Úvod

Poznámka (Organizační úvod)
Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (… PDE …).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

Poznámka (Konvence pro PDR)
$\Omega \subseteq \mathbb{R}^d$ je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

Poznámka
Dále se ukazovali konkrétní parciální rovnice.

Poznámka (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál: $L^p(\Omega)$, $L^p_{loc}(\Omega)$, $||u||_p$, $C^k(\Omega)$, $C^k(\overline{\Omega})$,
$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) | \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha} < \infty \right\}, ||u||_{C^{0,\alpha}} = \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha}.$$
- $\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$, $\vec{n} = (n_1, \dots, n_d)$.
- Funkcionální analýza 1: Banachův prostor, $u^n \to u$ silná konvergence, $u^n \rightharpoonup u$ slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita (L^p jsou separabilní až na $p = \infty$, $C^k(\overline{\Omega})$ je separabilní, $C^{0,\alpha}$ není separabilní pro $\alpha \in (0, 1]$).

Poznámka (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

1 Sobolevovy prostory

Definice 1.1 (Multiindex)

α je multiindex $\equiv d = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$. Délka α je $|\alpha| := \alpha_1 + \ldots + \alpha_d$. Pro $u \in C^k(\Omega)$ definujeme $D^\alpha u = \frac{\partial^{|d|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$.

Definice 1.2 (Slabá derivace)

Buď $u, v_\alpha \in L^1_{loc}(\Omega)$. Řekneme, že v_α je α-tá slabá derivace $u \equiv$

$$\equiv \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Příklad

$u = \text{sign} x$ nemá slabou derivaci.

Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

Důkaz

v_α^1, v_α^2 dvě α-té derivace u.

$$(-1)^{|\alpha|} \int v_\alpha^1 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int v_\alpha^2 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$\int_\Omega (v_\alpha^1 - v_\alpha^2) \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ skoro všude v Ω.

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají. □

Definice 1.3 (Sobolevův prostor)

$\omega \subseteq \mathbb{R}^d$ otevřená, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega) \right\}.$$

$$||u||_{W^{k,p}(\Omega)}||u||_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} ||D^\alpha u||_p^p\right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha|\leqslant k} ||D^\alpha u||_\infty, & p = \infty. \end{cases}$$

Poznámka
Od teď D^α nebo $\frac{\partial}{\partial x_1}$ nebo ∂_i značí slabou derivaci.

Lemma 1.2 (Základní vlastnosti slabých derivací a Sobolevových prostorů)
Nechť $u, v \in W^{k,p}(\Omega), k \in \mathbb{N}$, a α multiindex s délkou $\leqslant k$.

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ *a* $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, *pro* $|\alpha| + |\beta| \leqslant k$.
- $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ *a* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
- $\forall \tilde{\Omega} \subseteq \Omega$ *otevřená*
$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$
- $\forall \eta \in C^\infty(\Omega)$*:* $\eta u \in W^{k,p}(\Omega)$ *a* $D^\alpha(\eta u) = \sum_{\beta_i \leqslant \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$, *kde* $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Důkaz
Cvičení na doma. □

Věta 1.3 (Basic properties of Sobolev spaces)
Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then

- $W^{k,p}(\Omega)$ *is a Banach space;*
- *if* $p < \infty$ *it is separable space;*
- *if* $p \in (1, \infty)$ *it is reflexive space.*

Důkaz
BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness: u^n is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \to u$ in L^p. $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \to v_a \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

TODO

$$|\int_\Omega (v_\alpha - D^\alpha u^n)\varphi| \leqslant ||v_\alpha - D^\alpha u^n||_p ||\varphi||_{p'} \leqslant C||v_\alpha - D^\alpha u^n|| \to 0.$$

$$|\int_\Omega (u^n - u) D^\alpha \varphi| \leqslant ||u^n - u||_p ||D^\alpha \varphi||_{p'} \leqslant C||u^n - u||_p \to 0.$$

„2+3“: $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \ldots \times L^P(\Omega)$ $(d+1$ times), X closed subspace from first property. Lemma: if $X \subseteq Y$ is closed subspace then Y separable $\implies$ X separable and Y reflexive $\implies$ X reflexive. (From functional analysis and topology.) □

2 Approximation of Sobolev function

Věta 2.1

Let $\Omega \subseteq \mathbb{R}^d$ open, ?. $p \in [1, \infty)$.

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} = W^{k,p}(\Omega).$$

Pozor

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} \subsetneq W^{k,p}(\Omega).$$

Důkaz
Summer semester. □

Věta 2.2 (Local density)

$$\forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty$$
$$u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} open, \overline{\tilde{\Omega}} \subseteq \Omega$$
$$u^n \to u in W^{k,p}(\tilde{\Omega})$$

Důkaz

u is extended by 0 to $\mathbb{R}^d \backslash \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \qquad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \qquad \eta \in C_0^\infty(B_1), \eta \geqslant 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

$$u \in L^P(SET) \qquad u^\varepsilon \to uinL^p(SET).$$

We need: $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\forall \alpha, |\alpha| \leqslant k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leqslant \varepsilon_0$ (so that ball of radius ε_0 and center in $\tilde{\Omega}$ is in Ω):

$$(D^\alpha u)^\varepsilon(x) = \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} D^\alpha u(y) \eta_\varepsilon(x-u)dy =$$

$$= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y)dy.$$

$$D^\alpha u^\varepsilon = D_x^\alpha \int_{\mathbb{R}^d} u(y) \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)dy.$$

□

Tvrzení 2.3

Ω is open connected set, $u \in W^{1,1}(\Omega)$, then $u = \text{const}. \Leftrightarrow \frac{\partial u}{\partial x_i} = 0$ $\forall i \in [d]$.

$W^{1,1}(I) \hookrightarrow C(I)$ for I interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$.

Důkaz
„1. $\implies$ " trivial. „1. $\impliedby$ ": $\tilde{\Omega} \subseteq \Omega$ connected ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$. u^ε-modification of u is smooth, so

$$\frac{\partial u^\varepsilon}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^\varepsilon = 0 \quad in\tilde{\Omega}$$
$$\implies u^\varepsilon = \mathrm{const}(\varepsilon) \quad in\tilde{\Omega}.$$

$$c(\varepsilon) = \int_{\mathbb{R}} c(\varepsilon)\eta_\delta(x-y)dy = \int_{\mathbb{R}} u^\varepsilon(y)\eta_\delta(x-y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\eta_\varepsilon(y-z)\eta_\delta(x-y)dzdy =$$
$$\int\int u(z+y)\eta_\varepsilon(z)\eta_\delta(y-x)dzdy = \int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dzdw =$$
$$\int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dwdz = \int_{\mathbb{R}^d} u^\delta(z+x)\eta_\varepsilon(z)dz = \int c(\delta)\eta_\varepsilon(z)dz = c(\delta).$$

„2.": WLOG $I = (0,1)$. Define $v(x) = \int_0^x \frac{\partial u}{\partial y}(y)dy$. We show: $v \in W^{1,1}(I)$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$.

$$|v(x)| \leqslant \int_0^1 |\frac{\partial u}{\partial x}| \leqslant ||u||_{1,1}.$$
$$\varphi \in C_0^1(0,1) \qquad \int_0^1 v(x)\frac{\partial \varphi}{\partial x}(x)dx$$
$$= \int_0^1 \left(\int_0^x \frac{\partial u}{\partial y}(y)dy\right)\frac{\partial \varphi}{\partial x}(x)dx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dydx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dxdy =$$
$$= \int_0^1 \left(\int_y^1 \frac{\partial \varphi(x)}{\partial x}dx\right)\frac{\partial u}{\partial y}(y)dy = -\int_0^1 \varphi(y)\frac{\partial u}{\partial y}(y)dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}.$$

TODO.
$$x \to y \implies \int_y^x |\frac{\partial u}{\partial z}|^\alpha \to 0 \implies |u(x)-u(y)| \to 0$$
$$||u||_{C(I)} \leqslant ||v+c||_{C(I)} \leqslant ||u||_{1,1} + |c| = ||u||_{1,1} + |u(x)-v(x)| \forall x \in I$$
$$||u||_{C(I)} \leqslant ||u||_{1,1} + \int_0^1 |u(x)-v(x)|dx \leqslant -||-+\int_0^1 |u| + \int_0^1 |v| \leqslant ||u||_{1,1}.$$

„3." was shown without proof. □

3 Characterization of Sobolev function

Věta 3.1

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega | \operatorname{dist}(x, \delta\Omega) > \delta\}$. *Then*

$$\forall u \in W^{1,p}(\Omega) : ||\Delta_i^h u||_{L^p(\Omega_delta)} \leqslant ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}, \qquad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

$$u \in L^P \implies \forall \delta, h : ||\Delta_i^h u||_{L^p(\Omega_\delta)} \leqslant c.$$

$p > 1 \implies \frac{\partial u}{partial x_i}$ *exists and* $||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)} \leqslant c$.

Definice 3.1 (Class $C^{k,\mu}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \ldots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r = \left\{x'_r \in \mathbb{R}^{d-1} | \, |x_{r_i}| \leqslant \alpha\right\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting tr the orthogonal transformation from (x'_r, x_{r_d}) to (x', x_d), then $\forall x \in \partial\Omega$ $\exists r \in \{1, \ldots, M\}$ such that $x = \operatorname{tr}\left(x'_{r_1}, a(x_{r_d})\right)$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\right\}$$

$$V_r^- := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\right\}$$

$$\Lambda_r := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) = x_{r_d}\right\}$$

Then $\operatorname{tr}(V_r^+) \subset \Omega, \operatorname{tr}(V_r^-) \subset \mathbb{R}^d \backslash \overline{\Omega}, \operatorname{tr}(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M \operatorname{tr}(\Lambda_r) = \partial\Omega$.

