

Poznámka (Exam)
Oral, similar as in FA1.

Poznámka (Credit)
Similar as in FA1.

1 Banach algebras

1.1 Basic properties

Definice 1.1 (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over \mathbb{K} , if

- $(A, +, -, 0, \cdot_S)$ is vector space over \mathbb{K} ;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K} \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

Důsledek

1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ ($\forall a \in A$).

If e_1 is left unit and e_2 is right unit, then $e_1 = e_2$ is unit. ($e_1 = e_1 \cdot e_2 = e_2$)

2) (Algebra) homomorphism $\varphi : A \rightarrow B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

Tvrzení 1.1

Let A be algebra over \mathbb{K} . Put $A_e = A \times \mathbb{K}$ with operations A_e defined coordinate-wise and multiplication defined by

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \quad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

Then A_e is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if A is commutative, then A_e is commutative.

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Důkaz

We have A_e is vector space (from linear algebra). We easy proof from definition, that A_e is algebra, $(\mathbf{o}, 1)$ is a unit in A_e and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. \square

Definition 1.2 (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv A$ is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ ($\forall a, b \in A$).

Definition 1.3 (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv (A, \|\cdot\|)$ is normed algebra nad Banach space.

Například

$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If T is Hausdorff topological space, then $\mathcal{C}_b(T) = \{f : T \rightarrow \mathbb{K} | f \text{ is continuous and bounded}\} \subseteq l_\infty(T)$ is closed subalgebra.

If T is locally compact, Hausdorff, not compact. Then $\mathcal{C}_0(T) = \{f : T \rightarrow \mathbb{K} \text{ continuous} | \forall \varepsilon > 0 : \{t \in T : |f(t)| \geq \varepsilon\} \text{ is compact}\} \subseteq \mathcal{C}_b(T)$ is closed subalgebra, which doesn't have unit.

If X is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If X is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

Tvrzení 1.2

If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \rightarrow A$ is Lipschitz on bounded sets.

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Důkaz

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r) \quad \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) :$$

$$\|ab - cd\| \leq \|a(b-d)\| + \|(a-c) \cdot d\| \leq \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leq R \cdot (\|b-d\| + \|a-c\|) \leq 2R\|(a, b) - (c, d)\|.$$

└ \square

Tvrzení 1.3

Let $(A, \|\cdot\|)$ be a Banach algebra. On A_e we consider the norm

$$\|(a, \alpha)\| := \|a\| + |\alpha|, \quad (a, \alpha) \in A \times \mathbb{K} = A_e.$$

Then $(A_e, \|\cdot\|)$ is Banach algebra.

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Důkaz

It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a, \alpha) \cdot (b, \beta)\| \leq \|(a, \alpha)\| \cdot \|(b, \beta)\|,$$

└ which is easy. □

Poznámka

There is more (natural) ways to define norm on A_e (unlike \cdot on A_e , which is natural).

A has a unit ... we may still consider A_e .

If $e \in A \setminus \{\mathbf{o}\}$ is a unit, then $\|e\| \geq 1$, because $\|e\| = \|e^2\| \leq \|e\|^2$.

Věta 1.4

Let A be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a \cdot x$, $x \in A$. Then $I : A \rightarrow \mathbb{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leq 1$.

Moreover, if A has a unit e , then I is isomorphism into and $I(e) = \text{id}$.

If $\|x^2\| = \|x\|^2$, $x \in A$, then I is isometry into.

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Důkaz

„ $L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leq 1$ “: Linearity is obvious, $\|L_a(x)\| = \|a \cdot x\| \leq \|a\| \cdot \|x\|$, so $\|L_a\| \leq \|a\|$ and so $\|I\| \leq 1$. Since it is easily I preserves multiplication, so we are left to prove the „Moreover“ part.

„ A has a unit e “: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geq \|L_a\left(\frac{e}{\|e\|}\right)\| = \frac{\|a\|}{\|e\|} = \frac{1}{\|e\|} \cdot \|a\|.$$

So I is bounded from below, so I is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \text{id}.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$:

$$\|a\| \geq \|I(a)\| = \|L_a\| \geq \|L_a\left(\frac{a}{\|a\|}\right)\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So I is isometry.

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□

Poznámka

$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies \exists$ equivalent norm $\|\cdot\|$ on A such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

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Důkaz

Let $I : A \rightarrow \mathcal{L}(A)$ be as before. Put $\|x\| := \|I(x)\|$, $x \in A$. Since I is isomorphism, $\|\cdot\|$ is equivalent norm. Moreover, $\|x \cdot y\| = \|I(x \cdot y)\| \leq \|I(x)\| \cdot \|I(y)\| = \|x\| \cdot \|y\|$, $x, y \in A$. So $(A, \|\cdot\|)$ is a Banach algebra. Finally

$$\|e\| = \|I(e)\| = \|\text{id}\| = 1.$$

└

□

1.2 Inverse elements

Definice 1.4

(M, \cdot, e) is monoid (\cdot is associative, e is unit). Then invertible elements form a group ($e^{-1} = e$, $\exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^\times := \{x \in M \mid \exists x^{-1}\}$

Tvrzení 1.5

If (A, \cdot, e) is monoid and $x_1, \dots, x_n \in A$ commute, then $x_1 \dots x_n \in A^\times \Leftrightarrow \{x_1, \dots, x_n\} \subset A^\times$.

Důkaz

It suffices to prove it for $n = 2$ (and use induction). „If x^{-1} and y^{-1} exists, then $(xy)^{-1}$ “ is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so z is left inverse to y . Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so \tilde{z} is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. \square

Lemma 1.6

Let A be a Banach algebra with a unit.

- $\|x\| < 1 \implies \exists (e - x)^{-1} \wedge (e - x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x + h)^{-1} \wedge \|(x + h)^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

Důkaz

„First item“: We have $\|x^n\| \leq \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0}^{\infty} x^n \in A$. Moreover,

$$(e - x) \cdot \left(\sum_{n=0}^{\infty} x^n \right) = \lim_{N \rightarrow \infty} (e - x) \cdot (e + x + \dots + x^N) = \lim_{N \rightarrow \infty} e - x^{N+1} = e,$$

because $\lim_{N \rightarrow \infty} \|x^{N+1}\| \leq \lim_{N \rightarrow \infty} \|x\|^{N+1} = 0$. And similarly $(\sum x^n) \cdot (e - x) = e$.

„Second item“: $x + h = x \cdot (e + x^{-1}h)$ we have x^{-1} exists and $(e + x^{-1}h)^{-1}$ exists (from first item), so from previous fact $(x + h)^{-1}$ exists. Moreover

$$(x + h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \stackrel{1)}{=} \sum_{n=0}^{\infty} (-x^{-1}h)^n x^{-1},$$

so

$$\begin{aligned} \|(x + h)^{-1} - x^{-1}\| &= \left\| \sum_{n=1}^{\infty} (-x^{-1}h)^n x^{-1} \right\| \leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leq \\ &\leq \|x^{-1}\| \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} (\|x^{-1}\| \cdot \|h\|)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}. \end{aligned}$$

Důsledek

A Banach algebra with a unit $\implies A^x \subset A$ is open and A^x is topological group.

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Důkaz

$A^x \subset A$ is open by previous lemma (second item). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^x \ni x_n \rightarrow x \in A^x \stackrel{?}{\implies} x_n^{-1} \rightarrow x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \stackrel{h:=x_n-x}{\leq} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \rightarrow 0.$$

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□

1.3 Spectral theory

Definice 1.5 (Resolvent set, spectrum and resolvent)

A Banach algebra with a unit, $x \in A$. We define resolvent set of x as $\varrho_A(x) := \{\lambda \in \mathbb{K} \mid \exists (\lambda \cdot e - x)^{-1}\}$. Next we define spectrum of x as $\sigma_A(x) := \mathbb{K} \setminus \varrho_A(x)$. Finally we define resolvent of x as $R_x : \varrho(x) \rightarrow A$, $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$.

If A doesn't have a unit, then notions above are defined with respect to A_e .

Tvrzení 1.7

A Banach algebra

- a) $\forall x \in A : 0 \in \sigma_{A_e}(x)$ (in particular, if A has no unit, then $0 \in \sigma_A(x)$);
- b) A has unit $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

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Důkaz (a)

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\dots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

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□

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Důkaz (b))

By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in A and second means that $((0, \lambda) - (x, 0))^{-1} = (-x, \lambda)^{-1}$ exists in A . We take „ $x \rightarrow -x$ “.

„ \Rightarrow “: find $(b, \beta) \in A_e$ such that $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$. So $(x \cdot b + \lambda \cdot b + \beta \cdot x, \lambda \cdot \beta) = (\mathbf{o}, 1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e + x)^{-1} \cdot x$. Similarly we find left inverse $(-\frac{1}{\lambda}x(x + \lambda e)^{-1}, \frac{1}{\lambda}) (x, \lambda)$. And next we prove that they are really inverses.

„ \Leftarrow “: Put $(b, \beta) := (x, \lambda)^{-1}$. Then $(\lambda e + x)^{-1} = b + \beta \cdot e$. We have $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$, so $\lambda \cdot \beta = 1$ and $x \cdot b + \lambda \cdot b + \beta \cdot x = \mathbf{o}$. Then

$$(\lambda e + x) \cdot (b + \beta \cdot e) = \lambda \cdot b + \lambda \cdot \beta \cdot e + x \cdot b + \beta \cdot x = e.$$

└ Similarly second inverse. □

Věta 1.8

$\{\mathbf{o}\} \neq A$ complex Banach algebra, $x \in A$. Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is compact, nonempty.

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Důkaz

└ After theory. □

Definice 1.6 (Derivative)

Y Banach space, $\Omega \subset \mathbb{K}$, $f : \Omega \rightarrow Y$, $a \in \Omega$. Then

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of f at a .

Tvrzení 1.9 (Fact)

Y Banach, $\Omega \subset \mathbb{K}$, $f : \Omega \rightarrow Y$, $a \in \Omega$. Then $f'(a)$ exists $\implies f$ is continuous at $a \wedge \forall x^* \in Y^* : (x^* \circ f)'(a) = x^*(f'(a))$.

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Důkaz

Continuity: $\lim_{x \rightarrow a} f(x) - f(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = f'(a) \cdot 0 = 0$.

$x^* \in Y^*$ given, then

$$\lim_{x \rightarrow a} \frac{x^*(f(x)) - x^*(f(a))}{x - a} = \lim_{x \rightarrow a} x^* \left(\frac{f(x) - f(a)}{x - a} \right) = x^*(f'(a)).$$

└ □

Tvrzení 1.10

A Banach algebra with a unit, $x \in A$. Then

- $\varrho(x)$ is open set;
- $\forall |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$;
- (important!) $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$ has derivative at each $\lambda \in \varrho(x)$;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$.

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Důkaz

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left(e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right)^n.$$

For third we fix $\lambda \in \varrho(x)$ and $t \in (0, \delta)$ for δ small enough ($\lambda + t \in \varrho(x)$ and *). We shall prove that „ $R'_x(\lambda) = -R_x(\lambda)^2$ “:

$$0 \stackrel{?}{=} \left\| \frac{R_x(\lambda + t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| = \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leq$$

$$\begin{aligned} & \text{* for existence of the inverse} \\ & \leq \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \|(e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t\| = \end{aligned}$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leq$$

$$\stackrel{\|x^n\| \leq \|x\|^n}{\leq} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \stackrel{\text{* for denominator} \leq 1/2}{\leq} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \rightarrow 0.$$

Fourth: In general $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$ (proof: $u^{-1}v^{-1} = (vu)^{-1}$). And we apply it for $u = (\mu e - x)$ and $v = (\nu e - x)$.

Fifth: In general $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$ (proof: $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$) so:

$$\begin{aligned} R_x(\mu) - R_x(\nu) &= R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) \\ &= R_x(\mu)R_x(\nu) (R_x(\nu)^{-1} - R_x(\mu)^{-1}) = R_x(\mu)R_x(\nu)(\nu - \mu). \end{aligned}$$

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□

Věta 1.11 (Liouville for Banach space valued functions)

Y Banach space over \mathbb{C} , $f : \mathbb{C} \rightarrow Y$ has derivative at each point, f is bounded ($\equiv \|f\|$ is bounded). Then $f \equiv \text{const}$.

