

Poznámka
Topology...

Definice 0.1 (Topological vector space (TVS))

A Topological vector space over \mathbb{F} is a pair (X, τ) , where X is a vector space over \mathbb{F} and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X ;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X ;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of \mathbf{o} in (X, τ) .

Definice 0.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka
Two homework (in Moodle) and one presentation.

Například

Let $(X, \|\cdot\|)$ be a normed linear space. Let τ be the topology induced by $\|\cdot\|$. The (X, τ) is HLCS.

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Důkaz

$\varrho(x, y) = \|x - y\|$ metric induced by $\|\cdot\|$. τ induced by ϱ . This τ is Hausdorff. Continuity of the operations: (from Funkcionalka)

$$x_n \rightarrow x, y_n \rightarrow y, t_n \rightarrow t \implies x_n + y_n \rightarrow x + y \wedge t_n x_n \rightarrow tx.$$

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So, it is a HTVS. Base of neighbourhood of \mathbf{o} is e. g. $U(0, r), r > 0$, which is convex. \square

Let Γ be any nonempty set, $X = \mathbb{F}^\Gamma$ (= all functions $\Gamma \rightarrow \mathbb{F}$) with point-wise operations, so it is a vector space over \mathbb{F} . It is a HLCS.

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Důkaz

„Continuity of addition:“ $x, y \in \mathbb{F}^\Gamma$, U a neighbourhood of $x + y \implies \exists F \subset \Gamma$ finite $\exists \varepsilon > 0$ such that

$$U_0 = \{z \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |z(\gamma) - (x(\gamma) + y(\gamma))| < \varepsilon\} \subset U$$

$$U_x = \left\{z \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2}\right\}$$

$$U_y = \left\{z \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |z(\gamma) - y(\gamma)| < \frac{\varepsilon}{2}\right\}$$

$\implies V_x$ is neighbourhood of x , and V_y is neighbourhood of y , and $U_x + U_y \subset U_0 \subset U$.
Thus $z_1 \in V_x, z_2 \in V_y \implies z_1 + z_2 \in U_0 \subset U$.

„Continuity of multiplication:“ $\lambda \in \mathbb{F}, x \in \mathbb{F}^\Gamma$, U a neighbourhood of $\lambda x \implies \exists F \subset \Gamma$ finite $\exists \mu > 0$ such that

$$U_0 = \{z \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |z(\gamma) - \lambda x(\gamma)| < \varepsilon\} \subset U$$

$$|\mu z(\gamma) - \lambda x(\gamma)| \leq |\mu| \cdot |z(\gamma) - x(\gamma)| + |\mu - \lambda| \cdot |x(\gamma)|.$$

$$M := \max_{\gamma \in F} |x(\gamma)|.$$

$$V = \left\{\mu \in \mathbb{F} \mid |\mu - \lambda| < \frac{\varepsilon}{2(M+1)}\right\}, \quad W = \left\{z \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2(|\lambda| + \frac{\varepsilon}{2(M+1)})}\right\}.$$

$$\mu \in V, z \in W \implies \mu z \in U_0 \subset U.$$

„Local convexity“: Base of neighbourhoods of \mathbf{o} : $\{x \in \mathbb{F}^\Gamma \mid \forall \gamma \in F : |x(\gamma)| < \varepsilon\}$, $F \subset \Gamma$ finite, $\varepsilon > 0$, consists of convex sets.

„Hausdorff“: $x \neq y \implies \exists \gamma \in \Gamma : x(\gamma) \neq y(\gamma)$. Take $\varepsilon = \frac{|x(\gamma) - y(\gamma)|}{2}$.

$$U = \{z \in \mathbb{F}^\Gamma \mid |z(\gamma) - x(\gamma)| < \varepsilon\}, V = \{z \in \mathbb{F}^\Gamma \mid |z(\gamma) - y(\gamma)| < \varepsilon\} \implies U \cap V = \emptyset.$$

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□

$$X = C(\mathbb{R}, \mathbb{F}) = \{f : \mathbb{R} \rightarrow \mathbb{F} \text{ continuous}\},$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \max_{t \in [-n, n]} |f(t) - g(t)| \right\} =: \sum_{N=1}^{\infty} \frac{1}{2^N} \min \{1, p_N(f - g)\}$$

is translation invariant (that implies addition is continuous, see lecture) metric.