Věta 3.2 (Density of smooth functions)

Let $\Omega \in C^0$. *Then* $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{||\cdot||_{k,p}}$, $p \in [1, \infty)$.

Věta 3.3 (Extension of Sobolev functions)

Let $\Omega \in C^{0,1}$ *(*Ω *is Lipschitz) and* $k \in \mathbb{N}$, $p \in [1, \infty]$. *Then there exists a continuous linear operator* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ *such that:*

- $||Eu||_{W^{k,p}(\mathbb{R}^d)} \leqslant C||Eu||_{W^{k,p}(\Omega)}$ *(*C *is independent of* u*)*
- $Eu = u$ *almost everywhere in* Ω.

Věta 3.4 (Trace theorem)

Let $\Omega \in C^{0,1}$, $p \in [1, \infty]$. *Then there exists a continuous linear operator* tr : $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that:*

- $|| \operatorname{tr} u||_{L^p(\partial\Omega)} \leqslant c||u||_{1,p}$,
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}$.

Definice 3.2

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{k,p}}.$$

Věta 3.5

Let $\Omega \in C^{0,1}$ and let $p \in [1, \infty]$. Then

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$.*

Moreover

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1 - \frac{d}{p}$.*

$$X \hookrightarrow\hookrightarrow Y \Leftrightarrow X \leqslant Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y)\,.$$

$$X \hookrightarrow\hookrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^\infty, \exists c : ||u^n||_{1,p} \leqslant c \implies \exists u^{n_j} : u^{n_j} \to u \text{ in } Y)\,.$$

Důsledek (Trace theorem)
Let $\Omega \in C^{0,1}$. Then $\forall u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ we have integration by parts:

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\omega v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

Věta 3.6 (Poincaré)

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Let $\Omega_1, \Omega_2 \subseteq \Omega$, $|\Omega_i| > 0$ and $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$, $|\Gamma_i|_{d-1} > 0$. Let $\alpha_1, \alpha_2 \geqslant 0$ and $\beta_1, \beta_2 \geqslant 0$ and at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Then there exist $c_1, c_2 > 0$ *such that* $\forall u \in W^{1,p}(\Omega)$

$$c_1||u||_{1,p}^p \leqslant ||\nabla u||_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2| \int_{\Omega_2} u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2| \int_{\Gamma_2} u|^p \leqslant c_2||u||_{1,p}^p.$$

$$(||u||_{1,p}^p = ||u||_p^p + ||\nabla u||_p^p.)$$

Důkaz (Of the first (the only difficult) inequality)
TODO!!! □

4 Linear elliptic PDEs

Definice 4.1 (Elliptic)

Let $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$, where $\Omega \leqslant \mathbb{R}^d$ is bounded. We say that L is elliptic if $\exists c_1 > 0$ such that $\forall \zeta \in \mathbb{R}^d$ and almost all $x \in \Omega$

$$A\zeta \cdot \zeta \geqslant c_1|\zeta|^2.$$

Lemma 4.1

If u *is classical solution, then* $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$ *on* Γ_1*:* $B_{L,\delta}(u, \varphi) = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$.

Důkaz
TODO!!! □

Lemma 4.2

If $u \in C^2(\overline{\Omega})$ *and* $A, b, \mathbf{c}, \mathbf{d}$ *are smooth and previous lemma holds* $\forall \varphi \in C^1$, $\varphi|_{\Gamma_1} = 0$ *and* $u = u_0$ *on* Γ_1*, then* u *is a classical solution.*

Důkaz
TODO!!! □

Definice 4.2 (Weak solution)

Let $\Omega \subseteq \mathbb{R}^d$ Lipschitz, L be an elliptic operator, $u_0 \in W^{1,2}(\Omega)$, $f \in (W^{1,2}(\Omega))^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff

- $\operatorname{tr} u = \operatorname{tr} u_0$ on Γ_1 and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$, $\forall \varphi \in V$, where $V := \{\varphi \in W^{1,2}(\Omega) | \operatorname{tr} \varphi = 0 \text{ on } \Gamma_1\}$.

4.1 Existence of solution for coercive operators

Definice 4.3 (Elliptic form)

Let $B : V \times V \to \mathbb{R}$ bilinear nad V be a Hilbert space, $c_1, c_2 > 0$. We say that B is elliptic if it is

- V-bounded $\Leftrightarrow |B(u,\varphi)| \leqslant c_2||u||_V||\varphi||_V$ and
- V-coercive $\Leftrightarrow B(u,u) \geqslant c_1||u||_V^2$.

Věta 4.3 (Lax-Milgram)

Let B be a bilinear elliptic form. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V : B(u,\varphi) =< F, \varphi > .$$

Definice 4.4

Let $B : V \to V^*$. We say that B is

- Lipschitz $\equiv \forall u, v \in V : ||B(u) - B(v)||_{V*} \leqslant c_2||u - v||_V$, $c_2 > 0$;
- Uniformly monotone $\equiv \forall u, v \in V :< B(u) - B(v), u - v >_V \geqslant c_1||u - v||_V^2$, $c_1 > 0$.

Věta 4.4 (Non-linear Lax-Milgram)

Let B be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V :< B(u), \varphi >=< F, \varphi > .$$

Důkaz
TODO!!! □

Důkaz (Lax-Milgram)
TODO!!! □

Věta 4.5

If $B_{L,\sigma}$ is bilinear, V-bounded and V-elliptic. Then there exists a unique weak solution u.

Důkaz
TODO!!! □

4.2 Existence via Fredholm alternative

TODO!!!

Věta 4.6

Let $\Omega \in C^{0,1}$, L be an elliptic operator and $\Gamma_1 = \partial\Omega$. Then

1. *Σ is at most countable and if infinite $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \to \infty$;*
2. *$(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1\ \exists! u : Lu = f + \lambda u$;*
3. *$\forall \lambda \notin \Sigma\ \exists C > 0\ \forall f \in L^2\ \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$ and $||u||_{1,2} \leqslant c||f||_2$;*

Důkaz

3) TODO improve convergence of u^{n_k} and show

$$u^{n_k} \to u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show $\{u^{n_k}\}$ is Cauchy in $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1||\nabla(u^n - u^m)||_2^2 \leqslant \int_\Omega A\nabla v^{n,m}\nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_\Omega \mathbf{c}\nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d}\nabla v^{n,m} v^{n,m} =$$

$$= \int_\Omega (f^n - f^m)v^{n,m} + \lambda(v^{n,m})^2 \pm - ||- \leqslant$$

$$\leqslant ||v^{n,m}||_2(||f^n - f^m||_2 + \lambda||v^{n,m}||_2 + ||\mathbf{c}||_\infty||\nabla v^{n,m}||_2 + ||\mathbf{d}||_\infty||\nabla v^{n,m}||_2 + ||b||_\infty||v^{n,m}||_2) \leqslant$$

$$\leqslant ||v^{n,m}||C(\lambda) \overset{u^n \text{ is Cauchy}}{\leqslant} C(\lambda)\varepsilon$$

$$\implies \nabla u^n \text{ is Cauchy sequence} \implies u^n \to u \text{ in } W_0^{1,2}(\Omega) \implies ||?||_{n_k} = 1$$

$$\int_\Omega A\nabla a u^n \nabla a\varphi + bu^n\varphi + \mathbf{c}\nabla u^n\varphi - \mathbf{d}\nabla ?u^n = \int_\Omega f^n\varphi + \lambda u^n\varphi.$$

$$n \to \infty$$

$$\int_\Omega A\nabla u\nabla\varphi + bu\varphi + \mathbf{c}\nabla u\varphi - \mathbf{d}\nabla\varphi u = \lambda\int u\varphi \Leftrightarrow Lu = \lambda u$$

But $\lambda \notin \Sigma$. □

Poznámka
Next we discussed homework.

4.3 Variational approach – minimization

Poznámka
$B_{L,\sigma}(u,v)$ must be symmetric! ($B_{L,\sigma}(u,v) = B_{L,\sigma}(v,u)$)

$$L = -\operatorname{div}(A\nabla u) + bu + \mathbf{c}\nabla u + \operatorname{div}(\mathbf{d}u)$$
$$B_{L,\sigma}(u,v) := \int_\Omega A\nabla u \cdot \nabla v + Buv + \mathbf{c}\cdot\nabla uv - \mathbf{d}\nabla vu + \int_\Gamma \sigma uv$$
$$B_{L,\sigma}(v,u) := \int_\Omega A\nabla v \cdot \nabla u + Bvu + \mathbf{c}\cdot\nabla vu - \mathbf{d}\nabla uv + \int_\Gamma \sigma vu$$
$$\implies A = A^T, \qquad \mathbf{c} = -\mathbf{d}$$

Věta 4.7

Let $B_{L,\sigma}$ be linear symmetric V-elliptic and V-bounded. $f \in V^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$, $u \in ?$. Then the following is equivalent:

- $u - u_0 \in V$ *and* $B_{L,\sigma}(u,v) =< f, \varphi > + \int_{\Gamma_2\cup\Gamma_3} g\varphi$;
- $u - u_0 \in V$ $\forall v \in W^{1,2}(\Omega)$, $v, u_0 \in V$
$$\frac{1}{2}B_{L,\sigma}(u,u) - < f, u > - \int_{\Gamma_2\cup\Gamma_3} gu \leqslant \frac{1}{2}B_{L,\sigma}(v,v) - < f, v > - \int_{\Gamma_2\cup\Gamma_3} gv.$$

