Poznámka
At least 1 from (3-)4 homework (flexible deadlines – last lecture).

Poznámka
In this lecture, there was also the revision of topology. (Topological space, topology, basis of topology, continuous map, quotient space, product topology, Hausdorff spaces).

Poznámka
World Homotopy comes from homós (= same, simiar) and topos (place).

Definice 0.1 (Homotopic functions)

Given two topological spaces X and Y and two continuous functions $f, g : X \to Y$, we say that f is homotopic to g ($f \sim g$) if there is a 1-parametric family $f_t : X \to Y$: $f_0 = f$, $f_1 = g$ and the map $F : [0,1] \times X \to Y$ defined by $(t, x) \mapsto f_t(x)$ is continuous.

Definice 0.2 (Homotopy equivalent spaces)

Given two topological spaces X and Y we say that X and Y are homotopy equivalent if there is a pair of continuous maps (f, g) such that $f : X \to Y$ and $g : Y \to X$ and $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$, $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$.

Příklad
Given $\mathbb{R}$, $\mathbb{R}^2$ with the standard Euclidean topology and two maps $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto f(x) = (x, x^3)$, $g : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto g(x) = (x, e^x)$.

Are f and g homotopic? (Show that by constructing homotopy.)

Řešení

$$F(t, x) = (1-t)(x, x^3) + t(x, e^x) = (x, (1-t)x^3 + te^x).$$

Příklad
Given three topological spaces $(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$ and two pairs of continuous maps $f_1, g_1 : (X, \tau_X) \to (Y, \tau_Y)$ and $f_2, g_2 : (Y, \tau_Y) \to (Z, \tau_Z)$. Assume that f_1 is homotopic to g_1 and f_2 is homotopic to g_2. Show that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$.

Řešení

$$F(t, x) = F_2(t, F_1(t, x)).$$

Příklad
Take $B^n := \left\{x, \ldots, x_n | \sqrt{x_1^2 + \ldots + x_n^2} \leqslant 1\right\} \subseteq \mathbb{R}^n$. And take a map $f : B^n \to B^n$: $f(x) = (0, \ldots, 0) \in B^n$ for all $x \in B^n$. Shows that there is a homotopy from id to f.

Řešení

$$F : [0,1] \times B^n \to B^n, \qquad (t,x) \mapsto (1-t)x.$$

Příklad
Take a 2-ball B^2. B^2 is homotopy equivalent to its center by previous problem, but it is not homeomorphic to $(0,0)$.

Definice 0.3 (Deformation retraction)

A deformation retraction of a topological space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in [0,1]$: $f_0 = \mathrm{id}_X$, $f_1(X) = A$ and $f_t|_A = \mathrm{id}_A$. And family f_t is continuous in the following sense:

$$F : [0,1] \times X \to X, (t,x) \to f_t(x), \text{ is continuous.}$$

Tvrzení 0.1

Given a deformation retraction $f_t : X \to X$, there is a pair $(f,g) : X \xrightarrow{f} A \xrightarrow{g} X : g \circ f \sim \mathrm{id}_X, f \circ g \sim \mathrm{id}_A$.

Poznámka (Suggestion)
$f = f_1$, $g = f_i \circ i_A$ ($A \overset{i_A}{\hookrightarrow} X$), tj. $f \circ g : A \overset{i_A}{\hookrightarrow} X \xrightarrow{f_1} X \xrightarrow{f_1} X$, $a \mapsto a \mapsto a \mapsto a$ (or A) $\implies f \circ g = \mathrm{id}_A$. $g \circ f : X \xrightarrow{f_i} A \xrightarrow{i_A} X \implies f_1(x) \sim \mathrm{id}_X$.

Definice 0.4

Given two topological spaces X and Y and a continuous map $f : X \to Y$, the mapping cylinder M_f is defined to be the quotient space of $X \times [0,1] \coprod Y$ and $\sim$: $(x,1) \sim f(x)$. $M_f = X \times [0,1] \coprod Y / \sim$.

Tvrzení 0.2

Given X, Y and f, M_f deformation retracts to Y.

Důkaz (/ Idea of proof)
The way to construct $f_t = F(\cdot, t) : M_f \to M_f$ is to slide each point (x, t) along the segment $\{x\} \times [0, 1]$ to $f(x)$:

$$F : (x, t) \mapsto f_t(x), \qquad \forall y \in Y : y = F(y, t) \mapsto \{f_1 = \mathrm{id}\, Y \to Y\}$$

In your HW you will check that $F(x, t)$ is continuous. ▫

Poznámka
Cell complex (CW complex) is a topological space with a nice decomposition into small pieces.

1. Start with a discrete set X^0, whose points are called 0-cells.

2. We form the n-skeleton X^n from X^{n-1} by attaching cells $e^n_\alpha = I^n = [0, 1]^n$. By the attachment we mean ($e^n_a = B^n_\alpha, \partial e^n_a = S^n_\alpha$) $\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$. Hence we can view $X^n = X^{n-1} \coprod \coprod B^n_\alpha / \sim$, where $x \sim \varphi_\alpha(x)$ for $x \in \partial\partial B^n_\alpha$.

