

Úvod

Poznámka (Organizační úvod)

K ukončení předmětu je třeba pouze udělat zkoušku: 2 příklady na definice, 2 věta-důkaz.

Literatura:

- L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.
- W. Rudin, Analýza v reálném a komplexním oboru, Academia, 2003.

1 Differentiation of measures

1.1 Covering theorems

Definition 1.1 (Vitali cover)

Let $A \subset \mathbb{R}^n$ we say that a system \mathcal{V} consisting of closed balls from \mathbb{R}^n forms Vitali cover of A , if

$$\forall x \in A \forall \varepsilon > 0 \exists B \in \mathcal{V} : x \in B \wedge \text{diam } B < \varepsilon.$$

Definition 1.2 (Notation)

λ_n is Lebesgue measure on \mathbb{R}^n . λ_n^* is outer Lebesgue measure on \mathbb{R}^n . If $B \subset \mathbb{R}^n$ is a ball and $\alpha > 0$, then $\alpha \cdot B$ stands for the ball, which is concentric with B and with α -times greater radius than B .

Věta 1.1 (Vitali)

Let $A \subset \mathbb{R}^n$ and \mathcal{V} be a system of closed balls forming a Vitali cover of A . Then there exists a countable disjoint subsystem $\mathcal{A} \subseteq \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.

Dikaz

First assume that A is bounded. Take an open bounded set $G \subset \mathbb{R}^n$ with $A \subset G$. We set

$$\mathcal{V}^* = \{B \in \mathcal{V} \mid B \subset G\}.$$

Then \mathcal{V}^* is a Vitali cover of A . If there exists a finite disjoint subsystem of \mathcal{V}^* covering A , we are done. So Assume that there is no such subsystem. Mathematical induction:

First step: We set $s_1 = \sup \{\text{diam } B \mid B \in \mathcal{V}^*\}$. We choose a ball $B_1 \in \mathcal{V}^*$ such that $B_1 > \frac{1}{2}s_1$.

k -th step: Suppose that we have already constructed balls B_1, B_2, \dots, B_{k-1} . We set

$$s_k = \sup \left\{ \text{diam } B \mid B \in \mathcal{V}^* \wedge B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset \right\}.$$

We find $B_k \in \mathcal{V}^*$ such that $\text{diam } B_k > \frac{1}{2}s_k > 0$, $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$.

Let $\mathcal{A} = \{B_k \mid k \in \mathbb{N}\}$. It is disjoint, it is countable, it holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$:

$$\begin{aligned} \sum_{i=1}^{\infty} \lambda_n(B_i) &= \lambda_n\left(\bigcup_{i=1}^{\infty} B_i\right) \leq \lambda_n(G) < \infty \implies \\ \implies \lim_{i \rightarrow \infty} s_i &= 0 \implies \lim_{i \rightarrow \infty} \text{diam}(B_i) = 0 \implies \lim_{i \rightarrow \infty} s_i = 0. \end{aligned}$$

We show that

$$\begin{aligned} \forall x \in A \setminus \bigcup \mathcal{A} \quad \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j > i : x \in 5 \cdot B_j \\ \Leftrightarrow A \setminus \bigcup \mathcal{A} \subseteq \bigcup_{j=i+1}^{\infty} 5 \cdot B_j \end{aligned}$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbb{N}$. Denote $\delta = \text{dist}(x, \bigcup_{k=1}^i B_k) > 0$. There exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\text{diam } B < \delta \implies B \cap \bigcup_{k=1}^i B_k = \emptyset$. Then we have $\text{diam } B > s_p$ for some $p \in \mathbb{N}$.

Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geq \text{diam } B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\text{diam } B_j > \frac{1}{2}s_j \geq \frac{1}{2} \text{diam } B \implies 2 \text{diam } B_j \geq \text{diam } B$. This implies that $x \in B \subset 5 \cdot B_j$.

$$\lambda_n^*(A \setminus \bigcup \mathcal{A}) \leq \lambda_n\left(\bigcup_{j=i+1}^{\infty} 5 \cdot B_j\right) \leq \sum_{j=i+1}^{\infty} \lambda_n(5 \cdot B_j) = \sum_{j=i+1}^{\infty} 5^n \lambda_n(B_j) = 5^n \cdot \sum_{j=i+1}^{\infty} \lambda_n(B_j) \rightarrow 0 \implies \lambda_n(A \setminus \bigcup \mathcal{A}) = 0$$

