

1 Dynamické systémy

Definice 1.1 (Dynamický systém)

(φ, Ω) , $\Omega \subset \mathbb{R}^n$ otevřená, $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$ $\varphi(t, x)$.

- $\varphi(0, x) = x$;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- φ je spojitý.

Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geq 0\}$ je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leq 0\}$ je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$ je plný orbit.

Definice 1.3 (Pozitivně, negativně a úplně invariantní)

(φ, Ω) dynamický systém, $M \subset \Omega$.

M je pozitivně invariantní $\equiv \forall x \in M : \gamma^+(x) \subset M$.

M je negativně invariantní $\equiv \forall x \in M : \gamma^-(x) \subset M$.

M je úplně invariantní $\equiv \forall x \in M : \gamma(x) \subset M$.

Poznámka

$\gamma^+(x_0)$ je pozitivně invariantní, $\gamma^-(x_0)$ je negativně invariantní a $\gamma(x_0)$ je úplně invariantní.

Definice 1.4

$$\omega(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow \infty : \varphi(t_k, x_0) \rightarrow y\},$$

$$\alpha(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow -\infty : \varphi(t_k, x_0) \rightarrow y\}.$$

Poznámka (To je ekvivalentní)

$$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0 \ \forall T > 0 \ \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon\}.$$

Lemma 1.1

$$\omega(x_0) = \bigcap_{\tau \geq 0} \overline{\gamma^+(\tau, x_0)}.$$

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Důkaz„ \subseteq “: $y \in \omega(x_0)$: $\forall \varepsilon > 0 \forall T \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon$. Chceme:

$$\forall \tau \geq 0 \forall \varepsilon > 0 \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geq 0 \forall \varepsilon > 0 \exists s \geq \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

$$\text{„}\supseteq\text{“: } \forall \tau \geq 0 \ y \in \overline{\gamma^+(\tau, x_0)} \implies$$

$$\implies \forall \varepsilon \exists s \geq \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

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□

Věta 1.2 (Vlastnosti ω -limitní množiny)

Nechť (φ, Ω) je dynamický systém, $x_0 \in \Omega$. Potom

1. $\omega(x_0)$ je uzavřená, úplně invariantní.
2. Pokud $\gamma^+(x_0)$ je relativně kompaktní v \mathbb{R}^n , pak $\omega(x_0) \neq \emptyset$, $\omega(x_0)$ je kompaktní, souvislá.

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Důkaz

1. $\omega(x_0)$ je průnik uzavřených množin, tedy uzavřená. $y \in \omega(x_0) \exists t_k \nearrow \infty \varphi(t_k, x_0) \rightarrow y$.

$$s_k = t_k + t \quad \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0))$$

$$t_k \rightarrow \infty, \varphi \text{ spojitá} \quad \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \rightarrow \varphi(t, y)$$

2. $\exists K \subset \mathbb{R}^n$ kompaktní $\gamma^+(x_0) \subset K$. a) pokud $t_n \geq 0, t_n \rightarrow \infty \{\varphi(t_n, x_0)\}_{n=1}^\infty$ omezená posloupnost $\implies \exists \{t_{n_k}\}_{k=1}^\infty \subset \{t_n\}_{n=1}^\infty$, podposloupnost, $\exists y \in \Omega \varphi(t_{n_k}, x_0) \rightarrow y$. Pak $y \in \omega(x_0)$.

b) $\omega(x_0)$ je tedy úplná a omezená, takže kompaktní. c) ať $\omega(x_0)$ je nesouvislá, tedy $\omega(x_0) \subseteq U \cup V$, U, V otevřené disjunktní neprázdné, $U, V \subseteq K$. Vezměme $y \in \omega(x_0) \cap U$, $z \in \omega(x_0) \cap V$. Nechť t_n je posloupnost taková, že $\varphi(t_{2n}, x_0) \rightarrow y$, $\varphi(t_{2n+1}, x_0) \rightarrow z$, $t_{2n} < t_{2n+1}$, $\varphi(t_{2n}, x_0) \in U$, $\varphi(t_{2n+1}, x_0) \in V$. $F = K \setminus (U \cup V)$ uzavřená, tedy $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$. Tedy $\{\varphi(s_n, x_0)\}$ je omezená posloupnost $\implies \exists$ podposloupnost konvergující k $w \in F$. □

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Definice 1.5 (Topologická konjugovanost)

(φ, Ω) , ψ, Θ dynamické systémy. $\exists : \Omega \rightarrow \Theta$ homeomorfismus (bijekce, spojitá, spojitá inverze):

$$\forall x \in \Omega \forall t \in \mathbb{R} \quad h(\varphi(t, x)) = \psi(t, h(x)).$$

Poznámka

Dá se zobecnit ještě zobrazováním časů.

Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0, (\varphi, \Omega)$ příslušný dynamický systém. $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, y(0) = 0$ a (ψ, Θ) je

příslušný dynamický systém. Potom $(\varphi, \Omega), (\psi, \Theta)$ jsou lokálně topologicky konjugované ($\exists U$ okolí $x_0 \in \Omega$ a V okolí $\mathbf{o} \in \mathbb{R}^n$ taková, že $\exists g : U \rightarrow V$ homeomorfismus $g(\varphi(t, x)) = \psi(t, g(x))$ $\forall x \in U, \forall t : \varphi(t, x) \in U$).

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Důkaz

BÚNO $f_1(x_0) = \alpha \neq 0$ (první souřadnice funkce f) a $x_0 = \mathbf{o}$. Buď \tilde{V} okolí $\mathbf{o} \in \mathbb{R}^n$ $G : \tilde{V} \rightarrow \mathbb{R}^n, G(y_1, \dots, y_n) = \varphi(y, (0, y_2, \dots, y_n))$. Chceme ukázat, že G je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_1} \Big|_{(0, \dots, 0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \dots, y_n)) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(y_1(0, y_2, \dots, y_n))) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(0, \dots, 0)) = f(x_0) = \alpha \neq 0$$

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_j} \Big|_{(0, \dots, 0)} = \lim_{h \rightarrow 0} \frac{G(0, \dots, h, \dots, 0) - G(0, \dots, 0)}{h} = \lim_{h \rightarrow 0} \frac{(0, \dots, h, \dots, 0)^T - (0, \dots, 0)^T}{h} = (0, \dots, 1, \dots, 0)^T$$

Tedy $\nabla G(0, \dots, 0)$ je „jednotková matice, až na to, že a_{11} je α “, tudíž podle věty o inverzi funkce $\exists V \subseteq \tilde{V}$ okolí 0, $\exists U$ okolí bodu x_0 tak, že $G : V \rightarrow U$ je homeomorfismus. Položme $g = G^{-1}$.

Nyní už stačí $g(\varphi(t, x_0)) = \psi(t, g(x_0)) \forall x_0 \in U \forall t : \varphi(t, x_0) \in U. \varphi(t, x_0) = G(\psi(t, g(x_0)))$

3. $x \in U = G(V) \exists y \in V x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

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□

Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \rightarrow \Omega, f \text{loc.Lip.}$$

$$\exists V : \Omega \rightarrow \mathbb{R}, \text{ bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leq l\} - \text{bounded}$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^n \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leq 0 \quad \forall x \in \Omega_l.$$

$$R = \{x \in \Omega_l \mid \dot{V}_f(x) = 0\}, \quad M = \{y \in R \mid \gamma^+(y) \subset R\}.$$

Then $\forall x \in \Omega_l : \omega(x) \subset M$.

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Důkaz

Let $x \in \Omega_l$. $\forall y \in \omega(x) \exists t_k \nearrow \infty : x(t_k) \rightarrow y$. $\varphi(t, x_0) = x(t)$.

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leq 0.$$

$V(x(t)) \searrow$ and $\exists C : \forall x \in \Omega : V(x) > -C$ so $\exists \lim_{t \rightarrow \infty} V(x(t)) = c$.