3. We can either stop this inductive process at a certain finite steps or take an infinite number of steps. In the first case $X = X^n$ for some n, in the second one $X = \bigcup_{n \in \mathbb{N}_0} X^n$ with the weak topology ($A \subset X$ is open $\leftrightarrow$ $A \cap X^n$ is open for all n).

Například
Example of 1-skeleton is graph.

Definice 0.6

Given a cell complex X. Each cell e^n_α has a characteristic map $\Phi_\alpha : e^n_\alpha = B^n_\alpha \to X$ which extends the attaching map $\varphi_\alpha : \partial B^n_\alpha \to X^n$, it is homeomorphism from the interior of B^n_α onto e^n_α. Namely

$$B^n_\alpha \hookrightarrow X^{n-1} \coprod \coprod_\beta B^n_\beta \overset{quotient}{\to} X^n \to X, \qquad B^n_\alpha \to X$$

Definice 0.7

A subcomplex of CW complex is a closed subspace $A \subset X$ that is a union of cells with the corresponding attachments.

Příklad
Construct two different CW structures on S^2.

Řešení
$S^2 = e^0 \cup e^2$, $S^2 = e^0 \cup e^1 \cup \{e^2_1, e^2_2\}$. (See practicals.)

Příklad
We define $\mathbb{R}P^n$ to be the quotient of $S^n/\sim$, where $V \sim$ the antipodal point to V. TODO?

Definice 0.8

Consider a pair (X, A) where X is a CW complex and A is subcomplex. Then we define the quotient complex X/A to be the CW complex with the structure: There are all the cells of $X \backslash A$ with the corresponding attaching maps, and there is a extra 0-cell which is A in $X \backslash A$. For a cell e^n_α of $X \backslash A$ attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map in the corresponding cell in $X \backslash A$ is the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

Příklad
Show that $S^n = e^0 \cup e^n$ is $B^n/S^{n-1} = TODO/e^0 \cup e^{n-1}$.

TODO!!!

Tvrzení 0.3

There is an isomorphism $\pi_1(X, x_1) \to \pi_1(X, x_0)$ for x_0 and x_1 in the same path connected component.

Důkaz
Since x_0, x_1 are in one path connected component $\tilde{X}$, $\exists$ path $h : [0, 1] \to X$: h is in $\tilde{X}$ and $h(0) = x_0, h(1) = x_1$. $\overline{h}(s) := h^{-1}(s) := h(1 - s)$, $s \in [0, 1]$.

To each loop f based at x_1 we associate a loop $h \cdot f \cdot h^{-1}$. $h \cdot f \cdot h^{-1}$ is based at x_0. $\beta_h : \pi_1(x, x_1) \to \pi_1(x, x_0), [f] \mapsto [h \cdot f \cdot h^{-1}]$. We claim, that β_h is an isomorphism. „β_h is homomorhism“:

$$\beta_h([f \cdot h]) = [hfgh^{-1}] = [hfh^{-1}hgh^{-1}] = [hfh^{-1}] \cdot [hgh^{-1}] = \beta_h([f]) \cdot \beta_h([g]).$$

„β_h is isomorphism“: „the inverse of β_h is $\beta_{h^{-1}}$“ (which is homomorphism too by the argument we used for β_h):

$$\beta_{h^{-1}}(\beta_h([f])) = \beta_{h^{-1}}([hfh^{-1}]) = [h^{-1}hfh^{-1}h] = [f].$$

□

Věta 0.4 (Fundamental group of S^1)

S^1 is path connected, thus $\pi_1(S^1, x_0) = \pi_1(S^1)$.

$\pi_1(S^1) \simeq \mathbb{Z}$.

Důkaz
We claim that $\pi_1(S^1) \simeq \langle[\omega]\rangle$, where $\omega : [0,1] \to S^1$, $s \mapsto (\cos(2\pi s), \sin(2\pi s)) \in \mathbb{R}^2$, $s \in [0,1]$. $\omega_n(s) := (\cos(2\pi ns), \sin(2\pi ns)) \sim \omega^n$, so $[\omega]^n = [\omega_n]$.

Now our theorem is equivalent to the statement that every loop in S^1 based at $(1,0)$ is homotopic to the unique ω_n. We use the following two facts:

Fact 1: For every path $f : I \to X$ starting at $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to \tilde{X}$ starting at x_0.

Fact 2: For each homotopy $f_x : I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists$ unique lifted homotopy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

p that we need: $p : \mathbb{R} \to S^1$; $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If we define $\tilde{\omega}_n(s) = n \cdot s$. We will apply Facts 1 and 2 to $p : \mathbb{R} \to S^1$, $\tilde{\omega}_n$: Given $f : [0,1] \to S^1$ based at $(0,1)$ representing some element of $\pi_1(S^1)$. We take $\tilde{f}$. Since $p\tilde{f}(1) = f(1) = (1,0)$ (and $p^{-1}(1) \in \mathbb{Z}$), we can argument that if $\tilde{f}$ ends at u (i.e. $\tilde{f}(1) = f$), it is homotopoc to $\tilde{\omega}_n$ by the homotopy $\tilde{F} = (1-t)\tilde{f} + t\tilde{\omega}_n$.

From fact 1 exists $\tilde{f}$ starting at 0 and ending at $p^{-1}(1) \in \mathbb{Z}$.

Theorem: Exists homotopy $\tilde{F}$ from $\tilde{\omega}_k$ to $\tilde{f}$ denoted by $(*)$.

