

Poznámka

Credit for giving 'small lecture'. Oral exam.

1 Meromorphic functions

Definice 1.1

We say that a function f is holomorphic in a set $F \subset \mathbb{C}$ if there is an open $G \supseteq F$ such that f is holomorphic on G .

In particular, f is holomorphic at $z_0 \in \mathbb{C}$ if f is holomorphic in some neighbour ($= U(z_0) = U(z_0, \varepsilon)$) of z_0 .

Definice 1.2

Function f has at ∞ a removable singularity, if $f\left(\frac{1}{z}\right)$ has a removable singularity at 0. Similarly pole and essential singularity.

Function f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Let $G \subset \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any z_0 . Denote $\mathcal{H}(G) := \{f : G \rightarrow \mathbb{C} \mid f \text{ holomorphic}\}$.

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Například

From Liouville theorem $\mathcal{H}(\mathbb{S}) = \text{constant functions}$. So $\mathcal{H}(G)$ is interesting only for $G \subsetneq \mathbb{S}$,
└ so WLOG $G \subset \mathbb{C}$.

Definice 1.3 (Meromorphic function)

Let $G \subset \mathbb{S}$ be open. Then a function f on G is called meromorphic if at any $z_0 \in G$ the function f is either holomorphic at z_0 or has a pole at z_0 .

Denote $\mathcal{M}(G)$ the set of meromorphic functions on G .

Důsledek

- $\mathcal{H}(G) \subset \mathcal{M}(G)$.
- Denote $P_f := \{z_0 \in G \mid f \text{ has a pole at } z_0\}$. Then P_f has no limit points in G .
- If $f = \infty$ on P_f , then $f : G \rightarrow \mathbb{S}$ is continuous. (We always assume, that $f \in \mathcal{H}(G)$ has this property.)

Například

$$\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), \quad e^{\frac{1}{z}} \notin \mathcal{M}(\mathbb{C}), \quad \Gamma \in \mathcal{M}(\mathbb{C}), \quad \zeta \in \mathcal{M}(\mathbb{C}).$$

$\mathcal{M}(\mathbb{S}) = \text{rational functions}$. (One inclusion is clear, second: Let $f \in \mathcal{M}(\mathbb{S})$, then because \mathbb{S} is compact it holds that P_f is finite (has no limit point), $P_f \cap \mathbb{C} = \{z_1, \dots, z_n\}$, so from theorem from last semester there exists $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^n p_j \left(\frac{1}{z-z_j} \right)$ for some polynomials p_j . f has removable singularity or pole at infinity and p_j and $\frac{1}{z-z_j}$ have removable singularity there, so $h(z)$ is polynomial, otherwise $h(z)$ has infinity Taylor polynom and $h\left(\frac{1}{z}\right)$ has essential singularity at 0.)

So $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subset \mathbb{C}$.

If $G \subset \mathbb{C}$ is domain, $f, g \in \mathcal{H}(G)$ and $g \not\equiv 0$, then $f/g \in \mathcal{M}(G)$. The inverse is also true (we will prove it) (but not for $G = \mathbb{S}$).

Lemma 1.1

Let $G \subset \mathbb{C}$ be open. Then there are compacts K_n , $n \in \mathbb{N}$, in G such that $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \text{int}(K_{n+1})$ and for any compact K in G , $\exists n \in \mathbb{N} : K \subset K_n$.

Důkaz

Set $K_n := \{z \in G \mid \text{dist}(z, \mathbb{C} \setminus G) \geq \frac{1}{n}\} \cap \overline{U(0, n)}$. □

Tvrzení 1.2

Let $G \subset \mathbb{S}$ be open and $M \subset G$ has no limit point in G . Then

- $G \setminus M$ is open;
- if K is a compact in G , then $K \cap M$ is finite. In particular for $G = \mathbb{S}$ we have M is finite;
- M is at most countable. If M is infinite, then $\emptyset \neq M' \subset \partial G$;
- if $G \subset \mathbb{C}$ is domain (connected), then $G \setminus M$ is domain.

Věta 1.3 (Uniqueness of meromorphic functions)

Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ and $f \not\equiv 0$. Then $N_f := \{z \in G \mid f(z) = 0\}$ has no limit points in G .

Důkaz

We know this holds for holomorphic functions. Set $G_0 := G \setminus P_f$. Then $G_0 \subset \mathbb{C}$ is also domain and $f \in \mathcal{H}(G)$ and $f \not\equiv 0$ on G_0 . Then $N_f \subset G_0$ has no limit points in G_0 , nor in P_f . \square

Věta 1.4 (Residue theorem)

Let $G \subset \mathbb{C}$ be open, φ be a closed curve (or cycle) in G and

$$\text{int } \varphi := \{z_0 \in \mathbb{C} \setminus \langle \varphi \rangle \mid \text{ind}_\varphi z_0 \neq 0\} \subset G.$$

Let $M \subset G \setminus \langle \varphi \rangle$ be finite and $f \in \mathcal{H}(G \setminus M)$. Then $\int_\varphi f = 2\pi i \cdot \sum_{s \in M} \text{ind}_\varphi s \cdot \text{res}_s f$.

Poznámka

This holds true even if instead of finiteness of M , we assume only that $M \subset G \setminus \langle \varphi \rangle$ has no limit points in G . Indeed, we have $M_0 = M \cap \text{int } \varphi$ is finite, because $\langle \varphi \rangle \cup \text{int } \varphi$ is compact and $G_0 := G \setminus (M \setminus M_0)$ is open and f is holomorphic on $G_0 \setminus M_0$ and by R. theorem for G_0 and M_0 we get $\int_\varphi f = 2\pi i \sum_{s \in M_0} \text{res}_s f \cdot \text{ind}_\varphi s$.

1.1 Logarithmic integrals

Definice 1.4 (Logarithmic integral)

Let $\varphi : [a, b] \rightarrow \mathbb{C}$ be a (regular) curve and let f be a non-zero holomorphic function on $\langle \varphi \rangle$. Then we define logarithmic integral as

$$\begin{aligned} I &:= \frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \frac{1}{2\pi i} \int_a^b \frac{f'(\varphi(t))\varphi'(t)}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_{f \circ \varphi} \frac{dz}{z} = \\ &= \frac{1}{2\pi i} (\Phi(b) - \Phi(a)), \end{aligned}$$

where Φ is a branch (jednoznačná větev) of logarithm of $f \circ \varphi$. If φ is, in addition, closed, then $I = \text{ind}_{f \circ \varphi} 0 = \frac{1}{2\pi} (\Theta(b) - \Theta(a)) \in \mathbb{Z}$, where Θ is a branch of argument of $f \circ \varphi$.

($\frac{f'}{f}$ is called logarithmic derivative of f , because $'(\log f)' = \frac{f'}{f}$.)

Věta 1.5 (Argument principle)

Let $G \subseteq \mathbb{C}$ be a domain, φ be a closed curve in G and $f \in \mathcal{M}(G)$. Let $\text{int } \varphi \subset G$ and $\langle \varphi \rangle \cap N_f = \emptyset$, $\langle \varphi \rangle \cap P_f = \emptyset$. Then

$$\frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \sum_{s \in \text{int } \varphi, f(s)=0} n_f(s) \cdot \text{ind}_\varphi s - \sum_{s \in \text{int } \varphi, f(s)=\infty} p_f(s) \cdot \text{ind}_\varphi s,$$

where $n_f(s)$ is multiplicity of the zero point s of f and $p_f(s)$ is multiplicity of the pole s of f .

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Důkaz

By Residua theorem, we have

$$\frac{1}{2\pi i} \int_{\varphi} \frac{f'}{f} = \sum_{s \in \text{int } \varphi, s \in N_f \cup P_f} \text{res}_s \left(\frac{f'}{f} \right) \cdot \text{ind}_{\varphi} s.$$

If $s \in N_f$ then on $P(s)$:

$$\frac{f'(z)}{f(z)} = \frac{p \cdot c_p(z-s)^{p-1} + \dots}{c_p(z-s)^p + \dots} = \frac{p}{z-s} \cdot \frac{1 + \dots}{1 + \dots} \implies \text{res}_s \left(\frac{f'}{f} \right) = p = n_f(s).$$

If $s \in P_f$ then on $P(s)$

$$\frac{f'(z)}{f(z)} = \frac{p \cdot c_p(z-s)^{p-1} + \dots}{c_p(z-s)^p + \dots} = \frac{p}{z-s} \cdot \frac{1 + \dots}{1 + \dots} \implies \text{res}_s \left(\frac{f'}{f} \right) = p = -p_f(s).$$

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□

Definice 1.5

$$\Sigma(f, \varphi) := \sum_{s \in \text{int } \varphi, f(s)=0} n_f(s) \cdot \text{ind}_{\varphi} s - \sum_{s \in \text{int } \varphi, f(s)=\infty} p_f(s) \cdot \text{ind}_{\varphi} s.$$

Lemma 1.6

Let $\varphi_1, \varphi_2 : [a, b] \rightarrow \mathbb{C}$ be closed curve and $s \in \mathbb{C} \setminus (\langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle)$. Assume, for $t \in [a, b]$, $|\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - s|$. Then $\text{ind}_{\varphi_1} s = \text{ind}_{\varphi_2} s$.

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Důkaz

For $t \in [a, b]$, we have $|(\varphi_1(t) - s) - (\varphi_2(t) - s)| < |\varphi_1(t) - s|$. Divide by $|\varphi_1(t) - s|$:

$$|1 - \psi(t)| < 1, \quad \psi(t) := \frac{\varphi_2(t) - s}{\varphi_1(t) - s}.$$

Then ψ is a closed curve, $\langle \psi \rangle \subset U(1, 1)$, and so

$$\begin{aligned} 0 = \text{ind}_{\psi} 0 &= \frac{1}{2\pi i} \int_a^b \frac{\psi'}{\psi} = \frac{1}{2\pi i} \int_a^b \frac{\frac{\varphi_2'(\varphi_1-s) - \varphi_1'(\varphi_2-s)}{(\varphi_1-s)^2}}{\frac{\varphi_2-s}{\varphi_1-s}} = \frac{1}{2\pi i} \int_a^b \frac{\varphi_2'}{\varphi_2-s} - \frac{1}{2\pi i} \int_a^b \frac{\varphi_1'}{\varphi_1-s} = \\ &= \text{ind}_{\varphi_2} s - \text{ind}_{\varphi_1} s. \end{aligned}$$

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□

Věta 1.7 (Rouché)

Let $G \subset \mathbb{C}$ be a domain, $f_1, f_2 \in \mathcal{M}(G)$ and φ be closed curve in G such that $\text{int } \varphi \subset G$. Assume $\forall z \in \langle \varphi \rangle$:

$$|f_1(z) - f_2(z)| < |f_1(z)| < +\infty$$

Then $\Sigma(f_1, \varphi) = \Sigma(f_2, \varphi)$.

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Důkaz

Set $\varphi_j = f_j \circ \varphi$. Then

$$\text{ind}_{\varphi_j} 0 = \frac{1}{2\pi i} \int_{\varphi} \frac{f_j'}{f_j} = \Sigma(f_j, \varphi).$$

By previous lemma we have for $s = 0$: $\text{ind}_{\varphi_1} 0 = \text{ind}_{\varphi_2} 0$. □

Důsledek

Let f_1, f_2 be holomorphic functions on $\overline{U(z_0, r)}$ and $\forall z \in \partial U(z_0, r) : |f_1(z) - f_2(z)| < |f_1(z)|$. Then $\Sigma_1 = \Sigma_2$, where $\Sigma_j := \sum_{s \in U(z_0, r), f(s)=0} n_{f_j}(s)$.

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Důkaz

Apply Rouché's theorem to $\varphi(t) := z_0 + r \cdot e^{it}$, $t \in [0, 2\pi]$. □

Příklad

$f_2 = p$, $f_1(z) = a_0 z^n$ and big enough $U(0, r)$.

Definice 1.6 (Notation)

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. We say that $f(z_0) = w_0 \in \mathbb{C}$ p times for $p \in \mathbb{N}$ if z_0 is a zero point of $f - w_0$ of order p .

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Poznámka

Following statements are equivalent to each other:

- $f(z_0) = w_0$ p times;
- $f(z_0) = w_0$, $f'(z_0) = 0 = \dots = f^{(p-1)}(z_0)$, $f^{(p)}(z_0) \neq 0$;
- $f(z) = w_0 + \sum_{k=p}^{+\infty} c_k (z - z_0)^k$ on some neighbourhood of z_0 and $c_p \neq 0$.

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We say that $f(z_0) = \infty$ p times if z_0 is a zero point of $\frac{1}{f}$ of order p . (It's the same as z_0 is pole of f of order p .) And we say that $f(\infty) = w_0 \in \mathbb{S}$ p times if $f(1/z)$ attains w_0 p times at 0.

Věta 1.8 (On a multiple value)

Let $z_0, w_0 \in \mathbb{S}$, f be a holomorphic function on a $P(z_0)$ and $f(z_0) = w_0$ p times for some $p \in \mathbb{N}$. Let $\delta_0 > 0$. Then there are $\varepsilon > 0$ and $\delta \in (0, \delta_0)$ such that, for any $w \in P(w_0, \varepsilon)$ there are just p different points z_1, \dots, z_p in $P(z_0, \delta)$ with $f(z_j) = w$. In addition, $f(z_j) = w$ once.

┌ *Důkaz*

WLOG, assume $z_0 = 0 = w_0$. Then $z_0 = 0$ is a zero point of f of order p . Choose $\delta \in (0, \delta_0)$ such that $f \neq 0$ and $f' \neq 0$ on $P(0, 2\delta)$. Set $\varepsilon := \min_{|z|=\delta} |f(z)| > 0$.

Let $w \in P(0, \varepsilon)$. Use Rouché's theorem for $f_1 := f$, $f_2 := f - w$ and $\varphi := \delta e^{it}$, $t \in [0, 2\pi]$. Of course, $|f_1 - f_2| = |w| < \varepsilon < |f_1|$ on $\langle \varphi \rangle$.

