

# 1 Locally convex spaces

## Definition 1.1 (Topological vector space (TVS))

A Topological vector space over  $\mathbb{F}$  is a pair  $(X, \tau)$ , where  $X$  is a vector space over  $\mathbb{F}$  and  $\tau$  is a topology on  $X$  with the following two properties:

1. The mapping  $(x, y) \mapsto x + y$  is a continuous mapping of  $X \times X$  into  $X$ ;
2. The mapping  $(t, x) \mapsto tx$  is a continuous mapping of  $\mathbb{F} \times X$  into  $X$ ;

We also denote Hausdorff topological vector space by HTVS. And the symbol  $\tau(\mathbf{o})$  will denote the family of all the neighbourhoods of  $\mathbf{o}$  in  $(X, \tau)$ .

## Definition 1.2 (Locally convex (LCS, HLCS))

Let  $(X, \tau)$  be a TVS. The space  $X$  is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

## Tvrzení 1.1 (Observation)

If  $(X, \tau)$  is a LCS, then  $\tau$  is translation invariant ( $U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x+U \in \tau)$ ). Hence  $\tau$  is determined by  $\tau(\mathbf{o})$ .

## Definition 1.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

$X$  is a vector space,  $A \subset X$ . Then  $A$  is

- convex if  $tx + (1 - t)y \in A$  for  $x, y \in A, t \in [0, 1]$ ;
- symmetric if  $A = -A$ ;
- balanced if  $\alpha A \subset A$  for  $\alpha \in \mathbb{F}, |\alpha| \leq 1$ ;
- absolutely convex if it is convex and balanced;
- absorbing if  $\forall x \in X \exists t > 0 : \{sX | s \in [0, t]\} \subset A$ .

## Definice 1.4

$\text{co}(A)$  = convex hull,  $\text{b}(A)$  = balanced hull,  $\text{aco}(A)$  = absolutely convex hull.

## Tvrzení 1.2

$X$  is a metric space over  $\mathbb{F}$ ,  $A \subset X$ . Then:

- (a) If  $\mathbb{F} = \mathbb{R}$ , it holds  $A$  is absolutely convex  $\Leftrightarrow A$  is convex and symmetric.
- (b)  $\text{co } A = \{t_1x_1 + \dots + t_kx_k \mid x_1 \dots x_k \in A, t_1 \dots t_k \geq 0, t_1 + \dots + t_k = 1, k \in \mathbb{N}\}$ .
- (c)  $\text{b}(A) = \{\alpha x \mid x \in A, \alpha \in \mathbb{F}, |\alpha| \leq 1\}$ .
- (d)  $\text{aco}(A) = \text{co}(\text{b}(A))$ .
- (e)  $A$  is convex  $\Leftrightarrow (s+t)A = sA + tA$  for all  $s, t > 0$ .

*Důkaz (a)*

„ $\Rightarrow$ “: trivial (and it also holds for  $\mathbb{F} = \mathbb{C}$ ). „ $\Leftarrow$ “: Assume  $A$  is convex and symmetric. We show that  $A$  is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leq 1 \Rightarrow \alpha x \in A.$$

And  $x \in A, -x \in A$ , so the segment from  $x$  to  $-x$  is contained in  $A$  ( $\alpha x = \frac{1-\alpha}{2}(-x) + \frac{1+\alpha}{2}x \in A$ ).  $\square$

*Důkaz (b)*

„ $\subseteq$ “: by induction on  $k$ :

$$t_1x_1 + \dots + t_{k+1}x_{k+1} = (t_1 + \dots + t_k) \frac{t_1x_1 + \dots + t_kx_k}{t_1 + \dots + t_k} + t_{k+1}x_{k+1}.$$

„ $\supseteq$ “: the set on the RHS is convex and contain  $A$ .  $\square$

*Důkaz (c)*

„ $\supseteq$ “: clear. „ $\subseteq$ “: RHS is a balanced set.  $\square$

*Důkaz (d)*

„ $\supseteq$ “: clear. „ $\subseteq$ “ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c):

$$\text{co}(\text{b}(A)) = \{t_1\alpha_1x_1 + \dots + t_k\alpha_kx_k \mid x_1, \dots, x_k \in A, |\alpha_j| \leq 1, t_j \geq 0, t_1 + \dots + t_k = 1\}$$

is clearly balanced.)  $\square$

*Důkaz (e)*

„ $\Rightarrow$ “: „ $\subseteq$ “: always, „ $\supseteq$ “:  $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$ .

„ $\Leftarrow$ “: in particular  $\forall t \in (0, 1)$ :  $tA + (1-t)A \subset A$ , it is the definition of convexity.  $\square$

### Tvrzení 1.3

Let  $(X, \tau)$  be a LCS,  $U \in \tau(\mathbf{o})$ . Then

- (i)  $U$  is absorbing.
- (ii)  $\exists V \in T(0) : V + V \subset U$ .
- (iii)  $\exists V \in \tau(\mathbf{o})$  absolutely convex, open:  $V \subset U$ .

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Důkaz (i)

$$\begin{aligned} x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V \text{ a neighbourhood of } 0 \text{ in } \mathbb{F} : V \cdot x \subset U \implies \\ \implies \exists t > 0 : [0, t] \subset V. \end{aligned}$$

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Důkaz (ii)

$$\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W_2 \subset U.$$

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Take  $V = W_1 \cap W_2$ .

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Důkaz

$$\begin{aligned} \exists U_0 \in \tau(\mathbf{o}) \text{ convex, } U_0 \subset U : \mathbf{o} \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0 \exists W \in \tau(\mathbf{o}) \text{ open :} \\ \forall \lambda, |\lambda| < c : \lambda W \subset U_0. \end{aligned}$$

$V_1 := \bigcup_{0 < |\lambda| < 1} \lambda W$ . Then  $V_1 \in \tau(0)$  open, balanced,  $V_1 \subset U_0$ . Let  $V := \text{co } V_1$ . Then  $V$  is absolutely convex (the previous proposition (d)),  $V \subset U_0 \subset U$  (as  $V_0$  is convex).  $V \in \tau(\mathbf{o})$  as  $V \supset V_1$ . „ $V$  is open“:

$$V = \bigcup \{t_1 x_1 + \dots + t_n x_n + t_{n+1} V_1 \mid t_1, \dots, t_{n+1} \geq 0, t_1 + \dots + t_{n+1} = 1, x_1, \dots, x_n \in V_1\}$$

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□

### Věta 1.4

1. Let  $(X, \tau)$  be a LCS. Then there is  $\mathcal{U}$ , a base of neighbourhoods of  $\mathbf{o}$  with properties:

- the elements of  $\mathcal{U}$  are absorbing, open, absolutely convex;
- $\forall U \in \mathcal{U} \exists V \in \mathcal{U} : 2V \subset U$ .

If  $X$  is Hausdorff, then  $\bigcap \mathcal{U} = \{\mathbf{o}\}$ .

2. Let  $X$  be a vector space,  $\mathcal{U}$  a nonempty family of subsets of  $X$  satisfying:

- the elements of  $\mathcal{U}$  are absorbing and absolutely convex;
- $\forall U \in \mathcal{U} \exists V \in \mathcal{U} : 2V \subset U$ ;
- $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U} : W \subset U \cap V$ .

Then there is a unique topology  $\tau$  on  $X$  such that  $(X, \tau)$  is LCS and  $\mathcal{U}$  is a base of neighbourhoods of  $\mathbf{o}$ . Further, if  $\bigcap \mathcal{U} = \{\mathbf{o}\}$ , the  $\tau$  is Hausdorff.

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Důkaz (1.)

Let  $\mathcal{U}$  be the family of all open absolutely convex neighbourhoods of  $\mathbf{o}$ . The previous proposition (iii) gives us  $\mathcal{U}$  is a base of neighbourhoods of  $\mathbf{o}$ , (1) gives us elements of  $\mathcal{U}$  are absorbing, so the first item holds. (ii) gives us  $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$ .

Assume  $X$  is Hausdorff:  $x \in X \setminus \{\mathbf{o}\} \xrightarrow{\text{Hausdorff}} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$ . □

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Důkaz (2.)

Set  $\tau = \{G \subset X \mid \forall x \in G \exists U \in \mathcal{U} : x + U \subset G\}$ . This is a unique possibility so uniqueness is clear.

„ $\tau$  is topology“:  $\emptyset, X \in \tau$  and  $\tau$  is closed to arbitrary union (clear).  $\tau$  is closed to finite intersections by third item ( $G_1, G_2 \in \tau, x \in G_1 \cap G_2 \dots U_1, U_2 \in \tau, x + U_1 \subset G_1, x + U_2 \subset G_2; \exists V \in \mathcal{U} : V \subset U_1 \cap U_2$ , then  $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$ ).

„Elements of  $\mathcal{U}$  are neighbourhoods of  $\mathbf{o}$ “:  $U \in \mathcal{U}. V := \{x \in U \mid \exists W \in \mathcal{U} : x + W \subset U\}$ . Then  $V \subset U, 0 \in V$  (take  $W = U$ ).  $V \in \tau$  ( $x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$ ; let  $\tilde{W} \in \mathcal{U}$  such that  $2\tilde{W} \subset W$ , then  $x + \tilde{W} \subset V$ , because  $y \in \tilde{W} \implies x + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$ ).

„ $\mathcal{U}$  is a base of neighbourhood of  $\mathbf{o}$ “: now clear.

„ $(X, \tau)$  is a TVS“:  $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$ . Then  $(x + V) + (y + V) \subset x + y + 2V \subset x + y + U \subset G$ .  $\lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G$ ;  $\exists V \in \mathcal{U} : 2V \subset U$ ;  $V$  is absorbing  $\implies \exists c > 0 \forall t \in [0, c] : tx \in V$ ;  $V$  balanced  $\implies \forall \mu \in \mathbb{F}, |\mu| \leq c : \mu x \in V$ ; assume  $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1}V$ ,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda)y}_{(\mu - 1) \cdot (\mu + \frac{1}{|\lambda|+1})V} + \underbrace{\lambda(y - x)}_{\in \frac{\lambda}{|\lambda|+1}V \subset V}.$$

„Local convexity“: by first item:  $\forall U \in \mathcal{U} : U$  is convex.

Assume  $\bigcap \mathcal{U} = \{\mathbf{o}\}$ . Take  $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$ . Take  $V \in \mathcal{U} : 2V \subset U$ . Then if  $(x + V) \cap (y + V) = \emptyset, x + v_1 = y + v_2, x - y = v_2 - v_1 \in V + V = 2V \subset U \nmid$ . □

**Věta 1.5** (On the topology generated by a family of seminorms)

Let  $X$  be a vector space and let  $\mathcal{P}$  be a family of seminorms on  $X$ . Then there is a unique topology  $\tau$  on  $X$  such that  $(X, \tau)$  is a LCS and

$$\mathcal{U} = \{ \{x \in X \mid p_1(x) < c_1, \dots, p_k(x) < c_k\} \mid p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0 \}$$

is a base of neighbourhood of  $\mathbf{o}$ .

$$(X, \tau) \text{ is Hausdorff} \Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\} \exists p \in \mathcal{P}, p(x) > 0.$$

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*Důkaz*

Use the previous theorem 2. on  $\mathcal{U}$ : The sets are absolutely convex (by properties of seminorms). „Absorbing“:  $U = \{x \in X \mid p_1(x) < c_1, \dots, p_k(x) < c_k\}$ . Take  $x \in X$ ,  $j \in [k]$ . Then  $p_j(x) \in (0, \infty)$  as for  $t > 0$ :  $p_j(t \cdot x) = t \cdot p_j(x)$  and  $\exists c > 0$  such that  $c \cdot p_j(x) < c_j$  for  $j \in [k]$ . Now for  $t \in [0, c]$ :  $tx \in U$ .

$$U = \{x \in X \mid p_1(x) < c_1, \dots, p_k(x) < c_k\}. \text{ Take } V = \{x \in X \mid p_1(x) < \frac{c_1}{2}, \dots, p_k(x) < \frac{c_k}{2}\}.$$

$$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U} \text{ trivially.}$$

„Hausdorffness“:

$$\bigcap U = \{x \in X \mid \forall p \in \mathcal{P} : p(x) = 0\}.$$

„ $\supseteq$ “ clear. „ $\subseteq$ “: Assume  $y \in X$ ,  $p \in \mathcal{P} : p(y) > 0$ :  $U = \{x \in X \mid p(x) < p(y)\} \in \mathcal{U} \implies y \notin U$ . □

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*Například*

$(X, \|\cdot\|)$  is a normed space, then its topology is generated by  $\mathcal{P} = \{\|\cdot\|\}$ .

The topology on  $\mathbb{F}^\Gamma$  is generated by seminorms  $p_\gamma(f) = |f(\gamma)|$ ,  $f \in \mathbb{F}^\Gamma$  ( $\gamma \in \Gamma$ ).

$C(\mathbb{R}, \mathbb{F})$  + topology is generated by this sequence of seminorms:  $p_N(f) = \max_{x \in [-N, N]} |f(x)|$ .

**Definice 1.5** (Minkowski functional)

$X$  vector space,  $A \subset X$  convex absorbing. Then Minkowski functional of set  $A$  is defined by the formula  $p_A(x) := \inf \{\lambda > 0 \mid x \in \lambda \cdot A\}$ .

**Lemma 1.6**

Let  $X$  be LCS,  $A \subset X$  convex set.

$$x \in \overline{A}, y \in \text{int } A \implies \{tx + (1-t)y \mid t \in [0, 1]\} \subset \text{int } A.$$

*Důkaz*

WLOG  $y = 0$ .  $t = 0$  clear,  $0 \in \text{int } A$ .  $t \in (0, 1)$ :

Fix  $U$ , an open absolutely convex neighbourhood of  $\mathbf{o}$  such that  $U \subset A$ . Then  $x + \frac{1-t}{t}U$  is a neighbourhood of  $x \implies \exists z \in (x + \frac{1-t}{t}U) \cap A$ , i.e.  $\exists m \in U$   $z = x + \frac{1-t}{t}m \implies -m \in U \subset m + A$ . Find  $V$  an absolutely convex open neighbourhood of  $\mathbf{o}$  such that  $-m + V \subset U \subset A \implies tz + (1-t)(-m + V) \subset A$  (an open set containing  $tx$ ). (Because  $tx = tz - (1-t)a$ .) And that's it.  $\square$

### **Tvrzení 1.7** (on the Minkowski functional of a convex neighborhood of zero)

Let  $X$  be LCS,  $A \subset X$  a convex neighbourhood of  $\mathbf{o}$ . Then:

Clearly:  $[p_A < 1] \subset A \subset [p_A \leq 1]$ .

*Důkaz*

„ $[p_A < 1] = \text{int } A$ “: „ $\subseteq$ “:  $p_A(x) < 1 \implies \exists c > 1$  such that  $cx \in A \implies x = \frac{1}{c}cx \in \text{int } A$ . „ $\supseteq$ “:  $x \in \text{int } A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$ .  $U$  absorbing  $\implies \exists \alpha > 0 : \alpha x \in U$ . Then  $(1 + \alpha)x \in A \implies p(x) \leq \frac{1}{1+\alpha} < 1$ .

„ $[p_A \leq 1] = \overline{A}$ “: „ $\subseteq$ “:  $p_A(x) \leq 1 \implies \forall n \in \mathbb{N} : p_x((1 - \frac{1}{n})x) = (1 - \frac{1}{n})p_A(x) \leq 1$ .  $(1 - \frac{1}{n})x \in \text{int } A \implies x \in \overline{\text{int } A} \subset \overline{A}$ . „ $\supseteq$ “:  $x \in \overline{A} \implies \forall n \in \mathbb{N} : (1 - \frac{1}{n})x \in \text{int } A$ , so,  $p_A((1 - \frac{1}{n})x) < 1 \xrightarrow{n \rightarrow \infty} p_A(x) \leq 1$ .  $\square$

$p_A$  is continuous on  $X$ .

*Důkaz*

$[p_A < c] = \emptyset$  if  $c \leq 0$  and  $c \cdot \text{int } A$  if  $c > 0$ .  $[p_A > c] = X$  if  $c < 0$ ,  $X \setminus (c \cdot \overline{A})$  if  $c > 0$ , and  $\bigcup_{t>0} X \setminus t\overline{A}$  if  $c = 0$ . All these sets are open.  $\square$

$$p_A = p_{\overline{A}} = p_{\text{int } A}.$$

*Důkaz*

$\text{int } A \subset A \subset \overline{A} \implies p_{\overline{A}} \leq p_A \leq p_{\text{int } A}$ . „Conversely“: Assume that  $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : (1 - \frac{1}{n})x \in d \cdot \text{int } A \implies (1 - \frac{1}{n})p_{\text{int } A}(x) \leq d \implies p_{\text{int } A}(x) \leq d < c$ .  $\square$

*Důsledek*

Any LCS  $(X)$  is completely regular.

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Důkaz

$x \in X$ ,  $U$  an open neighbourhood of  $x$ . Take  $V$  a convex neighbourhood of  $\mathbf{o}$  such that  $x + V \in U$ .  $f(y) := \min \{1, p_V(y - x)\}$ . The  $f$  is continuous by the previous proposition,  $f(x) = 0$ .

$$y \in X \setminus U \implies y - x \notin V \implies p_V(y - x) \geq 1 \implies f(y) = 1.$$

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□

### Věta 1.8 (On generating of locally convex topologies)

Let  $(X, \tau)$  be a LCS. Let  $\mathcal{P}_\tau$  be the family of all continuous seminorms on  $(X, \tau)$ . Then topology generated by  $\mathcal{P}_\tau$  coincides with  $\tau$ .

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Důkaz

Let  $\tau_1$  be topology induced by  $\mathcal{P}_\tau$ .  $\tau_1 \subset \tau$  (seminorms from  $\mathcal{P}_\tau$  are  $\tau$ -continuous, hence the sets from theorem 5? are  $\tau$ -open). „ $\tau \subset \tau_1$ “: Let  $U \in \tau(\mathbf{o}) \implies \exists V$  a neighbourhood of  $\mathbf{o}$  such that  $V \subset U$ . The  $p_V \in \mathcal{P}_\tau$  (from the previous proposition is continuous)  $\implies [p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$ . □

### Tvrzení 1.9

$X$  a vector space.

1.  $p$  is seminorm  $\implies [p < 1]$  is absolutely convex, absorbing, and  $p_{[p < 1]} = p$ .
2.  $p, q$  are seminorms, then  $p \leq q \Leftrightarrow [p < 1] \supset [q < 1]$ .
3.  $\mathcal{P}$  a set of seminorms generated by a topology  $\tau$ .  $p$  a seminorm on  $X$ . Then  $p$  is  $\tau$ -continuous  $\Leftrightarrow \exists p_1, \dots, p_k \in \mathcal{P} \exists c > 0 : p \leq c \cdot \max \{p_1, \dots, p_k\}$ .

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Důkaz (1.)

Absolutely convex and absorbing is clear.

$$p_{[p < 1]}(x) = \inf \{ \lambda > 0 \mid x \in \lambda [p < 1] \} = \inf \{ \lambda > 0 \mid x \in [p < \lambda] \} = p(x).$$

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□

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Důkaz (2.)

„ $\implies$ “ trivial. „ $\Leftarrow$ “:  $[p < 1] \supset [q < 1] \implies p = p_{[p < 1]} \leq p_{[q < 1]} = q$ . □

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┌ *Důkaz* (3.)

„ $\Leftarrow$ “:  $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = [p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}]$ , which is a  $\tau$ -open set  $\implies A$  is a neighbourhood of  $\mathbf{o} \implies p = p_A$  is continuous (by 1. and the previous proposition).

„ $\implies$ “:  $p$  is continuous  $\implies [p < 1]$  is neighbourhood of  $\mathbf{o}$  ( $p(\mathbf{o}) = 0$ )  $\implies \exists p_1, \dots, p_k \in \mathcal{P} \exists c_1, \dots, c_k > 0$  such that  $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c} \max\{p_1, \dots, p_k\} < 1]$  ( $c = \min\{c_1, \dots, c_k\}$ ). Use 2. for seminorms  $p, \frac{1}{2 \max\{p_1, \dots, p_k\}}$  and get  $p \leq \frac{1}{c} \max\{p_1, \dots, p_k\}$ .  $\square$

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## 1.1 Continuous and bounded linear mapping

### Tvrzení 1.10

$(X, \tau), (Y, \mathcal{U})$  LCS,  $L : X \rightarrow Y$  linear. Then the following assertions are equivalent:

1.  $L$  is continuous;
2.  $L$  is continuous at  $\mathbf{o}$ ;
3.  $L$  is uniformly continuous.

┌ *Důkaz*

„1.  $\implies$  2.“ trivial, „2.  $\implies$  3.“ assume  $L$  continuous at  $\mathbf{o}$ . Then, given  $U \in \mathcal{U}(\mathbf{o})$ , there is  $V \in \tau(\mathbf{o})$  such that  $L(V) \subset U$ . Take  $x, y \in X$  such that  $x - y \in V$ . Then  $L(x) - L(y) = L(x - y) \in U$  and that's continuous. „3.  $\implies$  1.“ trivial.  $\square$

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### Tvrzení 1.11

$L : X \rightarrow Y$  linear.  $L$  is continuous  $\Leftrightarrow \forall q$  a continuous seminorm on  $Y \exists p$  a continuous seminorm on  $X : \forall x \in X : q(L(x)) \leq p(x)$ .

┌ *Důkaz*

„ $\implies$ “:  $L$  continuous,  $q$  a continuous seminorm on  $Y$ , the  $p(x) = q(L(x))$  is a continuous seminorm on  $X$ . „ $\Leftarrow$ “: By the previous proposition it is enough „ $L$  is continuous at  $\mathbf{o}$ “:  $U$  neighbourhood of  $\mathbf{o}$  in  $Y$ ,  $\exists V \subset U$  an absolutely convex neighbourhood of  $\mathbf{o}$ .  $q := p_V$  is a continuous seminorm. Let  $p$  be a continuous seminorm on  $X$  such that  $q \circ L \leq p$ .  $W := [p < 1]$  a neighbourhood of  $\mathbf{o}$  in  $X$  and  $L(W) \subset V \subset U$ .  $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$ .  $\square$

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### Tvrzení 1.12

Let  $(X, \tau)$  be a LCS over  $\mathbb{F}$  and let  $L : X \rightarrow \mathbb{F}$  be a linear mapping. Then the following assertions are equivalent:



- $L$  is continuous;
- $\text{Ker } L$  is a closed subspace of  $X$ ;
- there exists  $U \in \tau(\mathbf{o})$  such that  $L(U)$  is a bounded subset of  $\mathbb{F}$ .

If  $\mathcal{P}$  is a family of seminorms generating the topology of  $X$ , the continuity of  $L$  is also equivalent to

$$\exists p_1, \dots, p_k \in \mathcal{P} \exists c > 0 \forall x \in X : |L(x)| \leq c \cdot \max \{p_1(x), \dots, p_k(x)\}.$$

If  $L$  is discontinuous, then  $\text{Ker } L$  is a dense subspace of  $X$

┌ *Důkaz*

└ TODO!!!

□

### Definice 1.6

Let  $(X, \tau)$  be a LCS and let  $A \subset X$ . The set  $A$  is said to be bounded in  $(X, \tau)$ , if for any  $U \in \tau(\mathbf{o})$ , there exists  $\lambda > 0$  such that  $A \subset \lambda U$ .

### Lemma 1.13

Let  $(X, \tau)$  be a LCS and let  $A \subset X$ . Then the set  $A$  is bounded in  $X$  if and only if each continuous seminorm  $p$  on  $X$  is bounded on  $A$ . (It is enough to test it for a family of seminorms generating the topology of  $X$ .)

┌ *Důkaz*

└ TODO!!!

□

### Tvrzení 1.14

Let  $(X, \tau)$  and  $(Y, \sigma)$  be LCS over  $\mathbb{F}$  and let  $L : X \rightarrow Y$  be a linear mapping. Consider the following two assertions: 1.  $L$  is continuous. 2. for any bounded subset  $A \subset X$  its image  $L(A)$  is bounded in  $Y$  (i.e.,  $L$  is a bounded mapping).

Then 1.  $\implies$  2.. In case  $\tau$  is generated by a translation invariant metric on  $X$ , then 1.  $\Leftrightarrow$  2..

┌ *Důkaz*

└ TODO!!!

□

### Definice 1.7 (Isomorphism, isomorphic LCS)

Let  $(X, \tau)$  and  $(Y, \sigma)$  be LCS over  $\mathbb{F}$  and let  $L : X \rightarrow Y$  be a linear mapping. Then mapping  $L$  is said to be an isomorphism of  $X$  into  $Y$ , if  $L$  is continuous, one-to-one and  $L^{-1}$  is continuous on  $L(X)$ ; and isomorphism of  $X$  onto  $Y$  if  $L$  is continuous, one-to-one, onto and  $L^{-1}$  is continuous on  $Y$ .

