Poznámka
Stručný obsah: Diferencovatelnost v Banachových prostorech; Asplundovy prostory; slabé Asplundovy prostory; fragmentovanost a oddělovací spojitost; atd.

1 Diferencovatelnost

1.1 Základní pojmy

Poznámka
Většina by fungovala i pro NLP, ale my se pro jednoduchost zaměříme na Banachovy prostory.

Definice 1.1

X, Y reálné Banachovy prostory, $U \subset X$ otevřená, $f : U \to Y$, $x \in U$, $h \in X$:

$$\partial_h^+ f(x) = \lim_{t \to 0_+} \frac{f(x + t \cdot h) - f(x)}{t} \in Y, \text{ pokud existuje,}$$

$$\partial_h f(x) = \lim_{t \to 0} \frac{f(x + t \cdot h) - f(x)}{t} \in Y, \text{ pokud existuje.}$$

Poznámka
$\partial_{\mathbf{o}}^+ f(x) = \partial_{\mathbf{o}} f(x) = 0$. Pokud $\|h\| = 1$, pak je to směrová derivace.

Pokud $\alpha > 0$, pak $\partial_{\alpha h}^+ f(x) = \alpha \partial_h^+ f(x)$, má-li alespoň jedna strana smysl. Podobně pro $\alpha \in \mathbb{R} \setminus \{0\}$ je $\partial_{\alpha h} f(x) = \alpha \partial_h f(x)$, má-li alespoň jedna strana smysl (speciálně $\alpha = -1$).

$$\exists \partial_h f(x) \Leftrightarrow \exists \partial_{-h}^+ f(x) = -\partial_h^+ f(x).$$

Definice 1.2 (Gateauxova derivace)

X, Y reálné Banachovy prostory, $U \subset X$ otevřená, $f : U \to Y$, $x \in U$, $h \in X$: Pokud $\exists L \in \mathcal{L}(X, Y)$, že $\forall h \in X : L(h) = \partial_h f(x)$, značíme $f_g'(x) = L$.

Poznámka
Stačí, aby $\forall h \in X : L(h) = \partial_u^+ f(a)$. Znamená to, že $h \mapsto \partial_h^{(+)} f(x)$ je omezený lineární operátor.

Definice 1.3 (Fréchetova derivace)

f má v bodě $x \in U$ Fréchetovu derivaci, pokud $\exists L \in \mathcal{L}(X, Y)$:

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0.$$

Poznámka

Pokud takové L existuje, nutně platí $L = f'_g(x)$. Fréchetovu derivaci značíme $f'_F(x)$.

Poznámka

$$\exists f'_F(x) \Leftrightarrow \exists f'_g(x) \wedge \lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = \partial_h f(x) \text{ stejnoměrně pro } h \in B_X \text{ (resp. } h \in S_X).$$

Důkaz

$f'_F(x)$ existuje $\Leftrightarrow$

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall h \in X, \|h\| < \delta : \|f(x+h) - f(x) - \partial_h f(x)\| \leqslant \varepsilon \cdot \|h\|$$

Existenci $f'_g(x)$ máme, tedy: $\varepsilon > 0$ … najdeme to $\delta > 0$: $h \in B_x$, $t \in \mathbb{R}$, $0 < |t| < \delta$ $\implies$ $\|t \cdot h\| < \delta$:

$$\|f(x+th) - f(x) - \partial_{t \cdot h} f(x)\| \leqslant \varepsilon \|t \cdot h\| = \varepsilon \cdot |t|$$

$$\|\frac{f(x+th) - f(x)}{t} - \partial_h t(x)\| \leqslant \varepsilon$$

to dává stejnoměrnou konvergenci „ $\implies$ “.

„ $\impliedby$ “: Nechť $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall h \in \{x | \forall t \in P(\mathbf{o}, \delta)\}$:

$$\|\frac{f(x + t \cdot h) - f(x)}{t} - \partial_h f(x)\| \leqslant \varepsilon.$$

$\varepsilon > 0$ … najdeme to $\delta > 0$: Zvolíme $h \in X$, $0 < \|h\| < \delta \implies \frac{h}{\|h\|} \in S_X \implies$

$$\implies \|\frac{f(x+h) - f(h)}{\|h\|} - \frac{\partial_h f(x)}{\|h\|}\| \leqslant \varepsilon \implies$$

$$\implies \frac{\|f(x+h) - f(x) - \partial_h f(x)\|}{\|h\|} < \varepsilon.$$

$\square$

Poznámka

1. $X = \mathbb{R}$, pak je F. derivace, G. derivace a běžná derivace to samé.

