint, it is countable, it holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$:

$$\begin{aligned} \sum_{i=1}^{\infty} \lambda_n(B_i) &= \lambda_n\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \lambda_n(G) < \infty \implies \\ \implies \lim_{i \rightarrow \infty} s_i &= 0 \implies \lim_{i \rightarrow \infty} \text{diam}(B_i) = 0 \implies \lim_{i \rightarrow \infty} s_i = 0. \end{aligned}$$

We show that $\forall x \in A \setminus \bigcup \mathcal{A} \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j > i : x \in 5 \cdot B_j \left(\Leftrightarrow A \setminus \bigcup \mathcal{A} \subseteq \bigcup_{j=i+1}^{\infty} 5 \cdot B_j \right)$.

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbb{N}$. Denote $\delta = \text{dist}(x, \bigcup_{k=1}^i B_k) > 0$. There exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\text{diam } B < \delta \implies B \cap \bigcup_{k=1}^i B_k = \emptyset$. Then we have $\text{diam } B > s_p$ for some $p \in \mathbb{N}$.

Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geq \text{diam } B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\text{diam } B_j > \frac{1}{2}s_j \geq \frac{1}{2}\text{diam } B \implies 2\text{diam } B_j \geq \text{diam } B$ This implies that $x \in B \subset 5 \cdot B_j$.

$$\begin{aligned} \lambda_n^*(A \setminus \bigcup \mathcal{A}) &\leq \lambda_n\left(\bigcup_{j=i+1}^{\infty} 5 \cdot B_j\right) \leq \sum_{j=i+1}^{\infty} \lambda_n(5 \cdot B_j) = \sum_{j=i+1}^{\infty} 5^n \lambda_n(B_j) = 5^n \cdot \sum_{j=i+1}^{\infty} \lambda_n(B_j) \rightarrow 0 \\ &\implies \lambda_n(A \setminus \bigcup \mathcal{A}) = 0. \end{aligned}$$

General case (A not bounded): Let $(G_j)_{j=1}^{\infty}$ be a sequence of disjoint open sets such that $\lambda_n(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$. We define $\mathcal{V}_j = \{B \in \mathcal{V}_i, B \subseteq G_j\}$. \mathcal{V}_j is a Vitali cover of $A \cap G_j \implies \exists \mathcal{A}_j \subseteq \mathcal{V}_j$ countable disjoint and $\lambda_n(A \cap G_j \setminus \bigcup \mathcal{A}_j) = 0$. We set $\mathcal{A} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$. \mathcal{A} is countable, disjoint and $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$. \square

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Definice 1.3

We say that a measure μ on \mathbb{R}^n satisfies Vitali theorem, if for every Vitaly cover \mathcal{V} of $M \subseteq \mathbb{R}^n$ there exists a disjoint countable $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

Poznámka

If μ satisfies Vitali theorem and $\nu \ll \mu$, then ν satisfies Vitali theroem.

Věta 1.2

Set $E \subset \mathbb{R}^n$ be Lebesgue measurable and \mathcal{S} be a finite system of closed balls covering E . Then there exists a disjoint system $\mathcal{L} \subset \mathcal{S}$ such that $\lambda_n(E) \leq 3^n \cdot \sum_{B \in \mathcal{L}} \lambda_n(B)$.

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Důkaz

WLOG $\mathcal{S} \neq \emptyset$. Suppose $B_1 \in \mathcal{S}$ with maximal radius among balls from \mathcal{S} .

Suppose that we have already constructed $B_1, \dots, B_{k-1} \in \mathcal{S}$. If possible, choose $B_k \in \mathcal{S}$ disjoint with $\bigcup_{j < k} B_j$ and with maximal radius among balls satisfying this property.

We set $\mathcal{L} = \{B_1, \dots, B_N\}$. We show $E \subseteq \bigcup_{B \in \mathcal{L}} 3 * B = \bigcup_{i=1}^N 3 * B_i$. $x \in E$. Find $B \in \mathcal{S}$ with $x \in B$. Find smallest k with $B \cap B_k \neq \emptyset$. This means $\text{rad}(B) \leq \text{rad}(B_k) \implies x \in B \subseteq 3 * B_k$.

$$\text{Now } \lambda_n(E) \leq \lambda_n\left(\bigcup_{i=1}^N 3 * B_i\right) \leq \sum_{i=1}^N \lambda_n(3 * B_i) = 3^n \sum_{i=1}^N \lambda_n(B_i). \quad \square$$

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Věta 1.3 (Besicovitch theorem)

For each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with the following property:

If $A \subset \mathbb{R}^n$ and $\Delta : A \rightarrow (0, \infty)$ is a bounded function, then there exist sets $A_1, \dots, A_N \subseteq A$ such that

- $\{\overline{B}(x, \Delta x) | x \in A_j\}$ is disjoint for every $j \in [N]$;
- $A \subset \bigcup \left\{ \overline{B}(x, \Delta x) | x \in \bigcup_{i=1}^N A_i \right\}$.

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Důkaz (Case A is bounded)

Let $R := \sup_A \Delta$. Choose $B_1 := \overline{B}(a_1, \Delta(a_1))$ such that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$.

Assume that we already constructed B_1, \dots, B_{j-1} , $j \geq 2$. $B_{j-1} = \overline{B}(a_{j-1}, \Delta(a_{j-1})) = \overline{B}(a_{j-1}, r_{j-1})$. Let $F_j := A \setminus \bigcup_{i=1}^{j-1} B_i$. If $F_j = \emptyset$ we set $J := j$. If not $B_j := \overline{B}(a_j, \Delta(a_j)) = \overline{B}(a_j, r_j)$, $a_j \in F_j$ and $r_j > \frac{3}{4} \sup_{F_j} \Delta$.

If $F_j \neq \emptyset$ for every $j \in \mathbb{N}$, then we set $J := \infty$. So we have $(B_j)_{j < J}$. If $J < \infty$, then we covered A . „If $J = \infty$, then $A \subset \bigcup_{j < J} B_j$ “: \square

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Důkaz

„ $\lim_{j \rightarrow \infty} r_j = 0$ “: because A is bounded

$$\|a_i - a_j\| \geq r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j = \frac{1}{3}(r_i + r_j) \implies \frac{1}{3} * B_i \cap \frac{1}{3} * B_j = \emptyset.$$

$$\{\frac{1}{3}B_j | j < J\} \text{ is a disjoint family } \implies \sum_{j=1}^{\infty} \lambda_n(\frac{1}{3} * B_j) < \infty.$$

If $A \in A \setminus \bigcup_{j=1}^{\infty} B_j$, then $a \in \bigcap_{j=1}^{\infty} F_j$. We find $j_0 \in \mathbb{N}$ with $r_{j_0} \leq \frac{3}{4}\Delta(a)$. ∇ .

Fix $k < J$. We set $I = \{i < k | B_i \cap B_k \neq \emptyset\}$, $I_1 = \{i < k | B_i \cap B_k \neq \emptyset \wedge r_i < 10r_k\}$, $I_2 = \{i < k | B_i \cap B_k \wedge r_i \geq 10r_k\}$. The estimate of I_1 : „We have $\frac{1}{3}B_i \subseteq 15 * B_k$ for every $i \in I_1$ “: Take $x \in \frac{1}{3} * B_i$. Then

$$\|x - a_k\| \leq \|x - a_j\| + \|a_i - a_k\| \leq \frac{1}{3}r_i + r_i + r_k \leq \frac{10}{3}r_k + 10r_k + r_k \leq 15r_k$$

$$\begin{aligned} \lambda_n(\frac{1}{3} * B_i) &= \lambda(\overline{B}(0, 1)) \cdot (\frac{1}{3}r_i)^n \geq \lambda_n(\overline{B}(0, 1)) \cdot (\frac{1}{3} \cdot \frac{3}{4}r_k)^n = \lambda_n(\overline{B}(0, 1)) \cdot \frac{1}{4^n}r_k^n = \\ &= \frac{1}{60^n} \lambda_n(15 * B_k) \implies |I_1| \leq 60^n. \end{aligned}$$

Denote $b_i = a_i - a_k$, vector between centers of balls. Take a family $\{Q_m | 1 \leq m \leq (22n)^n\}$ of closed cubes with edge length $\frac{1}{11n}$ which cover $[-1, 1]^n$. We claim that „for each $1 \leq m \leq (22n)^n$ there is at most one $i \in I_2$ with $\frac{b_i}{\|b_i\|} \in Q_m$ “:

$$i, j \in I_2, i < j, \left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \leq \frac{1}{11}.$$

