

*Poznámka*

At least 1 from (3-)4 homework (flexible deadlines – last lecture).

*Poznámka*

In this lecture, there was also the revision of topology. (Topological space, topology, basis of topology, continuous map, quotient space, product topology, Hausdorff spaces).

*Poznámka*

World Homotopy comes from homós (= same, similar) and topos (place).

**Definition 0.1** (Homotopic functions)

Given two topological spaces  $X$  and  $Y$  and two continuous functions  $f, g : X \rightarrow Y$ , we say that  $f$  is homotopic to  $g$  ( $f \sim g$ ) if there is a 1-parametric family  $f_t : X \rightarrow Y$ :  $f_0 = f$ ,  $f_1 = g$  and the map  $F : [0, 1] \times X \rightarrow Y$  defined by  $(t, x) \mapsto f_t(x)$  is continuous.

**Definition 0.2** (Homotopy equivalent spaces)

Given two topological spaces  $X$  and  $Y$  we say that  $X$  and  $Y$  are homotopy equivalent if there is a pair of continuous maps  $(f, g)$  such that  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  and  $X \xrightarrow{f} Y$  and  $Y \xrightarrow{g} X$ ,  $g \circ f \sim \text{id}_X$ ,  $f \circ g \sim \text{id}_Y$ .

*Příklad*

Given  $\mathbb{R}$ ,  $\mathbb{R}^2$  with the standard Euclidean topology and two maps  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $x \mapsto f(x) = (x, x^3)$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $x \mapsto g(x) = (x, e^x)$ .

Are  $f$  and  $g$  homotopic? (Show that by constructing homotopy.)

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*Řešení*

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$$F(t, x) = (1 - t)(x, x^3) + t(x, e^x) = (x, (1 - t)x^3 + te^x).$$

*Příklad*

Given three topological spaces  $(X, \tau_X)$ ,  $(Y, \tau_Y)$ ,  $(Z, \tau_Z)$  and two pairs of continuous maps  $f_1, g_1 : (X, \tau_X) \rightarrow (Y, \tau_Y)$  and  $f_2, g_2 : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$ . Assume that  $f_1$  is homotopic to  $g_1$  and  $f_2$  is homotopic to  $g_2$ . Show that  $f_2 \circ f_1$  is homotopic to  $g_2 \circ g_1$ .

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*Řešení*

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$$F(t, x) = F_2(t, F_1(t, x)).$$

*Příklad*

Take  $B^n := \{x, \dots, x_n | \sqrt{x_1^2 + \dots + x_n^2} \leq 1\} \subseteq \mathbb{R}^n$ . And take a map  $f : B^n \rightarrow B^n$ :  $f(x) = (0, \dots, 0) \in B^n$  for all  $x \in B^n$ . Shows that there is a homotopy from  $\text{id}$  to  $f$ .

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*Řešení*

$$F : [0, 1] \times B^n \rightarrow B^n, \quad (t, x) \mapsto (1 - t)x.$$

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*Příklad*

Take a 2-ball  $B^2$ .  $B^2$  is homotopy equivalent to its center by previous problem, but it is not homeomorphic to  $(0, 0)$ .

### Definice 0.3 (Deformation retraction)

A deformation retraction of a topological space  $X$  onto a subspace  $A$  is a family of maps  $f_t : X \rightarrow X$ ,  $t \in [0, 1]$ :  $f_0 = \text{id}_X$ ,  $f_1(X) = A$  and  $f_t|_A = \text{id}_A$ . And family  $f_t$  is continuous in the following sense:

$$F : [0, 1] \times X \rightarrow X, (t, x) \mapsto f_t(x), \text{ is continuous.}$$

### Tvrzení 0.1

Given a deformation retraction  $f_t : X \rightarrow X$ , there is a pair  $(f, g) : X \xrightarrow{f} A \xrightarrow{g} X : g \circ f \sim \text{id}_X, f \circ g \sim \text{id}_A$ .

*Poznámka* (Suggestion)

$$f = f_1, g = f_0 \circ i_A \quad (A \xrightarrow{i_A} X), \text{ tj. } f \circ g : A \xrightarrow{i_A} X \xrightarrow{f_1} X \xrightarrow{f_1} X, a \mapsto a \mapsto a \mapsto a \text{ (or } A) \\ \implies f \circ g = \text{id}_A. g \circ f : X \xrightarrow{f_0} A \xrightarrow{i_A} X \implies f_1(x) \sim \text{id}_X.$$

### Definice 0.4

Given two topological spaces  $X$  and  $Y$  and a continuous map  $f : X \rightarrow Y$ , the mapping cylinder  $M_f$  is defined to be the quotient space of  $X \times [0, 1] \amalg Y$  and  $\sim : (x, 1) \sim f(x)$ .  $M_f = X \times [0, 1] \amalg Y / \sim$ .

### Tvrzení 0.2

Given  $X, Y$  and  $f$ ,  $M_f$  deformation retracts to  $Y$ .

┌ *Důkaz* (/ Idea of proof)

The way to construct  $f_t = F(\cdot, t) : M_f \rightarrow M_f$  is to slide each point  $(x, t)$  along the segment  $\{x\} \times [0, 1]$  to  $f(x)$ :

$$F : (x, t) \mapsto f_t(x), \quad \forall y \in Y : y = F(y, t) \mapsto \{f_1 = \text{id } Y \rightarrow Y\}$$

└ In your HW you will check that  $F(x, t)$  is continuous. □

## Definice 0.5 (Cell complex (CW complex))

┌ *Poznámka*

Cell complex (CW complex) is a topological space with a nice decomposition into small pieces.

1. Start with a discrete set  $X^0$ , whose points are called 0-cells.

2. We form the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching cells  $e_\alpha^n = I^n = [0, 1]^n$ . By the attachment we mean  $(e_\alpha^n = B_\alpha^n, \partial e_\alpha^n = S_\alpha^n) \varphi_\alpha : \partial e_\alpha^n \rightarrow X^{n-1}$ . Hence we can view  $X^n = X^{n-1} \coprod \coprod B_\alpha^n / \sim$ , where  $x \sim \varphi_\alpha(x)$  for  $x \in \partial B_\alpha^n$ .

3. We can either stop this inductive process at a certain finite steps or take an infinite number of steps. In the first case  $X = X^n$  for some  $n$ , in the second one  $X = \bigcup_{n \in \mathbb{N}_0} X^n$  with the weak topology ( $A \subset X$  is open  $\leftrightarrow A \cap X^n$  is open for all  $n$ ).

┌ *Například*

└ Example of 1-skeleton is graph.

## Definice 0.6

Given a cell complex  $X$ . Each cell  $e_\alpha^n$  has a characteristic map  $\Phi_\alpha : e_\alpha^n = B_\alpha^n \rightarrow X$  which extends the attaching map  $\varphi_\alpha : \partial B_\alpha^n \rightarrow X^{n-1}$ , it is homeomorphism from the interior of  $B_\alpha^n$  onto  $e_\alpha^n$ . Namely

$$B_\alpha^n \hookrightarrow X^{n-1} \coprod \coprod_\beta B_\beta^n \xrightarrow{\text{quotient}} X^n \rightarrow X, \quad B_\alpha^n \rightarrow X$$

## Definice 0.7

A subcomplex of CW complex is a closed subspace  $A \subset X$  that is a union of cells with the corresponding attachments.

┌ *Příklad*

└ Construct two different CW structures on  $S^2$ .

Řešení

$S^2 = e^0 \cup e^2$ ,  $S^2 = e^0 \cup e^1 \cup \{e_1^2, e_2^2\}$ . (See practicals.)

Příklad

We define  $\mathbb{R}P^n$  to be the quotient of  $S^n / \sim$ , where  $V \sim$  the antipodal point to  $V$ . TODO?