General case (A not bounded): Let $(G_j)_{j=1}^{\infty}$ be a sequence of disjoint open sets such that $\lambda_n(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$. We define $\mathcal{V}_j = \{B \in \mathcal{V}_i \mid B \subseteq G_j\}$. \mathcal{V}_j is a Vitali cover of $A \cap G_j \implies \exists \mathcal{A}_j \subseteq \mathcal{V}_j$ countable disjoint and $\lambda_n(A \cap G_j \setminus \bigcup \mathcal{A}_j) = 0$. We set $\mathcal{A} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$. \mathcal{A} is countable, disjoint and $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$. \square

Definice 1.3

We say that a measure μ on \mathbb{R}^n satisfies Vitali theorem, if for every Vitaly cover \mathcal{V} of $M \subseteq \mathbb{R}^n$ there exists a disjoint countable $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

Poznámka

If μ satisfies Vitali theorem and $\nu \ll \mu$, then ν satisfies Vitali theroem.

Věta 1.2

Set $E \subset \mathbb{R}^n$ be Lebesgue measurable and \mathcal{S} be a finite system of closed balls covering E . Then there exists a disjoint system $\mathcal{L} \subset \mathcal{S}$ such that $\lambda_n(E) \leq 3^n \cdot \sum_{B \in \mathcal{L}} \lambda_n(B)$.

┌

Důkaz

WLOG $\mathcal{S} \neq \emptyset$. Suppose $B_1 \in \mathcal{S}$ with maximal radius among balls from \mathcal{S} .

Suppose that we have already constructed $B_1, \dots, B_{k-1} \in \mathcal{S}$. If possible, choose $B_k \in \mathcal{S}$ disjoint with $\bigcup_{j < k} B_j$ and with maximal radius among balls satisfying this property.

We set $\mathcal{L} = \{B_1, \dots, B_N\}$. We show $E \subseteq \bigcup_{B \in \mathcal{L}} 3 * B = \bigcup_{i=1}^N 3 * B_i$. $x \in E$. Find $B \in \mathcal{S}$ with $x \in B$. Find smallest k with $B \cap B_k \neq \emptyset$. This means $\text{rad}(B) \leq \text{rad}(B_k) \implies x \in B \subseteq 3 * B_k$.

Now $\lambda_n(E) \leq \lambda_n\left(\bigcup_{i=1}^N 3 * B_i\right) \leq \sum_{i=1}^N \lambda_n(3 * B_i) = 3^n \sum_{i=1}^N \lambda_n(B_i)$. □

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Věta 1.3 (Besicovitch theorem)

For each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with the following property:

If $A \subset \mathbb{R}^n$ and $\Delta : A \rightarrow (0, \infty)$ is a bounded function, then there exist sets $A_1, \dots, A_N \subseteq A$ such that

- $\{\overline{B}(x, \Delta x) | x \in A_j\}$ is disjoint for every $j \in [N]$;
- $A \subset \bigcup \left\{ \overline{B}(x, \Delta x) | x \in \bigcup_{i=1}^N A_i \right\}$.

┌ *Důkaz* (Case A is bounded)

Let $R := \sup_A \Delta$. Choose $B_1 := \overline{B}(a_1, \Delta(a_1))$ such that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$.

Assume that we already constructed B_1, \dots, B_{j-1} , $j \geq 2$. $B_{j-1} = \overline{B}(a_{j-1}, \Delta(a_{j-1})) = \overline{B}(a_{j-1}, r_{j-1})$. Let $F_j := A \setminus \bigcup_{i=1}^{j-1} B_i$. If $F_j = \emptyset$ we set $J := j$. If not $B_j := \overline{B}(a_j, \Delta(a_j)) = \overline{B}(a_j, r_j)$, $a_j \in F_j$ and $r_j > \frac{3}{4} \sup_{F_j} \Delta$.

