Poznámka (Exam)
Oral, similar as in FA1.

Poznámka (Credit)
Similar as in FA1.

1 Banach algebras

1.1 Basic properties

Definice 1.1 (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over $\mathbb{K}$, if

- $(A, +, -, 0, \cdot_S)$ is vector space over $\mathbb{K}$;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K}\ \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

Důsledek
1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ ($\forall a \in A$).

If e_1 is left unit and e_2 is right unit, then $e_1 = e_2$ is unit. ($e_1 = e_1 \cdot e_2 = e_2$)

2) (Algebra) homomorphism $\varphi : A \to B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

Tvrzení 1.1

Let A be algebra over $\mathbb{K}$. Put $A_e = A \times \mathbb{K}$ with operations A_e defined coordinate-wise and multiplication defined by

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \qquad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

Then A_e is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if A is commutative, then A_e is commutative.

Důkaz
We have A_e is vector space (from linear algebra). We easy proof from definition, that A_e is algebra, $(\mathbf{o}, 1)$ is a unit in A_e and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. □

Definice 1.2 (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv$ A is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ $(\forall a, b \in A)$.

Definice 1.3 (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv$ $(A, \|\cdot\|)$ is normed algebra nad Banach space.

Například
$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If T is Hausdorff topological space, then $\mathcal{C}_b(T) = \{f : T \to \mathbb{K} | f$ is continuous and bounded$\} \subseteq l_\infty(T)$ is closed subalgebra.

If T is locally compact, Hausdorff, not compact. Then $\mathcal{C}_0(T) = \{f : T \to \mathbb{K}$ continuous $| \forall \varepsilon > 0 : \{t \in T$ $\mathcal{C}_b(T)$ is closed subalgebra, which doesn't have unit.

If X is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If X is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

Tvrzení 1.2

If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \to A$ is Lipschitz on bounded sets.

Důkaz

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r)\ \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) :$$

$$\|ab - cd\| \leqslant \|a(b-d)\| + \|(a-c) \cdot d\| \leqslant \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leqslant R \cdot (\|b-d\| + \|a-c\|) \leqslant 2R\|(a, b) - (c, d)\|.$$

□

Tvrzení 1.3

Let $(A, \|\cdot\|)$ be a Banach algebra. On A_e we consider the norm

$$\|(a,\alpha)\| := \|a\| + |\alpha|, \qquad (a,\alpha) \in A \times \mathbb{K} = A_e.$$

Then $(A_e, \|\cdot\|)$ is Banach algebra.

Důkaz

It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a,\alpha)\cdot(b,\beta)\| \leqslant \|(a,\alpha)\| \cdot \|(b,\beta)\|,$$

which is easy. □

Poznámka

There is more (natural) ways to define norm on A_e (unlike $\cdot$ on A_e, which is natural).

A has a unit … we may still consider A_e.

If $e \in A \setminus \{\mathbf{o}\}$ is a unit, then $\|e\| \geqslant 1$, because $\|e\| = \|e^2\| \leqslant \|e\|^2$.

Věta 1.4

Let A be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a \cdot x$, $x \in A$. Then $I : A \to \mathbb{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leqslant 1$.

Moreover, if A has a unit e, then I is isomorphism into and $I(e) = \mathrm{id}$.

If $\|x^2\| = \|x\|^2$, $x \in A$, then I is isometry into.

Důkaz

„$L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leqslant 1$“: Linearity is obvious, $\|L_a(x)\| = \|a \cdot x\| \leqslant \|a\| \cdot \|x\|$, so $\|L_a\| \leqslant \|a\|$ and so $\|I\| \leqslant 1$. Since it is easily I preserves multiplication, so we are left to prove the „Moreover“ part.

