1 Area formula and coarea formula

Věta 1.1

Let (P_1, ϱ_1), (P_2, ϱ_2) be metric spaces, $s > 0$, and $f : P_1 \to P_2$ be β-Lipschitz. Then $\varkappa^s(f(P_1)) \leqslant \beta^s \varkappa^s(P_1)$.

Důkaz
Choose $\delta > 0$. Let $P_1 = \bigcup_{i=1}^{\infty} A_j$, $\operatorname{diam} A_j < \delta$. Then we have $f(P_1) = \bigcup_{j=1}^{\infty} f(A_j)$, $\operatorname{diam} f(A_j) < \beta \cdot \delta$.

$$\varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \sum_{j=1}^{\infty} (\operatorname{diam} f(A_j))^s \leqslant \sum_{j=1}^{\infty} \beta^s \cdot (\operatorname{diam} A_j)^s = \beta^s \cdot \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s.$$

It holds for all possible choices of (A_j), so we can take infimum:

$$\varkappa^s(f(P_1)) \leftarrow \varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \beta^s \inf_{(A_j)} \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s = \beta^s \varkappa^s(P_1, \delta) \to \beta^s \varkappa^s(P_1).$$

□

Lemma 1.2

Let $k, n \in \mathbb{N}$, $k \leqslant n$, and $L : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear mapping. Then for every λ_k-measurable set $A \subset \mathbb{R}^k$ it holds $H^k(L(A)) = \sqrt{\det(L^T L)} \lambda_k(A)$.

Důkaz ($\dim L(\mathbb{R}^k) = k$)
We find linear isometry Q of $\mathbb{R}^k$ onto $L(\mathbb{R}^k)$, from last semester

$$H^k(L(A)) = H^k(Q^{-1} \circ L(A)) = \lambda^k(Q^{-1} \circ L(A)) = |\det(Q^{-1}L)| \cdot \lambda_k(A).$$

$$(\det(Q^{-1}L))^2 = \det((Q^{-1}L)^T) \cdot \det(Q^{-1}L) = \det((Q^{-1}L)^T \cdot (Q^{-1}L)) =$$
$$= \det((\langle Q^{-1}Le^i, Q^{-1}L^T e^j \rangle)_{i,j}).$$

And because Q is isometry ($\implies$ Q^{-1} is isometry), we can remove Q^{-1} from scalar product and we get $\det(L^T L)$. □

Lemma 1.3

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood V of the point x such that

- *the mapping $y \mapsto \varphi(\varphi'(x)^{-1}(y))$ is β-Lipschitz on $\varphi'(x)(V)$;*
- *the mapping $z \mapsto \varphi'(x)(\varphi^{-1}(z))$ is β-Lipschitz on $\varphi(V)$.*

Důkaz
x, β fixed. We know, that there exists $\eta > 0$ such that

$$\forall v \in \mathbb{R}^k : \|\varphi'(x)(v)\| \geqslant \eta \cdot \|v\|.$$

We find $\varepsilon \in \left(0, \frac{1}{2}\eta\right)$ such that $\frac{2\varepsilon}{\eta} + 1 < \beta$. We find a neighbourhood V of x such that $\forall y \in V : \|\varphi'(x) - \varphi'(y)\| \leqslant \varepsilon$.

We show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leqslant \varepsilon\|u - v\|.$$

Fix $v \in V$ and consider the mapping

$$g : w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v).$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$:

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| = \|g(w) - g(v)\| \leqslant \sup\left\{\|g'(w)\| \mid w \in V\right\} \cdot \|u - v\| \leqslant \varepsilon \cdot \|u - v\|.$$

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

For $u - v \in V$ we compute $\|\varphi(u) - \varphi(v)\| \geqslant$

$$\geqslant -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \geqslant -\varepsilon\|u - v\| + \eta\|u - v\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

„First point“: Choose $a, b \in \varphi'(x)(V)$. We find $u, v \in V$ such that $\varphi'(x)(u) = a$, $\varphi'(x)(v) = b$. We compute:

$$\|\varphi(\varphi^{-1}(x)(a)) - \varphi(\varphi^{-1}(x)(b))\| = \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|\varphi'(x)(y - v)\| \leqslant \varepsilon\frac{1}{\eta}\|a - b\| + \|a - b\| = \left(\frac{\varepsilon}{\eta} + 1\right)\|a - b\| \leqslant \beta \cdot \|a - b\|.$$

„Second point“: $k, q \in \varphi(V)$. We find $u, v \in V$ such that $\varphi(u) = p$ and $\varphi(v) = q$:

$$\|\varphi'(x)(\varphi^{-1}(p)) - \varphi'(x)(\varphi^{-1}(q))\| = \|\varphi'(x)(u) - \varphi'(x)(v)\| =$$

$$= \|\varphi'(x)(u - v)\| \leqslant \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|p - q\| \leqslant \frac{2\varepsilon}{\eta}\|\varphi(u) - \varphi(v)\| + \|p - q\| = \left(\frac{2\varepsilon}{\eta} + 1\right)\|p - q\| \leqslant \beta\|p - q\|.$$

□

Lemma 1.4

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\alpha > 1$. Then there exists a neighbourhood V of x such that for every λ^k-measurable $E \subset V$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

Důkaz

Find $\beta > 1$, $\tau > 1$ such that $\beta^k \tau < \alpha$. By previous lemma we find a neighbourhood V_1 of x such that the conclusion of the lemma holds for β. We find a neighbourhood V_2 of x such that

$$\forall t \in V_2 : \tau^{-1} \operatorname{vol} \varphi'(x) \leqslant \operatorname{vol} \varphi'(t) \leqslant \tau \operatorname{vol} \varphi'(x).$$

Set $V = V_1 \cap V_2$.

Assume that $E \subset V$ is a λ^k-measurable set. We have

$$\tau^{-1} \operatorname{vol} \varphi'(x) \cdot \lambda^k(E) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau \operatorname{vol} \varphi'(t) \lambda^k(E).$$

By lemma above we have $\operatorname{vol} \varphi'(t) \lambda^k(E) = H^k(\varphi'(x)(E))$:

$$\tau^{-1} H^k(\varphi'(x)(E)) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau H^k(\varphi'(x)(E)).$$

By previous lemma we get

$$H^k(\varphi(E)) = H^k\left(\left(\varphi \circ (\varphi'(x))^{-1} \circ \varphi'(x)\right)(E)\right) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant$$

$$\leqslant \beta^k \tau \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

By lemma above we get

$$H^k(\varphi(E)) \geqslant \beta^{-k} H^k\left(\left(\varphi'(x) \circ \varphi^{-1} \circ \varphi\right)(E)\right) = \beta^{-k} H^k(\varphi'(x)(E)) \geqslant$$

$$\geqslant \beta^{-k} \tau^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \geqslant \alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

□

Věta 1.5

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping and $f : \varphi(G) \to \mathbb{R}$ be H^k-measurable. Then we have

$$\int_{\varphi(G)} f(x) dH^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t),$$

if the integral at the right side converges.

Důkaz
„φ^{-1} is well defined“: If $H \subset G$ is open, then we can write $H = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact for every $n \in \mathbb{N}$. Then we have $\varphi(H) = \bigcup_{n=1}^{\infty} \underbrace{\varphi(K_n)}_{\text{compact}}$ is F_σ. This implies that φ^{-1} is Borel. The mappings φ, φ^{-1} are locally Lipschitz by lemma above. ($\varphi(G)$ is Borel.) $\varphi(G)$ is H^k-σ-finite.

1. „$f = \chi_L$, $L \subset \varphi(G)$ is H^k-measurable“: We show $H^k(L) = \int_{\varphi^{-1}(L)} \varphi'(t) d\lambda^k(t)$. Choose $\alpha > 1$. By previous lemma we find for every $y \in G$ neighbourhood $V_y \subset G$ of the point y such that for every λ^k-measurable set $E \subset V_y$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

We have $\subset \{V_y | y \in G\} = G$. There exists a sequence $\{y_j\}_{j=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} V_{y_j} = G$. Using lemma from previous semester we find Borel sets $B, N \subset \varphi(G)$ such that $B \subset L \subset B \cup N$, $H^k(N) = 0$.

$\lambda^k(\varphi^{-1}(N)) = 0$. $\varphi^{-1}(B) \subset \varphi^{-1}(L) \subset \varphi^{-1}(B) \cup \varphi^{-1}(N) \implies \varphi^{-1}(L)$ is λ^k-measurable. We set $A_j = \varphi^{-1}(L) \cap \left(V_{y_j} \setminus \bigcup_{i=1}^{j-1} V_{y_j} \right)$. Then we have

- A_j is λ^k-measurable;
- $A_j \subset V_{y_j}$ for every $j \in \mathbb{N}$;
- $\forall j, j' \in \mathbb{N}, j \neq j' : A_j \cap A_{j'} = \emptyset$;
- $\bigcup_{j=1}^{\infty} A_j = \varphi^{-1}(L)$;
- for every $j \in \mathbb{N}$ we have $\alpha^{-1} \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(A_j)) \leqslant \alpha \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t)$.