The spaces  $X$  and  $Y$  are said to be isomorphic if there is an isomorphism of  $X$  onto  $Y$ .

## 1.2 Spaces of finite and infinite dimension

### Tvrzení 1.15

Let  $X$  be a HLCS of finite dimension. If  $Y$  is any LCS and  $L : X \rightarrow Y$  is any linear mapping, then  $L$  is continuous. The space  $X$  is isomorphic to  $\mathbb{F}^n$ , where  $n = \dim X$ .

┌

*Důkaz*

If  $\dim X = 0$ , i.e.,  $X = \{\mathbf{o}\}$ , it is trivial. Assume  $n := \dim X \in \mathbb{N}$ . Fix a basis  $x_1, \dots, x_n$  of  $X$ . Define  $T : (\mathbb{F}^n, \|\cdot\|_2) \rightarrow X$  by  $T(\lambda_1, \dots, \lambda_n) = \lambda_1 x_1 + \dots + \lambda_n x_n$ .

$T$  is clearly a linear bijection of  $\mathbb{F}^n$  onto  $X$  (since  $x_1, \dots, x_n$  is a basis). „ $T$  is continuous“:

1. The mapping  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j$  is continuous  $\mathbb{F}^n \rightarrow \mathbb{F}$  for each  $j$ . 2.  $\forall x \in X$  the mapping  $x \mapsto \lambda \cdot x$  is continuous  $\mathbb{F} \rightarrow X$ . 3. By composition  $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j \cdot x_j$  is continuous  $\mathbb{F}^n \rightarrow X$ .

Hence  $T$  is the sum of  $n$  continuous mappings  $\mathbb{F}^n \rightarrow X$ . It is enough to show that the sum of two continuous mapping is continuous and use mathematical induction: „ $\Omega$  topological space,  $X$  LCS,  $f_1, f_2 : \Omega \rightarrow X$  continuous  $\implies f_1 + f_2$  is continuous“:  $t \in \Omega$  arbitrary, let  $G \subset X$  be open such that  $f_1(t) + f_2(t) \in G \implies \exists U$  neighbourhood of  $\mathbf{o}$ :  $V + V \subset U$ .  $f_i$  continuous at  $t \implies \exists W_i$  open in  $\Omega$ ,  $t \in W_i$ :  $f_i(W_i) \subset f_i(t) + V$ .  $W := W_1 \cap W_2$  open in  $\Omega$ ,  $t \in W$ .  $s \in W \implies f_1(s) + f_2(s) \in f_1(t) + V + f_2(t) + V \subset f_1(t) + f_2(t) + U \subset G$ .

So  $T$  is continuous. „ $T^{-1}$  is continuous as well“:  $S_{\mathbb{F}^n}$  the sphere of  $\mathbb{F}^n$  is compact in  $\mathbb{F}^n \implies T(S_{\mathbb{F}^n})$  is compact in  $X$ .  $X$  Hausdorff  $\implies T(S_{\mathbb{F}^n})$  is closed. Clearly  $\mathbf{o} \notin T(S_{\mathbb{F}^n})$  ( $T$  is a linear bijection and  $\mathbf{o} \notin S_{\mathbb{F}^n}$ )  $\implies \exists U$  an absolutely convex neighbourhood of  $\mathbf{o}$  in  $X$  such that  $U \cap T(S_{\mathbb{F}^n}) = \emptyset$ . We claim that  $U \subset T(U_{\mathbb{F}^n})$  (open unit ball). Assume  $x \in U \setminus T(U_{\mathbb{F}^n}) \implies z := T^{-1}(x)$  satisfies  $\|z\|_2 \geq 1$ . Then  $\frac{z}{\|z\|_2} \in S_{\mathbb{F}^n}$ ,  $T\left(\frac{z}{\|z\|_2}\right) = \frac{1}{\|z\|_2} \cdot T(z) = \frac{1}{\|z\|_2} \cdot x \in U$  ( $U$  is balanced)  $\implies \frac{1}{\|z\|_2} \cdot x \in U \cap T(S_{\mathbb{F}^n})$  a contradiction. So  $T^{-1}$  is continuous at  $\mathbf{o}$  ( $(T^{-1})^{-1}(U_{\mathbb{F}})$  is a neighbourhood of  $\mathbf{o}$  and the same for all multiplies).

$\implies T^{-1}$  is continuous. So,  $T$  is an isomorphism and second part is proven.

By the second part WLOG  $X = \mathbb{F}^n$ . Let  $L : \mathbb{F}^n \rightarrow Y$  be linear,  $Y$  LCS. Let  $e_1, \dots, e_n$  be the canonical basis of  $\mathbb{F}^n$ . Then  $L(\lambda_1, \dots, \lambda_n) = \lambda_1 L(e_1) + \dots + \lambda_n L(e_n)$ . This is continuous (by the same argument as in the second part for  $T$ ).  $\square$

└

*Důsledek*

Let  $X$  be a HLCS. Then any its finite-dimensional subspace is closed.

┌ *Důkaz*  
└ TODO!!!

□

### Definice 1.8

Let  $(X, \tau)$  be a LCS and let  $A \subset X$ . Then set  $A$  is said to be totally bounded (or precompact), if for any  $U \in \tau(\mathbf{o})$  there exists a finite set  $F \subset X$  such that  $A \subset F + U$ .

*Poznámka*

Any compact set in any LCS is totally bounded. Any totally bounded set is bounded.

### Lemma 1.16

Let  $(X, \tau)$  be a LCS and let  $A \subset X$ . Then following assertions are equivalent:

1.  $A$  is totally bounded in  $X$ .
2.  $A$  is totally bounded in  $(X, p)$ , for any continuous seminorm  $p$  on  $X$ .
3. For any continuous seminorm  $p$  on  $X$  and any sequence  $(x_n)$  in  $A$  there is a subsequence  $x_{n_k}$  which is Cauchy with respect to  $p$ , i.e.  $\forall \varepsilon > 0 \exists k_0 \forall k, l \geq k_0 : p(x_{n_k} - x_{n_l}) < \varepsilon$ .

┌ *Důkaz*  
└ TODO!!!

□

### Věta 1.17

Let  $X$  be a HLCS. Then following assertions are equivalent:

1.  $\dim X < \infty$ .
2. There exists a compact neighbourhood of zero in  $X$ .
3. There exists a totally bounded neighbourhood of zero in  $X$ .

┌  
Důkaz

„1.  $\implies$  2.“ by the proposition above. „2.  $\implies$  3.“ trivial. „3.  $\implies$  1.“: Let  $U$  be an absolutely convex totally bounded open neighbourhood of  $\mathbf{o}$ . Then  $\frac{1}{2}U$  is also a neighbourhood of  $\mathbf{o}$ , so, there is  $F \subset X$  finite with  $U \subset F + \frac{1}{2}U$ . Set  $Y := \text{LO } F$ . We claim that  $Y = X$ .

„ $\forall n \in \mathbb{N} : U \subset Y + 2^{-n} \cdot U$ “: By induction:  $n = 1$  follows by the previous.  $n \mapsto n + 1$ : suppose  $U \subset Y + 2^{-n} \cdot U$ . Then

$$\begin{aligned} U &\subset Y + 2^{-n} \cdot U = Y + 2^{-n+1} \cdot \left( \frac{1}{2}U \right) \subset Y + 2^{-n+1} \left( \frac{1}{2} \left( Y + \frac{1}{2}U \right) \right) = \\ &= Y + 2^{-n+1} \left( Y + \frac{1}{4} \cdot U \right) = Y + 2^{-n+1} \cdot Y + 2^{-n+1} \cdot \frac{1}{4} \cdot U = Y + 2^{-n+1} \cdot U. \end{aligned}$$

( $Y$  is linear subspace.)

If  $Y \neq X$ , then  $\exists x \in X \setminus Y$ . Since  $U$  is absorbing  $\exists t > 0$  such that  $tx \in U$ . So  $U \setminus Y \neq \emptyset$ . Fix  $x \in U \setminus Y$ .  $X$  Hausdorff,  $\dim Y < \infty \implies Y$  is closed. Hence there is  $V$ , an absolutely convex neighbourhood of  $\mathbf{o}$  such that  $x + V \subset U \setminus Y$ .

Since  $U$  is totally bounded, it is also bounded, so  $\exists n \in \mathbb{N} : U \subset 2^n \cdot V$ , i.e.  $\frac{1}{2^n} \cdot U \subset V$ . It follows that  $x + \frac{1}{2^n} \cdot U \cap Y = \emptyset \implies x \notin Y + \frac{1}{2^n} \cdot U$ . So,  $x \in U \setminus (Y + \frac{1}{2^n} \cdot U)$ , a contradiction.  $\square$

└

### 1.3 Metrizable of locally convex spaces

#### **Tvrzení 1.18**

1. Let  $(X, \tau)$  be a metrizable LCS. Then the topology  $\tau$  is generated by a sequence of seminorms  $(p_n)$  satisfying  $p_1 \leq p_2 \leq p_3 \leq \dots$

2. Let  $X$  be a vector space and let  $(p_n)$  be a sequence of seminorms on  $X$  satisfying conditions:  $p_1 \leq p_2 \leq p_3 \leq \dots$  and  $\forall x \in X \setminus \{\mathbf{o}\} \exists n : p_n(x) > 0$ . Then

$$\varrho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min \{1, p_n(x - y)\}, \quad x, y \in X$$

is a translation invariant metric on  $X$  which generates the locally convex topology on  $X$  generated by the sequence of seminorms  $(p_n)$ . Moreover, given a sequence  $(x_k)$  in  $X$  we have  $\varrho(x_k, x) \rightarrow 0 \iff \forall n \in \mathbb{N} : p_n(x_k - x) \rightarrow 0$ ; and the sequence  $(x_k)$  is Cauchy in  $\varrho$  if and only if it is Cauchy in each of the seminorms  $p_n$ .

┌  
Důkaz

└  
TODO!!!  $\square$

**Věta 1.19** (On metrizability of LCS)

Let  $(X, \tau)$  be a HLCS. Then the following assertions are equivalent:

1.  $X$  is metrizable (i.e., the topology  $\tau$  is generated by a metric on  $X$ ).
2. There exists a translation invariant metric on  $X$  generating the topology  $\tau$ .
3. There exists a countable base of neighbourhoods of  $\mathbf{o}$  in  $(X, \tau)$ .
4. The topology  $\tau$  is generated by a countable family of seminorms.

┌  
Důkaz

„2.  $\implies$  1.“ trivial. „1.  $\implies$  3.“ if  $\varrho$  a metric generating  $\tau$ , then  $U_n = \{x \in X \mid \varrho(x, 0) < \frac{1}{n}\} \implies (U_n)_n$  is a base of neighbourhoods of  $\mathbf{o}$ . „3.  $\implies$  4.“: (see the proof of the previous proposition, 1.)  $(U_n)$  base of neighbourhood of  $\mathbf{o}$ , take  $V_n \subset U_n$  absolutely convex neighbourhood of  $\mathbf{o}$ ,  $p_n = p_{V_n} \implies (p_n)$  generate  $\tau$ . „4.  $\implies$  2.“: the previous proposition

└ 2. □

**Věta 1.20** (A characterization of normable LCS)

$(X, \tau)$  is HLCS.  $X$  is normable  $\Leftrightarrow \exists U$ , a bounded neighbourhood of  $\mathbf{o}$ .

┌  
Důkaz

„ $\implies$  “:  $\tau$  generated by  $\|\cdot\|$ ,  $U := \{x \in X \mid \|x\| < 1\}$  is a bounded neighbourhood of  $\mathbf{o}$ .

„ $\Leftarrow$  “:  $U$  bounded neighbourhood of  $\mathbf{o}$ . WLOG  $U$  is absolutely convex. Then  $\frac{1}{n}U$ ,  $n \in \mathbb{N}$  is a base of neighbourhoods of  $\mathbf{o}$  ( $V$  neighbourhood of  $\mathbf{o}$ ,  $W \subset V$  an absolutely convex neighbourhood of  $\mathbf{o} \implies \exists \lambda > 0 : U \subset \lambda W$  Take  $n \in \mathbb{N}$  such that  $n > \lambda$ . Then  $U \subset n \cdot W$  so  $\frac{1}{n}U \subset W \subset V$ ). Finally,  $p_U$  is a norm generating the topology ( $U$  absolutely convex neighbourhood of  $\mathbf{o} \implies p_U$  is a continuous seminorm.  $\frac{1}{n}U = [p_U < \frac{1}{n}]$ ,  $n \in \mathbb{N}$  is a base of neighbourhood of  $\mathbf{o} \implies p_U$  generated topology of  $X$ . From  $X$  Hausdorff,  $p_U$  is a norm.) □

## 1.4 Fréchet spaces

**Definice 1.9** (Fréchet space)

A LCS whose topology is generated by a complete translation invariant metric is called Fréchet space.

**Tvrzení 1.21**

$(X, \tau)$  is a Fréchet space,  $\varrho$  any translation invariant metric on  $X$  generating  $\tau \implies \varrho$  is complete.

┌ *Důkaz*

$\varrho, d$  two translation invariant metrics generating by  $\tau$ . Idea: convergent sequences with respect to  $\varrho$  and  $d$  coincide, Cauchy sequences with respect to  $\varrho$  and  $d$  coincide.  $(x_n)$   $\varrho$ -Cauchy:  $\varepsilon > 0 \implies \{x | d(x, \mathbf{o}) < \varepsilon\}$  is a neighbourhood of  $\mathbf{o} \implies \exists \delta > 0 : \{x | \varrho(x, \mathbf{o}) < \delta\} \subset \{x | d(x, \mathbf{o}) < \varepsilon\}$ .  $\exists n_0 \forall m, n > n_0$ :

$\varrho(x_m - x_n, \mathbf{o}) = \varrho(x_m, x_n) < \delta \implies d(x_m - x_n, 0) = d(x_m, x_n) < \varepsilon \implies (x_n)$  is  $d$ -Cauchy.

└

□

## Tvrzení 1.22

$X$  Fréchet,  $A \subset X$ .  $A$  is compact  $\Leftrightarrow A$  is closed and totally bounded.

┌ *Důkaz*

Let  $\varrho$  be a complete translation invariant metric generating the topology.  $A$  is compact  $\Leftrightarrow A$  is closed and  $\varrho$ -totally bounded. But  $\varrho$ -totally boundedness = total boundedness in  $X$ .  $A$  is totally bounded in  $X \Leftrightarrow \forall U$  neighbourhood of  $\mathbf{o} \exists F \subset X$  finite  $A \subset F + U$ .  $A$  is totally bounded in  $\varrho \Leftrightarrow \forall \varepsilon > 0 \exists F \subset X$  finite such that  $A \subset \bigcup_{x \in F} U_\varrho(x, \varepsilon) = F + U_\varrho(0, \varepsilon)$ . □

## Tvrzení 1.23

$X$  LCS,  $A \subset X$  totally bounded  $\implies \text{aco } A$  is totally bounded.

┌ *Důkaz*

Let  $U$  be a neighbourhood of  $\mathbf{o}$ . Let  $V$  be an absolutely convex neighbourhood of  $\mathbf{o}$  such that  $2V \subset U \implies \exists F \subset X$  finite such that  $A \subset F + V$ . Then clearly  $\text{aco } A \subset (\text{aco } F) + V$ .  $\text{aco } F$  is compact,  $F = \{x_1, \dots, x_k\} \implies \text{aco}(F) = \text{co}(\text{b}(F)) =$

$$= \text{co} \{ \lambda x_j | j \in [k], |\lambda| \leq 1 \} = \left\{ t_1 \lambda_1 x_1 + \dots + t_n \lambda_n x_n \mid |\lambda_j| \leq 1, t_j \geq 0, \sum t_j = 1 \right\}.$$

$$B = \left\{ (\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mid |\lambda_j| \leq 1, t_j \geq 0, \sum t_j = 1 \right\}$$

a compact set in  $\mathbb{F}^n \times \mathbb{R}^n$ .  $(\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mapsto t_1 \lambda_1 x_1 + \dots + t_n \lambda_n x_n$  is a continuous map and maps  $B$  onto  $\text{aco } F$ .

$\text{aco } F$  compact  $\implies$  totally bounded  $\implies \exists H \subset X$  finite,  $\text{aco } F \subset H + V$  So  $\text{aco } A \subset \text{aco } F + V \subset H + V + V = H + 2V \subset H + U$ . □

┌ *Důsledek*

$X$  Fréchet space,  $A \subset X$  compact  $\implies \overline{\text{aco } A}$  is compact.

┌ *Důkaz*

$A$  compact  $\implies A$  is totally bounded  $\implies \text{aco } A$  is totally bounded  $\implies$  (because  $M \subset X$  any set  $\implies \overline{M} \subset M + U$ )  $\text{aco } \overline{A}$  is totally bounded  $\implies \text{aco } \overline{A}$  is compact.

( $M$  totally bounded, for any  $U \in \tau(\mathbf{o})$ :  $U$  is neighbourhood of  $\mathbf{o}$ ,  $x \in \overline{M}$ ,  $U$  absolutely convex neighbourhood of  $\mathbf{o}$ , then  $V \subseteq U$  absolutely convex such that  $2V \subset U \implies (x + U) \cap M \neq \emptyset \implies x \in M + U$ .)

Find  $F$  finite such that  $M \subset F + V \implies \overline{M} \subset M + V \subset F + V + V \subset F + U$ .  $\square$

### Věta 1.24 (Banach–Steinhaus)

Let  $X$  be a Fréchet space and let  $Y$  be LCS. Let  $(T_n)$  be a sequence of continuous linear mappings  $T_n : X \rightarrow Y$  such that  $\forall x \in X : \lim_{n \rightarrow \infty} T_n x$  exists in  $Y$ . Then  $Tx := \lim_{n \rightarrow \infty} T_n x$  define a continuous linear map  $X \rightarrow Y$ .

┌ *Důkaz*

Clear:  $T$  is a linear map  $X \rightarrow Y$ . „Continuous“: Fix  $q$  any continuous sequence on  $Y$ .

$$A_m = \{x \in X \mid \forall n \in \mathbb{N} : q(T_n x) \leq m\}.$$

Then  $A_m$  is closed, absolutely convex and  $\bigcup_{m=1}^{\infty} A_m = X$ .

TODO?

Baire category theorem  $\implies \exists m \in \mathbb{N} : \text{int } A_m \neq \emptyset \implies \exists x \in A_m \exists U$  an absolutely convex neighbourhood of  $\mathbf{o}$  such that  $x + U \subset A_m \implies -(x + U) \subset A_m \implies (A_m \text{ convex})$   
 $U \subset A_m$  ( $y \in U \implies y = \frac{1}{2}(x + y + (-x + y))$ ) „ $\implies q(Ty) \leq mp_U(y)$ “:

$$\begin{aligned} p_U(y) < c &\implies \frac{y}{c} \in U \subset A_m \implies \forall n \in \mathbb{N} : q\left(T_n \frac{y}{c}\right) \leq m \implies \\ &\implies q\left(T \frac{y}{c}\right) \leq m \implies q(Ty) \leq c \cdot m. \end{aligned}$$

└  $\square$

### Věta 1.25 (Open mapping theorem)

$X, Y$  Fréchet,  $T : X \rightarrow Y$  linear continuous surjective mapping. Then  $T$  is an open mapping

Dikaz

1. It is enough to show that  $\forall U$  neighbourhood of  $\mathbf{o}$  in  $X$ :  $T(U)$  is a neighbourhood of  $\mathbf{o}$  in  $Y$ .

2. „ $\forall U$  a neighbourhood of  $\mathbf{o}$  in  $X$ ,  $\overline{TU}$  is neighbourhood of  $\mathbf{o}$  in  $Y$ “:  $U$  an neighbourhood of  $\mathbf{o}$  in  $X$ .  $\exists V \subset U$  an absolutely convex neighbourhood of  $\mathbf{o}$ .  $V$  absorbing  $\implies$

$$\implies X = \bigcup_{n=1}^{\infty} nV \implies Y = T(X) = T\left(\bigcup_{n=1}^{\infty} n \cdot V\right) = \bigcup_{n=1}^{\infty} n \cdot T(V).$$

$Y$  Fréchet  $\implies$  by Baire category theorem

$$\exists n \in \mathbb{N} : \emptyset \neq \text{int } \overline{n \cdot T(V)} = \text{int } n \cdot \overline{T(V)} = n \cdot \text{int } \overline{T(V)} \implies \text{int } \overline{T(V)} \neq \emptyset \implies$$

$\implies \exists y \in Y \exists W$  an absolutely convex neighbourhood of  $\mathbf{o}$  in  $Y$  such that  $y + W \subset \overline{T(V)}$ .  $\overline{T(V)}$  is absolutely convex  $\implies -(y + w) \subset T(V) \implies W \subset T(V) \subset T(U)$ .

3. „ $\forall U$  neighbourhood of  $\mathbf{o}$  in  $X$ ,  $TU$  is a neighbourhood of  $\mathbf{o}$  in  $Y$ “:  $\varrho$  a translation invariant metric on  $X$ , complete, generating topology.  $U_n = \{x \in X \mid \varrho(0, x) < \frac{1}{2^n}\}$ . The  $U_n$  is a base of neighbourhoods of  $\mathbf{o}$ . It is enough to prove that  $T(U_n)$  is a neighbourhood of  $\mathbf{o}$  for each  $n \in \mathbb{N}$ . We know from 2. that  $\forall n : \overline{TU_n}$  is a neighbourhood of  $\mathbf{o}$  in  $Y$ . We will be done if we show that  $TU_{n-1} \supset \overline{TU_n}$  for each  $n \in \mathbb{N}$ .

We will prove it for  $n = 1$ : So we will ?  $TU_1 \subset TU_0$ . Fix  $y \in \overline{TU_1}$ . Since  $\overline{TU_2}$  is a neighbourhood of  $\mathbf{o}$   $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$ . So there is  $x_1 \in U_1$  such that  $y - Tx_1 \in \overline{TU_2}$ .  $\overline{TU_3}$  is a neighbourhood of  $\mathbf{o}$  in  $Y \implies y - Tx_1 - \overline{TU_3} \subset \overline{apTU_2} = \emptyset$  so, there is  $x_2 \in U_2$  such that  $y - Tx_1 - Tx_2 \in \overline{TU_3}$ .

By induction we find  $x_n \in U_n$  such that  $y - Tx_1 - Tx_2 - \dots - Tx_n \in \overline{TU_{n+1}}$  ( $n \in \mathbb{N}$ ).

$$x := \sum_{n=1}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n :$$

$$\begin{aligned} M > N &\implies \varrho\left(\sum_{n=1}^M x_n, \sum_{n=1}^N x_n\right) = \varrho\left(\sum_{n=N+1}^M x_n, \mathbf{o}\right) \leq \\ &\leq \underbrace{\varrho\left(\sum_{n=N+1}^M x_n, \sum_{n=N+2}^M x_n\right)}_{\varrho(x_{N+1}, \mathbf{o})} + \underbrace{\varrho\left(\sum_{n=N+2}^M x_n, \sum_{n=N+3}^M x_n\right)}_{\varrho(x_{N+2}, \mathbf{o})} + \dots + \varrho(x_M, \mathbf{o}). \end{aligned}$$

$$Tx = y : y - Tx = \lim_{n \rightarrow \infty} (y - Tx_1 - \dots - Tx_n)$$

$$y - Tx_1 - \dots - Tx_n \in \overline{TU_{n+1}} \subset \overline{TU_k} \quad \text{for } n+1 > k$$

so,  $y - Tx \in \overline{TU_k}$  for each  $k \in \mathbb{N}$ , so  $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k} = \{\mathbf{o}\}$ . „Last equality“:  $y \in Y \setminus \{\mathbf{o}\} \implies \exists V$  neighbourhood of  $\mathbf{o}$  in  $Y$  such that  $y \notin \overline{B}$ .  $T$  continuous  $\implies \exists k \in \mathbb{N}$  such that  $T(U_k) \subset V \implies \overline{T(U_1)} \subset \overline{V} \implies y \notin \overline{T(U_k)}$ .  $\square$



## 1.5 Extension and separation theorems

### Definice 1.10 (Dual space (the dual))

$X$  LCS,  $X^*$  is the vector space of continuous linear functions on  $X$ .

### Věta 1.26 (Hahn–Banach extension theorem)

$X$  LCS,  $Y \subseteq X$ ,  $f \in Y^*$ . Then  $\exists g \in X^*$  such that  $g|_Y = f$ .

┌

*Poznámka*

└ If topology of  $X$  is generated by  $\mathcal{P}$  a topology of seminorms TODO?

┌

*Důkaz*

1. Topology of  $Y$ :  $U \subset Y$  is open  $\Leftrightarrow \exists \tilde{U} \subset X$  open such that  $U = \tilde{U} \cap Y$ .  $U$  is a neighbourhood of  $\mathbf{o}$  in  $Y \Leftrightarrow \exists \tilde{U}$  a neighbourhood of  $\mathbf{o}$  in  $X$  such that  $U = \tilde{U} \cap Y$ . Lz.pat.  $Y$  is also a LCS.