Since in $U(0, \delta)$ the function $f = f_1$ has the only zero point of order p at origin, $f - w = f_2$ has just p simple zero points in $P(0, \delta)$. □

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Důsledek

Let $G \subset \mathbb{S}$ be a domain, $f \in \mathcal{M}(G)$ and f be not constant on G . Then $f : G \rightarrow \mathbb{S}$ is an open map (for any open $\Omega \subset G$, $f(\Omega)$ is open).

┌ *Důkaz*

Let $\Omega \subset G$ be open and $w_0 \in f(\Omega)$. Then there is a $z_0 \in \Omega$ and $p \in \mathbb{N}$ such that $f(z_0) = w_0$ p times. Choose $\delta_0 > 0$ such that $U(z_0, \delta_0) \subset \Omega$. By the previous theorem, there is $\varepsilon > 0$, $\delta \in (0, \delta_0)$ such that $P(w_0, \varepsilon) \subset f(P(z_0, \delta))$, so $U(w_0, \varepsilon) \subset f(U(z_0, \delta)) \subset f(\Omega)$. □

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┌ *Poznámka*

This is true for $\mathcal{H}(G)$ too.

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Důsledek

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. Then $f'(z_0) \neq 0$ if and only if there is $U(z_0)$ such that $f|_{U(z_0)}$ is one-to-one.

┌ *Důkaz*

„ \implies “: Let $f'(z_0) \neq 0$. Then $f(z_0) = w_0$ once, so we choose $\delta_0 > 0$ such that $f \neq w_0$ on a $P(z_0, \delta_0)$. By the previous theorem choose $\varepsilon > 0$, $\delta \in (0, \delta_0)$. Moreover, due to the continuity of f at z_0 choose $\delta_1 \in (0, \delta)$ such that $f(U(z_0, \delta_1)) \subset U(w_0, \varepsilon)$. Then $f|_{U(z_0, \delta_1)}$ is one-to-one.

„ \impliedby “: Let $f'(z_0) = 0$ and let f be not constant on any neighbourhood of z_0 . Then $f(z_0) = w_0$ p times ($p \in \mathbb{N} \setminus \{1\}$). By the previous theorem f is not one-to-one on any neighbourhood of z_0 . □

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Věta 1.9 (On holomorphic inverse)

Let $G \subset \mathbb{C}$ be open and $f : G \rightarrow \mathbb{C}$ be a one-to-one holomorphic^a function, then $f' \neq 0$ on G , $\Omega := f(G)$ is open and $f_{-1} : \Omega \xrightarrow{\text{onto}} G$ is holomorphic.

In addition, $(f_{-1})' = \frac{1}{f' \circ f_{-1}}$ on Ω .

Důkaz

WLOG, $G \subset \mathbb{C}$ is a domain. By the first „důsledek“ of the previous theorem f is an open map, so $\Omega := f(G)$ is open and $f_{-1} : \Omega \rightarrow G$ is continuous. Let $z_0 \in G$ and $w_0 = f(z_0)$. By second „důsledek“ we have $f'(z_0) \neq 0$, and

$$\frac{1}{f'(z_0)} = \lim_{z \rightarrow z_0} \frac{z - z_0}{f(z) - f(z_0)} \stackrel{*}{=} \lim_{w \rightarrow w_0} \frac{f_{-1}(w) - f_{-1}(w_0)}{w - w_0} = f'_{-1}(w_0).$$

The equality $*$ follows from theorem on limits of composite functions because f_{-1} is continuous and $f_{-1}(w) \neq f_{-1}(w_0)$ for $w \neq w_0$. \square

^aOne-to-one holomorphic function is sometimes called conformal.

Věta 1.10 (Hurwitz)

Let $G \subset \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(G)$, $f_n \xrightarrow{\text{loc.}} f$ on G and $f \not\equiv 0$. Let $z_0 \in G$ be a zero point of f . Then $\exists \{z_n\}_{n=1}^{\infty} \subset G$ and a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $z_n \rightarrow z_0$ and $f_{k_n}(z_n) = 0$.

Poznámka

Not true in \mathbb{R} ! The assumption $f \not\equiv 0$ is important! ($f_n(z) := z/n$)

Důsledek

Let $G \subset \mathbb{C}$ be a domain, f_n be one-to-one holomorphic functions on G and $f_n \xrightarrow{\text{loc.}} f$ on G . Then f is either one-to-one and holomorphic, or constant.

Důkaz (Hurwitz theorem)

Choose $\delta > 0$ such that $U(z_0, \delta) \subset G$ and $f \neq 0$ on $P(z_0, \delta)$. For $n \in \mathbb{N}$ put $\varrho_n := \frac{\delta}{n+1}$ and $\varphi_n(t) := z_0 + \varrho_n e^{it}$, $t \in [0, 2\pi]$. Of course, $\tau_n := \min_{\langle \varphi_n \rangle} |f| > 0$. For a given n , there is (from uniformly convergence) $k_n \in \mathbb{N}$ such that $\forall z \in \langle \varphi_n \rangle : |f_{k_n}(z) - f(z)| < \tau_n \leq |f|$.

By Rouché's theorem there is $z_n \in U(z_0, \varrho_n)$ such that $f_{k_n}(z_n) = 0$. Of course, we can choose $\{k_n\}$ to be increasing. \square

Důkaz (Corollary)

Assume that there is $w_0 \in \mathbb{C}$ such that $f \neq w_0$ but, for different $z', z'' \in G$ we have $f(z') = w_0 = f(z'')$. WLOG $w_0 = 0$. Choose $\delta > 0$ such that $U(z', \delta) \cap U(z'', \delta) = \emptyset$. By Hurwitz, there are $\{z'_n\} \subset U(z', \delta)$ and $\{f_{k'_n}\}$ of $\{f_n\}$ such that $z'_n \rightarrow z'$ and $f_{k'_n}(z'_n) = 0$. By Hurwitz, there are also $\{z''_n\} \subset U(z'', \delta)$ and $\{f_{k''_n}\} \subset \{f_{k'_n}\}$ such that $z''_n \rightarrow z''$ and $f_{k''_n}(z''_n) = 0$.

Every $f_{k''_n}$ has at least two different zero points which is contradiction. \square

Věta 1.11 (Mittag-Leffler)

Let $\{s_j\} \subset \mathbb{C}$ be one-to-one, $s_j \rightarrow \infty$ and $s_0 := 0 < |s_1| \leq |s_2| \leq |s_3| \leq \dots \leq |s_j| \leq \dots$

Let $P_0, P_1, \dots, P_j, \dots$ be polynomials such that $P_j(0) = 0$. Then the function

$$f(z) := P_0\left(\frac{1}{z}\right) + \sum_{j=1}^{\infty} \left(P_j\left(\frac{1}{z-s_j}\right) - Q_j(z) \right)$$

for some polynomials Q_j satisfies:

1. series in definition converges locally uniformly on \mathbb{C} , i. e., on any compact $K \subset \mathbb{C}$, the series converges uniformly if we omit finitely many terms which have poles.
2. $f \in \mathcal{M}(\mathbb{C})$ and f has poles just at $s_0, s_1, \dots, s_j, \dots$, while at s_j the function f has its principal part equal to $P_j\left(\frac{1}{z-s_j}\right)$.
3. If $g \in \mathcal{M}(\mathbb{C})$ satisfies previous property, then there is $h \in \mathcal{H}(\mathbb{C})$ such that $g = f + h$ on G .

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Důkaz

Let $k \in \mathbb{N}$. Then $H_k(z) := P_k\left(\frac{1}{z-s_k}\right) \in \mathcal{H}(U(0, |s_k|))$, $H_k(z) = \sum_{n=0}^{\infty} c_n^k z^n$ for $|z| < |s_k|$.

There is $n_k \in \mathbb{N}$ such that $Q_k(z) = \sum_{n=1}^{n_k} c_n^k z^n$ satisfies $|H_k(z) - Q_k(z)| < \frac{1}{2^k}$, $|z| \leq \frac{|s_k|}{2}$ (*).

Let $K \subset \mathbb{C}$ be a compact. Choose $k_0 \in \mathbb{N}$ such that $K \subset \overline{U(0, |s_{k_0}|/2)}$. If $k > k_0$, (*) holds on K which implies 1. obviously, 2. is valid.

3. follow from the fact that $g - f \in \mathcal{M}(\mathbb{C})$ has all isolated singularities removable. \square

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2 Zero points of holomorphic functions

Tvrzení 2.1

Let f be non-zero holomorphic function on a simply connected domain (G is domain, and $\mathbb{S} \setminus G$ is connected) $G \subset \mathbb{C}$. Then there is $L \in \mathcal{H}(G)$ such that $f = e^L$ on G .

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Důkaz

1) Let $L \in \mathcal{H}(G)$ and assume that $f = e^L$ on G . Then $f' = L' \cdot e^L$ and $f'/f = L'$.

2) Since G is a simply connected domain and $f'/f \in \mathcal{H}(G)$, by Cauchy theorem, there is $L_0 \in \mathcal{H}(G)$ such that $L'_0 = f'/f$.

3) On G we have $(f \cdot e^{-L_0})' = e^{-L_0} \cdot (f' - L'_0 \cdot f) = 0$ on G , hence $f \cdot e^{-L_0} = e^c$ is constant, i. e. $c \in \mathbb{C}$. Put $L := L_0 + c$. \square

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Poznámka

Polynomial $f(z) = \prod_{j=1}^n (z - z_j)$ has zero points just at z_1, \dots, z_n and their multiplicity corresponds to their occurrence.

Let $g \in \mathcal{H}(\mathbb{C})$ have the same zero points including multiplicity as f . Then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on \mathbb{C} . (Proof: use previous tvrzení for g/f .)

Poznámka (Notation)

Let $\{a_j\} \subset \mathbb{C}$. Then we define

$$\prod_{j=1}^{\infty} a_j := \lim_{n \rightarrow \infty} \prod_{j=1}^n a_j,$$

if the limit on the right-hand side exists.

Tvrzení 2.2

Let $0 \neq z_j \rightarrow \infty$ and $k \in \mathbb{N}_0$ (multiplicity of 0 as zero point). Then consider

$$f(z) := z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right).$$

It sometimes converges and then f has zero points in z_j with right multiplicities.

Věta 2.3 (On infinite product)

Let M be a set (in \mathbb{C}), $u_j : M \rightarrow \mathbb{C}$ be bounded and $\sum_{j=1}^{\infty} |u_j|$ converges uniformly on M . Then $p_n := \prod_{j=1}^n (1 + u_j)$ converge uniformly to a function $f : M \rightarrow \mathbb{C}$, and it holds that $f = \prod_{j=1}^{\infty} (1 + u_{n(j)})$ on M , where n is bijection onto \mathbb{N} .

If $z_0 \in M$, then $f(z_0) = 0$ if and only if $u_{j_0}(z_0) = -1$ for some $j_0 \in \mathbb{N}$.

Důkaz

Denote $p_n^* := \prod_{j=1}^n (1 + |u_j|)$. Then $p_n^* \leq \exp\left(\sum_{j=1}^n |u_j|\right)$ and $|p_n - 1| \leq p_n^* - 1$ (from $1 + x \leq e^x$ and the second inequality by induction on n : $n = 1$ yes, $p_{n+1} - 1 = p_n(1 + u_{n+1}) - 1 = (p_n - 1) \cdot (1 + u_{n+1}) + u_{n+1}$ so $|p_{n+1} - 1| \leq (p_n^* - 1) \cdot (1 + |u_{n+1}|) + |u_{n+1}| = p_{n+1}^* - 1$).

$\sum_{j=1}^{\infty} |u_j|$ is bounded on M , because there is $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^{\infty} |u_j| < 1$. By inequalities there is $C \in (0, +\infty)$ such that $|p_n| \leq C \forall n \in \mathbb{N}$.

Let $0 < \varepsilon < \frac{1}{2}$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} |u_n| < \varepsilon$ on M . Let $\{n_1, n_2, \dots\}$ be a permutation of \mathbb{N} and $q_m := \prod_{j=1}^m (1 + u_{n_j})$, $m \in \mathbb{N}$. Let $n \geq n_0$ and $m \in \mathbb{N}$ be such that $\{n_1, \dots, n_m\} \supseteq [n]$. Then

$$\begin{aligned} |q_m - p_n| &= |p_n| \cdot \left(\prod_{n_j > n, j \in [m]} (1 + u_{n_j}) - 1 \right) \leq |p_n| \left(\prod_{\dots} (1 + |u_{n_j}|) - 1 \right) \leq \\ &\leq |p_n| \cdot \left(e^{\sum_{\dots} |u_{n_j}|} - 1 \right) \leq |p_n| \cdot (e^{\varepsilon} - 1) \leq |p_n| \cdot 2\varepsilon \leq 2C\varepsilon. \end{aligned}$$

If $n_j = j \forall j \in \mathbb{N}$, then $q_m = p_m$ and we get $\forall m > n : |q_m - p_n| < 2C\varepsilon$, so $p_n \rightrightarrows f$ on M . Moreover we have, for $n \geq n_0$, $|p_n - p_{n_0}| \leq 2\varepsilon |p_{n_0}|$, so $|p_n| \geq |p_{n_0}| - |p_n - p_{n_0}| \geq (1 - 2\varepsilon) |p_{n_0}|$. For $n \rightarrow \infty$: $|f| \geq (1 - 2\varepsilon) |p_{n_0}|$, hence $f(z_0) = 0 \Leftrightarrow p_{n_0}(z_0) = 0$.

If n_j is any, then $q_m \rightrightarrows f$ on M . □

Důsledek

Let $G \subset \mathbb{C}$ be open, $f_n \in \mathcal{H}(G)$ and $f_n \not\equiv 0$ on any component of G . We assume $\sum_{n=1}^{\infty} |1 - f_n|$ converges locally uniformly on G . Then $f = \prod_{n=1}^{\infty} f_n$ converges locally uniformly on G , $f \in \mathcal{H}(G)$ and the resulting infinite product f does not depend on the order of functions f_n . Moreover, we have

$$n_f(s) = \sum_{k=1}^{\infty} n_{f_k}(s), \quad s \in G$$

where $n_f(s)$ is multiplicity of a zero point s of f . Here we put $n_f(s) = 0$ if $f(s) \neq 0$.