From all except for second point we have

$$\alpha^{-1} \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(L) \leqslant \underbrace{\sum_{j=1}^{\infty} H^k(\varphi(A_j))}_{=H^k\left(\bigcup_{j=1}^{\infty} \varphi(A_j)\right) = H^k(L)} \leqslant \alpha \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

2. „$f \geqslant 0$ simple H^k-measurable“: From linearity of integrals.

3. „$f \geqslant 0$ H^k-measurable“: we approximate f by $0 \leqslant f_j \leqslant f_{j+1}$ simple functions and from Levi:

$$\lim_{j \to \infty} \int_{\varphi(G)} f_j(x) dH^k(x) = \int_{\varphi(G)} f(x) dH^k(x),$$

$$\lim_{j \to \infty} \int_G f_j(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

3. „f H^k-measurable“: We add positive and negative part. □

Věta 1.6 (Coarea formula)

Let $k, n \in \mathbb{N}$, $k > n$, $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz mapping, $f : \mathbb{R}^k \to \mathbb{R}$ be λ^k-integrable function. Then we have

$$\int_{\mathbb{R}^k} f(x)\sqrt{\det(\varphi'(x) \cdot (\varphi'(x))^T)}d\lambda^k(x) = \int_{\mathbb{R}^n} \int_{\varphi^{-1}(\{y\})} f(x)dH^{k-n}(x)d\lambda^k(y)$$

Věta 1.7

Let $f : \mathbb{R}^k \to \mathbb{R}$ be λ^k-integrable function. Then we have

$$\int_{\mathbb{R}^k} f(x)d\lambda^k(x) = \int_0^\infty \left(\int_{x\in\mathbb{R}^k, \|x\|=z} f(x)dH^{k-1}(x)\right) d\lambda^1(z).$$

Důkaz

By Coarea formula. □

2 Semicontinuous functions

Definice 2.1

Let X be a topological space and $f : X \to \mathbb{R}^*$. We say that f is lower semicontinuous (lsc), if the set $\{x \in X | f(x) > a\}$ is open for every $a \in \mathbb{R}$. We say that f is upper semicontinuous (usc) if the set $\{x \in X | f(x) < a\}$ is open for every $a \in \mathbb{R}$.

Tvrzení 2.1 (Fact)

$f : \mathbb{R} \to \mathbb{R}$:

$$f \text{ is lsc } \Leftrightarrow \forall x \in \mathbb{R} : \liminf_{t\to x} f(t) \geqslant f(x).$$

Věta 2.2

Let X be a metrizable topological space and $f : X \to \mathbb{R}^*$ be a function bounded from below. Then f is lsc if and only if there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbb{R}$ such that $f_0 \leqslant f_1 \leqslant \ldots$ and $f_n \to f$.

Důkaz

„ $\Longleftarrow$ “: Choose $a \in \mathbb{R}$. Assume that $f(x_0) > a$. There exists $k \in \mathbb{N}$ such that $f_k(x_0) > a$. Then there is an open set $G \subset X$ such that $x_0 \in G$ and $f_k|_G > a$. Thus we have $f|_G \geqslant f_k|_G > a$. So $\{x \in X | f(x) > a\}$ is open.

„ $\Longrightarrow$ “ The case „$f \equiv \infty$“: Then we consider $f_n \equiv n$. The case „$f \not\equiv \infty$“. Fix a compatible metric ϱ on X. We set $f_n(x) = \inf\{f(y) + n \cdot \varrho(x,y) | y \in X\}$. Then we have $f_n : X \to \mathbb{R}$ and $f_0 \leqslant f_1 \leqslant \ldots$. We have

$$|f_n(x) - f_n(z)| \leqslant n \cdot \varrho(x,z) \impliedby f_n(x) - f_n(z) \leqslant$$

$$\leqslant f(y) + n \cdot \varrho(x,y) - (f(y) + n \cdot \varrho(y,z)) + \varepsilon = n(\varrho(x,y) - \varrho(y,z)) + \varepsilon \leqslant n \cdot \varrho(x,z) + \varepsilon.$$

So f_n is continuous.

„$f_n \to f$“: There exists $K \in \mathbb{R}$ such that $f(x) \geqslant K$ for every $x \in X$. Fix $x \in X$. Choose $\varepsilon > 0$. For every $n \in \mathbb{N}$ we find $y_n \in X$ such that $f(y_n) \leqslant f(y_n) + n \cdot \varrho(x, y_n) \leqslant f_n(x) + \varepsilon$. Then we have

$$\varrho(x, y_n) \leqslant \frac{1}{n}\left(f_n(x) + \varepsilon - f(y_n)\right) \leqslant \frac{1}{n}\left(f_n(x) + \varepsilon - K\right).$$

$f_n(x) \to \infty \implies f(x) = \infty$, since $f_n(x) \leqslant f(x)$. $f_n(x)$ is bounded $\implies y_n \to x$, so we can find $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0 : f(y_n) > f(x) - \varepsilon$. Then we have $f(x) < f(y_n) + \varepsilon \leqslant f_n(x) + 2\varepsilon$, $\lim f_n(x) \leqslant f(x) \leqslant \lim f_n(x) + 2\varepsilon$, thus $\lim f_n(x) = f(x)$. □

3 Function of Baire class 1

Definice 3.1

Let X and Y be metrizable topological spaces, a function $f : X \to Y$ is of Baire class 1 (B_1-function) if for every open set $U \subset Y$ the set $f^{-1}(U)$ is F_σ.

Věta 3.1 (Lebesgue–Hausdorff–Banach)

Let X be a metrizable topological space and $f : X \to \mathbb{R}$ be a B_1–function. Then there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbb{R}$ with $f_n \to f$.

Lemma 3.2

Let X be a metrizable topological space and $A \subset X$ be G_δ and F_σ. Then χ_A is point-wise limit of a sequence of continuous functions.

Důkaz

$A = \bigcup_{n\in\mathbb{N}} F_n$, $X \backslash A = \bigcup_{n\in\mathbb{N}} H_n$, $F_n \subseteq F_{n+1}$, $H_n \subseteq H_{n+1}$, F_n, H_n closed. By Urysohn lemma there exists continuous function $f_n : X \to [0,1]$ such that $f_n|_{H_n} = 0$ and $f_n|_{F_n} = 1$. Then $f_n(x) \to f(x)$. □

Lemma 3.3

Let X be a metrizable topological space, $p_n : X \to \mathbb{R}$, $n \in \omega$, be a point-wise limit of a sequence of continuous functions. If the sequence $\{p_n\}$ converges uniformly to p, then p is point-wise limit of continuous functions.

Důkaz

Claim: If $q_n : X \to \mathbb{R}$, $n \in \omega$, is point-wise limit of continuous functions, $\|q_n\|_\infty \leqslant 2^{-n}$, then $\sum_{n=0}^\infty q_n$ is a point-wise limit of continuous functions.

Corollary: One can assume $\|p - p_n\|_\infty \leqslant 2^{-(n+1)}$. $p = p_0 + \sum_{n=0}^\infty (p_{n+1} - p_n)$

$$\|p_{n+1} - p_n\|_\infty \leqslant \|p_{n+1} - p\| + \|p - p_n\| < 2^{-(n+2)} + 2^{-(n+1)} < 2^{-n}.$$

Proof of claim: For every $n \in \omega$, there exists a sequence of continuous functions $\{q_i^n\}_{i=0}^\infty$ such that $q_i^n \to q_n$ and moreover we may assume $\|q_i^n\|_\infty \leqslant 2^{-n}$. We set $r_i = \sum_{n=0}^\infty q_i^n$. The sum converges uniformly, so r_i is continuous for every $i \in \omega$.

Set $x \in X$ and $\varepsilon > 0$. We find $N \in \omega$ such that

$$\left|\sum_{n=N+1}^\infty q_i^n(x)\right| < \frac{1}{2}\varepsilon, \left|\sum_{n=N+1}^\infty q_n(x)\right| < \frac{1}{2}\varepsilon.$$

Then we have

$$\left|r_i(x) - \sum_{n=0}^\infty q_n(x)\right| = \left|\sum_{n=0}^\infty q_i^n(x) - \sum_{n=0}^\infty q_n(x)\right| \leqslant$$

$$\leqslant \left|\sum_{i=0}^N q_i^n(x) - q_n(x)\right| + \left|\sum_{n=N+1}^\infty q_i^n(x) - \sum_{n=N+1}^\infty q_n(x)\right| \leqslant \left|\sum_{n=0}^N (q_i^n(x)) - q_n(x)\right| + \varepsilon.$$

$$\limsup_{i\to\infty} |r_i(x) \sum_{n=0}^\infty q_n(x)| \leqslant \varepsilon \implies r_i(x) \to \sum_{n=0}^\infty q_n(x).$$

□

Lemma 3.4 (Reduction theorem for F_σ sets)

Let X be a metrizable topological space, $A_n \subset X$ be an F_σ set for every $n \in \omega$. Then there are F_σ sets $A_n^* \subset A_n$, such that $A_n^* \cap A_m^* = \varnothing$, whenever $n, m \in \omega$, $n \neq m$, and $\bigcup_{n=0}^\infty A_n = \bigcup_{n=0}^\infty A_n^*$.