2.  $f \in Y^* \implies \exists p$  a continuous seminorm on  $Y$  such that  $|f(y)| \leq p(y), y \in Y$ .  $U = [p < 1]$  a neighbourhood of  $\mathbf{o}$  in  $Y \implies \exists \tilde{U}$  a neighbourhood of  $\mathbf{o}$  in  $X$  such that  $\tilde{U} \cap Y = U \implies \exists \tilde{V} \subset \tilde{U}$  an absolutely convex neighbourhood of  $\mathbf{o}$  in  $X$ . The  $p_{\tilde{V}}$  is a continuous seminorm on  $X$ . Moreover,  $p_{\tilde{V}}|_Y \geq p$ . ( $[p_{\tilde{V}}|_Y < 1] \subset \tilde{V} \cap Y \subset U = [p < 1]$ ). So, for  $y \in Y : |f(y)| \leq p(y) \leq p_{\tilde{V}}(y) \implies$  (algebraic H–B for seminorms)  $\exists g : X \rightarrow \mathbb{F}$  linear,  $g|_Y = f$ ,  $|g(x)| \leq p_{\tilde{V}}(x)$  for  $x \in X \implies g$  is continuous by the proposition above.  $\square$

*Důsledek* (Separation from a subspace)

$X$  LCS,  $Y \subseteq X$  closed,  $x \in X \setminus Y$ . Then  $\exists f \in X^* : f|_Y = 0, f(x) = 1$ .

┌

*Důkaz*

Set  $\tilde{Y} = \text{LO}(Y \cup \{x\})$ . Define  $g(y + \lambda x) = \lambda, y \in Y, \lambda \in \mathbb{F} \implies g$  is linear functional on  $\tilde{Y}$ ,  $g|_Y = 0, g(x) = 1$ .  $\text{Ker } g = Y$  is closed  $\implies g$  is continuous  $\implies g$  can be extended to  $f \in X^*$ .  $\square$

*Důsledek* (A proof of density using Hahn–Banach theorem)

$X$  LCS,  $Z \subseteq Y \subseteq X$ .

$$\overline{Z} \supset Y \Leftrightarrow \forall f \in X^* : f|_Z = 0 \implies f|_Y = 0.$$

┌

*Důkaz*

└ „ $\implies$ “: clear. „ $\Leftarrow$ “:  $y \in Y \setminus \overline{Z} \implies \exists f \in X^* : f(y) = 1, f|_Z = 0$ .  $\square$

*Důsledek* (The dual separates the points)

$X$  HLCS,  $x \in X \setminus \{\mathbf{o}\} \implies \exists f \in X^* : f(x) \neq 0$ .

┌

*Důkaz*

$Y = \{\mathbf{o}\}$  is closed linear subspace and use the first corollary.

└

□

### **Věta 1.27** (Hahn–Banach separation theorem)

$X$  LCS,  $A, B \subset X$  nonempty convex,  $A \cap B = \emptyset$ .

- $\text{int } A \neq \emptyset \implies \exists f \in X^* \setminus \{0\} \exists c \in \mathbb{R} \forall a \in A \forall b \in B : \Re f(a) \leq c < \Re f(b)$ .
- $A$  compact,  $B$  closed  $\implies \exists f \in X^* \exists c, d \in \mathbb{R} \forall a \in A \forall b \in B : \Re f(a) \leq c < d \leq \Re f(b)$ .

┌ *Důkaz*

Analogous to the theorem above. Assume  $X$  is a real space ( $\mathbb{F} = \mathbb{R}$ ). „First item“:  $\text{int } A \neq \emptyset \implies \text{int}(B - A) \neq \emptyset$  and  $- \notin B - A$ . Fix  $z \in \text{int}(B - A)$ , set  $U := z - (B - A)$ . The  $U$  is a convex neighbourhood of  $\mathbf{o}$ ,  $z \notin U \implies p_U(z) \geq 1$ . Define  $g_0 : \text{LO}\{z\} \rightarrow \mathbb{R}$  by  $g_0(t \cdot z) = t \cdot p_U(z) \implies g_0$  is a linear functional,  $g_0 \leq p_U$  on  $\text{LO}\{z\}$  ( $t \geq 0 \implies g_0(t \cdot z) = t \cdot p_U(z) = p_U(t \cdot z)$ ,  $t < 0 \implies g_0(t \cdot z) = t \cdot p_U(z) < 0 \leq p_U(t \cdot z)$ ).

From algebraic Hahn–Banach  $\exists g : X \rightarrow \mathbb{R}$  linear,  $g|_{\text{LO}\{z\}} = g_0$ ,  $g \leq p_U$  on  $X$ .  $g$  is continuous ( $g \leq 1$  on  $U \implies g \geq -1$  on  $-U$ , so  $|g| \leq 1$  on  $U \cap (-U)$ , a neighbourhood of  $\mathbf{o}$ ).  $a \in A$ ,  $b \in B \implies$

$$\implies g(z) - g(b) + g(a) = g(z - (b - a)) \leq p_U(z - (b - a)) \leq 1, \quad g(a) \leq g(b) + 1 - \underbrace{g(z)}_{\leq 0}^{\leq p_U(z) \geq 1}.$$

So,  $\forall a \in A \forall b \in B : g(a) \leq g(b)$ ,  $c := \sup g(A)$ .

„Second item“:  $A$  compact,  $B$  closed. For  $x \in A \exists U_x$  an absolutely convex open neighbourhood of  $\mathbf{o}$  such that  $(x + U_x) \cap B = \emptyset$ . The  $(x + \frac{1}{2}U_x)_{x \in A}$ , is an open cover of  $A$ .  $A$  is compact  $\implies \exists x_1, \dots, x_n \in A : A \subset (x_1 + \frac{1}{2}U_{x_1}) \cup \dots \cup (x_n + \frac{1}{2}U_{x_n})$ . Set  $V := \frac{1}{2}U_{x_1} \cap \dots \cap \frac{1}{2}U_{x_n}$  an absolutely convex open neighbourhood of  $\mathbf{o}$ . Then  $(A + V) \cap B = \emptyset$

$$\left( a \in A \implies \exists j : a \in x_j + \frac{1}{2}U_{x_j} \implies a + V \subset x_j + \frac{1}{2}U_{x_j} + V \subset x_j + U_{x_j} \right).$$

Apply first item to  $A + V$  (open convex),  $B$  (convex)  $\implies \exists f \in X^* \setminus \{0\}$  such that

$$\sup f(A) + \sup f(V) = \sup(f(A) + f(V)) = \sup f(A + V) \leq \inf f(B),$$

observe that  $\sup f(V) > 0$  ( $f \neq 0$ ,  $V$  is neighbourhood of  $\mathbf{o}$ , hence absorbing).

$$c := \sup f(A), \quad d := \sup f(A) + \sup f(V).$$

„ $X$  complex“: look at  $X$  as a real space,  $f : X \rightarrow \mathbb{R}$  real-linear such that. Define  $f_c(x) = f(x) - if(ix)$ ,  $x \in X$ . □

*Důsledek*

$X$  LCS,  $\emptyset \neq A \subset X$ ,  $x \in X$ .

- $x \in X \setminus \overline{\text{co}}A \Leftrightarrow \exists f \in X^* : \Re f(x) > \sup \{\Re f(a) | a \in A\}$ . („ $\Leftarrow$ “: is clear because  $\{y \in X, \Re f(y) \leq \sup \{\Re f(a) | a \in A\}\}$  is closed convex set containing  $A$ . „ $\Rightarrow$ “: Apply the previous theorem to  $\{x\}$  and  $\overline{\text{co}}A$ , get  $f$  and take  $-f$ .)
- $x \in X \setminus \overline{\text{aco}}A \Leftrightarrow \exists f \in X^* : |f(x)| > \sup \{|f(a)| | a \in A\}$  („ $\Leftarrow$ “: Clear. „ $\Rightarrow$ “: Apply the previous theorem to  $\{x\}$  and  $\overline{\text{aco}}A$  (and multiply by  $-1$ ),  $f \in X^*$ :  $|f(x)| \geq \Re f(x) > \sup \{\Re f(y) | y \in \overline{\text{aco}}A\} = \sup \{|f(y)| | y \in \overline{\text{aco}}A\}$ . „ $\leq$ “ clear, „ $\geq$ “:  $y \in \overline{\text{aco}}A \implies \exists \alpha \in \mathbb{F}, |\alpha| = 1 : |f(y)| = \alpha f(y)$ , then  $|f(y)| = \lambda f(y) = \Re \alpha f(y) = \Re f(\alpha y)$ ,  $\alpha y \in \overline{\text{aco}}A$ ).

## 2 Weak topologies

### 2.1 General weak topologies and duality

#### Definice 2.1 (Algebraic dual, general weak topology)

$X$  vector space.  $X^\#$  is the algebraic dual of  $X$  (it means set of all linear functionals on  $X$ ).  $\emptyset \neq M \subset X^\#$ , then  $\sigma(X, M)$  is the topology on  $X$  generated by seminorms  $X \mapsto |f(x)|$ ,  $f \in M$ .

#### Tvrzení 2.1

*Properties:*

1.  $(X, \sigma(X, M))$  is LCS (by the theorem above).
2.  $(X, \sigma(X, M))$  is Hausdorff  $\Leftrightarrow \forall x \in X \setminus \{0\} \exists f \in M : f(x) \neq 0$  (i.e.  $M$  separates points) (by the theorem above).
3.  $f \in M \implies f$  is continuous on  $(X, \sigma(X, M))$  (fix  $f \in M$ ,  $p(x) = |f(x)|$ ,  $x \in X$  is a continuous seminorm and  $|f(x)| = p(x) \leq p(x)$ ).
4.  $\sigma(X, M)$  is the weakest topology on  $X$  making all  $f \in M$  continuous.
5.  $\sigma(X, M) = \sigma(X, \text{LO}(M))$ .
6.  $T$  a topological space,  $F : T \rightarrow X$  mapping. Then  $F$  is continuous  $T \rightarrow \sigma(X, M) \Leftrightarrow \forall f \in M : f \circ F$  is continuous ( $T \rightarrow \mathbb{F}$ ).

┌

*Důkaz (4.)*

Assume  $\tau$  is any topology on  $X$  such that all  $f \in M$  are  $\tau$ -continuous  $\implies$

$$\forall x \in X \forall f_1, \dots, f_n \in M \forall c_1, \dots, c_n > 0 : \{y \in X \mid |f_j(y) - f_j(x)| < c_j \forall j \in [n]\} \text{ is } \tau\text{-open}$$

but these sets form a base of  $\sigma(X, M) \implies \sigma(X, M) \subset \tau$ . □

└

┌

*Důkaz (5.)*

„ $\subseteq$ “: Clear. „ $\supseteq$ “:  $f \in \text{LO } M \implies f$  is  $\sigma(X, M)$ -continuous (the linear combination of continuous linear functionals is continuous)  $f = \alpha_1 f_1 + \dots + \alpha_n f_n$ ,  $f_1, \dots, f_n \in M$ ,  $x_1, \dots, x_n \in \mathbb{F}$ .

$$|f(x)| \leq |\alpha_1| \cdot |f_1(x)| + \dots + |\alpha_n| \cdot |f_n(x)| \leq (|\alpha_1| + \dots + |\alpha_n|) \cdot \max \{|f_1(x)|, \dots, |f_n(x)|\}.$$

So by the previous point we get  $\sigma(X, \text{LO } M) \subset \sigma(X, M)$ . □

└

┌ *Důkaz* (6.)

„ $\implies$ “:  $f \in M \implies f$  is  $\sigma(X, M)$ -continuous, so  $f \circ F$  is continuous. „ $\impliedby$ “:  $t \in T$ ,  $U$  neighbourhood of  $F(t)$  in  $\sigma(X, M) \implies \exists f_1, \dots, f_n \in M \exists c_1, \dots, c_n > 0$  such that

$$F(t) \in \{y \in X \mid \forall j \in [n] |f_j(y) - f_j(F(t))| < c_j\} \subset U.$$

Let  $G = \{u \in T \mid \forall j \in [n] : |(f_j \circ F)(u) - (f_j \circ F)(t)| < c_j\}$ . Then  $G$  is an open neighbourhood of  $t$  (by continuity of  $f_j \circ F$  and  $F(G) \subset U$ ). □

└

*Příklad*

$X$  LCS. Then  $X^* \subseteq X^\#$ . So, we may consider  $\sigma(X, X^*)$  „the weak topology of  $X$ “.  $\sigma(X, X^*)$  is Hausdorff when  $X$  is HLCS.

TODO?

## Lemma 2.2

Let  $X$  be a vector space and  $f, f_1, \dots, f_k \in X^\#$ . The following assertions are equivalent:

1.  $f \in \text{LO}\{f_1, \dots, f_k\}$ ;
2.  $\exists C > 0 \forall x \in X : |f(x)| \cdot C \cdot \max\{|f_1(x)|, \dots, |f_k(x)|\}$ ;
3.  $\bigcap_{j=1}^k \text{Ker } f_j \subset \text{Ker } f$ .

┌ *Důkaz*

└ TODO!!! □

## Věta 2.3

Let  $X$  be a vector space and let  $M \subset X^\#$  be a nonempty set. Then  $(X, \sigma(X, M))^* = \text{LO } M$ .

┌ *Důkaz*

└ TODO!!! □

*Důsledek*

1. Let  $X$  be a LCS and let  $f \in X^\#$ . Then  $f$  is continuous on  $X$  (i.e.,  $f \in X^*$ ), if and only if it is weakly continuous (i.e.,  $\sigma(X, X^*)$ -continuous) on  $X$ .
2. Let  $X$  be a LCS. Then  $(X^*, \sigma(X^*, X))^* = \mathfrak{K}(X)$ .
3. Let  $X$  be a normed linear space and let  $f \in X^{**}$ . Then  $f \in \mathfrak{K}(X)$ , if and only if  $f$  is weak\* continuous (i.e.,  $\sigma(X^*, X)$  continuous) on  $X^*$ .

┌ *Důkaz*  
└ TODO!!!

□

## 2.2 Weak topologies on locally convex spaces

### Věta 2.4 (Mazur theorem)

Let  $X$  be a LCS and let  $A \subset X$  be a convex set. Then a)  $\overline{A}^w = \overline{A}$  b)  $A$  is closed if and only if it is weakly closed.

┌ *Důkaz*  
└ TODO!!!

□

*Důsledek*

Let  $X$  be a metrizable LCS and let  $(x_n)$  be a sequence in  $X$  weakly converging to a point  $x \in X$ . Then there is a sequence  $(y_n)$  in  $X$  such that  $y_n \in \text{co}\{x_k, k \geq n\}$  for each  $n \in \mathbb{N}$  and  $y_n \rightarrow x$  in (the original topology of)  $X$ .

┌ *Důkaz*  
└ TODO!!!

□

### Věta 2.5 (Boundedness and weak boundedness)

Let  $X$  be a LCS and let  $A \subset X$ . Then  $A$  is bounded in  $X$  if and only if it is bounded in  $\sigma(X, X^*)$ .

┌ *Důkaz*  
└ „ $\implies$ “: trivial. „ $\impliedby$ “: TODO!!! (notes)

□

### Tvrzení 2.6 (Weak topology on a subspace)

Let  $X$  be a LCS and let  $Y \subseteq X$ . Then the weak topology  $\sigma(Y, Y^*)$  coincides with the restriction of the weak topology  $\sigma(X, X^*)$  to  $Y$ .

┌ *Důkaz*  
└ TODO!!!

□

## 2.3 Polars and their applications

### Definice 2.2 (Polars, absolute polars and annihilators)

Let  $X$  be a LCS. Let  $A \subset X$  and  $B \subset X^*$  be nonempty sets. We define

$$\begin{aligned} A^\triangleright &:= \{f \in X^* \mid \forall x \in A : \Re f(x) \leq 1\}, & B_\triangleright &:= \{x \in X \mid \forall f \in B : \Re f(x) \leq 1\}, \\ A^\circ &:= \{f \in X^* \mid \forall x \in A : |f(x)| \leq 1\}, & B_\circ &:= \{x \in X \mid \forall f \in B : |f(x)| \leq 1\}, \\ A^\perp &:= \{f \in X^* \mid \forall x \in A : f(x) = 0\}, & B_\perp &:= \{x \in X \mid \forall f \in B : f(x) = 0\}. \end{aligned}$$

*Příklad*

Let  $X$  be a normed linear space. Then  $(B_X)^\triangleright = (B_X)^\circ = B_{X^*}$ ,  $(B_{X^*})_\triangleright = (B_{X^*})_\circ = B_X$ .

## Tvrzení 2.7 (Polar calculus)

Let  $X$  be a LCS and let  $A \subset X$  be a nonempty set.

1. The set  $A^\triangleright$  is convex and contains the zero functional,  $A^\circ$  is absolutely convex and  $A^\perp$  is a subspace of  $X^*$ . All the three sets are moreover weak\* closed.
2.  $A^\perp \subset A^\circ \subset A^\triangleright$ .
3. If  $A$  is balanced, then  $A^\triangleright = A^\circ$ . If  $A \subseteq X$ , then  $A^\triangleright = A^\circ = A^\perp$ .
4.  $\{\mathbf{o}\}^\triangleright = \{\mathbf{o}\}^\circ = \{\mathbf{o}\}^\perp = X^*$ ,  $X^\triangleright = X^\circ = X^\perp = \{\mathbf{o}\}$ .
5.  $(c \cdot A)^\triangleright = \frac{1}{c} \cdot A^\triangleright$  and  $(c \cdot A)^\circ = \frac{1}{c} \cdot A^\circ$  whenever  $c > 0$ .
6. Let  $(A_i)_{i \in I}$  be a nonempty family of nonempty subsets of  $X$ . Then  $(\bigcup_{i \in I} A_i)^\circ = \bigcap_{i \in I} A_i^\circ$ . The analogous formulas hold for polars and annihilators too.

*Poznámka*

Analogous statements hold for  $B \subset X^*$ . There are just two differences: The sets  $B_\triangleright$ ,  $B_\circ$  and  $B_\perp$  are weakly closed and for the validity of the second statement in 4. one needs to assume that  $X$  is Hausdorff.

*Důkaz*

TODO!!!

□

## Věta 2.8 (Bipolar theorem)

Let  $X$  be a LCS and let  $A \subset X$  and  $B \subset X^*$  be nonempty sets. Then

$$\begin{aligned} (A^\triangleright)_\triangleright &= \overline{\text{co}}(A \cup \{\mathbf{o}\}) (= \overline{\text{co}}^w(A \cup \{\mathbf{o}\})), & (B_\triangleright)^\triangleright &= \overline{\text{co}}^w(B \cup \{\mathbf{o}\}), \\ (A^\circ)_\circ &= \overline{\text{aco}}A (= \overline{\text{aco}}^w A), & (B_\circ)^\circ &= \overline{\text{aco}}^w B, \\ (A^\perp)_\perp &= \overline{\text{LO}}A (= \overline{\text{LO}}^w A), & (B_\perp)^\perp &= \overline{\text{LO}}^w B. \end{aligned}$$

┌ *Důkaz*  
└ TODO!!!

□

*Důsledek*

Let  $X$  and  $Y$  be normed linear spaces and let  $T \in L(X, Y)$ . Then  $(\text{Ker } T)^\perp = \overline{T^*(Y^*)}^{w^*}$ .

┌ *Důkaz*  
└ TODO!!!

□

### Věta 2.9 (Goldstine)

Let  $X$  be a normed linear space and let  $\varkappa : X \rightarrow X^{**}$  be the canonical embedding. Then  $B_{X^{**}} = \overline{\varkappa(B_X)}^{\sigma(X^{**}, X^*)}$ .

┌ *Důkaz*  
└ TODO!!!

□

### Věta 2.10 (Banach–Alaoglu)

Let  $X$  be a LCS and let  $U \subset X$  be a neighbourhood of  $\mathbf{o}$ . Then

1.  $U^\circ$  is a weak\* compact subset of  $X^*$ .
2. If  $X$  is moreover separable,  $U^\circ$  is metrizable in the topology  $\sigma(X^*, X)$ .

┌ *Důkaz*  
└ TODO!!! (notes)

□

*Důsledek* (Banach–Alaoglu for normed spaces)

Let  $X$  be a normed linear space. Then  $(B_{X^*}, w^*)$  is compact. If  $X$  is separable  $(B_{X^*}, w^*)$  is moreover metrizable.

*Důsledek* (Reflexivity and weak compactness)

Let  $X$  be a Banach space. Then  $X$  is reflexive if and only if  $B_X$  is weakly compact. If  $X$  is reflexive and separable,  $(B_X, w)$  is moreover metrizable.

*Důsledek*

Let  $X$  be a reflexive Banach space and let  $f : X \rightarrow \mathbb{R}$  be a function with the following properties:

- $f$  is weakly sequentially lower semi-continuous, i.e.

$$\forall x \in X \quad \forall (x_n) \subset X : x_n \xrightarrow{w} x \implies f(x) \leq \liminf f(x_n);$$



- $\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$ .

Then  $f$  attains its minimum at some point of  $X$ .

┌  
Důkaz  
└  
TODO?

□

## 3 Elements of the theory of distributions

### 3.1 Space of test functions and weak derivatives

**Definition 3.1** (Support, test functions, the space of test functions, locally integrable, approximate unit (smoothing kernel))

Let  $d \in \mathbb{N}$  and let  $\Omega \subset \mathbb{R}^d$  be an open set.

- If  $f : \Omega \rightarrow \mathbb{F}$  is continuous, its support is the set

$$\text{supp } f = \overline{\{x \in \Omega \mid f(x) \neq 0\}},$$

where the closure is taken in  $\Omega$ .

- Let  $\mathcal{D}(\Omega, \mathbb{F}) := \{f \in C^\infty(\Omega, \mathbb{F}) \mid \text{supp } f \text{ is compact subset of } \Omega\}$ . Elements of  $\mathcal{D}(\Omega, \mathbb{F})$  are called test functions, the space  $\mathcal{D}(\Omega, \mathbb{F})$  is called the space of test functions.
- A measurable function  $f : \Omega \rightarrow \mathbb{F}$  is called locally integrable in  $\Omega$ , if for any  $x \in \Omega$  there exists  $r > 0$  such that  $f$  is Lebesgue integrable on  $U(\mathbf{x}, r)$  (i.e.  $\int_{U(\mathbf{x}, r)} |f| < \infty$ ). The space of all locally integrable functions in  $\Omega$  is denoted by  $L^1_{loc}(\Omega, \mathbb{F})$ . (More precisely, the space of equivalence classes.)
- Choose a non-negative  $h \in \mathcal{D}(\mathbb{R}^d)$  such that  $\text{supp } h \subset U(0, 1)$  and  $\int_{\mathbb{R}^d} h = 1$ . For  $j \in \mathbb{N}$  we define a function  $h_j$  by  $h_j(\mathbf{x}) := j^d \cdot h(j \cdot \mathbf{x})$  for  $\mathbf{x} \in \mathbb{R}^d$ . The sequence  $(h_j)$  obtained in this way is called an approximate unit in  $\mathcal{D}(\mathbb{R}^d)$  or a smoothing kernel.

#### Lemma 3.1

Let  $\Omega \subseteq \mathbb{R}^d$  be open. Then  $\mathcal{D}(\Omega)$  is a dense subspace of  $L^p(\Omega)$  for any  $p \in [1, \infty)$ .

┌  
Důkaz  
└  
TODO? (notes)

□

#### Lemma 3.2

Let  $\Omega \subset \mathbb{R}^d$  be an open set.

1. Let  $\mu$  be a (finite) signed or complex regular Borel measure on  $\Omega$ . If  $\int_{\Omega} \varphi d\mu = 0$  for any  $\varphi \in \mathcal{D}(\Omega)$ , then  $\mu = 0$ .
2. Let  $f \in L^1_{loc}(\Omega)$  and  $\int_{\Omega} f\varphi = 0$  for any  $\varphi \in \mathcal{D}(\Omega)$ . Then  $f = 0$  almost everywhere on  $\Omega$ .

┌ Důkaz

└ TODO? (notes)

□

### Definice 3.2 (Weak derivative)

Let  $(a, b) \subset \mathbb{R}$  be an open interval and let  $f \in L^1_{loc}((a, b))$ .

- A function  $g \in L^1_{loc}((a, b))$  is called a weak derivative of a function  $f$ , if for any  $\varphi \in \mathcal{D}((a, b))$  we have  $\int_a^b f\varphi' = -\int_a^b g\varphi$ .
- Let  $\mu$  be a finite regular Borel measure on  $(a, b)$  (signed or complex). The measure  $\mu$  is said to be a weak derivative of a function  $f$  if for any  $\varphi \in \mathcal{D}((a, b))$  we have  $\int_a^b f\varphi' = -\int_{(a,b)} \varphi d\mu$ .