2. TODO?

3. TODO?

Tvrzení 1.1

$\dim X < \infty$, $U \subset X$ otevřená; $f : U \to Y$ lipschitzovská, $x \in U$, $f'_g(x)$ existuje $\implies$ $f'_F(x)$ existuje.

Důkaz

f lipschitzovská $\implies$ existuje $L > 0 : \|f(x) - f(y)\| \leqslant L \cdot \|x - y\|$ $(x, y \in U)$. Nechť existuje $f'_g(x)$. Potom $\forall \varepsilon > 0$ existuje $h_1, \ldots, h_N \in S_X$ ε-síť. Nechť $\delta > 0$ je takové, že $B(x, \delta) \subset U$ a $0 < |t| < \delta \implies \|\frac{f(x+th_i)-f(x)}{t} - f'_g(x)(h_i)\| < \varepsilon$.

Vezmeme $h \in S_X$ libovolné, $0 < |t| < \delta$. Existuje i, že $\|h - h_i\| < \varepsilon$:

$$\|\frac{f(x+t\cdot h) - f(x)}{t} - f'_g(x)(h)\| \leqslant \|\frac{f(x+t\cdot h) - f(x+t\cdot h_i)}{t}\| + \|\frac{f(x+t\cdot h_i) - f(x)}{t} - f'_g(x)(h_i)\| + \|f'_g(x$$

□

Poznámka

Stačí lokálně lipschitzovská.

Tvrzení 1.2

$f : (a, b) \to \mathbb{R}$ konvexní $\implies$ $f'(x)$ existuje v každém bodě (a, b) až na spočetně mnoho.

Důkaz

1) $\forall x \in (a, b)$ existuje vlastní $f'_+(x)$, neboť $f'_+(x) = \lim_{y \to x+} \frac{f(y)-f(x)}{y-x}$, což je neklesající funkce v $y \in (x, b)$ a zdola omezená hodnotou $\frac{f(z)-f(x)}{z-x}$ pro $z \in (a, x)$.

2) $x \mapsto f'_+(x)$ je neklesající na (a, b). 3) Podobně pro f'_-. Tedy f je spojitá na (a, b). 4) $f'(x)$ neexistuje $\Leftrightarrow$ f'_+ má v bodě x skok. (f'_+ je spojitá v x $\implies$ $f'_x(x) = \lim_{y \to x-} f'_+(y) = \lim_{y \to x-} f'_-(y)$, $f'_-(y) \leqslant f'_+(y) \leqslant f'_-(z)$ pro $z > y$). □

Tvrzení 1.3

f convex and bounded from above on $B(x, r)$, $x \in X, r > 0$ $\implies$ f is Lipschitz on $B\left(x, \frac{1}{2}\right)$.

Důkaz
1) „$f \leqslant M$ on $B(x,r) \implies f \geqslant 2f(x) - M$ on $B(x,r)$“: $y \in B(x,r)$, $z := x + (x-y) \implies z \in B(x,r)$, $x = \frac{1}{2}(y+z)$. $f(x) \leqslant \frac{1}{2}(f(y)+f(z))$, $f(y) \geqslant 2f(x) - f(z) \geqslant 2f(x) - M$.

2) Assume $|f| \leqslant M$ on $B(x,r)$. Take $v, w \in B(x, \frac{r}{2})$, $v \neq w$, $z := w + \frac{z}{2}\frac{w-v}{\|w-v\|} \implies z \in B(x,r)$. $w(1 + \frac{z}{2\|w-v\|}) = z + \frac{z}{2\|w-v\|}v$,

$$f(w) \leqslant \frac{f(z) + \frac{z}{2\|w-v\|}f(v)}{1 + \frac{z}{2\|w-v\|}}$$

$$f(w) - f(v) \leqslant \frac{f(z) + f(v)}{1 + \frac{z}{2\|w-v\|}}$$

$$\frac{f(w) - f(v)}{\|w-v\|} \leqslant \frac{f(z) - f(v)}{\|w-v\| + 1/2} \leqslant \frac{2M}{\frac{r}{2}} = \frac{4M}{r}$$

$\implies f$ is $\frac{4M}{r}$-lipschitz on $B(x, \frac{y}{2})$. □

Důsledek

- $\dim X < \infty$, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex $\implies f$ is locally lipschitz on U. (WLOG: $X = (\mathbb{R}^n, \|\cdot\|_1)$. $x \in U \implies \exists r > 0 \overline{B_{\|\cdot\|_1}(x,r)} \subset U$. $\overline{B_{\|\cdot\|_1}(x,r)} = \operatorname{conv}\{x \pm r e_i | i \in [n]\}$. $f \leqslant \max_{i \in [n]} f(x \pm r \cdot e_i)$ on $\overline{B_{\|\cdot\|_1}(x,r)} \implies f$ is Lipschitz on $\overline{B_{\|\cdot\|_1}(x, \frac{r}{2})}$)
- $\dim X < \infty$, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex, $x \in U \implies f'_F(x)$ exists if and only if f'_g („$\implies$“ always, „$\impliedby$“ from first item and tvrzeni above).
- X Banach space, $U \subset X$ open convex, $f : U \to \mathbb{R}$ continuous convex, then f is locally Lipschitz on U (f continuous $\implies f$ is locally bounded $\implies f$ is locally Lipschitz).