Důkaz

$A_n = \bigcup_{j=0}^\infty A_{n,j}$, $A_{n,j}$ is closed. $k \mapsto (k', k'')$ bijection of ω onto $\omega \times \omega$.

$$Q_k = A_{(k)_0,(k)_1} \setminus \bigcup_{l<k} A_{(l)_0,(l)_1}.$$

$(Q_k)_{k\in\omega}$ is sequence of F_σ sets, which is disjoint. $A_n^* := \bigcup \{Q_k | (k)_0 = n\} \subseteq A_n$ is F_σ set, $A_n^* \cap A_m^* = \varnothing$ if $n \neq m$ and $\bigcup_{n=0}^\infty A_n^* = \bigcup_{k=0}^\infty Q_k = \bigcup_{n=0}^\infty A_n$. □

Důkaz (Of Lebesgue–Hausdorff–Banach theorem)
It is sufficient to prove result for $g : X \to (0,1)$. Because if $f \in B_1$, then we set $g = k \circ f$ where $k : \mathbb{R} \to (0,1)$ is homeomorphism. We find $g_n : X \to \mathbb{R}$, continuous and $g_n \to g$. $\tilde{g}_n := \min\left\{\max\left\{\frac{1}{n}, g_n\right\}, 1 - \frac{1}{n}\right\}$. $\tilde{g}_n(X) \subset (\frac{1}{n}, 1 - \frac{1}{n})$.

Let $g : X \to (0,1)$ be B_1. For $N \in \omega$, $N \geqslant 2$, and $i \in [N-2]$ we set

$$A_i^N := g^{-1}\left(\frac{i}{N}, \frac{i+2}{N}\right) \ldots F_\sigma, \qquad \bigcup_{i=0}^{N-2} A_i^N = X.$$

$B_i^N \subset A_i^N$ such that $\bigcup_{i=0}^{N-2} B_i^N = X$, B_i^N is F_σ and G_δ (why?) and $B_i^N \cap B_{i'}^N = \varnothing$, whenever $i \neq i'$. $g_N(x) := \sum_{i=0}^{N-2} \frac{1}{N}\chi_{B_i^n}(x)$. $g_N \rightrightarrows g$ ($\|g - g_N\|_\infty \leqslant \frac{2}{N}$). □

Věta 3.5 (Baire)

Let X be a metrizable topological space, Y be separable metrizable topological space, and $f : X \to Y$ be B_1-function. Then the set of points of continuity of f is G_δ and residual.

Důkaz
$\{V_n\}$ open countable basis of Y. f isn't continuous at $x \Leftrightarrow \exists n \in \omega : x \in f^{-1}(V_n) \setminus \operatorname{int} f^{-1}(V_n)$.
$D(f) = \{x \in X | f \text{ is not continuous at } x\} = \bigcup_{n\in\omega} \underbrace{(f^{-1}(V_n) \setminus \operatorname{int} f^{-1}(V_n))}_{\in F_\sigma}$.

$B = (f^{-1}(V_n) \setminus \operatorname{int} f^{-1}(V_n)) = \bigcup_{k\in\omega} F_{n,k}$ is closed and $\operatorname{int} F_{n,k} = \varnothing$, so $F_{n,k}$ is nowhere dense. So B is meager. And complement of meager is residual. □

Lemma 3.6

Let X be a Polish space, $A, B \subseteq X$ disjoint. If A cannot be separated from B by a set of type Δ_2^0, then there is non-empty closed set $F \subseteq X$ such that $A \cap F$ and $B \cap F$ are dense in F. (Opposite implication is also true.)

Důkaz
Let $\{B_n\}_{n\in I}$ be an open basis of X. We set $F_0 := X$,

$$F_{\alpha+1} := \overline{F_\alpha \cap A} \cap \overline{F_\alpha \cap B}, \qquad \alpha < \omega_1,$$

$$F_\eta := \bigcap_{\alpha<\eta} F_\alpha, \qquad \eta \text{ limit ordinal}, \eta < \omega_1.$$

The sequence $\{F_\alpha\}_{\alpha<\omega_1}$ is a non-increasing ($F_{\alpha+1} \subseteq \overline{F_\alpha} = F_\alpha$) sequence of closed sets.

„Observation 1: There exists $\eta < \omega_1$ such that $F_\eta = F_{\eta+1}$“: We will proceed by contradiction. Then for every $\alpha < \omega_1$ there exists $u(\alpha) \in \omega$ such that

$$B_{u(\alpha)} \cap F_\alpha \neq \varnothing \qquad \wedge \qquad B_{u(\alpha)} \cap F_{\alpha+1} = \varnothing.$$

Assume that $\alpha < \alpha' < \omega$, then we have $\varnothing = B_{u(\alpha)} \cap F_{\alpha+1} \supseteq B_{u(\alpha)} \cap F_{\alpha'}$ (monotonicity). Thus we have $B_{u(\alpha)} \cap F_{\alpha'} = \varnothing$ and $B_{u(\alpha')} \cap F_{\alpha'} \neq \varnothing$, so $u(\alpha) \neq u(\alpha')$. Thus $\omega_1 \to \omega$ is injective. $\lightning(\omega < \omega_1)$.

„Observation 2: We have this η from observation 1 and we want to show $F_\eta \neq \varnothing$“: Assume that (towards contradiction) $F_\eta = \varnothing$. Then we can write that $X = \bigcup_{\alpha<\eta}(F_\alpha \setminus F_{\alpha+1})$, $X \setminus F_\eta = X$. Then we have $A \subseteq \bigcup_{\alpha<\eta}(\overline{F_\alpha \cap A} \setminus F_{\alpha+1}) =: C$, $B \cap C = \varnothing$, $\implies$ C separates A from B and $C \in \Delta^0_2$.

Suppose that $x \in A$. $\implies$ $\exists\alpha < \eta$: $x \in F_\alpha \setminus F_{\alpha+1}$. $\implies$ $x \in \overline{F_\alpha \cap A}$ and $x \notin F_{\alpha+1} = \overline{F_\alpha \cap A} \cap \overline{F_\alpha \cap B}$ $\implies$ $x \notin \overline{F_\alpha \cap B}$. $\implies$ $x \in \overline{F_\alpha \cap A} \setminus F_{\alpha+1} \subseteq C$. $\implies$ $A \subseteq C$.

„$C \cap B = \varnothing$“: $x \in B$ $\implies$ $\exists\alpha < \eta : x \in F_\alpha \setminus F_{\alpha+1}, x \in \overline{F_\alpha \cap B}$. $x \notin \overline{F_\alpha \cap A}$. $\implies$ $x \notin \overline{F_\alpha \cap A} \setminus F_{\alpha+1}$ and $x \in F_\alpha \setminus F_{\alpha+1}$ $\implies$ $x \notin C$. $\implies$ $B \cap C = \varnothing$.

„$C \in \Sigma^0_2$“: $\overline{F_\alpha \cap A} \setminus F_{\alpha+1} \in \Sigma^0_2$ (the diference of two closed sets). $\implies$ $C \in \Sigma^0_2$ (countable union).

„$C \in \Pi^0_2$“: $G_\alpha := \left(\overline{F_\alpha \cap A} \setminus F_{\alpha+1}\right) \cup \left(F_\alpha^C \cup F_{\alpha+1}\right)$, $\alpha < \eta$, is G_δ set. $C = \bigcap_{\alpha<\eta} G_\alpha$ (countable intersection of G_δ sets $= \Pi^0_2$).

$F_\eta \neq \varnothing$ $\implies$ contradiction (C separates A from B). So F_η is non-empty closed set. We set $F := F_\eta = F_{\eta+1} = \overline{F_\eta \cap A} \cap \overline{F_\eta \cap B}$ $\implies$ $A \cap F_\eta$ and $B \cap F_\eta$ are dense in F. $\square$

Věta 3.7 (Baire)

Let X be a Polish space, Y be a separable metrizable topological space, and $f : X \to Y$. Then $f \in B_1(X, Y) \Leftrightarrow f|_F$ has a point of continuity for every closed non-empty set $F \subseteq X$.

Důkaz
F is also a Polish space (a closed subset of Polish space).

„1) $\implies$ 2)“: $f|_F : F \to Y$, $\mathcal{C}(f|_F)$ is G_δ and residual, especially non-empty.

„2) $\implies$ 1)“: Assume that $U \subseteq Y$ is open. We want to show that $f^{-1}(U) \in F|_G$. $U = \bigcup_{n=1}^{\infty} F_n$, F_n closed (metrizable) in Y. $f^{-1}(U) = \bigcup_{n=1}^{\infty} f^{-1}(F_n)$. Consider n fixed: $f^{-1}(Y \setminus U)$ and $f^{-1}(F_n)$.