Poznámka

Moreover the ? in previous sum contains only finitely many non-zero terms for any $s \in G$.

Důkaz

Sufficient to prove previous equality. Let $s \in G$. There is a neighbourhood V of s such that $f_n \rightrightarrows 1$ on V . Choose $n_0 \in \mathbb{N}$ such that $f_n \neq 0$ on V for $n > n_0$. By previous theorem, we get $\prod_{n=n_0+1}^{\infty} f_n \neq 0$ on V . Since $f = (\prod_{n=1}^{n_0} f_n) \cdot (\prod_{n=n_0+1}^{\infty} f_n)$ we get $n_f(s) = \sum_{k=1}^{n_0} n_{f_k}(s) = \sum_{k=1}^{\infty} n_{f_k}(s)$. □

Příklad (Homework)

Under the assumption of previous corollary prove that

$$\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f'_n}{f_n} \text{ on } G \setminus N_f.$$

Například (Euler formula)

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 2.4 (Weierstrass's factor)

Let $E_0(z) := (1-z)$ and $E_m(z) := (1-z) \cdot e^{z+\dots+\frac{z^m}{m}}$, $z \in \mathbb{C}$, $m \in \mathbb{N}$. Then $|1-E_m(z)| \leq |z|^{m+1}$, $|z| \leq 1$.

┌

Důkaz

$$E'_m(z) = e^{z+\dots+\frac{z^m}{m}} \cdot (-1 + (1-z) \cdot (1 + \dots + z^m)) = -z^m \cdot e^{z+\dots+\frac{z^m}{m}} = -z^m \cdot \sum_{k=0}^{\infty} b_k z^k,$$

where $b_0 = 1$, $b_k \geq 0$, $k \in \mathbb{N}$. Hence

$$E_m(0) - E_m(z) = 1 - E_m(z) = - \int_{[0,z]} E'_m(w) dw = + \sum_{k=0}^{\infty} c_k z^{k+m+1}$$

with $c_k = \frac{b_k}{m+k+1} \geq 0$.

By this, if $|z| \leq 1$, $z \neq 0$, then $\left| \frac{1-E_m(z)}{z^m} \right| \leq \sum_{k=0}^{\infty} c_k = 1 - E_m(1) = 1$. □

└

Věta 2.5 (Weierstrass factorization in \mathbb{C})

Let $k \in \mathbb{N}_0$ and $0 \neq z_i \rightarrow \infty$. Then there is $\{m_j\} \subset \mathbb{N}_0$ such that

$$f(z) = z^k \cdot \prod_{j=1}^{\infty} E_{m_j} \left(\frac{z}{z_j} \right)$$

converges locally uniformly on \mathbb{C} , $f \in \mathcal{H}(\mathbb{C})$ and f has at 0 zero point of multiplicity K and 'non-zero' zero points just at $z_1, z_2, \dots, z_j, \dots$, and their multiplicity corresponds to their occurrence in $\{z_j\}$. We can always take $m_j := j - 1$, $j \in \mathbb{N}$.

If $g \in \mathcal{H}(\mathbb{C})$ has the same zero points as f including multiplicities, then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on \mathbb{C} .

┌ *Důkaz*

By the previous corollary, we know the product converges locally uniformly in \mathbb{C} if

$$\sum_{j=1}^{\infty} \left| 1 - E_{m_j} \left(\frac{z}{z_j} \right) \right|$$

converges locally uniformly on \mathbb{C} . By lemma, this is true if $\sum_{j=1}^{\infty} \left| \frac{z}{z_j} \right|^{m_j+1}$ converges locally uniformly on \mathbb{C} .

Let $r > 0$ and $|z| \leq r$. Choose $j_0 \in \mathbb{N}$ such that $\frac{r}{|z_j|} < \frac{1}{2}$ for $j \geq j_0$. If $m_j := j - 1$, then $\left| \frac{z}{z_j} \right|^j \leq \frac{1}{2^j}$, $j \geq j_0$ and $|z| \leq r$. So, for $m_j := j - 1$, sum converges uniformly on $|z| \leq r$. \square

Poznámka

If $\sum_{j=1}^{\infty} \frac{1}{|z_j|} < +\infty$, take $m_j = 0$. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|^2} < +\infty$, take $m_j = 1$. Etc.

Věta 2.6 (Weierstrass factorization in a general open set)

Let $G \subsetneq \mathbb{S}$ be open, $N \subset G$ have no limit points in G and $n : N \rightarrow \mathbb{N}$. Then there is $f \in \mathcal{H}(G)$ such that $N_f = N$ and $n_f(s) = n(s)$, $s \in N_f$.

┌ *Důkaz*

WLOG $\infty \in G \setminus N$. Then $K := \mathbb{S} \setminus G = \mathbb{C} \setminus G$ is compact in \mathbb{C} . For a finite N it is obvious. Assume that N is (infinite) countable. We put points of N into the sequence s_1, s_2, \dots, s_n such that any $s \in N$ occurs in $\{s_n\}$ just $n(s)$ times. For any n , take $t_n \in K$ such that $|s_n - t_n| = \text{dist}(s_n, K)$, $n \in \mathbb{N}$.

Then „ $|s_n - t_n| \rightarrow 0$ “: Let $\varepsilon > 0$ and $\{n_k\} \subset \mathbb{N}$ such that $|s_{n_k} - t_{n_k}| \geq \varepsilon$, i. e., $\text{dist}(s_{n_k}, K) \geq \varepsilon$. If s_{∞} is a limit point of s_{n_k} , then $\text{dist}(s_{\infty}, K) \geq \varepsilon$. Hence $s_{\infty} \in G$, a contradiction.

Put $f(z) := \prod_{n=1}^{\infty} E_n \left(\frac{s_n - t_n}{z - t_n} \right)$, $z \in G$. The infinite product converges locally uniformly on G . In fact, let L be a compact in G . Put $r_n := 2 \cdot |s_n - t_n|$. Since $\text{dist}(L, K) > 0$, there is $n_0 \in \mathbb{N}$ such that $|z - t_n| > r_n$, $\forall z \in L$, $\forall n \geq n_0$. So

$$\left| \frac{s_n - t_n}{z - t_n} \right| < \frac{1}{2} \quad \forall z \in L \quad \forall n \geq n_0.$$

By lemma on Weierstrass factors, we get

$$\left| 1 - E_n \left(\frac{s_n - t_n}{z - t_n} \right) \right| < \frac{1}{2^n} \quad \forall z \in L \quad \forall n \geq n_0.$$

┌ Now use theorem on infinite product. \square

Lemma 2.7

If $G \subseteq \mathbb{C}$ is open and $f \in \mathcal{M}(G)$, then there are $g, h \in \mathcal{H}(G)$ such that $f = \frac{g}{h}$ on G .

┌

Důkaz

Let P_f be the set of poles of f . By Weierstrass factorization, we construct $h \in \mathcal{H}(G)$ such that $N_h = P_f$ and $n_h = p_f$ on P_f . Put $g := f \cdot h$. Then $g \in \mathcal{H}(G)$ because at the points of P_f g has a removable singularities. \square

└

3 The space $\mathcal{H}(G)$

Poznámka (Arzela–Ascoli theorem)

Let $\mathcal{F} \subset \mathcal{C}(K)$ and let the functions of \mathcal{F} be equibounded (i.e. $\exists M \in (0, +\infty) \forall f \in \mathcal{F} : |f| \leq M$ on K) and equicontinuous (i.e. $\forall \varepsilon > 0 \exists \delta > 0 \forall f \in \mathcal{F} \forall x, y \in K : \varrho(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$, where ϱ is metric on K). Then every $\{f_n\} \subset \mathcal{F}$ has $\{f_{n_k}\}$ which is uniformly convergent on K .

3.1 The space $\mathcal{C}(G)$

Definice 3.1

Let $G \subseteq \mathbb{C}$, then $\mathcal{C}(G) := \{f : G \rightarrow \mathbb{C} | f \text{ continuous}\}$.

Tvrzení 3.1

For $f_n, f \in \mathcal{C}(G)$ and K_m compact in G such that $\bigcup_{m=1}^{\infty} K_m = G$ and $\forall m \in \mathbb{N} : K_m \subseteq \text{int } K_{m+1}$, TSAE:

- $f_n \xrightarrow{\text{loc.}} f$ on G ;
- for any compact K in G , $\|f_n - f\|_K \rightarrow 0$, where $\|f\|_K := \sup_K |f|$ is a seminorm on $\mathcal{C}(G)$;
- $\forall m \in \mathbb{N} : \|f_n - f\|_{K_m} \rightarrow 0$ for $n \rightarrow \infty$;
- $\varrho(f_n, f) \rightarrow 0$, where $\varrho(f_n, f) := \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|f_n - f\|_{K_m}}{1 + \|f_n - f\|_{K_m}}$.

┌

Důkaz

„1 \Leftrightarrow 2 \implies 3“ is obvious. „2 \Leftarrow 3“: Let K be a compact in G . Then $K \subset K_{m_0}$ for some $m_0 \in \mathbb{N}$. Then $\|f_n - f\|_K \leq \|f_n - f\|_{K_{m_0}}$. „3 \Leftrightarrow 4“ homework. \square

└

Poznámka

$(\mathcal{C}(G), \varrho)$, where ϱ is defined in previous tvrzeni, is complete metric space and $\mathcal{H}(G)$ is closed subspace.

ϱ is not canonical, it depends on the choice of $\{K_m\}$.

The convergence / the topology on $\mathcal{C}(G)$ is given by the system of seminorms $\|\cdot\|_K$ for any compact K in G .

Věta 3.2 (Moore–Osgood, Montöl)

Let $G \subset \mathbb{C}$ be open and let $\{f_n\} \subset \mathcal{H}(G)$ be locally equibounded (i.e. on every compact K in G $\{f_n\}$ is equibounded). Then there is $\{f_{n_k}\}$ which converges locally uniformly on G .

┌

Důkaz

First step: Let $\overline{U(z_0, 2r)} \subset G$ and $\varphi(t) := z_0 + 2re^{it}$, $t \in [0, 2\pi]$. Let $z_1, z_2 \in \overline{U(z_0, r)}$. Then by the Cauchy formula we get $f_n(z_j) = \frac{1}{2\pi i} \int_{\varphi} \frac{f_n(z)}{z - z_j} dz$. There is $M \in (0, +\infty)$ such that $\forall n \in \mathbb{N} |f_n| \leq M$ on $\langle \varphi \rangle$. Then we have

$$\begin{aligned} |f_n(z_1) - f_n(z_2)| &= \frac{1}{2\pi} \left| \int_{\varphi} f_n(z) \cdot \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz \right| \leq \\ &\leq \frac{2\pi \cdot 2r}{2\pi} \cdot M \cdot \frac{|z_1 - z_2|}{r^2} \end{aligned}$$

$$\left(\left| \frac{1}{z - z_1} - \frac{1}{z - z_2} \right| = \left| \frac{z_1 - z_2}{(z - z_1)(z - z_2)} \right| \leq \frac{|z_1 - z_2|}{r^2} \right).$$

By this $\{f_n\}$ are equicontinuous on $\overline{U(z_0, r)}$, and by Arzela–Ascoli, there is $\{f_{n_k}\}$ which is uniformly convergent on $\overline{U(z_0, r)}$.

Second step: Let us cover the set G by $U_j = U(z_j, r_j)$, $j \in \mathbb{N}$, such that $\overline{U(z_j, 2r_j)} \subset G$. Then use a diagonal choice: 1. By first step choose $\{f_{n_k^1}\}$ of $\{f_n\}$ such that $\{f_{n_k^1}\}$ converges uniformly on $\overline{U_1}$. 2. By first step choose $\{f_{n_k^2}\}$ subsequence of $\{f_{n_k^1}\}$ such that $\{f_{n_k^2}\}$ converges uniformly on $\overline{U_2}$ and so on.

Then $\{f_{n_k^k}\}_{k=1}^{\infty}$ converges uniformly on any $\overline{U_j}$, i.e., locally uniformly on G . □

Definice 3.2

Let E be a (complex) linear space and let \mathcal{P} be a system of seminorms on E . Then (E, \mathcal{P}) is called locally convex space (LCS). In (E, \mathcal{P}) we define:

- convergence: $f_n \rightarrow f \Leftrightarrow \forall p \in \mathcal{P} : p(f_n - f) \rightarrow 0$;
- topology τ is the weakest topology on E for which all $p \in \mathcal{P}$ are continuous;

- $\mathcal{F} \subset E$ is bounded if \mathcal{F} is bounded with respect to any $p \in \mathcal{P}$, i.e.,

$$\forall p \in \mathcal{P} \exists C \in (0, +\infty) : p(f) \leq C \quad \forall f \in \mathcal{F};$$

- the dual space to (E, \mathcal{P}) is defined as

$$E^* := \{L : E \rightarrow \mathbb{C} \mid L \text{ linear and continuous}\}.$$

Poznámka

$\mathcal{C}(G)$ is the so-called Fréchet space, i.e., completely metrizable LCS. So is $\mathcal{H}(G)$ because $\mathcal{H}(G)$ is closed subspace of $\mathcal{C}(G)$.

Topology τ on $\mathcal{C}(G)$ is generated by the system of seminorms

$$\mathcal{P} := \{\|\cdot\|_K \mid K \text{ is compact in } G\}.$$

$U \subset \mathcal{C}(G)$ is neighbourhood of $f \in \mathcal{C}(G)$ iff there are a compact $K \in G$ and $\varepsilon > 0$ such that

$$U \supset U_{K,\varepsilon}(f) := \{g \in \mathcal{C}(G) \mid \|g - f\|_K < \varepsilon\}.$$

┌

Důkaz

„ \Leftarrow “: obvious. „ \Rightarrow “: There are $m \in \mathbb{N}$, compact, K_1, \dots, K_m in G and $\varepsilon_1, \dots, \varepsilon_m > 0$ such that

$$U \supset \bigcap_{j=1}^m U_{K_j, \varepsilon_j}(f) \supset U_{K, \varepsilon}(f),$$

where $K := K_1 \cup \dots \cup K_m$ and $\varepsilon := \min \{\varepsilon_1, \dots, \varepsilon_m\} > 0$. □

Poznámka

Let $X = \mathcal{H}(G)$. Then in the sense of (LCS) $\mathcal{F} \subset \mathcal{H}(G)$ is bounded iff in the functions of \mathcal{F} are locally equibounded on G . By the Montal theorem, we get $\overline{\mathcal{F}}$ is a compact in $\mathcal{H}(G)$. Easily we get that $\mathcal{F} \subset X$ is compact iff \mathcal{F} is closed and bounded in X .