We have $r_i < \|b_i\| \leq r_i + r_k$ and $r_j < \|b_j\| \leq r_j + r_k$. So $\|b_i\| - \|b_j\| \leq |r_i - r_j| + r_k$. $\|b_j\| \leq r_j + r_k \leq r_j + \frac{1}{10}r_j = \frac{11}{10}r_j$.

$$\begin{aligned} \|a_i - a_j\| &= \|b_i - b_j\| \leq \left\| b_i - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - b_j \right\| \leq \|b_i\| - \|b_j\| + \frac{1}{11}\|b_j\| \leq \\ &\leq |r_i - r_j| + r_k + \frac{1}{11} \cdot \frac{11}{10}r_j \leq |r_i - r_j| + \frac{1}{5}r_j. \end{aligned}$$

We distinguish two cases:

$$r_i > r_j : \|a_i - a_j\| \leq r_i - \frac{4}{5}r_j < r_i;$$

$$r_i \leq r_j : \|a_i - a_j\| \leq -r_i + r_j + \frac{1}{5}r_j = -r_i + \frac{6}{5}r_j \leq -r_i + \frac{8}{5}r_i < r_i \implies a_j \in \overline{B}(a_i, r_i) = B_i, \nabla.$$

Choose $N > (60)^n + (22n)^n$. $B_i \in A_i$, $i \in [N]$. From previous every ball intersect at most $(60)^n + (22n)^n$ previous balls, so we always have A_i where we can put ball. \square

Důkaz (Case A is not bounded)

Let $A^l := A \cap \{x \in \mathbb{R}^n | 3(l-1)R \leq \|x\| < 3lR\}$, $l \in \mathbb{N}$. We get A_i^l , $i \in [M]$ by the previous.
 $A_i = \bigcup_{l=2k+1} A_i^l$, $A_{M+i} = \bigcup_{l=2k} A_i^l$. □

Definice 1.4 (Radon measure)

Let P be a locally compact Hausdorff space and \mathcal{S} a σ -algebra of subsets of P . We say that μ is a Radon measure if

- \mathcal{S} contains all Borel sets,
- $\mu(K) < \infty$ for every compact $K \in P$,
- $\mu(G) = \sup \{\mu(K) | K \subset G \text{ is compact}\}$ for every $G \subset P$ open,
- $\mu(A) = \inf \{\mu(K) | A \subset K \text{ is compact}\}$ for every $A \in \mathcal{S}$,
- μ is complete.

Lemma 1.4

Let μ be a measure on X and $\{A_j\}_{j=1}^\infty$ be an increasing sequence of subsets of X . Then $\lim \mu^*(A_j) = \mu^*\left(\bigcup_{j=1}^\infty A_j\right)$.

Věta 1.5

Let μ be a Radon measure on \mathbb{R}^n and \mathcal{F} be a collection of closed balls in \mathbb{R}^n . Let A denote the set of centers of balls in \mathcal{F} . Assume $\inf \{r | B(a, r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus \bigcup \mathcal{G}) = 0$.

Důkaz (The case $\mu^*(A) < \infty$)

Let $N \in \mathbb{N}$ be the constant from Besicovitch theorem. We find Θ such that $1 - \frac{1}{N} < \Theta < 1$.

Claim: „Let $U \subset \mathbb{R}^n$ be an open set. Then there exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*((A \cap U) \setminus \bigcup \mathcal{H}) \leq \Theta \cdot \mu^*(A \cap U).$$

“

$$\mathcal{F}_1 \subset \mathbb{F}, \mathbb{F}_1 = \{B \in \mathbb{F}, \text{diam } B < 1 \wedge B \subset U\}$$

By theorem above there exists disjoint families $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}_1$ such $A \cap U \subseteq \bigcup_{i=1}^N \bigcup \mathcal{G}_i$. Thus $\mu^*(A \cap U) \leq \sum_{i=1}^N \mu^*(A \cap U \cap \bigcup \mathcal{G}_i)$. Consequently, there exists an integer $1 \leq j \leq N$ such that

$$\mu^*(A \cap U \cap \bigcup \mathcal{G}_j) \geq \frac{1}{N} \mu^*(A \cap U) > (1 - \Theta) \mu^*(A \cap U).$$

Using lemma above we find a finite system $\mathcal{H} \subset \mathcal{G}_j$ such that

$$\mu^*(A \cap U \cap \mathcal{H}) > (1 - \Theta) \mu^*(A \cap U).$$

The set $\bigcup \mathcal{H}$ is μ -measurable

$$\mu^*(A \cap U) = \mu^*(A \cap U \cap \bigcup \mathcal{H}) + \mu^*(A \cap U \setminus \bigcup \mathcal{H}) \geq (1 - \Theta) \mu^*(A \cap U) + \mu^*(A \cap U \setminus \bigcup \mathcal{H}).$$

Set $U_1 = \mathbb{R}^n$. Using claim we find a disjoint finite system $\mathcal{H}_1 \subset \mathcal{F}$ such that $\bigcup \mathcal{H}_1 \subset U_1$ and $\mu^*(A \cap U_1 \setminus \bigcup \mathcal{H}_1) \leq \Theta \mu^*(A \cap U_1)$. Continuing by induction we construct a sequence of open sets (U_j) and a sequence of disjoint finite families (\mathcal{H}_j) such that $U_{j+1} = U_j \setminus \bigcup \mathcal{H}_j$, $\bigcup \mathcal{H}_j \subset U_j$, $\mathcal{H}_j \subset \mathcal{F}$ and

$$\mu^*(A \cap U_{j+1}) = \mu^*((A \cap U_j) \setminus \bigcup \mathcal{H}_j) \leq \Theta \mu^*(A \cap U_j).$$

Since $\mu^*(A) < \infty$ we get $\mu^*(A \setminus \bigcup_{j=1}^{\infty} \bigcup \mathcal{H}_j) = 0$, since $\mu^*(A \cap U_{j+1}) \leq \Theta^j \mu^*(A)$.

$$\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$$

□

Důkaz (General case)

We find a sequence (G_j) of open sets, which are disjoint and $\mu(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$.

□

1.2 Differentiation of measures

Poznámka (Notation)

\mathcal{B} is set of closed balls in \mathbb{R}^n .

Definice 1.5 (Derivative of measure)

Let μ and ν be measures on \mathbb{R}^n and $x \in \mathbb{R}^n$. Then we define

- upper derivative of ν with respect to μ and x by

$$\overline{D}(\nu, \mu, x) = \lim_{r \rightarrow 0_+} \sup_{B \in \mathcal{B}, \text{diam } B < r} \frac{\nu(B)}{\mu(B)},$$

if the term at the right side is well-defined;

- lower derivative of ν with respect to μ and x by

$$\underline{D}(\nu, \mu, x) = \lim_{r \rightarrow 0_+} \inf_{B \in \mathcal{B}, \text{diam } B < r} \frac{\nu(B)}{\mu(B)},$$

if the term at the right side is well-defined;

- derivative of ν with respect to μ and x by

$$D(\nu, \mu, x) = \overline{D}(\nu, \mu, x) = \underline{D}(\nu, \mu, x),$$

if they are equal.

Věta 1.6

Let ν and μ be Radon measures and \mathbb{R}^n and μ satisfy Vitali theorem. Then $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$ exist μ -almost everywhere.

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Důkaz

$M := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) > 0\}$ and $\mathcal{V} := \{B \in \mathcal{B} \mid \mu(B) = 0\}$, \mathcal{V} is a Vitali cover of M . Then there exists a disjoint countable family $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

$$\mu\left(\bigcup \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \mu(B) = 0 \implies \mu(M) = 0.$$

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□

Věta 1.7

Let μ and ν be Radon measures, μ satisfy Vitali theorem, $C \in (0, \infty)$, and $M \subset \mathbb{R}^n$.

- If for every $x \in M$ we have $\overline{D}(\nu, \mu, x) > c$, then $\nu^*(M) \geq c\mu^*(M)$.
- If for every $x \in M$ we have $\underline{D}(\nu, \mu, x) < c$, then there exists $H \subset M$ such that $\mu(M \setminus H) = 0$ and $\nu^*(H) \leq c\mu^*(M)$.

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Důkaz (1.)