So $\exists c \forall y \in \omega(x_0) V(y) = c$. $V(x(t_k)) \rightarrow V(y) = c$.

$$\gamma^+(y) \subset \omega(x_0) \quad V(\varphi(t, y)) = c \quad \forall t \geq 0 \implies$$

$$\implies \frac{d}{dt}V(\varphi(t, y)) = 0.$$

└ $\gamma^+(y) \subset R$ in particular, $y \in R$. Hence $y \in M$. □

2 Poincaré-Bendixson theory

Věta 2.1 (Poincaré-Bendixson)

Let $p \in \Omega$, Ω open connected. $\omega(p)$ doesn't contain stationary points and $\gamma^+(p)$ is relatively compact ($\gamma^+(p)$ is compact). Then $\omega(p) = \Gamma$ -periodic orbit.

Věta 2.2 (Bendixson-Dulas)

Ω -simply connected (\forall closed Jordan curve γ in Ω , $\text{int}(\gamma) \subset \Omega$). $\exists B : \Omega \rightarrow \mathbb{R} : (\text{div } Bf)(x) = \frac{\partial Bf_1}{\partial x_1}(x_1, x_2) + \frac{\partial Bf_2}{\partial x_2}(x_1, x_2) > 0$ for almost every $x \in \Omega$. Then $x' = f(x)$ doesn't have nontrivial periodic solutions.

Definice 2.1 (Transverzála)

Σ segment on a line such that $\forall p \in \Sigma : \Sigma \nparallel f(p)$.

Lemma 2.3

Σ transversála, $p \in \Sigma \subset \Omega$. Then $\exists \tilde{U} \subset U$ neighborhood of p . $\exists \Delta > 0$ such that

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U \quad \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

┌ *Důkaz*

└ Use Th. of rect. □

Lemma 2.4

Let $p \in \Omega$ and assume that $|\gamma^+(p) \cap \Sigma| \geq 3$, i. e. $\exists t_1 < t_2 < t_3 \varphi(t_j, p) \in \Sigma, j = 1, 2, 3$. Then $\varphi(t_2, p)$ lie between $\varphi(t_1, p)$ and $\varphi(t_3, p)$.

┌ *Důkaz*

Simple closed curve:

$$\psi := \{\varphi(t, x), t \in [t_1, t_2]\} \cup \underbrace{\text{conv}\{z_1, z_2\}}_{\subset \Sigma}.$$

└ By uniqueness of φ and by the Jordan Lemma. □

Lemma 2.5

$\Sigma \subseteq \Omega \subseteq \mathbb{R}^2$ transversal, $p \in \Omega \implies |\omega(p) \cap \Sigma| \leq 1$.

┌ *Důkaz*

$$y \neq z \in \omega(p) \cap \Sigma \implies \exists t_k \nearrow \infty : x(t_{2k}) \rightarrow y \wedge x(t_{2k+1}) \rightarrow z.$$

From lemma above: $\exists \tilde{U} \subset U$ – neighbourhoods of y and $\exists \Delta$:

$$\exists k_0 : \forall k > k_0, (x(t_{2k}) \in \tilde{U}) \implies (\exists \tilde{t}_{2k} : |\tilde{t}_{2k} - t_{2k}| < \frac{\Delta}{2} \wedge x(\tilde{t}_{2k}) \in \Sigma \cap \tilde{U}.$$

Similarly $\exists \tilde{V}$ – neighbourhood of z , $\exists \tilde{t}_{2k+1} : |\tilde{t}_{2k+1} - t_{2k+1}| < \frac{\Delta}{2}$ and $x(\tilde{t}_{2k+1}) \in \Sigma \cap \tilde{V}$.

└ WLOG $\tilde{V} \cap \tilde{U} = \emptyset$. Now continue with Lemma 2 (not monotonic). □

Důkaz (Poincaré-Bendixson theorem)

Step 1, „ $q \in \omega(p)$ “: We want to show that q belongs to $q \in \Gamma$, where Γ is non-trivial periodic orbit.

$\exists x_0 \in \omega(q), \exists t_k \nearrow \infty : \varphi(t_k, q) \rightarrow x_0$. x_0 is not a stationary point ($q \in \omega(p) \implies \omega(q) \subseteq \omega(p)$). So there exists a transversal $\Sigma \subseteq \Omega, x_0 \in \Sigma$.

By lemma above $\exists \tilde{t}_k, \exists \Delta > 0 : |\tilde{t}_k - t_k| < \frac{\Delta}{2}, q \in \omega(o) \implies \varphi(\tilde{t}_k, q) \in \omega(p) \implies \varphi(\tilde{t}_k, q) \in \Sigma \cap \omega(p)$ at most 1-point set by theorem...

$$\varphi(\tilde{t}_k, q) \rightarrow x_0 \implies \varphi(\tilde{t}_k, q) = x_0.$$

Periodic orbit $\implies x_0 \in \Gamma = \{\varphi(t, q) | \tilde{t}_k < t < \tilde{t}_{k+1}, k \in \mathbb{N}\} \implies q \in \Gamma$ (uniqueness).

Now we want to show that $\omega(p) \subseteq \Gamma$. Let $M \neq \emptyset$, $M = \omega(p) \setminus \Gamma$: $\gamma^+(p)$ is bounded $\implies \omega(p)$ is connected

$$\exists x_0 \in \Gamma : \exists \{p_n\}_{n \in \mathbb{N}} \subseteq M, p_n \rightarrow x_0.$$

$\exists \Sigma$ transversal: $x_0 \in \Sigma$ (because not stationary). By lemma above we have

$$\exists \tilde{p}_n \in \gamma^+(p_n) : \tilde{p}_n \in \Sigma \cap \gamma^+(p_n) \cap \tilde{U}(x_0).$$

Since $p_n \in \omega(p)$, $n \in \mathbb{N}$, then $\gamma^+(p_n) \subseteq \omega(p) \implies \tilde{p}_n \in \omega(p)$.

By previous lemma $\tilde{p}_n = x_0$ and $p_n \in \gamma^-(\tilde{p}_n) = \gamma^-(x_0) \subseteq \Gamma$. ∇

□

Důkaz (Bendixson-Dulas theorem)

Let Γ be a non-trivial periodic orbit, $\Gamma \subset \Omega$, $\Gamma = \partial M$

$$0 < \int_M \operatorname{div}[B(x) \cdot f(x)] d\lambda^2 = \int_{\partial M} \langle B(x) \cdot f(x), \nu(x) \rangle dS = 0.$$

□

3 Caratheodory theory

Definition 3.1 (Caratheodory theory)

f measurable, $x(t)$ absolutely continuous, Lebesgue integral.

Definition 3.2

$\Omega \subseteq \mathbb{R}^{n+1}$, $f \in \operatorname{Car}(\Omega) \equiv \forall I \times B \subset \Omega$, $I \subseteq \mathbb{R}$ bounded interval, $B \subseteq \mathbb{R}^n$ bounded closed ball:

- $\forall x \in B$: $t \mapsto f(t, x(t))$ is measurable;
- for almost every $t \in I$: $x \mapsto f(t, x)$ is continuous;
- $\exists h \in L^1(I)$: $|f(t, x)| \leq |h(t)|$ for almost every $t \in I$ and $\forall x \in B$.

Definition 3.3 (*)

$x' = f(t, x)$, $x(t_0) = x_0$, $\Omega \subseteq \mathbb{R}^{n+1}$ open, $f : \Omega \rightarrow \mathbb{R}^n$, $f \in \operatorname{Car}(\Omega)$.

Definition 3.4

$x : I \rightarrow \mathbb{R}^n$ (I interval) is a solution of * in the sense of Caratheodory, if $x \in AC_{loc}(I)$ and $\operatorname{graph}(x) \subset \Omega$ and for almost every $t \in I$: $x'(t) = f(x(t), t)$ and $x(t_0) = x_0$.

Poznámka

$$\Leftrightarrow x(t) = x_0 + \int_{t_0}^t f(x(s), s) ds, \quad \text{for almost every } t.$$

Lemma 3.1

$f \in Car(\Omega)$, $x : I \rightarrow \mathbb{R}$ is continuous, $graph(f) \subseteq \Omega$, then $f(t, x(t)) \in L_{loc}^1(I)$.

