Poznámka
Credit for giving 'small lecture'. Oral exam.

1 Meromorphic functions

Definice 1.1

We say that a function f is holomorphic in a set $F \subset \mathbb{C}$ if there is an open $G \supseteq F$ such that f is holomorphic on G.

In particular, f is holomorphic at $z_0 \in \mathbb{C}$ if f is holomorphic in some neighbour ($= U(z_0) = U(z_0, \varepsilon)$) of z_0.

Definice 1.2

Function f has at ∞ a removable singularity, if $f\left(\frac{1}{z}\right)$ has a removable singularity at 0. Similarly pole and essential singularity.

Function f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Let $G \subset \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any z_0. Denote $\mathcal{H}(G) := \{f : G \to \mathbb{C} | f \text{ holomorphic}\}$.

Například
From Liouville theorem $\mathbb{H}(\mathbb{S})$ = constant functions. So $\mathbb{H}(G)$ is interesting only for $G \subsetneq \mathbb{S}$, so WLOG $G \subset \mathbb{C}$.

Definice 1.3 (Meromorphic function)

Let $G \subset \mathbb{S}$ be open. Then a function f on G is called meromorphic if at any $z_0 \in G$ the function f is either holomorphic at z_0 or has a pole at z_0.

Denote $\mathcal{M}(G)$ the set of meromorphic functions on G.

Důsledek

- $\mathcal{H}(G) \subset \mathcal{M}(G)$.
- Denote $P_f := \{z_0 \in G | f \text{ has a pole at } z_0\}$. Then P_f has no limit points in G.
- If $f = \infty$ on P_f, then $f : G \to \mathbb{S}$ is continuous. (We always assume, that $f \in \mathcal{H}(G)$ has this property.)

Například

$$\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), \qquad e^{\frac{1}{z}} \notin \mathcal{M}(\mathbb{C}), \qquad \Gamma \in \mathcal{M}(\mathbb{C}), \qquad \zeta \in \mathcal{M}(\mathbb{C}).$$

$\mathcal{M}(\mathbb{S})$ = rational functions. (One inclusion is clear, second: Let $f \in \mathcal{M}(\mathbb{S})$, then because $\mathbb{S}$ is compact it holds that P_f is finite (has no limit point), $P_f \cap \mathbb{C} = \{z_1, \dots, z_n\}$, so from theorem from last semester there exists $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^n p_j\left(\frac{1}{z-z_j}\right)$ for some polynomials p_j. f has removable singularity or pole at infinity and p_j and $\frac{1}{z-z_j}$ have removable singularity there, so $h(z)$ is polynomial, otherwise $h(z)$ has infinity Taylor polynom and $h\left(\frac{1}{z}\right)$ has essential singularity at 0.)

So $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subset \mathbb{C}$.

If $G \subset \mathbb{C}$ is domain, $f, g \in \mathbb{H}(G)$ and $g \equiv 0$, then $f/g \in \mathcal{M}(G)$. The inverse is also true (we will prove it) (but not for $G = \mathbb{S}$).

Lemma 1.1

Let $\mathbb{G} \subset \mathbb{C}$ be open. Then there are compacts K_n, $n \in \mathbb{N}$, in G such that $G = \bigcup_{n=1}^\infty K_n$, $K_n \subset \operatorname{int}(K_{n+1})$ and for any compact K in G, $\exists n \in \mathbb{N} : K \in K_n$.

Důkaz

Set $K_n := \left\{z \in G \mid \operatorname{dist}(z, \mathbb{C}\backslash G) \geqslant \frac{1}{n}\right\} \cap U(0, n)$. $\square$

Tvrzení 1.2

Let $G \subset \mathbb{S}$ be open and $M \subset G$ has no limit point in G. Then

- *$G\backslash M$ is open;*
- *if K is a compact in G, then $K \cap M$ is finite. In particular for $G = \mathbb{S}$ we have M is finite;*
- *M is at most countable. If M is infinite, then $\varnothing \neq M' \subset \partial G$;*
- *if $G \subset \mathbb{C}$ is domain (connected), then $G\backslash M$ is domain.*

Věta 1.3 (Uniqueness of meromorphic functions)

Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ and $f \not\equiv 0$. Then $N_f := \{z \in G \mid f(z) = 0\}$ has no limit points in G.

Důkaz
We know this holds for holomorphic functions. Set $G_0 := G \backslash P_f$. Then $G_0 \subset \mathbb{C}$ is also domain and $f \in \mathcal{H}(G)$ and $f \not\equiv 0$ on G_0. Then $N_f \subset G_0$ has no limit points in G_0, nor in P_f. □

Věta 1.4 (Residue theorem)

Let $G \subset \mathbb{C}$ be open, φ be a closed curve (or cycle) in G and $\operatorname{int} \varphi := \{z_0 \in \mathbb{C} \backslash \langle \varphi \rangle \,|\, \operatorname{ind}_\varphi z_0 \neq 0\} \subset G$. Let $M \subset G \backslash \langle \varphi \rangle$ be finite and $f \in \mathcal{H}(G \backslash M)$. Then $\int_\varphi f = 2\pi i \cdot \sum_{s \in M} \operatorname{ind}_\varphi s \cdot \operatorname{res}_s f$.