Věta 1.4

$X = l_1$, $f : X \to \mathbb{R}$, $f(x) = \|x\| = \sum_{n=1}^{\infty} |x_n|$.

$$\exists f'_g(x) \Leftrightarrow \forall n \in \mathbb{N} : x_n \neq 0. \qquad \implies f'_g(x) = (\operatorname{sgn} x_n)_{n=1}^{\infty} \in l_\infty,$$

$$\forall x \in l_1 \ \nexists f'_F(x).$$

Důkaz
1) $x \in l_1$, $n \in \mathbb{N}$, $x_n = 0$. Take $h = e_n \sum_{k \neq n} |x_k| + |t|$. $\partial_h f(x) = \lim_{t \to 0} \frac{\|x + t \cdot e_n\| - \|x\|}{t} = \lim_{t \to} \frac{|t|}{t}$ doesn't exist. This prove „ $\implies$ “.

„ $\impliedby$ “: Assume $\forall n \in \mathbb{N}$: $x_n \neq 0$, $h \in l_1$, $h \neq 0$, $\varepsilon > 0$:

$$\left| \frac{f(x + t \cdot h) - f(x)}{t} - \sum_{n=1}^{\infty} h_n \cdot \operatorname{sgn} x_n \right| = \left| \frac{1}{t} \sum_{n=1}^{\infty} (|x_n + t \cdot h_n| - |x_n| - t h_n \operatorname{sgn} x_n) \right| \leqslant \left| \frac{1}{t} \sum_{n=1}^{N} (\dots) \right| + \left| \frac{1}{t} \sum_{n>N} (\right.$$

□

TODO?

2 Subdiferential

Definice 2.1

X Banach, $U \subset X$ open + convex, $f : U \to \mathbb{R}$ convex + continuous ($\implies$ locally Lipschitz). $x \in U$,

$$\partial f(x) := \{x^* \in X^* | \forall y \in U : x^*(y - x) \leqslant f(y) - f(x)\}.$$

Poznámka

$$\forall h \in X \; \exists \partial_h^+ f(x)$$

$$x^* \in \partial f(x) \Leftrightarrow \forall h \in X : x^*(h) \leqslant \partial_h^+ f(x)$$

(„ $\implies$ “: Fix $h \in X$, find $\delta > 0$: $\forall |t| < \delta : x + t \cdot h \in U$. Then $\forall t \in (0, \delta) : x^*(x + t \cdot h - x) \leqslant f(x + t \cdot h) - f(x)$, $x^*(h) \leqslant \frac{f(x+t \cdot h) - f(x)}{t} \to \partial_h^+ f(x)$. „ $\impliedby$ “: Fix $y \in X$, put $h := y - x$. Then $x^*(y - x) = x^*(h) \leqslant \partial_h^+ f(x) \leqslant \frac{f(x+h) - f(x)}{1} = f(y) - f(x)$.)

$$U = X, f(x) = \|x\| \implies \partial f(x) = \{x^* \in B_{X^*} | x^*(x) = \|x\|\}.$$

(„$\subseteq$“ Let $x^* \in \partial f(x)$. Then $x^*(x) \leqslant \|x + x\| - \|x\| = \|x\|$, $x^*(-x) \leqslant \|0\| - \|x\| = -\|x\|$. Thus $x^*(x) = \|x\|$. And for $h \in X : x^*(h) \leqslant \|x + h\| - \|x\| \leqslant \|h\|$, therefore $\|x^*\| \leqslant 1$. „$\supseteq$“: Let $x^* \in B_{X^*}$, $\|x\| = x^*(x)$. Then $\forall y \in X : x^*(y - x) = x^*(y) - x^*(x) \leqslant \|y\| - \|x\|$.)

Tvrzení 2.1

$\forall x \in U : \partial f(x) \neq \varnothing$, convex, w^*-compact.

Důkaz
$h \mapsto \partial_h^+ f(x)$ is sublinear functional ($t \cdot \partial_h^+ f(x) = \partial_{t\cdot h}^+ f(x)$, $t > 0$, and

$$\partial_{h_1+h_2}^+ f(x) = \lim_{t\to 0_+} \frac{f(x+t\cdot(h_1+h_2)) - f(x)}{t} \leqslant \lim_{t\to 0_+} \left(\frac{f(x+2\cdot t\cdot h_1) - f(x)}{2t} + \frac{f(x+2\cdot t\cdot h_2) - f(x)}{2t}\right.$$

so it is sublinear functional).