Důkaz („1 $\implies$ 2“)

$$0 \overset{V-\text{elliptic}}{\leqslant} \frac{1}{2}B_{L,\sigma}(v-u, v-u) \overset{\text{linearity}}{=} \frac{1}{2}B_{L,\sigma}(v,v) + \frac{1}{2}B_{L,\sigma}(u,u) - \frac{1}{2}B_{L,\sigma}(u,v) - \frac{1}{2}B_{L,\sigma}(v,u) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u,u) - B_{L,\sigma}(u,v) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u, u-v) \overset{\text{weak formulation}}{=}$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + < f, u - v > + \int_{\Gamma_2\cup\Gamma_3} g(u-v)$$

□

Důkaz („2 $\implies$ 1“)
u is minimizer, so set $v = u + \varepsilon\varphi$, $\varphi \in V$

$$\frac{1}{2}B_{L,\sigma}(u,u) - < j, u > - \int gu \leqslant \frac{1}{2}B_{L,\sigma}(u+\varepsilon\varphi, u+\varepsilon\varphi) - < j, u+\varepsilon\varphi > - \int g(u+\varepsilon\varphi) =$$

$$= \frac{1}{2}B_{L,\sigma}(u,u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi,\varphi) + \varepsilon B_{L,\sigma}(u,\varphi) - < f, u > -\varepsilon < f, \varphi > - \int ga - \varepsilon\int g\varphi$$

divide by ε and $\varepsilon \to 0_+$

$$0 \leqslant B_{L,\sigma}(u,\varphi) - < j, \varphi > - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \qquad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for $-\varphi \implies 0 = -||- \implies u$ is weak solution. $\square$

Věta 4.8 (Duel formulation)

Let $Lu = -\operatorname{div}(A\nabla u)$ with A elliptic, bounded and symmetric, $\Gamma_1 \neq \emptyset$, $\Gamma = \emptyset$, $f \in V^*$, $g \in L^2(\Gamma_2)$, $u_0 \in W^{1,2}(\Omega)$. Then the f following are equivalent:

- *u is a weak solution;*
- *$\nabla u = A^{-1}\mathbf{T}$, where $\mathbf{T}$ minimizes $\int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} = \nabla u_0 \mathbf{T}$ over the set $\tilde{V} := \left\{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d)\right\}$, $\forall \varphi \in V$.*

$$\int_\Omega \mathbf{T}\cdot\nabla\varphi = < f, \varphi > + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div}\mathbf{T} = f \text{ in } \Omega, T\mathbf{u} = g \text{ on } \Gamma_2$$

Důkaz („1 $\implies$ 2“)
Let $\mathbf{V} \in \tilde{V}$ and $\mathbf{T} := A\nabla u \in \tilde{V}$.

$$0 \leqslant \frac{1}{2}\int_\Omega A^{-1}(\mathbf{V}-\mathbf{T})\cdot(\mathbf{V}-\mathbf{T}) = \int \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2}\int_\Omega A^{-1}\mathbf{T}\cdot\mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\left(\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0\mathbf{T}\right) + \int_\Omega \left(\nabla u_0(\mathbf{V}-\mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T}-\mathbf{V})\right) =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega \nabla(u-u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} + 0.$$

So $\mathbf{T}$ is minimizer of the formula above. $\square$

Důkaz („2 $\implies$ 1")
$\mathbf{T} \in \tilde{V}$ $\forall V \in \tilde{V}$: $\int_\Omega \frac{1}{2} A^{-1}\mathbf{T} \cdot \mathbf{T} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0 \mathbf{V}$. $\mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}$, $\mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$
$\forall \varphi \in V$: $\int_\Omega \mathbf{W} \cdot \nabla\varphi = 0$.

$$\int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T} + \varepsilon^2 A^{-1}\mathbf{W}\cdot\mathbf{W} + 2\varepsilon A^{-1}\mathbf{T}\cdot\mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by ε and $\varepsilon \to 0_+$:

$$0 \leqslant \int_\Omega A^{-1}\mathbf{T} \cdot \mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for $-\mathbf{W}$, co $0 = -||-$.

Now we find unique $u \in W^{1,2}$ $u - u_0 \in V$: $\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega A^{-1}\mathbf{T} \cdot \nabla\varphi$ $(< F, \varphi >_V)$.

$$\int_\Omega |A^{-1}\mathbf{T} - \nabla u|^2 = \int_\Omega (A^{-1}\mathbf{T} - \nabla u)(A^{-1}\mathbf{T} - \nabla u) =$$

$$= \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0) \cdot (A^{-1}\mathbf{T} - \nabla u) + \int_\Omega \nabla(u_0 - u)(A^{-1}\mathbf{T} - \nabla u) = 0 + 0 = 0$$

□

Lemma 4.9

Let X be a reflexive space and $\{u^n\}_{n=1}^\infty$ be a bounded sequence, $||u^n||_X \leqslant c < \infty$. Then $\exists u^{n_k}$, $\exists u \in x$: $u^{n_k} \rightharpoonup u$ ($\forall F \in X^* :< F, u^{n_k} > \to < F, u >$).

Věta 4.10 (Spectrum of symmetric operator)

V Hilbert infinity-dimensional space. Let B be linear, symmetric, V-elliptic and V-bonded operator. Then there exist $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$ and corresponding $\{u_i\}_{i=1}^\infty$ such that

- $B(u^k, \varphi) = \lambda_k \int_\Omega u^k \varphi$;
- $\lambda_k \to \infty$;
- $\{u^k\}_{k=1}^\infty$ *is basis in V and fulfils*

$$\int_\Omega u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_\Omega u u^i)$, *then* $\forall n : ||P^n u||_2 \leqslant ||u||_2$ *and* $B(P^n u, P^n u) \leqslant B(u, u)$.

Důkaz

Step 1: Construct λ_k, u^k: $\lambda_1 := \inf_{u\in V, ||u||_2=1} B(u,u)$ and denote u^1 function, where infimum is obtained. Then for $V^N = \left\{u \in V | \forall k \in [N] : B(u, u^k) = 0\right\}$ we do the same.

Step 2: The construction is OK:

$$0 < \lambda_1 = \lim_{n\to\infty} B(u^n, u^n), ||u^n||_2 = 1 \implies$$

$$\implies ||u^n||_V \leqslant C \implies u^{n_k} \rightharpoonup u \text{ in } V$$

$$V \hookrightarrow L^2 \implies u^{n_k} \to u \text{ in } L^2(\Omega) \implies ||u||_2 = 1$$

$$\lambda_1 = \lim_{n_k\to\infty} B(u^{n_k}, u^{n_k}) \geqslant B(u,u) \geqslant \lambda_1.$$

Step 3: λ_k, u^k eigenvalues, eigen functions: $\forall v \in V, ||v||_2 = 1$, $\lambda_1 = B(u^1, u^1) \leqslant B(v,v)$, $\quad ||u^1||_2 = 1$

$$v = \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leqslant B(\frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2})$$

$$\lambda_1 ||u^1+\varepsilon\psi||_2 \leqslant B(u^1+\varepsilon\psi, u^1+\varepsilon\psi) = B(u_1,u_1)+\varepsilon^2 B(\psi,\psi)+2\varepsilon B(u,\psi) \leqslant \lambda_1||u^1||_2^2+\lambda_1\varepsilon^2||\psi||_2^2+2\varepsilon\lambda_1 \int_\Omega u^1$$

$$\varepsilon \to 0_+ \implies 2\lambda_1 \int_\Omega u^1\psi \leqslant 2B(u,\psi).$$

So $\lambda_1 \int_\Omega u^1\psi = B(u,\psi)$.

The same way we obtain $\lambda_k \int_\Omega u^k\psi \leqslant B(u,\psi)$ for $\psi \in V^N$.

$$u^1 : \lambda_1 \int_\Omega u^1\psi = B(u^1,\psi) \implies \psi = u^k \int_\Omega u^1 u^k = V(u_1, u^k).$$

But $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$, so $\int u^1 u^k = B(u^1, u^k) = 0$.

$$\implies \forall i \in [k-1] : \int_\Omega u^k u^1 = B(u^k, u^i) = 0.$$

Step 4: $\lambda_k \nearrow \infty$. We already know $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$. Assume a contradiction $\lambda_k \leqslant C < \infty$. $c_1||u^k||_V^2 \leqslant B(u^k, u^k) = \lambda_k ||u^k||_2^2 = \lambda_k < C$.

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \to u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$||u^n - y^m||_2^2 = ||u^n||_2^2 + ||u^m||_2^2 - 2\int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_? n} B(u^n, u^m) = 2 \implies \text{ not Cauchy.}$$

Step 5: λ_k are all eigenvalues (u^k is basis of V and of L^2). Assume that $\lambda \neq \lambda_j$ is also eigenvalue, so $\exists u : B(u,\varphi) = \lambda \int_\Omega u\varphi \forall\varphi$. We can find $i \in \mathbb{N}$, so $\lambda_i < \lambda < \lambda_{i+1}$.

$$B(u, u^j) = \lambda \int uu^j \wedge B(u^j, u) = \lambda_j \int u^j u \implies B(u, u_j) = 0$$

4.4 Regularity of weak solution

Poznámka

We assume that we have $u \in W^{1,2}(\Omega)$ a weak solution

$$-\operatorname{div} A\nabla u + Vu + \mathbf{c}\cdot\nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When $u \in W^{2,2}_{loc}(\Omega)$, when $u \in W^{2,2}(\Omega)$, when $u \in W^{k,2}_{loc}(\Omega)$, $u \in W^{k,2}(\Omega)$.

Simplify $-\operatorname{div} A\nabla u = f - bu - \mathbf{c}\nabla u - u\operatorname{div}\mathbf{d} - \nabla u\cdot\mathbf{d} = \tilde{f}$. If $u \in W^{1,2}$, $f \in L^2$, $b \in L^\infty$, $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$.