Poznámka
This holds true even if instead of finiteness of M, we assume only that $M \subset G \backslash \langle \varphi \rangle$ has no limit points in G. Indeed, we have $M_0 = M \cap \operatorname{int} \varphi$ is finite, because $\langle \varphi \rangle \cup \operatorname{int} \varphi$ is compact and $G_0 := G \backslash (M \backslash M_0)$ is open and f is holomorphic on $G_0 \backslash M_0$ and by R. theorem for G_0 and M_0 we get $\int_\varphi f = 2\pi i \sum_{s \in M_0} \operatorname{res}_s f \cdot \operatorname{ind}_\varphi s$.

1.1 Logarithmic integrals

Definice 1.4 (Logarithmic integral)

Let $\varphi : [a, b] \to \mathbb{C}$ be a (regular) curve and let f be a non-zero holomorphic function on $\langle \varphi \rangle$. Then we define logarithmic integrals integral as

$$I := \frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \frac{1}{2\pi i} \int_a^b \frac{f'(\varphi(t))\varphi'(t)}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_{f \circ \varphi} \frac{dz}{z} = \frac{1}{2\pi i} (\Phi(b) - \Phi(a)),$$

where Φ is a branch (jednoznačná větev) of logarithm of $f \circ \varphi$. If φ is, in addition, closed, then $I = \operatorname{ind}_{f \circ \varphi} 0 = \frac{1}{2\pi}(\Theta(b) - \Theta(a)) \in \mathbb{Z}$, where Θ is a branch of argument of $f \circ \varphi$.

($\frac{f'}{f}$ is called logarithmic derivative of f, because '$(\log f)' = \frac{f'}{f}$'.)

Věta 1.5 (Argument principle)

Let $G \subseteq \mathbb{C}$ be a domain, φ be a closed curve in G and $f \in \mathcal{M}(G)$. Let $\operatorname{int} \varphi \subset G$ and $\langle \varphi \rangle \cap N_f = \emptyset$, $\langle \varphi \rangle \cap P_f = \emptyset$. Then

$$\frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \sum_{s \in \operatorname{int} \varphi, f(s) = 0} n_f(s) \cdot \operatorname{ind}_\varphi s - \sum_{s \in \operatorname{int} \varphi, f(s) = \infty} p_f(s) \cdot \operatorname{ind}_\varphi s,$$

where $n_f(s)$ is multiplicity of the zero point s of f and $p_f(s)$ is multiplicity of the pole s of f.

Důkaz
By Residua theorem, we have

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{s\in \operatorname{int}\varphi, s\in N_f\cup P_f} \operatorname{res}_s\left(\frac{f'}{f}\right)\cdot \operatorname{ind}_\varphi s.$$

If $s \in N_f$ then on $P(s)$:

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = n_f(s).$$

If $s \in P_f$ then on $P(s)$

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = -p_f(s).$$

□

Definice 1.5

$$\Sigma(f,\varphi) := \sum_{s\in\operatorname{int}\varphi, f(s)=0} n_f(s)\cdot \operatorname{ind}_\varphi s - \sum_{s\in\operatorname{int}\varphi, f(s)=\infty} p_f(s)\cdot\operatorname{ind}_\varphi s.$$

Lemma 1.6

Let $\varphi_1,\varphi_2 : [a,b]\to\mathbb{C}$ be closed curve and $s\in\mathbb{C}\backslash(\langle\varphi_1\rangle\cup\langle\varphi_2\rangle)$. Assume, for $t\in[a,b]$, $|\varphi_1(t)-\varphi_2(t)| < |\varphi_1(t)-s|$. Then $\operatorname{ind}_{\varphi_1} s = \operatorname{ind}_{\varphi_2} s$.

Důkaz
For $t\in[a,b]$, we have $|(\varphi_1(t)-s)-(\varphi_2(t)-s)| < |\varphi_1(t)-s|$. Divide by $|\varphi_1(t)-s|$:

$$|1-\psi(t)|<1, \qquad \psi(t) := \frac{\varphi_2(t)-s}{\varphi_1(t)-s}.$$

Then ψ is a closed curve, $<\psi>\subset U(1,1)$, and so

$$0 = \operatorname{ind}_\psi 0 = \frac{1}{2\pi i}\int_a^b \frac{\psi'}{\psi} = \frac{1}{2\pi i}\int_a^b \frac{\frac{\varphi_2'(\varphi_1-s)-\varphi_1'(\varphi_2-s)}{(\varphi_1-s)^2}}{\frac{\varphi_2-s}{\varphi_1-s}} = \frac{1}{2\pi i}\int_a^b\frac{\varphi_2'}{\varphi_2-s} - \frac{1}{2\pi i}\int_a^b \frac{\varphi_1'}{\varphi_1-s} = \operatorname{ind}_{\varphi_2} s - \operatorname{ind}_{\varphi_1} s$$

□

Věta 1.7 (Rouché)

Let $G\subset\mathbb{C}$ be a domain, $f_1,f_2\in\mathcal{M}(G)$ and φ be closed curve in G such that $\operatorname{int}\varphi\subset G$. Assume $\forall z\in\langle\varphi\rangle$:

$$|f_1(z)-f_2(z)| < |f_1(z)| < +\infty$$

Then $\Sigma(f_1,\varphi) = \Sigma(f_2,\varphi)$.

