Poznámka (Literature)
Kechris.

1 Polish topological space

Definice 1.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka
Complete compatible metric is not unique: $\tilde{\varrho} = \min\{1, \varrho\}$.

Například
$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $2 := \{0, 1\}$, $\omega := \{0, 1, 2, \ldots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, 2^ω, ω^ω (both with product topology).

Věta 1.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X.

Důkaz
Without proof. (We should know it already.) □

Věta 1.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X, such that $\operatorname{diam}(F_n) \to 0$. *Then* $|\bigcap F_n| = 1$.

Důkaz
Without proof. (We should know it already.) □

Věta 1.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n\in\omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow$ $H \in \mathcal{G}_\delta(X)$

Důkaz ((i))
Let d_n be CCM (complete compatible metric) on X_n, $n \in \omega$. Then

$$d(x,y) := \sum_{n=0}^{\infty} \min\left\{2^{-n}, d_n(x_n, y_n)\right\}$$

is CCM on $X = \prod_{n\in\omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „d is metric“ straightforward, „d is complete“ also easy, compatibility too). $\square$

Důkaz ((ii))
$H = \varnothing$, $H = X$ trivial. Assume $H \neq \varnothing, X$.

„ $\Longrightarrow$ “: Fix CCM ϱ on H.

$$V_n := \bigcup \left\{V \subset X | V \text{ open in } X \wedge V \cap H \neq \varnothing \wedge \operatorname{diam}_\varrho(V \cap H) < 2^{-n}\right\}, \qquad n \in \omega.$$

We want to show $H \overset{?}{=} \bigcap_{n\in\omega}(V_n \cap \overline{H}) \in \mathcal{G}_\delta$: „$\subseteq$“: $x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n$. „$\supseteq$“: $x \in V_n \cap \overline{H}$ for every $n \in \omega \implies \exists$ open sets G_n: $x \in G_n$, $G \cap H \neq \varnothing$, $\operatorname{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \overset{?}{\neq} \varnothing \impliedby x \in G_n \cap G_{n+1} \cap \overline{H} \neq \varnothing$).

$\{y\} := \bigcap_{n\in\omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O$, $G_n \cap H \subset B(y, 2^{-n})$, $n \in \omega$. $\implies \exists n \in \omega G_n \cap H \subset O$, $x \in G_n \cap (X \backslash \overline{O}) \cap \overline{H} \implies G_n \cap (X \backslash \overline{O}) \cap H \neq \varnothing$.

„ $\Longleftarrow$ “: fix CCM d on X, $H = \bigcap_{n\in\omega} U_n$, $\varnothing \neq U_n \neq X$. $F_n := X \backslash U_n$, $\tilde{d}(x,y) = d(x,y) + \sum_{n=0}^{\infty} \min\left\{2^{-n}, \left|\frac{1}{\operatorname{dist}(x,F_n)} - \frac{1}{\operatorname{dist}(y,F_n)}\right|\right\}$, $x, y \in H$. Next we verified that $\tilde{d}$ is metric, that $\tilde{d}$ is equivalent with d on H (by convergence), and that $(H, \tilde{d})$ is complete metric space and separable. TODO? $\square$

Definice 1.2 (Notation)

$A \neq 0$:

- $A^{<\omega}$:= finite sequence of elements of $A = \bigcup_{n\in\omega} A^n$;
- $s \in A^k$, $t \in A^{<\omega} \cup A^\omega$: $s^\wedge t := (s_0, s_1, \ldots, s_{k-1}, t_0, t_1, \ldots)$, where $s = (s_0, \ldots, s_{k-1})$, $t = (t_0, t_1, \ldots)$;
- $s \in A^{<\omega} \cup A^\omega$: $|s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega$, $k \in \omega$, $|s| \geqslant k$, then we denote restriction of s on first k elements as s/k;
- $s \prec t$ iff $|t| \geqslant |s|$ and $s = t/|s|$ ($s \in A^{<\omega}$, $t \in A^{<\omega} \cup A^\omega$).

1.1 Baire space ω^ω

Definice 1.3 (Baire interval)

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega | s \prec \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \setminus \bigcup \{\mathcal{N}(t) || t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N} | s \in \omega^{<\omega}\}$ is base of topology of ω^ω.

Věta 1.4 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz
Bez důkazu. □

Důsledek
ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Věta 1.5

Let $X \neq \emptyset$, PTS. Then X is continuous image of ω^ω.

Poznámka
$X \neq \emptyset$ PTS. Then there $\exists F \subset \omega^\omega$, F closed, and continuous injection $\varphi : F \to X$.

Důkaz
Find CCM on X such that $\operatorname{diam} X \leqslant 1$. We inductively construct closed $\emptyset \neq A_s \subset X$ for every $s \in \omega^{<\omega}$ such that 1. $A_\emptyset = X$; 2. $\operatorname{diam}(A_s) \leqslant 2^{-|s|}$; 3. $A_s = \bigcup_{i \in \omega} A_{s^\wedge i}$.

Empty set is trivial. Assume we already have A_s. Find $\{x_i | i \in \omega\} \subset A_s$ dense in A_s. $A_{s^\wedge i} := A_s \cap \overline{B(x_i, 2^{-|s|-2})} \neq \emptyset$ closed.

Fix $\forall \nu \in \omega^\omega : f(\nu) := x$, where $\{x\} = \bigcap_{k \in \omega} A_{\nu/k} \neq \emptyset$ (intersection of closed nonempty non-increasing sequence of sets). „f is surjection“: $x \in A_s \overset{3.}{\Longrightarrow} \exists n \in \omega : x \in A_{s^\wedge n} \overset{1.}{\Longrightarrow} \forall x \in X\ \exists \alpha \in \omega^\omega\ \forall k \in \omega : x \in A_{\alpha/k} \implies x = f(\alpha)$.

„f continuous“: $f(\mathcal{N}_{\nu/k}) \subset A_{\nu/k}$ for every $\nu \in \omega^\omega$, $k \in \omega$, $\operatorname{diam} A_{\nu/k} \leqslant 2^{-k}$. □

1.2 Cantor set 2^ω

Tvrzení 1.6 (Brouwer)

2^ω is up to homeomorphism unique nonempty nuldimensional compact metrizable space without isolated points (space without isolated points is called perfect space).

Tvrzení 1.7

Let $X \neq \emptyset$ metrizable, compact. Then X is continuous image of 2^ω.

Důkaz
Without proof, but it is similar to the previous one. □

1.3 Hilbert cube $[0,1]^\omega$

Tvrzení 1.8

Let X be PTS. Then X is homeomorphic to G_δ subset of $[0,1]^\omega$.

Důkaz
X PTS, case $\emptyset$ is trivial, so assume $X \neq \emptyset$, ϱ is CCM on X, $\varrho \leqslant 1$. Let $\{x_n, n \in \omega\}$ be dense in X. Define $f : X \to [0,1]^\omega : f(x) = (\varrho(x, x_n))_{n\in\omega}$. $\varrho \leqslant 1 \implies f(x) \in [0,1]^\omega$.

„Continuity of f“: $f^{-1}(U) = \bigcap_{i=1}^n B(x_i, b_i)\backslash\overline{B(x_i, a_i)}$ open.

„Injective“: $x \neq y \implies \exists n \in \omega : \varrho(x, x_n) < \varrho(y, x_n) \implies f(x) \neq f(y)$.

„Continuity of f^{-1}“ $f(y^n) \to f(y) \overset{?}{\implies} y^n \to y$.

$$f(y^n) \to f(y) \overset{?}{\Leftrightarrow} \forall k \in \omega : \varrho(y^n, x_k) \to \varrho(y, x_k).$$

Let $\varepsilon > 0$ be arbitrary:

$$\exists k \in \omega : \varrho(y, x_k) < \frac{\varepsilon}{3}.\ \exists n_0\ \forall n \geqslant n_0 : \varrho(y^n, x_k) < \frac{2\varepsilon}{3}.$$

Then

$$\forall n \geqslant n_0 : \varrho(y^n, y) \leqslant \varrho(y^n, x_k) + \varrho(x_k, y) < \varepsilon.$$

So $f(X)$ is homeomorphism to $X \implies f(X)$ is PTS $\implies f(X) \in \mathcal{G}_\delta([0,1]^\omega)$. □

Důsledek
Let X be compact metrizable space. Then X is homeomorphic to some closed subset of $[0,1]^\omega$.

Důkaz
Compact metrizable space is Polish. And compact subset must be closed. □

1.4 $\mathcal{K}(X)$: Hyperspace of compact subsets of X

Definice 1.4 (Vietoris topology)

Let X be PTS, denote $\mathcal{K}(X) := \{K \subset X | K \text{ is compact}\}$. Vietoris topology on $\mathcal{K}(X)$ is generated by $\{K \in \mathcal{K}(X) | K \subset V\}$ for V open and

$$\{K \in \mathcal{K}(X) | K \cap V \neq \emptyset\} = \mathcal{K}(X) \setminus \{K \in \mathcal{K}(X) | K \subset X \setminus V\} \text{ for } V \text{ open.}$$

Tvrzení 1.9 (Hausdorff metric)

Let X be PTS, ϱ CCM on X, $\varrho \leqslant 1$. Then mapping $h : \mathcal{K}(X) \times \mathcal{K}(X) \mapsto [0, +\infty)$ defined as:

$$h(K, L) = \begin{cases} 0, & K = L = \emptyset, \\ \max\left\{\sup_{x \in K} \varrho(x, L), \sup_{y \in L} \varrho(y, K)\right\}, & K, L \neq \emptyset, \\ 1, & \textit{other cases,} \end{cases}$$

is CCM on $\mathcal{K}(X)$ with Vietoris topology. h is known as Hausdorff metric.

$\mathcal{K}(X)$ is separable if X is PTS. X is compact metrizable $\implies$ $\mathcal{K}(X)$ is compact (totally bounded).

Důkaz

($\emptyset \neq K, L, P \in \mathcal{K}(X)$.) h is metric, definition is correct, $h \geqslant 0$ trivial, $h(K, L) = h(L, K)$ trivial, $h(K, L) = 0 \implies K = L$ ($x \notin L \implies \varrho(x, L) > 0 \implies K \subset L \wedge L \subset K$).

„$\triangle$“ aka „$h(K, L) \leqslant h(K, P) + h(P, L)$“: Let $x \in K, y \in L, p \in P$. Then

$$\varrho(x, L) \leqslant \varrho(x, y) \leqslant \varrho(x, p) + \varrho(p, y) \qquad \inf y \in L$$

$$\varrho(x, L) \leqslant \varrho(x, p) + \varrho(p, L) \qquad \sup p \in P$$
$$\varrho(x, L) \leqslant \varrho(x, p) + h(P, L) \qquad \inf p \in P$$
$$\varrho(x, L) \leqslant \varrho(x, P) + h(P, L) \qquad \inf p \in P$$
$$\sup_{x \in K} \varrho(x, L) \leqslant h(K, P) + h(P, L).$$

Similarly $\sup_{y \in L} \varrho(y, K) \leqslant h(K, P) + h(P, L)$. □

Důkaz (Kompatibilita s Vietorisovou topologií)
Označme Vietorisovu topologii jako $\mathcal{V}$ a topologii indukovanou Hausdorffovou metrikou jako $\mathcal{H}$. Chceme dokázat $\mathcal{H} = \mathcal{V}$.

„$\subset$“: Zvolme $K \in \mathcal{K}(X)$ a $\varepsilon > 0$. Množina K je kompaktní, takže můžeme nalézt konečný systém $\mathcal{B}$ otevřených koulí s diametrem $< \varepsilon$ protínajících K, který pokrývá K. Potom platí

$$B_h(K, \varepsilon) \supset \left\{ L \in \mathcal{K}(X) \middle| L \subset \bigcup \mathcal{B} \wedge \forall B \in \mathcal{B} : L \cap B \neq \varnothing \right\} \ni K.$$

„$\supset$“: Nechť $V \subset X$ je otevřená a $K \in \mathcal{K}(X)$ splňuje $K \cap V \neq \varnothing$. Zvolme $x \in K \cap V$ a k němu nalezněme $\varepsilon > 0$, že $B(x, \varepsilon) \subset V$. Potom $B_h(K, \varepsilon) \subset \{L \in \mathcal{K}(X) | L \cap V \neq \varnothing\}$.

