

TODO!!!

Definice 0.1 (Dot product on the space of matrices)

$$\mathbb{A} : \mathbb{B} = \text{tr}(\mathbb{A}\mathbb{B}^T).$$

Definice 0.2 (Norm of matrix)

$$|\mathbb{A}| = (\mathbb{A} : \mathbb{A})^{\frac{1}{2}}.$$

Příklad

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

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Důkaz

$$\mathbf{u} \cdot (\mathbf{a} \otimes \mathbf{b})^T \mathbf{v} = (\mathbf{a} \otimes \mathbf{b}) \mathbf{u} \cdot \mathbf{v} = (\mathbf{a}(\mathbf{b} \cdot \mathbf{u})) \mathbf{v} = (\mathbf{b} \cdot \mathbf{u})(\mathbf{a} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{b}(\mathbf{a} \cdot \mathbf{v})) = \mathbf{u} \cdot (\mathbf{b} \otimes \mathbf{a}) \mathbf{v}.$$

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Příklad

$$\det(e^{\mathbb{A}}) = e^{\text{tr} \mathbb{A}}.$$

┌ *Důkaz*

$$e^{\mathbb{A}} = \lim \left(\mathbb{I} + \frac{\mathbb{A}}{n} \right)^n.$$

$$\det e^{\mathbb{A}} = \lim_{n \rightarrow \infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n} \right)^n \right) = \lim_{n \rightarrow \infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n} \right) \right)^n = ?$$

Subtask: Is there an approximation for $\det(\mathbb{I} + \mathbb{S})$, where \mathbb{S} is a „small“ matrix. Yes, we did it (KontinuumDU1.pdf) for $\mathbb{S} \in \mathbb{R}^{3 \times 3}$:

$$\det(\mathbb{I} + \mathbb{S}) = \det \mathbb{I} + \text{tr}(\mathbb{I} \text{ cof } \mathbb{S}) + \text{tr}(\mathbb{S}^T \text{ cof } \mathbb{I}) + \det \mathbb{S} \approx 1 + \text{tr}(\mathbb{S}^T \text{ cof } \mathbb{I}) + o(\mathbb{S}^2) = 1 + \text{tr}(\mathbb{S}) + o(\mathbb{S}^2).$$

And for $\mathbb{S} \in \mathbb{R}^{n \times n}$, one can see that:

$$\begin{aligned} \det(\mathbb{I} + \mathbb{S}) &= \det \begin{pmatrix} 1 + s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & 1 + s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \dots & 1 + s_{nn} \end{pmatrix} = (1 + s_{11})(1 + s_{22}) \cdot \dots \cdot (1 + s_{nn}) + o(\mathbb{S}^2) = \\ &= 1 + s_{11} + s_{22} + \dots + s_{nn} + o(\mathbb{S}^2) = 1 + \text{tr } \mathbb{S} + o(\mathbb{S}^2). \\ &? = \lim_{n \rightarrow \infty} \left(1 + \frac{\text{tr } \mathbb{A}}{n} + \dots \right)^n = e^{\text{tr } \mathbb{A}}. \end{aligned}$$

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Tvrzení 0.1

$$\det(\mathbb{I} + \mathbb{S}) = 1 + \text{tr } \mathbb{S} + \dots$$

Definice 0.3 (Gateaux derivative)

$$Df(\mathbf{x})[\mathbf{y}] := \frac{d}{d\tau} f(\mathbf{x} + \tau \mathbf{y})|_{\tau=0}.$$

Definice 0.4 (Fréchet derivative)

$f: U \rightarrow V$:

$$\lim_{\|\mathbf{y}\|_U \rightarrow 0} \frac{\|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - Df(\mathbf{x})[\mathbf{y}]\|_V}{\|\mathbf{y}\|_V} = 0.$$

┌ *Poznámka*

Sometimes we write $\nabla f(\mathbf{x}) \cdot \mathbf{y}$ instead of $Df(\mathbf{x})[\mathbf{y}]$ (from Riesz representation theorem).

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For matrices ($\varphi: \mathbb{A} \in \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$):

$$\frac{\|\varphi(\mathbb{A} + \mathbb{B}) - \varphi(\mathbb{A}) - D\varphi(\mathbb{A})[\mathbb{B}]\|_{\mathbb{R}}}{\|\mathbb{B}\|_{\mathbb{R}^{3 \times 3}}}.$$

Poznámka

We write $\frac{\partial \varphi}{\partial \mathbb{A}}(\mathbb{A}) : \mathbb{B}$ instead of $D\varphi(\mathbb{A})[\mathbb{B}]$, where $\frac{\partial \varphi}{\partial \mathbb{A}}(\mathbb{A})$ is right matrix. Warning $\frac{\partial \varphi}{\partial \mathbb{A}}(\mathbb{A}) \neq D\varphi(\mathbb{A})$, because of transposition ($\mathbb{A} : \mathbb{B} = \text{tr}(\mathbb{A}\mathbb{B}^T) = \text{tr}(\mathbb{A}^T\mathbb{B})$).

Příklad

$$\frac{\partial \text{tr } \mathbb{A}}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\text{tr}(\mathbb{A} + \tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}(\text{tr } \mathbb{A} + \tau \text{tr } \mathbb{B})|_{\tau=0} = \text{tr } \mathbb{B} = \mathbb{I} : \mathbb{B}.$$

So $\frac{\partial \text{tr } \mathbb{A}}{\partial \mathbb{A}} = \mathbb{I}$.

Příklad

$$\begin{aligned} \frac{\partial \det \mathbb{A}}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] &= \frac{d}{d\tau}(\det(\mathbb{A} + \tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}(\det(\mathbb{A}) \cdot \det(\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B}))|_{\tau=0} = \\ &= \frac{d}{d\tau}((\det \mathbb{A}) \cdot (1 + \tau \text{tr}(\mathbb{A}^{-1}\mathbb{B}) + o(\tau^2)))|_{\tau=0} = (\det \mathbb{A}) \text{tr}(\mathbb{A}^{-1}\mathbb{B}) = \\ &= (\det \mathbb{A}) \text{tr}((\mathbb{A}^{-T})^T \mathbb{B}) = ((\det \mathbb{A})\mathbb{A}^{-T}) : \mathbb{B}. \end{aligned}$$

So $\frac{\partial \det \mathbb{A}}{\partial \mathbb{A}} = (\det \mathbb{A})\mathbb{A}^{-T} = \text{cof}(\mathbb{A})$.

Příklad

$\mathbb{A} : \mathbb{R} \rightarrow \mathbb{R}^{3 \times 3}$.

$$\frac{d}{dt}(\det \mathbb{A}(t)) = (\det \mathbb{A}(t)) \text{tr} \left(\mathbb{A}(t)^{-1} \frac{d\mathbb{A}(t)}{dt} \right).$$

Příklad

$\mathbb{F} : \mathbb{A} \in \mathbb{R}^{3 \times 3} \rightarrow \mathbb{F}(\mathbb{A}) \in \mathbb{R}^{3 \times 3}$. $\mathbb{F}(\mathbb{A}) = \mathbb{A}^{-1}$. (We know $\frac{1}{1+x} = 1 - x + \dots$)

$$\begin{aligned} \frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] &= \frac{d}{d\tau}((\mathbb{A} + \tau\mathbb{B})^{-1})|_{\tau=0} = \frac{d}{d\tau}((\mathbb{A}(\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B}))^{-1})|_{\tau=0} = \\ &= \frac{d}{d\tau}((\mathbb{I} + \tau\mathbb{A}^{-1}\mathbb{B})^{-1} \mathbb{A}^{-1})|_{\tau=0} = \frac{d}{d\tau}((\mathbb{I} - \tau\mathbb{A}^{-1}\mathbb{B} + \dots) \mathbb{A}^{-1})|_{\tau=0} = -\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}. \end{aligned}$$

So we have $\frac{\partial (\mathbb{A}^{-1})_{ij}}{\partial (\mathbb{A})_{kl}}(\mathbb{B})_{kl}$.