### Tvrzení 3.3

Let  $(a, b) \subset \mathbb{R}$  be an open interval and let  $f \in L^1_{loc}((a, b))$ . If  $\int_a^b f\varphi' = 0$  for any  $\varphi \in \mathcal{D}((a, b))$ , the function  $f$  is constant.

┌ Důkaz

└ TODO? (notes)

□

### Věta 3.4

Let  $f \in L^1_{loc}((a, b))$ .

1. The weak derivative of  $f$  is uniquely determined.
2. If  $f$  is absolutely continuous on  $[a, b]$ , it has a finite derivative almost everywhere,  $f' \in L^1((a, b))$  and  $f'$  is the weak derivative of  $f$ . Conversely, if a function  $f$  has a weak derivative  $g \in L^1((a, b))$ , there exists a function  $f_0$  absolutely continuous on  $[a, b]$ , equal to  $f$  almost everywhere on  $(a, b)$ . In this case  $g = f'_0$  almost everywhere.  
More generally, a function  $f$  has a weak derivative in  $L^1_{loc}((a, b))$  if and only if there exists function  $f_0$  locally absolutely continuous on  $(a, b)$  such that  $f_0 = f$  almost everywhere.
3. There exists a finite measure  $\mu$ , which is a weak derivative of function  $f$  if and only if there exists a function  $f_0$  of bounded variation on  $[a, b]$  such that  $f_0 = f$  almost everywhere on  $(a, b)$ . In this case for each subinterval  $(c, d) \subset (a, b)$  we have  $\mu((c, d)) = \lim_{x \rightarrow d-} f_0(x) - \lim_{x \rightarrow c+} f_0(x)$ .

Moreover,  $\mu$  is real-valued if and only if  $f_0$  may be real-valued and  $\mu$  is non-negative if and only if  $f_0$  may be non-increasing.

Důkaz

TODO? (notes)

□

## 3.2 Distributions – basic properties and operations

### Definice 3.3 (Convergence in $\mathcal{D}$ )

Let  $\Omega \subset \mathbb{R}^d$  be an open set,  $(\varphi_n)$  a sequence in  $\mathcal{D}(\Omega)$  and  $\varphi \in \mathcal{D}(\Omega)$ . We say that the sequence  $(\varphi_n)$  converges to  $\varphi$  in  $\mathcal{D}(\Omega)$ , if the following two conditions are fulfilled:

- There exists  $K \subset \Omega$  compact such that  $\text{supp } \varphi_n \subset K$  for each  $n \in \mathbb{N}$ .
- $D^\alpha \varphi_n \rightrightarrows D^\alpha \varphi$  on  $K$  for each multiindex  $\alpha \in \mathbb{N}_0^d$ .

This is expressed by writing  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\Omega)$ .

### Lemma 3.5

Let  $\Omega \subset \mathbb{R}^d$  be an open set.

- $\|\cdot\|_N$  is a norm on  $\mathcal{D}(\Omega)$ ;
- $\mathcal{D}_K(\Omega)$  is a Fréchet space when equipped with  $(\|\cdot\|_N)_{N \in \mathbb{N}_0}$ .

Důkaz (a))

TODO!!!

□

Důkaz (b))

$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots \implies \mathcal{D}_K(\Omega)$  is a metrizable LCS (by translation invarinat metric  $\varrho$  from the proposition above).

$(\varphi_n) \subset \mathcal{D}_K(\Omega)$   $\varrho$ -cauchy, then  $\forall N \in \mathbb{N}_0$ :  $(\varphi_n)$  is  $\|\cdot\|_N$ -cauchy  $\implies \forall \alpha$ :  $(D^\alpha \varphi_n)$  is  $\|\cdot\|_\infty$ -cauchy  $\implies \forall \alpha \exists \psi_\alpha$  such that  $D^\alpha \varphi_n \rightrightarrows \psi_\alpha$  on  $\Omega$ . The  $\psi_\alpha$  is continuous,  $\varphi_\alpha = 0$  on  $\Omega \setminus K$ . Fix  $\alpha \in \mathbb{N}_0^d$  and  $j \in [d]$ . Then

$$D^\alpha \varphi_n \rightrightarrows \psi_\alpha \wedge \frac{\partial}{\partial x_j} D^\alpha \varphi_n = D^{\alpha+e_j} \varphi_n \rightrightarrows \psi_{\alpha+e_j} \implies \psi_{\alpha+e_j} = \frac{\partial}{\partial x_j} \psi_\alpha.$$

$$\implies \psi_\alpha = D^\alpha \psi_0.$$

$$\implies \psi_0 \in \mathcal{D}_K(\Omega), \forall \alpha : D^\alpha \varphi_n \rightrightarrows D^\alpha \psi_0, \text{ i.e. } \forall N : \varphi_n \rightarrow \psi_0 \text{ in } \|\cdot\|_N. \implies \varphi_n \rightarrow \psi_0 \text{ in } \varrho.$$

□

### Tvrzení 3.6

$\Lambda : \mathcal{D}(\Omega) \rightarrow \mathbb{F}$  linear then following assertions are equivalent:

1.  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \rightarrow \Lambda(\varphi)$ ;
2.  $\varphi_n \rightarrow 0$  in  $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \rightarrow 0$ ;
3.  $\forall K \subset \Omega$  compact and  $\Lambda|_{\mathcal{D}_K(\Omega)}$  is continuous;
4.  $\forall K \subset \Omega$  compact  $\exists N \in \mathbb{N}_0 \exists C > 0$  such that

$$|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \quad \varphi \in \mathcal{D}_K(\Omega).$$

┌

*Důkaz*

„1.  $\implies$  2.“ is trivial. „2.  $\implies$  3.“: Fix  $K \subset \Omega$  compact.  $\varphi_n \rightarrow 0$  on  $\mathcal{D}_K(\Omega) \implies \varphi_n \rightarrow 0$  in  $\mathcal{D}(\Omega) \xrightarrow{2.} \Lambda(\varphi_n) \rightarrow 0$ . Thus  $\Lambda|_{\mathcal{D}_K(\Omega)}$  is continuous at  $\mathbf{0}$ , so it is continuous.

„3.  $\implies$  1.“  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\Omega) \implies \exists K \subset \Omega$  compact such that  $\text{supp } \varphi_n \subset K$  for each  $n$ . Then  $(\varphi_n) \subset \mathcal{D}_K(\Omega) \implies \varphi_n \rightarrow \varphi$  in  $\mathcal{D}_K(\Omega) \xrightarrow{3.} \Lambda(\varphi_n) \rightarrow \Lambda(\varphi)$ .

└

„3.  $\Leftrightarrow$  4.“ By the proposition above. □

### Definice 3.4 (Distribution, finite order)

A distribution on  $\Omega$  is a linear functional  $\Lambda : \mathcal{D}(\Omega) \rightarrow \mathbb{F}$  satisfying assertions from the previous proposition. We will denote distributions on  $\Omega$  by  $\mathcal{D}'(\Omega)$ .

$\Lambda \in \mathcal{D}'(\Omega)$  is of finite order, if  $N \in \mathbb{N}_0$  in 4. of the previous proposition can be chosen independently on  $K$ .

*Například*

$f \in L^1_{loc}(\Omega)$ .  $\Lambda_f(\varphi) = \int_{\Omega} f \cdot \varphi$  ( $\varphi \in \mathcal{D}(\Omega)$ )  $\implies \Lambda_f$  is a distribution of order 0. Because  $K \subset \Omega$  compact  $\implies \int_K |f| < \infty$ ,  $\varphi \in \mathcal{D}_K(\Omega)$ :

$$|\Lambda_f(\varphi)| = \left| \int_{\Omega} f \cdot \varphi \right| = \left| \int_K f \cdot \varphi \right| \leq \int_K |f \varphi| \leq \|\varphi\|_{\infty} \cdot \int_K |f| = \|\varphi\|_0 \cdot \int_K |f|.$$

$\mu \geq 0$  regular Borel measure, finite on compacts.  $\Lambda_{\mu}(\varphi) = \int_{\Omega} \varphi d\mu$  is a distribution on  $\Omega$  of order 0. Because if  $K \subset \Omega$ ,  $\varphi \in \mathcal{D}_K(\Omega)$ , then

$$|\Lambda_{\mu}(\varphi)| = \left| \int_{\Omega} \varphi d\mu \right| = \left| \int_K \varphi d\mu \right| \leq \|\varphi\|_{\infty} \mu(K).$$

$\mu$  is a signed (or complex) finite measure  $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$  is a distribution of order 0:

$$\left| \int_K \varphi d\mu \right| \leq \int_K |\varphi| d|\mu| \leq |\mu|(K) \cdot \|\varphi\|_\infty \leq \|\mu\| \cdot \|\varphi\|_\infty.$$

$\Lambda(\varphi) = \varphi'(0)$ ,  $\varphi \in \mathcal{D}(\mathbb{R})$  is a distribution of order 1. (Clearly  $|\Lambda(\varphi)| \leq \|\varphi'\|_\infty \leq \|\varphi\|_1$ .)  
 $\Lambda$  not of order 0: Find  $\varphi \in \mathcal{D}(\mathbb{R})$  such that  $\varphi'(0) = 1$ ,  $\text{supp } \varphi \subset [-c, c]$  for some  $c > 0$ .  
 $\varphi_n(x) = \varphi(nx)$ ,  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$ ,  $\implies \varphi_n \in \mathcal{D}(\mathbb{R})$ .  $\text{supp } \varphi_n \subset [-c/n, c/n] \subset [-c, c]$ .  
 $\|\varphi_n\|_0 = \|\varphi\|_0$ .  $\Lambda(\varphi_n) = \varphi'_n(0) = \varphi'(0) \cdot n = n$ .

$\Lambda(\varphi) = \sum_{n=0}^{\infty} \varphi^{(n)}(n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}) \implies \Lambda$  is a distribution on  $\mathbb{R}$ , not of finite order  
 $(\text{supp } \varphi \subset [-k, k], k \in \mathbb{N}, \implies |\Lambda(\varphi)| \leq (k+1)\|\varphi\|_K)$

*Poznámka*

If  $f, g \in L^1_{loc}(\Omega)$ ,  $\Lambda_f = \Lambda_g$ , then  $f = g$  almost everywhere. If  $\mu, \nu$  measures,  $\Lambda_\mu = \Lambda_\nu$ , then  $\mu = \nu$ .

If  $f \in L^1(\Omega)$ ,  $\mu$  finite measure,  $\Lambda_f = \Lambda_\mu$ , then  $\mu(A) = \int_A f$ , for each  $A \subset \Omega$  Borel.

### Definice 3.5

$\Lambda \in \mathcal{D}'(\Omega)$ .

- For  $\alpha \in \mathbb{N}_0^d$  define  $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$ . (For any  $\varphi \in \mathcal{D}(\Omega)$ .)
- For  $f \in C^\infty(\Omega)$  define  $(f\Lambda)(\varphi) = \Lambda(f\varphi)$ . (For any  $\varphi \in \mathcal{D}(\Omega)$ .)

### Tvrzení 3.7

a)  $\Lambda \in \mathcal{D}'(\Omega)$ ,  $\alpha \in \mathbb{N}_0^d \implies D^\alpha \Lambda \in \mathcal{D}'(\Omega)$ .

┌

*Důkaz*

Clear:  $D^\alpha \Lambda : \mathcal{D}(\Omega) \rightarrow \mathbb{F}$  linear,  $K \subset \Omega$  compact  $\implies \exists N \in \mathbb{N}_0, C > 0 : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega)$ . Then  $\forall \varphi \in \mathcal{D}_k(\Omega)$ :

$$|D^\alpha \Lambda(\varphi)| = |\Lambda(D^\alpha \varphi)| \leq C \cdot \|D^\alpha \varphi\|_N \leq C \cdot \|\varphi\|_{|\alpha|+N}$$

└

□

b)  $f \in C^\infty(\Omega) \implies D^\alpha \Lambda_f = \Lambda_{D^\alpha f}$

┌ Důkaz (For  $\partial/\partial x_1$ )

$$\frac{\partial}{\partial x_1} \Lambda_f(\varphi) = -\Lambda_f \left( \frac{\partial \varphi}{\partial x_1} \right) =? = - \int_{\Omega} f \cdot \frac{\partial \varphi}{\partial x_1}$$

TODO!!!

$$\begin{aligned} &= \int_{\bigcup_j (a_j, b_j)} f \frac{\partial \varphi}{\partial x_1} dx_1 = \\ &= \sum_{j=1}^n \left( [f \cdot \varphi]_{a_j}^{b_j} - \int_{a_j}^{b_j} \frac{\partial f}{\partial x_1} \varphi dx_1 \right) = \int_{\Omega} \frac{\partial f}{\partial x_1} \varphi = \Lambda_{\frac{\partial f}{\partial x_1}}(\varphi). \end{aligned}$$

└

□

c)  $d = 1$ ,  $\Omega = (a, b)$ ,  $f \in L^1_{loc}(\Omega)$ . Then  $(\Lambda_f)' = \Lambda_g \Leftrightarrow g$  is the weak derivative of  $f$ . And  $(\Lambda_f)' = \Lambda_\mu \Leftrightarrow \mu$  is the weak derivative of  $f$ .

┌ Důkaz

└ By definitions.

□

$$d) \Lambda \in \mathcal{D}'(\Omega), f \in C^\infty(\Omega) \implies f\Lambda \in \mathcal{D}'(\Omega).$$

┌ Důkaz

└ clear:  $f\Lambda : \mathcal{D}(\Omega) \implies$  IF linear

□

### Tvrzení 3.8

a)  $\Lambda \in \mathcal{D}'((a, b))$ ,  $\Lambda' = 0 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$ .

┌ Důkaz

We will prove  $\text{Ker } \Lambda_1 \subset \text{Ker } \Lambda$ . Then  $\exists c : \Lambda = c \cdot \Lambda_1 = \Lambda_c$ .

$$\varphi \in \text{Ker } \Lambda_1 \implies \Lambda_1(\varphi) = 0, \text{ i.e. } \int_a^b \varphi = 0.$$

Define  $\varphi(t) = \int_a^t \varphi$ ,  $t \in (a, b)$ . Then  $\psi \in \mathcal{D}((a, b))$ ,  $\psi' = \varphi$  ( $\psi' = \varphi$  ... differentiation of indefinite integral  $\implies \psi \in C^\infty((a, b))$ ,  $\psi = 0$  on  $(a, \min \text{supp } \varphi)$  and  $(\max \text{supp } \varphi, b)$   $\implies \psi \in \mathcal{D}((a, b))$ ). Hence  $\Lambda(\varphi) = \Lambda(\psi') = -\Lambda'(\psi) = 0$ , so  $\varphi \in \text{Ker } \Lambda$ . □

└

b)  $\Omega \subset \mathbb{R}^d$  open connected,  $\Lambda \in \mathcal{D}'(\Omega)$ ,  $D^\alpha \Lambda = 0$  for  $|\alpha| = 1 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$ .

„Důkaz

„Step 1:  $\Omega = \prod_{j=1}^d (a_j, b_j)$ “: Induction on  $d$ . For  $d = 1$  use a). Assume it holds for  $d - 1$ , denote  $\Omega' = \prod_{j=1}^{d-1} (a_j, b_j)$ ,  $x \in \Omega \implies x = (x', x_d)$  ( $x' \in \mathbb{R}^{d-1}$ ,  $x_d \in \mathbb{R}$ ),  $\alpha \in N_0^d \implies \alpha = (\alpha', \alpha_d)$ .

$\Lambda \in \mathcal{D}'(\Omega)$ ,  $D^\alpha \Lambda = 0$  for  $|\alpha| = 1$ . It means:  $\forall \varphi \in \mathcal{D}(\Omega) \forall j \in [d] : \Lambda \left( \frac{\partial \varphi}{\partial x_j} \right) = 0$ .

Claim:  $\psi \in \mathcal{D}(\Omega)$ . Then  $\exists \varphi \in \mathcal{D}(\Omega) : \frac{\partial \varphi}{\partial x_d} = \psi \iff \forall x' \in \Omega' : \int_{a_d}^{b_d} \psi(x', x_d) dx_d = 0$ . („ $\implies$ “ clear, „ $\impliedby$ “: define  $\varphi(x', x_d) = \int_{a_d}^{x_d} \psi(x', t) dt$ ). Define

$$T : \mathcal{D}(\Omega) \rightarrow \mathcal{D}(\Omega'), \quad T\varphi(x') = \int_{a_d}^{b_d} \varphi(x', x_d) dx_d, \quad \varphi \in \mathcal{D}(\Omega).$$

$T$  is linear,  $\text{Ker } T \subset \text{Ker } \Lambda$  ( $T\varphi = 0 \implies \exists \psi \in \mathcal{D}(\Omega) : \varphi = \frac{\partial \psi}{\partial x_d}$ , thus  $\Lambda(\varphi) = 0$ ). Fix  $\eta \in \mathcal{D}((a_d, b_d))$ ,  $\int_{a_d}^{b_d} \eta = 1$ . For  $\varphi \in \mathcal{D}(\Omega')$  define  $(\varphi\eta)(x) = \varphi(x')\eta(x_d)$ . Then  $\varphi\eta \in \mathcal{D}(\Omega)$ .  $\tilde{\Lambda}(\varphi) = \Lambda(\varphi\eta)$ ,  $\varphi \in \mathcal{D}(\Omega')$ . Then  $\tilde{\Lambda} \in \mathcal{D}'(\Omega')$ .

Moreover,  $\forall \alpha'$  with  $|\alpha'| = 1 : D^{\alpha'} \tilde{\Lambda} = 0$ .

$$\left( \forall j \in [d-1] : \frac{\partial}{\partial x_j} \tilde{\Lambda}(\varphi) = -\tilde{\Lambda} \left( \frac{\partial \varphi}{\partial x_j} \right) = -\Lambda \left( \frac{\partial \varphi}{\partial x_j} \eta \right) = -\Lambda \left( \frac{\partial}{\partial x_j} (\varphi\eta) \right) = 0. \right)$$

$\implies \exists c \in \mathbb{F} : \tilde{\Lambda} = \Lambda_c$  in  $\mathcal{D}'(\Omega')$ . Then  $\Lambda = \Lambda_c$  (in  $\mathcal{D}(\Omega)$ ) cause

$$\begin{aligned} \varphi \in \mathcal{D}(\Omega) &\implies \varphi - (T\varphi)\eta \in \mathcal{D}(\Omega), \varphi - (T\varphi)\eta \in \text{Ker } T \subset \text{Ker } \Lambda, \text{ so, } \Lambda(\varphi) = \Lambda((T\varphi)\eta) = \\ &= \tilde{\Lambda}(T\varphi) = \Lambda_c(T\varphi) = \int_{\Omega'} c \cdot T\varphi = \int_{\Omega'} c \cdot \int_{a_d}^{b_d} \varphi(x', x_d) dx_d dx' \stackrel{\text{FUBINI}}{=} \int_{\Omega} c \cdot \varphi = \Lambda_c(\varphi). \end{aligned}$$

„Step 2:  $\Omega$  is open connected,  $\Lambda \in \mathcal{D}'(\Omega)$ ,  $D^\alpha \Lambda = 0$ ,  $|\alpha| = 1$ .“: Step 1  $\implies \forall Q \subset \Omega$  cuboid  $\exists c : \Lambda|_{\mathcal{D}(Q)} = \Lambda_c$ . Fix one cuboid  $Q_0 \subset \Omega$  and the respective  $c$ .

$$A := \{x \in \Omega | \exists Q \subset \Omega \text{ cuboid}, x \in Q, \Lambda|_{\mathcal{D}(Q)} = \Lambda_c\}.$$

Fix  $A \neq \emptyset$  ( $Q_0 \subset A$ ),  $A$  is open,  $A$  is closed in  $\Omega$  ( $x \in \bar{A} \cap \Omega$ ,  $Q \cap A \neq \emptyset$ ,  $\Lambda|_{\mathcal{D}(Q)} = \Lambda_d$ ,  $y \in Q \cap A \implies \Lambda|_{\mathcal{D}(Q_y)} = \Lambda_c \implies$  on  $\mathcal{D}(Q \cap Q_y) : \Lambda = \Lambda_c = \Lambda_d \implies c = d \implies x \in A$ ). So  $A = \Omega$  as  $\Omega$  is connected. The  $\Lambda = \Lambda_c$  in  $\mathcal{D}'(\Omega)$ . (Proof of this was skipped, it remains that for every  $\varphi \in \mathcal{D}(\Omega)$ , not only for every  $\varphi \in \mathcal{D}(Q)$ , it holds  $\Lambda(\varphi) = \Lambda_c(\varphi)$ .)  $\square$

### 3.3 A bit more on distributions

**Definition 3.6** (Convergence in distributions (in  $\mathcal{D}'$ ))

$$\Lambda_n \rightarrow \Lambda \text{ in } \mathcal{D}(\Omega) \equiv \forall \varphi \in \mathcal{D}(\Omega) : \Lambda_n(\varphi) = \Lambda(\varphi).$$

### **Tvrzení 3.9** (On the convergence of distributions)

a)  $\Lambda_n \rightarrow \Lambda$  in  $\mathcal{D}(\Omega)$ , then:

- $\forall \alpha : D^\alpha \Lambda_n \rightarrow D^\alpha \Lambda;$

┌

*Důkaz*

$$D^\alpha \Lambda_n(\varphi) = (-1)^{|\alpha|} \Lambda_n(D^\alpha \varphi) \rightarrow (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = D^\alpha \Lambda(\varphi).$$

└

□

- $f \in C^\infty(\Omega) : f \Lambda_n \rightarrow f \Lambda.$

┌

*Důkaz*

$$f \Lambda_n(\varphi) = \Lambda_n(f \varphi) \rightarrow \Lambda(f \varphi) = f \Lambda(\varphi).$$

└

□

b)  $f_n \rightarrow f$  in  $L^1_{loc}(\Omega)$  ( $\forall K \subset \Omega$  compact:  $\int_K |f_n - f| \rightarrow 0$ ). Then  $\Lambda_{f_n} \rightarrow \Lambda_f$  in  $\mathcal{D}'(\Omega)$ .

┌

*Důkaz*

$$\begin{aligned} \varphi \in \mathcal{D}(\Omega) : |\Lambda_{f_n}(\varphi) - \Lambda_f(\varphi)| &= \left| \int_{\Omega} f_n \varphi - \int_{\Omega} f \varphi \right| \leq \int_{\Omega} |f_n - f| \cdot |\varphi| = \\ &= \int_{\text{supp } \varphi} |f_n - f| \cdot |\varphi| \leq \|\varphi\|_{\infty} \int_{\text{supp } \varphi} |f_n - f| \rightarrow 0. \end{aligned}$$

└

□

c)  $f_n \rightarrow f$  in  $L^p(\Omega)$  for some  $p \in [1, \infty]$ . Then  $\Lambda_{f_n} \rightarrow \Lambda_f$ .

┌

*Důkaz*

Let  $K \subset \Omega$  be compact,  $q$  the dual exponent. Then use b) with

$$\int_K |f_n - f| \leq \|f_n - f\|_{L^p(K)} \cdot \|1\|_{L^q(K)} \rightarrow 0.$$

└

□

d)  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\Omega)$ . Then  $\Lambda_{\varphi_n} \rightarrow \Lambda_{\varphi}$  in  $\mathcal{D}'(\Omega)$ .

┌

*Důkaz*

$$\varphi_n \rightarrow \varphi \text{ in } \mathcal{D}(\Omega) \implies \varphi_n \rightarrow \varphi \text{ in } C^\infty(\Omega), \text{ and use c).}$$

└

□



**Věta 3.10**

$(\Lambda_n) \subset \mathcal{D}'(\Omega)$  and  $\forall \varphi \in \mathcal{D}(\Omega) : (\Lambda_n(\varphi))$  converges in  $\mathbb{F}$ . Then  $\Lambda(\varphi) = \lim_{n \rightarrow \infty} \Lambda_n(\varphi)$  is a distribution on  $\Omega$ .

┌

*Důkaz*

Clearly  $\Lambda$  is a linear functional on  $\mathcal{D}(\Omega)$ . Further:  $K \subset \Omega$  compact  $\implies \forall n : \Lambda_n|_{\mathcal{D}_K(\Omega)}$  is continuous.  $\mathcal{D}_K(\Omega)$  is a Fréchet space  $\xRightarrow{\text{the lemma above, b)}} \Lambda|_{\mathcal{D}_K(\Omega)}$  continuous  $\implies \Lambda \in \mathcal{D}'(\Omega)$ . □

└

**Definice 3.7** (Vanishing of distributions, support of distribution, distribution with compact support)

$\Lambda \in \mathcal{D}'(\Omega)$ .

