Úvod

Poznámka (Organizační úvod)
K ukončení předmětu je třeba pouze udělat zkoušku: 2 příklady na definice, 2 věta-důkaz.

Literatura:

- L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.
- W. Rudin, Analýza v reálném a komplexním oboru, Academia, 2003.

1 Differentiation of measures

1.1 Covering theorems

Definice 1.1 (Vitali cover)

Let $A \subset \mathbb{R}^n$ we say that a system $\mathcal{V}$ consisting of closed balls from $\mathbb{R}^n$ forms Vitali cover of A, if

$$\forall x \in A\ \forall \varepsilon > 0 \exists B \in \mathcal{V} : x \in B \wedge \operatorname{diam} B < \varepsilon.$$

Definice 1.2 (Notation)

λ_n is Lebesgue measure on $\mathbb{R}^n$. λ_n^* is outer Lebesgue measure on $\mathbb{R}^n$. If $B \subset \mathbb{R}^n$ is a ball and $\alpha > 0$, then $\alpha \cdot B$ stands for the ball, which is concentric with B and with α-times greater radius than B.

Věta 1.1 (Vitali)

Let $A \subset \mathbb{R}^n$ and $\mathcal{V}$ be a system of closed balls forming a Vitali cover of A. Then there exists a countable disjoint subsystem $\mathcal{A} \subseteq \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.

Důkaz
First assume that A is bounded. Take an open bounded set $G \subset \mathbb{R}^n$ with $A \subset G$. We set

$$\mathcal{V}^* = \{B \in \mathcal{V} |\; V \subset G\}\,.$$

Then $\mathcal{V}^*$ is a Vitali cover of A. If there exists a finite disjoint subsystem of $\mathcal{V}^*$ covering A, we are done. So Assume that there is no such subsystem. Mathematical induction:

First step: We set $s_1 = \sup\{\operatorname{diam} B | B \in \mathcal{V}^*\}$. We choose a ball $B_1 \in \mathcal{V}^*$ such that $B_1 > \frac{1}{2}s_1$.

k-th step: Suppose that we have already constructed balls $B_1, B_2, \ldots, B_{k-1}$. We set

$$s_k = \sup\left\{\operatorname{diam} B | B \in \mathcal{V}^* \wedge B \cap \bigcup_{i=1}^{k-1} B_i = \varnothing\right\}.$$

We find $B_k \in \mathcal{V}^*$ such that $\operatorname{diam} B_k > \frac{1}{2}s_k > 0$, $B_k \cap \bigcup_{i=1}^{k-1} B_i = \varnothing$.

Let $\mathcal{A} = \{B_k | k \in \mathbb{N}\}$. It is disjoint, it is countable, it holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$:

$$\sum_{i=1}^{\infty} \lambda_n(B_i) = \lambda_n(\bigcup_{i=1}^{\infty} B_i) \leqslant \lambda_n(G) < \infty \implies$$
$$\implies \lim_{i\to\infty} = 0 \implies \lim_{i\to\infty} \operatorname{diam}(B_i) = 0 \implies \lim_{i\to\infty} s_i = 0.$$

We show that

$$\forall x \in A \setminus \bigcup \mathcal{A}\ \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j > i : x \in 5 \cdot B_j$$
$$\Leftrightarrow A \setminus \bigcup \mathcal{A} \subseteq \bigcup_{j=i+1}^{\infty} 5 \cdot B_j$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbb{N}$. Denote $\delta = \operatorname{dist}(x, \bigcup_{k=1}^{i} B_k) > 0$. There exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\operatorname{diam} B < \delta \implies B \cap \bigcup_{k=1}^{i} B_k = \varnothing$. Then we have $\operatorname{diam} B > s_p$ for some $p \in \mathbb{N}$.

Therefore there exists $j > i$ with $B_j \cap B \neq \varnothing$. Let j be the smallest number with this property. Then we have $s_j \geqslant \operatorname{diam} B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \varnothing$. Further we have $\operatorname{diam} B_j > \frac{1}{2}s_j \geqslant \frac{1}{2} \operatorname{diam} B \implies 2 \operatorname{diam} B_j \geqslant \operatorname{diam} B$ This implies that $x \in B \subset 5 \cdot B_j$.

$$\lambda_n^*(A \setminus \bigcup A) \leqslant \lambda_n\left(\bigcup_{j=i+1}^{\infty} 5 \cdot B_j\right) \leqslant \sum_{j=i+1}^{\infty} \lambda_n(5 \cdot B_j) = \sum_{j=i+1}^{\infty} 5^n \lambda_n(B_j) = 5^n \cdot \sum_{j=i+1}^{\infty} \lambda_n(B_j) \to 0 \implies \lambda_n(A \setminus \bigcup$$

General case (A not bounded): Let $(G_j)_{j=1}^{\infty}$ be a sequence of disjoint open sets such that $\lambda_n(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$. We define $\mathcal{V}_j = \{B \in \mathcal{V}_i, B \subseteq G_j\}$. $\mathcal{V}_j$ is a Vitali cover of $A \cap G_j$ $\implies \exists \mathcal{A}_j \subseteq \mathcal{V}_j$ countable disjoint and $\lambda_n(A \cap G_j \setminus \bigcup A_j) = 0$. We set $\mathcal{A} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$. $\mathcal{A}$ is countable, disjoint and $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$. □