Nyní nechť $V \subset X$ je otevřená a $K \in \mathcal{K}(X)$, $K \neq \varnothing$, splňuje $K \subset V$. Nalezneme $\varepsilon > 0$ takové, že $\{y \in X | \varrho(y, K) < \varepsilon\} \subset U$. Potom platí $B_h(K, \varepsilon) \subset \{L \in \mathcal{K}(X) | L \subset V\}$. Pro $K = \varnothing$ stačí položit $\varepsilon = 1/2$. □

Důkaz (Separabilita $\mathcal{K}(X)$)
Nalezněme $D \subset X$ spočetnou a hustou. Položme $\mathcal{D} := \{K \in \mathcal{K}(X) | K$ konečná $\wedge\ K \subset D\}$. Množina $\mathcal{D}$ je spočetná. Jestliže $V_0, V_1, \ldots, V_n$ jsou otevřené podmnožiny X takové, že

$$\mathcal{G} := \{K \in \mathcal{K}(X) | K \subset V_0 \wedge \forall i \in [n] : K \cap V_i \neq \varnothing\}$$

je neprázdná, pak pro každé $i \in [n]$ zvolme $x_i \in V_0 \cap V_i \cap D$ a položme $L := \{x_1, \ldots, x_n\}$. Potom $L \in \mathcal{G} \cap \mathcal{D}$, takže $\mathcal{D}$ je hustá v $\mathcal{K}(X)$. □

Důkaz (Úplnost $(\mathcal{K}(X), h)$)
Nechť (K_n) je cauchyovská posloupnost v $(\mathcal{K}(X), h)$. Položme $K := \bigcap_{n\in\omega} \overline{\bigcup_{j\geqslant n} K_j}$. Odvodíme 1. $K \in \mathcal{K}(X)$, 2. $K_n \to K$.

„1.“: Stačí ukázat, že množina $\bigcup_{n\in\omega} K_n$ je totálně omezená, pak je totiž množina $\overline{\bigcup_{n\in\omega} K_n}$ kompaktní. Vezměme $\varepsilon > 0$. Pak existuje $n_0 \in \omega$ takové, že $\forall n \in \omega, n \geqslant n_0 : h(K_n, K_{n_0}) < \varepsilon/2$. Nechť S je konečná $\varepsilon/2$-síť v K_{n_0}. Potom

$$\bigcup_{n\geqslant n_0} K_n \subset \{y \in X | \varrho(y, K_{n_0}) < \varepsilon/2\}.$$

Odtud plyne, že S je ε-síť $\bigcup_{n\geqslant n_0} K_n$. Existuje tedy ε-konečná síť množin $\bigcup_{n\in\omega} K_n$, protože $\bigcup_{n<n_0} K_n$ je kompaktní.

„2.“ Nechť množina V je otevřená v X a $K \subset V$. Platí $\bigcap_{n\in\omega} \overline{\bigcup_{j\geqslant n} K_j} \cap V^c = K \cap V^c = \emptyset$, a tedy existuje $n_0 \in \omega$ takové, že $\overline{\bigcup_{j\geqslant n_0} K_j} \subset V$. Pro $n \geqslant n_0$ tedy platí $K_n \subset V$.

Nyní nechť množina $V \subset X$ je opět otevřená a $K \cap V \neq \emptyset$. Nalezneme $\varepsilon > 0$ a $x \in K \cap V$ takové, že $B(x, \varepsilon) \subset V$. K tomuto ε nalezneme $n_0 \in \omega$ takové, že $\forall n, m \geqslant n_0 : h(K_n, K_m) < \varepsilon/2$. Dále existuje $m_0 \geqslant n_0$ takové, že $K_{m_0} \cap B(x, \varepsilon/2) \neq \emptyset$. Potom pro každé $n \geqslant m_0$ platí $K_n \cap B(x, \varepsilon) \neq \emptyset$, a tedy $K_n \cap V \neq \emptyset$. □

1.5 Rozšiřování spojitých zobrazení

Věta 1.10 (Kuratowski)

Nechť X je metrický prostor, Y je úplný metrický prostor, $A \subset X$ a $f : A \to Y$ je spojité zobrazení. Potom existuje G_δ množina G taková, že $A \subset G \subset \overline{A}$, a existuje spojité rozšíření $g : G \to Y$ zobrazení f.

Důkaz
Definujeme oscilaci zobrazení f v bodě $x \in \overline{A}$ jako $\operatorname{osc}(f, x) = \inf\{\operatorname{diam} f(U) | U \text{ je okolí } x\}$. Pro $x \in A$ platí $\operatorname{osc}(f, x) = 0$. Položme $G = \left\{x \in \overline{A} | \operatorname{osc}(f, x) = 0\right\}$. Ověříme požadované vlastnosti:

„Inkluze $A \subset G \subset \overline{A}$“ zřejmě platí. „Množina G je typu G_δ“ plyne z rovnosti $G = \bigcap_{n\in\omega} \left\{x \in \overline{A} | \operatorname{osc}(f, x) < 1/n\right\}$, neboť tyto množiny jsou otevřené v $\overline{A}$.

Zobrazení g definujeme takto: $\{g(x)\} := \bigcap_{k\in\omega} \overline{f(B(x, 2^{-k}))}$, $x \in G$. Ověříme:

„Korektnost definice g“ plyne z jedné z vět výše. „Spojitost g“ plyne z rovnosti $\operatorname{osc}(g, x) = \operatorname{osc}(f, x) = 0$ pro $x \in G$. „Zobrazení g rozšiřuje f“ je snadné. □

Věta 1.11 (Lavrentěv)

Nechť X, Y jsou úplné metrické prostory, $A \subset X$, $B \subset Y$ a $f : A \to B$ je homeomorfismus A na B. Pak existují G_δ množiny G, H takové, že $A \subset G \subset X$, $B \subset H \subset Y$, a $\tilde{f} : G \to H$

homeomorfismus G na H splňující $\tilde{f}|_A = f$.

Důkaz

Podle předchozí věty existuje G_δ množina G_1 a spojité zobrazení $f_1 : G_1 \to Y$ takové, že $A \subset G_1$ a $f_1|_A = f$. Podobně nalezneme G_δ množinu H_1 a spojité zobrazení $g_1 : H_1 \to X$ takové, že $B \subset H_1$ a $g_1|_B = f^{-1}$. Označme

$$R := \{(x, y) \in G_1 \times Y | f_1(x) = y\}, \qquad S := \{(x, y) \in X \times H_1 | x = g_1(y)\}.$$

Položíme $G := \pi_X(R \cap S)$, $H := \pi_Y(R \cap S)$ a $\tilde{f} := f_1|_G$, a tedy $\tilde{f}$ je homeomorfismus G na H. Zobrazení $\psi(x) = (x, f_1(x))$ je spojité na G_1, S je uzavřená v $X \times H_1$ a $G = \psi^{-1}(S)$. Množina G je tedy G_δ v X. Podobně lze odvodit, že H je G_δ v Y. □

2 Základní vlastnosti borelovských a analytických množin

2.1 Zavedení borelovské hierarchie a její vlastnosti

Definice 2.1 (Borel hierarchy)

X is metrizable space, $1 \leqslant \alpha < \omega_1$. We define $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, and $\Delta^0_\alpha(X)$ by induction:

$$\Sigma^0_1(X) := \{U \subset X | U \text{ open}\},$$

$$\Pi^0_\alpha(X) := \left\{A \subset X | X \backslash A \in \Sigma^0_\alpha(X)\right\},$$

$$\Sigma^0_\alpha(X) := \left\{\bigcup_{n\in\omega} A_n \,\middle|\, A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \alpha, n \in \omega\right\},$$

$$\Delta^0_\alpha(X) := \Sigma^0_\alpha(X) \cap \Pi^0_\alpha(X).$$

Poznámka (By induction it can be prooven)

$$\Sigma^0_\alpha(X) \subset \Sigma^0_\beta(X), \Pi^0_\alpha(X) \subset \Pi^0_\beta(X), \qquad 1 \leqslant \alpha < \beta < \omega_1.$$

Poznámka

$$\forall \alpha, \beta : 1 \leqslant \alpha < \beta < \omega_1 : \Sigma^0_\alpha(X) \subset \Pi^0_\beta(X).$$

Poznámka

If X contains homeomorphic copy of 2^ω then all inclusions are strict.

We denote $Borel(X)$ as σ-algebra of Borel sets (σ-algebra generated by $\Sigma^0_1(X)$).

Poznámka (Also non-trivial theorem)

$$Borel(X) = \bigcup_{1 \leqslant \alpha < \omega_1} \Sigma^0_\alpha(X) = \bigcup_{1 \leqslant \alpha < \omega_1} \Pi^0_\alpha(X) = \bigcup_{1 \leqslant \alpha < \omega_1} \Delta^0_\alpha(X).$$

$$A_n \in \bigcup_{1 \leqslant \alpha < \omega_1} \Sigma^0_\alpha(X) \implies \exists 1 \leqslant \alpha_n < \omega_1 : A_n \in \Sigma^0_{\alpha_n}(X) \implies A_n \in \Sigma^0_{\sup\{\alpha_n | n \in \omega\}} \implies$$

$$\implies \bigcup_{n \in \omega} A_n \in \Sigma^0_{\sup\{\alpha_n, n \in \omega\}} \implies \bigcup A_n \in \bigcup_{1 \leqslant \alpha < \omega_1} \Sigma^0_\alpha(X).$$

Poznámka
$F_\sigma = \Sigma^0_2$, $G_\delta = \Pi^0_2$, $F_{\sigma\delta} = \Pi^0_3$, $G_{\delta\sigma} = \Sigma^0_3$.

$\Sigma^0_\alpha(X)$ is closed under countable union and $\Pi^0_\alpha(X)$ under countable intersection.

Věta 2.1

X be metrizable, $1 \leqslant \alpha < \omega_1$. Then

1. *$\Sigma^0_\alpha(X)$ is closed under finite intersection;*
2. *$\Pi^0_\alpha(X)$ is closed under finite union.*

Důkaz
„1.“ Firstly for $\alpha = 1$, it is trivial. Then let $A, B \in \Sigma^0_\alpha(X)$, $\alpha > 1$. Then $A = \bigcup_{n \in \omega} A_n$, $A_n \in \Pi^0_{\alpha_n}(X)$, $\alpha_n < \alpha$, $B = \bigcup_{m \in \omega} B_m$, $B_m \in \Pi^0_{\beta_m}(X)$, $\beta_n < \alpha$. $A \cap B = \bigcup_{(m,n) \in \omega^2} A_n \cap B_m$, $A_n \cap B_m \in \Pi^0_{\max\{\alpha_n, \beta_n\}}(X) \implies A \cap B \in \Sigma^0_\alpha(X)$. „2.“ $\impliedby$ de Morgan and 1. □

Věta 2.2

X be metrizable, $A \subset Z \subset X$, $1 \leqslant \alpha < \omega_1$. Then $A \in \Sigma^0_\alpha(Z) \Leftrightarrow$ there exists $\tilde{A} \in \Sigma^0_\alpha(X)$: $A = \tilde{A} \cap Z$. Similarly for $\Pi^0_\alpha, \Delta^0_\alpha$.

Důkaz

Firstly $\alpha = 1$ from definition of subspace. Then assume that it is all true for all $\beta < \alpha$. We want to prove it for α. „ $\implies$ “:

$$A \in \Sigma^0_\alpha(Z) \implies A = \bigcup A_n, A_n \in \Pi^0_{\beta_n}(Z), \beta_n < \alpha \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X) : \tilde{A}_n \cap Z = A_n.$$

$$\tilde{A} = \bigcup \tilde{A}_n \in \Sigma^0_\alpha(X), \tilde{A} \cap Z = Z \cap \bigcup \tilde{A}_n = \bigcup (Z \cap \tilde{A}_n) = \bigcup A_n = A.$$

„ $\impliedby$ “:

$$\tilde{A} \in \Sigma^0_\alpha(X), A = \tilde{A} \cap Z \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X), \beta_n < \alpha, \bigcup \tilde{A}_n = \tilde{A}.$$

$$\tilde{A} \cap Z \in \Pi^0_{\beta_n}(Z) \implies A = \tilde{A} \cap Z = \left(\bigcup \tilde{A}_n\right) \cap Z = \bigcup \left(\tilde{A}_n \cap Z\right) = \bigcup A_n \in \Sigma^0_\alpha(Z).$$

□

Věta 2.3

X, Y be metric spaces, $f : X \to Y$ is continuous. If $A \in \Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$, $\Delta^0_\alpha(Y)$) then $f^{-1}(A) \in \Sigma^0_\alpha(X)$ ($\Pi^0_\alpha(X)$, $\Delta^0_\alpha(Y)$).