From chain rule (but this is easily solvable by differentiating $\mathbb{A}^{-1}(t)\mathbb{A}(t) = \mathbb{I}$):

$$\frac{d}{dt}(\mathbb{A}^{-1}) = -\mathbb{A}^{-1} \frac{d\mathbb{A}}{dt} \mathbb{A}^{-1}.$$

Příklad

$$\mathbb{F}(\mathbb{A}) = e^{\mathbb{A}}$$

$$\frac{\partial e^{\mathbb{A}}}{\partial \mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}(e^{\mathbb{A}+\tau\mathbb{B}})|_{\tau=0} = \frac{d}{d\tau} \left(\mathbb{I} + \frac{\mathbb{A} + \tau\mathbb{B}}{1!} + \frac{(\mathbb{A} + \tau\mathbb{B})^2}{2!} \right) |_{\tau=0}.$$

Věta 0.2 (Daleckii–Krein)

\mathbb{A} real symmetric matrix, $\mathbb{A} \in \mathbb{R}^{k \times k}$, $\mathbb{A} = \sum_{i=1}^k \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$, where λ_i are eigenvalues and \mathbf{v}_i are normalised orthogonal ($\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$) eigenvectors.

f continuously differentiable real function defined on open set containing the spectrum of \mathbb{A}

$$\mathbb{F}(\mathbb{A}) := \sum_{i=1}^k f(\lambda_i) \mathbf{v}_i \otimes \mathbf{v}_i =: \sum_{i=1}^k f(\lambda_i) \mathbb{P}_i.$$

Then the formula for the Gateaux derivative of f at point \mathbb{A} in direction \mathbb{X} reads

$$D\mathbb{F}(\mathbb{A})[\mathbb{X}] = \frac{\partial \mathbb{F}}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{i=1}^k \frac{df}{d\lambda} \Big|_{\lambda=\lambda_i} \mathbb{P}_i \mathbb{X} \mathbb{P}_i + \sum_{i=1}^k \sum_{j=1, j \neq i}^k \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \mathbb{P}_i \mathbb{X} \mathbb{P}_j.$$

Sometimes we write $D\mathbb{F}(\mathbb{A})[\mathbb{X}] = f^{[1]}(\mathbb{A}) \circ \mathbb{X}$ (Schur product of matrices, it is point-wise multiplication). Then

$$[f^{[1]}(\mathbb{A})]_{ij} = \begin{cases} \frac{df}{d\lambda} \Big|_{\lambda=\lambda_i}, & i = j, \\ \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, & i \neq j. \end{cases}$$

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Důkaz

No summation conventions, all sums are stated explicitly!

$$\begin{aligned}\mathbb{F}(\mathbb{A}) &= \sum_{i=1}^k f(\lambda_i) \mathbf{v}_i \otimes \mathbf{v}_i = \\ &= \sum_{i=1}^k f(\lambda_i(a_{11}, a_{12}, \dots, a_{21}, \dots)) \mathbf{v}_i(a_{11}, a_{12}, \dots, a_{21}, \dots) \otimes \mathbf{v}_i(a_{11}, a_{12}, \dots, a_{21}, \dots). \\ \frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}} &= \sum_{i=1}^k \left(\frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i \otimes \mathbf{v}_i + f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + f(\lambda_i) \mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \right) = ?.\end{aligned}$$

We derivate $\mathbb{A} \mathbf{v}_i = \lambda_i \mathbf{v}_i$:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}.$$

We multiply (with dot product) it by \mathbf{v}_i :

$$\begin{aligned}\mathbb{P}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbb{A}^T \mathbf{v}_i &= \frac{\partial \lambda_i}{\partial \mathbb{A}} \cdot 1 + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \mathbb{A} \cdot \mathbf{v}_i. \\ \frac{\partial \lambda_i}{\partial \mathbb{A}} &= \mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i.\end{aligned}$$

We again multiply derivative of $\mathbb{A} \mathbf{v}_i = \lambda_i \mathbf{v}_i$, but this time by \mathbf{v}_j :

$$\begin{aligned}\mathbf{v}_j \otimes \mathbf{v}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \lambda_j \mathbf{v}_j &= 0 + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j. \\ (\lambda_j - \lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j &= -\mathbf{v}_j \otimes \mathbf{v}_i.\end{aligned}$$

We also need $(\mathbf{v}_j \otimes \mathbf{v}_i) \mathbb{X}_{ij} = \dots = \mathbb{P}_i \mathbb{X} \mathbb{P}_j$:

$$\dots = (\mathbf{v}_j \otimes \mathbf{v}_i)(\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j) = (\mathbf{v}_j \otimes \mathbf{v}_i) \mathbb{X}(\mathbf{v}_j \otimes \mathbf{v}_j).$$

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TODO!!!

1 Kinematics

Definice 1.1

We have some abstract body with point P . We can look at it in reference configuration (some point in past), where $K_0(P) = \mathbf{X}$ (K_0 = placer), $t = t_0$. Or in current configuration

(how it is situated now), where $K_t(P) = \mathbf{x}$.

The change of configuration, χ in $\mathbf{x} = \chi(\mathbf{X}, t)$ is called deformation (but it contains translation and rotation too!).

Definice 1.2

Let us consider quantity θ that describes the given material point. We can describe it by:

- $\theta(P, t)$;
- $\hat{\theta}(\mathbf{X}, t)$ (referential/Lagrangian description, commonly used for solids because deformation is with respect to reference configuration);
- $\tilde{\theta}(\mathbf{x}, t)$ (spatial/Eulerian description, commonly used for fluids because velocity is time-local property).

But people write those functions without $\hat{\cdot}$ or $\tilde{\cdot}$

Poznámka

$$\tilde{\theta}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} = \hat{\theta}(\mathbf{X}, t).$$

Definice 1.3 (Deformation gradient)

$$\begin{aligned} d\mathbf{x} &= \mathbf{x}_2 - \mathbf{x}_1 = \chi(\mathbf{X}_2, t) - \chi(\mathbf{X}_1, t) = \\ &= \chi(\mathbf{X}_1 + d\mathbf{X}, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1, t) + \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X} + \dots - \chi(\mathbf{X}_1, t) = \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}. \end{aligned}$$

$$\mathbb{F}(\mathbf{X}, t) := \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}. \quad d\mathbf{x} = \mathbb{F}d\mathbf{X}$$

Poznámka

It can be derived by derivatives on curves (see lecture).

Důsledek

Transformation of infinitesimal line segment: $d\mathbf{x} = \mathbb{F}d\mathbf{X}$.

Transformation of infinitesimal surface elements: $d\mathbf{s} = (\det \mathbb{F})\mathbb{F}^{-T}d\mathbf{S} = \text{cof } \mathbb{F}d\mathbf{S}$.

Transformation of infinitesimal volume: $dv = (\det \mathbb{F})dV$.

Důsledek (In tangent spaces)

$$F(\mathbf{X}, t_0) = f(\chi(\mathbf{X}, t), t).$$

Representation theorem:

$$(GradF)\mathbf{W} = \mathbf{U}_{GradF} \cdot \mathbf{W}$$

$$(Gradf)\mathbf{w} = \mathbf{u}_{Gradf} \cdot \mathbf{w}$$

$$f(\chi(\mathbf{X}, t), t) = F(\mathbf{X}, t_0)$$

$$Gradf(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} = GradF(\mathbf{X}, t_0)$$

$$\mathbf{U}_{GradF} \cdot \mathbf{W} = (GradF)\mathbf{W} = (Gradf)\mathbb{F}\mathbf{W} = (gradf)(\mathbb{F}\mathbf{W}) = \mathbf{u}_{gradf} \cdot \mathbb{F}\mathbf{W} = \mathbb{F}^T \mathbf{u}_{Gradf} \cdot \mathbf{W}.$$

$$\mathbf{u}_{gradf} = \mathbb{F}^{-T} \mathbf{U}_{GradF}.$$

Příklad (Hollow cylinder)

$$r = f(R), \varphi = \Phi, z = Z.$$

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Řešení

$$\mathbb{F} = \frac{\partial \chi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{E}_j$$

$$X_1 = R \cos \Phi, \quad X_2 = R \sin \Phi, \quad x_1 = r \cos \Phi, \quad x_2 = r \sin \Phi.$$

$$x_1 = \chi_1(X_1, X_2, t), \quad x_2 = \chi_2(X_1, X_2, t), \quad x_i = \chi_i(X_j, t).$$

By chain rule:

$$\frac{\partial x_1}{\partial X_2} = \frac{\partial r \cos \Phi}{\partial \partial X_2} = \frac{\partial}{\partial X_2} f(R) \cos \Phi.$$

$$\mathbb{F} = F_{rR} \mathbf{e}_r \otimes \mathbf{E}_R + F_{r\Phi} \mathbf{e}_r \otimes \mathbf{E}_\Phi + \dots$$

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Řešení

From image:

$$\mathbf{E}_R \xrightarrow{\mathbb{F}} F_{rR} \mathbf{e}_r.$$

$$\mathbf{E}_\Phi \xrightarrow{\mathbb{F}} F_{\varphi\Phi} \mathbf{e}_\varphi$$

$$\text{So } \mathbb{F} = \begin{pmatrix} F_{rR} & 0 \\ 0 & F_{\varphi\Phi} \end{pmatrix}$$

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TODO? (Solution by curve)

Poznámka

How to differentiate in time tensorial quantities related to the current configuration?