Důkaz
Set $\varphi_j = f_j \circ \varphi$. Then
$$\operatorname{ind}_{\varphi_j} 0 = \frac{1}{2\pi i} \int_\varphi \frac{f_j'}{f_j} = \Sigma(f_j, \varphi).$$
By previous lemma we have for $s = 0$: $\operatorname{ind}_{\varphi_1} 0 = \operatorname{ind}_{\varphi_2} 0$. □

Důsledek
Let f_1, f_2 be holomorphic functions on $\overline{U(z_0, r)}$ and $\forall z \in \partial U(z_0, r) : |f_1(z) - f_2(z)| < |f_1(z)|$. Then $\Sigma_1 = \Sigma_2$, where $\Sigma_j := \sum_{s \in U(z_0,r), f(s)=0} n_{f_j}(s)$.

Důkaz
Apply Rouché's theorem to $\varphi(t) := z_0 + r \cdot e^{it}$, $t \in [0, 2\pi]$. □

Příklad
$f_2 = p$, $f_1(z) = a_0 z^n$ and big enough $U(0, r)$.

Definice 1.6 (Notation)

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. We say that $f(z_0) = w_0 \in \mathbb{C}$ p times for $p \in \mathbb{N}$ if z_0 is a zero point of $f - w_0$ of order p.

Poznámka
Following statements are equivalent to each other:

- $f(z_0) = w_0$ p times;
- $f(z_0) = w_0$, $f'(z_0) = 0 = \ldots = f^{(p-1)}(z_0)$, $f^{(p)}(z_0) \neq 0$;
- $f(z) = w_0 + \sum_{k=p}^{+\infty} c_k (z - z_0)^k$ on some neighbourhood of z_0 and $c_p \neq 0$.

We say that $f(z_0) = \infty$ p times if z_0 is a zero point of $\frac{1}{f}$ of order p. (It's the same as z_0 is pole of f of order p.) And we say that $f(\infty) = w_0 \in \mathbb{S}$ p times if $f(1/z)$ attains w_0 p times at 0.

Věta 1.8 (On a multiple value)

Let $z_0, w_0 \in \mathbb{S}$, f be a holomorphic function on a $P(z_0)$ and $f(z_0) = w_0$ p times for some $p \in \mathbb{N}$. Let $\delta_0 > 0$. Then there are $\varepsilon > 0$ and $\delta \in (0, \delta_0)$ such that, for any $w \in P(w_0, \varepsilon)$ there are just p different points $z_1, \ldots, z_p$ in $P(z_0, \delta)$ with $f(z_j) = w$. In addition, $f(z_j) = 0$ once.

Důkaz

WLOG, assume $z_0 = 0 = w_0$.Then $z_0 = 0$ is a zero point of f of order p. Choose $\delta \in (0, \delta_0)$ such that $f \neq 0$ and $f' \neq 0$ on $P(0, 2\delta)$. Set $\varepsilon := \min_{|z|=\delta} |f(z)| > 0$.

Let $w \in P(0, \varepsilon)$. Use Rouché's theorem for $f_1 := f$, $f_2 := f - w$ and $\varphi := \delta e^{it}$, $t \in [0, 2\pi]$. Of course, $|f_1 - f_2| = |w| < \varepsilon < |f_1|$ on $\langle \varphi \rangle$.

Since in $U(0, \delta)$ the function $f = f_1$ has the only zero point of order p at origin, $f - w = f_2$ has just p simple zero points in $P(0, \delta)$. □

Důsledek

Let $G \subset \mathbb{S}$ be a domain, $f \in \mathcal{M}(G)$ and f be not constant on G. Then $f : G \to \mathbb{S}$ is an open map (for any open $\Omega \subset G$, $f(\Omega)$ is open).

Důkaz

Let $\Omega \subset G$ be open and $w_0 \in f(\Omega)$. Then there is a $z_0 \in \Omega$ and $p \in \mathbb{N}$ such that $f(z_0) = w_0$ p times. Choose $\delta_0 > 0$ such that $U(z_0, \delta_0) \subset \Omega$. By the previous theorem, there is $\varepsilon > 0$, $\delta \in (0, \delta_0)$ such that $P(w_0, \varepsilon) \subset f(P(z_0, \delta))$, so $U(w_0, \varepsilon) \subset f(U(z_0, \delta)) \subset f(\Omega)$. □

Poznámka

This is true for $\mathcal{H}(G)$ too.

Důsledek

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. Then $f'(z_0) \neq 0$ if and only if there is $U(z_0)$ such that $f|_{U(z_0)}$ is one-to-one.

Důkaz

„ $\implies$ “: Let $f'(z_0) \neq 0$. Then $f(z_0) = w_0$ once, so we choose $\delta_0 > 0$ such that $f \neq w_0$ on a $P(z_0, \delta_0)$. By the previous theorem choose $\varepsilon > 0$, $\delta \in (0, \delta_0)$. Moreover, due to the continuity of f at z_0 choose $\delta_1 \in (0, \delta)$ such that $f(U(z_0, \delta_1)) \subset U(w_0, \varepsilon)$. Then $f|_{U(z_0, \delta_1)}$ is one-to-one.

