

Prerequisites

0.1 Regularization

Definition 0.1 (Regularization kernel)

$\eta \in C_0^\infty(B_1(\mathbf{o}))$, non-negative, radially symmetric, $\int_{B_1(\mathbf{o})} \eta(x) dx = 1$.

Definition 0.2 (Regularization of function)

Let $f \in L^p(\Omega)$. We extend f by zero to $\mathbb{R}^d \setminus \Omega$ and define $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) := \frac{1}{\varepsilon^d} \eta(x/\varepsilon)$.

Poznámka

$f_\varepsilon \in C^\infty(\mathbb{R}^d)$, $f_\varepsilon \rightarrow f$ in $L^p(\Omega)$ if $p \in [1, \infty)$ and $f_\varepsilon \rightharpoonup^* f$ in L^∞ .

Věta 0.1

$L^p(\Omega)$ is a Banach space, separable for $p \in [1, \infty)$, reflexive for $p \in (1, \infty)$.

Důsledek

f^n is a bounded sequence in $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ measurable bounded. Then

1. $p \in (1, \infty)$: $\exists f^{n_k}, f: f^{n_k} \rightharpoonup f$ in $L^p(\Omega)$. ($\Leftrightarrow \forall g \in L^{p'}(\Omega) : \lim_{k \rightarrow \infty} \int_\Omega f^{n_k} g = \int_\Omega f g$, where $\frac{1}{p} + \frac{1}{p'} = 1$).
2. $p = \infty$: $\exists f^{n_k}, f: f^{n_k} \rightharpoonup^* f$ in $L^\infty(\Omega)$. ($\Leftrightarrow g \in L^1(\Omega) : \lim \int_\Omega f^{n_k} g = \int_\Omega f g$).
3. $p = 1$: $\exists f^{n_k}, f: f^{n_k} \rightharpoonup^* f$ in $M(\overline{\Omega})$ (Radon measures). ($\Leftrightarrow \forall g \in C(\overline{\Omega}) : \int_\Omega f^{n_k} g \rightarrow \langle f, g \rangle_M = \int_{\overline{\Omega}} g df$).
4. $p = 1$: $\exists f^{n_k}, \tilde{f} \exists \Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \dots, |\Omega \setminus \Omega_l| \rightarrow 0$ as $l \rightarrow \infty$: $\forall l \in \mathbb{N} : f^{n_k} \rightharpoonup \tilde{f}$ in $L^1(\Omega)$. (\tilde{f} is called biting limit.)

0.2 Fixpoint theorems

Věta 0.2

$F : X \rightarrow X$, where X is a Banach space, F is continuous and compact. Let there exists closed convex non-empty set $U \subseteq X$ such that $F(U) \subset U$. Then $\exists x \in U : F(x) = x$.

Věta 0.3

$F : \mathbb{R}^d \rightarrow \mathbb{R}^d$, F is continuous. Let there exists closed, convex non-empty set $U \subseteq \mathbb{R}^d$: $F(U) \subseteq U$. Then $\exists x \in U : F(x) = x$.

0.3 Nemytskii operator

Věta 0.4

Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is Carathéodory (i.e. $\forall y \in \mathbb{R}^N : f(\cdot, y)$ is measurable and for almost all $x \in \Omega$: $f(x, \cdot)$ is continuous). Assume that $|f(x, y)| \leq g(x) + c \cdot \sum_{i=1}^N |y_i|^{p_i/p}$ for some $p_1 \in [1, \infty)$, $p \in [1, \infty)$ with $y \in L^p(\Omega)$.

Then $\forall u_i \in L^{p_i}(\Omega)$, the function $f(x, u_1(x), \dots, u_N(x))$ is measurable and the mapping (named Nemytskii operator) $(u_1, \dots, u_N) \mapsto f(\cdot, u_1, u_2, \dots, u_N)$ is continuous from $L^{p_1}(\Omega) \times \dots \times L^{p_N}(\Omega)$ to $L^p(\Omega)$.

TODO!!!

TODO!!!

Věta 0.5

Let $\Omega \subseteq \mathbb{R}^d$ open bounded, $\Omega_\delta := \{x \in \Omega \mid B_\delta(x) \subseteq \Omega\}$, $u_i^h(x) := \frac{u(x+h \cdot e_i) - u(x)}{h}$, and $p \in [1, \infty]$. Then

1. if $u \in W^{1,p}(\Omega)$ then $\forall \delta > 0 \forall h \leq \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$;
2. if $p \in (1, \infty]$ and $\sup_{\delta > 0} \sup_{h < \delta/2} \|u_i^h\|_{L^p(\Omega_\delta)} \leq K$ then $\frac{\partial u}{\partial x_i}$ exists and $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq K$;
3. if $p \in [1, \infty)$ and $u \in W^{1,p}(\Omega)$ then $u_i^h \rightarrow \frac{\partial u}{\partial x_i}$ as $h \rightarrow 0_+$ in $L^p_{loc}(\Omega)$.