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Důkaz

Step 1: „ f is measurable“: We approximate $x(t)$ by step function on $I = I_1 \cup \dots \cup I_n \cup \dots$, $\{x_n(x)\}_{n=1}^\infty$, piecewise constant functions, $x_n(t) \Rightarrow x(t)$ on I_k , $I = \bigcup_j I_{j,n}$ disjoint union, $x_n(x) = \xi_{j,n}$ for $t \in I_{j,n}$. $f(t, x_n(t)) = f(t, \xi_{j,n})$ for $t \in I_{j,n}$, $f(t, \xi_{j,n})$ is measurable.

$f(t, x_n(t)) \rightarrow f(t, x(t))$ for almost every $t \in I \implies f(t, x(t))$ is measurable.

Step 2: $|f(t, x(t))| \leq m(t)$ for almost every $t \implies f \in L_{loc}^1(I)$. □

Lemma 3.2

$x : I \rightarrow \mathbb{R}^n$ continuous, $graph(x) \subseteq \Omega$, $f \in Car(\Omega)$ then x is solution of $*$ $\Leftrightarrow \forall t_1, t_2 : x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s)) ds$.

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Důkaz

„ \implies “ $x \in AC_{loc}(I)$, $x'(t) = f(t, x)$ for almost every $t \in I$, add \int :

$$\int_{t_1}^{t_2} x'(t) dt = \int_{t_1}^{t_2} f(s, x(s)) ds.$$

„ \Leftarrow “ $t_1 = t_0$, $t_2 = t$:

$$x(t) - x_0 = \int_{t_0}^t f(s, x(s)) ds, \quad x(0) = x_0.$$

($f \in L_{loc}^1$, so it make sense).

$\implies x$ is AC, $graph(x) \subseteq \Omega$, $x'(t) = f(t, x)$ for almost every $t \in I$. □

Věta 3.3 (Uniform contraction theorem (generalized Banach theorem))

Λ, X metric spaces, $X \neq \emptyset$ complete, $\Phi : \Lambda \times X \rightarrow X$. $\forall x \in X : \Phi(\cdot, x)$ is continuous, $\exists \varkappa \in (0, 1) : \varrho(\Phi(\lambda, x), \Phi(\lambda, y)) \leq \varkappa \cdot \varrho(x, y) \forall \lambda \in \Lambda, \forall x, y \in X$.

Then $\forall \lambda \in \Lambda \exists ! x(\lambda) \in X$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$, $\lambda \mapsto x(\lambda)$ is continuous and

$$\varrho(y, x(\lambda)) \leq \frac{\varrho(y, \Phi(y))}{1 - \varkappa} \quad \forall y \in X \quad \forall \lambda \in \Lambda.$$

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Důkaz

Let $x_0 \in X$, $x_1 = x_1(\lambda, x_0) := \Phi(\lambda, x_0)$, $x_{n+1} = x_{n+1}(\lambda, x_0) := \Phi(\lambda x_n)$. $\lambda \in \Lambda$ fixed:

$$\begin{aligned}
 \varrho(x_n(\lambda, x_0), x(\lambda, x_0)) &\leq \sum_{k=n}^{m-1} \varrho(x_k(\lambda, x_0), x_{k+1}(\lambda, x_0)) = \\
 &= \sum_{k=n}^{m-1} \varrho(\Phi(\lambda, x_{k-1}(x, x_0)), \Phi(\lambda, x_k(\lambda, x_0))) \leq \\
 &\leq \sum_{k=n}^{m-1} \varkappa \varrho(x_k, x_{k-1}) \leq \\
 &(\varrho(x_k, x_{k-1}) \leq \varkappa \varrho(x_k, x_{k-1}) \leq \dots \leq \varkappa^k \varrho(x_0, x_1).) \\
 &\leq \sum_{k=n}^{m-1} \varkappa^k \varrho(x_0, x_1(\lambda, x_0)) \leq \varrho(x_0, x_1(\lambda, x_0)) \underbrace{\sum_{k=n}^{\infty} \varkappa^k}_{\frac{\varkappa^m}{1-\varkappa}}.
 \end{aligned}$$

\implies sequence $\{x_n(\lambda, x_0)\}_{k=1}^{\infty}$ is Cauchy \implies it has a limit:

$$\exists x(\lambda, x_0) : \lim_{n \rightarrow \infty} x_n(\lambda, x_0) = x(\lambda, x_0).$$

We want to show that $x(\lambda, x_0)$ does not depend on x_0 . $\tilde{x}_0 : x_n(\lambda, \tilde{x}_0) =: \tilde{x}_n$.

$$\varrho(x_n, \tilde{x}_n) = \varrho(\Phi(\lambda x_{n-1}), \Phi(x, \tilde{x}_{n-1})) \leq \varkappa^n \varrho(x_0, \tilde{x}_0) \rightarrow 0 \implies x = \tilde{x}.$$

$$\Phi(\lambda, x(\lambda)) = x(\lambda).$$

$$\begin{aligned}
 \varrho(\Phi(\lambda, x(\lambda)), x(\lambda)) &= \lim_{n \rightarrow \infty} \varrho(\Phi(\lambda, x(\lambda)), x_n(\lambda)) = \\
 &= \lim_{n \rightarrow \infty} \varrho(\Phi(\lambda, x(\lambda)), \Phi(\lambda, x_{n-1}(\lambda))) \leq \varkappa \varrho(x(\lambda, x_{n-1})) = 0.
 \end{aligned}$$

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□

┌ *Důkaz* („ $\lambda \mapsto x(\lambda)$ is continuous“)

$$\begin{aligned} \varrho(x(\lambda), x(\mu)) &= \varrho(\Phi(\lambda, x(\lambda)), \Phi(\mu, x(\mu))) \leq \varrho(\Phi(\lambda, x(\lambda)), \Phi(\lambda, x(\mu))) + \varrho(\Phi(\mu, x(\mu)), \Phi(\lambda, x(\mu))) \leq \\ &\leq \kappa \varrho(x(\lambda), x(\mu)) + \varrho(x(\lambda), x(\mu)) \leq (1 - \kappa)^{-1} \varrho(\Phi(\lambda, x(\mu)), \Phi(\mu, x(\mu))) \rightarrow 0. \end{aligned}$$

So Φ is continuous in the first variable.

$$\mu \text{ fixed, } \lambda = \mu_n \rightarrow \mu. \ y, \ \lambda \text{ fixed, } x_1 = \Phi(\lambda, y), \ x_n = \Phi(\lambda, x_{n-1}), \ n \geq 2.$$

$$n = 0 \ (y = x_0): \ m \rightarrow \infty$$

$$\varrho(y, \lambda(x)) \leq \frac{\kappa^0}{1 - \kappa} \varrho(y, \Phi(\lambda, y)) = \frac{\kappa^0}{1 - \kappa} \varrho(y, x_1).$$

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□

TODO!!!

3.1 Controllability

Definition 3.5 (Control theory)

$$x' = f(x, u), \ f : \Omega \times U, \ \Omega \subset \mathbb{R}^n, \ U \subset \mathbb{R}^n,$$

$$u \in \mathcal{U} := \{u : [0, T] \rightarrow \mathbb{R}^n | \text{measurable, } \|u\|_\infty < \infty\} = L^\infty(0, T, \mathbb{R}^n).$$

(\mathcal{U} is admissible functions).

Definition 3.6 (Linear task)

$$x' = Ax + Bu, \ A, B \in \mathbb{R}^{n \times m}, \ m < n.$$

Definition 3.7

$$x_0 \xrightarrow[u(0)]{t} 0 \text{ iff } x(0) = x_0, \ x(t) = 0.$$

Definition 3.8 (Area of controllability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in L^\infty(0, t, \mathbb{R}^n) : x_0 \xrightarrow[u(0)]{t} 0 \right\}$$

Definition 3.9 (Kalman matrix)

$$\mathcal{K}(A, B) := (B | AB | A^2 B | \dots | A^{n-1} B)$$

Věta 3.4

For linear problem $\mathcal{R}(t) = \text{LO}(g_1, g_2, \dots, g_{n-m})$, where $\mathcal{K}(A, B) = (g_1|g_2|\dots|g_{n-m})$

Tvrzení 3.5 (Observation)

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

$$x_0 \xrightarrow[u(0)]{t} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = - \int_0^t e^{-As}Bu(s)ds \quad (KO)$$

Lemma 3.6 (1)

$$A^k \in \text{LO}(I, A, A^2, \dots, A^{n-1}), k \in \mathbb{N}_0$$

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Důkaz
Cayley-Hamilton.
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Důkaz

1) $\mathcal{R}(t)$ is vector subspace of \mathbb{R}^n from definition $x_0 + x_1 \xrightarrow[(u_1+u_2)(0)]{t} 0, \alpha x \xrightarrow[\alpha u(0)]{t} 0$.