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Důkaz

$f_n \rightarrow f$ in $\varrho \Leftrightarrow \forall N : f_n \rightrightarrows f$ on $[-N, N]$.

„ $f_n \rightarrow f, \lambda_n \rightarrow \lambda \implies \lambda_n f_n \rightarrow \lambda f$ “: Let $N \in \mathbb{N}$. We will show $\lambda_n f_n \rightrightarrows \lambda f$ in $[-N, N]$.
 $x \in [-N, N]$:

$$|\lambda_n f_n(x) - \lambda f(x)| \leq |\lambda_n| \cdot |f_n(x) - f(x)| + |\lambda_n - \lambda| \cdot |f(x)| \leq c \cdot p_N(f_n - f) + |\lambda_n - \lambda| \cdot p_N(f) \rightarrow 0.$$

Hence, X is HTVS. „Local convexity“: $U_{N,\varepsilon} = \{f \in X | p_N(t) < \varepsilon\}$, clearly $U_{N,\varepsilon}$ is a convex set and $U_{N,\varepsilon}$ is neighbourhood of \mathbf{o} . If $\varepsilon < \lambda$, then $\{f | \varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}\} \subset U_{N,\varepsilon}$, because for $\varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}$ it is $\frac{1}{2^N} p_N(f) < \frac{\varepsilon}{2^N}$. „they form a base“: $f \in U_{N,\varepsilon} \implies \varrho(f, \mathbf{o}) < \varepsilon + \frac{1}{2^N}$. Hence fix $r > 0$ and take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$. Then $U_{N,\frac{r}{2}} \subset \{f | \varrho(f, \mathbf{o}) < r\}$ \square

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(Ω, Σ, μ) a measure space, $p \in (0, 1)$. $L^p(\Omega, \Sigma, \mu) = \{f : \Omega \rightarrow \mathbb{F} \text{ measurable} | \int |f|^p d\mu < \infty\}$ (we identify functions equal almost everywhere). $\varrho(f, g) = \int |f - g|^p d\mu$ is a metric making $X = L^p(\Omega, \Sigma, \mu)$ a HTVS (but not locally convex).

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Důkaz

„ ϱ is a metric“: „ Δ -inequality“: $a, b \in [0, \infty) : (a + b)^p \leq a^p + b^p$. (Fix $a \geq 0$, take $\varphi_a(b) = (a + b)^p - a^p - b^p \implies \varphi_a$ is continuous on $[0, \infty)$, $\varphi_a(0) = 0$. For $b > 0$: $\varphi_a(b) = p(a + b)^{p-1} - pb^{p-1} = p \cdot ((a + b)^{p-1} - b^{p-1}) < 0$ as $p - 1 < 0 \implies \varphi_a$ decreasing on $[0, \infty)$ and $\varphi_a \leq 0$.)

φ is translation invariant \implies addition is continuous. „Multiplication“: We can see that $\varrho(\lambda f, \mathbf{o}) = |\lambda|^p \varrho(f, \mathbf{o})$. $f_n \rightarrow f, \lambda_n \rightarrow \lambda$:

$$\varrho(\lambda_n f_n, \lambda f) \leq \varrho(\lambda_n f_n, \lambda_n f) + \varrho(\lambda_n f, \lambda f) = |\lambda_n|^p \varrho(f_n, f) + |\lambda_n - \lambda|^p \varrho(f, \mathbf{o}) \rightarrow 0.$$

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Hence, we have a HTVS. \square

Tvrzení 0.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x + U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 0.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1 - t)y \in A$ for $x, y \in A, t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}, |\alpha| \leq 1$;
- absolutely convex if it is convex and balanced;

- absorbing if $\forall x \in X \exists t > 0 : \{sX | s \in [0, t]\} \subset A$.

Definice 0.4

$\text{co}(A)$ = convex hull, $\text{b}(A)$ = balanced hull, $\text{aco}(A)$ = absolutely convex hull.