So we define homotopy F from ω_n to f by $F = p \circ \tilde{F}$, homotopy from ω_n to f. Since $[\omega_n] = n \cdot [\omega]$, $\pi_1(S^1) \simeq \mathbb{Z}$.

Now we would like to show that $[f]$ is uniformly determined. Assume that $f \sim \omega_n$ and $f \sim \omega_m$, then using Facts 1 and 2 we have $[\omega_n] = [\omega_m]$ which lends to contradiction since they have different endpoints on $\mathbb{R}$. □

Definice 0.9

Given a topological space X, a covering space of X consists of a topological space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ satisfying that $\forall x \in X$ $\exists$ open neighbourhood U of x in X such that $p^{-1}(U)$ is a disjoint union of open subsets U_α each of which is homeomorphically mapped to U.

Definice 0.10

Given a map $[0,1] \xrightarrow{f} X$ and $p : \tilde{X} \to X$ we say that $\tilde{f} : [0,1] \to \tilde{X}$ is a lift of f if $p \circ \tilde{f} = f$.

The same construction can be defined for homotopy.

Tvrzení 0.5 (*)

Given a map $F : Y \times [0,1] \to X$ and a map $\tilde{F} : Y \times \{\mathbf{o}\} \to \tilde{X}$, where $p : \tilde{X} \to X$ is a covering space, and $\tilde{F}$ lifts $F|_{Y \times \{\mathbf{o}\}}$; there restricting to $\tilde{F}$ on $Y \times \{\mathbf{o}\}$.

Poznámka (Corollary: Fact 1 and Fact 2 from the previous proof)
Fact 1 is free, it comes when $Y = \{\text{point}\}$, Fact 2 also follows.

Příklad
We say that a topological (path-connected) space is simply connected $\Leftrightarrow \pi_1(X) = \{e\}$. Examples of simply connected topological spaces: $\mathbb{R}, \mathbb{R}^2, \ldots$. S^1 is not simply connected.

Příklad
Given X, Y path-connected and $x_0 \in X$, $y_0 \in Y$. Show that $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Řešení
Product topology is defined to be such that a map $f : Z \to X \times Y$ is continuous $\Leftrightarrow$ $(p_x : X \times Y \to X,\ p_y : X \times Y \to Y)$ $p_x \circ f$ and $p_y \circ f$ are continuous.

A loop $\gamma : [0,1] \to X \times Y$ based at (x_0, y_0) splits at two loops $\gamma_1 : [0,1] \to X$, $\gamma_2 : [0,1] \to Y$. The same holds for homotopy, i.e. F from γ to $\tilde{\gamma}$ splits into (F_1, F_2), where F_1 is a homotopy on X from γ_1 to $\tilde{\gamma}_1$ and F_2 is a homotopy on Y from γ_2 to $\tilde{\gamma}_2$.

Důsledek
$\pi_1(T^n) := \pi_1(S^1 \times S^1 \times \ldots \times S^1) = \mathbb{Z}^n$.

Příklad
Show that TODO!!! is a covering space for $S^1 V S^1$.

TODO!!!

Důkaz (Proposition *)
To prove our proposition we need to construct $\tilde{F} : N \times I \to \tilde{X}$, where N is an open neighbourhood in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighbourhood of $F(y_0, t)$.

By compactness of $\{y_0\} \times I$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times I$. This implies that we can choose a single neighbourhood N of y_0, and a partition of $[0,1]$ $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ such that $F|_{N \times [t_i, t_{i+1}]}$ is contained in an evenly covered neighbourhood U_i.

Assume inductively that $\tilde{F}$ has been constructed for $N \times [0, t_i]$ starting at a given $\tilde{F}$ on $N \times \{0\}$. We have that $F(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, there is an open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically to U_i via p and $\tilde{F}((y_0, t_i)) \in \tilde{U}_i$. After replacing N by a smaller neighbourhood of y_0. (We replace $N \times \{t_i\}$ with the intersection with ?) we may assume that $\tilde{F}(N \times \{t_0\}) \in \tilde{U}_i$. Now we define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F

with $p^{-1} : U_i \to \tilde{U}_i$. After a finite number of steps, we eventually get a lift $\tilde{F} : N \times I \to \tilde{X}$ for N (some neighbourhood of y_0).

Next we show the uniqueness for $Y = \{\text{point}\}$. In this case we ? Suppose there are two lifts $\tilde{F} : I \to \tilde{X}$, $\tilde{F}' : I \to \tilde{X}$. As before we choose a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ so that $\forall i : F([t_i, t_{i+1}])$ is contained in some evenly covered neighbourhood U_i. Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, co is $\tilde{F}([t_i, t_{i+1}])$, which must therefore lie in one of the disjoint open sets $\tilde{U}_i$ projecting homeomorphically to U_i. By the same token, $\tilde{F}'([t_i, t_{i+1}])$ lies in a single $\tilde{U}_i$, in fact in the same containing $\tilde{F}([t_i, t_{i+1}])$ (by the assumption of induction). Since p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$ and the induction step follows.

