

# 1 Introduction

*Poznámka* (Literature)

„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

## 1.1 Differentiability

### Definition 1.1 (Differentiable)

A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable (also holomorphic) at a point  $z_0 \in \mathbb{C}$  if the following limit exists

$$\lim_{|h| \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call  $f'(z_0)$  the derivative of  $f$  at  $z_0$ . A function  $f$  is differentiable on a domain (open connected subset of  $\mathbb{C}$ ) if its differentiable for all points of this domain.

*Poznámka* (Writing complex numbers in cartesian coordinates)

$z = x + iy$ , for  $x, y \in \mathbb{R}$ , we can write a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  in terms of two functions  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

### Věta 1.1 (Cauchy–Riemann equations)

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function on an open subset of  $\mathbb{C}$ . Considering  $f = u + iv$ , then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

### Definition 1.2 (Orientability, orientation-preserving function)

Define an equivalence relation on the set of all bases of  $\mathbb{R}^2$  by saying that  $B_1 \sim B_2$  iff the determinant of the change of basis matrix is positive.

A function  $f : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}^2$  is said to be orientation-preserving if on an open dense subset of  $U$ , the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

*Důsledek*

Let  $f$  be a non-constant holomorphic function, then  $f$  is orientation-preserving.

*Důsledek*

Since  $f$  is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \stackrel{\text{C-R}}{=} \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \geq 0.$$

Since  $f$  is non-constant, the inequality is strict on a dense open subset of the domain of definition.

### **Věta 1.2** (Open mapping theorem)

*A non-constant holomorphic function  $f$  is open (that is if  $U$  is an open subset of  $\mathbb{C}$ , then  $f(U)$  is also open).*

## 1.2 Integration

### **Definice 1.3**

For a path  $\gamma$  (smooth function,  $\gamma : \mathbb{R} \supset [a, b] \rightarrow \mathbb{C}$ ) we define

$$\int_{\gamma} f(x) dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

### **Definice 1.4** (Continuous deformation)

For  $\gamma, \mu : [a, b] \rightarrow U$  ( $U$  simply connected), paths with the same endpoints ( $\gamma(a) = \mu(a)$  and  $\gamma(b) = \mu(b)$ ). Then a continuous deformation  $\gamma$  into  $\mu$  is a continuous function  $H : [a, b] \times [0, 1] \rightarrow U \subseteq \mathbb{C}$  such that  $H(s, 0) = \gamma(s)$ ,  $H(s, 1) = \mu(s)$ ,  $H(a, t) = z_a := \gamma(a) = \mu(a)$  and  $H(b, t) = z_b := \gamma(b) = \mu(b)$ .

### **Věta 1.3**

*Suppose that  $\gamma, \mu : [a, b] \rightarrow U$  ( $U$  simply connected) are related by a continuous deformation of paths  $H$ . Then for any holomorphic function  $f$  on  $U$  we have*

$$\int_{\gamma} f(z) dz = \int_{\mu} f(z) dz.$$

┌

*Důkaz* (Partial proof assuming  $H$  admits partial derivatives)

For any  $t \in [0, 1]$  we integrate the function  $INT(t) = \int_{H(\cdot, t)} f(z) dz$ . Consider the derivative of  $INT(t)$  with respect to  $t$ :

$$\begin{aligned} \frac{d}{dt}(INT(t)) &= \frac{d}{dt} \left( \int_a^b f(H(s, t)) \frac{\partial H}{\partial s}(s, t) ds \right) \stackrel{\text{Leibniz}}{=} \stackrel{\text{chain rule}}{=} \\ &= \int_a^b f'(H(s, t)) \frac{\partial H}{\partial t}(s, t) \cdot \frac{\partial H}{\partial s}(s, t) + f(H(s, t)) \frac{\partial^2 H}{\partial s \partial t}(s, t) ds = \\ &= \int_a^b \frac{d}{ds} \left[ f(H(s, t)) \frac{\partial H}{\partial t} \right] ds = \\ &= f(H(s, t)) \frac{\partial H}{\partial t} \Big|_{s=a}^{s=b} \stackrel{\text{constant endpoints}}{=} 0. \end{aligned}$$

Having derivative identically equal to 0, means that  $INT(t)$  is a constant function and  $\int_{\gamma} f(z) dz = INT(0) = INT(1) = \int_{\mu} f(z) dz$ . □

*Důsledek*

Let  $U$  be a simply connected subset of  $\mathbb{C}$  and  $f : U \rightarrow \mathbb{C}$  a holomorphic function. For any closed path whose image is inside  $U$ ,  $\int_{\gamma} f(z) dz = 0$ .