### Definice 0.8

Consider a pair  $(X, A)$  where  $X$  is a CW complex and  $A$  is subcomplex. Then we define the quotient complex  $X/A$  to be the CW complex with the structure: There are all the cells of  $X \setminus A$  with the corresponding attaching maps, and there is a extra 0-cell which is  $A$  in  $X \setminus A$ . For a cell  $e_\alpha^n$  of  $X \setminus A$  attached by  $\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$ , the attaching map in the corresponding cell in  $X/A$  is the composition  $S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$ .

Příklad

Show that  $S^n = e^0 \cup e^n$  is  $B^n / S^{n-1} = \text{TODO} / e^0 \cup e^{n-1}$ .

### Definice 0.9 (Suspension)

For a space  $X$ , the suspension  $SX$  is the quotient of  $X \times T := X \times [0, 1]$  by collapsing  $X \times \{0\}$  to one point and  $X \times \{1\}$  to another point.

Například (Motivation)

$$X = S^n \implies SX = X \times [0, 1] / \sim = S^{n+1}.$$

### Definice 0.10 (Wedge sum)

Given spaces  $X$  and  $Y$  with chosen points  $x_0 \in X$  and  $y_0 \in Y$ , the wedge sum  $X \vee Y$  is the quotient of the disjoint union  $X \coprod Y$  obtained by identifying  $x_0$  and  $y_0$  to a single point.

## 1 Fundamental group

### 1.1 Paths and homotopy

#### Definice 1.1 (Path, endpoints)

Given a topological space  $X$ , a path in  $X$  is a continuous map  $f : [0, 1] \rightarrow X$ . Then points  $f(0)$  and  $f(1)$  are usually called endpoints of  $f$ .

### Definice 1.2 (Homotopy)

A homotopy of paths in  $X$  is a family  $f_t : [0, 1] \rightarrow X$ ,  $t \in [0, 1]$ , such that the endpoints  $f_t(0) = x_0$ ,  $f_t(1) = x_1$  for all  $t \in [0, 1]$ , i.e. the endpoints are independent of  $t$ ; the associated map  $F : I \times I \rightarrow X$  defined by  $F(s, t) := f_t(s)$  is continuous.

### Tvrzení 1.1

*The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

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*Důkaz*

„Reflexivity“:  $X$  topological space. We want  $f : [0, 1] \rightarrow X \implies f$  is homotopic to  $f$ . The homotopy is trivial, i.e. we take  $f_t(s) = f(s) \forall t$ .

„Symmetry“: We have  $f_0 : [0, 1] \rightarrow X$ ,  $f_1 : [0, 1] \rightarrow X$ .  $f_0$  is homotopic to  $f_1$  via  $f_t(s)$ . We want to have a homotopy  $g_t$  from  $f_1$  to  $f_0$ . We can define  $g_t(s) = f_{1-t}(s)$ .

„Transitivity“:  $f_0$  homotopic to  $f_1$  via  $f_t$ , and  $f_1$  to  $f_2$  via  $g_t$ . We define homotopy from  $f_0$  to  $f_2$  by  $h_t(s) = f_{2t}(s)$  if  $t \in [0, 1/2]$  and  $h_t(s) = g_{2t-1}(s)$  if  $t \in [1/2, 1]$ .  $h_t$  is continuous since  $f_t, g_t$  are continuous. □

### Definice 1.3 (Composition of paths (product of paths))

Given two paths  $f, g : I \rightarrow X$  such that  $f(1) = g(0)$ . We can take a composition of paths also called the product of paths

$$f \cdot g(s) := \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2}, \\ g(2s - 1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

*Poznámka* (Loop, constant loop)

From now on, we will consider not all paths but loops, i.e.  $f : [0, 1] \rightarrow X$  continuous such that  $f(0) = f(1)$ .

We take a fixed point  $x_0$ . There is a special loop – constant loop –  $f : [0, 1] \rightarrow X$ :  $f(s) = x_0 \forall s \in [0, 1]$ .

*Poznámka* (Homotopy equivalence, product of equivalence classes)

When we take product  $f \cdot g$  for two loops  $f$  and  $g$  starting and ending at  $x_0$ , the product  $f \cdot g$  up to homotopy (denoted  $[f \cdot g]$ ) depends only on  $[f]$  and  $[g]$  ( $f$  and  $g$  up to homotopy).

$$\text{I.e. } f \sim f' \wedge g \sim g' \implies f \cdot g \sim f' \cdot g'.$$

**Definition 1.4** (Fundamental group)

Given a topological space  $X$  with a fixed point  $x_0 \in X$ , the fundamental group  $\pi_1(X, x_0)$  as a set is a set of homotopy classes of loops with endpoints at  $x_0$  and the group structure is given by the product of equivalence classes of loops.

**Definition 1.5** (Reparametrization of path)

A reparametrization of path  $f$  is a composition  $f \circ \varphi$ , where  $\varphi : I \rightarrow I$  continuous, such that  $\varphi(0) = 0$ ,  $\varphi(1) = 1$ .

*Důsledek*

Reparametrization a path preserves its homotopy class since  $f \circ \varphi \simeq f$  via the homotopy  $\varphi_t(s) = (1 - t)\varphi(s) + t \cdot s$  so that  $\varphi_0 = \varphi$  and  $\varphi_1(s) = s$

**Definition 1.6** (Inverse loop)

Given a loop  $f : [0, 1] \rightarrow X$  with  $f(0) = f(1) = x_0$ , the inverse loop  $f^{-1}$  (or  $\bar{f}$ ) is  $f^{-1}(s) = f(1 - s)$ .

**Tvrzení 1.2**

$\pi_1(X, x_0)$  is a group.

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*Důkaz*

So far we guaranteed that the product  $[f \cdot g]$  of two loops  $[f]$  and  $[g]$  is defined. ( $[f \cdot g] = [f] \cdot [g]$ .) In order to be a group  $(\pi_1(X, x_0), \cdot)$  should satisfy group axioms.

„Constant“: Given a loop  $f : I \rightarrow X$  with  $f(0) = f(1) = x_0$  and a constant loop  $c : I \rightarrow X$  with  $c(s) = x_0$ . We take  $f \cdot c$ .  $f \cdot c$  is a reparametrization of  $f$  via function  $\varphi(s) = \min(2s, 1)$ . For  $c \cdot f$  we see that  $c \cdot f = f \circ \varphi$ , where  $\varphi(s) = \max(0, 2s - 1)$ . So  $c \cdot f \simeq f$  and  $f \cdot c \simeq f$ , so  $[c \cdot f] = [f] = [f \cdot c]$ .

„Associativity“: If we take three loops  $f, g, h$ , then take products  $(f \cdot g) \cdot h$  and  $f \cdot (g \cdot h)$ . Observe that  $f \cdot (g \cdot h)$  is a reparametrization of  $[f \cdot g] \cdot h$  via the map  $\varphi(s) = \max(s/2, s - 1/2, 2s - 1)$ . Hence  $[f] \cdot [g \cdot h] = [f \cdot g] \cdot [h]$ .

„Inverse“: Take  $f \cdot f^{-1}$  and construct a homotopy to a constant loop  $c$  by taking  $h_t = f_t \cdot g_t$ , where  $f_t(s) = f(s)$  for  $s \in [0, 1 - t]$ , and  $f_t(s) = f(1 - t)$  for  $s \in [1 - t, 1]$ . ( $f_0 = f$ ,  $f_1 = c$ ).  $g_t = f_t^{-1}$  for all  $t$ .  $h_0 = f_0 \cdot f_0^{-1} = f \cdot f^{-1}$ ,  $h_1 = c \cdot c^{-1} = c$ .