- $G \subset \Omega$  open.  $\Lambda$  vanishes on  $G$  if  $\Lambda(\varphi) = 0$  whenever  $\varphi \in \mathcal{D}(\Omega)$ ,  $\text{supp } \varphi \subset G$ .
- $\text{supp } \Lambda = \Omega \setminus \{G \subset \Omega \text{ open} \mid \Lambda \text{ vanishes on } G\} =$   
 $= \{x \in \Omega \mid \forall \varepsilon > 0 \exists \varphi \in \mathcal{D}(\Omega) : \text{supp } \varphi \subset U(x, \varepsilon) \wedge \Lambda(\varphi) \neq 0\}.$
- $\Lambda$  has compact support if  $\text{supp } \Lambda$  is a compact subset of  $\Omega$ .

**Tvrzení 3.11** (On the support of a distribution)

a)  $\Lambda = \Lambda_f$  for some  $f \in L^1_{loc}(\Omega)$ . Then

$$\text{supp } \Lambda_f = \text{supp } f = \{x \in \Omega \mid \forall \varepsilon > 0 : \lambda^d(\{y \in U(x, \varepsilon) \cap \Omega \mid f(y) \neq 0\}) > 0\}$$

┌

*Důkaz*

„ $\Leftarrow$ “:  $x \notin \text{supp } f \implies \exists \varepsilon > 0 : f = 0$  almost everywhere on  $U(x, \varepsilon) \cap \Omega \implies \Lambda_f$  vanishes on  $U(x, \varepsilon) \cap \Omega \implies x \notin \text{supp } \Lambda_f$ .

„ $\Rightarrow$ “:  $x \in \text{supp } \Lambda$ . Let  $\varepsilon > 0$ . Then  $f$  is not 0 almost everywhere on  $U(x, \varepsilon) \cap \Omega \implies \exists \varphi \in \mathcal{D}(U(x, \varepsilon) \cap \Omega)$  □

└

b)  $\Lambda = \Lambda_\mu$ . Then  $\text{supp } \Lambda = \text{supp } \mu = \Omega \setminus \bigcup \{G \subset \Omega \text{ open} \mid \forall B \subset G \text{ Borel } \mu(B) = 0\}$ .

┌

*Důkaz*

$G \subset \Omega$  open the  $\forall B \subset G$  Borel  $\mu(B) = 0 \Leftrightarrow \forall \varphi \in \mathcal{D}(G) : \int \varphi d\mu = 0 \Leftrightarrow \Lambda_\mu$  vanishes on  $G$ . □

└

*Poznámka*

$f$  is continuous  $\implies \text{supp } f = \overline{\{x \mid f(x) \neq 0\}} \cap \Omega$ .

└

c)  $\varphi \in \mathcal{D}(\Omega)$ ,  $\text{supp } \varphi \cap \text{supp } \Lambda = \emptyset \implies \Lambda(\varphi) = 0$ .

*Důkaz*

Máme  $\text{supp } \varphi \cap \text{supp } \Lambda = \emptyset \implies \text{supp } \varphi \subset \bigcup \{G \subset \Omega \text{ open} \mid \Lambda \text{ vanishes on } G\} \implies \exists G_1, G_2, \dots, G_n \subset \Omega \text{ open such that } \Lambda \text{ vanishes on each } G_j \text{ and } \text{supp } \varphi \subset G_1 \cup \dots \cup G_n.$   
We will be done if we show that  $\Lambda$  vanishes on  $G_1 \cup \dots \cup G_n$ .  $\square$

*Důkaz* ( $\Lambda$  vanishes on  $G_1, G_2 \implies$  vanishes on  $G_1 \cup G_2$ )

$\psi \in \mathcal{D}(\Omega)$ ,  $\text{supp } \psi \subset G_1 \cup G_2$ . If  $\text{supp } \psi \subset G_1$  or  $\text{supp } \psi \subset G_2$ , then  $\Lambda(\psi) = 0$ . Assume  $\text{supp } \psi \not\subset G_1$  and  $\text{supp } \psi \not\subset G_2$ . Then  $L := \text{supp } \varphi \setminus G_2 \implies L$  is compact, nonempty,  $L \subset G_1$ . Fix  $\delta > 0$  such that  $3\delta < \text{dist}(L, \mathbb{R}^d \setminus G_1)$ ,  $h_k$  smooth kernel.

Fix  $k \in \mathbb{N}$  such that  $\frac{1}{k} < \delta$ ,  $\xi := h_k * \chi_{L+B(0,2\delta)} \implies \xi \in C^\infty(\mathbb{R}^d)$ .  $\text{supp } \xi \subset L + B(0, 2\delta) + U(0, 1/k) \subset L + U(0, 3\delta) \subset G_1$ ,  $\xi = 1$  on  $L + B(0, \delta)$ . Set  $\psi_1 = \xi \cdot \psi$ ,  $\psi_2 = (1 - \xi)\psi \implies \psi_1, \psi_2 \in \mathcal{D}(\Omega)$ ,  $\text{supp } \psi_1 \subset \xi \subset G_1$ ,  $\text{supp } \psi_2 \subset \text{supp } \psi \setminus (L + B(0, \delta)) \subset \text{supp } \psi \setminus (L + U(0, \delta)) \subset \text{supp } \psi \setminus L \subset G_2 \implies \Lambda(\psi_1) = \Lambda(\psi_2) = 0$ .  $\psi = \psi_1 + \psi_2 \implies \Lambda(\psi) = \Lambda(\psi_1) + \Lambda(\psi_2) = 0$ .  $\square$

d)  $\Lambda$  has compact support  $\implies \exists N \in \mathbb{N}_0 \exists c > 0 : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N$  for  $\varphi \in \mathcal{D}(\Omega)$ . In particular,  $\Lambda$  has finite order.

*Důkaz*

$\text{supp } \Lambda$  is a compact subset of  $\Omega \implies \exists \delta > 0 : K := \text{supp } \Lambda + B(0, 3\delta) \subset \Omega \implies K \subset \Omega$  is compact  $\implies$

$$\exists N \in \mathbb{N}_0 \exists c > 0 : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$\xi := h_k * \chi_{\text{supp } \Lambda + B(0, 2\delta)}$ . ( $1/k < \delta$ .)  $\xi \in C^\infty(\mathbb{R}^d)$ ,  $\text{supp } \xi \subset \text{supp } \Lambda + B(0, 2\delta) + U(0, 1/k) \subset K$ .  $\xi = 1$  on  $\text{supp } \Lambda + B(0, \delta)$ .

$\forall \varphi \in \mathcal{D}(\Omega) : \Lambda(\varphi) = \Lambda(\varphi\xi)$ .  $(1 - \xi)\varphi \in \mathcal{D}(\Omega) = 0$  on  $\text{supp } \Lambda + B(0, \delta) \implies \text{supp}(1 - \xi)\varphi \cap \text{supp } \Lambda = \emptyset \implies \Lambda((1 - \xi)\varphi) = 0 \implies \Lambda(\varphi) = \Lambda(\xi\varphi)$ .

Then

$$|\Lambda(\varphi)| = |\Lambda(\varphi\xi)| \leq C \cdot \|\xi \cdot \varphi\|_N \leq C \cdot 2^N \cdot \|\xi\|_N \cdot \|\varphi\|_N.$$

$\square$

e)  $\text{supp } \Lambda = \{p\} \Leftrightarrow \exists N \in \mathbb{N}_0, C_\alpha \in \mathbb{F}, |\alpha| \leq N, \Lambda = \sum_{|\alpha| \leq N} C_\alpha D^\alpha \Lambda_{\delta_p}$ .

*Důkaz*

„ $\Leftarrow$ “: trivial. „ $\Rightarrow$ “:  $\{p\}$  is compact  $\implies \exists N, C : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}(\Omega)$ . The  $\Lambda$  is a linear combination of  $D^\alpha \Lambda_{\delta_p}$ ,  $|\alpha| \leq N$ . To prove this, we use lemma above and show

$$\bigcap_{|\alpha| \leq N} \text{Ker } D^\alpha \Lambda_{\delta_p} \subset \text{Ker } \Lambda,$$

i.e.  $\forall \varphi \in \mathcal{D}(\Omega) : D^\alpha \varphi(p) = 0$  for each  $|\alpha| \leq N \implies \Lambda(\varphi) = 0$ .  $\square$

### 3.4 Convolution of distribution

**Definice 3.8** (Notation: translate, reflexion and derivative in direction)

$M \subset \mathbb{R}^d$ ,  $f : M \rightarrow \mathbb{F}$

- $y \in \mathbb{R}^d$ ,  $\tau_y f(x) = f(x - y)$ ,  $x \in y + M$ ;
- $\hat{f}(x) = f(-x)$ ,  $x \in -M$ ;
- $a, e \in \mathbb{R}^d$ :  $\partial_e f(a) = \lim_{r \rightarrow 0} : \frac{f(a+re) - f(a)}{r}$ .

**Lemma 3.12**

$\varphi \in \mathcal{D}(\mathbb{R}^d)$ .

a)  $x_n \rightarrow x$  in  $\mathbb{R}^d \implies \tau_{x_n} \varphi \rightarrow \tau_x \varphi$  in  $\mathcal{D}(\mathbb{R}^d)$ .

┌

*Důkaz*

$\text{supp } \varphi \subset U(0, r_1)$  for some  $r_1 > 0$ ,  $\{x_n, n \in \mathbb{N}\} \subset U(0, r_2)$  for some  $r_2 > 0$ .  $K := \overline{U(0, r_1 + r_2)} \implies K$  is compact and  $\text{supp } \tau_{x_n} \varphi \subset K$  for each  $n$ .  $\alpha \in \mathbb{N}_0^d$ :

$$\|D^\alpha \tau_{x_n} \varphi - D^\alpha \tau_x \varphi\|_\infty = \sup_{y \in \mathbb{R}^d} |D^\alpha \varphi(y - x_n) - D^\alpha \varphi(y - x)| = \sup_{y \in K} |D^\alpha \varphi(y - x_n) - D^\alpha \varphi(y - x)|.$$

Thus  $D^\alpha \varphi$  is continuous, so it is uniformly continuous on  $\overline{U(2r_2 + r_1)}$ .

$$\varepsilon > 0 \implies \exists \delta > 0 \forall y_1, y_2 \in \overline{U(2r_2 + r_1)} : (\|y_1 - y_2\| < \delta \implies |D^\alpha \varphi(y_1) - D^\alpha \varphi(y_2)| < \varepsilon).$$

$$x_n \rightarrow x \implies \exists n_0 \forall n \geq n_0 : \|x_n - x\| < \delta.$$

$$\begin{aligned} n \geq n_0, y \in K &\implies y - x_n, y - x \in \overline{U(2r_2 + r_1)}, \|(y - x_n) - (y - x)\| = \|x_n - x\| < \delta \implies \\ &\implies |D^\alpha \varphi(y - x_n) - D^\alpha \varphi(y - x)| < \varepsilon \implies D^\alpha \tau_{x_n} \varphi \rightrightarrows D^\alpha \tau_x \varphi. \end{aligned}$$

└

□

b)  $e \in \mathbb{R}^d \implies \partial_e \varphi \in \mathcal{D}(\mathbb{R}^d)$ . Moreover, set

$$\varphi_r(x) := \frac{1}{r}(\varphi(x + re) - \varphi(x)), \quad x \in \mathbb{R}^d,$$

then  $\varphi_r \xrightarrow{r \rightarrow 0} \partial_e \varphi$  in  $\mathcal{D}(\mathbb{R}^d)$ .

┌ *Důkaz* ( $e \in \mathbb{R}^d \implies \partial_e \varphi \in \mathcal{D}(\mathbb{R}^d)$ )  
 $x \in \mathbb{R}^d$ .  $g_x(t) := \varphi(x + te)$ ,  $t \in \mathbb{R}$ . Then  $g_x \in C^\infty(\mathbb{R})$ .

$$\begin{aligned} \partial_e \varphi(x) &= g'_x(0) = \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x + te) \cdot e_j|_{t=0} = \\ &= \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x) e_j \implies \partial_e \varphi = \sum_{j=1}^d e_j \frac{\partial \varphi}{\partial x_j} \in \mathcal{D}(\mathbb{R}^d). \end{aligned}$$

└ *Důkaz* (Moreover part)

Fix  $c > 0$ , such that  $\text{supp } \varphi \subset U(0, c)$ , and  $0 < |r| < 1$ . Then  $\text{supp } \varphi_r \subset \overline{U(0, c + \|e\|)}$ .

$$\begin{aligned} |\varphi_r(x) - \partial_e \varphi(x)| &= \left| \frac{1}{r}(g_x(r) - g_x(0)) - g'_x(0) \right| = \left| \frac{1}{r} \int_0^r g'_x - g'_x(0) \right| = \\ &= \left| \frac{1}{r} \int_0^r (g'_x(t) - g'_x(0)) dt \right| = \left| \frac{1}{r} \int_0^r \sum_{j=1}^d e_j \left( \frac{\partial \varphi}{\partial x_j}(x + te) - \frac{\partial \varphi}{\partial x_j}(x) \right) dt \right| \leq \\ &\leq \left| \frac{1}{r} \int_0^r \|e\| \left( \sum_{j=1}^d \left\| \frac{\partial \varphi}{\partial x_j}(x + te) - \frac{\partial \varphi}{\partial x_j}(x) \right\|^2 \right)^{1/2} dt \right| \leq \\ &\leq \left| \frac{1}{r} \int_0^r \|e\| \left( \sum_{j=1}^d \left\| \tau_{-te} \frac{\partial \varphi}{\partial x_j} - \frac{\partial \varphi}{\partial x} \right\|_\infty^2 \right)^{1/2} dt \right|. \end{aligned}$$

$$\varepsilon > 0 \implies \exists \delta \forall y, \|y\| < \delta : \left\| \tau_{-te} \frac{\partial \varphi}{\partial x_j} - \frac{\partial \varphi}{\partial x} \right\|_\infty < \varepsilon.$$

If  $0 < |t| \cdot \|e\| \cdot c$ , then

$$\|e\| \left( \sum_{j=1}^d \left\| \tau_{-te} \frac{\partial \varphi}{\partial x_j} - \frac{\partial \varphi}{\partial x} \right\|_\infty^2 \right)^{1/2} \leq \|e\| \cdot \sqrt{d} \cdot \varepsilon.$$

└ So  $\varphi_r \rightrightarrows \partial_e \varphi$ ,  $D^\alpha \varphi_r = (D^\alpha \varphi)_r \rightrightarrows \partial_e(D^\alpha \varphi) = D^\alpha(\partial_e \varphi)$ . □

### Tvrzení 3.13

$\varphi \in \mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ .

a)  $\Lambda \in \mathcal{D}'(\mathbb{R}^{d_1})$ . Define  $\psi(y) = \Lambda(x \mapsto \varphi(x, y))$  ( $y \in \mathbb{R}^{d_2}$ ). Then  $\psi \in \mathcal{D}(\mathbb{R}^{d_2})$ .

Důkaz

Fix  $c > 0$  such that  $\text{supp } \varphi \subset \overline{U(\mathbf{o}, c)}$ . 1. „ $\psi$  is well defined“: given  $y \in \mathbb{R}^{d_2}$ ,  $x \mapsto \varphi(x, y)$  belongs to  $\mathcal{D}(\mathbb{R}^{d_1})$ , i.e. it is  $C^\infty$  and  $\text{supp } \subset \overline{U(0, c)}$ . 2.  $\text{supp } \psi \subset \overline{U(\mathbf{o}, c)}$ , so it is compact.

3.  $y \in \mathbb{R}^{d_2}$ ,  $\varphi_y(x) = \varphi(x, y)$  ( $x \in \mathbb{R}^{d_1}$ ). Then „ $y_n \rightarrow y$  in  $\mathbb{R}^{d_2} \implies \varphi_{y_n} \rightarrow \varphi_y$  in  $\mathcal{D}(\mathbb{R}^{d_1})$ “: Assume  $y_n \rightarrow y$  in  $\mathbb{R}^{d_2}$ . WLOG  $\|y_n\| \leq c$  for each  $n$ .  $\forall n : \text{supp } \varphi_{y_n} \subset \overline{U(\mathbf{o}, c)}$ . Fix  $\alpha \in \mathbb{N}_0^{d_1}$ . Then „ $D^\alpha \varphi_{y_n} \rightrightarrows D^\alpha \varphi_y$ “:

$D^\alpha \varphi_{y_n}(x) = D^{(\alpha, 0)} \varphi(x, y_n)$ .  $D^{(\alpha, 0)} \varphi$  is continuous, hence uniformly continuous on  $\overline{U(\mathbf{o}, c)}$ . So, give  $\varepsilon > 0 \exists \delta > 0 \forall (u_1, u_2), (v_1, v_2) \in \overline{U(\mathbf{o}, c)}$ :

$$\|(u_1, v_1) - (u_2, v_2)\| < \delta \implies |D^{(\alpha, 0)} \varphi(u_1, v_1) - D^{(\alpha, 0)} \varphi(u_2, v_2)| < \varepsilon.$$

Fix  $n_0 \in \mathbb{N}$  such that  $\forall n \geq n_0 : \|y - y_n\| < \delta$ . If  $n \geq n_0$  and  $x \in \overline{U_{\mathbb{R}^{d_1}}(\mathbf{o}, c)}$ , then

$$|D^{(\alpha, 0)} \varphi(x, y_n) - D^{(\alpha, 0)} \varphi(x, y)| < \varepsilon \iff \|(x, y_n) - (x, y)\| < \delta.$$

Hence  $\|D^\alpha \varphi_{y_n} - D^\alpha \varphi_y\| \leq \varepsilon$  for  $n \geq n_0$ .

4. „ $\psi$  is continuous“:

$$y_n \rightarrow y \xrightarrow{3.} \varphi_{y_n} \rightarrow \varphi_y \text{ in } \mathcal{D}(\mathbb{R}^{d_1}) \implies \psi(y_n) = \Lambda(\varphi_{y_n}) \rightarrow \Lambda(\varphi_y) = \psi(y).$$

5. „ $\frac{\partial \psi}{\partial y_j}(y) = \Lambda(x \mapsto \frac{\partial \varphi}{\partial y_j}(x, y))$ “:

$$\begin{aligned} \frac{\partial \psi}{\partial y_j}(y) &= \lim_{t \rightarrow 0} \frac{\psi(y + te_j) - \psi(y)}{t} \stackrel{\Lambda \text{ linear}}{=} \lim_{t \rightarrow 0} \Lambda \left( x \mapsto \frac{\varphi(x, y + te_j) - \varphi(x, y)}{t} \right) = \\ &= \lim_{t \rightarrow 0} \Lambda(x \mapsto \varphi_t(x, y)). \end{aligned}$$

We know  $\varphi_t \rightarrow \partial_{(0, y_j)} \varphi$  in  $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ . So we have  $\varphi_t \rightarrow \frac{\partial \varphi}{\partial y_j}$  in  $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ . Hence, for each  $y \in \mathbb{R}^{d_2}$ :  $(\varphi_t)_y \rightarrow \left( \frac{\partial \varphi}{\partial y_j} \right)_y$  in  $\mathcal{D}(\mathbb{R}^{d_1}) \implies \Lambda((\varphi_t)_y) \rightarrow \Lambda \left( \left( \frac{\partial \varphi}{\partial y_j} \right)_y \right)$ .

$$(*) = \Lambda \left( \left( \frac{\partial \varphi}{\partial y_j} \right)_y \right) = \Lambda(x \mapsto \frac{\partial \varphi}{\partial y_j}(x, y)).$$

6. „ $\psi \in C^\infty(\mathbb{R}^{d_2})$  and  $\forall \alpha : D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0, \alpha)} \varphi(x, y))$ “: 5.  $\implies$  for  $|\alpha| = 1$ . 4. applied to  $\frac{\partial \varphi}{\partial y_j}$  implies  $\psi \in C^1(\mathbb{R}^{d_2})$ . Induction: Assume it holds for  $|\alpha| \leq k$ , take  $|\alpha| = k+1$ . Then  $\alpha = \beta + e_j$ ,  $|\beta| = k$ ,  $j \in [d]$ .

$$\begin{aligned} D^\alpha \psi(y) &= \frac{\partial}{\partial y_j} (D^\beta \psi)(y) = \frac{\partial}{\partial y_j} \left( y \mapsto \Lambda \left( x \mapsto D^{(0, \beta)} \varphi(x, y) \right) \right) \stackrel{5.}{=} \\ &= \Lambda(x \mapsto \frac{\partial}{\partial y_j} D^{(0, \beta)} \varphi(x, y)) = \Lambda(x \mapsto D^{(0, \alpha)} \varphi(x, y)). \end{aligned}$$

□

**Lemma 3.14**

$\Omega \subset \mathbb{R}^d$  open,  $\Lambda \in \mathcal{D}(\Omega)$ ,  $K \subset \Omega$  compact. Then  $\exists N \in \mathbb{N}_0$ ,  $\exists \mu_\alpha$ ,  $|\alpha| \leq N$ , finite (signed or complex) Borel measure on  $K$  such that

$$\Lambda(\varphi) = \sum_{|\alpha| \leq N} \int_K D^\alpha \varphi d\mu_\alpha, \quad \varphi \in \mathcal{D}_K(\Omega).$$

*Důkaz* (of lemma, sketch)

From the proposition above  $\exists N, C$  such that

$$|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$X := (C(K))^{\{|\alpha| \leq N\}}$ .  $T : \mathcal{D}_K(\Omega) \rightarrow X$  by  $T\varphi = (D^\alpha \varphi)_{|\alpha| \leq N} \implies \Lambda \circ T^{-1}$  is continuous on  $T(\mathcal{D}_K(\Omega)) \implies$  extend to  $X \implies$  (by Riesz) find  $\mu_\alpha, |\alpha| \leq N$ .  $\square$

b)  $\Lambda_1 \in \mathcal{D}'(\mathbb{R}^{d_1})$ ,  $\Lambda_2 \in \mathcal{D}'(\mathbb{R}^{d_2})$ . Then

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x, y))) = \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x, y))).$$

*Důkaz*

By a) both sides are well defined.  $\text{supp } \varphi \subset \overline{U(\mathbf{o}, c)}$ . From the previous lemma:  $\Lambda_1$  (resp.  $\Lambda_2$ ) on  $\overline{U(\mathbf{o}, c)}$  is equal to  $\mu_\alpha$  (resp.  $\nu_\alpha$ ) for some  $|\alpha| \leq N_1$  (resp.  $|\alpha| \leq N_2$ ).

$$\begin{aligned} \Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x, y))) &= \sum_{|\beta| \leq N_2} \int D^\beta \lambda_1(x \mapsto \varphi(x, y)) d\nu_\beta(y) = \\ &= \sum_{|\beta| \leq N_2} \int \Lambda_1(x \mapsto D^{(0, \beta)} \varphi(x, y)) d\nu_\beta(y) = \\ &= \sum_{|\beta| \leq N_2} \sum_{|\alpha| \leq N_1} \int \int D^{(\alpha, \beta)} \varphi(x, y) d\mu_\alpha(x) d\nu_\beta(y) \stackrel{\text{FUBINI}}{=} \\ &= \sum_{|\beta| \leq N_2} \sum_{|\alpha| \leq N_1} \int \int D^{(\alpha, \beta)} \varphi(x, y) d\nu_\beta(y) d\mu_\alpha(x) \dots \end{aligned}$$

**Definice 3.9** (Convolution in distributions)

$U \in \mathcal{D}'(\mathbb{R}^d)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ ,  $U * \varphi(x) = U(\tau_x \tilde{\varphi}) = U(y \mapsto \varphi(x - y))$  ( $x \in \mathbb{R}^d$ ).

**Věta 3.15** (On the convolution of a distribution and a test function)

a)  $f \in L^1_{loc} \implies \Lambda_f * \varphi = f * \varphi$ .

┌  
Důkaz

$$\Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x - y)) = \int_{\mathbb{R}^d} f(y) \varphi(x - y) dy = f * \varphi(x).$$

└

□

$$b) U * \varphi \in C^\infty(\mathbb{R}^d), D^\alpha(U * \varphi) = D^\alpha U * \varphi = U * D^\alpha \varphi.$$

┌  
Důkaz

„ $U * \varphi$  is continuous“:

$$x_n \rightarrow x \text{ in } \mathbb{R}^d \implies \tau_{x_n} \check{\varphi} \rightarrow \tau_x \check{\varphi} \text{ in } \mathcal{D}(\mathbb{R}^d) \implies U * \varphi(x_n) = U(\tau_{x_n} \check{\varphi}) \rightarrow U(\tau_x \check{\varphi}) = U * \varphi(x).$$

$$\begin{aligned} \frac{\partial}{\partial x_j}(U * \varphi)(x) &= \lim_{t \rightarrow 0} \frac{U * \varphi(x + te_j) - U * \varphi(x)}{t} = \\ &= \lim_{t \rightarrow 0} U \left( \frac{\tau_{x+te_j} \check{\varphi} - \tau_x \check{\varphi}}{t} \right) \stackrel{\psi := \tau_x \check{\varphi}}{=} \lim_{t \rightarrow 0} U \left( \frac{\tau_{te_j} \psi - \psi}{t} \right) = U(\partial_{-e_j} \psi) = \\ &= U \left( \tau_x \left( \frac{\partial \varphi}{\partial x_j} \right) \right) = U * \frac{\partial \varphi}{\partial x_j}(x). \end{aligned}$$

$$\partial_{-e_j} \psi = -\partial_{e_j} \psi = -\frac{\partial \psi}{\partial y_j} = -\frac{\partial}{\partial y_j}(\tau_x \check{\varphi}) = \tau_x \left( \frac{\partial \varphi}{\partial y_j} \right)^v.$$

$$\frac{\partial}{\partial x_j}(U * \varphi) = U * \frac{\partial \varphi}{\partial x_j}.$$

$$\frac{\partial U}{\partial x_j} * \varphi(x) = \frac{\partial U}{\partial x_j} \tau_x \check{\varphi} = -U \left( \frac{\partial \tau_x \check{\varphi}}{\partial x} \right) = U * \frac{\partial \varphi}{\partial x_j}(x).$$

└ So, we have it for  $|\alpha| = 1$ . The general case by induction.