If $F_j \neq \emptyset$ for every $j \in \mathbb{N}$, then we set $J := \infty$. So we have $(B_j)_{j < J}$. If $J < \infty$, then we covered A . „If $J = \infty$, then $A \subset \bigcup_{j < J} B_j$ “:

„ $\lim_{j \rightarrow \infty} r_j = 0$ “: because A is bounded

$$\|a_i - a_j\| \geq r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j = \frac{1}{3}(r_i + r_j) \implies \frac{1}{3} * B_i \cap \frac{1}{3} * B_j = \emptyset.$$

$$\{\frac{1}{3}B_j | j < J\} \text{ is a disjoint family } \implies \sum_{j=1}^{\infty} \lambda_n(\frac{1}{3} * B_j) < \infty.$$

If $A \in A \setminus \bigcup_{j=1}^{\infty} B_j$, then $a \in \bigcap_{j=1}^{\infty} F_j$. We find $j_0 \in \mathbb{N}$ with $r_{j_0} \leq \frac{3}{4}\Delta(a)$. \nless

Fix $k < J$. We set $I = \{i < k | B_i \cap B_k \neq \emptyset\}$, $I_1 = \{i < k_i | B_i \cap B_k \neq \emptyset \wedge r_i < 10r_k\}$, $I_2 = \{i < k_i | B_i \cap B_k \wedge r_i \geq 10r_k\}$. The estimate of I_1 : „We have $\frac{1}{3}B_i \subseteq 15 * B_k$ for every $i \in I_1$ “: Take $x \in \frac{1}{3} * B_i$. Then

$$\|x - a_k\| \leq \|x - a_j\| + \|a_i - a_k\| \leq \frac{1}{3}r_i + r_i + r_k \leq \frac{10}{3}r_k + 10r_k + r_k \leq 15r_k$$

$$\begin{aligned} \lambda_n(\frac{1}{3} * B_i) &= \lambda(\overline{B}(0, 1)) \cdot (\frac{1}{3}r_i)^n \geq \lambda_n(\overline{B}(0, 1)) \cdot (\frac{1}{3} \cdot \frac{3}{4}r_k)^n = \lambda_n(\overline{B}(0, 1)) \cdot \frac{1}{4^n}r_k^n = \\ &= \frac{1}{60^n} \lambda_n(15 * B_k) \implies |I_1| \leq 60^n. \end{aligned}$$

Denote $b_i = a_i - a_k$, vector between centers of balls. Take a family $\{Q_m | 1 \leq m \leq (22n)^n\}$ of closed cubes with edge length $\frac{1}{11n}$ which cover $[-1, 1]^n$. We claim that „for each $1 \leq m \leq (22n)^n$ there is at most one $i \in I_2$ with $\frac{b_i}{\|b_i\|} \in Q_m$ “:

$$i, j \in I_2, i < j, \left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \leq \frac{1}{11}.$$

We have $r_i < \|b_i\| \leq r_i + r_k$ and $r_j < \|b_j\| \leq r_j + r_k$. So $\|b_i\| - \|b_j\| \leq |r_i - r_j| + r_k$. $\|b_j\| \leq r_j + r_k \leq r_j + \frac{1}{10}r_j = \frac{11}{10}r_j$.

$$\begin{aligned} \|a_i - a_j\| &= \|b_i - b_j\| \leq \left\| b_i - \frac{\|b_j\|}{\|b_i\|} b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|} b_i - b_j \right\| \leq \|b_i\| - \|b_j\| + \frac{1}{11} \|b_j\| \leq \\ &\leq |r_i - r_j| + r_k + \frac{1}{11} \cdot \frac{11}{10} r_j \leq |r_i - r_j| + \frac{1}{5} r_j. \end{aligned}$$

We distinguish two cases:

$$(1) r_i > r_j : \|a_i - a_j\| \leq r_i - \frac{4}{5}r_j < r_i;$$

$$(2) r_i \leq r_j : \|a_i - a_j\| \leq -r_i + r_j + \frac{1}{5}r_j = -r_i + \frac{6}{5}r_j \leq -r_i + \frac{8}{5}r_i < r_i \implies a_j \in \overline{B}(a_i, r_i) = B_i, \nless$$

Důkaz (Case A is not bounded)

Let $A^l := A \cap \{x \in \mathbb{R}^n | 3(l-1)R \leq \|x\| < 3lR\}$, $l \in \mathbb{N}$. We get A_i^l , $i \in [M]$ by the previous.
 $A_i = \bigcup_{l=2k+1} A_i^l$, $A_{M+i} = \bigcup_{l=2k} A_i^l$. □

Definice 1.4 (Radon measure)

Let P be a locally compact Hausdorff space and \mathcal{S} a σ -algebra of subsets of P . We say that μ is a Radon measure if

- \mathcal{S} contains all Borel sets,
- $\mu(K) < \infty$ for every compact $K \in P$,
- $\mu(G) = \sup \{\mu(K) | K \subset G \text{ is compact}\}$ for every $G \subset P$ open,
- $\mu(A) = \inf \{\mu(K) | A \subset K \text{ is compact}\}$ for every $A \in \mathcal{S}$,
- μ is complete.