By Hahn–Banach theorem, $\exists x^* \in X^\# : x^*(h) \leqslant \partial_h^+ f(x)$, $h \in X$. Moreover x^* is continuous ($x^* \in X^*$), because f is locally Lipschitz, so $\exists r > 0\ \exists L > 0 : f|_{B(x,r)}$ is L–Lipschitz, so $\left|\frac{f(x+t\cdot h)-f(x)}{t}\right| \leqslant L \cdot \|h\|$ and so $x^*(h) \leqslant \partial_h^+ f(x) \leqslant L \cdot \|h\|$, $h \in X$.

So by remark $x^* \in \partial f(x)$. Thus $\partial f(x) \neq \emptyset$. And also $\forall y^* \in \partial f(x)$. $\|?x\| \leqslant L$. Thus $\partial f(x)$ is bounded, so $\subseteq R(B_{X^*}, w^*)$ for some $R > 0$, which is w^*-compact. So since $\partial f(x)$ is w^*-closed, it is w^*-compact. (It is closed, because $\partial f(x) = \bigcap_{y\in U} \{x^* \in X^* | x^*(y-x) \leqslant f(y) - f(x)\}$).

Finally „$\partial f(x)$ is convex“: For $x^*, y^* \in \partial f(x)$, $\lambda \in (0,1)$:

$$\forall y \in U : (\lambda x^* + (1-\lambda)y^*)(y-x) \leqslant \lambda(f(y)-f(x)) + (1-\lambda)(f(y)-f(x)) = f(y) - f(x).$$

□

Tvrzení 2.2

$x \in U$. Then following is equivalent:

- $\exists f'_G(x)$;
- $|\partial f(x)| = 1$;
- $\forall h \in X : \partial_h^+ f(x) = -\partial_{-h}^+ f(x)$.

Moreover $\partial f(x) = \{f'_G(x)\}$, if one of item is true.

Důkaz
„1. $\implies$ 2.“: We have $\forall h \in X : f'_G(x)(h) = \partial_h^+ f(x) \implies f'_G(x) \in \partial f(x)$. Moreover

$$\forall x^* \in \partial f(x)\, \forall h \in X : x^*(h) \leqslant \partial_h^+ f(x) = f'_G(x)(h) \wedge -x^*(h) = x^*(-h) \leqslant f'_G(x)(-h) = -f'_G(x)(h) \implies x$$

„2. $\implies$ 3.“: Let $\exists h \in X : \partial_h^+ f(x) \neq -\partial_{-h}^+ f(x)$. Always holds $\partial_h^+ f(x) \geqslant -\partial_{-h}^+ f(x)$ ($\varphi(t) = f(x + t \cdot h)$ is convex, then $-\partial_{-h}^+ f(x) = \partial'_-(0) \leqslant \partial'_+(0) = \partial_h^+ f(x)$). So $\partial_h^+ f(x) > -\partial_{-h}^+ f(x)$.

Define $x_1^*(t \cdot h) := t \cdot \partial_h^+ f(x)$ and $x_2^*(t \cdot h) := -t \partial_{-h}^+ f(x)$, $t \in \mathbb{R}$. Then $x_1^*, x_2^* \in (\mathrm{LO}(h))^*$. And for $j = 1, 2$:

$$x_j^*(t \cdot h) \leqslant \partial_{t \cdot h}^+ f(x), \qquad t \in \mathbb{R}.$$

For $t \geqslant 0 : x_1^*(t \cdot h) = t \partial_h^+ f(x) = \partial_{t \cdot h}^+ f(x)$. For $t < 0 : x_1^*(t \cdot h) = t \cdot x_1^*(h) = t \cdot \partial_h^+ f(x) < -t \cdot \partial_{-h}^+ f(x) = \partial_{t \cdot h} f(x)$. Same for x_2^*. By Hahn–Banach theorem, we extend x_j^*, $j \in \{1, 2\}$ to $x_j^* \in X^\#$ satisfying $x_j^*(z) \leqslant \partial_z^+ f(x)$, $z \in X$. And because f is locally Lipschitz, similarly as before we have $x_1^*, x_2^* \in X^*$. Thus $x_1^*, x_2^* \in \partial f(x)$ and $x_1^* \neq x_2^*$.

