

Poznámka

At least 1 from (3-)4 homework (flexible deadlines – last lecture).

Poznámka

In this lecture, there was also the revision of topology. (Topological space, topology, basis of topology, continuous map, quotient space, product topology, Hausdorff spaces).

Poznámka

World Homotopy comes from homós (= same, similar) and topos (place).

Definition 0.1 (Homotopic functions)

Given two topological spaces X and Y and two continuous functions $f, g : X \rightarrow Y$, we say that f is homotopic to g ($f \sim g$) if there is a 1-parametric family $f_t : X \rightarrow Y$: $f_0 = f$, $f_1 = g$ and the map $F : [0, 1] \times X \rightarrow Y$ defined by $(t, x) \mapsto f_t(x)$ is continuous.

Definition 0.2 (Homotopy equivalent spaces)

Given two topological spaces X and Y we say that X and Y are homotopy equivalent if there is a pair of continuous maps (f, g) such that $f : X \rightarrow Y$ and $g : Y \rightarrow X$ and $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$, $g \circ f \sim \text{id}_X$, $f \circ g \sim \text{id}_Y$.

Příklad

Given \mathbb{R} , \mathbb{R}^2 with the standard Euclidean topology and two maps $f : \mathbb{R} \rightarrow \mathbb{R}^2$, $x \mapsto f(x) = (x, x^3)$, $g : \mathbb{R} \rightarrow \mathbb{R}^2$, $x \mapsto g(x) = (x, e^x)$.

Are f and g homotopic? (Show that by constructing homotopy.)

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Řešení

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$$F(t, x) = (1 - t)(x, x^3) + t(x, e^x) = (x, (1 - t)x^3 + te^x).$$

Příklad

Given three topological spaces (X, τ_X) , (Y, τ_Y) , (Z, τ_Z) and two pairs of continuous maps $f_1, g_1 : (X, \tau_X) \rightarrow (Y, \tau_Y)$ and $f_2, g_2 : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$. Assume that f_1 is homotopic to g_1 and f_2 is homotopic to g_2 . Show that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$.

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Řešení

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$$F(t, x) = F_2(t, F_1(t, x)).$$

Příklad

Take $B^n := \{x, \dots, x_n | \sqrt{x_1^2 + \dots + x_n^2} \leq 1\} \subseteq \mathbb{R}^n$. And take a map $f : B^n \rightarrow B^n$: $f(x) = (0, \dots, 0) \in B^n$ for all $x \in B^n$. Shows that there is a homotopy from id to f .

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Řešení

$$F : [0, 1] \times B^n \rightarrow B^n, \quad (t, x) \mapsto (1 - t)x.$$

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Příklad

Take a 2-ball B^2 . B^2 is homotopy equivalent to its center by previous problem, but it is not homeomorphic to $(0, 0)$.

Definice 0.3 (Deformation retraction)

A deformation retraction of a topological space X onto a subspace A is a family of maps $f_t : X \rightarrow X$, $t \in [0, 1]$: $f_0 = \text{id}_X$, $f_1(X) = A$ and $f_t|_A = \text{id}_A$. And family f_t is continuous in the following sense:

$$F : [0, 1] \times X \rightarrow X, (t, x) \mapsto f_t(x), \text{ is continuous.}$$

Tvrzení 0.1

Given a deformation retraction $f_t : X \rightarrow X$, there is a pair $(f, g) : X \xrightarrow{f} A \xrightarrow{g} X : g \circ f \sim \text{id}_X, f \circ g \sim \text{id}_A$.

Poznámka (Suggestion)

$$f = f_1, g = f_0 \circ i_A \quad (A \xrightarrow{i_A} X), \text{ tj. } f \circ g : A \xrightarrow{i_A} X \xrightarrow{f_1} X \xrightarrow{f_1} X, a \mapsto a \mapsto a \mapsto a \text{ (or } A) \\ \implies f \circ g = \text{id}_A. g \circ f : X \xrightarrow{f_0} A \xrightarrow{i_A} X \implies f_1(x) \sim \text{id}_X.$$

Definice 0.4

Given two topological spaces X and Y and a continuous map $f : X \rightarrow Y$, the mapping cylinder M_f is defined to be the quotient space of $X \times [0, 1] \amalg Y$ and $\sim : (x, 1) \sim f(x)$. $M_f = X \times [0, 1] \amalg Y / \sim$.

Tvrzení 0.2

Given X, Y and f , M_f deformation retracts to Y .

Důkaz (/ Idea of proof)

The way to construct $f_t = F(\cdot, t) : M_f \rightarrow M_f$ is to slide each point (x, t) along the segment $\{x\} \times [0, 1]$ to $f(x)$:

$$F : (x, t) \mapsto f_t(x), \quad \forall y \in Y : y = F(y, t) \mapsto \{f_1 = \text{id } Y \rightarrow Y\}$$

In your HW you will check that $F(x, t)$ is continuous. □

Poznámka

Cell complex (CW complex) is a topological space with a nice decomposition into small pieces.

1. Start with a discrete set X^0 , whose points are called 0-cells.

2. We form the n -skeleton X^n from X^{n-1} by attaching cells $e_\alpha^n = I^n = [0, 1]^n$. By the attachment we mean $(e_\alpha^n = B_\alpha^n, \partial e_\alpha^n = S_\alpha^n) \varphi_\alpha : \partial e_\alpha^n \rightarrow X^{n-1}$. Hence we can view $X^n = X^{n-1} \coprod \coprod B_\alpha^n / \sim$, where $x \sim \varphi_\alpha(x)$ for $x \in \partial B_\alpha^n$.

3. We can either stop this inductive process at a certain finite steps or take an infinite number of steps. In the first case $X = X^n$ for some n , in the second one $X = \bigcup_{n \in \mathbb{N}_0} X^n$ with the weak topology ($A \subset X$ is open $\leftrightarrow A \cap X^n$ is open for all n).

Například

Example of 1-skeleton is graph.

Definition 0.6

Given a cell complex X . Each cell e_α^n has a characteristic map $\Phi_\alpha : e_\alpha^n = B_\alpha^n \rightarrow X$ which extends the attaching map $\varphi_\alpha : \partial B_\alpha^n \rightarrow X^{n-1}$, it is homeomorphism from the interior of B_α^n onto e_α^n . Namely

$$B_\alpha^n \hookrightarrow X^{n-1} \coprod \coprod_{\beta} B_\beta^n \xrightarrow{\text{quotient}} X^n \rightarrow X, \quad B_\alpha^n \rightarrow X$$

Definition 0.7

A subcomplex of CW complex is a closed subspace $A \subset X$ that is a union of cells with the corresponding attachments.