Důkaz

„2.“: $p \in (1, \infty)$ then L^p is reflexive. $p = \infty$ then L^∞ has separate procedure. Fix $\Omega' \subseteq \Omega$. $\|u_i^h\|_{L^p(\Omega')} \leq K \implies$

$$p \in (1, \infty) : \exists h_n : u_i^{h_n} \rightharpoonup \bar{u}_i \text{ in } L^p(\Omega'),$$

$$p = \infty : \exists h_n : u_i^{h_n} \rightharpoonup^* \bar{u}_i \text{ in } L^p(\Omega').$$

$$\implies \|\bar{u}_i\|_{L^p(\Omega')} \leq \lim_{h_n \rightarrow 0} \|u_i^{h_n}\|_{L^p(\Omega')} \leq K. \quad \Omega' \nearrow \Omega \implies \|\bar{u}_i\|_{L^p(\Omega)} \leq K.$$

Remain to show: $\bar{u}_i = \frac{\partial u}{\partial x_i}$.

TODO!!!

„1.“: $u \in W^{1,p}(\Omega)$. Mollify to u_ε . $u_\varepsilon \rightarrow u$ in $W_{loc}^{1,p}(\Omega)$, for $p \neq \infty$, and $u_\varepsilon \rightharpoonup^*$ in $W_{loc}^{1,\infty}(\Omega)$ for $p = \infty$. $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon$ in Ω_ε for $p = \infty$, $D^\alpha u_\varepsilon \rightarrow D^\alpha u$ in $L_{loc}^p(\Omega)$ for $p \neq \infty$. $x \in \Omega_\varepsilon, h \leq \delta/2$:

$$\frac{u_\varepsilon(x + h \cdot e_i) - u_\varepsilon(x)}{h} = \frac{1}{h} \int_0^1 \frac{d}{dt} u_\varepsilon(x + h \cdot t \cdot e_i) dt = \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot t \cdot e_i) dt.$$

$$\begin{aligned} \int_{\Omega_\delta} \left| \frac{u_\varepsilon(x + h \cdot e_i) - u_\varepsilon(x)}{h} \right|^p dx &\leq \int_{\Omega_\delta} \left| \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot t \cdot e_i) dt \right|^p dx \stackrel{\text{Jensen}}{\leq} \int_{\Omega_\delta} \int_0^1 \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot t \cdot e_i) \right|^p dt dx \leq \\ &\leq \int_0^1 \int_{\Omega_\delta} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot t \cdot e_i) \right|^p dx dt \leq \int_0^1 \int_{\Omega_{\delta/2}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x) \right|^p dx dt \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}^p. \end{aligned}$$

„3.“: It is enough to show that $u_i^{h_n}$ is Cauchy in $L_{loc}^p(\Omega)$:

$$u_i^{h^m} - u_i^{h^n} = \int_0^1 \frac{\partial u}{\partial x_i}(x + h^m \cdot t \cdot e_i) - \frac{\partial u}{\partial x_i}(x + h^n \cdot t \cdot e_i) dt.$$

$$\int_{\Omega_\delta} |u_i^{h^m} - u_i^{h^n}|^p \leq \int_0^1 \int_{\Omega} \left| \frac{\partial u}{\partial x_i}(x + h^m \cdot t \cdot e_i) - \frac{\partial u}{\partial x_i}(x + h^n \cdot t \cdot e_i) \right|^p dx dt \leq \varepsilon \text{ provided } h^n, h^m \ll 1.$$

$\implies u_i^h$ is Cauchy. □

0.4 Properties up to the boundary

Věta 0.6

Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open and $p \in [1, \infty)$. Then $\forall u \in W^{1,p}(\Omega)$:

1. $\exists \{u^n\}_{n=1}^\infty \subseteq C^\infty(\Omega)$ such that $u^n \rightarrow u$ in $W^{1,p}(\Omega)$;
2. if $\Omega \in C^0$ then $\exists \{u^n\}_{n=1}^\infty \subseteq C^\infty(\bar{\Omega})$ such that $u^n \rightarrow u$ in $W^{1,p}(\Omega)$.