Tvrzení 0.2

X is a metric space over \mathbb{F} , $A \subset X$. Then:

- (a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow A$ is convex and symmetric.
- (b) $\text{co } A = \{t_1x_1 + \dots + t_kx_k | x_1 \dots x_k \in A, t_1 \dots t_k \geq 0, t_1 + \dots + t_k = 1, k \in \mathbb{N}\}$.
- (c) $\text{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leq 1\}$.
- (d) $\text{aco}(A) = \text{co}(\text{b}(A))$.
- (e) A is convex $\Leftrightarrow (s+t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)

„ \Rightarrow “: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „ \Leftarrow “: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leq 1 \implies \alpha x \in A.$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{1+\alpha}{2}x \in A$). \square

Důkaz (b)

„ \subseteq “: by induction on k :

$$t_1x_1 + \dots + t_{k+1}x_{k+1} = (t_1 + \dots + t_k) \frac{t_1x_1 + \dots + t_kx_k}{t_1 + \dots + t_k} + t_{k+1}x_{k+1}.$$

„ \supseteq “: the set on the RHS is convex and contain A . \square

Důkaz (c)

„ \supseteq “: clear. „ \subseteq “: RHS is a balanced set. \square

Důkaz (d)

„ \supseteq “: clear. „ \subseteq “ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c): $\text{co}(\text{b}(A)) = \{t_1\alpha_1x_1 + \dots + t_k\alpha_kx_k | x_1, \dots, x_k \in A, |\alpha_j| \leq 1, t_j \geq 0, t_1 + \dots + t_k = 1\}$ is clearly balanced.) \square

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Důkaz (e)„ \implies “: „ \subseteq “: always, „ \supseteq “: $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$.„ \Leftarrow “: in particular $\forall t \in (0, 1): tA + (1-t)A \subset A$, it is the definition of convexity. \square **Tvrzení 0.3***Let (X, τ) be a LCS, $U \in \tau(\mathbf{o})$. Then**(i) U is absorbing.**(ii) $\exists V \in T(0) : V + V \subset U$.**(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.*

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Důkaz (i) $x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V$ a neighbourhood of 0 in $\mathbb{F} : V \cdot x \subset U \implies \exists t > 0 : [0, t] \subset V$

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Důkaz (ii) $\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2$ neighbourhoods of $\mathbf{o} : W_1 + W \subset U$.Take $V = W_1 \cap W_2$.

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Důkaz $\exists U_0 \in \tau(\mathbf{o})$ convex, $U_0 \subset U : \mathbf{o} \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0 \exists W \in \tau(\mathbf{o})$ open : $\forall \lambda, |\lambda| < c : \lambda W \subset U_0$. $V_1 := \bigcup_{0 < |\lambda| < 1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \text{co } V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „ V is open“:

$$V = \bigcup \{t_1 x_1 + \dots + t_n x_n + t_{n+1} V_1 \mid t_1, \dots, t_{n+1} \geq 0, t_1 + \dots + t_{n+1} = 1, x_1, \dots, x_n \in V_1\}$$

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Věta 0.4*1. Let (X, τ) be a LCS. Then there is \mathcal{U} , a base of neighbourhoods of \mathbf{o} with properties:*

- the elements of \mathcal{U} are absorbing, open, absolutely convex;
- $\forall U \in \mathcal{U} \exists V \in \mathcal{U} : 2V \subset U$.

If X is Hausdorff, then $\bigcap \mathcal{U} = \{\mathbf{o}\}$.

2. Let X be a vector space, \mathcal{U} a nonempty family of subsets of X satisfying:

- the elements of \mathcal{U} are absorbing and absolutely convex;
- $\forall U \in \mathcal{U} \exists V \in \mathcal{U} : 2V \subset U$;
- $\forall U, V \in \mathcal{U} \exists W \in \mathcal{U} : W \subset U \cap V$.

Then there is a unique topology τ on X such that (X, τ) is LCS and \mathcal{U} is a base of neighbourhoods of \mathbf{o} . Further, if $\bigcap \mathcal{U} = \{\mathbf{o}\}$, the τ is Hausdorff.

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Důkaz (1.)

