Příklad (1.)
(We assume $e = e(\eta, \varrho)$.) Let us assume that our substance of interest is the calorically perfect ideal gas. We know that the engineering equation of state for this substance reads

$$p_{th} = c_{V,ref}(\gamma - 1)\varrho\theta,$$

where $c_{V,ref}$ is a positive constant (specific heat at constant volume) and γ is a positive constant greater that one (adiabatic exponent). Further, we also know that the internal energy of the substance is proportional to the temperature

$$e = c_{V,ref}\theta.$$

Use this characterisations, solve the partial differential equations for e and identify – for our particular substance – the formula for the internal energy e as a function of the entropy and the density. Once you find the function $e(\eta, \varrho)$, find also the explicit formula for the entropy η as a function of the temperature and the density, $\eta(\theta, \varrho)$.

Řešení

Z přednášky / celého znění zadání máme diferenciální rovnice pro e, kam dosadíme rovnosti výše:

$$\frac{\partial e}{\partial \eta}(\eta, \varrho) = \theta = \frac{e}{c_{V,ref}},$$

$$\varrho^2 \frac{\partial e}{\partial \varrho}(\eta, \varrho) = p_{th} = c_{V,ref}(\gamma - 1)\varrho\theta = (\gamma - 1)\varrho e.$$

Takže když se na funkci e podíváme ve směru η, dostaneme $e(\eta, |\varrho) = C \cdot \exp\left(\frac{\eta}{c_{V,ref}}\right)$. Když se podíváme ve směru ϱ, dostaneme $e(|\eta, \varrho) = C \cdot \varrho^{\gamma-1}$. Tudíž máme $e(\eta, \varrho) = C \cdot \exp\left(\frac{\eta}{c_{V,ref}}\right) \cdot \varrho^{\gamma-1}$.

Abychom splnili počáteční podmínky, tak $\frac{\partial e}{\partial \eta}(0, \varrho_{ref}) = \theta_{ref}$, tedy $C \cdot \frac{1}{c_{V,ref}} \cdot \varrho_{ref}^{\gamma-1} = \theta_{ref}$, tj.

$$e(\eta, \varrho) = \frac{c_{V,ref} \cdot \theta_{ref}}{\varrho_{ref}^{\gamma-1}} \cdot \exp\left(\frac{\eta}{c_{V,ref}}\right) \cdot \varrho^{\gamma-1}.$$

Pokud z toho vyjádříme η, tak dostaneme:

$$\frac{e(\eta, \varrho)}{c_{V,ref} \cdot \theta_{ref}} \cdot \left(\frac{\varrho}{\varrho_{ref}}\right)^{1-\gamma} = \exp\left(\frac{\eta}{c_{V,ref}}\right),$$

$$\ln\left(\frac{e(\eta, \varrho)}{c_{V,ref} \cdot \theta_{ref}} \cdot \left(\frac{\varrho}{\varrho_{ref}}\right)^{1-\gamma}\right) = \frac{\eta}{c_{V,ref}},$$

$$c_{V,ref} \cdot \ln\left(\frac{e(\eta, \varrho)}{c_{V,ref} \cdot \theta_{ref}} \cdot \left(\frac{\varrho}{\varrho_{ref}}\right)^{1-\gamma}\right) = \eta.$$

Nyní už stačí dosadit $\frac{e}{c_{V,ref}} = \theta$:

$$\eta(\theta, \varrho) = c_{V,ref} \cdot \ln\left(\frac{\theta}{\theta_{ref}} \cdot \left(\frac{\varrho}{\varrho_{ref}}\right)^{1-\gamma}\right) = \eta.$$