2) We want $\mathcal{R}(t)^\perp = (\text{LO}(g_1, \dots, g_n))^\perp$. „ \supseteq “: $p \in (\text{LO}(g_1, \dots, g_n))^\perp$. $x_0 \in \mathcal{R}(t)$ arbitrary. From KO:

$$0 \stackrel{?}{=} p^T x_0 = - \int_0^t p^T e^{-As} Bu(s) ds = - \int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k Bu(s) ds$$

We know $(p, g_j) = 0, p^T g_j = 0, p^T \mathcal{K}(A, B) = 0, p^T A^k B = 0, k \in [n-1]$. And from lemma 1 $k \in \mathbb{N}$. „ \subseteq “: $p \in \mathbb{R}^n, p \in \mathcal{R}(t)^\perp$. We want to prove $p \perp B, AB, A^2B, \dots, A^{n-1}B$. $B = (b_1|\dots|b_m)$. $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$. $\varphi \in L^\infty(0, T, \mathbb{R}), u(t) = \varphi(t) \cdot \mathbf{e}_j$, where $x_0 = - \int_0^t e^{-As} Bu(s) ds$. We have $x_0 \xrightarrow[u(0)]{t} 0$, hence $x_0 \in \mathcal{R}(t)$.

$$0 = p^T x_0 = -p^T \int_0^t e^{-As} Bu(s) ds = - \int_0^t p^T e^{-As} b_j \varphi(s) ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have $p^T e^{-As} b_j \equiv 0$, we derivate it, $p^T A^n e^{-As} b_j \equiv 0$, and set $s = 0$.
└

□

Důsledek

$\mathcal{R}(t)$ doesn't depend on time.

Definice 3.10 (Locally and globally controllable)

Linear problem is called locally controllable, iff $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 \mid |x_0| < \delta\} \subset \mathcal{R}(t)$. And globally if $\mathbb{R}^n = \mathcal{R}(t)$.

Důsledek

Linear problem is controllable $\Leftrightarrow \text{rank } K(A, B) = n$.

3.2 Observability

Definition 3.11 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, y = g(x), g : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, m < n.$$

Definition 3.12

We say that system $x' = f(x)$ is observable through $g(\cdot)$ on $[0, t]$, iff $\forall x_1(\cdot), x_2(\cdot) : [0, T] \rightarrow \mathbb{R}^n : g(x_1(t)) = g(x_2(t)) \forall t \in [0, T] \implies x_1(0) = x_2(0)$.

Definition 3.13 (Linear observability)

$$x' = Ax, y = Bx, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}.$$

Věta 3.7

$x' = Ax$ is observable on $[0, T]$ through $y = Bx \Leftrightarrow x' = A^T x + B^T u$ is controllable.

┌

Důkaz

(We will prove equivalence with $\text{rank } \mathcal{K}(A^T, B^T) = n$.) „ \Leftarrow “: For contradiction

$$\exists x_1(t), x_2(t), [0, T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At}x_0, Bx(t) = B^T e^{At}x_0 \equiv 0 \quad \forall t \in [0, T].$$

We differentiate it, set $t = 0$ and get $Bx_0 = 0, BAx_0 = 0, \dots, BA^{n-1}x_0 = 0$. So $x_0^T B^T = 0, \dots, x_0^T (A^T)^{n-1} B^T = 0$. $x_0^T \mathcal{K}(A^T, B^T) = 0, x_0 \perp \mathcal{K}(A^T, B^T)$, \nexists .

„ \implies “: For contradiction $\text{rank}(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T \mathcal{K}(A^T, B^T) = 0$. $x_0^T (A^T)^k B^T = 0 \forall k \in [n-1]$ and from lemma 1 $\forall k \in \mathbb{N}$. $BA^T x_0 = 0, Be^{At}x_0 = 0 \forall t \in [0, T]$. \nexists . □

└

Věta 3.8 (Local controllability)

$V \subset \mathbb{R}^n$ neighbourhood of 0, $U \subset \mathbb{R}^n$ neighbourhood of 0, $f : V \times U \rightarrow \mathbb{R}^n$ C^1 smooth, $f(0, 0) = 0, \mathcal{U} = \{u : [0, T] \rightarrow U \text{ measurable}\}$, $A = \nabla_x f(0, 0), B = \nabla_u f(0, 0), \text{rank } \mathcal{K}(A, B) = n$. Then

$$x' = f(x, u), x(0) = x_0 \text{ is locally controllable } \forall t \in (0, T].$$

┌ *Důkaz*

Fix $t > 0$, consider $x' = Ax + Bu$. Since $\text{rank}(A, B) = n$, the linear problem is globally controllable. Take initial condition y_1, \dots, y_n linearly independent.

$$\exists u_i \in L^\infty(0, t, \mathbb{R}^n) : y_j \xrightarrow{u_\lambda(0)}^t 0$$

$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ denote by $u_\lambda(t) = \sum_{j=1}^n \lambda_j u_j(t)$. We know $\sum_{j=1}^n \lambda_j y_j \xrightarrow{u_\lambda(0)}^t 0$.

Step 2:

$$x'_\lambda = f(x_\lambda, u_\lambda), \quad x_\lambda(t) = 0$$

If $\lambda = 0$, then $u_\lambda \equiv 0$, then $x_\lambda \equiv 0$.

$$\psi(\lambda) := x_\lambda(0), \psi : U_\lambda(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We want to prove $\psi(U_\lambda(0)) \supseteq \tilde{V}$, for some $\tilde{V} \subset \mathbb{R}^n$ open, $0 \in \tilde{V}$. We will prove that ψ is C^1 smooth, and that $\nabla \psi(0)$ is regular (if this is proved, then ψ is local diffeomorphism).

Step 3:

$$x_\lambda(s) = x_\lambda(t) + \int_t^s f(x_\lambda(s), u_\lambda(s)) ds.$$

Formally differentiate:

$$\frac{\partial x_\lambda(s)}{\partial \lambda_j} = \int_t^s (\nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot \frac{\partial x_\lambda(s)}{\partial \lambda_j} + \nabla_u f(x_\lambda(s), u_\lambda(s))) ds.$$

Denote $y_{\lambda,j}(s) = \frac{\partial x_\lambda(s)}{\partial \lambda_j}$.

$$y'_{\lambda,j}(s) = \nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot y_{\lambda,j}(s) + \nabla_u f(x_\lambda(s), u_\lambda(s)) \cdot u_j(s).$$

$$y_{\lambda,j}(t) = 0.$$

Consider $(LPy) \rightarrow y_{\lambda,j}(\cdot)$.

$$x_{\lambda+\Delta\lambda}(s) - x_\lambda(s) - \Delta\lambda \cdot y_{\lambda,j}(s) = 0$$

(as in Thm? of differentiability w. r. t. initial condition)

$$\frac{\partial \psi}{\partial \lambda_j}(\lambda = 0) = \frac{\partial x_\lambda(s=0)}{\partial \lambda_j}|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_j.$$

If $\lambda = 0$, then (LPy) : $y'_{0,j}(s) = Ay_{0,t}(s) + Bu_j(s)$, $y_{0,j}(t) = 0$. From uniq.: $y_{0,j}(0) = y_{j,0}$.

$$\nabla \psi(0) = \left(\frac{\partial \psi}{\partial \lambda_1}(0) \dots \frac{\partial \psi}{\partial \lambda_n}(0) \right) = (y_1, \dots, y_n)$$

└ regular matrix. □

Poznámka

$$x' = Ax + Bu, u \in \mathcal{U} = \{u : [0, T] \rightarrow [-1, 1] \text{ measurable}\}, x(0) = x_0.$$

Definice 3.14

$$\mathcal{R}(t) = \{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U} \wedge x_0 \rightarrow_{u(0)}^t 0\}.$$

Definice 3.15

$$u_n \in \mathcal{U}_0: u_n \rightarrow^* u \in \mathcal{U} \equiv \forall f \in L(0, T, \mathbb{R}^n) : \int_0^T f(s)u_n(s)ds \rightarrow \int_0^T f(s)u^*(s)ds.$$

Věta 3.9 (Alaoglu)

\mathcal{U} is weak-* sequentially compact (i. e. $\forall \{u_n\}_{n=1}^\infty \in \mathcal{U} \exists \{u_{n_k}\}$ weakly-* convergent).