Let \mathcal{U} be the family of all open absolutely convex neighbourhoods of \mathbf{o} . The previous proposition (iii) gives us \mathcal{U} is a base of neighbourhoods of \mathbf{o} , (1) gives us elements of \mathcal{U} are absorbing, so the first item holds. (ii) gives us $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$.

Assume X is Hausdorff: $x \in X \setminus \{\mathbf{o}\} \xrightarrow{\text{Hausdorff}} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$. □

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┌ *Důkaz* (2.)

Set $\tau = \{G \subset X \mid \forall x \in G \exists U \in \mathcal{U} : x + U \subset G\}$. This is a unique possibility so uniqueness is clear.

„ τ is topology“: $\emptyset, X \in \tau$ and τ is closed to arbitrary union (clear). τ is closed to finite intersections by third item ($G_1, G_2 \in \tau, x \in G_1 \cap G_2 \dots U_1, U_2 \in \tau, x + U_1 \subset G_1, x + U_2 \subset G_2; \exists V \in \mathcal{U} : V \subset U_1 \cap U_2$, then $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$).

„Elements of \mathcal{U} are neighbourhoods of \mathbf{o} “: $U \in \mathcal{U}. V := \{x \in U \mid \exists W \in \mathcal{U} : x + W \subset U\}$. Then $V \subset U, 0 \in V$ (take $W = U$). $V \in \tau$ ($x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$; let $\tilde{W} \in \mathcal{U}$ such that $2\tilde{W} \subset W$, then $x + \tilde{W} \subset V$, because $y \in \tilde{W} \implies x + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$).

„ \mathcal{U} is a base of neighbourhood of \mathbf{o} “: now clear.

„ (X, τ) is a TVS“: $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$. Then $(x + V) + (y + V) \subset x + y + 2V \subset x + y + U \subset G. \lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G; \exists V \in \mathcal{U} : 2V \subset U; V$ is absorbing $\implies \exists c > 0 \forall t \in [0, c] : tx \in V; V$ balanced $\implies \forall \mu \in \mathbb{F}, |\mu| \leq c : \mu x \in V$; assume $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1}V$,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda)y}_{(\mu - 1) \cdot (\mu + \frac{1}{|\lambda|+1})V} + \underbrace{\lambda(y - x)}_{\in \frac{\lambda}{|\lambda|+1}V \subset V}.$$

„Local convexity“: by first item: $\forall U \in \mathcal{U} : U$ is convex.

Assume $\bigcap \mathcal{U} = \{\mathbf{o}\}$. Take $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$. Take $V \in \mathcal{U} : 2V \subset U$. Then if $(x + V) \cap (y + V) = \emptyset, x + v_1 = y + v_2, x - y = v_2 - v_1 \in V + V = 2V \subset U \nmid$. \square

Věta 0.5

Let X be a vector space and let \mathcal{P} be a family of seminorms on X . Then there is a unique topology τ on X such that (X, τ) is a LCS and $\mathcal{U} = \{\{x \in X \mid p_1(x) < c_1, \dots, p_k(x) < c_k\} \mid p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0\}$ is a base of neighbourhood of \mathbf{o} .

(X, τ) is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\} \exists p \in \mathcal{P}, p(x) > 0$.

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Důkaz

Use the previous theorem (2.) on \mathcal{U} : The sets are absolutely convex (by properties of seminorms). „Absorbing“: $U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $x \in X$?, $j \in [k]$. Then $p_j(x) \in (0, \infty)$ as for $t > 0$: $p_j(t \cdot x) = t \cdot p_j(x)$ and $\exists c > 0$ such that $c \cdot p_j(x) < c_j$ for $j \in [k]$. Now for $t \in [0, c] : tx \in U$.

$U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $V = \{x \in X | p_1(x) < \frac{c_1}{2}, \dots, p_k(x) < \frac{c_k}{2}\}$.

$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$ trivially.

„Hausdorffness“:

$$\bigcap U = \{x \in X | \forall p \in \mathcal{P} : p(x) = 0\}.$$

„ \supseteq “ clear. „ \subseteq “: Assume $y \in X$, $p \in \mathcal{P} : p(y) > 0$: $U = \{x \in X | p(x) < p(y)\} \in \mathcal{U} \implies y \notin U$. □

Například

$(X, \|\cdot\|)$ is a normed space, then its topology is generated by $\mathcal{P} = \{\|\cdot\|\}$.