□

$$c) \text{supp}(U * \varphi) \subset \text{supp } U + \text{supp } \varphi.$$

┌  
Důkaz

$$U * \varphi(x) \neq 0 \implies U(\tau_x \check{\varphi}) \neq 0 \implies \text{supp}(\tau_x \check{\varphi}) \cap \text{supp } U \neq \emptyset \implies x \in \text{supp } \varphi + \text{supp } U.$$

└

□

┌  
Důsledek

So  $U$  has compact support  $\implies U * \varphi$  has compact support.

└

$$d) h_j \text{ smoothing kernel. Then } \Lambda_{U * h_j} \rightarrow U \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

┌  
Důkaz

$$\begin{aligned}\Lambda_{U * h_j}(\varphi) &= \int (U * h_j)(x) \varphi(x) dx = \int U(y \mapsto h_j(x - y)) \varphi(x) dx = \\ &= \int U(y \mapsto \varphi(x) h_j(x - y)) dx = \Lambda_1(y \mapsto \varphi(x) h_j(x - y)) = \\ &= U(y \mapsto \Lambda_1(x \mapsto \varphi(x) h_j(x - y))) = U(y \mapsto \int \varphi(x) h_j(x - y) dx) = U(\varphi * \check{h}_j) \rightarrow \Lambda(\varphi).\end{aligned}$$

Because  $\varphi * \check{h}_j \rightarrow \varphi$  in  $\mathcal{D}(\mathbb{R}^d)$  and

$$\begin{aligned}\text{supp}(\varphi * \check{h}_j) &\subset \text{supp } \varphi + U(0, 1/j) \subset \varphi + \overline{U(0, 1)}, \\ D^\alpha(\varphi * \check{h}_j) &= (D^\alpha \varphi) * h_j \rightrightarrows D^\alpha \varphi.\end{aligned}$$

└

$$e) \tau_x(U * \varphi) = \tau_x U * \varphi = U * \tau_x \varphi$$

┌  
Důkaz

$$\begin{aligned}\tau_x(U * \varphi)(z) &= (U * \varphi)(z - x) = U(\tau_{z-x} \check{\varphi}) = U(\tau_{-x} \tau_z \check{\varphi}) = \tau_x U(\tau_z \check{\varphi}) = \tau_x U * \varphi(z). \\ \tau_x(U * \varphi)(z) &= (U * \varphi)(z - x) = U(\tau_{z-x} \check{\varphi}) = U(\tau_z(\tau_{-x} \check{\varphi})) = U(\tau_z(\widetilde{\tau_x \varphi})) = U * \tau_x \varphi(z). \\ (\tau_{-x} \check{\varphi}(y) &= \check{\varphi}(y + x) = \varphi(-y - x) = \tau_x \varphi(-y) = (\widetilde{\tau_x \varphi})(y).)\end{aligned}$$

└

$$f) U * (\varphi * \psi) = (U * \varphi) * \psi \quad (U \in \mathcal{D}'(\mathbb{R}^d), \varphi, \psi \in \mathcal{D}(\mathbb{R}^d)).$$

┌  
Důkaz

$$\begin{aligned}U * (\varphi * \psi)(x) &= U(y \mapsto (\varphi * \psi)(x - y)) = U(y \mapsto \int_{\mathbb{R}^d} \varphi(x - y - z) \psi(z) dz) = \\ &= U(y \mapsto \Lambda_1(z \mapsto \varphi(x - y - z) \psi(z))) = \Lambda_1(z \mapsto U(y \mapsto \varphi(x - y - z) \psi(z))) = \\ &= \Lambda_1(z \mapsto \psi(z) \cdot U(y \mapsto \varphi(x - y - z))) = \Lambda_1(z \mapsto \psi(z) \cdot (U * \varphi)(x - z)) = \\ &= \int \psi(z) \cdot (U * \varphi)(x - z) dz = (U * f) * \psi(x).\end{aligned}$$

└

*Poznámka*

$$\check{U}(\varphi) = U(\check{\varphi}), \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\tau_x U$  and  $\check{U}$  are distributions,  $\tau_x \Lambda_f = \Lambda_{\tau_x f}$ ,  $\check{\Lambda}_f = \Lambda_{\check{f}}$ ,  $f \in L^1_{loc}(\mathbb{R}^d)$  (standard one page of computations or less).



*Poznámka*

$U, V$  distributions,  $U * V(\varphi) = U(\check{V} * \varphi)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ :

- It is natural formula:

$$V = \Lambda_\psi, \psi \in \mathcal{D}(\mathbb{R}^d) \implies \Lambda_{U*\psi}(\varphi) = U(\check{\psi} * \varphi).$$

┌

*Důkaz*

$$\begin{aligned} \Lambda_{U*\psi}(\varphi) &= \int_{\mathbb{R}^d} U * \psi(x) \varphi(x) dx = \int_{\mathbb{R}^d} U(y \mapsto \psi(x - y)) \varphi(x) dx = \\ &= \int_{\mathbb{R}^d} U(y \mapsto \psi(x - y) \varphi(x)) dx = U(y \mapsto \int_{\mathbb{R}^d} \psi(x - y) \varphi(x) dx) = U(y \mapsto \check{\psi} * \varphi(y)). \end{aligned}$$

└

□

- This formula does not work in general because  $\check{V} * \varphi$  is a  $C^\infty$ -function but it need not have compact support.

*Poznámka (1.)*

$\text{supp } V$  is compact, then  $V * \varphi \in \mathcal{D}(\mathbb{R}^n)$  for each  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  ( $\text{supp } \check{V} * \varphi \subset \text{supp } \check{V} + \text{supp } \varphi$ , so it is compact). Then  $U * V$  is linear functional on  $\mathcal{D}(\mathbb{R}^d)$ . Moreover, „it is a distribution“:

Fix  $K \subset \mathbb{R}^d$  compact. Set  $L := \text{supp } \check{V} + K \implies$

$$\implies \exists C > 0, N \in \mathbb{N}_0 : |V(\psi)| \leq C \cdot \|\psi\|, \quad \forall \psi \in \mathcal{D}_L(\mathbb{R}^d).$$

$$\varphi \in \mathcal{D}_K(\mathbb{R}^d) \implies \check{V} * \varphi \in \mathcal{D}_L(\mathbb{R}^d) \implies |(U * V)(\varphi)| = |U(\check{V} * \varphi)| \leq C \cdot \|\check{V} * \varphi\|_N \leq C \cdot D \cdot \|\varphi\|_{N+M}.$$

$$(\check{V} * \varphi)(x) = V(y \mapsto \varphi(x + y)), V \text{ has compact support} \implies \exists D, M : |V(\eta)| \leq D \cdot \|\eta\|_M, \quad \forall \eta \in \mathcal{D}(\mathbb{R}^d).$$

*Poznámka (2.)*

$\text{supp } U$  is compact  $\implies \exists \psi \in \mathcal{D}(\mathbb{R}^d)$  such that  $U(\varphi) = U(\psi \cdot \varphi)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ . (Proof of the theorem above item d.) So, define  $(U * V)(\varphi) = U(\psi \cdot (\check{V} * \varphi))$ . Again  $U * V \in \mathcal{D}'(\mathbb{R}^d)$ . (Proof skipped.)

*Poznámka (3.)*

$\forall r > 0 : (\overline{U(\mathbf{o}, r)} - \text{supp } V) \cap \text{supp } U$  is compact. For  $r > 0$  let  $\psi_r \in \mathcal{D}(\mathbb{R}^d)$ ,  $\psi_r = 1$  on a neighbourhood of this set. Then  $U$  may be extended to

$$Y = \left\{ f \in C^\infty(\mathbb{R}^d) \mid \text{supp } f \subset \overline{U(\mathbf{o}, r)} - \text{supp } V \text{ for some } r > 0 \right\}.$$

$\tilde{U}(f) = U(\psi_r \cdot f)$  if  $\text{supp } f \subset \overline{U(\mathbf{o}, r)} - \text{supp } V$ .

Then define  $U * V(\varphi) = \tilde{U}(\check{V} * \varphi)$  ( $\text{supp } \check{V} * \varphi \subset \text{supp } \varphi - \text{supp } V$ ).

*Poznámka (4.)*

Assume  $\exists m, n \in \mathbb{N}_0, c, d > 0$ :

$$|U(\varphi)| \leq c \cdot \|\varphi\|_n \wedge |V(\varphi)| \leq d \cdot \|\varphi\|_m, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\implies \mu_\alpha, |\alpha| \leq n$  measures (finite ...):

$$U(\varphi) = \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} D^\alpha \varphi d\mu_\alpha, \varphi \in \mathcal{D}(\mathbb{R}^d) \implies$$

$$\implies (U * V)(\varphi) = \sum_{|\alpha| \leq n} \int_{\mathbb{R}^d} D^\alpha (\check{V} * \varphi) d\mu_\alpha.$$

$$|(U * V)(\varphi)| \leq c \cdot d \cdot \|\varphi\|_{n+m}.$$

### 3.5 Tempered distributions

**Definice 3.10** (Schwartz space)

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \forall \alpha \in \mathbb{N}_0^d \ \forall N \in \mathbb{N} : x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \text{ is bounded on } \mathbb{R}^d\}.$$

$$f \in \mathcal{S}(\mathbb{R}^d), \quad N \in \mathbb{N}_0, \quad p_N(f) := \max_{|\alpha| \leq N} \|x \mapsto (1 + \|x\|^2)^N D^\alpha f(x)\|_\infty.$$

Then  $(p_N)_{N=0}^\infty$  is sequence of norms on  $\mathcal{S}(\mathbb{R}^d)$ ,  $p_0 \leq p_1 \leq p_2 \leq \dots$  ( $p_0(f) = \|f\|_\infty$ )

**Tvrzení 3.16**

a)  $\mathcal{S}(\mathbb{R}^d)$  is a Fréchet space when equipped with  $(p_N)_{N=0}^\infty$ .

┌

*Důkaz*

$\mathcal{S}(\mathbb{R}^d)$  is a metrizable LCS. Let  $\varrho$  be the respective translation invariant metric. „Completeness“: Assume  $(f_n)$  is  $\varrho$ -Cauchy  $\implies \forall N: (f_n)$  is  $p_N$ -Cauchy  $\implies \forall N \ \forall \alpha, |\alpha| \leq N : (x \mapsto (1 + \|x\|^2)^N D^\alpha f_k(x))_{k=1}^\infty$  is  $\|\cdot\|_\infty$ -Cauchy  $\implies \forall N, \alpha, |\alpha| \leq N \ \exists g_{N,\alpha}$  such that  $(1 + \|x\|^2)^N D^\alpha f_n(x) \rightrightarrows g_{N,\alpha}(x)$  on  $\mathbb{R}^d$ .  $D^\alpha f_k(x) \rightrightarrows \frac{g_{N,\alpha}(x)}{(1 + \|x\|^2)^N}$ .  $\implies \forall \alpha \ \exists h_\alpha$  continuous such that  $g_{N,\alpha}(x) = (1 + \|x\|^2)^N h_\alpha(x)$  if  $N \geq |\alpha|$ .  $D^\alpha f_k \rightrightarrows h_\alpha \implies h_\alpha = D^\alpha h_\alpha \implies h_\alpha \in C^\infty(\mathbb{R}^d)$ .

„ $h_0 \in \mathcal{S}(\mathbb{R}^d)$ “:

$$(1 + \|x\|^2)^N D^\alpha h_0(x) = g_{N,\alpha}(x),$$

which is bounded (uniform limit of bounded functions). Moreover  $f_k \rightarrow h_0$  in  $p_N$ , hence by the theorem above  $f_n \rightarrow h_0$  in  $\mathcal{S}(\mathbb{R}^d)$  (in  $\varrho$ ).  $\square$

└

b)  $\mathcal{D}(\mathbb{R}^d)$  is a dense subset of  $\mathcal{S}(\mathbb{R}^d)$ .

*Důkaz*

Clearly  $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$ . „Density“: Fix  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi = 1$  na  $U(\mathbf{o}, 1)$ . Let  $f \in \mathcal{S}(\mathbb{R}^d)$ . Let  $f_n(x) = f(x) \cdot \varphi(x/n)$ ,  $x \in \mathbb{R}^d$ . Then  $f_n \in \mathcal{D}(\mathbb{R}^d)$ . Moreover, „ $f_n \rightarrow f$  in  $\mathcal{S}(\mathbb{R}^d)$ “: Let  $N \in \mathbb{N}_0$ ,  $d \in \mathbb{N}_0^d$ ,  $|\alpha| \leq N$ :

$$\begin{aligned} & |(1 + \|x\|^2)^N (D^\alpha f(x) - D^\alpha f_n(x))| = (1 + \|x\|^2)^N |D^\alpha((1 - \varphi(x/n))f(x))| = \\ & = (1 + \|x\|^2)^N \left| (1 - \varphi(x/n))D^\alpha f(x) + \sum_{0 \neq \beta \leq \alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} (-1)^{\frac{1}{n^{|\beta|}}} D^\beta \varphi(x/n) D^{\alpha-\beta} f(x) \right| \\ & \begin{cases} = 0, & \|x\| \leq n \\ \leq \sup_{\|x\| \geq n, |\gamma| \leq N} \frac{(1 + \|x\|^2)^{N+1} |D^\gamma f(x)|}{1 + \|x\|^2}, & \|x\| > n \end{cases} \\ & \left( \sup_{\|x\| \geq n} \left( 1 + \sum_{0 \neq \beta \leq \alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} \cdot \underbrace{\frac{1}{n^{|\beta|}}}_{\leq 1} \underbrace{|D^\beta \varphi(x/n)|}_{\leq \|\varphi\|_N} \right) \right) \leq 1 + 2^N \|\varphi\|_N. \\ & \leq (1 + 2^N \cdot \|\varphi\|_N) \cdot \frac{p_{N+1}(f)}{1 + n^2} \rightarrow 0. \end{aligned}$$

└

c)  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\mathbb{R}^d) \implies \varphi_n \rightarrow \varphi$  in  $\mathcal{S}(\mathbb{R}^d)$ .

*Důkaz*

Assume  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}(\mathbb{R}^d) \implies \exists R > 0$  such that  $\text{supp } \varphi_n \subset \overline{U(\mathbf{o}, R)}$ . Then

$$p_n(\varphi_n - \varphi) = \max_{|\alpha| \leq N} \|x \mapsto (1 + \|x\|^2)^N (D^\alpha \varphi_n(x) - D^\alpha \varphi(x))\|_\infty \leq (1 + R^2)^N \cdot \|\varphi_n - \varphi\|_N \rightarrow 0.$$

└

### Definice 3.11 (A tempered distribution on $\mathbb{R}^d$ )

A tempered distribution on  $\mathbb{R}^d$  is a continuous linear functional on  $\mathcal{S}(\mathbb{R}^d)$ . Notation:  $\mathcal{S}'(\mathbb{R}^d)$ .

*Poznámka*

$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \implies \Lambda|_{\mathcal{D}(\mathbb{R}^d)} \in \mathcal{D}'(\mathbb{R}^d)$ . (By the previous theorem item c.)

$\mathcal{D}'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ . (By item a. and b.)

We say that distribution is tempered, if it can be extended to  $\mathcal{S}(\mathbb{R}^d)$ .

### Tvrzení 3.17 (A characterization of tempered distributions)

a)  $\Lambda : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{F}$  linear. Then

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0 \exists C > 0 : |\Lambda(\varphi)| \leq C \cdot p_N(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

┌ *Důkaz*

By the proposition above. □

└

b) Assume  $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ . Then  $\Lambda$  is tempered iff

$$\exists N \in \mathbb{N}_0 \exists c > 0 : |\Lambda(\varphi)| \leq C \cdot p_N(\varphi), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

┌ *Důkaz*

„ $\Rightarrow$ “: a). „ $\Leftarrow$ “: For example by Hahn–Banach and a). □

└

### Definice 3.12 (Convergence in $\mathcal{S}'$ )

$\Lambda_n \rightarrow \Lambda$  in  $\mathcal{S}'(\mathbb{R}^d) \equiv \forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \Lambda_n(\varphi) \rightarrow \Lambda(\varphi)$ , i.e.  $\Lambda_n \xrightarrow{w^*} \Lambda$ .

### Věta 3.18 (BVanach–Steinhaus theorem for tempered distribution)

$(\Lambda_n) \subset \mathcal{S}'(\mathbb{R}^d)$ ,  $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : (\Lambda_n(\varphi))$  converges in  $\mathbb{F}$ . Then  $\Lambda(\varphi) = \lim_{n \rightarrow \infty} \Lambda_n(\varphi)$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  is tempered distribution.

┌ *Důkaz*

Use the previous proposition item a) and the theorem above. □

└

### Tvrzení 3.19 (Examples of tempered distributions)

a)  $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ ,  $\text{supp } \Lambda$  is compact  $\Rightarrow \Lambda$  is tempered.

┌ *Důkaz*

$\Lambda$  has compact support  $\Rightarrow \exists C > 0 \exists N \in \mathbb{N}_0 : |\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N \leq C \cdot p_N(\varphi)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^d)$ . □

└

b)  $f \in L^p(\mathbb{R}^d)$  for some  $p \in [1, \infty]$ . Then  $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$  and, moreover,  $L_f(\varphi) = \int_{\mathbb{R}^d} f \varphi$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

┌  
Dikaz

Theorem IV.11(a)  $\implies \mathcal{S}(\mathbb{R}^d) \subset \bigcap_{p \in [1, \infty]} L^p(\mathbb{R}^d)$ . (It was stated and almost proven at chapter IV, but full proof is not easy.) So, fix  $p \in [1, \infty]$  and  $f \in L^p(\mathbb{R}^d)$ . Let  $p'$  be the dual exponent. Then  $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \varphi \in L^{p'}(\mathbb{R}^d)$ , hence  $f\varphi \in L^1(\mathbb{R}^d)$ .

So  $\tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  is a well-defined linear functional on  $\mathcal{S}(\mathbb{R}^d)$ : „continuity“:

$$p = 1 : |\tilde{\Lambda}(\varphi)| = \left| \int_{\mathbb{R}^d} f\varphi \right| \leq \|f\|_1 \cdot \|\varphi\|_\infty = \|f\|_1 \cdot p_0(\varphi);$$

$p > 1 : \forall n \in \mathbb{N} : f \cdot \chi_{U(\mathbf{o}, n)} \in L^1(\mathbb{R}^d) \implies \Lambda_{f \cdot \chi_{U(\mathbf{o}, n)}} \in \mathcal{S}(\mathbb{R}^d)$  by the first case  $\implies$

$$\implies \tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f \cdot \chi_{U(\mathbf{o}, n)} \varphi = \lim_{n \rightarrow \infty} \Lambda_{f \cdot \chi_{U(\mathbf{o}, n)}}(\varphi) = \Lambda(\varphi).$$

└

c)  $f$  measurable on  $\mathbb{R}^d$ ,  $|f| \leq |p|$  for some polynomial  $p$  on  $\mathbb{R}^d$ . Then  $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$  and  $\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi$ ,  $f \in \mathcal{S}(\mathbb{R}^d)$ .

┌  
Dikaz

$p$  polynomial  $\implies p(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$  ( $c_\alpha \in \mathbb{F}$ ,  $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_d^{\alpha_d}$ ).

$$\implies |p(x)| \leq c \cdot (\sqrt{2})^{dN} (1 + \|x\|^2)^{N \cdot \frac{d}{2}}, \quad c = \max_\alpha |c_\alpha|.$$

So, if  $|f| \leq |p|$ , then  $\frac{|f(x)|}{(1 + \|x\|^2)^m} \leq c \cdot (\sqrt{2})^{dN} \cdot (1 + \|x\|^2)^{N \cdot \frac{d}{2} - m}$ . If  $m$  is large enough (such that  $N \cdot \frac{d}{2} - m < -\frac{d}{2}$ ), then  $f(x)/(1 + \|x\|^2)^m$  is integrable in  $\mathbb{R}^d$ . ( $1/(1 + \|x\|^2)^k$  is integrable for  $k > \frac{d}{2}$  see the comment before theorem IV.11). Then:

$$\left| \int_{\mathbb{R}^d} f \cdot \varphi \right| = \left| \int_{\mathbb{R}^d} \frac{f(x) \cdot (1 + \|x\|^2)^m}{(1 + \|x\|^2)^m} \right| \leq \left( \int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + \|x\|^2)^m} \right) \cdot p_m(f).$$

└

d)  $\mu$  is a finite measure  $\implies \Lambda_\mu \in \mathcal{S}'(\mathbb{R}^d)$ ,  $\Lambda_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

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Dikaz

$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \varphi$  is continuous and bounded.

$$\left| \int_{\mathbb{R}^d} \varphi d\mu \right| \leq \int_{\mathbb{R}^d} |\varphi| d|\mu| \leq \|f\|_\infty \cdot \|\mu\| = p_0(\varphi) \cdot \|\mu\|.$$

└

**Lemma 3.20** (Continuity of operations on the Schwartz space)

$f \mapsto D^\alpha f$  is continuous  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ .

┌

*Důkaz*

$f \in \mathcal{S}(\mathbb{R}^d)$ ,  $\alpha \in \mathbb{N}_0^d \implies D^\alpha f \in L^\infty(\mathbb{R}^d)$ . Fix  $N \in \mathbb{N}_0$  and  $\beta$ ,  $|\beta| \leq N$ :

$$|(1 + \|x\|^2)^N D^\beta(D^\alpha f)(x)| = (1 + \|x\|^2)^N |D^{\beta+\alpha} f(x)| \leq p_{N+|\alpha|}(f) \implies$$

$$\implies p_N(D^\alpha f) \leq p_{N+|\alpha|}(f).$$

└

□

$p$  is polynomial  $\implies f \mapsto p \cdot f$  is continuous  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ .

┌

*Důkaz*

Clearly  $p \cdot f \in C^\infty(\mathbb{R}^d)$ . Fix  $N \in \mathbb{N}_0$ . Then  $\exists c > 0, m \in \mathbb{N}$  such that

$$\forall \alpha, |\alpha| \leq N, \forall x \in \mathbb{R}^d |D^\alpha p(x)| \leq c \cdot (1 + \|x\|^2)^m.$$

Fix  $\alpha$ ,  $|\alpha| \leq N$ ,  $x \in \mathbb{R}^d$ :

$$\begin{aligned} |(1 + \|x\|^2)^N D^\alpha(p \cdot f)(x)| &= (1 + \|x\|^2)^N \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta p(x) D^{\alpha-\beta} f(x) \right| \leq \\ &\leq c \cdot (1 + \|x\|^2)^{N+m} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta} f(x)| \leq c \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} p_{N+M}(f) \leq c \cdot 2^N p_{N+M}(f) \implies \\ &\implies p_N(p \cdot f) \leq c \cdot 2^N p_{N+M}(f). \end{aligned}$$

└

□

$g \in \mathcal{S}(\mathbb{R}^d) \implies f \mapsto f \cdot g$  is continuous  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ .

┌

*Důkaz*

$g \in \mathcal{S}(\mathbb{R}^d) \implies \forall \alpha : D^\alpha g$  is bounded on  $\mathbb{R}^d$ . Fix  $N \in \mathbb{N}_0$ . Set  $C := \max_{|\alpha| \leq N} \|D^\alpha g\|_\infty$ . Fix  $\alpha$ ,  $|\alpha| \leq N$ ,  $x \in \mathbb{R}^d$ .

$$|(1 + \|x\|^2)^N D^\alpha(f \cdot g)(x)| = (1 + \|x\|^2)^N \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta g(x) D^{\alpha-\beta} f(x) \right| \leq$$

$$C \cdot \sum_{\beta \leq \alpha} p_N(f) \leq C \cdot 2^N \cdot p_N(f) \implies p_N(g \cdot f) \leq C \cdot 2^N p_N(f).$$

└

□

┌

*Poznámka*

Similarly one may prove that:  $g \in C^\infty(\mathbb{R}^d)$ ,  $\forall \alpha \exists P_\alpha$  polynomial:  $|D^\alpha g| \leq |P_\alpha|$  on  $\mathbb{R}^d \implies f \mapsto f \cdot g$  is a continuous mapping  $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$ .