„ $\impliedby$ “: Let $f'(z_0) = 0$ and let f be not constant on any neighbourhood of z_0. Then $f(z_0) = w_0$ p times ($p \in \mathbb{N} \setminus \{1\}$). By the previous theorem f is not one-to-one on any neighbourhood of z_0. □

Věta 1.9 (On holomorphic inverse)

Let $G \subset \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be a one-to-one holomorphic[a] *function, then $f' \neq 0$ on G, $\Omega := f(G)$ is open and $f_{-1} : \Omega \overset{onto}{\to} G$ is holomorphic.*

In addition, $(f_{-1})' = \frac{1}{f' \circ f_{-1}}$ on Ω.

Důkaz
WLOG, $G \subset \mathbb{C}$ is a domain. By first „dusledek“ of previous theorem f is an open map, so $\Omega := f(G)$ is open and $f_{-1} : \Omega \to G$ is continuous. Let $z_0 \in G$ and $w_0 = f(z_0)$. By second „dusledek“ we have $f'(z_0) \neq 0$, and

$$\frac{1}{f'(z_0)} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} \overset{*}{=} \lim_{w \to w_0} \frac{f_{-1}(w) - f_{-1}(w_0)}{w - w_0} = f'_{-1}(w_0).$$

The equality * follows from theorem on limits of composite functions because f_{-1} is continuous and $f_{-1}(w) \neq f_{-1}(w_0)$ for $w \neq w_0$. □

[a] One-to-one holomorphic function is sometimes called conformal.

Věta 1.10 (Hurwitz)

Let $G \subset \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(G)$, $f_n \overset{loc.}{\rightrightarrows} f$ on G and $f \not\equiv 0$. Let $z_0 \in G$ be a zero point of f. Then $\exists \, \{z_n\}_{n=1}^{\infty} \subset G$ and a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $z_n \to 0$ and $f_{k_n}(z_n) = 0$.

Poznámka
Not true in $\mathbb{R}$! The assumption $f \not\equiv 0$ is important! ($f_n(z) := z/n$)

Důsledek
Let $G \subset \mathbb{C}$ be a domain, f_n be one-to-one holomorphic functions on G and $f_n \overset{\text{loc}}{\rightrightarrows} f$ on G. Then f is either one-to-one and holomorphic, or constant.

Důkaz (Hurwitz theorem)
Choose $\delta > 0$ such that $U(z_0, \delta) \subset G$ and $f \neq 0$ on $P(z_0, \delta)$. For $n \in \mathbb{N}$ put $\varrho_n := \frac{\delta}{n+1}$ and $\varphi_n(t) := z_0 + \varrho_n e^{it}$, $t \in [0, 2\pi]$. Of course, $\tau_n := \min_{\langle \varphi_n \rangle} |f| > 0$. For a given n, there is (from uniformly convergence) $k_n \in \mathbb{N}$ such that $\forall z \in \langle \varphi_n \rangle : |f_{k_n}(z) - f(z)| < \tau_n \leqslant |f|$.

By Rouché's theorem there is $z_n \in U(z_0, \varrho_n)$ such that $f_{k_n}(z_n) = 0$. Of course, we can choose $\{k_n\}$ to be increasing. □

Důkaz (Corollary)
Assume that there is $w_0 \in \mathbb{C}$ such that $f \neq w_0$ but, for different $z', z'' \in G$ we have $f(z') = w_0 = f(z'')$. WLOG $w_0 = 0$. Choose $\delta > 0$ such that $U(z', \delta) \cap U(z'', \delta) = \varnothing$. By Hurwitz, there are $\{z'_n\} \subset U(z', \delta)$ and $\left\{f_{k'_n}\right\}$ of $\{f_n\}$ such that $z'_n \to z'$ and $f_{k'_n}(z'_n) = 0$. By Hurwitz, there are also $\{z''_n\} \subset U(z'', \delta)$ and $\left\{f_{k''_n}\right\} \subset \left\{f_{k'_n}\right\}$ such that $z''_n \to z''$ and $f_{k''_n}(z''_n) = 0$.

Every $f_{k''_n}$ has at least two different zero points which is contradiction. □

Věta 1.11 (Mittag-Leffler)

Let $\{s_j\} \subset \mathbb{C}$ be one-to-one, $s_j \to \infty$ and

$$s_0 := 0 < |s_1| \leqslant |s_2| \leqslant |s_3| \leqslant \ldots \leqslant |s_j| \leqslant \ldots$$

Let $P_0, P_1, \ldots, P_j, \ldots$ be polynomials such that $P_j(0) = 0$. Then the function

$$f(z) := P_0\left(\frac{1}{z}\right) + \sum_{j=1}^{\infty}\left(P_j\left(\frac{1}{z-s_j}\right) - Q_j(z)\right)$$

for some polynomials Q_j satisfies:

1. *series in definition converges locally uniformly on $\mathbb{C}$, i. e., on any compact $K \subset \mathbb{C}$, the series converges uniformly if we omit finitely many terms which have poles.*
2. *$f \in \mathcal{M}(\mathbb{C})$ and f has poles just at $s_0, s_1, \ldots, s_j, \ldots$, while at s_j the function f has its principal part equal to $P_j\left(\frac{1}{z-s_j}\right)$.*
3. *If $g \in \mathcal{M}(\mathbb{C})$ satisfies previous property, then there is $h \in \mathcal{H}(\mathbb{C})$ such that $g = f + h$ on G.*

Důkaz

Let $k \in \mathbb{N}$. Then $H_k(z) := P_k\left(\frac{1}{z-s_k}\right) \in \mathcal{H}(U(0,|s_k|))$, $H_k(z) = \sum_{n=0}^{\infty} c_n^k z^n$ for $|z| < |s_k|$. There is $n_k \in \mathbb{N}$ such that $Q_k(z) = \sum_{n=1}^{n_k} c_n^k z^n$ satisfies $|H_k(z) - Q_k(z)| < \frac{1}{2^k}$, $|z| \leqslant \frac{|s_k|}{2}$ (*).

