Poznámka
Topology...

Definice 0.1 (Topological vector space (TVS))

A Topological vector space over $\mathbb{F}$ is a pair (X, τ), where X is a vector space over $\mathbb{F}$ and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of $\mathbf{o}$ in (X, τ).

Definice 0.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka
Two homework (in Moodle) and one presentation.

Například
Let $(X, \|\cdot\|)$ be a normed linear space. Let τ be the topology induced by $\|\cdot\|$. The (X, τ) is HLCS.

Důkaz
$\varrho(x, y) = \|x - y\|$ metric induced by $\|\cdot\|$. τ induced by ϱ. This τ is Hausdorff. Continuity of the operations: (from Funkcionalka)

$$x_n \to x, y_n \to y, t_n \to t \implies x_n + y_n \to x + y \wedge t_n x_n \to tx.$$

So, it is a HTVS. Base of neighbourhood of $\mathbf{o}$ is e. g. $U(0, r), r > 0$, which is convex. □

Let Γ be any nonempty set, $X = \mathbb{F}^\Gamma$ (= all functions $\Gamma \to \mathbb{F}$) with point-wise operations, so it is a vector space over $\mathbb{F}$. It is a HLCS.

Důkaz
„Continuity of addition:“ $x, y \in \mathbb{F}^\Gamma$, U a neighbourhood of $x+y \implies \exists F \subset \Gamma$ finite $\exists \varepsilon > 0$ such that

$$U_{\mathbf{o}} = \left\{ z \in \mathbb{F}^\Gamma \middle| \, \forall \gamma \in F : |z(\gamma) - (x(\gamma) + y(\gamma))| < \varepsilon \right\} \subset U$$

$$U_x = \left\{ z \in \mathbb{F}^\Gamma \middle| \, \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$$U_y = \left\{ z \in \mathbb{F}^\Gamma \middle| \, \forall \gamma \in F : |z(\gamma) - y(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$\implies$ V_x is neighbourhood of x, and V_y is neighbourhood of y, and $U_x + U_y \subset U_0 \subset U$. Thus $z_1 \in V_x$, $z_2 \in V_y \implies z_1 + z_2 \in U_0 \subset U$.

„Continuity of multiplication“: $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^\Gamma$, U a neighbourhood of $\lambda x \implies \exists F \subset \Gamma$ finite $\exists \mu > 0$ such that

$$U_0 = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - \lambda x(\gamma)| < \varepsilon \right\} \subset U$$

$$|\mu z(\gamma) - \lambda x(\gamma)| \leqslant |\mu| \cdot |z(\gamma) - x(\gamma)| + |\mu - \lambda| \cdot |x(f)|.$$

$$M := \max_{\gamma \in F} |x(\gamma)|.$$

$$V = \left\{ \mu \in \mathbb{F} \middle| |\mu - \lambda| < \frac{\varepsilon}{2(M+1)} \right\}, \qquad W = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2(|\lambda| + \frac{\varepsilon}{2(M+1)})} \right\}.$$

$$\mu \in V, z \in W \implies \mu z \in U_0 \subset U.$$

„Local convexity“: Base of neighbourhoods of $\mathbf{o}$: $\left\{ x \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |x(\gamma)| < \varepsilon \right\}$, $F \subset \Gamma$ finite, $\varepsilon > 0$, consists of convex sets.

„Hausdorff“: $x \neq y \implies \exists \gamma \in \Gamma : x(\gamma) \neq y(\gamma)$. Take $\varepsilon = \frac{|x(\gamma) - y(\gamma)|}{2}$.

$$U = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - x(\gamma)| < \varepsilon \right\}, V = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - y(\gamma)| < \varepsilon \right\} \implies U \cap V = \emptyset.$$

□

$$X = C(\mathbb{R}, \mathbb{F}) = \{ f : \mathbb{R} \to \mathbb{F} \text{ continuous} \},$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \max_{t \in [-n,n]} \{ |f(t) - g(t)| \} \right\} =: \sum_{N=1}^{\infty} \frac{1}{2^N} \min \{ 1, p_N(f - g) \}$$

is translation invariant (that implies addition is continuous, see lecture) metric.

Důkaz
$f_n \to f$ in $\varrho \Leftrightarrow \forall N : f_n \rightrightarrows f$ on $[-N, N]$.