Důkaz

$\alpha = 1$ trivial. Assume it holds true for $\Sigma^0_\beta(Y)$, $\Pi^0_\beta(Y)$, $\beta < \alpha$, and we want to show for $\Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$). Let $A \in \Sigma^0_\alpha(Y)$, $\alpha > 1 \implies A = \bigcup_{n\in\omega} A_n$, $A_n \in \Pi^0_{\beta_n}(Y)$, $\beta_n < \alpha$.

$$f^{-1}(A) = f^{-1}\left(\bigcup A_n\right) = \bigcup \underbrace{f^{-1}(A_n)}_{\Pi^0_{\beta n}(X)} \in \Sigma^0_\alpha(X),$$

$$f^{-1}(Y \backslash A) = f^{-1}(Y) \backslash f^{-1}(A) = X \backslash f^{-1}(A).$$

□

Věta 2.4 (Borel classes in PTS)

X, Y be PTS, $A \in \Sigma^0_\alpha(X)$, $\alpha \geqslant 3$ (resp. $A \in \Pi^0_\alpha(X)$, $\alpha \geqslant 2$), $B \subset Y$. If B and A are homeomorphic then $B \in \Sigma^0_\alpha(Y)$ (resp. Π^0_α).

Důkaz

$f : A \to B$ is homeomorphism A onto B. The theorem Lavrentěv there is extension $\tilde{f}$ of f, $\tilde{f}$ is homeomorphism $\tilde{A}$ onto $\tilde{B}$, $A \subset \tilde{A}$, $B \subset \tilde{B}$, $\tilde{A} \in \Pi^0_2(X)$, $\tilde{B} \in \Pi^0_2(Y)$. Then $B \in \Sigma^0_\alpha(\tilde{B})$ (because $B = (f^{-1})^{-1}(A)$). From the theorem above, $\exists \hat{B} \in \Sigma^0_\alpha(Y) : B = \hat{B} \cap \tilde{B} \in \Sigma^0_\alpha(Y) \impliedby \alpha \geqslant 3$. □

2.2 Analytic sets

Definice 2.2 (Analytic set, coanalytic set)

X PTS, $A \subset X$. We say that A is analytic set in X if there exists PTS Y and continuous mapping $\varphi : Y \to X$ such that $\varphi(Y) = A$. We denote collection of analytic subsets of X as $\Sigma_1^1(X)$. We say that A is coanalytic in X if $X \backslash A \in \Sigma_1^1(X)$ and we denote this collection as $\Pi_1^1(X)$. $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Například $Q = \{\alpha \in 2^\omega | \exists n \in \omega \ \forall j \geqslant n : \alpha_j = 0\} = 2^{<\omega} \in \Sigma_2^0(2^\omega) \backslash \Pi_2^0(2^\omega)$, TODO?

Poznámka
X PTS, $F : X \to \mathcal{K}(X)$ by $F(x) = \{x\}$. Then F is continuous, $F^{-1}(\mathcal{K}(A)) = A \implies$ if $\mathcal{K}(A) \in \Sigma_\alpha^0(\mathcal{K}(X))$ (Π_α^0, Δ_α^0) then $A \in \Sigma_\alpha^0(X)$ (Π_α^0, Δ_α^0). A open $\implies$ $\mathcal{K}(A)$ is open, A is closed $\implies$ $\mathcal{K}(A)$ is closed. $\mathcal{K}(\bigcap A_n) = \bigcap \mathcal{K}(A_n)$. Thus for $A \in \Pi_2^0(X) : \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X))$. $A \in \Sigma_1^0(X)$ ($\Pi_1^0(X)$, $\Pi_2^0(X)$) $\Leftrightarrow \mathcal{K}(A) \in \Sigma_1^0(\mathcal{K}(X))$ ($\Pi_1^0(\mathcal{K}(X))$, $\Pi_2^0(\mathcal{K}(X))$).

Věta 2.5

X PTS, $|X| > \omega$. Assume $I \subset \mathcal{K}(X)$, I is σ-ideal ($K \in I, L \subset K \implies L \in I$; $K_n \in I, \bigcup K_n \in \mathcal{K}(X) \implies \bigcup K_n \in I$). If $I \in \Pi_2^0(\mathcal{K}(X))$, then $I \in \Sigma_1^1(\mathcal{K}(X))$.

Důsledek $$A \notin \Pi_2^0(X) \implies \mathcal{K}(A) \notin \Sigma_1^1(\mathcal{K}(X)).$$

Poznámka $A \in \Pi_1^1(X)$, $\quad \mathcal{K}(A) = \mathcal{K}(X) \backslash \{K \in \mathcal{K}(X) | \exists x \in (X \backslash A) \cap K\}$

$\{(K, x) \in \mathcal{K}(X) \times X | x \in K\}$ is closed.

Definice 2.3

$$\Sigma_1^1(X) := \{A \subset X | \exists Y \text{ PTS}, f : Y \to X \text{ continuous} : f(Y) = A\}.$$

Poznámka
- $\varnothing \in \Sigma_1^1$;
- X PTS $\implies \Pi_2^0(X) \subset \Sigma_1^1(X)$, $f = \mathrm{id}$;
- X, Z PTS, $\psi : X \to Z$ continuous, $A \in \Sigma_1^1(X) \implies \psi(A) \in \Sigma_1^1(Z)$;
- $\Sigma_{n+1}^1(X) = \{A \subset X | \exists Y \text{ PTS}, \psi : Y \to X \text{ cont}, B \in \Pi_n^1(X), A = \psi(B)\}$, $n \in \omega \backslash \{0\}$;
- $\Pi_n^1(X) = \{A \subset X | X \backslash A \in \Sigma_n^1(X)\}$, $\Delta_n^1(X) = \sum_n^1(X) \cap \Pi_n^1(X)$;
- $\bigcup_{n \in \mathbb{N}} \Sigma_n^1(X) = \bigcup_{n \in \mathbb{N}} \Pi_n^1 = \bigcup_{n \in \mathbb{N}} \Delta_n^1(x) = \mathbb{P}(X)$;
- $\#\mathbb{P}(X) \leqslant 2^\omega$, $\mathbb{P}(X)$ is closed under continuous images and inverse images;
- $\Sigma_1^1(X) = \{A \subset X | \exists \psi : \omega^\omega \to X \text{ continuous} : \psi(\omega^\omega) = A\}$; Y PTS, $f : Y \to X : f(Y) = A$, $g : \omega^\omega \to Y : g(\omega^\omega) = Y$, g, f are constant. So $\psi = f \circ g$.

Věta 2.6

X PTS, $A_n \in \Sigma^1_1(X)$, $n \in \omega$. Then $\bigcup_{n\in\omega} A_n, \bigcap_{n\in\omega} A_n \in \Sigma^1_1(X)$.

Důsledek
Similar for $\Pi^1_1(X)$.

Důkaz
„Union“: Assume $A_n \neq \emptyset$, $n \in \omega \implies \varphi_n : \omega^\omega \to X : \varphi_n(\omega^\omega) = A_n$ continuous. Define $\varphi : \omega^\omega \to X$ by $\varphi(\nu_0, \nu_1, \dots) = \varphi_{\nu_0}(\nu_1, \nu_2, \dots)$. „$\varphi$ is continuous“: $\nu^j \to \nu \implies \exists n_0 \in \omega\ \forall j \geqslant n_9 : \nu^j_0 = \nu_0$.

$$\lim_{j\to\infty} \varphi(\nu^j) = \lim_{j\to\infty} \varphi_{\nu^j_0}(\nu^j_1, \nu^j_2, \dots) = \lim_{j\to\infty} \varphi_{\nu_0}(\nu^j_1, \dots) = \varphi_{\nu_0}(\nu_1, \dots) = \varphi(\nu).$$

„$\varphi(\omega^\omega) = \bigcup_{n\in\omega} A_n$“:

$$x \in \bigcup A_n \implies \exists n \in \omega : x \in A_n \implies \exists \nu \in \omega^\omega : \varphi_n(\nu) = x \implies \varphi(n^\wedge \nu) = x.$$

$$x \in \varphi(\omega^\omega) \implies \exists \tilde{\nu} \in \omega^\omega : \varphi(\tilde{\nu}) = x \implies x = \varphi_{\tilde{\nu}_0}(\tilde{\nu}_1, \dots) \implies z \in A_{\tilde{\nu}_0} \implies x \in \bigcup A_n.$$

□

Důkaz
WLOG: $A_n \neq \emptyset$, $n \in \omega$. $Y := (\omega^\omega)^\omega$, Y PTS by the theorem above (first item). $\varphi_n : \omega^\omega \to X$, meh that $\varphi_n(\omega^\omega) = A_n$.

$$F := \{y = (y_0, y_1, \dots) \in Y | \forall n, m \in \omega : \varphi_n(y_n) = \varphi_m(y_m)\} =$$

$$= \bigcap_{n,m\in\omega} \{y \in Y | \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap))n, m \in \omega \{y \in Y | \varphi_n \circ \pi_n(y) = \varphi_m \circ \pi_m(y)\}$$

intersection of closed, so F is closed and is PTS.

„$\varphi_0 \circ \pi_0(F) = \bigcap_{n\in\omega} A_n$“:

$$x \in \varphi_0 \circ \pi_0(F) \implies \exists y \in F : x = \varphi_0(y_0) = \varphi_1(y_1) = \varphi_2(y_2) = \dots \implies x \in \bigcap_{n\in\omega} A_n.$$

$$x \in \bigcap A_n \implies \exists y_0, y_1, \dots \in \omega^\omega : \varphi_0(y_0) = x, \varphi_1(y_1) = x, \dots \implies$$
$$\implies y = (y_0, y_1, \dots) \in F, \varphi_0 \circ \pi_0(y) = x \implies x \in \varphi_0 \circ \pi_0(F).$$

□

Poznámka
$\Sigma^1_1(X)$ is not closed under complement: $\sigma(\Sigma^1_1(X)) \supset \Sigma^1_1(X) \cup \Pi^1_1(x)$.

$Borel(X) \subset \Sigma_1^1(X) \cap \Pi_1^1(X) = \Delta_1^1(X).$

Věta 2.7

X, Y PTS, $A \in \Sigma_1^1(X)$ (respective $\Pi_1^1(X)$), $B \subset Y$, A and B are homeomorphism. Then $B \in \Sigma_1^1(Y)$ (resp. $\Pi_1^1(Y)$).

Důkaz
For Σ_1^1 trivial. $A \in \Pi_1^1(X)$, $\varphi : A \to B$ homeomorphism. Then from the theorem above, $\exists \tilde{A} \in \Pi_2^0(X), \tilde{B} \in \Pi_2^0(Y)$ and $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ homeomorphism extending φ, $A \subset \tilde{A}$, $B \subset \tilde{B}$. Then $\tilde{A} \setminus A = (X \setminus A) \cap \tilde{A} \in \Sigma_1^1(X) \implies \tilde{B} \setminus B \in \Sigma_1^1(Y)$. $B = Y \setminus (\tilde{B} \setminus B \cup Y \setminus \tilde{B}) \in \Pi_1^1(Y)$. □

Věta 2.8

X PTS. Then $Borel(X) \subset \Delta_1^1(X)$.

Důkaz
Trivial. □

2.3 Luzin theorem

Věta 2.9 (Luzin)

X PTS, $A_1, A_2 \in \Sigma_1^1(X)$, $A_1 \cap A_2 = \varnothing$. Then there exists $B \in Borel(X)$, such that $A_1 \subset B \subset X \setminus A_2$.

Důsledek
X PTS. $\Delta_1^1(X) = Borel(X)$.

Důkaz
$\Delta_1^1(X) \subseteq Borel(X)$ we already have. $A \in \Delta_1^1(X) \implies A \in \Sigma_1^1(X), X \setminus A \in \Sigma_1^1 \implies$

$$\implies \exists B \in Borel(X) : A \subset B \subset X \setminus (X \setminus A) = A \implies A = B \implies A \in Borel(X).$$

□

Lemma 2.10

$C_n, D_n \subset X$, $n, m \in \omega$ and $\forall n, m \in \omega$ we can separate C_n, D_m by some Borel set. Then we can separate $\bigcup_{n \in \omega} C_n$ and $\bigcup_{m \in \omega} D_m$ by Borel set.

Důkaz
Let $B_{n,m} \in Borel(X)$ separating C_n from D_m ($C_n \subset B_{n,m} \subset X \backslash D_m$).