The topology on \mathbb{F}^Γ is generated by seminorms $p_\gamma(f) = |f(\gamma)|$, $f \in \mathbb{F}^\Gamma$ ($\gamma \in \Gamma$).

$C(\mathbb{R}, \mathbb{F})$ the topology is generated by this sequence of seminorms: $p_N(f) = \max_{x \in [-N, N]} |f(x)|$.

Definition 0.5 (Minkowski functional)

X vector space, $A \subset X$ convex absorbing. Then

$$p_A(x) := \inf \{\lambda > 0 | x \in \lambda \cdot A\}.$$

Lemma 0.6

Let X be LCS, $A \subset X$ convex set.

$$x \in \overline{A}, y \in \text{int } A \implies \{tx + (1-t)y | t \in [0, 1]\} \subset \text{int } A.$$

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Důkaz

WLOG $y = 0$. $t = 0$ clear, $0 \in \text{int } A$. $t \in (0, 1)$:

Fix U , an open absolutely convex neighbourhood of $\mathbf{0}$ such that $U \subset A$. Then $x + \frac{1-t}{t}U$ is a neighbourhood of $x \implies \exists$

TODO!!! □

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TODO!!!

Důkaz (Continuity of multiplication? Theorem 4. TODO?)

„ U is a neighbourhood of \mathbf{o} in τ , $\lambda > 0 \implies \lambda U$ is neighbourhood of \mathbf{o} “: $\lambda \geq 1$: $\exists V \in \mathcal{U} : V \subset U \implies V \subset \lambda V \subset \lambda U$ (V is absolutely convex) $\implies \lambda U$ is neighbourhood of \mathbf{o} . $\lambda = \frac{1}{2}$: $\exists V \in \mathcal{U} : V \subset U$, then $\exists W \in \mathcal{U} : 2W \subset V$, then $W \subset \frac{1}{2}V \subset \frac{1}{2}U \implies \frac{1}{2}U$ is a neighbourhood of \mathbf{o} . Now by induction for $\lambda = \frac{1}{2^n}$. For $\lambda > 0$ find $n \in \mathbb{N}$ such that $\lambda > \frac{1}{2^n}$.

$\lambda x \in G$ ($\lambda \in \mathbb{F}, x \in X, G \in \tau$) $\implies \exists U \in \mathcal{U} : \lambda x + U \in G$. Find $V \in \mathcal{U} : 2V \subset U$ such that V is absorbing ($\implies \exists c > 0 \forall t \in [0, c] : tx \in V$) and V is balanced ($\implies \forall \mu \in \mathbb{F}, |\mu| \leq c : \mu x \in V$). Let $\mu \in F, y \in X$ such that

$$|\mu - \lambda| < c \wedge y \in x + \frac{1}{|\lambda| + c}V \text{ (a neighbourhood of } \mathbf{o})$$

$$\implies \mu y - \lambda x = \mu(y - x) + (\mu - \lambda)x \in V + V = 2V \subset U \implies \mu y \in \lambda x + U \subset G.$$

□

Tvrzení 0.7 (8. see notes of lecturer)

Let X be LCS, $A \subset X$ a convex neighbourhood of \mathbf{o} .

Clearly: $[p_A < 1] \subset A \subset [p_A \leq 1]$.

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Důkaz

„ $[p_A < 1] = \text{int } A$ “: „ \subseteq “: $p_A(x) < 1 \implies \exists c > 1$ such that $cx \in A \implies x = \frac{1}{c}cx \in \text{int } A$. „ \supseteq “: $x \in \text{int } A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$. U absorbing $\implies \exists \alpha > 0 : \alpha x \in U$. Then $(1 + \alpha)x \in A \implies p(x) \leq \frac{1}{1 + \alpha} < 1$.