In the same construction we take  $f^{-1}$  instead of  $f$  and we get  $f^{-1} \cdot f \simeq c$ . This way we checked  $[f] \cdot [f^{-1}] = [f^{-1}] \cdot [f] = [c]$ . □

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TODO? (Definition of category)

### Tvrzení 1.3

There is an isomorphism  $\pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  for  $x_0$  and  $x_1$  in the same path connected component.

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*Důkaz*

Since  $x_0, x_1$  are in one path connected component  $\tilde{X}$ ,  $\exists$  path  $h : [0, 1] \rightarrow X$ :  $h$  is in  $\tilde{X}$  and  $h(0) = x_0, h(1) = x_1$ .  $\bar{h}(s) := h^{-1}(s) := h(1 - s), s \in [0, 1]$ .

To each loop  $f$  based at  $x_1$  we associate a loop  $h \cdot f \cdot h^{-1}$ .  $h \cdot f \cdot h^{-1}$  is based at  $x_0$ .  $\beta_h : \pi_1(x, x_1) \rightarrow \pi_1(x, x_0), [f] \mapsto [h \cdot f \cdot h^{-1}]$ . We claim, that  $\beta_h$  is an isomorphism. „ $\beta_h$  is homomorphism“:

$$\beta_h([f \cdot h]) = [hfg h^{-1}] = [hfh^{-1}hgh^{-1}] = [hfh^{-1}] \cdot [hgh^{-1}] = \beta_h([f]) \cdot \beta_h([g]).$$

„ $\beta_h$  is isomorphism“: „the inverse of  $\beta_h$  is  $\beta_{h^{-1}}$ “ (which is homomorphism too by the argument we used for  $\beta_h$ ):

$$\beta_{h^{-1}}(\beta_h([f])) = \beta_{h^{-1}}([hfh^{-1}]) = [h^{-1}hfh^{-1}h] = [f].$$

└

□

### Věta 1.4 (Fundamental group of $S^1$ )

$S^1$  is path connected, thus  $\pi_1(S^1, x_0) = \pi_1(S^1)$ .

$$\pi_1(S^1) \simeq \mathbb{Z}.$$

*Důkaz*

We claim that  $\pi_1(S^1) \simeq \langle [\omega] \rangle$ , where  $\omega : [0, 1] \rightarrow S^1$ ,  $s \mapsto (\cos(2\pi s), \sin(2\pi s)) \in \mathbb{R}^2$ ,  $s \in [0, 1]$ .  $\omega_n(s) := (\cos(2\pi ns), \sin(2\pi ns)) \sim \omega^n$ , so  $[\omega]^n = [\omega_n]$ .

Now our theorem is equivalent to the statement that every loop in  $S^1$  based at  $(1, 0)$  is homotopic to the unique  $\omega_n$ . We use the following two facts:

Fact 1: For every path  $f : I \rightarrow X$  starting at  $x_0 \in X$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$  there is a unique lift  $\tilde{f} : I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .

Fact 2: For each homotopy  $f_x : I \rightarrow X$  of paths starting at  $x_0$  and each  $\tilde{x}_0 \in p^{-1}(x_0) \exists$  unique lifted homotopy  $\tilde{f}_t : I \rightarrow \tilde{X}$  of paths starting at  $\tilde{x}_0$ .

$p$  that we need:  $p : \mathbb{R} \rightarrow S^1$ ;  $p(s) = (\cos 2\pi s, \sin 2\pi s)$ . If we define  $\tilde{\omega}_n(s) = n \cdot s$ . We will apply Facts 1 and 2 to  $p : \mathbb{R} \rightarrow S^1$ ,  $\tilde{\omega}_n$ : Given  $f : [0, 1] \rightarrow S^1$  based at  $(0, 1)$  representing some element of  $\pi_1(S^1)$ . We take  $\tilde{f}$ . Since  $p\tilde{f}(1) = f(1) = (1, 0)$  (and  $p^{-1}(1) \in \mathbb{Z}$ ), we can argue that if  $\tilde{f}$  ends at  $u$  (i.e.  $\tilde{f}(1) = f$ ), it is homotopic to  $\tilde{\omega}_n$  by the homotopy  $\tilde{F} = (1 - t)\tilde{f} + t\tilde{\omega}_n$ .

From fact 1 exists  $\tilde{f}$  starting at 0 and ending at  $p^{-1}(1) \in \mathbb{Z}$ .

Theorem: Exists homotopy  $\tilde{F}$  from  $\tilde{\omega}_k$  to  $\tilde{f}$  denoted by  $(*)$ .

So we define homotopy  $F$  from  $\omega_n$  to  $f$  by  $F = p \circ \tilde{F}$ , homotopy from  $\omega_n$  to  $f$ . Since  $[\omega_n] = n \cdot [\omega]$ ,  $\pi_1(S^1) \simeq \mathbb{Z}$ .

Now we would like to show that  $[f]$  is uniformly determined. Assume that  $f \sim \omega_n$  and  $f \sim \omega_m$ , then using Facts 1 and 2 we have  $[\omega_n] = [\omega_m]$  which leads to contradiction since they have different endpoints on  $\mathbb{R}$ .  $\square$

## Definition 1.7

Given a topological space  $X$ , a covering space of  $X$  consists of a topological space  $\tilde{X}$  and a continuous map  $p : \tilde{X} \rightarrow X$  satisfying that  $\forall x \in X \exists$  open neighbourhood  $U$  of  $x$  in  $X$  such that  $p^{-1}(U)$  is a disjoint union of open subsets  $U_\alpha$  each of which is homeomorphically mapped to  $U$ .

## Definition 1.8

Given a map  $[0, 1] \xrightarrow{f} X$  and  $p : \tilde{X} \rightarrow X$  we say that  $\tilde{f} : [0, 1] \rightarrow \tilde{X}$  is a lift of  $f$  if  $p \circ \tilde{f} = f$ .

The same construction can be defined for homotopy.

## Tvrzení 1.5 (\*)

Given a map  $F : Y \times [0, 1] \rightarrow X$  and a map  $\tilde{F} : Y \times \{\mathbf{o}\} \rightarrow \tilde{X}$ , where  $p : \tilde{X} \rightarrow X$  is a covering space, and  $\tilde{F}$  lifts  $F|_{Y \times \{\mathbf{o}\}}$ ; there restricting to  $\tilde{F}$  on  $Y \times \{\mathbf{o}\}$ .



┌ *Poznámka* (Corollary: Fact 1 and Fact 2 from the previous proof)

└ Fact 1 is free, it comes when  $Y = \{\text{point}\}$ , Fact 2 also follows.

*Příklad*

We say that a topological (path-connected) space is simply connected  $\Leftrightarrow \pi_1(X) = \{e\}$ .  
Examples of simply connected topological spaces:  $\mathbb{R}, \mathbb{R}^2, \dots$   $S^1$  is not simply connected.

*Příklad*

Given  $X, Y$  path-connected and  $x_0 \in X, y_0 \in Y$ . Show that  $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$ .

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*Řešení*

Product topology is defined to be such that a map  $f : Z \rightarrow X \times Y$  is continuous  $\Leftrightarrow (p_x : X \times Y \rightarrow X, p_y : X \times Y \rightarrow Y)$   $p_x \circ f$  and  $p_y \circ f$  are continuous.

A loop  $\gamma : [0, 1] \rightarrow X \times Y$  based at  $(x_0, y_0)$  splits at two loops  $\gamma_1 : [0, 1] \rightarrow X, \gamma_2 : [0, 1] \rightarrow Y$ . The same holds for homotopy, i.e.  $F$  from  $\gamma$  to  $\tilde{\gamma}$  splits into  $(F_1, F_2)$ , where  $F_1$  is a homotopy on  $X$  from  $\gamma_1$  to  $\tilde{\gamma}_1$  and  $F_2$  is a homotopy on  $Y$  from  $\gamma_2$  to  $\tilde{\gamma}_2$ .