Upper convected derivative:

$$\frac{\nabla}{\Delta}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} = \det \mathbb{F}(\mathbf{X}, t) \left[\frac{d}{dt} (\mathbb{F}^{-1}(\mathbf{X}, t) \mathbb{A}(\chi(\mathbf{X}, t), t) \mathbb{F}^{-T}(\mathbf{X}, t)) \right] \mathbb{F}^T(\mathbf{X}, t).$$

1.1 Derivatives

Definition 1.4 (Lagrangian velocity)

$$\mathbf{V}(\mathbf{X}, t) = \frac{d\chi(\mathbf{X}, t)}{dt}.$$

$$\mathbf{v}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)}$$

Definition 1.5 (Eulerian velocity)

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}(\mathbf{X}, t)|_{\mathbf{x}=\chi^{-1}(\mathbf{x}, t)}.$$

Definition 1.6 (Material time derivative)

$\frac{d}{dt}$ = keep \mathbf{X} fixed, and differentiate with respect to time.

$$\begin{aligned} \psi(\mathbf{X}, t) &\rightarrow \frac{d}{dt}\psi(\mathbf{X}, t) = \frac{\partial \psi}{\partial t}(\mathbf{X}, t) \\ \psi(\mathbf{x}, t) &\rightarrow \frac{d}{dt}\psi(\chi(\mathbf{X}, t), t) = \frac{\partial \psi}{\partial t}|_{\mathbf{x}=\chi(\mathbf{X}, t)} + \frac{\partial \psi}{\partial x_i}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \frac{d\chi_i}{dt}(\mathbf{X}, t) = \\ &= \left(\frac{\partial \psi}{\partial t}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} + V_i(\mathbf{X}, t) \frac{\partial \psi}{\partial x_i}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \right) = \\ &= \left(\frac{\partial \psi}{\partial t}(\mathbf{x}, t) + v_i(\mathbf{x}, t) \frac{\partial \psi}{\partial x_i}(\mathbf{x}, t) \right) |_{\mathbf{x}=\chi(\mathbf{X}, t)} \\ \frac{d}{dt}\psi(\mathbf{x}, t) &= \frac{\partial \psi}{\partial t}(\mathbf{x}, t) + (\mathbf{v}(\mathbf{x}, t) \cdot \nabla)\psi(\mathbf{x}, t). \end{aligned}$$

Definition 1.7 (Time derivative of deformation gradient \mathbb{F})

$$\frac{d}{dt}\mathbb{F}(\mathbf{X}, t) = \frac{d}{dt} \left(\frac{\partial \chi(\mathbf{X}, t)}{\partial \mathbf{X}} \right) = \frac{\partial}{\partial \mathbf{X}} \frac{d\chi(\mathbf{X}, t)}{dt} = \frac{\partial}{\partial \mathbf{X}} \mathbf{V}(\mathbf{X}, t) =$$

$$= \frac{\partial}{\partial \mathbf{X}} \mathbf{v}(\chi(\mathbf{X}, t), t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}|_{\mathbf{x}=\chi(\mathbf{X}, t)} \mathbb{F}(\mathbf{X}, t).$$

$$\mathbb{L}(\mathbf{x}, t) := \nabla \mathbf{V}(\mathbf{x}, t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x}, t).$$

Důsledek

$$\frac{d\mathbb{F}}{dt} = \mathbb{L}\mathbb{F}$$

Důsledek

$$\frac{\nabla}{\mathbb{A}} = \frac{d\mathbb{A}}{dt} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^T$$

TODO!!!

Poznámka (Balance laws in Eulerian description (revision, the last lecture))

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0;$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \quad \mathbb{T} = \mathbb{T}^T;$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q;$$

or

$$\varrho \frac{d}{dt} \left(e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) = \operatorname{div}(\mathbb{T}^T \mathbf{v}) + \varrho \mathbf{b} \cdot \mathbf{v} - \operatorname{div} \mathbf{j}_q.$$

Poznámka (Balance laws in Lagrangian description)

Starting with $\frac{d}{dt} \int_{V(t)} \varrho(\mathbf{x}, t) dv = \frac{d}{dt} m_{V(t)} = 0$ i.e. mass remains same: $m_{V(t)} = m_{V(t_0)}$. We integrate over volume:

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) dV = \int_{V(t)} \varrho(\mathbf{x}, t) dv = \int_{V(t_0)} \varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \det \mathbb{F} dV.$$

Localization principle:

$$\varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \det \mathbb{F} = \varrho_R(\mathbf{X}).$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} + \int_{V(t)} \varrho \mathbf{b} dv.$$

$$\int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} = \int_{\partial V(t)} \mathbb{T} \mathbf{n} ds = \int_{\partial V(t_0)} \mathbb{T}(\det \mathbb{F}) \mathbb{F}^{-T} \mathbf{N} dS \stackrel{\text{Stokes}}{=} \int_{V(t_0)} \operatorname{div}_{\mathbf{X}}((\det \mathbb{F}) \mathbb{T} \mathbb{F}^{-T}) dV =: \int_{V(t_0)} (\mathbb{T}_R) dV.$$

$$\mathbb{T}_R(\mathbf{X}, t) := (\det \mathbb{F}(\mathbf{X}, t)) \mathbb{T}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \mathbb{F}^{-T}(\mathbf{X}, t)$$

is first Piola–Kirchhoff stress tensor. Cauchy (\mathbb{T}) is current \rightarrow current. P–K (\mathbb{T}_R) is reference \rightarrow current.

$$\int_{\partial V(t)} dv \rightarrow \int_{\partial V(t_0)} (\operatorname{div}_{\mathbf{X}} \mathbb{T}_R) dV.$$

$$\int_{V(t)} \varrho \mathbf{b} dv = \int_{V(t_0)} \varrho(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \mathbf{b}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R(\mathbf{X}) \mathbf{b} dV.$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t_0)} \varrho \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X}, t) \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R \frac{\partial^2 \chi}{\partial t^2} dV.$$

Altogether:

$$\varrho_R \frac{\partial^2 \chi}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbb{T}_R + \varrho_R \mathbf{b} \quad (\text{Solve for } \chi).$$

$$\mathbb{T} = \mathbb{T}^T \rightarrow \mathbb{T}_R \mathbb{F}^T = \mathbb{F} \mathbb{T}_R^T \text{ (P–K is not symmetric!).}$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q \rightarrow \int_{V(t)} \varrho \frac{de}{dt} dv = \int_{V(t)} \mathbb{T} : \mathbb{L} dv - \int_{V(t)} \operatorname{div} \mathbf{j}_q dv.$$

$$\begin{aligned} \int_{V(t)} \operatorname{div} \mathbf{j}_q dv &= \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds = \int_{\partial V(t_0)} \mathbf{j}_q(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} \cdot \det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-T}(\mathbf{X}, t) \mathbf{N} dS = \\ &= \int_{\partial V(t_0)} (\det \mathbb{F}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t) \mathbf{j}_q(\mathbf{x}, t)) \cdot \mathbf{N} dS = \\ &= \int_{V(t_0)} \operatorname{div}((\det \mathbb{F}) \mathbb{F}^{-1} \mathbf{j}_q) dV. \end{aligned}$$

$\mathbf{J}_q = (\det \mathbb{F}) \mathbb{F}^{-1} \mathbf{j}_q$ is called referential heat flux. (It cannot be given by Fourier's law ($\mathbf{j}_q = k \nabla_{\mathbf{x}} \theta$, $\operatorname{div} \mathbf{j}_q = \operatorname{div}(k \nabla \theta)$).)

$$\begin{aligned} \int_{V(t)} \underbrace{\mathbb{T} : \overbrace{\mathbb{L}}^{\nabla_{\mathbf{x}} \mathbf{v}}}_{\operatorname{tr}(\mathbb{T} \mathbb{L}^T) = \operatorname{tr}(\mathbb{L} \mathbb{T}^T)} dv &= \int_{V(t_0)} (\det \mathbb{F}) \mathbb{T} : \mathbb{L} dV = \\ &= \int_{V(t_0)} \operatorname{tr}((\det \mathbb{F}) \mathbb{T} \mathbb{L}^T) dV = \int_{V(t_0)} \operatorname{tr} \left((\det \mathbb{F}) \mathbb{T} \mathbb{F}^{-T} \left(\frac{d\mathbb{F}}{dt} \right)^T \right) dV = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV. \end{aligned}$$

Altogether

$$\varrho_R \frac{\partial e}{\partial t} = \mathbb{T}_R : \dot{\mathbb{F}} - \operatorname{div}_{\mathbf{X}} \mathbf{J}_q.$$

2 Entropy

Poznámka (Objective)

Find quantity that is increasing/decreasing in time.