Problem is reduced to

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma_1,$$
$$(A\nabla u)\cdot\mathbf{v} = g \text{ on } \Gamma_2,$$
$$(A\nabla u)\cdot\mathbf{v} + \sigma u = g \text{ on } \Gamma_3.$$

Definice 4.5 (Interior regularity)

$u \in W^{2,2}_{loc}(\Omega)$; assumptions: $A \in W^{k+1,\infty}$, $f \in W^{k,2}(\Omega) \implies u \in W^{k+1,2}_{loc}(\Omega)$.

Definice 4.6 (Boundary regulariry)

$u \in W^{2,2}(\Omega)$; assumptions: on $\Omega \in C^{k+1,\infty}$, $g \in W^{\frac{1}{2}?,2}(\partial\Omega)$ and $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\varnothing\} \implies u \in W^{2,2}(\Omega)$.

Věta 4.11 (Interior regularity)

Let A be an elliptic operator and $u \in W^{1,2}$ solves

$$\int_\Omega A\nabla u\cdot\nabla\varphi = \int_\Omega f\varphi, \qquad \forall\varphi \in W^{1,2}_0(\Omega)\ \forall f \in L^2(\Omega).$$

Then if $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$, $f \in W^{k,2}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.

Moreover $\forall\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega\ \exists c(\tilde{\Omega}, A)$:

$$||u||_{W^{k+2,2}}(\tilde{\Omega}) \leqslant c(||f||_2 + ||u||_{W^{1,2}(\Omega)}).$$

Důkaz

$k = 0$: Recall $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x+he_k) - v(x)|^2}{h^2} \leqslant c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{ u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2 \right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left| \frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j} \right|^2}{h^2} \leqslant c,$$

$$\int_{\Omega_h} \left| \frac{\nabla u(x+he_i) - \nabla u(x)}{h} \right|^2 \leqslant c.$$

$\int_\Omega A \nabla u \nabla \varphi = \int_\Omega f \varphi$

$$h > 0, \varphi \in W_0^{1,2}(\Omega), \varphi(x) = 0 \text{ if } \operatorname{dist}(x, \partial\Omega) \subset h.$$

Set $\varphi(x) := \psi(x - he_k)$.

$$\implies \int_\Omega A(x) \nabla u(x) \nabla \psi(x - he_k) = \int_\Omega f(x) \psi(x - he_k) =$$

$$= \int_\Omega A(x+he_k) \nabla u(x+he_k) \cdot \nabla \psi(x) dx.$$

Set $\varphi(x) := \psi(x)$:

$$\int_\Omega A(x) \cdot \nabla u(x) \cdot \nabla \psi(x) = \int_\Omega f(x) \psi(x) dx.$$

$$\int_\Omega A(x+he_k)(\nabla u(x+he_k) - \nabla u(x)) \cdot \nabla \psi(x) =$$

$$= -\int (A(x+he_k) - A(x)) \nabla u(x) \cdot \nabla \psi(x) + \int_\Omega f(x)(\psi(x - he_k) - \psi(x)).$$

Set $\psi := (u(x+he_k) - u(x))\tau^2(x)$, $\tau(x) = 0$, if dist $\in (x, \partial\Omega)$, $\tau \in C^1(\tilde{\Omega})$.

Evaluate all terms ($w^{h,i} = u(x+he^i) - u(x)$):

$$\int_\Omega A(x+he_i) \nabla w^{h,i} \cdot (\nabla w^{h,i} \tau^2 + 2 w^{h,i} \tau \nabla \tau) \geqslant$$

$$\overset{ellip.}{\geqslant} c_1 \int_\Omega |\nabla w^{hi}|^2 \tau^2 - \int_\Omega \frac{2||A||_\infty |w^{h,i}| - |\nabla \tau|(|\nabla w^{hi}| \sqrt{c_1} \tau)}{\sqrt{c_1}} \geqslant$$

$$\geqslant \frac{c_1}{2} \int_\omega |\nabla w^{h,i}|^2 \tau^2 - \frac{2}{c_1} ||A||_\infty^2 ||\nabla \tau||_\infty^2 h^2 \int_{\Omega_h} \frac{|u(x+he_i) - u(x)|^2}{h^2} \geqslant$$

$$\geqslant \frac{c_1}{2} \int_\Omega |\nabla w^{h,i}|^2 \tau^2 - \frac{2||A||_\infty^2 ||\nabla \tau||_\infty^2}{c_1} h^2 c ||\nabla u||_2^2$$

TODO?

Věta 4.12 (Regularity up to the boundary)

Let u be a weak solution $-\operatorname{div}(A\nabla u) = f$ in Ω, $A\nabla u \cdot \mathbf{v} = g$ on Γ_2, $A\nabla u \cdot \mathbf{v} + \sigma u = g$ on Γ_3, $u = u_0$ on Γ_1.

Assume that $\Omega \in C^{k+1,\infty}$, $A \in W^{k,\infty}$, $f \in W^{k-1,2}$, $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega)$, $\sigma \in W^{k,\infty}(\partial\Omega)$ and Γ_1, Γ_2, Γ_3 are smooth open (in partialΩ) and $\overline{\Gamma_i} \cap \overline{\Gamma_j} = \varnothing$ $\forall i \neq j$.

Then $u \in W^{k+1,2}(\Omega)$.

Důkaz (Step 1: Flat boundary)
$\Omega = (-1,1)^{d-1} \times (0,1)$. Assume that $u \in W^{1,2}(\Omega)$ and $u = 0$ on $(x,0)$. We want that $u \in W^{2,3}((-1+\delta, 1-\delta)^{d-1} \times (0, 1-\delta)$.

1a tangential derivatives $\frac{\partial u}{\partial x_1} \in W^{1,2}(-||-)$. 1b normal derivative $\frac{\partial^2 u}{\partial x_d^2} \in L^2(-||-)$.

1a: WF $-\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi$ $\forall \varphi \in W_0^{1,2}(\Omega)$. Take continuous $\tau = 1$ in $-||-$ and $\tau = 0$ in $\Omega\setminus$ "inflated" $-||-$.

$$\varphi(x) = \psi(x - he_i)\tau, \qquad i \in [d-1], \psi \in W_0^{1,2}(\Omega\setminus \text{"inflated"} \; -||-)$$

Redefiny interior regularity

$$\int_\Omega (A(x+he_i)\nabla u(x+he_i) - A(x)\nabla u(x))\nabla\varphi(x) = \int_\Omega f(\psi(x-he_i) - \psi(x)).$$

Set $\psi = (u(x+he_i) - u(x))\tau^2 \in W_0^{1,2}$ and apply local regularity.

1b: $\varphi \in C_0^\infty(-||-)$

$$-\int_\Omega \sum_{i,j}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})\varphi = -\int_\Omega \operatorname{div}(A\nabla u)\varphi = \int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi$$

$$-\int_\Omega a_{dd}\frac{\partial^2 u}{\partial x_d^2}\varphi = \underbrace{\int_\Omega f\varphi}_{\in L^2(\Omega)} + \int_\Omega \varphi \left(\sum_{i,j=1}^{d} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1,\neg(i=j=d}^{d}) \right) a_{ij}\frac{\partial u}{\partial x_i x_j}.$$

$$||a_{dd}\frac{\partial^2 u}{\partial x_d^2}||_2^2 \leqslant ||f + \sum \frac{\partial a_{ij}}{\partial x_1}\frac{\partial u}{\partial x_j} + \sum_{\neg(i=j=d)} a_{ij}\frac{\partial^2 u}{\partial x_i x_j}||_2^2 \leqslant C.$$

A is elliptic

$$c_1|\zeta|^2 \leqslant a_{ij}\zeta_i\zeta_j$$

Special choice $\zeta = (0, \ldots, 0, 1)$, $0 < c_1 \leqslant a_{dd}(x) \implies ||\frac{\partial^2 u}{\partial x_d^2}||_{L^2}^2 \leqslant C(DATA?)$ □

Důkaz (Step 2: Transfer from flat to small parts of $\partial\Omega$)
TODO!!!

TODO!!! □

Důkaz (Step 3: Introduce a proper covering of $\partial\Omega$ and use step 2)

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

? Ω $u \in W^{2,2}_{loc}(\Omega)$. ? of $\partial\Omega$, apply step 2.

Define $w := u - u_0 \in W_0^{1,2}(\Omega)$.

$$-\operatorname{div}(A\nabla w) = f + \operatorname{div}(A\nabla u_0)$$

if $f \in L^2$ and $\operatorname{div}(A\nabla u_0) \in L^2$, e.g. $A \in W^{1,\infty} \wedge u_0 \in W^{2,2}(\Omega)$. □

5 Bochner integral

Definice 5.1 (Measurability)

We say that $f : I \to X$ is measurable (strongly, Bochner) if $\exists \{s_j\}_{j=1}^{\infty}$ simple functions, $||f(t) - s_n(t)||_X \to 0$ as $n \to \infty$ for almost every $t \in I$.