Put $B := \bigcup_{n\in\omega} \bigcap_{m\in\omega} B_{n,m}$. □

Důkaz (Luzin theorem)
Assume $A_1, A_2 \neq \emptyset$. Then exists $\varphi_1, \varphi_2 : \omega^\omega \to X$ $\varphi_i(\omega^\omega) = A_i$. We assume A_1 can't be separated from A_2 by any Borel set. $A_i = \varphi_i(\omega^\omega) \implies A_i = \bigcup_{n\in\omega} \varphi_i(\mathcal{N}(n)) \implies$

$$\implies \exists \nu_0, \mu_0 \in \omega : \varphi_i(\mathcal{N}(\mu_0)) \text{ can't be separated from } \varphi_2(\mathcal{N}(\nu_0)).$$

We use lemma again and obtain $\mu, \nu \in \omega^\omega$ such that $\forall k \in \omega : \varphi_1(\mathcal{N}(\mu/k))$ can't be separated from $\varphi_2(\mathcal{N}(\nu/k))$

$$\varphi_1(\mu) \in A_1, \varphi_2(\nu) \in A_2 \implies \varphi_1(\mu) \neq \varphi_2(\nu) \implies \exists G_1, G_2 \text{ open }, G_1 \cap G_2 = \emptyset$$

such that $\varphi_1(\mu) \in G_1$, $\varphi_2(\nu) \in G_2$, φ_1, φ_2 are continuous $\implies \exists k \in \omega : \varphi_1(\mathcal{N}(\mu/k)) \subset G_1$, $\varphi_2(\mathcal{N}(\nu/k)) \subset G_2$ which is continuous. □

Například
$\{f \in C([0,1]) | \forall x \in [0,1] : f'(x) \in \mathbb{R}\} \in \Pi^1_1 \backslash \Delta^1_1$.

$\{f \in C([0,2\pi)|$ Fourier series converges to f for every $x \in [0,2\pi]\} \in \Pi^1_1 \backslash \Delta^1_1$.

$\{K \in \mathcal{K}([0,1]) | |K| \leqslant \omega\}, \{K \in \mathcal{K}(\mathbb{R}) | K \subset \mathbb{Q}\} \in \Pi^1_1 \backslash \Delta^1_1$.

Například
$\{x \in X | \exists y \in Y : (x,y) \in B\} \in \Sigma^1_1(X)$.

2.4 Suslinova operace

Definice 2.4 (Suslinovo schéma, Suslinova operace)

Suslinovým schématem (na množině X) rozumíme systém podmnožin X indexovaný prvky $\omega^{<\omega}$. Suslinova operace aplikovaná na schéma $(P_s)_{s\in\omega^{<\omega}}$ je $\mathcal{A}_s P_s = \bigcup_{\nu\in\omega^\omega} \bigcap_{n\in\omega} P_{\nu|n}$.

Věta 2.11

Nechť X je polský, $A \subset X$. Pak je ekvivalentní:

1. *$A \in \Sigma^1_1(X)$;*
2. *existuje uzavřená $F \subset X \times \omega^\omega$ taková, že $\pi_X(F) = A$;*
3. *existuje Suslinovo schéma $(F_s)_{s\in\omega^{<\omega}}$ z uzavřených podmnožin X takové, že $A = \mathcal{A}_s F_s$.*

Důkaz
Předpokládejme, že A je neprázdná, jinak je tvrzení zřejmé. „1. $\implies$ 3.“: Nechť $\varphi : \omega^\omega \to X$ je spojité zobrazení na A. Položme $F_s = \overline{\varphi(\mathcal{N}(s))}$. Pokud $x \in A$, pak existuje $\mu \in \omega^\omega$ takové, že $x = \varphi(\mu)$. Pak máme $x \in \varphi(\mathcal{N}(\mu|k)) \subset F_{\mu|k}$ pro každé $k \in \omega$. Odtud $x \in \bigcap_{k\in\omega} F_{\mu|k} \subset \bigcup_{\nu\in\omega^\omega} \bigcap_{n\in\omega} F_{\nu|n}$. Pokud existuje $\mu \in \omega^\omega$ taková, že $x \in \overline{\varphi(\mathcal{N}(\mu|k))}$, pak $x = \varphi(x)$ díky spojitosti φ.

„3. $\implies$ 2.“: Položme $F_n = \bigcup_{s\in\omega^{<\omega},|s|=n}(F_s \times \mathcal{N}(s))$, $F = \bigcap_{n\in\omega} F_n$. Pak F_n jsou uzavřené, a tedy i F je uzavřená. Navíc $\pi(F) = A$. „2. $\implies$ 1.“ zřejmé. □

2.5 Obrazy a vzory při borelovských zobrazeních

Věta 2.12

Nechť X je polský. Potom existuje uzavřená množina $F \subset \omega^\omega$ a spojitá bijekce $f : F \to X$.

Důkaz
Zafixujeme úplnou kompatibilní metriku na X splňující $\operatorname{diam} X \leqslant 1$. Budeme konstruovat Suslinovo schéma $(F_s)_{s\in\omega^{<\omega}}$ takové, že pro každé $s \in \omega^{<\omega}$ platí

- $F_\varnothing = X$,
- F_s je F_σ,
- $F_s = \bigcup_{j\in\omega} F_{s^\wedge j} = \bigcup_{j\in\omega} \overline{F_{s^\wedge j}}$,
- $\operatorname{diam} F_s \leqslant 2^{-|s|}$,
- $\forall i, j \in \omega : F_{s^\wedge i} \cap F_{s^\wedge j} = \varnothing$.

□

Konstrukce schématu: Pro libovolnou F_σ množinu D a libovolné $\varepsilon > 0$ stačí nalézt spočetný disjunktní systém $\mathcal{D}$ obsahující F_σ množiny o diametru menším než ε a splňující $D = \bigcup \mathcal{D}$, $\forall E \in \mathcal{D} : \overline{E} \subset D$.

Napišme nejprve D jako jako sjednocení rostoucí posloupnosti uzavřených množin C_j, $j \in \omega$, přičemž $C_0 = \varnothing$. Potom každý rozdíl $C_{j+1} \backslash C_j$ vyjádříme jako spočetné sjednocení disjunktního systému F_σ množin $(E_i^j)_{i\in\omega}$, kde $\operatorname{diam} E_i^j < \varepsilon$. Položme $\mathcal{D} = \left\{E_i^j, i \in \omega, j \in \omega\right\}$. Požadované podmínky jsou splněny včetně poslední, platí totiž $\overline{E_i^j} \subset C_j \subset D$.

Konstrukce F a f: Položme

$$F := \left\{\nu \in \omega^\omega | \bigcap_{n\in\omega} F_{\nu|n} \neq \varnothing\right\}, \qquad \{f(\nu)\} = \bigcap_{n\in\omega} F_{\nu|n}.$$

Není těžké ověřit, že zobrazení $f : F \to X$ je dobře definováno, je spojité a je na. Ukážeme, že množina F je uzavřená. Vezměme posloupnost (ν^j) prvků množiny F s limitou ν. Pro každé $j \in \omega$ nalezneme $x_j \in \bigcap_{n\in\omega} F_{\nu^j|n}$. Posloupnost (x^j) je cauchyovská, a tedy konvergentní. Označme x^* limitu této posloupnosti. Potom $x^* \in \bigcap_{n\in\omega} \overline{F_{\nu|n}} = \bigcap_{n\in\omega} F_{\nu|n}$, a tedy $\nu \in F$.

Lemma 2.13

(X, τ) PTS, $F \in \Pi^0_1(X)$. Let τ_F be topology generated by $\tau \cup \{F\}$. Then τ_F is Polish, $F \in \Delta^0_1(X, \tau_F)$, $\Delta^1_1(X, \tau_F) = \Delta^1_1(X, \tau)$.

Důkaz

(X, τ_F) is homeomorphic with $((X \backslash F) \times \{0\}) \cup (F \times \{1\}) \subset X \times \{0, 1\}$ which is PTS and those two subsets are G_δ in $X \times \{0, 1\}$, so, (X, τ_F) is Polish.

$$\Delta^1_1(((X\backslash F) \times \{0\}) \cup (F \times \{1\})) \Leftrightarrow$$

$$\Leftrightarrow \Delta^1_1(\tau_F) = \left\{A \cup B | A \in \Delta^1_1(X\backslash F, \tau), B \in \Delta^1_1(F, \tau)\right\} \subset \Delta^1_1(\tau) \subset \Delta^1_1(\tau_F).$$

□

Lemma 2.14

(X, τ) PTS, $(\tau_n)_{n\in\omega}$ Polish topology, $\tau \subset \tau_n$, $n \in \omega$. Then topology τ_∞ generated by $\bigcup_{n\in\omega} \tau_n$ is polish. If $\forall n \in \omega : \tau_n \subset \Delta^1_1(\tau)$, then $\Delta^1_1(\tau) = \Delta^1_1(\tau_\infty)$.

Důkaz

Set $X_n := (X, \tau_n)$, $\varphi : X \to \prod_{n\in\omega} X_n$, $\varphi(x) = (x, x, x, x, \ldots)$. φ is homomorphism (X, τ_∞) on $\varphi(X)$. ($U \in$ base of $\tau_\infty \implies \exists n \in \omega : U \in \tau_n, \varphi(U) = x_1 \times x_2 \times \ldots \times x_{n-1} \times U \times x_{n+1} \times \ldots \cap \varphi(X)$ is open. $\varphi(X) \in \Pi^0_1(\prod X_n) \implies \varphi(X)$ PTS $\implies (X, \tau_\infty)$ PTS.)

$$\Delta^1_1(\tau) = \Delta^1_1(\tau_\infty) \impliedby \sigma(\sigma(M)) = \sigma(M). \ (\tau_\infty \subset \Delta^1_1(\tau) = \Delta^1_1(\tau_n).) \ \tau_\infty \subset \bigcup \Delta^1_1(\tau_n).$$ □

Věta 2.15

(X, τ) PTS, $A \in \Delta^1_1(X, \tau)$. There exists polish topology τ_A such that $\tau \subset \tau_A$, $\Delta^1_1(\tau_A) = \Delta^1_1(\tau)$ and $A \in \Delta^0_1(X, \tau_A)$.

Důkaz

$$\mathcal{S} := \left\{D \in \Delta^1_1(X) | \text{ exists polish topology } \tau_D \supset \tau \text{ and } \Delta^1_1(\tau_D) = \Delta^1_1(\tau), D \in \Delta^0_1(X, \tau_D)\right\}.$$

We know that $\tau \subset \mathcal{S}$ and that $\mathcal{S}$ is closed under complements. Moreover, $\mathcal{S}$ is closed under countable union ($A_n \in \mathcal{S} \to \tau_{A_n} \to \tau_\infty = \tau_{\bigcup A_n}$). So $\mathcal{S} = \Delta^1_1(X, \tau)$. □

Lemma 2.16

X, Y PTS. $f : X \to Y$ Borel. Then graph$(f) \in \Delta^1_1(X \times Y)$.

Důkaz

Fix compatible complete metric ϱ on Y. U_n, $n \in \omega$, countable colection of open balls with diam $< 2^{-n}$ covering Y.

$$\operatorname{graph} f \stackrel{?}{=} \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U \in \Delta^1_1(X \times Y).$$

„$\subseteq$“: $(x, y) \in \operatorname{graph}(f) \Leftrightarrow f(x) = y \implies \forall n \in \omega\ \exists U \in U_n : y \in U \wedge x \in f^{-1}(U) \implies (x, y) \in \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U$.

„$\supseteq$“: $(x, y) \notin \operatorname{graph}(f) \Leftrightarrow f(x) \neq y \implies \exists n \in \omega : \varrho(f(x), y) > \frac{1}{n} \implies \exists n \in \omega \forall U \in U_n \neg(x \in f^{-1}(U) \cap y \in U) \implies (x, y) \notin \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U$. □

Poznámka (Notation)
If f is Borel, we write $f \in \Delta^1_1$.

Věta 2.17

X, Y PTS, $f \in \Delta^1_1(X \times Y)$. If $A \in \Delta^1_1(X)$ and $f|_A$ is injective, then $f(A) \in \Delta^1_1(Y)$.

Důkaz
If $f : X \to Y$ is injective, then $f(A) = \prod_Y(\mathrm{graph}(f) \cap A \times Y) \in \Sigma_1^1(Y)$.

$$Y \backslash F(A) = \prod_Y (\mathrm{graph}(f) \cap (X \backslash A) \times Y) \in \Sigma_1^1(Y) \implies f(A) \in \Delta_1^1(Y).$$

Assume f is continuous, $A \in \Pi_1^0(X)$. From the theorem above $A \subset \omega^\omega$, $B_s := f(\mathcal{N}(s) capA)$. $\forall s \in \omega^{<\omega}\ \forall i, j, i \neq j : B_{s^\wedge i} \cap B_{s^\wedge j} = \emptyset \impliedby f$ is injection. $\forall s \in \omega^{<\omega} : B_s = \bigcup_{i \in \omega} B_{s^\wedge i}$.