„A has a unit e“: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geqslant \|L_a\left(\frac{e}{\|e\|}\right) = \frac{a}{\|e\|} = \frac{1}{\|e\|} \cdot a.$$

So I is bounded from below, so I is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \mathrm{id}\,.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$:

$$\|a\| \geqslant \|I(a)\| = \|L_a\| \geqslant \|L_a\left(\frac{a}{\|a\|}\right)\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So I is isometry. □

Poznámka

$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies$ $\exists$ equivalent norm $\|\cdot\|$ on A such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

Důkaz

Let $I : A \to \mathcal{L}(A)$ be as before. Put $|\|x\|| := \|I(x)\|$, $x \in A$. Since I is isomorphism, $|\|\cdot\||$ is equivalent norm. Moreover, $|\|x \cdot y\|| = \|I(x \cdot y)\| \leqslant \|I(x)\| \cdot \|I(y)\| = |\|x\|| \cdot |\|y\||$, $x, y \in A$. So $(A, |\|\cdot\||)$ is a Banach algebra. Finally

$$|\|e\|| = \|I(e)\| = \|\,\mathrm{id}\,\| = 1.$$

□

1.2 Inverse elements

Definice 1.4

$(M, \cdot, e)$ is monoid ($\cdot$ is associative, e is unit). Then invertible elements form a group ($e^{-1} = e, \exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^\times := \{x \in M \mid \exists x^{-1}\}$

Tvrzení 1.5

If $(A, \cdot, e)$ is monoid and $x_1, \ldots, x_n \in A$ commute, then $x_1 \cdot \ldots \cdot x_n \in A^x \Leftrightarrow \{x_1, \ldots, x_n\} \subset A^x$.

Důkaz

It suffices to prove it for $n = 2$ (and use induction). „If x^{-1} and y^{-1} exists, then $(xy)^{-1}$“ is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so z is left inverse to y. Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so $\tilde{z}$ is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. □

Lemma 1.6

Let A be a Banach algebra with a unit.

- $\|x\| < 1 \implies \exists (e-x)^{-1} \wedge (e-x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x+e)^{-1} \wedge \|(x+h)^{-1} - x^{-1}\| \leqslant \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

Důkaz

„First item“: We have $\|x^n\| \leqslant \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0} |^{\infty} x^n \in A$. Moreover,

$$(e-x) \cdot \left(\sum_{n=0}^{\infty} x^n\right) = \lim_{N \to \infty} (e-x) \cdot (e + x + \ldots + x^N) = \lim_{N \to \infty} e - x^{N+1} = e,$$

because $\lim_{N \to \infty} \|x^{n+1}\| \leqslant \lim_{N \to \infty} \|x\|^N = 0$. And similarly $(\sum x^n) \cdot (e-x) = e$.

„Second item“: $x + h = x \cdot (e + x^{-1}h)$ we have x^{-1} exists and $(e + x^{-1}h)^{-1}$ exists (from first item), so from previous fact $(x+h)^{-1}$ exists. Moreover

$$(x+h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \overset{1)}{=} \sum_{n=0}^{\infty} \left(-x^{-1}h\right)^n x^{-1},$$

so

$$\|(x+h)^{-1} - x^{-1}\| = \| \sum_{n=1}^{\infty} \left(-x^{-1}h\right)^n x^{-1} \| \leqslant \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leqslant$$

$$\leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left(\|x^{-1}\| \cdot \|h\|\right)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}.$$

□

Důsledek
A Banach algebra with a unit $\implies A^x \subset A$ is open and A^x is topological group.

Důkaz
$A^x \subset A$ is open by previous lemma (second item). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^x \ni x_n \to x \in A^x \overset{?}{\implies} x_n^{-1} \to x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \overset{h:=x_n-x}{\leqslant} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \to 0.$$

□

1.3 Spectral theory

Definice 1.5 (Resolvent set, spectrum and resolvent)

A Banach algebra with a unit, $x \in A$. We define resolvent set of x as $\varrho_A(x) := \{\lambda \in \mathbb{K} | \exists (\lambda \cdot e - x)^{-1}\}$. Next we define spectrum of x as $\sigma_A(x) := \mathbb{K} \backslash \varrho_A(x)$. Finally we define resolvent of x as $R_x : \varrho(x) \to A$, $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$.

If A doesn't have a unit, then notions above are defined with respect to A_e.