Let $K \subset \mathbb{C}$ be a compact. Choose $k_0 \in \mathbb{N}$ such that $K \subset \overline{U(0,|s_{k_0}|/2)}$. If $k > k_0$, (*) holds on K which implies 1. obviously, 2. is valid.

3. follow from the fact that $g - f \in \mathcal{M}(\mathbb{C})$ has all isolated singularities removable. □

2 Zero points of holomorphic functions

Tvrzení 2.1

Let f be non-zero holomorphic function on a simply connected domain (G is domain, and $\mathbb{S}\backslash G$ is connected) $G \subset \mathbb{C}$. Then there is $L \in \mathcal{H}(G)$ such that $f = e^L$ on G.

Důkaz
1) Let $L \in \mathcal{H}(G)$ and $f = e^L$ on G. Then $f' = L' \cdot e^L$ and $f'/f = L'$.

2) Since G is a simply connected domain and $f'/f \in \mathcal{H}(G)$, by Cauchy theorem, there is $L_0 \in \mathcal{H}(G)$ such that $L_0' = f'/f$.

3) On G we have $\left(f \cdot e^{-L_0}\right)' = e^{-L_0} \cdot (f' - L_0' \cdot f) = 0$ on G, hence $f \cdot e^{-L_0} = e^c$ is constant, i. e. $c \in \mathbb{C}$. Put $L := L_0 + c$. □

Poznámka
Polynomial $f(z) = \prod_{j=1}^n (z - z_j)$ has zero points just at $z_1, \dots, z_n$ and their multiplicity corresponds to their occurrence.

Let $g \in \mathcal{H}(\mathbb{C})$ have the same zero points including multiplicity as f. Then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on $\mathbb{C}$. (Proof: use previous tvrzeni for g/f.)

Poznámka (Notation)
Let $\{a_j\} \subset \mathbb{C}$. Then we define

$$\prod_{j=1}^{\infty} a_j := \lim_{n\to\infty} \prod_{j=1}^{n} a_j,$$

if the limit on the right-hand side exists.

Tvrzení 2.2

Let $0 \neq z_j \to \infty$ and $k \in \mathbb{N}_0$ (multiplicity of 0 as zero point). Then consider

$$f(z) := z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right).$$

It sometimes converges and then f has zero points in z_j with right multiplicities.

Věta 2.3 (On infinite product)

Let M be a set (in $\mathbb{C}$), $u_j : M \to \mathbb{C}$ be bounded and $\sum_{j=1}^{\infty} |u_j|$ converges uniformly on M. Then $p_n := \prod_{j=1}^n (1 + u_j)$ converge uniformly to a function $f : M \to \mathbb{C}$, and it holds that $f = \prod_{j=1}^{\infty} (1 + u_{n(j)})$ on M, where n is bijection onto $\mathbb{N}$.

If $z_0 \in M$, then $f(z_0) = 0$ if and only if $u_{j_0}(z_0) = -1$ for some $j_0 \in \mathbb{N}$.

Důkaz
Denote $p_n^* := \prod_{j=1}^n (1+|u_j|)$. Then $p_n^* \leqslant \exp\left(\sum_{j=1}^n |u_j|\right)$ and $|p_n - 1| \leqslant p_n^* - 1$ (from $1+x \leqslant e^x$ and the second inequality by induction on n: $n=1$ yes, $p_{n+1} - 1 = p_n(1+u_{n+1}) - 1 = (p_n - 1)\cdot(1+u_{n+1}) + u_{n+1}$ so $|p_{n+1} - 1| \leqslant (p_n^* - 1)\cdot(1+|u_{n+1}|) + |u_{n+1}| = p_{n+1}^* - 1$).

$\sum_{j=1}^\infty |u_j|$ is bounded on M, because there is $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^\infty |u_j| < 1$. By inequalities there is $C \in (0, +\infty)$ such that $|p_n| \leqslant C \; \forall n \in \mathbb{N}$.

Let $0 < \varepsilon < \frac{1}{2}$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty |u_n| < \varepsilon$ on M. Let $\{n_1, n_2, \ldots\}$ be a permutation of $\mathbb{N}$ and $q_m := \prod_{j=1}^m (1+u_{n_j})$, $m \in \mathbb{N}$. Let $n \geqslant n_0$ and $m \in \mathbb{N}$ be such that $\{n_1, \ldots, n_m\} \supseteq [n]$. Then

$$|q_m - p_n| = |p_n \cdot \left(\prod_{n_j > n, j \in [m]} (1+u_{n_j}) - 1\right) \leqslant |p_n| \left(\prod_{\ldots}(1+|u_{n_j}|) - 1\right) \leqslant |p_n| \cdot \left(e^{\sum_{\ldots} |u_{n_j}|} - 1\right) \leqslant |p_n| \cdot (e^{\varepsilon} -$$

If $n_j = j \; \forall j \in \mathbb{N}$, then $q_m = p_m$ and we get $\forall m > n : |q_m - p_n| < 2C\varepsilon$, so $p_n \rightrightarrows f$ on M. Moreover we have, for $n \geqslant n_0$, $|p_n - p_{n_0}| \leqslant 2\varepsilon |p_{n_0}|$, so $|p_n| \geqslant |p_{n_0}| - |p_n - p_{n_0}| \geqslant (1-2\varepsilon)|p_{n_0}|$. For $n \to \infty$: $|f| \geqslant (1-2\varepsilon)|p_{n_0}|$, hence $f(z_0) = 0 \Leftrightarrow p_{n_0}(z_0) = 0$.

