Příklad

In the discussion of compatibility conditions we have used several identities. It remains to prove them. I recall that we have decomposed the displacement gradient to the symmetric and the skew-symmetric part as

$$\nabla \mathbf{U} = \varepsilon + \omega,$$

and we have also solved the equation

$$(\operatorname{rot} \varepsilon)^T = \nabla \mathbf{a}$$

for the vector field $\mathbf{a}$. Furthermore, using the vector field $\mathbf{a}$ and the concept of the axial vector we have defined the skew-symmetric matrix $\mathbb{A}_{\mathbf{a}}$ such that $\mathbb{A}_{\mathbf{a}} \mathbf{w} = \mathbf{a} \times \mathbf{w}$ holds for any fixed vector $\mathbf{w}$. Show that

$$\operatorname{rot} \varepsilon = \frac{1}{2} (\nabla (\operatorname{rot} \mathbf{U}))^T,$$

Důkaz

ε je symetrická část $\nabla \mathbf{U}$, tedy $\varepsilon = \frac{1}{2} \nabla \mathbf{U} + \frac{1}{2} (\nabla \mathbf{U})^T$. Tudíž

$$(\operatorname{rot} \varepsilon)_{ij} \overset{\text{def}}{=} \varepsilon_{jkl} \frac{\partial \varepsilon_{il}}{\partial x_k} = \varepsilon_{jkl} \frac{\partial \left(\frac{1}{2} \frac{\partial u_i}{\partial x_l} + \frac{1}{2} \frac{\partial u_l}{\partial x_i} \right)}{\partial x_k} =$$

$$\frac{1}{2} \varepsilon_{jkl} \frac{\partial^2 u_i}{\partial x_k \partial x_l} + \frac{1}{2} \varepsilon_{jkl} \frac{\partial^2 u_l}{\partial x_k \partial x_i} = 0 + \frac{1}{2} \frac{\partial}{\partial x_i} \left(\varepsilon_{jkl} \frac{\partial u_l}{\partial x_k} \right) = \frac{1}{2} \left(\nabla \left(\operatorname{rot} \mathbf{U} \right) \right)_{li} = \frac{1}{2} \left(\left(\nabla \left(\operatorname{rot} \mathbf{U} \right) \right)^T \right)_{il}.$$

□

$$\operatorname{rot} \mathbb{A}_{\mathbf{a}} = (\operatorname{div} \mathbf{a}) \mathbb{I} - (\nabla \mathbf{a})^T.$$

Důkaz

Pro nějaké fixní $\mathbf{w}$ máme ($\mathbb{A}_{\mathbf{a}}^T \mathbf{w} = \mathbf{w} \times \mathbf{a}$ díky antisymetrii $\mathbb{A}_{\mathbf{a}}$ a $\times$)

$$(\operatorname{rot} \mathbb{A}_{\mathbf{a}})^T \mathbf{w} \overset{\text{def}}{=} \operatorname{rot}(\mathbb{A}_{\mathbf{a}}^T \mathbf{w}) = \operatorname{rot}(\mathbf{w} \times \mathbf{a}) =$$

(podle vzorců, které jsme dokazovali v druhém domácím úkolu, a linearity div)

$$= \operatorname{div}(\mathbf{w} \otimes \mathbf{a} - \mathbf{a} \otimes \mathbf{w}) = \operatorname{div}(\mathbf{w} \otimes \mathbf{a}) - \operatorname{div}(\mathbf{a} \otimes \mathbf{w}) = [\nabla \mathbf{w}]\mathbf{a} + \mathbf{w} \operatorname{div} \mathbf{a} - [\nabla \mathbf{a}]\mathbf{w} + \mathbf{a} \operatorname{div} \mathbf{w} =$$

($\mathbf{w}$ je konstantní)

$$= \mathbf{o} + \mathbf{w} \operatorname{div} \mathbf{a} - [\nabla \mathbf{a}]\mathbf{w} + \mathbf{o} = ((\operatorname{div} \mathbf{a}) \cdot \mathbb{I} - (\nabla \mathbf{a}))\, \mathbf{w}.$$

Tedy $\operatorname{rot} \mathbb{A}_{\mathbf{a}} = ((\operatorname{div} \mathbf{a}) \cdot \mathbb{I} - (\nabla \mathbf{a}))^T = (\operatorname{div} \mathbf{a}) \mathbb{I} - (\nabla \mathbf{a})^T$. □