└

### **Tvrzení 3.21** (Operations with tempered distributions)

Let  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ .

a)  $\forall \alpha : D^\alpha \Lambda \in \mathcal{S}'(\mathbb{R}^d)$  and  $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

┌ *Důkaz*

$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies D^\alpha \varphi \in \mathcal{S}(\mathbb{R}^d)$ . So,  $\tilde{\Lambda}(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ , is well-defined linear functional on  $\mathcal{S}(\mathbb{R}^d)$  whose restriction to  $\mathcal{D}(\mathbb{R}^d)$  is  $D^\alpha \Lambda$ . „Continuity:“  $\varphi_n \rightarrow \varphi$  in  $\mathcal{S}(\mathbb{R}^d) \implies$  (by the previous lemma)  $D^\alpha \varphi_n \rightarrow D^\alpha \varphi$  in  $\mathcal{S}(\mathbb{R}^d)$ , so  $\tilde{\Lambda}(\varphi_n) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi_n) \rightarrow (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = \tilde{\Lambda}(\varphi)$ .  $\square$

└

b)  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $f$  is a polynomial  $\implies f \cdot \Lambda \in \mathcal{S}'(\mathbb{R}^d)$  and  $f \Lambda(\varphi) = \Lambda(f \varphi)$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

┌ *Důkaz* (Skipped on lecture)

Completely analogous to a).  $\square$

└

c)  $y \in \mathbb{R}^d \implies \tau_y \Lambda \in \mathcal{S}'(\mathbb{R}^d)$ ,  $\tau_y \Lambda(\varphi) = \Lambda(\tau_{-y} \varphi)$ ,  $\varphi \in \mathcal{S}'(\mathbb{R}^d)$ .

┌ *Důkaz*

„ $\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \tau_{-y} \varphi \in \mathcal{S}(\mathbb{R}^d)$ ,  $\tau_{-y} \varphi(x) = \varphi(x - y)$ “: Clearly  $\tau_{-y} \varphi \in C^\infty(\mathbb{R}^d)$ .

$$|\alpha| \leq N : (1 + \|x\|^2)^N D^\alpha \tau_{-y} \varphi(x) = (1 + \|x\|^2)^N D^\alpha \varphi(x + y) =$$

$$= \left( \frac{1 + \|x\|^2}{1 + \|x + y\|^2} \right)^N \cdot (1 + \|x + y\|^2)^N D^\alpha \varphi(x + y) \leq \left( \frac{1 + \|x\|^2}{1 + \|x + y\|^2} \right)^N \cdot p_N(\varphi) \leq M^N,$$

where  $M = \sup_{t \in [0, \infty)} \frac{1+t^2}{1+(t-\|y\|)^2} < \infty$ .  $\implies \tau_{-y} \varphi \in \mathcal{S}(\mathbb{R}^d)$  and  $p_N(\tau_{-y} \varphi) \leq M^N p_N(\varphi)$ .

So  $\varphi \mapsto \tau_{-y} \varphi$  is continuous and then continue as in a).  $\square$

└

d)  $\check{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$ ,  $\check{\Lambda}(\varphi) = \Lambda(\check{\varphi})$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .

┌ *Důkaz*

Observe that  $\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \check{\varphi} \in \mathcal{S}(\mathbb{R}^d)$  ( $\check{\varphi}(x) = \varphi(-x)$ ) and  $p_N(\check{\varphi}) = p_N(\varphi)$ .  $\square$

└

### Tvrzení 3.22

$\Lambda_n \rightarrow \Lambda$  in  $\mathcal{S}'(\mathbb{R}^d)$ . a)  $\forall \alpha : D^\alpha \Lambda_n \rightarrow D^\alpha \Lambda$ , b)  $f \in \mathcal{S}(\mathbb{R}^d)$  and  $f$  is polynomial  $\implies f \Lambda_n \rightarrow f \Lambda$ .

┌ *Důkaz*

„a“:  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ :

$$D^\alpha \Lambda_n(\varphi) = (-1)^{|\alpha|} \Lambda_n(D^\alpha \varphi) \rightarrow (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = D^\alpha \Lambda(\varphi).$$

„b“ similarly.  $\square$

└

## 3.6 Convolution and the Fourier transform of tempered distributions

*Poznámka* (Recall)

$$f \in L^1(\mathbb{R}^d) \implies \hat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-i\langle t, x \rangle} dm_d(x).$$

Fourier transform maps  $L^1(\mathbb{R}^d)$  into  $C_0(\mathbb{R}^d)$  and  $\mathcal{S}(\mathbb{R}^d)$  onto  $\mathcal{S}(\mathbb{R}^d)$ .

$$\hat{\hat{f}} = \check{f}, \quad f \in \mathcal{S}(\mathbb{R}^d), \quad \left( \hat{\hat{\hat{f}}} = f \right).$$

### Lemma 3.23

Fourier transform is an isomorphism of  $\mathcal{S}(\mathbb{R}^d)$  onto  $\mathcal{S}(\mathbb{R}^d)$ .

┌

*Důkaz*

1. The theorem above  $\implies$  Fourier transform is a linear bijection  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}(\mathbb{R}^d)$ .

2.  $m := \lfloor \frac{d}{2} \rfloor + 1$ . Then  $C := \int_{\mathbb{R}^d} \frac{1}{(1+\|x\|^2)^m} dm_d(x) < \infty$ .  $f \in \mathcal{S}(\mathbb{R}^d) \implies$

$$\implies \|\hat{f}\|_\infty \leq \|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)| dm_d(x) \leq \int_{\mathbb{R}^d} \frac{(1+\|x\|^2)^m |f(x)|}{(1+\|x\|^2)^m} dm_d(x) \leq C \cdot p_m(f).$$

TODO!!!

3. Fix  $N \in \mathbb{N}_0$ ,  $\alpha$ ,  $|\alpha| \leq N$ .

$$\begin{aligned} f \in \mathcal{S}(\mathbb{R}^d) : (1+\|x\|^2)^N D^\alpha \hat{f}(x) &= (1+\|x\|^2)^N (y \mapsto \widehat{(-1)^{|\alpha|} y^\alpha f(y)})(x) = \\ &= (-1)^{|\alpha|} (y \mapsto \check{\widehat{(D)(y^\alpha f(y))}})(x) = (-1)^{|\alpha|} (y \mapsto \sum_{|\beta| \leq 2N} \widehat{a_\beta D^\beta (y^\alpha f(y))})(x), \end{aligned}$$

where  $\check{p}(x) = p(ix)$ ,  $p(D)f = \sum_{c_\alpha} D^\alpha f$  if  $p(x) = \sum c_\alpha x^\alpha$ .  $\check{p}(x) = (1 - \sum_{j=1}^d x_j^2)^N$  a polynomial of degree  $2N$ .

So,  $\|x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \leq C \cdot p_m(y \mapsto \sum_{|\beta| \leq 2N} a_\beta D^\beta (y^\alpha f(y)))$ . From the previous lemma  $f \mapsto \sum_{|\beta| \leq 2N} a_\beta D^\beta (y^\alpha f(y))$  is continuous.

So,  $\exists M = M_{N,\alpha} > 0$ ,  $\exists m = m_{N,\alpha} \in \mathbb{N}_0$ :

$$\begin{aligned} p_m(y \mapsto \sum_{|\beta| \leq 2N} a_\beta D^\beta (y^\alpha f(y))) &\leq M \cdot p_m(f) \implies \\ \implies \|x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty &\leq C \cdot M \cdot p_m(f). \end{aligned}$$

4. So,  $p_N(\hat{f}) \leq C \cdot \tilde{M} \cdot p_{\tilde{m}}(f)$ , where  $\tilde{M} = \max_{|\alpha| \leq N} M_{N,\alpha}$ ,  $\tilde{m} = \max_{|\alpha| \leq N} m_{N,\alpha}$ . □

### Definition 3.13 (Fourier transform of tempered distribution)

$\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ :  $\hat{\Lambda}(\varphi) = \Lambda(\hat{\varphi})$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ .



┌ Poznámka

$\hat{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$ :  $\varphi_n \rightarrow \varphi$  in  $\mathcal{S}(\mathbb{R}^d) \implies \hat{\varphi}_n \rightarrow \hat{\varphi}$  in  $\mathcal{S}(\mathbb{R}^d) \implies \hat{\Lambda}(\varphi_n) = \Lambda(\hat{\varphi}_n) \rightarrow \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi)$ .

### Věta 3.24

a) Fourier transform is a linear bijection of  $\mathcal{S}'(\mathbb{R}^d)$  onto  $\mathcal{S}'(\mathbb{R}^d)$ .

┌ Důkaz

$\hat{\hat{\Lambda}} = \check{\Lambda}$ ,  $\hat{\hat{\Lambda}} = \Lambda$  for  $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ ,  $(\check{\Lambda})(\varphi) = \Lambda(\check{\varphi})$ ,  $\hat{\Lambda}_1 = \hat{\Lambda}_2 \implies \hat{\hat{\Lambda}}_1 = \hat{\hat{\Lambda}}_2 \implies \Lambda_1 = \Lambda_2$ . □

b)  $\Lambda_n \rightarrow \Lambda$  in  $\mathcal{S}'(\mathbb{R}^d) \implies \hat{\Lambda}_n \rightarrow \hat{\Lambda}$  in  $\mathcal{S}'(\mathbb{R}^d)$ .

┌ Důkaz

$$\hat{\Lambda}_n(\varphi) = \Lambda_n(\hat{\varphi}) \rightarrow \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi).$$

└ □

c)  $f \in C^1(\mathbb{R}^d) \implies \hat{\Lambda}_f = \Lambda_{\hat{f}}$ .

┌ Důkaz

$$\hat{\Lambda}_f(\varphi) = \Lambda_f(\hat{\varphi}) = \int f \hat{\varphi} dm_d = \int \hat{f} \varphi dm_d = \Lambda_{\hat{f}}(\varphi).$$

└ □

d)  $f \in L^2(\mathbb{R}^d) \implies \hat{\Lambda}_f = \Lambda_{\mathcal{P}(f)}$ , where  $\mathcal{P}$  is the Plancherel transform.

┌ Důkaz

$f_n := f \cdot \chi_{U(0,n)}$ . Then  $f_n \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ,  $f_n \rightarrow f$  in  $L^2(\mathbb{R}^d)$ , and, moreover,  $\hat{f}_n \rightarrow \mathcal{P}(f)$  in  $L^2(\mathbb{R}^d)$ . So,  $\hat{\Lambda}_f(\varphi) = \Lambda_f(\hat{\varphi}) =$

$$= \int_{\mathbb{R}^d} f \hat{\varphi} dm_d = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \hat{\varphi} dm_d = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \hat{f}_n \varphi = \int_{\mathbb{R}^d} \mathcal{P}(f) \varphi dm_d = \Lambda_{\mathcal{P}(f)}(\varphi).$$

└ □

e)  $p$  polynomial  $\implies \widehat{p(D)\Lambda} = \check{p}\hat{\Lambda}$ ,  $\widehat{p \cdot \Lambda} = \check{p}(D)\hat{\Lambda}$ .

$$\left( \check{p}(t) = p(it), \quad \check{p}(t) = p(-t), \quad p(t) = \sum c_\alpha t^\alpha \implies p(D)f = \sum c_\alpha D^\alpha f. \right)$$

┌  
Důkaz

$$\begin{aligned} \widehat{p(D)\Lambda}(\varphi) &= p(D)\Lambda(\hat{\varphi}) = \Lambda(\check{(D)}\hat{\varphi}) = \Lambda\left(\widehat{\check{\varphi}}\right) = \hat{\Lambda}(\check{p}\varphi) = \check{p}\hat{\Lambda}(\varphi). \\ \widehat{p \cdot \Lambda}(\varphi) &= p \cdot \Lambda(\hat{\varphi}) = \Lambda(p\hat{\varphi}) = \Lambda\left(\widehat{\check{\varphi}}(D)\right) = \hat{\Lambda}(\check{p}(D)\varphi) = \check{p}(D)\hat{\Lambda}(\varphi). \end{aligned}$$

└

┌  
Poznámka

In particular

$$\widehat{D^\alpha \Lambda} = (x \mapsto c^{|\alpha|} x^\alpha) \hat{\Lambda}, \quad (x \mapsto \widehat{x^\alpha}) L = c^{|\alpha|} D^\alpha \hat{\Lambda}.$$

└

□

Poznámka

Next two lemmata are analogues of Lemmata above.

### Lemma 3.25

a)  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ ,  $x_n \rightarrow x$  in  $\mathbb{R}^d \implies \tau_{x_n}\varphi \rightarrow \tau_x\varphi$  in  $\mathcal{S}(\mathbb{R}^d)$ .

┌  
Důkaz

Fix  $N \in \mathbb{N}_0$ ,  $\alpha \in \mathbb{N}_0^d$ ,  $|\alpha| \leq N$ .  $x, z \in \mathbb{R}^d$ :

$$\begin{aligned} & |(1 + \|x\|^2)^N D^\alpha \tau_z \varphi(y) - (1 + \|y\|^2)^N D^\alpha \tau_x \varphi(y)| = \\ &= (1 + \|y\|^2)^N \left| \int_0^1 \frac{d}{dt} D^\alpha \varphi(y - x - t(z - x)) dt \right| \leq \\ &\leq (1 + \|y\|^2)^N \int_0^1 \left| \sum_{j=1}^d D^{\alpha+e_j} \varphi(y - x - t(z - x)) (x_j - z_j) \right| dt \leq \\ &\leq (1 + \|y\|^2)^N \int_0^1 \left| \sum_{j=1}^d D^{\alpha+e_j} \varphi(y - x - t(z - x))^2 \right|^{1/2} \|x - z\| dt \leq \\ &\leq \|x - z\| p_{N+1}(\varphi) (1 + \|y\|^2)^N \int_0^1 \frac{\sqrt{d} dt}{(1 + \|y - x - t(z - x)\|^2)^{N+1}} \leq \\ &\leq \|x - z\| p_{N+1}(\varphi) \cdot \sqrt{d} \frac{(1 + \|y\|^2)^N}{(1 + \|y - x\|^2 - \|y - x\|)^{N+1}}. \\ &\left( 1 + \|y - x\| - t\|z - x\| \geq 1 + \|y - x\|^2 - \|y - x\|, \text{ if } \|z - x\| \leq \frac{1}{2} \right) \end{aligned}$$

└

□

b) skipped and proof skipped too.

**Lemma 3.26** (RT)

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0, \mu_\alpha, |\alpha| \leq N_0 : \Lambda(\varphi) = \sum_{|\Lambda| \leq N} \int_{\mathbb{R}^d} (1 + \|x\|^2)^N D^\alpha \varphi(x) d\mu_\alpha(x), \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(Finite signed/complex measure on  $\mathbb{R}^d$ .)

**Definice 3.14** (Convolution of function and tempered distribution)

$$U \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{S}(\mathbb{R}^d) \implies U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x - y)).$$

**Věta 3.27** (Analogues to the theorem above)

a) *skipped.*

b)  $\Lambda_{U*\varphi}$  is a tempered distribution.

┌

*Důkaz*

The proposition above  $\implies \exists N, C$  such that  $|U(\psi)| \leq c \cdot p_N(\psi)$

$$\implies |(U * \varphi)(x)| = |U(\tau_x \check{\varphi})| \leq C \cdot p_N(\tau_x \check{\varphi}).$$

$$|\alpha| \leq N : |(1 + \|y\|^2)^N D^\alpha \varphi(x - y)| \leq p_N(\varphi) \cdot \left( \frac{1 + \|y\|^2}{1 + \|x - y\|^2} \right)^N \leq p_N(\varphi) (1 + \|y\| + \|x\|^2)^N \implies$$

$\implies \Lambda'_{U*\varphi}$  is tempered.

$$\begin{aligned} \frac{1 + \|y\|^2}{1 + \|x - y\|^2} &= \frac{1 + \|y - x\|^2 + 2\langle y - x, x \rangle + \|x\|^2}{1 + \|x - y\|^2} = \\ &= 1 + \frac{2 \cdot \|y - x\| \cdot \|x\|}{1 + \|x - y\|^2} + \frac{\|x\|^2}{1 + \|x - y\|^2} \leq 1 + \|x\| + \|x\|^2. \end{aligned}$$

└

□

c) *skipped.*

$$d) \widehat{\Lambda_{U*\varphi}} = \hat{\varphi} \cdot \hat{U}, \widehat{\varphi \cdot U} = \Lambda_{\hat{\varphi} \cdot \hat{U}}.$$

┌  
Důkaz

$$\begin{aligned}\widehat{\Lambda_{U*\varphi}}(\psi) &= \Lambda_{U*\varphi}(\hat{\psi}) = \int_{\mathbb{R}^d} (U * \varphi)(x) \hat{\psi}(x) f m_d(x) = \int_{\mathbb{R}^d} U(y \mapsto \varphi(x-y)) \hat{\psi}(x) dm_d(x) = \\ &= U(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y) \hat{\psi}(x) dm_d(x)) = U(\check{\varphi} * \hat{\psi}) = U(\hat{\varphi} + \hat{\psi}) = U(\widehat{\varphi + \psi}) = \hat{U}(\hat{\varphi} \cdot \psi) = \hat{\varphi} \cdot \hat{U}(\psi).\end{aligned}$$

$$\Lambda_{\hat{\varphi} * \hat{U}} = \widehat{\widehat{\Lambda_{\varphi + U}}} = \widehat{\hat{\varphi} \cdot \hat{U}} = \widehat{\check{\varphi} \cdot \check{U}} = \widehat{\varphi \cdot U}.$$

└

□

## 4 Elements of vector integration

*Poznámka*

$(M, \mathcal{A})$  is measure space,  $(\Omega, \Sigma, \mu)$  is a complete measure space ( $\mu \geq 0$ ),  $X$  is a Banach space.

### 4.1 Measurability

#### Definice 4.1

$f : M \rightarrow X$ .

- $f$  is simple, if  $f(M)$  is finite, i.e.  $f = \sum_{j=1}^k x_j \chi_{A_j}$ , where  $x_j \in X$ ,  $A_j \subset M$  pairwise disjoint;
- $f$  is simple measurable, if  $f$  is a simple and, moreover,  $A_j \in \mathcal{A}$ ;
- $f$  is (strongly)  $\mathcal{A}$ -measurable if  $\exists (u_n)$  simple measurable:  $u_n \rightarrow f$  point-wise, i.e.  $\forall x \in M : u_n(x) \rightarrow f(x)$  in  $(X, \|\cdot\|)$ ;
- $f$  is Borel  $\mathcal{A}$ -measurable, if  $\forall U \subset X$  open:  $f^{-1}(U) \in \mathcal{A}$ ;
- $f$  is weakly  $\mathcal{A}$ -measurable if  $\forall \varphi \in X^* : \varphi \circ f$  is  $\mathcal{A}$ -measurable.

#### Tvrzení 4.1

a) Simple functions, simple measurable functions, strongly  $\mathcal{A}$ -measurable functions, and weakly  $\mathcal{A}$ -measurable functions form vector spaces.

┌

*Důkaz*

$f, g : M \rightarrow X, \alpha, \beta \in \mathbb{F}$ .

- „ $f, g$  simple  $\implies \alpha f + \beta g$  is simple“:  $(\alpha f + \beta g)(M) \subset \alpha f(M) + \beta g(M)$ .
- „ $f, g$  simple measurable  $\alpha f + \beta g$  is simple measurable“:

$$f = \sum_{j=1}^k x_j \chi_{A_j}, \quad g = \sum_{l=1}^m y_l \chi_{B_l}, \quad \alpha f + \beta g = \sum_{j=1}^k \sum_{l=1}^m (\alpha x_j + \beta y_l) \cdot \chi_{A_j \cap B_l},$$

$$A_j, B_l \in \mathcal{A} \implies A_j \cap B_l \in \mathcal{A}.$$

- „ $f, g$  strongly  $\mathcal{A}$ -measurable  $\implies \alpha f + \beta g$  is strongly  $\mathcal{A}$ -measurable“:  $f = \lim u_n, g = \lim v_n, u_n, v_n$  simple measurable,  $\alpha f + \beta g = \lim(\alpha u_n + \beta v_n)$ .
- „ $f, g$  weakly  $\mathcal{A}$ -measurable  $\implies \alpha f + \beta g$  weakly  $\mathcal{A}$ -measurable“:  $\forall \varphi \in X^* : \varphi \circ (\alpha f + \beta g) = \alpha \varphi \circ f + \beta \varphi \circ g$  (measurable by the scakercah?).

└

□

b)  $f_n \rightarrow f$  point-wise,  $f_n$  Borel  $\mathcal{A}$ -measurable (resp. weakly  $\mathcal{A}$ -measurable)  $\implies f$  is Borel  $\mathcal{A}$ -measurable (resp. weakly  $\mathcal{A}$ -measurable).

┌

*Důkaz*

Assume that  $\forall n: f_n$  is Borel  $\mathcal{A}$ -measurable.  $U \subset X$  open:

$$f^{-1}(U) = \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k=m}^{\infty} f_k^{-1} \left( \left\{ x \in X \mid \text{dist}(x, X \setminus U) > \frac{1}{n} \right\} \right),$$

$$f(x) \in U \Leftrightarrow \exists n \in \mathbb{N} \exists m \in \mathbb{N} \forall k \geq m : \text{dist}(f_k(x), X \setminus U) > \frac{1}{n}.$$

$f_n$  are weakly  $\mathcal{A}$ -measurable,  $\varphi \in X^* \implies \forall n : \varphi \circ f_n$  is Borel  $\mathcal{A}$ -measurable and  $\varphi \circ f_n \rightarrow \varphi \circ f$ , so,  $\varphi \circ f$  is Borel  $\mathcal{A}$ -measurable. □

└

c)  $f$  is strongly  $\mathcal{A}$ -measurable  $\implies f$  is Borel  $\mathcal{A}$ -measurable  $\implies f$  is weakly  $\mathcal{A}$ -measurable.

┌ *Důkaz*

$f$  is simple  $\implies (f \text{ is simple measurable} \Leftrightarrow f \text{ is Borel } \mathcal{A}\text{-measurable})$ .

„ $f$  strongly  $\mathcal{A}$ -measurable  $\implies f$  Borel  $\mathcal{A}$  measurable“:  $f = \lim u_n$ ,  $u_n$  simple measurable, then  $u_n$  are Borel  $\mathcal{A}$ -measurable, so by b),  $f$  is Borel  $\mathcal{A}$ -measurable.

„ $f$  Borel  $\mathcal{A}$ -measurable  $\implies f$  is weakly  $\mathcal{A}$ -measurable“:  $\varphi \in X^*$ ,  $U \subset \mathbb{F}$  open  $\implies (\varphi \circ f)^{-1}(U) = f^{-1}(\underbrace{\varphi^{-1}(U)}_{\text{open}}) \in \mathcal{A}$ .

„ $f$  simple, weakly  $\mathcal{A}$ -measurable  $\implies f$  is simple measurable“:  $f(M) = \{x_1, \dots, x_k\}$ , distinct points. „ $f^{-1}(x_1) \in \mathcal{A}$ “: for  $j \in \{2, \dots, k\}$  find  $\varphi_j \in X^*$ ,  $\varphi_j(x_1) \neq \varphi_j(x_j)$ . Then

$$f^{-1}(x_1) = \bigcap_{j=2}^k \{t \in M \mid \varphi_j(f(t) - x_j) \neq 0\} = \bigcap_{j=2}^k \overbrace{(\varphi_j \circ f)^{-1}(\underbrace{\mathbb{F} \setminus \{\varphi_j(x_j)\}}_{\text{open}})}^{\in \mathcal{A}}.$$

└

□

d)  $f : M \rightarrow X$  strongly  $\mathcal{A}$ -measurable  $\implies f(M)$  is separable.

┌ *Důkaz*

$f = \lim u_n$ ,  $u_n$  simple measurable.  $f(M) \subset \overline{\bigcup_n u_n(M)}$ .

└

□

e)  $f$  Borel  $\mathcal{A}$ -measurable  $\implies t \mapsto \|f(t)\|$  measurable.

┌ *Důkaz*

$h(x) = \|x\|$ ,  $x \in X$ , is continuous, hence  $h \circ f$  is measurable:

$$U \text{ open} : (h \circ f)^{-1}(U) = f^{-1}(\underbrace{h^{-1}}_{\text{open}}) \in \mathcal{A}.$$

└

□

## Lemma 4.2

$(f_n)$  strongly  $\mathcal{A}$ -measurable,  $f_n \rightarrow f$  point-wise  $\implies f$  is strongly  $\mathcal{A}$ -measurable.