We choose $\varepsilon > 0$. There exists an open set $G \subset \mathbb{R}^n$ with $M \subset G$ and $\nu(G) \leq \nu^*(M) + \varepsilon$. We define

$$\mathcal{V} := \{B \in \mathcal{B} \mid B \subset G, \nu(B) > c \cdot \mu(B)\}.$$

The family \mathcal{V} is a Vitali cover of M . There exists a disjoint countable family $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$. Then we have

$$\nu^*(M) + \varepsilon \geq \nu(G) \geq \nu\left(\bigcup \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \nu(B) \leq \sum_{B \in \mathcal{A}} c\mu(B) = c\mu\left(\bigcup \mathcal{A}\right) \geq c\mu^*(M)$$

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Důkaz (2.)

For every $k \in \mathbb{N}$ we find an open set $G_k \supset M$ and $\mu(G_k) \leq \mu^*(M) + \frac{1}{k}$.

$$\mathcal{V}_k := \{B \in \mathcal{B} \mid B \subset G_k \wedge \nu(B) < c \cdot \mu(B)\}.$$

The system \mathcal{V}_k is a Vitali cover of M . Thus there exist a countable disjoint system $\mathcal{A}_k \subset \mathcal{V}_k$ such that $\mu(M \setminus \bigcup \mathcal{A}_k) = 0$. Set $H_k = M \cap \bigcup \mathcal{A}_k$. Then $\mu(M \setminus H_k) = 0$, $H_k \subset M$. We have

$$\nu^*(H_k) \leq \nu\left(\bigcup \mathcal{A}_k\right) = \sum_{B \in \mathcal{A}_k} \nu(B) \leq c \sum_{B \in \mathcal{A}_k} \mu(B) = c\mu\left(\bigcup \mathcal{A}_k\right) \leq c \cdot \mu(G_k) \leq c\left(\mu^*(M) + \frac{1}{k}\right).$$

$$H := \bigcap H_k : \quad \nu^*(H) \leq c\mu^*(M).$$

$$\mu(M \setminus H) \leq \sum_{k=1}^{\infty} \underbrace{\mu(M \setminus H_k)}_{=0} = 0.$$

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Věta 1.8

Let ν and μ be Radon measures on \mathbb{R}^n and μ satisfies Vitali theorem. Then $D(\nu, \mu, x)$ exists finite μ almost everywhere.

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Důkaz

$$D := \{x \in \mathbb{R}^n \mid D(\nu, \mu, x) \in [0, \infty)\}$$

$$N_1 := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) \text{ is not defined}\}, \quad N_3 = \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) = \infty\},$$

$$N_2 := \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) \text{ is not defined}\}, \quad N_4 = \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) = \infty\}.$$

We already showed that $\mu(N_1) = \mu(N_2) = 0$.

$$A_k := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) > k\}, k \in \mathbb{N}$$

$$A(r, s) = \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) < s < r < \overline{D}(\nu, \mu, x)\}, \quad s, r \in \mathbb{Q}^+, s < r$$

$$N_3 = \bigcap_{k=1}^{\infty} A_k, \quad N_4 = \bigcup \{A(r, s), r, s \in \mathbb{Q}^+, s < r\}$$

„ $\mu(N_3) = 0$ “: Choose $Q \subset N_3$ bounded. By previous theorem (1.) $k\mu^*(Q) \leq \nu^*(Q)$ for every $k \in \mathbb{N}$.

$$\implies \mu^*(Q) = 0 \implies \mu^*(N_3) = 0 \implies \mu(N_3) = 0.$$

„ $\mu(N_4) = 0$ “: It is sufficient to prove $\mu(A(r, s)) = 0$ for any $r, s \in \mathbb{Q}^+, r > s$. Choose $Q \subset A(r, s)$ bounded. By previous theorem (2.) there exists $H \subset Q$ such that $\mu(Q \setminus H) = 0$ and $\nu^*(H) \leq s\mu^*(Q)$. By previous theorem (1.) we have $r\mu^*(H) \leq \nu^*(H)$.

$$r\mu^*(Q) = r\mu^*(H) \leq \nu^*(H) \leq s\mu^*(Q) < \infty.$$

$$\implies \mu^*(Q) = 0 \implies \mu(A(r, s)) = 0.$$

└

□

Lemma 1.9

Let ν and μ be as before. Then the mappings $x \mapsto \overline{D}(\nu, \mu, \lambda)$, $x \mapsto \underline{D}(\nu, \mu, \lambda)$ are μ -measurable.

┌ *Důkaz*

$$M(r, \alpha) = \left\{ x \in \mathbb{R}^n \mid \exists B \in \mathcal{B} : \text{diam } B < r, x \in B, \frac{\nu(B)}{\mu(B)} < \alpha \right\}, \quad r > 0, \alpha > 0.$$

„ $M(r, \alpha)$ is open“: Assume $x \in M(r, \alpha)$ we find $y \in \mathbb{R}^n$, $s > 0$ such that $x \in \overline{B}(y, s)$, $2s < r$,

$$\frac{\nu(\overline{B}(y, s))}{\mu(\overline{B}(y, s))}.$$

We find $s' > s$, $2s' < r$, $\frac{\nu(\overline{B}(y, s'))}{\mu(\overline{B}(y, s'))} < \alpha$. Then $B(y, s') \subset M(r, \alpha)$.

$$D := \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) \text{ exists finite}\}.$$

For every $x \in D$ we have

$$\underline{D}(\nu, \mu, x) < \alpha \Leftrightarrow \exists \tau \in \mathcal{Q}, \tau > 0 \forall r \in \mathcal{Q}, r > 0 \exists B \in \mathcal{B} : \text{diam } B < r, x \in B, \frac{\nu(B)}{\mu(B)} < \alpha - \tau,$$

$$\underline{D}(\nu, \mu, x) < \alpha \Leftrightarrow \exists \tau \in \mathcal{Q}, \tau > 0 \forall r \in \mathcal{Q}, r > 0 : x \in M(r, \alpha - \tau).$$

└ $\{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) < \alpha\}$ is μ -measurable. □

Věta 1.10

Let ν and μ be as before, $\nu \ll \mu$, and $B \subset \mathbb{R}^n$ is μ -measurable. Then we have $\nu(B) = \int_B D(\nu, \mu, x) d\mu(x)$.

Dikaz

Let $B \subset \mathbb{R}^n$ be μ -measurable. Choose $\beta > 1$.

$$B_k := \{x \in B \mid \beta^k < D(\nu, \mu, x) \leq \beta^{k+1}\}, k \in \mathbb{Z}.$$

$$N := \{x \in B \mid D(\nu, \mu, x) = 0\}.$$

$$\mu(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)) = 0.$$

$$\begin{aligned} \int_B D(\nu, \mu, x) d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu, \mu, x) d\mu(x) \leq \\ &\leq \sum_{k=-\infty}^{\infty} \beta^{k+1} \mu(B_k) = \sum_{k=-\infty}^{\infty} \beta^{k+1} \cdot \beta^{-k} \nu(B_k) = \beta \cdot \sum_{k=-\infty}^{\infty} \nu(B_k) \leq \beta \cdot \nu(B). \end{aligned}$$

$$\beta \rightarrow 1_+ : \int_B D(\nu, \mu, x) d\mu(x) \leq \nu(B).$$

Using absolute continuity: $\nu(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)) = 0$. We use theorem above to get $\nu^*(Q) \leq C\mu^*(Q)$ for any $c > 0$ and $Q \subset N$ bounded. $\implies \nu^*(Q) = 0 \implies \nu(N) = 0$.

$$\begin{aligned} \int_B D(\nu, \mu, x) d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu, \mu, x) d\mu(x) \geq \\ &\geq \sum_{k=-\infty}^{\infty} \beta^k \cdot \mu(B_k) \geq \sum_{k=-\infty}^{\infty} \beta^k \cdot \beta^{-(k+1)} \nu(B_k) = \frac{1}{\beta} \cdot \nu(B). \end{aligned}$$

$$\beta \rightarrow 1_+ : \int_B D(\nu, \mu, x) d\mu(x) \geq \nu(B).$$

□

1.3 Lebesgue points

Definition 1.6 (\mathcal{L}_{loc}^1)

Let μ be a Radon measure on \mathbb{R}^n . The symbol $\mathcal{L}_{loc}^1(\mu)$ denotes the set of all functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$, which are μ -measurable and for every $x \in \mathbb{R}^n$ there exists $r > 0$ such that $\int_{B(x,r)} |f| d\mu < \infty$.