Poznámka (With no interior)

$$\varrho \frac{d}{dt} \left(e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) = \operatorname{div} \mathbb{T} + \varrho \mathbf{b} - \operatorname{div} \mathbf{j}_q = \operatorname{div} \mathbb{T} + 0 + \operatorname{div}(k \nabla \theta).$$

Let us work with^a $\operatorname{div} \mathbb{T} = -p_{th} \mathbb{I} + \tilde{\lambda}(\operatorname{div} \mathbf{v}) + 2\mu \mathbb{D}_\delta$, and assume that $\mathbb{T} = -p_{th}(\varrho, \theta) \mathbb{I} + \tilde{\lambda}(\operatorname{div} \mathbf{v}) \mathbb{I} + 2\mu \mathbb{D}_\delta$ (from $\frac{pV}{T} = \text{const}$).

$$\varrho \frac{d}{dt} (e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}) = \operatorname{div}(\mathbb{T}^T \mathbf{v}) - \operatorname{div} \mathbf{j}_q.$$

$$\frac{d}{dt} \int_{V(t)} \varrho (e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}) dv = \int_{\partial V(t)} \mathbb{T}^T \mathbf{v} \cdot \mathbf{n} ds - \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds.$$

The first part is work and it is zero if we have boundary condition $\mathbf{v}|_{\partial V} = 0$. The second part is heat exchange which is zero if we have boundary condition $\mathbf{j}_q \cdot \mathbf{n}|_{\partial V} = 0$. Both boundary conditions together are math way to say system with *no interactions*.

$$\varrho, \theta, \mathbf{v} \rightarrow \varrho, e, \theta.$$

Assume $\eta = \eta(\varrho, e) \rightarrow e = e(\eta, \varrho)$. (We will write $e = e(\eta, \varrho) = e(\eta(\mathbf{x}, t), \varrho(\mathbf{x}, t)) = e(\mathbf{x}, t)$.)

We have 1. Balance of internal energy

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q;$$

2. Chain rule

$$\varrho \frac{de}{dt} = \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} + \frac{\partial e}{\partial \varrho}(\eta, \varrho) \frac{d\varrho}{dt}.$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q - \frac{\partial e}{\partial \varrho}(\eta, \varrho) \frac{d\varrho}{dt},$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q + \frac{\partial e}{\partial \varrho}(\eta, \varrho) \varrho \operatorname{div} \mathbf{v},$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = (-p_{th} \mathbb{I} + (\tilde{\lambda}(\operatorname{div} \mathbf{v}) \mathbb{I} + 2\mu \mathbb{D}_\delta)) : \mathbb{D} - \operatorname{div} \mathbf{j}_q + \frac{\partial e}{\partial \varrho}(\eta, \varrho) \varrho \operatorname{div} \mathbf{v},$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = \left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho) \right) \operatorname{div} \mathbf{v} + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta - \operatorname{div} \mathbf{j}_q,$$

$$\varrho \frac{d\eta}{dt} = \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho) \right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} - \frac{\operatorname{div} \mathbf{j}_q}{\frac{\partial e}{\partial \eta}} + \frac{\tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu |\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial \eta}}.$$

There is no chance that this could be positive. (Its obvious, because the value can flow, so point-wise ≥ 0 is lost case.) But we can integrate over volume. Thus instead of $\frac{d\eta}{dt} \geq 0$

we want just $\frac{d}{dt} \int_{V(t)} \varrho \eta dv \geq 0$.

$$\frac{d}{dt} \int_{V(t)} \varrho \eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} dv - \int_{V(t)} \frac{\operatorname{div} \mathbf{j}_q}{\frac{\partial e}{\partial \eta}} dv + \int_{V(t)} \frac{\tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu |\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial \eta}} dv.$$

The third integral OK, if $\frac{\partial e}{\partial \eta} > 0$.

$$\operatorname{div} \left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}} \right) = \frac{\operatorname{div} \mathbf{j}_q}{\frac{\partial e}{\partial \eta}} + \nabla \left(\frac{1}{\frac{\partial e}{\partial \eta}} \right) \cdot \mathbf{j}_q.$$

$$\frac{d}{dt} \int_{V(t)} \varrho \eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} dv - \int_{V(t)} \operatorname{div} \left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}} \right) dv + \int_{V(t)} \nabla \left(\frac{1}{\frac{\partial e}{\partial \eta}} \right) \cdot \mathbf{j}_q dv + REST.$$

The second integral is zero from Stokes and boundary condition $\mathbf{j}_q \cdot \mathbf{n}|_{\partial V} = 0$. On the third integral, we can use derivative of inverse value:

$$\begin{aligned} \frac{d}{dt} \int_{V(t)} \varrho \eta dv &= \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} dv - \int_{V(t)} \frac{\nabla \left(\frac{\partial e}{\partial \eta} \right) \cdot \mathbf{j}_q}{\left(\frac{\partial e}{\partial \eta} \right)^2} dv + REST = \\ &= \int_{V(t)} \frac{\left(-p_{th} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div} \mathbf{v} dv + k \int_{V(t)} \frac{\nabla \left(\frac{\partial e}{\partial \eta} \right) \cdot \nabla \theta}{\left(\frac{\partial e}{\partial \eta} \right)^2} dv + REST. \end{aligned}$$

If we set $\frac{\partial e}{\partial \eta}(\eta, \varrho) = \theta$, the second integral is non-negative. Moreover, for $\theta \geq 0$ we satisfy the assumption for the "first third integral". Moreover if we enforce $\varrho^2 \frac{\partial e}{\partial \varrho}(\varrho, \eta) = p_{th}(\theta, \varrho)$, the first integral is zero, so we win.

$${}^a\mathbb{D}_\delta := \mathbb{D} - \frac{1}{3}(\operatorname{tr} \mathbb{D})\mathbb{I}. \text{ (Traceless part of } \mathbb{D}.)$$

Poznámka

Volíme si tedy $\tilde{\lambda}, \mu > 0$.

Důsledek

$$\frac{d}{dt} \int_{V(t)} \varrho \eta dv \geq 0$$

is granted for quantity that solves equations

$$e = e(\eta, \varrho), \quad \frac{\partial e}{\partial \eta} = \theta, \quad \varrho^2 \frac{de}{d\varrho} = p_{th}(\theta, \varrho).$$

Příklad

$$p_{th}(\theta, \varrho) = c_V(\gamma - 1)\varrho\theta, \quad e(\theta, \varrho) = c_V\theta.$$

Poznámka (?)

1. Energy is constant.
2. Energy is function of entropy and volume.
3. Entropy increases.

Poznámka

$e = e(\eta, \varrho)$ is given \rightarrow we know everything $\theta = \frac{\partial e}{\partial \eta}(\eta, \varrho)$, $p_{th} = \varrho^2 \frac{\partial e}{\partial \varrho}(\eta, \varrho)$. (Warning: $e = e(\varrho, \theta)$ is not enough!)

Poznámka

Is there a better function that will allow us to do something like this?

Definice 2.1 (Helmholtz free energy density)

$$\psi(\theta, \varrho) := e(\eta, \varrho)|_{\eta=\eta(\theta, \varrho)} - \theta\eta|_{\eta=\eta(\theta, \varrho)}.$$

Poznámka

This is the Legendre transformation of internal energy.

Důsledek

$$\frac{\partial \psi}{\partial \theta}(\theta, \varrho) = -\eta, \quad \frac{\partial \psi}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta, \varrho)} \quad \left(= \frac{p_{th}}{\varrho^2} \right).$$

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Důkaz

$$\begin{aligned} \frac{\partial \psi}{\partial \theta}(\theta, \varrho) &= \frac{\partial e(\eta, \varrho)}{\partial \eta}|_{\eta=\eta(\theta, \varrho)} \frac{\partial \eta}{\partial \theta}(\theta, \varrho) - \eta|_{\eta=\eta(\theta, \varrho)} - \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\eta|_{\eta=\eta(\theta, \varrho)}. \\ \frac{\partial \psi}{\partial \varrho}(\theta, \varrho) &= \frac{\partial e}{\partial \eta}(\eta, \theta) \frac{\partial \eta}{\partial \varrho}(\theta, \varrho) + \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta, \varrho)} - \theta \frac{\partial \eta}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta, \varrho)}. \end{aligned}$$

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□

Poznámka (Why do we call $c_{V,ref}$ the specific heat at constant volume?)

(Constant volume = constant density.) $\mathbf{j}_q = -k\nabla\theta$, $\mathbb{T} \approx -p_{th}\mathbb{I}$, $\mathbb{D} = \frac{1}{2}((\nabla\mathbf{v}) + (\nabla\mathbf{v})^T)$.

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q,$$

$$\varrho \frac{\partial e}{\partial \theta}(\theta, \varrho) \frac{d\theta}{dt} + \varrho \frac{\partial e}{\partial \varrho} \frac{d\varrho}{dt} = -p_{th}(\underbrace{\mathbf{Div} \mathbf{v}}_{\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho \frac{\partial e}{\partial \theta}(\theta, \varrho) \frac{d\theta}{dt} + 0 = 0 + \operatorname{div}(k\nabla\theta),$$

$$\int_V \varrho \frac{\partial e}{\partial \theta}(\theta, \varrho) \frac{d\theta}{dt} dv = \int_{\partial V} (k\nabla\theta) \mathbf{n} ds.$$

So in left we multiply ϱ , difference of temperature and some $c_V(\theta, \varrho) := \frac{\partial e}{\partial \theta}(\theta, \varrho)$. (On the right there is flow of heat, \mathbf{j}_q , through boundary).

For calorically perfect ideal gas: $e = e(\varrho, \theta) = c_{V,ref} \cdot \theta$.