„ $[p_A \leq 1] = \overline{A}$ “: „ \subseteq “: $p_A(x) \leq 1 \implies \forall n \in \mathbb{N} : p_x((1 - \frac{1}{n})x) = (1 - \frac{1}{n})p_A(x) \leq 1$. $(1 - \frac{1}{n})x \in \text{int } A \implies x \in \overline{\text{int } A} \subset \overline{A}$. „ \supseteq “: $x \in \overline{A} \implies \forall n \in \mathbb{N} : (1 - \frac{1}{n})x \in \text{int } A$, so, $p_A((1 - \frac{1}{n})x) < 1 \xrightarrow{n \rightarrow \infty} p_A(x) \leq 1$. □

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p_A is continuous on X .

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Důkaz

$[p_A < c] = \emptyset$ if $c \leq 0$ and $c \cdot \text{int } A$ if $c > 0$. $[p_A > c] = X$ if $c < 0$, $X \setminus (c \cdot \overline{A})$ if $c > 0$, and $\bigcup_{t>0} X \setminus t\overline{A}$ if $c = 0$. All these sets are open. □

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$$p_A = p_{\overline{A}} = p_{\text{int } A}.$$

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Důkaz

$\text{int } A \subset A \subset \overline{A} \implies p_{\overline{A}} \leq p_A \leq p_{\text{int } A}$. „Conversely“: Assume that $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : (1 - \frac{1}{n})x \in d \cdot \text{int } A \implies (1 - \frac{1}{n})p_{\text{int } A}(x) \leq d \implies p_{\text{int } A}(x) \leq d < c$. □

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Důsledek

Any LCS (X) is completely regular.

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Důkaz

$x \in X$, U an open neighbourhood of x . Take V a convex neighbourhood of \mathbf{o} such that $x + V \in U$. $f(y) := \min \{1, p_V(y - x)\}$. The f is continuous by the previous proposition, $f(x) = 0$.

$$y \in X \setminus U \implies y - x \notin V \implies p_V(y - x) \geq 1 \implies f(y) = 1.$$

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□

Věta 0.8

TODO!!! The topology generated by \mathcal{P}_τ coincides with τ .

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Důkaz

Let τ_1 be topology induced by \mathcal{P}_τ . $\tau_1 \subset \tau$ (seminorms from \mathcal{P}_τ are τ -continuous, hence the sets from theorem 5? are τ -open). „ $\tau \subset \tau_1$ “: Let $U \in \tau(\mathbf{o}) \implies \exists V$ a neighbourhood of \mathbf{o} such that $V \subset U$. The $p_V \in \mathcal{P}_\tau$ (from the previous proposition is continuous) $\implies [p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$.

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□

Tvrzení 0.9

X a vector space.

1. p is seminorm $\implies [p < 1]$ is absolutely convex, absorbing, and $p_{[p < 1]} = p$.
2. p, q are seminorms, then $p \leq q \Leftrightarrow [p < 1] \supset [q < 1]$.
3. \mathcal{P} a set of seminorms generated by a topology τ . p a seminorm on X . Then p is τ -continuous $\Leftrightarrow \exists p_1, \dots, p_k \in \mathcal{P} \exists c > 0 : p \leq c \cdot \max \{p_1, \dots, p_k\}$.

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Důkaz (1.)

Absolutely convex and absorbing is clear.

$$p_{[p < 1]}(x) = \inf \{ \lambda > 0 \mid x \in \lambda [p < 1] \} = \inf \{ \lambda > 0 \mid x \in [p < \lambda] \} = p(x).$$

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□

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Důkaz (2.)

„ \implies “ trivial. „ \Leftarrow “: $[p < 1] \supset [q < 1] \implies p = p_{[p < 1]} \leq p_{[q < 1]} = q$.

└

□

┌ *Důkaz* (3.)

„ \Leftarrow “: $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = [p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}]$, which is a τ -open set $\implies A$ is a neighbourhood of $\mathbf{o} \implies p = p_A$ is continuous (by 1. and the previous proposition).