Příklad

Construct two different CW structures on S^2 .

Řešení

$S^2 = e^0 \cup e^2$, $S^2 = e^0 \cup e^1 \cup \{e_1^2, e_2^2\}$. (See practicals.)

Příklad

We define $\mathbb{R}P^n$ to be the quotient of S^n / \sim , where $V \sim$ the antipodal point to V . TODO?

Definice 0.8

Consider a pair (X, A) where X is a CW complex and A is subcomplex. Then we define the quotient complex X/A to be the CW complex with the structure: There are all the cells of $X \setminus A$ with the corresponding attaching maps, and there is a extra 0-cell which is A in $X \setminus A$. For a cell e_α^n of $X \setminus A$ attached by $\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$, the attaching map in the corresponding cell in X/A is the composition $S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$.

Příklad

Show that $S^n = e^0 \cup e^n$ is $B^n/S^{n-1} = \text{TODO}/e^0 \cup e^{n-1}$.

Definice 0.9 (Cuspension)

For a space X , the suspension SX is the quotient of $X \times T := X \times [0, 1]$ by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

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Například (Motivation)

$$X = S^n \implies SX = X \times [0, 1] / \sim = S^{n+1}.$$

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Definice 0.10 (Wedge sum)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum $X \wedge Y$ is the quotient of the disjoint union $X \coprod Y$ obtained by identifying x_0 and y_0 to a single point.

1 Fundamental group

1.1 Paths and homotopy

Definice 1.1 (Path, endpoints)

Given a topological space X , a path in X is a continuous map $f : [0, 1] \rightarrow X$. Then points $f(0)$ and $f(1)$ are usually called endpoints of f .

Definice 1.2 (Homotopy)

A homotopy of paths in X is a family $f_t : [0, 1] \rightarrow X$, $t \in [0, 1]$, such that the endpoints $f_t(0) = x_0$, $f_t(1) = x_1$ for all $t \in [0, 1]$, i.e. the endpoints are independent of t ; the associated map $F : I \times I \rightarrow X$ defined by $F(s, t) := f_t(s)$ is continuous.

Tvrzení 1.1

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.

┌ *Důkaz*

„Reflexivity“: X topological space. We want $f : [0, 1] \rightarrow X \implies f$ is homotopic to f . The homotopy is trivial, i.e. we take $f_t(s) = f(s) \forall t$.

„Symmetry“: We have $f_0 : [0, 1] \rightarrow X$, $f_1 : [0, 1] \rightarrow X$. f_0 is homotopic to f_1 via $f_t(s)$. We want to have a homotopy g_t from f_1 to f_0 . We can define $g_t(s) = f_{1-t}(s)$.

„Transitivity“: f_0 homotopic to f_1 via f_t , and f_1 to f_2 via g_t . We define homotopy from f_0 to f_2 by $h_t(s) = f_{2t}(s)$ if $t \in [0, 1/2]$ and $h_t(s) = g_{2t-1}(s)$ if $t \in [1/2, 1]$. h_t is continuous since f_t, g_t are continuous. □

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Definice 1.3 (Composition of paths (product of paths))

Given two paths $f, g : I \rightarrow X$ such that $f(1) = g(0)$. We can take a composition of paths also called the product of paths

$$f \cdot g(s) := \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2}, \\ g(2s - 1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Poznámka (Loop, constant loop)

From now on, we will consider not all paths but loops, i.e. $f : [0, 1] \rightarrow X$ continuous such that $f(0) = f(1)$.

We take a fixed point x_0 . There is a special loop – constant loop – $f : [0, 1] \rightarrow X$: $f(s) = x_0 \forall s \in [0, 1]$.

Poznámka (Homotopy equivalence, product of equivalence classes)

When we take product $f \cdot g$ for two loops f and g starting and ending at x_0 , the product $f \cdot g$ up to homotopy (denoted $[f \cdot g]$) depends only on $[f]$ and $[g]$ (f and g up to homotopy).

$$\text{I.e. } f \sim f' \wedge g \sim g' \implies f \cdot g \sim f' \cdot g'.$$

Definice 1.4 (Fundamental group)

Given a topological space X with a fixed point $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes of loops with endpoints at x_0 and the group structure is given by the product of equivalence classes of loops.

Definice 1.5 (Reparametrization of path)

A reparametrization of path f is a composition $f \circ \varphi$, where $\varphi : I \rightarrow I$ continuous, such that $\varphi(0) = 0$, $\varphi(1) = 1$.

Důsledek

Reparametrization a path preserves its homotopy class since $f \circ \varphi \simeq f$ via the homotopy $\varphi_t(s) = (1-t)\varphi(s) + t \cdot s$ so that $\varphi_0 = \varphi$ and $\varphi_1(s) = s$

Definice 1.6 (Inverse loop)

Given a loop $f : [0, 1] \rightarrow X$ with $f(0) = f(1) = x_0$, the inverse loop f^{-1} (or \overline{f}) is $f^{-1}(s) = f(1-s)$.

Tvrzení 1.2

$\pi_1(X, x_0)$ is a group.

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Důkaz

So far we guaranteed that the product $[f \cdot g]$ of two loops $[f]$ and $[g]$ is defined. ($[f \cdot g] = [f] \cdot [g]$.) In order to be a group $(\pi_1(X, x_0), \cdot)$ should satisfy group axioms.

„Constant“: Given a loop $f : I \rightarrow X$ with $f(0) = f(1) = x_0$ and a constant loop $c : I \rightarrow X$ with $c(s) = x_0$. We take $f \cdot c$. $f \cdot c$ is a reparametrization of f via function $\varphi(s) = \min(2s, 1)$. For $c \cdot f$ we see that $c \cdot f = f \circ \varphi$, where $\varphi(s) = \max(0, 2s - 1)$. So $c \cdot f \simeq f$ and $f \cdot c \simeq f$, so $[c \cdot f] = [f] = [f \cdot c]$.