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Důkaz

Assume $f \not\equiv \text{const}$, so there are $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$ (by Hahn–Banach theorem) $\exists x^* \in Y^* : x^*(f(a)) \neq x^*(f(b))$. From fact $x^* \in f : \mathbb{C} \rightarrow \mathbb{C}$ has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions. \square

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Důkaz (Theorem before theory)

First case: „ A has a unit“: Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is closed, so $\sigma(x)$ is compact. Assume that $\varrho(x) = \mathbb{C}$. By previous tvrzení we have $R_x : \mathbb{C} \rightarrow A$ has derivative everywhere, and it is bounded because $\lim_{|\lambda| \rightarrow \infty} \lambda \rightarrow \infty R_x(\lambda) = \lim_{|\lambda| \rightarrow \infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$. From previous theorem $R_x \equiv \text{const}$ so $\lim_{|\lambda| \rightarrow \infty} R_x(\lambda) = 0 \implies R_x \equiv 0$. In particular $0 = R_x(0) = (-x)^{-1}$. \nexists (If $A \neq \{0\}$ then $x^{-1} \neq 0$ for $x \in A$.)

Second case: „ A hasn't a unit“, then $\sigma(x) := \sigma_{A_e}((x, 0))$ so we apply the already proven case. \square

Poznámka (Convention)

If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

Věta 1.12 (Gelfand–Mazur)

$\{\mathbf{o}\} \neq A$ Banach algebra with a unit. Assume $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$. Then A is isomorphic to \mathbb{C} . If moreover e is a unit in A and $\|e\| = 1$, then A is isometrically isomorphic to \mathbb{C} .

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Důkaz

Consider $\psi : \mathbb{C} \rightarrow A$ defined as $\psi(\lambda) := \lambda \cdot e$. This is algebraic homomorphism and $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$, so it is isomorphism (and isometry, if $\|e\| = 1$).

It remains „ φ is surjective“: Pick $a \in A$. From previously proved theorem $\exists \lambda \in \sigma(a)$, then $(\lambda e - a) \notin A^x$. So, $\lambda \cdot e - a = 0$, then $\psi(\lambda) = a$. \square

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Definice 1.7 (Spectral radius)

A Banach algebra, $x \in A$. Then $r(x) := \sup \{|\lambda|, \lambda \in \sigma(x)\}$ is called spectral radius of x .

Věta 1.13 (Beurling–Gelfand)

A Banach algebra, $x \in A \implies r(x) = \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$.

Lemma 1.14

A Banach algebra with a unit, $x \in A$. For $p(z) = \sum_{j=1}^n \alpha_j z^j \in \mathbb{C}$ a polynom (with complex coefficients) we put $p(x) = \sum_{j=1}^n \alpha_j x^j \in A$. Then $\sigma(p(x)) = p(\sigma(x))$.

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Důkaz

Fix $\lambda \in \mathbb{C}$ and write $(\lambda - p)(z) = c \cdot \prod_{i=1}^m (z - z_i)$, where z_1, \dots, z_m are roots of $\lambda - p$. Then $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$ does not exist. $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^m (x - z_i \cdot e)$, so it doesn't exist if and only if $\exists i \in [m]$, such that $(x - z_i \cdot e)^{-1}$ doesn't exist $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow \exists$ root ν of $\lambda - p$ such that $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$. \square

Důkaz (Beurling–Gelfand)

WLOG A has a unit. Step 1, „ $r(x) \leq \inf_n \sqrt[n]{\|x^n\|}$ “: fix $\lambda \in \sigma(x)$. By previous lemma $\forall n : \lambda^n \in \sigma(x^n)$. By theorem 'Before theory' we have $\forall n : |\lambda|^n \leq \|x^n\|$.

Step 2, „ $r(x) \geq \limsup_n \sqrt[n]{\|x^n\|}$ “: Pick $r > r(x)$. Claim: „ $\frac{x^n}{r^n} \rightarrow^w 0$ “: Fix $x^* \in A^*$ and put $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$. By fact and tvrzení after it, f has derivative at each $\lambda \in \rho(x)$. Moreover for $|\lambda| \geq \|x\|$ we have $f(\lambda) = \lambda \cdot x^* \left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$. Thus $f(\lambda) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$, $\lambda \in P(0, r(x), \infty)$. From Complex analysis $f \in H(P(0, r, \infty))$ is uniquely given by Laurent series. In particular $f(r) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{r^n}$, so $x^* \left(\frac{x^n}{r^n} \right) \rightarrow 0$.

From principle of unique boundedness (last semester): $\frac{x^n}{r^n}$ if $\|\cdot\|$ -bounded, so $\exists c > 0 : \|x^n\| \leq cr^n$, $\sqrt[n]{\|x^n\|} \leq \sqrt[n]{c} \cdot r \rightarrow r$. So $\limsup \sqrt[n]{\|x^n\|} \leq r$. \square

Důsledek

A Banach algebra, $x \in A$ and $|\lambda| > r(x)$. Then $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ is absolutely convergent and $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

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Důkaz

Fix q , such that $\frac{r(x)}{|\lambda|} < q < 1$. By previous theorem, $\exists n_0 \forall n \geq n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$, so $\frac{\|x^n\|}{|\lambda|^n} < q^n$, $n \geq n_0$. Thus $\sum \left\| \frac{x^n}{\lambda^n} \right\| \leq \infty$, so the sum is absolutely convergent.

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Now we easily check that $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$. \square

1.4 Subalgebra

Věta 1.15

A Banach algebra with a unit e , $B \subset A$ is closed subalgebra such that $e \in B$. Fix $x \in B$. Then

- $C \subset \rho_A(x)$ is component (maximum connected subset) $\implies C \subseteq \sigma_B(x)$ or $C \cap \sigma_B(x) = \emptyset$;

- $\partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$;
- $\varrho_A(x)$ is connected $\implies \sigma_A(x) = \sigma_B(x)$;
- $\text{int } \sigma_B(x) = \emptyset \implies \sigma_A(x) = \sigma_B(x)$.

Důkaz

„ $\sigma_A(x) \subseteq \sigma_B(x)$ “: $(\lambda e - x)^{-1}$ exists in B implies it exists (it's same) in A .

„First item“: Let $C \subset \varrho_A(x)$ be component. Pick $\lambda_0 \in C \cap \sigma_B(x)$. Wanted: „ $C \setminus \sigma_B(x) = \emptyset$ “. Pick $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$ (separate B and $R_x(\lambda) \notin B$). Then $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is holomorphic function on open (because maximum) connected set C . Which is zero^a on $C \setminus \sigma_B(x)$.

Since $C \setminus \sigma_B(x)$ is open, if it is nonempty it contains a ball, so it has cluster point. Thus $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is such that $\{\lambda \in C \mid x^*(R_x(\lambda))\} = 0$ has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero. \nLeftarrow with $x^*(R_x(\lambda_0)) = 1$.

„Second item“: Pick $\lambda \in \sigma_B(x) \setminus \sigma_A(x)$ and let $C \subset \varrho_A(x)$ be a component containing λ . By first item, $C \subseteq \sigma_B(x)$, C is open, so $\lambda \in C \subseteq \text{int}(\sigma_B(x))$.

„Third item“: If $\varrho_A(x)$ is connected, we can apply first item to $C = \varrho_A(x)$, we have either $\varrho_A(x) \subseteq \sigma_B(x)$ or $\varrho_A(x) \cap \sigma_B(x) = \emptyset$. But first is not possible, because $\varrho_A(x)$ is unbounded and $\sigma_B(x)$ is bounded. Therefore $\sigma_B(x) \subseteq \sigma_A(x)$.

„Fourth item“: If $\text{int}(\sigma_B(x)) = \emptyset$, then (by second item) $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$. □

^aFor $\lambda \in C \setminus \sigma_B(x)$, $(\lambda e - x)^{-1}$ exists in B so $R_x(\lambda) \in B$ and therefore, $x^*(R_x(\lambda)) = 0$

Důsledek

A Banach algebra, $B \subseteq A$ closed subalgebra, $x \in B$. Then all items from previous theorem hold as well if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.

Důkaz

Without proof. (Basically same that previous; we add unit to A and B , so this unit is same $((\mathbf{o}, 1))$, etc.) □

1.5 Holomorphic calculus

Definice 1.8

X Banach, $\gamma : [a, b] \rightarrow \mathbb{C}$ path (continuous, piecewise smooth (C^1)), $f : \langle \gamma \rangle \rightarrow X$ continuous. Then

$$\int_{\gamma} f := \int_{[a,b]} \gamma'(t) f(\gamma(t)) dt. \quad (\text{As Bochner integral.})$$

If $\Gamma = \gamma_1 + \dots + \gamma_n$ is chain in \mathbb{C} , $f : \langle \Gamma \rangle \rightarrow X$ continuous, then

$$\int_{\Gamma} f := \sum_{i=1}^n \int_{\gamma_i} f.$$

Lemma 1.16

Γ chain in \mathbb{C} , X Banach, $f : \langle \Gamma \rangle \rightarrow X$, $x \in X$. Then

$$\int_{\Gamma} f = x \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_{\Gamma} x^* \circ f.$$

┌

Důkaz

„ \Leftarrow “ by Hahn–Banach theorem. „ \Rightarrow “: (by previous semester x^* and \int “commutes”)

$$x^* \left(\int_{\Gamma} f \right) = \sum_{i=1}^n x^* \left(\int_{\gamma_i} f \right) = \sum_{i=1}^n \int_{[a_i, b_i]} \gamma_i'(t) x^*(f(\gamma_i(t))) dt = \int_{\Gamma} x^* \circ f.$$

└

□

Poznámka (Recall)

If $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact. Then there is a cycle Γ such that $\langle \Gamma \rangle \subset \Omega \setminus K$ and $\text{ind}_{\Gamma} z = 1$ if $z \in K$ and 0 if $z \notin \Omega$.

Then we say that Γ circulates K in Ω .

Definice 1.9

Let A be a Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open and $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $f(x) := \frac{1}{2\pi i} \int_{\Gamma} f \cdot R_x$, where is any cycle which circulates $\sigma(x)$ in Ω .

Poznámka

$f(x)$ exists ($f \cdot R_x$ is continuous on $\langle \Gamma \rangle$), $f(x)$ does not depend on the choice of Γ (Pick $x^* \in X^*$, then $(x^* \circ f \cdot R_x)(\lambda) = f(\lambda) \cdot x^*(R_x(\lambda))$ is holomorphic. Pick Γ_1, Γ_2 cycles circulating $\sigma(x)$ in Ω , then $\int_{\Gamma_1 - \Gamma_2} x^* \circ f \cdot R_x = 0$ from Cauchy).

Věta 1.17 (Holomorphic calculus)

A Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open such that $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $\Phi : \mathcal{H}(\Omega) \rightarrow A$ defined as $\Phi(f) = f(x)$ (from definition above) has the following properties:

- Φ is algebra homomorphism, $\Phi(1) = e$, $\Phi(\text{id}) = x$;
- $f_n \xrightarrow{\text{loc.}} f$ in $H(\Omega)$, then $f_n(x) \rightarrow f(x)$;
- $f(x)^{-1}$ exists $\Leftrightarrow f \neq 0$ on $\sigma(x)$, in this case $f(x)^{-1} = \frac{1}{f}(x)$;

- $\sigma(f(x)) = f(\sigma(x))$;
- if Ω_1 is open and $f(\sigma(x)) \in \Omega_1$, $g \in \mathcal{H}(\Omega_1)$, then $(g \circ f)(x) = g(f(x))$;
- if $y \in A$ commutes with x , then y commutes with $f(x)$.

Moreover, if $\psi : \mathcal{H}(\Omega) \rightarrow A$ satisfy first two item, then $\psi = \Phi$.