┌ *Důkaz*

$u_{m,n}$  simple measurable,  $u_{m,n} \xrightarrow{m} f_n$ .  $C = \bigcup_{m,n} u(m,n)(M)$  is countable, so,  $C = \{x_k, k \in \mathbb{N}\}$ .

For  $k \in \mathbb{N}$  define  $g_k : M \rightarrow X$  by  $g_k(x) =$  the point from  $\{x_1, \dots, x_k\}$  nearest to  $f(x)$  (the first such point). Then  $g_k$  is simple,  $g_k \rightarrow f$  point-wise ( $t \in M, \varepsilon > 0 \implies \exists n_0 \forall n \geq n_0 : \|f_n(t) - f(t)\| < \varepsilon/2$ ). Fix one  $n \geq n_0 \implies \exists m_0 \forall m \geq m_0 : \|u_{m,n}(t) - f_n(t)\| < \varepsilon/2$ . Fix one  $m \geq m_0 \implies \|u_{m,n}(t) - f(t)\| < \varepsilon$ , and there is  $k_0$  such that  $u_{m,n}(t) = x_{k_0}$ . Then for  $k \geq k_0 : \|f(t) - g_k(t)\| \leq \|f(t) - x_{k_0}\| < \varepsilon$ .

„ $g_k$  are also simple measurable“:  $f$  is Borel  $\mathcal{A}$ -measurable  $\implies \forall x \in X : f - x$  is Borel  $\mathcal{A}$ -measurable  $\implies \forall x \in X : t \mapsto \|f(t) - x\|$  is measurable,  $g_k(t) = x_j$

$\Leftrightarrow \forall i \in [k] : \|x_j - f(x)\| \leq \|x_i - f(x)\| \wedge \forall i < j : \|x_j - f(x)\| < \|x_i - f(x)\| \Leftrightarrow \forall i \in [k] \forall g \in \mathbb{Q} :$

$\|x_j - f(x)\| \leq q \vee \|x_i - f(x)\| \geq q \wedge \forall i < j \exists q \in \mathbb{Q} : \|x_j - f(x)\| < q \wedge \|x_i - f(x)\| > q.$

└

### Věta 4.3 (Pettis)

$f : M \rightarrow X$ . Then following assertions are equivalent:

1.  $f$  is strongly  $\mathcal{A}$ -measurable;
2.  $f$  is Borel  $\mathcal{A}$ -measurable and  $f(M)$  is separable;
3.  $f$  is weakly  $\mathcal{A}$ -measurable and  $f(M)$  is separable.

┌ *Důkaz*

„1.  $\implies$  2.  $\implies$  3.“ from the previous proposition. „3.  $\implies$  1.“: Firstly WLOG  $X$  is separable (replace  $X$  by  $\overline{\text{LO } f(M)}$ ). Secondly let  $(x_n)$  be a dense sequence in  $X$ .

$$\forall n : \text{fix } \varphi_n \in X^*, \|\varphi_n\| = 1, \varphi_n(x_n) = \|x_n\|.$$

Thirdly  $\forall x \in X : \|x\| = \sup_n |\varphi_n(x)|$  („ $\geq$ “: clear as  $\|\varphi_n\| = 1$ , „ $\leq$ “: it holds for  $x = x_n$ , so on dense set, LHS is continuous, RHS is continuous (supremum of 1-Lipschitz functions), so it holds on  $X$ ). Fourthly  $\forall x \in X : t \mapsto \|f(t) - x\|$  is measurable ( $\|f(t) - x\| = \sup_n |(\varphi_n \circ f)(t) - \varphi_n(x)|$ , so supremum from  $\mathcal{A}$ -measurable functions).

Fifthly  $k, n \in \mathbb{N} : A_n^k := f^{-1}(U(x_n, 1/k)) = \{t \in M : \|f(t) - x_n\| < 1/k\} \in \mathcal{A}$  by fourthly.

$$\bigcup_n A_n^k = M, \quad B_n^k = A_n^k \setminus \bigcup_{j < n} A_j^k \in \mathcal{A}, \quad \bigcup_n B_n^k = M,$$

and  $\{B_n^k, n \in \mathbb{N}\}$  is pair-wise disjoint.

Define  $g_k(t) = x_n, t \in B_n^k$ . Then  $\|g_k(t) - f(t)\| < \frac{1}{k}$ . So  $g_k \rightrightarrows f$  on  $M$ .  $g_n$  is strongly measurable.  $g_k = \lim_{n \rightarrow \infty} \sum_{j=1}^n x_j \chi_{B_j^k}$ , so, by the previous lemma  $f$  is strongly  $\mathcal{A}$ -measurable. □

└

**Definice 4.2** (Strongly  $\mu$ -measurable)

$(\Omega, \Sigma, \mu)$  complete measure space,  $f : \Omega \rightarrow X$  is strongly  $\mu$ -measurable if  $\exists (u_n)$  simple measurable such that  $u_n \rightarrow f$  point-wise  $\mu$ -almost everywhere.

*Poznámka*

$f$  is strongly  $\mu$ -measurable  $\Leftrightarrow \exists g$  strongly  $\Sigma$ -measurable:  $f = g$  almost everywhere.

┌

*Důkaz*

„ $\Leftarrow$ “ obvious. „ $\Rightarrow$ “:  $u_n$  simple measurable,  $u_n \rightarrow f$  almost everywhere.  $\exists N, \mu(N) = 0 : u_n \rightarrow f$  on  $\Omega \setminus N$ . Modify  $u_n, f$ :  $v_n = 0$  on  $N$  and  $u_n$  on  $\Omega \setminus N$ ,  $g = 0$  on  $N$  and  $f$  on  $\Omega \setminus N$ .  $v_n$  simple measurable,  $v_n \rightarrow g$ . □

└

**Definice 4.3** (Essentially separably valued)

(If  $f : \Omega \rightarrow X$  is (strongly)  $\mu$ -measurable, then)

$$\exists Y \leq X \text{ separable } \exists N \in \Sigma : \mu(N) = 0 \wedge f(\Omega \setminus N) \subset Y.$$

**Lemma 4.4**

Let  $(f_n)$  be a sequence of strongly  $\mu$ -measurable functions  $f_n : M \rightarrow X$  almost everywhere converging to a function  $f : M \rightarrow X$ . Then  $f$  is strongly  $\mu$ -measurable as well.

┌

*Důkaz*

TODO!!! (notes) □

└

**Věta 4.5** (Pettis)

Let  $f : \Omega \rightarrow X$  be a function. Then following assertions are equivalent

1.  $f$  is strongly  $\mu$ -measurable.
2.  $f$  is Borel  $\mu$ -measurable and essentially separably valued.
3.  $f$  is weakly  $\mu$ -measurable and essentially separably valued.

┌

*Důkaz*

TODO!!! (Notes) □

└

## 4.2 Integrability of vector-valued functions



**Definice 4.4** (Integrable over set, integral over set, integrable, Bochner integrable, Bochner integral, weakly integrable)

Let  $f : \Omega \rightarrow X$  be a simple measurable function of the form  $f = \sum_{j=1}^k x_j \chi_{E_j}$ . Let  $E \in \Sigma$ . We say that  $f$  is integrable over  $E$ , if for each  $j \in [k]$  one has either  $\mu(E \cap E_j) < \infty$  or  $x_j = \mathbf{o}$ . By the integral of  $f$  over  $E$  we mean the element of  $X$  defined by the formula

$$\int_E f d\mu = \sum_{j=1}^k \mu(E \cap E_j) x_j, \quad (\infty \cdot \mathbf{o} = \mathbf{o}).$$

If  $f$  is integrable over  $\Omega$ , it is called integrable.

Let  $f : \Omega \rightarrow X$  be strongly  $\mu$ -measurable. Then function  $f$  is said to be Bochner integrable if there exists a sequence  $(f_n)$  of simple integrable functions such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

By Bochner integral of  $f$  we then mean the element of  $X$  defined by

$$(B) \int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} f_n d\mu.$$

A function  $f : \Omega \rightarrow X$  is said to be weakly integrable if  $\varphi \circ f$  is integrable (i.e.,  $\varphi \circ f \in L^1(\mu)$ ) for each  $\varphi \in X^*$ .

**Tvrzení 4.6** (Basic properties of the Bochner integral)

a) Integrable functions form a vector space, and the mapping assigning to a simple integrable function  $f$  its integral  $\int_{\Omega} f d\mu$  is linear.

b) Let  $f$  be a simple measurable function. Then  $f$  is integrable if and only if the function  $\omega \mapsto \|f(\omega)\|$  is integrable. In this case  $\|\int_{\Omega} f d\mu\| \leq \int_{\Omega} \|f(\omega)\| d\mu(\omega)$ .

c) The limit defining the Bochner integral does exist and does not depend on the choice of the sequence  $(f_n)$ .

d) Bochner integrable functions form a vector space and the mapping assigning to a Bochner integrable function its Bochner integral is linear.

e) If  $f : \Omega \rightarrow X$  is Bochner integrable, then the function  $\omega \mapsto \|f(\omega)\|$  is integrable and  $\|(B) \int_{\Omega} f d\mu\| \leq \int_{\Omega} \|f(\omega)\| d\mu(\omega)$ .

f) If  $f : \Omega \rightarrow X$  Bochner integrable, then  $\chi_E \cdot f$  is Bochner integrable for each  $E \in \Sigma$ . (The value  $(B) \int_{\Omega} \chi_E \cdot f d\mu =: (B) \int_E f d\mu$  is called the Bochner integral of  $f$  over  $E$ .)

┌

Důkaz

└ TODO!!!

□

**Věta 4.7** (A characterization of Bochner integrability)

Let  $f : \Omega \rightarrow X$  be a strongly  $\mu$ -measurable function. Then  $f$  is Bochner integrable if and only if  $\int_{\Omega} \|f(\omega)\| d\mu(\omega) < \infty$ .

┌ Důkaz

└ TODO!!! (notes) □

**Věta 4.8** (Lebesgue dominated convergence theorem for Bochner integral)

Let  $(f_n)$  be a sequence of Bochner integrable functions  $f_n : \Omega \rightarrow X$  almost everywhere converging to a function  $f : \Omega \rightarrow X$ . Let  $g : \Omega \rightarrow \mathbb{R}$  be an integrable function such that for each  $n \in \mathbb{N}$  one has  $\|f_n(\omega)\| \leq g(\omega)$  for almost all  $\omega \in \Omega$ . Then  $f$  is Bochner integrable and  $(B) \int_{\Omega} f d\mu = \lim_{n \rightarrow \infty} (B) \int_{\Omega} f_n d\mu$ .

┌ Důkaz

└ TODO!!! (notes) □

**Tvrzení 4.9** (Absolute continuity of Bochner integral)

Let  $f : \Omega \rightarrow X$  be Bochner integrable. Then

$$\forall \varepsilon > 0 \exists \delta > 0 \forall E \in \Sigma : \mu(E) < \delta \implies \left\| \int_E f d\mu \right\| < \varepsilon.$$

┌ Důkaz

└ TODO!!! □

*Poznámka*

$(\Omega, \Sigma, \mu)$  complete measure space,  $X$  Banach space,  $f : \Omega \rightarrow X$  is weakly integrable  $\equiv \forall \varphi \in X^* : \varphi \circ f$  is integrable.

**Tvrzení 4.10** (Weak integral)

$f$  weakly integrable  $\implies F(\varphi) = \int_{\Omega} \varphi \circ f d\mu$ ,  $\varphi \in X^*$  belongs to  $X^{**}$ .

Then

- $(D) \int_{\Omega} f d\mu := F$ ;
- $E \in \Sigma$ , then  $(D) \int_E f d\mu := (D) \int_{\Omega} \chi_E \cdot f d\mu$ ;
- $f$  is Pettis-integrable if  $\forall E \in \Sigma : (D) \int_E f d\mu \in \mathfrak{K}(X)$ .

### Tvrzení 4.11

Let  $f : \Omega \rightarrow X$  be Bochner integrable. Let  $L : X \rightarrow Y$  be bounded linear operator. Then  $L \circ f$  is Bochner integrable and  $(B) \int_{\Omega} L \circ f d\mu = L((B) \int_{\Omega} f d\mu)$ .

*Důkaz*

„1.  $L \circ f$  is strongly  $\mu$ -measurable.“:  $f$  strongly  $\mu$ -measurable  $\implies \exists s_n$  simple measurable  $s_n \rightarrow f$  almost everywhere. The  $L \circ s_n$  are simple measurable and  $L \circ s_n \rightarrow L \circ f$  almost everywhere.

2.  $f$  Bochner integrable  $\implies \exists (s_n)$  simple integrable such that  $\int_{\Omega} \|s_n - f\| d\mu \rightarrow 0$ . The  $L \circ s_n$  are simple integrable

$$\begin{aligned} \int \| (L \circ s_n)(\omega) - (L \circ f)(\omega) \| d\mu(\omega) &= \int \| L(s_n(\omega) - f(\omega)) \| d\mu(\omega) \leq \\ &\leq \int \| L \| \cdot \| s_n(\omega) - f(\omega) \| d\mu(\omega) \rightarrow 0 \implies \end{aligned}$$

$\implies L \circ f$  is Bochner integrable.

$$3.(B) \int_{\Omega} L \circ f d\mu = \lim_{n \rightarrow \infty} \overbrace{\int_{\Omega} L \circ s_n d\mu}^{\sum L(x_j)\mu(E_j)} = \lim_{n \rightarrow \infty} L \left( \overbrace{\int_{\Omega} s_n d\mu}^{\sum x_j \mu(E_j)} \right) = L \left( (B) \int_{\Omega} f d\mu \right),$$

$\xrightarrow{(B) \int_{\Omega} f d\mu}$

where  $s_n = \sum_{j=1}^k x_j \chi_{E_j}$ ,  $L \circ s_n = \sum_{j=1}^n L(x_j) \chi_{E_j}$ . □

*Důsledek*

$f$  Bochner integrable  $\implies f$  is Pettis integrable.

*Příklad*

$\Omega = \mathbb{N}$ ,  $\Sigma = \mathcal{P}(\mathbb{N})$ ,  $\mu$  the counting measure.  $f : \Omega \rightarrow X$

a)  $f$  is Bochner integrable  $\iff \sum f(n)$  is absolutely convergent.

*Důkaz*

$\sum f(n)$  absolutely convergent  $\iff \int_{\Omega} \|f(n)\| d\mu(n) = \sum \|f(n)\| < \infty \iff f$  Bochner integrable. □

b) If  $\sum f(n)$  is unconditionally convergent then  $f$  is Pettis integrable

*Důkaz*

$\sum_{n \in \mathbb{N}} f(n)$  is unconditionally convergent  $\implies \forall A \subset \mathbb{N} : \sum_{n \in A} f(n)$  unconditionally convergent  $\implies$

$$\implies \forall A \subset \mathbb{N} \forall \varphi \in X^* : \sum_{n \in A} \varphi(f(n)) \text{ unconditionally convergent} \implies \sum_{n \in A} |\varphi(f(n))| < \infty.$$

$\implies f$  is weakly integrable. Moreover  $\sum_{n \in A} \varphi(f(n)) = \varphi(\sum_{n \in A} f(n)) \implies f$  is Pettis integrable.  $\square$

*Příklad*

$f : \mathbb{N} \rightarrow l_2$ ,  $f(n) = \frac{e_n}{n}$ . Then  $f$  is Pettis integrable, not Bochner integrable ( $f$  strongly measurable).

$g : \mathbb{N} \rightarrow C_0$ ,  $g(n) = e_n$ . Then  $g$  is weakly integrable not Pettis integrable. The weak integral is  $(1)_{n=1}^\infty \in l^\infty$ .

## 4.3 Lebesgue–Bochner spaces

### Definice 4.5

$f : \Omega \rightarrow X$  strongly  $\mu$ -measurable.

$$p \in [1, \infty): f \in L^p(\mu; X) \text{ if } \int_\Omega \|f(\omega)\|^p d\mu(\omega) < \infty, \|f\|_p = \left( \int_\Omega \|f(\omega)\|^p d\mu(\omega) \right)^{1/p}.$$

$$p = \infty: f \in L^\infty(\mu; X) \text{ if } \omega \mapsto \|f(\omega)\|^p \text{ is essentially bounded, } \|f\|_\infty = \text{esssup}_{\omega \in \Omega} \|f(\omega)\|^p.$$

*Poznámka*

1. Simple integrable functions belong to  $L^p(\mu; X)$ ,  $p \in [1, \infty)$ .

$$\left\| \sum_{j=1}^n x_j \chi_{E_j} \right\|_p = \left( \sum_{j=1}^n \|x_j\|^p \mu(E_j) \right)^{1/p}.$$

2. Simple measurable functions belong to  $L^\infty(\mu; X)$ .

$$\left\| \sum_{j=1}^n x_j \chi_{E_j} \right\|_\infty = \max \{ \|x_j\|, \mu(E_j) > 0 \}.$$

3.  $p \in [1, \infty]$ ,  $h \in C^p(\mu)$ ,  $x \in X$ . Then  $f(\omega) = h(\omega) \cdot x$ ,  $\omega \in \Omega$  ( $f = h \cdot x$ ),  $f \in L^p(\mu; X)$ ,  $\|f\|_p = \|h\|_p \cdot \|x\|$ .

## Věta 4.12

a)  $(L^p(\mu; X), \|\cdot\|_p)$  is a Banach space.

┌

*Důkaz*

It is a NLS by the Minkowski inequality. Completeness:  $p = \infty$  easy,  $p \in [1, \infty)$ : Assume  $(f_n) \subset L^p(\mu; X)$ ,  $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$ . Define  $g_n(\omega) = \|f_n(\omega)\|$ ,  $\omega \in \Omega$ . Then  $g_n \in L^p(\mu)$ ,  $\|g_n\|_p = \|f_n\|_p$ . So,  $\sum_{n=1}^{\infty} \|g_n\|_p < \infty$ ,  $L^p(\mu)$  complete  $\implies g := \sum_{n=1}^{\infty} g_n \in L^p(\mu) \implies \exists$  a subsequence of the sequence of partial sums which converges almost everywhere. But  $g_n \geq 0$ , so  $g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$  almost everywhere. So, for almost all  $\omega \in \Omega$ . We have  $\sum g_n(\omega) < \infty$ , hence  $\sum \|f_n(\omega)\| < \infty$ , so  $f(n) = \sum_{n=1}^{\infty} f_n(\omega)$  is defined almost everywhere.

Then  $f$  is strongly  $\mu$ -measurable.  $\|f(\omega)\| \leq \sum_{n=1}^{\infty} \|f_n(\omega)\| = g(\omega)$ . Since  $g \in L^p(\mu)$ , we deduce  $f \in L^p(\mu; X)$ .

$$\begin{aligned} \|f(\omega) - \sum_{k=1}^n f_k(\omega)\| &= \left\| \sum_{k>n} f_k(\omega) \right\| \leq \sum_{k>n} \|f_k(\omega)\| \text{ almost everywhere } \implies \\ &\implies \|f - \sum_{k=1}^n f_k\|_p \leq \|\omega \mapsto \sum_{k>n} \|f_k(\omega)\|\|_p \leq \\ &\leq \sum_{k>n} \|\omega \mapsto \|f_k(\omega)\|\|_p = \sum_{k>n} \|f_k\|_p \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

So  $f = \sum_{k=1}^{\infty} f_k$  is  $L^p(\mu; X)$ . □

└

b)  $L^1(\mu; X) = \text{Bochner-integrable functions. (see the definition and a theorem above.)}$

c) If  $X$  is a Hilbert space. Then  $L^2(\mu; X)$  is also a Hilbert space.

$$\langle f, g \rangle := \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu(\omega).$$

┌

*Důkaz*

$\omega \mapsto \langle f(\omega), g(\omega) \rangle$  is (strongly) measurable.  $\omega \mapsto \langle f(\omega), g(\omega) \rangle$  is integrable

$$\int |\langle f(\omega), g(\omega) \rangle| \leq \int \|f(\omega)\| \cdot \|g(\omega)\| \leq \left( \int \|f(\omega)\|^2 \right)^{\frac{1}{2}} \cdot \left( \int \|g(\omega)\|^2 \right)^{\frac{1}{2}} = \|f\|_2 \cdot \|g\|_2 < \infty.$$

└

□

## Věta 4.13

$p \in [1, \infty)$ .

a) Simple integrable functions are dense in  $L^p(\mu; X)$ .

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*Důkaz*

$f \in L^p(\mu; X) \implies f$  is strongly measurable. Find  $(s_n)$  simple measurable,  $s_n \rightarrow f$  almost everywhere. Define  $f_n : \Omega \rightarrow X$ :

$$f_n(\omega) := \begin{cases} s_n(\omega) & \text{if } \|f(\omega) - s_n(\omega)\| < 2 \cdot \|f(\omega)\|, \\ 0 & \text{otherwise.} \end{cases}$$

Then  $f_n$  are simple measurable,  $f_n \rightarrow f$  almost everywhere (a proof above).  $\|f_n(\omega) - f(\omega)\| \leq 2 \cdot \|f(\omega)\|$ . So,

$$f_n - f \in L^p(\mu; X) \implies f_n \in L^p(\mu; X) \implies f_n \text{ simple integrable.}$$

$\|f_n(\omega) - f(\omega)\|^p \rightarrow 0$  almost everywhere,  $2 \cdot \|f(\omega)\|^p$  is a majorant  $\implies$

$$\int \|f_n(\omega) - f(\omega)\|^p d\mu(\omega) \rightarrow 0 \quad f_n \rightarrow f \text{ in } L^p(p; X).$$

└

□

b)  $L^p(\mu)$ ,  $X$  separable  $\implies L^p(m; X)$  is separable.

┌

*Důkaz*

$\{z_n, n \in \mathbb{N}\}$  dense in  $X$ ,  $\{k_n, n \in \mathbb{N}\}$  dense in  $L^p(\mu)$ .

$$A = \left\{ \sum_{j=1}^k z_{n_k} \cdot k_{m_j}, n_1, \dots, n_k \in \mathbb{N}, m_1, \dots, m_k \in \mathbb{N}, k \in p \right\}$$

is a countable dense set of  $L^p(\mu; X)$

1.  $A$  is countable subset of  $L^p(\mu; X)$ .

2.  $f \in L^p(\mu; X)$ ,  $\varepsilon > 0$  by a):  $\exists g_1$  simple integrable  $\|f - g_1\|_p < \frac{\varepsilon}{3}$ . Then  $g_1 = \sum_{j=1}^k x_j \chi_{E_j}$  ( $E_j \in \Sigma$ , disjoint,  $x_j \in X$ ,  $\mu(E_j) < \infty$ ).

3. Find  $n_1, \dots, n_k \in \mathbb{N}$  such that  $\|x_j - z_{n_j}\|$  is so small, such that

$$\sum_{j=1}^k \|x_j - z_{n_j}\|^p \mu(E_j) < \left(\frac{\varepsilon}{3}\right)^p.$$

Set  $g_2 := \sum_{j=1}^k z_{n_j} \chi_{E_j}$ . Then  $\|g_2 - g_1\|_p < \frac{\varepsilon}{3}$ .

4. Find  $m_1, \dots, m_k \in \mathbb{N}$  such that  $\|\chi_{E_j} - k \cdot m_j\|_p$  is so small that

$$\sum_{j=1}^k \|z_{n_j}\| \cdot \|\chi_{E_j} - k_{m_j}\|_p \leq \frac{\varepsilon}{3}.$$

Then  $g_3 := \sum_{j=1}^k z_{n_j} \cdot k_{m_j} \in A$ ,  $\|g_3 - g_2\|_p < \frac{\varepsilon}{3}$ .

To summarize:  $\|g_3 - f\| < \varepsilon$ .

└

□

*Příklad*

1.  $G \subset \mathbb{R}^n$  Lebesgue measurable.  $\mu := \lambda^n|_G$  for some  $\mu(G) > 0$ .  $L^p(G; X) := L^p(\mu; X)$ .  $p \in [1, \infty)$ ,  $X$  separable  $\implies L^p(G; X)$  is separable.

2.  $\mu =$  the counting measure on  $\mathbb{N} \implies L^p(\mu, X) = l^p(X)$ .  $X$  separable,  $p \in [1, \infty) \implies l^p(X)$  separable.

*Poznámka* (Representation of duals)

$p, p^*$  dual exponents,  $p \in [1, \infty)$  ( $p^* \in (1, \infty]$ ).

$$(l^p(X))^* \approx l^{p^*}(X^*).$$

$X$  reflexive,  $\mu$   $\sigma$ -finite:

$$(L^p(\mu; X))^* = L^{p^*}(\mu; X^*).$$

$X$  reflexive,  $\mu$   $\sigma$ -finite,  $p \in (1, \infty)$ :  $L^p(\mu; X)$  is also reflexive

## 5 Compact convex sets

TODO!!!