TODO!!!

Definice 0.1 (Dot product on the space of matrices)

$$\mathbb{A} : \mathbb{B} = \mathrm{tr}(\mathbb{A}\mathbb{B}^T).$$

Definice 0.2 (Norm of matrix)

$$|\mathbb{A}| = (\mathbb{A} : \mathbb{A})^{\frac{1}{2}}.$$

Příklad

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

Důkaz

$$\mathbf{u} \cdot (\mathbf{a} \otimes \mathbf{b})^T \mathbf{v} = (\mathbf{a} \otimes \mathbf{b})\mathbf{u} \cdot \mathbf{v} = (\mathbf{a}(\mathbf{b} \cdot \mathbf{u}))\mathbf{v} = (\mathbf{b} \cdot \mathbf{u})(\mathbf{a} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{b}(\mathbf{a} \cdot \mathbf{v})) = \mathbf{u} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{v}.$$

$\square$

Příklad

$$\det(e^{\mathbb{A}}) = e^{\mathrm{tr}\,\mathbb{A}}.$$

1

Tvrzení 0.1

$$\det(\mathbb{I} + \mathbb{S}) = 1 + \operatorname{tr}\mathbb{S} + \ldots$$

Definice 0.3 (Gateaux derivative)

$$D\mathbf{f}(\mathbf{x})[\mathbf{y}] := \frac{d}{d\tau}\mathbf{f}(\mathbf{x} + \tau\mathbf{y})|_{\tau=0}.$$

Definice 0.4 (Fréchet derivative)

$\mathbf{f}: U \to V$:
$$\lim_{\|\mathbf{y}\|_U\to 0} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})[\mathbf{y}]\|_V}{\|\mathbf{y}\|_V} = 0.$$

Poznámka
Sometimes we write $\nabla f(\mathbf{x}) \cdot \mathbf{y}$ instead of $Df(\mathbf{x})[\mathbf{y}]$ (from Riesz representation theorem).

For matrices ($\varphi : \mathbb{A} \in \mathbb{R}^{3\times3} \to \mathbb{R}$):

$$\frac{\|\varphi(\mathbb{A}+\mathbb{B}) - \varphi(\mathbb{A}) - D\varphi(\mathbb{A})[\mathbb{B}]\|_{\mathbb{R}}}{\|\mathbb{B}\|_{\mathbb{R}^{3\times3}}}.$$

Příklad

$$\frac{\partial \, \text{tr} \, \mathbb{A}}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\text{tr}(\mathbb{A} + \tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\text{tr} \, \mathbb{A} + \tau \, \text{tr} \, \mathbb{B}\right)|_{\tau=0} = \text{tr} \, \mathbb{B} = \mathbb{I} : \mathbb{B}.$$

So $\frac{\partial \, \text{tr} \, \mathbb{A}}{\partial \mathbb{A}} = \mathbb{I}$.

Příklad

$$\frac{\partial \det \mathbb{A}}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\det(\mathbb{A} + \tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\det(\mathbb{A}) \cdot \det\left(\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B}\right)\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left((\det \mathbb{A}) \cdot \left(1 + \tau \, \text{tr}(\mathbb{A}^{-1}\mathbb{B}) + o(\tau^2)\right)\right)|_{\tau=0} = (\det \mathbb{A}) \, \text{tr}\left(\mathbb{A}^{-1}\mathbb{B}\right) =$$

$$= (\det \mathbb{A}) \, \text{tr}\left(\left(\mathbb{A}^{-T}\right)^T \mathbb{B}\right) = \left((\det \mathbb{A})\mathbb{A}^{-T}\right) : \mathbb{B}.$$

So $\frac{\partial \det \mathbb{A}}{\partial \mathbb{A}} = (\det \mathbb{A})\mathbb{A}^{-T} = \text{cof}(\mathbb{A})$.

Příklad

$\mathbb{A} : \mathbb{R} \to \mathbb{R}^{3\times 3}$.

$$\frac{d}{dt}(\det \mathbb{A}(t)) = (\det \mathbb{A}(t)) \, \text{tr}\left(\mathbb{A}(t)^{-1}\frac{d\mathbb{A}(t)}{dt}\right).$$

Příklad

$\mathbb{F} : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{F}(\mathbb{A}) \in \mathbb{R}^{3\times 3}$. $\mathbb{F}(\mathbb{A}) = \mathbb{A}^{-1}$. (We know $\frac{1}{1+x} = 1 - x + \ldots$.)

$$\frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}\left((\mathbb{A} + \tau\mathbb{B})^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{A}\left(\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B}\right)\right)^{-1}\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left(\left(\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B}\right)^{-1}\mathbb{A}^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{I} - \tau\mathbb{A}^{-1}\mathbb{B} + \ldots\right)\mathbb{A}^{-1}\right)|_{\tau=0} = -\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}.$$

So we have $\frac{\partial(\mathbb{A}^{-1})_{ij}}{\partial(\mathbb{A})_{kl}}(\mathbb{B})_{kl}$.

From chain rule (but this is easily solvable by differentiating $\mathbb{A}^{-1}(t)\mathbb{A}(t) = \mathbb{I}$):

$$\frac{d}{dt}\left(\mathbb{A}^{-1}\right) = -\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\mathbb{A}^{-1}.$$

Příklad

$\mathbb{F}(\mathbb{A}) = e^{\mathbb{A}}$

$$\frac{\partial e^{\mathbb{A}}}{\partial \mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}(e^{\mathbb{A}+\tau\mathbb{B}})|_{\tau=0} = \frac{d}{d\tau}\left(\mathbb{I} + \frac{\mathbb{A}+\tau\mathbb{B}}{1!} + \frac{(\mathbb{A}+\tau\mathbb{B})^2}{2!}\right)|_{\tau=0}.$$

Věta 0.2 (Daleckii–Krein)

$\mathbb{A}$ *real symmetric matrix,* $\mathbb{A} \in \mathbb{R}^{k \times k}$, $\mathbb{A} = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$, *where* λ_i *are eigenvalues and* $\mathbf{v}_i$ *are normalised orthogonal* ($\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$) *eigenvectors.*

f continuously differentiable real function defined on open set containing the spectrum of $\mathbb{A}$

$$\mathbb{F}(\mathbb{A}) := \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =: \sum_{i=1}^{k} f(\lambda_i)\mathbb{P}_i.$$

Then the formula for the Gateaux derivative of f at point $\mathbb{A}$ *in direction* $\mathbb{X}$ *reads*

$$D\mathbb{F}(\mathbb{A})[\mathbb{X}] = \frac{\partial \mathbb{F}}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{i=1}^{k} \frac{df}{d\lambda}|_{\lambda=\lambda_i}\mathbb{P}_i\mathbb{X}\mathbb{P}_i + \sum_{i=1}^{k}\sum_{j=1,j\neq i}^{k} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}\mathbb{P}_i\mathbb{X}\mathbb{P}_j.$$