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*Důsledek*

$$\pi_1(T^n) := \pi_1(S^1 \times S^1 \times \dots \times S^1) = \mathbb{Z}^n.$$

*Příklad*

Show that  $\text{TODO!!!}$  is a covering space for  $S^1 \vee S^1$ .

$\text{TODO!!!}$

*Důkaz* (Proposition \*)

To prove our proposition we need to construct  $\tilde{F} : N \times I \rightarrow \tilde{X}$ , where  $N$  is an open neighbourhood in  $Y$  of a given point  $y_0 \in Y$ . Since  $F$  is continuous, every point  $(y_0, t) \in Y \times I$  has a product neighbourhood  $N_t \times (a_t, b_t)$  such that  $F(N_t \times (a_t, b_t))$  is contained in an evenly covered neighbourhood of  $F(y_0, t)$ .

By compactness of  $\{y_0\} \times I$ , finitely many such products  $N_t \times (a_t, b_t)$  cover  $\{y_0\} \times I$ . This implies that we can choose a single neighbourhood  $N$  of  $y_0$ , and a partition of  $[0, 1]$   $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$  such that  $F|_{N \times [t_i, t_{i+1}]}$  is contained in an evenly covered neighbourhood  $U_i$ .

Assume inductively that  $\tilde{F}$  has been constructed for  $N \times [0, t_i]$  starting at a given  $\tilde{F}$  on  $N \times \{0\}$ . We have that  $F(N \times [t_i, t_{i+1}]) \subset U_i$ , so since  $U_i$  is evenly covered, there is an open set  $\tilde{U}_i \subset \tilde{X}$  projecting homeomorphically to  $U_i$  via  $p$  and  $\tilde{F}((y_0, t_i)) \in \tilde{U}_i$ . After replacing  $N$  by a smaller neighbourhood of  $y_0$ . (We replace  $N \times \{t_i\}$  with the intersection with  $?$ ) we may assume that  $\tilde{F}(N \times \{t_0\}) \in \tilde{U}_i$ . Now we define  $\tilde{F}$  on  $N \times [t_i, t_{i+1}]$  to be the composition of  $F$

with  $p^{-1} : U_i \rightarrow \tilde{U}_i$ . After a finite number of steps, we eventually get a lift  $\tilde{F} : N \times I \rightarrow \tilde{X}$  for  $N$  (some neighbourhood of  $y_0$ ).

Next we show the uniqueness for  $Y = \{\text{point}\}$ . In this case we ? Suppose there are two lifts  $\tilde{F} : I \rightarrow \tilde{X}$ ,  $\tilde{F}' : I \rightarrow \tilde{X}$ . As before we choose a partition  $0 = t_0 < t_1 < \dots < t_n = 1$  of  $[0, 1]$  so that  $\forall i : F([t_i, t_{i+1}])$  is contained in some evenly covered neighbourhood  $U_i$ . Assume inductively that  $\tilde{F} = \tilde{F}'$  on  $[0, t_i]$ . Since  $[t_i, t_{i+1}]$  is connected,  $\text{co}$  is  $\tilde{F}([t_i, t_{i+1}])$ , which must therefore lie in one of the disjoint open sets  $\tilde{U}_i$  projecting homeomorphically to  $U_i$ . By the same token,  $\tilde{F}'([t_i, t_{i+1}])$  lies in a single  $\tilde{U}_i$ , in fact in the same containing  $\tilde{F}([t_i, t_{i+1}])$  (by the assumption of induction). Since  $p$  is injective on  $\tilde{U}_i$  and  $p\tilde{F} = p\tilde{F}'$ , it follows that  $\tilde{F} = \tilde{F}'$  on  $[t_i, t_{i+1}]$  and the induction step follows.

The last step of the prove is to observe that since  $\tilde{F}$ ,  $\tilde{F}'$  are constructed on the sets of form  $N \times I$  and are unique when we restrict to each segment  $\{y\} \times I$ , they must agree whenever two such sets  $N \times I$  overlap, so we get in fact a well-defined lift  $\tilde{F}$  on  $Y \times I$ . This  $\tilde{F}$  is continuous since it is continuous on each segment  $\{y\} \times I$ .  $\square$

### Poznámka

We would like to see  $\pi_1$  as a functor  $\pi_1 : \text{Top} \rightarrow \text{Grp}$ . In order for  $\pi_1$  to be a functor, we want for  $\varphi : (X, x_0) \xrightarrow{\text{cont.}} (Y, y_0)$  associate  $\varphi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ . How to get  $\varphi_*$ ? Given a loop  $\gamma$  on  $X$  based at  $x_0$ . We have a loop  $\varphi \circ \gamma$  on  $Y$  based at  $y_0$ .  $\varphi_*([\gamma]) := [\varphi \circ \gamma]$ . Is  $\varphi_*$  a homomorphism?  $\varphi_*([\gamma; \gamma_2]) = [\varphi \circ (\gamma_1 \cdot \gamma_2)] = [\varphi \circ \gamma_1 \cdot \varphi \circ \gamma_2] = [\varphi \circ \gamma_1] \cdot [\varphi \circ \gamma_2]$ . Hence  $\varphi_*$  is a homomorphism.

$$\forall (X, x_0) \in \text{Fb}(\text{Top}) \exists \varphi = 1 \in \text{Hom}((X, x_0), (X, x_0)).$$

We want  $\pi_1(1) = \text{id}_{\pi_1(X, x_0)}$ . This because of the definition of  $\pi_1(1)$ , which maps  $[\gamma] \rightarrow [\gamma] \implies \pi_1(1) : [\gamma] \mapsto [\gamma]$ , so it is  $\text{id}_{\pi_1(X, x_0)}$ .

In order for  $\pi_1$  to be a functor we also need: given  $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$  then  $i(\psi \circ \varphi)_* = \pi_1(\psi \circ \varphi) = \pi_1(\psi) \circ \pi_1(\varphi) = (\psi)_* \cdot (\varphi)_*$ . This is because

$$\pi_1(\psi \circ \varphi)([\gamma]) = (\psi \circ \varphi)_*([\gamma]) = [\psi \circ \varphi \circ \gamma] = \psi_*([\varphi \circ \gamma]) = \pi_1(\psi)(\varphi_*([\gamma])) = \pi_1(\psi) \circ \pi_1(\varphi)([\gamma]).$$

Hence it holds and  $\pi_1$  is functor  $\text{TOP} \rightarrow \text{GRP}$ .

### Tvrzení 1.6

Given a topological space  $X$ . If  $X = \bigcup A_\alpha$ , where each of  $A_\alpha$  is a path connected subspace of  $X$  and  $A_\alpha \cap A_\beta$  is path connected  $\forall \alpha, \beta$ , then each loop in  $X$  based at  $x_0$  can be decomposed as a product of loops each of which is in some  $A_\alpha$ .

*Důkaz*

Given  $f : I \rightarrow X$  with the basepoint  $x_0$ , we claim that there is a partition  $0 = s_0 < s_1 < \dots < s_m = 1$  of  $I$  such that  $[s_{i-1}, s_i]$  is mapped by  $f$  to a single  $A_\alpha$  (that by call  $A_i$ ). Since  $f$  is continuous, each  $s \in I$  has an open neighbourhood  $V_i$  in  $I$  mapped by  $f$  to some  $A_\alpha$ . We may in fact take  $V_s$  to be an interval whose closure is mapped to a single  $A_\alpha$ . By compactness of  $I$ , we see that a finite number of such intervals cover  $I$ . The endpoints of these intervals form a partition of  $I$ :  $0 = s_0 < s_1 < \dots < s_m = 1$ . Again  $A_i$  we call  $A_\alpha : f([s_{i-1}, s_i]) \subset A_\alpha$ . Let  $f_i$  be denoted as  $f|_{[s_{i-1}, s_i]}$ . Then  $f = f_i \cdot \dots \cdot f_m$ , where  $f_i$  is a path in  $A_i$ .