„3. $\implies$ 2.“: We know $\varphi : h \mapsto \partial_h^+ f(x)$ is sublinear and we know $\varphi(h) = -\varphi(-h)$. This implies, that φ is linear ($\varphi(t \cdot h) = t \cdot \varphi(h), t \in \mathbb{R}$ arbitrary, $\varphi(h_1 + h_2) \leqslant \varphi(h_1) + \varphi(h_2)$, $\varphi(h_1 + h_2) = -\varphi(-h_1 - h_2) \geqslant -(\varphi(-h_1) + \varphi(-h_2)) = \varphi(h_1) + \varphi(h_2)$). Moreover, φ is continuous, because $\varphi(h) \leqslant \varphi_h^+ f(x)$ and f is Lipschitz. □

Důsledek
$f(x) = \|x\|$, $x \in X$. Then $f'_G(x)$ exists $\Leftrightarrow \exists! x^* \in Bx^* : x^*(x) = \|x\|$.

TODO?

TODO?

Důsledek
$X = \mathbb{R}^n$, $U \subset X$ open, $f : U \to \mathbb{R}$ convex, $x \in U$. Then $f'_F(x)$ exists $\Leftrightarrow \forall i \in [n] : \frac{\partial f}{\partial x_i}(x)$ exists.

Důkaz
„ $\implies$ “ ?. „ $\impliedby$ “:

$$x^* \in \partial f(x) \implies x^*(e_i) \leqslant \partial_{e_i}^+ f(x) = \frac{\partial f}{\partial x_i}(x) \wedge x^*(-e_i) \leqslant \partial_{-e_i}^+ f(x) = -\frac{\partial f}{\partial x_i}(x) \implies x^*(e_i) = \frac{\partial f}{\partial x_i}(x) \implies$$

$\implies \partial f(x)$ contains at most one point $\implies f$ contains exactly one point $\implies f'_G(x)$ exists $\implies$ (locally Lipschitz, $\dim \mathbb{R}^n < \infty$) $f'_F(x)$ exists. □

Definice 2.2 (Monotone, upper semi-continuous (usc))

X Banach space, $D \subset X$, $T : D \to 2^{x^*}$ is monotone if $\forall x \in D : Tx \subset X^*$, $Tx \neq \emptyset$ and $\forall x, y \in D\ \forall x^* \in Tx\ \forall y^* \in Ty : \langle x^* - y^*, x - y \rangle \geqslant 0$.

> *Poznámka*
> $f : (a, b) \to \mathbb{R}$ is non-decreasing $\Leftrightarrow \forall x, y \in (a, b) : (f(x) - f(y))(x - y) \geqslant 0$

Let S and T be topological spaces. Then $\varphi : S \to 2^T$ is usc (upper semi-continuous) $\equiv$ $\forall U \subset T$ open: $\{x \in S | \varphi(x) \subset U\}$ is open in S.

> *Poznámka* (This we will not use)
> lsc $\equiv \forall U \subset T$ open $\{x \in S | \varphi(x) \cap U \neq \emptyset\}$ is open.

Tvrzení 2.3

X Banach space, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous. Then $\partial f : U \to 2^{X^*}$ is

a) *monotone;*

b) *locally bounded;*

c) *usc from $\|\cdot\|$ to w^*.*

Důkaz
„a)“: $x, y \in U$, $x^* \in \partial f(x)$, $y^* \in \partial f(y)$. Then $x^*(y-x) \leqslant f(y)-f(x)$, $y^*(x-y) \leqslant f(x)-f(y)$.

$$x^*(y-x) + y^*(x-y) \leqslant 0, \qquad (x^* - y^*)(x-y) \geqslant 0.$$

„b)“: f is locally Lipschitz:

$$x \in U \implies \exists z > 0, L > 0, B(x, z) \subset U,$$

f is L-Lipschitz on $B(x, z) \implies$

$$\forall y \in B(x, z) : \partial f(y) \subset L \cdot B_{X^*}.$$

„c)“ $G \subset X^*$ w^*-open, $x \in U$, $\partial f(x) \subset G$. We want: „$\exists z > 0 : B(x, r) \subset I$ and $\forall y \in B(x, z) : \partial f(y) \subset G$“. It's enough to show „$\forall (x_n) \subset U, x_n \to x, \exists n_0\ \forall n \geqslant n_0 : \partial f(x_n) \subset G$“.

We show it by contradiction: Assume not, i.e., $\exists (y_n) \subset U, y_n \to x$, $\forall n : \partial f(y_n) \neq 0$. Fix $y_n^* \in \partial f(y_n) \backslash G$. By b) we know ? $\implies \exists R > 0 : \forall n : y_n^* \in \overline{B(0, R)}$ (in X^*).