From Luzin separation theorem, there exists (by induction) $(B'_s)_{s \in \omega^{<\omega}}$ of Borel sets:

$$\forall s \in \omega^{<\omega}\ \forall i, j \in \omega, i \neq j B'_{s^\wedge i} \cap B'_{s^\wedge j} = \emptyset.$$

(separation $B_{s^\wedge i}$, $\bigcup_{j<i} B_{s^\wedge j} \cup \bigcup_{l>i} B_{s^\wedge l}$) $\forall s \in \omega^{<\omega} : B_s \subset B'_s$.

Put: $B^*_\emptyset = Y$, $B^*_{s^\wedge j} = B_{s^\wedge j} \cap \overline{B_{s^\wedge j}} \cap B^*_s$. $\forall s \in \omega^{<\omega} : B^*_s \in \Delta_1^1(Y), B_s \subset B^*_s \subset \overline{B_s}$, $B^*_{s^\wedge j} \subset B^*_s$, $B^*_{s^\wedge j} \cap B^*_{s^\wedge i} = \emptyset$, $s \in \omega^{<\omega}$, $i, j \in \omega, i \neq j$. We proof: $f(A) \stackrel{?}{=} \bigcup_{s \in \omega^{<\omega}} \bigcap_{k \in \omega} B^*_{s/k} = \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B^*_s \in \Delta_1^1(Y)$.

$B^*_s, s \in \omega^{<\omega}, B^*_s \in \Delta_1^1(Y)$. $f(A) = \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B^*_s$:

„$\subseteq$“: $x \in f(A) \implies \exists \nu \in A : f(\nu) = x$. Then $x \in f(\mathcal{N}_{\nu/k} \cap A) = B_{\nu/k} \subset B^*_{\nu/k}$, $k \in \omega$ $\implies x \in \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B^*_s$.

„$\supseteq$“: $x \in \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B^*_s \implies \forall k \in \omega\ \exists \nu^k \in \omega^\omega : x \in B^*_{\nu/k}$. $\exists \nu \in \omega^\omega : \nu^k = \nu$, $k \in \omega$. $\implies \forall k \in \omega\ \exists \nu \in \omega^\omega : f(\mathcal{N}(\nu/k) \cap A) \neq \emptyset$. $\implies \exists \nu^k \in \mathcal{N}(\nu/k) \cap A, \nu^k \to \nu \implies \nu \in A$ (A is closed). $f(\nu) = x$? Assume $f(\nu) \neq x \implies \exists U$ $\underline{\text{neighbourhood}}$ of $f(\nu)$, such that $x \notin \overline{U} \implies$ (f is continuous) $\exists k_0 \in \omega : x \in B^*_{\nu/k_0} \subset \overline{f(\mathcal{N}_{\nu/k_0}) \cap A} = \overline{B_{\nu/k_0}} \subset \overline{U}$ which is contradiction.

a) Let f is continuous and $A \in \Delta_1^1(X)$. On X we find Polish topology τ_A such that $A \in \Delta_1^0(\tau_A)$, $\tau \subset \tau_A$ (so f is continuous with respect to τ_A), $\Delta_1^1(\tau) = \Delta_1^1(\tau_A)$.

b) Let $f \in \Delta_1^1$. Then $f(A) = \pi_Y(\mathrm{graph}(f) \cap A \times Y)$. Observe that π_Y is injective on $(\mathrm{graph}(f) \cap A \times Y$ if f is injective on A. $\square$

Věta 2.18

X, Y PTS, $f \in \Delta_1^1(X \times Y)$.

1. $A \in \Sigma_1^1(X) \implies f(A) \in \Sigma_1^1(Y)$;
2. $B \in \Sigma_1^1(Y) \implies f^{-1}(B) \in \Sigma_1^1(X)$;
3. $B \in \Pi_1^1(Y) \implies f^{-1}(B) \in \Pi_1^1(X)$.

Důkaz
„1.“: $f(A) = \pi_Y((\mathrm{graph}(f) \cap A \times Y)$ is continuous image of Σ_1^1 set.

„2.“: $f^{-1}(B) = \pi_X((\mathrm{graph}(f) \cap X \times B)$ is continuous image of Σ_1^1 set.

„3.“: $f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(Y \backslash B)$. □

2.6 Standard Borel spaces (SBS)

Definice 2.5 (Standard Borel space (SBS))

Measurable space $(X, \mathcal{S})$ is called standard Borel space (SBS) if there exists Polish topology τ on X such that $\Delta_1^1(X, \tau) = \mathcal{S}$.

Definice 2.6 (Effros Borel space)

Let X be PTS and $\mathcal{F}(X) := \Pi_1^0(X)$. Let $\mathcal{S}$ be σ-algebra generated by sets of form $\{F \in \mathcal{F}(X) | F \cap U \neq \varnothing\} =: M_U$, where $U \in \Sigma_1^0(X)$. $(\mathcal{F}(X), \mathcal{S})$ is called Effros Borel space.

Věta 2.19

X PTS. Then $(\mathcal{F}(X), \mathcal{S})$ is SBS.

Důkaz
Without proof. □

Poznámka
X be measurable compact. Then $\mathcal{F}(X)$ can be equipped by Vietoris topology.

Příklad
$SB := \{Y \in \mathcal{F}(C([0,1])) | Y$ is Banach subspace of $C([0,1])\}$. If we restrict Effros σ-algebra on SB then SB is SBS.

$$SD = \{Y \in SB | Y \text{ has separable dual}\},$$
$$NU = \{Y \in SB | Y \text{ is not universal}\},$$
$$REFL = \{Y \in SB | Y \text{ is reflexive}\},$$
$$NL_1 = \{Y \in SB | Y \text{ does not contain } l_1\}.$$

3 Regularity of Σ^1_1 sets

3.1 Sets with Baire property (BP)

Definice 3.1 (Baire property (BP))

X TS, $A \subset X$ has Baire property (BP) in X if there exists open $U \subset X$ and set of 1. category $M \subset X$ such that $A = U \triangle M := (U \backslash M) \cup (M \backslash U)$. Collection of all sets with BP we denote as $Baire(X)$.

Věta 3.1

X TS. Then $Baire(X)$ is σ-algebra and $Baire(X) \supset Borel(X)$.

Důkaz

1. „$Baire(X) \supset \Sigma^0_1(X)$“ trivial. 2. „$Baire(X)$ is σ-algebra“: a) „$A \in Baire(X) \overset{?}{\Longrightarrow} X \backslash A \in Baire(X)$“: $A \in Baire(X) \implies \exists G \in \Sigma^0_1(X)$ and M meager such that $A = G \triangle M$.

$$X \backslash A = X \backslash (G \triangle M) = (X \backslash G) \triangle M = (\operatorname{int}(X \backslash G) \cup (X \backslash G) \backslash \operatorname{int}(X \backslash G)) \triangle M =$$

$$= (V \cup M_1) \triangle M_2 = V \triangle M \qquad (M = M_1 \triangle M_2).$$

b) „$A_n \in Baire(X) \overset{?}{\Longrightarrow} \bigcup A_n \in Baire(X)$“: $A_n = G_n \triangle M_n$, $G_n \in \Sigma^0_1(X)$, M_n meager. $M'_n = G_n \cap M_n$ (meager), $M''_n = M_n \backslash G_n$ (meager).

$$\bigcup A_n = \bigcup ((G_n \backslash M'_n) \cup M''_n) = ((\bigcup G_n) \backslash M''') \cup \bigcup M''_n,$$

where $M''' \subset \bigcup_{n \in \omega} M'_n$. □

Lemma 3.2

X TS, $A \subset X$. Then A is meager iff $\forall x \in A\ \exists V \in \Sigma^0_1(X)$ such that $x \in V$ and $A \cap V$ is meager.

Důkaz
„ $\Longrightarrow$ “ trivial. „ $\Longleftarrow$ “ $\mathcal{U}$ denote as maximal collection of disjoint Σ_1^0 sets such that $U \cap A$ is meager for $U \in \mathcal{U}$. We show that $A \cap \bigcup \mathcal{U}$ is meager, $X \backslash \bigcup \mathcal{U}$ is nowhere dense, so meager.

„$X \backslash \bigcup \mathcal{U}$ is nowhere dense“: By contradiction we assume that there exists $\emptyset \neq V \in \Sigma_1^0(X)$, $V \subset X \backslash \bigcup \mathcal{U}$. Now we have 2 cases: $A \cap V = \emptyset \implies V \in \mathcal{U}$ contradiction, or $A \cap V \neq \emptyset \implies \exists x \in A \cap V \implies \exists W \in \Sigma_1^0(X) : x \in W, W \cap A$ is meager $\implies x \in W \cap V \neq \emptyset, W \cap V \cap A$ is meager $\implies W \cap V \in \mathcal{U}$ contradiction.

„$\bigcup \mathcal{U} \cap A$ is meager“: $\mathcal{U} := \{U_\alpha | \alpha \in I\}$, $U_\alpha \cap A$ meager $\implies$ exist? $F_n^\alpha \in \Pi_1^0(X)$ nowhere dense: $U_\alpha \cap A \subset \bigcup F_n^\alpha \subset \overline{U_\alpha}$. We show that $\bigcup_{\alpha \in I} F_n^\alpha$ is nowhere dense:

$$a) \bigcup_{\alpha \in I} U_\alpha \backslash F_n^\alpha \in \Sigma_1^0(X), \quad (\bigcup_{\alpha \in I} U_\alpha \backslash F_n^\alpha) \cap (\bigcup_{\alpha \in I} F_n^\alpha) = \emptyset \impliedby F_n^\alpha \subset \overline{U_\alpha}, \quad \overline{U_\alpha} \cap U_\beta = \emptyset, \alpha \neq \beta$$

So $\mathcal{U}$ is disjoint collection, so $\bigcup_{\alpha \in I} U_\alpha F_n^\alpha \cap \overline{\bigcup_{\alpha \in I} F_n^\alpha} = \emptyset$.

$$\implies \overline{\bigcup_{\alpha \in I} F_n^\alpha} \subset (\bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha)) \cup (X \backslash \bigcup \mathcal{U}).$$

b) We assume $\exists V \in \Sigma_1^0(X)$, $V \neq \emptyset$, $V \subset \overline{\bigcup_{\alpha \in I} F_n^\alpha}$.

$$? \implies V \not\subset X \backslash \bigcup \mathcal{U} \overset{a)}{\implies} V \cap \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \neq \emptyset \implies \exists \alpha \in I : V \cap U_\alpha \neq \emptyset.$$

$$a) \implies V \cap U_\alpha \subset \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \overset{\mathcal{U} \text{ disjoint}}{\implies} V \cap U_\alpha \subset F_n^\alpha \lightning.$$

□

Definice 3.2 ($\mathcal{S}$-Obal)

Nechť $(X, \mathcal{S})$ je měřitelný prostor a $A \subset X$. Řekneme, že $\hat{A} \in \mathcal{S}$ je $\mathcal{S}$-obalem množiny A, jestliže $A \subset \hat{A}$ a jestliže $\forall\ B \in \mathcal{S}$ splňující $A \subset B$ je každá podmnožina $\hat{A} \backslash B$ je prvkem $\mathcal{S}$.

Věta 3.3

Nechť X je topologický prostor. Pak každá množina v X má Baire(X)-obal.

Důkaz

Položme $E(A) = X \backslash \bigcup \{V \subset X | V \text{ je otevřená} \wedge A \text{ je první kategorie ve } V\}$. Potom množina $A \backslash E(A)$ je první kategorie podle předchozího lemmatu. Existuje tedy $W \subset X$ první kategorie a F_σ taková, že $A \backslash E(A) \subset W$. Položme $\hat{A} = E(A) \cup W$. Potom $\hat{A} \in Baire(X)$.

Nechť $B \in Baire(X)$ splňující $A \subset B$. Zřejmě $E(A) \subset E(B)$. Takže

$$\hat{A} \backslash B = (E(A) \cup W) \backslash B = (E(A) \backslash B) \cup (W \backslash B) \subset (E(B) \backslash B) \cup (W \backslash B).$$

Množina $W \backslash B$ je první kategorie. Množina B má Baierovu vlastnost, a proto lze psát $B = H \triangle M$, kde H je otevřená množina a M je množina první kategorie. Potom máme

$$E(B) \backslash B \subset (E(B) \backslash H) \cup M \subset (\overline{H} \backslash H) \cup M.$$

Poslední množina je první kategorie, takže $\hat{A} \backslash B$ je také první kategorie, což jsme měli dokázat. □

Věta 3.4 (Szpilrajn-Marszewski (1907–1976))

$(X, \mathcal{S})$ measurable space, such that for every $A \subset X$ there exists $\mathcal{S}$-cover of A. Then $\mathcal{S}$ is closed under Suslin operation.