Since  $f([s_{i-1}, s_i]) \subset A_i \wedge f([s_i, s_{i+1}]) \subset A_{i+1} \implies f(s_i) \in A_i \cap A_{i+1}$ . Since  $A_i \cap A_{i+1}$  is path connected, we can choose path  $g_i \subset A_i \cap A_{i+1}$ ,  $g_i$  starts at  $x_0$  and ends at  $f(s_i)$ . Hence  $[f][f_1 g_1^{-1}][g_1 f_2 g_2^{-1}] \dots [g_{n-1} f_n g_n^{-1}]$ , loops in  $A_1, A_2, \dots, A_n$ .  $\square$

TODO?

TODO!!!

**Tvrzení 1.7** (\*, should be somewhere above)

If  $X = \bigcup_{\alpha \in I} A_\alpha$ ,  $x_0 \in \bigcap_{\alpha \in I} A_\alpha$ ,  $A_\alpha \cap A_\beta$  is path connected for all  $\alpha, \beta$ . Then for each loop  $\gamma$  in  $X$  based at  $x_0$ , there is a sequence of loops  $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_k} : [\gamma] = [\gamma_{\alpha_1}] \cdot \dots \cdot [\gamma_{\alpha_k}]$ , where  $\gamma_{\alpha_i}$  is a loop in  $A_{\alpha_i}$ .

**Věta 1.8** (Van Kampen)

Given a topological space  $X$  such that  $X = \bigcup A_\alpha$ , where  $A_\alpha$ 's are path connected subspaces of  $X$ ,  $x_0 \in \bigcap A_\alpha \neq \emptyset$ . Besides that we assume  $A_\alpha \cap A_\beta$  is connected  $\forall \alpha, \beta$ . Then  $\exists \Phi : *_{\alpha \in I} \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$ , which is a surjective homomorphism. If in addition  $\forall \alpha, \beta, \gamma : A_\alpha \cap A_\beta \cap A_\gamma$  is path connected, then  $\text{Ker } \Phi = \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle$ .

$$*_{\alpha \in I} \pi_1(A_\alpha, x_0) / \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle \simeq \pi_1(X, x_0).$$

*Důkaz*

The proof uses Proposition \*. Same terminology: Given a loop  $f$  in  $X$  based at  $x_0$ , by a factorization of  $[f] \in \pi_1(X, x_0)$  we mean a formal product  $[f_1] \cdot \dots \cdot [f_k]$ , where each  $f_i$  is a loop in some  $A_\alpha$  based at  $x_0$  and  $[f_i] \in \pi_1(A_\alpha, x_0)$ , and where  $f$  is homotopic to  $f_1 \cdot \dots \cdot f_k$ . (Factorization is a presentation from Proposition \*. Factorization can be seen as a word in  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$ , but not necessary reduced word.)

We construct  $\Phi$  by sending  $[f]$  to  $[f_1] \cdot \dots \cdot [f_k]$  coming from Proposition \* with the potential reduction. We will be concerned with the uniqueness of such factorizations. We define an equivalence relation on factorizations. We say that two sequences are equivalent iff they are related by a sequence of moves: combine adjacent terms:  $[f_i] \cdot [f_j] = [f_i f_j]$  if  $[f_i], [f_j] \in \pi_1(A_\alpha, x_0)$  for some  $\alpha$ ; and regard  $[f_i] \in \pi_1(A_\alpha)$  as  $[f_i] \in \pi_1(A_\beta)$  if  $f_i$  is a loop in  $A_\beta$ ; and we do not write constant loops in the decomposition.

This equivalence relation mimics, what is necessary for words in  $*\pi_1(A_\alpha, x_0)$ . We modify

this equivalence relation by substituting second condition (regard ...) by: For  $f_i, f_j: [f_i] \in \pi_1(A_\alpha, x_0)$ ,  $[f_j] \in \pi_1(A_\beta, x_0)$ , we view  $[f_i], [f_j]$  as elements of  $\pi_1(A_\alpha \cap A_\beta, x_0)$ . So first and third condition do not change the elements in  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$  and the new second one is such that it does not change elements  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N$ .

So equivalent factorizations supposed to give the same element  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$ . Hence we can show that any two factorizations of  $f$  are equivalent, this will imply  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N \rightarrow \pi_1(X, x_0)$  is injective, and therefore  $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N \rightarrow *_{\alpha \in I} \pi_1(A_\alpha, x_0)$  is trivial, and this map is an isomorphism.

„Equivalence of two factorizations“: We take two factorizations  $[f_1] \cdot \dots \cdot [f_k]$  and  $[f'_1] \cdot \dots \cdot [f'_l]$  of  $[f] \in \pi_1(X, x_0)$ . We would like to show that  $[f_1] \cdot \dots \cdot [f_k] \sim [f'_1] \cdot \dots \cdot [f'_l]$ . Since they represent  $[f]$ , they are homotopic. Let  $F : I \times I \rightarrow X$  is a homotopy from  $f_1 \cdot \dots \cdot f_k$  to  $f'_1 \cdot \dots \cdot f'_l$ . Then we say that  $\exists$  decompositions  $0 = s_0 < s_1 < \dots < s_m = 1$ ,  $0 = t_0 < t_1 < \dots < t_n = 1$  such that each  $R_{ij} := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$  is mapped by  $F$  to a single  $A_\alpha$ , which we label by  $A_{ij}$ . These partitions can be obtained from  $I \times I$  by finitely many rectangles  $[a, b] \times [c, d]$  mapping to a single  $A_\alpha$  by compactness of  $I \times I$ .

With this type of description, we can assume that  $R_{ij}$ 's are such that they are to the subdivision giving homotopies between  $f_i$ 's and  $f_j$ 's.

Since  $F$  maps a neighbourhood of  $R_{ij}$  into  $A_{ij}$ , we may perturb the vertical sides of  $R_{ij}$  so that each point of  $I \times I$  lies in at most three  $R_{ij}$ 's. After this modification we do the relabelling by the following scheme: label row by row starting from the first one, and label from left to right.

If  $\gamma$  is path in  $I \times I$  from the left edge (i.e.  $\{0\} \times [0, 1]$ ) to the right edge (i.e.  $\{1\} \times [0, 1]$ ), then  $F|_\gamma$  is a loop from  $x_0$  to  $x_0$ . We write  $\gamma_r$  to be the path (through edges of rectangles) separating the first  $r$  rectangles  $R_1, \dots, R_r$  from the remaining rectangles.  $\gamma_0$  is bottom edge,  $\gamma_m$  is the top edge.

Let us call the corners of  $R_i$ 's vertices. So for each vertex  $v$  with  $F(v) \neq x_0$ , there is a path  $g_v$  from  $x_0$  to  $F(v)$  that lies in the intersection of the two or three  $A_{ij}$ 's corresponding to  $R_i$ 's containing  $v$ . (This is by the choice of  $R_i$ 's and the fact that  $A_\alpha$ 's are path connected and contains  $x_0$  and  $F(v)$ . And by path connectedness of disjoint of  $A$ 's.)