3.2 The dual space $\mathcal{H}^*(G)$

Poznámka

1. Let $G = \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Let $L \in \mathcal{H}^*(\mathbb{D})$. Let $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, and $R := \frac{1}{\limsup_{n \rightarrow +\infty} \sqrt[n]{|a_n|}} \geq 1$. Then

$$L(f) = L\left(\sum_{n=0}^{\infty} a_n z^n\right) = L\left(\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k z^k\right) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k L(z^k) = \sum_{n=0}^{\infty} a_n \cdot b_n,$$

where $b_n := L(z^n) \in \mathbb{C}$. We show $r := \limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} < 1$:

If $r > 1$, then for $a_n := 1$, $n \in \mathbb{N}_0$, we get $\sum_{n=0}^{\infty} a_n \cdot b_n$ is divergent. If $r = 1$, then there is $\{n_k\}$ such that $0 \neq \sqrt[n_k]{|b_{n_k}|} \rightarrow 1$. Putting $a_n = \frac{1}{b_{n_k}}$, $n = n_k$, we get $\sum_{n=0}^{\infty} a_n b_n$ is divergent.

Conclusion: $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\{b_n\} \subset \mathbb{C}$ such that $\limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} < 1$ and $L(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$. In addition, $b_n = L(z^n)$, $n \in \mathbb{N}_0$. (\Leftarrow obvious, HW.)

Poznámka (Integral form of L)

Let $\{b_n\} \subset \mathbb{C}$ and $r := \limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} < 1$. Define

$$\lambda(z) := \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}, \quad |z| > r.$$

Of course, $\lambda \in \mathcal{H}(\mathbb{S} \setminus \overline{U(0, r)})$, $\lambda(\infty) = 0$ and $b_n = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$. Here $\lambda^{(k)}(\infty) := \left(\lambda\left(\frac{1}{z}\right)\right)^{(k)}(0)$.

Let $R \in (r, 1)$ and $\varphi(t) := Re^{it}$, $t \in [0, 2\pi]$. Let $f \in \mathcal{H}(\mathbb{D})$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_{\varphi} f(z) \cdot \lambda(z) dz &= \frac{1}{2\pi i} \int_{\varphi} \left(\sum_{n=0}^{\infty} a_n \cdot z^n \right) \cdot \left(\sum_{m=0}^{\infty} \frac{b_m}{z^{m+1}} \right) dz = \\ &= \frac{1}{2\pi i} \int_{\varphi} \sum_{n,m=0}^{\infty} a_n b_m z^{n-m-1} dz = \sum_{n,m=0}^{\infty} a_n \cdot b_m \cdot \frac{1}{2\pi i} \int_{\varphi} z^{n-m-1} dz = \sum_{n=0}^{\infty} a_n \cdot b_n = L(f). \end{aligned}$$

Definition 3.3 (Notation)

Let $A \subset \mathbb{S}$. Then a function f is holomorphic on A if f is holomorphic on some open superset $U \supset A$. Let f_1, f_2 be holomorphic function on A . We say that $f_1 \sim f_2$ if there are open $U_1, U_2 \subset \mathbb{S}$ such that $A \subset U_1 \cap U_2$, $f_1 \in \mathcal{H}(U_1)$, $f_2 \in \mathcal{H}(U_2)$ and $f_1 = f_2$ on $U_1 \cap U_2$. Denote $\mathcal{H}(A) := \{[f] \mid f \text{ is holomorphic on } A\}$, where $[f]$ is an equivalence class for \sim . As usual, we do not often distinguish between $[f]$ and f .

We have that $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D}) := \{\mu \in \mathcal{H}(\mathbb{S} \setminus \mathbb{D}) \mid \mu(\infty) = 0\}$. Moreover, we have

$$(*) L(f) = \frac{1}{2\pi i} \int_{\varphi} f(z) \cdot \lambda(z) dz, \quad f \in \mathcal{H}(\mathbb{D});$$

$$L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, \quad n \in \mathbb{N}_0;$$

$$\lambda(w) = L\left(\frac{1}{w-z}\right), \quad |w| \geq 1.$$

┌ *Důkaz*

In fact, we have

$$L\left(\frac{1}{w-z}\right) = L\left(\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{b_n}{w^{n+1}} = \lambda(w),$$

because $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{z}{w}} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$, $z \in \mathbb{D}$. □

Poznámka (Conclusion)

$$\mathcal{H}^*(\mathbb{D}) = \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D}).$$

In particular, $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D})$ such that $(*)$ hold true.

Příklad (Birkhoff)

There is a universal entire function, i.e., $f \in \mathcal{H}(\mathbb{C})$ such that $\overline{\{\tau_\gamma(f) | \gamma \in \mathbb{C}\}} = \mathcal{H}(\mathbb{C})$, where $\tau_\gamma(f) := f(z - \gamma)$, $z, \gamma \in \mathbb{C}$.

┌ *Řešení*

HW.

Poznámka

2. Let $G = \bigcup_{j=1}^n D_j$ with $D_j = U(z_j, r_j)$ and $D_j \cap D_k = \emptyset$ for $j \neq k$.

Let $L \in \mathcal{H}^*(G)$. For $j \in [n]$, put $L_j(d) := L(\tilde{f})$ for $f \in \mathcal{H}(D_j)$ and $\tilde{f} := f$ on D_j and $\tilde{f} := 0$ on D_k , $k \neq j$. Then

$$L(f) = \sum_{j=1}^n L_j(f|_{D_j}), \quad f \in \mathcal{H}(G).$$

By 1., for each $j \in [n]$, there are $\tilde{r}_j \in (0, r_j)$ and $\lambda_j \in \mathcal{H}_0(\mathbb{S} \setminus \overline{U(z_j, \tilde{r}_j)})$ such that

$$L_j(f) = \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz, \quad f \in \mathcal{H}(D_j),$$

where $\varphi_j(t) := z_j + R_j e^{it}$, $t \in [0, 2\pi]$ for some $R_j \in (\tilde{r}_j, r_j)$.

In addition, we have

$$L_j(z^n) = \frac{\lambda_j^{(n+1)}(\infty)}{(n+1)!}, \quad n \in \mathbb{N}_0.$$

If $f \in \mathcal{H}(G)$, then $L(f) = \sum_{j=1}^n \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz$.

$\stackrel{?}{\implies} L(f) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \cdot \lambda(z) dz$, where $\Gamma := \{\varphi_1, \dots, \varphi_n\}$ and $\lambda := \sum_{j=1}^n \lambda_j$.

? holds true because $\int_{\varphi_j} f(z) \cdot \lambda_k(z) dz = 0$ for $k \neq j$ by Cauchy ($f(z) \cdot \lambda_k(z) \in \mathcal{H}(D_j)$).

We have $L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$.

Poznámka (Conclusion)

$(G = \bigcup_{j=0}^n D_j) \mathcal{H}^*(G) = \mathcal{H}_0(\mathbb{S} \setminus G)$. Indeed, $L \in \mathcal{H}^*(G)$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ such that last 2 equation hold true.

4 Hahn–Banach theorem

Lemma 4.1

Let $L : E \rightarrow \mathbb{C}$ be linear. Then $L \in E^*$ iff there is a compact K in G and $M \in [0, +\infty)$ such that $|L(f)| \leq M \cdot \|f\|_K$, $f \in E$.

┌

Důkaz

„ \Leftarrow “ from continuity of $\|\cdot\|_K$. „ \implies “: Since $U := L^{-1}(\mathbb{D})$ is a neighbourhood of \mathbf{o} in E , there are a compact K in G and $\varepsilon > 0$ such that $U \supseteq U_{K,\varepsilon}(\mathbf{o}) := \{f \in E \mid \|f\|_K < \varepsilon\}$. Let $f \in E$.

1. Let $\|f\|_K \neq 0$. Then

$$\left| L\left(\frac{f}{\|f\|_K} \cdot \frac{\varepsilon}{2}\right) \right| < 1,$$

hence $|L(f)| < \frac{2}{\varepsilon} \|f\|_K$. Put $M := \frac{2}{\varepsilon}$.

2. Let $\|f\|_K = 0$. Then for any $n \in \mathbb{N}$, we have $\|nf\|_K = 0$, so $|L(n \cdot f)| < 1$, $|L(f)| < \frac{1}{n} \rightarrow 0$, $L(f) = 0$. □

Věta 4.2 (Hahn–Banach)

Let A be a linear subspace of E . Then

- if $L \in A^*$, then there is $\tilde{L} \in E^*$ such that $\tilde{L}|_A = L$;
- if A is closed and $0 \neq b \in E \setminus A$, then there is $L \in E^*$ such that $L(b) = 1$ and $L = 0$ on A ;
- $\overline{A} = E$ iff $(L \in E^*, L = 0 \text{ on } A \implies L = 0 \text{ on } E)$.

┌ *Důkaz*

„1.“ Use lemma and algebraic version of HB theorem.

└ „2. + 3.“ can be proved as for Banach space. □

Věta 4.3 (Runge (special))

Let $G \subset \mathbb{C}$ be a finite union of pairwise disjoint open discs as in above "poznámka"s. Then for each $f \in \mathcal{H}(G)$ there are polynomials P_n , $n \in \mathbb{N}$, such that $P_n \xrightarrow{loc.} f$ on G .

┌ *Důkaz*

Let $\mathcal{P} := \text{LO}\{1, z, \dots\}$ be the space of polynomials. Then $\mathcal{P} \subset \mathcal{H}(G)$. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on \mathcal{P} . We know that there is $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ such that ? is valid. So, $\lambda^{(n)}(\infty) = 0$, $n \in \mathbb{N}_0$. By the uniqueness theorem, we get $\lambda \equiv 0$, so $L = 0$ on $\mathcal{H}(G)$ (because $L = 0$ fits and is uniquely determined by λ). By HB theorem, $\overline{\mathcal{P}} = \mathcal{H}(G)$. □

Věta 4.4 (Cauchy formula for compact)

Let $G \subset \mathbb{C}$ be open, $K \subset G$ compact. Then there is a cycle $\Gamma \subset G$, $K \subseteq \text{int } \Gamma \subseteq G$ and $\forall a \in \text{int } \Gamma : \text{ind}_\Gamma a = 1$.

In addition

$$\forall f \in \mathcal{H}(G) : \int_\Gamma f = 0 \wedge \forall a \in \text{int } \Gamma : f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} dz.$$

┌ *Poznámka*

„In addition“ follows from the properties of Γ and residue's theorem for cycles, but we prove it directly.

Důkaz (Cauchy formula for compact)

Choose $0 < \delta < \frac{1}{2} \text{dist}(K, \mathbb{C} \setminus G)$, if $G \subsetneq \mathbb{C}$, otherwise, if $G = \mathbb{C}$, take $\delta := 1$. For $m, n \in \mathbb{Z}$ let $Q_{m,n}$ be the closed square with edges (parallel to the axes) with length δ , and such that $m\delta + in\delta$ is the lower left vertex of $Q_{m,n}$.

Denote $Q^* := \{Q_{n,m} | Q_{n,m} \cap K \neq \emptyset\}$, $U := \bigcup Q^*$. Q^* is finite because of compactness of K . Of course, $K \subseteq U \subseteq \bigcup Q^* \subseteq G$ (by choice of δ).

We understand $\partial Q_{m,n}$ as a positively oriented curve (piece-wise linear curve). Let Γ be the system of all edges $\Gamma_1, \dots, \Gamma_k$ of squares of Q^* when we omit those edges which occur twice (\pm). Of course, $U = \bigcup Q^* \setminus \text{Im } \Gamma$.

a) Let $f \in \mathcal{H}(G)$. Then $\int_\Gamma f := \sum_{j=1}^k \int_{\Gamma_j} f = \sum_{Q_{m,n} \subset Q^*} \int_{\partial Q_{m,n}} f = 0$.

b) Γ can be viewed as a cycle. In fact the edges $\Gamma_1, \dots, \Gamma_k$ form finitely many closed simple piece-wise linear curves.

For $j \in [k]$ put $\Gamma_j =: [a_j, b_j]$.

(*) „Every point $c \in \mathbb{C}$ is the starting point of some edge of Γ as many times as it is the ending point of some edge in Γ “:

Take a polynomial P such that $p(c) = 1$ and $p(a) = 0$, if $a \neq c$ and $[a, b] \in \Gamma$ for some b . $p(b) = 0$, if $b \neq c$ and $[a, b] \in \Gamma$ for some a . By a):

$$\begin{aligned} 0 &= \int_{\Gamma} p' = \sum_{j=1}^k \int_{\Gamma_j} p' = \sum_{j=1}^k (p(b_j) - p(a_j)) = \sum_{j=1}^k p(b_j) - \sum_{j=1}^k p(a_j) = \\ &= \# c \text{ is the ending point} - \# c \text{ is the starting point.} \end{aligned}$$

„ Γ can be viewed as a cycle“: Let L be longest (one of the longest) simple piecewise linear curve consisting of edges of Γ which begins with Γ_1 , i. e.,

- $L = [c_1, c_2, \dots, c_l] := [c_1, c_2] + [c_2, c_3] + \dots + [c_{l-1}, c_l]$;
- $\Gamma_1 = [c_1, c_2]$;
- $c_i \neq c_j$ for $i \neq j$ (simple curve);
- l is the biggest.