*Důkaz* (Sketch)

The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path  $c$ .

$$\int_{\gamma} f(z) dz = \int_c f(z) dz = \int_a^b f(c(z)) \cdot c'(z) dz = \int_a^b f(c(z)) \cdot 0 dz = 0$$

└

□

*Příklad*

Let  $U$  be a simple connected domain and  $f : U \rightarrow \mathbb{C}$  a holomorphic function on  $U \setminus \{z_0\}$ . For  $j = 1, 2$ , let  $\gamma_j$  be a path parametrizing a circle centered at  $z_0$  of radius  $r_j$ , oriented counterclockwise and completely contained in  $U$ . Show that  $\oint_{\gamma_1} f(z) dz = \oint_{\gamma_2} f(z) dz$ .

## 1.3 Cauchy's integral formula

**Věta 1.4** (Cauchy's integral formula)

Let  $\gamma$  be a loop around  $z \in \mathbb{C}$ , and  $f : U \rightarrow \mathbb{C}$  a holomorphic function. For  $U$  a neighbourhood of  $\gamma$ ,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} dw.$$

Důkaz

Conway 1978, Chapter IV.

□

Důsledek

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0 + z_0 - z} dw = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \cdot \left( \frac{1}{1 - \frac{z - z_0}{w - z_0}} \right) dw = \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \left( \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} \right) dw = \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^n} \right) (z - z_0)^n. \end{aligned}$$

For sufficiently "small" (shrunk)  $\gamma$ . So  $f$  is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around  $z_0$  converges to  $f$  in a neighbourhood of  $z_0$ ).

### Definition 1.5 (Pole)

Given a positive integer  $n$ , a complex function  $f$  has pole of order  $n$  at the point  $z_0 \in \mathbb{C}$  if  $(z - z_0)^n f(z)$  is holomorphic at  $z_0$  but  $(z - z_0)^{n-1} f(z)$  is not.

Příklad

Show that if  $f$  has a pole of order  $n$  at  $z_0 \in \mathbb{C}$ . Then it admits a Laurent expansion  $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$  with  $a_{-n} \neq 0$ .

### Definition 1.6 (Residue)

Let  $f$  have a pole of order  $n$  at the point  $z_0 \in \mathbb{C}$ . Then the residue of  $f$  at  $z_0$  is the  $k = -1$  coefficient of the Laurent expansion of  $f$  at  $z_0$ .

Příklad

Show that if  $f$  has a pole of order 1 at  $z_0$ , then the residue of  $f$  at  $z_0$  can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)

Let  $\gamma : [a, b] \rightarrow U \subset \mathbb{C}$  be a simple closed path, bounding a domain  $W$  containing the points  $z_1, \dots, z_m$ . Assume that  $f$  is holomorphic on  $U \setminus \{z_1, \dots, z_m\}$  and has poles at  $\{z_1, \dots, z_m\}$ .

Show that

$$\oint_{\gamma} f(z)dz = 2\pi i \sum_{j=1}^m \text{res}_{z=z_j} f(z).$$

TODO!!!

## 1.4 (Real) Projective space

*Poznámka* (Building structures)

Set  $\rightarrow$  Topology  $\rightarrow$  Differential structure (atlas)  $\rightarrow$  Riemann metric  $\rightarrow$  Connection...

### Definition 1.7 (Real projective space)

The set  $\mathbb{P}^n(\mathbb{R})$  is defined to be either of the following bijective sets: Lines through the origin in  $\mathbb{R}^{n+1}$ ; Equivalence classes of  $(n+1)$ -tuples of real numbers  $(x_0, \dots, x_n) \neq (0, \dots, 0)$ , such that for any real number  $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ :  $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ .

