Poznámka (Note of Me – autor of notes)
Bad English in this text is my fault, not lecturer's one.

Úvod

Poznámka
3 part exam: theorem -> proof; scientific paper -> understand + explain; therms + concepts -> explain

credits: homework (time demanding)

Microsoft teams

0.1 Matrix analysis / linear algebra

Poznámka
Scalar product: $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$: $\mathbf{u} \cdot \mathbf{v}$, cross product: $\mathbf{u} \times \mathbf{v}$, and more: $\mathbf{u} = u^i \mathbf{e}_i$ $\mathbf{u} \cdot \mathbf{v} = \delta_{ij}(u^i v^j)$, $(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k$ (where ε_{ijk}, Levi-Civita symbol, does expecting thing).

Definice 0.1 (Tensor product)

$$\mathbf{u} \otimes \mathbf{v} \qquad (\mathbf{u} \otimes \mathbf{v})\mathbf{w} := \mathbf{u}(\mathbf{v} \cdot \mathbf{w})$$

Tvrzení 0.1 (Identities for Levi-Civita symbol)

$$\varepsilon_{ijk}\varepsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}$$

$$\varepsilon_{ijk} \cdot \delta_{lm} = \varepsilon_{jkm} \cdot \delta_{il} + \varepsilon_{klm} \cdot \delta_{jl} + \varepsilon_{ijm} \cdot \delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{ijm}\varepsilon_{ijn} = 2\delta_{mn}$$

Definice 0.2 (Transpose matrix)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$, $\mathbb{A}^T$ is defined as $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 : \mathbb{A}^T \mathbf{u} \cdot \mathbf{v} := \mathbf{u} \cdot \mathbb{A}\mathbf{v}$.

Definice 0.3 (Trace of matrix)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$, $\operatorname{tr} \mathbb{A}$ is defined as $\operatorname{tr}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

Poznámka
Matrix, tensor and linear operator is the same.

$$\mathbb{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbb{A}\mathbf{v} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(v_m\mathbf{e}_m) = A_{ij}v_m\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_m) = (A_{ij}v_j)\mathbf{e}_i.$$

Definice 0.4 (Axial vector)

$\mathbb{A} \in \mathbb{R}^{3\times3}$, $\mathbb{A}$ is stew-symetric ($-\mathbb{A} = \mathbb{A}^T$). Then we can prove that $\forall \mathbf{w} \in \mathbb{R}^3 : \mathbb{A}\mathbf{w} = \mathbf{v}_{\mathbb{A}} \times \mathbf{w}$. We call $\mathbf{v}$ the axial vector.

Poznámka
$\mathbf{v}_{\mathbb{A}} = (A_{23}, A_{13}, A_{12})^T$.

Tvrzení 0.2

$\mathbb{A}\mathbf{v}_{\mathbb{A}} = \mathbf{o}$ *and* $(\mathbf{u} \otimes \mathbf{v})^T = (\mathbf{v} \otimes \mathbf{u})$.

Definice 0.5 (Determinant in 3D)

$\det \mathbb{A} := \frac{\mathbb{A}\mathbf{u}\cdot(\mathbb{A}\mathbf{v}\times\mathbb{A}\mathbf{w})}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}$ for three arbitrary vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

Poznámka (Nanson formula)

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\det \mathbb{A})^{-1}\mathbb{A}\mathbf{w} \cdot (\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) = \mathbf{w} \cdot (\det \mathbb{A})^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) \implies$$
$$\implies \mathbf{u} \times \mathbf{v} = (\det A)^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v})$$
$$\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v} = (\det \mathbb{A})\mathbb{A}^{-T}(\mathbf{u} \times \mathbf{v})$$

Definice 0.6 (Cofactor)

$$\operatorname{cof} \mathbb{A} := (\det \mathbb{A})\mathbb{A}^{-T}.$$

(Change of surface area under linear mapping $\mathbb{A}$.)

