Příklad (1.)
Let $\mathbb{A} \in \mathbb{R}^{3\times3}$ be an invertible matrix and let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary fixed vectors in $\mathbb{R}^3$ such that $\mathbf{v} \cdot A^{-1}\mathbf{u} \neq -1$. Show that

$$(\mathbb{A} + \mathbf{u} \otimes \mathbf{v})^{-1} = \mathbb{A}^{-1} - \frac{1}{1 + \mathbf{v} \cdot A^{-1}\mathbf{u}} (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v}).$$

Důkaz
Z předpokladů máme, že obě strany existují, tedy nám stačí ukázat, že $(\mathbb{A} + \mathbf{u} \otimes \mathbf{v})$ krát pravá strana je $\mathbb{I}$.

$$(\mathbb{A} + \mathbf{u} \otimes \mathbf{v}) \cdot \left(\mathbb{A}^{-1} - \frac{1}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})\right) =$$

$$= \mathbb{I} + \mathbf{u} \otimes \mathbf{v} \cdot A^{-1} - \frac{\mathbb{A} \cdot (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} - \frac{(\mathbf{u} \otimes \mathbf{v}) \cdot (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}}$$

Nyní

$$\forall \mathbf{a} : \mathbb{A} \cdot \big((\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})\big)\, \mathbf{a} = \mathbb{A} \cdot \mathbb{A}^{-1}\mathbf{u}(\mathbb{A}^{-T}\mathbf{v} \cdot \mathbf{a}) = \mathbf{u}(\mathbb{A}^{-T}\mathbf{v} \cdot \mathbf{a}) = \mathbf{u}(\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{a}) =$$

$$= (\mathbf{u} \otimes (\mathbf{v})) \cdot \mathbb{A}^{-1}\mathbf{a},$$

tedy

$$\mathbb{A} \cdot (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v}) = \mathbf{u} \otimes \mathbf{v} \cdot \mathbb{A}^{-1}.$$

A

$$\forall \mathbf{a} : (\mathbf{u} \otimes \mathbf{v}) \big((\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})\big)\, \mathbf{a} = (\mathbf{u} \otimes \mathbf{v})\mathbb{A}^{-1}\mathbf{u}(\mathbb{A}^{-T}\mathbf{v} \cdot \mathbf{a}) = \mathbf{u}(\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u})(\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{a}) =$$

$$= (\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}) \cdot \mathbf{u}(\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{a}) = (\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}) \cdot (\mathbf{u} \otimes \mathbf{v}) \cdot A^{-1}\mathbf{a},$$

tedy

$$(\mathbf{u} \otimes \mathbf{v}) \big((\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})\big) = (\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}) \cdot (\mathbf{u} \otimes \mathbf{v}) \cdot A^{-1}.$$

Dosazením do původního součinu dostaneme chtěnou rovnost:

$$(\mathbb{A} + \mathbf{u} \otimes \mathbf{v}) \cdot \left(\mathbb{A}^{-1} - \frac{1}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} (\mathbb{A}^{-1}\mathbf{u}) \otimes (\mathbb{A}^{-T}\mathbf{v})\right) =$$

$$= \mathbb{I} + \mathbf{u} \otimes \mathbf{v} \cdot A^{-1} - \frac{(\mathbf{u} \otimes \mathbf{v}) \cdot \mathbb{A}^{-1}}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} - \frac{(\mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}) \cdot (\mathbf{u} \otimes \mathbf{v}) \cdot \mathbb{A}^{-1}}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} =$$

$$= \mathbb{I} + \mathbf{u} \otimes \mathbf{v} \cdot A^{-1} - \frac{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}}{1 + \mathbf{v} \cdot \mathbb{A}^{-1}\mathbf{u}} (\mathbf{u} \otimes \mathbf{v} \cdot A^{-1}) = \mathbb{I}$$

□

Příklad (2.)
Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be arbitrary fixed vectors in $\mathbb{R}^3$. Show that

$$\operatorname{tr}((\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d})) = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}).$$

Důkaz
Z definice tenzoru víme, že

$$\forall\mathbf{v} : (\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d})\mathbf{v} = (\mathbf{a}\otimes\mathbf{b})\mathbf{c}(\mathbf{d}\cdot\mathbf{v}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{v}) = (\mathbf{b}\cdot\mathbf{c})\cdot\mathbf{a}(\mathbf{d}\cdot\mathbf{v}) = (\mathbf{b}\cdot\mathbf{c})\cdot(\mathbf{a}\otimes\mathbf{d})\cdot\mathbf{v},$$

tedy

$$(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})\cdot(\mathbf{a}\otimes\mathbf{d}).$$