Since we have (*) there is an index $j \in [l-2]$ such that $[c_l, c_j] \in \Gamma$ (otherwise we would have a longer curve).

$$L' := [c_j, c_{j+1}] + \dots + [c_{l-2}, c_l] + [c_l, c_j] \subseteq L$$

$\implies L'$ is simple closed piece-wise linear curve. The proper subset Γ' , which we get from Γ_k by omitting the edges of L' has again (*). We can process in this fashion for Γ' , by finitely many steps we get what we want.

c) Let $f \in \mathcal{H}(G)$ and $a \in U = \text{int}(\bigcup Q^*)$. c1) $a \in \text{int}(\tilde{Q})$ for some $\tilde{Q} \in Q^*$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a} dz = \sum_{Q_{m,n} \in Q^*} \frac{1}{2\pi i} \int_{\partial Q_{m,n}} \frac{f(z)}{z-a} dz = f(a)$$

(residue's theorem for $Q_{m,n} = \tilde{Q}$, Cauchy for $Q_{m,n} \neq \tilde{Q}$).

c2) $a \in \partial \tilde{Q}$, $\tilde{Q} \in Q^*$ ($a \notin \partial Q^* \implies a \notin \text{Im}(\Gamma)$). Take $a_j \in \text{int} \tilde{Q}$, $a_j \rightarrow a$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a_j} dt \rightarrow f(a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-a} dz.$$

d) „ $U = \text{int}(\Gamma)$ “: Let $a \in \mathbb{C} \setminus (U \cup \text{Im} \Gamma) \implies a \in \mathbb{C} \setminus \bigcup Q^*$. As in a), we show easily that $\text{ind}_{\Gamma} a = 0$. If $a \in U$, then by c) we have that $\text{ind}_{\Gamma}(1) = 1$. \square

Věta 4.5 (Description of $\mathcal{H}^*(G)$)

Let $G \subset \mathbb{C}$ be open subset. Then $\mathcal{H}^*(G) \simeq \mathcal{H}_0(\mathbb{S} \setminus G)$.

In more detail, let $L \in \mathcal{H}^*(G)$. Then there are a compact $K \subset G$ and $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus K)$ such that

$$L(f) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \lambda(z) dz, \quad f \in \mathcal{H}(G),$$

where Γ is a cycle in $G \setminus K$ with $K \subset \text{int } \Gamma \subset G$ and $\forall z_0 \in \text{int } \Gamma : \text{ind}_{\Gamma} z_0 = 1$.

In addition, as an element of $\mathcal{H}_0(\mathbb{S} \setminus G)$, λ is uniquely determined by

$$\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k), k \in \mathbb{N}_0, \frac{\lambda^{(k)}(z_0)}{k!} = -L\left(\frac{1}{(z - z_0)^{k+1}}\right), z_0 \in \mathbb{C} \setminus G, k \in \mathbb{N}_0.$$

┌

Důkaz (Step 1)

Let $L \in \mathcal{H}^*(G)$.

Step 1: There are a compact $K \subset G$ and $L_1 \in (\mathcal{C}(K))^* =: \mathcal{C}^*(K)$ such that $L(f) = L_1(f|_K), f \in \mathcal{H}(G)$.

We know that there are a compact $K \subseteq G$ and $C \in (0, +\infty)$ such that $\forall f \in \mathcal{H}(G) : |L(f)| \leq \|f\|_K \cdot C$.

By the Hahn–Banach theorem we can extend L (from $\mathcal{H}^*(G)$ to $\mathcal{C}^*(G)$) to $\tilde{L} \Leftrightarrow \tilde{L} \in \mathcal{C}^*(G)$ such that $\tilde{L}|_{\mathcal{H}(G)} = L$ and $|L(f)| \leq \|f\|_K \cdot C, f \in \mathcal{C}(G)$.

For each $f \in \mathcal{C}(K)$ put $L_1(f) := \tilde{L}_1(\tilde{f})$, where $\tilde{f} \in \mathcal{C}(G)$ and $\tilde{f}|_K = f$.

Is definition of L_1 correct?

i) by Tietze theorem: $f \in \mathcal{C}(K)$ can be extended to $f \in \mathcal{C}(G)$,

$$\forall f \in \mathcal{C}(K) \exists \tilde{f} \in \mathcal{C}(\mathbb{C} \setminus (G)) : \tilde{f}|_K = f;$$

ii) for any extension we want to get the same result. $\tilde{f}_1, \tilde{f}_2 \in \mathcal{C}(G), \tilde{f}_i|_K = f, i = 1, 2$.

$$\implies |\tilde{L}_1(\tilde{f}_1) - \tilde{L}_1(\tilde{f}_2)| = |\tilde{L}_1(\tilde{f}_1 - \tilde{f}_2)| \leq C \cdot \|\tilde{f}_1 - \tilde{f}_2\|_K = C\|f - f\|_K = 0.$$

└

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Poznámka ($\mathcal{C}^*(K)$)

By the Riesz representation theorem, for each $L_1 \in \mathcal{C}^*(K)$ there is a unique complex Borel measure μ on K such that

$$L_1(f) = \int_K f d\mu, \quad \forall f \in \mathcal{C}(K).$$

└

Důkaz

Step 2: By the Cauchy formula for compact, there is a cycle $\Gamma \subset G$ such that $K \subset \text{int } \Gamma \subset G$, $\forall a \in \text{int } \Gamma : \text{ind}_\Gamma a = 1$ and we have, $\forall f \in \mathcal{H}(G)$:

$$f(z_1) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z_2) dz_2}{z_2 - z_1}, \quad z_1 \in K.$$

Denote

$$L_2(f) := \frac{1}{2\pi i} \int_\Gamma f(z_2) dz_2, f \in \mathcal{C}(\langle \Gamma \rangle), \quad F(z_1, z_2) := \frac{f(z_2)}{z_2 - z_1}.$$

Of course $L_2 \in \mathcal{C}^*(\langle \Gamma \rangle)$ and $f(z_1) = L_2(F(z_1, z_3))$, $z_1 \in K$.

Step 3: For a given $f \in \mathcal{H}(G)$,

$$L(f) = L_1(f(z_1)) = L_1(L_2(F(z_1, z_2))) \stackrel{\text{Fubini}}{=} L_2(L_1(F(z_1, z_2))),$$

hence

$$L(f) = \frac{1}{2\pi i} \int_\Gamma f(z_2) \cdot \lambda(z_2) dz_2,$$

where

$$\lambda(z_2) := L_1\left(\frac{1}{z_2 - z_1}\right), \quad z_2 \in \mathbb{C} \setminus K.$$

Step 4: $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus K)$ satisfies „in addition“: Let $U(\infty, \varepsilon) \subset \mathbb{S} \setminus K$. For $u \in P(0, \varepsilon)$, we have

$$\lambda\left(\frac{1}{u}\right) = L_1\left(\frac{u}{1 - u \cdot z_1}\right) = L_1\left(\sum_{k=0}^{\infty} z_1^k u^{k+1}\right) = \sum_{k=0}^{\infty} L_1(z_1^k) u^{k+1},$$

hence $\lambda(\infty) = 0$ and

$$\forall k \in \mathbb{N}_0 : \frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L_1(z_1^k).$$

Let $U(z_0, \varepsilon) \subset \mathbb{C} \setminus K$. Then $\forall z_2 \in U(z_0, \varepsilon)$:

$$\lambda(z_2) = L_1\left(\frac{1}{z_2 - z_1}\right) = -L_1\left(\sum_{k=0}^{\infty} \frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}\right) = -\sum_{k=0}^{\infty} L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right) (z_2 - z_0)^k;$$

$$\forall z_1 \in K : \frac{1}{z_2 - z_1} = \frac{1}{(z_2 - z_0) - (z_1 - z_0)} = -\frac{1}{z_1 - z_0} \cdot \frac{1}{1 - \frac{z_2 - z_0}{z_1 - z_0}} = -\sum_{k=0}^{\infty} \frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}.$$

Hence $\frac{\lambda^{(k)}(z_0)}{k!} = -L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right)$, $k \in \mathbb{N}_0$.

Step 5: As an element of $\mathcal{H}_0(\mathbb{S} \setminus G)$, λ is uniquely determined by „in addition“. (Proof below.) \square

Lemma 4.6

Let $G \subset \mathbb{C}$ be open and K be a compact in G . There is a compact K_1 such that $K \subset K_1 \subset G$ and each component of $\mathbb{S} \setminus K_1$ contains some component of $\mathbb{S} \setminus G$.

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Důkaz

Take $n \in \mathbb{N}$ such that $K_1 := \{z \in G \mid \text{dist}(z, \mathbb{C} \setminus G) \geq \frac{1}{n}\} \cap \overline{U(0, n)} \supset K$. In addition, we have

$$\mathbb{S} \setminus K_1 = \bigcup_{z_0 \in \mathbb{S} \setminus G} U(z_0, \frac{1}{n}).$$

Let V be a component of $\mathbb{S} \setminus K_1$. There is $z_0 \in \mathbb{S} \setminus G$ such that $U(z_0, \frac{1}{n}) \subset V$. If W is a component of $\mathbb{S} \setminus G$ containing z_0 , then $W \subset V$. □

└

Důkaz (Step 5)

Let $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S} \setminus G)$ satisfying „in addition“. Then there is a compact $K \subset G$ such that $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S} \setminus K)$.

By the previous lemma, WLOG we assume that each component V of $\mathbb{S} \setminus K$ intersect $\mathbb{S} \setminus G$. We show $\lambda_1 = \lambda_2$ on $\mathbb{S} \setminus K$.

Let V be any component of $\mathbb{S} \setminus K$ and $z_0 \in V \cap (\mathbb{S} \setminus G) \neq \emptyset$. By „in addition“ we have $\lambda_1^{(k)}(z_0) = \lambda_2^{(k)}(z_0) \forall k \in \mathbb{N}_0$. By the uniqueness theorem $\lambda_1 = \lambda_2$ on the domain B , so $\lambda_1 = \lambda_2$ on $\mathbb{S} \setminus K$. □

Lemma 4.7 (Fubini)

Let $K_1, K_2 \subset \mathbb{C}$ be compact, $L_j \in \mathcal{C}^*(K_j)$ for $j = 1, 2$ and $F \in \mathcal{C}(K_1 \times K_2)$. Then we have

$$L_1(L_2(F(z_1, z_2))) = L_2(L_1(F(z_1, z_2))).$$

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Důkaz (Sketch)

Obviously it holds true for the functions of the following form: $F(z_1, z_2) = f(z_1) \cdot g(z_2)$ for $f \in \mathcal{C}(K_1)$, $g \in \mathcal{C}(K_2)$.

Now we can use the Stone–Weierstrass theorem which shows that the linear span of the functions of this form is dense in $\mathcal{C}(K_1 \times K_2)$. □

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5 Runge's theorem

Definition 5.1 (Notation)

Let $E \subset \mathbb{C}$ and $m : E \rightarrow \mathbb{N} \cup \{\infty\}$. We call $m(e)$ the multiplicity of $e \in E$. We say that (E, m) has a limit point $e \in \mathbb{S}$ if e is a limit point of E , or $e \in E$ with $m(e) = \infty$.

Denote by $\mathcal{F}(E, m)$ system of functions which consists of

- $\frac{1}{z-e}$ if $e \in E \cap \mathbb{C}$, $m(e) < \infty$;
- $\frac{1}{(z-e)^k}$, $k \in \mathbb{N}$ if $e \in E \cap \mathbb{C}$, $m(e) = \infty$;
- z^k , $k \in \mathbb{N}_0$ if $\infty \in E$, $m(\infty) = \infty$.

Věta 5.1 (Runge)

Let $G \subset \mathbb{C}$ be open, $E \subset \mathbb{S} \setminus G$ and $m : E \rightarrow \mathbb{N} \cup \{\infty\}$. If (E, m) has a limit point in every component of $\mathbb{S} \setminus G$, then the linear span of $\mathcal{F}(E, m)$ is dense in $\mathcal{H}(G)$.

Důkaz

We shall use Hahn–Banach theorem. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathcal{F}(E, m)$. We need to show $L = 0$ on $\mathcal{H}(G)$. Let $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ which represents L in the sense of theorem describing $\mathcal{H}^*(G)$.

If $e \in E \cap \mathbb{C}$, $m(e) < \infty$, then $\lambda(e) = -L\left(\frac{1}{z-e}\right) = 0$. If $e \in E \cap \mathbb{C}$, $m(e) = \infty$, then $\frac{\lambda^{(k)}(e)}{k!} = -L\left(\frac{1}{(z-e)^k}\right) = 0 \ \forall k \in \mathbb{N}_0$. If $\infty \in E$, $m(\infty) = \infty$, then $\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k) = 0 \ \forall k \in \mathbb{N}_0$.

We show that $\lambda = 0$ in $\mathcal{H}_0(\mathbb{S} \setminus G)$. There is a compact $K \subset G$ such that $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus K)$ and every component of $\mathbb{S} \setminus K$ contains some component of $\mathbb{S} \setminus G$.

Let V be any component of $\mathbb{S} \setminus K$. Then V is domain and V contains a limit point e of (E, m) . By the uniqueness theorem, we get $\lambda = 0$ on V , so on $\mathbb{S} \setminus K$. \square

Věta 5.2 (Runge, classical version)

Let $G \subset \mathbb{C}$ be open and $f \in \mathcal{H}(G)$. Then there are rational functions R_n , $n \in \mathbb{N}$ with poles outside G such that $R_n \xrightarrow{Loc.} f$ on G .

If, in addition, $\mathbb{S} \setminus G$ is connected, then there are polynomials P_n , $n \in \mathbb{N}$, such that $P_n \xrightarrow{Loc.} f$ on G .

Důkaz

„Second part“: Let $E = \{\infty\}$ and put $m(\infty) = \infty$. Then

$$\mathcal{F}(E, m) = \{1, z, \dots, z^k, \dots\}$$

and by the previous theorem, the polynomials are dense in $\mathcal{H}(G)$.

„First part“: Let $E \subset \mathbb{S} \setminus G$ containing at least one point of every component of $\mathbb{S} \setminus G$. Put $m = \infty$ on E . Then $\text{LO}(\mathcal{F}(E, m))$ is dense in $\mathcal{H}(G)$ and it is a subspace of rational functions with poles outside G . \square

Důsledek (Cauchy's theorem for simply connected domains)

Let $G \subset \mathbb{C}$ be open and $\mathbb{S} \setminus G$ be connected. If $f \in \mathcal{H}(G)$ and φ is a closed curve in G , then $\int_{\varphi} f = 0$.