It is sufficient to show that $\exists D_n \in \Delta_2^0$ which separates $f^{-1}(F_n)$ from $f^{-1}(Y\setminus U)$ such that:

$$f^{-1}(F_n) \subseteq D_n, \qquad D_n \cap f^{-1}(Y\setminus U) = \varnothing, \qquad D_n \in \Delta_2^0$$
$$\implies \bigcup_{n=1}^{\infty} D_n \in F_G$$
$$f^{-1}(U) = \bigcup_{n=1}^{\infty} D_n, \qquad (D_n \cap f^{-1}(Y\setminus U) = \varnothing)$$
$$\implies \mathcal{F}_G \to f \in B_1(X, Y).$$

Suppose towards contradiction that there is $n \in \mathbb{N}$: $f^{-1}(F_{n_0})$ cannot be separated from $f^{-1}(Y\setminus U)$ by a Δ_2^0 set. By the previous lemma, we get that $\exists F \subseteq X$ closed (non-empty) such that $F \cap f^{-1}(F_n)$ and $F \cap f^{-1}(Y\setminus U)$ are dense in F.

$$2 \implies \exists x_0 \in \mathcal{C}(f|_F),$$
$$\exists a_n : a_n \to x_0, a_n \in f^{-1}(F_{n_0}) \cap F, \qquad \exists b_n : b_n \to x_0, b_n \in f^{-1}(Y\setminus U) \cap F.$$

Point of continuity $\implies$ $f(a_n) \to f(x_0)$ and $f(b_n) \to f(x_0)$. So $f(a_n) \in F_{n_0}$ are closed $\implies$ $f(x_0) \in F_{n_0} \subseteq U$ and $f(b_n) \in Y\setminus U$ is open $\implies$ $f(x_0) \in Y\setminus U$. So $f(x_0) \in F_{n_0} \cap Y\setminus U = \varnothing$. $\lightning$. □

4 Density topology, approximative continuity, differentiability

Definice 4.1

Let $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$, $L \in \mathbb{R}$. We say that f has at the point a an approximate limit L, if we have:

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \forall B \in \mathcal{B}, a \subset B, \operatorname{diam} B < \delta : \lambda_n^* \{x \in B | |f(x) - L| \geqslant \varepsilon\} < \varepsilon \lambda_n(B).$$

$\mathcal{B}$ are closed balls.

Poznámka

If $\lim_{y\to x} f(y) = L$, then $\{x \in B \| f(x) - L| \geqslant \varepsilon\}$ has at most one point (for sufficiently small δ).

Věta 4.1

Let $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$. Then f has at most one approximate limit at a.

Důkaz

Assume that $L, L' \in \mathbb{R}$ are approximate limits of f at a, $L \neq L'$. Choose $\varepsilon > 0$ such that $3\varepsilon = |L - L'|$. Then $\exists \delta > 0\ \forall B \in \mathcal{B}$, $a \in B$, $\operatorname{diam} B < \delta$:

$$\lambda_1^* \{x \in B \| f(x) - L| \geqslant \varepsilon\} < \varepsilon\lambda_1(B) \wedge \lambda_1^* \{x \in B \| f(x) - L'| \geqslant \varepsilon\} < \varepsilon\lambda_1(B).$$

WLOG $\varepsilon < \frac{1}{2}$, $\frac{\lambda_1^*\{x \in B \| f(x) - L| \geqslant \varepsilon\}}{\lambda_1(B)} < \frac{1}{2}$ and $\frac{\lambda_1^*\{x \in B \| f(x) - L'| \geqslant \varepsilon\}}{\lambda_1(B)} < \frac{1}{2}$.

Choose $B \in \mathcal{B}$, $a \in B$, $\operatorname{diam} B < \delta$:

$$B \subseteq \underbrace{\{x \in B \| f(x) - L| \geqslant \varepsilon\}}_{=:C_1} \cup \underbrace{\{x \in B \| f(x) - L'| \geqslant \varepsilon\}}_{=:C_2}.$$

Because $y \in B \implies |f(x) - L| \geqslant \varepsilon \vee |f(x) - L'| \geqslant 2\varepsilon$, it is $B = C_1 \cup C_2$.

$$\lambda_1(B) \leqslant \lambda_1^*(C_1 \cup C_2) \leqslant \lambda_1^*(C_1) + \lambda_1^*(C_2) \leqslant$$

$$\leqslant \frac{1}{2}\lambda_1(B) + \frac{1}{2}\lambda_1(B) = \lambda_1(B) \implies \lambda_1(B) < \lambda_1(B). \lightning.$$

□

Definice 4.2 (Notation)

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. Then approximate limit of f at $a \in \mathbb{R}$ is denoted by $\operatorname{ap-lim}_{x\to a} f(x)$.

Definice 4.3

A function from $\mathbb{R}$ to $\mathbb{R}$ is approximately continuous at $a \in \mathbb{R}$ if $\operatorname{ap-lim}_{x\to a} f(x) = f(a)$.

Definice 4.4

We say that a λ^*-measurable set $A \subset \mathbb{R}$ is d-open, if every point $x \in A$ is a point of density A.

Například

Every open set is d-open.

Věta 4.2

The system of d-open sets in $\mathbb{R}$ forms a topology on $\mathbb{R}$.

Důkaz

We denote $\tau_d := \{E \subseteq \mathbb{R} | E \text{ is } d\text{-open}\}$. Clearly $\emptyset, \mathbb{R} \in \tau_d$ (interior point is a point of density).

„$G_1, G_2 \in \tau_d \implies G_1 \cap G_2 \in \tau_d$“: The set $G_1 \cap G_2$ is λ measurable. Assume that $x \in G_1 \cap G_2 \implies$ „x is point of density of $G_1 \cap G_2$“: choose $\varepsilon > 0$, we find $\delta > 0$ such that (x is a point of density of G_1 and G_2)

$$\forall B \in \mathcal{B}, x \in B, \operatorname{diam} B < \delta : \frac{\lambda(B \cap G_1)}{\lambda(B)} > 1 - \varepsilon \wedge \frac{\lambda(B \cap G_2)}{\lambda(B)} > 1 - \varepsilon.$$

Take $B \in \mathcal{B}$ with $x \in B$ and $\operatorname{diam} B < \delta$. Since $B \subseteq (B \cap G_1 \cap G_2) \cup (B \backslash G_1) \cup (B \backslash G_2)$, we get $\lambda(B) \leqslant \lambda(B \cap G_1 \cap G_2) + \lambda(B \backslash G_1) + \lambda(B \backslash G_2)$. We have

$$\frac{\lambda(B \cap G_1 \cap G_2)}{\lambda(B)} \geqslant \frac{\lambda(B) - \lambda(B \backslash G_1) - \lambda(B \backslash G_2)}{\lambda(B)} = 1 - \frac{\lambda(B \backslash G_1)}{\lambda(B)} - \frac{\lambda(B \backslash G_2)}{\lambda(B)} > 1 - 2\varepsilon.$$

$$d_\lambda(x, G_1 \cap G_2) = 1 \implies G_1 \cap G_2 \in \tau_d.$$

„$\mathcal{A} \subseteq \tau_d \implies \bigcup \mathcal{A} \in \tau_d$“: Denote $T := \bigcup \mathcal{A}$.

„T is measurable“: WLOG T is bounded, since otherwise we consider $T \cap U$, where U is any open ball (density is local notion $\implies$ T is measurable).

Denote $\mathcal{S} := \{\bigcup \mathcal{A}_0 | \mathcal{A}_0 \subset \mathcal{A} \text{ is countable}\}$. Then there exists $S \in \mathcal{S}$ such that $\lambda(S) = \sup\{\lambda(M) | M \in \mathcal{S}\}$. T is bounded $\implies$ $\sup\{\lambda(M) | M \in \mathcal{S}\} < \infty$. Using definition of supremum we get $\{M_i\}_{i=1}^\infty \in \mathcal{S}^\mathbb{N}$: $\lambda(M_i) \to \sup$. Then $\bigcup_{i=1}^\infty M_i =: S$, then $\lambda(S) = \lim_{i \to \infty} \lambda(M_i) = \sup$.