*Příklad*

Confirm that the sets above are in bijection with each other.

*Poznámka* (Notation)

We will often denote a point in  $\mathbb{P}^n(\mathbb{R})$  as the equivalence class  $[x_0, \dots, x_n]$ .

### Definition 1.8 (Topology of $\mathbb{P}^n(\mathbb{R})$ )

We give a topology to  $\mathbb{P}^n(\mathbb{R})$  by endowing it with following quotient topology: consider the surjection  $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \rightarrow \mathbb{P}^n(\mathbb{R})$ ,  $(x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]$ . A set  $U \subset \mathbb{P}^n(\mathbb{R})$  is defined to be open if  $\pi^{-1}(U) := \{x \in \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \mid \pi(x) \in U\}$  is open in  $\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}$ .

That is we give  $\mathbb{P}^n(\mathbb{R})$  the finest topology that makes  $\pi$  continuous.

*Příklad*

Check that for  $\mathbb{C}$  we can define  $\mathbb{P}^n(\mathbb{C})$  or  $\mathbb{CP}^n$  the same way.

*Příklad* (Projective space)

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  is an abelian group. Let  $\mathbb{R}^*$  act on  $\mathbb{R}^{n+1}$  by component wise multiplication. When a general group  $G$  acts on a set  $X$  we have equivalence relation  $x \sim y$  if  $y = g \circ x$ . We call the equivalence classes the orbits of  $G$ . So  $\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}) / \mathbb{R}^*$ .

Sphere quotient: Let  $S^n \subseteq \mathbb{R}^{n+1}$ . Denote the unit sphere. Then the group  $\mathbb{Z}_2 = \{+1, -1\}$  act on the sphere by  $\pm 1(x_0, \dots, x_n) = (\pm x_0, \dots, \pm x_n)$ . Then  $S^n / \mathbb{Z}_2 = \mathbb{P}^n(\mathbb{R})$ .

Disk model: Consider the  $n$ -dimensional closed unit disk  $\overline{\mathbb{D}^n} \subseteq \mathbb{R}^n$ , and the equivalence

relation on the points of the boundary:  $x \sim -x$  if  $\|x\| = 1$ . Then  $\mathbb{P}^n(\mathbb{R})$  is the quotient (collection of equivalence classes), i.e.  $\overline{D^n} \setminus \sim \simeq \mathbb{P}^n(\mathbb{R})$ .

*Příklad*

Conclude from either of these models of  $\mathbb{P}^n(\mathbb{R})$  that as a topological space,  $\mathbb{R}^n(\mathbb{P})$  is compact and Hausdorff.

*Poznámka*

Now we come to the smooth manifold structures. Let's start with  $\mathbb{P}^1(\mathbb{R})$ . Define

$$U_x := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x = 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y]) = \frac{x}{y}.$$

Similarly, we define a second chart:

$$U_y := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | y = 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y]) = \frac{y}{x}.$$

┌ *Příklad*

Check that  $U_x, U_y$  are open and that  $\varphi_x, \varphi_y$  are homeomorphisms.

┌ *Důkaz*

Consider the translation functions:

$$U = U_x \cap U_y = \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x, y \neq 0\}, \quad \varphi_x(U) = \varphi_y(U) = \mathbb{R} \setminus \{0\}.$$

The translation function  $T_{x,y} := \varphi_y \circ (\varphi_x)^{-1}$  sends, for  $y \neq 0$ :

$$T_{x,y} : y \xrightarrow{(\varphi_x)^{-1}} [1, y] = \left[ \frac{1}{y}, 1 \right] \xrightarrow{\varphi_y} \frac{1}{y}.$$

Which is smooth on the domain  $\mathbb{R} \setminus \{0\}$ .

└ TODO smooth. Thus  $\mathbb{P}^1(\mathbb{R})$  is a smooth manifold. □

*Příklad*

Show that  $\mathbb{P}^1(\mathbb{R})$  is homomorphic to the circle  $S^1$ . We call  $\mathbb{P}^1(\mathbb{R})$  the real projective line.