Sometimes we write $D\mathbb{F}(\mathbb{A})[\mathbb{X}] = f^{[1]}(\mathbb{A}) \bigcirc \mathbb{X}$ *(Schur product of matrices, it is point-wise multiplication). Then*

$$[f^{[1]}(\mathbb{A})]_{ij} = \begin{cases} \frac{df}{d\lambda}|_{\lambda=\lambda_i}, & i = j, \\ \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, & i \neq j. \end{cases}$$

Důkaz

No summation conventions, all sums are stated explicitly!

$$\mathbb{F}(\mathbb{A}) = \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =$$

$$= \sum_{i=1}^{k} f(\lambda_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots))\mathbf{v}_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots) \otimes \mathbf{v}_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots).$$

$$\frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}} = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i \otimes \mathbf{v}_i + f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + f(\lambda_i) \mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \right) = ?.$$

We derivate $\mathbb{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}.$$

We multiply (with dot product) it by $\mathbf{v}_i$:

$$\mathbb{P}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbb{A}^T \mathbf{v}_i = \frac{\partial \lambda_i}{\partial \mathbb{A}} \cdot 1 + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \mathbb{A} \cdot \mathbf{v}_i.$$

$$\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i.$$

We again multiply derivative of $\mathbb{A}|\mathbf{v}_i = \lambda\mathbf{v}_i$, but this time by $\mathbf{v}_j$:

$$\mathbf{v}_j \otimes \mathbf{v}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \lambda_j \mathbf{v}_j = 0 + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j.$$

$$(\lambda_j - \lambda_i)\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = -\mathbf{v}_j \otimes \mathbf{v}_i.$$

We also need $(\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}_{ij} = \ldots = \mathbb{P}_i \mathbb{X} \mathbb{P}_j$:

$$\ldots = (\mathbf{v}_j \otimes \mathbf{v}_i)(\mathbf{v}_i \cdot \mathbb{X}\mathbf{v}_j) = (\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}(\mathbf{v}_j \otimes \mathbf{v}_j).$$

TODO!!!

1 Kinematics

Definice 1.1

We have some abstract body with point P. We can look at it in reference configuration (some point in past), where $K_0(P) = \mathbf{X}$ (K_0 = placer), $t = t_0$. Or in current configuration

(how it is situated now), where $K_t(P) = \mathbf{x}$.

The change of configuration, χ in $\mathbf{x} = \chi(\mathbf{X}, t)$ is called deformation (but it contains translation and rotation too!).

Definice 1.2

Let us consider quantity θ that describes the given material point. We can describe it by:

- $\theta(P, t)$;

- $\hat{\theta}(\mathbf{X}, t)$ (referential/Lagrangian description, commonly used for solids because deformation is with respect to reference configuration);

- $\tilde{\theta}(\mathbf{x}, t)$ (spatial/Eulerian description, commonly used for fluids because velocity is time-local property).

But people write those functions without ˆ or ˜

Poznámka

$$\tilde{\theta}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \hat{\theta}(\mathbf{X}, t).$$

Definice 1.3 (Deformation gradient)

$$d\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = \chi(\mathbf{X}_2, t) - \chi(\mathbf{X}_1, t) =$$

$$= \chi(\mathbf{X}_1 + d\mathbf{X}, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1, t) + \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X} + \ldots - \chi(\mathbf{X}_1, t) = \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}.$$

$$\mathbb{F}(\mathbf{X}, t) := \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}. \qquad d\mathbf{x} = \mathbb{F}d\mathbf{X}$$

Poznámka
It can be derived by derivatives on curves (see lecture).

Důsledek
Transformation of infinitesimal line segment: $d\mathbf{x} = \mathbb{F}d\mathbf{X}$.

Transformation of infinitesimal surface elements: $d\mathbf{s} = (\det \mathbb{F})\mathbb{F}^{-T}d\mathbf{S} = \operatorname{cof}\mathbb{F}d\mathbf{S}$.

Transformation of infinitesimal volume: $dv = (\det \mathbb{F})dV$.

Důsledek (In tangent spaces)

$$F(\mathbf{X}, t_0) = f(\chi(\mathbf{X}, t), t).$$

Representation theorem:

$$(GradF)\mathbf{W} = \mathbf{U}_{GradF} \cdot \mathbf{W}$$

$$(Gradf)\mathbf{w} = \mathbf{u}_{Gradf} \cdot \mathbf{w}$$

$$f(\chi(\mathbf{X}, t), t) = F(\mathbf{X}, t_0)$$

$$Gradf(\mathbf{x}, t)|_{\mathbf{x} = \chi(\mathbf{X}, t)} = GradF(\mathbf{X}, t_0)$$

$$\mathbf{U}_{GradF} \cdot \mathbf{W} = (GradF)\mathbf{W} = (Gradf)\mathbb{F}\mathbf{W} = (gradf)(\mathbb{F}\mathbf{W}) = \mathbf{u}_{gradf} \cdot \mathbb{F}\mathbf{W} = \mathbb{F}^T \mathbf{u}_{Gradf} \cdot \mathbf{W}.$$

$$\mathbf{u}_{gradf} = \mathbb{F}^{-T} \mathbf{U}_{GradF}.$$

Příklad (Hollow cylinder)
$r = f(R)$, $\varphi = \Phi$, $z = Z$.

$\ulcorner$
Řešení

$$\mathbb{F} = \frac{\partial \chi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{E}_j$$

$$X_1 = R\cos\Phi, \qquad X_2 = R\sin\Phi, x_1 = r\cos\Phi, x_2 = r\sin\Phi.$$

$$x_1 = \chi_1(X_1, X_2, t), \qquad x_2 = \chi(x_1, x_2, t), x_i = \chi_i(X_j, t).$$

By chain rule:
$$\frac{\partial x_1}{\partial X_2} = \frac{\partial r\cos\Phi}{\partial \partial X_2} = \frac{\partial}{\partial X_2} f(R)\cos\Phi.$$

$$\mathbb{F} = F_{rR}\mathbf{e}_r \otimes \mathbf{E}_R + F_{r\Phi}\mathbf{e}_r \otimes \mathbf{E}_\Phi + \dots$$

$\llcorner$
$\ulcorner$
Řešení
From image:

$$\mathbf{E}_R \xrightarrow{\mathbb{F}} F_{rR}\mathbf{e}_r.$$

$$\mathbf{E}_\Phi \xrightarrow{\mathbb{F}} F_{\varphi\Phi}\mathbf{e}_\varphi$$

So $\mathbb{F} = \begin{pmatrix} F_{rR} & 0 \\ 0 & F_{\varphi\Phi} \end{pmatrix}$

$\llcorner$

TODO? (Solution by curve)

Poznámka

How to differentiate in time tensorial quantities related to the current configuration?