Assume $x \in T \implies$ there exists $A \in \mathcal{A}$: $x \in A$ is d-open. We have $d_\lambda(x, A) = 1$. By the choice of S we have $\lambda(S) = \lambda(S \cup A)$. Since $S \subseteq T$, T bounded: $\lambda(S) < \infty$. Then we have $0 = \lambda(A \backslash S) = \lambda(A \cup S) - \lambda(S) = \lambda(S) - \lambda(S)$. This implies $d_\lambda(x, S) = 1$, since $\lambda(S \cap B) \geqslant \lambda(A \cap B)$ and $d_\lambda(x, A) = 1$.

$$\lambda(S \cap B) = \lambda(S \cap B \cap A) + \lambda(S \cap B \backslash A) = \lambda(S \cap B \cap A) + 0.$$

$$\lambda(A \cap B) = \lambda(A \cap B \cap S) + \lambda((A \cap B) \backslash S) = \lambda(A \cap B \cap S) + 0.$$

This implies $\lambda(T \backslash S) = 0$ by Lebesgue density theorem. $\forall x \in T$: x is a point of density of S. We can write $T = (T \backslash S) \cup S$, which is countable union of measurable sets. $\implies$ T is measurable.

„$T \in \tau_d$“: Take $y \in T \implies \exists A \in \mathcal{A} : y \in A$, $d_\lambda(y, A) = 1$, $A \subseteq T \implies d_{\lambda^*}(y, T) = 1$. T is an d-open set.

So τ_d forms a topology. □

Poznámka (Properties of τ_d)
$\tau_e \subseteq \tau_d$. τ_d is not metrizable. $K \subset \mathbb{R}$ is τ_d-compact $\Leftrightarrow$ K is finite. Baire theorem holds in $(\mathbb{R}, \tau_d)$.

Věta 4.3

The topology τ_d is completely regular, i.e., if $F \subseteq \mathbb{R}$ is closed with respect to τ_d and $x_0 \notin F$, then $\exists \tau_d$-continuous function $f : \mathbb{R} \to [0,1]$ such that $f(x_0) = 0$ and $f(F) \subseteq \{1\}$.

Lemma 4.4

Let $E \subseteq \mathbb{R}$ be measurable, $X \subseteq E$ be τ_d-closed and $d(x, E) = 1$, $\forall x \in X$. Then there exists closed $P \subseteq \mathbb{R} : X \subseteq P \subseteq E$, $\forall x \in X : d(x, P) = 1$, $\forall p \in P : d(p, E) = 1$.

Důkaz
Denote $\tilde{E} := \{x \in E | d(x, E) = 1\}$. By Lebesgue density theorem $\lambda(E \backslash \tilde{E}) = 0$, $X \subseteq \tilde{E}$ and $d(x, \tilde{E}) = 1$ for every $x \in X$. We denote $R_j := \left\{x \in \tilde{E} \middle| 2^{-j} < \operatorname{dist}(x, X) \leqslant 2^{-j+1}\right\}$, $j \in \mathbb{N}$. Then we have $X \cup \bigcup_{j=1}^{\infty} R_j = \left\{x \in \tilde{E} \middle| \operatorname{dist}(x, X) \leqslant 1\right\}$.

Then we find ($\forall j \in \mathbb{N}$) a closed set $P_j \subseteq R_j$ with $\lambda(R_j \backslash P_j) < 4^{-j}$ (regularity of λ measure). We set $P := X \cup \bigcup_{j=1}^{\infty} P_j$ (using limits). P is closed, $X \subseteq P \subseteq \tilde{E} \subseteq E \implies$

$$\implies \forall x \in X : d(x, P) = 1.$$

Assume that, choose $\varepsilon > 0$. We find $\delta > 0$ such that $\forall B \in \mathcal{B}$, $x \in B$, $\operatorname{diam} B < \delta$: $\frac{\lambda(B \cap E)}{\lambda(B)} > 1 - \varepsilon$ and there is $j_0 \in \mathbb{N} : \delta < 2^{-j_0+1} < \varepsilon$.

Choose $B \in \mathcal{B}$, $x \in B$ and $\eta := \operatorname{diam} B < \delta$. We find $j_1 \in \mathbb{N}$: $2^{-j_1} < \eta \leqslant 2^{-j_1+1} \implies j_1 \geqslant j_0$. Then we have: $B \cap P \subseteq X \cup \bigcup_{j=j_1}^{\infty} P_j$. Further we have:

$$\lambda(B \cap (E \backslash P)) \leqslant \lambda(\bigcup_{j=j_1}^{\infty} (R_j \backslash P_j)) \leqslant \sum_{j=j_1}^{\infty} \lambda(R_j \backslash P_j) \leqslant \sum_{j=j_1}^{\infty} 4^{-j} = 4^{-j_1} \cdot \frac{4}{3} = \frac{1}{3} \cdot 4^{-j_0+1}.$$

We compute $\frac{\lambda(B \cap P)}{\lambda(B)} = \frac{\lambda(B \cap E) - \lambda(B \cap (E \backslash P))}{\lambda(B)} \geqslant 1 - \varepsilon - \frac{\lambda(B \cap (E \backslash P))}{\lambda(B)} \geqslant$

$$\geqslant 1 - \varepsilon - \frac{\frac{1}{3} 4^{-j_1+1}}{\lambda(B)} \geqslant 1 - \varepsilon - \frac{1}{3} 4^{-j_1+1} \cdot 2^{j_1-1} = 1 - \varepsilon - \frac{1}{3} 2^{-j_1+1} > 1 - 2\varepsilon \implies d_\lambda(x, P) = 1.$$

It remains to verify the last property:

$$P \subseteq \tilde{E} \implies d_\lambda(x, E) = 1 \text{ for each } x \in P.$$

$\square$

Důkaz (The previous theorem)
Let $F \subseteq \mathbb{R}$ be d-closed, „$x_0 \notin F \implies \exists\tau_d$-continuous $f : \mathbb{R} \to [0,1]$ such that $f(F) \subseteq \{0\} \wedge f(x_0) = 1$“: We find a set $E \in \mathcal{F}_G(\mathbb{R})$ such that $x_0 \in E$, $E \cap F = \emptyset$: $\lambda((R\backslash F)\backslash E) = 0$. F is τ_d-closed, hence measurable, F^c is τ_d-open, hence measurable.

$$\exists E \in \mathcal{F}_G : \lambda((R\backslash F)\backslash E) = 0$$

if necessary $E := E \cup \{x_0\}$.

$E = \bigcup_{n=1}^{\infty} F_n$, F_n closed, $n \in \mathbb{N}$. We may assume that $x_0 \in F_1$. Then we have $F_1 \subseteq E$, $\forall x \in F_1 : d_\lambda(x, E) = 1$. $x \in X : d_\lambda(x, F_1) = 1$. We set $\Phi(1) := F_1$. Now assume that we have already constructed $\Phi(1) \subseteq \Phi(2) \subseteq \ldots \subseteq \Phi(k) \subseteq F$ closed sets and $F_j \subseteq \Phi(j)$, $j \leqslant k$.

$$\forall j < k \ \forall x \in \Phi(j) : d(x, \Phi(j_1)) = 1, \qquad \forall x \in \Phi(k) : d(x, E) = 1.$$

$(k+1)$-th term: We use the previous lemma and find a set P such that $\Phi(k) \subseteq P \subseteq E$, $\forall x \in \Phi(k) : d_\lambda(x, P) = 1$, $\forall x \in P : d_\lambda(x, E) = 1$.

$\Phi(k+1) := P \cup F_{k+1}$ closed, then $F_{k+1} \subseteq \Phi(k+1) \subseteq E$. $j = k : \forall x \in \Phi(k) : d(x, \Phi(k)) = 1$. F^c is d-open.

We have
$$\bigcup_{k=1}^{\infty} \Phi(k) = E \qquad (F_j \subseteq \Phi(j) \subseteq E).$$
Now we define $\Phi\left(\frac{n}{2^m}\right)$, $n \in \mathbb{N}$, $n \geqslant 2^m$, $m \in \mathbb{N}_0$, $n/2^m \geqslant 1$: If $m = 0$ we have already constructed $\Phi(k)$.

„$m \mapsto m+1$“: $\Phi\left(\frac{2n}{2^{m+1}}\right) := \Phi\left(\frac{n}{2^m}\right)$ (numerator even) and $\Phi\left(\frac{2n+1}{2^{m+1}}\right)$ (numerator odd) is constructed so that

- $\Phi\left(\frac{n}{2^m}\right) \subseteq \Phi\left(\frac{2n+1}{2^{m+1}}\right) \subseteq \Phi\left(\frac{n+1}{2^m}\right)$;
- $\forall x \in \Phi\left(\frac{n}{2^m}\right) : d_\lambda\left(x, \Phi\left(\frac{2n+1}{2^{m+1}}\right)\right) = 1$;
- $\forall \Phi\left(\frac{2n+1}{2^{m+1}}\right) : d_\lambda\left(x, \Phi\left(\frac{n+1}{2^m}\right)\right) = 1$.

For $\lambda \in [1, +\infty)$ we set $\Phi(\lambda) = \bigcup_{\frac{n}{2^m} \geqslant \lambda} \Phi\left(\frac{n}{2^m}\right)$ closed, compatible with previous definition.

For $1 \leqslant \lambda_1 < \lambda_2$, we have $\Phi(\lambda_1) \subseteq \Phi(\lambda_2)$, if λ_1, λ_2 is dyadic numbers, by definition, if $\lambda_1 < \frac{n}{2^m} < \lambda_2$, then $\Phi(\lambda_1) \subseteq \Phi\left(\frac{n}{2^m}\right) \subseteq \Phi(\lambda_2)$.