Definice 0.7 (Eigenvalues, eigenvectors)

$\mathbb{A}\mathbf{v} = \lambda\mathbf{v}$.

Characteristic polynomial: $\det(\mathbb{A} - \mu\mathbb{I}) = -\mu^3 + c_1\mu^2 - c_2\mu + c_3$.

Věta 0.3 (Cayley-Hamilton)

$$-\mathbb{A}^3 + c_1\mathbb{A}^2 - c_2\mathbb{A} + c_3\mathbb{I} = \mathbb{O}$$

Tvrzení 0.4

$$c_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det \mathbb{A}$$
$$c_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \operatorname{tr}\operatorname{cof}\mathbb{A} = \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}(\mathbb{A}^2))$$
$$c_1 = \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}\mathbb{A}$$

Důkaz
With definition of characteristic polynomial, Cayley-Hamilton and Schur decomposition. Schur decomposition: $\mathbb{A} \in \mathbb{R}^{3\times3}$. There exists an invertible matrix $\mathbb{U}$ and upper triangular matrix $\mathbb{T}$ such that

$$\mathbb{A} = \mathbb{U}^{-1}\mathbb{T}\mathbb{U}, \qquad \mathbb{T} = \begin{pmatrix} \lambda_1 & T_{12} & T_{13} \\ 0 & \lambda_2 & T_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

□

Tvrzení 0.5 (Useful identity from CH)

$$\mathbb{A}^{-1} = \frac{1}{c_3}\mathbb{A}^2 - \frac{c_1}{c_3}\mathbb{A} + \frac{c_2}{c_3}\mathbb{I} = \frac{1}{\det\mathbb{A}}\mathbb{A}^2 - \frac{\operatorname{tr}\mathbb{A}}{\det\mathbb{A}}\mathbb{A} + \frac{\operatorname{tr}\operatorname{cof}\mathbb{A}}{\det\mathbb{A}}\mathbb{I}$$

Poznámka (Functions of matrices)
$\exp\mathbb{A}$, $\ln\mathbb{A}$, $\sin\mathbb{A}$, ...

There are several ways of define it: Analytics calculus = Taylor series, Borel calculus: $\mathbb{A} = \sum \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$ () $\implies f(\mathbb{A}) := \sum f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i$, Holomorphic calculus ($f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$) $f(\mathbb{A}) = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta\mathbb{I} - \mathbb{A})^{-1} d\zeta$ (where curve ζ envelops eigenvalues of $\mathbb{A}$)

Tvrzení 0.6 (Useful identities for functions)

$$\det(\exp\mathbb{A}) = \exp(\operatorname{tr}\mathbb{A})$$
$$\exp\mathbb{A} = \lim_{n\to\infty}\left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n$$

Definice 0.8 (Invariants of matrix)

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}\mathbb{A} = I_1;$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \operatorname{tr}\operatorname{cof}\mathbb{A} = \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}(\mathbb{A}^2)) = I_2;$$
$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det\mathbb{A} = I_3$$

0.2 Representation theorems for isotropic functions

Definice 0.9 (Isotropic function)

$\varphi : \mathbb{R}^{3\times3} \to \mathbb{R}$ is isotropic $\equiv \varphi(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \varphi(\mathbb{A})$ for all proper orthogonal matrices ($\mathbb{Q}\mathbb{Q}^T = \mathbb{I}$, $\det\mathbb{Q} > 0$).

$f : \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ is isotropic $\equiv f(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \mathbb{Q}f(\mathbb{A})\mathbb{Q}^T$ for all proper orthogonal matrices.

Věta 0.7

A scalar function $\varphi : \mathbb{A} \in \mathbb{R}^{3\times3} \to \mathbb{R}$ of symmetric matrices is isotropic if and only if it can be rewritten as a function of invariants of $\mathbb{A}$.