Důkaz

„1.“: Prose? covering of Ω : $\Omega_i := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \frac{1}{i}\}$. $\Omega_i \subseteq \Omega_j$ for $i \leq j$. $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. Define $V_i = \Omega_{i+3} \setminus \overline{\Omega_{i+1}} = \{x \in \Omega \mid \frac{1}{i+1} > \text{dist}(x, \partial\Omega) > \frac{1}{i+3}\}$. Find $V_0 \subseteq \Omega$ such that $\bigcup_{i=0}^{\infty} V_i = \Omega$.

$u_i = u\varphi_i$, where φ_i is partition of unity (from next lemma). So $\forall i \exists j$ such that $u_i \subset V_j$. $\forall \varepsilon$ find (by convolution) $u_i^n \in C^\infty(\mathbb{R}^d) : \|u_i - u_i^n\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{2^i}$. (Such that $u_i^n \subseteq \Omega_{i+n} \setminus \overline{\Omega_i}$)

Define $u^n := \sum_{i=0}^{\infty} u_i^n$. $K \subseteq \Omega$ compact, then

$$\|u - u^n\|_{W^{1,p}(\Omega)} = \left\| \sum u\varphi_i - \sum u_i^n \right\|_{W^{1,p}(K)} = \left\| \sum (u_i - u_i^n) \right\|_{W^{1,p}(\Omega)} \leq \sum \|u_i - u_i^n\|_{W^{1,p}(\Omega)} \leq \varepsilon \cdot \sum \frac{1}{2^i} \leq 2\varepsilon.$$

$$\implies \|u^n - u\|_{W^{1,p}(\Omega)} \leq 2\varepsilon.$$

„2.“ TODO!!!?

Start with u_{M+1} . $u_{M+1} = u \cdot \varphi_{M+1}$. $\text{supp } \varphi_{M+1} \subset \Omega \implies u_{M+1} \in W^{1,p}(\mathbb{R}^d)$. $u_{M+1}^\varepsilon := u_{M+1} * \eta_{\sigma(\varepsilon)}$, where δ is taken such that $\|u_{M+1}^\varepsilon - u_{M+1}\|_{W^{1,p}(\Omega)} \leq \frac{\varepsilon}{M+1}$.

u_1 : assume $T_r = \text{id}$. $u_1 = u \cdot \varphi_1$. $u_1^h(x; x_d) := u_1(x; x_d + h)$. $h \leq h_0$:

$$\|u_1^n - u_1\|_{W^{1,p}(\Omega)} = \|u_1^n - u_1\|_{W^{1,p}(V^+)} \leq \frac{\varepsilon}{2 \cdot (M+1)}.$$

$u_1^\varepsilon = u_1^h * \eta_{\delta(\varepsilon, h, \varphi_1, a_1)} \in C^\infty(\overline{\Omega})$. $\|u_1^\varepsilon - u_1^h\|_{W^{1,p}(V^+)} \leq \frac{\varepsilon}{2(M+1)}$. Find δ : $(x; x_d) \in \Lambda$, $y \in B_\sigma(x; x_d)$, $a(y') > y_d - h$.

$$a(y') \geq a(x') - |a(x') - a(y')| = x_d - |a(x') - a(y')| \geq y_d - (|a(x') - a(y')| + |y_d - x_d|).$$

Find $\delta_0 > 0$: $\forall x, y, |x - y| \leq \delta_0 : |a(x') - a(y')| + |y_d - x_d| < h$. □

Lemma 0.7 (For the previous proof: Partition of unity I)

Let $\Omega \subseteq \mathbb{R}^d$ be open set. Assume that $\{V_i\}_{i \in I}$ be (uncountable) covering-?. Then there exists countable system $\{\varphi_j\}_{j=1}^{\infty}$ such that $\varphi_j \in C_0^\infty(\mathbb{R}^d)$, $\forall j \in \mathbb{N} \exists i \in I : \text{supp } \varphi_j \subset V_i$, $0 \leq \varphi_j \leq 1$, and $\forall x \in \Omega : \sum_{j=1}^{\infty} \varphi_j(x) = 1$. Moreover, for any compact $K \subseteq \Omega$, we have that $\varphi_j(x) \neq 0$ for finitely many j 's.

Lemma 0.8 (For the previous proof: Partition of unity II)

Let $\overline{\Omega}$ be a compact set and $\{\tilde{V}_i\}_{i=1}^N$ be its open covering ($\overline{\Omega} \subseteq \bigcup_{i=1}^N \tilde{V}_i$). Then $\exists \varphi_i \in C_0^\infty(\tilde{V}_i)$, $0 \leq \varphi_i \leq 1$, such that $\forall x \in \overline{\Omega} : \sum_{i=1}^N \varphi_i(x) = 1$.

TODO!!!?