„ \implies “: p is continuous $\implies [p < 1]$ is neighbourhood of \mathbf{o} ($p(\mathbf{o}) = 0$) $\implies \exists p_1, \dots, p_k \in \mathcal{P} \exists c_1, \dots, c_k > 0$ such that $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c} \max\{p_1, \dots, p_k\} < 1]$ ($c = \min\{c_1, \dots, c_k\}$). Use 2. for seminorms $p, \frac{1}{2 \max\{p_1, \dots, p_k\}}$ and get $p \leq \frac{1}{c} \max\{p_1, \dots, p_k\}$. \square

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1 Continuous and bounded linear mapping

Tvrzení 1.1

$(X, \tau), (Y, \mathcal{U})$ LCS, $L : X \rightarrow Y$ linear. Then the following assertions are equivalent:

1. L is continuous;
2. L is continuous at \mathbf{o} ;
3. L is uniformly continuous.

┌ *Důkaz*

„1. \implies 2.“ trivial, „2. \implies 3.“ assume L continuous at \mathbf{o} . Then, given $U \in \mathcal{U}(\mathbf{o})$, there is $V \in \tau(\mathbf{o})$ such that $L(V) \subset U$. Take $x, y \in X$ such that $x - y \in V$. Then $L(x) - L(y) = L(x - y) \in U$ and that's continuous. „3. \implies 1.“ trivial. \square

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Tvrzení 1.2

$L : X \rightarrow Y$ linear. L is continuous $\Leftrightarrow \forall q$ a continuous seminorm on $Y \exists p$ a continuous seminorm on $X : \forall x \in X : q(L(x)) \leq p(x)$.

┌ *Důkaz*

„ \implies “: L continuous, q a continuous seminorm on Y , the $p(x) = q(L(x))$ is a continuous seminorm on X . „ \Leftarrow “: By the previous proposition it is enough „ L is continuous at \mathbf{o} “: U neighbourhood of \mathbf{o} in Y , $\exists V \subset U$ an absolutely convex neighbourhood of \mathbf{o} . $q := p_V$ is a continuous seminorm. Let p be a continuous seminorm on X such that $q \circ L \leq p$. $W := [p < 1]$ a neighbourhood of \mathbf{o} in X and $L(W) \subset V \subset U$. $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$. \square

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TODO!!!

TODO!!!

Věta 1.3

TODO[Theorem 22]!!!

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Důkaz

„2. \implies 1.“ trivial. „1. \implies 3.“ if ϱ a metric generating τ , then $U_n = \{x \in X \mid \varrho(x, 0) < \frac{1}{n}\} \implies (U_n)_n$ is a base of neighbourhoods of \mathbf{o} . „3. \implies 4.“: (see the proof of the previous proposition, 1.) (U_n) base of neighbourhood of \mathbf{o} , take $V_n \subset U_n$ absolutely convex neighbourhood of \mathbf{o} , $p_n = p_{V_n} \implies (p_n)$ generate τ . „4. \implies 2.“: the previous proposition 2. □

Věta 1.4

(X, τ) is HLCS. X is normable $\Leftrightarrow \exists U$, a bounded neighbourhood of \mathbf{o} .

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Důkaz

„ \implies “: τ generated by $\|\cdot\|$, $U := \{x \in X \mid \|x\| < 1\}$ is a bounded neighbourhood of \mathbf{o} .

„ \Leftarrow “: U bounded neighbourhood of \mathbf{o} . WLOG U is absolutely convex. Then $\frac{1}{n}U$, $n \in \mathbb{N}$ is a base of neighbourhoods of \mathbf{o} (V neighbourhood of \mathbf{o} , $W \subset V$ an absolutely convex neighbourhood of $\mathbf{o} \implies \exists \lambda > 0 : U \subset \lambda W$ Take $n \in \mathbb{N}$ such that $n > \lambda$. Then $U \subset n \cdot W$ so $\frac{1}{n}U \subset W \subset V$). Finally, p_U is a norm generating the topology (U absolutely convex neighbourhood of $\mathbf{o} \implies p_U$ is a continuous seminorm. $\frac{1}{n}U = [p_U < \frac{1}{n}]$, $n \in \mathbb{N}$ is a base of neighbourhood of $\mathbf{o} \implies p_U$ generated topology of X . From X Hausdorff, p_U is a norm.) □

2 Fréchet spaces

Definice 2.1 (Fréchet space)

A LCS whose topology is generated by a complete translation invariant metric is called Fréchet space.

Například

X Banach space $\implies X$ Fréchet space. $\mathbb{F}^{\mathbb{N}}, C(\mathbb{R}, \mathbb{F}), H(\Omega)$ are Fréchet spaces.

