Příklad (1.)
Show that the Leibniz integral rule (LIR)

$$\frac{d}{dt}\int_{\xi=a(t)}^{b(t)} f(\xi,t)d\xi = \int_{\xi=a(t)}^{b(t)} \frac{\partial f(\xi,t)}{\partial t}d\xi + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$$

where f, a and b are some smooth scalar valued functions, is a special case of Reynolds transport theorem (RTT).

Důkaz
V RTT zvolíme $\forall \mathbf{x}, t : \varphi(\mathbf{x}, t) = 1$, $\chi : [0,1]^3 \times \mathbb{R} \to \mathbb{R}^3$ $\chi(X_1, X_2, X_3, t) = (a(t) + X_1(b(t) - a(t)), X_2 f(a(t)+X_1(b(t)-a(t)), t), X_3)$, tedy nebudeme integrovat „žádnou funkci“, jen nás zajímá změna objemu, který právě v první souřadnici odpovídá proměnné v LIR, v druhé funkční hodnotě v LIR a ve třetí souřadnici se nemění.

Nejdříve dosadíme a pomocí Gaussovy věty a linearity integrálu převedeme RTT na

$$\frac{d}{dt}\int_{V(t)} 1dv = \int_{V(t)} \frac{d1}{dt} + \int_{\partial V(t)} 1 \cdot (\mathbf{v}\cdot\mathbf{n})ds = 0 + \int_{\partial V(t)} 1 \cdot (\mathbf{v}\cdot\mathbf{n})ds.$$

Dále můžeme použít Lagrangeovo kritérium pro vyjádření $\mathbf{v}\cdot\mathbf{n}$:

$$\frac{d}{dt}\int_{V(t)} dv = \int_{\partial V(t)} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|}ds,$$

pro diferencovatelnou funkci g, která je na $\operatorname{int} V(t)$ kladná a na $\partial V(t)$ nulová (oproti přednášce je tedy gradient opačný vůči normále, tedy jsme dostali výraz bez mínus).

Teď bychom si chtěli zvolit správnou funkci g. Můžeme využít toho, že nulový činitel nám zaručuje nulový součin, tedy podmínky $x_i \leqslant h$ zapíšeme jako $(h - x_i)$ a vynásobíme:

$$g(x_1, x_2, x_3, t) = (x_1 - a(t)) \cdot (b(t) - x_1) \cdot x_2 \cdot (f(x_1, t) - x_2) \cdot x_3 \cdot (1 - x_3).$$

Teď můžeme spočítat (podle vzorců pro derivování) vyžadované $\frac{\partial g}{\partial t}$, $\nabla_{\mathbf{x}}$, $g/\ldots$ označuji g bez tohoto členu (tedy v $\ldots = 0$, kde nás tento výraz reálně zajímá, je to dodefinováno intuitivně):

$$\frac{\partial g}{\partial t} = -\frac{da}{dt}\cdot\frac{g}{x_1 - a(t)} + \frac{db}{dt}\cdot\frac{g}{b(t) - x_1} + \frac{\partial f}{\partial t}\cdot\frac{g}{f(x_1,t) - x_2},$$

$$\nabla_{\mathbf{x}} g = \left(\frac{g}{x_1 - a(t)}) - \frac{g}{b(t) - x_1} + \frac{\partial f(x_1,t)}{\partial x_1}\cdot\frac{g}{f(x_1,t) - x_2}, \frac{g}{x_2} - \frac{g}{f(x_1,t) - x_2}, \frac{g}{x_3} - \frac{g}{1 - x_3}\right).$$

Teď se zase vrátíme k RTT. Vždy když $g(\mathbf{x}, t) = 0$, tak musí být nulový jeden z činitelů, tedy integrál přes povrch můžeme rozložit na jednotlivé případy:

- $x_3 = 0$, pak $\frac{\partial g}{\partial t} = 0$, neboť ve všech členech je g nevydělené x_3. Tedy

$$\int_{\partial V(t), x_3=0} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|} ds = \int 0 = 0.$$

- $x_3 = 1$, pak ze stejného důvodu $\int_{\partial V(t), x_3=1} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|} ds = 0$.

- $x_2 = 0$ taktéž dává $\int_{\partial V(t), x_2=0} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|} ds = 0$.