Věta 0.8

A matrix valued function $f : \mathbb{A} \in \mathbb{R}^{3\times3} \to \mathbb{R}^{3\times3}$ (from symmetric matrices to symmetric matrices) is isotropic if and only if it can be rewritten as

$$f(\mathbb{A}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{A} + \alpha_2\mathbb{A}^2,$$

where $\{\alpha_i\}_{i=1}^3$ are scalar function of the invariants.

Důsledek
$\mathbb{A} \mapsto \mathbb{A}^{-1}$ is isotropic function.

Poznámka (Notation)

$$\mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times3}, \qquad \mathbb{A} : \mathbb{B} := \operatorname{tr}(\mathbb{A}\mathbb{B}^T), \qquad ||\mathbb{A}|| := (\operatorname{tr}(\mathbb{A}\mathbb{A}^T))^{1/2}$$

0.3 Calculus

Definice 0.10 (Gateaux derivative)

$$Df(x)[y] = \left(\frac{d}{d\tau}f(x+\tau y)\right)|_{\tau=0}.$$

Definice 0.11 (Fréchet derivative)

$$\lim_{||y||\to 0} \frac{||f(x+y) - f(x) - Df(x)[y]||}{||y||} = 0.$$

Poznámka
$Df(\mathbb{A})[\mathbb{B}] \sim \frac{\partial f}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] \sim \frac{\partial f}{\partial \mathbb{A}}(\mathbb{A}) : \mathbb{B}$.

Příklad

$$D(I_2(\mathbb{A}))[B] = D(-\frac{1}{2}\operatorname{tr}\mathbb{A}^2 + \frac{1}{2}(\operatorname{tr}\mathbb{A})^2)[B] = \frac{d}{d\tau}(-\frac{1}{2}\operatorname{tr}(\mathbb{A}+\tau\mathbb{B})^2 + \frac{1}{2}(\operatorname{tr}(\mathbb{A}+\tau\mathbb{B}))^2)|_{\tau=0} =$$
$$= -\operatorname{tr}(\mathbb{A}\mathbb{B}) + (\operatorname{tr}\mathbb{A})(\operatorname{tr}\mathbb{B}) = (\operatorname{tr}\mathbb{A})\mathbb{I} : \mathbb{B} - \mathbb{A}^T : \mathbb{B} = ((\operatorname{tr}\mathbb{A})\mathbb{I} - \mathbb{A}^T) : \mathbb{B}.$$

$$D(\det\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\det(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = (\det\mathbb{A})\frac{d}{d\tau}(\det(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}))_{\tau=0} = \det\mathbb{A}\frac{d}{d\tau}(1+\tau\operatorname{tr}(\mathbb{A}^{-1}\mathbb{B})+\ldots)|_{\tau=0} =$$

Poznámka
Chain rule works as usual.

Příklad

$$\frac{d}{dt}(\det\mathbb{A}(t)) = (\det\mathbb{A})\operatorname{tr}\left(\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\right).$$

Příklad

$$\frac{\partial\mathbb{A}^{-1}}{\partial\mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}\left((\mathbb{A}+\tau\mathbb{B})^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left((\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{A}^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left((\mathbb{I}-\tau\mathbb{A}^{-1}\mathbb{B}+\ldots)\mathbb{A}^{-1}\right)|_{\tau=0} = -$$

Příklad

$$\frac{\partial e^{\mathbb{A}}}{\partial\mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}\left(e^{\mathbb{A}+\tau\mathbb{B}}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\mathbb{I} + (\mathbb{A}+\tau\mathbb{B}) + \frac{(\mathbb{A}+\tau\mathbb{B})^2}{2!} + \ldots\right)|_{\tau=0} =$$
$$= \frac{d}{d\tau}\left(\mathbb{I} + (\mathbb{A}\tau\mathbb{B}) + \ldots + \tau(\mathbb{A}\mathbb{B}+\mathbb{B}\mathbb{A}) + \tau(\mathbb{A}\mathbb{A}\mathbb{B}+\mathbb{A}\mathbb{B}\mathbb{A}+\mathbb{B}\mathbb{A}\mathbb{A}) + \ldots\right)$$