Věta 0.9 (Extension)

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Then there exists continuous linear operator $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ such that $\forall u \in W^{1,p}(\Omega)$:

1. $Eu = u$ in Ω ;
2. $\exists B_R \subseteq \mathbb{R}^d : Eu = 0$ in $\mathbb{R}^d \setminus B_R$;
3. $\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq c(\Omega, p, d) \cdot \|u\|_{W^{1,p}(\Omega)}$.

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Důkaz

„1.“ By picture. „2.“: $u = \sum_{r=1}^{M+1} u_r$, where $u_r := u\varphi_r \in W^{1,p}(\Omega)$. Step 0: extension of u_{M+1} by zero is trivial. Step 1: $u_1, T_1 = \text{id}, u_1 \in W^{1,p}(V_1^+)$. $F : V_1 \rightarrow \overline{V_1}, (x', x_d) \mapsto (y', y_d)$, $y' = x', y_d = x_d - a_1(x')$. $F^{-1} : \overline{V_1} \rightarrow V_1, x' = y', x_d = y_d + a(y')$.

TODO!!!

Proof of (*): It is enough to show that $\frac{\partial Ev(y)}{\partial y_1} = \frac{\partial v(y)}{\partial y_1}$ for $y_d > 0$ and $\dots = \frac{\partial v}{\partial y_i}(y', -y_d)$ for $y_d < 0$; and $\frac{\partial E(y)}{\partial d} = \frac{\partial v(y)}{\partial y_d}$ for $y_d > 0$ and $\dots = -\frac{\partial v}{\partial y_d}(y'; y_d)$ for $y_d < 0$.

We know $Ev \in W^{1,p}(\overline{V_1^+})$ and $Ev \in W^{1,p}(\overline{V_1^-})$. $\|Ev\|_{W^{1,p}(V_1^-)} = \|Ev\|_{W^{1,p}(V_1^+)}$.

TODO!!!!???

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□

1 Embeddings

Věta 1.1

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Then:

- $W^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega)$ if $p < d$;
- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all q if $p = d$;
- $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ if $p > d$;
- $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega}) \hookrightarrow C^{0,\beta}(\overline{\Omega})$ if $p > d$ (for $\beta < 1 - \frac{d}{p}$);
- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q < \frac{dp}{d-p}$ if $p < d$ (respectively $< \infty$ if $p = d$).

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Dukaz (Case $p > d$)

Lemma (Marey): Let $u \in W^{1,1}(B_R)$ and \mathbf{o} be the Lebesgue point of u . Then

$$\left| \int_{B_R} u dx - u(\mathbf{o}) \right| \leq c(d, A) R^A \sup_{\varrho \leq R} \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} dx \quad \forall A \in (0, 1).$$

Proof of lemma:

$$\begin{aligned} \int_{B_R} u dx - u(\mathbf{o}) &= \lim_{r \rightarrow 0+} \left(\int_{B_R} u - \int_{B_r} u \right) = \lim_{r \rightarrow 0+} \int_r^R \frac{d}{d\varrho} \int_{B_\varrho} u(x) dx d\varrho = \\ &= \lim_{r \rightarrow 0+} \int_r^R \frac{d}{d\varrho} \int_{B_1} u(\varrho x) dx = \lim_{r \rightarrow 0+} \int_r^R \int_{B_1} \frac{\nabla u(\varrho x) \cdot x}{\sum_{i=1}^d \frac{\partial u}{\partial y_i}(\varrho x) \cdot x_i} dx d\varrho \leq \\ &\leq \lim_{r \rightarrow 0+} \int_r^R \int_{B_1} |\nabla u(\varrho x)| dx d\varrho = \lim_{r \rightarrow 0+} \int_r^R \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} \frac{\varrho^{d-1+A}}{\varrho^d} dx d\varrho \kappa(\varrho) d\varrho = \\ &= \lim_{r \rightarrow 0+} \int_r^R \varrho^{A-1} \left(\int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} dx \right) d\varrho \leq \left(\sup_{0 \leq \varrho \leq R} \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} \right) \kappa(d) \int_0^R \varrho^{A-1} d\varrho = \frac{\kappa(d)}{A} R^A \sup_{r \leq R} \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} \end{aligned}$$

Lemma (Marey II) Let $u \in W_{loc}^{1,1}(\mathbb{R}^d)$ and x, y be Lebesgue points. Then

$$|u(x) - u(y)| \leq c(d, A) |x - y|^A \sup_{\varrho \leq R, z \in [x, y]} \int_{B_\varrho(z)} \frac{|\nabla u|}{\varrho^{d-1+A}} dx.$$