Upper convected derivative:

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \det \mathbb{F}(\mathbf{X},t) \left[\frac{d}{dt} \left(\mathbb{F}^{-1}(\mathbf{X},t)\mathbb{A}(\chi(\mathbf{X},t),t)\mathbb{F}^{-T}(\mathbf{X},t) \right) \right] \mathbb{F}^{T}(\mathbf{X},t).$$

1.1 Derivatives

Definice 1.4 (Lagragian velocity)

$$\mathbf{V}(\mathbf{X},t) = \frac{d\chi(\mathbf{X},t)}{dt}.$$

$$\mathbf{v}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}$$

Definice 1.5 (Eulerian velocity)

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X},t)|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)}.$$

Definice 1.6 (Material time derivative)

$\frac{d}{dt}$ = keep $\mathbf{X}$ fixed, and differentiate with respect to time.

$$\psi(\mathbf{X},t) \rightarrow \frac{d}{dt}\psi(\mathbf{X},t) = \frac{\partial \psi}{\partial t}(\mathbf{X},t)$$

$$\psi(\mathbf{x},t) \rightarrow \frac{d}{dt}\psi(\chi(\mathbf{X},t),t) = \frac{\partial \psi}{\partial t}|_{\mathbf{x}=\chi(\mathbf{X},t)} + \frac{\partial \psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{d\chi_i}{dt}(\mathbf{X},t) =$$

$$= \left(\frac{\partial \psi}{\partial t}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} + V_i(\mathbf{X},t)\frac{\partial \psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \right) =$$

$$= \left(\frac{\partial \psi}{\partial t}(\mathbf{x},t) + v_i(\mathbf{x},t)\frac{\partial \psi}{\partial x_i}(\mathbf{x},t) \right)|_{\mathbf{x}=\chi(\mathbf{x},t)}$$

$$\frac{d}{dt}\psi(\mathbf{x},t) = \frac{\partial \psi}{\partial t}(\mathbf{x},t) + (\mathbf{v}(\mathbf{x},t) \cdot \nabla)\psi(\mathbf{x},t).$$

Definice 1.7 (Time derivative of deformation gradient $\mathbb{F}$)

$$\frac{d}{dt}\mathbb{F}(\mathbf{X},t) = \frac{d}{dt}\left(\frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}} \right) = \frac{\partial}{\partial \mathbf{X}}\frac{d\chi(\mathbf{X},t)}{dt} = \frac{\partial}{\partial \mathbf{X}}\mathbf{V}(\mathbf{X},t) =$$

$$= \frac{\partial}{\partial \mathbf{X}} \mathbf{v}(\chi(\mathbf{X}, t), t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t) = \frac{\partial \mathbf{v}}{\partial x}|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbb{F}(\mathbf{X}, t).$$

$$\mathbb{L}(\mathbf{x}, t) := \nabla \mathbf{V}(\mathbf{x}, t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}, t).$$

Důsledek

$$\frac{d\mathbb{F}}{dt} = \mathbb{L}\mathbb{F}$$

Důsledek

$$\overset{\triangledown}{\mathbb{A}} = \frac{d\mathbb{A}}{dt} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^T$$

TODO!!!

Poznámka (Balance laws in Eulerian description (revision, the last lecture))

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0;$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \mathbb{T} = \mathbb{T}^T;$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q;$$

or

$$\varrho \frac{d}{dt}(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = \operatorname{div}(\mathbb{T}^T \mathbf{v}) + \varrho \mathbf{b} \cdot \mathbf{v} - \operatorname{div} \mathbf{j}_q.$$

Poznámka (Balance laws in Lagrangian description)
Starting with $\frac{d}{dt}_{V(t)}\varrho(\mathbf{x}, t)dv = \frac{d}{dt}m_{V(t)} = 0$ i.e. mass remains same: $m_{V(t)} = m_{V(t_0)}$). We integrate over volume:

$$\int_{V(t_0)} \varrho_R(\mathbf{X})dV = \int_{V(t)} \varrho(\mathbf{x}, t)dv = \int_{V(t_0)} \varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F}dV.$$

Localization principle:

$$\varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} = \varrho_R(\mathbf{X}).$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt}dv = \int_{V(t)} \operatorname{div} \mathbb{T}dv + \int_{V(t)} \varrho \mathbf{b}dv.$$

$$\int_{V(t)} \operatorname{div} \mathbb{T}dv = \int_{\partial V(t)} \mathbb{T}\mathbf{n} \, ds = \int_{\partial V(t_0)} \mathbb{T}(\det \mathbb{F})\mathbb{F}^{-T}\mathbf{N} \, dS \overset{\text{Stokes}}{\int_{V(t_0)}} \operatorname{div}_{\mathbf{X}}((\det\mathbb{F})\mathbb{T}\mathbb{F}^{-T})dV =: \int_{V(t_0)} (\mathbb{T}_R)dV.$$

$$\mathbb{T}_R(\mathbf{X}, t) := (\det \mathbb{F}(\mathbf{X}, t))\mathbb{T}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbb{F}^{-T}(\mathbf{X}, t)$$

is first Piola–Kirchhoff stress tensor. Cauchy ($\mathbb{T}$) is current $\to$ current. P–K ($\mathbb{T}_R$) is reference $\to$ current.

$$\int_{\partial V(t)} dv \to \int_{\partial V(t_0)} (\operatorname{div}_{\mathbf{X}} \mathbb{T}_R) dV.$$

$$\int_{V(t)} \varrho \mathbf{b} dv = \int_{V(t_0)} \varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbf{b}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R(\mathbf{X}) \mathbf{b} dV.$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t_0)} \varrho \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X}, t) \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R \frac{\partial^2 \chi}{\partial t^2} dV.$$

Altogether:

$$\varrho_R \frac{\partial^2 \chi}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbb{T}_R + \varrho_R \mathbf{b} \qquad \text{(Solve for } \chi\text{)}.$$

$$\mathbb{T} = \mathbb{T}^T \to \mathbb{T}_R \mathbb{F}^T = \mathbb{F} \mathbb{T}_R^T \text{ (P–K is not symmetric!)}.$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q \to \int_{V(t)} \varrho \frac{de}{dt} dv = \int_{V(t)} \mathbb{T} : \mathbb{L} dv - \int_{V(t)} \operatorname{div} \mathbf{j}_q dv.$$

$$\int_{V(t)} \operatorname{div} \mathbf{j}_q dv = \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds = \int_{\partial V(t_0)} \mathbf{j}_q(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} dS =$$

$$= \int_{\partial V(t_0)} (\det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{j}_q(\mathbf{x}, t)) \cdot \mathbf{N} dS =$$

$$= \int_{V(t_0)} \operatorname{div}((\det \mathbb{F}) \mathbb{F}^{-1} \mathbf{j}_q) dV.$$

$\mathbf{J}_q = (\det \mathbb{F}) \mathbb{F}^{-1} \mathbf{j}_q$ is called referential heat flux. (It cannot be given by Fouriers law ($\mathbf{j}_q = k\nabla_{\mathbf{x}}\theta$, $\operatorname{div} \mathbf{j}_q = \operatorname{div}(k\nabla\theta)$).)

$$\int_{V(t)} \mathbb{T} : \overbrace{\underbrace{\mathbb{L}}_{\operatorname{tr}(\mathbb{T}\mathbb{L}^T)=\operatorname{tr}(\mathbb{L}\mathbb{T}^T)}}^{\nabla_{\mathbf{x}}\mathbf{v}} dv = \int_{V(t_0)} (\det \mathbb{F}) \mathbb{T} : \mathbb{L} dV =$$

$$= \int_{V(t_0)} \operatorname{tr}((\det \mathbb{F}) \mathbb{T} \mathbb{L}^T) dV = \int_{V(t_0)} \operatorname{tr}\left((\det \mathbb{F}) \mathbb{T} \mathbb{F}^{-T} \left(\frac{d\mathbb{F}}{dt}\right)^T\right) dV = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV.$$

Altogether

$$\varrho_R \frac{\partial e}{\partial t} = \mathbb{T}_R : \dot{\mathbb{F}} - \operatorname{div}_{\mathbf{X}} \mathbf{J}_q.$$

2 Entropy

Poznámka (Objective)
Find quantity that is increasing/decreasing in time.