Věta 5.1 (Measurability)

$f : I \to X$ is measurable iff

1. *f is almost separably valued;*

$$\exists E \subset I : |E| = 0, f(I \backslash E) \text{ is separable.}$$

2. *f is weakly measurable;*

$$\forall F \in X^* :< F^*, u(t) >_X \text{ is Lebesgue measure w.r.t } t \in I.$$

Definice 5.2 (Bochner integral for simple function)

Let $s : I \to X$ be a simple function on ?. We define

$$\int_I s(t)dt := \sum_{j=1}^{n} X_j |I_j|$$

Definice 5.3 (Bochner integral for measurable functions)

Let $s : I \to X$ be a Bochner measurable function. We say that f is Bochner integrable if $\exists \{s^n\}_{n=1}^{\infty}$ such that $s^n(t) \to f(t)$ a. a. t and $\int_I ||s^n(t) - f(t)||_X dt \to 0$ as $n \to \infty$ and we set

$$X \ni \int_I f(t)dt = \lim_{n\to\infty} \int_I s^n(t)dt.$$

$$\int_I s(t)dt := \sum_{j=1}^{n} X_j |I_j|$$

Definice 5.4 ($L^p(O,T,X)$ space)

Let X be a Banach space

$$L^p(O,T,X) = \left\{ f : (O,T) \to X \text{ bochner integrable} \middle| \int_I ||f(t)||_X^p < \infty \right\}$$

$$||f||_{L^p(O,T,X)} = \left(\int_I ||f(t)||_X^P dt \right)^{\frac{1}{p}}.$$

Věta 5.2 (Dual space)

Let X be a Banach space, separable and $p \in [1,\infty)$, then

$$(L^p(O,T,X))^* = L^{p'}(O,T,X^*)$$

5.1 Sobolev-Bochner spaces

Definice 5.5

Let $f : I \to X$ be Bochner integrable. We say that $g : I \to X$ is a weak derivative of f w. r. t. iff g is Bochner integrable and $\forall \tau \in C_0^\infty(I) : \int_I f(t)\tau'(t)dt = -\int_I g(t)\tau(t)dt$.

Poznámka

If $f \in L^1(I,x)$ and $\frac{\partial f}{\partial t} \in L^1(I,x)$, then $f \in C(I,x)$.

Věta 5.3

$$W^{1,p}(I,X) := \{f \in L^p(I,x), \partial_t f \in L^p(I,X)\}, \qquad ||f||_{W^{1,p}(I,X)} = \begin{cases} \left(\int_I ||f||_X^p + ||\partial_t f||_X^p\right)^{\frac{1}{p}}, & p \in [1 \\ \text{esssup}_{t\in I}(||f(t)||_X + ||\partial_t f||_X), & p = \infty \end{cases}$$

Then $W^{1,p}(I,X)$ is a Banach space, is separable for $p < \infty$ and X separable and *is reflexive if $p \in (1, \infty)$ and X is reflexive and separable.*

5.2 Time derivative in heat/wave equations – Gelfand triple

Poznámka (Motivation)

$$\partial_t u - \Delta u = 0 \text{ in } (0,T) \times \Omega, u = 0 \text{ on } (0,T) \times \partial\Omega, x(0,x) = u_0(x) \text{ for } x \in \Omega, \qquad \Omega \subseteq \mathbb{R}^d$$

.

Definice 5.6 (Gelfand triple)

We say that X, H, X^* is Gelfand triple iff $X \overset{\text{dense}}{\hookrightarrow} H \cong H^* \overset{\text{dense}}{\hookrightarrow} X^*$.

Například

$$X = W_0^{1,2}(\Omega), H = L^2(\Omega), X^* = (W_0^{1,2}(\Omega))^*,$$

Neboť $W_0^{1,2}$ is dense in $C_0 \overset{\text{dense}}{\hookrightarrow} L^2(\Omega)$ and $f \in (W_0^{1,2}(\Omega))^* \implies \exists! u \in W_0^{1,2}(\Omega) : -\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$.

$$\forall \varphi \in W_0^{1,2}(\Omega) < f, \varphi >= \int_\Omega \nabla u \cdot \nabla \varphi = \lim_{n\to\infty} \int_\Omega \nabla u^n \nabla \varphi = \lim_{n\to\infty} - \int_\Omega \Delta u^n \varphi = \lim_{n\to\infty} (f^n, \varphi)_{L^2(\Omega)},$$

where $\{u^n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$, $u^n \to u$ in $W_0^{1,2}(\Omega)$,

$$(X = W_0^{1,p}(\Omega \cap L^2(\Omega)), H = L^2(\Omega))$$

Definice 5.7

Let X, H, X^* be Gelfand triple, $\varphi : H \to H^*$ is Riesz representation and define $i : X \to X^*$, such that $\forall x_0, x \in X$:

$$< i(x_0), x >_X := (\mathrm{id}(x_0), \mathrm{id}(x))_H =< \varphi\, \mathrm{id}(x_0), \mathrm{id}(x) >_H,$$

i maps X densely onto X^*.

Lemma 5.4

Let $u \in L^1(0,T,H)$, $\partial_t u \in L^1(0,T,X^*)$ and X, H, X^* be a Gelfand triple. Then $\forall w \in X$ $\forall \tau \in C_0^1(0,T)$ we have

$$\int_0^T < \partial_t u, w > \tau dt =< \int_0^T \partial_t u \tau dt, w >_X =$$

$$= - < \int_0^T u\tau' dt, w >_X = - \int_0^T < u\tau', w >_X \, dt =$$

$$= -\int_0^T (u\tau', w)_H dt \overset{if\ \partial_t u \in L^1(0,T)}{=} \int_0^T (\partial_t u\tau, w)_H.$$

Věta 5.5 (Integration by parts for Sobolev-Bochner function)

Let $p \in (1, \infty)$, X, H, X^* a Gelfond triple, $u, v \in L^p(0, T, X)$, $\partial_t u, \partial_t v \in L^{p'}(0, T, X^*)$. Then $u, v \in C([0, T], H)$ and $\forall 0 \leqslant t_1 < t_2 \leqslant T$.

$$\int_{t_1}^{t_2} < \partial_t u, v >_X + < \partial_t v, u >_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Důkaz

Step 1) Modify u, v in terms of the Steklov ar? $u_h = \fint_t^{t+h} u(\tau) d\tau$.

Step 2) Prove for u_h, v_h from step 1).

Step 3) $h \to 0_+$. □

Důkaz („Step 1)“)
Define $u_h(t) := \frac{1}{h}\int_t^{t+h} u(\tau)d\tau$, $\forall t \in (0, T-h)$. $u_h \to h$ $L^p(0, T-h_0, X)$, $\forall h_0 \in (0,T)$. We want „$(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$“.

$$(\partial_t u)_h \to \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \qquad \forall h_0 \in (0,T).$$

$$\begin{aligned}\varphi \in C_0^\infty(0,T-h) : \int_0^{T-h} u_h(t)\varphi'(t)dt = \frac{1}{h}\int_0^{T-h}\varphi'(t)\int_t^{t-h} u(t)d\tau dt = \\ = \frac{1}{h}\int_0^{T-h}\varphi'(t)\left(\int_0^{t+h} u(\tau)d\tau - \int_0^t u(\tau)d\tau\right) = \\ = -\frac{1}{h}\int_0^{T-h}\varphi(t)(u(t+h)-u(t)) \Leftrightarrow \partial_t u_h = \frac{u(t+h)-u(t)}{h}.\end{aligned}$$

$$\begin{aligned}\varphi \in C_0^\infty(0,T-h) : \int_0^T \varphi(t)(\partial_t u)_h(t)dt = \frac{1}{h}\int_0^{T-h}\varphi(t)\int_t^{t+h}\partial_t u(\tau)d\tau dt = \\ = \frac{1}{h}\int_0^{T-h}\varphi(t)\left(\int_0^{t+h}\partial_t u(\tau)d\tau - \int_0^t \partial_t u(\tau)d\tau\right)dt = (*)\end{aligned}$$

$$\frac{1}{h}\int_0^{T-h}\varphi(t)\left(\int_0^t \partial_t u(\tau)d\tau\right)dt = \int_0^{T-h}\int_0^{T-h}\varphi(t)\partial_t u(\tau)\chi_{\tau\leqslant t}d\tau dt = \frac{1}{h}\int_0^{T-h}\partial_t u(\tau)\left(\int_t^{T-h}\varphi(t)dt\right)d\tau.$$

$$(*) = \frac{1}{h}\int_0^{T-h}\partial_t u(\tau)\underbrace{\left(\int_{\tau-h}^{\tau}\varphi(t)dt\right)}_{C_0^\infty(0,T)}d\tau = -\frac{1}{h}\int_0^{T-h}u(\tau)\left(\varphi(\tau)-\varphi(\tau-h)\right)d\tau dt.$$

□

Důkaz („Step 2)“)
We want

$$\int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X \, dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \Leftrightarrow$$

$$\leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

$$\int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau)_H dt = \frac{1}{h_1 h_2} \int_{t_1}^{t} (u(t+h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t} v(\tau+h_2) d\tau - \int_{t_1}^{t} v(\tau) d\tau)_H =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1}^{t_2} v(\tau+h_2) - v(\tau) d\tau)_H dt + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau) d\tau, \int_{t_1}^{t_2} u(t+h_1) - u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau) d\tau, \int_{t_2}^{t_2+h_1} u(t) - \int_{t_2}^{t_2+h_1} u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)$$

$$\int_{t_1}^{t_2} \left(\frac{v(\tau+h) - v(\tau)}{h_2}, \,| \int_{\tau}^{\tau+h_1} u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h} u(t) dt)_H + \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau) d\tau)_H =$$

$$- \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau)) d\tau + REST$$

$$REST = \frac{1}{h_1 h_2} \left(\int_{t_2}^{t_2+h_2} v(t) dt - \int_{t_1}^{t_1+h_2} v(t) dt, \int_{t_2}^{t_2+h} u(t) dt \right) + SIMILAR =$$

$$= (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \dots$$

□

Důkaz („Step 3)“)
We have

$$\int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let $h_1 \to 0_+$ and $h_2 \to 0_+$. We have $\partial_t u_{h_1} \to \partial_t u$ in $L^{p'}(0,T,X^*)$, $\partial_t v_{h_2} \to \partial_t v$ in $L^{p'}(0,T,X^*)$, $u_{h_1} \to u$ in $L^p(0,T,X)$, $V_{h_2} \to v$ in $L^p(0,T,X)$. So for almost all t in $(0,T)$: $v_{h_2}(t) \to v(t)$ in $X \hookrightarrow H$ and $u_{h_1}(t) \to u(t)$ in $X \hookrightarrow H$.