Definition 1.7 (Lebesgue point)

Let $f \in \mathcal{L}_{loc}^1(\mu)$. We say that $x \in \mathbb{R}^n$ is Lebesgue point of f at x (with respect to μ) if we have

$$\forall \varepsilon > 0 \exists \delta > 0 \forall B \in \mathbb{B}, x \in B, \text{diam } B < \delta : \frac{\int_B |f(t) - f(x)| d\mu(t)}{\mu(B)} < \varepsilon.$$

Věta 1.11

Let μ be a Radon measure on \mathbb{R}^n satisfying Vitali theorem and $f \in \mathcal{L}_{loc}^1(\mu)$. Then μ -almost every point are Lebesgue point of f (with respect to μ).

┌

Důkaz

WLOG $\mu(\mathbb{R}^n) < \infty$ and $f \in \mathcal{L}^1(\mu)$. Set $(C_k)_{k=1}^\infty$ be a sequence of closed balls in \mathbb{C} forming a basis of topology in \mathbb{C} . We define

$$g_k(x) := \text{dist}(f(x), C_k), \quad x \in \mathbb{R}^n, k \in \mathbb{N}.$$

The function g_k is non-negative, μ -measurable, $g_k \in \mathcal{L}^1(\mu)$. Set $\nu_k = \int g_k d\mu$. We set $P_k := \{x \in f^{-1}(C_k) \mid \neg(D(\nu_k, \mu, x) = 0)\}$. We have $g_k = 0$ on $f^{-1}(C_k) \implies \mu(P_k) = 0$.

$$\nu_k = \int D(\nu_k, \mu, x) d\mu(x).$$

For $x \in \mathbb{R}^n \setminus \bigcup_{k=1}^\infty P_k$ we choose $\varepsilon > 0$ and we find C_k such that $f(x) \in C_k$ and $C_k \subset B(f(x), \frac{1}{2}\varepsilon)$. For any $t \in \mathbb{R}^n$ it holds $|f(t) - f(x)| \leq g_k(t) + \varepsilon$.

$x \in f^{-1}(C_k) \implies D(\nu_k, \mu, x) = 0$. We find $\delta > 0$ such that

$$\forall B \in \mathbb{B}, x \in B, \text{diam } B < \delta : \frac{\nu_k(B)}{\mu(B)} = \frac{\int_B g_k d\mu}{\mu(B)} < \varepsilon.$$

Let $B \in \mathbb{B}$, $x \in B$ and $\text{diam } B < \delta$. We get

$$\frac{\int_B |f(t) - f(x)| d\mu(t)}{\mu(B)} \leq \frac{\int_B (g_k(t) + \varepsilon) d\mu(t)}{\mu(B)} < \varepsilon + \varepsilon = 2\varepsilon.$$

└

□

1.4 Density theorem

Definice 1.8

Let μ be a measure on \mathbb{R}^n , $A \subset \mathbb{R}^n$ be μ -measurable and $x \in \mathbb{R}^n$. We say that $c \in [0, 1]$ is μ -density of A at x if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall B \in \mathcal{B}, x \in B, \text{diam } B < \delta : \left| \frac{\mu(A \cap B)}{\mu(B)} - c \right| < \varepsilon.$$

Věta 1.12 (Density theorem)

Let μ be a Radon measure on \mathbb{R}^n satisfying Vitali theorem and $M \subset \mathbb{R}^n$ be μ -measurable. Then

$$\begin{aligned} d_\mu(x, M) &= 1 \text{ for almost every } x \in M, \\ d_\mu(x, M) &= 0 \text{ for almost every } x \in \mathbb{R}^n \setminus M. \end{aligned}$$

┌
Důkaz

Define ν on \mathbb{R}^n by $\nu(A) = \mu(A \cap M)$ for every μ -measurable $A \subset \mathbb{R}^n$. Thus we have $d_\mu(M, X) = D(\nu, \mu, X)$, if at least one term is well-defined, $\nu \ll \mu$, $\nu = \int \chi_M d\mu$. From theorem above $\nu = \int D(\nu, \mu, x) d\mu(x) \implies \chi_M = D(\nu, \mu, x)$ μ -almost everywhere. \square

└

1.5 AC and BV functions

Věta 1.13

Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$. Then f is absolutely continuous on $[a, b]$ if and only if f is difference of two non-decreasing absolutely continuous functions on $[a, b]$.

┌
Důkaz

„ \implies “ choose $c \in (a, b)$. We define $v(x) = V_c^x f$, $x \in [c, b]$, and $v(x) = -V_x^c f$, $x \in [a, c]$. For every $y, d \in [a, b]$, $y < d$, we have $v(d) - v(y) = V_y^d f$. The function v is non-decreasing.

$$x, y \in [a, b], x < y:$$

$$(v(y) - f(y)) - (v(x) - f(x)) = (v(y) - v(x)) - (f(y) - f(x)) = V_x^y f - (f(y) - f(x)) \geq 0.$$

$v \in AC([a, b])$: Choose $\varepsilon > 0$. We find $\delta > 0$ such that $\sum_{j=1}^m |f(b_j) - f(a_j)| < \varepsilon$, whenever $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_m < b_m \leq b$ and $\sum_{j=1}^m (b_j - a_j) < \delta$. Assume that $a \leq A_1 < B_1 \leq A_2 < B_2 \leq \dots \leq A_p < B_p \leq b$ with $\sum_{j=1}^p (B_j - A_j) < \delta$. For each $j \in [p]$ we find points

$$A_j = a_1^j < b_1^j = a_2^j < b_2^j < \dots < a_{m_j}^j < b_{m_j}^j = B_j$$

such that

$$v(B_j) - v(A_j) = V_{A_j}^{B_j} f < \sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| + \frac{\varepsilon}{p}.$$

$$\sum_{j=1}^p |v(B_j) - v(A_j)| < \sum_{j=1}^p \left(\left(\sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| \right) + \frac{\varepsilon}{p} \right) < \varepsilon + p \cdot \frac{\varepsilon}{p} = 2\varepsilon.$$

$$f = v - (v - f).$$

└

\square

Lemma 1.14

Let $f : (a, b) \rightarrow \mathbb{R}$, $x_0 \in (a, b)$, and $f'(x_0) \in \mathbb{R}$. Then we have

$$\lim_{[x_1, x_2] \rightarrow [x_0, x_0], x_1 \leq x_0 \leq x_2, x_1 \neq x_2} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0).$$

┌
Důkaz

WLOG $f'(x_0) = 0$ ($x \mapsto f(x) - f'(x_0) \cdot x$). Choose $\varepsilon > 0$. We find $\delta > 0$ such that

$$\forall x \in (a, b), 0 < |x - x_0| < \delta : \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \varepsilon.$$

For any $x_1 \in (x_0 - \delta, x_0]$, $x_2 \in [x_0, x_0 + \delta]$ we have

$$|f(x_1) - f(x_0)| \leq \varepsilon |x_1 - x_0|, \quad |f(x_2) - f(x_0)| \leq \varepsilon |x_2 - x_0|.$$

We get

$$|f(x_2) - f(x_1)| \leq |f(x_2) - f(x_0)| + |f(x_1) - f(x_0)| \leq \varepsilon |x_1 - x_0| + \varepsilon |x_2 - x_0| \leq \varepsilon |x_2 - x_1|.$$

$$\left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| \leq \varepsilon, \quad x_2 \neq x_1.$$

└

□

Lemma 1.15

Let $f : (a, b) \rightarrow \mathbb{R}$, be non-decreasing on (a, b) , $C(f)$ be the set of all points of continuity of f , and $A \in \mathbb{R}$. Then for every $x_0 \in C(f)$ it hold:

$$f'(x_0) = A \Leftrightarrow \lim_{[x_1, x_2] \rightarrow [x_0, x_0], x_1 \leq x_0 \leq x_2, x_1 \neq x_2, x_1, x_2 \in C(f)} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = A.$$

┌
Důkaz

„ \Rightarrow “: This follows from the previous lemma.