- $x_2 = f(x_1, t)$ je složitější, neboť $\frac{\partial g}{\partial t} = \frac{\partial f(x_1,t)}{\partial t} \frac{g}{f(x_1,t)-x_2}$, jelikož je to zase jediný nenulový člen. Stejně tak $\nabla \mathbf{x} g = (\frac{\partial f(x_1,t)}{\partial x_1} \frac{g}{f(x_1,t)-x_2}, -\frac{g}{f(x_1,t)-x_2}, 0)$. Takže v $\frac{\frac{\partial g}{\partial t}}{|\nabla_x g|}$ můžeme zkrátit $\frac{g}{f(x_1,t)-x_2}$ a zbude nám:

$$\int_{\partial V(t), x_2=f(x_1,t)} \frac{\frac{\partial f(x_1,t)}{\partial t}}{\left|\left(\frac{\partial f(x_1,t)}{\partial x_1}, -1, 0\right)\right|} dv = \int_{\partial V(t), x_2=f(x_1,t)} \frac{\frac{\partial f(x_1,t)}{\partial t}}{\sqrt{\left(\frac{\partial f(x_1,t)}{\partial x_1}\right)^2 + 1}} dv.$$

  Což můžeme z Fubiniovy věty rozložit na nezajímavý integrál přes z a křivkový integrál přes křivku $f(x_1, t)$ tedy

$$\ldots = \int_0^1 \int_{a(t)}^{b(t)} \frac{\frac{\partial f(x_1,t)}{\partial t}}{\sqrt{\left(\frac{\partial f(x_1,t)}{\partial x_1}\right)^2 + 1}} \cdot \sqrt{\left(\frac{\partial f(x_1,t)}{\partial x_1}\right)^2 + 1} dx_1 dx_3 = \int_{a(t)}^{b(t)} \frac{\partial f(x_1,t)}{\partial t} dx_1.$$

- $x_1 = a(t)$, potom (jediné nenulové, Fubini, ...)

$$\frac{\frac{\partial g}{\partial t}}{\nabla_{\mathbf{x}} g} = \frac{-\frac{da}{dt} \cdot \frac{g}{x_1 - a(t)}}{\left|\left(\frac{g}{x-a(t)}, 0, 0\right)\right|} = -\frac{da}{dt} \implies$$

$$\implies \int_{\partial V(t), x_1=a(t)} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|} ds = \int_0^1 \int_0^{f(x_1,t)} -\frac{da}{dt} dx_2 dx_3 = -\frac{da}{dt} f(a(t), t).$$

- $x_1 = b(t)$, potom úplně stejně jako v předchozím

$$\frac{\frac{\partial g}{\partial t}}{\nabla_{\mathbf{x}} g} = \frac{\frac{db}{dt} \cdot \frac{g}{b(t) - x_1}}{\left|\left(-\frac{g}{b(t)-x_1}, 0, 0\right)\right|} = -\frac{da}{dt} \implies$$

$$\implies \int_{\partial V(t), x_1=b(t)} \frac{\frac{\partial g}{\partial t}}{|\nabla_x g|} ds = \int_0^1 \int_0^{f(x_1,t)} \frac{db}{dt} dx_2 dx_3 = \frac{db}{dt} f(b(t), t).$$

Tedy máme

$$\frac{d}{dt} \int_{V(t)} dv = \int_{a(t)}^{b(t)} \frac{\partial f(x_1,t)}{\partial t} dx_1 + \frac{db}{dt} f(b(t), t) - \frac{da}{dt} f(a(t), t),$$

což už je skoro to, co chceme, stačí jen rozložit integrál na levé straně pomocí Fubiniovy věty:

$$\frac{d}{dt}\int_{V(t)} dv = \frac{d}{dt}\int_0^1 \int_{a(t)}^{b(t)} \int_0^{f(x_1,t)} dx_2 dx_1 dx_3 = \frac{d}{dt}\int_{a(t)}^{b(t)} f(x_1,t)dx_1.$$

$\square$

Příklad
Consider the deformation χ given by the following formulae

$$\begin{aligned} r &= f(R), \\ \varphi &= \Phi, \\ z &= Z. \end{aligned}$$

Show that the deformation gradient $\mathbb{F}$ is given by the formula

$$\mathbb{F} = \frac{df}{dR}\mathbf{e}_{\hat{r}} \otimes \mathbf{E}_{\hat{R}} + \frac{f}{R}\mathbf{e}_{\hat{\varphi}} \otimes \mathbf{E}_{\hat{\varphi}} + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}_{\hat{Z}}.$$