Lemma 1.18

(Ω, μ) complete measurable space, A Banach algebra, $f \in L_1(\mu, A)$. Let $x \in A$ and $E \subset \Omega$ is measurable. Then

$$x \cdot \left(\int_E f(t) d\mu(t) \right) = \int_E x \cdot f(t) d\mu(t), \quad \left(\int_E f(t) d\mu(t) \right) \cdot x = \int_E f(t) \cdot x d\mu(t).$$

┌ *Důkaz*

Easy (by commutation of integral and linear operator from last semester), skipped. □

Důkaz (Holomorphic calculus)

„1st item“: „ Φ is linear“ is easy, „ Φ is multiplicative“: Pick $f, g \in \mathcal{H}(\Omega)$, open set U such that $\sigma(x) \subset U \subset \bar{U} \subset \Omega$. Let Γ cycle circulating $\sigma(x)$ in U , Λ cycle circulating \bar{U} in Ω . Then

$$\begin{aligned} f(x) \cdot g(x) &= \left(\frac{1}{2\pi i} \int_{\Gamma} f \cdot R_x \right) \cdot g(x) \stackrel{\text{lemma}}{=} \frac{1}{2\pi i} \int_{\Gamma} f(t) R_x(t) g(x) dt = \frac{1}{2\pi i} \int_{\Gamma} f(t) \cdot R_x(t) \cdot \frac{1}{2\pi i} \int_{\Lambda} g(s) \cdot R_x(s) ds \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(t) \cdot \frac{1}{2\pi i} \int_{\Lambda} g(s) \cdot R_x(t) \cdot R_x(s) ds dt = \end{aligned}$$

because $\langle \Lambda \rangle \cap \langle \Gamma \rangle = \emptyset$, we can use theorem after definition of R_x :

$$\begin{aligned} &= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Lambda} f(t) \cdot g(s) \cdot \frac{R_x(t) - R_x(s)}{s - t} ds dt \stackrel{\text{Fubini to } x^*(\dots) \text{ and lemma}}{=} \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma} f(t) \left(\int_{\Lambda} \frac{g(s)}{s - t} ds \right) R_x(t) dt - \frac{1}{(2\pi i)^2} \int_{\Lambda} g(s) \left(\int_{\Gamma} \frac{f(t)}{s - t} dt \right) R_x(s) ds = \end{aligned}$$

(Now we use Cauchy theorem ($f(z) \text{ ind}_{\Gamma} z = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw$). $\forall s \in \langle \Lambda \rangle : (t \mapsto \frac{f(t)}{s - t}) \in \mathcal{H}(U) \wedge \text{ind}_{\Gamma} z = 0, z \notin U$, so $\int_{\Gamma} \frac{f(t)}{s - t} dt = 0$. $\forall t \in \langle \Gamma \rangle : \text{ind}_{\Lambda} t = 1 \wedge (s \mapsto g(s)) \in \mathcal{H}(\Omega) \implies g(t) = \frac{1}{2\pi i} \int_{\Lambda} \frac{g(s)}{s - t} ds$.)

$$= \frac{1}{2\pi i} \int_{\Gamma} f(t) g(t) R_x(t) dt - 0.$$

It remains that „if $f(z) = z^k$, $k \in \mathbb{N} \cup \{0\}$ then $f(x) = x^k$ “ (we want it for $k = 0$ and

$k = 1$). Put $\Gamma(t) = r \cdot e^{it}$, $t \in [0, 2\pi]$, where $r > \|x\|$ arbitrary. By some theorem:

$$R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}, \quad |\lambda| > \|x\|.$$

Thus (we switch integral and sum, because later we realize that sum of integral of absolute value is finite)

$$\forall x^* \in A^* : x^*(f(x)) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k x^* \left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} \right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=k-1}^{\infty} \int_{\Gamma} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda$$

because Γ (is 2π periodic).

„2nd item“: For $\Gamma = \gamma_1 + \dots + \gamma_N$:

$$\|f_n(x)-f(x)\| = \frac{1}{2\pi i} \left\| \int_{\Gamma} (f_n(\lambda)-f(\lambda))R_x(\lambda)d\lambda \right\| \leq \frac{1}{2\pi} \int_{\Gamma} |f_n(\lambda)-f(\lambda)| \cdot \|R_x(\lambda)\|d\lambda \leq \frac{1}{2\pi} \sum_{i=1}^N \int_{a_i}^{b_i} |\gamma_i'(t)| \sup_{z \in \langle \Gamma \rangle} \|R_x(z)\| dt$$

„Moreover part“: By Runge theorem (and second item) it is enough prove it for rational functions. If R was polynomial, then $\Phi(R) = \Psi(R)$ by second item. So it suffices „ $\forall p$ polynomial: $\frac{1}{p} \in \mathcal{H}(\Omega) \implies \Phi(\frac{1}{p}) = \psi(\frac{1}{p})$ “. Pick p polynomial. Then $e = \psi(1) = \psi(p \cdot \frac{1}{p}) = \psi(p) \cdot \psi(\frac{1}{p}) = \Phi(p) \cdot \psi(\frac{1}{p})$ (similarly for $\frac{1}{p} \cdot p$). So $\psi(\frac{1}{p}) = \Phi(p)^{-1} = \Phi(\frac{1}{p})$.

„3rd item“: „ \implies “ Let $f(z) = 0$ for some $z \in \sigma(x)$. Then exists $g \in H(\Omega) : f(u) = (z - u)g(z)$. By item one, we have $(ze - x)g(x) = f(x) = g(x)(ze - x)$. But $(ze - x)^{-1}$ does not exist, so $f(x)^{-1}$ does not exist.

„ \Leftarrow “ Suppose $f \neq 0$ on $\sigma(x)$ by compactness. $\exists \Omega_1 \subset \Omega$ open: $\sigma(x) \subset \Omega_1$ and $f \neq 0$ on Ω_1 . Then $\frac{1}{f} \in H(\Omega_1)$ and by first item we have $e = (f \cdot \frac{1}{f})(x) = f(x)\frac{1}{f}(x) = \dots = \frac{1}{f}(x) \cdot f(x) \implies f(x)^{-1} = \frac{1}{f}(x)$. \square

Poznámka

$f = g$ on a neighbourhood of $\sigma(x) \implies f(x) = g(x)$ (from definition), other implication doesn't hold!

1.6 Multiplicative functionals

Definition 1.10 (Multiplicative functional)

Let A be a Banach algebra. We say $\varphi : A \rightarrow \mathbb{C}$ is multiplicative linear functional $\equiv \varphi$ preserves $+, \cdot, \cdot^*$.

$$\Delta(A) := \{\varphi : A \rightarrow \mathbb{C} \mid \varphi \text{ multiplicative linear functional, } \varphi \not\equiv 0\}.$$

Tvrzení 1.19

A Banach algebra, $\varphi \in \Delta(A) \cup \{0\}$. Then

- $\exists! \tilde{\varphi} \in \Delta(A_e) : \tilde{\varphi}((x, 0)) = \varphi(x), \forall x \in A$. It is given by

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda.$$

Moreover, $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$.

- $\forall x \in A : \varphi(x) \in \sigma(x)$ whenever $\varphi \neq 0$.
- $\Delta(A) \subseteq B_{A^*}$.
- *A has a unit, $\varphi \neq 0 \implies \|\varphi\| \geq \frac{1}{\|e\|}$. In particular if $\|e\| = 1$, then $\|\varphi\| = 1$.*

┌

Důkaz

„1. uniqueness“: For $\tilde{\varphi} \in \Delta(A_e)$ such that $\tilde{\varphi}((x, 0)) = \varphi(x), x \in A$:

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda \tilde{\varphi}((\mathbf{o}, 1)) = \varphi(x) + \lambda,$$

second equality by $\varphi \in \Delta(A) \implies \varphi(e) = \varphi(e^2) = \varphi^2(e)$. „1. existence“ is proven by check that defined $\tilde{\varphi}$ is multiplicative linear functional (and it is nonzero, but $\tilde{\varphi}((0, 1)) = 1 \neq 0$). This is easy (omitted).

„ $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$ “: „ \subseteq “: $\varphi \in LHS$, put $\varphi(x) := \psi((x, 0))$. Then $\varphi \in \Delta(A) \cup \{0\}$ and $\tilde{\varphi} = \psi$ became:

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda = \psi((x, 0)) + \lambda = \psi((x, \lambda)).$$

„ \supseteq “: We know already that $\tilde{\varphi} \in \Delta(A_e)$.

„2. with A has unit e “: $\varphi \neq 0, \varphi \in \Delta(A)$: If $\lambda \in \varrho(x)$, then $\varphi(\lambda e - x) \neq 0$ ($\varphi(x) \neq 0$ if x^{-1} exists). $0 \neq \varphi(\lambda e - x) = \lambda - \varphi(x) \implies \lambda \neq \varphi(x)$. Thus $\varphi(x) \notin \varrho(x)$, so $\varphi(x) \in \sigma(x)$. „2. with A hasn't unit“, then $\varphi(x) = \tilde{\varphi}((x, 0)) \in \sigma_{A_e}((x, 0)) = \sigma_A(x)$.

„3.“: $\varphi \in \Delta(A)$. Then $\forall x \in A : \varphi(x) \in \sigma(x) \subseteq B(\mathbf{o}, \|x\|)$, so $|\varphi(x)| \leq \|x\|$.

„4.“: A has a unit e , then $\|\varphi\| \geq \left| \varphi\left(\frac{e}{\|e\|}\right) \right| = \frac{1}{\|e\|}$. □

└

Věta 1.20

A Banach algebra, $M := \Delta(A) \cup \{0\}$. Then $M \subset (B_{A^}, w^*)$ is compact, $\Delta(A)$ is locally compact and if A has a unit, then $\Delta(A)$ is compact. The mapping $\Phi : M \rightarrow \Delta(A_e)$, $\Phi(\varphi) = \tilde{\varphi}$ is w^*-w^* homeomorphism.*

Důkaz

By previous proposition, $M \subset (B_{A^*}, w^*)$ ((B_{A^*}, w^*) is compact by previous semester). So, it suffices to check that M is w^* -closed.

$$M = \bigcap_{x, y \in A} \{\varphi \in A^* \mid \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}.$$

Sets from RHS is closed by previous semester, so, M is closed. Thus M is compact.

$\Delta \subset M$ is open, so $\Delta(A)$ is locally compact (and M is 1-point compactification of $\Delta(A)$). If Δ has a unit, then $\Delta(A) = \{\varphi \in M \mid \varphi(e) = 1\}$ is w^* -closed, so $\Delta(A)$ is compact (and 0 is isolated in M).

Finally, by previous proposition, Φ is bijection. Φ is w^* -continuous:

$$\varphi_i \xrightarrow{w^*} \varphi \implies \forall (x, \lambda) : \tilde{\varphi}_i((x, \lambda)) = \varphi_i(x) + \lambda \rightarrow \varphi(x) + \lambda = \tilde{\varphi}((x, \lambda)) \implies \tilde{\varphi}_i \xrightarrow{w^*} \tilde{\varphi}$$

So, Φ is homeomorphism (continuous bijection on compact, last semester?). \square

Například

$\Delta(\mathcal{C}(K)) = \{\delta_x \mid x \in K\}$. ($f \mapsto f(x)$ is multiplicative. Suppose $\varphi \in \Delta(\mathcal{C}(K))$, $\varphi \notin \{\delta_x \mid x \in K\}$. So for $x \in K$ there is $g_x \in \mathcal{C}(K) : \varphi(g_x) \neq g_x(x)$. Consider $f_x = g_x - \varphi(g_x)$. Then $\varphi(f_x) = 0$, $f_x(x) \neq 0$. So there is U_x open neighbourhood of x such that $f_x \neq 0$ on U_x . Compactness implies $\exists x_1, \dots, x_n \in K : K \subset \bigcup_{i=1}^n U_{x_i}$. Consider $h := \sum_{i=1}^n |f_{x_i}|^2$. Then $h > 0$ on K , so h^{-1} exists and therefore $\varphi(h) \neq 0$. But $\varphi(h) = \sum_{i=1}^n \varphi(f_{x_i}) \overline{\varphi(f_{x_i})} = 0$.)