*Příklad*

Try to show that  $\mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$  is a smooth manifold.

*Příklad*

For  $\mathbb{P}^2(\mathbb{R})$  the followings charts form atlas:

$$U_x := \{[x, y, z] | x \neq 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y, z]) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

$$U_y := \{[x, y, z] | y \neq 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y, z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$U_z := \{[x, y, z] | z \neq 0\}, \quad \varphi_z : U_z \rightarrow \mathbb{R}, \quad \varphi_z([x, y, z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Check these are open subsets and homeomorphisms, with smooth transformation functions. And extend this to  $\mathbb{P}^n(\mathbb{R})$ .

## 1.5 Compact surfaces

### Definition 1.9 (Surface)

A surface is a manifold of real dimension 2.

*Například*

$\mathbb{R}^2$ ,  $\mathbb{C}$ , and any of their open subsets are surfaces.  $S^2$  is a compact surface, as is  $\mathbb{P}^2(\mathbb{R})$ .

### Definition 1.10 (Connected surface)

Given two connected surfaces  $S_1$  and  $S_2$ , the connected surface  $S_1 \# S_2$  is the surface obtained by removing an open disc from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

*Příklad*

At the level of topological spaces, show that the operation  $\#$  is well defined up to homeomorphism, that is, show that the choice of disks in  $S_1$  and  $S_2$  does not change the definition of  $S_1 \# S_2$  / homeomorphism.

*Příklad*

Show that  $\#$  gives the set of homeomorphism classes of connected compact surfaces the structure of a monoid. (Which surface is the identity of the monoid?)

### Věta 1.5 (Classification of compact surfaces)

Any connected, compact surfaces is homeomorphic to exactly one surface in the following list:

- $S^2$ ;
- $T^{\#g} := T \# \dots \# T, g \in \mathbb{N}_0$ ;

- $\mathbb{P}^2(\mathbb{R})^{\#n} := \mathbb{P}^2(\mathbb{R}) \# \dots \# \mathbb{P}^2(\mathbb{R}), n \in \mathbb{N}_0.$

*Poznámka* (Deep fact)

For  $d \leq 3$ , if two  $d$ -dimensional manifolds are homeomorphic, then they are diffeomorphic.

## 2 Riemann surfaces

### Definice 2.1 (Riemann surface)

A Riemann surface is a complex analytic manifold of dimension 1:

- $X$  is a Hausdorff, connected topological space;
- for all  $x \in X$ , there is a homeomorphism  $\varphi_x : U_x \rightarrow V_x$ , where  $U_x$  is an open neighbourhood of  $x \in X$ ,  $V_x$  is an open set in  $\mathbb{C}$ ;
- for any  $U_x, U_y$  such that  $U_x \cap U_y \neq \emptyset$ , the transition function  $T_{x,y} := \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \rightarrow \varphi_y(U_x \cap U_y)$  is holomorphic.

*Poznámka*

We saw in the first lecture that a holomorphic preserves orientation when thought of as a function from the real plane to itself. Since our transition functions are holomorphic, any Riemann surface is orientable.

*Příklad* (The complex projective line)

Just as for  $\mathbb{P}^1(\mathbb{R})$ , we define  $\mathbb{P}^1(\mathbb{C})$  to be the set whose elements are complex 1-dimensional subspaces of  $\mathbb{C}^2$ .

Let  $U_1 = U_2 := \mathbb{C}$  and define  $g : U_1 \setminus \{0\} \rightarrow U_2 \setminus \{0\}, z \mapsto \frac{1}{z}$ . We define  $\mathbb{P}^1(\mathbb{C})$  to be the quotient  $\mathbb{P}^1(\mathbb{C}) := U_1 \coprod U_2 / (z \sim g(z))$ .

*Příklad* (Show that)

As a set  $\mathbb{P}^1(\mathbb{C})$  is  $\mathbb{C}$  plus a point.

As a topological space  $\mathbb{P}^1(\mathbb{C})$  is the one point compactification of  $\mathbb{C}$ .

Conclude from the previous sentence that  $\mathbb{P}^1(\mathbb{C})$  is homeomorphic to the two sphere.