Věta 0.9 (Daleckii-Krein theorem)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$ real symmetric matrix. $\mathbb{A} = \sum_{i=1}^{3} \lambda_i \mathbb{P}_i$, $\mathbb{P}_i$-projector to i-th eigenvector, $\mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i$. f real valued function $f : \mathbb{R} \to \mathbb{R}$ differentiable.

$$f(\mathbb{A}) := \sum_{i=1}^{3} f(\lambda_i)\mathbb{P}_i = \sum_{i=1}^{3} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i.$$

$$Df(\mathbb{A})[\mathbb{B}] = \sum_{i=1}^{3} \frac{df}{d\lambda}|_{\lambda=\lambda_i} \mathbb{P}_i \mathbb{B} \mathbb{P}_i + \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \mathbb{P}_i \mathbb{B} \mathbb{P}_j$$

$$(Df(\mathbb{A})[\mathbb{B}])_{ij} = \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} B_{ij}, if i \neq j, (Df(\mathbb{A})[\mathbb{B}])_{ij} = \frac{df}{d\lambda}|_{\lambda=\lambda_j} B_{ij}, if i = j.$$

Důkaz
From chain rule:

$$\frac{\partial f(\mathbb{A})}{\partial \mathbb{A}} = \sum_{i=1}^{3} \frac{df(\lambda_i)}{d\lambda}|_{\lambda=\lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i \otimes \mathbf{v}_i + \sum_{i=1}^{3} f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + \sum_{i=1}^{3} f(\lambda_i) \mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}$$

First derivative at right side:

$$\mathbb{A}\mathbf{v}_i = \lambda_i \mathbf{v}$$

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \qquad \cdot \mathbf{v}_i$$

$$\frac{\partial A_{mn}}{\partial A_{kl}} (\mathbf{v}_i)_n + (A_{mn}) \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} = \frac{\partial \lambda_i}{\partial A_{kl}} (\mathbf{v}_i)_m + \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}$$

$$\delta_{mk}\delta_{nl}(\mathbf{v}_i)_n + A_{mn} \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} = \frac{\partial \lambda_i}{\partial A_{kl}} (\mathbf{v}_i)_m + \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} \qquad \cdot (\mathbf{v}_i)_m \sum_m$$

$$\sum_m \frac{\partial \lambda_i}{\partial A_{kl}} (\mathbf{v}_i)_m (\mathbf{v}_i)_m = \frac{\partial \lambda_i}{\partial A_{kl}}$$

From symmetry of $\mathbb{A}$ and definition of eigenvector:

$$\sum_m A_{mn} \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} (\mathbf{v}_i)_m = \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} (\mathbf{v}_i)_n$$

$$\sum_m \delta_{mk}\delta_{nl}(\mathbf{v}_i)_n (\mathbf{v}_i)_m = \delta_{nl}(\mathbf{v}_i)_n (\mathbf{v}_i)_k$$

So

$$\lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} (\mathbf{v}_i)_n + \delta_{nl}(\mathbf{v}_i)_n (\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}} + \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} \qquad \sum_n$$

$$\sum_n \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} (\mathbf{v}_i)_n + (\mathbf{v}_i)_l (\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}} + \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}$$

$$(\mathbf{v}_i)_l (\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}}$$

$$\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbf{v}_i \otimes \mathbf{v}_j$$

Second derivative at right side:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \qquad \cdot \mathbf{v}_j$$

$$\mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j$$

...

$$\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = \frac{\mathbf{v}_j \otimes \mathbf{v}_i}{\lambda_i - \lambda_j} i = \frac{\delta_{kj}\delta_{il}}{\lambda_i - \lambda_j}$$

$$\left(\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} [\mathbb{X}] \right)_j = \frac{\delta_{im}\delta_{jn}}{\lambda_i - \lambda_j} = \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j}.$$

Poznámka (V dokončení důkazu se ještě použije)

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}).$$