Příklad (1.)
Let φ, ψ, $\mathbf{u}$, $\mathbf{v}$ and $\mathbb{A}$ be smooth scalar, vector and tensor fields in $\mathbb{R}^3$. Show that:

$$\operatorname{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\nabla\varphi) + \varphi\operatorname{div}\mathbf{v}$$

Důkaz
Ze vzorce pro (parciální) derivaci součinu (φv_i je skalární funkce):

$$\operatorname{div}(\varphi\mathbf{v}) = \nabla(\varphi\mathbf{v}) = \sum_i \frac{\partial\varphi v_i}{\partial x_i} = \sum_i\left(\frac{\partial\varphi}{\partial x_i}v_i + \varphi\frac{\partial v_i}{\partial x_i}\right) = \sum_i\frac{\partial\varphi}{\partial x_i}v_i + \sum_i\varphi\frac{\partial v_i}{\partial x_i} =$$

$$= \mathbf{v}\cdot(\nabla\varphi) + \varphi(\nabla\cdot\mathbf{v}) = \mathbf{v}\cdot(\nabla\varphi) + \varphi\operatorname{div}\mathbf{v}$$

□

$$\operatorname{div}(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v}$$

Důkaz
Ze vzorce pro derivaci součinu a $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$:

$$\operatorname{div}(u\times v) = \nabla\cdot(u\times\mathbf{v}) = \sum_k\left(\sum_{i,j}\frac{\partial\varepsilon_{ijk}u_i\cdot v_j}{\partial x_k}\right) = \sum_{i,j,k}\left(\varepsilon_{ijk}\frac{\partial u_i}{\partial x_k}v_j + \varepsilon_{ijk}u_i\frac{\partial v_j}{\partial x_k}\right) =$$

$$= \sum_j\left(\sum_{k,i}\varepsilon_{kij}\frac{\partial u_i}{\partial x_k}v_j\right) + \sum_i\left(\sum_{j,k}\varepsilon_{jki}u_i\frac{\partial v_j}{\partial x_k}\right) = (\nabla\times\mathbf{u})\cdot\mathbf{v} + \mathbf{u}\cdot(\mathbf{v}\times\nabla) =$$

$$= \mathbf{v}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{v}) = \mathbf{v}\cdot\operatorname{rot}\mathbf{u} - \mathbf{u}\cdot\operatorname{rot}\mathbf{v}$$

□

$$\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) = [\nabla\mathbf{u}]\mathbf{v} + \mathbf{u}\operatorname{div}\mathbf{v}$$

Důkaz

$$\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) = \sum_i\frac{\partial u_j v_i}{\partial x_i} = \sum_i\frac{\partial u_j}{\partial x_i}v_i + \sum_i u_j\frac{\partial v_i}{\partial x_i} = [\nabla\mathbf{u}]\mathbf{v} + \mathbf{u}(\nabla\cdot\mathbf{v}) = [\nabla\mathbf{u}]\mathbf{v} + \mathbf{u}\operatorname{div}\mathbf{v}$$

□

$$\operatorname{div}(\varphi\mathbb{A}) = \mathbb{A}(\nabla\varphi) + \varphi \operatorname{div}\mathbb{A}$$

Důkaz

$$\operatorname{div}(\varphi\mathbb{A}) = \sum_k \frac{\partial \varphi A_{ik}}{\partial x_k} = \sum_k \left(\frac{\partial \varphi}{\partial x_k} A_{ik} + \varphi \frac{\partial A_{ik}}{\partial x_k} \right) = \sum_k \frac{\partial \varphi}{\partial x_k} A_{ik} + \sum_k \frac{\partial A_{ik}}{\partial x_k} \varphi =$$
$$= \mathbb{A}(\nabla\varphi) + \varphi \operatorname{div}\mathbb{A}$$

$\square$

Further, show that:

$$\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$$

Důkaz

$$\nabla(\varphi\psi) = \frac{\partial \varphi\psi}{\partial x_i} = \frac{\partial \varphi}{\partial x_i}\psi + \frac{\partial \psi}{\partial x_i}\varphi = \psi\nabla\varphi + \varphi\nabla\psi.$$

$\square$

$$\nabla(\varphi\mathbf{v}) = \mathbf{v} \otimes \nabla\varphi + \varphi\nabla\mathbf{v}$$

Důkaz

$$\nabla(\varphi\mathbf{v})) = (\frac{\partial \varphi v_i}{\partial x_j} e_i) e_j = (\frac{\partial \varphi}{\partial x_j} v_i \cdot e_i) e_j + (\varphi \frac{\partial v_i}{\partial x_j} e_i) e_j = \mathbf{v} \otimes \nabla\varphi + \varphi\nabla\mathbf{v}$$

$\square$

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = (\nabla\mathbf{u})^T\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{u}$$

Důkaz

$$\nabla(\mathbf{u}\cdot\mathbf{v}) = \frac{\partial \sum_j u_j \cdot v_j}{\partial x_i} = \sum_j \frac{\partial u_j \cdot v_j}{\partial x_i} = \sum_j \left(\frac{\partial u_j}{\partial x_i} v_j + u_j \cdot \frac{\partial v_j}{\partial x_i} \right) =$$
$$= \sum_j (\nabla u_j) v_j + \sum_j (\nabla v_j) u_j = (\nabla\mathbf{u})^T\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{u}.$$

$\square$

$$\operatorname{rot}(\varphi\mathbf{v}) = \varphi \operatorname{rot} \mathbf{v} - \mathbf{v} \times \nabla\varphi$$