Tvrzení 1.7

A Banach algebra

a) $\forall x \in A : 0 \in \sigma_{A_e}(x)$ *(in particular, if A has no unit, then $0 \in \sigma_A(x)$);*

b) *A has unit* $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

Důkaz (a))

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\ldots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

□

Důkaz (b))
By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in A and second means that $((0, \lambda) - (x, 0))^{-1} = (-x, \lambda)^{-1}$ exists in A. We take „$x \to -x$“.

„ $\Longrightarrow$ “: find $(b, \beta) \in A_e$ such that $(x, \lambda)\cdot(b, \beta) = (\mathbf{o}, 1)$. So $(x\cdot b+\lambda\cdot b+\beta\cdot x, \lambda\cdot\beta) = (\mathbf{o}, 1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e+x)^{-1}\cdot x$. Similarly we find left inverse $\left(-\frac{1}{\lambda}x(x + \lambda e)^{-1}, \frac{1}{\lambda}\right)(x, \lambda)$. And next we prove that they are really inverses.

„ $\Longleftarrow$ “: Put $(b, \beta) := (x, \lambda)^{-1}$. Then $(\lambda e+x)^{-1} = b+\beta\cdot e$. We have $(x, \lambda)\cdot(b, \beta) = (\mathbf{o}, 1)$, so $\lambda \cdot \beta = 1$ and $x \cdot b + \lambda \cdot b + \beta \cdot x = \mathbf{o}$. Then

$$(\lambda e + x) \cdot (b + \beta \cdot e) = \lambda \cdot b + \lambda \cdot \beta \cdot e + x \cdot b + \beta \cdot x = e.$$

Similarly second inverse. □

Věta 1.8

$\{\mathbf{o}\} \neq A$ *complex Banach algebra,* $x \in A$. *Then* $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ *is compact, nonempty.*

Důkaz
After theory. □

Definice 1.6 (Derivative)

Y Banach space, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of f at a.

Tvrzení 1.9 (Fact)

Y Banach, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then $f'(a)$ exists $\Longrightarrow$ f is continuous at a $\wedge$ $\forall x^* \in Y^*$: $(x^* \circ f)'(a) = x^*(f'(a))$.

Důkaz
Continuity: $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x - a) = f'(a) \cdot 0 = 0$.

$x^* \in Y^*$ given, then

$$\lim_{x\to a} \frac{x^*(f(x)) - x^*(f(a))}{x - a} = \lim_{x\to a} x^* \left(\frac{f(x) - f(a)}{x - a}\right) = x^*(f'(a)).$$

□

Tvrzení 1.10

A Banach algebra with a unit, $x \in A$. Then

- $\varrho(x)$ *is open set;*
- $\forall |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$;
- *!* $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$ *has derivative at each* $\lambda \in \varrho(x)$;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$.

Důkaz

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left(e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right)^n .$$

For third we fix $\lambda \in \varrho(x)$ and $t \in (0, \delta)$ for δ small enough ($\lambda + t \in \varrho(x)$ and *). We shall prove that „$R_x'(\lambda) = -R_x(\lambda)^2$“:

$$0 \overset{?}{=} \left\| \frac{R_x(\lambda + t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| = \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leqslant$$

$$\overset{*\text{ for existence of the inverse}}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| (e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t \right\| =$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leqslant$$

$$\overset{\|x^n\| \leqslant \|x\|^n}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \overset{*\text{ for denominator } \leqslant 1/2}{\leqslant} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \to 0.$$

Fourth: In general $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$ (proof: $u^{-1}v^{-1} = (vu)^{-1}$). And we apply it for $u = (\mu e - x)$ and $v = (\nu e - x)$.

Fifth: In general $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$ (proof: $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$) so:

$$R_x(\mu) - R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))$$

$$= R_x(\mu)R_x(\nu) \left(R_x(\nu)^{-1} - R_x(\mu)^{-1} \right) = R_x(\mu)R_x(\nu)(\nu - \mu).$$

□

Věta 1.11 (Liouville for Banach space valued functions)

Y Banach space over $\mathbb{C}$, $f : \mathbb{C} \to Y$ has derivative at each point, f is bounded ($\equiv \|f\|$ is bounded). Then $f \equiv \mathrm{const}$.