Poznámka (With no interior)

$$\varrho\frac{d}{dt}\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \operatorname{div}\mathbb{T} + \varrho\mathbf{b} - \operatorname{div}\mathbf{j}_q = \operatorname{div}\mathbb{T} + 0 + \operatorname{div}(k\nabla\theta).$$

Let us work with[a] $\operatorname{div}\mathbb{T} = -p_{th}\mathbb{I} + \tilde{\lambda}(\operatorname{div}\mathbf{v}) + 2\mu\mathbb{D}_\delta$, and assume that $\mathbb{T} = -p_{th}(\varrho, \theta)\mathbb{I} + \tilde{\lambda}(\operatorname{div}\mathbf{v})\mathbb{I} + 2\mu\mathbb{D}_\delta$ (from $\frac{pV}{T} = \text{const}$).

$$\varrho\frac{d}{dt}(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}) = \operatorname{div}(\mathbb{T}^T\mathbf{v}) - \operatorname{div}\mathbf{j}_q.$$

$$\frac{d}{dt}\int_{V(t)}\varrho(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v})dv = \int_{\partial V(t)}\mathbb{T}^T\mathbf{v}\cdot\mathbf{n}ds - \int_{\partial V(t)}\mathbf{j}_q\cdot\mathbf{n}ds.$$

The first part is work and it is zero if we have boundary condition $\mathbf{v}|_{\partial V} = 0$. The second part is heat exchange which is zero if we have boundary condition $\mathbf{j}_q\cdot\mathbf{n}|_{\partial V} = 0$. Both boundary conditions together are math way to say system with *no interactions*.

$$\varrho, \theta, \mathbf{v} \to \varrho, e, \theta.$$

Assume $\eta = \eta(\varrho, e) \to e = e(\eta, \varrho)$. (We will write $e = e(\eta, \varrho) = e(\eta(\mathbf{x}, t), \varrho(\mathbf{x}, t)) = e(\mathbf{x}, t)$.)

We have 1. Balance of internal energy

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q;$$

2. Chain rule

$$\varrho\frac{de}{dt} = \frac{\partial e}{\partial\eta}(\eta, \varrho)\frac{d\eta}{dt} + \frac{\partial e}{\partial\varrho}(\eta, \varrho)\frac{d\varrho}{dt}.$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta, \varrho)\frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q - \frac{\partial e}{\partial\varrho}(\eta, \varrho)\frac{d\varrho}{dt},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta, \varrho)\frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q + \frac{\partial e}{\partial\varrho}(\eta, \varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta, \varrho)\frac{d\eta}{dt} = (-p_{th}\mathbb{I} + \tilde{(}\operatorname{div}\mathbf{v})\mathbb{I} + 2\mu\mathbb{D}_\delta) : \mathbb{D} - \operatorname{div}\mathbf{j}_q + \frac{\partial e}{\partial\varrho}(\eta, \varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta, \varrho)\frac{d\eta}{dt} = \left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta, \varrho)\right)\operatorname{div}\mathbf{v} + \tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta - \operatorname{div}\mathbf{j}_q,$$

$$\varrho\frac{d\eta}{dt} = \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial\eta}}\operatorname{div}\mathbf{v} - \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial\eta}} + \frac{\tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu|\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial\eta}}.$$

There is no chance that this could be positive. (Its obvious, because the value can flow, so point-wise $\geqslant 0$ is lost case.) But we can integrate over volume. Thus instead of $\frac{d\eta}{dt} \geqslant 0$

we want just $\frac{d}{dt} \int_{V(t)} \varrho \eta \, dv \geqslant 0$.

$$\frac{d}{dt} \int_{V(t)} \varrho \eta \, dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} \, dv - \int_{V(t)} \frac{\operatorname{div} \mathbf{j}_q}{\frac{\partial e}{\partial \eta}} \, dv + \int_{V(t)} \frac{\tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu |\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial \eta}} \, dv.$$

The third integral OK, if $\frac{\partial e}{\partial \eta} > 0$.

$$\operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}}\right) = \frac{\operatorname{div} \mathbf{j}_q}{\frac{\partial e}{\partial \eta}} + \nabla\left(\frac{1}{\frac{\partial e}{\partial \eta}}\right) \cdot \mathbf{j}_q.$$

$$\frac{d}{dt} \int_{V(t)} \varrho \eta \, dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} \, dv - \int_{V(t)} \operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}}\right) dv + \int_{V(t)} \nabla\left(\frac{1}{\frac{\partial e}{\partial \eta}}\right) \cdot \mathbf{j}_q \, dv + REST.$$

The second integral is zero from Stokes and boundary condition $\mathbf{j}_q \cdot \mathbf{n}|_{\partial V} = 0$. On the third integral, we can use derivative of inverse value:

$$\frac{d}{dt} \int_{V(t)} \varrho \eta \, dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} \, dv - \int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial \eta}\right) \cdot \mathbf{j}_q}{\left(\frac{\partial e}{\partial \eta}\right)^2} dv + REST =$$

$$= \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} \, dv + k \int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial \eta}\right) \cdot \nabla \theta}{\left(\frac{\partial e}{\partial \eta}\right)^2} dv + REST.$$

If we set $\frac{\partial e}{\partial \eta}(\eta, \varrho) = \theta$, the second integral is non-negative. Moreover, for $\theta \geqslant 0$ we satisfy the assumption for the "first third integral". Moreover if we enforce $\varrho^2 \frac{\partial e}{\partial \varrho}(\varrho, \eta) = p_{th}(\theta, \varrho)$, the first integral is zero, so we win.

---

[a] $\mathbb{D}_\delta := \mathbb{D} - \frac{1}{3}(\operatorname{tr} \mathbb{D})\mathbb{I}$. (Traceless part of $\mathbb{D}$.)

Poznámka
Volíme si tedy $\tilde{\lambda}, \mu > 0$.