Lemma 1.4

Let μ be a measure on X and $\{A_j\}_{j=1}^\infty$ be an increasing sequence of subsets of X . Then $\lim \mu^*(A_j) = \mu^*\left(\bigcup_{j=1}^\infty A_j\right)$.

Věta 1.5

Let μ be a Radon measure on \mathbb{R}^n and \mathcal{F} be a collection of closed balls in \mathbb{R}^n . Let A denote the set of centers of balls in \mathcal{F} . Assume $\inf \{r | B(a, r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus \bigcup \mathcal{G}) = 0$.

┌ *Důkaz* (The case $\mu^*(A) < \infty$)

Let $N \in \mathbb{N}$ be the constant from Besicovitch theorem. We find Θ such that $1 - \frac{1}{N} < \Theta < 1$.

Claim: „Let $U \subset \mathbb{R}^n$ be an open set. Then there exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*((A \cap U) \setminus \bigcup \mathcal{H}) \leq \Theta \cdot \mu^*(A \cap U).$$

“

$$\mathcal{F}_1 \subset \mathbb{F}, \mathbb{F}_1 = \{B \in \mathbb{F}, \text{diam } B < 1 \wedge B \subset U\}$$

By theorem above there exists disjoint families $\mathcal{G}_1, \dots, \mathcal{G}_N \subset \mathcal{F}_1$ such $A \cap U \subseteq \bigcup_{i=1}^N \bigcup \mathcal{G}_i$. Thus $\mu^*(A \cap U) \leq \sum_{i=1}^N \mu^*(A \cap U \cap \bigcup \mathcal{G}_i)$. Consequently, there exists an integer $1 \leq j \leq N$ such that

$$\mu^*(A \cap U \cap \bigcup \mathcal{G}_j) \geq \frac{1}{N} \mu^*(A \cap U) > (1 - \Theta) \mu^*(A \cap U).$$

Using lemma above we find a finite system $\mathcal{H} \subset \mathcal{G}_j$ such that

$$\mu^*(A \cap U \cap \mathcal{H}) > (1 - \Theta) \mu^*(A \cap U).$$

The set $\bigcup \mathcal{H}$ is μ -measurable

$$\mu^*(A \cap U) = \mu^*(A \cap U \cap \bigcup \mathcal{H}) + \mu^*(A \cap U \setminus \bigcup \mathcal{H}) \geq (1 - \Theta) \mu^*(A \cap U) + \mu^*(A \cap U \setminus \bigcup \mathcal{H}).$$

Set $U_1 = \mathbb{R}^n$. Using claim we find a disjoint finite system $\mathcal{H}_1 \subset \mathcal{F}$ such that $\bigcup \mathcal{H}_1 \subset U_1$ and $\mu^*(A \cap U_1 \setminus \bigcup \mathcal{H}_1) \leq \Theta \mu^*(A \cap U_1)$. Continuing by induction we construct a sequence of open sets (U_j) and a sequence of disjoint finite families (\mathcal{H}_j) such that $U_{j+1} = U_j \setminus \bigcup \mathcal{H}_j$, $\bigcup \mathcal{H}_j \subset U_j$, $\mathcal{H}_j \subset \mathcal{F}$ and

$$\mu^*(A \cap U_{j+1}) = \mu^*((A \cap U_j) \setminus \bigcup \mathcal{H}_j) \leq \Theta \mu^*(A \cap U_j).$$

Since $\mu^*(A) < \infty$ we get $\mu^*(A \setminus \bigcup_{j=1}^{\infty} \bigcup \mathcal{H}_j) = 0$, since $\mu^*(A \cap U_{j+1}) \leq \Theta^j \mu^*(A)$.

$$\mathcal{G} = \bigcup_{j=1}^{\infty} \mathcal{H}_j$$

□

└ *Důkaz* (General case)

We find a sequence (G_j) of open sets, which are disjoint and $\mu(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$. □

1.2 Differentiation of measures

Poznámka (Notation)

\mathcal{B} is set of closed balls in \mathbb{R}^n .