$$\int_{t_1}^{t_2} < \partial_t u, v >_X + < \partial_t v, u >_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1)).$$

Now, it is enough to show $u, v \in C([0,T), H)$. We show that u_h is Cauchy in $C([0,T], H)$. Use IBP $u_{h_1} = u_{h^n} - u_{h^m}$, $v_{h_2} = u_{h^n} - u_{h^m}$:

$$||u_{h^n}(t_2) - u_{h^m}(t_2)||_H = ||u_{h^m}(t_1) - u_{h^m}(t_1) + 2\int_{t_1}^{t_2} < \partial_t(u_h^m - u_h^n), u_{h^n} - u_{h^m} >_X ||$$

$$||u_{h^n} - u_{h^m}||^2_{C([\frac{T}{4},T],L^2(\Omega)} = \sup_{t_2 \in (\frac{T}{2},T)} ||u_{h^n}(t_2) - u_{h^m}(t_2)||^2_H \leqslant$$

$$\leqslant ||u_{h^m}(t_1) - u_{h^n}(t_1)||^2_H + \int_0^T ||\partial_t(u_{h^n}) - \partial u_{h^m}||_{X*} ||u_{h^m} - u_{h^n}||_X dt.$$

Choose t_1 such that $u_h(t_1) \to u(t_1)$ in H:

$$\leqslant ||u_h(t_1) - u_{h^m}(t_1)||^2_H + ||\partial_t u_{h^m} - \partial_2 u_{h^n}||_{L^p(X^*)} \cdot \ldots$$

$$u \in C([\frac{T}{4}, T], L^2(\Omega)) \wedge u \in C([0, \frac{3T}{4}], L^2(\Omega)) \to u \in C([0,T], L^2(\Omega))(u(t_1), v(t_1))_H$$

□

6 Parabolic equations

Poznámka
Ω open set in $\mathbb{R}^d$, $T > 0$, L elliptic operator,

$$\partial_t u + Lu = f \text{ in } Q = (0,T) \times \Omega, \qquad u = 0 \text{ on } (0,T) \times \partial\Omega, \qquad u(0,x) = u_0(x) x \in \Omega.$$

$$Lu = -\operatorname{div}_x(A(t,x)\nabla_x u(t,x)) + b(t,x)u(t,x) + \mathbf{c}(t,x)\nabla u(t,x) + \operatorname{div}(\mathbf{d}(t,x)u(t,x)),$$

$$A, b, \mathbf{c}, \mathbf{d} \in L^\infty(\Omega).$$

$$A(t,x) \cdot \xi \cdot \xi \geqslant c_1|\xi|^2, \forall \xi \in \mathbb{R}^d \text{ and almost all } (t,x) \in Q.$$

6.1 Formal a priory estimates

Poznámka
Multiply by u and $\int_\Omega dx$ and use IBP.

$$\int_\Omega \partial_t uu + \int_\Omega A\nabla u \nabla u = \int_\Omega fu - bu^2 - \mathbf{c}\nabla uu + \mathbf{d}u\nabla u.$$

Hölder's inequality:

$$\frac{d}{dt}\frac{||u||_2^2}{2} + C_1||\nabla u||_2^2 \leqslant ||f||_2||u||_2 + ||b||_\infty ||u||_2^2 + ||\mathbf{c}||_\infty ||\nabla u||_2 ||\nabla u||_2 + ||\mathbf{d}||_\infty ||\nabla u||_2 ||\nabla u||_2 \leqslant C_1 \frac{||\nabla u||_2^2}{2} + C(\mathbf{c}$$

Poincaré's inequality:

$$\frac{d}{dt}||u||_2^2 + \mathbf{c}||u||_{1,2}^2 \leqslant C(\mathbf{c}, \mathbf{d}, b)||f||_2^2 + K||u||_2^2.$$

Grönwall's inequality:

$$\sup_{t \in (0,T)} ||u(t)||_2^2 \leqslant + \int_0^T ||f||_2^2)$$

$$\int_0^T ||u||_{1,2}^2 dt \leqslant C$$

$$||\partial_t u||_{(W_0^{1,2})^*} = \sup_{||\varphi|| \leqslant 1} < \partial_t u, \varphi >= \sup < f - Lu, \varphi >=$$

$$= \sup_{||\varphi|| \leqslant 1} \int_\Omega (f - ? - bu - \mathbf{c}\nabla u)\varphi - \int_\Omega (A\nabla u - \mathbf{d}u)\nabla\varphi \leqslant \int_0^T ||f||_2^2 + c(||u||_2^2 + ||\nabla u||_2^2).$$

Definice 6.1

Let $\Omega \subseteq \mathbb{R}^d$ open and bounded, L be an elliptic operator, $u_0 \in L^2(\Omega)$, $f \in L^2(0, T, V^*)$ ($V = W_0^{1,2}(\Omega)$). We say that u is a weak solution to

$$\partial_t u + Lu = f \text{ in } (0,T) \times \Omega,$$

$$u = 0 \text{ on } (0,T) \times \partial\Omega,$$

$$u(0) = u_0 \text{ in } \Omega$$

iff $u \in L^2(0,T,V) \cap W^{1,2}(0,T,V^*)$, $u(0) = u_0$ and for almost all $t \in (0,T)$ and $\forall \varphi \in V$:

$$< \partial_t u, \varphi >_V + \int_\Omega A\nabla u \cdot \nabla\varphi + bu\varphi + \mathbf{c} \cdot \nabla u\varphi - \mathbf{d}\nabla\varphi u =< f, \varphi >_V .$$

6.2 Existence and uniqueness

Věta 6.1

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^2(0,T,V^*)$, $u_0 \in L^2(\Omega)$ and L be elliptic operator. Then $\exists! u$ – weak solution.

Důkaz (Uniqueness)
u_1, u_2 are weak solutions. Define $w := u_1 - u_2 \in L^2(0,T,V) \cap W^{1,2}(0,T,V^*)$. WF for u_1 – WF for u_2:

$$< \partial_t w, \varphi > + \int_\Omega A\nabla w \cdot \nabla\varphi = \int_\Omega -bw\varphi - \mathbf{c}\nabla w\varphi + \mathbf{d}\cdot\nabla\varphi w.$$

Follow almost everywhere, replace u by w. Set $\varphi = w \implies$

$$< \partial_t w, w > + \mathbf{c}\|w\|_{1,2}^2 \leqslant c\|w\|_2^2.$$

Integrate in respect of time, use IBP-formula for $< \cdot, \cdot >$:

$$\int_0^t < \partial_t w, w > = \frac{1}{2}\|w(t)\|_2^2 - \frac{1}{2}\|w(0)\|_2^2 = \frac{1}{2}\|w(t)\|_2^2.$$

$$\implies \|w(t)\|_2^2 \leqslant c\int_0^t \|w(\tau)\|_2^2 d\tau$$

$$\frac{d}{dt}\underbrace{\int_0^t \|w(\tau)\|_2^2 d\tau}_{=:g(t)\geqslant 0} \leqslant c\int_0^t \|w(\tau)\|_2^2 d\tau$$

$$g' \leqslant c\cdot g$$

From Grönwall's inequality

$$g(t) \leqslant e^{c\cdot t}g(0) \implies \int_0^t \|w(\tau)\|_2^2 d\tau \leqslant e^{ct}\int_0^0 \|w(\tau)\|_2^2 d\tau = 0 \implies w(t) = 0.$$

□

Důkaz (Existence (via Galerkim approximation))
We know $\exists\, \{w_j\}_{j=1}^\infty$ basis of $\mathbf{V}$, which is ortonormal in L^2 and $\|P^N u\|_V \leqslant c\|u\|_V$, where P^N is orthogonal projection in $L^2(\Omega)$ onto $\{w_j\}_{j=1}^N$.

We solve for $u^n(t,x) = \sum_{i=1}^n a_i^n(t)w_i(x)$. We want

$$< \partial u^n, w_j > = -\int_\Omega A\nabla u^n\nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n + < f, w_j >$$

for $j \in [n]$ for almost all $t \in (0,T)$ (weak formulation of the problem for n, call it WFn). □

Důkaz („Existence of u^n“)
LHS of WFn:

$$\sum_{i=1}^{n} < \partial_t a_i^n w_i, w_j >_V = \sum_{i=1}^{n} \partial_t a_i^n(t) < w_i, w_j >_V = \sum_{i=1}^{n} \partial_t a_i^n \delta_{ij} = \partial_t a_j^n(t).$$

RHS of WFn:

$$\sum_{i=1}^{n} a_i^n(t) \left(\underbrace{-\int_\Omega A\nabla w_i \nabla w_j + b w_i w_j + \mathbf{c}\nabla w_i w_j - \mathbf{d}\nabla w_j w_i}_{G_{ij}(k) \text{ – bounded and measurable?}} \right) + \underbrace{< f(t), w_j >}_{g_j^{(t)} \text{ – measurable? on } g \in L^2(0,T)} .$$

So

$$\frac{d}{dt} a_j^n(t) = \sum_{i=1}^{n} a_i^n(t) G_{ij}(t) + g_j(t), \qquad j \in [n].$$

Initial data: $u^n(0) := P^n u_0$ ($a_j^n(0) := \int_\Omega u_0 w_j$).