The last step of the prove is to observe that since $\tilde{F}$, $\tilde{F}'$ are constructed on the sets of form $N \times I$ and are unique when we restrict to each segment $\{y\} \times I$, they must agree whenever two such sets $N \times I$ overlap, so we get in fact a well-defined lift $\tilde{F}$ on $Y \times I$. This $\tilde{F}$ is continuous since it is continuous on each segment $\{y\} \times I$. □

Poznámka

We would like to see π, as a functor $\pi_1 : Top \to Grp$. In order for π_1 to be a functor, we want for $\varphi : (X, x_0) \overset{\text{cont.}}{\to} (Y, y_0)$ associate $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$. How to get φ_*? Given a loop γ on X based at x_0. We have a loop $\varphi \circ \gamma$ on Y based at y_0. $\varphi_*([\gamma]) := [\varphi \circ \gamma]$. Is φ_* a homomorphism? $\varphi_*([\gamma; \gamma_2]) = [\varphi \circ (\gamma_1 \cdot \gamma_2)] = [\varphi \circ \gamma_1 \cdot \varphi \cdot \gamma_2] = [\varphi \circ \gamma_1] \cdot [\varphi \circ \gamma_2]$. Hence φ_* is a homomorphism.

$$\forall (X, x_0) \in Fb(Top) \exists \varphi = 1 \in Hom((X, x_0), (X, x_0)).$$

We want $\pi_1(1) = \mathrm{id}_{\pi_1(X, x_0)}$. This because of the definition of $\pi_1(1)$, which maps $[\gamma] \to [\gamma]$ $\implies \pi_1(1) : [\gamma] \mapsto [\gamma]$, so it is $\mathrm{id}_{\pi_1(X, x_0)}$.

In order for π_1 to be a functor we also need: given $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$ then $i(\psi \circ \varphi)_* = \pi_1(\psi \circ \varphi) = \pi_1(\psi) \circ \pi_1(\varphi) = (\psi)_* \cdot (\varphi)_*$. This is because

$$\pi_1(\psi \circ \varphi)([\gamma]) = (\psi \circ \varphi)_*([\gamma]) = [\psi \circ \varphi \circ \gamma] = \psi_*([\varphi \circ \gamma]) = \pi_1(\psi)(\varphi_*([\gamma])) = \pi_1(\psi) \circ \pi_1(\varphi)([\gamma]).$$

Hence it holds and π_1 is functor $TOP \to GRP$.

Tvrzení 0.6

Given a topological space X. If $X = \bigcup A_\alpha$, where each of A_α is a path connected subspace of X and $A_\alpha \cap A_\beta$ is path connected $\forall \alpha, \beta$, then each loop in X based at x_0 can be decomposed as a product of loops each of which is in some A_α.

Důkaz

Given $f : I \to X$ with the basepoint x_0, we claim that there is a partition $0 = s_0 < s_1 < \ldots < s_m = 1$ of I such that $[s_{i-1}, s_i]$ is mapped by f to a single A_α (that by call A_i). Since f is continuous, each $s \in I$ has an open neighbourhood V_i in I mapped by f to some A_α. We may in fact take V_s to be an interval whose closure is mapped to a single A_α. By compactness of I, we see that a finite number of such intervals cover I. The endpoints of these intervals form a partition of I: $0 = s_0 < s_1 < \ldots < s_m = 1$. Again A_i we call $A_\alpha : f([s_{i-1}, s_i]) \subset A_\alpha$. Let f_i be denoted as $f|_{[s_{i-1},s_i]}$. Then $f = f_i, \ldots, f_m$, where f_i is a path in A_i.

Since $f([s_{i-1}, s_i]) \subset A_i \wedge f([s_i, s_{i+1}]) \subset A_{i+1} \implies f(s_i) \in A_i \cap A_{i-1}$. Since $A_i \cap A_{i+1}$ is path connected, we can choose path $g_i \subset A_i \cap A_{i+1}$, g_i starts at x_0 and ends at $f(s_i)$. Hence $[f][f_1 g_1^{-1}][g_1 f_2 g_2^{-1}] \ldots [g_{n-1} f_n g_n^{-1}]$, loops in $A_1, A_2, \ldots, A_n$. □

TODO?

TODO!!!

Tvrzení 0.7 (*, should be somewhere above)

If $X = \bigcup_{\alpha \in I} A_\alpha$, $x_0 \in \bigcap_{\alpha \in I} A_\alpha$, $A_\alpha \cap A_\beta$ is path connected for all α, β. Then for each loop γ in X based at x_0, there is a sequence of loops $\gamma_{\alpha_1}, \ldots \gamma_{\alpha_k} : [\gamma] = [\gamma_{\alpha_1}] \cdot \ldots \cdot [\gamma_{\alpha_k}]$, where γ_{α_i} is a loop in A_{α_i}.

Věta 0.8 (Van Kampen)

Given a topological space X such that $X = \bigcup A_\alpha$, where A_α's are path connected subspaces of X, $x_0 \in \bigcap A_\alpha \neq \emptyset$. Besides that we assume $A_\alpha \cap A_\beta$ is connected $\forall \alpha, \beta$. Then $\exists \Phi : *_{\alpha \in I} \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$, which is a surjective homomorphism. If in addition $\forall \alpha, \beta, \gamma : A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\operatorname{Ker} \Phi = \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle$.

$$*_{\alpha \in I} \pi_1(A_\alpha, x_0) / < i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} > \simeq \pi_1(X, x_0).$$

Důkaz

The proof uses Proposition *. Same terminology: Given a loop f in X based at x_0, by a factorization of $[f] \in \pi_1(X, x_0)$ we mean a formal product $[f_1] \cdot \ldots \cdot [f_k]$, where each f_i is a loop in some A_α based at x_0 and $[f_i] \in \pi_1(A_\alpha, x_0)$, and where f is homotopic to $f_1 \cdot \ldots \cdot f_k$. (Factorization is a presentation from Proposition *. Factorization can be seen as a word in $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$, but not necessary reduced word.)

