Úvod

Poznámka (Organizační úvod)
Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (… PDE …).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

Poznámka (Konvence pro PDR)
$\Omega \subseteq \mathbb{R}^d$ je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

Poznámka
Dále se ukazovali konkrétní parciální rovnice.

Poznámka (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál: $L^p(\Omega)$, $L^p_{loc}(\Omega)$, $||u||_p$, $C^k(\Omega)$, $C^k(\overline{\Omega})$,
$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) | \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha} < \infty \right\}, ||u||_{C^{0,\alpha}} = \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha}.$$
- $\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$, $\vec{n} = (n_1, \ldots, n_d)$.
- Funkcionální analýza 1: Banachův prostor, $u^n \to u$ silná konvergence, $u^n \rightharpoonup u$ slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita (L^p jsou separabilní až na $p = \infty$, $C^k(\overline{\Omega})$ je separabilní, $C^{0,\alpha}$ není separabilní pro $\alpha \in (0,1]$).

Poznámka (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

1 Sobolevovy prostory

Definice 1.1 (Multiindex)

α je multiindex $\equiv d = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$. Délka α je $|\alpha| := \alpha_1 + \ldots + \alpha_d$. Pro $u \in C^k(\Omega)$ definujeme $D^\alpha u = \frac{\partial^{|d|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$.

Definice 1.2 (Slabá derivace)

Buď $u, v_\alpha \in L^1_{loc}(\Omega)$. Řekneme, že v_α je α-tá slabá derivace $u \equiv$

$$\equiv \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Příklad
$u = \operatorname{sign} x$ nemá slabou derivaci.

Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

Důkaz
v_α^1, v_α^2 dvě α-té derivace u.

$$(-1)^{|\alpha|} \int v_\alpha^1 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int v_\alpha^2 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$\int_\Omega (v_\alpha^1 - v_\alpha^2) \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ skoro všude v Ω.

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají. □

Definice 1.3 (Sobolevův prostor)

$\omega \subseteq \mathbb{R}^d$ otevřená, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}.$$

$$||u||_{W^{k,p}(\Omega)}||u||_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} ||D^\alpha u||_p^p\right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha|\leqslant k} ||D^\alpha u||_\infty, & p = \infty. \end{cases}$$

Poznámka

Od teď D^α nebo $\frac{\partial}{\partial x_1}$ nebo ∂_i značí slabou derivaci.

Lemma 1.2 (Základní vlastnosti slabých derivací a Sobolevových prostorů)

Nechť $u, v \in W^{k,p}(\Omega), k \in \mathbb{N}$, a α multiindex s délkou $\leqslant k$.

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ *a* $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, *pro* $|\alpha| + |\beta| \leqslant k$.
- $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ *a* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
- $\forall \tilde{\Omega} \subseteq \Omega$ *otevřená*
$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$
- $\forall \eta \in C^\infty(\Omega)$: $\eta u \in W^{k,p}(\Omega)$ *a* $D^\alpha(\eta u) = \sum_{\beta_i \leqslant \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$, *kde* $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Důkaz

Cvičení na doma. □

Věta 1.3 (Basic properties of Sobolev spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then

- $W^{k,p}(\Omega)$ *is a Banach space;*
- *if* $p < \infty$ *it is separable space;*
- *if* $p \in (1, \infty)$ *it is reflexive space.*

Důkaz
BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness: u^n is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \to u$ in L^p. $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \to v_a \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

TODO

$$|\int_\Omega (v_\alpha - D^\alpha u^n)\varphi| \leqslant ||v_\alpha - D^\alpha u^n||_p ||\varphi||_{p'} \leqslant C||v_\alpha - D^\alpha u^n|| \to 0.$$

$$|\int_\Omega (u^n - u) D^\alpha \varphi| \leqslant ||u^n - u||_p ||D^\alpha \varphi||_{p'} \leqslant C||u^n - u||_p \to 0.$$

„2+3“: $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \ldots \times L^P(\Omega)$ ($d+1$ times), X closed subspace from first property. Lemma: if $X \subseteq Y$ is closed subspace then Y separable $\implies$ X separable and Y reflexive $\implies$ X reflexive. (From functional analysis and topology.) □

2 Approximation of Sobolev function

Věta 2.1

Let $\Omega \subseteq \mathbb{R}^d$ open, ?. $p \in [1, \infty)$.