If n_j is any, then $q_m \rightrightarrows f$ on M. □

Důsledek
Let $G \subset \mathbb{C}$ be open, $f_n \in \mathcal{H}(G)$ and $f_n \not\equiv 0$ on any component of G. We assume $\sum_{n=1}^\infty |1 - f_n|$ converges locally uniformly on G. Then $f = \prod_{n=1}^\infty f_n$ converges locally uniformly on G, $f \in \mathcal{H}(G)$ and the resulting infinite product f does not depend on the order of functions f_n. Moreover, we have

$$n_f(s) = \sum_{k=1}^\infty n_{f_k}(s), \qquad s \in G$$

where $n_f(s)$ is multiplicity of a zero point s of f. Here we put $n_f(s) = 0$ if $f(s) \neq 0$.

Poznámka
Moreover the ? in previous sum contains only finitely many non-zero terms for any $s \in G$.

Důkaz
Sufficient to prove previous equality. Let $s \in G$. There is a neighbourhood V of s such that $f_n \rightrightarrows 1$ on V. Choose $n_0 \in \mathbb{N}$ such that $f_n \neq 0$ on V for $n > n_0$. By previous theorem, we get $\prod_{n=n_0+1}^\infty f_n \neq 0$ on V. Since $f = (\prod_{n=1}^{n_0} f_n) \cdot \left(\prod_{n=n_0+1}^\infty f_n\right)$ we get $n_f(s) = \sum_{k=1}^{n_0} n_{f_k}(s) = \sum_{k=1}^\infty n_{f_k}(s)$. □

Příklad (Homework)

Under the assumption of previous corollary prove that

$$\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n} \text{ on } G \backslash N_f.$$

Například (Euler formula)

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 2.4 (Weierstrass's factor)

Let $E_0(z) := (1-z)$ and $E_m(z) := (1-z)\cdot e^{z+\ldots+\frac{z^m}{m}}$, $z \in \mathbb{C}$, $m \in \mathbb{N}$. Then $|1-E_m(z)| \leqslant |z|^{m+1}$, $|z| \leqslant 1$.

Důkaz

$$E_m'(z) = e^{z+\ldots+\frac{z^m}{m}} \cdot (-1 + (1-z)\cdot(1+\ldots+z^m)) = -z^m \cdot e^{z+\ldots+\frac{z^m}{m}} = -z^m \cdot \sum_{k=0}^{\infty} b_k z^k,$$

where $b_0 = 1$, $b_k \geqslant 0$, $k \in \mathbb{N}$. Hence

$$E_m(0) - E_m(z) = 1 - E_m(z) = -\int_{[0,z]} E_m'(w)dw = +\sum_{k=0}^{\infty} c_k z^{k+m+1}$$

with $c_k = \frac{b_k}{m+k+1} \geqslant 0$.

By this, if $|z| \leqslant 1$, $z \neq 0$, then $\left|\frac{1-E_m(z)}{z^m}\right| \leqslant \sum_{k=0}^{\infty} c_k = 1 - E_m(1) = 1.$ □

Věta 2.5 (Weierstrass factorization in $\mathbb{C}$)

Let $k \in \mathbb{N}_0$ and $0 \neq z_i \to \infty$. Then there is $\{m_j\} \subset \mathbb{N}_0$ such that

$$f(z) = z^k \cdot \prod_{j=1}^{\infty} E_{m_j}\left(\frac{z}{z_j}\right)$$

converges locally uniformly on $\mathbb{C}$, $f \in \mathcal{H}(\mathbb{C})$ and f has at 0 zero point of multiplicity K and 'non-zero' zero points just at $z_1, z_2, \ldots, z_j, \ldots$, and their multiplicity corresponds to their occurrence in $\{z_j\}$. We can always take $m_j := j-1$, $j \in \mathbb{N}$.

If $g \in \mathcal{H}(\mathbb{C})$ has the same zero points as f including multiplicities, then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on $\mathbb{C}$.

Důkaz
By the previous corollary, we know the product converges locally uniformly in $\mathbb{C}$ if $\sum_{j=1}^{\infty} |1-E_{m_j}\left(\frac{z}{z_j}\right)|$ converges locally uniformly on $\mathbb{C}$. By lemma, this is true if $\sum_{j=1}^{\infty} \left|\frac{z}{z_j}\right|^{m_j+1}$ converges locally uniformly on $\mathbb{C}$.

Let $r > 0$ and $|z| \leqslant r$. Choose $j_0 \in \mathbb{N}$ such that $\frac{r}{|z_j|} < \frac{1}{2}$ for $j \geqslant j_0$. If $m_j := j-1$, then $\left|\frac{z}{z_j}\right|^j \leqslant \frac{1}{2^j}$, $j \geqslant j_0$ and $|z| \leqslant r$. So, for $m_j := j-1$, sum converges uniformly on $|z| \leqslant r$. □

Poznámka
If $\sum_{j=1}^{\infty} \frac{1}{|z_j|} < +\infty$, take $m_j = 0$. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|^2} < +\infty$, take $m_j = 1$. Etc.