Poznámka (How to get specific heat at constant pressure?)

$$e = e(\eta(\theta, p_{th}), \varrho(\theta, p_{th})), \quad \varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q = -p_{th} \cdot (\operatorname{div} \mathbf{v}) + \operatorname{div}(k\nabla\theta).$$

Chain rule:

$$\frac{de}{dt} = \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{\partial \eta}{\partial \theta}(\theta, p_{th}) \frac{d\theta}{dt} + \frac{\partial e}{\partial \varrho}(\eta, \varrho) \frac{d\varrho}{dt} + \dots \frac{dp_{th}}{dt} = \theta \frac{\partial \eta}{\partial \theta}(\theta, p_{th}) \frac{d\theta}{dt} + \frac{\partial e}{\partial \varrho}(\eta, \varrho) \frac{d\varrho}{dt}.$$

$$\varrho \theta \frac{\partial \eta}{\partial \theta}(\theta, p_{th}) \frac{d\theta}{dt} + \varrho \frac{\partial e}{\partial \varrho}(\eta, \varrho) \frac{d\varrho}{dt} = -p_{th} \cdot (\operatorname{div} \mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho \theta \frac{\partial \eta}{\partial \theta}(\theta, p_{th}) \frac{d\theta}{dt} + \varrho \frac{p_{th}}{\varrho^2} (\varrho \cdot (\operatorname{div} \mathbf{v})) = -p_{th} \cdot (\operatorname{div} \mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho \theta \frac{\partial \eta}{\partial \theta}(\theta, p_{th}) \frac{d\theta}{dt} = \operatorname{div}(k\nabla\theta).$$

So $c_p(\theta, p_{th}) := \theta \frac{\partial \eta}{\partial \theta}(\theta, p_{th})$ is specific heat at constant pressure.

Poznámka (Alternative formula for the specific heat at constant volume)

Chain rule:

$$\theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = \frac{\partial}{\partial \theta} e(\theta, \varrho) = c_V(\theta, \varrho).$$

$$c_V(\theta, \varrho) := \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho).$$

Poznámka (Another alternative formula for the specific heat at constant volume, for usage in practice)

$$c_V(\theta, \varrho) = \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho),$$

because $\eta(\theta, \varrho) = -\frac{\partial \psi}{\partial \theta}(\theta, \varrho)$ (property of Helmholtz free energy).

Conclusion: If $\psi(\theta, \varrho)$ is given, then

$$c_V(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho), \quad p_{th}(\theta, \varrho) = \varrho^2 \frac{\partial \psi}{\partial \varrho}(\theta, \varrho).$$

Poznámka (Where are my evolution equations?)

Unknowns: $\mathbf{v}, \varrho, \theta$.

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \quad \mathbb{T} = -p_{th}(\theta, \varrho) \mathbb{I} + \tilde{\lambda}(\operatorname{div} \mathbf{v}) \mathbb{I} + 2\mu \mathbb{D}_\delta.$$

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Poznámka (Third equation)

TODO!!!

$$\varrho \theta \frac{d\eta}{dt} = \operatorname{div}(k \nabla \theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$

$$\frac{d\eta}{dt} = -\frac{d}{dt} \left(\frac{\partial \psi}{\partial \theta}(\theta, \varrho) \right) = -\frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho) \frac{d\theta}{dt} - \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho) \frac{d\varrho}{dt}.$$

$$\theta c_V(\theta, \varrho) \frac{d\theta}{dt} = -\varrho^2 \theta \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho) \operatorname{div} \mathbf{v} + \operatorname{div}(k \nabla \theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta,$$

$$\theta c_V(\theta, \varrho) \frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta} \left(\varrho^2 \frac{\partial \psi}{\partial \varrho} \right) \operatorname{div} \mathbf{v} + \operatorname{div}(k \nabla \theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$

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$$\theta c_V(\theta, \varrho) \frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta} \left(\varrho^2 \frac{\partial \psi}{\partial \varrho} \right) \operatorname{div} \mathbf{v} + \operatorname{div}(k \nabla \theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta.$$

And we can get $c_V(\theta, \varrho)$ and p_{th} from $\psi = \psi(\theta, \varrho)$.

Poznámka (Why do we call γ the adiabatic exponent?)

(This applies only to the ideal gas.)

$$p_{th} = c_{V,ref}(\gamma - 1) \varrho \theta,$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q = \mathbb{T} : \mathbb{D}, \quad \mathbb{T} = -p_{th} \mathbb{I},$$

$$\varrho \frac{de}{dt} = -p_{th} \operatorname{div} \mathbf{v},$$

$$e = c_{V,ref} \cdot \theta$$

$$\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt} \iff \frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0.$$

$$\frac{dp_{th}}{dt} = c_{V,ref}(\gamma - 1) \frac{d\varrho}{dt} \theta + c_{V,ref}(\gamma - 1) \varrho \frac{d\theta}{dt},$$

$$\left(\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref} \theta \frac{d\varrho}{dt} \right)$$

$$\frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref} \theta \frac{d\varrho}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{c_{V,ref} \theta}{p_{th}} + \frac{1}{\varrho} \right) \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{1}{\varrho} \left(\frac{1}{\gamma - 1} + 1 \right) \right) \frac{d\varrho}{dt}.$$

$$\left(\frac{c_{V,ref} \theta}{p_{th}} + \frac{1}{\varrho} = ? \frac{c_{V,ref} \theta}{c_{V,ref} \theta \varrho (\gamma - 1)} \right)$$

$$\frac{d}{dt}(L_n p_{th}) = \gamma \frac{d}{dt} L_n \varrho.$$

$$p_{th} = p_{th,ref} \left(\frac{\varrho}{\varrho_{ref}} \right)^\gamma.$$

Poznámka (Where is my hyperbolic equation?)

Assuming isentropic process ($\eta = \text{const}$). $\mathbb{T} = -p_{th} \mathbb{I} = -p_{th}(\varrho, \eta) \mathbb{I}$. ($\operatorname{div} -p_{th} \mathbb{I} = -\nabla p_{th}$.)

$\varrho(\mathbf{x}, t) = \hat{\varrho} + \tilde{\varrho}(\mathbf{x}, t)$ = referential density + small perturbations, similarly $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}} + \tilde{\mathbf{v}}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\mathbf{x}, t)$.

$$\left(\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \quad \varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta) \right)$$

$$\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + (\mathbf{v} \cdot \nabla) \varrho = \frac{\partial}{\partial t} (\hat{\varrho} \tilde{\varrho}) + (\hat{\mathbf{v}} + \tilde{\mathbf{v}}) \cdot \nabla (\hat{\varrho} + \tilde{\varrho}) \approx \frac{\partial \tilde{\varrho}}{\partial t}.$$

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0 \xrightarrow{\text{Linearize}} \frac{\partial \tilde{\varrho}}{\partial t} + \hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta) \xrightarrow{\text{Linearize}} \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial \varrho}(\varrho, \eta)|_{\varrho=\hat{\varrho}} \nabla \tilde{\varrho}.$$

Differentiate by time:

$$\begin{aligned}\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \hat{\varrho} \operatorname{div} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} \right) &= 0 \\ \frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \operatorname{div} \left(\frac{\partial p_{th}}{\partial \varrho} \Big|_{\varrho=\hat{\varrho}} \nabla \tilde{\varrho} \right) &= 0. \\ \frac{\partial^2 \tilde{\varrho}}{\partial t^2} &= \left(\frac{\partial p_{th}}{\partial \varrho} \Big|_{\varrho=\hat{\varrho}} \right) \Delta \tilde{\varrho}.\end{aligned}$$

(Speed of sound: $c = \sqrt{\frac{\partial p_{th}}{\partial \varrho} \Big|_{\varrho=\hat{\varrho}}(\varrho, \eta)}$.)