Důkaz

$(P_s)_{s \in \omega^{<\omega}}$ Suslin scheme of elements of $\mathcal{S}$. Put $P := \mathcal{A}_s P_s$. We want $P \in \mathcal{S}$. We can assume $P_s \supset P_t$, $s < t$ ($\tilde{P}_t = \bigcap_{s \leqslant t} P_s \in \mathcal{S}$). Denote $A^s := \bigcup_{\nu \in \mathcal{N}(s)} \bigcap_{n \in \omega} P_{\nu/n} = \mathcal{A}_t P_{s^\wedge t}$.

$A^s \subset P_s$ trivial. $A^\varnothing = P$, $A^s = \bigcup_{n \in \omega} A^{s^\wedge n}$ trivial. Set $\overline{A^s}$ be $\mathcal{S}$-cover of A^s such that $A^s \subset \overline{A^s} \subset P_s (\overline{A^s} \cap P_s \in \mathcal{S})$. $Q_s := \overline{A^s} \backslash \left(\bigcup_{n \in \omega} \hat{A^{s^\wedge n}} \right)$, $Q = \bigcup_{s \in \omega^{<\omega}} Q_s$. $\bigcup_{n \in \omega} \hat{A^{s^\wedge n}}$ is $\mathcal{S}$-cover of A^s. $\left(\hat{A^{s^\wedge n}} \supset A^{s^\wedge n} \implies \bigcup \hat{A^{s^\wedge n}} \supset \bigcup A^{s^\wedge n} = A^s \right)$. $\implies$ any subset of Q_s is in $\mathcal{S}$ $\implies$ any subset of Q is in $\mathcal{S}$.

$$\hat{A^\varnothing} \backslash P = \hat{P} \backslash P \subset Q \impliedby A^\varnothing \backslash Q \subset P.$$

$$x \in \hat{A^\varnothing} \backslash Q \implies x \notin Q_\varnothing, x \in \hat{A^\varnothing} \implies \exists n_0 \in \omega : x \in \hat{A^{n_0}}, x \notin Q_{n_0} \implies$$

$$\implies \exists n_1 \in \omega : x \in \hat{A^{n_0, n_1}}, x \in Q_{n_0, n_1}.$$

Etc. $\implies \exists \nu \in \omega^\omega \; \forall n \in \omega : x \in \hat{A^{\nu/n}} \subset P^{\nu|n} \implies x \in \mathcal{A}_s P_s = P_\varnothing$. □

Důsledek

Nechť X je polský. Potom $\Sigma^1_1(X) \subset Baire(X)$.

Důkaz

Platí $\Pi^0_1(X) \subset Baire(X)$, a tedy $\Sigma^1_1(X) = \mathcal{A}\Pi^0_1(X) \subset \mathcal{A}Baire(X) = Baire(X)$. □

Definice 3.3 (Univerzálně měřitelná)

Nechť X je polský. Řekneme, že $A \subset X$ je univerzálně měřitelná, jestliže pro každou σ-konečnou borelovskou míru μ na X je množina A μ-měřitelná.

Věta 3.5 (Luzin–Sierpiński)

Nechť X je polský a $A \in \Sigma_1^1(X)$. Potom A je univerzálně měřitelná.

Důkaz

Nechť μ je σ-konečná borelovská míra na X a $\mathcal{S}$ je σ-algebra μ-měřitelných množin. Bez újmy na obecnosti můžeme předpokládat, že μ je pravděpodobnostní. Pro $A \subset X$ proložme $\mu^*(A) = \inf\{\mu(B), B \in Borel(X) \wedge A \subset B\}$.

Pak existuje $\hat{A} \in Borel(X)$ takové, že $\mu^*(A) = \mu(\hat{A})$. Jestliže $A \subset B$ a B je měřitelná, pak $\mu(\hat{A} \backslash B) = 0$. Jinak by totiž existovala $C \subset \overline{A} \backslash B \subset \overline{A} \backslash A$ taková, že $C \in Borel(X)$ a $\mu(C) > 0$, což nelze. Odtud a z předchozí věty plyne dokazované tvrzení. □

3.2 Solecki theorem

Poznámka (Notation)

X PTS, $\mathcal{I} \subset \Pi_1^0(X)$.

$$\mathcal{I}^{ext} := \left\{A \subset X | \exists \mathcal{F} \subset \mathcal{I}, |\mathcal{F}| = \omega, A \subset \bigcup \mathcal{F}\right\}.$$

Například

$\mathcal{I} = \{A \subset X | |A| < \omega\}$, $\mathcal{I} = \{A \subset X | A \text{ nowhere dense}\}$.

$$\mathcal{I}^{perf} = \left\{A \subset X | A \neq \varnothing, \forall U \in \Sigma_1^0(X) : U \cap A \neq \varnothing \implies U \cap A \notin \mathcal{I}^{ext}\right\}.$$

$$\operatorname{Ker} A := A \backslash \bigcup \left\{U \subset X | U \in \Sigma_1^0(X), U \cap A \in \mathcal{I}^{ext}\right\} =$$

$= \text{max perfect subset of } A \impliedby X \text{ has countable base.}$

$$MGR(A) = \{Z \subset A | Z \text{ be meager in } A\}, \qquad A \subset X.$$

Věta 3.6 (Solecki)

X PTS, $A \in \Sigma_1^1(A)$, $\mathcal{I} \subset \Pi_1^0(X)$. $A \notin \mathcal{I}^{ext} \implies \exists H \in \Pi_2^0(X), H \subset A, H \notin \mathcal{I}^{ext}$

Lemma 3.7 (For proof of Solecki)

$A \in \Sigma_1^1(X) \backslash \mathcal{I}^{ext}$. Then there exists Suslin scheme $(A_s)_{s \in \omega^{<\omega}}$ of closed subsets of X such

that:

$$A_\varnothing = \varnothing, \quad a_s A_s \subset A, \quad A_s \neq \varnothing \implies A \cap A_s \in \mathcal{I}^{perf}, \overline{A \cap A_s} = A_s, \quad \overline{\bigcup_{n\in\omega} A_{s^\wedge n}} = A_s.$$

Důkaz
$(H_s)_{s\in\omega^{<\omega}}$ closed subsets of X, decreasing $(H_s \supset H_{s^\wedge n}, n \in \omega)$, $A = a_s H_s \impliedby A \in \Sigma^1_1(X)$.
For $s \in \omega^{<\omega} : L_s := a_t H_{s^\wedge t}, A_s := \overline{\mathrm{Ker}(L_s)}$.

1. $A_\varnothing = \overline{\mathrm{Ker}(L_\varnothing)} = \overline{\mathrm{Ker}(A)} \neq \varnothing \impliedby A \notin \mathcal{I}^{ext}$ (X has countable base).
2. $H_s \searrow \implies L_s \subset H_s \implies \mathrm{Ker}(L_s) \subset H_s \overset{H_s\in\Pi^0_1(X)}{\implies} A_s \subset H_s \implies a_s A_s \subset a_s H_s = A$.
3. $\mathrm{Ker}(L_s) \subset A_s, L_s \subset A : (A = \bigcup_{|s|=k} L_s, k \in \omega \impliedby H_s \searrow) \implies \mathrm{Ker}(L_s) \subset A_s \cap A$, $\overline{\mathrm{Ker}(L_s)} = A_s$.
$$A_s = \overline{\mathrm{Ker}(L_s)} \subset \overline{A_s \cap A} \subset \overline{A_s} = A_s.$$
Assume $A_s \neq \varnothing \implies A \cap A_s \neq \varnothing$. $U \in \Sigma^0_1(X)$, $U \cap A \cap A_s \neq \varnothing \implies U \cap \mathrm{Ker}(L_s) \neq \varnothing \implies U \cap \mathrm{Ker}(L_s) \notin \mathcal{I}^{ext}$. $\implies U \cap A \cap A_s \notin \mathcal{I}^{ext}$.
4. $\bigcup_{n\in\omega} A_{s^\wedge n} \subset A_s \impliedby (H_s \searrow \implies L_s \searrow \implies A_s \searrow)$. Let $U \in \Sigma^0_1(X)$, $U \cap A_s \neq \varnothing \implies U \cap \mathrm{Ker}(L_s) \neq \varnothing \implies U \cap L_s \notin \mathcal{I}^{ext}$.
$$L_s = \bigcup_{n\in\omega} L_{s^\wedge n} \implies \exists n_0 \in \omega : U \cap L_{s^\wedge n_0} \notin \mathcal{I}^{ext} \implies U \cap \mathrm{Ker}(L_{s^\wedge n_0}) \notin \mathcal{I}^{ext} \implies$$
$$\implies U \cap A_{s^\wedge n_0} \neq \varnothing.$$

□

Důkaz (Solecki theorem, not in exam)
$A \in \Sigma^1_1(X) \backslash \mathcal{I}^{ext}$, $(A_s)_{s\in\omega^{<\omega}}$ from the previous lemma. There are 2 cases:

„1st case $\exists s \in \omega^{<\omega}\ \exists U \in \Sigma^0_1(X) : A_s \cap U \neq \varnothing \wedge MGR(A_s \cap U) \subset \mathcal{I}^{ext}$“: Put $\tilde{A} := A \cap A_s \cap U$. Then from the third item of the previous lemma $\tilde{A} \in \mathcal{I}^{perf}$, $\tilde{A} \in \Sigma^1_1(X)$. $A_s \neq \varnothing$, $A \cap A_s \in \mathcal{I}^{perf}$, $U \cap A_s \neq \varnothing \implies U \cap A \cap A_s \neq \varnothing \impliedby \overline{A \cap A_s} = A_s$.

$$\implies \tilde{A} \in Baire(A_s \cap U) \impliedby (A_s \cap U \in \Pi^0_2(X)), A_s \cap U \text{ PTS}.$$

$$\tilde{A} = H \cup M, H \in \Pi^0_2(A_s \cup U), M \in MGR(A_s \cap U) \subset \mathcal{I}^{ext} \implies H \notin \mathcal{I}^{ext}, H \subset A.$$

„2nd case $\forall s \in \omega^{<\omega}\ \forall U \in \Sigma^0_1(X), U \cap A_s \neq \varnothing : MGR(A_s \cap U) \backslash \mathcal{I}^{ext} \neq \varnothing$“: Notation: $\mathcal{F} \subset 2^X : \mathcal{F}^d := \overline{\bigcup \mathcal{F}} \backslash \bigcup \{\overline{F} | F \in \mathcal{F}\}$. Choose CCM $\leqslant 1$ on X. We will inductively construct $\varphi : \omega^{<\omega} \to \omega^{<\omega}$, $U_s \subset X$, $s \in \omega^{<\omega}$ such that:

1. $|\varphi(s)| = |s|$; $\varphi(s) \prec \varphi(t)$, for every $s \prec t$;

2. $U_s \in \Sigma^0_1(X)$;

3. $\operatorname{diam} U_s \leqslant 2^{-|s|}$;

4. $\lim_{n\to\infty} \operatorname{diam}(U_{s^\wedge n}) = 0$;

5. $\forall t, s \in \omega^{<\omega}, t < s, t \neq s : \overline{U_s} \subset U_t$;

6. $\forall s \in \omega^{<\omega}\ \forall m, n \in \omega, m \neq n : U_{s^\wedge m \cap U_{s^\wedge n}} = \varnothing$;

7. $U_a \cap A_{\varphi(s)} \neq \varnothing$;

8. $\{U_{s^\wedge n} | n \in \omega\}^d \notin \mathcal{I}^{ext}$;

9. $\{U_{s^\wedge n} | n \in \omega\}^d \subset U_s$;

10. (9. + 5.) $\overline{\bigcup_{n\in\omega} U_{s^\wedge n}} \subset U_s$.

Construction: $\varphi(\varnothing) = \varnothing$, $U_\varnothing$ be arbitrary open subset of X: $U_\varnothing \cap A_\varnothing \neq \varnothing$. Then all items are satisfied. We assume that U_s, φ_s are constructed for all $s \in \omega^{<\omega}$, $|s| \leqslant N \in \omega$. Let $s \in \omega^{<\omega}$, $|s| \leqslant N$ be arbitrary. From 7th item $U_s \cap A_{\varphi(s)} \neq \varnothing$, $MGR(A_{\varphi(s)} \cap U_s) \notin \mathcal{I}^{ext}$ $\implies$ $\exists K \subset A_{\varphi(s)} \cap U_s, K \in \Pi^0_1(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s$, $K \notin \mathcal{I}^{ext}$. Because

$$\exists L \in MGR(A_{\varphi(s)} \cap U_s) \setminus \mathcal{I}^{ext} \implies \exists H \in \Sigma^0_2(X), H \supset L, H \in \Sigma^0_2(A_{\varphi(s)} \cap U_s), H \notin \mathcal{I}^{ext},$$

so $H = \bigcup F_n$, $F_n \in \Pi^0_1(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s \implies \exists n_0 \in \omega : F_{n_0} = K \notin \mathcal{I}^{ext}$.