Věta 3.10

$\mathcal{R}(t)$ convex, symmetric, closed

$$0 < t_1 < t_2 \implies \mathcal{R}(t_1) \subset \mathcal{R}(t_2).$$

┌

Důkaz

Convex: $x_{01}, x_{02} \in \mathcal{H}(t), \alpha \in [0, 1] \implies \alpha x_{01} + (1 - \alpha)x_{02} \in \mathcal{R}(t).$

$$x(t) = e^{At}x_0 + \int_0^t e^{As}Bu(s)ds. x_{01} \rightarrow_u^t 0 \wedge x_{02} \rightarrow_u^t 0 \Leftrightarrow x_1 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

$$\text{Symmetry: } x_0 \in \mathcal{R}(t) \implies -x_0 \in \mathcal{R}(t), x_0 \rightarrow_u^t 0 \implies -x_0 \rightarrow_u^t 0.$$

Closedness: $x_{0n} \in \mathcal{R}(t), x_{0n} \rightarrow x_0. x_0 \in \mathcal{R}(t)? \exists u_n(0) \in \mathcal{U}, x_{0n} = - \int_0^t e^{(s-t)A}Bu_n(s)ds \rightarrow - \int_0^t e^{(s-t)A}Bu(s)ds. \text{ WLOG } u_n \rightarrow^* u \in \mathcal{U}. \text{ Then } x_0 \rightarrow_u^t 0.$

$$\mathcal{R}(t_1) \subset \mathcal{R}(t_2) 0 < t_1 < t_2 < T x_0$$

$$\exists u_1 \in \mathcal{U} \quad x_0 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

Define $u_2(s) = u_1(s)$ if $0 \leq s \leq t$, else 0.

└

□

Definice 3.16 (Area of controllability)

$$\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t).$$

Věta 3.11

$$\text{rank } \mathcal{K}(A, B) = n \Leftrightarrow \forall t > 0 \mathcal{R}(t) \supseteq U(0),$$

where $U(0) \subset \mathbb{R}^n$ is some neighbourhood of 0.

┌

Důkaz

„ \Leftarrow “: If $\exists t > 0 \mathcal{R}(t) \supset U(0)$ open, $0 \in U(0)$. $\tilde{\mathcal{R}} : u \in L^\infty, \mathcal{R} : \|u\|_\infty \leq 1$, then $\tilde{\mathcal{R}}(t) \supset \mathcal{R}(t) \supset U(0) \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n. \implies \text{rank } \mathcal{K}(A, B) = n$.

„ \Rightarrow “: $\text{rank } \mathcal{K}(A, B) = n \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n$. From theorem of local controllability. \square

Věta 3.12 (Minimal time)

$$x' = Ax + Bu$$

$$\forall x_0 \in \mathcal{R} = \bigcup_{s>0} \mathcal{R}(s)$$

$$\exists t > 0 \exists u(0) \in \mathcal{U} : x_0 \xrightarrow{u}^t 0$$

$$t = \inf \{s > 0 | x_0 \in \mathcal{R}(s)\}.$$

┌

Důkaz

$$t > 0, \exists t_n \searrow t, t_n \in (0, T], \exists u_k \in U, x_0 = - \int_0^{t_n} e^{(t_n-s)A} B u_n(s) ds.$$

Alaoglu: WLOG $u_n \xrightarrow{*} u \in U$.

$$x_0 = - \int_0^{t_n} e^{(t-s)A} B u_n(s) ds - \int_0^{t_n} [e^{(t-s)A} - e^{(t_n-s)A}] B u_n(s) ds$$

$$x_0 = - \underbrace{\int_0^t e^{(t-s)A} B u_n(s) ds}_{\xrightarrow{*} \int_0^t e^{(t-s)A} B u(s) ds} - \underbrace{\int_t^{t_n} e^{(t-s)A} B u_n(s) ds}_{\rightarrow 0} - \underbrace{\int_0^T [e^{(t-s)A} - e^{(t_n-s)A}] B u_n(s) ds}_{\rightarrow 0(DLCT)}.$$

└

\square

Definice 3.17 (Bang-bang)

We say that a regulation $u \in U(0)$ is of type bang-bang, if for almost every $t \in [0, T]$: $u(t) = \pm 1$.

Věta 3.13 (Bang-bang)

If $x_0 \in \mathcal{R}(t) \implies \exists \tilde{u}(0)$ of type bang-bang $x_0 \xrightarrow{\tilde{u}}^t 0$.

Definice 3.18 (Extremal point)

X vector space, $K \subset X$. $x \in K$ is called an extremal point, if it cannot be written as $x = \frac{y+z}{2}$, $y, z \in K$, $y \neq z$. We denote $ex(K)$ the set of extremal points.

Tvrzení 3.14 (Krein-Milman theorem)

X locally convex vector space, $K \subset X$: $K \neq \emptyset$, K convex and compact. Then $ex(K) \cap K \neq \emptyset$.

Důkaz (Bang-bang)

$$K = \{u \in \mathcal{U} | x_0 \rightarrow_{u(0)}^t 0\}, \quad X = L^\infty(0, T, \mathbb{R}^n).$$

$K \neq \emptyset$ ($u \in \mathcal{R}(t)$), K convex, K is compact (sequential compactness: Alaoglu theorem? $L'(0, T, \mathbb{R}^n)$ separable $\implies L^\infty(0, T, \mathbb{R}^n)$ with locale $*$ topology is metrizable \implies sequential compactness \implies compactness).

It remains to check that $\tilde{u}_j(s) = \pm 1$, $\forall j \in [n]$ for almost every $s \in (0, t)$. For contradiction: for some $j \in [n]$ $\exists E \subset (0, t)$, $\lambda(E) > 0$ $\forall s \in E$ $|\tilde{u}_j(s)| < 1$. WLOG

$$\exists \varepsilon > 0 \quad \forall s \in E |\tilde{u}_j(s)| < 1 - \varepsilon \cdot \left[E = \bigcup_{n \in \mathbb{N}} \left\{ s \in (0, t) | |\tilde{u}_j(s)| \leq 1 - \frac{1}{n} \right\} \right].$$

$$x_0 = - \int_0^t e^{-sA} B \tilde{u}(s) ds$$

We want to find $\varphi \in L^\infty(0, T, \mathbb{R})$ such that:

1. $\text{supp } \varphi \subset E$;
2. $\int_E e^{-sA} B(0, \dots, 0, \varphi(s), 0, \dots, 0)^T ds = 0$;
3. $\forall s \in E |\varphi(s)| < \varepsilon$.

Define $u_1(s) = \tilde{u}(s) + (0, \dots, 0, \varphi(s), 0, \dots, 0)^T$ and $u_2(s) = \tilde{u}(s) - (0, \dots, 0, \varphi(s), 0, \dots, 0)^T$. Then $x_0 \rightarrow_{u_{1,2}(0)}^t 0$, and $u_1, u_2 \in \mathcal{K}$. □

Věta 3.15 (Global controllability)

We have (LTP) $x' = Ax + Bu$, $x(0) = x_0$, $u \in \mathcal{U}$.

1. $\text{rank } \mathcal{K}(A, B) = n \implies$ (LTP) is locally controllable.
2. $\text{rank } \mathcal{K}(A, B) = n$ and $\Re \lambda \leq 0$ $\forall \lambda$ -eigenvalues of A . Then (LTP) is globally controllable $\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t) = \mathbb{R}^n$.