We construct Φ by sending $[f]$ to $[f_1] \cdot \ldots \cdot [f_k]$ coming from Proposition * with the potential reduction. We will be concerned with the uniqueness of such factorizations. We define an equivalence relation on factorizations. We say that two sequences are equivalent iff they are related by a sequence of moves: combine adjacent terms: $[f_i] \cdot [f_j] = [f_i f_j]$ if $[f_i], [f_j] \in \pi_1(A_\alpha, x_0)$ for some α; and regard $[f_i] \in \pi_1(A_\alpha)$ as $[f_i] \in \pi_1(A_\beta)$ if f_i is a loop in A_β; and we do not write constant loops in the decomposition.

This equivalence relation mimics, what is necessary for words in $*\pi_1(A_\alpha, x_0)$. We modify this equivalence relation by substituting second condition (regard ...) by: For f_i, f_j: $[f_i] \in \pi_1(A_\alpha, x_0)$, $[f_j] \in \pi_1(A_\beta, x_0)$, we view $[f_i]$, $[f_j]$ as elements of $\pi_1(A_\alpha \cap A_\beta, x_0)$. So first and third condition do not change the elements in $*_{\alpha\in I}\pi_1(A_\alpha, x_0)$ and the new second one is such that it does not change elements $*_{\alpha\in I}\pi_1(A_\alpha, x_0)/N$.

So equivalent factorizations supposed to give the same element $*_{\alpha\in I}\pi_1(A_\alpha, x_0)$. Hence we can show that any two factorizations of f are equivalent, this will imply $*_{\alpha\in I}\pi_1(A_\alpha, x_0)/N \to \pi_1(X, x_0)$ is injective, and therefore $*_{\alpha\in I}\pi_1(A_\alpha, x_0)/N \to *_{\alpha\in I}\pi_1(A_\alpha, x_0)$ is trivial, and this map is an isomorphism.

„Equivalence of two factorizations“: We take two factorizations $[f_1] \cdot \ldots \cdot [f_k]$ and $[f_1'] \cdot \ldots \cdot [f_l']$ of $[f] \in \pi_1(X, x_0)$. We would like to show that $[f_1] \cdot \ldots \cdot [f_k] \sim [f_1'] \cdot \ldots \cdot [f_l']$. Since they represent $[f]$, they are homotopic. Let $F : I \times I \to X$ is a homotopy from $f_1 \cdot \ldots \cdot f_k$ to $f_1' \cdot \ldots \cdot f_l'$. Then we say that $\exists$ decompositions $0 = s_0 < s_1 < \ldots < s_m = 1$, $0 = t_0 < t_1 < \ldots < t_n = 1$ such that each $R_{ij} := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F to a single A_α, which we label by A_{ij}. These partitions can be obtained from $I \times I$ by finitely many rectangles $[a, b] \times [c, d]$ mapping to a single A_α by compactness of $I \times I$.

With this type of description, we can assume that R_{ij}'s are such that they are to the subdivision giving homotopies between f_i's and f_j's.

Since F maps a neighbourhood of R_{ij} into A_{ij}, we may perturb the vertical sides of R_{ij} so that each point of $I \times I$ lies in at most three R_{ij}'s. After this modification we do the relabelling by the following scheme: label row by row starting from the first one, and label from left to right.

If γ is path in $I \times I$ from the left edge (i.e. $\{0\} \times [0, 1]$) to the right edge (i.e. $\{1\} \times [0, 1]$), then $F|_\gamma$ is a loop from x_0 to x_0. We write γ_r to be the path (through edges of rectangles) separating the first r rectangles $R_1, \ldots, R_r$ from the remaining rectangles. γ_0 is bottom edge, γ_m is the top edge.

Let us call the corners of R_i's vertices. So for each vertex v with $F(v) \neq x_0$, there is a path g_v from x_0 to $F(v)$ that lies in the intersection of the two or three A_{ij}'s corresponding to R_i's containing v. (This is by the choice of R_i's and the fact that A_α's are path connected and contains x_0 and $F(v)$. And by path connectedness of disjoint of A's.)

Then we obtain a factorization $[F|_{\gamma_r}]$ by inserting the appropriate path connection x_0 with $F|_{\text{vertices corresponds of } R_i\text{'s}}$ then we can say that $[F|_{\gamma_r}]$ has a decomposition that depends on the choice of A_{ij} that corresponds to the edge of R_i on which the component of the decomposition lies.

$[F|_{\gamma_r}]$ has a decomposition by inserting $g_v^{-1}g_v$. Different choices of A_{ij}'s will change the factorization of $[F|_{\gamma_r}]$. (Our equivalence relation $\sim$ is designed to make the choice of A_{ij}'s for such paths irrelevant. I.e. the different factorizations coming from different choices are equivalent.)