Find $D \subset A_{\varphi(s)} \cap U_s$: D is discrete in $X \setminus K$. $D \cap K = \varnothing$. $\overline{D} = K \cup D$. Let $\{y_n\} \subset K$, $\overline{\{y_n\}} = K$, and every element of $\{y_n\}$ repeats infinitely many times. Find $x_n \in (A_{\varphi(s)} \cap U_s) \setminus K$ such that $\varrho(x_n, y_n) < \frac{1}{n}$ (it exists $\impliedby$ K is nowhere dense in $A_{\varphi(s)} \cap U_s$). Then $D = \{x_n | n \in \omega\}$, $D \cap K = \varnothing$, $\overline{D} \supset \overline{D \cup \{y_n | n \in \omega\}} \supset D \cup K$, $x \notin K \cup D \implies \exists n \in \omega \setminus \{0\}$: $\varrho(x, K) > \frac{1}{n} \implies \#(B(x, 1/2n) \cap D) \leqslant 2n \implies x \notin \overline{D}$. $\implies \overline{D} = D \cup K$, D is discrete in $X \setminus K$. Assume $x_n \neq x_m$, $n \neq m$.

Define $U_{s^\wedge n}$ as open ball with center x_n: $\overline{U_{s^\wedge n}} \subset U_s$. $U_{s^\wedge n} \cap U_{s^\wedge m} = \varnothing$ (D is discrete), $\operatorname{diam} U_{s^\wedge n} \leqslant 2^{-|s|-1}$, $\lim_{n\to\infty} \operatorname{diam} U_{s^\wedge n} = 0$, $\overline{\bigcup_{n\in\omega} U_{s^\wedge n}} \setminus \bigcup_{n\in\omega} \overline{U_{s^\wedge n}} = \overline{\{U_{s^\wedge n} | n \in \omega\}} = K \impliedby \overline{U_{s^\wedge n}} \cap K = \varnothing$, $\overline{D} = K \cup D$. $x_n \in A_{\varphi(s)} \implies U_{s^\wedge n} \cap A_{\varphi(s)} \neq \varnothing$, $\overline{\bigcup_{k\in\omega} A_{\varphi(s)^\wedge k}} = A_{\varphi(s)} \implies \exists k \in \omega : U_{s^\wedge n} \cap A_{\varphi(s)^\wedge k} \neq \varnothing$.

Put $\varphi(s^\wedge n) = \varphi(s)^\wedge k$. And then all items are satisfied. $H = \bigcap_{n\in\omega} \bigcup_{|s|=n, s\in\omega^{<\omega}} U_s \in \Pi^0_2(X)$, $H \subset A$, $H \notin \mathcal{I}^{ext}$.

$$H := \bigcap_{n\in\omega} \bigcup \{U_s | s \in \omega^{<\omega}, |s| = n\} \in \Pi^0_2(\impliedby 2.).$$

$H \subset A$?, $H \notin \mathcal{I}^{ext}$. „$H \subset A$“: 5. and 6. $\implies H = \bigcup_{\nu\in\omega^\omega} \bigcap_{n\in\omega} \bigcup_{\nu/m} = a_s U_s$?

$$x \in H \Leftrightarrow \forall n \in \omega\ \exists \omega^{<\omega}, |s| = n : x \in U_s \overset{5.\wedge 6.}{\Leftrightarrow} \exists s \in \omega^\omega\ \forall n \in \omega x \in U_{s/n} \Leftrightarrow a_s U_s.$$

$$\big(3. \implies \operatorname{diam}(U_{\nu/n}) \leqslant 2^{-n}, \big(7. \implies U_{\nu/m} \cap A_{\varphi(\nu/n)} \neq \varnothing\big)\big) \implies$$

$$\implies \bigcap_{n\in\omega} \overline{U_{\nu/n}} \subset \bigcap_{n\in\omega} A_{\varphi(\nu/n)} \subset A \implies H \subset A.$$

„$H \notin \mathcal{I}^{ext}$": $\forall \nu \in \omega^\omega : \bigcap_{n\in\omega} U_{\nu/n} \neq \emptyset \impliedby$ 3. $\wedge$ 5., so $\forall s \in \omega^{<\omega} : U_s \cap H \neq \emptyset$. Assume $H \subset \bigcup_{m\in\omega} F_m$, $F_m \in \mathcal{I}$. $H \in G_\delta \implies \exists n_0 \in \omega : F_{n_0}$ is not meager in $H \implies \exists U \neq \emptyset$ open in X: $\emptyset \neq U \cap H \subset F_{n_0}$. Let $x \in U \cap H$. Then there exist $\nu \in \omega^\omega$ $\forall n \in \omega : x \in U_{\nu/n}$ $\implies \exists m_0 \in \omega : U_{\nu/m_0} \subset U \implies \emptyset \neq U_{\nu/m_0} \cap H \subset F_{n_0}$. Denote $\nu/m_0 =: s$. For contradiction assume 9. $\wedge$ 3. $\overset{?}{\implies} \{U_{s^\wedge n}, n \in \omega\}^d \subset F_{n_0}$ which contradicts 8. So 9. $\implies$ $\{U_{s^\wedge n}, n \in \omega\}^d \subset U_s$,

$$x \in \{U_{s^\wedge n}, n \in \omega\}^d \implies \exists n_k \nearrow, x_k \in U_{s^\wedge n_k} : x_k \to x, y_k \in U_{s^\wedge n_k} \cap H \subset F_{n_0} \implies$$

$$\implies 4. \implies y_k \to x \implies x \in F_{n_0}.$$ □

Věta 3.8 (Perfect set theorem)

X PTS, $A \in \Sigma^1_1(X)$, $|A| > \omega$. Then there exists homeomorphic copy C of 2^ω in A.

Důkaz

$\mathcal{I} = \{\{x\}, x \in X\}$, $\mathcal{I}^{ext}$ countable sets $\implies A \in \Sigma^1_1(X) \setminus \mathcal{I}^{ext} \implies \exists H \in \Pi^0_2(X)$, $H \notin \mathcal{I}^{ext}$ $\implies |H| > \omega$, H be PTS $\implies H$ contains 2^ω.

$G := \operatorname{Ker}(H) \implies G \in \mathcal{I}^{perf}$, $|G| > \omega$, $G \in \Pi^0_2$. ϱ ccm on G, $s \in 2^{<\omega}$. Find B_1 such that $\overline{B_{s^\wedge 0}} \cap \overline{B_{s^\wedge 1}} \neq \emptyset$, $\overline{B_{s^\wedge 0}} \cap \overline{B_{s^\wedge 1}} \subset B_s$. $\operatorname{diam} B_s \leqslant 2^{-|s|}$. $C = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} B_s := \bigcup_{s\in 2^\omega} \bigcap_{n\in\omega} B_{s/n}$. □

Poznámka

Tuto větu nelze pro koanalytické množiny v ZFC dokázat ani vyvrátit.

4 Infinite games

4.1 Baire definitions

Definice 4.1

Assume $A \neq \emptyset$, $X \subset A^\omega$. In game $G(X)$, there are 2 players I, II and those players choose $a_i, i \in \omega, a_i \in A$:

$$I : a_0, a_2, a_4, \ldots$$

$$II : a_1, a_3, a_5, \ldots$$

Player I wins $\equiv (a_i) \in X$, otherwise player II wins.

Strategy for I is mapping $\mathcal{S} : A^{<\omega} \to A^{<\omega}$ such that $\forall s \in A^{<\omega} : |\mathcal{S}(s)| = |s| + 1$ and $\forall s, t \in A^{<\omega}$, $s < t$: $\mathcal{S}(s) < \mathcal{S}(t)$.

Definice 4.2 (Notation)

$\sigma \subset A^{<\omega}$ be tree iff $\forall s, t \in A^{<\omega}$, $s < t : t \in \sigma \implies s \in \sigma$

Let σ be tree, $s \in \sigma$. Then s is leaf iff $\forall n \in A : s^\wedge n \notin \sigma$.

Let σ be tree. Then σ is prunned $\equiv$ σ does not have leaves ($\forall s \in \sigma : \exists a \in A : s^\wedge a \in \sigma$).

$$[\sigma] = \{\nu \in A^\omega | \forall n \in \omega : \nu/n \in \sigma\}, \qquad [\sigma] \in \Pi^0_1(A^\omega).$$

$\forall F \in \Pi^0_1(A^\omega)$ $\exists!$ prunned tree $\sigma \subset A^{<\omega}$: $[\sigma] = F$.

Strategy for I is prunned tree $\sigma \subset A^{<\omega}$ such that $\sigma \neq \emptyset$, $(a_0, a_1, \ldots, a_{2j}) \in \sigma \implies \forall a \in A : (a_0, \ldots, a_{2j}, a) \in \sigma$, and $(a_0, \ldots, a_{2j+1}) \in \sigma \implies \exists! a \in A : (a_0, \ldots, a_{2j+1}, a) \in \sigma$.

Definice 4.3

Strategy σ for I is winning $\Leftrightarrow$ I wins if the follows this strategy $[\sigma] \subset X$.

Poznámka
In Game $G(A, X)$ at most one player has winning strategy. It can happen (ZFC) that nobody has winning strategy.

Definice 4.4 (Game with rule)

$T \subset A^{<\omega}$ be prunned tree, $X \subset [T]$.

$$I : a_0, a_2, a_4, \ldots$$

$$II : a_1, a_3, a_5, \ldots$$

such that $\forall n \in \omega : (a_0, \ldots, a_n) \in T$ (T is tree of rules). Other notions are similar.

Poznámka
Assume

$$X = \{x \in A^\omega | x \in X \cap [T] \cup (\exists n \in \omega : x/n \notin T \text{ and the least } n \in \omega : x/n \notin T \text{ is odd})\}$$

then I (resp. II) has wining strategy in $G(X')$ $\Leftrightarrow$ I (resp. II) has winning strategy in $G(T, X)$.

Například
$SG(A, B_0, B_1)$. Let S, T be nonempty prunned trees on ω, $A \subset [S]$, $B_0, B_1 \subset [T]$.

$$I : x(0), x(1), x(2), \ldots$$

$$II : y(0), y(1), y(2), \ldots$$

$x(i), y(i) \in \omega, x/n \in S, y/n \in T$. Player II wins $\Leftrightarrow (x \in A \implies y \in B_0) \wedge (x \notin A \implies y \in B_1)$.

I has winning strategy $\Leftrightarrow \exists f : [T] \to [S]$ continuous: $f(B_0) \cap A = \emptyset$, $f(B_1) \subset A \Leftrightarrow f^{-1}(A)$ separates B_1 from B_2.

II has winning strategy $\Leftrightarrow \exists g : [S] \to [T]$ continuous: $g(A) \subset B_0$, $g(A^c) \subset B_1$.

Věta 4.1 (H?)

X Polish topological space, $A \in \Sigma^1_1(X) \backslash \Sigma^0_2(X)$. Then there exists C, the cantor set, $C \subset X$, such that $\overline{C \backslash A} = C$, $|C \backslash A| = \omega$ ($C \cap A$ is homeomorphic to $\mathcal{N}$).

Poznámka (Similarly)
$A \in \Pi^1_1(X)$. Than A is polish space or it contains rel. closed set homeomorphic to $\mathbb{Q}$.

4.2 Banach–Mazur game

Definice 4.5 (Banach–Mazur game $G^{**}(M, Y)$)

Let Y be PTS, $M \subset Y$, $Y \neq \emptyset$.

$$I : U_0, U_1, U_2, \dots$$

$$II : V_0, V_1, V_2, \dots$$

$U_i, V_i \neq \emptyset$ base open sets (some $\mathcal{V}$, the countable base of Y, $V_i, U_i \in \mathcal{V}$).

Poznámka
There is some ccm on Y and $\operatorname{diam}(U_i) \to 0$.

$U_i \supset V_i \supset U_{i+1}, i \in \omega$.

II player wins $\equiv \bigcap V_n = \bigcap U_n \subset M$.

Věta 4.2

$X \neq \emptyset$ topological space: 1. II has wining strategy $\Leftrightarrow$ M is co-meager. 2. X Polish topological space: I has wining strategy $\Leftrightarrow$ ($\exists U \subset X$ open, $U \neq \emptyset$: $M \cap U$ is meager).