ODE $\implies$ $\exists \tilde{T} \leqslant T$ and $a_i^n(t) \in AC$ on $[0, \tilde{T})$ and solve for almost all $t \in (0, \tilde{T})$. Moreover either we can set $\tilde{T} = T$ or $|a^n(t)| \overset{t \to \tilde{T}}{\to} \infty$.

Now we prove $\tilde{T} = T$. We show $|a^n(t)| \leqslant c$.

Multiply WFn for j by $a_j^n(t)$ and sum it:

$$LHS = \sum_{j=1}^{n} a_j^n < \partial_t u_j^n, w_j > = < \partial_t u^n, \sum_{j=1}^{n} a_j^n w_j > = < \partial u^n, u^n > .$$

$$RHS = \underbrace{< \partial_t u^n, u^n >}_{= \frac{d}{dt} \|u^n\|_2^2} + c_1 \|u^n\|_V^2 \leqslant c(\|f\|_{V*}^2 + \|u^n\|_2^2).$$

Grönwall: $\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leqslant c(\|u^n(0)\|_2^2) + \int_0^T \|f\|_{V*}^2$.

$$\forall t < \tilde{T} : \|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leqslant c \left(\int_0^{\tilde{T}} \|f\|_{V*}^2 + \|u_0\|_2^2 \right) \leqslant \tilde{c}.$$

$$\lim_{t \to \tilde{T}_-} |a^n(t)|^2 = \lim_{t \to \tilde{T}_-} \|u^n(t)\|_2^2 \leqslant \tilde{c}.$$

□

Důkaz

$$\|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \leqslant c(f,u_0).$$

Time derivative

$$\|\partial u^n\|_{V*} = \sup_{w\in V, \|w\|\leqslant 1} \langle \partial_t u^n, w\rangle = \sup_{w\in V, \|w\|\leqslant 1} \int_\Omega \partial u^n w = \sup_{\dots} \int_\Omega \partial u^n P^n w.$$

WFn:

$$\leqslant \sup_{\dots} c(\|f\|_{V*} + \|u^n\|_V)\|P^n w\|_V \leqslant \sup_{\dots} \tilde{c}(\|f\|_{V*} + \|u^n\|_V)\|w\|_V \leqslant$$

$$\leqslant \tilde{(}\|f\|_{V*} + \|u^n\|_V).$$

$$\int_0^T \|\partial_t u^n(t)\|_{V*}^2 \leqslant \tilde{\tilde{c}} \int_0^T (\|f(t)\|_{V*}^2 + \|u^n(t)\|_V^2) \leqslant c(f,u_0).$$

u^n is a bounded sequence in $L^2(0,T,V) \cap W^{1,2}(0,T,V^*)$ so $\exists$ a subsequence u^{m_n}:

$$u^{m_n} \rightharpoonup u \text{ in } L^2(0,T,V), \qquad \partial_t u^n \rightharpoonup \partial_t u \text{ in } L^2(0,T,V^*).$$

To show u is a weak solution.

TODO? □

TODO!!!

Důkaz (Initial condition)
$\tau \in C_0^\infty(-\infty,T)$:

$$-\int_0^T \int_\Omega u^n w_j \partial_t \tau - \int_\Omega u^n(0) w_j \tau(0) + \int_0^T (\dots)\tau \dots = 0.$$

$$\to -\int_0^T \int_\Omega u w_j \partial_t \tau - \int_\Omega u_0 w_j \tau(0) + \int_0^T (\dots)\tau = 0.$$

Integration by parts in time:

$$u \in L^2(W_0^{1,2}(\Omega)) \ni \tau w_j,$$

$$\partial_u \in L^2((W_0^{1,2}(\Omega))^*) \ni \partial_t(\tau w_i) = \partial_t \tau w_i \in L^2,$$

$$-\int_0^T \int_\Omega u w_j \partial_t \tau - \int_\Omega w_j \tau(0) = -\int_0^T < u, \partial_t(\tau w_j) > - \int_\Omega u_0 w_j \tau(0) =$$

$$= \int_0^T < \partial_t u, \tau w_j > + \int_\Omega u(0)\tau(0) w_j - \int_\Omega u_0 w_j \tau(0).$$

□

6.3 Regularity of parabolic equations

TODO Example?

Věta 6.2

Let $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in L^\infty$, $\operatorname{div} \mathbf{d} \in L^\infty$, A, ∇A, $\partial_t A \in L^\infty$, $f \in L^2(0, T, L^2(\Omega))$, *then* $\forall \delta > 0$*:*

$$\int_\delta^T \|\partial_t u\|_2^2 + \sup_{t \geqslant \delta} \|\nabla u(t)\|_2^2 \leqslant \frac{c}{\delta}$$

.

Moreover if $u_0 \in W_0^{1,2}(\Omega)$, *then* $\partial_t u \in L^2(0, T, L^2(\Omega))$, $u \in L^\infty(0, T, W_0^{1,2}(\Omega))$.

Moreover $u \in L^2(0, T, W_{loc}^{1,2}(\Omega))$ *and if* $\Omega \in C^{1,1}$, *then* $u \in L^2(0, T, W^{2,2}(\Omega))$.

Důkaz
Consider u^n–Galeikin approximation

$$u^n(t,x) = \sum_{i=1}^n a_i^n w_i : \int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n \nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n =< f, w_j > .$$

Multiply by $\partial_t a_j^n(t)$ and

$$\sum_{i=1}^n \int_\Omega \partial_t u^n \partial_k a_i^n w_j = \int_\Omega \partial_t u^n \left(\sum \partial_t a_i^n w_j\right) = \int_\Omega \partial_t u^n (\partial_t u^n).$$

$$\int_\Omega \partial_t u^n \partial_t u^n + \int_\Omega A\nabla u^n \cdot \nabla \partial_t u^n + bu^n \partial_t u^n + \mathbf{c}\cdot\nabla u^n \partial_t u^n - \mathbf{d}\nabla\partial_t u^n u^n =< f, \partial_t u^n > .$$

Good guy: $\int_\Omega \partial_t u^n \partial_t u^n = \|\partial_t u^n\|_2^2$.

First half of other guy: $\int_\Omega A\nabla u \nabla \partial_t u =$

$$\int_\Omega \frac{(A+A^T)}{2}\nabla u \nabla \partial_t u + \int_\Omega \frac{A-A^T}{2}\nabla u\nabla\partial_t u =$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \frac{A+A^T}{2}\nabla u\nabla u - \frac{1}{2}\int_\Omega \frac{\partial_t(A-A^T)}{2}\nabla u \nabla u - \int_\Omega \sum_{i,j}\frac{\partial}{\partial x_i}\frac{(A_{ij}-A_{ji})}{2}\frac{\partial u}{\partial x_j}\partial_t u - \underbrace{\sum_{i,j}\frac{A_{ij}-A_{ji}}{2}\frac{\partial^2 u}{\partial x_i \partial x}}_{\text{sum with symmetric dot antis}}$$

Worst guy: $\int_\Omega \mathbf{d}\cdot\nabla\partial_t u^n u^n =$

$$= -\int_\Omega \partial_t u^n \operatorname{div}(\mathbf{d}u^n) = -\int_\Omega \partial_t u^n(\operatorname{div}\mathbf{d}u^n + \mathbf{d}\cdot\nabla u^n).$$

$$\|\partial_t\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \frac{A+A^T}{2}\nabla u\nabla u \leqslant \int_\Omega |\partial_t u^n|((|f| + \ldots) + |\nabla u|^2|\partial_t A|.$$

From Young's inequality:

$$\leqslant \frac{1}{2}\int_\Omega |\partial_t u^n|^2 + C\int_\Omega |b|^2|u^n|^2 + |\mathbf{c}|^2|\nabla u^n|^2 + |\operatorname{div}\mathbf{d}|^2|u^n|^2 + |\mathbf{d}|^2|\nabla u^n|^2 + |f|^2 + \left|\nabla\frac{A-A^T}{2}\right|^2|\nabla u^n|^2 + \left|\partial_t\frac{A+A}{2}\right.$$

$$\leqslant \frac{1}{2}\|\partial_t u^n\|_2^2 \leqslant c(b,\mathbf{c},\mathbf{d})(\|f\|_2^2 + \|u^n\|_{1,2}^2).$$

$$\implies \|\partial u^n\|_2^2 + \frac{d}{dt}\int_\Omega A\ \nabla u^n\cdot\nabla u^n \leqslant c(\ldots)\cdot(\|f\|_2^2 + \|u^n\|_{1,2}^2).$$

We want to know, if right hand side is integrable in time:

$$\int_\tau^t \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t)\nabla u^n(t) \leqslant \int_\Omega A\nabla u^n(\tau)\nabla u^n(\tau) + c\cdot\int_\tau^t \|f\|_2^2 + \|u^n\|_{1,2}^2.$$

With $\tau \leqslant \delta$ we add $\int_0^\delta \cdot d\tau$:

$$\int_\delta^t \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t)\nabla u^n(t) \leqslant \int_0^\delta\int_\Omega A\nabla u^n(\tau)\nabla u^n(\tau)d\tau + C(DATA)$$

Věta 6.3

Let $\partial_t f \in L^2(0,T,L^2(\Omega))$, $\partial_t A, \partial_t b, \partial_t \mathbf{c}, \partial_t d \in L^\infty$. Then $\forall \delta > 0 : \partial_{tt} u \in L^2(\delta, T, V^*), \partial_t u \in L^2(\delta, T, W_0^{1,2}(\Omega))$. If $-Lu_0 + f(0) \in L^2(\Omega)$, then

$$\partial_{tt} u \in L^2(0,T,V^*), \qquad \partial_t u \in L^2(0,T,W_0^{1,2}(\Omega)).$$

Důkaz (Sketch)