Důsledek

$$\frac{d}{dt} \int_{V(t)} \varrho \eta \, dv \geqslant 0$$

is granted for quantity that solves equations

$$e = e(\eta, \varrho), \qquad \frac{\partial e}{\partial \eta} = \theta, \qquad \varrho^2 \frac{de}{d\varrho} = p_{th}(\theta, \varrho).$$

$$p_{th}(\theta, \varrho) = c_V(\gamma - 1)\varrho\theta, \qquad e(\theta, \varrho) = c_V\theta.$$

Poznámka (?)

1. Energy is constant.

2. Energy is function of entropy and volume.

3. Entropy increases.

Poznámka
$e = e(\eta, \varrho)$ is given $\to$ we know everything $\theta = \frac{\partial e}{\partial \eta}(\eta, \varrho)$, $p_{th} = \varrho^2 \frac{\partial e}{\partial \varrho}(\eta, \varrho)$. (Warning: $e = e(\varrho, \theta)$ is not enough!)

Poznámka
Is there a better function that will allow us to do something like this?

Definice 2.1 (Helmholtz free energy density)

$$\psi(\theta, \varrho) := e(\eta, \varrho)|_{\eta = \eta(\theta, \varrho)} - \theta\eta|_{\eta = \eta(\theta, \varrho)}.$$

Poznámka
This is le Legendre transformation of internal energy.

Důsledek

$$\frac{\partial \psi}{\partial \theta}(\theta, \varrho) = -\eta, \qquad \frac{\partial \psi}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta = \eta(\theta, \varrho)} \quad \left(= \frac{p_{th}}{\varrho^2} \right).$$

Důkaz

$$\frac{\partial \psi}{\partial \theta}(\theta, \varrho) = \frac{\partial e(\eta, \varrho)}{\partial \eta}|_{\eta = \eta(\theta, \varrho)} \frac{\partial \eta}{\partial \theta}(\theta, \varrho) - \eta|_{\eta = \eta(\theta, \varrho)} - \theta\frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\eta|_{\eta = \eta(\theta, \varrho)}.$$

$$\frac{\partial \psi}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \eta}(\eta, \theta)\frac{\partial \eta}{\partial \varrho}(\theta, \varrho) + \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta = \eta(\theta, \varrho)} - \theta\frac{\partial \eta}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta = \eta(\theta, \varrho)}.$$

$\square$

Poznámka (Why do we call $c_{V,ref}$ the specific heat at constant volume?)

(Constant volume = constant density.) $\mathbf{j}_q = -k\nabla\theta$, $\mathbb{T} \approx -p_{th}\mathbb{I}$, $\mathbb{D} = \frac{1}{2}\left((\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\right)$.

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q,$$

$$\varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt} + \varrho\frac{\partial e}{\partial\varrho}\frac{d\varrho}{dt} = -p_{th}(\underbrace{\mathbf{D}iv\mathbf{v}}_{\frac{d\varrho}{dt}+\varrho\operatorname{div}\mathbf{v}=0}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt} + 0 = 0 + \operatorname{div}(k\nabla\theta),$$

$$\int_V \varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt}dv = \int_{\partial V}(k\nabla\theta)\mathbf{n}ds.$$

So in left we multiply ϱ, difference of temperature and some $c_V(\theta,\varrho) := \frac{\partial e}{\partial\theta}(\theta,\varrho)$. (On the right there is flow of heat, $\mathbf{j}_q$, through boundary).

For calorically perfect ideal gas: $e = e(\varrho,\theta) = c_{V,ref} \cdot \theta$.

Poznámka (How to get specific heat at constant pressure?)

$$e = e(\eta(\theta,p_{th}),\varrho(\theta,p_{th})), \qquad \varrho\frac{de}{dt} = \mathbb{T}:\mathbb{D} - \operatorname{div}\mathbf{j}_q = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta).$$

Chain rule:

$$\frac{de}{dt} = \frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt} + \ldots \frac{dp_{th}}{dt} = \theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt}.$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt} = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \varrho\frac{p_{th}}{\varrho^2}(\varrho\cdot(\operatorname{div}\mathbf{v})) = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} = \operatorname{div}(k\nabla\theta).$$

So $c_p(\theta,p_{th}) := \theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})$ is specific heat at constant pressure.

Poznámka (Alternative formula for the specific heat at constant volume)

Chain rule:

$$\theta\frac{\partial\eta}{\partial\theta}(\theta,\varrho) = \frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{\partial\eta}{\partial\theta}(\theta,\varrho) = \frac{\partial}{\partial\theta}e(\theta,\varrho) = c_V(\theta,\varrho).$$

$$c_V(\theta,\varrho) := \theta\frac{\partial\eta}{\partial\theta}(\theta,\varrho).$$

Poznámka (Another alternative formula for the specific heat at constant volume, for usage in practice)

$$c_V(\theta, \varrho) = \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho),$$

because $\eta(\theta, \varrho) = -\frac{\partial \psi}{\partial \theta}(\theta, \varrho)$ (property of Helmholtz free energy).

Conclusion: If $\psi(\theta, \varrho)$ is given, then

$$c_V(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho), \qquad p_{th}(\theta, \varrho) = \varrho^2 \frac{\partial \psi}{\partial \varrho}(\theta, \varrho).$$

Poznámka (Where are my evolution equations?)

Unknowns: $\mathbf{v}, \varrho, \theta$.

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \mathbb{T} = -p_{th}(\theta, \varrho)\mathbb{I} + \tilde{\lambda}(\operatorname{div} \mathbf{v})\mathbb{I} + 2\mu \mathbb{D}_\delta.$$

> *Poznámka* (Third equation)
> TODO!!!
>
> $$\varrho\theta \frac{d\eta}{dt} = \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$
>
> $$\frac{d\eta}{dt} = -\frac{d}{dt}\left(\frac{\partial \psi}{\partial \theta}(\theta, \varrho)\right) = -\frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho)\frac{d\theta}{dt} - \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho)\frac{d\varrho}{dt}.$$
>
> $$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\varrho^2 \theta \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho)\operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta,$$
>
> $$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta}\left(\varrho^2 \frac{\partial \psi}{\partial \varrho}\right)\operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$

$$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta}\left(\varrho^2 \frac{\partial \psi}{\partial \varrho}\right)\operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$

And we can get $c_V(\theta, \varrho)$ and p_{th} from $\psi = \psi(\theta, \varrho)$.

Poznámka (Why do we call γ the adiabatic exponent?)