Věta 2.6 (Weierstrass factorization in a general open set)

Let $G \subsetneq \mathbb{S}$ be open, $N \subset G$ have no limit points in G and $n : N \to \mathbb{N}$. Then there is $f \in \mathcal{H}(G)$ such that $N_f = N$ and $n_f(s) = n(s)$, $s \in N_f$.

Důkaz
WLOG $\infty \in G \backslash N$. Then $K := \mathbb{S}\backslash G = \mathbb{C}\backslash G$ is compact in $\mathbb{C}$. For a finite N it is obvious. Assume that N is (infinite) countable. We put points of N into the sequence $s_1, s_2, \ldots, s_n$ such that any $s \in N$ occurs in $\{s_n\}$ just $n(s)$ times. For any n, take $t_n \in K$ such that $|s_n - t_n| = \operatorname{dist}(s_n, K)$, $n \in \mathbb{N}$.

Then „$|s_n - t_n| \to 0$“: Let $\varepsilon > 0$ and $\{n_k\} \subset \mathbb{N}$ such that $|s_{n_k} - t_{n_k}| \geqslant \varepsilon$, i. e., $\operatorname{dist}(s_{n_k}, K) \geqslant \varepsilon$. If s_∞ is a limit point of s_{n_k}, then $\operatorname{dist}(s_\infty, K) \geqslant \varepsilon$. Hence $s_\infty \in G$, a contradiction.

Put $f(z) := \prod_{n=1}^{\infty} E_n\left(\frac{s_n - t_n}{z - t_n}\right)$, $z \in G$. The infinite product converges locally uniformly on G. In fact, let L be a compact in G. Put $r_n := 2 \cdot |s_n - t_n|$. Since $\operatorname{dist}(L, K) > 0$, there is $n_0 \in \mathbb{N}$ such that $|z - t_n| > r_n$, $\forall z \in L$, $\forall n \geqslant n_0$. So

$$\left|\frac{s_n - t_n}{z - t_n}\right| < \frac{1}{2} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

By lemma on Weierstrass factors, we get

$$\left|1 - E_n\left(\frac{s_n - t_n}{z - t_n}\right)\right| < \frac{1}{2^n} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

Now use theorem on infinite product. □

Lemma 2.7

If $G \subseteq \mathbb{C}$ is open and $f \in \mathcal{M}(G)$, then there are $g, h \in \mathcal{H}(G)$ such that $f = \frac{g}{h}$ on G.

Důkaz
Let P_f be the set of poles of f. By Weierstrass factorization, we construct $h \in \mathcal{H}(G)$ such that $N_h = P_f$ and $n_h = p_f$ on P_f. Put $g := f \cdot h$. Then $g \in \mathcal{H}(G)$ because at the points of P_f g has a removable singularities. □

3 The space H(G)

Poznámka (Arzela–Ascoli theorem)
Let $\mathcal{F} \subset \mathcal{C}(K)$ and let the functions of $\mathcal{F}$ be equibounded (i.e. $\exists M \in (0, +\infty)\ \forall f \in \mathcal{F} : |f| \leqslant M$ on K) and equicontinuous (i.e. $\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in \mathcal{F}\ \forall x, y \in K : \varrho(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$, where ϱ is metric on K). Then every $\{f_n\} \subset \mathcal{F}$ has $\{f_{n_k}\}$ which is uniformly convergent on K.

3.1 The space C(G)

Definice 3.1

Let $G \subseteq \mathbb{C}$, then $\mathcal{C}(G) := \{f : G \to \mathbb{C} | f \text{ continuous}\}$.

Tvrzení 3.1

For $f_n, f \in \mathcal{C}(G)$ and K_m compact in G such that $\bigcup_{m=1}^{\infty} K_m = G$ and $\forall m \in \mathbb{N} : K_m \subseteq \operatorname{int} K_{m+1}$, TSAE:

- $f_n \overset{loc.}{\rightrightarrows}$ *on G;*
- *for any compact K in G, $\|f_n - f\| \to 0$, where $\|f\|_K := \sup_K |f|$ is a seminorm on $\mathcal{C}(G)$;*
- $\forall m \in \mathbb{N} : \|f_n - f\|_{K_m} \to 0$ *for $u \to \infty$;*
- $\varrho(f_n, f) \to 0$, *where* $\varrho(f_n, f) := \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|f_n - f\|_{K_m}}{1 + \|f_n - f\|_{K_m}}$.

Důkaz
„1 ⇔ 2 $\implies$ 3“ is obvious. „2 $\impliedby$ 3“: Let K be a compact in G. Then $K \subset K_{m_0}$ for come $m_0 \in \mathbb{N}$. Then $\|f_n - f\|_K \leqslant \|f_n - f\|_{K_{m_0}}$. „3 ⇔ 4“ homework. □

Poznámka
$(\mathcal{C}(G), \varrho)$, where ϱ is defined in previous tvrzeni, is complete metric space and $\mathcal{H}(G)$ is closed subspace.

ϱ is not canonical, it depends on the choice of $\{K_m\}$.