Take u^n – Galerkin approximation. Apply ∂_t to it:

$$\int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n \nabla u^n w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n = \int_\Omega f w_j, \qquad \forall j \in [n] \text{ and almost every } t \in (0,$$

$$\int_\Omega \partial_{tt} u^n w_j + \int_\Omega A\nabla \partial_t u^n \nabla w_j = \int_\Omega -\partial_t A \nabla u^n \nabla w_j + (\partial_t b u^n + b\partial_t u^n) w_j + \partial_t \mathbf{c} \nabla u^n + \mathbf{c}\nabla \partial_t u^n w_j.$$

Similar as before we replace w_j, b_j, $\partial_t u^n$:

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u^n\|_2^2 + c_1 \|\nabla \partial_t u^n\|_2^2 \leqslant \int_\Omega \|\nabla \partial_t u^n\|(SOMETHING).$$

$$\implies \frac{d}{dt}\|\partial_t u^n\|_2^2 + \|\nabla \partial_t u^n\|_2^2 \leqslant C(\|\partial_t u^n\|_2^2 + \ldots).$$

$$t \geqslant 2\delta : \|\partial_t u(t)\|_2^2 + \int_\tau^t \|\partial u^n\|_2^2 \leqslant C(1 + \int_\tau^t \|\partial_t u^n\|_2^2) + \|\partial u^n(\tau)\|_2^2.$$

Add $\int_\delta^2 \delta d\tau$:

$$\|\partial_t u(t)\|_2^2 + \int_{2\delta}^T \|\nabla u\|_2^2 \leqslant X(\int_\delta^T \|\partial_t u^n\|_2^2 + 1 + \ |\int_\delta^{2\delta} \|\partial_t u^n(\tau)\|_2^2) \leqslant C(1 + \frac{c}{\delta} + \frac{c}{\delta^2}).$$

$$\to C(\int_0^T \|\partial_t u^n\|_2^2 + \|\partial_t u^n(0)\|_2^2 + 1) \leqslant$$

$$\leqslant C + C\|\partial_t u^n(0)\|_2^2 = C + C\| - Lu_0^n + f(0)\|_2^2 \leqslant \text{const}.$$

□

7 Linear hyperbolic equations

Poznámka (Prototype)

$$\frac{\partial u^2}{\partial t^2} - \Delta u = 0 \text{ in } (0,T)\times\Omega, \qquad u = 0 \text{ on } (0,T)\times\partial\Omega.$$

$$u(0,x) = u_0(x) \in W_0^{1,2}(\Omega), \qquad \partial_t u(0,x) = u_1(x) \in L^2(\Omega).$$

Poznámka (Formal a priory estimate)
Test by $\partial_t u$:

$$\int_\Omega \partial_{tt}u\partial_t u - \Delta u \partial_t u = 0$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_2^2 + \int_\Omega \underbrace{\nabla u \nabla \partial_t u}_{\frac{1}{2}\partial_t\|\nabla u\|^2} = 0$$

$$\frac{d}{dt}\left(\|\partial_t u\|_2^2 + \|\nabla u\|_2^2\right) = 0$$

$$\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leqslant \|\partial_t u(0)\|_2^2 + \|\nabla u(0)\|_2^2 = \|u_1\|_2^2 + \|\nabla u_0\|_2^2.$$

$$\|\partial_{tt}^2 u\|_{(W_0^{1,2}(\Omega))^*} = \sup_{\|\varphi\|\leqslant 1} < \partial_{tt}^2 u, \varphi > \sim \sup \int_\Omega \partial_{tt}^2 u\varphi = \sup\int_\Omega \nabla u\varphi.$$

Věta 7.1

L be an elliptic operator such that $\int_0^T(\|\partial_t u\|_\infty + \|A\|_{1,\infty} + \|b\|_\infty + \|\mathbf{c}\|_\infty + \|\mathbf{u}\|_{1,\infty}) < \infty$ and $f \in L^2(0,T,L^2(\Omega))$. Assume that $u_0 \in W_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there $\exists! u \in L^2(0,T,W_0^{1,2}(\Omega)) \cap W^{1,2}(0,T,L^2(\Omega)) \cap W^{2,2}(0,T < V^*)$.

And $u(t) \to u_0$ in $L^2(\Omega)$, $\partial_t u(t) \to u_1$ in V^*.

Důkaz (Existence)
Step one: Galleikin approximation. Step two: Uniform estimates. Step three: $n \to \infty$. □

Důkaz (Step one)
$\{w_j\}_{j=1}^\infty$ base of $W_0^{1,2}$ ($\|P^n u\|_{1,2} \leqslant c\|u\|_{1,2}$). ? for $u^n(t,x) = \sum_{j=1}^n a_j^n(t)w_j(x)$.

$$\int_\Omega \partial_{tt}u^n w_j + \int_\Omega A\nabla u^n \nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j + \mathbf{d}\nabla w_j u^n = \int_\Omega f w_j.$$

Weak formulation for n-th coord. (WFn). $\partial_t u^n(0) = P^n u_1$ and $u^n(0) = P^n u_0$.

$$(a_j^n)'(0) = \int u_1 w_j, \qquad a_j^n(0) = \int u_0 w_j, (a_j^n)''(t) = F_j(a^n,t) + b_j(t).$$

Assume there exists u^n a solution to (WFn). □

Důkaz (Step two)
Uniform (N-independent) estimates: Multiply WFn by $(e_j^n)'(t)$ and $\sum_{j=1}^n$:

$$\int_\Omega \partial_{tt}u^n \partial_t u^n + \int_\Omega A\nabla u^n \partial\nabla u^n = \int_\Omega f\partial_t u^n + \mathbf{d}\nabla\partial_t u^n u^n - \mathbf{c}\nabla u^n \partial_t u^n - bu^n\partial_t u^n.$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\partial_t u^n|^2 + \frac{A+A^T}{2}\nabla u^n \nabla u^n\right) = -||-\int_\Omega \partial_t\left(\frac{A+A^T}{2}\nabla u^n\nabla u^n\right) - \underbrace{\int_\Omega \frac{A-A^T}{2}\nabla u^n \partial \nabla u^n}_{\text{by part}} \overset{\text{Hölder}}{\leqslant}$$

$$\leqslant c(\|f\|_2^2 + \|\partial_t u^n\|_2^2 + \|\nabla u^n\|_2^2) \leqslant \tilde{c}\left(\|f\|_2^2 + \int_\Omega |\partial_t u^n|^2 + \int_\Omega \frac{A+A^T}{2}\nabla u^n \nabla u^n\right).$$

Gronwall's lemma:

$$\int_\Omega |\partial_t u(t)|^2 + \frac{A+A^T}{2}\nabla u^n(t)\nabla u^n(t) \leqslant C(T)\cdot\left(\int_0^T \|f\|_2^2 + \int_\Omega |\partial_t u^n(0)|^2 + \frac{A+A^T}{2}\nabla u^n(0)\nabla u^n(0)\right) \leqslant C_t\cdot($$

$$\leqslant \tilde{C}_T(1 + \|v_0\|_2^2 + \|u_0\|_{1,2}^2).$$

$$\sup_{t\in(0,T)} \|\partial_t u^n(t)\|_2^2 + \|u^n(t)\|_{1,2}^2 \leqslant C(DATA).$$

$$\|\partial_{tt}u^n\|_{(W_0^{1,2}(\Omega))^*} := \sup_{\varphi\in W_0^{1,2}(\Omega)} < \partial_{tt}u^n, \varphi > \overset{\text{Gelfand}}{=}$$

$$= \sup_\varphi \int_\Omega \partial_{tt}u^n\varphi \overset{\text{basis}}{=} \sup_\varphi \int_\Omega \partial_{tt}u^n P^n(\varphi) \overset{\text{WFn}}{=}$$

$$= -\int_\Omega A\nabla u^n \nabla P^n(\varphi)\ldots \overset{\text{Hölder}}{\leqslant} C\cdot\|P^n(\varphi)\|_{1,2}\left(\|f\|_2 + \|u^n\|_{1,2}\right) \leqslant$$

$$\leqslant \tilde{c}\|\varphi\|_{1,2}(\|f\|_2 + \|u^n\|_{1,2}).$$

$$\int_0^T \|\partial_{tt}u^n\|_{(W_0^{1,2}(\Omega))^*}^2 \leqslant \tilde{c}\cdot 2\cdot\int_0^T (\|f\|_2^2 + \|u^n\|_{1,2}^2) \leqslant C(DATA).$$

□

Důkaz (Step three)
$u^n \rightharpoonup^* u$ in $W^{1,\infty}(0,T,L^2(\Omega)) \cap L^\infty(0,T,W_0^{1,2}(\Omega))$. $u^n \rightharpoonup u$ in $W^{2,2}(0,T,(W_0^{1,2}(\Omega))^*)$.

Limits:

$$\lim_{n\to\infty} \int_0^T \int_\Omega \partial_{tt} u^n w_j \tau dx dt = \int_0^T < \partial_{tt} u^n, w_j\tau > dt =< \partial_{tt} u^n, w_j\tau >_{L^2(0,T,W_0^{1,2}(\Omega))} .$$

$$\lim_{n\to\infty} \int_0^T \int_\Omega A\nabla u^n \nabla w_j \tau =< \nabla u^n, A^T \nabla w_j \tau >_{L^2(0,T,L^2(\Omega))} \to < \nabla u, A^T \nabla w_j \tau >= \int_0^T \int_\Omega A\nabla u \nabla w_j \tau.$$

$$WF : \int_0^T < \partial_{tt} u, w_j > \tau + \int_\Omega A\nabla u \nabla w_j \tau + b u w_j \tau + \mathbf{c}\nabla u w_j \tau - \mathbf{d}\nabla w_j \tau = \int_0^T \int_\Omega f w_j \tau.$$

TODO? □