*Poznámka*

In complex analysis  $\mathbb{P}^1(\mathbb{C})$  is known as the Riemann sphere.



### *Poznámka*

For  $i = 1, 2$ , we denote the image of  $U_i$  in the quotient  $U_1 \coprod U_2 / (z \sim g(z))$  by  $[U_i]$ . Note that  $U_i$  define the local coordinate functions:  $\varphi_i : [U_i] \rightarrow U_i, p \mapsto z_i$ , where  $z_i$  is the complex numbers in  $U_i$  such that  $[z_i] = p_i$ . Both  $\varphi_1, \varphi_2$  are homeomorphisms.

We now consider the transition functions: the intersection

$$[U_1] \cap [U_2] = [U_1 \setminus \{0\}] = [U_2 \setminus \{0\}].$$

The image of the intersection under  $\varphi_1$  is  $\varphi_1([U_1] \cap [U_2]) = \mathbb{C} \setminus \{0\}$  (\*). Thus (\*) is the domain of our single transition function  $T = \varphi_2 \circ \varphi_1^{-1}$ . For  $z_1 \neq 0$ , we have  $T : z_1 \xrightarrow{\varphi_1^{-1}} [z_1] = [z_2 := g(z_1) = 1/z_1] \xrightarrow{\varphi_2} z_2 = 1/z_1$ . Thus

$$T : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \quad z \mapsto \frac{1}{z}.$$

Since  $T$  has a pole only at  $z_1 = 0$ , we see that it is holomorphic on  $\mathbb{C} \setminus \{0\}$ . A symmetric (exchange 1 for 2) calculation shows that  $T^{-1}$  is also holomorphic. So  $\mathbb{P}^1(\mathbb{C})$  is a Riemann surface.

### *Příklad (Hopf fibration)*

Consider the 3-dimensional real sphere  $S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4$ . Given a point  $p \in S^3$ ,  $\exists$  a unique line  $l_p$  through the origin and  $p$ . Thus we got a function  $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C}), p \mapsto l_p$ .

Check that  $H$  is continuous and surjective. Since  $S^3$  is closed and bounded, it is compact. Moreover, since the image of a compact set under a continuous function is compact,  $\mathbb{P}^1(\mathbb{C})$  is compact.

What is the fiber of the surjective map  $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C})$ , i.e. what is  $H^{-1}(p)$ , for any point  $p \in \mathbb{P}^1(\mathbb{C})$ . (Hint: It is  $S^1$ .)

This gives us  $S^2 \times S^1 = S^3$  as set. (Not as topological space!)

### *Příklad (Complex tori)*

Definition: Let  $\tau_1$  and  $\tau_2$  be two complex numbers, that are linearly independent. The set of all integral linear combinations of  $\tau_1$  and  $\tau_2$ :

$$:= \{n\tau_1, m\tau_2 | n, m \in \mathbb{Z}\} \subseteq \mathbb{C}$$

is called the lattice of  $\tau_1$  and  $\tau_2$ .

Observe that we can assume that  $\tau_1 = 1$ , and  $\Im(\tau_2) > 0$ , allowing to make simplifying assumption: that our lattice has the form  $= \{n + m\tau | n, m \in \mathbb{Z}, \tau \in \mathbb{H}\}$ , where  $\mathbb{H}$  is the upper half plane.

Consider the quotient space  $T = \mathbb{C}/$ . That is the quotient space with respect to the equivalence relation  $z_2 \sim z_1 \Leftrightarrow z_2 = z_1 + w$  for  $w \in$ .

The canonical projection map  $\pi : \mathbb{C} \rightarrow T$  (i.e.  $\pi(z) = [z]$ ) induces a quotient topology on  $T$  (i.e.  $V \subseteq T$  is open iff  $p^{-1}(V)$  is open in  $\mathbb{C}$ ).

*Příklad*

For  $P$  the closed parallelogram with vertices  $0, 1, \tau, 1 + \tau$ , show that for any  $z \in \mathbb{C}$   $\exists z' \in P$  that  $\pi|_p \rightarrow T$  is surjective. Hence we can restrict our attention to  $p$ .