„Associativity“: If we take three loops f, g, h , then take products $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$. Observe that $f \cdot (g \cdot h)$ is a reparametrization of $[f \cdot g] \cdot h$ via the map $\varphi(s) = \max(s/2, s - 1/2, 2s - 1)$. Hence $[f] \cdot [g \cdot h] = [f \cdot g] \cdot [h]$.

„Inverse“: Take $f \cdot f^{-1}$ and construct a homotopy to a constant loop c by taking $h_t = f_t \cdot g_t$, where $f_t(s) = f(s)$ for $s \in [0, 1-t]$, and $f_t(s) = f(1-t)$ for $s \in [1-t, 1]$. ($f_0 = f$, $f_1 = c$). $g_t = f_t^{-1}$ for all t . $h_0 = f_0 \cdot f_0^{-1} = f \cdot f^{-1}$, $h_1 = c \cdot c^{-1} = c$.

In the same construction we take f^{-1} instead of f and we get $f^{-1} \cdot f \simeq c$. This way we checked $[f] \cdot [f^{-1}] = [f^{-1}] \cdot [f] = [c]$. □

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TODO? (Definition of category)

Tvrzení 1.3

There is an isomorphism $\pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ for x_0 and x_1 in the same path connected component.

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Důkaz

Since x_0, x_1 are in one path connected component \tilde{X} , \exists path $h : [0, 1] \rightarrow X$: h is in \tilde{X} and $h(0) = x_0, h(1) = x_1$. $\bar{h}(s) := h^{-1}(s) := h(1 - s)$, $s \in [0, 1]$.

To each loop f based at x_1 we associate a loop $h \cdot f \cdot h^{-1}$. $h \cdot f \cdot h^{-1}$ is based at x_0 . $\beta_h : \pi_1(x, x_1) \rightarrow \pi_1(x, x_0), [f] \mapsto [h \cdot f \cdot h^{-1}]$. We claim, that β_h is an isomorphism. „ β_h is homomorphism“:

$$\beta_h([f \cdot g]) = [hfg h^{-1}] = [hfh^{-1}hgh^{-1}] = [hfh^{-1}] \cdot [hgh^{-1}] = \beta_h([f]) \cdot \beta_h([g]).$$

„ β_h is isomorphism“: „the inverse of β_h is $\beta_{h^{-1}}$ “ (which is homomorphism too by the argument we used for β_h):

$$\beta_{h^{-1}}(\beta_h([f])) = \beta_{h^{-1}}([hfh^{-1}]) = [h^{-1}hfh^{-1}h] = [f].$$

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□

Věta 1.4 (Fundamental group of S^1)

S^1 is path connected, thus $\pi_1(S^1, x_0) = \pi_1(S^1)$.

$$\pi_1(S^1) \simeq \mathbb{Z}.$$

Důkaz

We claim that $\pi_1(S^1) \simeq \langle [\omega] \rangle$, where $\omega : [0, 1] \rightarrow S^1$, $s \mapsto (\cos(2\pi s), \sin(2\pi s)) \in \mathbb{R}^2$, $s \in [0, 1]$. $\omega_n(s) := (\cos(2\pi ns), \sin(2\pi ns)) \sim \omega^n$, so $[\omega]^n = [\omega_n]$.

Now our theorem is equivalent to the statement that every loop in S^1 based at $(1, 0)$ is homotopic to the unique ω_n . We use the following two facts:

Fact 1: For every path $f : I \rightarrow X$ starting at $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \rightarrow \tilde{X}$ starting at \tilde{x}_0 .

Fact 2: For each homotopy $f_x : I \rightarrow X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0) \exists$ unique lifted homotopy $\tilde{f}_t : I \rightarrow \tilde{X}$ of paths starting at \tilde{x}_0 .

p that we need: $p : \mathbb{R} \rightarrow S^1$; $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If we define $\tilde{\omega}_n(s) = n \cdot s$. We will apply Facts 1 and 2 to $p : \mathbb{R} \rightarrow S^1$, $\tilde{\omega}_n$: Given $f : [0, 1] \rightarrow S^1$ based at $(0, 1)$ representing some element of $\pi_1(S^1)$. We take \tilde{f} . Since $p\tilde{f}(1) = f(1) = (1, 0)$ (and $p^{-1}(1) \in \mathbb{Z}$), we can argue that if \tilde{f} ends at u (i.e. $\tilde{f}(1) = f$), it is homotopic to $\tilde{\omega}_n$ by the homotopy $\tilde{F} = (1 - t)\tilde{f} + t\tilde{\omega}_n$.

From fact 1 exists \tilde{f} starting at 0 and ending at $p^{-1}(1) \in \mathbb{Z}$.

Theorem: Exists homotopy \tilde{F} from $\tilde{\omega}_k$ to \tilde{f} denoted by $(*)$.

So we define homotopy F from ω_n to f by $F = p \circ \tilde{F}$, homotopy from ω_n to f . Since $[\omega_n] = n \cdot [\omega]$, $\pi_1(S^1) \simeq \mathbb{Z}$.

Now we would like to show that $[f]$ is uniformly determined. Assume that $f \sim \omega_n$ and $f \sim \omega_m$, then using Facts 1 and 2 we have $[\omega_n] = [\omega_m]$ which leads to contradiction since they have different endpoints on \mathbb{R} . \square

Definition 1.7

Given a topological space X , a covering space of X consists of a topological space \tilde{X} and a continuous map $p : \tilde{X} \rightarrow X$ satisfying that $\forall x \in X \exists$ open neighbourhood U of x in X such that $p^{-1}(U)$ is a disjoint union of open subsets U_α each of which is homeomorphically mapped to U .

Definition 1.8

Given a map $[0, 1] \xrightarrow{f} X$ and $p : \tilde{X} \rightarrow X$ we say that $\tilde{f} : [0, 1] \rightarrow \tilde{X}$ is a lift of f if $p \circ \tilde{f} = f$.

The same construction can be defined for homotopy.

Tvrzení 1.5 (*)

Given a map $F : Y \times [0, 1] \rightarrow X$ and a map $\tilde{F} : Y \times \{\mathbf{o}\} \rightarrow \tilde{X}$, where $p : \tilde{X} \rightarrow X$ is a covering space, and \tilde{F} lifts $F|_{Y \times \{\mathbf{o}\}}$; there restricting to \tilde{F} on $Y \times \{\mathbf{o}\}$.