Z přednášky (nebo aplikováním na e_i a skalárním vynásobením s e_i) víme, že $\operatorname{tr}(\mathbf{a}\otimes\mathbf{d}) = \mathbf{a}\cdot\mathbf{d}$. Navíc $\operatorname{tr}(x\cdot\mathbb{A}) = x\cdot\operatorname{tr}(\mathbb{A})$, tedy

$$\operatorname{tr}((\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d})) = \operatorname{tr}((\mathbf{b}\cdot\mathbf{c})\cdot(\mathbf{a}\otimes\mathbf{d})) = (\mathbf{b}\cdot\mathbf{c})\cdot\operatorname{tr}(\mathbf{a}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})\cdot(\mathbf{a}\cdot\mathbf{d}).$$

□

Příklad (3.)
Let φ, ψ, $\mathbf{u}$, $\mathbf{v}$ and $\mathbb{A}$ be smooth scalar, vector and tensor fields in $\mathbb{R}^3$. Show that:

$$\operatorname{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\nabla\varphi) + \varphi\operatorname{div}\mathbf{v}$$

Důkaz
Ze vzorce pro (parciální) derivaci součinu (φv_i je skalární funkce):

$$\operatorname{div}(\varphi\mathbf{v}) = \nabla(\varphi\mathbf{v}) = \sum_i \frac{\partial\varphi v_i}{\partial x_i} = \sum_i \left(\frac{\partial\varphi}{\partial x_i}v_i + \varphi\frac{\partial v_i}{\partial x_i}\right) = \sum_i \frac{\partial\varphi}{\partial x_i}v_i + \sum_i \varphi\frac{\partial v_i}{\partial x_i} =$$

$$= \mathbf{v}\cdot(\nabla\varphi) + \varphi(\nabla\cdot\mathbf{v}) = \mathbf{v}\cdot(\nabla\varphi) + \varphi\operatorname{div}\mathbf{v}$$

□

$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v}$$

Důkaz
Z vzorce pro derivaci součinu a $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$:

$$\operatorname{div}(u \times v) = \nabla \cdot (u \times \mathbf{v}) = \sum_k \left(\sum_{i,j} \frac{\partial \varepsilon_{ijk} u_i \cdot v_j}{\partial x_k} \right) = \sum_{i,j,k} \left(\varepsilon_{ijk} \frac{\partial u_i}{\partial x_k} v_j + \varepsilon_{ijk} u_i \frac{\partial v_j}{\partial x_k} \right) =$$

$$= \sum_j \left(\sum_{k,i} \varepsilon_{kij} \frac{\partial u_i}{\partial x_k} v_j \right) + \sum_i \left(\sum_{j,k} \varepsilon_{jki} u_i \frac{\partial v_j}{\partial x_k} \right) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} + \mathbf{u} \cdot (\mathbf{v} \times \nabla) =$$

$$= \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{rot} \mathbf{u} - \mathbf{u} \cdot \operatorname{rot} \mathbf{v}$$

□

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{v}) = [\nabla \mathbf{u}]\mathbf{v} + \mathbf{u} \operatorname{div} \mathbf{v}$$

Důkaz

$$\operatorname{div}(\mathbf{u} \otimes \mathbf{v}) = \sum_i \frac{\partial u_j v_i}{\partial x_i} = \sum_i \frac{\partial u_j}{\partial x_i} v_i + \sum_i u_j \frac{\partial v_i}{\partial x_i} = [\nabla \mathbf{u}]\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) = [\nabla \mathbf{u}]\mathbf{v} + \mathbf{u} \operatorname{div} \mathbf{v}$$

□

$$\operatorname{div}(\varphi \mathbb{A}) = \mathbb{A}(\nabla \varphi) + \varphi \operatorname{div} \mathbb{A}$$

Důkaz

$$\operatorname{div}(\varphi \mathbb{A}) = \sum_k \frac{\partial \varphi A_{ik}}{\partial x_k} = \sum_k \left(\frac{\partial \varphi}{\partial x_k} A_{ik} + \varphi \frac{\partial A_{ik}}{\partial x_k} \right) = \sum_k \frac{\partial \varphi}{\partial x_k} A_{ik} + \sum_k \frac{\partial A_{ik}}{\partial x_k} \varphi =$$
$$= \mathbb{A}(\nabla \varphi) + \varphi \operatorname{div} \mathbb{A}$$

□

Further, show that the following identities hold for the gradient of various products:

$$\nabla(\varphi\psi) = \psi \nabla \varphi + \varphi \nabla \psi$$

Důkaz

$$\nabla(\varphi\psi) = \frac{\partial \varphi\psi}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}\psi + \frac{\partial \psi}{\partial x_i}\varphi = \psi \nabla \varphi + \varphi \nabla \psi.$$

□

$$\nabla(\varphi \mathbf{v}) = \mathbf{v} \otimes \nabla \varphi + \varphi \nabla \mathbf{v}$$

Důkaz

$$\nabla(\varphi\mathbf{v})) = (\frac{\partial\varphi v_i}{\partial x_j}e_i)e_j = (\frac{\partial\varphi}{\partial x_j}v_i \cdot e_i)e_j + (\varphi\frac{\partial v_i}{\partial x_j}e_i)e_j = \mathbf{v}\otimes\nabla\varphi + \varphi\nabla\mathbf{v}$$

