

TODO!!!

### Definition 0.1 (WLOG)

$$D := U(0, 1), \quad T = \partial D.$$

TODO!!!

### Definition 0.2

$f \in \mathcal{H}(D)$ . We say that the boundary  $T$  is a natural boundary of  $f$  if  $R_f = \emptyset$ .

*Například*

$f(z) = \sum_{n=0}^{\infty} z^{2^n}$ . Radius of convergence is equal to 1 and  $f$  has natural boundary.

┌

*Důkaz*

$K = \{\exp(\frac{2\pi i k}{n}) \mid k, n \in \mathbb{N}\}$  is dense in  $T$ .  $f$  is "diverges on" this set, because  $f(z^{2^N}) = f(z) - \sum_{n=1}^N z^{2^n}$ . For  $\alpha \in (0, 1)$  we have parametrization of one "line"  $\alpha \cdot \exp(\frac{2k\pi i}{2^n})$  (for  $k, n$  fixed).

$$f(\alpha^{2^N}) = f\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right) + p\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right).$$

└

□

For every domain  $\Omega \subseteq \mathbb{C}$ , there exists  $f \in \mathcal{H}(\Omega)$  such that  $\partial\Omega$  is natural boundary of  $f$ .

┌

*Důkaz*

We use theorem (15.11 from Rudin or TODO from lecture).

└

□

TODO!!!

TODO!!!

## 1 Eulerův vzorec

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

### Lemma 1.1

$$z \neq \frac{k}{2}, k \in \mathbb{Z} : 2\pi \cotg(2\pi z) = \pi \cotg(\pi x) + \pi \cotg(\pi(z + \frac{1}{2})).$$

┌  
Důkaz

$$2\pi \cotg(2\pi z) = 2\pi \frac{\cos(2\pi z)}{\sin(2\pi z)} = \pi \frac{\cos^2(\pi z) - \sin^2(\pi z)}{\sin(\pi z) \cos(\pi z)} = \pi \left( \cotg(\pi z) - \frac{\sin(\pi z)}{\cos(\pi z)} \right) = \pi \left( \cotg(\pi z) + \frac{\cos \pi}{\sin \pi} \right)$$

└

□

### Lemma 1.2 (Herglotz)

$r > 1$ ,  $G$  oblast,  $G \supset [0, r)$ ,  $h$  funkce holomorfní na  $G$ ,  $z, z + \frac{1}{2}, 2z \in [0, r) : 2h(2z) = h(z) + h(z + \frac{1}{2})$ . Pak  $h$  je konstantní na  $G$ .

┌  
Důkaz

Zvol  $t \in (1, r)$ .

$$M := \max \{ |h'(z)|, z \in [0, t] \}, \quad 4|h'(2z)| \leq |h'(z)| + |h'(z + \frac{1}{2})| \implies$$

$$\implies 4|h'(z)| \leq |h'(\frac{z}{2})| + |h'(\frac{z}{2} + \frac{1}{2})| < 2M \implies 4M \leq 2M \implies M = 0.$$

└

□

### Lemma 1.3

$g$  holomorfní funkce na  $\mathbb{C} \setminus \mathbb{Z}$ , hlavní část Laurentovy řady  $g$  na  $P(k)$ ,  $k \in \mathbb{Z}$ , je rovna  $\frac{1}{z-k}$ ,  $g$  lichá,  $2g(2z) = g(z) + g(z + \frac{1}{2})$ ,  $z \neq \frac{k}{2}$ ,  $k \in \mathbb{Z}$ . Pak  $g(z) = \pi \cotg(\pi z)$ .

┌  
Důkaz

$h(z) := g(z) - \pi \cotg(\pi z)$ .  $h$  rozšíříme spojitě a holomorfně na  $\mathbb{C}$ . Z Herglotzova lemmatu je  $h$  konstantní na  $\mathbb{C}$  (obě funkce splňují podmínky). Navíc  $h(0) = 0$ . □

└

Důsledek (Eisenstein)

$$\pi \cotg(\pi z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}, \quad z \notin \mathbb{Z}.$$

┌ *Důkaz*

$z \mapsto \frac{2z}{z^2 - k^2}$  jsou holomorfní na  $U(0, n)$ ,  $k > n$ .

$$\left| \frac{2z}{z^2 - k^2} \right| \leq \frac{2n}{k^2 - n^2} \quad \wedge \quad \sum_{k=n+1}^{\infty} \frac{2n}{k^2 - n^2} K \implies \sum \in \mathcal{M}.$$