Poznámka (Stability of test state)

Set

$$\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}}, \quad \hat{\mathbf{v}} = 0, \quad \theta = \hat{\theta} + \tilde{\theta}, \quad \hat{\theta} \neq \hat{\theta}(\mathbf{x}, t), \quad \varrho = \hat{\varrho} + \tilde{\varrho}, \quad \hat{\varrho} \neq \hat{\varrho}(\mathbf{x}, t).$$

Is it $\tilde{\mathbf{v}}, \tilde{\varrho}, \tilde{\theta} \rightarrow 0$?

$$\begin{aligned}\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} &= 0 \longrightarrow \frac{\partial \tilde{\varrho}}{\partial t} + \hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} = 0, \\ \varrho \frac{d\mathbf{v}}{dt} &= \operatorname{div} \mathbb{T} + \varrho \mathbf{b} = \operatorname{div} \mathbb{T} \longrightarrow \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial \varrho} \Big|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta} \Big|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \nabla \tilde{\theta} + \operatorname{div} \left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right), \\ \varrho_{cV} \frac{d\theta}{dt} &= -\theta \frac{\partial p_{th}}{\partial \theta}(\theta, \varrho) \operatorname{div} \mathbf{v} + \operatorname{div}(k \nabla \theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta = 0 \longrightarrow \\ &\longrightarrow \hat{\varrho}_{cV} \Big|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \frac{\partial \tilde{\theta}}{\partial t} = \operatorname{div}(k \nabla \tilde{\theta}) - \hat{\theta} \frac{\partial p_{th}}{\partial \varrho} \Big|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \operatorname{div} \tilde{\mathbf{v}}.\end{aligned}$$

Test by $\tilde{\mathbf{v}}$:

$$\begin{aligned}\int_V \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \tilde{\mathbf{v}} &= \int_V \left(-\frac{\partial p_{th}}{\partial \varrho} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta} \nabla \tilde{\theta} \right) \cdot \tilde{\mathbf{v}} dv + \int_V \operatorname{div} \left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right) \cdot \tilde{\mathbf{v}} dv = \\ &= \dots + \int_V \operatorname{div} \left(\left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right) \tilde{\mathbf{v}} \right) - \int_V \left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \operatorname{div} \mathbf{v} \right) = \dots + 0 - \dots \\ \frac{d}{dt} \frac{1}{2} \int_V \hat{\varrho}(\tilde{\mathbf{v}})^2 dv &= \int_V \left(-\frac{\partial p_{th}}{\partial \varrho} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta} \nabla \tilde{\theta} \right) \cdot \tilde{\mathbf{v}} dv - \int_V \left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \right) dv.\end{aligned}$$

See that viscosity kills kinetic energy. Now we use $\nabla \varphi \cdot \tilde{\mathbf{v}} = \operatorname{div}(\varphi \tilde{\mathbf{v}}) - \varphi \operatorname{div} \tilde{\mathbf{v}}$ and on $\operatorname{div}(\dots)$ we use Stokes theorem.

$$\frac{d}{dt} \frac{1}{2} \int_V \hat{\varrho}(\tilde{\mathbf{v}})^2 dv = \int_V \frac{\partial p_{th}}{\partial \varrho} \tilde{\varrho} \operatorname{div} \tilde{\mathbf{v}} dv + \int_V \frac{\partial p_{th}}{\partial \theta} \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} dv - \int_V \left(\tilde{\lambda}(\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \right) dv. (1)$$

$$\frac{\partial \tilde{\varrho}}{\partial t} = -\hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} \quad / \cdot \frac{\partial p_{th}}{\partial \varrho} \frac{\tilde{\varrho}}{\hat{\varrho}}, \int_V$$

$$\frac{d}{dt} \int_V \frac{1}{2} (\tilde{\varrho})^2 \text{TODO!!!} dv = - \int_V \frac{\partial p_{th}}{\partial \varrho} \tilde{\varrho} \operatorname{div} \tilde{\mathbf{v}} dv. (2)$$

$$\begin{aligned} \hat{\varrho} c_V|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \frac{\partial \tilde{\theta}}{\partial t} &= \operatorname{div}(k \nabla \tilde{\theta}) - \hat{\theta} \frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} \operatorname{div} \tilde{\mathbf{v}} \quad / \cdot \frac{\tilde{\theta}}{\hat{\theta}}, \int_V \\ \frac{d}{dt} \int_V \frac{\hat{\varrho} c_V}{\hat{\theta}} (\tilde{\theta})^2 dv &= - \frac{1}{\hat{\theta}} \int_V \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} dv - \int_V \frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} (\operatorname{div} \tilde{\mathbf{v}}) \tilde{\theta} dv. (3) \end{aligned}$$

(1) + (2) + (3):

$$\begin{aligned} \frac{d}{dt} \int_V \frac{1}{2} \hat{\varrho} (\tilde{\mathbf{v}})^2 + \frac{1}{2} \frac{1}{\hat{\varrho}} \frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} (\tilde{\varrho})^2 + \frac{\hat{\varrho} c_V}{\hat{\theta}}|_{\varrho=\hat{\varrho}, \theta=\hat{\theta}} (\tilde{\theta})^2 &= \\ &= - \frac{1}{\hat{\theta}} \int_V \nabla \tilde{\theta} \nabla \tilde{\theta} dv - \int_V \tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta dv. \end{aligned}$$

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When we rightly choose Helmholtz free energy, $\frac{\partial p_{th}}{\partial \varrho, \theta} > 0$ and c_V will be positive and we win (perturbations go to zero).

Z $c_V > 0$ máme $\frac{\partial^2 \psi}{\partial \theta^2} < 0$.

TODO!!!

Poznámka (For Homework)

$\frac{d\mathbb{H}}{dt} \neq \frac{1}{2} \mathbb{B}^{-1} \frac{d\mathbb{B}}{dt}$, because there is no commutativity. We must use $\mathbb{T} : \dots$

Poznámka (*)

$$\mathbb{T} = \underbrace{-p_{th}(\varrho, \tau)I}_{\text{Why this?}} + \underbrace{\tilde{\lambda}(\operatorname{div} \mathbf{v}) + 2\mu \mathbb{D}_\delta}_{\text{Why this?}}.$$

$$\mathbb{T} = -p_{th}(\varrho, \theta)\mathbb{I} + \mathbb{S}(\mathbf{v}).$$

Now we use Galilei principle of relativity „ $\mathbf{x} = \mathbf{x} + \mathbf{w}t$ “. We see that this don't work. So we use $\nabla \mathbf{v}$ ($\nabla(\mathbf{v} + \mathbf{w}) = \nabla \mathbf{v} + \nabla \mathbf{w} = \nabla \mathbf{v}$):

$$\mathbb{T} = -p_{th}(\varrho, \theta)\mathbb{I} + \mathbb{S}(\nabla \mathbf{v}).$$

\mathbb{T} is symmetric matrix, so:

$$\mathbb{T} = -p_{th}(\varrho, \theta)\mathbb{I} + \mathbb{S}(\mathbb{D}).$$

2.1 Representation theorems for isotropic functions

Definice 2.2

We say that $\varphi : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}$ is isotropic $\equiv \varphi(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \varphi(\mathbb{A})$ holds for any $\mathbb{Q} \in Orth^+$ (= proper orthogonal matrices, i.e. $\mathbb{Q}\mathbb{Q}^T = \mathbb{I}$, $\det \mathbb{Q} > 0$).

Například

tr is isotropic, $\mathbb{A} \mapsto a_{11}$ is not isotropic.

We say that $\mathbb{F} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ is isotropic $\equiv \mathcal{F}(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \mathbb{Q}\mathbb{F}(\mathbb{A})\mathbb{Q}^T$ holds for any $\mathbb{Q} \in Orth^+$.

Například

id , $^{-1}$, \exp , \ln , ... is isotropic. $\mathbb{A} \mapsto a_{ii}\mathbb{I}$ is not isotropic.

Věta 2.1

$\varphi : Sym(\mathbb{R}^{3 \times 3}) \rightarrow \mathbb{R}$, φ isotropic $\implies \varphi(\mathbb{A}) = \varphi(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

$f : Sym(\mathbb{R}^{3 \times 3}) \rightarrow Sym(\mathbb{R}^{3 \times 3})$, f isotropic $\implies f(\mathbb{A}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{A} + \alpha_2\mathbb{A}^2$, where $\alpha_i = \alpha_i(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

Poznámka

The second one goes from: Assume $f(\mathbb{A}) = \sum_{i=0}^{+\infty} f_i \mathbb{A}^i$. Then by Cayley–Hamilton $f(\mathbb{A}) = \sum_{i=0}^2 f_i \mathbb{A}^i$. Furthermore, $f_i(\mathbb{A})$ must be isotropic...

Poznámka (Continuation of *)

Isotropic fluid: $\mathbb{S}(\mathbb{D})$ is isotropic function, $\mathbb{S}(\mathbb{D}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2$. What if we want Linear relation (i.e. $\mathbb{S}(\mathbb{D})$ is linear function of \mathbb{D})? Then $\mathbb{S}(\mathbb{D}) = c_0(\text{tr } \mathbb{D})\mathbb{I} + c_1\mathbb{D}$, where $c_0, c_1 = \text{const.}$

Pozor

This remarks (not theorem) works only for fluids, where 0 velocity means 0 stress.

Poznámka (Governing equations for incompressible isotropic fluids)

TODO?

$$\begin{aligned}\text{div } \mathbf{v} &= 0 \\ \varrho \frac{d\mathbf{v}}{dt} &= -\nabla p + \text{div}(2\mu\mathbb{D}) + \varrho\mathbf{b} \\ \varrho c_V \frac{d\theta}{dt} &= \dots\end{aligned}$$

So

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}, \quad \operatorname{div} \mathbb{T} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}).$$

The kind of this p is other than p_{th} . This is (artificial) pressure maintaining the incompressibility, and we solve! for it. (p_{th} is function of ϱ and θ). \mathbb{T} is not obtained by simply substitution, but from equations above!

Příklad (Archimedes law)

Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

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Poznámka

Fluid = incompressible fluid. So $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$.