Důkaz

Assume $f \not\equiv \mathrm{const}$, so there are $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$ (by Hahn–Banach theorem) $\exists x^* \in Y^* : x^*(f(x)) \neq x^*(f(x))$. From fact $x^* \in f : \mathbb{C} \to \mathbb{C}$ has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions. □

Důkaz (Theorem before theory)

First case: „A has a unit“: Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is closed, so $\sigma(x)$ is compact. Assume that $\varrho(x) = \mathbb{C}$. By previous tvrzeni we have $R_x : \mathbb{C} \to A$ has derivative everywhere, and it is bounded because $\lim_{|} \lambda| \to \infty R_x(\lambda) = \lim_{|\lambda| \to \infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$. From previous theorem $R_x \equiv \mathrm{const}$ so $\lim_{|\lambda|\to\infty} R_x(\lambda) = 0 \implies R_x \equiv 0$. In particular $0 = R_x(0) = (-x)^{-1}$. ↯(If $A \neq \{0\}$ then $x^{-1} \neq 0$ for $x \in A$.)

Second case: „A hasn't a unit“, then $\sigma(x) := \sigma_{A_e}((x, 0))$ so we apply the already proven case. □

Poznámka (Convention)

If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

Věta 1.12 (Gelfand–Mazur)

$\{\mathbf{o}\} \neq A$ Banach algebra with a unit. Assume $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$. Then A is isomorphic to $\mathbb{C}$. If moreover e is a unit in A and $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.

Důkaz

Consider $\psi : \mathbb{C} \to A$ defined as $\psi(\lambda) := \lambda \cdot e$. This is algebraic homomorphism and $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$, so it is isomorphism (and isometry, if $\|e\| = 1$).

It remains „φ is surjective“: Pick $a \in A$. From previously proved theorem $\exists \lambda \in \sigma(a)$, then $(\lambda e - a) \notin A^x$. So, $\lambda \cdot e - a = 0$, then $\psi(\lambda) = a$. □

Definice 1.7 (Spectral radius)

A Banach algebra, $x \in A$. Then $r(x) := \sup\{|\lambda|, \lambda \in \sigma(x)\}$ is called spectral radius of x.

Věta 1.13 (Beurling–Gelfand)

A Banach algebra, $x \in A \implies r(x) = \inf_{n\in\mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$.

Lemma 1.14

A Banach algebra with a unit, $x \in A$. For $p(z) = \sum_{j=1}^{n} \alpha_j z^j \in \mathbb{C}$ a polynom (with complex coefficients) we put $p(x) = \sum_{j=1}^{n} \alpha_j x^j \in A$. Then $\sigma(p(x)) = p(a(x))$.

Důkaz
Fix $\lambda \in \mathbb{C}$ and write $(\lambda - p)(z) = c \cdot \prod_{i=1}^{m}(z - z_i)$, where $z_1, \ldots, z_m$ are roots of $\lambda - p$. Then $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$ does not exists. $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^{m}(x - z_i \cdot e)$, so it does'nt exists if and only if $\exists i \in [m]$, such that $(x - z_i \cdot e)^{-1}$ doesn't exists $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow$ $\exists$ root ν of $\lambda - p$ such that $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$. □

Důkaz (Beurling–Gelfand)
WLOG A has a unit. Step 1, „$r(x) \leqslant \inf_n \sqrt[n]{\|x^n\|}$“: fix $\lambda \in \sigma(x)$. By previous lemma $\forall n : \lambda^n \in \sigma(x^n)$. By theorem 'Before theory' we have $\forall n : |\lambda|^n \leqslant \|x^n\|$.

Step 2, „$r(x) \geqslant \limsup_n \sqrt[n]{\|x^n\|}$“: Pick $r > r(x)$. Claim: „$\frac{x^n}{r^n} \to^w 0$“: Fix $x^* \in A^*$ and put $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$. By fact and tvrzeni after it, f has derivative at each $\lambda \in \varrho(x)$. Moreover for $|\lambda| \geqslant \|x\|$ we have $f(\lambda) = \lambda \cdot x^*\left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$. Thus $f(\lambda) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$, $\lambda \in P(0, r(x), \infty)$. From Complex analysis $f \in H(P(0, r, \infty))$ is uniquely given by Laurent series. In particular $f(r) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{r^n}$, so $x^*\left(\frac{x^n}{r^n}\right) \to 0$.