Důkaz
Z celého znění zadání máme vzorec pro $\mathbb{F}$ v kartézských souřadnicích. Následně budeme počítat jednotlivé členy:

$$\mathbb{F} = \frac{\partial \chi^{\hat{x}}}{\partial X}\mathbf{e}_{\hat{x}} \otimes \mathbf{E}^{\hat{X}} + \frac{\partial \chi^{\hat{x}}}{\partial Y}\mathbf{e}_{\hat{x}} \otimes \mathbf{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{x}}}{\partial Z}\mathbf{e}_{\hat{x}} \otimes \mathbf{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{y}}}{\partial X}\mathbf{e}_{\hat{y}} \otimes \mathbf{E}^{\hat{X}} + \frac{\partial \chi^{\hat{y}}}{\partial Y}\mathbf{e}_{\hat{y}} \otimes \mathbf{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{y}}}{\partial Z}\mathbf{e}_{\hat{y}} \otimes \mathbf{E}^{\hat{Z}} + \frac{\partial \chi^{\hat{z}}}{\partial X}\mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{X}} + \frac{\partial \chi^{\hat{z}}}{\partial Y}\mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{z}}}{\partial Z}\mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Z}}$$

$$\begin{aligned}
\frac{\partial \chi^{\hat{x}}}{\partial X} &= \frac{\partial}{\partial X}(r\cos\varphi) = \frac{\partial r}{\partial X}\cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial X} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial X}\right)\cos\varphi - r\sin\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial X}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial X} + 0 + 0\right)\cos\varphi - r\sin\varphi\left(0 + \frac{\partial \Phi}{\partial X} + 0\right) \\
\frac{\partial \chi^{\hat{x}}}{\partial Y} &= \frac{\partial}{\partial Y}(r\cos\varphi) = \frac{\partial r}{\partial Y}\cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial Y} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial Y} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial Y} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial Y}\right)\cos\varphi - r\sin\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial Y} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial Y} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial Y}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial Y} + 0 + 0\right)\cos\varphi - r\sin\varphi\left(0 + \frac{\partial \Phi}{\partial Y} + 0\right) \\
\frac{\partial \chi^{\hat{x}}}{\partial Z} &= \frac{\partial}{\partial Z}(r\cos\varphi) = \frac{\partial r}{\partial Z}\cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial Z} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial Z} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial Z} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial Z}\right)\cos\varphi - r\sin\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial Z} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial Z} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial Z}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial Z} + 0 + 0\right)\cos\varphi - r\sin\varphi\left(0 + \frac{\partial \Phi}{\partial Z} + 0\right) = 0 \\
\frac{\partial \chi^{\hat{y}}}{\partial X} &= \frac{\partial}{\partial X}(r\sin\varphi) = \frac{\partial r}{\partial X}\sin\varphi + r\cos\varphi\frac{\partial \varphi}{\partial X} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial X}\right)\sin\varphi + r\cos\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial X} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial X} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial X}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial X} + 0 + 0\right)\sin\varphi + r\cos\varphi\left(0 + \frac{\partial \Phi}{\partial X} + 0\right) \\
\frac{\partial \chi^{\hat{y}}}{\partial Y} &= \frac{\partial}{\partial Y}(r\sin\varphi) = \frac{\partial r}{\partial Y}\sin\varphi + r\cos\varphi\frac{\partial \varphi}{\partial Y} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial Y} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial Y} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial Y}\right)\sin\varphi + r\cos\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial Y} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial Y} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial Y}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial Y} + 0 + 0\right)\sin\varphi + r\cos\varphi\left(0 + \frac{\partial \Phi}{\partial Y} + 0\right) \\
\frac{\partial \chi^{\hat{y}}}{\partial Z} &= \frac{\partial}{\partial Z}(r\sin\varphi) = \frac{\partial r}{\partial Z}\sin\varphi + r\cos\varphi\frac{\partial \varphi}{\partial Z} = \left(\frac{\partial r}{\partial R}\frac{\partial R}{\partial Z} + \frac{\partial r}{\partial \Phi}\frac{\partial \Phi}{\partial Z} + \frac{\partial r}{\partial Z}\frac{\partial Z}{\partial Z}\right)\sin\varphi + r\cos\varphi\left(\frac{\partial \varphi}{\partial R}\frac{\partial R}{\partial Z} + \frac{\partial \varphi}{\partial \Phi}\frac{\partial \Phi}{\partial Z} + \frac{\partial \varphi}{\partial Z}\frac{\partial Z}{\partial Z}\right) = \\
&= \left(\frac{df}{dR}\frac{\partial R}{\partial Z} + 0 + 0\right)\sin\varphi + r\cos\varphi\left(0 + \frac{\partial \Phi}{\partial Z} + 0\right) = 0
\end{aligned}$$