$\Delta\{M_n\} = \emptyset$, $n \geq 2$, where M_n is (non-commutative) algebra of $n \times n$ matrices. ($M_n = \text{LO}\{E^{i,j}\}$. $E^{ij} \cdot E^{kl} = E^{il}$ if $j = k$, else 0. So $\varphi(E^{ij}) \cdot \varphi(E^{ij}) = \varphi(E^{ij} \cdot E^{ij}) = \varphi(E^{ij}) = 0$ if $i \neq j$. $\varphi(E^{ii}) = \varphi(E^{in} E^{ni}) = \varphi(E^{in}) \varphi(E^{ni}) = 0$. $\varphi(E^{nn}) = \varphi(E^{n1} E^{1n}) = 0$.)

Definition 1.11 (Ideal, maximal ideal)

A Banach algebra. Ideal in A is a subspace $I \subset A$ if $\forall x \in I \forall y \in A : x \cdot y \in I \wedge y \cdot x \in I$.

Maximal ideal \equiv proper ($I \neq A$) ideal and it is maximal proper ideal with respect to inclusion.

Například (2021, Johnson-Schetman, Acta mathematica)

$\mathcal{L}(L_p)$ has 2^{2^ω} non-isomorphic closed ideals.

Tvrzení 1.21

A Banach algebra with a unit. Then:

- Any proper ideal is contained in a maximum ideal. (From Zorn's lemma. And $I \subset A$ ideal is proper $\Leftrightarrow e \notin I$.)

- $I \subset A$ proper ideal $\implies \bar{I} \in A$ is proper ideal. In particular, maximal ideals are closed. (Easy: \bar{I} is ideal. Moreover, $I \cap A^* = \emptyset$ (if $x \in I$ was invertible thus $e = x \cdot x^{-1} \in I$, but $e \notin I$). So (A^* is open) $\bar{I} \cap A^* = \emptyset$ and therefore $e \notin \bar{I}$.)

Tvrzení 1.22

A Banach algebra, $I \subseteq A$ closed ideal $\implies A/I$ is Banach algebra ($[x] \cdot [y] := [x \cdot y]$).

Důkaz

Straightforward from definition. (Omitted.) □

Poznámka

From now on, A will be commutative.

Step 1: „Hahn-Banach“: $I \subset A$ closed ideal $\implies \exists \varphi \in \Delta(A) : \varphi|_I \equiv \dots$

Věta 1.23

A commutative Banach algebra with a unit. Then $\Phi : \Delta(A) \rightarrow \{\text{maximal ideals in } A\}$, $\Phi(\varphi) := \text{Ker } \varphi$, is bijection.

Důkaz

Pick $\varphi \in \Delta(A)$. Then „Ker φ is maximal ideal“: ideal: $y \in \text{Ker } \varphi, x \in A : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \dots \cdot 0 = 0$, proper: $\varphi \not\equiv 0$, maximal: $\text{codim Ker } \varphi = 1$: pick $x_0 : \varphi(x_0) \neq 0$, $a = a - \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} + \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} \in \text{Ker } \varphi \oplus \mathbb{R}$.

„ Φ is one-to-one“: Pick $\varphi, \psi \in \Delta(A) : \text{Ker } \varphi = \text{Ker } \psi$. Then (by lemma from previous semester) $\varphi = c \cdot \psi$ for some $c \in \mathbb{K}$. But $\varphi(e) = 1 = \psi(1)$ so $\varphi = \psi$.

„ Φ is surjective“: Let $I \subset A$ be maximal ideal (\implies closed). Step 1 „Any nonzero element in A/I is invertible“: For contradiction assume $\exists q(x) \in A/I$ ($q(x) = [x]$), $q(x) \neq 0 \wedge q(x)^{-1}$ does not exist. By next lemma $q(x)(A/I)$ is proper ideal. Then $q^{-1}(q(x)(A/I))$ is an ideal in A which is proper and $I \subsetneq q^{-1}(q(x)(A/I))$, which contradicts maximality of I . It follows from: ideal: follows from the fact that q is algebra homomorphism; proper: $q(e) = [e] \notin q(x)A/I$; $I \subseteq q^{-1}(\dots) : 0 \in q(x)A/I$; $I \neq q^{-1}(\dots) : q(x) \neq 0 \implies x \notin I$, but $q(x) = q(x)q(e) \in q(x)(A/I)$, so $x \in q^{-1}(\dots)$.

From Gelfand–Mazur theorem \exists surjective isomorphism $j : A/I \rightarrow \mathbb{C}$. Then $\varphi := j \circ q \in \Delta(A)$. It remains „ $I = \text{Ker } \varphi$ “: $x \in \text{Ker } \varphi \Leftrightarrow j(q(x)) = 0 \Leftrightarrow q(x) = 0 \Leftrightarrow x \in I$. □

Lemma 1.24

A commutative Banach algebra with a unit, $x \in A$ does not have inverse $\implies xA$ is proper ideal.

Důkaz

xA is ideal, because A is commutative. Then xA is proper ($e \notin xA$). \square

Důsledek (Hahn–Banach like theorem)

A is commutative Banach algebra with a unit, $I \subset A$ proper ideal. Then $\exists \varphi \in \Delta(A) : \varphi/I \equiv 0$.

Důkaz

Let $\tilde{I} \supseteq I$ be maximal ideal. By previous theorem there is $\varphi \in \Delta(A) : \tilde{I} = \text{Ker } \varphi$. \square

Tvrzení 1.25

A, B Banach algebras, $\Phi : A \rightarrow B$ algebraic isomorphism. Then $\Phi^\# : \Delta(B) \rightarrow \Delta(A)$ defined as $\Phi^\#(\varphi) := \varphi \circ \Phi$ is homeomorphism.

Důkaz

„ $\Phi^\#(\varphi) \in \Delta(A)$ “: $\Phi^\#(\varphi) = \varphi \circ \Phi \in \Delta(A) \cup \{0\}$ and $\varphi \not\equiv 0 \wedge \Phi$ is onto $\implies \varphi \circ \Phi \neq 0$.

„ $\Phi^\#$ is w^* - W^* continuous“: $\varphi_i \xrightarrow{w^*} \varphi \implies \varphi_i \circ \Phi \xrightarrow{w^*} \varphi \circ \Phi$.

Apply the proven part to Φ^{-1} , obtain that $(\Phi^{-1})^\# : \Delta(A) \rightarrow \Delta(B)$ is w^* - W^* continuous. Moreover we have $\Phi^\# \circ (\Phi^{-1})^\# = \text{id} \wedge (\Phi^{-1})^\# \circ \Phi^\#$. \square

Tvrzení 1.26

L locally compact T_2 . Then $\delta : L \rightarrow \Delta(C_0(L)), x \mapsto \delta_x$ is homeomorphism onto.

Důkaz

„Case 1: L is compact“: By example δ is onto. Of course, δ is one-to-one, continuous. So δ is homeomorphism.

„Case 2: L is not compact“: Then there is $K = L \cup \{\infty\}$, one-point compactification, and $\{f \in \mathcal{C}(K) | f(\infty) = 0\} \ni f \mapsto f|_L \in C_0(L)$ is isometric isomorphism. Moreover $\Phi : C_0(L)_e \rightarrow \mathcal{C}(K)$, $\Phi(f, \lambda) := f + \lambda$, is algebraic isomorphism.

So, we have $K \xrightarrow{\eta} \Delta(C(K)) \xrightarrow{\Phi^\#} \Delta(C_0(L)_e) \xrightarrow{\psi} \Delta(C_0(L)) \cup \{0\}$, where η is homeomorphism from case 1 and $\psi(\varphi) := \varphi|_{C_0(L)}$.

Thus $\delta := \psi \circ \Phi^\# \circ \eta$ is homeomorphism between $L \cup \{\infty\}$ and $\Delta(C_0(L)) \cup \{0\}$. Finally, for $x \in K$ and $f \in C_0(L)$:

$$\Phi^\# \circ \eta(x)(f) = (\eta(x) \circ \Phi)(f) = f(x),$$

so $\psi \circ \Phi^\# \circ \eta(x) = \Phi^\# \circ \eta(x)|_{C_0(L)} = \delta_x|_{C_0(L)}$. \square

Věta 1.27

K, L locally compact T_2 . Then following is equivalent

- $\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as Banach algebra;
- $\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as algebras;
- $K \approx L$ as topological spaces.

Důkaz

„1 \implies 2“ trivial. „2 \implies 3“: $K \approx \Delta(\mathcal{C}_0(K)) \approx \Delta(\mathcal{C}_0(L)) \approx L$ from previous two tvrzení.
„3 \implies 1“: Given $h : K \rightarrow L$ homeomorphism, $f \mapsto f \circ h$ is isometry between Banach algebras. \square

Definice 1.12 (Semi-simple Banach algebra)

A commutative Banach algebra. It is semi-simple $\equiv \Delta(A)$ separates points of A . ($\Leftrightarrow \bigcap \{\text{Ker } \varphi \mid \varphi \in \Delta(A)\} = \{\mathbf{o}\}$.)

Poznámka

Semi-simple \implies commutative. (Semi-simple and $x \cdot y \neq y \cdot x \implies \exists \varphi \in \Delta(A) : \varphi(x) \cdot \varphi(y) = \varphi(x \cdot y) \neq \varphi(y \cdot x) = \varphi(y) \cdot \varphi(x) \nmid$.)

Věta 1.28

A, B Banach algebras, B is semi-simple, then every (algebra) homomorphism $\Phi : A \rightarrow B$ is continuous.

Důkaz

Use Closed graph theorem. Pick $x_n \rightarrow x$, $\varphi(x_n) \rightarrow y$. Wanted „ $\Phi(x) = y$ “ ($\Leftrightarrow \forall \varphi \in \Delta(B) : \varphi(\Phi(x)) = \varphi(y)$). For $\varphi \in \Delta(B)$ we have $\varphi(y) = \lim_n \varphi(\Phi(x_n)) \stackrel{\varphi \circ \Phi \in \Delta(A) \subseteq A^*}{=} \varphi \circ \Phi(\lim_n x_n) = \varphi(\Phi(x))$. \square

Důsledek

$(A, \|\cdot\|)$ semi-simple Banach algebra and $(A, \|\cdot\|)$ is Banach algebra (with other norm), then $\|\cdot\|$ and $\|\cdot\|$ are equivalent.

Důkaz

We have that $\text{id} : (A, \|\cdot\|) \rightarrow (A, \|\cdot\|)$ is algebra homomorphism, so continuous by previous theorem. Similarly inverse is continuous (semi-simplicity doesn't depend on norm). So, id is isomorphism. \square

2 Gelfand transformation

Definice 2.1 (Gelfand transformation)

A Banach algebra. For $x \in A$ we define $\hat{x} : \Delta(A) \rightarrow \mathbb{C}$, $\hat{x}(\varphi) := \varphi(x)$. We say that \hat{x} is Gelfand transformation of x .

Poznámka

$\hat{x} \in \mathcal{C}_0(\Delta(A))$.

$A = \mathcal{C}_0(L) \implies \Delta(A) = \{\delta_x | x \in L\} \implies \forall f \in A : \hat{f}(\delta_x) = f(x), x \in L$. So, $\hat{f} = f$.

$A = L_1(\mathbb{R}^d) \implies \Delta(A) = \{e^{it \cdot x} | x \in \mathbb{R}^d\} \subseteq L_\infty(\mathbb{R}^d) = A^*$ and \hat{f} is Fourier transformation.

Věta 2.1

A commutative Banach algebra, $x \in A$. Then

- A has a unit $\implies \sigma(x) = \text{Rng } \hat{x}$;
- A doesn't have a unit $\implies \sigma(x) = \text{Rng } \hat{x} \cup \{0\}$;
- $\|\hat{x}\|_\infty = r(x) = \sup \{|\lambda| | \lambda \in \sigma(x)\}$.

┌

Důkaz

„a) \subseteq “: $\lambda \in \sigma(x) \Leftrightarrow (\lambda \cdot e - x)^{-1}$ does not exist \implies (Lemma above) $(\lambda e - x)A$ is proper ideal $\implies \exists \varphi \in \Delta(A) : \varphi|_{(\lambda e - x)A} \equiv 0 \implies \exists \varphi \in \Delta(A) : 0 = \varphi(\lambda e - x) = \lambda - \varphi(x) = \lambda - \hat{x}(\varphi) \implies \lambda \in \text{Rng } \hat{x}$.