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Důkaz

By Runge, there are polynomials P_n such that $P_n \xrightarrow{\text{Loc.}} f$ on G . Then (P_n) has a primitive function in \mathbb{C}) $0 = \int_{\varphi} P_n \rightarrow \int_{\varphi} f$. \square

└

Důsledek (Cauchy's theorem for cycles)

Let $G \subset \mathbb{C}$ be open and Γ be a cycle in G (i.e., $\langle \Gamma \rangle \subset G$). Then $(\forall f \in \mathcal{H}(G) : \int_{\Gamma} f = 0) \Leftrightarrow \text{int } \Gamma \subset G$.

┌

Důkaz

„ \Rightarrow “: If $z_0 \in \mathbb{C} \setminus G$, then $f(z) := \frac{1}{z - z_0} \in \mathcal{H}(G)$ and $\text{ind}_{\Gamma} z_0 = \frac{1}{2\pi i} \int_{\Gamma} f = 0$.

„ \Leftarrow “: Let $f \in \mathcal{H}(G)$. By Runge, there are rational R_n with poles outside G such that $R_n \xrightarrow{\text{Loc.}} f$. Then $0 = \int_{\Gamma} R_n \rightarrow \int_{\Gamma} f$. (First equality is from: Let $\Gamma = \{\varphi_1, \dots, \varphi_m\}$, where φ_j are closed curves in G . Then $\int_{\Gamma} R_n = \sum_{j=1}^m \int_{\varphi_j} R_n = \sum_{j=1}^m 2\pi i \sum_{R_n(s)=\infty} \text{res}_s R_n \text{ind}_{\varphi_j} s = 2\pi i \cdot \sum_{R_n(s)=\infty} \text{res}_s R_n \cdot \text{ind}_{\Gamma} s$, but s lies outside of G , so it is equal to 0.) \square

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Věta 5.3 (Runge, for compacts)

Let K be a compact in \mathbb{C} and let $S \subset \mathbb{S} \setminus K$ contain at least one point of any component of $\mathbb{S} \setminus K$. Let f be a holomorphic function on K . Then there are rational functions R_n with poles in S such that $R_n \rightrightarrows f$ on K .

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Poznámka (Technique: pushing poles)

Each rational function R can be uniquely expressed in the form (rational function has $n \in \mathbb{N}$ poles, and we will write the principal part of Laurent expansion around the pole z_k):

$$R(z) = \sum_{k=1}^n \sum_{j=1}^{n_k} \frac{A_j^k}{(z - z_k)^j} + C_0 + C_1 z + \dots + C_m z^m,$$

where $n, m, n_k \in \mathbb{N}$, $z_k \in \mathbb{C}$ and $A_{n_k}^k \neq 0$, $C_m \neq 0$. Then z_k is a pole of R of multiplicity n_k and ∞ is a pole of R of multiplicity m . A rational function R is a polynomial iff R has a pole at most at ∞ .

Notation: Let K be a compact in \mathbb{C} , $U \subset \mathbb{S}$ and $U \cap K = \emptyset$. Put $B(K, U) = \overline{\{R|_K | R \text{ is rational with poles in } U\}}^{\mathcal{C}(K)}$. ($B(K, U)$ is a closed subalgebra of $\mathcal{C}(K)$.)

Theorem (pushing poles): Let K be a compact in \mathbb{C} , $U \subset \mathbb{S}$ be a domain, $K \cap U = \emptyset$ and $z_0 \in U$. If R is rational function with poles in U , then $R \in B(K, \{z_0\})$.

Corollary: By theorem, we have $B(K, U) = B(K, z_0)$.

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┌ *Poznámka* (Technique: pushing poles)

Proof: Put $V := \left\{ \xi \in U \mid \frac{1}{z-\xi} \in B(K, z_0), \text{ for } \xi \in \mathbb{C} \text{ and } z \in B(K, z_0) \text{ for } \xi = \infty \right\}$. Of course $B(K, z_0) = B(K, V)$. Indeed, if $\xi \in V$, then $\frac{1}{(z-\xi)^k} \in B(K, z_0)$, for $\xi \in \mathbb{C}$ and $k \in \mathbb{N}$, and $z^k \in B(K, z_0)$ for $\xi = \infty$, $k \in \mathbb{N}$.

Then each rational R with poles in V is contained in $B(K, z_0)$. Hence $B(K, V) \subset B(K, z_0)$. Since $z_0 \in V$, we have $B(K, z_0) \subset B(K, V)$.

„ V is closed in U “: Let $\xi_n \in V$, $\xi_n \rightarrow \xi_0$ and $\xi_0 \in U$. We need to show that $\xi_0 \in V$. WLOG $\forall n \in \mathbb{N} : \xi_n \in \mathbb{C}$.

„ $\xi_0 \in \mathbb{C}$ “: Then put $\delta := \text{dist}(\xi_0, K) > 0$. Choose $n_0 \in \mathbb{N}$ such that $\text{dist}(\xi_n, K) \geq \frac{\delta}{2}$ for $n > n_0$. Then $\frac{1}{z-\xi_n} \rightrightarrows \frac{1}{z-\xi_0}$, $z \in K$,

$$\Longleftarrow \left| \frac{1}{z-\xi_n} - \frac{1}{z-\xi_0} \right| = \frac{|\xi_n - \xi_0|}{|z-\xi_n| \cdot |z-\xi_0|} \leq \frac{2}{\delta^2} \cdot |\xi_n - \xi_0| \rightarrow 0,$$

if $n > n_0$ and $z \in K$. Hence $\frac{1}{z-\xi_n} \in B(K, z_0)$, so $\xi_0 \in V$.

„ $\xi_0 = \infty$ “: Then

$$\frac{\xi_n z}{\xi_n - z} = -\xi_n \left(\frac{\xi_n}{z-\xi_n} + 1 \right) \in B(K, z_0).$$

Take $C > 0$ with $\forall z \in K : |z| \leq C$. Take $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : |\xi_n| > C$. Then $\forall z \in K : \frac{\xi_n z}{\xi_n - z} \rightrightarrows z$, because

$$\left| \frac{\xi_n z}{\xi_n - z} - z \right| = \frac{|z|^2}{|\xi_n - z|} \leq \frac{C^2}{|\xi_n| - C} \rightarrow 0.$$

if $n > n_0$ and $z \in K$. Hence $z \in B(K, z_0)$, so $\infty \in V$.

„ V is open (so $V = U$)“: Let $\xi_0 \in V$.

„ $\xi \in \mathbb{C}$ “: Put $\delta := \text{dist}(\xi_0, K) > 0$. Let $\xi \in U(\xi_0, \delta/2)$. Then

$$\frac{1}{z-\xi} = \frac{1}{(z-\xi_0) - (\xi-\xi_0)} = \frac{1}{z-\xi_0} \cdot \frac{1}{1 - \frac{\xi-\xi_0}{z-\xi_0}} = \sum_{k=0}^{\infty} \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}}$$

converges uniformly for $z \in K$ because $\left| \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}} \right| \leq \frac{(\frac{\delta}{2})^k}{\delta^{k+1}} = \frac{1}{\delta \cdot 2^k}$, for $z \in K$ and $k \in \mathbb{N}_0$. Hence $\frac{1}{z-\xi} \in B(K, \xi_0) \subset B(K, V) = B(K, z_0)$. So $U(\xi_0, \delta/2) \subset V$.

„ $\xi_0 = \infty$ “: Take $C > 0$ with $\forall z \in K : |z| \leq C$. Let $\xi \in \mathbb{C}$ with $|\xi| > 2C$. Then $\frac{1}{z-\xi} = -\frac{1}{\xi} \cdot \frac{1}{1 - \frac{z}{\xi}} = -\sum_{k=0}^{\infty} \frac{z^k}{\xi^{k+1}}$ converges uniformly for $z \in K$, because $|z^k/\xi^{k+1}| \leq C^k/(2C)^{k+1} = \frac{1}{C \cdot 2^{k+1}}$ for $z \in K$ and $k \in \mathbb{N}_0$. Hence $\frac{1}{z-\xi} \in B(K, \infty) \subset B(K, z_0)$. So $U(\infty, \frac{1}{2C}) \subset V$.

Důkaz

Let f be a holomorphic function on an open set $G \supset K$. Using Runge's theorem for "open sets", there are rational functions \tilde{R}_n with poles outside G such that $\tilde{R}_n \rightrightarrows f$ on K .

„ $\tilde{R}_n \in B(K, S)$ “: All poles of \tilde{R}_n are contained in a finitely many components C_1, \dots, C_k of $\mathbb{S} \setminus K$. Express $\tilde{R}_n = \tilde{Q}_1 + \dots + \tilde{Q}_k$, where \tilde{Q}_j is a rational function with poles in the domain C_j . For $j \in [k]$ take $s_j \in S \cap C_j$. By pushing poles we have $\tilde{Q}_j \in B(K, s_j)$. For given $\varepsilon > 0$ and $j \in [k]$, there is a rational function Q_j with a pole at s_j such that $|\tilde{Q}_j - Q_j| \leq \frac{\varepsilon}{k}$ on K . Put $R_n := Q_1 + \dots + Q_k \in B(K, S)$. Then $|R_n - \tilde{R}_n| \leq \varepsilon$ on K . Hence $\tilde{R}_n \in B(K, S)$. \square

6 Characterization of simple connectedness

Tvrzení 6.1

Let $G \subset \mathbb{C}$ be open. FSAE:

SC1 If φ is closed (regular) curve in G , then $\text{int } \varphi \subset G$;

SC2 $\mathbb{S} \setminus G$ is connected;

SC3 $\forall f \in \mathcal{H}(G) \exists$ polynomials $P_n: P_n \xrightarrow{\text{loc.}} f$ on G ;

SC4 $\forall f \in \mathcal{H}(G): \int_{\varphi} f = 0$ for any closed regular curve φ in G ;

SC5 $\forall f \in \mathcal{H}(G) \exists F \in \mathcal{H}(G): F' = f$ on G ;

SC6 $\forall f \in \mathcal{H}(G), f \neq 0$ on $G, \exists g \in \mathcal{H}(G): f = e^g$ on G ;

SC7 $\forall f \in \mathcal{H}(G), f \neq 0$ on $G, \exists h \in \mathcal{H}(G): h^2 = f$ on G .

Důkaz (SC1 \implies SC2)

Assume that $\mathbb{S} \setminus G$ is not connected. Then there are disjoint closed sets $\emptyset \neq K, L \subset \mathbb{S}$ such that $\mathbb{S} \setminus G = K \cup L$. WLOG $\infty \notin K$. Then K is compact in \mathbb{C} , $G_0 := G \cup K$ is an open set in \mathbb{C} and, by theorem Cauchy formula for compact we know, there is cycle Γ in G_0 such that $K \subset \text{int } \Gamma \subset G_0$.

Let $z_0 \in K$. Since $\text{ind}_{\Gamma} z_0 \neq 0$, there is $\varphi \in \Gamma$ with $\text{ind}_{\varphi} z_0 \neq 0$. Of course, $z_0 \in (\mathbb{C} \setminus G) \cap \text{int } \varphi$. \square

Důkaz (SC2 \implies SC3, SC3 \implies SC4)

See Runge's theorem (classical version).

See the proof of the Cauchy theorem for simply connected domains. \square

┌ *Důkaz* (SC4 \Leftrightarrow SC5, SC5 \implies SC6, SC6 \implies SC7)

We know from introduction to complex analysis.

See the proof of proposition about non-zero holomorphic function.

└ Put $h := e^{\frac{1}{2}g}$. □

6.1 The right topological definition

Definition 6.1 (Loop)

Let $G \subset \mathbb{C}$ be open. WLOG: We assume that all curves are defined on $[0, 1]$ (otherwise we can make linear reparametrization). A continuous closed curve $\varphi : [0, 1] \rightarrow G$ is called a loop in G .

Definition 6.2 (Homotopic loops)

We say that two loops φ and ψ are homotopic (in G) provided that there is a continuous map $H : [0, 1] \times [0, 1] \rightarrow G$ such that $\varphi_0(t) = \varphi(t)$ and $\varphi_1(t) = \psi(t)$ and $\varphi_s(0) = \varphi_s(1)$, where $\varphi_s(t) := H(s, t)$.

Tvrzení 6.2 (Continuation of the previous tvrzení)

SC8: Every loop φ in G is homotopic in G to a constant loop.

┌ *Důkaz* (SC7 \implies SC8)

Let φ be a loop in G . Let G_0 be a component of G containing $\langle \varphi \rangle$. If $G_0 = \mathbb{C}$, then (all star-like domain has the property SC8) φ is homotopic to a constant loop. So assume $G_0 \subsetneq \mathbb{C}$. Then $\emptyset \neq G_0 \subsetneq \mathbb{C}$ is a domain with the property SC7. By Riemann (next) theorem, G_0 is homeomorphic to \mathbb{D} and hence G_0 has SC8 property (all star-like domains have SC8 and homomorphism preserve homotopic loops). □

┌ *Důkaz* (SC8 \implies SC1)

Of course, every constant loop ψ has $\text{int } \psi = \emptyset$. Hence this implication follows from the theorem after Riemann theorem. □

Věta 6.3 (Riemann)

Let $\emptyset \neq G_0 \subsetneq \mathbb{C}$ be a domain with SC7. Then there is a one-to-one holomorphic function $h : G_0 \xrightarrow{\text{onto}} \mathbb{D} := U(0, 1)$.

Poznámka (Recall)

Proposition: Let $\varphi_1, \varphi_2 : [0, 1] \rightarrow \mathbb{C}$ be closed (regular) curves and $z_0 \in \mathbb{C} \setminus (\langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle)$. If

$\forall t \in [0, 1]:$

$$|\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - z_0|,$$

then $\text{ind}_{\varphi_1} z_0 = \text{ind}_{\varphi_2} z_0$.

Věta 6.4

Let φ, ψ be two loops homotopic in an open set $G \subset \mathbb{C}$. Then $\text{ind}_{\varphi} z_0 = \text{ind}_{\psi} z_0 \quad \forall z_0 \in \mathbb{C} \setminus G$.