□

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\nabla\mathbf{u})^T\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{u}$$

Důkaz

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \frac{\partial\sum_j u_j\cdot v_j}{\partial x_i} = \sum_j\frac{\partial u_j\cdot v_j}{\partial x_i} = \sum_j\left(\frac{\partial u_j}{\partial x_i}v_j + u_j\cdot\frac{\partial v_j}{\partial x_i}\right) =$$
$$= \sum_j(\nabla u_j)v_j + \sum_j(\nabla v_j)u_j = (\nabla\mathbf{u})^T\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{u}.$$

□

and, finally, show that the following identities hold for rot operator applied on products of various fields,

$$\operatorname{rot}(\mathbf{u}\times\mathbf{v}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$$

Důkaz

Vyjdeme z $\varepsilon_{ijk}\cdot\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, což si buď pamatujeme z přednášky, nebo si rozmyslíme, že pro nenulovost musí být $j = m$ a $n = k$, nebo $j = n$ a $k = m$ a pak si uvědomíme, že stejná pořadí se „vykrátí“ a opačná ne:

$$\operatorname{rot}(\mathbf{u}\times\mathbf{v}) = \nabla\times(\sum_{i,j}\varepsilon_{ijk}u_iv_j) = \sum_{l,k}\varepsilon_{lkm}\frac{\partial}{\partial x_l}\sum_{i,j}u_iv_j = \sum_{i,j,k,l}\varepsilon_{kml}\varepsilon_{kij}\frac{\partial u_iv_j}{\partial x_l} =$$

$$= \sum_{i,j,k,l}\left(\delta_{mi}\delta_{lj}\left(\frac{\partial u_i}{\partial x_l}v_j + u_i\frac{\partial v_j}{\partial x_l}\right) - \delta_{mj}\delta_{li}\left(\frac{\partial u_i}{\partial x_l}v_j + u_i\frac{\partial v_j}{\partial x_l}\right)\right) =$$

$$= \left(\sum_l\frac{\partial u_m}{x_l}v_l + \sum_l u_m\frac{\partial v_l}{\partial x_l}\right) - \left(\sum_l\frac{\partial u_l}{x_l}v_m + \sum_l u_l\frac{\partial v_m}{\partial x_l}\right) =$$

$$= ([\nabla\mathbf{u}]\mathbf{v} + \mathbf{u}\operatorname{div}\mathbf{v}) - (\mathbf{v}\operatorname{div}u + [\nabla\mathbf{v}]\mathbf{u}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{v}) - \operatorname{div}(\mathbf{v}\otimes\mathbf{u}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$$

□

$$\operatorname{rot}(\varphi\mathbf{v}) = \varphi\operatorname{rot}\mathbf{v} - \mathbf{v}\times\nabla\varphi$$

Důkaz

$$\operatorname{rot}(\varphi\mathbf{v}) = \nabla \times (\varphi\mathbf{v}) = \sum_{ij} \varepsilon_{ijk} \frac{\partial \varphi v_j}{\partial x_i} = \sum_{ij} \varepsilon_{ijk} \left(\varphi \frac{\partial v_j}{\partial x_i} + \frac{\partial \varphi}{\partial x_i} v_j \right) =$$

$$= \sum_{ij} \varepsilon_{ijk} \varphi \frac{\partial v_j}{\partial x_i} - \sum_{ij} \varepsilon_{jik} v_j \frac{\partial \varphi}{\partial x_i} = \varphi(\nabla \times \mathbf{v}) - \mathbf{v} \times \nabla\varphi = \varphi \operatorname{rot} \mathbf{v} - \mathbf{v} \times \nabla\varphi$$

□

Two successive applications of rot operator on vector field v can be expressed as follows

$$\operatorname{rot}(\operatorname{rot} \mathbf{v}) = \nabla(\operatorname{div} \mathbf{v}) - \Delta\mathbf{v}.$$

Důkaz

$$\operatorname{rot}(\operatorname{rot} \mathbf{v}) = \nabla\times(\nabla\times\mathbf{v}) = \nabla\times\left(\sum_{i,j} \varepsilon_{ijk} \frac{\partial v_j}{\partial x_i} \right) = \sum_{l,k,i,j} \varepsilon_{lkm} \frac{\partial}{\partial x_l} \varepsilon_{ijk} \frac{\partial v_j}{\partial x_i} = \sum_{i,j,k,l} \varepsilon_{kml}\varepsilon_{kij} \frac{\partial^2 v_j}{\partial x_l \partial x_i} =$$

$$= \sum_{ijkl} \delta_{mi}\delta_{kj} \frac{\partial^2 v_j}{\partial x_l \partial x_i} - \delta_{mj}\delta_{li} \frac{\partial^2 v_j}{\partial x_l \partial x_i} = \frac{\partial}{\partial x_m} \left(\sum_l \frac{\partial v_l}{\partial x_l} \right) - \sum_l \frac{\partial^2 v_m}{\partial x_l^2} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$$

□