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Důkaz

1) follows from „In theorem of local controllability for the problem $x' = f(x, u)$ we could take $u \in \mathcal{U}$.“

2a) If $\forall \lambda$ eigenvalue of A we have $\Re \lambda < 0 \implies$ theorem follows from text above: first, set $u = 0$. Then we arrive at a neighbourhood of zero.

2b) For contradiction $x_0 \in \mathbb{R}^n \setminus \mathcal{R}$. \mathcal{R} convex $\exists z_0 \in \partial \mathcal{R}$, n normal vector. $\forall x_1 \in \mathcal{R} : n^T(x_1 - x_0) \leq 0$, $n^T x_1 \leq n^T x_0 =: M$.

$$x_1 = - \int_0^t e^{-sA} B u(s) ds$$

$$n^T x_1 = - \int_0^t \underbrace{n^T e^{-sA} B}_{v(s)} u(s) ds$$

$$\tilde{u}(s) := \begin{cases} 0, & v(s) = 0, \\ \frac{-v(s)}{\|v(s)\|_2}, & v(s) \neq 0. \end{cases}$$

If $v(s) \equiv -$, then apply $\frac{d^p}{(ds)^p}$, $n^T A^p e^{-sA} B \equiv 0$, then $n^T \mathcal{K}(A, B) = 0$. \nmid

$$\int_0^\infty \|v(s)\|_2 ds = \infty.$$

If this is true, then $t_k \nearrow \infty$, $u_k = \tilde{u}|_{[0, t_k]}$, $x_{1,k} = - \int_0^{t_k} e^{-sA} B u_k(s) ds$.

$$n^T x_{1,k} = - \int_0^{t_k} v^t(s) \cdot \tilde{u}(s) ds = \int_0^{t_k} \|v(s)\|_2 ds \rightarrow \infty. \nmid$$

$v(s)$ is linear combination of $s^j e^{-s\lambda_p}$, $\Re \lambda_p \leq 0$. Then $\int_0^\infty |v(s)| ds = \infty$. □

Věta 3.16 (Pontrjagin maximum)

$$x' = Ax + Bu, \|u\|_\infty \leq 1, x(0) = x_0.$$

Let $x_0 \rightarrow_{u^*(0)}^{t^*} 0$, t^* is the minimal. Then $\exists h \in \mathbb{R}^n \setminus \{\mathbf{0}\} : h^T \cdot e^{-sA} B u^*(s) = \max_{\eta \in [-1, 1]^m} h^T e^{-sA} B \eta$ for almost every $s \in (0, t^*)$.

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Důkaz

$x_0 \in \partial \mathcal{R}(t^*)$.

Step 2 – contradiction: $\exists E \subset (0, t^*), \lambda(E) > 0, \forall s \in E \exists \eta_s \in [-1, 1]^m h^T e^{-sA} B u^*(s) < h^T e^{-sA} B \eta_s$. But $x_j(\delta) \in \mathcal{R}(t^* - \delta)$, hence $x_0 \in \mathcal{R}(t^* - \delta)$ and t^* is not minimal.

Step 1: $x_0 \in \partial \mathcal{R}(t^*)$. For contradiction $x_0 \in \text{int } \mathcal{R}(t^*)$.

$$\exists x_1, \dots, x_{n+1} \in \mathcal{R}(t^*), x_0 \in CO(x_1, \dots, x_{n+1}).$$

$$\exists u_1, \dots, u_{n+1} \in U, x_j \xrightarrow{u_j(\cdot)}^{t^*} 0 \quad \forall j \in [n+1].$$

Let $\tilde{u}_j(t)$ are the corresponding solutions

TODO!!!

└

□

Věta 3.17 (Pontrjagin)

$x'(f, u), x(0) = x_0, u \in \mathcal{U} = \{u : (0, T) \rightarrow U \subset \mathbb{R}^n\}, T \text{ fixed},$

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s)) ds \rightarrow \text{maximum}.$$

$f, g, r, \nabla_x f, \nabla_x g, \nabla_x r$ are continuous.

Let u is a local maximum of this problem (it maximizes P), then for p solving $(H(x, p, u) := p^T f(x, u) + r(x, u))$:

$$p' = -\nabla_x H(x, p, u),$$

$$p(T) = \nabla_x g(x(T)),$$

we have

$$H(x, p, u) = \max_{\eta \in U} H(x, p, \eta) \text{ for almost every } t \in (0, T).$$

┌ *Důkaz*

Step one „WLOG $r = 0$ “: We set

$$x' = f(x, u), \quad x'_{n+1} = r(x, u), \quad x_{n+1}(0) = 0, \quad P[u(\cdot)] = \hat{g}(\hat{x}(T)) = g(x(T)) + x_{n+1}(T).$$

Step 2: Fix $\tau \in (0, T)$, $\eta \in U$, $u_\varepsilon(T) = \begin{cases} \eta, & t \in (\tau - \varepsilon, \tau), \\ u(t), & \text{elsewhere,} \end{cases}$ and corresponding $x_\varepsilon(t)$.

$$u \text{ "best"} \implies P[u_\varepsilon(0)] \leq P[u(0)] \implies g(x_\varepsilon(T)) \leq g(x(T)).$$

$$D = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0+} Dg(x_\varepsilon(T)) \Big|_{\varepsilon=0+} \leq 0$$

$$\nabla_x g(x(T)) \cdot Dx_\varepsilon(T) \Big|_{\varepsilon=0+} \leq 0.$$

Step 2.2: $x_\varepsilon(t) = x_0 + \int_0^t f(x_\varepsilon(s), u_\varepsilon(s)) ds$. If $t < \tau$, then $u_\varepsilon \equiv u$, $x_\varepsilon \equiv x$, $Dx_\varepsilon(t) \equiv 0$ on $[0, t]$. If $t > \tau$, then $x_\varepsilon(t) =: y(t)$, $y'(t) = f(y(t), u(t))$, $u(\tau) = x_\varepsilon(\tau)$,

$$Dx_\varepsilon(t) \equiv z(t) : z' = \nabla_x f(y(t), u(t))z, \quad z(\tau) = Dx_\varepsilon(\tau), \quad \text{variational equation.}$$

Statement: $z' = A(t)z$, $p' = -A^T(t)p \implies p^T z = \text{const.}$ Proof: $(p^T z)' = (p^T)'z + p^T z' = -p^T A z + p^T A z = 0$.

Step 2.3: $p' = -(\nabla_x f(y(t), u(t)))^T p$, $p(T) = (\nabla_x g(x(T)))^T$. Then $p^T(t)z(t)$ constant on (τ, T) , $p^T(\tau)z(\tau) \leq 0$.

Step 2.4: $Dx_\varepsilon(\tau) \Big|_{\varepsilon=0+} \stackrel{?}{=} f(x(\tau), \eta) - f(x(\tau), u(\tau))$. Then

$$p^T(\tau) (f(x(\tau), \eta) - f(x(\tau), u(\tau))) \leq 0$$

$$\frac{1}{\varepsilon} (x_\varepsilon(\tau) - x(\tau)) = \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), u(s))] ds =$$

$$\frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), \eta)] ds + \int_{\tau-\varepsilon}^{\tau} [f(x(s), \eta) - f(x(s), u(s))] ds.$$

First converge to zero from Lebesgue theorem about average value. Second to $f(x(\tau), \eta) - f(x(\tau), u(\tau)) \rightarrow 0$. □

Věta 3.18 (Potrjagin for fixed point („fixed finish“))

Same as previous, but T is not fixed, $x(T)$ is fixed $\implies g \equiv 0$ (we don't "rate" final point, because it's the same for all u).

4 Bifurcation

Definice 4.1

$x' = \mu - x^2$ is saddle-node bifurcation, $x' = \mu x - x^2 = x(\mu - x)$ is transcritical bifurcation, $x' = \mu x - x^3 = x(\mu - x^2)$ is fork bifurcation, in $x' = x - \sin \mu$ there is no bifurcation.