„$f_n \to f$, $\lambda_n \to \lambda \implies \lambda_n f_n \to \lambda f$": Let $N \in \mathbb{N}$. We will show $\lambda_n f_n \rightrightarrows \lambda f$ in $[-N, N]$. $x \in [-N, N]$:

$$|\lambda_n f_n(x) - \lambda f(x)| \leqslant |\lambda_n| \cdot |f_n(x) - f(x)| + |\lambda_n - \lambda| \cdot |f(x)| \leqslant c \cdot p_N(f_n - f) + |\lambda_n - \lambda| \cdot p_N(f) \to 0.$$

Hence, X is HTVS. „Local convexity": $U_{N,\varepsilon} = \{f \in X | p_N(t) < \varepsilon\}$, clearly $U_{N,\varepsilon}$ is a convex set and $U_{N,\varepsilon}$ is neighbourhood of $\mathbf{o}$. If $\varepsilon < \lambda$, then $\left\{f | \varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}\right\} \subset U_{N,\varepsilon}$, because for $\varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}$ it is $\frac{1}{2^N} p_N(f) < \frac{\varepsilon}{2^N}$. „they form a base": $f \in U_{N,\varepsilon} \implies \varrho(f, \mathbf{o}) < \varepsilon + \frac{1}{2^N}$. Hence fix $r > 0$ and take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$. Then $U_{N,\frac{r}{2}} \subset \{f | \varrho(f, \mathbf{o}) < r\}$ □

(Ω, Σ, μ) a measure space, $p \in (0, 1)$. $L^p(\Omega, \Sigma, \mu) = \left\{f : \Omega \to \mathbb{F} \text{ measurable} | \int |f|^p d\mu < \infty\right\}$ (we identify functions equal almost everywhere). $\varrho(f, g) = \int |f - g|^p d\mu$ is a metric making $X = L^p(\Omega, \Sigma, \mu)$ a HTVS (but not locally convex).

Důkaz
„ϱ is a metric": „$\triangle$-inequality": $a, b \in [0, \infty) : (a + b)^p \leqslant a^p + b^p$. (Fix $a \geqslant 0$, take $\varphi_a(b) = (a + b)^p - a^p - b^p \implies \varphi_a$ is continuous on $[0, \infty)$, $\varphi_a(0) = 0$. For $b > 0$: $\varphi_a(b) = p(a + b)^{p-1} - pb^{p-1} = p \cdot ((a + b)^{p-1} - b^{p-1}) < 0$ as $p - 1 < 0 \implies \varphi_a$ decreasing on $[0, \infty)$ and $\varphi_a \leqslant 0$.)

φ is translation invariant $\implies$ addition is continuous. „Multiplication": We can see that $\varrho(\lambda f, \mathbf{o}) = |\lambda|^p \varrho(f, \mathbf{o})$. $f_n \to f$, $\lambda_n \to \lambda$:

$$\varrho(\lambda_n f_n, \lambda f) \leqslant \varrho(\lambda_n f_n, \lambda_n f) + \varrho(\lambda_n f, \lambda f) = |\lambda_n|^p \varrho(f_n, f) + |\lambda_n - \lambda|^p \varrho(f, \mathbf{o}) \to 0.$$

Hence, we have a HTVS. □

Tvrzení 0.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x + U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 0.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1 - t)y \in A$ for $x, y \in A$, $t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}$, $|\alpha| \leqslant 1$;
- absolutely convex if it is convex and balanced;

- absorbing if $\forall x \in X \ \exists t > 0 : \{sX | s \in [0, t]\} \subset A$.

Definice 0.4

$\mathrm{co}(A)$ = convex hull, $\mathrm{b}(A)$ = balanced hull, $\mathrm{aco}(A)$ = absolutely convex hull.

Tvrzení 0.2

X is a metric space over $\mathbb{F}$, $A \subset X$. Then:

(a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow$ A is convex and symmetric.

(b) $\mathrm{co}\, A = \{t_1 x_1 + \ldots + t_k x_k | x_1 \ldots x_k \in A, t_1 \ldots t_k \geqslant 0, t_1 + \ldots + t_k = 1, k \in \mathbb{N}\}$.

(c) $\mathrm{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leqslant 1\}$.

(d) $\mathrm{aco}(A) = \mathrm{co}(\mathrm{b}(A))$.