„ \supseteq “ follows from Tvzení above, $\varphi(x) \in \sigma(x)$ for $\varphi \in \Delta(A)$.

„b)“ For $x \in A$:

$$\begin{aligned} \sigma(x) &= \sigma_{A_e}((x, 0)) \stackrel{\text{a)}}{=} \text{Rng } (\hat{x}, 0) = (\{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}) = \\ &= \{\varphi(x) | \varphi \in \Delta(A) \cup \{0\}\} = \text{Rng } \hat{x} \cup \{0\}. \end{aligned}$$

„c)“ $\|\hat{x}\|_\infty = \sup \{|\lambda| | \lambda \in \text{Rng } \hat{x}\} = \sup \{|\lambda| | \lambda \in \text{Rng } \hat{x} \cup \{0\}\} = \sup \{|\lambda| | \lambda \in \sigma(x)\} = r(x)$. □

Definice 2.2 (Gelfand transformation of algebra)

A Banach algebra, then $\Gamma : A \rightarrow \mathcal{C}_0(\Delta(A))$, $\Gamma(x) := \hat{x}$ is the Gelfand transformation of A .

Věta 2.2

A commutative Banach algebra, Γ Gelfand transformation. Then

- Γ is algebra transformation, continuous, $\|\Gamma\| \leq 1$;
- $\Gamma(A)$ separates the points of $\Delta(A)$;
- Γ is one-to-one $\Leftrightarrow A$ is semi-simple;
- Γ is an isomorphism into $\Leftrightarrow \exists K > 0 : \|x^2\| \geq K \cdot \|x\|^2, x \in A$; ($\Leftrightarrow \Gamma$ is one-to-one and $\Gamma(A)$ is closed;)
- Γ is an isometry into $\Leftrightarrow \|x^2\| = \|x\|^2, x \in A$.

┌

Důkaz

„a“: Γ is linear (obvious), Γ preserves multiplication (obvious). Finally, $\|\Gamma(x)\|_\infty = \|\hat{x}\|_\infty = r(x) \leq \|x\|$. So $\|\Gamma\| \leq 1$.

„b“: Let $\varphi \neq \psi \in \Delta(A)$ and $x \in A : \hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$.

„c“: $\Gamma(x) = 0 \Leftrightarrow \hat{x}(\varphi) = 0, \varphi \in \Delta(A) \Leftrightarrow \varphi(x) = 0, \varphi \in \Delta(A)$. So, Γ is one-to-one $\Leftrightarrow \forall x \neq 0 \exists \varphi \in \Delta(A) : \varphi(x) \neq 0 \Leftrightarrow A$ is semi-simple.

„d) second“: Γ is isomorphism into $\Leftrightarrow \Gamma$ is bijection between A and $\Gamma(A) \wedge \Gamma(A)$ is closed. ($\Gamma(A)$ is closed, then we use Open mapping theorem; if Γ is isomorphism, $\Gamma(A)$ is a Banach space.).

„d) + e), \Rightarrow “: Suppose $\exists c > 0 : \|\Gamma(x)\| \geq c \cdot \|x\|, x \in A$. Then $\forall x \in A : \|x^2\| \stackrel{a)}{\geq} \|\Gamma(x^2)\| = \|\Gamma(x)\|^2 \geq c^2 \cdot \|x\|^2$.

„d) + e), \Leftarrow “: Let d) hold with K (this holds in every algebra). Then (proven by induction)

$$\begin{aligned} \forall x \in A : \|x^{2^n}\| &\geq K^{2^{n-1}} \|x\|^{2^n}, \quad n \in \mathbb{N}. \\ \Rightarrow \sqrt[n]{\|x^{2^n}\|} &\geq K^{1-2^{-n}} \|x\|, \end{aligned}$$

where left side converges (by Beurling) to $r(x)$ and right side converges to $\|x\|$. So $r(x) \geq K \cdot \|x\|$ and from previous theorem $r(x) \geq \|\hat{x}\|_\infty = \|\Gamma(x)\|$. \square

└

2.1 C^* -algebras

Definice 2.3 (Involution)

A is a Banach algebra. Involution is a mapping $*$: $A \rightarrow A$ such that

$$\forall x, y \in A \quad \forall \lambda \in \mathbb{C} :$$

$$(x + y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda}x^*, \quad (xy)^* = y^* \cdot x^*, \quad (x^*)^* = x.$$

Definition 2.4 (C^* -algebra)

Banach algebra with involution $*$ is a C^* -algebra, if

$$\forall x \in A : \|x \cdot x^*\| = \|x\|^2, x \in A.$$

Definition 2.5 (Self-adjoint element, normal element)

For A with involution $*$ and $x \in A$ we say that x is self-adjoint $\equiv x = x^*$, and x is normal $\equiv x \cdot x^* = x^* \cdot x$.

Tvrzení 2.3 (Properties)

A Banach algebra with involution, $x \in A$. Then

- e is left/right unit $\implies e$ is unit and $e = e^*$. (e is left unit $\Leftrightarrow e^*$ is right unit. So there is unit.)*
- A is C^* -algebra $\Leftrightarrow \|x \cdot x^*\| \geq \|x\|^2, x \in A$. Then $\|x^*\| = \|x\|, x \in A$. („ \implies “: clear, „ \Leftarrow “: Then $\forall x \in A : \|x\|^2 \leq \|x \cdot x^*\| \leq \|x\| \cdot \|x^*\|$, so $\|x\| \leq \|x^*\|$, and applying to x^* we get $\|x^*\| \leq \|x\|$. But then we have $\|x \cdot x^*\| \leq \|x\| \cdot \|x^*\| = \|x\|^2$.)*
- Let A has a unit. then x^{-1} exists $\Leftrightarrow (x^*)^{-1}$ exists. Then $(x^*)^{-1} = (x^{-1})^*$. („ \implies “: $x^* \cdot (x^{-1})^* = (x^{-1}x)^* = e^* = e$, analogically $(x^{-1})^*x^* = e$. „ \Leftarrow “: Apply the proven part to x^* .)*
- $\lambda \in \sigma(x) \Leftrightarrow \bar{\lambda} \in \sigma(x^*)$. (A has a unit: $\lambda \notin \sigma(x) \Leftrightarrow \exists(\lambda e - x)^{-1} \Leftrightarrow \exists((\lambda e - x)^*)^{-1} \Leftrightarrow \bar{\lambda} \notin \sigma(x^*)$. If A has not a unit, then we use previous sentence and next theorem?)*
- $x + x^*, x^* \cdot x, x \cdot x^*, i \cdot (x - x^*)$ are self-adjoint. (Easy, omitted.)*
- $\exists! u, v \in A$ self-adjoint: $x = u + i \cdot v$. Then $x^* = u - i \cdot v$, and x is normal $\Leftrightarrow uv = vu$. („Existence“: $u := \frac{1}{2}(x + x^*), v := \frac{1}{2i}(x - x^*)$. Then $x = u + i \cdot v$. „Formulas“: $(u + i \cdot v)^* = u^* + i \cdot v^*$. „Uniqueness“: Pick $a, b \in A_{sa} : x = a + i \cdot b$. Then $a + i \cdot b = x = u + i \cdot v, a - i \cdot b = x^* = u - i \cdot v$. By subtracting or summing equation we get $a = u$ and $b = v$. „Normality“: x normal $\Leftrightarrow (u + i \cdot v)(u - i \cdot v) = (u - i \cdot v)(u + i \cdot v) \Leftrightarrow -i \cdot u \cdot v + i \cdot v \cdot u = i \cdot u \cdot v - i \cdot v \cdot u \Leftrightarrow u \cdot v = v \cdot u$.)*

Věta 2.4

A is C^ -algebra, $x \in A$ is normal. Then $r(x) = \|x\|$.*

┌

Důkaz„Step 1: $\|x^2\| = \|x\|^2$ “:

$$\|x\|^4 = \|x^*x\|^2 = \|(x^*x)^*(x^*x)\| = \|x^*xx^*x\| = \|x^*x^*xx\| = \|(xx)^*xx\| = \|xx\|^2 = \|x^2\|^2.$$

Thus inductively, we obtain $\|x^{2^k}\| = \|x\|^{2^k}$, $k \in \mathbb{N}$. Thus, Beurling gives $r(x) = \lim_k \sqrt[2^k]{\|x^{2^k}\|} = \|x\|$. □

Důsledek

A (Banach) algebra with involution. Then there is at most one norm $\|\cdot\|$ on A , such that $(A, \|\cdot\|)$ is C^* -algebra.

┌

Důkaz

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on A such that $(A, \|\cdot\|)$ is C^* -algebra, then by previous theorem

$$\forall x \in A : \|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \|x^*x\|_2 = \|x\|_2^2.$$

└

□

Věta 2.5

$(A, \|\cdot\|)$ Banach algebra.

- $(a, \lambda)^* = (a^*, \bar{\lambda})$, $(a, \lambda) \in A_e$ defines an involution on A_e . (Trivial.)
- If A is C^* -algebra, then on A_e there exists a norm $\|\cdot\|$ (equivalent to the norm from $A \oplus_1 \mathbb{K}$) such that $(A_e, \|\cdot\|)$ is C^* -algebra and $\|(a, 0)\| = \|a\|$, $a \in A$.

Věta 2.6

A is C^* -algebra, $x \in A$. Then

- $x = x^* \implies \sigma(x) \subseteq \mathbb{R}$;
- A has a unit and $x^* = x^{-1}$ (that is, x is unitary) $\implies \sigma(x) \subseteq \{\lambda \mid |\lambda| = 1\}$.

┌

*Důkaz*By previous theorem, WLOG A has a unit.„a)“: Let $\alpha + i\beta \in \sigma(x)$, $\alpha, \beta \in \mathbb{R}$. We want $\beta = 0$. Trick: $x_t := x + i \cdot t \cdot e$, $t \in \mathbb{R}$. Then

$$\alpha + i \cdot (\beta + t) \in \sigma(x_t) \iff (\alpha + i(\beta + t))e - x_t = (\alpha + i \cdot \beta)e - x,$$

$$\alpha^2 + (\beta + t)^2 = |\alpha + i(\beta + t)|^2 \leq \|x_t\|^2 = \|x_t^* x_t\| = \|(x - i \cdot t \cdot e) \cdot (x + i \cdot t \cdot e)\| = \|x^2 + (t \cdot e)^2\| \leq \|x^2\| + t^2.$$

So, $\alpha^2 + (\beta + t)^2 - t^2 \leq \|x^2\|$, $t \in \mathbb{R} \implies \beta = 0$ (Otherwise $LHS \rightarrow +\infty$ for $t \rightarrow \pm\infty$.)

„b)“: ($\|e\| = \|e^2\| = \|e\|^2$.) $1 = \|e\| = \|x^* x\| = \|x\|^2$, so $\|x\| = 1$. Then, for $\lambda \in \sigma(x)$, we have $|\lambda| \leq \|x\| = 1$. On the other hand $\frac{1}{\lambda} \in \sigma(x^{-1})$ (because if not, then $\frac{1}{\lambda}e - x^{-1}$ has inverse $\implies \lambda e - x = (\lambda e - x)x^{-1}x = (\lambda x^{-1} - e)x = -\lambda(\frac{1}{\lambda}e - x^{-1})x \implies \lambda e - x$ has inverse.) So

$$\left| \frac{1}{\lambda} \right| \leq \|x^{-1}\| = \|x^*\| = \|x\| = 1.$$

└

□

Definice 2.6

A, B are C^* -algebras, then $\Phi : A \rightarrow B$ is $*$ -homomorphism if Φ is homomorphism preserving $*$ (that is, $\Phi(x^*) = (\Phi(x))^*$).

*Důsledek*Let A be a C^* -algebra and $\Phi \in \Delta_A$. Then Φ is $*$ -homomorphism.