$$\frac{\partial \chi^{\hat{z}}}{\partial X} = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial X} = 0, \quad \frac{\partial \chi^{\hat{z}}}{\partial Y} = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial Y} = 0, \quad \frac{\partial \chi^{\hat{z}}}{\partial Z} = \frac{\partial z}{\partial Z} = 1$$

$$\implies \mathbb{F} = \frac{\partial \chi^{\hat{x}}}{\partial X}\mathbf{e}_{\hat{x}} \otimes \mathbf{E}^{\hat{X}} + \frac{\partial \chi^{\hat{x}}}{\partial Y}\mathbf{e}_{\hat{x}} \otimes \mathbf{E}^{\hat{Y}} + \frac{\partial \chi^{\hat{y}}}{\partial X}\mathbf{e}_{\hat{y}} \otimes \mathbf{E}^{\hat{X}} + \frac{\partial \chi^{\hat{y}}}{\partial Y}\mathbf{e}_{\hat{y}} \otimes \mathbf{E}^{\hat{Y}} + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Z}} = \left(\frac{\partial \chi^{\hat{x}}}{\partial X}\mathbf{e}_{\hat{x}} + \frac{\partial \chi^{\hat{y}}}{\partial X}\mathbf{e}_{\hat{y}}\right) \otimes \mathbf{E}^{\hat{X}} + \left(\frac{\partial \chi^{\hat{x}}}{\partial Y}\mathbf{e}_{\hat{x}} + \frac{\partial \chi^{\hat{y}}}{\partial Y}\mathbf{e}_{\hat{y}}\right) \otimes \mathbf{E}^{\hat{Y}} + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Z}}$$

$$\mathbf{e}_{\hat{x}} = (\cos\varphi)\mathbf{e}_{\hat{r}} - (\sin\varphi)\mathbf{e}_{\hat{\varphi}}, \quad \mathbf{e}_{\hat{y}} = (\sin\varphi)\mathbf{e}_{\hat{r}} + (\cos\varphi)\mathbf{e}_{\hat{\varphi}}$$

$$\frac{\partial \chi^{\hat{x}}}{\partial X}\mathbf{e}_{\hat{x}} + \frac{\partial \chi^{\hat{y}}}{\partial X}\mathbf{e}_{\hat{y}} = \left(\frac{df}{dR}\frac{\partial R}{\partial X}\cos\varphi - \frac{\partial \Phi}{\partial X} r\sin\varphi\right)((\cos\varphi)\mathbf{e}_{\hat{r}} - (\sin\varphi)\mathbf{e}_{\hat{\varphi}}) + \left(\frac{df}{dR}\frac{\partial R}{\partial X}\sin\varphi + \frac{\partial \Phi}{\partial X} r\cos\varphi\right)((\sin\varphi)\mathbf{e}_{\hat{r}} + (\cos\varphi)\mathbf{e}_{\hat{\varphi}}) =$$

$$= \frac{df}{dR}\frac{\partial R}{\partial X}\mathbf{e}_{\hat{r}} + 0 + 0 + \frac{\partial \Phi}{\partial X} r\mathbf{e}_{\hat{\varphi}}$$

$$\frac{\partial \chi^{\hat{x}}}{\partial Y}\mathbf{e}_{\hat{x}} + \frac{\partial \chi^{\hat{y}}}{\partial Y}\mathbf{e}_{\hat{y}} = \left(\frac{df}{dR}\frac{\partial R}{\partial Y}\cos\varphi - \frac{\partial \Phi}{\partial Y} r\sin\varphi\right)((\cos\varphi)\mathbf{e}_{\hat{r}} - (\sin\varphi)\mathbf{e}_{\hat{\varphi}}) + \left(\frac{df}{dR}\frac{\partial R}{\partial Y}\sin\varphi + \frac{\partial \Phi}{\partial Y} r\cos\varphi\right)((\sin\varphi)\mathbf{e}_{\hat{r}} + (\cos\varphi)\mathbf{e}_{\hat{\varphi}}) =$$

$$= \frac{df}{dR}\frac{\partial R}{\partial Y}\mathbf{e}_{\hat{r}} + 0 + 0 + \frac{\partial \Phi}{\partial Y} r\mathbf{e}_{\hat{\varphi}}$$