Factorizations for two consecutive paths γ_r and γ_{r+1} are equivalent since pushing along R_{r+1} from γ_r to γ_{r+1} changing $F|_{\gamma_r}$ to $F|_{\gamma_{r+1}}$ by a homotopy within A_{ij} corresponding to R_{r+1}, so we can choose this A_{ij} for all segments of γ_r.

We can arrange that the factorization associated to γ_0 is equivalent to $[f_1] \cdot \ldots \cdot [f_k]$ by choosing g_r's for each vertex v along the lower edge of $I \times I$ to lie not just in the two A_{ij}'s corresponding to R_s's containing v, but also to lie in the A_α for f_i containing v in its domain. In the case when v is the common point of two domains for two consecutive f_i and f_{i+1} we have $F(v) = x_0$, so there is no need to choose g_v's for such v's. In this fashion we can assume that the factorization for $[f_1] \cdot \ldots \cdot [f_k]$ and $[f_1'] \cdot \ldots \cdot [f_l']$ are equivalent. □

TODO? (Examples of covering space)

Tvrzení 0.9

Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\tilde{f}_\sigma : Y \to \tilde{X}$ lifting f_0, then there is a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Poznámka
This statement in a more general form has already appeared on practicals.

Poznámka
One can use this proposition in two ways: We have a lifting property for paths which says that $\forall$path $f : [0,1] = I \to X$and each lift $\tilde{x}_0$ of $f(0) = x_0 \in X$ there is a unique $\tilde{f} : I \to \tilde{X}$ lifting f starting at $\tilde{x}_0$.

If $Y = I$, we get that every homotopy f_t of a path f_0 in X lifts to a homotopy $\tilde{f}_t$ of each lift $\tilde{f}_0$ of f_0.

As a corollary of Proposition (*) we can write:

Důsledek
The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by the covering space map $p : \tilde{X} \to X$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Důkaz (Of corollary)
An element of the kernel of p_* is represented by a loop $\tilde{f}_0 : I \to \tilde{X}$ with a homotopy $f_t : I \to X$ of $f_0 = p \cdot \tilde{f}_0$ to the trivial loop. By ? the applications of proposition (*) there is a lifted homotopy of loops $\tilde{f}_t$ starting at f_0 and ending with a constant loop. Hence $[\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{x}_0)$ and p_* is injective. □

Tvrzení 0.10

The number of sheets of a covering space $p : \tilde{X} \to X, \tilde{x} \mapsto x$ with X and $\tilde{X}$ path connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Důkaz
For a loop g in X based at x_0 let $\tilde{g}$ be its lift in $\tilde{X}$ based at $\tilde{x}_0$. A product $h \cdot g$, $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has a lift $\tilde{h} \cdot \tilde{g}$ ending at the same point as $\tilde{g}$ (since $\tilde{h}$ is a loop). Hence we can define a map Φ from cosets $H[g]$ to $p^{-1}(x_0)$ implies that Φ is surjective since $\tilde{x}_0$ can be lowed by a path $\tilde{g}$ to an arbitrary point in $p^{-1}(x_0)$ and $\tilde{g}$ projects to a loop g at x_0.

To show that Φ is injective, observe that $\Phi(H[g_1]) = \Phi(H[g_2]) \implies g_1 \cdot g_2^{-1}$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$, so $[g_1][g_2]^{-1} \in H \implies H[g_1] = H[g_2] \implies$ injectivity + surjectivity $\implies$ bijectivity of Φ. □

TODO? (Problems)

TODO!!!

TODO? (Problems)

Poznámka
Typically homology is easier to compute than homotopy.

Definice 0.11 (Simplex defined by points, standard simplex, direction of simplex)

Take $\mathbb{R}^m$ and points $v_0, \dots, v_m \in \mathbb{R}^m$ such that $v_1 - v_0$, $v_2 - v_0$, ..., $v_n - v_0$ are linearly independent (this condition is necessary to say that $v_0, \dots, v_m$ will not live on hyperplane of dimension $< n$). We define

$$[v_0, v_1, \dots, v_n] := \left\{ \sum_{i=1}^{n} t_i v_i | \sum t_i = 1, t_i \geqslant 0 \right\},$$

the smallest convex set in $\mathbb{R}^m$ containing $v_0, \dots, v_n$. We call $[v_0, v_1, \dots, v_n]$ a simplex defined by points $v_0, \dots, v_n$.

Standard simplex:

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \right\} : \sum_{i=0}^{n} t_i = 1 \wedge \forall i \in [0, n] t_i \geqslant 0$$

The order we use for points in particular determines directions of 1-simplices. Specifying the direction in such a way we got a canonical homeomorphism $\Delta^n \to [v_0, \dots, v_n]$ given by $(t_0, \dots, t_n) \to \sum_{i=1}^{n} t_i v_i$.

Definice 0.12 (Faces, boundary, and interior of simplex)

Given $[v_0, \dots, v_n]$, if we remove v_i, i.e. we write $[v_0, \dots, v_{i-1}, v_{i-2}, \dots, v_n]$, then such a simplex is a face of $[v_0, \dots, v_n]$. Same way we can remove more vectors and get faces of lower dimensions.