(e) A is convex $\Leftrightarrow$ $(s + t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)
„$\implies$“: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „$\impliedby$“: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \implies \alpha \in [-1, 1].$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{(1+\alpha)}{2} x \in A$). □

Důkaz (b)
„$\subseteq$“: by induction on k:

$$t_1 x_1 + \ldots + t_{k+1} x_{k+1} = (t_1 + \ldots + t_k) \frac{t_1 x_1 + \ldots + t_k x_k}{t_1 + \ldots + t_k} + t_{k+1} x_{k+1}.$$

„$\supseteq$“: the set on the RHS is convex and contain A. □

Důkaz (c)
„$\supseteq$“: clear. „$\subseteq$“: RHS is a balanced set. □

Důkaz (d)
„$\supseteq$“: clear. „$\subseteq$“ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c): $\mathrm{co}(\mathrm{b}(A)) = \{t_1 \alpha_1 x_1 + \ldots + t_k \alpha_k x_k | x_1, \ldots, x_k \in A, |\alpha_j| \leqslant 1, t_j \geqslant 0, t_1 + \ldots + t$ is clearly balanced.) □

Důkaz (e)
„ $\Longrightarrow$ “: „$\subseteq$“: always, „$\supseteq$“: $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$.

„ $\Longleftarrow$ “: in particular $\forall t \in (0,1)$: $tA + (1-t)A \subset A$, it is the definition of convexity. □

Tvrzení 0.3

Let (X,τ) be a LCS, $U \in \tau(\mathbf{o})$. Then

(i) U is absorbing.

(ii) $\exists V \in T(0) : V + V \subset U$.

(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.

Důkaz (i)

$$x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V \text{ a neighbourhood of } 0 \text{ in } \mathbb{F} : V \cdot x \subset U \implies \exists t > 0 : [0,t] \subset V$$

□

Důkaz (ii)

$$\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W \subset U.$$

Take $V = W_1 \cap W_2$. □

Důkaz

$$\exists U_0 \in \tau(\mathbf{o}) \text{ convex}, U_0 \subset U : \mathbf{o} \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0\ \exists W \in \tau(\mathbf{o}) \text{ open} :$$
$$\forall \lambda, |\lambda| < c : \lambda W \subset U_0.$$

$V_1 := \bigcup_{0<|\lambda|<1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \operatorname{co} V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „V is open“:

$$V = \bigcup \{t_1 x_1 + \ldots + t_n x_n + t_{n+1} V_1 | t_1, \ldots, t_{n+1} \geqslant 0, t_1 + \ldots + t_{n+1} = 1, x_1, \ldots, x_n \in V_1\}$$

□

Věta 0.4

1. Let (X,τ) be a LCS. Then there is $\mathcal{U}$, a base of neighbourhoods of $\mathbf{o}$ with properties:

- *the elements of $\mathcal{U}$ are absorbing, open, absolutely convex;*
- $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U$.

If X is Hausdorff, then $\bigcap \mathcal{U} = \{\mathbf{o}\}$.

2. Let X be a vector space, $\mathcal{U}$ a nonempty family of subsets of X satisfying:

- *the elements of $\mathcal{U}$ are absorbing and absolutely convex;*
- $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U$;
- $\forall U, V \in \mathcal{U}\ \exists W \in \mathcal{U} : W \subset U \cap V$.

Then there is a unique topology τ on X such that (X, τ) is LCS and $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$. *Further, if* $\bigcap \mathcal{U} = \{\mathbf{o}\}$, *the τ is Hausdorff.*

Důkaz (1.)
Let $\mathcal{U}$ be the family of all open absolutely convex neighbourhoods of $\mathbf{o}$. The previous proposition (iii) gives us $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$, (1) gives us elements of $\mathcal{U}$ are absorbing, so the first item holds. (ii) gives us $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$.

Assume X i Hausdorff: $x \in X \setminus \{\mathbf{o}\} \overset{\text{Hausdorff}}{\implies} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$. □

Důkaz (2.)
Set $\tau = \{G \subset X | \forall x \in G \ \exists U \in \mathcal{U} : x + U \subset G\}$. This is a unique possibility so uniqueness is clear.

„τ is topology“: $\emptyset, X \in \tau$ and τ is closed to arbitrary union (clear). τ is closed to finite intersections by third item ($G_1, g_2 \in \tau$, $x \in G_1 \cap G_2 \dots U_1, U_2 \in \tau$, $x + U_1 \subset G_1$, $x + U_2 \subset G_2$; $\exists V \in \mathcal{U} : V \subset U_1 \cap U_2$, then $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$).