The convergence / the topology on $\mathcal{C}(G)$ is given by the system of seminorms $\|\cdot\|_K$ for any compact K in G.

Věta 3.2 (Moore–Osgood, Montöl)

Let $G \subset \mathbb{C}$ be open and let $\{f_n\} \subset \mathcal{H}(G)$ be locally equibounded (i.e. on every compact K in G $\{f_n\}$ is equibounded). Then there is $\{f_{n_k}\}$ which converges locally uniformly on G.

Důkaz

First step: Let $\overline{U(z_0, 2r)} \subset G$ and $\varphi(t) := z_0 + 2re^{it}$, $t \in [0, 2\pi]$. Let $z_1, z_2 \in \overline{U(z_0, r)}$. Then by the Cauchy formula we get $f_n(z_j) = \frac{1}{2\pi i} \int_\varphi \frac{f_n(z)}{z-z_j} dz$. There is $M \in (0, +\infty)$ such that $\forall n \in \mathbb{N}$ $|f_n| \leqslant M$ on $\langle\varphi\rangle$. Then we have

$$|f_n(z_1) - f_n(z_2)| = \frac{1}{2\pi} \left| \int_\varphi f_n(z) \cdot \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz \right| \leqslant$$

$$\leqslant \frac{2\pi \cdot 2r}{2\pi} \cdot M \cdot \frac{|z_1 - z_2|}{r^2}$$

$\left(\left|\frac{1}{z-z_1} - \frac{1}{z-z_2}\right| = \left|\frac{z_1-z_2}{(z-z_1)\cdot(z-z_2)}\right| \leqslant \frac{|z_1-z_2|}{r^2}\right)$.

By this $\{f_n\}$ are equicontinuous on $\overline{U(z_0, r)}$, and by Arzela–Ascoli, there is $\{f_{n_k}\}$ which is uniformly convergent on $\overline{U(z_0, r)}$.

Second step: Let us cover the set G by $U_j = U(z_j, r_j)$, $j \in \mathbb{N}$, such that $\overline{U(z_j, 2r_j)} \subset G$. Then use a diagonal choice: 1. By first step choose $\left\{f_{n_k^1}\right\}$ of $\{f_n\}$ such that $\left\{f_{n_k^1}\right\}$ converges uniformly on $\overline{U_1}$. 2. By first step choose $\left\{f_{n_k^2}\right\}$ subsequence of $\left\{f_{n_k^1}\right\}$ such that $\left\{f_{n_k^2}\right\}$ converges uniformly on $\overline{U_2}$ and so on.

Then $\left\{f_{n_k^k}\right\}_{k=1}^\infty$ converges uniformly on any $\overline{U_j}$, i.e., locally uniformly on G. □

Definice 3.2

Let E be a (complex) linear space and let $\mathcal{P}$ be a system of seminorms on E. Then $(E, \mathcal{P})$ is called locally convex space (LCS). In $(E, \mathcal{P})$ we define:

- convergence: $f_n \to f \Leftrightarrow \forall p \in \mathcal{P} : p(f_n - f) \to 0$;
- topology τ is the weakest topology on E for which all $p \in \mathcal{P}$ are continuous;
- $\mathcal{F} \subset E$ is bounded if $\mathcal{F}$ is bounded with respect to any $p \in \mathcal{P}$, i.e.,

$$\forall p \in \mathcal{P}\ \exists C \in (0, +\infty) : p(f) \leqslant C\ \forall f \in \mathcal{F};$$

- the dual space to $(E, \mathbb{P})$ is defined as

$$E^* := \{L : E \to \mathbb{C} | L \text{ linear and continuous}\}.$$

Poznámka
$\mathcal{C}(G)$ is the so-called Fréchet space, i.e., completely metrizable LCS. So is $\mathcal{H}(G)$ because $\mathcal{H}(G)$ is closed subspace of $\mathcal{C}(G)$.

Topology τ on $\mathcal{C}(G)$ is generated by the system of seminorms

$$\mathcal{P} := \{\|\cdot\|_K | K \text{ is compact in } G\}.$$

$U \subset \mathcal{C}(G)$ is neighbourhood of $f \in \mathcal{C}(G)$ iff there are a compact $K \in G$ and $\varepsilon > 0$ such that

$$U \supset U_{K,\varepsilon}(f) := \{g \in \mathcal{C}(G) | \|g - f\|_K < \varepsilon\}.$$

Důkaz
„ $\Longleftarrow$ “: obvious. „ $\Longrightarrow$ “: There are $m \in \mathbb{N}$, compact, $K_1, \ldots, K_m$ in G and $\varepsilon_1, \ldots, \varepsilon_m > 0$ such that

$$U \supset \bigcap_{j=1}^{m} U_{K_j,\varepsilon_j}(f) \supset U_{K,\varepsilon}(f),$$

where $K := K_1 \cup \ldots \cup K_m$ and $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_m\} > 0$. $\square$

Poznámka
Let $X = \mathcal{H}(G)$. Then in the sense of (LCS) $\mathcal{F} \subset \mathcal{H}(G)$ is bounded iff in the functions of $\mathcal{F}$ are locally equibounded on G. By the Montal theorem, we get $\overline{\mathcal{F}}$ is a compact in $\mathcal{H}(G)$. Easily we get that $\mathcal{F} \subset X$ is compact iff $\mathcal{F}$ is closed and bounded in X.