Proof of lemma: ($R = |x - y|$)

$$\begin{aligned} |u(x) - u(y)| &\leq \left| \int_{B_R(x)} u(z) dz - u(x) \right| + \left| \int_{B_R(y)} u(z) dz - u(y) \right| + \left| \int_{B_R(x)} u(z) dz - \int_{B_R(y)} u(z) dz \right| \leq \\ &\leq c(d, A) R^A \left(\sup_{\varrho \leq R} \int_{B_\varrho(x)} \frac{|\nabla u|}{\varrho^{d-1+A}} + \sup_{\varrho \leq R} \int_{B_\varrho(y)} \frac{|\nabla u|}{\varrho^{d-1+A}} \right) + \left| \int_0^1 \frac{d}{dt} \int_{B_R(tx + (1-t)y)} u(z) dz dt \right| = \\ &= \dots + \left| \int_0^1 \frac{d}{dt} \int_{B_R(\mathbf{o})} u(tx + (1-t)y + z) dz \right| \leq \dots + \left| \int_0^1 \int_{B_R(\mathbf{o})} \nabla u(tx + (1-t)y + z) \cdot (x - y) dz \right| \leq \\ &\leq \dots + \int_0^1 R^A \int_0^1 \kappa^{-1}() \int_{B_R(tx + (1-t)y)} \frac{|\nabla u|}{R^{d-1+A}} dz dt \leq \\ &\leq \tilde{c}(d, A) R^A \sup_{\varrho \leq R} \sup_{z \in [x, y]} \int_{B_\varrho(z)} \frac{|\nabla u|}{\varrho^{d-1+A}}. \end{aligned}$$

Proof of theorem: We have $\|u\|_{C^{0,\alpha}} \leq c \cdot \|u\|_{1,p}$ for $u \in C^1(\overline{\Omega})$

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|Eu\|_{C^{0,\alpha}(\overline{\Omega})} \leq \|Eu\|_{C^{0,\alpha}(B_R)} \stackrel{1.}{\leq} c(\overline{B_R}, p, d) \cdot \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \stackrel{\text{Extension}}{\leq} C(\overline{B_R}, p, d, \Omega) \|u\|_{W^{1,p}(\Omega)},$$

where $\overline{B_R}$ is support of E .

$$u \in C_0^1(\overline{B_R})$$

$$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^A} \leq \sup_{x \neq y} c(d, A) \sup_{\varrho \leq |x-y|, z \in [x, y]} \int_{B_\varrho(z)} \frac{|\nabla u|}{\varrho^{d-1+A}} dx \leq$$

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Důkaz (Case $d > p$ ($d = p$), only for $u \in C_0^\infty(\mathbb{R}^d)$)

$$(*) : \quad \|u\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \leq c(d, p) \|\nabla u\|_{L^p(\mathbb{R}^d)} \quad \text{At home}^{1,p} \quad W(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega) \quad p < d, \Omega \in C^{0,1}.$$

„Step 1: If $(*)$ is true for $p = 1$, then $(*)$ is true for $p \in [1, d)$ “: (Set $v := |u|^q$)

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{q \cdot d}{d-1}} \right)^{\frac{d-1}{d}} = \|v\|_{L^{\frac{d}{d-1}}} \leq c(d) \cdot \|\nabla v\|_{L^1} \leq c(d) \int_{\mathbb{R}^d} q \cdot |u|^{q-1} \cdot |\nabla u| \leq c(d, q) \|\nabla u\|_{L^p} \cdot \|u\|_{L^{p'(q-1)}}^{q-1}.$$

Set $q := \frac{p \cdot (d-1)}{d-p}$:

$$\begin{aligned} \left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}} \right)^{\frac{d-1}{d}} &\leq c(d, p) \|\nabla u\|_p \cdot \|u\|_{L^{\frac{dp}{d-p}}}^{\frac{p \cdot (d-1)}{d-p} - 1} \\ &\left(\frac{p}{p-1} \cdot \left(\frac{p \cdot (d-1)}{d-p} - 1 \right) = \frac{dp}{d-p} \right). \end{aligned}$$

Lemma (Gagliardo): Let $u_i \in C_0^\infty(\mathbb{R}^{d-1})$, $i \in [d]$. Define $v_i(x_1, \dots, x_d) := u_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$. Then

$$\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)| dx \leq \prod_{i=1}^d \|u_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