„ \Leftarrow “: We check that $f'_+(x_0) = A$: We choose a sequence $\{x_n\}_{n=1}^\infty$ such that

$$x_n \in (a, b) \setminus \{x_0\}, x_n > x_0, \quad \lim_{x_n} = x_0.$$

We want:

$$\lim_{n \rightarrow \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = A.$$

For each $n \in \mathbb{N}$ we find z_n, y_n such that

$$y_n \leq x_n \leq z_n, n \in \mathbb{N}, \quad \frac{x_n - x_0}{y_n - x_0} \in B\left(1, \frac{1}{n}\right), \quad \frac{x_n - x_0}{z_n - x_0} \in B\left(1, \frac{1}{n}\right), \quad y_n, z_n \in C(f).$$

$$\underbrace{\frac{f(y_n) - f(x_0)}{y_n - x_0}}_{\rightarrow A} \cdot \underbrace{\frac{y_n - x_0}{x_n - x_0}}_{\rightarrow 1} \leq \frac{f(x_n) - f(x_0)}{x_n - x_0} \leq \underbrace{\frac{f(z_n) - f(x_0)}{z_n - x_0}}_{\rightarrow A} \cdot \underbrace{\frac{z_n - x_0}{x_n - x_0}}_{\rightarrow 1}.$$

└

□

Lemma 1.16

Let f be a distribution function of a measure μ on \mathbb{R} , $x_0 \in C(f)$, $A \in \mathbb{R}$. Then

$$f'(x_0) = A \Leftrightarrow D(\mu, \lambda_1, x_0) = A.$$

┌

Důkaz

We choose sequences $\{x_n^1\}_n, \{x_n^2\}_n$ such that

$$x_n^1 \leq x_0 \leq x_n^2, \quad \lim(x_n^2 - x_n^1) = 0, \quad x_n^1 \neq x_n^2.$$

We want:

$$\frac{\mu([x_n^1, x_n^2])}{\lambda([x_n^1, x_n^2])} \rightarrow A.$$

For every $n \in \mathbb{N}$ we find $y_n^1, y_n^2 \in C(f)$ such that

$$y_n^1 \leq x_0 \leq y_n^2, \quad \frac{y_n^2 - y_n^1}{x_n^2 - x_n^1} \in B\left(1, \frac{1}{n}\right), \quad y_n^1 < x_n^1 \leq x_0 \leq x_n^2 < y_n^2, \quad \lim(y_n^2 - y_n^1) = 0.$$

$$\lim_{n \rightarrow \infty} \frac{\mu([y_n^1, y_n^2])}{y_n^2 - y_n^1} = \lim_{n \rightarrow \infty} \frac{f(y_n^2) - f(y_n^1)}{y_n^2 - y_n^1} = A.$$

$$\lim_{n \rightarrow \infty} \frac{\mu([x_n^1, x_n^2])}{x_n^2 - x_n^1} = \lim_{n \rightarrow \infty} \left(\underbrace{\frac{\mu([y_n^1, y_n^2])}{y_n^2 - y_n^1}}_{\rightarrow A} \cdot \underbrace{\frac{y_n^2 - y_n^1}{x_n^2 - x_n^1}}_{\rightarrow 1} + \underbrace{\frac{\mu([x_n^1, x_n^2]) - \mu([y_n^1, y_n^2])}{x_n^2 - x_n^1}}_{|\cdot| < \frac{1}{n}} \right) = A.$$

└

□

Věta 1.17 (Lebesgue)

Let f be a monotone function on an interval I . Then we have

- $f'(x)$ exists almost everywhere in I ;
- f' is measurable and $|\int_a^b f'| \leq |f(b) - f(a)|$, whenever $a, b \in I, a < b$;
- $f' \in L_{loc}^1(I)$.

┌
Důkaz

WLOG f is non-decreasing. Let $a, b \in I$, $a < b$. We define $g : \mathbb{R} \rightarrow \mathbb{R}$:

$$g(x) = \begin{cases} \lim_{t \rightarrow a+} f(t), & x \in (-\infty, a], \\ \lim_{t \rightarrow x+} f(t), & x \in (a, b), \\ f(b), & x \in [b, \infty). \end{cases}$$

g is non-decreasing and continuous from the right, $\{x \in (a, b) | f(x) \neq g(x)\}$ is countable.

There exists a Radon measure ν on \mathbb{R} such that

$$\forall c, d \in \mathbb{R}, c < d : \nu((c, d]) = g(d) - g(c).$$

$\nu = \mu + \sigma$, where μ, σ are Radon measures, $\mu \perp \lambda$, $\sigma \ll \lambda$.

Claim: „ $D(\mu, \lambda, x) = 0$ λ -almost everywhere.“ $N \subset \mathbb{R}$ measurable, $\lambda(N) = 0$ and $\mu(\mathbb{R} \setminus N) = 0$. $c > 0 : D := \{x \in \mathbb{R} \setminus N | D(\mu, \lambda, x) > c\}$.

$$0 = \mu(D) \geq c \cdot \lambda(D) \implies \lambda(D) = 0.$$

Previous lemma gives $g'(x) = D(\nu, \lambda, x)$ λ -almost everywhere, since g is continuous at each point of $[a, b]$ except on countable set $x_0 \in (a, b) \cap C(f)$, then $f'(x_0) = A \in \mathbb{R} \Leftrightarrow g'(x_0) = A \implies f'1$ exists almost everywhere in $[a, b]$.

$$\begin{aligned} f(b) - f(a) &\geq g(b) - g(a) = \nu((a, b]) \geq \sigma((a, b]) = \int_a^b D(\sigma, \lambda, x) d\lambda(x) = \\ &= \int_a^b D(\nu, \lambda, x) d\lambda(x). \end{aligned}$$

└

□

Věta 1.18

Let I be a nonempty interval and $f \in BV(I)$. Then $f'(x)$ exists finite almost everywhere in I .

┌
Důkaz

$f = f_1 - f_2$, where f_1, f_2 are non-decreasing. And we use previous. └

□

Věta 1.19

Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$ the following are equivalent:

- $f \in AC([a, b])$;
- We have $\varphi \in L^1([a, b])$ such that $f(x) = f(a) + \int_a^x \varphi(t) dt$, $x \in [a, b]$;

- $f'(x)$ exists almost everywhere $f' \in L^1([a, b])$, and $f(x) = f(a) + \int_a^x f'(t)dt$, $x \in [a, b]$.

┌
Důkaz

„1. \implies 3.“ WLOG f is absolutely continuous and non-decreasing. We define an extension \tilde{f} (which we denote by f again) by a constant on $(-\infty, a)$ and on (b, ∞) to keep continuity. Let ν be a measure satisfying $\nu([x, y]) = f(y) - f(x)$, $x, y \in \mathbb{R}$, $x \leq y$. Then we have $\nu|_{[a, b]} \ll \lambda_1|_{[a, b]}$.

Then

$$\nu([a, x]) = f(x) - f(a) = \int_a^x D(\nu, \lambda_1, t) d\lambda_1(t) = \int_a^x f'(t) d\lambda_1(t).$$

„3. \implies 2.“ triviální. „2. \implies 1.“: $\varphi = \varphi^+ - \varphi^-$, $\varphi^+, \varphi^- \in L^1([a, b])$. We set

$$f_1(x) := \int_a^x \varphi^+(t) dt, \quad f_2(x) := \int_a^x \varphi^-(t) dt,$$

$$\nu(M) = \int_M \varphi^+(t) dt, \quad M \subset [a, b] \text{ measurable.}$$

Then we have $\nu \ll \lambda_1|_{[a, b]}$, $\nu([x, y]) = \int_x^y \varphi^+(t) dt = f_1(y) - f_1(x)$, $f_1, f_2 \in AC([a, b])$, $f(x) = f(a) + f_1(x) - f_2(x) \implies f \in AC([a, b])$. \square

Věta 1.20 (Per partes for Lebesgue integral)

Let $f, g \in AC([a, b])$, $a < b$. Then $\int_a^b f'g = [fg]_a^b - \int_a^b fg'$.

┌
Důkaz

$f', g' \in L^1([a, b])$. $(fg)' = f'g + fg'$ almost everywhere in $[a, b]$. $\int_a^b (fg)' = \int_a^b (f'g + fg') = \int_a^b f'g + \int_a^b fg'$.

$$a \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n \leq b :$$

$$\sum_{i=1}^n |f(b_i)g(b_i) - f(a_i)g(a_i)| \leq M \cdot \sum_{i=1}^n |g(b_i) - g(a_i)| + M \cdot \sum_{i=1}^n |f(b_i) - f(a_i)| \leq M \cdot \varepsilon$$

$$\begin{aligned} & (|f(b_i)g(b_i) - f(b_i)g(a_i) + f(b_i)g(a_i) - f(a_i)g(a_i)| \leq \\ & \leq |f(b_i)| \cdot |g(b_i) - g(a_i)| + |g(a_i)| \cdot |f(b_i) - f(a_i)|. \end{aligned}$$

└

\square

Věta 1.21

Let g be a non-negative function on $[a, b]$ with $g \in L^1([a, b])$ and f be a continuous function on $[a, b]$. Then there exists $\xi \in [a, b]$ such that

$$\int_a^b fg = f(\xi) \int_a^b g.$$

┌
Důkaz

We set $m := \min_{[a,b]} f$, $M := \max_{[a,b]} f$.

$$mg(x) \leq f(x)g(x) \leq Mg(x), x \in [a, b].$$

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g.$$

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

If $\int_a^b g = 0$, then we are done, else $\exists \xi \in [a, b] : f(\xi) = \frac{\int_a^b fg}{\int_a^b g}$. □

Věta 1.22

Let $f \in L^1([a, b])$ and g be a monotone function on $[a, b]$. Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f = g(a) \int_a^\xi f + g(b) \int_\xi^b f.$$

┌
Důkaz

WLOG g is non-decreasing.