Let y^* be a w^* cluster point of (y_n^*). Thus „$y^* \in \partial f(x)$“: If not, $\exists y \in U : y^*(y-x) > f(y) - f(x) \implies \varepsilon > 0 : y^*(y-x) \geqslant f(y) - f(x) + \varepsilon$, now $y_n^*(y - x + y_n - y_n) \leqslant f(y - x + y_n) - f(y_n)$ ($y - x + y_n \in U$ for large n).

So, for n large enough:

$$y_n^*(y-x) \leqslant f(y - x + y_n) - f(y_n)$$

with $n \to \infty$ LHS has cluster point $y^*(y-x)$ and $RHS \to f(y) - f(x)$

$$\implies y^*(y-x) \leqslant f(y) - f(x). \lightning$$

(But $y^* \in X^* \backslash S \wedge y^* \in \partial f(y) \subset S$.) □

Definice 2.3 (Maximal monotone operator)

X Banach space, $U \subset X$. $T : U \to 2^{X^*}$ is maximal monotone operator, if T is monotone and graph T is maximal within graphs of monotone operators on U.

Definice 2.4

$graphT := \{(x, x^*) \in U \times X^* | x^* \in Tx\}$.

T monotone: $x, y \in U$, $x^* \in Tx$, $y^* \in Ty$, then $\langle x^* - y^*, x - y\rangle \geqslant 0$.

Maximality: $x \in U$, $x^* \in X^*$, $\forall y \in U\ \forall y^* \in Ty : \langle x^* - y^*, x - y\rangle \geqslant 0$, then $x^* \in Tx$.

Lemma 2.4

$U \subset X$ open, $T : U \to 2^{X^*}$ monotone usc $\|\cdot\| \to w^*$, $\forall x \in U : Tx \neq \emptyset$, convex, w^*-closed. Then T is maximal monotone.

Důkaz

$y \in U$, $y^* \in X^*$, $\forall x \in U\ \forall x \in Tx : \langle x^* - y^*, x - y\rangle \geqslant 0$. Assume $y^* \notin Ty \implies$ (by HB) $\exists z \in X : y^*(z) > \sup\langle Ty, z\rangle$.

So $\forall x^* \in Ty$, $x^*(z) < y^*(z) \implies Ty \subset \{z^* \in x^* | z^*(z) < y^*(z)\} =: W$ w^* open.

T usc $\implies \exists r > 0 : B(y, r) \subset U$ and $\forall x \in B(y, r) : Tx \subset W$. In particular: if $t > 0$ is small enough, then $y + t \cdot z \in B(y, z)$ and thus $T(y + t \cdot z) \subset W$.

$w^* \in T(y + t \cdot z) \implies w^*(z) < y^*(z)$ and $t \cdot \langle w^* - y^*, z\rangle = \langle w^* - y^*, y + t \cdot z - y\rangle \geqslant 0$. $\lightning$. □

Definice 2.5

S topological space, X Banach space. $\varphi : S \to 2^{X^*}$ is a minimal convex-valued usc if

- $\forall x \in S : \varphi(x) \neq \emptyset$, convex, w^*-compact;
- φ is usco $S \to w^*$;
- φ is minimal among maps satisfying two first conditions (i.e., φ satisfies two first conditions and $\psi(x) \subset \varphi(x)$ for all $x \in S \implies \psi = \varphi$).

Věta 2.5

$U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous. Then $\partial f : U \to 2^{X^*}$ is a maximal monotone operator and a minimal convex valued usco.

Důkaz

From tvrzeni above: ∂f is monotone operator. Assume $T \subset \partial f$ ($\forall x \in U : T(x) \subset \partial f(x)$) is a convex valued onto. Clearly T is monotone. Then from previous lemma T is maximal operator. □

Tvrzení 2.6

$U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous. Then $f'_F(x)$ exists $\Leftrightarrow \partial f(x)$ is a singleton and $\partial f(x)$ is usc $\|\cdot\| \to \|\cdot\|$ at X ($\forall G \subset X^*$ $\|\cdot\|$-open, $\partial f(x) \subset G$, $\exists r > 0 : B(x, r) \subset U \wedge \forall y \in B(x, r) : \partial f(y) \subset G$).