Then we obtain a factorization  $[F|_{\gamma_r}]$  by inserting the appropriate path connection  $x_0$  with  $F|_{\text{vertices}}$  corresponds of  $R_i$ 's then we can say that  $[F|_{\gamma_r}]$  has a decomposition that depends on the choice of  $A_{ij}$  that corresponds to the edge of  $R_i$  on which the component of the decomposition lies.

$[F|_{\gamma_r}]$  has a decomposition by inserting  $g_v^{-1} g_v$ . Different choices of  $A_{ij}$ 's will change the factorization of  $[F|_{\gamma_r}]$ . (Our equivalence relation  $\sim$  is designed to make the choice of  $A_{ij}$ 's for such paths irrelevant. I.e. the different factorizations coming from different choices are equivalent.)

Factorizations for two consecutive paths  $\gamma_r$  and  $\gamma_{r+1}$  are equivalent since pushing along  $R_{r+1}$  from  $\gamma_r$  to  $\gamma_{r+1}$  changing  $F|_{\gamma_r}$  to  $F|_{\gamma_{r+1}}$  by a homotopy within  $A_{ij}$  corresponding to  $R_{r+1}$ , so we can choose this  $A_{ij}$  for all segments of  $\gamma_r$ .

We can arrange that the factorization associated to  $\gamma_0$  is equivalent to  $[f_1] \cdot \dots \cdot [f_k]$  by choosing  $g_r$ 's for each vertex  $v$  along the lower edge of  $I \times I$  to lie not just in the two  $A_{ij}$ 's corresponding to  $R_s$ 's containing  $v$ , but also to lie in the  $A_\alpha$  for  $f_i$  containing  $v$  in its domain. In the case when  $v$  is the common point of two domains for two consecutive  $f_i$  and  $f_{i+1}$  we have  $F(v) = x_0$ , so there is no need to choose  $g_v$ 's for such  $v$ 's. In this fashion we can assume that the factorization for  $[f_1] \cdot \dots \cdot [f_k]$  and  $[f'_1] \cdot \dots \cdot [f'_l]$  are equivalent.  $\square$

TODO? (Examples of covering space)

### Tvrzení 1.9

Given a covering space  $p : \tilde{X} \rightarrow X$ , a homotopy  $f_t : Y \rightarrow X$  and a map  $\tilde{f}_\sigma : Y \rightarrow \tilde{X}$  lifting  $f_0$ , then there is a unique homotopy  $\tilde{f}_t : Y \rightarrow \tilde{X}$  of  $\tilde{f}_0$  that lifts  $f_t$ .

*Poznámka*

This statement in a more general form has already appeared on practicals.

*Poznámka*

One can use this proposition in two ways: We have a lifting property for paths which says that  $\forall \text{path } f : [0, 1] = I \rightarrow X$  and each lift  $\tilde{x}_0$  of  $f(0) = x_0 \in X$  there is a unique  $\tilde{f} : I \rightarrow \tilde{X}$  lifting  $f$  starting at  $\tilde{x}_0$ .

If  $Y = I$ , we get that every homotopy  $f_t$  of a path  $f_0$  in  $X$  lifts to a homotopy  $\tilde{f}_t$  of each lift  $\tilde{f}_0$  of  $f_0$ .

As a corollary of Proposition (\*) we can write:

*Důsledek*

The map  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  induced by the covering space map  $p : \tilde{X} \rightarrow X$  is injective. The image subgroup  $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  in  $\pi_1(X, x_0)$  consists of the homotopy classes of loops in  $X$  based at  $x_0$  whose lifts to  $\tilde{X}$  starting at  $\tilde{x}_0$  are loops.

┌

*Důkaz* (Of corollary)

An element of the kernel of  $p_*$  is represented by a loop  $\tilde{f}_0 : I \rightarrow \tilde{X}$  with a homotopy  $f_t : I \rightarrow X$  of  $f_0 = p \cdot \tilde{f}_0$  to the trivial loop. By ? the applications of proposition (\*) there is a lifted homotopy of loops  $\tilde{f}_t$  starting at  $\tilde{f}_0$  and ending with a constant loop. Hence  $[\tilde{f}_0] = 0$  in  $\pi_1(\tilde{X}, \tilde{x}_0)$  and  $p_*$  is injective.  $\square$

└

### Tvrzení 1.10

The number of sheets of a covering space  $p : \tilde{X} \rightarrow X, \tilde{x} \mapsto x$  with  $X$  and  $\tilde{X}$  path connected equals the index of  $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  in  $\pi_1(X, x_0)$ .

*Důkaz*

For a loop  $g$  in  $X$  based at  $x_0$  let  $\tilde{g}$  be its lift in  $\tilde{X}$  based at  $\tilde{x}_0$ . A product  $h \cdot g$ ,  $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  has a lift  $\tilde{h} \cdot \tilde{g}$  ending at the same point as  $\tilde{g}$  (since  $\tilde{h}$  is a loop). Hence we can define a map  $\Phi$  from cosets  $H[g]$  to  $p^{-1}(x_0)$  implies that  $\Phi$  is surjective since  $\tilde{x}_0$  can be lowered by a path  $\tilde{g}$  to an arbitrary point in  $p^{-1}(x_0)$  and  $\tilde{g}$  projects to a loop  $g$  at  $x_0$ .

To show that  $\Phi$  is injective, observe that  $\Phi(H[g_1]) = \Phi(H[g_2]) \implies g_1 \cdot g_2^{-1}$  lifts to a loop in  $\tilde{X}$  based at  $\tilde{x}_0$ , so  $[g_1][g_2]^{-1} \in H \implies H[g_1] = H[g_2] \implies$  injectivity + surjectivity  $\implies$  bijectivity of  $\Phi$ .  $\square$

TODO? (Problems)

### Definition 1.9

We say that a topological space is locally path connected if for each  $x \in X$  and each  $U$  neighbourhood of  $X$  exists  $V \subset U$  such that  $V$  is path connected.

### Tvrzení 1.11

*Given a covering space  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  and a map  $f : (Y, y_0) \rightarrow (X, x_0)$  with  $Y$  path and locally path connected. Then a lift of  $f$  that we denote by  $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$  exists  $\Leftrightarrow f_* : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$  has a property that  $f_*(\pi_1(Y, y_0)) \subset p_* \left( \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \right)$ .*

┌ *Důkaz*

„ $\implies$ “: simply follows from the definition of lifting, i.e.  $f_* = p_* \tilde{f}_*$ .

„ $\impliedby$ “ Let  $y \in Y$  and let  $\gamma$  be a path in  $U$  from  $y_0$  to  $y$ . The path  $f\gamma = f(\gamma)$  in  $X$  starts at  $x_0$  and has unique lift  $\tilde{f}\gamma$  which is a path that starts at  $\tilde{x}_0$ . Define  $\tilde{f}(y) = \tilde{f}\gamma(1)$ . To show that this is well-defined and independent of the choice of  $\gamma$ , let  $\gamma'$  be another path from  $y_0$  to  $y$  in  $Y$ .

Then  $(f\gamma') \cdot (f\gamma)$  is a loop that we call  $h_0$  in  $X$  with

$$[h_0] \in f_* (\pi_1(Y, y_0)) \subseteq p_* \left( \pi_1 \left( \tilde{X}, \tilde{x}_0 \right) \right).$$

This means that there is a homotopy  $h_t$  of  $h_0$  to a loop  $h_1$  that lifts to a loop  $\tilde{h}_1$  in  $\tilde{X}$  based at  $\tilde{x}_0$ . We apply the covering homotopy property (proposition above) to  $h_t$  to get a lift  $\tilde{h}_t$ . Since  $\tilde{h}_1$  is a loop at  $\tilde{x}_t$ , so is  $\tilde{h}_0$ . By the uniqueness of lifted paths, the first half of  $\tilde{h}_0$  is  $\tilde{f}\gamma'$  and the second half is  $(\tilde{f}\gamma)^{-1}$  with the common midpoint  $\tilde{f}\gamma(1) = \tilde{f}\gamma'(1)$ .