(This applies only to the ideal gas.)

$$p_{th} = c_{V,ref}(\gamma - 1)\varrho\theta,$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q = \mathbb{T} : \mathbb{D}, \qquad \mathbb{T} = -p_{th}\mathbb{I},$$

$$\varrho \frac{de}{dt} = -p_{th} \operatorname{div} \mathbf{v},$$

$$e = c_{V,ref} \cdot \theta$$

$$\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt} \quad \Longleftarrow \quad \frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0.$$

$$\frac{dp_{th}}{dt} = c_{V,ref}(\gamma - 1)\frac{d\varrho}{dt}\theta + c_{V,ref}(\gamma - 1)\varrho\frac{d\theta}{dt},$$

$$\left(\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref}\theta \frac{d\varrho}{dt}. \right)$$

$$\frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref}\theta \frac{d\varrho}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{c_{V,ref}\theta}{p_{th}} + \frac{1}{\varrho} \right) \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{1}{\varrho} \left(\frac{1}{\gamma - 1} + 1 \right) \right) \frac{d\varrho}{dt}.$$

$$\left(\frac{c_{V,ref}\theta}{p_{th}} + \frac{1}{\varrho} =? \frac{c_{V,ref}\theta}{c_{V,ref}\theta\varrho(\gamma - 1)}. \right)$$

$$\frac{d}{dt}(L_n p_{th}) = \gamma \frac{d}{dt} L_n \varrho.$$

$$p_{th} = p_{th,ref} \left(\frac{\varrho}{\varrho_{ref}} \right)^{\gamma}.$$

Poznámka (Where is my hyperbolic equation?)
Assuming isentropic process ($\eta = \text{const}$). $\mathbb{T} = -p_{th}\mathbb{I} = -p_{th}(\varrho, \eta)\mathbb{I}$. ($\operatorname{div} -p_{th}\mathbb{I} = -\nabla p_{th}$.)

$\varrho(\mathbf{x}, t) = \hat{\varrho} + \tilde{\varrho}(\mathbf{x}, t) = $ referential density + small perturbations, similarly $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}} + \tilde{\mathbf{v}}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\mathbf{x}, t)$.

$$\left(\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \qquad \varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta). \right)$$

$$\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + (\mathbf{v} \cdot \nabla)\varrho = \frac{\partial}{\partial t}(\hat{\varrho}\tilde{\varrho}) + (\hat{\mathbf{v}} + \tilde{\mathbf{v}}) \cdot \nabla(\hat{\varrho} + \tilde{\varrho}) \approx \frac{\partial \tilde{\varrho}}{\partial t}.$$

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0 \xrightarrow{\text{Linearize}} \frac{\partial \tilde{\varrho}}{\partial t} + \hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta) \xrightarrow{\text{Linearize}} \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial \varrho}(\varrho, \eta)|_{\varrho = \hat{\varrho}} \nabla \tilde{\varrho}.$$

Differentiate by time:

$$\frac{\partial^2 \tilde{\varrho}}{\partial t} - \hat{\varrho}\operatorname{div}\left(\frac{\partial \tilde{\mathbf{v}}}{\partial t}\right) = 0$$

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \operatorname{div}\left(\frac{\partial p_{th}}{\partial \varrho}\Big|_{\varrho=\hat{\varrho}}\nabla\tilde{\varrho}\right) = 0.$$

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} = \left(\frac{\partial p_{th}}{\partial \varrho}\Big|_{\varrho=\hat{\varrho}}\right)\Delta\tilde{\varrho}.$$

(Speed of sound: $c = \sqrt{\frac{\partial p_{th}}{\partial \varrho}\big|_{\varrho=\hat{\varrho}}(\varrho,\eta)}$.)

Poznámka (Stability of test state)
Set

$$\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}}, \quad \hat{\mathbf{v}} = 0, \qquad \theta = \hat{\theta} + \tilde{\theta}, \quad \hat{\theta} \neq \hat{\theta}(\mathbf{x},t), \qquad \varrho = \hat{\varrho} + \tilde{\varrho}, \quad \hat{\varrho} \neq \hat{\varrho}(\mathbf{x},t).$$

Is it $\tilde{\mathbf{v}}, \tilde{\varrho}, \tilde{\theta} \to 0$?

$$\frac{d\varrho}{dt} + \varrho\operatorname{div}\mathbf{v} = 0 \longrightarrow \frac{\partial\tilde{\varrho}}{\partial t} + \hat{\varrho}\operatorname{div}\tilde{\mathbf{v}} = 0,$$

$$\varrho\frac{d\mathbf{v}}{dt} = \operatorname{div}\mathbb{T} + \varrho\mathbf{b} = \operatorname{div}\mathbb{T} \longrightarrow \hat{\varrho}\frac{\partial\tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial\varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\nabla\tilde{\varrho} - \frac{\partial p_{th}}{\partial\theta}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\nabla\tilde{\theta} + \operatorname{div}\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})\mathbb{I} + 2\mu\tilde{\mathbb{D}}_\delta\right),$$

$$\varrho c_V\frac{d\theta}{dt} = -\theta\frac{\partial p_{th}}{\partial\theta}(\theta,\varrho)\operatorname{div}\mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta = 0 \longrightarrow$$

$$\longrightarrow \hat{\varrho}c_V|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\frac{\partial\tilde{\theta}}{\partial t} = \operatorname{div}(k\nabla\tilde{\theta}) - \hat{\theta}\frac{\partial p_{th}}{\partial\varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\operatorname{div}\tilde{\mathbf{v}}.$$

Test by $\tilde{\mathbf{v}}$:

$$\int_V \hat{\varrho}\frac{\partial\tilde{\mathbf{v}}}{\partial t}\cdot\tilde{\mathbf{v}} = \int_V\left(-\frac{\partial p_{th}}{\partial\varrho}\nabla\tilde{\varrho} - \frac{\partial p_{th}}{\partial\theta}\nabla\tilde{\theta}\right)\cdot\tilde{\mathbf{v}}dv + \int_V\operatorname{div}\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})\mathbb{I} + 2\mu\tilde{\mathbb{D}}_\delta\right)\cdot\tilde{\mathbf{v}}dv =$$

$$= \ldots + \int_V\operatorname{div}\left(\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})\mathbb{I} + 2\mu\tilde{\mathbb{D}}_\delta\right)\tilde{\mathbf{v}}\right) - \int_V\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})^2 + 2\mu\tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta\operatorname{div}\mathbf{v}\right) = \ldots + 0 - \ldots.$$

$$\frac{d}{dt}\frac{1}{2}\int_V\hat{\varrho}(\tilde{\mathbf{v}})^2dv = \int_V\left(-\frac{\partial p_{th}}{\partial\varrho}\nabla\tilde{\varrho} - \frac{\partial p_{th}}{\partial\theta}\nabla\tilde{\theta}\right)\cdot\tilde{\mathbf{v}}dv - \int_V\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})^2 + 2\mu\tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta\right)dv.$$

See that viscosity kills kinetic energy. Now we use $\nabla\varphi\cdot\tilde{\mathbf{v}} = \operatorname{div}(\varphi\tilde{\mathbf{v}}) - \varphi\operatorname{div}\tilde{\mathbf{v}}$ and on $\operatorname{div}(\ldots)$ we use Stokes theorem.