Důkaz
„1.“: „ $\Longleftarrow$ “: Let W_n be open, dense: $M \supseteq \bigcap W_n$. II plays playes $V_n \subseteq U_n \cap W_n \wedge \bigcap V_n \subset \bigcap W_n \subset M$.

„ $\Longrightarrow$ “:

$$\mathcal{M}_0 := \{V_0 | V_0 \text{ is answer of II on some } U_n \text{ from I}\},$$

$\tilde{\mathcal{M}}_0$ maximal disjoin subsystem of $\mathcal{M}_0$. $W_0 := \bigcup \tilde{\mathcal{M}}_0$ is open and dense.

$$\mathcal{M}_n := \left\{V_n | V_n \text{ is answer of II on some } U_n \text{ from base which is continued in some element of } \tilde{\mathcal{M}}_{n-1}\right\},$$

$\tilde{\mathcal{M}}_n$ is maximal disjoin subsystem of $\mathcal{M}_n$. $W_n := \bigcup \tilde{\mathcal{M}}_n$ is open and dense.

$\bigcap W_n \subset M \implies M$ is co-meager.

„2.“: „ $\Longrightarrow$ “: I has wining strategy. Put $U := V_0$. Then rest of the game is played inside of U_0 and I wins $\Leftrightarrow \bigcap U_n \subset U_0 \backslash A \implies U_0 \backslash A$ is co-meager in $U_0 \implies A$ is meager in $U_0 = U$.

„ $\Longleftarrow$ “: $M \subset \bigcup F_n$, F_n are closed, nowhere dense in $U_0 = U$. $U_i \subseteq V_{i-1} \backslash F_{i-1}$, $i \geqslant 1$. $\bigcap U_n \not\subset \bigcup F_n \implies \bigcap U_n \not\subset M$. $\overline{U_i \subset V_i}, |V_i| \to 0$. □

4.3 Game of J. Malý (point-line game)

Definice 4.6 (Game of J. Malý)

Let B be unit ball in $\mathbb{R}^2$.

$$I : a_0, a_1, a_2, \ldots$$

$$II : p_0, p_1, \ldots$$

$a_i \in B$, p_i line in $\mathbb{R}^2$, $a_i, a_{i+1} \in p_i$. II player wins $\equiv (a_i)$ is convergent.

Věta 4.3 (From Real functions)

II player has wining strategy in this game.

4.4 Determinacy of games

Definice 4.7

We say that game $G(A, X)$ is determined, if I or II player has winning strategy.

Poznámka (Remark)
I has wining strategy if $\exists a_0\ \forall a_1\ \exists a_2\ \forall a_3 \ldots (a_0, a_1, \ldots) \in X$.

II has wining strategy if $\forall a_0\ \exists a_1\ \forall a_2\ \exists a_3 \ldots (a_0, a_1, \ldots) \notin X$.

Věta 4.4 (Gale–Steward, 1953)

$A \neq \emptyset$, $T \subset A^{<\omega}$, $T \neq \emptyset$ pruned tree, $X \subset [T] \subset A^\omega$ be closed in $[T]$. Then $G(T, X)$ is determined.

Důkaz
Assume II does not have winning strategy in $G(T, X)$. We say that $p = (a_0, a_1, \ldots, a_{2n-1}) \in T$ is non-loosing for I, if II player does not have winning strategy in game with this beginning. II does not have winning strategy $\implies$ $\emptyset$ is non-loosing. If p is non-loosing than there exist $a_{2n} \in A$ such that $(a_0, a_1, \ldots, a_{2n}) \in T$ and $\forall a = a_{2n+1} \in A$ such that $(a_0, a_1, \ldots, a_{2n+1}) \in T$ this position is non-loosing for I. The strategy for I player is choosing non-loosing positions. Assume that this strategy is not winning strategy. So, there exists $(a_0, a_1, \ldots) \in [T] \backslash X$ and for every $n \in \omega$: $(a_0, \ldots, a_{2n+1})$ are non-loosing position. But from fact that $[T] \backslash X$ is open, $\exists n \in \omega : \mathcal{N}(a_0, a_1, \ldots, a_{2n+1}) \cap [T] \subset [T] \backslash X \implies$ this is loosing position for I. □

Příklad („Closed“ is nessesary)
$A = \{0, 1\}$, $T = 2^{<\omega}$, $X = 2^\omega \backslash \{0, 0, \ldots\}$. This game is determined, but the previous proof cannot be used:

$$I : 0, 0, 0, 0, \ldots$$

$$II : 0, 0, 0, 0, \ldots$$

is always non-loosing, but $(0, 0, 0, \ldots) \notin X$.

Důsledek
$T \neq \emptyset$ pruned tree on A, $X \subset [T]$ is open in $[T]$. Then $G(T, X)$ is determined.

Důkaz
I player playes a_0 and we assume game where II player starts and we play with set $(\mathcal{N}(a_0) \cap A^{<\omega})$. Using the previous theorem we have that game is determined. □

Poznámka (Historical remark)
$G(T, X)$ be determined for:

- $X \in \Pi^0_1$: 1953 Gale-Stewnar.
- $X \in \Pi^0_2$: 1955 Woolfe.
- $X \in \Pi^0_3$: 1964 Davis.
- $X \in \Pi^0_4$: 1972 Paris.
- Martin: $T \neq \emptyset$ pruned tree on A, $X \subset [T]$, $X \in \Delta^1_1([T])$. Then $G(T, X)$ is determined.

Analytic determinacy is independent of ZFC.

4.5 Game $G^*(A)$

Definice 4.8 (Game $G^*(A)$)

$X \neq \emptyset$ Polish topological space, d is compatible compact metric on X, $V = (V_n)$ countable base of X, $V_n \neq \emptyset$, $A \subset X$.

$$I : (U_0^{(0)}, U_1^{(0)}), (U_0^{(1)}, U_1^{(1)}), \ldots$$
$$II : i_0, i_1, \ldots$$

$i_n \in \{0,1\}$. $U_i^{(n)} \in V$, $\operatorname{diam} U_i^{(n)} < 2^{-n}$. $\overline{U_0^{(n)}} \cap \overline{U_1^{(n)}} = \emptyset$, $\overline{U_0^{(n+1)} \cup U_1^{(n+1)}} \subset U_{i_n}^{(n)}$.

We define $x \in X$ by $\{x\} = \bigcap_{n\in\omega} \overline{U_{i_n}^{(n)}}$, then I wins $\Leftrightarrow x \in A$.

Poznámka

X Polish topological space, $|x| > \omega$. Then there exist $\tilde{X} \in \Pi_1^0(X)$ such that $|X \backslash \tilde{X}| = \omega$, $\tilde{X}$ is perfect. $(\operatorname{Ker}(X), I = \{M \subset X, |M| < \omega\})$.

Věta 4.5

$X \neq \emptyset$, perfect Polish topological space, $A \subset X$. Then 1. I player has winning strategy $\Leftrightarrow$ A contains homomorphic copy of 2^ω. 2. II player has winning strategy $\Leftrightarrow$ $|A| \leqslant \omega$.

Důkaz

„1. $\implies$ “: Winning strategy for I gives scheme $(U_s)_{s\in 2^\omega\backslash\{\emptyset\}}$ such that $U_s \neq \emptyset$ open $\overline{U_{s^\frown 0} \cup U_{s^\frown 1}} \subset U_s$, $\overline{U_{s^\frown 0}} \cap \overline{U_{s^\frown 1}} = \emptyset$, $\operatorname{diam} U_s < \sum 2^{|s|+1}$ $\forall y \in 2^\omega$: $\bigcap_{n\in\omega} \bigcup y|_m = \{x\} \subset A$. ($\impliedby$ $f(y) = x$ clearly homeomorphism.).

$$C = \bigcap_{n=1}^{\infty} \bigcup_{|s|=n} \bigcup s = \bigcup_{s\in 2^\omega} \bigcap_{k\in\omega\backslash\{0\}} \bigcup s/k \subset A.$$

„1. $\impliedby$ “: A continuous copy C of 2^ω (homeomorphic). I player choose $U_i^{(n+1)}$: $\operatorname{diam}(U_i^{(n+1)}) < 2^{-n-1}$, $U_i^{(n+1)} \cap C \neq \emptyset$, $\overline{U_0^{(n+1)}} \cap \overline{U_1^{(n+1)}} = \emptyset$, $\overline{U_0^{(n+1)}} \cup U_1^{(n+1)} \subset U_{i_n}^{(n)}$.

„2. $\impliedby$ “: $|A| \leqslant \omega$: $A = \{x_0, x_1, \ldots\}$. II in n-th move choose $i_n \in \{0,1\}$ to have $x_n \notin U_{i_n}^{(n)}$. $\implies$ $\bigcap_{n\in\omega} \bigcup_{i_n}^{(n)} \not\subset A$. $\square$

Důkaz (2. $\implies$)
Let σ be winning strategy for II player. Let $x \in A$. We say that position

$$p = ((U_0^{(0)}, U_1^{(0)}), i_0, \ldots, (U_0^{(n-1)}, U_1^{(n-1)}), i_{n-1})$$

is good for x if we get to this position following one winning strategy and $x \in U_{i_{n-1}}^{(n-1)}$. We assume empty sequence be good for x. Assume that every good position p for x has nontrivial extension, which is still good for x. Then there exists some play following σ, such that the result is $x \in A$, which is const. So $\forall x \in A$ $\exists$maximal good position p for x. $A_p = \Big\{ y \in U_{i_{(n-1)}}^{(n-1)} |$ for every extension following rules $(U_0^{(n)}, U_1^{(n)})$ if i is the move following σ, then $y \notin U_i^{(n)}$ $\Big\} = \{y | p \text{ maximal good for } y\}$.

$A \subset \bigcup_p A_p$, $|A_p| = 1$? By constr. $y_0 \neq y_1$, $y_0, y_1 \in A_p$. We choose $U_0^{(n)}, U_1^{(n)} : y_i \in U_i^{(n)}$ $\implies$ y_0 or $y_1 \in U_{i_n}^{(n)}$ and p is not maximal for this element. $\implies$ $|A| = \omega$. □

4.6 Game $G_u^*(F)$

Definice 4.9 (Game $G_u^*(F)$)

$X \neq \emptyset$, perfect, Polish topological space, $F \subset X \times \omega^\omega$.

$$I : y(0), (U_0^{(0)}, U_1^{(0)}), y(1), (U_0^{(1)}, U_1^{(1)}), \ldots$$

$$II : i_0, i_1, \ldots$$

$U_i^{(n)}, i_n$ are same like in the previous game, $U_i^{(n)} \in V$ (fix constant base), $y(u) \in \omega$, $x \in X$ like before, $y = (y(0), y(1), \ldots)$. I wins $\Leftrightarrow$ $(x, y) \in F$.

Věta 4.6

$X \neq \emptyset$, perfect, Polish topological space, $F \subset X \times \omega^\omega$, $A = \prod_X(F)$. Then 1. I has winning strategy in G_u^* $\implies$ A contains 2^ω. 2. II has winning strategy in $G_u^*(F)$ $\implies$ $|A| \leqslant \omega$.

Důkaz
„1.“ proof is same like proof of 1. in the previous theorem.

„2.“ Let II has winning strategy in $G_n^*(F)$. Assume $(x, y) \in F$. Similarly like before we find maximal good position for (x, y)

$$\Big(p = \Big(y(0), (U_0^{(0)}, U_1^{(0)}), i_0, y(1), (U_0^{(1)}, U_1^{(1)}), i_1, \ldots, y(n-1), (U_0^{(n-1)}, U_1^{(n-1)}), i_{n-1}\Big)\Big).$$

$p \in \sigma$, $x \in U_{i_n}^{(n-1)}$. Put $A_{p,\sigma} = \Big\{ z \in U_{i_{n-1}}^{(n-1)} |$ for any possible extension $(a, U_0^{(n)}, U_1^{(n)})$, if i is played winning σ, then $z \notin U_i^{(n)} \Big\}$. If $a = y_0(n)$ and p as above, there $x \in A_{p,a}$.

Similarly like before $|A'_{p,n}| \leqslant 1$, $A \subset \bigcup_{p \in \sigma, a \in \omega} A_{p,a}$ $\implies$ $|A| \leqslant \omega$. □

Definice 4.10

Let Γ be class of subsets of Polish topological spaces, $A \subset X$ (X Polish topological space) be Γ-hard $\equiv \forall B \in \Gamma(\omega^\omega)\ \exists f : \omega^\omega \to X : f^{-1}(A) = B$.

A is Γ-complete $\equiv A \in \Gamma \wedge A$ be Γ-hard.