Proof of lemma: By induction (with respect to d):

$$d = 2 : \quad \int_{\mathbb{R}^d} |v_1(x)| \cdot |v_2(x)| dx = \int_{\mathbb{R}^2} |u_1(x_2)| \cdot |u_2(x_1)| dx_1 dx_2 = \|u_1\|_{L^1(\mathbb{R})} \cdot \|u_2\|_{L^1(\mathbb{R})}.$$

$$\begin{aligned} d \implies d+1 : \quad & \int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1} |v_i(x)| dx = \int_{\mathbb{R}^d} |v_{d+1}(x)| \cdot \left(\int_{\mathbb{R}} \prod_{i=1}^d |v_i(x)| dx_{d+1} \right) dx_1 \dots dx_d \stackrel{\text{Hölder}}{\leq} \\ & \leq \left(\int_{\mathbb{R}^d} |v_{d+1}(x)|^d dx_1 \dots dx_d \right)^{1/d} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \prod_{i=1}^d |v_i| dx_{d+1} \right)^{d'} dx_1 \dots dx_d \right)^{1/d'} \stackrel{\text{Hölder}}{\leq} \\ & \leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \left(\prod_{i=1}^d \left(\int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{1/d} \right)^{d'} dx_1 \dots dx_d \right)^{1/d'} \leq \\ & = \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \cdot \left(\int_{\mathbb{R}^d} \prod_{i=1}^d \underbrace{\left(\int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{\frac{1}{d-1}}}_{=: z_i} dx_1 \dots dx_d \right)^{1/d'} \leq \\ & \leq \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \cdot \left(\int_{\mathbb{R}^d} \prod_{i=1}^d |z_i| dz \right)^{1/d'} \stackrel{\text{Induction hypothesis}}{\leq} \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^d \|z_i\|_{L^{\frac{d}{d-1}}}^{\frac{d-1}{d}} = \\ & = \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^d \text{TODO} = \prod_{i=1}^{d+1} \|u_i\|_{L^d}. \end{aligned}$$

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Gagliardo–Nirenberg inequality

Proof of theorem: We want $\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq c(d) \|\nabla u\|_{L^1(\mathbb{R}^d)} \quad \forall u \in C_0^\infty(\mathbb{R}^d).$

┌ *Důkaz* (Compact embeddings)

Step 1: $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$. Step 2: $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $q < \frac{dp}{d-p}$.

„Step 1 \implies Step 2“: $p \leq q \leq z$:

$$\|u\|_{L^q} \leq \|u\|_{L^p}^\alpha \cdot \|u\|_z^{1-\alpha}, \quad \frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{z}.$$

S bounded set in $W^{1,p}(\Omega)$. Goal: $\forall \varepsilon > 0 \exists \{u_i\}_{i=1}^N \subset L^q(\Omega) \forall u \in S : \min_i \|u - u_i\| < \varepsilon$.
 $W^{1,p}(\Omega) \hookrightarrow W^{1,1}(\Omega) \xrightarrow{\text{Step 1}} \forall \tilde{\varepsilon} > 0 \exists \{u_i\}_{i=1}^{N(\tilde{\varepsilon}, S)} : \min_i \|u - u_i\|_{L^1(\Omega)} \leq \tilde{\varepsilon}$.

$$\|u - u_i\|_{L^q(\Omega)} \leq \|u - u_i\|_{L^1(\Omega)}^\alpha \cdot \|u - u_i\|_{\frac{dp}{d-p}}^{1-\alpha}, \quad \frac{1}{q} = \frac{\alpha}{1} + \frac{(1-\alpha) \cdot (d-p)}{dp} \leq$$

$$\leq c(\Omega, p) \|u - u_i\|_{L^1(\Omega)}^\alpha \cdot \|u - u_i\|_{W^{1,p}(\Omega)}^{1-\alpha} \leq c(\Omega, p, S) \cdot \|u - u_i\|_{L^1(\Omega)}^\alpha.$$

$$\min \|u - u_i\|_{L^q(\Omega)} \leq c(\Omega, p, S) \tilde{\varepsilon}^\alpha.$$

Given $\varepsilon > 0$. $\tilde{\varepsilon} := \frac{\varepsilon^{1/\alpha}}{c(\Omega, p, S)^{1/\alpha}}$, find $\{u_i\}$ from Step 1 $\implies \min_i \|u - u_i\|_{L^q} \leq \varepsilon$.

„Step 1“: Enough $W_0^{1,1}(B_R) \hookrightarrow L^1(B_R)$. $u \in W_0^{1,1}(B_R)$, $u_\delta := u * \eta_\delta$.

$$\begin{aligned} \int_{\mathbb{R}^d} |u - u_\delta| dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (u(y) - u(x)) \eta_\delta(x - y) dy \right| dx = \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|}{|y|} \eta_\delta(y) |y| dx dy, \\ &\leq \|\nabla u\|_{L^1(\mathbb{R}^d)} \cdot \int_{\mathbb{R}^d} |y| \eta_\delta(y) dy \leq \delta \|\nabla u\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

Recall: $C^1(\overline{B_R}) \hookrightarrow C^0(\overline{B_R}) \hookrightarrow L^1(\overline{B_R})$ (Arzela–Ascoli + Hölder).