Definice 1.5 (Derivative of measure)

Let μ and ν be measures on \mathbb{R}^n and $x \in \mathbb{R}^n$. Then we define

- upper derivative of ν with respect to μ and x by

$$\overline{D}(\nu, \mu, x) = \lim_{r \rightarrow 0_+} \sup_{B \in \mathcal{B}, \text{diam } B < r} \frac{\nu(B)}{\mu(B)},$$

if the term at the right side is well-defined;

- lower derivative of ν with respect to μ and x by

$$\underline{D}(\nu, \mu, x) = \lim_{r \rightarrow 0_+} \inf_{B \in \mathcal{B}, \text{diam } B < r} \frac{\nu(B)}{\mu(B)},$$

if the term at the right side is well-defined;

- derivative of ν with respect to μ and x by

$$D(\nu, \mu, x) = \overline{D}(\nu, \mu, x) = \underline{D}(\nu, \mu, x),$$

if they are equal.

Věta 1.6

Let ν and μ be Radon measures and \mathbb{R}^n and μ satisfy Vitali theorem. Then $\overline{D}(\nu, \mu, x)$ and $\underline{D}(\nu, \mu, x)$ exist μ -almost everywhere.

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Důkaz

$M := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) < \infty\}$ and $\mathcal{V} := \{B \in \mathcal{B} \mid \mu(B) < \infty\}$, \mathcal{V} is a Vitali cover of M . Then there exists a disjoint countable family $\mathcal{A} \subset \mathcal{V}$ such that $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

$$\mu\left(\bigcup \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \mu(B) < \infty \implies \mu(M) < \infty.$$

└

□

Věta 1.7

Let μ and ν be Radon measures, μ satisfy Vitali theorem, $C \in (0, \infty)$, and $M \subset \mathbb{R}^n$.

- If for every $x \in M$ we have $\overline{D}(\nu, \mu, x) > c$, then $\nu^*(M) \geq c\mu^*(M)$.
- If for every $x \in M$ we have $\underline{D}(\nu, \mu, x) < c$, then there exists $H \subset M$ such that $\mu(M \setminus H) = 0$ and $\nu^*(H) \leq c\mu^*(M)$.

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Důkaz (1.)

We choose $\varepsilon > 0$. There exists an open set $G \subset \mathbb{R}^n$ with $M \subset G$ and $\nu(G) \leq \nu^*(M) + \varepsilon$. We define

$$\mathcal{V} := \{B \in \mathcal{B} \mid B \subset G, \nu(B) > c \cdot \mu(B)\}.$$

The family \mathcal{V} is a Vitali cover of M . There exists a disjoint countable family $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$. Then we have

$$\nu^*(M) + \varepsilon \geq \nu(G) \geq \nu\left(\bigcup \mathcal{A}\right) = \sum_{B \in \mathcal{A}} \nu(B) \leq \sum_{B \in \mathcal{A}} c\mu(B) = c\mu\left(\bigcup \mathcal{A}\right) \geq c\mu^*(M)$$

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Důkaz (2.)

For every $k \in \mathbb{N}$ we find an open set $G_k \supset M$ and $\mu(G_k) \leq \mu^*(M) + \frac{1}{k}$.

$$\mathcal{V}_k := \{B \in \mathcal{B} \mid B \subset G_k \wedge \nu(B) < c \cdot \mu(B)\}.$$

TODO(1 řádek)!!! a countable disjoint system $\mathcal{A}_k \subset \mathcal{V}_k$ such that $\mu(M \setminus \bigcup \mathcal{A}_k) = 0$. Set $H_k = M \cap \bigcup \mathcal{A}_k$. Then $\mu(M \setminus H_k) = 0$, $H_k \subset M$. We have

$$\nu^*(H_k) \leq \nu\left(\bigcup \mathcal{A}_k\right) = \sum_{B \in \mathcal{A}_k} \nu(B) \leq c \sum_{B \in \mathcal{A}_k} \mu(B) = c\mu\left(\bigcup \mathcal{A}_k\right) \leq c \cdot \mu(G_k) \leq c(\mu^*(M) + \frac{1}{k}).$$

$$H := \bigcap H_k : \quad \nu^*(H) \leq c\mu^*(M).$$

$$\mu(M \setminus H) \leq \sum_{k=1}^{\infty} \underbrace{\mu(M \setminus H_k)}_{=0} = 0.$$

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Věta 1.8

Let ν and μ be Radon measures on \mathbb{R}^n and μ satisfies Vitali theorem. Then $D(\nu, \mu, x)$ exists finite μ almost everywhere.