Pozorování

$f(x_0, \mu_0) \neq 0 \implies$ no bifurcation. (From lemma of rect.) (Bifurcation $\implies f = 0$.)

Pozorování

$$f(x_0, \mu_0) = 0, \sigma(\nabla_x f(x_0, \mu_0)) = \{\lambda_j | \Re \lambda_j \neq 0\}.$$

Definice 4.2

Point from previous observation is called hyperbolic stationary point.

Věta 4.1

$f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ be C^1 , (x_0, μ_0) is a hyperbolic stationary point. Then $\exists \Delta > 0 \exists \delta > 0 \forall \mu \in U_\delta(\mu_0) \exists x = x(\mu) \in U_\Delta(x_0)$, stationary point $x(\mu)$ is a hyperbolic stationary point of $\mu \mapsto x(\mu)$, which is C^1 .

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Důkaz

IFT:

$$f(x_0, \mu_0) = 0 \wedge \nabla_x f(x_0, \mu_0) \text{ regular?} \wedge f \in C^1 \implies x = x(\mu), f(x(\mu), \mu) = 0.$$

Hyperbolic? Eigenvalues of $A = \nabla_x f(x(\mu), \mu)$, $\det(\lambda I - A(\mu))$ – polynomial in λ , $\deg = n$. □

└

Věta 4.2

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^2 on neighborhood $(0, 0) \in \mathbb{R}^2$.

$$f(0, 0) = 0, \quad f_\mu(0, 0) \neq 0, \quad f_x(0, 0) = 0, \quad f_{xx}(0, 0) \neq 0.$$

Then f has bifurcation at $(0, 0)$ of the type saddle-node.

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Důkaz

Without proof. □

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Věta 4.3

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^2 on neighborhood $(0, 0) \in \mathbb{R}^2$.

$$f(0, 0) = 0, \quad f_x(0, 0) = 0, \quad f(0, \mu) = 0 \forall \mu \in U_\delta(0), \quad f_{xx}(0, 0) \neq 0, \quad f_{x\mu}(0, 0) \neq 0.$$

Then f has bifurcation at $(0, 0)$ of the type transcritical.

┌ *Důkaz*

└ Without proof. □

Věta 4.4

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^2 on neighborhood $(0, 0) \in \mathbb{R}^2$.

$$f(0, 0) = 0, \quad f_x(0, 0) \neq 0, \quad f_{xx}(0, 0) = 0, \quad f_{xxx}(0, 0) \neq 0,$$

$$f(0, \mu) = 0 \quad \forall \mu \in U_\delta(0), \quad f_{x\mu}(0, 0) \neq 0.$$

Then f has bifurcation at $(0, 0)$ of the type fork.

┌ *Důkaz*

└ Without proof. □

Lemma 4.5 (About division)

$h : U(0, 0) \rightarrow \mathbb{R}$ be C^k for some $k \in \mathbb{N}$. $h(0, \lambda) = 0 \quad \forall \lambda \in U_\delta(0)$. Then

$$h(x, \lambda) = xH(x, \lambda), \quad H \in C^{k-1}(U(0, 0), \mathbb{R}).$$

$$H(0, 0) = h_x(0, 0), \quad H_x(0, 0) = \frac{1}{2}h_{xx}(0, 0), \quad H_\lambda(0, 0) = h_{x\lambda}(0, 0),$$

$$H_{xx}(0, 0) = \frac{1}{3}h_{xxx}(0, 0),$$

if k is sufficiently large.

┌ *Důkaz*

$$H(x, \lambda) := \int_0^1 \partial_x h(\sigma x, \lambda) d\sigma.$$

└ □

TODO?

Důkaz (Theorem of transcritical bifurcation)

$f(x, \mu) = xF(x, \mu)$. $F_\mu(0, 0) \neq 0$?

$$F(x, \mu(x)) = 0? \rightarrow \frac{d}{dt} : \mu'(x) = \frac{-\partial_x F(x, \mu(x))}{\partial_\mu(F(x, \mu(x)))}.$$

$$f_{x\mu}(x, \mu) = F_\mu(x, \mu) + xF_{x\mu}(x, \mu) \implies F_\mu(0, 0) = f_{x\mu}(0, 0) \neq 0.$$

□

Věta 4.6 (Fork)

$$f \in C^3(U), \quad f(0,0) = f_x(0,0) = f_{xx}(0,0) = 0, \quad f_{xxx}(0,0) \neq 0, \quad f(0,\mu) = 0 \quad \forall (0,\mu) \in U,$$

$$(f_\mu(0,0) = 0), \quad f_{x\mu}(0,0) \neq 0.$$

Then ?

┌

Důkaz

$$\mu'(0) = 0, \quad \mu''(0) = \frac{\dots}{-\partial_\mu F(x, \mu(x))}, \quad \partial_{x,x} F(0,0) = \frac{1}{3} f_{xxx}(0,0) \neq 0.$$

$$\mu''(0) \neq 0 \implies \mu''(x) \text{ doesn't change sign} \implies \mu(x) \text{ has a local extreme at } (0,0).$$

└

□

Věta 4.7 (?)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f(x, y, \mu), \quad f \in C^2, \quad f(0,0,\mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \sigma(\nabla f(0,0,\mu)) = \{\alpha(\mu) \pm i\omega(\mu)\}.$$

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \omega(0) \neq 0, \quad \alpha, \omega \in C^1.$$

$$\text{Then } \exists \delta > 0, \varepsilon > 0 : \mu = \mu(a), \quad a \in (0, \varepsilon) \mapsto \mu(a) \in (-\delta, \delta).$$

$$\forall a \in (0, \varepsilon) \exists \text{ nontrivial periodic solution passing through } (a, 0).$$

┌ *Důkaz*

Rotation: $x' = d(\mu)x - \omega(\mu)y + f_1(x, y, \mu)$, $y' = \omega(\mu)x + \alpha(\mu)y + f_2(x, y, \mu)$,
 $f_2(x, y, \mu) = O(x^2 + y^2)$.

Polar coords:

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta, x' = g_1(x, y), y' = g_2(x, y), \\ r' \cos \theta - r \sin \theta \cdot \theta' &= g_1, \quad r' \sin \theta + r \cos \theta \cdot \theta' = g_2. \\ r' &= g_1 \cdot \cos \theta + g_2 \sin \theta, \quad r \cdot \theta' = -g_1 \sin \theta + g_2 \cos \theta. \\ r; &= \alpha \cdot r + \underbrace{f_1 \cdot \cos \theta + f_2 \cdot \sin \theta}_{=R}, \quad r\theta' = \omega(\mu) \cdot r + \underbrace{(-f_1 \cdot \sin \theta + f_2 \cdot \cos \theta)}_{=r \cdot Q}. \end{aligned}$$

$$r' = \alpha(\mu)r + R(r, \theta, \mu), R = O(r^2), \quad \theta' = \Omega(\mu) + Q(r, \theta, \mu), Q = O(r).$$

WLOG $\omega(0) > 0$. $\exists \varepsilon, \delta > 0 \forall r \leq \varepsilon \forall \mu \in [-\delta, \delta], \theta'(t) > 0$. $r(t) \mapsto \hat{r}(\theta) := r(t(\theta))$.
 $t \mapsto \theta(t)$ is simple $\implies \exists t = t(\theta)$.

$$\frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{\alpha(\mu)r + R}{\omega(\mu) + Q} = \frac{\alpha(\mu)}{\omega(\mu)}r + T(r, \theta, \mu).$$

$$\lambda(\mu) := \frac{\alpha(\mu)}{\omega(\mu)} : r'(\theta) = \lambda(\mu)r(\theta) + T(r, \theta, \mu),$$

$$(r(\theta)e^{-\lambda(\mu)\theta})' = T(r, \theta, \mu)e^{-\lambda(\mu)\theta}$$

$$r(\theta) \cdot e^{-\lambda(\mu)\theta} = r'(\theta_0) \cdot e^{-\lambda(\mu)\theta_0} + \int_{\theta_0}^{\theta} T(r(\psi), \psi, \mu) \cdot e^{-\lambda(\mu)\psi} d\psi.$$