For $1 \leqslant \lambda_1 < \lambda_2$, we have $\forall x \in \Phi(\lambda_1) : d_\lambda(x, \Phi(\lambda 2)) = 1$. We find n, m such that

$$\lambda_1 < \frac{2n}{2^{m+1}} < \frac{2n+1}{2^{m+1}} < \lambda_2.$$

Pick $x \in \Phi(\lambda_1) \subseteq \Phi\left(\frac{2n}{2^{m+1}}\right) \subseteq \Phi(\lambda_2) \implies d_\lambda\left(x, \Phi\left(\frac{2n+1}{2^m}\right)\right) = 1 \implies d_\lambda(x, \Phi(x_2)) = 1.$

We define $f(x) = \frac{\chi_E(x)}{\inf\{\lambda | x \in \Phi(\lambda)\}}$.

$\forall x \in F : f(x) = 0$ ($E \cap F = \varnothing \implies F \subseteq (\mathbb{R} \backslash E)$). So $f(F) \subseteq \{0\}$.

$f(x_0) = \frac{\chi_E(x_0)}{\inf\{\lambda | x_0 \in \Phi(\lambda)\}} = \frac{1}{1} = 1$. ($x_0 \in F_1 \subseteq \Phi(1)$.) Also $\operatorname{Im} f \subseteq [0,1]$.

„Continuity of f with respect to τ_d“: Assume that $b \in (0,1)$ (otherwise obvious), $a \in (0,1]$.

„$B := \{x \in \mathbb{R} | f(x) > b\}$ is d-open“: $f(x) > b > 0 \implies \frac{1}{b} > \inf\{\lambda | x \in \Phi(\lambda)\}$. We find $\lambda \geqslant 1$ such that $\frac{1}{b} > \lambda$ and $x \in \Phi(\lambda)$. We find $\lambda' \geqslant 1$ with $\frac{1}{b} > \lambda' > \lambda$. Then we have $d(x, \Phi(\lambda')) = 1$ and $x \in \Phi(\lambda) \subseteq \Phi(\lambda') \subseteq B \implies d_\lambda(x, B) = 1 \implies B$ is d-open.

„$A := \{x \in \mathbb{R} | f(x) < a\}$ is d-open“: Choose $x \in A : f(x) < a$, then $\frac{1}{a} < \inf\{\lambda | \lambda \in \Phi(\lambda)\}$. Take $\lambda_0 \geqslant 1 : \frac{1}{a} < \lambda_0$, $x \notin \Phi(\lambda_0)$.

Then we have $\Phi(x_0)^c \subseteq A$, $y \notin \mathbb{R} \backslash \Phi(x_0) \implies \lambda_0 \notin \{\lambda | y \in \Phi(x)\} \implies \inf\{\lambda | y \in \Phi(\lambda)\} \geqslant \lambda_0 \implies f(y) \leqslant \frac{1}{\lambda_0} < a \implies y \in A$.

$\Phi(x_0)^c$ is ϱ_E-open $\implies \Phi(x_0)^c \in \tau_d \implies A$ is d-open. ▫

Poznámka
Approximate continuity is equivalent to continuity with respect to τ_d ($f : \mathbb{R} \to \mathbb{R}$ is approximately continuous at $x_0 \Leftrightarrow f$ is τ_d continuous at x_0).

$f : \mathbb{R} \to \mathbb{R}$ is approximately continuous $\Leftrightarrow \forall M \subseteq \mathbb{R}$ λ_1-measurable: $d_\lambda(x_0, M) = 1$ and $\lim_{x \to x_0, x \in M} f(x) = f(x_0)$.

Věta 4.5 (Denjoy)

Let $f : \mathbb{R} \to \mathbb{R}$. Then the function f is approximately continuous λ-almost everywhere iff f is λ-measurable.

Důkaz

„ $\implies$ “ $N := \{x \in \mathbb{R} | f$ is not approximately continuous function$\}$. $\implies$ $\lambda(N) = 0$. It is sufficient to show that sub/super level sets are measure.

$$c \in \mathbb{R}, M := \{x \in \mathbb{R} | f(x) > c\}.$$

$$y \in M \backslash N \Leftrightarrow f(y) > c \wedge f \text{ is approximately continuous at } y.$$

$\exists \tau_d$-open set G such that $f|_G > c \implies M, N$ τ_d-open $\implies M \backslash N$ is τ_d-open $M = (M \backslash N) \cup (N \cap M)$.

„ $\impliedby$ “: Luzin: $\forall \varepsilon > 0 \ \exists G : \lambda(G) < \varepsilon$ and $f|_G$ is continuous.

Let $\varepsilon > 0$, Luzin theorem gives us $F \subseteq \mathbb{R}$ closed (ϱ_E) such that $\lambda(\mathbb{R} \backslash F) < \varepsilon$ and $f|_F$ is continuous (ϱ_E).

By Lebesgue density theorem λ-almost every point of F is a point of density. Let $\tilde{F}$ is set of those points. $\lambda(F \backslash \tilde{F}) = 0 \implies \tilde{F}$ is d-open.

f is approximately continuous λ-almost everywhere in $\mathbb{R}$. □

Věta 4.6

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded approximately continuous function then f has an antiderivative on $\mathbb{R}$.

Důkaz

$\exists K \in \mathbb{R} \ \forall x \in \mathbb{R} : |f(x)| \leqslant K$. $F(X) = \int_0^x f d\lambda$ (λ-measurable $\implies$ well-defined).

We have $\frac{1}{h}\lambda\left(\{y \in [x, x+h] || f(y) - f(x)| \geqslant \varepsilon\}\right) < \varepsilon \impliedby$ approximately continuity at x.

Fix $h \in (0, \delta)$. Denote $M = \{y \in [x, x+h] || f(y) - f(x)| \geqslant \varepsilon\}$.

$$\left| \frac{1}{h} |F(x+h) - F(x)| - f(x) \right| = \left| \frac{1}{h} \right| \cdot \left| \int_x^{x+h} (f(t) - f(x)) dt \right| \leqslant$$

$$\leqslant \frac{1}{h} \int_M |f(t) - f(x)| dt + \frac{1}{h} \int_{[x,x+h] \backslash M} (f(t) - f(x)) dt \leqslant$$

$$\leqslant \frac{2K}{h} \lambda(M) + \frac{h}{h} \varepsilon \leqslant (2K+1)\varepsilon \implies$$

$\implies F'_+(x) = f(x)$. Analogously $F'_-(x) = f(x) \implies f$ has an antiderivative. □

Důsledek
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded approximately continuous function. Then f has Darboux property and is in B_1.

Důkaz
The previous theorem gives that there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ for every $x \in \mathbb{R}$. So f has Darboux property.

„$f \in B_1$“: $f(x) = F'(x) = \lim_{n\to\infty} \frac{F(x+\frac{1}{n})-F(x)}{\frac{1}{n}}$. □

Věta 4.7

There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the sets $\{x \in \mathbb{R} | f'(x) > 0\}$ and $\{x \in \mathbb{R} | f'(x) < 0\}$ are dense.

Důkaz
Let $A, B \subset \mathbb{R}$ be countable, dense, and disjoint. $A = \{a_n, n \in \mathbb{N}\}$, $B = \{b_n, n \in \mathbb{N}\}$. Observe that A and B are d-closed. Using theorem above we find for every $n \in \mathbb{N}$ approximately continuous g_n and h_n such that $g_n(a_n) = 1$, $0 \leqslant g_n \leqslant 1$, $g_n|_B = 0$, similarly $h_n(b_n) = 1$, $0 \leqslant h_n \leqslant 1$, $h_n|_A = 0$.

We define $\psi = \sum_{n=1}^{\infty} 2^{-n} g_n - \sum_{n=1}^{\infty} 2^{-n} h_n$. ψ is bounded. ψ is approximately continuous. ψ is positive on A and negative on B. By the previous theorem $\exists f : \mathbb{R} \to \mathbb{R}$ such that $f' = \psi$. □

Poznámka
We say that differentiable function g is of Köpcke type if g' is bounded and the sets $\{g' > 0\}$, $\{g' < 0\}$ are dense.

Poznámka
A and B are countable disjoint $\implies$ A and B are τ_d-closed. Towards contradiction assume that there exists $f : \mathbb{R} \to [0, 1]$ approximately continuous such that $f|_A = 0$ and $f|_B = 1$ $\implies$ $f \in B_1$ $\implies$ f has comeagerly many points of continuity.

5 More on derivatives

Definice 5.1 (Notation)

Let $I \subset \mathbb{R}$ be a nonempty open interval. We denote

$$\Delta'(I) = \{f : I \to \mathbb{R} | f \text{ has an antiderivative on I}\}$$

Věta 5.1 (Denjoy-Clarkson)

Let I be a nonempty open interval and $f \in \Delta'(I)$. Then f has Denjoy–Clarkson property, i.e., for every open $G \subset \mathbb{R}$ we have that either $f^{-1}(G) = \varnothing$ or $\lambda(f^{-1}(G)) > 0$.