$$\implies \mathbb{F} = \left(\frac{df}{dR}\frac{\partial R}{\partial X}\mathbf{e}_{\hat{r}} + \frac{\partial \Phi}{\partial X} r\mathbf{e}_{\hat{\varphi}}\right) \otimes \mathbf{E}^{\hat{X}} + \left(\frac{df}{dR}\frac{\partial R}{\partial Y}\mathbf{e}_{\hat{r}} + \frac{\partial \Phi}{\partial Y} r\mathbf{e}_{\hat{\varphi}}\right) \otimes \mathbf{E}^{\hat{Y}} + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Z}} = \frac{df}{dR}\mathbf{e}_{\hat{r}} \otimes \left(\frac{\partial R}{\partial X}\mathbf{E}^{\hat{X}} + \frac{\partial R}{\partial Y}\mathbf{E}^{\hat{Y}}\right) + r\mathbf{e}_{\hat{\varphi}} \otimes \left(\frac{\partial \Phi}{\partial X}\mathbf{E}^{\hat{X}} + \frac{\partial \Phi}{\partial Y}\mathbf{E}^{\hat{Y}}\right) + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}^{\hat{Z}}$$

$$\mathbf{E}^{\hat{X}} = (\cos\Phi)\mathbf{E}^{\hat{R}} - (\sin\Phi)\mathbf{E}^{\hat{\Phi}}, \quad \mathbf{E}^{\hat{Y}} = (\sin\Phi)\mathbf{E}^{\hat{R}} + (\cos\Phi)\mathbf{E}^{\hat{\Phi}}, \quad R^2 = X^2 + Y^2, \quad R = \sqrt{X^2 + Y^2}, \quad \sin\Phi = \frac{Y}{R}, \quad \cos\Phi = \frac{X}{R}$$

$$\frac{\partial R}{\partial X}\mathbf{E}^{\hat{X}} + \frac{\partial R}{\partial Y}\mathbf{E}^{\hat{Y}} = \frac{2X\mathbf{E}^{\hat{X}} + 2Y\mathbf{E}^{\hat{Y}}}{2\sqrt{X^2 + Y^2}} = \frac{X\mathbf{E}^{\hat{X}} + Y\mathbf{E}^{\hat{Y}}}{R} = \frac{X(\cos\Phi)\mathbf{E}^{\hat{R}} - X(\sin\Phi)\mathbf{E}^{\hat{\Phi}} + Y(\sin\Phi)\mathbf{E}^{\hat{R}} + Y(\cos\Phi)\mathbf{E}^{\hat{\Phi}}}{R} =$$

$$= \frac{\frac{X^2}{R}\mathbf{E}^{\hat{R}} - \frac{XY}{R}\mathbf{E}^{\hat{\Phi}} + \frac{Y^2}{R}\mathbf{E}^{\hat{R}} + \frac{XY}{R}\mathbf{E}^{\hat{\Phi}}}{R} = \frac{X^2 + Y^2}{R^2}\mathbf{E}^{\hat{R}} = \mathbf{E}^{\hat{R}}$$

$$\frac{\partial \Phi}{\partial X}\mathbf{E}^{\hat{X}} + \frac{\partial \Phi}{\partial Y}\mathbf{E}^{\hat{Y}} = \frac{-Y\mathbf{E}^{\hat{X}} + X\mathbf{E}^{\hat{Y}}}{X^2 + Y^2} = \frac{-Y\mathbf{E}^{\hat{X}} + X\mathbf{E}^{\hat{Y}}}{R^2} = \frac{-Y(\cos\Phi)\mathbf{E}^{\hat{R}} + Y(\sin\Phi)\mathbf{E}^{\hat{\Phi}} + X(\sin\Phi)\mathbf{E}^{\hat{R}} + X(\cos\Phi)\mathbf{E}^{\hat{\Phi}}}{R^2} =$$

$$= \frac{-\frac{XY}{R}\mathbf{E}^{\hat{R}} + \frac{Y^2}{R}\mathbf{E}^{\hat{\Phi}} + \frac{XY}{R}\mathbf{E}^{\hat{R}} + \frac{X^2}{R}\mathbf{E}^{\hat{\Phi}}}{R^2} = \frac{Y^2 + X^2}{R^3}\mathbf{E}^{\hat{\Phi}} = \frac{1}{R}\mathbf{E}^{\hat{\varphi}}$$

$$\implies \mathbb{F} = \frac{df}{dR}\mathbf{e}_{\hat{r}} \otimes \mathbf{E}_{\hat{R}} + \frac{r}{R}\mathbf{e}_{\hat{\varphi}} \otimes \mathbf{E}_{\hat{\varphi}} + \mathbf{e}_{\hat{z}} \otimes \mathbf{E}_{\hat{Z}}.$$

$\square$