We get $r(\theta_0) = r(\theta_0 + 2\pi)$ – periodicity. If we denote $r(\theta_0) = a$, we get

$$(e^{2\pi\lambda(\mu)} - 1)a + \int_{\theta_0}^{\theta_0+2\pi} T(r(\psi), \psi, \mu) \cdot e^{-\lambda(\mu)(\psi-\theta_0)} d\psi.$$

$$h_\mu(0, 0) \neq 0? \quad h''(a, \mu), a = 0 \implies r = 0, T = 0, h(0, \mu) = 0.$$

$$h(a, \mu) =: a \cdot H(a, \mu).$$

$$H(0, 0) = 0? \quad H_\mu(0, 0) \neq 0? \quad H \in C^1 \quad H(0, 0) = \partial_\mu h(0, 0).$$

└

□

5 Central manifolds

Poznámka

$$\begin{aligned}
x' &= Ax + f(x, y), & y' &= By + g(x, y), \\
\sigma(B) &\subset \{z \in \mathbb{C} | \Re z < -\beta\}, & \beta &> 0, & \sigma(A) &\subset \{z \in \mathbb{C} | \Re z \geq 0\}, \\
x &\in \mathbb{R}^n, & y &\in \mathbb{R}^n, & f, g &\in C^1(\mathbb{R}^{n+m}), \\
f(0) &= 0, & g(0) &= 0, & \nabla f(0) &= 0, & \nabla g(0) &= 0, \\
|f| &\leq \varrho, & |g| &\leq \varrho, & |\nabla f| &\leq \sigma, & |\nabla g| &\leq \sigma.
\end{aligned}$$

Goal: $\exists \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ Lipschitz, $\varphi(0) = 0$, $\nabla \varphi(0) = 0$:

INV: if $(x(t), y(t))$ solution to previous and $y(0) = \varphi(x(0))$, then $\forall t : y(t) = \varphi(x(t))$.

Definice 5.1 (Reduced equation)

$$p'(t) = A \cdot p(t) + f(p(t), \varphi(p(t))), p(t) \in \mathbb{R}^n.$$

Lemma 5.1

φ satisfies INV iff φ satisfies RED: (if $p(t)$ satisfies reduced equation then $(x(t), y(t)) := (p(t), \varphi(p(t)))$ satisfies INV equation)

┌

Důkaz

Straightforward.

└

□

Definice 5.2 (RED')

If $p(t)$ satisfies reduced equation, then $y(t) := \varphi(p(t))$ satisfies

$$y'(t) = By(t) + g(p(t), \varphi(p(t))).$$

Lemma 5.2

$\gamma(t)$ is bounded on $(-\infty, 0]$. Then $\exists! y(t) : y'(t) = By(t) + \gamma(t)$, such that $y(t)$ is bounded on $(-\infty, 0]$. For this y , $y(0) = \int_{-\infty}^0 e^{-Bs} \gamma(s) ds$.

┌ *Důkaz*

$$e^{-Bt}y'(t) - Be^{-Bt}y(t) = e^{-Bt}\gamma(t) \cdot (e^{-Bt}y(t))' = e^{-Bt}\gamma(t).$$

$$e^{-Bt}y(t) = y(0) + \int_0^t e^{-Bs}\gamma(s)ds.$$

If y is bounded on $(-\infty, 0]$, then $y(0) + \int_0^{-\infty} e^{-Bs}\gamma(s)ds = 0$, $y(0) = \int_{-\infty}^0 e^{-Bs}\gamma(s)ds$.

Take y with (i. c.). Then

$$y(t) = e^{Bt} \left(\int_{-\infty}^t e^{-Bs}\gamma(s)ds \right) = \int_{-\infty}^t e^{B(t-s)}\gamma(s)ds.$$

$$|e^{s \cdot B}| \leq c_0 e^{-\beta s}, \quad c_0 > 0, \forall s$$

$$|y(t)| \leq \int_{-\infty}^t |e^{B(t-s)}| \cdot |\gamma(s)|ds \leq \|\gamma\|_{\infty} \int_{-\infty}^t c_0 e^{-\beta(t-s)}ds = \frac{\|\gamma\|_{\infty} c_0}{\beta}.$$

└

□

Lemma 5.3

φ satisfies INV $\Leftrightarrow \varphi$ satisfies (RED) $\Leftrightarrow \varphi$ satisfies (RED') $\Leftrightarrow \varphi$ satisfies FP:

$$\forall p_0 \in \mathbb{R}^n : \varphi(p_0) = \int_{-\infty}^0 e^{-Bs} g(p(s), \varphi(p(s)))ds,$$

where p satisfies reduced equation with $p(0) = p_0$.

┌

Důkaz

„ \implies “: φ RED’ \implies y satisfies $y'(t) = By(t) + g(p(t), \varphi(p(t)))$ and y is bounded. Then previous lemma:

$$\varphi(p_0) = \varphi(p(0)) = y(0) = \int_{-\infty}^0 e^{-B \cdot s} g(p(s), \varphi(p(s))) ds.$$

„ \Leftarrow “: Let y satisfy $y'(t) = By(t) + g(p(t), \varphi(p(t)))$, $y(0) = \varphi(p(0))$. $\forall t_1 : y(t_1) = \varphi(p(t_1))$?

t_1 arbitrary $y_1 := y(t+t_1), p_1(t) := p(t+t_1). p_1'(t) = Ap_1(t) + f(p_1(t), \varphi(p, t)), p_1(0) = p(t_1),$

$$y_1'(t) = B \cdot y_1(t) + g(p_1(t), \varphi(p, t)), y_1(0) = y(t_1).$$

$y(0) = \varphi(p(0)) = \int_{-\infty}^0 e^{-B \cdot s} g(p(s), \varphi(p(s))) ds \implies y$ is bounded on $(-\infty, 0] \implies y_1$ is bounded. From lemma:

$$y(t_1) = y_1(0) = \int_{-\infty}^0 e^{-B \cdot s} g(p(s), \varphi(p, s)) ds = \varphi(p(0)) = \varphi(p(t_1)).$$

└

□

Věta 5.4 (Existence of central manifold)

$\forall \beta \exists \varrho > 0, \sigma > 0, b > 0, l > 0 : \exists ! p \in \mathcal{X}$ satisfying INV.

┌
Důkaz

$$\mathcal{X} \subset C(\mathbb{R}^n, \mathbb{R}^n), \|\varphi\|_{\mathcal{X}} = \sup_{x \in \mathbb{R}^n} |\varphi(x)|,$$

$$T : \mathcal{X} \rightarrow \mathcal{X}, \varphi \mapsto T\varphi, \quad (T\varphi)(p_0) = \int_{-\infty}^0 e^{-B \cdot s} g(p(s), \varphi(p(s))) ds.$$

Step 1: T is well-defined $\forall \varphi \in \mathcal{X}: T\varphi \in \mathcal{X}$.

Step 2: T is contraction.

Step 1 + 2 \implies (Banach) $\exists! \varphi \in \mathcal{X}: T\varphi = \varphi$.

- $(T\varphi)(0) = 0$? Take $p(0)$ satisfying reduced equation, $p(0) = 0 \implies p(t) \equiv 0$.
- $|(T\varphi)(p_0)| \leq \int_{-\infty}^0 |e^{-B \cdot s}| \cdot |g(p(s), \varphi(p(s)))| ds \leq \frac{\varrho c_0}{\beta} \stackrel{?}{\leq} f$ (true for sufficiently small ϱ).
- $(T\varphi)(p_0) - (T\varphi)(q_0) = \int_{-\infty}^0 e^{-B \cdot s} [g(p(s), \varphi(p(s))) - g(q(s), \varphi(q(s)))] ds$.

└

□

Definition 5.3 (Central manifold)

$\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called a central manifold if $\varphi(0) = 0$, $\nabla \varphi(0) = 0$, $\varphi \in C^1$, it satisfies INV.

Definition 5.4

$$M[\varphi](x) = \nabla \varphi(x)(Ax + f(x, \varphi(x))) - B\varphi(x) - g(x, \varphi(x)).$$

Důsledek

φ is a central manifold $\Leftrightarrow M[\varphi] = 0$.