Příklad (1.)
Recall that the Euler–Almansi strain tensor is defined as

$$\boldsymbol{\tau}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} := \frac{1}{2}\left(\mathbb{I} - \mathbb{F}^{-T}(\mathbf{X},t)\mathbb{F}^{-1}(\mathbf{X},t)\right).$$

Show that the material time derivative of Euler–Almansi strain tensor is given by the formula $\frac{d\mathbb{e}}{dt} = \mathbb{D} - \mathbb{L}^T\mathbb{e} - \mathbb{e}\mathbb{L}$.

Důkaz

$$\frac{d\boldsymbol{\tau}}{dt} = \frac{1}{2}\frac{d}{dt}\left(\mathbb{I} - \mathbb{F}^{-T}\mathbb{F}^{-1}\right) = -\frac{1}{2}\frac{d}{dt}\left(\mathbb{F}^{-T}\mathbb{F}^{-1}\right) =$$

$$= -\frac{1}{2}\frac{d\mathbb{F}^{-T}}{dt}\mathbb{F}^{-1} - \frac{1}{2}\mathbb{F}^{-T}\frac{d\mathbb{F}^{-1}}{dt} = -\frac{1}{2}\frac{d\mathbb{F}^{-T}}{dt}\mathbb{F}^{T}\mathbb{F}^{-T}\mathbb{F}^{-1} - \frac{1}{2}\mathbb{F}^{-T}\mathbb{F}^{-1}\mathbb{F}\frac{d\mathbb{F}^{-1}}{dt} =$$

(Z minulého domácího úkolu už víme, že $\frac{d\mathbb{F}^{-T}}{dt}\mathbb{F}^{T} = -\mathbb{L}^T$ a $\mathbb{F}\frac{d\mathbb{F}^{-1}}{dt} = -\mathbb{L}$.)

$$= \mathbb{L}^T\frac{1}{2}\mathbb{F}^{-T}\mathbb{F}^{-1} + \frac{1}{2}\mathbb{F}^{-T}\mathbb{F}^{-1}\mathbb{L} = \mathbb{L}^T\left(-\mathbb{e} + \frac{1}{2}\mathbb{I}\right) + \left(-\mathbb{e} + \frac{1}{2}\mathbb{I}\right)\mathbb{L} = \frac{1}{2}\mathbb{L}^T + \frac{1}{2}\mathbb{L} - \mathbb{L}^T\mathbb{e} - \mathbb{e}\mathbb{L} =$$

$$= \mathbb{D} - \mathbb{L}^T\mathbb{e} - \mathbb{e}\mathbb{L}.$$ □

Příklad (2.)
Let $\mathbf{v}$ denote the Eulerian velocity field. Show that $\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\operatorname{rot}\mathbf{v})\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)$, where $\frac{d}{dt}$ in the material time derivative.

Důkaz
Víme, že „material time derivative“ $\implies \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$. První členy se shodují, tedy se podíváme na druhý a třetí člen ze zadání:

$$\left((\operatorname{rot}\mathbf{v})\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\right)_i = \left((\nabla\times\mathbf{v})\times\mathbf{v} + \sum_{j=1}^{3}\frac{1}{2}\frac{\partial v_j^2}{\partial x_i}\right)_i =$$

$$= \left(\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\varepsilon_{ijk}\varepsilon_{jlm}\frac{\partial v_m}{\partial x_l}v_k\right) + \sum_{j=1}^{3}\frac{1}{2}\cdot 2\frac{\partial v_j}{\partial x_i}v_j = \left(\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3} -\varepsilon_{jik}\varepsilon_{jlm}\frac{\partial v_m}{\partial x_l}v_k\right) + \sum_{j=1}^{3}\frac{\partial v_j}{\partial x_i}v_j.$$