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Důkaz

$$D := \{x \in \mathbb{R}^n \mid D(\nu, \mu, x) \in [0, \infty)\}$$

$$N_1 := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) \text{ is not defined}\}, \quad N_3 = \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) = \infty\},$$

$$N_2 := \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) \text{ is not defined}\}, \quad N_4 = \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) = \infty\}.$$

We already showed that $\mu(N_1) = \mu(N_2) = 0$.

$$A_k := \{x \in \mathbb{R}^n \mid \overline{D}(\nu, \mu, x) > k\}, k \in \mathbb{N}$$

$$A(r, s) = \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) < s < r < \overline{D}(\nu, \mu, x)\}, \quad s, r \in \mathbb{Q}^+, s < r$$

$$N_3 = \bigcap_{k=1}^{\infty} A_k, \quad N_4 = \bigcup \{A(r, s), r, s \in \mathbb{Q}^+, s < r\}$$

„ $\mu(N_3) = 0$ “: Choose $Q \subset N_3$ bounded. By previous theorem (1.) $k\mu^*(Q) \leq \nu^*(Q)$ for every $k \in \mathbb{N}$.

$$\implies \mu^*(Q) = 0 \implies \mu^*(N_3) = 0 \implies \mu(N_3) = 0.$$

„ $\mu(N_4) = 0$ “: It is sufficient to prove $\mu(A(r, s)) = 0$ for any $r, s \in \mathbb{Q}^+, r > s$. Choose $Q \subset A(r, s)$ bounded. By previous theorem (2.) there exists $H \subset Q$ such that $\mu(Q \setminus H) = 0$ and $\nu^*(H) \leq s\mu^*(Q)$. By previous theorem (1.) we have $r\mu^*(H) \leq \nu^*(H)$.

$$r\mu^*(Q) = r\mu^*(H) \leq \nu^*(H) \leq s\mu^*(Q) < \infty.$$

$$\implies \mu^*(Q) = 0 \implies \mu(A(r, s)) = 0.$$

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□

Lemma 1.9

Let ν and μ be as before. Then the mappings $x \mapsto \overline{D}(\nu, \mu, \lambda)$, $x \mapsto \underline{D}(\nu, \mu, \lambda)$ are μ -measurable.

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Důkaz

$$M(r, \alpha) = \left\{ x \in \mathbb{R}^n \mid \exists B \in \mathcal{B} : \text{diam } B < r, x \in B, \frac{\nu(B)}{\mu(B)} < \alpha \right\}, \quad r > 0, \alpha > 0.$$

„ $M(r, \alpha)$ is open“: Assume $x \in M(r, \alpha)$ we find $y \in \mathbb{R}^n$, $s > 0$ such that $x \in \overline{B}(y, s)$, $2s < r$,

$$\frac{\nu(\overline{B}(y, s))}{\mu(\overline{B}(y, s))}.$$

We find $s' > s$, $2s' < r$, $\frac{\nu(\overline{B}(y, s'))}{\mu(\overline{B}(y, s'))} < \alpha$. Then $B(y, s') \subset M(r, \alpha)$.

$$D := \{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) \text{ exists finite}\}.$$

For every $x \in D$ we have

$$\underline{D}(\nu, \mu, x) < \alpha \Leftrightarrow \exists \tau \in \mathcal{Q}, \tau > 0 \forall r \in \mathcal{Q}, r > 0 \exists B \in \mathcal{B} : \text{diam } B < r, x \in B, \frac{\nu(B)}{\mu(B)} < \alpha - \tau,$$

$$\underline{D}(\nu, \mu, x) < \alpha \Leftrightarrow \exists \tau \in \mathcal{Q}, \tau > 0 \forall r \in \mathcal{Q}, r > 0 : x \in M(r, \alpha - \tau).$$

└ $\{x \in \mathbb{R}^n \mid \underline{D}(\nu, \mu, x) < \alpha\}$ is μ -measurable. □

Věta 1.10

Let ν and μ be as before, $\nu \ll \mu$, and $B \subset \mathbb{R}^n$ is μ -measurable. Then we have $\nu(B) = \int_B D(\nu, \mu, x) d\mu(x)$.

