Poznámka
At least 1 from (3-)4 homework (flexible deadlines – last lecture).

Poznámka
In this lecture, there was also the revision of topology. (Topological space, topology, basis of topology, continuous map, quotient space, product topology, Hausdorff spaces).

Poznámka
World Homotopy comes from homós (= same, simiar) and topos (place).

Definice 0.1 (Homotopic functions)

Given two topological spaces X and Y and two continuous functions $f, g : X \to Y$, we say that f is homotopic to g ($f \sim g$) if there is a 1-parametric family $f_t : X \to Y$: $f_0 = f$, $f_1 = g$ and the map $F : [0,1] \times X \to Y$ defined by $(t, x) \mapsto f_t(x)$ is continuous.

Definice 0.2 (Homotopy equivalent spaces)

Given two topological spaces X and Y we say that X and Y are homotopy equivalent if there is a pair of continuous maps (f, g) such that $f : X \to Y$ and $g : Y \to X$ and $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$, $g \circ f \sim \mathrm{id}_X$, $f \circ g \sim \mathrm{id}_Y$.

Příklad
Given $\mathbb{R}$, $\mathbb{R}^2$ with the standard Euclidean topology and two maps $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto f(x) = (x, x^3)$, $g : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto g(x) = (x, e^x)$.

Are f and g homotopic? (Show that by constructing homotopy.)

Řešení

$$F(t, x) = (1 - t)(x, x^3) + t(x, e^x) = (x, (1 - t)x^3 + te^x).$$

Příklad
Given three topological spaces $(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$ and two pairs of continuous maps $f_1, g_1 : (X, \tau_X) \to (Y, \tau_Y)$ and $f_2, g_2 : (Y, \tau_Y) \to (Z, \tau_Z)$. Assume that f_1 is homotopic to g_1 and f_2 is homotopic to g_2. Show that $f_2 \circ f_1$ is homotopic to $g_2 \circ g_1$.

Řešení

$$F(t, x) = F_2(t, F_1(t, x)).$$

Příklad

Take $B^n := \left\{x, \dots, x_n | \sqrt{x_1^2 + \dots + x_n^2} \leqslant 1\right\} \subseteq \mathbb{R}^n$. And take a map $f : B^n \to B^n$: $f(x) = (0, \dots, 0) \in B^n$ for all $x \in B^n$. Shows that there is a homotopy from id to f.

Řešení

$$F : [0,1] \times B^n \to B^n, \qquad (t, x) \mapsto (1-t)x.$$

Příklad

Take a 2-ball B^2. B^2 is homotopy equivalent to its center by previous problem, but it is not homeomorphic to $(0,0)$.

Definice 0.3 (Deformation retraction)

A deformation retraction of a topological space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in [0,1]$: $f_0 = \mathrm{id}_X$, $f_1(X) = A$ and $f_t|_A = \mathrm{id}_A$. And family f_t is continuous in the following sense:

$$F : [0,1] \times X \to X, (t, x) \to f_t(x), \text{ is continuous.}$$

Tvrzení 0.1

Given a deformation retraction $f_t : X \to X$, there is a pair $(f, g) : X \xrightarrow{f} A \xrightarrow{g} X : g \circ f \sim \mathrm{id}_X, f \circ g \sim \mathrm{id}_A$.

Poznámka (Suggestion)

$f = f_1$, $g = f_i \circ i_A$ $(A \overset{i_A}{\hookrightarrow} X)$, tj. $f \circ g : A \overset{i_A}{\hookrightarrow} X \xrightarrow{f_1} X \xrightarrow{f_1} X$, $a \mapsto a \mapsto a \mapsto a$ (or A) $\implies f \circ g = \mathrm{id}_A$. $g \circ f : X \xrightarrow{f_i} A \xrightarrow{i_A} X \implies f_1(x) \sim \mathrm{id}_X$.

Definice 0.4

Given two topological spaces X and Y and a continuous map $f : X \to Y$, the mapping cylinder M_f is defined to be the quotient space of $X \times [0,1] \coprod Y$ and $\sim$: $(x, 1) \sim f(x)$. $M_f = X \times [0,1] \times Y / \sim$.

Tvrzení 0.2

Given X, Y and f, M_f deformation retracts to Y.

Důkaz (/ Idea of proof)
The way to construct $f_t = F(\cdot, t) : M_f \to M_f$ is to slide each point (x, t) along the segment $\{x\} \times [0, 1]$ to $f(x)$:

$$F : (x, t) \mapsto f(x), \qquad \forall y \in Y : y = F \mapsto \{f_1 = \mathrm{id}\, Y \to Y\}$$

In your HW you will check that $F(x, t)$ is continuous. □

Poznámka
Cell complex (CW complex) is a topological space with a nice decomposition into small pieces.

1. Start with a discrete set X^0, whose points are called 0-cells.

2. We form the n-skeleton X^n from X^{n-1} by attaching cells $e^n_\alpha = I^n = [0, 1]^n$. By the attachment we mean $(e^n_a = B^n_\alpha, \partial e^n_a = S^n_\alpha)$ $\varphi_\alpha : \partial e^n_\alpha \to X^{n-1}$. Hence we can view $X^n = X^{n-1} \coprod \coprod B^n_\alpha / \sim$, where $x \sim \varphi_\alpha(x)$ for $x \in \partial\partial B^n_\alpha$.

3. We can either stop this inductive process at a certain finite steps or take an infinite number of steps. In the first case $X = X^n$ for some n, in the second one $X = \bigcup_{n \in \mathbb{N}_0} X^n$ with the weak topology ($A \subset X$ is open $\leftrightarrow$ $A \cap X^n$ is open for all n).

Například
Example of 1-skeleton is graph.

Definice 0.6

Given a cell complex X. Each cell e^n_α has a characteristic map $\Phi_\alpha : e^n_\alpha = B^n_\alpha \to X$ which extends the attaching map $\varphi_\alpha : \partial B^n_\alpha \to X^n$, it is homeomorphism from the interior of B^n_α onto e^n_α. Namely

$$B^n_\alpha \hookrightarrow X^{n-1} \coprod \coprod_\beta B^n_\beta \overset{quotient}{\to} X^n \to X, \qquad B^n_\alpha \to X$$

Definice 0.7

A subcomplex of CW complex is a closed subspace $A \subset X$ that is a union of cells with the corresponding attachments.

Příklad
Construct two different CW structures on S^2.

Řešení
$S^2 = e^0 \cup e^2$, $S^2 = e^0 \cup e^1 \cup \{e^2_1, e^2_2\}$. (See practicals.)

Příklad

We define $\mathbb{R}P^n$ to be the quotient of $S^n/\sim$, where $V \sim$ the antipodal point to V. TODO?

Definice 0.8

Consider a pair (X, A) where X is a CW complex and A is subcomplex. Then we define the quotient complex X/A to be the CW complex with the structure: There are all the cells of $X\backslash A$ with the corresponding attaching maps, and there is a extra 0-cell which is A in $X\backslash A$. For a cell e^n_α of $X\backslash A$ attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map in the corresponding cell in $X\backslash A$ is the composition $S^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

Příklad

Show that $S^n = e^0 \cup e^n$ is $B^n/S^{n-1} = TODO/e^0 \cup e^{n-1}$.