┌

Důkaz„If $x = x^*$ “, then $\Phi(x) \in \sigma(x) \subseteq \mathbb{R}$, so $\Phi(x^*) = \Phi(x) = \overline{\Phi(x)}$.

„In general“, if $x = u + i \cdot v$ ($u = u^*, v = v^*$), then $\Phi(x^*) = \Phi(u - i \cdot v) = \Phi(u) - i \cdot \Phi(v) = \overline{\Phi(u) + i \cdot \Phi(v)} = \overline{\Phi(x)}$. □

└

Tvrzení 2.7 (Automatical continuous)

Let A, B be C^* -algebras, $\Phi : A \rightarrow B$ is $*$ -homomorphism. Then Φ is continuous and $\|\Phi\| \leq 1$.

┌
Důkaz

$$\forall x \in A : \|\Phi(x)\|^2 = \|\Phi(x)^* \cdot \Phi(x)\| = r(\Phi(x)^* \cdot \Phi(x)) = r(\Phi(x^*x)) \stackrel{*}{=} r(x^*x) = \|x^*x\| = \|x\|^2.$$

Thus it suffices to show that (by following lemma)

$$\sigma(\Phi(x^*x)) \subseteq \sigma(x^*x) \cup \{0\}.$$

└

□

Lemma 2.8

Let A, B be Banach algebras, $\Phi : A \rightarrow B$ algebra homomorphism. Then $\forall x \in A : \sigma_B(\Phi(x)) \subseteq \sigma_A(x) \cup \{0\}$.

┌
Důkaz

Consider $\tilde{\Phi} : A_e \rightarrow B_e$ defined as $\tilde{\Phi}(a, \lambda) := (\Phi(a), \lambda)$. Then $\tilde{\Phi}$ is algebra homomorphism preserving unit. Moreover $\sigma_B(\Phi(x)) \subseteq \sigma_{B_e}((\Phi(x), 0)) \cup \{0\}$ and $\sigma_{A_e}((x, 0)) \subseteq \sigma_A(x) \cup \{0\}$. Thus, WLOG A, B have units and $\Phi(e_A) = e_B$.

But then for $\lambda \neq 0$ and $x \in A : \lambda e - x$ has inverse in A , then $\Phi(\lambda e - x) = \lambda \Phi(e) - \Phi(x)$ has inverse in B . So, $\lambda \notin \sigma_A(x) \cup \{0\} \implies \lambda \notin \sigma_B(\Phi(x))$.
└

□

Věta 2.9 (Gelfand–Naimark)

A commutative C^* -algebra. Then the Gelfand transformation $\Gamma : A \rightarrow \mathcal{C}_0(\Delta(A))$ is isometric *-isomorphism onto.

┌
Důkaz

By proposition above, Γ is algebra homomorphism, $\|\Gamma\| \leq 1$ and from theorem above $\|\Gamma(x)\|_\infty = r(x)$, $x \in A$. „ Γ is *-homomorphism“:

$$\forall a \in A \forall \varphi \in \Delta(A) : \Gamma(a^*)(\varphi) = \varphi(a^*) = \overline{(\varphi(a))} = \overline{\Gamma(a)(\varphi)}.$$

„ Γ is isometry“:

$$\forall x \in A : \|\Gamma(x)\|^2 = \|\overline{\Gamma(x)} \cdot \Gamma(x)\| = \|\Gamma(x^*x)\| = r(x^*x) = \|x^*x\| = \|x\|^2.$$

„ Γ is onto“: $\Gamma(A)$ is Banach space so $\Gamma(A) \subseteq \mathcal{C}_0(\Delta(A))$ is closed and *-subalgebra. And $\Gamma(A)$ separates points of $\Delta(A)$. So from Stone–Weierstrass theorem ($A \subset \mathcal{C}_0(K)$ is *-subalgebra separating the points, then $\overline{A}^{\|\cdot\|} = \mathcal{C}_0(K)$) $\Gamma(A) = \mathcal{C}_0(\Delta(A))$.
└

□

Důsledek

A, B commutative C^* -algebras. Then the following items are equivalent:

- A and B are isometrically *-isomorphic;

- A and B are algebraically isomorphic;
- $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

Důkaz

„2. \Leftrightarrow 3.“ follows from theorem above (where it is proved for $\mathcal{C}_0(K)$ -spaces). „1. \Rightarrow 2.“: trivial.

„3. \Rightarrow 1.“: easy for $\mathcal{C}_0(K)$ -spaces, because if $h : K \rightarrow L$ is homeomorphism, then $f \mapsto f \circ h$ is isometrical $*$ -isomorphism. \square

Definice 2.7

A Banach algebra, $M \subset A$. Then $\text{alg}(M) = \bigcap \{B \supseteq M \mid B \text{ is subalgebra of } A\}$.

Poznámka (Easy)

$$= \left\{ \sum_{i=1}^n \alpha_i \prod_{j=1}^m x_{ij} \mid n, m \in \mathbb{N}, \alpha_i \in \mathbb{C}, x_{ij} \in M \right\}.$$

Moreover $\overline{\text{alg}}M = \bigcap \{B \supseteq M \mid B \text{ is closed subalgebra of } A\}$.

Poznámka (Easy)

$$= \overline{\text{alg}}M^{\|\cdot\|}.$$

Tvrzení 2.10 (Fact)

A is C^* -algebra, $M \subset A$ is commutative and closed under $*$, then $\overline{\text{alg}}M$ is commutative C^* -subalgebra of A .

Věta 2.11

A, B are C^* -algebras, $h : A \rightarrow B$ is $*$ -homomorphism, one-to-one. Then h is isometry.

┌ *Důkaz*

WLOG A and B have units and $h(e)$ is a unit ($(a, \lambda) \mapsto (h(a), \lambda)$ is one-to-one $*$ -homomorphism). Suffices: $\forall x \in A$ self-adjoint: $\|x\| = \|h(x)\|$ ($\forall y \in A : \|h(y)\|^2 = \|h(y^*y)\| = \|y^*y\| = \|y\|^2$). Let $x \in A$ be self-adjoint. Put $A_0 := \overline{\text{alg}}\{e, x\} = \overline{\text{LO}}\{e, x, x^2, x^3, \dots\}$ is commutative and C^* -subalgebra.

$$B_y := \overline{\text{alg}}\{e, h(x)\} = \overline{\text{LO}}\{e, h(x), h(x^2), \dots\}$$

is commutative and C^* -subalgebra. So, we have $A_0 \xrightarrow{h} B_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(B_0))$, $A_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(A_0))$. So there is $\tilde{h} : \mathcal{C}(\Delta(A_0)) \rightarrow \mathcal{C}(\Delta(B_0))$ one-to-one $*$ -homeomorphism, $\tilde{h}(1) = 1$. So, it suffices to prove the following lemma. \square

Lemma 2.12

Let K, L be T_2 compact spaces, $\varphi : \mathcal{C}(K) \rightarrow \mathcal{C}(L)$ $*$ -homomorphism, $\varphi(1) = 1$. Then $\exists \alpha : L \rightarrow K$ continuous mapping such that $\varphi(f) := f \circ \alpha$, $f \in \mathcal{C}(K)$.

Moreover, if φ is one-to-one, then α is onto and so φ is isometry.

┌ *Důkaz*

By proposition above $\|\varphi\| \leq 1$ and φ is continuous. Consider $\varphi^* : \mathcal{M}(L) \rightarrow \mathcal{M}(K)$. Then „ $\varphi^*(\Delta(\mathcal{C}(L))) \subseteq \Delta(\mathcal{C}(K))$ “:

$$\forall h \in \Delta(\mathcal{C}(L)) \forall f, g : \varphi^*h(fg) = h(\varphi(fg)) = h(\varphi(f))h(\varphi(g)) = \varphi^*h(f)\varphi^*h(g).$$

So, we have: $L \xrightarrow{\delta} \Delta(\mathcal{C}(L)) \xrightarrow{\varphi^*} \Delta(\mathcal{C}(K)) \xrightarrow{\delta^{-1}} K$. So, $\alpha(x) := \delta^{-1}(\varphi^*(\delta(x)))$, $x \in L$ is continuous from L to K .

For this α we have:

$$\forall x \in L \forall f \in \mathcal{C}(K) : \varphi(f)(x) = \delta_x(\varphi(f)) = (\varphi^* \circ \delta_x)(f) = f(\delta^{-1}\varphi^*\delta_x) = f(\alpha(x)).$$

Moreover, „if φ is one-to-one, then α is onto“: Suppose $\alpha(L) \subsetneq K \implies \exists f \in \mathcal{C}(K) \setminus \{0\} : f|_{\alpha(L)} \equiv 0$. But then $\varphi(f) \equiv 0$, but $f \neq 0$. \nexists (φ should be one-to-one.) Thus φ is isometry. \square

Poznámka (GNS construction)

A is C^* -algebra $\implies \exists H$ Hilbert $\exists \varphi : A \rightarrow B(H)$ $*$ -isomorphism into.

┌ *Důkaz* (Sketch)

$f \geq 0$ ($\sigma(f) \geq 0$) on $A|_{\{a|f(a*a)=0\}}$ constructs inner product $\langle [x], [y] \rangle := f(y^*x)$. Put $H := \overline{A|_{\{a|f(a*a)=0\}}}$. Then $\varphi(a)([x]) = [ax]$. \square

3 Continuous calculus for formal elements of C^* -algebras

Poznámka

Idea: $\varphi(\sigma(x)) \ni f \mapsto f(x) \in A$.

For $A = C(K)$:

$$g \in \mathcal{C}(K), \varphi(\sigma(x)) \ni f \implies g \circ f \in C(K).$$

Let A be C^* -algebra with a unit, $x \in A$ normal. Consider

$$B = \overline{\text{alg}}\{e, x, x^*\} \in A \implies \Gamma_B : B \rightarrow \mathcal{C}(\Delta(B)) \wedge f(x) := \Gamma_B^{-1}(f \circ \Gamma_B(x)), f \in \mathcal{C}(\sigma_A(x)).$$

Problem is when $\Gamma_B(x) \subseteq \sigma_A(x)$.

Lemma 3.1

A is C^* -algebra, $B \subset A$ is C^* -algebra. Then

- If A and B have the same unit $\implies \forall x \in B : \exists x^{-1} \in B \Leftrightarrow \exists x^{-1} \in A$;
- $\forall x \in B : B$ has a unit, which is not a unit in $A \implies \sigma_A(x) = \sigma_B(x) \cup \{0\}$, otherwise $\sigma_A(x) = \sigma_B(x)$;
- (In any case $\sigma_B(x) = \sigma_A(x)$).

┌

Důkaz

„1.“: Pick $x \in B$. „ \implies “: easy. „ \Leftarrow “: If x^{-1} exists in A , then $(x^*x)^{-1}$ exists in A . So $0 \notin \sigma_A(x^*x) = \sigma_B(x^*x) \implies (x^*x)^{-1}$ exists in B . $x^{-1} = x^{-1}(x^*)^{-1}x^* = x^{-1}(x^*x)^{-1}x^*$.

„2.“: If A and B have the same unit, we have $\sigma_A(x) = \sigma_B(x)$. WLOG A has a unit $e \notin B$ (Because $B \in A_e$ and $\sigma_{A_e}(x) = \sigma_A(x)$ if A has not unit). Then $\sigma_A(x) = \sigma_{B+\text{LO}(e)}(x) \stackrel{*}{=} \sigma_{B_e}((x, 0)) = \sigma_B(x)$ if B has no unit and $\sigma_B(x) \cup \{0\}$ if B has a unit.

*: $\varphi : B + \text{LO}(e) \rightarrow B_e, b + \lambda e \mapsto (b, \lambda)$ is algebra homomorphism. □

└

Věta 3.2

Let A be a C^* -algebra with a unit, $x \in A$ normal, $f \in \mathcal{C}(\sigma(x))$. Then the mapping

$$\Phi : \mathcal{C}(\sigma(x)) \rightarrow A, \quad \Phi(g) := g(x) := \Gamma_B^{-1}(g \circ \Gamma_B(x))$$

has the following properties:

1. Φ is isometric $*$ -isomorphism onto $B = \overline{\text{alg}}\{e, x, x^*\}$, $\Phi(1) = e$ and $\Phi(\text{id}) = x$.