$\partial\Delta_n$ = boundary of Δ_n = union of all faces (including lower dimensional) of Δ_n. (The same for $[v_0,\dots,v_n]$.)

$\mathring{\Delta}_n := \Delta_n \backslash \partial\Delta_n$ is interior of Δ_n. (The same for $[v_0,\dots,v_n]$.)

Definice 0.13 (Δ complex structure (Δ-complex))

A Δ complex structure on a topological space X is a collection of maps $\delta_\alpha : \Delta^n \to X$ with n independent of α (i.e. n is fixed, α indexes collection) such that

- $\delta_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X can be in the image of exactly one such restriction;
- $\delta_\alpha|_{\text{face of } \Delta^n}$ is one of the maps $\delta_\beta : \Delta^{n-1} \to X$. Here we identify the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism that preserves the ordering of vertices.
- $A \subset X$ is open $\Leftrightarrow$ $\partial_\alpha^{-1}(A)$ is open in Δ^n for each δ_α.

Poznámka
Δ-complex can be built from the collection of disjoint simplices by identifying various subcomplexes spanned by the subsets of the vertices, where the identification can be done by canonical linear homeomorphisms that preserves the ordering of vertices.

From the discussion about CW complexes it follows that interiors of Δ^n that define $\delta_\alpha(\mathring{\Delta}_n)$ (open simplices on X) are the e_α^n of a CW complex structure on X.

Definice 0.14 (Simplicial homology)

Given a topological space with a fixed Δ-complex structure on it. We define $\Delta_n(X) = C_n(X)$ which is by definition the free Abelian group with basis consist of $\sigma_n(\mathring{\Delta}_n) = e_\alpha^n$ of X. (Free module over $\{e_\alpha^n\}$.)

Poznámka
Elements of $\Delta_n(X) = C_n(X)$ can be written as formal sums $\sum_\alpha n_\alpha e_\alpha^n$, where $n_\alpha \in \mathbb{Z}$. Elements of $\Delta_n(X) = C_n(X)$ are typically called chains.

Equivalently we could write $\sum_\alpha n_\alpha \delta_\alpha$, where δ_α are the defining on characteristic maps of e_α^n. We remember that $\partial([v_0, v_n])$ consists of $n-1$-simplices.

TODO? (Last remarks)

TODO? (Problems)

1 Singular homology

Definice 1.1 (Singular n-simplex, chain, chain group)

A singular n-simplex in a topological space X is a continuous map $\delta : \Delta^n \to X$.

$C_n(X)$ = free Abelian group generated by the set of singular n-simplices in X.

Poznámka
The generators of $C_n(X)$ is in general an uncountable set.

If $X = \{point\}$, then all possible maps $\Delta^n \to X$ are just constant maps and there is only one constant map. Hence the generating set of $|C_n(X)| = 1$ and thus $C_n(X) \simeq \mathbb{Z}$.

Elements of $C_n(X)$ are called chains, and $C_n(X)$ is called a n-th chain group. We define

$$\partial_n : C_n(X) \to C_{n-1}(X), \qquad \partial_n(\delta) = \sum_{i=0}^{n} (-1)^i \delta|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

Poznámka
This is the same formula that we used for simplicial homology.

Since the formulas for ∂_n and ∂_{n-1} coincide with the correspoding formulas for simplicial homology, we see that $\partial_{n-1} \circ \partial_n = 0$. This folows from the proof for simplicial homology. (The standard notation for singular complex will be $C(X)$, for simplicial $C^\Delta(X)$.)

Therefore, we can form a complex $C(X)$:

$$\ldots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \to \ldots \to 0$$

On the left side, a priori, we can't say that a certain level we start getting only zeroes. ($C_n = 0 \equiv C_n = \{0\}$).

$H^\Delta(X)$ = simplicial homology. Then we define $H_n(X) = \operatorname{Ker} \partial_n / \Im \partial_{n+1}$. ($n$-th singular homology group.)

Poznámka
On one hand singular homology looks more general than simplicial homology. On the other hand, singular homology can be seen as a particular case of simplicial homology by the following construction:

For an arbitrary space X, we define the singular complex $S(X)$ to be the Δ-complex with one n-simplex Δ^n_δ for each singular n-simplex $\delta : \Delta^n \to X$ with Δ^n_δ attached to the $n-1$ simplices of $S(X)$, that are restrictions of δ to the $n-1$ simplices in the boundary $\partial\Delta^n$ by the restriction maps. $H^\Delta_n(S(X))$ is identified with $H_n(X)$.

Poznámka
There are two variants of singular homology:

1. Reduced homology: Assume that we have a singular chain complex

$$\to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \to 0 \to \dots$$

We modify this complex, ? will be called the reduced complex (reduced from $C(X)$):

$$\to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \to \dots,$$

where ε is called an augmentation map and it is defined by $\varepsilon(\sum_{i \text{ finite}} n \cdot i \cdot \delta_i) = \sum_{i \text{ finite}} n \cdot i \implies \varepsilon$ is well defined. (Exercise: check that ε vanishes on ∂_1.) Denote the homology of the reduced complex by $\widetilde{H}(X)$. TODO!!! (Asi dva řádky rovnic.)