First case „ $g \in AC([a, b])$ “: $F(z) = \int_a^b f$, $F \in AC([a, b])$, $\int_a^b fg = \int_a^b F'g =$

$$[Fg]_a^b - \int_a^b Fg' = F(b)g(b) - F(a)g(a) - F(\xi) \int_a^b g' = \underbrace{\left(\int_a^b f \right)}_{\int_a^\xi f + \int_\xi^b f} \cdot g(b) - \left(\int_a^\xi f \right) \cdot (g(b) - g(a)).$$

General case: $(D_n)_{n=1}^\infty$ sequence of partition of $[a, b]$, $\nu(D_n) \rightarrow 0$. g_n piece wise affine function: $g_n(x_j^n) = g(x_j^n)$, $j \in [k_n]$. $\lim_{n \rightarrow \infty} g_n(x) = g(x)$, whenever $x \in [a, b]$ is a point of continuity of g .

Using first case we find for every $n \in \mathbb{N}$ a point $\xi_n \in [a, b]$, such that

$$\int_a^b fg_n = g_n(a) \int_a^{\xi_n} f + g_n(b) \int_{\xi_n}^b f.$$

We may assume, by going to a subsequence, that $\lim \xi_n = \xi \in [a, b]$.

$$\sup \{|g_n(x)| \mid x \in [a, b], n \in \mathbb{N}\} \leq \max \{|g(a)|, |g(b)|\}$$

$$\int_a^b fg_n \rightarrow \int_a^b fg \stackrel{?}{=} g(a) \int_a^\xi f + g(b) \int_\xi^b f \leftarrow g_n(a) \int_a^\xi f + g_n(b) \int_{\xi_n}^b f = \int_a^b fg_n.$$

└

□

Věta 1.23 (?)

Let $G \subset \mathbb{R}^n$ be open nonempty and $f : G \rightarrow \mathbb{R}$ be Lipschitz on G . Then f is differentiable almost everywhere on G .

Lemma 1.24

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous and $i \in \{1, \dots, n\}$. Then the set

$$D_i := \left\{ x \in \mathbb{R}^n \mid \frac{\partial f}{\partial x_i}(x) \text{ exists} \right\}$$

is Borel.

┌

Důkaz

$$\begin{aligned} \exists \frac{\partial f}{\partial x_i}(x) &\Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0 \forall t_1, t_2 \in (-\delta, \delta) \setminus \{0\} : \left| \frac{f(x + t_1 e^i) - f(x)}{t_1} - \frac{f(x + t_2 e^i) - f(x)}{t_2} \right| < \varepsilon \Leftrightarrow \\ &\Leftrightarrow \forall \varepsilon \in \mathbb{Q}^+ \exists \delta \in \mathbb{Q}^+ \forall t_1, t_2 \in ((-\delta, \delta) \cap \mathbb{Q}) \setminus \{0\} : \left| \frac{f(x + t_1 e^i) - f(x)}{t_1} - \frac{f(x + t_2 e^i) - f(x)}{t_2} \right| < \varepsilon. \end{aligned}$$

└

□

Lemma 1.25

Let $\beta > 0$. Let $A \neq \emptyset$ open, f_α , $\alpha \in A$, be β -Lipschitz on \mathbb{R}^n and there exists $x \in \mathbb{R}^n$ such that $\sup_{\alpha \in A} f_\alpha(x) < \infty$. Then the function $z \mapsto \sup_{\alpha \in A} f_\alpha(z)$ is β -Lipschitz on \mathbb{R}^n .

┌

Důkaz

Let $u, v \in \mathbb{R}^n$. Then we have $|f_\gamma(u) - f_\gamma(v)| \leq \beta \cdot \|u - v\|$ for any $\gamma \in A$.

$$f_\gamma(u) \leq f_\gamma(v) + \beta \|u - v\|,$$

$$f_\gamma(u) \leq f_\gamma(x) + \beta \|u - v\| \leq \sup_{\alpha \in A} f_\alpha(x) + \beta \|u - x\| \implies$$

$$\implies \sup_{\gamma \in A} f_\gamma(u) \leq \sup_{\alpha \in A} f_\alpha(x) + \beta \|u - x\| < \infty,$$

So $z \mapsto \sup_{\gamma \in A} f_\gamma(z)$ is well defined.

$$f_\gamma(u) \leq f_\gamma(v) + \beta \|u - v\| \leq \sup_{\alpha \in A} f_\alpha(v) + \beta \|u - v\|,$$

$$\sup_{\gamma \in A} f_\gamma(u) \leq \sup_{\alpha \in A} f_\alpha(v) + \beta \|u - v\|,$$

$$\sup_{\gamma \in A} f_\gamma(u) - \sup_{\alpha \in A} f_\alpha(v) \leq \beta \|u - v\| \wedge \sup_{\gamma \in A} f_\gamma(v) - \sup_{\alpha \in A} f_\alpha(u) \leq \beta \|v - u\| \implies \beta\text{-Lipschitzness.}$$

└

□

Lemma 1.26

Let $E \subset \mathbb{R}^n$ be nonempty and $f_n : E \rightarrow \mathbb{R}$ be β -Lipschitz. Then there exists β -Lipschitz function $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\tilde{f}|_E = f$.

┌

Důkaz

$\forall x \in E$ we define $f_x : y \mapsto f(x) - \beta\|y - x\|$. „ f_x is β -Lipschitz“:

$$\begin{aligned} |f_x(u) - f_x(v)| &= |f(x) - \beta\|u - x\| - f(x) + \beta\|v - x\|| = \\ &= |\beta| \cdot |\|v - x\| - \|u - x\|| \leq |\beta| \cdot \|v - u\|. \end{aligned}$$

$$\sup_{x \in E} (f(x) - \beta\|y - x\|) = \sup_{x \in E} f_x(y) \leq f(y).$$

We set $\tilde{f}(y) = \sup_{x \in E} f_x(y)$. By previous lemma \tilde{f} is β -Lipschitz on \mathbb{R}^n .

It remains to prove that „ $\tilde{f}|_E = f$ “: $z \in E : \tilde{f}(z) \geq f_z(z) = f(z)$,

$$f_x(z) = f(x) - \beta\|z - x\| \leq f(z) \implies \tilde{f}(z) \leq f(z).$$

└

□

Důkaz (Theorem ?)

From previous lemma WLOG f is defined on \mathbb{R}^n . Let E be the set of those points, where at least one partial derivative doesn't exist.

$$E := \bigcup_{i=1}^n (\mathbb{R}^n \setminus D_i).$$

Using 1-dimensional Radmachor theorem, Fubini theorem, and measurability of D_i , we get

$$\lambda_n(\mathbb{R}^n \setminus D_i) = 0, \quad \forall i \in [n].$$

So $\lambda_n(E) = 0$.

$$p, q \in \mathbb{Q}^n, m \in \mathbb{N} :$$

$$S(p, q, m) := \left\{ x \in \mathbb{R}^n \mid \forall i \in [n] \ \forall t \in \left(-\frac{1}{m}, \frac{1}{m} \right) \setminus \{0\} : p_i \leq \frac{f(x + te_i) - f(x)}{t} \leq q_i \right\}.$$

$S(p, q, m)$ is Borel. $\tilde{S}(p, q, m)$ be the set of $x \in S(p, q, m)$ such that x is a point of density of $S(p, q, m)$. From theorem above $\lambda_n(S(p, q, m) \setminus \tilde{S}(p, q, m)) = 0$.

$$N := \bigcup \left\{ S(p, q, m) \setminus \tilde{S}(p, q, m) \mid p, q \in \mathbb{Q}, m \in \mathbb{N} \right\}. \quad \lambda_n(N) = 0.$$

$x \in \mathbb{R}^n \setminus (N \cup E)$, $\varepsilon \in (0, 1)$. Choose $p, q \in \mathbb{Q}^n$ such that $q_i - \varepsilon \leq p_i < \frac{\partial f}{\partial x_i}(x) < q_i$, $i \in [n]$. We can find $m \in \mathbb{N}$ such that $x \in S(p, q, m) =: S$. We find $\delta \in (0, \frac{1}{m})$ such that $\lambda_n(B(x, r) \setminus S) \leq (\frac{\varepsilon}{2})^n \lambda_n(B(x, r))$ for every $r \in (0, 2\delta)$.