„Elements of $\mathcal{U}$ are neighbourhoods of $\mathbf{o}$“: $U \in \mathcal{U}$. $V := \{x \in U | \exists W \in \mathcal{U} : x + W \subset U\}$. Then $V \subset U$, $0 \in V$ (take $W = U$). $V \in \tau$ ($x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$; let $\tilde{W} \in \mathcal{U}$ such that $2\tilde{W} \subset W$, then $x + \tilde{W} \subset V$, because $y \in \tilde{W} \implies X + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$).

„$\mathcal{U}$ is a base of neighbourhood of $\mathbf{o}$“: now clear.

„(X, τ) is a TVS“: $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$. Then $(x+V)+(y+V) \subset x+y+2V \subset x+y+U \subset G$. $\lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G$; $\exists V \in \mathcal{U} : 2V \subset U$; V is absorbing $\implies \exists c > 0 \ \forall t \in [0, c] : tx \in V$; V balanced $\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$; assume $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1}V$,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda) y}_{(\mu-1)\cdot\left(\mu + \frac{1}{|\lambda|+1}\right)V} + \underbrace{\lambda(y - x)}_{\in \frac{\lambda}{|\lambda|+1}V \subset V}.$$

„Local convexity“: by first item: $\forall U \in \mathcal{U} : U$ is convex.

Assume $\bigcap \mathcal{U} = \{\mathbf{o}\}$. Take $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$. Take $V \in \mathcal{U} : 2V \subset U$. Then if $(x + V) \cap (y + V) = \emptyset$, $x + v_1 = y + v_2$, $x - y = v_2 - v_1 \in V + V = 2V \subset U$ ↯. □

Věta 0.5

Let X be a vector space and let $\mathcal{P}$ be a family of seminorms on X. The there is a unique topology τ on X such that (X, τ) is a LCS and $\mathcal{U} = \{\{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\} | p_1, \dots, p_k \in \mathcal{P}, c_1, \dots,$ is a base of neighbourhood of $\mathbf{o}$.

(X, τ) is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\} \ \exists p \in \mathcal{P}, p(x) > 0$.

Důkaz
Use the previous theorem (2.) on $\mathcal{U}$: The sets are absolutely convex (by properties of seminorms). „Absorbing“: $U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $x \in X$?, $j \in [k]$. Then $p_j(x) \in (0, \infty)$ as for $t > 0 : p_j(t \cdot x) = t \cdot p_j)_x$ and $\exists c > 0$ such that $c \cdot p_j(x) < c_j$ for $j \in [k]$. Now for $t \in [0, c] : tx \in U$.

$U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $V = \left\{x \in X | p_1(x) \subset \frac{c_1}{2}, \dots, p_k(x) < \frac{c_k}{2}\right\}$.

$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$ trivially.

„Hausdorffness“:

$$\bigcap U = \{x \in X | \forall p \in \mathcal{P} : p(x) = 0\}.$$

„$\supseteq$“ clear. „$\subseteq$“: Assume $y \in X$, $p \in \mathcal{P} : p(y) > 0$: $U = \{x \in X | p(x) < p(y)\} in \mathcal{U} \implies y \notin U$. □

Například
$(X, \|\cdot\|)$ is a normed space, then its topology is generated by $\mathcal{P} = \{\|\cdot\|\}$.

The topology on $\mathbb{F}^\Gamma$ is generated by seminorms $p_\gamma(f) = |f(\gamma)|$, $f \in \mathbb{F}^\Gamma$ $(\gamma \in \Gamma)$.

$C(\mathbb{R}, \mathbb{F})$ the topology is generated by this sequence of seminorms: $p_N(f) = \max_{x \in [-N,N]} |f(x)|$.

Definice 0.5 (Minkowski functional)

X vector space, $A \subset X$ convex absorbing. Then

$$p_A(x) := \inf\{\lambda > 0 | x \in \lambda \cdot A\}.$$

Lemma 0.6

Let X be LCS, $A \subset X$ convex set.

$$x \in \overline{A}, y \in \operatorname{int} A \implies \{tx + (1-t)y | t \in [0,1)\} \subset \operatorname{int} A.$$

Důkaz
WLOG $y = 0$. $t = 0$ clear, $0 \in \operatorname{int} A$. $t \in (0,1)$:

Fix U, an open absolutely convex neighbourhood of $\mathbf{o}$ such that $U \subset A$. Then $x + \frac{1-t}{t} U$ is a neighbourhood of $x \implies \exists$

TODO!!! □

TODO!!!