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Poznámka

Weight = body force = gravitation force

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Poznámka

It talks about floating, so nothing moves!

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Poznámka (Governing equations)

$$\operatorname{div} \mathbf{v} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \quad \mathbf{b} = -g\mathbf{e}_z,$$

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D},$$

$$\mathbf{F} = \int_{\partial B} \mathbb{T} \mathbf{n} ds.$$

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Řešení

1. Solve for \mathbf{v} and p .

2. Evaluate \mathbf{F} .

„1.“ is easy, we have static problem, so $\mathbf{v} = 0$.

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b} \wedge \operatorname{div} \mathbb{T} = -\nabla p + \operatorname{div}(2\mu \mathbb{D}) \implies \implies 0 = -\nabla p - \varrho_{fluid} g \mathbf{e}_z \implies$$

$$\implies \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) = (0, 0, -\varrho g) \implies p = -\varrho g z + p_0.$$

„2.“:

$$\int_{\partial \mathcal{B}} \mathbb{T} \mathbf{n} ds = \int_{\partial \mathcal{B}} -p \mathbb{I} \mathbf{n} ds = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z - p_0) \mathbf{n} ds = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{n}) ds - p_0 \int_{\partial \mathcal{B}} \mathbf{n} ds =: *.$$

By stokes (for constant \mathbf{w})

$$\mathbf{w} \cdot \int_{\partial \mathcal{B}} \mathbf{n} ds = \int_{\partial \mathcal{B}} \mathbf{w} \cdot \mathbf{n} = \int_{\mathcal{B}} \operatorname{div} \mathbf{w} dv = 0.$$

So $\int_{\partial \mathcal{B}} \mathbf{n} ds = 0$. Continue:

$$* = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{n}) ds,$$

for constant \mathbf{w} :

$$\mathbf{F} \cdot \mathbf{w} = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{w}) \cdot \mathbf{n} ds = \int_{\mathcal{B}} \varrho_{fluid} g \operatorname{div}(z \mathbf{w}) dv + \int_{\mathcal{B}} \varrho_{fluid} g ((\nabla z) \cdot \mathbf{w} + z \operatorname{div} \mathbf{w}) dv = \varrho_{fluid} g \int_{\mathcal{B}} 1 dv \mathbf{e}_z \cdot \mathbf{w}.$$

So $\mathbf{F} = \varrho_{fluid} g V \mathbf{e}_z$.

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Příklad (Stability of flow in a container)

$\mathbf{v}|_{\partial \Omega} = \mathbf{0}$, $\mathbb{T} = -p \mathbb{I} + 2\mu \mathbb{D}$.

$$t \rightarrow +\infty \implies \mathbf{v} \rightarrow \mathbf{0}.$$

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Příklad

$$\operatorname{div} \mathbb{T} = -\nabla p + \mu \Delta \mathbf{v}.$$

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Poznámka (Governing equations)

$$\begin{aligned}\operatorname{div} \mathbf{v} &= 0, \\ \varrho \frac{d\mathbf{v}}{dt} &= \operatorname{div} \mathbb{T} = -\nabla p + \mu \Delta \mathbf{v}, \\ \mathbf{v}|_{\partial\Omega} &= \mathbf{0}, \\ \mathbf{v}|_{t_0} &= \mathbf{v}_0.\end{aligned}$$

Řešení

Test equation by solution:

$$\begin{aligned}\int_{\Omega} \varrho \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} &= \int_{\Omega} (-(\nabla p) \cdot \mathbf{v} + \mu(\Delta \mathbf{v}) \cdot \mathbf{v}) dv. \\ \varrho \int_{\Omega} ((\mathbf{v} \cdot \nabla) \mathbf{v}) \cdot \mathbf{v} dv &= \varrho \int_{\Omega} v_i \frac{\partial v_j}{\partial x_i} v_j dv = \frac{1}{2} \varrho \int_{\Omega} v_i \frac{\partial}{\partial x_i} (v_j)^2 dv = \\ &= \varrho \int_{\Omega} \mathbf{v} \cdot \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) dv = \varrho \int_{\Omega} \operatorname{div}(\mathbf{v} \frac{|\mathbf{v}|^2}{2}) dv = \varrho \int_{\partial\Omega} \frac{|\mathbf{v}|^2}{2} (\mathbf{v} \cdot \mathbf{n}) ds = 0. \\ \frac{d}{dt} \int_{\Omega} \frac{1}{2} \varrho (\mathbf{v})^2 dv &= \dots - \mu \int_{\Omega} (\nabla \mathbf{v}) : \nabla \mathbf{v} dv.\end{aligned}$$

Poincaré inequality (we live in Dirichlet zero \iff boundary conditions):

$$\begin{aligned}\int |\nabla \mathbf{v}|^2 &\leq C_p^2 \int |\mathbf{v}|^2. \\ \frac{1}{2} \varrho \frac{d}{dt} \int_{\Omega} |\mathbf{v}|^2 &\leq -\frac{\mu}{C_p} \int_{\Omega} |\mathbf{v}|^2 dv \\ \frac{d}{dt} (\|\mathbf{v}\|_{L^2(\Omega)}^2) &\leq \frac{-2\mu}{\varrho C_p} \|\mathbf{v}\|_{L^2(\Omega)}^2.\end{aligned}$$

Příklad

$$\operatorname{div} \mathbf{v} = 0 \quad \varrho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \Delta \mathbf{v}.$$

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Poznámka (Dimensionless form) l_{char} = Characteristic length = arbitrary chosen length. v_{char} = characteristic velocity. $t_{char} = l_{char}/v_{char}$.Dimensionless variables: $\mathbf{x}^* = \mathbf{x}/l_{char}$, $t^* = t/t_{char}$, $\mathbf{v}^* = \mathbf{v}/v_{char}$.

$$\text{div}^* \mathbf{v}^* = 0.$$

$$\varrho \frac{v_{char}}{t_{char}} \frac{d\mathbf{v}^*}{dt^*} = -\frac{1}{l_{char}} (\nabla^* p^*) p_{char} + \frac{\mu}{l_{char}^2} (\Delta^* \mathbf{v}^*) v_{char},$$

$$\frac{d\mathbf{v}^*}{dt^*} = \frac{t_{char}}{\varrho v_{char} l_{char}} (\nabla^* p^*) p_{char} + \frac{\mu v_{char}}{l_{char}^2} \frac{t_{char}}{\varrho v_{char}} \Delta^* \mathbf{v}^*,$$

$$\frac{dv^*}{dt^*} = -\frac{p_{char}}{\varrho v_{char}^2} (\nabla^* p^*) + \frac{\mu}{\varrho l_{char} v_{char}} \Delta^* \mathbf{v}^*.$$

$$p_{char} := \varrho v_{char}^2$$

$$\implies \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* p^* + \frac{1}{Re} \Delta^* \mathbf{v}^*,$$

where $\frac{1}{Re} := \frac{\mu}{\varrho l_{char} v_{char}}$ is Reynold's number.

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3 Solids

TODO!!!

Příklad

TODO!!!

Řešení
TODO!!!

$$0 = \operatorname{div} \boldsymbol{\tau} = \begin{pmatrix} \frac{\partial \tau_{\hat{r}\hat{r}}}{\partial r} + \frac{1}{r} \left(\frac{\partial \tau_{\hat{r}\hat{\varphi}}}{\partial \varphi} - \tau_{\hat{\varphi}\hat{\varphi}} + \tau_{\hat{r}\hat{r}} \right) + \frac{\partial \tau_{\hat{r}\hat{z}}}{\partial z} \\ \frac{\partial \tau_{\hat{\varphi}\hat{r}}}{\partial r} + \frac{1}{r} \left(\frac{\partial \tau_{\hat{\varphi}\hat{\varphi}}}{\partial \varphi} + \tau_{\hat{r}\hat{\varphi}} + \tau_{\hat{\varphi}\hat{r}} \right) + \frac{\partial \tau_{\hat{\varphi}\hat{z}}}{\partial z} \\ \frac{\partial \tau_{\hat{z}\hat{r}}}{\partial r} + \frac{1}{r} \left(\frac{\partial \tau_{\hat{z}\hat{\varphi}}}{\partial \varphi} + \tau_{\hat{z}\hat{r}} \right) + \frac{\partial \tau_{\hat{z}\hat{z}}}{\partial z} \end{pmatrix},$$

where $\boldsymbol{\tau} = \begin{pmatrix} \tau_{\hat{r}\hat{r}} & \tau_{\hat{r}\hat{\varphi}} & \tau_{\hat{r}\hat{z}} \\ \tau_{\hat{\varphi}\hat{r}} & \tau_{\hat{\varphi}\hat{\varphi}} & \tau_{\hat{\varphi}\hat{z}} \\ \tau_{\hat{z}\hat{r}} & \tau_{\hat{z}\hat{\varphi}} & \tau_{\hat{z}\hat{z}} \end{pmatrix}$. So $\boldsymbol{\tau} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tau_{\hat{z}\hat{z}} & 0 & T \end{pmatrix} =: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix}$.