Notice that the set $B(x, (1 + \varepsilon)\tau) \setminus S$ does not contain a ball with radius $\varepsilon\tau$ whenever $\tau \in (0, \delta)$. So for contradiction assume, that we can find ball with radius $\varepsilon\tau$.

$$C_n \cdot (\varepsilon\tau)^n = \lambda_n(B(d, \varepsilon\tau)) \leq \lambda_n(B(x, (1 + \varepsilon)\tau) \setminus S) \leq \left(\frac{\varepsilon}{2}\right)^n C_n (1 + \varepsilon)^n \tau^n$$

$$1 \leq \left(\frac{1}{2}\right)^n (1 + \varepsilon)^n < 1.$$

$$y^i := [y_1, y_2, \dots, y_i, x_{i+1}, \dots, x_n], i \in [n]. \quad B_i := B(y^i, \varepsilon \|y - x\|).$$

Then $B_i \cap S \neq \emptyset$. Find points $z^i \in S \cap B_i$, $i \in [n - 1]$ and denote $w^i = z^{i-1} + (y_i - x_i)e_i$, $i \in [n]$.

We have $p_i \leq \frac{f(w^i) - f(z^{i-1})}{y_i - x_i} \leq q_i$, if $x_i \neq y_i$, $p_i < \frac{\partial f}{\partial x_i}(x) < q_i$. Therefore we have

$$\left| f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x) \cdot (y_i - x_i) \right| \leq (q_i - p_i) \cdot |y_i - x_i| \leq \varepsilon \|y - x\|.$$

$$\left| f(y) - f(x) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) \cdot (y_i - x_i) \right| \leq *.$$

$$f(y) - f(x) = \sum_{i=1}^n (f(y^i) - f(y^{i-1})) =$$

$$= \sum_{i=1}^n ((f(w^i) + f(y^i) - f(w^i)) - (f(z^{i-1}) + f(y^{i-1}) - f(z^{i-1}))) =$$

$$= \sum_{i=1}^n (f(w^i) - f(z^{i-1})) + \sum_{i=1}^n (f(y^i) - f(w^i)) - \sum_{i=1}^n (f(y^{i-1}) - f(z^{i-1})) \implies * =$$

$$\left| \sum_{i=1}^n \left(f(w^i) - f(z^{i-1}) - \frac{\partial f}{\partial x_i}(x) \cdot (y_i - x_i) \right) + \sum_{i=1}^n (f(y^i) - f(w^i)) - \sum_{i=1}^n (f(y^{i-1}) - f(z^{i-1})) \right| \leq$$

$$\leq n \cdot \varepsilon \|y - x\| + \sum_{i=1}^n |f(y^i) - f(w^i)| + \sum_{i=1}^n |f(y^{i-1}) - f(z^{i-1})| \leq$$

$$\leq n \cdot \varepsilon \|y - x\| + 2n \cdot 2\varepsilon \|y - x\| \cdot \beta = \varepsilon(n + 4n\beta) \|y - x\|.$$

□

Poznámka

$$|f(y^i) - f(w^i)| \leq \beta \cdot \|y^i - w^i\| = ? < \varepsilon \|y - x\|.$$

If H is Hilbert space and $f : H \rightarrow \mathbb{R}$ is Lipschitz, then there exists $x \in H$ such that $f'(x)$ in Fréchet sence. $\exists L : H \rightarrow \mathbb{R}$ linear and continuous

$$\lim_{y \rightarrow x} \frac{|f(y) - f(x) - L(y - x)|}{\|y - x\|} = 0.$$

┌ *Důkaz*

└ Difficult. □

Poznámka

There exists a closed measure zero set $F \subset \mathbb{R}^2$ such that any Lipschitz function in \mathbb{R}^2 is differentiable at same point of F .

┌ *Důkaz*

└ Difficult. □

1.6 Maximal operator (was in 3rd lecture)

Definice 1.9

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be measurable. For $x \in \mathbb{R}^n$ we define

$$Mf(x) = \sup_{B \in \mathcal{B}, x \in B} \frac{1}{\lambda_n(B)} \int_B |f|.$$

Věta 1.27 (Hardy-Littlewood-Weiner)

- If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then Mf is finite almost everywhere.
- There exists $c > 0$ such that for every $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ we have

$$\lambda_n(\{x \in \mathbb{R}^n | Mf(x) > \alpha\}) \leq \frac{c}{\alpha} \|f\|_1.$$

- Let $p \in (1, \infty]$. Then there exists A such that for every $f \in L^p(\mathbb{R}^n)$ we have $\|Mf\|_p \leq A \|f\|_p$.

┌ *Důkaz*

└ TODO? □

1.7 Lipschitz functions and Sobolev spaces

Věta 1.28

Let $U \subset \mathbb{R}^n$ be open. Then $f : U \rightarrow \mathbb{R}$ is local Lipschitz on U if and only if $f \in W_{loc}^{1,\infty}(U)$.

┌
Důkaz
Skipped.
└

□

2 Hausdorff measures

2.1 Basic notions

Poznámka
 (P, ϱ) metric space.

Definice 2.1 (Hausdorff measure)

Let $p > 0$, $A \subset P$. Denote

$$\varkappa_p(A, \delta) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } A_j)^p \mid A \subseteq \bigcup_{j=1}^{\infty} A_j, \text{diam } A_j \leq \delta \right\},$$
$$\varkappa_p(A) = \lim_{\delta \rightarrow 0_+} \varkappa_p(A, \delta).$$

The function is called p -dimensional Hausdorff measure.

Definice 2.2

An outer measure γ on P is called metric outer measure, if for every $A, B \subset P$ with $\inf \{ \varrho(a, b) \mid a \in A, b \in B \} > 0$ we have $\mu(A \cup B) = \mu(A) + \mu(B)$.

Věta 2.1

Let γ be a metric outer measure on P . Then every Borel subset of P is γ measurable.

┌

Důkaz

It is sufficient to prove that any closed set $F \subset P$ is γ -measurable:

$$\gamma(T) = \gamma(T \setminus F) + \gamma(T \cap F).$$

$$P_0 = \{x \in T \mid \text{dist}(x, F) \geq 1\}, \quad P_j = \left\{x \in T \mid \frac{1}{j+1} \leq \text{dist}(x, F) < \frac{1}{j}\right\}, j \in \mathbb{N}.$$

$$\begin{aligned} \sum_{j=0}^m \gamma(P_{2j}) &= \gamma\left(\bigcup_{j=0}^m P_{2j}\right) \leq \gamma(T) \wedge \\ \wedge \sum_{j=0}^m \gamma(P_{2j-1}) &= \gamma\left(\bigcup_{j=0}^m P_{2j-1}\right) \leq \gamma(T) \implies \\ &\implies \sum_{j=0}^{\infty} \gamma(P_j) < \infty. \end{aligned}$$

$$\begin{aligned} \gamma(T \cap F) + \gamma(T \setminus F) &= \gamma(T \cap F) + \gamma\left(\bigcup_{j=0}^{\infty} P_j\right) \leq \\ &\leq \gamma(T \cap F) + \gamma\left(\bigcup_{j=0}^m P_j\right) + \gamma\left(\bigcup_{j=m+1}^{\infty} P_j\right) = \\ &= \gamma\left((T \cap F) \cup \bigcup_{j=0}^m P_j\right) + \gamma\left(\bigcup_{j=m+1}^{\infty} P_j\right) \leq \\ &\leq \gamma(T) + \sum_{j=m+1}^{\infty} \gamma(P_j) \rightarrow \gamma(T) \implies \\ &\implies \gamma(T \cap F) + \gamma(T \setminus F) \leq \gamma(T). \end{aligned}$$

└

□

Věta 2.2

κ_p is a metric outer measure.