We still need to show that  $\tilde{f}$  is continuous. To see this we take open neighbourhood  $U \subset X$  of  $f(y)$  having a lift  $\tilde{U} \subset \tilde{X}$  containing  $\tilde{f}(y)$  such that  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism. Choose a path connected open neighbourhood  $V$  of  $y$  with  $f(V) \subset U$ . For paths from  $y_0$  to point  $y'$  in  $V$  we take a fixed path  $\gamma$  from  $y_0$  to  $y$  followed by paths  $\eta$  in  $V$  from  $y$  to the points  $y'$ . Then the paths  $(f\gamma) \cdot (f\eta)$  in  $X$  have lifts  $(\tilde{f}\gamma)(\tilde{f}\eta)$ , where  $\tilde{f}\eta = p^{-1}f\eta$  and  $p^{-1} : U \rightarrow \tilde{U}$ . Thus  $\tilde{f}(V) \subset \tilde{U}$  and  $\tilde{f}|_V = ?$ ,  $p$  is homeomorphism  $\implies p^{-1}$  is continuous and  $f$  is continuous  $\implies \tilde{f}|_V$  is continuous.  $\square$

TODO? (Problems)

*Poznámka*

Typically homology is easier to compute than homotopy.

**Definition 1.10** (Simplex defined by points, standard simplex, direction of simplex)

Take  $\mathbb{R}^m$  and points  $v_0, \dots, v_m \in \mathbb{R}^m$  such that  $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$  are linearly independent (this condition is necessary to say that  $v_0, \dots, v_m$  will not live on hyperplane of dimension  $< n$ ). We define

$$[v_0, v_1, \dots, v_n] := \left\{ \sum_{i=1}^n t_i v_i \mid \sum t_i = 1, t_i \geq 0 \right\},$$

the smallest convex set in  $\mathbb{R}^m$  containing  $v_0, \dots, v_n$ . We call  $[v_0, v_1, \dots, v_n]$  a simplex defined by points  $v_0, \dots, v_n$ .

Standard simplex:

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1 \wedge \forall i \in [0, n] t_i \geq 0 \right\}$$

The order we use for points in particular determines directions of 1-simplices. Specifying the direction in such a way we got a canonical homeomorphism  $\Delta^n \rightarrow [v_0, \dots, v_n]$  given by  $(t_0, \dots, t_n) \rightarrow \sum_{i=1}^n t_i v_i$ .

### Definition 1.11 (Faces, boundary, and interior of simplex)

Given  $[v_0, \dots, v_n]$ , if we remove  $v_i$ , i.e. we write  $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$ , then such a simplex is a face of  $[v_0, \dots, v_n]$ . Same way we can remove more vectors and get faces of lower dimensions.

$\partial\Delta_n$  = boundary of  $\Delta_n$  = union of all faces (including lower dimensional) of  $\Delta_n$ . (The same for  $[v_0, \dots, v_n]$ .)

$\mathring{\Delta}_n := \Delta_n \setminus \partial\Delta_n$  is interior of  $\Delta_n$ . (The same for  $[v_0, \dots, v_n]$ .)

### Definition 1.12 ( $\Delta$ complex structure ( $\Delta$ -complex))

A  $\Delta$  complex structure on a topological space  $X$  is a collection of maps  $\delta_\alpha : \Delta^n \rightarrow X$  with  $n$  independent of  $\alpha$  (i.e.  $n$  is fixed,  $\alpha$  indexes collection) such that

- $\delta_\alpha|_{\mathring{\Delta}_n}$  is injective, and each point of  $X$  can be in the image of exactly one such restriction;
- $\delta_\alpha|_{\text{face of } \Delta^n}$  is one of the maps  $\delta_\beta : \Delta^{n-1} \rightarrow X$ . Here we identify the face of  $\Delta^n$  with  $\Delta^{n-1}$  by the canonical linear homeomorphism that preserves the ordering of vertices.
- $A \subset X$  is open  $\Leftrightarrow \partial_\alpha^{-1}(A)$  is open in  $\Delta^n$  for each  $\delta_\alpha$ .

*Poznámka*

$\Delta$ -complex can be built from the collection of disjoint simplices by identifying various subcomplexes spanned by the subsets of the vertices, where the identification can be done by canonical linear homeomorphisms that preserves the ordering of vertices.

From the discussion about CW complexes it follows that interiors of  $\Delta^n$  that define  $\delta_\alpha(\mathring{\Delta}_n)$  (open simplices on  $X$ ) are the  $e_\alpha^n$  of a CW complex structure on  $X$ .

### Definition 1.13 (Simplicial homology)

Given a topological space with a fixed  $\Delta$ -complex structure on it. We define  $\Delta_n(X) = C_n(X)$  which is by definition the free Abelian group with basis consist of  $\sigma_n(\mathring{\Delta}_n) = e_\alpha^n$  of  $X$ . (Free module over  $\{e_\alpha^n\}$ .)



*Poznámka*

Elements of  $\Delta_n(X) = C_n(X)$  can be written as formal sums  $\sum_{\alpha} n_{\alpha} e_{\alpha}^n$ , where  $n_{\alpha} \in \mathbb{Z}$ . Elements of  $\Delta_n(X) = C_n(X)$  are typically called chains.

Equivalently we could write  $\sum_{\alpha} n_{\alpha} \delta_{\alpha}$ , where  $\delta_{\alpha}$  are the defining on characteristic maps of  $e_{\alpha}^n$ . We remember that  $\partial([v_0, v_n])$  consists of  $n - 1$ -simplices.

TODO? (Last remarks)

TODO? (Problems)

## 2 Singular homology

### Definice 2.1 (Singular $n$ -simplex, chain, chain group)

A singular  $n$ -simplex in a topological space  $X$  is a continuous map  $\delta : \Delta^n \rightarrow X$ .

$C_n(X)$  = free Abelian group generated by the set of singular  $n$ -simplices in  $X$ .

*Poznámka*

The generators of  $C_n(X)$  is in general an uncountable set.

If  $X = \{\text{point}\}$ , then all possible maps  $\Delta^n \rightarrow X$  are just constant maps and there is only one constant map. Hence the generating set of  $|C_n(X)| = 1$  and thus  $C_n(X) \simeq \mathbb{Z}$ .

Elements of  $C_n(X)$  are called chains, and  $C_n(X)$  is called a  $n$ -th chain group. We define

$$\partial_n : C_n(X) \rightarrow C_{n-1}(X), \quad \partial_n(\delta) = \sum_{i=0}^n (-1)^i \delta|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

*Poznámka*

This is the same formula that we used for simplicial homology.

Since the formulas for  $\partial_n$  and  $\partial_{n-1}$  coincide with the corresponding formulas for simplicial homology, we see that  $\partial_{n-1} \circ \partial_n = 0$ . This follows from the proof for simplicial homology. (The standard notation for singular complex will be  $C(X)$ , for simplicial  $C^{\Delta}(X)$ .)

Therefore, we can form a complex  $C(X)$ :

$$\dots \rightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \rightarrow \dots \rightarrow 0$$

On the left side, a priori, we can't say that a certain level we start getting only zeroes. ( $C_n = 0 \equiv C_n = \{0\}$ ).

$H^\Delta(X)$  = simplicial homology. Then we define  $H_n(X) = \text{Ker } \partial_n / \Im \partial_{n+1}$ . ( $n$ -th singular homology group.)