Důkaz (Denjoy–Clarkson)
Let $F : I \to \mathbb{R}$ satisfy $F' = f$ on I. Let $G \subseteq \mathbb{R}$ be open. WLOG $G = (\alpha, \beta)$ (otherwise we consider countable union).

Let

$$E := \{x \in I | f(x) \in (\alpha, \beta)\} = f^{-1}(G).$$

Assume that $E \neq \varnothing$ and $\lambda(E) = 0$. Choose $x_0 \in E$ and find $\alpha_1, \beta_1 \in \mathbb{R}$ such that $\alpha < \alpha_1 < f(x_0) < \beta_1 < \beta$. Define

$$E_1 := \{x \in I | f(x) \in (\alpha_1, \beta_1)\} \ni x_0.$$

So $E_1 \subseteq E \implies \lambda(E_1) = 0$.

We set $P = \overline{E_1}$, $f \in B_1 \implies \exists y \in \mathcal{C}(f|_P)$. $x_1 \in P \cap I$. We find an open interval $I_1 \subseteq I$ such that $x_1 \in I_1$ and (by continuity)

$$\forall x \in I_1 \cap P : |f(x) - f(x_1)| < \max\{\alpha_1 - \alpha, \beta - \beta_1\} \leqslant \varepsilon,$$

and E_1 is dense in P and $I_1 \cap P$ is open $\implies \exists x_2 \in I_1 \cap E_1 \subseteq I_1 \cap P$. Then we have: $|f(x_2) - f(x_1)| < \varepsilon \implies f(x_2) \in (\alpha_1, \beta_1) \implies f(x_1) \in (\alpha, \beta)$ (triangle inequality).

We can find an open interval $I_2 \subseteq I$ such that $x_1 \in I_2$ and $\forall x \in I_2 \cap P$ (continuity) $f(x) \in (\alpha, \beta)$. Then we have $I_2 \cap P \subseteq E$ and therefore $\lambda(I_2 \cap P) = 0$ closed in I_2.

$\implies$ $I_2 \cap P$ is nowhere dense in I_2. We find a countable disjoint family $\mathcal{I}$ of nonempty open intervals such that $I_2 \backslash P = \bigcup \mathcal{I}$ open in $\mathbb{R}$.

For every $I \in \mathcal{I}$ we have:

$$\forall x \in I : f(x) \leqslant \alpha_1 \vee f(x) \geqslant \beta_1$$

outside of P and outside of E_1.

Since f has Darboux property we have for every $J \in \mathcal{I}$ either $\forall x \in \overline{J} \cap I_2 : f(x) \leqslant \alpha_1$ or $\forall x \in \overline{J} \cap I_2 : f(x) \geqslant \beta_1$.

$\implies$ We can split $\mathcal{I}$ into two subfamilies

$$\mathcal{I}_1 := \{J \in \mathcal{I} | \forall x \in J : f(x) \leqslant \alpha_1\}, \qquad \mathcal{I}_2 := \{J \in \mathcal{I} | \forall x \in J : f(x) \geqslant \beta_1\}.$$

Now the set $\bigcup\{\partial J | J \in \mathcal{I}\}$ is dense in P since P is nowhere dense.

Using this and continuity of f, at x_1 we can find a closed interval I_3 such that $\mathrm{int}(I_3) \ni x_1$. And $I_3 \subseteq I_2$ and $\bigcup \mathcal{I}_1 \cap I_3 = \varnothing$ or $\bigcup \mathcal{I}_2 \cap I_3 = \varnothing$ otherwise ↯with continuity.

Assume that $*?$ holds true. Then for every $x \in I_3$ we have $P \cap I_3 \subseteq I_2 \cap P \implies f(x) \in (\alpha, \beta)$. If $x \in P^c \implies \exists I \in \mathcal{I} : x \in I : f(x) \geqslant \beta_1 \implies f(x) \geqslant \alpha$. $F' = f$ bounded from below.

By the previous lemma, we have $F \in AC(I_3)$. Further we have that λ-almost everywhere $x \in I_3$:

$$F'(x) = f(x) \geqslant \beta_1, \qquad \lambda(P) = 0$$

but

$$\operatorname{int} I_3 \cap \neq \varnothing \implies \operatorname{int} I_3 \cap E_1 \neq \varnothing.$$

Pick $x_3 \in I_3 \cap E_1$. Then

$$f(x_3) = \lim_{x \to x_3^+} \frac{F(x) - F(x_3)}{x - x_3} = \lim_{x \to x_3} \frac{(L) \int_{x_3}^{x} f(t)dt}{x - x_3} \geqslant \beta_1 > f(x_3).$$

$\lightning x_3 \in E_1 \Leftrightarrow f(x_3) \in (\alpha_1, \beta_1)$. □

Lemma 5.2

Let F be differentiable at each point of $[a, b] \subset \mathbb{R}$ and F' is bounded from below. Then F is absolutely continuous on $[a, b]$.

Důkaz

Let $K \in \mathbb{R}$ be such that $F'(x) \geqslant K$ for every $x \in [a, b]$. Then $x \mapsto F(x) - K \cdot x$ is non--decreasing on $[a, b]$. By theorem above we have that $F' \in L^1([a, b])$. For every $x \in [a, b]$ we have

$$F(x) - F(a) = (N) \int_a^x F'(t)dt = (L) \int_a^x F'(t)dt \implies F \in AC([a, b]).$$

□

Věta 5.3

Let f be differentiable at each point of $[a, b]$ and $f' \in L^1([a, b])$. Then we have

$$f(x) - f(a) = (L) \int_a^x f'(t)dt, \qquad x \in [a, b].$$

Věta 5.4 (Vitali–Caratheodory)

Let $f : \mathbb{R} \to \mathbb{R}$, $f \in L^1(\mathbb{R})$, and $\varepsilon > 0$. Then there exist $u, v : \mathbb{R} \to \mathbb{R}^*$ such that

1. *$u \leqslant f \leqslant v$;*
2. *u is usc and bounded from above;*
3. *v is lsc and bounded from below;*

4. $\int (v-u) < \varepsilon$.

Důkaz (The previous previous theorem)
We may assume that $x = l$. Choose $\varepsilon > 0$. Using the previous theorem we find a lsc function g on $[a,b]$ such that $g > f'$ and $\int_a^b g < \int_a^b f' + \varepsilon$. We set

$$G_\eta(x) = \int_a^x g - f(x) + f(a) + \eta(x-a), \qquad x \in [a,b], \eta > 0.$$

Fix $\eta > 0$. For every $x \in [a,b)$ there is $\delta_x > 0$ such that

$$g(t) > f'(x) \qquad \wedge \qquad \frac{f(t)-f(x)}{t-x} < f'(x) + \eta$$

for every $t \in (x, x+\delta_x)$.

For $x \in [a,b)$ and $t \in (x, x+\delta_x)$ we have

$$G_\eta(t) - G_\eta(x) = \int_x^t g - (f(t)-f(x)) + \eta(t-x) > (t-x)f'(x) - (f'(x)+\eta)(t-x) + \eta(t-x) = 0.$$

$\implies G_\eta \geqslant 0$ on $[a,b]$:

$$G_\eta(b) = \int_a^b g - f(b) + f(a) + \eta(b-a) \geqslant 0.$$

$$\int_a^b f' + \varepsilon > \int_a^b g > f(b) - f(a) - \eta(b-a).$$

$$\int_a^b f' + \varepsilon \geqslant f(b) - f(a).$$

$$(L)\int_a^b f' \geqslant f(b) - f(a) \wedge (L)\int_a^b -f' \geqslant -f(b) + f(a) \wedge f(b) - f(a) \geqslant (L)\int_a^b f.$$

□

Důkaz (Vitali–Caratheodory)
We assume that $f \geqslant 0$ and $f \not\equiv 0$. $0 \leqslant s_n \nearrow f$.

$$f = \sum_{n=1}^{\infty} (s_n - s_{n-1}), \qquad s_0 = 0.$$

$$f = \sum_{i=1}^{\infty} c_i \chi_{E_i}, \qquad c_i > 0, \qquad E_i \text{ measurable.}$$

$$\int f = \sum_{i=1}^{\infty} c_i \lambda(E_i) < \infty.$$

For each $i \in \mathbb{N}$ we find K_i compact and V_i open such that $K_i \subset E_i \subset V_i$ and $c_i\lambda(V_i \backslash K_i) < 2^{-(i-1)}\varepsilon$. We define

$$v = \sum_{i=1}^{\infty} c_i \chi_{V_i}, u = \sum_{i=1}^{N} c_i \chi_{K_i},$$

where N is chosen such that $\sum_{i=N+1}^{\infty} c_i \mu(E_i) < \frac{\varepsilon}{2}$.