$$\frac{d}{dt}\frac{1}{2}\int_V\hat{\varrho}(\tilde{\mathbf{v}})^2dv = \int_V\frac{\partial p_{th}}{\partial\varrho}\tilde{\varrho}\operatorname{div}\tilde{\mathbf{v}}dv + \int_V\frac{\partial p_{th}}{\partial\theta}\tilde{\theta}\operatorname{div}\tilde{\mathbf{v}}dv - \int_V\left(\tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})^2 + 2\mu\tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta\right)dv.\text{(1)}$$

$$\frac{\partial\tilde{\varrho}}{\partial t} = -\hat{\varrho}\operatorname{div}\tilde{\mathbf{v}} \qquad /\cdot\frac{\partial p_{th}}{\partial\varrho}\frac{\tilde{\varrho}}{\hat{\varrho}}, \int_V$$

17

$$\frac{d}{dt}\int_V \frac{1}{2}(\tilde{\varrho})^2 TODO!!! dv = -\int_V \frac{\partial p_{th}}{\partial \varrho}\tilde{\varrho}\,\mathrm{div}\,\tilde{\mathbf{v}}dv. (2)$$

$$\hat{\varrho}c_V|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\frac{\partial \tilde{\theta}}{\partial t} = \mathrm{div}(k\nabla\tilde{\theta}) - \hat{\theta}\frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\,\mathrm{div}\,\tilde{\mathbf{v}} \qquad / \cdot \frac{\tilde{\theta}}{\hat{\theta}}, \int_V$$

$$\frac{d}{dt}\int_V \frac{\hat{\varrho}c_V}{\hat{\theta}}(\tilde{\theta})^2 dv = -\frac{1}{\hat{\theta}}\int_V \nabla\tilde{\theta}\cdot\nabla\tilde{\theta}dv - \int_V \frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\mathrm{div}\,\tilde{\mathbf{v}})\tilde{\theta}dv. (3)$$

$(1) + (2) + (3)$:

$$\frac{d}{dt}\int_V \frac{1}{2}\hat{\varrho}(\tilde{\mathbf{v}})^2 + \frac{1}{2}\frac{1}{\hat{\varrho}}\frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\tilde{\varrho})^2 + \frac{\hat{\varrho}c_V}{\hat{\theta}}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\tilde{\theta})^2 =$$

$$= -\frac{1}{\hat{\theta}}\int_V \nabla\tilde{\theta}\nabla\tilde{\theta}dv - \int_V \tilde{\lambda}(\mathrm{div}\,\tilde{\mathbf{v}})^2 + 2\mu\tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta dv.$$

o

When we rightly choose Helmholtz free energy, $\frac{\partial p_{th}}{\varrho,\theta} > 0$ and c_V will be positive and we win (perturbations go to zero).

Z $c_V > 0$ máme $\frac{\partial^2 \psi}{\partial \theta^2} < 0$.

TODO!!!

Poznámka (For Homework)
$\frac{d\mathbb{H}}{dt} \neq \frac{1}{2}\mathbb{B}^{-1}\frac{d\mathbb{B}}{dt}$, because there is no commutativity. We must use $\mathbb{T} : \ldots$.

Poznámka (*)
$$\mathbb{T} = \underbrace{-p_{th}(\varrho,\tau)I}_{} + \underbrace{\tilde{\lambda}(\mathrm{div}\,\mathbf{v}) + 2\mu\mathbb{D}_\delta}_{\text{Why this?}}.$$

$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\mathbf{v}).$$

Now we use Galilei principle of relativity „$\mathbf{x} = \mathbf{x} + \mathbf{w}t$". We see that this don't work. So we use $\nabla\mathbf{v}$ ($\nabla(\mathbf{v} + \mathbf{w}) = \nabla\mathbf{v} + \nabla\mathbf{w} = \nabla\mathbf{v}$):

$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\nabla\mathbf{v}).$$

$\mathbb{T}$ is symmetric matrix, so:
$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\mathbb{D}).$$

2.1 Representation theorems for isotropic functions

Definice 2.2

We say that $\varphi : \mathbb{R}^{3\times3} \to \mathbb{R}$ is isotropic $\equiv \varphi(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \varphi(\mathbb{A})$ holds for any $\mathbb{Q} \in Orth^+$ (= proper orthogonal matrices, i.e. $\mathbb{Q}\mathbb{Q}^T = \mathbb{I}$, $\det\mathbb{Q} > 0$).

> *Například*
> tr is isotropic, $\mathbb{A} \mapsto a_{11}$ is not isotropic.

We say that $\mathbb{F} : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is isotropic $\equiv \mathcal{F}(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \mathbb{Q}\mathbb{F}(\mathbb{A})\mathbb{Q}^T$ holds for any $\mathbb{Q} \in Orth^+$.

> *Například*
> id, $^{-1}$, exp, ln, … is isotropic. $\mathbb{A} \mapsto a_{ii}\mathbb{I}$ is not isotropic.

Věta 2.1

$\varphi : Sym(\mathbb{R}^{3\times3}) \to \mathbb{R}$, φ *isotropic* $\implies$ $\varphi(\mathbb{A}) = \varphi(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

$f : Sym(\mathbb{R}^{3\times3}) \to Sym(\mathbb{R}^{3\times3})$, f *isotropic* $\implies$ $f(\mathbb{A}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{A} + \alpha_2\mathbb{A}$, *where* $\alpha_i = \alpha_i(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

> *Poznámka*
> The second one goes from: Assume $f(\mathbb{A}) = \sum_{i=0}^{+\infty} f_i\mathbb{A}^i$. Then by Cayley–Hamilton $f(\mathbb{A}) = \sum_{i=0}^{2} f_i\mathbb{A}^i$. Furthermore, $f_i(\mathbb{A})$ must be isotropic…

Poznámka (Continuation of *)
Isotropic fluid: $\mathbb{S}(\mathbb{D})$ is isotropic function, $\mathbb{S}(\mathbb{D}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2$. What if we want Linear relation (i.e. $\mathbb{S}(\mathbb{D})$ is linear function of $\mathbb{D}$)? Then $\mathbb{S}(\mathbb{D}) = c_0(\operatorname{tr}\mathbb{D})\mathbb{I} + c_1\mathbb{D}$, where $c_0, c_1 = \text{const.}$

Pozor
This remarks (not theorem) works only for fluids, where 0 velocity means 0 stress.

Poznámka (Governing equations for incompressible isotropic fluids)
TODO?

$$\operatorname{div}\mathbf{v} = 0$$

$$\varrho\frac{d\mathbf{v}}{dt} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}) + \varrho\mathbf{b}$$

$$\varrho c_V \frac{d\theta}{dt} = \ldots$$

So
$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}, \qquad \operatorname{div}\mathbb{T} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}).$$

The kind of this p is other than p_{th}. This is (artificial) pressure maintaining the incompressibility, and we solve! for it. (p_{th} is function of ϱ and θ). $\mathbb{T}$ is not obtained by simply substitution, but from equations above!

Příklad (Archimedes law)
Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

Poznámka
Fluid = incompressible fluid. So $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$.

Poznámka
Weight = body force = gravitation force

Poznámka
It talks about floating, so nothing moves!