2. Relative complex (complex of a pair): Given a space X and a subspace $A \subset X$, we define $C_n(X, A) := C_n(X)/C_n(A)$ (the chains in A will be trivial in $C_n(X, A)$). This way we get a complex

$$\dots \to C_n(X, A) \overset{\partial_n(X,A)}{\to} C_{n-1}(X, A) \overset{\partial_{n-1}(X,A)}{\to} C_{n-2}(X, A) \to \dots \to 0,$$

here $\partial_n(X, A) : C_n(X, A) \to C_{n-1}(X, A)$ are quotient maps of ∂_n's for $\partial_n : C_n(X) \to C_{n-1}(X)$.

Since we know that $\partial_{n-1} \circ \partial_n = 0$, then for quotient maps the corresponding compositions $\partial_{n-1}(X, A) \circ \partial_n(X, A) = 0$. Hence we really see that this is a complex. We can define the homology of it $H_n(X, A) := \operatorname{Ker} \partial_n(X, A)/\Im \partial_{n+1}(X, A)$.

1.1 Homotopy equivalence of singular homology

Definice 1.2

For a map $f : X \to Y$, an induced homomorphism $f_n^c : C_n(X) \to C_n(Y)$ is defined by

$$(\Delta^n \xrightarrow{\partial_X} X) \to (\Delta^n \xrightarrow{\partial_X} X \xrightarrow{f} Y)$$

$\delta_X \mapsto f \circ \delta_X = \delta_Y$. These maps f_n^c satisfy $f_c^{n-1} \circ \partial_n^X = \partial_n^Y \circ f_n^c$. TODO?(This is since…)

Typically maps $f_c = (f_c^n)_n$ are called chain maps. f_c induces a map $f_* : H_n(X) \simeq H_n(Y)$.

TODO?

TODO? (Problems)

Tvrzení 1.1

Given two topological spaces X and Y that are homotopy equivalent. Then $H_n(X) \simeq H_n(Y)$ $\forall n$.

Důkaz
The essential procedure is a subdivision (simplex decomposition) of $I \times \Delta^n$. Let $\{0\} \times \Delta^n$ be given by $[v_0, \dots, v_n]$ and $\{1\} \times \Delta^n$ be given by $[w_0, \dots, w_n]$, where v_i and w_i have the same image under the standard/canonical projection $I \times \Delta^n \to \Delta^n$. We would like to pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n-simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. The first step for us is to ? $[v_0, \dots, v_{n-1}, v_n]$ to $[v_0, \dots, v_{n-1}, w_n]$. The second step is to move this up to $[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$. Then we keep going and proceed with other (remaining) steps.

The typical step for us is to $[v_0, \dots, v_{i-1}, v_i, w_{i+1}, \dots, w_n]$ moves up to $[v_0, \dots, v_{i-1}, w_i, w_{i+1}, \dots, w_n]$. The region between simplices $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ and $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ is a $(n+1)$-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$.

This way for each of the steps, we get $(n+1)$-simplex. This leads to the decomposition of $I \times \Delta^n$ into $(n+1)$-simplices, each intersecting the previous one with the n-simplex face. Since X is homotopy equivalent to Y, there is a homotopy $F : I \times X \to Y$ and for each simplex $\delta : \Delta^n \to X$ we get the composition $F \circ (\delta \times 1) : \Delta^n \times I \to X \times I \to Y$. Using this map we can define the prism operator $P : C_n(X) \to C_{n+1}(Y)$. $P(\delta) = \sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$.

We will show that P satisfies the following property: $\partial_Y P = g_c - f_c - P\partial_X$. To prove this, we compute:

$$\partial P(\delta) = \partial(\sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}) = \sum_{j \leqslant i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots, w_n]} + \sum_{j \geqslant i} (-1)^{i+(j+1)} F \circ (\delta$$

The terms with $i = j$ will cancel out except of $F \circ (\delta \times 1)|_{[\hat{v_0}, w_0, \dots, w_n]}$, which is $g \circ \delta = g_c(\delta)$, and $-F \circ (\delta \times 1)|_{[v_0, \dots, v_n, \hat{w_n}]}$, which corresponds to $-f \circ \delta = -f_c(\delta)$.

Now observe that $i \neq j$, then part of that corresponds to $-P\partial$, since $\partial P(\delta) = (g_c - f_c - P\partial)(\delta)$, where $\delta = [v_0, \dots, v_n]$.

$$\partial P(\delta) = \sum_{j<i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, \hat{v_j}, \dots, v_i, w_i, \dots, w_n]} + \sum_{j>i} (-1)^{i+(j+1)} F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, \hat{w_j} \dots, w_n]}.$$

$$P(\partial(\delta)) = P(\sum_j)(-1)^i TODO!!!$$

We finish the proof by saying the following: If $\alpha \in C_n(X)$, then $g_c(\alpha) - f_c(\alpha) - P\partial(\alpha) = \partial P(\alpha) \implies$

$$\implies g_c(\alpha) - f_c(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha),$$

because α is a ?. Hence $g_c(\alpha) - f_c(\alpha)$ is a boundary (it is $\partial(P(\alpha))$). So $g_c(\alpha)$ and $f_c(\alpha)$ determine the same class in homotopy $\implies f_* = g_* \implies$ homotopy equivalence of X and $Y \implies H_n(X) \simeq H_n(Y)$. □

TODO? (Problems)