Z prvního domácího úkolu (tedy spíše z přednášky před ním) víme, že $\sum_{j=1}^{3}\varepsilon_{jik}\varepsilon_{jlm} = \delta_{il}\delta_{km} - \delta_{im}\delta_{kl}$. Tedy levý člen je:

$$\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{j=1}^{3} -\varepsilon_{jik}\varepsilon_{jlm}\frac{\partial v_m}{\partial x_l}v_k = \sum_{k=1}^{3} -\frac{\partial v_k}{\partial x_i}v_k + \frac{\partial v_i}{\partial x_k}v_k.$$

Po sečtení s druhým členem nám zbude $\sum_{k=1}^{3}\frac{\partial v_i}{\partial x_k}v_k$, což je přesně $(\mathbf{v}\cdot\nabla)v_i$, tedy i-tá složka druhého členu v „material time derivative“. □

Příklad (3.)
Prove Zorawski lemma. The lemma claims that

$$\frac{d}{t}\int_{s(t)} \mathbf{q}\cdot\mathbf{n}\,ds = \int_{s(t)} \left(\frac{d\mathbf{q}}{dt} + \mathbf{q}\operatorname{div}\mathbf{v} - \mathbb{L}\mathbf{q}\right)\cdot\mathbf{n}\,ds,$$

where $s(t)$ is material surface.

Důkaz

$$\frac{d}{dt}\int_{s(t)} \mathbf{q}\cdot\mathbf{n}\,ds = \frac{d}{dt}\int_{s(t_0)} \mathbf{q}\cdot(\det\mathbb{F})\mathbb{F}^{-T}\mathbf{N}\,dS = \int_{s(t_0)} \frac{d}{dt}\left(\mathbf{q}\cdot(\det\mathbb{F})\mathbb{F}^{-T}\mathbf{N}\right)dS =$$

$$= \int_{s(t_0)} \left(\frac{d}{dt}\mathbf{q}\right)\cdot(\det\mathbb{F})\mathbb{F}^{-T}\mathbf{N} + 0 + \mathbf{q}\cdot\left(\frac{d}{dt}\det\mathbb{F}\right)(\det\mathbb{F})^{-1}(\det\mathbb{F})\mathbb{F}^{-T}\mathbf{N} +$$

$$+\mathbf{q}\cdot(\det\mathbb{F})\left(\frac{d}{dt}\mathbb{F}^{-T}\right)\mathbb{F}^{T}\mathbb{F}^{-T}\mathbf{N}\,dS =$$

$$= \int_{s(t)} \left(\frac{d}{dt}\mathbf{q}\right)\cdot\mathbf{n} + \mathbf{q}\cdot\left(\frac{d}{dt}\det\mathbb{F}\right)(\det\mathbb{F})^{-1}\mathbf{n} + \mathbf{q}\cdot\left(\frac{d}{dt}\mathbb{F}^{-T}\right)\mathbb{F}^{T}\mathbf{n}\,ds =$$

(První člen: Necháme. Druhý člen: $\frac{d}{dt}(\det\mathbb{A}) = (\det\mathbb{A})\operatorname{tr}\left(\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\right)$. Třetí člen: Z minulého DU máme $\left(\frac{d}{dt}\mathbb{F}^{-T}\right)\mathbb{F}^{T} = -\mathbb{L}^{T}$ a použijeme definici transpozice.)

$$= \int_{s(t)} \left(\frac{d}{dt}\mathbf{q}\right)\cdot\mathbf{n} + \mathbf{q}\cdot\operatorname{tr}\left(\mathbb{F}^{-1}\frac{d\mathbb{F}}{dt}\right)\mathbf{n} - \mathbb{L}\mathbf{q}\cdot\mathbf{n}\,ds.$$

Tedy zbývá prostřední člen. Tj. $\operatorname{tr}\left(\mathbb{F}^{-1}\frac{d\mathbb{F}}{dt}\right) \overset{?}{=} \operatorname{div}\mathbf{v}$. Z minulého úkolu už víme, že $\operatorname{tr}(\dots) = \operatorname{tr}(\mathbb{L})$. Ale my víme, že $\mathbb{L} = \nabla\mathbf{v}$, tedy $\operatorname{tr}(\mathbb{L}) = \nabla\cdot\mathbf{v} = \operatorname{div}\mathbf{v}$. □