Důkaz (Continuity of multiplication? Theorem 4. TODO?)
„U is a neighbourhood of $\mathbf{o}$ in τ, $\lambda > 0 \implies \lambda U$ is neighbourhood of $\mathbf{o}$“: $\lambda \geqslant 1$: $\exists V \in \mathcal{U} : V \subset U \implies V \subset \lambda V \subset \lambda U$ (V is absolutely convex) $\implies \lambda U$ is neighbourhood of $\mathbf{o}$. $\lambda = \frac{1}{2}$: $\exists V \in \mathcal{U} : V \subset U$, then $\exists W \in \mathcal{U} : 2W \subset V$, then $W \subset \frac{1}{2}V \subset \frac{1}{2}U \implies \frac{1}{2}U$ is a neighbourhood of $\mathbf{o}$. Now by induction for $\lambda = \frac{1}{2^n}$. For $\lambda > 0$ find $n \in \mathbb{N}$ such that $\lambda > \frac{1}{2^n}$.

$\lambda x \in G$ ($\lambda \in \mathbb{F}, x \in X, G \in \tau$) $\implies \exists U \in \mathcal{U} : \lambda x + U \in G$. Find $V \in \mathcal{U} : 2V \subset U$ such that V is absorbing ($\implies \exists c > 0\ \forall t \in [0, c] : tx \in V$) and V is balanced ($\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$). Let $\mu \in F, y \in X$ such that

$$|\mu - \lambda| < c \wedge y \in x + \frac{1}{|\lambda| + c} V \text{ (a neighbourhood of } \mathbf{o}\text{)}$$

$$\implies \mu y - \lambda x = \mu(y - x) + (\mu - \lambda)x \in V + V = 2V \subset U \implies \mu y \in \lambda x + U \subset G.$$

□

Tvrzení 0.7 (8. see notes of lecturer)

Let X be LCS, $A \subset X$ a convex neighbourhood of $\mathbf{o}$.

Clearly: $[p_A \subset 1] \subset A \subset [p_A \leqslant 1]$.

Důkaz
„$[p_a < 1] = \operatorname{int} A$“: „$\subseteq$“: $p_A(x) < 1 \implies \exists c > 1$ such that $cx \in A \implies x = \frac{1}{c}cx \in \operatorname{int} A$. „$\supseteq$“: $x \in \operatorname{int} A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$. U absorbing $\implies \exists \alpha > 0 : \alpha x \in U$. Then $(1+\alpha)x \in A \implies p(x) \leqslant \frac{1}{1+\alpha} < 1$.

„$[p_A \leqslant 1] = \overline{A}$“: „$\subseteq$“: $p_A(x) \leqslant 1 \implies \forall n \in \mathbb{N} : p_x\left(\left(1 - \frac{1}{n}\right)x\right) = \left(1 - \frac{1}{n}\right)p_A(x) \leqslant 1$. $\left(1 - \frac{1}{n}\right)x \in \operatorname{int} A \implies x \in \overline{\operatorname{int} A} \subset \overline{A}$. „$\supseteq$“: $x \in \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in \operatorname{int} A$, so, $p_A\left(\left(1 - \frac{1}{n}\right)x\right) < 1 \overset{n \to \infty}{\implies} p_A(x) \leqslant 1$. □

p_A is continuous on X.

Důkaz
$[p_A < c] = \emptyset$ if $c \leqslant 0$ and $c \cdot \operatorname{int} A$ if $c > 0$. $[p_A > c] = X$ if $c < 0$, $X \backslash (c \cdot \overline{A})$ if $c > 0$, and $\bigcup_{t>0} X \backslash t\overline{A}$ if $c = 0$. All these sets are open. □

$p_A = p_{\overline{A}} = p_{\operatorname{int} A}$.

Důkaz
$\operatorname{int} A \subset A \subset \overline{A} \implies p_{\overline{A}} \leqslant p_A \leqslant p_{\operatorname{int} A}$. „Conversely“: Assume that $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in d \operatorname{int} A \implies \left(1 - \frac{1}{n}\right)p_{\operatorname{int} A}(x) \leqslant d \implies p_{\operatorname{int} A}(x) \leqslant d < c$. □

Důsledek
Any LCS (X) is completely regular.