S set bounded in $W^{1,1}(\overline{B_R})$. $S_\delta := \{u_\delta, u \in S\}$. $\|u_\delta\|_{C^1(\overline{B_R})} \leq \frac{C(B_R) \cdot \|u\|_{W^{1,1}(B_R)}}{\delta}$ (for δ small).

Given $\varepsilon > 0 \exists \delta$: $\|u - u_\delta\|_{L^1} \leq \frac{\varepsilon}{2} \forall u \in S$. $u_\delta \in S_\delta$ (bounded set in $C^1(\overline{B_R})$), $\|u_\delta\|_{C^1} \leq \frac{\varepsilon}{\delta} = c(\varepsilon)$. Find $\{u_\delta^i\}_{i=1}^{N(\varepsilon)}$: $\min \|u_\delta - u_\delta^i\|_{L^1} \leq \frac{\varepsilon}{2}$.

$$\|u - u_\delta^i\|_{L^1} \leq \|u - u_\delta\|_{L^1} + \|u_\delta - u_\delta^i\| \leq \varepsilon.$$

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□

1.1 Traces

Poznámka (C^1 functions on cube)

$\Omega = (-1, 1)^{d-1} \times (0, 1)$, $x' = (x_1, \dots, x_{d-1})$. $u \in C^1(\overline{\Omega})$, $u(x', 1) = 0$.

Optimal q such that $\int_{(-1,1)^{d-1}} |u(x', \mathbf{o})|^q dx_1 \dots dx_{d-1} \leq c \cdot \|\nabla u\|_{L^p(\Omega)}^q$?

$$\begin{aligned} \int_{(-1,1)^{d-1}} |u(x', 0)|^q dx' &= \int_{(-1,1)^{d-1}} - \int_0^1 \frac{\partial}{\partial x_d} |u(x', x_d)|^q dx_d dx' \leq q \cdot \int_{\Omega} |u|^{q-1} \cdot |\nabla u| dx \leq \\ &\leq q \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \| |u|^{q-1} \|_{L^{p'}(\Omega)} \stackrel{?}{\leq} q \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \|u\|_{L^{\frac{dp}{d-p}}(\Omega)}^{q-1}. \end{aligned}$$

Set q : $(q-1)p' = \frac{dp}{d-p} \implies q = \frac{d(p-1)}{d-p} + 1 = \frac{dp-p}{d-p} = \frac{p \cdot (d-1)}{d-p}$.

$$\|u\|_{L^{\frac{p(d-1)}{d-p}}((-1,1)^{d-1})} \leq C(\Omega, p) \cdot \|u\|_{W^{1,p}(\Omega)}.$$

Poznámka (Integral on boundary for $\Omega \in C^{0,1}$)

$$\int_{\partial\Omega} f ds := \int_{\partial\Omega} \sum_{i=1}^N f \varphi_i = \sum_{i=1}^N \int_{(-1,1)^{d-1}} f(T_i(y)) \varphi_i(T_i(y)) \sqrt{1 + |\nabla y|^2} dy',$$

where φ_i is partition of unity corresponding to $C^{0,1}$ and T_i .

We should show independence on φ_i , V_i . Also we can show $\int_{\Omega} \frac{\partial f}{\partial x_i} dx = \int_{\partial\Omega} f n_i dS$.
($\forall f \in C^1(\overline{\Omega})$.)

TODO!!!

Poznámka (On spaces with non-integer derivative)

tr is not onto $L^{\frac{(d-1)p}{d-p}}(\partial\Omega)$.

Věta 1.2 (Inverse trace theorem)

$\Omega \in C^{0,1}$, $p \in (1, \infty]$, $s \in (1/p, 1]$. Then tr is bounded linear from $W^{s,p}(\Omega)$ to $W^{s-\frac{1}{p},p}(\partial\Omega)$.
Moreover $\exists \text{tr}^{-1} : W^{s-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{s,p}(\Omega)$ linear bounded, such that $\text{tr}(\text{tr}^{-1}) = u$ on $\partial\Omega$.

Definice 1.1 (Sobolev–Slobodeckij spaces)

We say that $u \in W^{s,p}(\Omega)$, $s \in (0, 1)$, iff

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{d+ps}} dx dy < \infty \quad \wedge \quad u \in L^p(\Omega).$$

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)}.$$

Definice 1.2 (Nikólskii spaces)

We say that $u \in N^{s,p}(\Omega)$, $p \in [1, \infty]$, $s \in (0, 1]$, iff

$$\sup_{h,i} \int_{\Omega_h} \frac{|u(x + he_i) - u(x)|^p}{h^{p \cdot s}} dx < \infty.$$

Lemma 1.3

$$W^{s,p}(\Omega) \hookrightarrow N^{s,p}(\Omega) \hookrightarrow W^{s-\varepsilon,p}(\Omega), \quad \forall 0 < \varepsilon < s.$$

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Důkaz

└ At home.