2. If $\psi : \mathcal{C}(\sigma(x)) \rightarrow A$ is $*$ -homomorphism, $\psi(1) = e$, $\psi(\text{id}) = x$, then $\psi = \Phi$.
3. If $g \in \mathcal{H}(\Omega)$, where $\Omega \subset \mathbb{C}$ open, $\sigma(x) \subset \Omega$, then $\Phi(g|_{\sigma(x)}) = \psi(g)$, where ψ is from holomorphic calculus.
4. $f(x)^{-1}$ exists in $A \Leftrightarrow f \neq 0$ on $\sigma(x)$. In this case $f(x)^{-1} = \left(\frac{1}{f}\right)(x)$.
5. $\sigma(f(x)) = f(\sigma(x))$.
6. $\forall g \in \mathcal{C}(f(\sigma(x))) : (g \circ f)(x) = g(f(x))$.
7. $\forall y \in A : yx = xy : yf(x) = f(x)y$.

┌

Důkaz

„1.“: Recall theorem above $\Gamma_B(x) : \Delta(B) \rightarrow \mathcal{C}$ continuous and onto $\sigma_B(x)$. And it is „one-to-one“:

$$\forall \varphi_1, \varphi_2 \in \Delta(B) : \varphi_1(x) = \varphi_2(x) \implies \varphi_1 = \varphi_2 \text{ on } B.$$

So $\Gamma_B(x) : \Delta(B) \rightarrow \sigma(x)$ is homeomorphism, then $\mathcal{C}(\sigma(x)) \ni g \mapsto g \circ \Gamma_B(x) \in \mathcal{C}(\Delta(A))$ is isometric $*$ -isomorphism onto. Thus $\mathcal{C}(\sigma(x)) \ni g \mapsto \Gamma_B^{-1}(g \circ \Gamma_B(x)) \in B$ is isometric $*$ -isomorphism onto.

Moreover, $\Phi(1) = \Gamma_B^{-1}(1) = e$ ($\forall \varphi \in \Delta(B) : \varphi(e) = 1$). $\Phi(\text{id}) = \Gamma_B^{-1}(\Gamma_B(x)) = x$.

„2.“: By theorem above, ψ is continuous (because it is $*$ -isomorphism), moreover $\psi = \Phi$ on complex polynomials. Since complex polynomials are dense in $\mathcal{C}(\sigma(x))$ by (S-W), by continuity $\Phi = \psi$ everywhere.

„3.“: Omitted (on polynomials, on inverse, on rationals, rationals are dense in \mathcal{H}).

„4.“: Since $f(x) \in B$, we have $f(x)^{-1}$ exists in $B \Leftrightarrow f(x)^{-1}$ exists in $A \stackrel{\Phi \text{ is } ?}{\Leftrightarrow} f^{-1}$ exists in $\mathcal{C}(\sigma(x)) \Leftrightarrow f \neq 0$ on $\sigma(x)$. And if $f \neq 0$ on $\sigma(x)$, then $f(x)^{-1} = \Phi(f^{-1}) = \Phi\left(\frac{1}{f}\right) = \left(\frac{1}{f}\right)(x)$.

„5.“: $f(x) \in B$, so $\sigma_A(f(x)) \stackrel{\text{Lemma}}{=} \sigma_B(f(x)) = \sigma_B(\Phi(f)) \stackrel{\Phi \text{ is isomorphism}}{=} \sigma_{\mathcal{C}(\sigma(x))} = \text{Rng } f = f(\sigma(x))$.

„6.“: Omitted.

„7.“: Next time. □

└

TODO!!!

4 Borel measurable calculus

Lemma 4.1 (Lax–Milgram)

H Hilbert, $S : H \times H \rightarrow \mathbb{C}$ sesquilinear, $\|S\| := \sup_{x,y \in S_H} |S(x,y)| < \infty$. Then $\exists! T \in \mathcal{L}(H) : \|T\| = \|S\| \wedge \langle Tx, y \rangle = S(x, y)$.

┌

Důkaz

Fix $y \in H$. Then $H \ni x \mapsto S(x, y)$ is a point in $H^* \implies \exists! U(y) \in H : S(x, y) = \langle x, U(y) \rangle$, $x \in H$. Then $U \in \mathcal{L}(H)$, $\|U\| = \|S\|$.

„Linearity“: Easy:

$$\forall y, z \in H, \alpha \in \mathbb{K} \implies \forall x \in H : \langle x, U(\alpha y + z) \rangle = S(x, \alpha y + z) = \bar{\alpha} S(x, y) + S(x, z) = \bar{\alpha} \langle x, U y \rangle + \langle x, U z \rangle.$$

„ $\|U\| \leq \|S\|$ “:

$$\forall y \in H : \|U y\|^2 = \langle U y, U y \rangle = S(U y, y) \leq \|S\| \cdot \|U y\| \cdot \|y\| \implies \|U y\| \leq \|S\| \cdot \|y\|.$$

„ $\|U\| \geq \|S\|$ “:

$$\forall x, y \in S_H : |S(x, y)| = |\langle x, U y \rangle| \leq \|x\| \cdot \|U\| \cdot \|y\| = \|U\|.$$

„Uniqueness“: Bounded operator is given by values of $\langle T x, y \rangle$.

└

□

Definice 4.1

H Hilbert, $T \in \mathcal{L}(H)$ normal, $\Phi : \mathcal{C}(\sigma(T)) \rightarrow \mathcal{L}(H)$ continuous ?.

- $\forall x, y \in H : \mu_{x,y} \in M(\sigma(T))$ is the unique measure satisfying

$$\int_{\sigma(T)} f d\mu_{x,y} = \langle \Phi(f)x, y \rangle, \quad f \in \mathcal{C}(\sigma(T)).$$

- $\forall f \in \text{Bor}_0(\sigma(T))$ (bounded, Borel) we get $\Phi(f) \in \mathcal{L}(H)$ be the unique operator such that

$$\langle \Phi(f)x, y \rangle = \int_{\sigma(T)} f d\mu_{x,y}, \quad x, y \in H$$

„Důkaz

„1.“: $f \mapsto \langle \Phi(f)x, y \rangle$ is linear and $|\langle \Phi(f)x, y \rangle| \leq \|\Phi(f)\| \cdot \|x\| \cdot \|y\|$. So $f \mapsto \langle \Phi(f)x, y \rangle \in \mathcal{C}(\sigma(T))^* = M(\sigma(T)) \implies \mu$ exists by Riesz representation theorem.

„2.“:

$$\forall x, x_2, y \in H \forall \alpha \in \mathbb{K} \forall f \in \mathcal{C}(\sigma(T)) : \langle \Phi(f)(\alpha x_1 + x_2), y \rangle = \alpha \langle \Phi(f)x_1, y \rangle + \langle \Phi(f)x_2, y \rangle = \alpha \mu_{x_1, y}(f) + \mu_{x_2, y}(f)$$

Thus $\cdot \mapsto \mu_{\cdot, y}$ is linear (for each y). Analogously $\cdot \mapsto \mu_{x, \cdot}$ is conjugate-linear.

Thus, $(x, y) \mapsto \mu_{x, y}(f) \in \mathbb{C}$ is sesquilinear form.

$$\forall x, y \in S_H : |\mu_{x, y}(f)| \leq \int |f| d|\mu_{x, y}| \leq \|f\|_\infty \cdot \|x\| \cdot \|y\| = \|f\|_\infty.$$

And from Lax–Milgram:

$$\exists! \Phi(f) \in \mathcal{L}(H) : \langle \Phi(f)x, y \rangle = \mu_{x, y}.$$

Moreover $\|\Phi(f)\| \leq \|f\|_\infty$.

□

Poznámka

H Hilbert, $T \in \mathcal{L}(H)$ normal:

- Mapping $H \times H \ni (x, y) \mapsto \mu_{x, y}$ is sesquilinear, so

$$\mu_{x, y} = \frac{1}{4} (\mu_{x+y, x+y} - \mu_{x-y, x-y} + i\mu_{x+iy, x+iy} - i\mu_{x-iy, x-iy}).$$

- $\forall x \in H : \mu_{x, x} \geq 0$. (Proof: „ $f \geq 0 \implies \mu_{x, x}(f) \geq 0, f \in \mathcal{C}(\sigma(T))$ “: $f \geq 0 \implies \Phi(f) \geq 0$ ($\sigma(\Phi(f)) = f(\sigma(T)) \subseteq [0, \infty) \implies \Phi(f) \geq 0$.) So $\int_{\sigma(T)} f d\mu_{x, x} = \Phi(f)x, x \geq 0$.)
- $Bor_b(\sigma(T)) \subseteq l_\infty(\sigma(T)) \mapsto \mathcal{L}(H)$ is C^* -subalgebra.
- The mapping $\Phi : Bor_b(\sigma(T)) \rightarrow \mathcal{L}(H)$ from previous definition, is extension of continuous calculus from theorem above.

Věta 4.2

Let P be a metric space, Φ be the smallest system of functions such that $\mathcal{C}_b(P) \subset \Phi$ and Φ is closed under point-wise bounded convergent sequences. Then $\Phi = Bor_b(P)$.

┌ *Důkaz* (Sketch)

Suffices: „ $\forall A \subset P$ Borel: $\chi_A \in \Phi$.“

$$\mathcal{F} := \{A \subset P \text{ Borel} \mid \chi_A \in \Phi\}$$

└ is σ -algebra containing closed sets $\implies \mathcal{F} = \text{Bor}(P)$. □

Definice 4.2

Let X, Y be normed linear spaces. On $\mathcal{L}(X, Y)$ we define the following two Hausdorff locally convex topologies:

- τ_{SOT} generated by pseudonorms $\{P_x(T) = \|T_x\| \mid x \in X\}$ (so, $T_i \xrightarrow{SOT} T \Leftrightarrow \forall x \in X : T_i x \xrightarrow{\|\cdot\|} T x$);
- τ_{WOT} generated by pseudonorms $\{P_{x,y^*}(T) = y^*(T x) \mid x \in X \wedge y^* \in Y^*\}$ (so, $T_i \xrightarrow{WOT} T \Leftrightarrow \forall x \in X : T_i x \xrightarrow{w} T x$) (in $X = Y = H$ Hilbert: $\Leftrightarrow \forall x, y \in H : \langle T_i x, y \rangle \rightarrow \langle T x, y \rangle$).

Poznámka

$$T_i \xrightarrow{\|\cdot\|} T \implies T_i \xrightarrow{SOT} T \implies T_i \xrightarrow{WOT} T.$$

┌ *Například*

$R_n x := (0, 0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$, $x \in l_2$. Then $R_n \in \mathcal{L}(l_2)$, $n \in \mathbb{N}$, and $R_n \xrightarrow{\|\cdot\|} 0$, because $\|R_n(e_{n+1})\| = 1$, $n \in \mathbb{N}$. But $R_n \xrightarrow{SOT} 0$, because $\forall x \in l_2 : \|R_n x\|_2^2 = \sum_{i=n+1}^{\infty} |x_i|^2 \rightarrow 0$.

$S_n x := (0, 0, \dots, 0, x_1, x_2, \dots)$, $x \in l_2$. Then $S_n \in \mathcal{L}(l_2)$ is isometry $\forall n \in \mathbb{N}$. But $S_n \xrightarrow{SOT} 0$, because $\|S_n(e_1)\| = 1 \rightarrow 0$. But $S_n \xrightarrow{WOT} 0$, because $\forall x, y \in l_2$:

$$|\langle S_n x, y \rangle| = \left| \sum_{i=1}^{\infty} x_i y_{n+i} \right| \leq \|x\|_2 \sqrt{\sum_{i=n+1}^{\infty} |y_i|^2} \rightarrow 0.$$

Věta 4.3

H Hilbert, $T \in \mathcal{L}(H)$ normal, $f \in \text{Bor}_b(\sigma(T))$, $\Phi : \text{Bor}_b(\sigma(T)) \rightarrow \mathcal{L}(H)$ as in definition above. Then

Φ is continuous $*$ -homomorphism and $\|\Phi\| = 1$;

┌
Důkaz

Φ is linear (easy from definition). $\|\Phi\| \leq 1$ follows from the second point of the previous theorem, and $\|\Phi(1)\| = \|\text{id}\| = 1$, so $\|\Phi\| = 1$.