Důkaz
„ $\Longrightarrow$ “: Assume $f'_F(x) =: x^*$. Then we know the $\partial f(x) = \{x^*\}$. Fix $\varepsilon > 0$. We want $r > 0$ such that $B(x,r) \subset U$ and $\forall y \in B(x,r) : \partial f(x) \subset B(x^*, \varepsilon)$. By contradiction: Assume there is $(x_n) \subset B$, $x_n \to x$, $x_n^* \in \partial f(x_n)$, $\|x_n^* - x^*\| > 2\varepsilon$. Thus $\exists h_n \in X, \|h_n\| = 1 : \langle x_n^* - x^*, h_n \rangle > 2\varepsilon$.

$$x^* = f'_F(x) \implies \exists \delta > 0 : \overline{B(x,\delta)} \subset U \ \forall h \in X, \|h\| \leqslant \delta : f(x+h) - f(x) - x^*(h) \leqslant \varepsilon \|h\|.$$

$$x_n^* \in \partial f(x_n) \implies x_n^*(x + \delta h_n - x_n) \leqslant f(x + \delta h_n) - f(x_n)$$
$$x_n^*(\delta h_n) \leqslant f(x + \delta h_n) - f(x_n) + x_n^*(x_n - x).$$

$$2\varepsilon\delta < \langle x_n^* - x^*, \delta h_n \rangle \leqslant f(x + \delta h_n) - f(x) + f(x) - f(x_n) - x^*(\delta h_n) + x_n^*(x_n - x) \leqslant$$

$$\leqslant \varepsilon\delta + f(x) - f(x_n) + x_n^*(x_n - x) \to \varepsilon\delta \implies 2\varepsilon\delta \leqslant \varepsilon\delta.$$

„ $\Longleftarrow$ “: $\partial f(x) = \{x^*\} \implies$ we know that $x^* = f'_G(x)$. We will show: $x^* = f'_F(x)$. $\varepsilon > 0 \implies \exists \delta > 0 : B(x,\delta) \subset U \wedge \forall y \in B(x,\delta) : \partial f(y) \subset B(x^*, \varepsilon)$. $y \in B(x,\delta)$, $y^* \in \partial f(y)$. Then $x^*(y - x) \leqslant f(y) - f(x)$, $y^*(x - y) \leqslant f(x) - f(y)$. So

$$0 \leqslant f(y) - f(x) - x^*(y-x) \leqslant y^*(y-x) - x^*(y-x) = (y^* - x^*)(y-x) \leqslant \|y^* - x^*\| \cdot \|y - x\| \leqslant \varepsilon \cdot \|y - x\|.$$

So, $x^* = f'_F(x)$. □

Tvrzení 2.7

$U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous. $U_F = \{x \in U | \exists f'_F(x)\}$ is G_δ set and $f'_F : U_F \to X^*$ is continuous $\|\cdot\| \to \|\cdot\|$.

Důkaz

$$U_F = \bigcap_{n \in \mathbb{N}} \left\{ x \in U | \exists V \text{ a neighbourhood of } x : \operatorname{diam} \bigcup \{\partial f(y) | y \in V\} \leqslant \frac{1}{n} \right\}.$$

„$\subset$“: $x \in U_F$, $x^* = f'_F(x)$. From previous tvrzeni $\forall n \in \mathbb{N}\ \exists V$ a neighbourhood of x such that $\forall y \in V : \partial f(y) \subset B(x^*, \frac{1}{2n}) \implies x^* \in RHS$.

„$\supset$“: $x \in RHS$, $n \in \mathbb{N}$, V_n ? $\partial f(x) = 0 \implies \partial f(x) = \{x^*\}$ and $\forall y \in V_n : \partial f(y) \subset \overline{B(x^*, \frac{1}{n})} \implies x \in U_F$. □

Poznámka
Continuity of f'_F on U_F: „$f'_F = \partial f|_{U_F}$“.

Tvrzení 2.8

$U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous, $U_G = \{x \in U | \exists f'_G(x)\}$. Then

- *X separable $\implies$ U is G_δ;*
- *$f'_G : U_G \to X^*$ is continuous $\|\cdot\| \to w^*$.*

Důkaz

$$U_G = \{x \in U | |\partial f(x)| = 1\}.$$

$x \in U_G \implies \partial f(x) = \{f'_G(x)\}$. We know the ∂f is usc $\|\cdot\| \to w^*$. Thus the restriction to U_G is continuous $\|\cdot\| \to w^*$. □

TODO!!!

Tvrzení 2.9

M a topological space (usually Baire), X Banach space.

- *If $\Phi : M \to 2^{X^*}$ is usco (to w^* topology), then $\psi(m) = \overline{\operatorname{conv}}^{w^*} \Phi(m)$, $m \in M$, is also usco.*
- *If $\Psi : M \to 2^{X^*}$ minimal convex-valued usco, $\Phi : M \to 2^{X^*}$ minimal usco, $\Phi \subset \Psi$. Then $\forall m \in M : (|\Phi(m)| = 1 \Leftrightarrow |\Psi(m)| = 1)$.*

Důkaz

Clearly for $m \in M$: $\Phi(m) \neq \emptyset$, $\psi(m)$ convex and w^*-compact.

„ψ is usco“: $U \subset X^*$ w^*-open, $m \in M$, $\psi(m) \subset U \implies \exists V$ w^*-open: $\psi(m) \subset V \subset \overline{V}^{w^*} \subset U$ (by regularity of w^* topology). $x^* \in \psi(m) \implies \exists H$ TODO!!!