$u \leqslant f \leqslant v$, u is usc, v is lsc, u is bounded from above, v is bounded from below

$$v - u = \sum_{i=1}^{N} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{V_i} \leqslant \sum_{i=1}^{\infty} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{i=N+1}^{\infty} c_i \chi_{E_i}.$$

$\int (v-u) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$.

„General case:“ $f = f^+ - f^-$: $u_1 \leqslant f^+ \leqslant v_1$, $u_2 \leqslant f^- \leqslant v_2$, $v := v_1 - u_2$, $u := u_1 - v_2$. □

Poznámka (Buczolich)
$\forall n \geqslant 2$ exists a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\{x \in \mathbb{R}^n | \nabla f(x) \in B(0,1)\}$ is nonempty and has λ-measurable zero.

5.1 Zahorski classes

Definice 5.2 (Zahorski conditions?)

Let $E \subset \mathbb{R}$ be an F_σ set. We say that E belongs to class:

- M_0 if every point of E is a point of bilateral accumulation of E;
- M_1 if every point of E is a point of bilateral condensation of E;
- M_2 if each one-sided neighbourhood of each $x \in E$ intersects E in a set of positive measure;
- M_3 if for each $x \in E$ and each sequence $\{I_n\}$ of closed intervals converging to x such that $\lambda(I_n \cap E) = 0$ for each n we have $\lim_{n\to\infty} \frac{\lambda(I_n)}{\operatorname{dist}(x, I_n)} = 0$;
- M_4 if there exists a sequence of closed sets $\{K_n\}$ and a sequence of positive numbers η_n such that $E = \bigcup_{n=1}^{\infty} K_n$ and $\forall x \in K_n$ $\forall c > 0$ there exists a number $\varepsilon > 0$ such that if k and H_1 satisfy $k \cdot h_1 > 0$, $\frac{k}{h_1} < c$, $|k + h_1| < \varepsilon$ then $\frac{\lambda(E \cap (x+k, x+k+h_1))}{|h_1|} > \eta_n$;
- M_5 if every point of E is a point of density of E.

Definice 5.3 (Zahorski classes)

Let $k \in [5]$, $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$. We say that f is in a class $\mathcal{M}_k$ if every associated set, i.e., $\{f > \alpha\}$, $\{f < \alpha\}$, is in M_k.

Věta 5.5

$\mathcal{D}B_1 = \mathcal{M}_0 = \mathcal{M}_1 \supsetneq \mathcal{M}_2 \supsetneq \mathcal{M}_3 \supsetneq \mathcal{M}_4 \supsetneq \mathcal{M}_5 =$ *approximately continuous functions.*

Důsledek
$\Delta' \subset \mathcal{M}_2$.

Poznámka
$\Delta' \subset \mathcal{M}_3$, bounded $\Delta' \subset \mathcal{M}_4$.

6 Sets with finite perimeter and divergence theorem

Lemma 6.1

Let F be a distribution function on a signed Radon measure μ and $\varphi \in \mathcal{C}_c^1(\mathbb{R})$. Then $\int \varphi d\mu = -\int F\varphi' d\lambda$.

Důkaz

WLOG $\mu \geqslant 0$. Suppose that $\varphi \in \mathcal{C}^1(\mathbb{R})$ and spt $\varphi \subseteq [a,b]$, $a,b \in \mathbb{R}$, $a < b$. Choose $\varepsilon > 0$. Find $\delta > 0$ such that

- $\forall x, y \in [a,b], |x-y| < \delta : |\varphi(x) - \varphi(y)| < \varepsilon;$
- $\forall x, y \in [a,b], |x-y| < \delta : |\varphi'(x) - \varphi'(y)| < \varepsilon;$
- for all partition $D = \{x_i\}_{i=0}^n$ of interval $[a,b]$, $\nu(D) < \delta$ and for each $\xi_1, \ldots, \xi_n$ such that $\xi_i \in [x_{i-1}, x_i]$, $i \in [n]$, it holds

$$\left|\sum_{i=1}^n F(\xi_i)\varphi'(\xi_i)(x_i - x_{i-1}) - \int_a^b F\varphi'\right| < \varepsilon.$$

$$0 = F(b)\varphi(b) - F(a)\varphi(a).$$

Let D be a partition of $[a,b]$ with $\nu(D) < \delta$, $D = \{x_i\}_{i=1}^n$.

$$0 = F(b)\varphi(b) - F(a)\varphi(a) = \sum_{i=1}^n (F(x_i)\varphi(x_i) - F(x_{i-1})\varphi(x_{i-1})) =$$

$$= \sum_{i=1}^n F(x_i)(\varphi(x_i) - \varphi(x_{i-1})) + \varphi(x_{i-1})(F(x_i) - F(x_{i-1})).$$

$$\left|\sum_{i=1}^n F(x_i)(\varphi(x_i) - \varphi(x_{i-1})) - \int_a^b F\varphi'\right| =$$

$$\left|\sum_{i=1}^n F(x_i)\varphi'(\eta_i)(x_i - x_{i-1}) - \int_a^b F\varphi'\right| \leqslant$$

$$\leqslant \left|\sum_{i=1}^n F(x_i)\varphi'(x_i)(x_i - x_{i-1}) - \int_a^b F\varphi'\right| + \sum_{i=1}^n |F(x_i)| \cdot |\varphi'(x_i) - \varphi'(\eta_i)| \cdot (x_i - x_{i-1}) <$$

$$< \varepsilon + \sum_{i=1}^n K \cdot \varepsilon \cdot (x_i - x_{i-1}) = \varepsilon \cdot (1 + K \cdot (b-a)),$$

where $K := \sup_{[a,b]} |F|$.

$$\left|\sum_{i=1}^n \varphi(x_{i-1})(F(x_i) - F(x_{i-1})) - \int \varphi d\mu\right| = \left|\sum_{i=1}^n \varphi(x_{i-1}) \cdot \mu((x_{i-1}, x_i]) - \int \varphi d\mu\right| =$$

$$= \left|\int_{[a,b]} \int_{i=1}^n \varphi(x_{i-1})\chi_{(x_{i-1},x_i]} d\mu - \int_{[a,b]} \varphi d\mu\right| \leqslant \int \left|\sum_{i=1}^n \varphi(x_{i-1}) \cdot \chi_{(x_{i-1},x_i]} - \varphi(x)\right| d\mu \leqslant$$

$$\leqslant \varepsilon \cdot \mu([a,b]). \qquad \Longrightarrow \left|\int F\varphi - \int \varphi d\mu\right| \leqslant C \cdot \varepsilon \implies \int \varphi d\mu = \in F\varphi'.$$

$\square$

Věta 6.2

Let $u \in L^1(\mathbb{R})$. Then following assumptions are equivalent

- *there exists a signed Radon measure μ such that $Du = \mu$;*
- *there exists $v : \mathbb{R} \to \mathbb{R}$ such that $v \in BV([a,b])$ for every $a, b \in \mathbb{R}$, $a < b$, and $u = v$ almost everywhere.*

Důkaz

„ $\Longrightarrow$ “: $F : \mathbb{R} \to \mathbb{R}$ a distribution function of μ, i.e., $F(y) - F(x) = \mu((x,y])$, $x < y$. For $a < b$, take D, a partition of $[a,b]$, $D = \{x_i\}_{i=0}^n$:

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \int_{i=1}^{n} |\mu((x_{i-1}, x_i])| \leqslant \sum_{i=1}^{n} |\mu|((x_{i-1}, x_i]) \leqslant |\mu|([a,b]).$$

So $F \in BV([a,b])$, $DF = \mu$, $\varphi \in \mathcal{D}(\mathbb{R})$, $DF(\varphi) = -F(\varphi') = -\int F\varphi' d\lambda = \int \varphi d\mu = \mu(\varphi)$.

„ $\Longleftarrow$ “: TODO!!!

$$\overline{v} = \lim_{t \to x_+} v(t) = \inf\{v(t) | t > x\}$$

v is non-decreasing. x is a point of continuity of $v \implies v(x) = \overline{v}(x)$. $v = \overline{v}$ almost everywhere. $\overline{v}$ is continuous form the right of each $x \in \mathbb{R}$. $Du = Dv = D\overline{v} = \mu$. $\implies \exists!$ Radon measure μ: $\overline{v}(y) - \overline{v}(x) - \mu((x,y])$, $x < y$. □

Věta 6.3 (Gauss divergence theorem)

Let $n > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded open nonempty set with $\mathcal{H}^{n-1}(\partial\Omega) < \varepsilon$, $\mathcal{H}^{n-1}(\partial\Omega \backslash \partial_r\Omega) = 0$, $f \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}^n)$. Then we have

$$\int_{\partial\Omega} \langle f(y), \nu_\Omega(y) \rangle \, d\mathcal{H}^{n-1}(y) = \int_{\Omega} \operatorname{div} f(x) d\lambda^n(x).$$