Poznámka (Corollary: Fact 1 and Fact 2 from the previous proof)
Fact 1 is free, it comes when $Y = \{\text{point}\}$, Fact 2 also follows.

Příklad

We say that a topological (path-connected) space is simply connected $\Leftrightarrow \pi_1(X) = \{e\}$.
Examples of simply connected topological spaces: $\mathbb{R}, \mathbb{R}^2, \dots$. S^1 is not simply connected.

Příklad

Given X, Y path-connected and $x_0 \in X, y_0 \in Y$. Show that $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

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Řešení

Product topology is defined to be such that a map $f : Z \rightarrow X \times Y$ is continuous $\Leftrightarrow (p_x : X \times Y \rightarrow X, p_y : X \times Y \rightarrow Y) p_x \circ f$ and $p_y \circ f$ are continuous.

A loop $\gamma : [0, 1] \rightarrow X \times Y$ based at (x_0, y_0) splits at two loops $\gamma_1 : [0, 1] \rightarrow X, \gamma_2 : [0, 1] \rightarrow Y$. The same holds for homotopy, i.e. F from γ to $\tilde{\gamma}$ splits into (F_1, F_2) , where F_1 is a homotopy on X from γ_1 to $\tilde{\gamma}_1$ and F_2 is a homotopy on Y from γ_2 to $\tilde{\gamma}_2$.

Důsledek

$$\pi_1(T^n) := \pi_1(S^1 \times S^1 \times \dots \times S^1) = \mathbb{Z}^n.$$

Příklad

Show that TODO!!! is a covering space for $S^1 \vee S^1$.

TODO!!!

Důkaz (Proposition *)

To prove our proposition we need to construct $\tilde{F} : N \times I \rightarrow \tilde{X}$, where N is an open neighbourhood in Y of a given point $y_0 \in Y$. Since F is continuous, every point $(y_0, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ such that $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighbourhood of $F(y_0, t)$.

By compactness of $\{y_0\} \times I$, finitely many such products $N_t \times (a_t, b_t)$ cover $\{y_0\} \times I$. This implies that we can choose a single neighbourhood N of y_0 , and a partition of $[0, 1]$ $0 = t_0 < t_1 < t_2 < \dots < t_n = 1$ such that $F|_{N \times [t_i, t_{i+1}]}$ is contained in an evenly covered neighbourhood U_i .

Assume inductively that \tilde{F} has been constructed for $N \times [0, t_i]$ starting at a given \tilde{F} on $N \times \{0\}$. We have that $F(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, there is an open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically to U_i via p and $\tilde{F}((y_0, t_i)) \in \tilde{U}_i$. After replacing N by a smaller neighbourhood of y_0 . (We replace $N \times \{t_i\}$ with the intersection with $?$) we may assume that $\tilde{F}(N \times \{t_0\}) \in \tilde{U}_i$. Now we define \tilde{F} on $N \times [t_i, t_{i+1}]$ to be the composition of F

with $p^{-1} : U_i \rightarrow \tilde{U}_i$. After a finite number of steps, we eventually get a lift $\tilde{F} : N \times I \rightarrow \tilde{X}$ for N (some neighbourhood of y_0).

Next we show the uniqueness for $Y = \{\text{point}\}$. In this case we ? Suppose there are two lifts $\tilde{F} : I \rightarrow \tilde{X}$, $\tilde{F}' : I \rightarrow \tilde{X}$. As before we choose a partition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$ so that $\forall i : F([t_i, t_{i+1}])$ is contained in some evenly covered neighbourhood U_i . Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, co is $\tilde{F}([t_i, t_{i+1}])$, which must therefore lie in one of the disjoint open sets \tilde{U}_i projecting homeomorphically to U_i . By the same token, $\tilde{F}'([t_i, t_{i+1}])$ lies in a single \tilde{U}_i , in fact in the same containing $\tilde{F}([t_i, t_{i+1}])$ (by the assumption of induction). Since p is injective on \tilde{U}_i and $p\tilde{F} = p\tilde{F}'$, it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$ and the induction step follows.

The last step of the prove is to observe that since \tilde{F} , \tilde{F}' are constructed on the sets of form $N \times I$ and are unique when we restrict to each segment $\{y\} \times I$, they must agree whenever two such sets $N \times I$ overlap, so we get in fact a well-defined lift \tilde{F} on $Y \times I$. This \tilde{F} is continuous since it is continuous on each segment $\{y\} \times I$. \square

Poznámka

We would like to see π_* as a functor $\pi_1 : \text{Top} \rightarrow \text{Grp}$. In order for π_1 to be a functor, we want for $\varphi : (X, x_0) \xrightarrow{\text{cont.}} (Y, y_0)$ associate $\varphi_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$. How to get φ_* ? Given a loop γ on X based at x_0 . We have a loop $\varphi \circ \gamma$ on Y based at y_0 . $\varphi_*([\gamma]) := [\varphi \circ \gamma]$. Is φ_* a homomorphism? $\varphi_*([\gamma; \gamma_2]) = [\varphi \circ (\gamma_1 \cdot \gamma_2)] = [\varphi \circ \gamma_1 \cdot \varphi \circ \gamma_2] = [\varphi \circ \gamma_1] \cdot [\varphi \circ \gamma_2]$. Hence φ_* is a homomorphism.

$$\forall (X, x_0) \in \text{Fb}(\text{Top}) \exists \varphi = 1 \in \text{Hom}((X, x_0), (X, x_0)).$$

We want $\pi_1(1) = \text{id}_{\pi_1(X, x_0)}$. This because of the definition of $\pi_1(1)$, which maps $[\gamma] \rightarrow [\gamma] \implies \pi_1(1) : [\gamma] \mapsto [\gamma]$, so it is $\text{id}_{\pi_1(X, x_0)}$.

In order for π_1 to be a functor we also need: given $(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0)$ then $i(\psi \circ \varphi)_* = \pi_1(\psi \circ \varphi) = \pi_1(\psi) \circ \pi_1(\varphi) = (\psi)_* \cdot (\varphi)_*$. This is because

$$\pi_1(\psi \circ \varphi)([\gamma]) = (\psi \circ \varphi)_*([\gamma]) = [\psi \circ \varphi \circ \gamma] = \psi_*([\varphi \circ \gamma]) = \pi_1(\psi)(\varphi_*([\gamma])) = \pi_1(\psi) \circ \pi_1(\varphi)([\gamma]).$$

Hence it holds and π_1 is functor $\text{TOP} \rightarrow \text{GRP}$.