Definition 6.3 (Index of (non-regular) loop)

Let $\varphi : [0, 1] \rightarrow \mathbb{C}$ be a loop and $z_0 \in \mathbb{C} \setminus \langle \varphi \rangle$. There are regular closed curves $\varphi_n : [0, 1] \rightarrow \mathbb{C}$ such that $\varphi_n \rightrightarrows \varphi$. Indeed using the uniform continuity of φ , φ can be uniformly approximated by piecewise linear closed curves with vertices on φ given by sufficiently fine partitions of $[0, 1]$.

Define $\text{ind}_{\varphi} z_0 := \lim_{n \rightarrow \infty} \text{ind}_{\varphi_n} z_0$.

By recalled proposition, the definition is correct because there is $n_0 \in \mathbb{N}$ such that $\text{ind}_{\varphi_n} z_0$, $n \geq n_0$, are constant and $\text{ind}_{\varphi} z_0$ does not depend of the choice of $\{\varphi_n\}$.

Poznámka

„Proposition from recall holds true for (non-regular) loops φ_1, φ_2 .“: Indeed, let loops φ_1 and φ_2 satisfy

$$\forall t \in [0, 1] : |\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - z_0|.$$

Then, by definition, there are approximations $\tilde{\varphi}_1, \tilde{\varphi}_2$ which are regular, satisfy the assumptions of proposition from recall and $\text{ind}_{\varphi_j} z_0 = \text{ind}_{\tilde{\varphi}_j} z_0$, $j = 1, 2$.

Důkaz

Let $H : [0, 1] \times [0, 1] \rightarrow G$ be continuous, $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_s(0) = \varphi_s(1)$, $\forall s \in [0, 1]$, where $\varphi_s(t) = H(s, t)$. Put $\varepsilon := \text{dist}(z_0, H([0, 1]^2)) > 0$ ($H([0, 1]^2)$ is compact).

Since H is uniformly continuous, there is $n \in \mathbb{N}$ such that for each $k \in [n - 1]$ and $t \in [0, 1]$ we have

$$\left| \varphi_{\frac{k}{n}}(t) - \varphi_{\frac{k+1}{n}}(t) \right| = \left| H\left(\frac{k}{n}, t\right) - H\left(\frac{k+1}{n}, t\right) \right| < \varepsilon.$$

In particular, $\varphi_{\frac{k}{n}}$ and $\varphi_{\frac{k+1}{n}}$ satisfy the assumptions of proposition. Hence

$$\text{ind}_{\varphi_0} z_0 = \text{ind}_{\varphi_{\frac{1}{n}}} z_0 = \text{ind}_{\varphi_{\frac{2}{n}}} z_0 = \dots = \text{ind}_{\varphi_1} z_0.$$

□

Věta 6.5 (The Schwarz lemma)

Let $f \in \mathcal{H}(\mathbb{D})$, $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Then $|f(z)| \leq |z|$, $z \in \mathbb{D}$ and $|f'(0)| \leq 1$. If the equality occurs in first inequality got some $z \in \mathbb{D} \setminus \{0\}$ or in second inequality, then f is a rotation, i.e., $f(z) = \lambda z$, $z \in \mathbb{D}$ for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

┌
Důkaz

Put $g(z) := \frac{f(z)}{z}$, for $z \in \mathbb{D} \setminus \{0\}$, and $f'(0)$, $z = 0$. Then $g \in \mathcal{H}(\mathbb{D})$. Let $0 < r < 1$. Then $|g(r)| \leq \frac{1}{r}$, $|z| = r$. By the maximum modulus theorem, we get $|g(z)| \leq \frac{1}{z}$, $|z| \leq r$.

Let $z \in \mathbb{D}$. Then, for r close enough to $1-$, we have this inequality and letting $r \rightarrow 1-$ we get $|g(z)| \leq 1$. If $|g(z)| = 1$ for some $z \in \mathbb{D}$, then by maximum modulus theorem g is constant on \mathbb{D} . □

Lemma 6.6

For $\alpha \in \mathbb{D}$ put $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$. Then

i $\varphi_\alpha \in \mathcal{H}(\mathbb{C} \setminus \{\frac{1}{\alpha}\})$, φ_α is one-to-one, $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$, $\varphi_\alpha(\mathbb{T}) = \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$;

ii $(\varphi_\alpha)_{-1} = \varphi_{-\alpha}$;

iii $\varphi_\alpha(\alpha) = 0$, $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$, $\varphi'_\alpha(0) = 1 - |\alpha|^2$.

┌
Důkaz (ii)

$$w = \frac{z - \alpha}{1 - \bar{\alpha}z} : w - \bar{\alpha}wz = z - \alpha, \quad w + \alpha = z \cdot (1 + \bar{\alpha}w), \quad z = \frac{w + \alpha}{1 + \bar{\alpha}w} = \varphi_{-\alpha}(w).$$

□

┌
Důkaz (i)

If $z \in \mathbb{T}$, then

$$\left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| = \frac{|z - \alpha|}{|z - \alpha| \cdot |z|} = 1.$$

Hence $\varphi_\alpha(\mathbb{T}) \subseteq \mathbb{T}$. The same is true for $(\varphi_\alpha)_{-1} = \varphi_{-\alpha}$, so $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$. By the fact $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$ and maximum modulus theorem, we get $\varphi_\alpha(\mathbb{D}), \varphi_{-\alpha}(\mathbb{D}) \subset \mathbb{D}$, so $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$. □

┌
Důkaz (iii)

$$\varphi'_\alpha(\alpha) = \lim_{z \rightarrow \alpha} \frac{\varphi_\alpha(z)}{z - \alpha} = \frac{1}{1 - |\alpha|^2}.$$

$$\varphi'_\alpha(0) = \frac{1 - \bar{\alpha}z + (z - \alpha)\bar{\alpha}}{(1 - \bar{\alpha}z)^2} \Big|_{z=0} = 1 - |\alpha|^2.$$

□

Věta 6.7 (Conformal transformations of \mathbb{D})

A function f is one-to-one holomorphic map of \mathbb{D} onto \mathbb{D} , iff there are $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z} (= \text{rot}_\theta \cdot \varphi_\alpha), \quad z \in \mathbb{D}.$$

┌ *Důkaz*

„ \Leftarrow “: by the previous lemma. „ \Rightarrow “: Let $\alpha \in \mathbb{D}$ and $f(\alpha) = 0$. Then $g := f \circ \varphi_{-\alpha}$ is a one-to-one holomorphic map of \mathbb{D} onto \mathbb{D} and $g(0) = 0$. By the Schwarz lemma, for $z \in \mathbb{D}$, $|g(z)| \leq |z|$, $|g_{-1}(z)| \leq |z|$, so $|g(z)| = |z|$. By Schwarz lemma, g is a rotation. \square

Lemma 6.8 (Schwarz–Pick)

Let $F \in \mathcal{H}(\mathbb{D})$, $F(\mathbb{D}) \subset \mathbb{D}$ and $F(\alpha) = \beta$. Then $|F'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. If the equality occurs, then $F(z) = \varphi_{-\beta}(\lambda \varphi_{\alpha}(z))$, $z \in \mathbb{D}$, for $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

In particular $|F'(0)| < 1$ unless F is a rotation.

┌ *Důkaz*

Use the Schwarz lemma for $f := \varphi_{\beta} \circ F \circ \varphi_{-\alpha}$. Then $|f'(0)| \leq 1$ and

$$f'(0) = \varphi'_{\beta}(\beta) \circ F'(\alpha) \circ \varphi'_{-\alpha}(0) = \frac{1}{1-|\beta|^2} \circ F'(\alpha) \circ (1-|\alpha|^2)$$

└ If $\alpha = 0 = \beta$ and F is not rotation, then $|F'(0)| < 1$. \square

Věta 6.9 (Riemann)

Let $\emptyset \neq G \subsetneq \mathbb{C}$ be a simply connected domain. Then there is a one-to-one holomorphic map $f : G \xrightarrow{\text{onto}} \mathbb{D}$.

┌ *Poznámka*

By this, we finish our proof that conditions (SC1)-(SC8) are equivalent with each other. (In proof we use SC7.)

Důkaz

Let $\emptyset \neq G \subsetneq \mathbb{C}$ be a domain with (SC7). Take a point $z_0 \in G$. Denote by Σ the set of all one-to-one holomorphic maps $\psi : G \rightarrow \mathbb{D}$. Then we have (we will prove it later)

- $\Sigma \neq \emptyset$;
- If $\psi \in \Sigma$ and $\psi(G) \neq \mathbb{D}$, then there is $\tilde{\psi} \in \Sigma$ such that $|\tilde{\psi}'(z_0)| > |\psi'(z_0)|$.

Put $\eta := \sup_{\psi \in \Sigma} |\psi'(z_0)|$. Take $\psi \in \Sigma \neq \emptyset$. Since ψ is one-to-one we have $\psi'(z_0) \neq 0$ and hence $\eta > 0$.

By the definition of η , there are $\psi_n \in \Sigma$, $n \in \mathbb{N}$, such that $|\psi_n'(z_0)| \rightarrow \eta$. Since φ_n , $n \in \mathbb{N}$, are uniformly bounded, by the Montol theorem, there is a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k} \xrightarrow{\text{loc.}} f$ on G . By the Weierstrass theorem, $f \in \mathcal{H}(G)$ and $|f'(z_0)| = \eta > 0$ thus f is not constant (but it is limit of one-to-one functions), so by Hurwitz theorem f is one-to-one.

Of course, $f(G) \subset \overline{\mathbb{D}}$ but by openness of f we have $f(G) \subset \mathbb{D}$. Hence $f \in \Sigma$ and $f(G) = \mathbb{D}$. \square

Důkaz (First property of Σ)

Let $w_0 \in \mathbb{C} \setminus G$. Then by (SC7), there is $\varphi \in \mathcal{H}(G)$ such that $z - w_0 = \varphi^2(z)$, $z \in G$. „If $\varphi(z_1) = \pm \varphi(z_2)$, then $z_1 = z_2$.“: Indeed, $z_1 - w_0 = \varphi^2(z_1) = \varphi^2(z_2) = z_2 - w_0$.

By this, φ is one-to-one and $0 \neq w \in \varphi(G) \implies -w \notin \varphi(G)$. Since $\emptyset \neq \varphi(G)$ is open, there is $0 \notin U(a, r) \subset \varphi(G)$. By the previous implication, we have $U(-a, r) \cap \varphi(G) = \emptyset$, i.e., $|\varphi(z) + a| \geq r$, $\forall z \in G$.

Put $\psi := \frac{r}{2(\varphi(z)+a)}$, $z \in G$. Then $|\psi| \leq \frac{1}{2}$ on G , so $\psi \in \Sigma$. \square

Důkaz (Second property of Σ)

Let $\psi \in \Sigma$ and $\alpha \in \mathbb{D} \setminus \psi(G)$. Consider the map $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$, $z \in \mathbb{D}$. Then $\varphi_\alpha \circ \psi \in \Sigma$ and $\varphi_\alpha \circ \psi \neq 0$ on G .

By (SC7), there is $g \in \mathcal{H}(G)$ such that $\varphi_\alpha \circ \psi = g^2$ on G . Then g is one-to-one, because

$$g(z_1) = g(z_2) \implies g^2(z_1) = g^2(z_2) \implies \varphi_\alpha \circ \psi(z_1) = \varphi_\alpha \circ \psi(z_2) \implies z_1 = z_2.$$

Hence $g \in \Sigma$. If $\beta := g(z_0)$, then put $\tilde{\psi} := \varphi$. Of course, $\tilde{\psi} \in \Sigma$ and $\tilde{\psi}(z_0) = 0$.

Denoting $s(w) := w^2$, $w \in \mathbb{C}$, we have that $\Psi = (\varphi_{-\alpha} \circ s \circ \varphi_{-\beta}) \circ z\tilde{\psi} = F \circ \tilde{\psi}$, where $F := \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. We have $F \in \mathcal{H}(\mathbb{D})$, $F(\mathbb{D}) \subset \mathbb{D}$ and F is not a rotation (because F is not one-to-one). By the Schwarz–Pick lemma, we have $|F'(0)| < 1$. Since $\psi'(z_0) = F'(0) \cdot \tilde{\psi}'(z_0)$, we have $0 < |\psi'(z_0)| < |\tilde{\psi}'(z_0)|$. \square

Definition 6.4

Let $G \subset \mathbb{S}$ be open. We say that $f : G \rightarrow \mathbb{S}$ is a conformal map if f is one-to-one meromorphic on G .

Definition 6.5

Let $\Omega, G \subset \mathbb{S}$ be open. We say that G and Ω are conformally equivalent (we write $G \sim \Omega$) if there is a conformal map $f : G \xrightarrow{\text{onto}} \Omega$.

Příklad

Show that there are just 4 classes of conform equivalent simply connected domains in \mathbb{S} , namely:

$$\emptyset, \quad \mathbb{S}, \quad [\mathbb{C}] := \{\mathbb{S} \setminus \{z_0\} \mid z_0 \in \mathbb{S}\}, \quad [\mathbb{D}].$$

7 Preservation of angles

Definition 7.1

For $z \in \mathbb{C} \setminus \{0\}$, put $A(z) := \frac{z}{|z|}$.

Definition 7.2 (Map preserving angles)

Let $G \subseteq \mathbb{C}$ be open, $f : G \rightarrow \mathbb{C}$, $z_0 \in G$ have $P(z_0) \subset G$ such that $f(z) \neq f(z_0) \forall z \in P(z_0)$. Then we say that f preserves angles (with orientation) at z_0 if $\forall \theta \in \mathbb{R}$:

$$\lim_{r \rightarrow 0^+} e^{-k*\theta} \cdot A[f(z_0 + re^{i\theta}) - f(z_0)]$$

exists and is independent of θ .