Důkaz
$x \in X$, U an open neighbourhood of x. Take V a convex neighbourhood of $\mathbf{o}$ such that $x + V \in U$. $f(y) := \min\{1, p_V(y - x)\}$. The f is continuous by the previous proposition, $f(x) = 0$.

$$y \in X \backslash U \implies y - x \notin V \implies p_V(y - x) \geqslant 1 \implies f(y) = 1.$$

□

Věta 0.8

TODO!!! The topology generated by $\mathcal{P}_\tau$ coincides with τ.

Důkaz
Let τ_1 be topology induced by $\mathcal{P}_\tau$. $\tau_1 \subset \tau$ (seminorms from $\mathcal{P}_\tau$ are τ-continuous, hence the sets from theorem 5? are τ-open). „$\tau \subset \tau_1$“: Let $U \in \tau(\mathbf{o}) \implies \exists V$ a neighbourhood of $\mathbf{o}$ such that $V \subset U$. The $p_V \in \mathcal{P}_\tau$ (from the previous proposition is continuous) $\implies [p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$. □

Tvrzení 0.9

X a vector space.

1. *p is seminorm $\implies [p < 1]$ is absolutely convex, absorbing, and $p_{[p<1]} = p$.*
2. *p, q are seminorms, then $p \leqslant q \Leftrightarrow [p < 1] \supset [q < 1]$.*
3. *$\mathcal{P}$ a set of seminorms generated by a topology τ. p a seminorm on X. Then p is τ-continuous $\Leftrightarrow \exists p_1, \ldots, p_k \in \mathcal{P}\ \exists c > 0 : p \leqslant c \cdot \max\{p_1, \ldots, p_k\}$.*

Důkaz (1.)
Absolutely convex and absorbing is clear.

$$p_{[p<1]}(x) = \inf\{\lambda > 0 | x \in \lambda[p < 1]\} = \inf\{\lambda > 0 | x \in [p < \lambda]\} = p(x).$$

□

Důkaz (2.)
„ $\implies$ “ trivial. „ $\impliedby$ “: $[p < 1] \supset [q < 1] \implies p = p_{[p<1]} \leqslant p_{[q<1]} = q$. □

Důkaz (3.)
„ $\Longleftarrow$ “: $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = \left[p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}\right]$, which is a τ-open set $\implies A$ is a neighbourhood of $\mathbf{o} \implies p = p_A$ is continuous (by 1. and the previous proposition).

„ $\implies$ “: p is continuous $\implies [p < 1]$ is neighbourhood of $\mathbf{o}$ ($p(\mathbf{o}) = 0$) $\implies \exists p_1, \dots, p_k \in \mathcal{P}\ \exists c_1, \dots, c_k > 0$ such that $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c} \max\{p_1, \dots, p_k\} < 1]$ ($c = \min\{c_1, \dots, c_k\}$). Use 2. for seminorms $p, \frac{1}{2\max\{p_1, \dots, p_k\}}$ and get $p \leqslant \frac{1}{c}\max\{p_1, \dots, p_k\}$. □

1 Continuous and bounded linear mapping

Tvrzení 1.1

$(X, \tau), (Y, \mathcal{U})$ LCS, $L : X \to Y$ linear. Then the following assertions are equivalent:

1. *L is continuous;*
2. *L is continuous at $\mathbf{o}$;*
3. *L is uniformly continuous.*

Důkaz
„1. $\implies$ 2.“ trivial, „2. $\implies$ 3.“ assume L continuous at $\mathbf{o}$. Then, given $U \in \mathcal{U}(\mathbf{o})$, there is $V \in \tau(\mathbf{o})$ such that $L(V) \subset U$. Take $x, y \in X$ such that $x - y \in V$. Then $L(x) - L(y) = L(x - y) \in U$ and that's continuous. „3. $\implies$ 1.“ trivial. □

Tvrzení 1.2

$L : X \to Y$ linear. L is continuous $\Leftrightarrow \forall q$ a continuous seminorm on Y $\exists p$ a continuous seminorm on X: $\forall x \in X : q(L(x)) \leqslant p(x)$.

Důkaz
„ $\implies$ “: L continuous, q a continuous seminorm on Y, the $p(x) = q(L(x))$ is a continuous seminorm on X. „ $\Longleftarrow$ “: By the previous proposition it is enough „L is continuous at $\mathbf{o}$“: U neighbourhood of $\mathbf{o}$ in Y, $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. $q := p_V$ is a continuous seminorm. Let p be a continuous seminorm on X such that $q \circ L \leqslant p$. $W := [p < 1]$ a neighbourhood of $\mathbf{o}$ in X and $L(W) \subset V \subset U$. $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$. □