Tvrzení 1.6

Given a topological space X . If $X = \bigcup A_\alpha$, where each of A_α is a path connected subspace of X and $A_\alpha \cap A_\beta$ is path connected $\forall \alpha, \beta$, then each loop in X based at x_0 can be decomposed as a product of loops each of which is in some A_α .

Důkaz

Given $f : I \rightarrow X$ with the basepoint x_0 , we claim that there is a partition $0 = s_0 < s_1 < \dots < s_m = 1$ of I such that $[s_{i-1}, s_i]$ is mapped by f to a single A_α (that by call A_i). Since f is continuous, each $s \in I$ has an open neighbourhood V_i in I mapped by f to some A_α . We may in fact take V_s to be an interval whose closure is mapped to a single A_α . By compactness of I , we see that a finite number of such intervals cover I . The endpoints of these intervals form a partition of I : $0 = s_0 < s_1 < \dots < s_m = 1$. Again A_i we call $A_\alpha : f([s_{i-1}, s_i]) \subset A_\alpha$. Let f_i be denoted as $f|_{[s_{i-1}, s_i]}$. Then $f = f_i \cdot \dots \cdot f_m$, where f_i is a path in A_i .

Since $f([s_{i-1}, s_i]) \subset A_i \wedge f([s_i, s_{i+1}]) \subset A_{i+1} \implies f(s_i) \in A_i \cap A_{i+1}$. Since $A_i \cap A_{i+1}$ is path connected, we can choose path $g_i \subset A_i \cap A_{i+1}$, g_i starts at x_0 and ends at $f(s_i)$. Hence $[f][f_1 g_1^{-1}][g_1 f_2 g_2^{-1}] \dots [g_{n-1} f_n g_n^{-1}]$, loops in A_1, A_2, \dots, A_n . \square

TODO?

TODO!!!

Tvrzení 1.7 (*, should be somewhere above)

If $X = \bigcup_{\alpha \in I} A_\alpha$, $x_0 \in \bigcap_{\alpha \in I} A_\alpha$, $A_\alpha \cap A_\beta$ is path connected for all α, β . Then for each loop γ in X based at x_0 , there is a sequence of loops $\gamma_{\alpha_1}, \dots, \gamma_{\alpha_k} : [\gamma] = [\gamma_{\alpha_1}] \cdot \dots \cdot [\gamma_{\alpha_k}]$, where γ_{α_i} is a loop in A_{α_i} .

Věta 1.8 (Van Kampen)

Given a topological space X such that $X = \bigcup A_\alpha$, where A_α 's are path connected subspaces of X , $x_0 \in \bigcap A_\alpha \neq \emptyset$. Besides that we assume $A_\alpha \cap A_\beta$ is connected $\forall \alpha, \beta$. Then $\exists \Phi : *_{\alpha \in I} \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$, which is a surjective homomorphism. If in addition $\forall \alpha, \beta, \gamma : A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\text{Ker } \Phi = \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle$.

$$*_{\alpha \in I} \pi_1(A_\alpha, x_0) / \langle i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1} \rangle \simeq \pi_1(X, x_0).$$

Důkaz

The proof uses Proposition *. Same terminology: Given a loop f in X based at x_0 , by a factorization of $[f] \in \pi_1(X, x_0)$ we mean a formal product $[f_1] \cdot \dots \cdot [f_k]$, where each f_i is a loop in some A_α based at x_0 and $[f_i] \in \pi_1(A_\alpha, x_0)$, and where f is homotopic to $f_1 \cdot \dots \cdot f_k$. (Factorization is a presentation from Proposition *. Factorization can be seen as a word in $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$, but not necessary reduced word.)

We construct Φ by sending $[f]$ to $[f_1] \cdot \dots \cdot [f_k]$ coming from Proposition * with the potential reduction. We will be concerned with the uniqueness of such factorizations. We define an equivalence relation on factorizations. We say that two sequences are equivalent iff they are related by a sequence of moves: combine adjacent terms: $[f_i] \cdot [f_j] = [f_i f_j]$ if $[f_i], [f_j] \in \pi_1(A_\alpha, x_0)$ for some α ; and regard $[f_i] \in \pi_1(A_\alpha)$ as $[f_i] \in \pi_1(A_\beta)$ if f_i is a loop in A_β ; and we do not write constant loops in the decomposition.

This equivalence relation mimics, what is necessary for words in $*\pi_1(A_\alpha, x_0)$. We modify

this equivalence relation by substituting second condition (regard ...) by: For $f_i, f_j: [f_i] \in \pi_1(A_\alpha, x_0)$, $[f_j] \in \pi_1(A_\beta, x_0)$, we view $[f_i], [f_j]$ as elements of $\pi_1(A_\alpha \cap A_\beta, x_0)$. So first and third condition do not change the elements in $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$ and the new second one is such that it does not change elements $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N$.

So equivalent factorizations supposed to give the same element $*_{\alpha \in I} \pi_1(A_\alpha, x_0)$. Hence we can show that any two factorizations of f are equivalent, this will imply $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N \rightarrow \pi_1(X, x_0)$ is injective, and therefore $*_{\alpha \in I} \pi_1(A_\alpha, x_0)/N \rightarrow *_{\alpha \in I} \pi_1(A_\alpha, x_0)$ is trivial, and this map is an isomorphism.

„Equivalence of two factorizations“: We take two factorizations $[f_1] \cdot \dots \cdot [f_k]$ and $[f'_1] \cdot \dots \cdot [f'_l]$ of $[f] \in \pi_1(X, x_0)$. We would like to show that $[f_1] \cdot \dots \cdot [f_k] \sim [f'_1] \cdot \dots \cdot [f'_l]$. Since they represent $[f]$, they are homotopic. Let $F : I \times I \rightarrow X$ is a homotopy from $f_1 \cdot \dots \cdot f_k$ to $f'_1 \cdot \dots \cdot f'_l$. Then we say that \exists decompositions $0 = s_0 < s_1 < \dots < s_m = 1$, $0 = t_0 < t_1 < \dots < t_n = 1$ such that each $R_{ij} := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F to a single A_α , which we label by A_{ij} . These partitions can be obtained from $I \times I$ by finitely many rectangles $[a, b] \times [c, d]$ mapping to a single A_α by compactness of $I \times I$.