Definition 7.3 (Notation)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $\mathbb{R}^2 = \mathbb{C}$, have the total differential $df(z_0)$ at $z_0 \in \mathbb{R}^2 = \mathbb{C}$, i.e.,

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0) - df(z_0)h}{|h|} = 0.$$

Then

$$df(z_0)h = \frac{\partial f}{\partial x}(z_0)h_1 + \frac{\partial f}{\partial y}(z_0)h_2,$$

$$h = (h_1, h_2) = h_1 + ih_2 \in \mathbb{R}^2 = \mathbb{C}.$$

We have $h_1 = \frac{h+\bar{h}}{2}$, $h_2 = \frac{h-\bar{h}}{2i}$ and

$$df(z_0)h = \partial f(z_0)h + \bar{\partial} f(z_0)\bar{h},$$

where

$$\partial f(z_0) := \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) - i \cdot \frac{\partial f}{\partial y}(z_0) \right),$$

$$\bar{\partial} f(z_0) := \frac{1}{2} \left(\frac{\partial f}{\partial x}(z_0) + i \cdot \frac{\partial f}{\partial y}(z_0) \right).$$

Poznámka

We know $f'(z_0)$ exists iff $df(z_0)$ exists and $\bar{\partial} f(z_0) = 0$, in this case $f'(z_0) = \partial f(z_0)$.

Věta 7.1

Let $G, \Omega \subset \mathbb{C}$ be open. Then $f : G \rightarrow \Omega$ is conformal iff f is a diffeomorphism of G onto Ω preserving angles at any point of G .

TODO!!!

8 Harmonic functions

Poznámka

We study $f : \mathbb{C} \rightarrow \mathbb{C}$. Since $\mathbb{C} = \mathbb{R}^2$ we have $z = x + iy$, $x = \Re z$, $y = \Im z$ and $f = u + iv$ with $u = \Re f$, $v = \Im f$.

Poznámka (Observation)

If $G \subseteq \mathbb{C}$ is a domain, $f, g \in \mathcal{H}(G)$ and $\Re f = \Re g$ on G , then there is $c \in \mathbb{C}$ such that $\Im f = \Im g + c$ on G .

┌

Důkaz

It follows from CR condition:

$$\frac{\partial v}{\partial y} = + \frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}.$$

└

□

What are the real parts of holomorphic functions?

Lemma 8.1

If $f \in \mathcal{C}^2(G)$, $G \subseteq \mathbb{C}$ open, then $\partial \bar{\partial} f = \bar{\partial} \partial f = \frac{1}{4} \Delta f$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

┌
Důkaz

$$\partial\bar{\partial}f = \frac{1}{4} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = \frac{1}{4} \Delta f.$$

└

□

Definition 8.1

If $G \subseteq \mathbb{C}$ is open, we say that $u \in \mathcal{C}^2(G)$ is harmonic if $\Delta u = 0$ on G .

TODO example?

┌
Příklad

If $G \subset \mathbb{C}$ is a simply connected domain, $f \in \mathcal{H}(G)$ and $f \neq 0$ on G , then $\log |f| = \Re F$ for some $F \in \mathcal{H}(G)$. In particular, $\log |f|$ is harmonic.

┌
Důkaz

We know that there is a holomorphic branch F of $\log f$, i.e., $F \in \mathcal{H}(G)$ and $f = e^F$. Then $|f| = e^{\Re F}$. □

┌
Důsledek (From examples)

If $G \subset \mathbb{C}$ is open, $0 \neq f \in \mathcal{H}(G)$, then $\log |f|$ is harmonic.

┌
Důkaz

By the previous exercise, $\log |f|$ is harmonic on any open ball U in G . □

TODO example?

Věta 8.2

If $G \subset \mathbb{C}$ is a simply connected domain and $u : G \rightarrow \mathbb{R}$ is harmonic, then there is $f \in \mathcal{H}(G)$ such that $\Re f = u$.

┌
Poznámka

Every harmonic function is locally (but not globally) the real part of a holomorphic function.

┌ If $f \in \mathcal{H}(G)$, then $f' = \partial f = \partial(f + \bar{f}) = 2\partial(\Re f)$, because $\partial\bar{f} = \overline{(\partial f)} = 0$.

└

Důkaz

We have $\partial u \in \mathcal{H}(G)$, because $\bar{\partial}(\partial u) = \frac{1}{4}\Delta u = 0$. Then there is $f_0 \in \mathcal{H}(G)$ such that $f'_0 = 2\partial u$.

By remark, $2\partial(\Re f_0) = f'_0 = 2\partial u$, $\partial(\Re f_0 - u) = 0 \implies \frac{\partial}{\partial x}(\Re f_0 - u) = 0$, $\frac{\partial}{\partial y}(\Re f_0 - u) = 0$ on the domain G . \square

Důsledek

Let $G \subset \mathbb{C}$ be open and $u : G \rightarrow \mathbb{R}$ be harmonic. Then $u \in \mathcal{C}^\infty(G)$ and u satisfies the mean value property:

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt = u(z_0),$$

whenever $\overline{U(z_0, r)} \subset G$.

Důkaz

Let $\overline{U(z_0, r)} \subset G$. Take $R \in (r, +\infty)$ such that $U(z_0, R) \subset G$. Then $u = \Re F$ for some $F \in \mathcal{H}(U(z_0, R))$. So $u \in \mathcal{C}^\infty(U(z_0, R))$ and, by the Cauchy integral formula,

$$F(z_0) = \frac{1}{2\pi i} \int_{\varphi} \frac{F(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} F(z_0 + re^{it}) \cdot \frac{ire^{it}}{re^{it}} dt,$$

where $\varphi(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. This implies mean value property. \square

Věta 8.3 (The maximum principle)

Let $G \subset \mathbb{C}$ be a domain and $u : G \rightarrow \mathbb{R}$ be continuous and u satisfies mean value property. If u is not constant, then u does not attain an extreme in G .

Důkaz

Let us assume that $z_0 \in G$ and, say, $u(z_0) \geq u$ on G . Put $M := \{z \in G | u(z) = u(z_0)\}$. Obviously, $\emptyset \neq M$ is closed in G . If we show that M is open, then $M = G$. Let $z_1 \in M$ and $U(z_1, r) \subset G$. We show that $U(z_1, r) \subset M$. Assume that there is $z_2 \in U(z_1, r) \setminus M$.

By mean value property, $u(z_0) = u(z_1) = \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + \varrho e^{it}) dt < u(z_0)$, where $\varrho := |z_1 - z_2|$. The Part inequality is strict because $u \leq u(z_0)$ on G and $u < u(z_0)$ in a neighbourhood of z_2 . \square

Důsledek

Let $G \subset \mathbb{C}$ be bounded and open, $u \in \mathcal{C}(\overline{G})$ and u be harmonic on G . Then

$$\min_{\partial G} u \leq u \leq \max_{\partial G} u \text{ on } G.$$

Poznámka

The assumption of harmonicity can be replaced with mean value property.

Důkaz

Let $z_0 \in G$ and $u(z_0) = \max_{\overline{G}} u$. Then u is constant on the component G_0 of G containing z_0 . So u attains the maximum at the boundary. \square

9 Poisson integral

Poznámka

Let $u : \mathbb{D} \rightarrow \mathbb{R}$ be harmonic, i.e. u is harmonic on $U(0, r)$ for some $r > 1$. Then there is $f \in \mathcal{H}(\mathbb{D})$ such that $u = \Re f$ on \mathbb{D} and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| \leq 1$.

For $|z| \leq 1$, $z = re^{i\theta}$, we have

$$u(z) = \Re f(z) = \Re a_0 + \sum_{n=0}^{+\infty} \frac{1}{2} (a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta}).$$

Hence $u(z) = \sum_{n=-\infty}^{\infty} b_n \cdot r^{|n|} e^{in\theta}$, where $b_n = \Re a_0$ for $n = 0$, $b_n = \frac{1}{2} a_n$ for $n > 0$ and $b_n = \frac{1}{2} \overline{a_{-n}}$ for $n < 0$.

In addition, we have

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{-imt} dt &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} b_n e^{int} \right) e^{-imt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} b_n e^{i(n-m)t} dt = \\ &= \sum_{n=-\infty}^{\infty} b_n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t} dt = b_m. \end{aligned}$$

Putting this equations together, we get, for $z = re^{i\theta}$ with $r \in [0, 1)$,

$$u(z) = \sum_{n=-\infty}^{\infty} r^{|n|} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{in(\theta-t)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} \right) u(e^{it}) dt.$$

Definice 9.1 (Poisson kernel, poisson integral)

For $0 \leq r < 1$, $\theta \in \mathbb{R}$, put

$$P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

$$[Pu](re^{i\theta}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it}) dt$$

for $0 \leq r < 1$, $\theta \in \mathbb{R}$.

Příklad

$$P_r(\theta) = \Re \left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}.$$

┌
Důkaz
└ TODO?

□

Věta 9.1

If u is harmonic on $\overline{\mathbb{D}}$, then $u = Pu$ on \mathbb{D} .

┌
Poznámka

The Poisson formula for harmonic functions is an analogue of the Cauchy integral formula.

└ $P1 = 1$.

Věta 9.2 (Properties of PI)

Let $g \in L^1(\mathbb{T})$. Then Pg is harmonic on \mathbb{D} and if g is continuous at $z_0 \in \mathbb{T}$, then $\lim_{z \rightarrow z_0, z \in \mathbb{D}} [Pg](z) = g(z_0)$. In particular, if $g \in \mathcal{C}(\mathbb{T})$, then the function $u := Pg$ in \mathbb{D} and $u := g$ in \mathbb{T} is continuous on $\overline{\mathbb{D}}$ and harmonic in \mathbb{D} .

Dikaz

WLOG, we assume that g is real-valued. For $|z| < 1$, $z = re^{i\theta}$, we have

$$\begin{aligned} Pg(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left(\frac{1 + re^{i(\theta-t)}}{1 - re^{i(\theta-t)}} \right) g(e^{it}) dt = \\ &= \Re \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} g(e^{it}) dt \right). \end{aligned}$$

So we have $f \in \mathcal{H}(\mathbb{D})$ and $Pg\Re f$ is harmonic on \mathbb{D} .

WLOG, assume $g(z_0) = 0$ (otherwise, consider $g - g(z_0)$ and use the fact that $P1|_{\mathbb{T}} = 1|_{\mathbb{D}}$). Let $\varepsilon > 0$. Take $\delta \in (0, \pi)$ such that $|g(e^{it})| < \varepsilon \forall t \in (\theta_0 - \delta, \theta_0 + \delta)$. Let $z \in \mathbb{D}$, $z = r \cdot e^{i\theta}$ and $|\theta - \theta_0| < \frac{\delta}{2}$.

Then

$$Pg(z) = \frac{1}{2\pi} \int_{\theta_0 - \delta}^{\theta_0 + \delta} P_r(\theta - t) g(e^{it}) dt + \frac{1}{2\pi} \int_A P_r(\theta - t) g(e^{it}) dt = I_1 + I_2$$

with $A := (\theta_0 - \pi, \theta_0 - \delta) \cup (\theta_0 + \delta, \theta_0 + \pi)$. Of course, $P_r > 0$ and

$$P1(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) dt = 1$$

for $0 \leq r < 1$, $\theta \in \mathbb{R}$. Hence $|I_1| < \varepsilon$.

Next we have

$$|I_2| \leq \frac{1}{2\pi} \int_A P_r(\theta - t) |g(e^{it})| dt \leq P_r \left(\frac{\delta}{2} \right) \|g\|_{L^1(\mathbb{T})} \rightarrow 0,$$

because

$$0 < P_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - t)} \stackrel{*}{\leq} P_r \left(\frac{\delta}{2} \right),$$

here $\|g\|_{L^1} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{it})| dt$.

„*“: For $t \in A$, $\frac{\delta}{2} \leq |\theta - t| \leq \frac{3\pi}{2}$. Indeed, if, say, $t \in (\theta_0 + \delta, \theta_0 + \pi)$, then

$$\theta_0 + \delta < t < \theta_0 + \pi,$$

$$-\frac{\delta}{2} < \theta_0 - \theta < \frac{\delta}{2},$$

$$\delta < t - \theta_0 < \pi.$$

By $|I_2| \rightarrow 0$, take $r_0 \in (0, 1)$ such that $|I_2| < \varepsilon \forall r \in (r_0, 1)$. Then for all $z = re^{i\theta}$ with $|\theta - \theta_0| < \frac{\delta}{2}$ and $r \in (r_0, 1)$, we have $|Pg(z)| \leq |I_1| + |I_2| \leq 2\varepsilon$. \square

Definice 9.2 (The Dirichlet problem)

Let $G \subset \mathbb{C}$ be open and bounded. Let $g \in \mathcal{C}(\partial G)$. The Dirichlet problem (DP) is to find $u \in \mathcal{C}(\overline{G})$ such that u is harmonic on G and $u = g$ on ∂G .

┌ *Poznámka*

DP compounds to many problems from physics, e.g., a ? distribution g on ∂G gives an electric potential u on G .

By maximum principle (DP) has at most one solution (indeed, let u_1, u_2 be solutions of DP, then, for $u := u_1 - u_2$ we have $0 = \min_{\partial G} u \leq u \leq \max_{\partial G} u = 0$ on G).

DP has not always (classical) solution, e.g. for $G := \mathbb{D} \setminus \{0\}$ and $g := 0$ on \mathbb{T} , 1 at 0.

DP has a unique solution on "nice" domains, e.g., on domains with Lipschitz boundaries.

In particular, DP has a unique solution on any open disc in \mathbb{C} . (For $G = \mathbb{D}$ we use the previous theorem. For general disc, use a simple transformations of the form $z \mapsto rz + b$ for $r > 0$ and $b \in \mathbb{C}$.)

Věta 9.3

Let $G \subseteq \mathbb{C}$ be open and $u : G \rightarrow \mathbb{C}$. Then u is harmonic on G iff u is continuous on G and satisfies the mean value property.

┌ *Důkaz*

„ \implies “: we already know. „ \impliedby “ Let $u \in \mathcal{C}(G)$ satisfy mean value property on G . Let $U := U(z_0, r)$ with $\overline{U} \subset G$. Let h be a unique solution of DP on U with the boundary data $u|_{\partial U}$. Then $v := u - h$ is continuous on \overline{U} , satisfies mean value property on U and $v = 0$ on ∂U .

By the maximum principle we get $v = 0$ on U , $u = h$ on U . In particular, u is harmonic in U , so in G . □