Důkaz

$$\operatorname{rot}(\varphi\mathbf{v}) = \nabla \times (\varphi\mathbf{v}) = \sum_{ij} \varepsilon_{ijk} \frac{\partial \varphi v_j}{\partial x_i} = \sum_{ij} \varepsilon_{ijk} \left(\varphi \frac{\partial v_j}{\partial x_i} + \frac{\partial \varphi}{\partial x_i} v_j \right) =$$

$$= \sum_{ij} \varepsilon_{ijk} \varphi \frac{\partial v_j}{\partial x_i} - \sum_{ij} \varepsilon_{jik} v_j \frac{\partial \varphi}{\partial x_i} = \varphi(\nabla \times \mathbf{v}) - \mathbf{v} \times \nabla\varphi = \varphi \operatorname{rot} \mathbf{v} - \mathbf{v} \times \nabla\varphi$$

□

Příklad (2.)
Let v be a smooth vector field. Show that

$$\operatorname{rot}(\operatorname{rot} \mathbf{v}) = \nabla(\operatorname{div} \mathbf{v}) - \Delta\mathbf{v}.$$

Důkaz

$$\operatorname{rot}(\operatorname{rot} \mathbf{v}) = \nabla \times (\nabla \times \mathbf{v}) = \nabla \times \left(\sum_{i,j} \varepsilon_{ijk} \frac{\partial v_j}{\partial x_i} \right) = \sum_{l,k,i,j} \varepsilon_{lkm} \frac{\partial}{\partial x_l} \varepsilon_{ijk} \frac{\partial v_j}{\partial x_i} = \sum_{i,j,k,l} \varepsilon_{kml} \varepsilon_{kij} \frac{\partial^2 v_j}{\partial x_l \partial x_i} =$$

$$= \sum_{ijkl} \delta_{mi} \delta_{kj} \frac{\partial^2 v_j}{\partial x_l \partial x_i} - \delta_{mj} \delta_{li} \frac{\partial^2 v_j}{\partial x_l \partial x_i} = \frac{\partial}{\partial x_m} \left(\sum_l \frac{\partial v_l}{\partial x_l} \right) - \sum_l \frac{\partial^2 v_m}{\partial x_l^2} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$$

□

Příklad (3.)
Let $\mathbb{A}$ be a sufficiently smooth tensor/matrix field, and let $\mathbb{A}$ be (at every point $\mathbf{x}$) a *symmetric matrix*. Show that

$$\operatorname{rot}\left((\operatorname{rot}\mathbb{A})^T\right) = [\Delta \operatorname{tr}\mathbb{A} - \operatorname{div}(\operatorname{div}\mathbb{A})]\mathbb{I} + \nabla(\operatorname{div}\mathbb{A}) + [\nabla(\operatorname{div}\mathbb{A})]^T - \nabla(\nabla \operatorname{tr}\mathbb{A}) - \Delta\mathbb{A}.$$

Důkaz
Z definice:

$$\operatorname{rot}\left((\operatorname{rot}\mathbb{A})^T\right) = \operatorname{rot}\left(\varepsilon_{jkl}\frac{\partial A_{il}}{\partial x_k}\cdot \mathbf{e}_j \otimes \mathbf{e}_i\right) = \varepsilon_{nop}\varepsilon_{mkl}\frac{\partial^2 A_{pl}}{\partial x_k \partial x_o}\mathbf{e}_m \otimes \mathbf{e}_n$$

Z rovnosti

$$\varepsilon_{nop}\varepsilon_{mkl} = \det\begin{pmatrix} \delta_{nm} & \delta_{nk} & \delta_{nl} \\ \delta_{om} & \delta_{ok} & \delta_{ol} \\ \delta_{pm} & \delta_{pk} & \delta_{pl} \end{pmatrix}$$

dostáváme na pravé straně 6 členů:

- $\frac{\partial^2 A_{pp}}{\partial x_k \partial x_o}\mathbf{e}_m \otimes \mathbf{e}_n = (\Delta \operatorname{tr}\mathbb{A})\mathbb{I}$, neboť
$$A_{pp} = \operatorname{tr}\mathbb{A}, \qquad \frac{\partial^2}{\partial x_o^2} = \nabla\cdot\nabla = \Delta, \qquad \mathbf{e}_n \otimes \mathbf{e}_n = \mathbb{I};$$
- $-\frac{\partial^2 A_{po}}{\partial x_p \partial x_o}\mathbf{e}_n \otimes \mathbf{e}_n = -\operatorname{div}(\operatorname{div}\mathbb{A})$;
- $\frac{\partial^2 A_{po}}{\partial x_n \partial x_o}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A}^T)_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A})_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\operatorname{div}\mathbb{A})_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \nabla(\operatorname{div}\mathbb{A})$;
- $\frac{\partial^2 A_{pn}}{\partial x_p \partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A})_n}{\partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = \frac{\partial(\operatorname{div}\mathbb{A})_n}{\partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = [\nabla(\operatorname{div}\mathbb{A})]^T$;
- $-\frac{\partial^2 A_{pp}}{\partial x_n \partial x_o}e_n \otimes e_o = -\frac{\partial^2 \operatorname{tr}\mathbb{A}}{\partial x_n \partial x_o}e_n \otimes e_o = -\nabla(\nabla \operatorname{tr}\mathbb{A})$;
- $-\frac{\partial^2}{\partial x_o^2}A_{pm}\mathbf{e}_p \otimes \mathbf{e}_n = -\frac{\partial^2}{\partial x_o^2}\mathbb{A} = -\Delta\mathbb{A}$.

Tím jsme dokázali rovnost. □