*Poznámka*

On one hand singular homology looks more general than simplicial homology. On the other hand, singular homology can be seen as a particular case of simplicial homology by the following construction:

For an arbitrary space  $X$ , we define the singular complex  $S(X)$  to be the  $\Delta$ -complex with one  $n$ -simplex  $\Delta_\delta^n$  for each singular  $n$ -simplex  $\delta : \Delta^n \rightarrow X$  with  $\Delta_\delta^n$  attached to the  $n - 1$  simplices of  $S(X)$ , that are restrictions of  $\delta$  to the  $n - 1$  simplices in the boundary  $\partial\Delta^n$  by the restriction maps.  $H_n^\Delta(S(X))$  is identified with  $H_n(X)$ .

*Poznámka*

There are two variants of singular homology:

1. Reduced homology: Assume that we have a singular chain complex

$$\rightarrow C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \rightarrow 0 \rightarrow \dots$$

We modify this complex, ? will be called the reduced complex (reduced from  $C(X)$ ):

$$\rightarrow C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0 \rightarrow \dots,$$

where  $\varepsilon$  is called an augmentation map and it is defined by  $\varepsilon(\sum_{i \text{ finite}} n \cdot i \cdot \delta_i) = \sum_{i \text{ finite}} n \cdot i \implies \varepsilon$  is well defined. (Exercise: check that  $\varepsilon$  vanishes on  $\partial_1$ .) Denote the homology of the reduced complex by  $\tilde{H}(X)$ . TODO!!! (Asi řádek rovnic.) If  $X \neq \emptyset$ , then  $H_{-1}(X) \simeq 0$  and  $H_0(X) \simeq \tilde{H}_0(X) \oplus \mathbb{Z}$ .

2. Relative complex (complex of a pair): Given a space  $X$  and a subspace  $A \subset X$ , we define  $C_n(X, A) := C_n(X)/C_n(A)$  (the chains in  $A$  will be trivial in  $C_n(X, A)$ ). This way we get a complex

$$\dots \rightarrow C_n(X, A) \xrightarrow{\partial_n(X, A)} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}(X, A)} C_{n-2}(X, A) \rightarrow \dots \rightarrow 0,$$

here  $\partial_n(X, A) : C_n(X, A) \rightarrow C_{n-1}(X, A)$  are quotient maps of  $\partial_n$ 's for  $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ .

Since we know that  $\partial_{n-1} \circ \partial_n = 0$ , then for quotient maps the corresponding compositions  $\partial_{n-1}(X, A) \circ \partial_n(X, A) = 0$ . Hence we really see that this is a complex. We can define the homology of it  $H_n(X, A) := \text{Ker } \partial_n(X, A) / \Im \partial_{n+1}(X, A)$ .

## 2.1 Homotopy equivalence of singular homology

**Definice 2.2**

For a map  $f : X \rightarrow Y$ , an induced homomorphism  $f_n^c : C_n(X) \rightarrow C_n(Y)$  is defined by

$$(\Delta^n \xrightarrow{\partial_X} X) \rightarrow (\Delta^n \xrightarrow{\partial_X} X \xrightarrow{f} Y)$$

$\delta_X \mapsto f \circ \delta_X = \delta_Y$ . These maps  $f_n^c$  satisfy  $f_c^{n-1} \circ \partial_n^X = \partial_n^Y \circ f_n^c$ . TODO?(This is since...)

Typically maps  $f_c = (f_c^n)_n$  are called chain maps.  $f_c$  induces a map  $f_* : H_n(X) \simeq H_n(Y)$ .

TODO?

TODO? (Problems)

**Tvrzení 2.1**

*Given two topological spaces  $X$  and  $Y$  that are homotopy equivalent. Then  $H_n(X) \simeq H_n(Y) \forall n$ .*

*Důkaz*

The essential procedure is a subdivision (simplex decomposition) of  $I \times \Delta^n$ . Let  $\{0\} \times \Delta^n$  be given by  $[v_0, \dots, v_n]$  and  $\{1\} \times \Delta^n$  be given by  $[w_0, \dots, w_n]$ , where  $v_i$  and  $w_i$  have the same image under the standard/canonical projection  $I \times \Delta^n \rightarrow \Delta^n$ . We would like to pass from  $[v_0, \dots, v_n]$  to  $[w_0, \dots, w_n]$  by interpolating a sequence of  $n$ -simplices, each obtained from the preceding one by moving one vertex  $v_i$  up to  $w_i$ , starting with  $v_n$  and working backwards to  $v_0$ . The first step for us is to ?  $[v_0, \dots, v_{n-1}, v_n]$  to  $[v_0, \dots, v_{n-1}, w_n]$ . The second step is to move this up to  $[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$ . Then we keep going and proceed with other (remaining) steps.

The typical step for us is to  $[v_0, \dots, v_{i-1}, v_i, w_{i+1}, \dots, w_n]$  moves up to  $[v_0, \dots, v_{i-1}, w_i, w_{i+1}, \dots, w_n]$ . The region between simplices  $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$  and  $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$  is a  $(n+1)$ -simplex  $[v_0, \dots, v_i, w_i, \dots, w_n]$ .

This way for each of the steps, we get  $(n+1)$ -simplex. This leads to the decomposition of  $I \times \Delta^n$  into  $(n+1)$ -simplices, each intersecting the previous one with the  $n$ -simplex face. Since  $X$  is homotopy equivalent to  $Y$ , there is a homotopy  $F : I \times X \rightarrow Y$  and for each simplex  $\delta : \Delta^n \rightarrow X$  we get the composition  $F \circ (\delta \times 1) : \Delta^n \times I \rightarrow X \times I \rightarrow Y$ . Using this map we can define the prism operator  $P : C_n(X) \rightarrow C_{n+1}(Y)$ .  $P(\delta) = \sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$ .

We will show that  $P$  satisfies the following property:  $\partial_Y P = g_c - f_c - P\partial_X$ . To prove this, we compute:

$$\partial P(\delta) = \partial \left( \sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) = \sum_{j \leq i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} + \sum_{j \geq i} (-1)^{i+(j+1)} F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_{j+1}, \dots, w_n]}$$

The terms with  $i = j$  will cancel out except of  $F \circ (\delta \times 1)|_{[v_0, w_0, \dots, w_n]}$ , which is  $g \circ \delta = g_c(\delta)$ , and  $-F \circ (\delta \times 1)|_{[v_0, \dots, v_n, w_n]}$ , which corresponds to  $-f \circ \delta = -f_c(\delta)$ .

Now observe that  $i \neq j$ , then part of that corresponds to  $-P\partial$ , since  $\partial P(\delta) = (g_c - f_c - P\partial)(\delta)$ , where  $\delta = [v_0, \dots, v_n]$ .

$$\partial P(\delta) = \sum_{j < i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} + \sum_{j > i} (-1)^{i+(j+1)} F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_{j+1}, \dots, w_n]}.$$

$$P(\partial(\delta)) = P\left(\sum_j (-1)^j \partial(\delta)|_{[v_0, \dots, v_j, \dots, v_n]}\right)$$

We finish the proof by saying the following: If  $\alpha \in C_n(X)$ , then  $g_c(\alpha) - f_c(\alpha) - P\partial(\alpha) = \partial P(\alpha) \implies$

$$\implies g_c(\alpha) - f_c(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha),$$

because  $\alpha$  is a ?. Hence  $g_c(\alpha) - f_c(\alpha)$  is a boundary (it is  $\partial(P(\alpha))$ ). So  $g_c(\alpha)$  and  $f_c(\alpha)$  determine the same class in homotopy  $\implies f_* = g_* \implies$  homotopy equivalence of  $X$  and  $Y \implies H_n(X) \simeq H_n(Y)$ .  $\square$

TODO? (Problems)

TODO!!!