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} = W^{k,p}(\Omega).$$

Pozor

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} \subsetneq W^{k,p}(\Omega).$$

Důkaz
Summer semester. □

Věta 2.2 (Local density)

$$\forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty$$
$$u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} open, \overline{\tilde{\Omega}} \subseteq \Omega$$
$$u^n \to u in W^{k,p}(\tilde{\Omega})$$

Důkaz
u is extended by 0 to $\mathbb{R}^d \backslash \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \qquad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \qquad \eta \in C_0^\infty(B_1), \eta \geqslant 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

$$u \in L^P(SET) \qquad u^\varepsilon \to u in L^p(SET).$$

We need: $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\forall \alpha, |\alpha| \leqslant k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leqslant \varepsilon_0$ (so that ball of radius ε_0 and center in $\tilde{\Omega}$ is in Ω):

$$(D^\alpha u)^\varepsilon(x) = \int_{\mathbb{R}^d} D^\alpha u(y)\eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} D^\alpha u(y)\eta_\varepsilon(x-u)dy =$$

$$= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y)dy.$$

$$D^\alpha u^\varepsilon = D_x^\alpha \int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)dy.$$

□

Tvrzení 2.3

Ω is open connected set, $u \in W^{1,1}(\Omega)$, then $u = \text{const}. \Leftrightarrow \frac{\partial u}{\partial x_i} = 0$ $\forall i \in [d]$.

$W^{1,1}(I) \hookrightarrow C(I)$ for I interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$.

Důkaz
„1. $\implies$ " trivial. „1. $\impliedby$ ": $\tilde{\Omega} \subseteq \Omega$ connected ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$. u^ε-modification of u is smooth, so

$$\frac{\partial u^\varepsilon}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^\varepsilon = 0 \quad in \tilde{\Omega}$$

$$\implies u^\varepsilon = \mathrm{const}(\varepsilon) \quad in \tilde{\Omega}.$$

$$c(\varepsilon) = \int_{\mathbb{R}} c(\varepsilon)\eta_\delta(x-y)dy = \int_{\mathbb{R}} u^\varepsilon(y)\eta_\delta(x-y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\eta_\varepsilon(y-z)\eta_\delta(x-y)dzdy =$$

$$\int\int u(z+y)\eta_\varepsilon(z)\eta_\delta(y-x)dzdy = \int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dzdw =$$

$$\int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dwdz = \int_{\mathbb{R}^d} u^\delta(z+x)\eta_\varepsilon(z)dz = \int c(\delta)\eta_\varepsilon(z)dz = c(\delta).$$

„2.": WLOG $I = (0,1)$. Define $v(x) = \int_0^x \frac{\partial u}{\partial y}(y)dy$. We show: $v \in W^{1,1}(I)$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$.

$$|v(x)| \leqslant \int_0^1 |\frac{\partial u}{\partial x}| \leqslant ||u||_{1,1}.$$

$$\varphi \in C_0^1(0,1) \qquad \int_0^1 v(x)\frac{\partial \varphi}{\partial x}(x)dx$$

$$= \int_0^1 \left(\int_0^x \frac{\partial u}{\partial y}(y)dy\right)\frac{\partial \varphi}{\partial x}(x)dx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dydx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dxdy =$$

$$= \int_0^1 \left(\int_y^1 \frac{\partial \varphi(x)}{\partial x}dx\right)\frac{\partial u}{\partial y}(y)dy = -\int_0^1 \varphi(y)\frac{\partial u}{\partial y}(y)dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}.$$

TODO.

$$x \to y \implies \int_y^x |\frac{\partial u}{\partial z}|^\alpha \to 0 \implies |u(x) - u(y)| \to 0$$

$$||u||_{C(I)} \leqslant \; ||v+c||_{C(I)} \leqslant ||u||_{1,1} + |c| = ||u||_{1,1} + |u(x) - v(x)| \forall x \in I$$

$$||u||_{C(I)} \leqslant ||u||_{1,1} + \int_0^1 |u(x) - v(x)|dx \leqslant -|| - + \int_0^1 |u| + \int_0^1 |v| \leqslant ||u||_{1,1}.$$

„3." was shown without proof. □

3 Characterization of Sobolev function

Věta 3.1

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega | \operatorname{dist}(x, \delta\Omega) > \delta\}$? *Then*

$$\forall u \in W^{1,p}(\Omega) : ||\Delta_i^h u||_{L^p(\Omega_delta)} \leqslant ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}$$

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

$$u \in L^P \implies \forall \delta, h : ||\Delta_i^h u||_{L^p(\Omega_\delta)} \leqslant c.$$

$p > 1 \implies \frac{\partial u}{partial x_i}$ *exists and* $||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)} \leqslant c$.

TODO!