From princip of unique boundedness (last semester): $\frac{x^n}{r^n}$ if $\|\cdot\|$-bounded, so $\exists c > 0$: $\|x^n\| \leqslant cr^n$, $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{c} \cdot r \to r$. So $\limsup \sqrt[n]{\|x^n\|} \leqslant r$. □

Důsledek
A Banach algebra, $x \in A$ and $|\lambda| > r(x)$. Then $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ is absolutely convergent and $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

Důkaz
Fix q, such that $\frac{r(x)}{|\lambda|} < q < 1$. By previous theorem, $\exists n_0\ \forall n \geqslant n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$, so $\frac{\|x^n\|}{|\lambda|^n} < q^n$, $n \geqslant n_0$. Thus $\sum \left\|\frac{x^n}{\lambda^n}\right\| \leqslant \infty$, so the sum is absolutely convergent.

Now we easily check that $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$. □

1.4 Subalgebra

Věta 1.15

A Banach algebra with a unit e, $B \subset A$ is closed subalgebra such that $e \in B$. Fix $x \in B$. Then

- *$C \subset \varrho_A(x)$ is component (maximum connected subset) $\implies$ $C \subseteq \sigma_B(x)$ or $C \cap \sigma_B(x) = \varnothing$;*

- $\partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$;
- $\varrho_A(x)$ *is connected* $\implies \sigma_A(x) = \sigma_B(x)$;
- $\operatorname{int}\sigma_B(x) = \varnothing \implies \sigma_A(x) = \sigma_B(x)$.

Důkaz
„$\sigma_A(x) \subseteq \sigma_B(x)$“: $(\lambda e - x)^{-1}$ exists in B implies it exists (it's same) in A.

„First item“: Let $C \subset \varrho_A(x)$ be component. Pick $\lambda_0 \in C \cap \sigma_B(x)$. Wanted: „$C \backslash \sigma_B(x) = \varnothing$“. Pick $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$ (separate B and $R_x(\lambda) \notin B$). Then $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is holomorphic function on open (because maximum) connected set C. Which is zero[a] on $C \backslash \sigma_B(x)$.

Since $C \backslash \sigma_B(x)$ is open, if it is nonempty it contains a ball, so it has cluster point. Thus $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is such that $\{\lambda \in C | x^*(R_x(\lambda))\} = 0$ has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero. $\lightning$with $x^*(R_x(\lambda_0)) = 1$.

„Second item“: Pick $\lambda \in \sigma_B(x) \backslash \sigma_A(x)$ and let $C \subset \varrho_A(x)$ be a component containing λ. By first item, $C \subseteq \sigma_B(x)$, C is open, so $\lambda \in C \subseteq \operatorname{int}(\sigma_B(x))$.

„Third item“: If $\varrho_A(x)$ is connected, we can apply first item to $C = \varrho_A(x)$, we have either $\varrho_A(x) \subseteq \sigma_B(x)$ or $\varrho_A(x) \cap \sigma_B(x) = \varnothing$. But first is not possible, because $\varrho_A(x)$ is unbounded and $\sigma_B(x)$ is bounded. Therefore $\sigma_B(x) \subseteq \sigma_A(x)$.

„Fourth item“: If $\operatorname{int}(\sigma_B(x)) = \varnothing$, then (by second item) $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$. □

[a] For $\lambda \in C \backslash \sigma_B(x)$, $(\lambda e - x)^{-1}$ exists in B so $R_x(\lambda) \in B$ and therefore, $x^*(R_x(\lambda)) = 0$

Důsledek
A Banach algebra, $B \subseteq A$ closed subalgebra, $x \in B$. Then all items from previous theorem hold as well if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.

Důkaz
Without proof. (Basically same that previous; we add unit to A and B, so this unit is same $((\mathbf{o}, 1))$, etc.) □