With this type of description, we can assume that R_{ij} 's are such that they are to the subdivision giving homotopies between f_i 's and f_j 's.

Since F maps a neighbourhood of R_{ij} into A_{ij} , we may perturb the vertical sides of R_{ij} so that each point of $I \times I$ lies in at most three R_{ij} 's. After this modification we do the relabelling by the following scheme: label row by row starting from the first one, and label from left to right.

If γ is path in $I \times I$ from the left edge (i.e. $\{0\} \times [0, 1]$) to the right edge (i.e. $\{1\} \times [0, 1]$), then $F|_\gamma$ is a loop from x_0 to x_0 . We write γ_r to be the path (through edges of rectangles) separating the first r rectangles R_1, \dots, R_r from the remaining rectangles. γ_0 is bottom edge, γ_m is the top edge.

Let us call the corners of R_i 's vertices. So for each vertex v with $F(v) \neq x_0$, there is a path g_v from x_0 to $F(v)$ that lies in the intersection of the two or three A_{ij} 's corresponding to R_i 's containing v . (This is by the choice of R_i 's and the fact that A_α 's are path connected and contains x_0 and $F(v)$. And by path connectedness of disjoint of A 's.)

Then we obtain a factorization $[F|_{\gamma_r}]$ by inserting the appropriate path connection x_0 with $F|_{\text{vertices corresponds of } R_i\text{'s}}$ then we can say that $[F|_{\gamma_r}]$ has a decomposition that depends on the choice of A_{ij} that corresponds to the edge of R_i on which the component of the decomposition lies.

$[F|_{\gamma_r}]$ has a decomposition by inserting $g_v^{-1} g_v$. Different choices of A_{ij} 's will change the factorization of $[F|_{\gamma_r}]$. (Our equivalence relation \sim is designed to make the choice of A_{ij} 's for such paths irrelevant. I.e. the different factorizations coming from different choices are equivalent.)

Factorizations for two consecutive paths γ_r and γ_{r+1} are equivalent since pushing along R_{r+1} from γ_r to γ_{r+1} changing $F|_{\gamma_r}$ to $F|_{\gamma_{r+1}}$ by a homotopy within A_{ij} corresponding to R_{r+1} , so we can choose this A_{ij} for all segments of γ_r .

We can arrange that the factorization associated to γ_0 is equivalent to $[f_1] \cdot \dots \cdot [f_k]$ by choosing g_r 's for each vertex v along the lower edge of $I \times I$ to lie not just in the two A_{ij} 's corresponding to R_s 's containing v , but also to lie in the A_α for f_i containing v in its domain. In the case when v is the common point of two domains for two consecutive f_i and f_{i+1} we have $F(v) = x_0$, so there is no need to choose g_v 's for such v 's. In this fashion we can assume that the factorization for $[f_1] \cdot \dots \cdot [f_k]$ and $[f'_1] \cdot \dots \cdot [f'_l]$ are equivalent. \square

TODO? (Examples of covering space)

Tvrzení 1.9

Given a covering space $p : \tilde{X} \rightarrow X$, a homotopy $f_t : Y \rightarrow X$ and a map $\tilde{f}_\sigma : Y \rightarrow \tilde{X}$ lifting f_0 , then there is a unique homotopy $\tilde{f}_t : Y \rightarrow \tilde{X}$ of \tilde{f}_0 that lifts f_t .

Poznámka

This statement in a more general form has already appeared on practicals.

Poznámka

One can use this proposition in two ways: We have a lifting property for paths which says that $\forall \text{path } f : [0, 1] = I \rightarrow X$ and each lift \tilde{x}_0 of $f(0) = x_0 \in X$ there is a unique $\tilde{f} : I \rightarrow \tilde{X}$ lifting f starting at \tilde{x}_0 .

If $Y = I$, we get that every homotopy f_t of a path f_0 in X lifts to a homotopy \tilde{f}_t of each lift \tilde{f}_0 of f_0 .

As a corollary of Proposition (*) we can write:

Důsledek

The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ induced by the covering space map $p : \tilde{X} \rightarrow X$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to \tilde{X} starting at \tilde{x}_0 are loops.

┌

Důkaz (Of corollary)

An element of the kernel of p_* is represented by a loop $\tilde{f}_0 : I \rightarrow \tilde{X}$ with a homotopy $f_t : I \rightarrow X$ of $f_0 = p \cdot \tilde{f}_0$ to the trivial loop. By ? the applications of proposition (*) there is a lifted homotopy of loops \tilde{f}_t starting at \tilde{f}_0 and ending with a constant loop. Hence $[\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{x}_0)$ and p_* is injective. \square

└

Tvrzení 1.10

The number of sheets of a covering space $p : \tilde{X} \rightarrow X, \tilde{x} \mapsto x$ with X and \tilde{X} path connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.

Důkaz

For a loop g in X based at x_0 let \tilde{g} be its lift in \tilde{X} based at \tilde{x}_0 . A product $h \cdot g$, $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has a lift $\tilde{h} \cdot \tilde{g}$ ending at the same point as \tilde{g} (since \tilde{h} is a loop). Hence we can define a map Φ from cosets $H[g]$ to $p^{-1}(x_0)$ implies that Φ is surjective since \tilde{x}_0 can be lowered by a path \tilde{g} to an arbitrary point in $p^{-1}(x_0)$ and \tilde{g} projects to a loop g at x_0 .

To show that Φ is injective, observe that $\Phi(H[g_1]) = \Phi(H[g_2]) \implies g_1 \cdot g_2^{-1}$ lifts to a loop in \tilde{X} based at \tilde{x}_0 , so $[g_1][g_2]^{-1} \in H \implies H[g_1] = H[g_2] \implies$ injectivity + surjectivity \implies bijectivity of Φ . \square

TODO? (Problems)

TODO!!!

TODO? (Problems)

Poznámka

Typically homology is easier to compute than homotopy.