„ Φ is multiplicative“: Step 1: „ $\mathcal{F} := \{g \in \text{Bor}_b(\sigma(T)) \mid \forall f \in \mathcal{C}(\sigma(t)) : \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$ “, then $\mathcal{F} = \text{Bor}_b(\sigma(T))$ “: $\mathcal{F} \subseteq \mathcal{C}(\sigma(T))$ follows from continuous calculus, „ \mathcal{F} closed under point-wise limits of bounded sequences“: Let $\mathcal{F} \ni g_n \rightarrow g$ and $f \in \mathcal{C}(\sigma(T))$. Then $g_n f \rightarrow gf$ point-wise. So, for $x, y \in H$:

$$\langle \Phi(g, f)x, y \rangle = \int_{\sigma(T)} gf d\mu_{x,y} = \lim \int g_n f d\mu_{x,y} = \lim \langle \Phi(g_n)x, y \rangle = \lim \langle \Phi(g_n)(\Phi(f)x), y \rangle = \lim \int g_n d\mu_{x, \Phi(f)x}$$

$$\implies \mathcal{F} = \text{Bor}_b(\sigma(T)).$$

Step 2: „ $\mathcal{H} := \{g \in \text{Bor}_b(\sigma(T)) \mid \forall f \in \text{Bor}_b(\sigma(t)) : \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$ “, then $\mathcal{H} = \text{Bor}_b(\sigma(T))$ “: „ \mathcal{H} is closed under point-wise limits of bounded sequences“: $\mathcal{H} \ni f_n \xrightarrow{\tau_p} f$, f_n bounded, then $\forall x, y \in H \forall g \in \text{Bor}_b(\sigma(T))$:

$$\langle \Phi(gf)x, y \rangle \stackrel{\text{Lebesgue}}{=} \lim_n \langle \varphi(gf_n)x, y \rangle = \lim_n \langle \Phi(g)\Phi(f_n)x, y \rangle = \lim_n \langle \Phi(f_n)x, \Phi(g)^*y \rangle = \lim_n \int f_n d\mu_{x, \Phi(g)^*y}$$

Thus $\Phi(gf) = \Phi(g)\Phi(f)$.

„ Φ preserves $*$ “: $\mathcal{F} := \{g \in \text{Bor}_b(\sigma(T)) \mid \Phi(g)^* = \Phi(\bar{g})\}$. Then $\mathcal{F} \subseteq \mathcal{C}(\sigma)$ by continuous calculus and \mathcal{F} is ”closed under taking limits” analogously as above. $\implies \mathcal{F} = \text{Bor}_b(\sigma(T))$. \square

└

$(f_n) \in \text{Bor}_b(\sigma(T))^{\mathbb{N}}$ bounded and $f_n \xrightarrow{\tau_p} f$, then $\Phi(f_n) \xrightarrow{SOT} \Phi(f)$.

┌
Důkaz

Step 1: „ $\Phi(f_n) \xrightarrow{\text{WOT}} \Phi(f)$ “:

$$\forall x, y \in H : \langle \Phi(f_n)x, y \rangle \xrightarrow{\text{Lebesgue}} \langle \Phi(f)x, y \rangle.$$

Step 2: „ $\|\Phi(f_n)x\| \rightarrow \|\Phi(f)x\|, x \in H$ “:

$$\|\Phi(f_n)x\|^2 = \langle \Phi(\bar{f}_n)\Phi(f_n)x, x \rangle = \langle \Phi(\bar{f}_n f_n)x, x \rangle \xrightarrow{\text{Lebesgue}} \langle \Phi(\bar{f}f)x, x \rangle = \|\Phi(f)x\|^2.$$

Step 3: From steps 1 and 2:

$$\|\Phi(f_n)x - \Phi(f)x\|^2 \stackrel{\text{Cos. v\u011bta}}{=} \|\Phi(f_n)x\|^2 + \|\Phi(f)x\|^2 - 2\Re \langle \Phi(f_n)x, \Phi(f)x \rangle \rightarrow 0.$$

└

If $K \subset \mathbb{C}$ is compact, $K \ni \sigma(T)$, $\psi : \text{Bor}_b(K) \rightarrow \mathcal{L}(H)$ is continuous $*$ -homomorphism, $\psi(1) = \text{id}$, $\psi(\text{id}) = T$ and $f_n \xrightarrow{\tau_p} f \implies \psi(f_n) \xrightarrow{\text{WOT}} \psi(f)$. Then $\psi(g) = \Phi(g|_{\sigma(T)})$,

$g \in \text{Bor}_b(K)$.

Důkaz

Skipped. Using characterization of Bor_b . □

$\Phi(f)$ is normal, $\Phi(f)$ is self-adjoint $\Leftrightarrow f$ is real.

Důkaz

Skipped. Easy from first part of theorem. □

$$\sigma(\Phi(f)) \subseteq \overline{f(\sigma(T))}.$$

$$g \in \text{Bor}_b(\overline{\text{Rng } f}) \implies (g \circ f)(T) = g(f(T)).$$

$$\forall S \in \mathcal{L}(H), ST = TS : Sf(T) = f(T)S.$$

5 Spectral decomposition of normal operator

Definice 5.1 (Spectral measure)

H Hilbert space, (X, \mathcal{A}) measurable space. Then $E : \mathcal{A} \mapsto \mathcal{L}(H)$ is spectral measure for (X, \mathcal{A}, H) if

- $\forall A \in \mathcal{A} : E(A)$ is orthogonal projection;
- $E(X) = \text{id}, E(\emptyset) = 0$;
- if $\{A_n, n \in \mathbb{N}\} \subset \mathcal{A}$ is point-wise disjoint, then

$$E\left(\bigcup_{n=1}^{\infty} A_n\right)x = \sum_{n=1}^{\infty} E(A_n)x, x \in H.$$

Tvrzení 5.1 (Properties of spectral measure)

H Hilbert, (X, \mathcal{A}) measurable space, E is spectral measure for (X, \mathcal{A}, H) . Then

1. $\forall A, B \in \mathcal{A}, A \subset B : E(A) \leq E(B)$ (that's $E(B) - E(A) \geq 0$);
2. $\forall A, B \in \mathcal{A} : E(A \cap B) = E(A) \cdot E(B)$, in particular, if $A \cap B = \emptyset$, then $E(A) \cdot E(B) = \emptyset$.
3. $\forall x, y \in H : \mathcal{A} \ni A \mapsto \langle E(A)x, y \rangle$ is a complex measure (denoted by $E_{x,y}$), with total variation $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$.
4. $(x, y) \mapsto E_{x,y}$ is sesquilinear mapping.

5. $\forall x, y \in H \ \forall A \in \mathcal{A}$:

$$|E_{x,y}(A)| \leq \frac{1}{2} (E_{x,x}(A) + E_{y,y}(A)) .$$

6. $\forall x, y \in H$:

$$E_{x+y,x+y} \leq 2 (E_{x,x} + E_{y,y}) .$$

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Důkaz

„1.“: $E(A) + E(B \setminus A) = E(B)$, so $E(B) - E(A) = E(B \setminus A) \geq 0$.

„2.“: „Step 1: $A \cap B = \emptyset$ “:

$$\text{id} = E(X) = E(A) + E(A^c) \geq E(A) + E(B),$$

so $E(B) \leq \text{id} - E(A)$, which is orthogonal projection onto $(\text{Rng } E(A))^\perp$. Thus („ $P, Q \in \mathcal{L}(A)$ orthogonal projections, $Q - P \geq 0$, then $\text{Rng } P \subset (\text{Rng } Q)^\perp$ “:

$$\|Px\|^2 = \|QPx\|^2 + \|(\text{id} - Q)Px\|^2 = \langle QPx, Px \rangle + \|(\text{id} - Q)Px\|^2 \geq \underbrace{\langle PPx, Px \rangle}_{\|Px\|^2} + \|(\text{id} - Q)Px\|^2,$$

thus, $(\text{id} - Q)Px = 0$, so $\text{Rng } P \subseteq \text{Ker}(\text{id} - Q) = \text{Rng } Q$. $\text{Rng } E(B) \subseteq (\text{Rng } E(A))^\perp$. Thus $\text{Rng } E(A) \perp \text{Rng } E(B)$, so $E(A) \cdot E(B) = 0$.

„Step 2: In general“:

$$E(A) = E(A \cap B) + E(A \setminus B), \quad E(B) = E(A \cap B) + E(B \setminus A) \implies$$

$$\implies E(A) \cdot E(B) = (E(A \setminus B) + E(A \cap B)) \cdot (E(A \cap B) + E(B \setminus A)) = E^2(A \cap B) + 0 + 0 + 0 = E(A \cap B).$$

„3.“: „ $E_{x,y}$ is countably additive“ is easy. By this it is a complex measure. „Calculation of $\|E_{x,y}\|$ “: Fix $A_1, \dots, A_n \in \mathcal{A}$ disjoint such that $\bigcup_{i=1}^n A_i = X$. For $i \in [n]$ pick $\alpha_i \in S_{\mathbb{C}}$: $\alpha_i \langle E(A_i)x, y \rangle = |\langle E(A_i)x, y \rangle|$. Then

$$\sum_{i=1}^n |E_{x,y}(A_i)| = \sum_{i=1}^n \alpha_i \langle E(A_i)x, y \rangle \stackrel{\text{Cauchy-Schwartz}}{\leq} \left\| \sum_{i=1}^n \alpha_i E(A_i)x \right\| \cdot \|y\|.$$

$$\left\| \sum_{i=1}^n E(A_i)(\alpha_i x) \right\|^2 \stackrel{\text{Pythagoras}}{=} \sum_{i=1}^n \|E(A_i)(\alpha_i x)\|^2 = \sum_{i=1}^n \|E(A_i)(x)\|^2 = \sum_{i=1}^n \langle E(A_i)x, x \rangle = \left\langle E\left(\bigcup_{i=1}^n A_i\right)x, x \right\rangle = \langle x, x \rangle$$

„4.“: Easy, using definition. „5.“:

$$|E_{x,y}(A)| = |\langle E(A)x, y \rangle| = |\langle E(A)x, E(A)y \rangle| \stackrel{\text{Cauchy-Schwartz}}{\leq} \|E(A)x\| \cdot \|E(A)y\| = \sqrt{E_{x,x}(A)} \cdot \sqrt{E_{y,y}(A)}$$

„6.“:

$$E_{x+y, x+y}(A) = E_{x,x}(A) + E_{y,x}(A) + E_{x,y}(A) + E_{y,y}(A) \leq E_{x,x}(A) + 2\Re E_{y,x}(A) + E_{y,y}(A) \leq E_{x,x}(A) + 2 \cdot \frac{1}{2} (E_{x,x}(A) + E_{y,y}(A))$$

└

□

Poznámka

From 4. we get $E_{x,y}(A) = \frac{1}{4} \sum_{k=0}^3 i^k \langle E(A)(x + i^k y), x + i^k y \rangle$. Thus 3. is equivalent to $\forall x \in H : E_{x,x} \geq 0$ is measure.

Definice 5.2 (Integral)

H Hilbert space, (X, \mathcal{A}) measurable space, E spectral measure for (X, \mathcal{A}, H) . $f : X \rightarrow \mathbb{C}$ bounded \mathcal{A} -measurable function. Then integral of f with respect to E is the operator $T \in \mathcal{L}(H)$ such that

$$\langle Tx, y \rangle = \int_X f dE_{x,y}, \quad x, y \in H.$$

Notation: Then $\int f dE := T$.

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Poznámka

It always exists due to Lax–Milgram: $(x, y) \mapsto \int f dE_{x,y}$ is sesquilinear and $|\int f dE_{x,y}| \leq \|f\|_\infty \cdot \|E_{x,y}\| \leq \|f\|_\infty \cdot \|x\| \cdot \|y\|$. So T exists and $\|T\| \leq \|f\|_\infty$.

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