$f$  je holomorfní na  $\mathbb{C} \setminus \mathbb{Z}$ . Také je lichá. Nakonec

$$s_n(z) = \frac{1}{2} + \sum_{k=1}^n \frac{2z}{z^2 - k^2} = \frac{1}{2} + \sum_{k=1}^n \frac{1}{z + k} + \sum_{k=1}^n \frac{1}{z - k} = \sum_{-n}^n \frac{1}{z + k},$$

$$s_n\left(\frac{z}{2}\right) + s_n\left(\frac{z+1}{2}\right) = \sum_{k=-n}^n \frac{2}{z + 2k} + \frac{2}{z + 2k + 1} = 2 \sum_{k=-2n}^{2n} \frac{1}{z + k} + 2 \cdot \frac{1}{z + 2n + 1} = 2s_{2n}(k) + \frac{2}{z + 2n + 1},$$

$$n \rightarrow \infty : f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right) = 2f(z) + 0.$$

└ Z předchozího lemmatu vyplývá důkaz. □

## Věta 1.4 (Euler)

$$\underbrace{\sin(\pi z)}_g = \underbrace{\pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}_f.$$

┌ *Důkaz*

$\forall z \in \mathbb{C} \exists$  neighbourhood  $U$ :  $\sum_{k=1}^{\infty} \left\| \left( z \mapsto \frac{z^2}{k^2} \right) \right\|_{\infty}$  is convergent  $\implies \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$  is holomorphic.

$$f_k(z) := 1 - \frac{z^2}{k^2}, \quad \frac{f'_k(z)}{f_k(z)} = \frac{\frac{-2z}{k^2}}{\frac{k^2 - z^2}{k^2}} = \frac{2z}{z^2 - k^2}.$$

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi \prod (1 - \frac{z^2}{k^2})} \left( \pi \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) + \pi z \left( \prod_{k=1}^{\infty} \dots \right)' \right) = \frac{1}{2} = \frac{\Pi'}{\Pi} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{f'_k(z)}{f_k(z)} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}$$

$$\left( \frac{g'(z)}{g(z)} \right) = \frac{\pi \cos(\pi z)}{\sin \pi z} = \pi \cot g(\pi z), \quad \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} = 0 \implies \frac{f}{g} \text{ is constant}.$$

$$\lim_{z \rightarrow 0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow 0} \frac{\pi \sin(z\pi)}{TODO} TODO(online).$$

└ □

## 2 Gamma function

## Definice 2.1

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in (0, \infty).$$

*Poznámka* (Notion)

$$I_n = \int_0^1 1 \cdot (-\log x)^n dx \stackrel{\text{IBP}}{=} [x(-\log x)^n]_0^1 - \int_0^1 xn(-\log x)^{n-1} \left(-\frac{1}{x}\right) dx = n \cdot I_{n-1}.$$

So  $I_n = n!$ . Set  $\log x = t$ ,  $e^t = x$ :

$$I_n = \int_{-\infty}^0 (-t)^n \cdot e^t dt = \int_0^\infty t^n e^{-t} dt.$$

## Lemma 2.1

$\forall n \in \mathbb{N}$  we define  $\Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt$ , then  $\Gamma_n(x) \rightarrow \Gamma(x)$ .

┌

*Důkaz*

$$„0 \leq e^t - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n} e^{-t}“:$$

$$0 \leq e^{-t} \cdot \left(1 - \frac{t}{n}\right)^n = e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) \leq e^{-t} \left(1 - \left(1 - \frac{t}{n}\right)^n \cdot \left(1 + \frac{t}{n}\right)^n\right) = e^{-t} \left(1 - \left(1 - \frac{t^2}{n^2}\right)^n\right)$$

(Last inequality from Bernoulli:  $(1+x)^n \geq 1+n \cdot x$ ,  $x \geq -1$ .)

$$\begin{aligned} |\Gamma(x) - \Gamma_n(x)| &\leq \left| \int_0^n \left(e^{-t} \left(1 - \frac{t}{n}\right)^n\right) \cdot t^{x-1} dt \right| + \left| \int_n^\infty e^{-t} t^{x-1} dt \right| \leq \\ &\leq \frac{1}{n} \int_0^n e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \leq \frac{1}{n} \int_0^\infty e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \rightarrow 0. \end{aligned}$$

└

□

## Lemma 2.2

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)}.$$

—

*Důkaz*

$$, \Gamma_n(x) = \frac{n!n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)} \text{“: } \Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{1}{n^n} \cdot \int_0^n t^{x-1} (n - A^t) dt \stackrel{\text{IBP}}{=}$$

$$= \left(\frac{1}{n}\right)^n \left( \left[ \frac{1}{x} \cdot t \cdot (n-t)^n \right]_0^n + \int_0^n \frac{n}{x} t^x (n-t)^{n-1} dt \right) = \frac{1}{n^n} \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{x \cdot (x+1) \cdot (x+2) \cdot \dots \cdot (x+n-1)} \cdot \int_0^n t^{x+n-1} dt$$

---

☐