Definition 1.9 (Simplex defined by points, standard simplex, direction of simplex)

Take \mathbb{R}^m and points $v_0, \dots, v_m \in \mathbb{R}^m$ such that $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are linearly independent (this condition is necessary to say that v_0, \dots, v_m will not live on hyperplane of dimension $< n$). We define

$$[v_0, v_1, \dots, v_n] := \left\{ \sum_{i=1}^n t_i v_i \mid \sum t_i = 1, t_i \geq 0 \right\},$$

the smallest convex set in \mathbb{R}^m containing v_0, \dots, v_n . We call $[v_0, v_1, \dots, v_n]$ a simplex defined by points v_0, \dots, v_n .

Standard simplex:

$$\Delta_n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1 \wedge \forall i \in [0, n] t_i \geq 0 \right\}$$

The order we use for points in particular determines directions of 1-simplices. Specifying the direction in such a way we got a canonical homeomorphism $\Delta^n \rightarrow [v_0, \dots, v_n]$ given by $(t_0, \dots, t_n) \rightarrow \sum_{i=0}^n t_i v_i$.

Definition 1.10 (Faces, boundary, and interior of simplex)

Given $[v_0, \dots, v_n]$, if we remove v_i , i.e. we write $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$, then such a simplex is a face of $[v_0, \dots, v_n]$. Same way we can remove more vectors and get faces of lower dimensions.

$\partial\Delta_n$ = boundary of Δ_n = union of all faces (including lower dimensional) of Δ_n . (The same for $[v_0, \dots, v_n]$.)

$\mathring{\Delta}_n := \Delta_n \setminus \partial\Delta_n$ is interior of Δ_n . (The same for $[v_0, \dots, v_n]$.)

Definice 1.11 (Δ complex structure (Δ -complex))

A Δ complex structure on a topological space X is a collection of maps $\delta_\alpha : \Delta^n \rightarrow X$ with n independent of α (i.e. n is fixed, α indexes collection) such that

- $\delta_\alpha|_{\mathring{\Delta}^n}$ is injective, and each point of X can be in the image of exactly one such restriction;
- $\delta_\alpha|_{\text{face of } \Delta^n}$ is one of the maps $\delta_\beta : \Delta^{n-1} \rightarrow X$. Here we identify the face of Δ^n with Δ^{n-1} by the canonical linear homeomorphism that preserves the ordering of vertices.
- $A \subset X$ is open $\Leftrightarrow \partial_\alpha^{-1}(A)$ is open in Δ^n for each δ_α .

Poznámka

Δ -complex can be built from the collection of disjoint simplices by identifying various subcomplexes spanned by the subsets of the vertices, where the identification can be done by canonical linear homeomorphisms that preserves the ordering of vertices.

From the discussion about CW complexes it follows that interiors of Δ^n that define $\delta_\alpha(\mathring{\Delta}_n)$ (open simplices on X) are the e_α^n of a CW complex structure on X .

Definice 1.12 (Simplicial homology)

Given a topological space with a fixed Δ -complex structure on it. We define $\Delta_n(X) = C_n(X)$ which is by definition the free Abelian group with basis consist of $\sigma_n(\mathring{\Delta}_n) = e_\alpha^n$ of X . (Free module over $\{e_\alpha^n\}$.)

Poznámka

Elements of $\Delta_n(X) = C_n(X)$ can be written as formal sums $\sum_\alpha n_\alpha e_\alpha^n$, where $n_\alpha \in \mathbb{Z}$. Elements of $\Delta_n(X) = C_n(X)$ are typically called chains.

Equivalently we could write $\sum_\alpha n_\alpha \delta_\alpha$, where δ_α are the defining on characteristic maps of e_α^n . We remember that $\partial([v_0, v_n])$ consists of $n - 1$ -simplices.

TODO? (Last remarks)

TODO? (Problems)

2 Singular homology

Definice 2.1 (Singular n -simplex, chain, chain group)

A singular n -simplex in a topological space X is a continuous map $\delta : \Delta^n \rightarrow X$.

$C_n(X)$ = free Abelian group generated by the set of singular n -simplices in X .

Poznámka

The generators of $C_n(X)$ is in general an uncountable set.

If $X = \{point\}$, then all possible maps $\Delta^n \rightarrow X$ are just constant maps and there is only one constant map. Hence the generating set of $|C_n(X)| = 1$ and thus $C_n(X) \simeq \mathbb{Z}$.

Elements of $C_n(X)$ are called chains, and $C_n(X)$ is called a n -th chain group. We define

$$\partial_n : C_n(X) \rightarrow C_{n-1}(X), \quad \partial_n(\delta) = \sum_{i=0}^n (-1)^i \delta|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}.$$

Poznámka

This is the same formula that we used for simplicial homology.

Since the formulas for ∂_n and ∂_{n-1} coincide with the corresponding formulas for simplicial homology, we see that $\partial_{n-1} \circ \partial_n = 0$. This follows from the proof for simplicial homology. (The standard notation for singular complex will be $C(X)$, for simplicial $C^\Delta(X)$.)

Therefore, we can form a complex $C(X)$:

$$\dots \rightarrow C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \rightarrow \dots \rightarrow 0$$

On the left side, a priori, we can't say that a certain level we start getting only zeroes. ($C_n = 0 \equiv C_n = \{0\}$).

$H^\Delta(X)$ = simplicial homology. Then we define $H_n(X) = \text{Ker } \partial_n / \Im \partial_{n+1}$. (n -th singular homology group.)

Poznámka

On one hand singular homology looks more general than simplicial homology. On the other hand, singular homology can be seen as a particular case of simplicial homology by the following construction:

For an arbitrary space X , we define the singular complex $S(X)$ to be the Δ -complex with one n -simplex Δ_δ^n for each singular n -simplex $\delta : \Delta^n \rightarrow X$ with Δ_δ^n attached to the $n - 1$ simplices of $S(X)$, that are restrictions of δ to the $n - 1$ simplices in the boundary $\partial\Delta^n$ by the restriction maps. $H_n^\Delta(S(X))$ is identified with $H_n(X)$.

Poznámka

There are two variants of singular homology:

1. Reduced homology: Assume that we have a singular chain complex

$$\rightarrow C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \rightarrow 0 \rightarrow \dots$$

We modify this complex, ? will be called the reduced complex (reduced from $C(X)$):

$$\rightarrow C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0 \rightarrow \dots,$$

where ε is called an augmentation map and it is defined by $\varepsilon(\sum_{i \text{ finite}} n \cdot i \cdot \delta_i) = \sum_{i \text{ finite}} n \cdot i \implies \varepsilon$ is well defined. (Exercise: check that ε vanishes on ∂_1 .) Denote the homology of the reduced complex by $\tilde{H}(X)$. TODO!!! (Asi dva řádky rovnic.)