„Second item“: „ $\Longleftarrow$ “ obvious. „ $\implies$ “: TODO!!! □

Tvrzení 2.10

X is a Banach space, X^* separable, M a Baire topological space, $\Phi : M \to 2^{X^*}$ minimal convex-valued usco. Then $G := \{m \in M \mid |\Phi(m)| = 1 \wedge \Phi$ is usco to $\|\cdot\|$ at $x\}$ is a dense G_δ subset of M.

Důkaz

$$A_n = \left\{ m \in M | \forall U \text{ neighbourhood of } m : \operatorname{diam} \Phi(U) > \frac{1}{n} \right\}$$

$\implies A_n$ is closed in M and $M \backslash \bigcup_n A_n = G$.

We will check that each A_n is meager (first category). $\{x_k^*, k \in \mathbb{N}\}$ dense in X^*, $A_{n,k} := \left\{ m \in A_n | \operatorname{dist}(\Phi(m), x_k^*) < \frac{1}{8n} \right\} \implies A_n = \bigcup_{k \in \mathbb{N}} A_{n,k}$. We will show that each $A_{n,k}$ is nowhere dense.

Fix n, k, $m \in A_{n,k}$, U neighbourhood of m. Choose $n^* \in \Phi(m)$ with $\|m^* - x_k^*\| < \frac{1}{8n}$. $m \in A_n \implies \exists z_1, z_2 \in U, z_1^* \in \Phi(z_1), z_2^* \in \Phi(z_2), \|z_1^* - z_2^*\| > \frac{1}{2n} - \frac{1}{8n} > \frac{1}{4n} \implies \exists x \in X$, $\|x\| = 1$, $\langle z^* - x_k^*, x \rangle > \frac{1}{4n}$.

Hence $\langle z^*, x \rangle > \langle x_k^*, x \rangle + \frac{1}{4n}$.

U is neighbourhood of z. $\implies \exists v \in U : \forall v^* \in \varphi(V) : \langle v^*, x \rangle > \langle x_k^*, x \rangle + \frac{1}{4n}$.

Assume not, i.e., $\forall v \in U$:

$$\varphi(v) \cap \left\{ y^* | y^*(x) \leqslant x_k^*(x) + \frac{1}{4n} \right\} \neq \varnothing.$$

Define $\tilde{\varphi}(m)' := \varphi(m) \cdot ?$, $m \in U$ and $\varphi(m)$, $m \in M \backslash U$. Assume the proof of P1, $\tilde{\varphi}$ is a convex-valued usco, $\tilde{\varphi} \subset \varphi \implies \tilde{\varphi} = \varphi \implies$

$$\forall v \in U : \varphi(v) \subset \left\{ y^* | y^*(x) \leqslant x_k^*(x) + \frac{1}{4n} \right\}$$

but this fault. ↯.

We get $v \in U$:

$$\varphi(v) \subset \left\{ y^* | y^*(x) > x_k^*(x) + \frac{1}{4n} \right\} \; w^*\text{-open}$$

$\overset{\varphi \text{ is usco}}{\implies} \exists V$, a neighbourhood of v, $V \subset U$,

$$\varphi(V) \subset \left\{ y^* | y^*(x) > x_k^*(x) + \frac{1}{4n} \right\}.$$

Then $V \cap A_{n,k} = \varnothing$. ($w \in V \implies \forall w^* \in \varphi(w) : \|w^* - x_k^*\| \geqslant \langle w^* - x_k^*, x \rangle > \frac{1}{4n} \implies \operatorname{dist}(\varphi(w), x_k^*) \geqslant \frac{1}{4n} > \frac{1}{8n}$.)

So $A_{n,m}$ is nowhere dense, hence A_n are meager. Thus $\bigcup_n A_n$ is meager, hence $M \backslash \bigcup_n A_n$ is dense (G_δ as? are closed). □

Věta 2.11

X a separable Banach space, X is Asplud? $\Leftrightarrow$ X^* is separable.

Důkaz
„ $\Longleftarrow$ “ Proposition above aplied to ∂f.

„ $\Longrightarrow$ “ X separable, X^* not separable. The B_{X*} is nonseparable $\implies \exists M_0 \subset B_{X*}$ uncountable such that $\exists \varepsilon > 0 \ \forall m_1, m_2 \in M_0 \ m_1 \neq m_2, \|m_1 - m_2\| > \varepsilon$.

(B_{X*}, w^*) is compact metrizable $\implies \exists M \subset M_0$ uncountable with w^* isolated points.

$$U = \{TODO!!!\}$$

TODO!!! □