Důkaz

Let $B \subset \mathbb{R}^n$ be μ -measurable. Choose $\beta > 1$.

$$B_k := \{x \in B \mid \beta^k < D(\nu, \mu, x) \leq \beta^{k+1}\}, k \in \mathbb{Z}.$$

$$N := \{x \in B \mid D(\nu, \mu, x) = 0\}.$$

$$\mu(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)) = 0.$$

$$\begin{aligned} \int_B D(\nu, \mu, x) d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu, \mu, x) d\mu(x) \leq \\ &\leq \sum_{k=-\infty}^{\infty} \beta^{k+1} \mu(B_k) = \sum_{k=-\infty}^{\infty} \beta^{k+1} \cdot \beta^{-k} \nu(B_k) = \beta \cdot \sum_{k=-\infty}^{\infty} \nu(B_k) \leq \beta \cdot \nu(B). \end{aligned}$$

$$\beta \rightarrow 1_+ : \int_B D(\nu, \mu, x) d\mu(x) \leq \nu(B).$$

Using absolute continuity: $\nu(B \setminus (\bigcup_{k=-\infty}^{\infty} B_k \cup N)) = 0$. We use theorem above to get $\nu^*(Q) \leq C\mu^*(Q)$ for any $c > 0$ and $Q \subset N$ bounded. $\implies \nu^*(Q) = 0 \implies \nu(N) = 0$.

$$\begin{aligned} \int_B D(\nu, \mu, x) d\mu(x) &= \sum_{k=-\infty}^{\infty} \int_{B_k} D(\nu, \mu, x) d\mu(x) \geq \\ &\geq \sum_{k=-\infty}^{\infty} \beta^k \cdot \mu(B_k) \geq \sum_{k=-\infty}^{\infty} \beta^k \cdot \beta^{-(k+1)} \nu(B_k) = \frac{1}{\beta} \cdot \nu(B). \end{aligned}$$

$$\beta \rightarrow 1_+ : \int_B D(\nu, \mu, x) d\mu(x) \geq \nu(B).$$

□

1.3 Lebesgue points

Definition 1.6 (\mathcal{L}_{loc}^1)

Let μ be a Radon measure on \mathbb{R}^n . The symbol $\mathcal{L}_{loc}^1(\mu)$ denotes the set of all functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$, which are μ -measurable and for every $x \in \mathbb{R}^n$ there exists $r > 0$ such that $\int_{B(x,r)} |f| d\mu < \infty$.

Definition 1.7 (Lebesgue point)

Let $f \in \mathcal{L}_{loc}^1(\mu)$. We say that $x \in \mathbb{R}^n$ is Lebesgue point of f at x (with respect to μ) if we have

$$\forall \varepsilon > 0 \exists \delta > 0 \forall B \in \mathbb{B}, x \in B, \text{diam } B < \delta : \frac{\int_B |f(t) - f(x)| d\mu(t)}{\mu(B)} < \varepsilon.$$

Věta 1.11

Let μ be a Radon measure on \mathbb{R}^n satisfying Vitali theorem and $f \in \mathcal{L}_{loc}^1(\mu)$. Then μ -almost every point are Lebesgue point of f (with respect to μ).

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Důkaz

WLOG $\mu(\mathbb{R}^n) < \infty$ and $f \in \mathcal{L}^1(\mu)$. Set $(C_k)_{k=1}^\infty$ be a sequence of closed balls in \mathbb{C} forming a basis of topology in \mathbb{C} . We define

$$g_k(x) := \text{dist}(f(x), C_k), \quad x \in \mathbb{R}^n, k \in \mathbb{N}.$$

The function g_k is non-negative, μ -measurable, $g_k \in \mathcal{L}^1(\mu)$. Set $\nu_k = \int g_k d\mu$. We set $P_k := \{x \in f^{-1}(C_k) \mid \neg(D(\nu_k, \mu, x) = 0)\}$. We have $g_k = 0$ on $f^{-1}(C_k) \implies \mu(P_k) = 0$.

$$\nu_k = \int D(\nu_k, \mu, x) d\mu(x).$$

For $x \in \mathbb{R}^n \setminus \bigcup_{k=1}^\infty P_k$ we choose $\varepsilon > 0$ and we find C_k such that $f(x) \in C_k$ and $C_k \subset B(f(x), \frac{1}{2}\varepsilon)$. For any $t \in \mathbb{R}^n$ it holds $|f(t) - f(x)| \leq g_k(t) + \varepsilon$.