┌

Důkaz

„ $\kappa_p(\emptyset) = 0$ “: $\kappa_p(\emptyset, \delta) = 0 \implies \kappa_p(\emptyset) = 0$.

„ σ -subadditivity of κ_p “: $M_i \subset P$, $i \in \mathbb{N}$. If $\kappa_p(M_{i_0}) = \infty$ for some $i_0 \in \mathbb{N}$, then we have $\kappa_p(\bigcup_{i \in \mathbb{N}} M_i) \leq \sum_{i=1}^{\infty} \kappa_p(M_i)$.

So we will assume that $\kappa_p(M_i) < \infty$ for every $i \in \mathbb{N}$. We choose $\varepsilon > 0$, $\delta > 0$. For every $i \in \mathbb{N}$ we find sets $A_{i,j}$, $j \in \mathbb{N}$ such that

$$M_i \subset \bigcup_{j \in \mathbb{N}} A_{i,j}, \quad \sum_{j=1}^{\infty} (\text{diam } A_{i,j})^p < \kappa_p(M_i, \delta) + \frac{\varepsilon}{2^i}, \quad \text{diam } A_{i,j} < \delta.$$

Then

$$\kappa_p \left(\bigcup_{i=1}^{\infty} M_i, \delta \right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\text{diam } A_{i,j})^p \leq \sum_{i=1}^{\infty} \left(\kappa_p(M_i, \delta) + \frac{\varepsilon}{2^i} \right) \leq \left(\sum_{i=1}^{\infty} \kappa_p(M_i) \right) + \varepsilon.$$

$$\kappa_p \left(\sum_{i=1}^{\infty} M_i \right) \leq \sum_{i=1}^{\infty} \kappa_p(M_i) + \varepsilon.$$

„ κ_p is a metric outer measure“: Let $A, B \subset P$ with $\inf \{\varrho(a, b) | a \in A, b \in B\} = \delta_0 > 0$. Take $\delta \in (0, \delta_0)$. For $M \subseteq A \cup B$ with $\text{diam } M < \delta$, we have either $M \subseteq A$ or $M \subseteq B$. This implies

$$\kappa_p(A \cup B, \delta) = \kappa_p(A, \delta) + \kappa_p(B, \delta).$$

└

□

Důsledek

Every Borel subset of P is κ_p -measurable.

┌

Důkaz

By previous two theorems.

└

□

Důsledek (???)

TODO? $P = \mathbb{R}^n$ is translation invariant.

Věta 2.3

Let $k, n \in \mathbb{N}$, $k \leq n$, $K = [0, 1]^k \times \{0\}^{n-k} \subset \mathbb{R}^n$. Then $0 < \kappa_k(K) < \infty$.

┌
Důkaz

„ $\varkappa_k(K) < \infty$ “: Let $\delta > 0$. We find $m \in \mathbb{N}$ such that $\frac{\sqrt{k}}{m} < \delta$. Then set K will be splitted into „ $\frac{1}{m^k}$ cubes“, K_j for $j \in [m^k]$.

$$\varkappa_k(k, \delta) \leq m^k \left(\frac{\sqrt{k}}{m} \right)^k = (\sqrt{k})^k.$$

$$\varkappa_k(K) \leq (\sqrt{k})^k < \infty.$$

„ $\varkappa_k(K) > 0$ “: $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $\Pi : (x_1, \dots, x_n) = (x_1, \dots, x_k)$, $\mu(A) = \lambda^{k*}(\Pi(A))$, $A \subset \mathbb{R}^n$. We have $\mu(A) \leq (\text{diam } A)^k$. Let $\{A_j\}_{j=1}^\infty$ be a sequence of sets in \mathbb{R}^n such that $\bigcup_{j=1}^\infty A_j = K$.

$$\sum_{j=1}^\infty (\text{diam } A_j)^k \geq \sum_{j=1}^\infty \mu(A_j) \geq \mu(K) = 1 \implies \varkappa_k(K) \geq 1.$$

└

□

Definice 2.3 (Notation)

$k, n \in \mathbb{N}$, $k \leq n$:

$$\varkappa_k := \varkappa_k(K) \in (0, \infty), \quad H^k := \frac{1}{\varkappa_k} \varkappa_k, \quad \varkappa_k := \frac{\Gamma\left(\frac{1}{2}\right)^k}{\Gamma\left(\frac{k}{2} + 1\right) 2^k}$$

$$\left(\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \quad s > 0 \right).$$

Věta 2.4

Let $k, n \in \mathbb{N}$, $k \leq n$, and $A \subset \mathbb{R}^n$. Then there exists a Borel set $B \subset \mathbb{R}^n$ such that $A \subset B$ and $\varkappa_k(A) = \varkappa_k(B)$.

┌
Důkaz

If $\varkappa_k(A) = \infty$, then we set $B = \mathbb{R}^n$. Assume $\varkappa_k(A) < \infty$. For each $j \in \mathbb{N}$ we find an F_j set $f_j \subset \mathbb{R}^n$ such that $\varkappa_k\left(F_j, \frac{1}{j}\right) < \varkappa_k\left(A, \frac{1}{j}\right) + \frac{1}{j}$ and $A \subset F_j$. $A \subseteq \bigcup \overline{A_i} \sum (\text{diam } A_i)^k$.

We set $B = \bigcap_{j=1}^\infty F_j$. Then B is Borel and $A \subset B$. We estimate

$$\varkappa_k\left(A, \frac{1}{j}\right) \leq \varkappa_k\left(B, \frac{1}{j}\right) \leq \varkappa_k\left(F_j, \frac{1}{j}\right) < \varkappa_k\left(A, \frac{1}{j}\right) + \frac{1}{j} :$$

$$\varkappa_k(A) \leq \varkappa_k(B) \leq \varkappa_k(A) \implies \varkappa_k(A) = \varkappa_k(B).$$

└

□

Věta 2.5

Let $n \in \mathbb{N}$ and $A \subset \mathbb{R}^n$. Then $H^n(A) = \lambda^{n*}(A)$.

┌

Důkaz

We have $H^n([0, 1]^n) = \lambda^{n*}([0, 1]^n)$. Since H^n and λ^{n*} are translation invariant, we obtain $H^n(Q) = \lambda^{n*}(Q)$ for any $Q \subset \mathbb{R}^n$ of the form

$$Q = \prod_{i=1}^m \left[\frac{l_i}{2^m}, \frac{l_i + 1}{2^m} \right],$$

$l_i \in \mathbb{Z}$, $i \in [m]$, $m \in \mathbb{N}_0$. Denote this sets by \mathcal{Q} .

Let $G \subseteq \mathbb{R}^n$ be open. Then there exists $\tilde{\mathcal{Q}} \subset \mathcal{Q}$ such that $\tilde{\mathcal{Q}}$ is a disjoint family and $\bigcup \tilde{\mathcal{Q}} = G$. (Proof: $Q_1, Q_2 \in \mathcal{Q} \implies Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$ or $Q_1 \cap Q_2 = \emptyset$. So $G = \bigcup \{Q \in \mathcal{Q} | Q \subset G\} =: \bigcup \mathcal{S}$. $Q \in \mathcal{S} \implies M(Q) \in \mathcal{S}$ maximal with inclusion. Then $\tilde{\mathcal{Q}} = \{M(Q) | Q \in \mathcal{S}\}$.)

$$\implies H^n(G) = \lambda^{n*}(G) \implies H^n = \lambda^{n*} \text{ on Borel sets.}$$

For $A \subseteq \mathbb{R}^n$ we find Borel sets $B_1, B_2 \subseteq \mathbb{R}^n$, $A \subseteq B_1$, $H^n(A) = H^n(B_1)$, $A \subseteq B_2$, $\lambda^{n*}(A) = \lambda^{n*}(B_2)$. We set $B = B_1 \cap B_2$. Then we have $H^n(A) = H^n(B) = \lambda^{n*}(B) = \lambda^{n*}(A)$. □

└

Věta 2.6 (Area formula)

Let $k, n \in \mathbb{N}$, $k \leq n$, $G \subseteq \mathbb{R}^k$ be an open set, $\varphi : G \rightarrow \mathbb{R}^n$ be an injective regular mapping and $f : \varphi(G) \rightarrow \mathbb{R}$ be H^k -measurable. Then we have

$$\int_{\varphi(G)} f(x) dH^k(x) = \int_G f(\varphi(t)) \text{vol} \varphi'(t) d\lambda^k(t),$$

if the integral on the right side converges. ($\text{vol} L = \sqrt{\det(L^T L)}$ for linear mapping L .)