Thus $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix} \Big|_{z=L} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{S} \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}$, so $T = \frac{F}{S}$.

$$\boldsymbol{\tau} = \lambda(\operatorname{tr} \boldsymbol{\varepsilon}) \mathbb{I} + 2\mu \boldsymbol{\varepsilon}. \operatorname{tr} \boldsymbol{\tau} = (3\lambda + 2\mu) \operatorname{tr} \boldsymbol{\varepsilon} \implies \operatorname{tr} \boldsymbol{\varepsilon} = \frac{\operatorname{tr} \boldsymbol{\tau}}{3\lambda + 2\mu}.$$

$$\boldsymbol{\varepsilon} = f(\boldsymbol{\tau}). \quad \boldsymbol{\varepsilon} = \frac{1}{2\mu} \left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda + 2\mu} (\operatorname{tr} \boldsymbol{\tau}) \mathbb{I} \right).$$

$$RHS = \frac{1}{2\mu} \left(\boldsymbol{\varepsilon} - \frac{\lambda}{3\lambda + 2\mu} (\operatorname{tr} \boldsymbol{\tau}) \mathbb{I} \right) = \begin{pmatrix} -\frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{F}{S} & 0 & 0 \\ 0 & -\frac{\lambda}{2\mu(3\lambda+2\mu)} \frac{F}{S} & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \frac{F}{S} \end{pmatrix}.$$

$$LHS = \frac{1}{2} (\nabla \mathbf{U} + (\nabla \mathbf{U})^T).$$

Symmetric gradient in cylindrical coordinate system is another long formula, so expect:
 $\mathbf{U} = (U_{\hat{r}}(r), 0, U_{\hat{z}}(z))^T$.

$$\frac{1}{2} \nabla \mathbf{U} + \frac{1}{2} (\nabla \mathbf{U})^T = \begin{pmatrix} \frac{dU_{\hat{r}}}{dr} & 0 & 0 \\ 0 & \frac{U_{\hat{r}}}{r} & 0 \\ 0 & 0 & \frac{dU_{\hat{z}}}{dz} \end{pmatrix}.$$

Together: $U_{\hat{r}} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot r$, $U_{\hat{z}} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot z$.

Length of cylinder: $\varepsilon := \frac{\Delta L}{L} = \frac{\mathbf{U}|_{z=L}}{L} = \frac{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot L}{L} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} =: \frac{1}{E} \sigma$, where $\sigma = E\varepsilon$ is Hooke law, where E is Young modulus, σ is „applied force“ and ε is change of length.

Change of radius:

$$-\frac{\frac{\Delta R}{R}}{\frac{\Delta L}{L}} = -\frac{-\frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot R}{\frac{\Delta L}{L}} = \frac{\frac{\lambda}{2\mu(3\lambda+2\mu)}}{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}} = \frac{\lambda}{2(\lambda + \mu)} =: \nu.$$

Poznámka

This is easy to measure -> we can measure Young modulus $E := \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and Poisson ration $\nu := \frac{\lambda}{2(\lambda+\mu)}$.

Poznámka

Young modulus is certainly positive. However, Poisson ration can be negative. (Move V s horizontally in VAVAVAVAVAV.)

Poznámka (Fixing negativity of constants (Poisson ration))

$$\begin{aligned}\boldsymbol{\tau} &= \frac{1}{2\mu} \left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda + 2\mu} (\text{tr } \boldsymbol{\tau}) \mathbb{I} \right), & \boldsymbol{\tau} &= \lambda (\text{tr } \boldsymbol{\epsilon}) \mathbb{I} + 2\mu \boldsymbol{\epsilon}. \\ \boldsymbol{\epsilon} &= \frac{1}{2\mu} \left(\left(\boldsymbol{\tau} - \frac{1}{3} (\text{tr } \boldsymbol{\tau}) \mathbb{I} \right) + \left(\frac{1}{3} - \frac{\lambda}{3\lambda + 2\mu} \right) (\text{tr } \boldsymbol{\tau}) \mathbb{I} \right). \\ \boldsymbol{\epsilon} &= \frac{1}{2\mu} \boldsymbol{\tau}_\delta + \frac{1}{9(\lambda + \frac{2}{3}\mu)} (\text{tr } \boldsymbol{\tau}) \mathbb{I}.\end{aligned}$$

Experiment: compress material. Then we see, that $K := \lambda + \frac{2}{3}\mu$ must be positive. It is called bulk modulus (related to change of volume). $G := \mu > 0$ is shear modulus.

Důsledek

$$\begin{aligned}\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}, & \mu &= \frac{E}{2(1+\nu)}, & K &= \frac{E}{3(1-2\nu)}, \\ \mu &= \frac{E}{2(1+\nu)}, & E &= \frac{9K\mu}{\mu + 3K}, & \nu &= \frac{3K - 2\mu}{2(3K + \mu)}.\end{aligned}$$

$$\implies -1 < \nu < \frac{1}{2}.$$

$\nu = \frac{1}{2}$ means incompressible (solid) material (= no $\boldsymbol{\epsilon}$).

Definice 3.1

Spherical stress: $\boldsymbol{\tau} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix}$. Shear stress: $\boldsymbol{\tau} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3.1 Elastic materials

Definice 3.2

Elastic material is a material that does not produce entropy. No energy loss/gain in cyclic mechanical processes.

We have $\mathbb{T} = \mathbb{T}(\mathbb{B}) = \alpha_0 \mathbb{I} + \alpha_1 \mathbb{B} + \alpha_2 \mathbb{B}^2$ (for solids, isotropic solids) (hence $\mathbb{T}\mathbb{B} = \mathbb{B}\mathbb{T}$), $e(\eta, \varrho)$, $\psi(\theta, \varrho)$, and from $\varrho \det \mathbb{F} = \varrho_R$ ($\varrho(\det \mathbb{B})^{1/2} = \varrho_R$), we get $\psi(\theta, \det \mathbb{B})$. So $\psi = \psi(\theta, \mathbb{B}) = \psi(\theta, I_1(\mathbb{B}), I_2(\mathbb{B}), I_3(\mathbb{B}))$.

From $\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q$ we get $\psi(\theta, \mathbb{B}) = (e(\eta, \mathbb{B}) - \theta \eta)|_{\eta=\eta(\theta, \mathbb{B})}$.

$$\frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt} + \frac{\partial \psi}{\partial \theta} \frac{d\theta}{dt} = \frac{de}{dt} - \frac{d\theta}{dt} \eta - \theta \cdot \frac{d\eta}{dt}.$$

Důsledek ($\overset{\nabla}{\mathbb{B}} = \mathbb{O}$)

$$\begin{aligned} \frac{de}{dt} &= \frac{\partial \psi}{\partial \mathbb{B}}(\theta, \mathbb{B}) : \frac{d\mathbb{B}}{dt} + \theta \frac{d\eta}{dt}, \\ \varrho \theta \frac{d\eta}{dt} - \varrho \frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt} &= \end{aligned}$$

$$\begin{aligned} \varrho \frac{de}{dt} &= \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q \rightarrow \varrho \theta \frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q - \varrho \frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt} = \\ &= \left(\mathbb{T} : \mathbb{D} - \varrho \frac{\partial \psi}{\partial \mathbb{B}} : (\mathbb{L}\mathbb{B} + \mathbb{B}\mathbb{L}^T) \right) - \operatorname{div} \mathbf{j}_q = \mathbb{T} : \mathbb{D} - 2\varrho \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}} : \mathbb{D} - \operatorname{div} \mathbf{j}_q. \\ \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B} &= \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}}, \end{aligned}$$

because ψ is isotropic function of \mathbb{B} . Thus we get evolution equation for entropy.

$$\varrho \theta \frac{d\eta}{dt} = \left(\mathbb{T} - 2\varrho \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}} \right) : \mathbb{D} - \operatorname{div} \mathbf{j}_q.$$

Důsledek

No entropy production can be done (for all processes at once) "only" by setting $\mathbb{T} = 2\varrho \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}}$.

Důsledek

$$\begin{aligned} 0 &= \left(\int_V \varrho \psi(\theta, \mathbb{B}) \right) |_{t_{end}} - \left(\int_V \varrho \psi(\theta, \mathbb{B}) \right) |_{t_{start}} = \\ &= \int_{t_{start}}^{t_{end}} \left(\int_V \mathbb{T} : \mathbb{D} dv \right) dt = \int_{t_{start}}^{t_{end}} \int_V 2\varrho \mathbb{B} \frac{\partial \psi}{\partial \mathbb{B}} : \mathbb{D} dv dt = \\ &= \int_{t_{start}}^{t_{end}} \int_V \left(\varrho \frac{\partial \psi}{\partial \mathbb{B}} \frac{d\mathbb{B}}{dt} dv \right) dt = \int_{t_{start}}^{t_{end}} \frac{d}{dt} \int_V \varrho \psi dv dt. \end{aligned}$$

TODO!!!