Poznámka (Governing equations)

$$\operatorname{div}\mathbf{v} = 0,$$
$$\varrho\frac{d\mathbf{v}}{dt} = \operatorname{div}\mathbb{T} + \varrho\mathbf{b}, \qquad \mathbf{b} = -g\mathbf{e}_{\hat{z}},$$
$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D},$$
$$\mathbf{F} = \int_{\partial\mathcal{B}} \mathbb{T}\mathbf{n}\,ds.$$

Řešení

1. Solve for $\mathbf{v}$ and p.

2. Evaluate $\mathbf{F}$.

„1.“ is easy, we have static problem, so $\mathbf{v} = 0$.

$$\varrho\frac{d\mathbf{v}}{dt} = \operatorname{div}\mathbb{T} + \varrho\mathbf{b} \wedge \operatorname{div}\mathbb{T} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}) \implies \implies 0 = -\nabla p - \varrho_{fluid}g\mathbf{e}_{\hat{z}} \implies$$

$$\implies \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right) = (0, 0, -\varrho g) \implies p = -\varrho gz + p_0.$$

„2.“:

$$\int_{\partial\mathcal{B}} \mathbb{T}\mathbf{n}ds = \int_{\partial\mathcal{B}} -p\mathbb{I}\mathbf{n}ds = \int_{\partial\mathcal{B}} (\varrho_{fluid}gz - p_0) = \int_{\partial\mathcal{B}} (\varrho_{fluid}gz\mathbf{n})ds - p_0\int_{\partial\mathcal{B}} \mathbf{n}ds =: *.$$

By stokes (for constant $\mathbf{w}$)

$$\mathbf{w}\cdot\int_{\partial\mathcal{B}} \mathbf{n}ds = \int_{\partial\mathcal{B}} \mathbf{w}\cdot\mathbf{n} = \int_{\mathcal{B}} \operatorname{div}\mathbf{w}dv = 0.$$

So $\int_{\partial\mathcal{B}} \mathbf{n}ds = 0$. Continue:

$$* = \int_{\partial\mathcal{B}} (\varrho_{fluid}gz\mathbf{n})ds,$$

for constant $\mathbf{w}$:

$$\mathbf{F}\cdot\mathbf{w} = \int_{\partial\mathcal{B}} (\varrho_{fluid}gz\mathbf{w})\cdot\mathbf{n}ds = \int_{\mathcal{B}} \varrho_{fluid}g\operatorname{div}(z\mathbf{w})dv + \int_{\mathcal{B}} \varrho_{fluid}g((\nabla z)\cdot\mathbf{w} + z\operatorname{div}\mathbf{w}dv = \varrho_{fluid}g\int_{\mathcal{B}} 1dv\mathbf{e}_{\hat{z}}\cdot\mathbf{w}.$$

So $\mathbf{F} = \varrho_{fluid}gV\mathbf{e}_{\hat{z}}$.

Příklad (Stability of flow in a container)

$\mathbf{v}|_{\partial\Omega} = \mathbf{o}$, $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$.

$t \to +\infty \implies \mathbf{v} \to \mathbf{o}$.

Příklad

$$\operatorname{div}\mathbb{T} = -\nabla p + \mu\Delta\mathbf{v}.$$

Poznámka (Governing equations)

$$\operatorname{div} \mathbf{v} = 0,$$
$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} = -\nabla p + \mu \Delta \mathbf{v},$$
$$\mathbf{v}|_{\partial\Omega} = \mathbf{o},$$
$$\mathbf{v}|_{t_0} = \mathbf{v}_0.$$

Řešení

Test equation by solution:

$$\int_\Omega \varrho \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \int_\Omega (-(\nabla p) \cdot \mathbf{v} + \mu(\Delta \mathbf{v}) \cdot \mathbf{v}) dv.$$

$$\varrho \int_\Omega ((\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{v} dv = \varrho \int_\Omega v_i \frac{\partial v_j}{\partial x_i} v_j dv = \frac{1}{2}\varrho \int_\Omega v_i \frac{\partial}{\partial x_i} (v_j)^2 dv =$$

$$= \varrho \in_\Omega \mathbf{v} \cdot \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) dv = \varrho \int_\Omega \operatorname{div}(\mathbf{v}\frac{|\mathbf{v}|^2}{2}) dv = \varrho \int_{\partial\Omega} \frac{|\mathbf{v}|^2}{2} (\mathbf{v} \cdot \mathbf{n}) ds = 0.$$

$$\frac{d}{dt} \int_\mathcal{B} \frac{1}{2}\varrho(\mathbf{v})^2 dv = \ldots - \mu \int_\mathcal{B} (\nabla\mathbf{v}) : \nabla\mathbf{v} dv.$$

Poincaré inequality (we live in Dirichlet zero $\Longleftarrow$ boundary conditions):

$$\int |\nabla\mathbf{v}|^2 \leqslant C_p^2 \int |\nabla\mathbf{v}|^2.$$

$$\frac{1}{2}\varrho\frac{d}{dt} \int_\Omega \leqslant -\frac{\mu}{C_p} \int_\Omega |\mathbf{v}|^2 dv$$

$$\frac{d}{dt}(\|\mathbf{v}\|^2_{L^2(\Omega)}) \leqslant \frac{-2\mu}{\varrho C_p} \|\mathbf{v}\|^2_{L^2(\Omega)}.$$

Příklad

$$\operatorname{div} \mathbf{v} = 0 \qquad \varrho\frac{d\mathbf{v}}{dt} = -\nabla p + \mu\Delta\mathbf{v}.$$

Poznámka (Dimensionless form)

l_{char} = Characteristic length = arbitrary chosen length. v_{char} = characteristic velocity. $t_{char} = l_{char}/v_{char}$.

Dimensionless variables: $\mathbf{x}^* = \mathbf{x}/l_{char}$, $t^* = t/t_{char}$, $\mathbf{v}^* = \mathbf{v}/v_{char}$.

$$\operatorname{div}^* \mathbf{v}^* = 0.$$

$$\varrho \frac{v_{char}}{t_{char}} \frac{d\mathbf{v}^*}{dt^*} = -\frac{1}{l_{char}}(\nabla^* p^*)p_{char} + \frac{\mu}{l_{char}^2}(\Delta^* \mathbf{v}^*)v_{char},$$

$$\frac{d\mathbf{v}^*}{dt^*} = \frac{t_{char}}{\varrho v_{char} l_{char}}(\nabla^* p^*)p_{char} + \frac{\mu v_{char}}{l_{char}^2}\frac{t_{char}}{\varrho v_{char}}\Delta^* \mathbf{v}^*,$$

$$\frac{dv^*}{dt^*} = -\frac{p_{char}}{\varrho v_{char}^2}(\nabla^* p^*) + \frac{\mu}{\varrho l_{char} v_{char}}\Delta^* \mathbf{v}^*.$$

$p_{char} := \varrho v_{char}^2$

$$\implies \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* p^* + \frac{1}{Re}\Delta^* \mathbf{v}^*,$$

where $\frac{1}{Re} := \frac{\mu}{\varrho l_{char} v_{char}}$ is Reinold's number.