$x \in f^{-1}(C_k) \implies D(\nu_k, \mu, x) = 0$. We find $\delta > 0$ such that

$$\forall B \in \mathbb{B}, x \in B, \text{diam } B < \delta : \frac{\nu_k(B)}{\mu(B)} = \frac{\int_B g_k d\mu}{\mu(B)} < \varepsilon.$$

Let $B \in \mathbb{B}$, $x \in B$ and $\text{diam } B < \delta$. We get

$$\frac{\int_B |f(t) - f(x)| d\mu(t)}{\mu(B)} \leq \frac{\int_B (g_k(t) + \varepsilon) d\mu(t)}{\mu(B)} < \varepsilon + \varepsilon = 2\varepsilon.$$

└

□

1.4 Density theorem

Definice 1.8

Let μ be a measure on \mathbb{R}^n , $A \subset \mathbb{R}^n$ be μ -measurable and $x \in \mathbb{R}^n$. We say that $c \in [0, 1]$ is μ -density of A at x if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall B \in \mathcal{B}, x \in B, \text{diam } B < \delta : \left| \frac{\mu(A \cap B)}{\mu(B)} - c \right| < \varepsilon.$$

Věta 1.12 (Density theorem)

Let μ be a Radon measure on \mathbb{R}^n satisfying Vitali theorem and $M \subset \mathbb{R}^n$ be μ -measurable. Then

$$\begin{aligned} d_\mu(x, M) &= 1 \text{ for almost every } x \in M, \\ d_\mu(x, M) &= 0 \text{ for almost every } x \in \mathbb{R}^n \setminus M. \end{aligned}$$

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Důkaz

Define ν on \mathbb{R}^n by $\nu(A) = \mu(A \cap M)$ for every μ -measurable $A \subset \mathbb{R}^n$. Thus we have $d_\mu(M, X) = D(\nu, \mu, X)$, if at least one term is well-defined, $\nu \ll \mu$, $\nu = \int \chi_M d\mu$. From theorem above $\nu = \int D(\nu, \mu, x) d\mu(x) \implies \chi_M = D(\nu, \mu, x)$ μ -almost everywhere. \square

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1.5 AC and BV functions

Věta 1.13

Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$. Then f is absolutely continuous on $[a, b]$ if and only if f is difference of two non-decreasing absolutely continuous functions on $[a, b]$.

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Důkaz

„ \implies “ choose $c \in (a, b)$. We define $v(x) = V_c^x f$, $x \in [c, b]$, and $v(x) = -V_x^c f$, $x \in [a, c]$. For every $y, d \in [a, b]$, $y < d$, we have $v(d) - v(y) = V_y^d f$. The function v is non-decreasing.

$x, y \in [a, b]$, $x < y$:

$$(v(y) - f(y)) - (v(x) - f(x)) = (v(y) - v(x)) - (f(y) - f(x)) = V_x^y f - (f(y) - f(x)) \geq 0.$$

$v \in AC([a, b])$: Choose $\varepsilon > 0$. We find $\delta > 0$ such that $\sum_{j=1}^m |f(b_j) - f(a_j)| < \varepsilon$, whenever $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_m < b_m \leq b$ and $\sum_{j=1}^m (b_j - a_j) < \delta$. Assume that $a \leq A_1 < B_1 \leq A_2 < B_2 \leq \dots \leq A_p < B_p \leq b$ with $\sum_{j=1}^p (B_j - A_j) < \delta$. For each $j \in [p]$ we find points

$$A_j = a_1^j < b_1^j = a_2^j < b_2^j < \dots < a_{m_j}^j < b_{m_j}^j = B_j.$$

TODO!!!

$$\sum_{j=1}^n |v(B_j) - v(A_j)| < \sum_{j=1}^p \left(\left(\sum_{i=1}^{m_j} |f(b_i^j) - f(a_i^j)| \right) + \frac{\varepsilon}{p} \right) < \varepsilon + p \cdot \frac{\varepsilon}{p} = 2\varepsilon.$$

$$f = v - (v - f).$$

\square

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