□

2 Nonlinear elliptic equations as a „compact“ perturbation of linear PDE

Poznámka

$-\Delta u = f(x, u, \nabla u)$ in Ω , $u = 0$ on $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^d$ is Caratheódory, $|f(x, u, \xi)| \leq c$ ($\forall u \in \mathbb{R}, \xi \in \mathbb{R}^d$, and almost all $x \in \Omega$).

Lemma 2.1

$$\exists u \in W_0^{1,2}(\Omega) : \int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f(\cdot, u, \nabla u) \varphi \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

┌ *Důkaz*

Consider $M : W_0^{1,2}(\Omega) \rightarrow W_0^{1,2}(\Omega), v \mapsto u$:

$$\int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f(x, v, \nabla v) \varphi \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Linear theory $\implies \forall v \in W_0^{1,2}(\Omega) \exists! u \in W_0^{1,2}(\Omega)$ satisfying this equation.

Does M have a fixpoint? Schauder fixpoint theorem assumptions: (A1) $\exists K$ convex set in $W_0^{1,2}(\Omega)$ such that $M(K) \subseteq K$, (A2) M compact, continuous.

Ad (A1): Set $\varphi := u$.

$$C_1 \|u\|_{1,2}^2 \leq \|\nabla u\|_2^2 = \int_{\Omega} \nabla u \nabla u = \int_{\Omega} f(x, v, \nabla v) u \leq C \int_{\Omega} |u| \leq \tilde{C}(\Omega) \|u\|_{1,2} \implies \|u\|_{1,2} \leq \tilde{C}.$$

$$K \subseteq W_0^{1,2}(\Omega), K = \left\{ u \in W_0^{1,2}(\Omega) \mid \|u\|_{1,2} \leq \tilde{C} \right\}.$$

Ad (A2): $v^n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$, $u^n = M(v^n)$. $\exists u^{n_k} \rightarrow u$ strongly in $W_0^{1,2}(\Omega)$?

$$\|u^n\|_{1,2} \leq \tilde{C}, u^{n_k} \rightarrow u \text{ in } W_0^{1,2}(\Omega), u^{n_k} \rightarrow u \text{ in } L^2(\Omega).$$

Set $\varphi := u^n - u \in W_0^{1,2}(\Omega)$.

$$\begin{aligned} \lim_{n_k \rightarrow 0} \|u^{n_k} - u\|_{1,2}^2 &\leq C \cdot \lim_{n_k \rightarrow 0} \|\nabla u^{n_k} - \nabla u\|_2^2 = C \cdot \lim_{n_k \rightarrow 0} \int_{\Omega} \nabla u^{n_k} \cdot (\nabla u^{n_k} - \nabla u) - C \cdot \lim_{n_k \rightarrow 0} \int_{\Omega} \nabla u \cdot (\nabla u^{n_k} - \nabla u) = \\ &= C \cdot \lim_{n_k \rightarrow 0} \int_{\Omega} f(\cdot, v^n, \nabla v^n) (u^{n_k} - u) - C \cdot \lim_{n_k \rightarrow 0} \int_{\Omega} \nabla u \cdot (\nabla u^{n_k} - \nabla u), \end{aligned}$$

TOOD!!!

TODO!!!

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Poznámka

Works also if $|f(x, u, \xi)| \leq C(g(x) + |u|^\alpha + |\xi|^\alpha)$, where $0 \leq \alpha < 1$ and $g \in L^2(\Omega)$.

No estimates available (in principle): $-\Delta u = l|u|^{p-2}u$ in Ω , $u = 0$ on $\partial\Omega$. Does the problem have nontrivial solution? TODO!!!

$$-\Delta v = \lambda A^{2-p} \cdot |v|^{p-2}v$$

if $\exists \lambda > 0$ $u \not\equiv 0$ solving $-\Delta u = \lambda |u|^{p-2}u$, then $\forall B > 0 \exists v : -\Delta v = B |v|^{p-2}v$.

Lemma 2.2

Let $p < \frac{2d}{d-2}$. Then $\exists \lambda > 0 \exists u \in W_0^{1,2}(\Omega)$ such that $-\Delta u = \lambda |u|^{p-2}u$ in Ω .x TODO!!!

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Dūkaz
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TODO!!!

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