2. Relative complex (complex of a pair): Given a space X and a subspace $A \subset X$, we define $C_n(X, A) := C_n(X)/C_n(A)$ (the chains in A will be trivial in $C_n(X, A)$). This way we get a complex

$$\dots \rightarrow C_n(X, A) \xrightarrow{\partial_n(X, A)} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}(X, A)} C_{n-2}(X, A) \rightarrow \dots \rightarrow 0,$$

here $\partial_n(X, A) : C_n(X, A) \rightarrow C_{n-1}(X, A)$ are quotient maps of ∂_n 's for $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$.

Since we know that $\partial_{n-1} \circ \partial_n = 0$, then for quotient maps the corresponding compositions $\partial_{n-1}(X, A) \circ \partial_n(X, A) = 0$. Hence we really see that this is a complex. We can define the homology of it $H_n(X, A) := \text{Ker } \partial_n(X, A) / \text{Im } \partial_{n+1}(X, A)$.

2.1 Homotopy equivalence of singular homology

Definice 2.2

For a map $f : X \rightarrow Y$, an induced homomorphism $f_n^c : C_n(X) \rightarrow C_n(Y)$ is defined by

$$(\Delta^n \xrightarrow{\partial_X} X) \rightarrow (\Delta^n \xrightarrow{\partial_X} X \xrightarrow{f} Y)$$

$\delta_X \mapsto f \circ \delta_X = \delta_Y$. These maps f_n^c satisfy $f_c^{n-1} \circ \partial_n^X = \partial_n^Y \circ f_n^c$. TODO?(This is since...)

Typically maps $f_c = (f_c^n)_n$ are called chain maps. f_c induces a map $f_* : H_n(X) \simeq H_n(Y)$.

TODO?

TODO? (Problems)

Tvrzení 2.1

Given two topological spaces X and Y that are homotopy equivalent. Then $H_n(X) \simeq H_n(Y) \forall n$.

Důkaz

The essential procedure is a subdivision (simplex decomposition) of $I \times \Delta^n$. Let $\{0\} \times \Delta^n$ be given by $[v_0, \dots, v_n]$ and $\{1\} \times \Delta^n$ be given by $[w_0, \dots, w_n]$, where v_i and w_i have the same image under the standard/canonical projection $I \times \Delta^n \rightarrow \Delta^n$. We would like to pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n -simplices, each obtained from the preceding one by moving one vertex v_i up to w_i , starting with v_n and working backwards to v_0 . The first step for us is to ? $[v_0, \dots, v_{n-1}, v_n]$ to $[v_0, \dots, v_{n-1}, w_n]$. The second step is to move this up to $[v_0, \dots, v_{n-2}, w_{n-1}, w_n]$. Then we keep going and proceed with other (remaining) steps.

The typical step for us is to $[v_0, \dots, v_{i-1}, v_i, w_{i+1}, \dots, w_n]$ moves up to $[v_0, \dots, v_{i-1}, w_i, w_{i+1}, \dots, w_n]$. The region between simplices $[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$ and $[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$ is a $(n+1)$ -simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$.

This way for each of the steps, we get $(n+1)$ -simplex. This leads to the decomposition of $I \times \Delta^n$ into $(n+1)$ -simplices, each intersecting the previous one with the n -simplex face. Since X is homotopy equivalent to Y , there is a homotopy $F : I \times X \rightarrow Y$ and for each simplex $\delta : \Delta^n \rightarrow X$ we get the composition $F \circ (\delta \times 1) : \Delta^n \times I \rightarrow X \times I \rightarrow Y$. Using this map we can define the prism operator $P : C_n(X) \rightarrow C_{n+1}(Y)$. $P(\delta) = \sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$.

We will show that P satisfies the following property: $\partial_Y P = g_c - f_c - P\partial_X$. To prove this, we compute:

$$\partial P(\delta) = \partial \left(\sum_i (-1)^i F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_n]} \right) = \sum_{j \leq i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} + \sum_{j \geq i} (-1)^{i+(j+1)} F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_{j+1}, \dots, w_n]}$$

The terms with $i = j$ will cancel out except of $F \circ (\delta \times 1)|_{[v_0, w_0, \dots, w_n]}$, which is $g \circ \delta = g_c(\delta)$, and $-F \circ (\delta \times 1)|_{[v_0, \dots, v_n, w_n]}$, which corresponds to $-f \circ \delta = -f_c(\delta)$.

Now observe that $i \neq j$, then part of that corresponds to $-P\partial$, since $\partial P(\delta) = (g_c - f_c - P\partial)(\delta)$, where $\delta = [v_0, \dots, v_n]$.

$$\partial P(\delta) = \sum_{j < i} (-1)^{i+j} F \circ (\delta \times 1)|_{[v_0, \dots, v_j, \dots, v_i, w_i, \dots, w_n]} + \sum_{j > i} (-1)^{i+(j+1)} F \circ (\delta \times 1)|_{[v_0, \dots, v_i, w_i, \dots, w_{j+1}, \dots, w_n]}.$$

$$P(\partial(\delta)) = P\left(\sum_j (-1)^j \partial(\delta)|_{[v_0, \dots, v_j, \dots, v_n]}\right) \text{ TODO!!!}$$

We finish the proof by saying the following: If $\alpha \in C_n(X)$, then $g_c(\alpha) - f_c(\alpha) - P\partial(\alpha) = \partial P(\alpha) \implies$

$$\implies g_c(\alpha) - f_c(\alpha) = \partial P(\alpha) + P\partial(\alpha) = \partial P(\alpha),$$

because α is a ?. Hence $g_c(\alpha) - f_c(\alpha)$ is a boundary (it is $\partial(P(\alpha))$). So $g_c(\alpha)$ and $f_c(\alpha)$ determine the same class in homotopy $\implies f_* = g_* \implies$ homotopy equivalence of X and $Y \implies H_n(X) \simeq H_n(Y)$. \square

TODO? (Problems)

TODO!!!