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Důkaz ($\mathbb{F}^{\mathbb{N}}$)

$$p_n((x_k)) = \max \{|x_k| \mid k \in [n]\}$$

seminorms generating the topology, $p_1 \leq p_2 \leq \dots$

$$\varrho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \{1, p_n(x - y)\}$$

is translation invariant metric generating the topology. It is complete: $((x_k^m)_k)_{m=1}^{\infty}$ a ϱ -Cauchy sequence $\implies \forall n \in \mathbb{N} : ((x_k^m)_m)$ is p_n -Cauchy \implies it is $\|\cdot\|_{\infty}$ -Cauchy in $\mathbb{F}^{\mathbb{N}}$ \implies (because $\mathbb{F}^{\mathbb{N}}$ is complete) $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \xrightarrow{m \rightarrow \infty} (y_1^n, \dots, y_n^n) \in \mathbb{F}^n$.

Moreover, if $i \leq n_1 \leq n_2$, then $y_i^{n_1} = y_i^{n_2}$. So, we have $y = (y_k)_{k=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$, such that $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \xrightarrow{m} (y_k)_{k=1}^n$

$$\implies \forall n \in \mathbb{N} : p_n(x^n - y) \xrightarrow{m} 0 \implies \varrho(x^n, y) \rightarrow 0, \text{ i.e. } x^n \rightarrow y \text{ in } X.$$

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□

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Důkaz ($\mathbb{C}(\mathbb{R}, \mathbb{F})$)

$$p_n(f) = \max_{x \in [-n, n]} |f(x)|.$$

(f_k) ϱ -Cauchy $\implies \forall n : (f_k)$ is p_n -Cauchy $\implies \forall n : (f_k|_{[-n, n]})$ is $\|\cdot\|_{\infty}$ -Cauchy in $C([-n, n]) \implies \forall n \exists g_n \in C([-n, n])$ such that $f_k|_{[-n, n]} \xrightarrow{k} g_n$ in $C([-n, n])$.

$\forall n_1 \leq n_2 : g_{n_2}|_{[-n_1, n_1]} = g_{n_1}$ so, we have one function $g : \mathbb{R} \rightarrow \mathbb{F}$ such that $\forall n \in \mathbb{N} : g|_{[-n, n]} = g_n$. Then g is continuous, i.e. $g \in C(\mathbb{R}, \mathbb{F})$ and $\forall n \in \mathbb{N} : p_n(f_k - g) \xrightarrow{k} 0$. So $p_n(f_k, g) \rightarrow 0 \implies f_n \rightarrow g$.
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□

Tvrzení 2.1

(X, τ) is a Fréchet space, ϱ any translation invariant metric on X generating $\tau \implies \varrho$ is complete.

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Důkaz

ϱ, d two translation invariant metrics generating by τ . Idea: convergent sequences with respect to ϱ and d coincide, Cauchy sequences with respect to ϱ and d coincide. (x_n) ϱ -Cauchy: $\varepsilon > 0 \implies \{x \mid d(x, \mathbf{o}) < \varepsilon\}$ is a neighbourhood of $\mathbf{o} \implies \exists \delta > 0 : \{x \mid \varrho(x, \mathbf{o}) < \delta\} \subset \{x \mid d(x, \mathbf{o}) < \varepsilon\}$.

$\exists n_0 \forall m, n > n_0 : \varrho(x_m - x_n, \mathbf{o}) = \varrho(x_m, x_n) < \delta \implies d(x_m - x_n, 0) = d(x_m, x_n) < \varepsilon \implies (x_n)$ is d -Cauchy

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□

Tvrzení 2.2

X Fréchet, $A \subset X$. A is compact $\Leftrightarrow A$ is closed and totally bounded.

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Důkaz

Let ϱ be a complete translation invariant metric generating the topology. A is compact $\Leftrightarrow A$ is closed and ϱ -totally bounded. But ϱ -totally boundedness = total boundedness in X . A is totally bounded in $X \Leftrightarrow \forall U$ neighbourhood of $\mathbf{o} \exists F \subset X$ finite $A \subset F + U$. A is totally bounded in $\varrho \Leftrightarrow \forall \varepsilon > 0 \exists F \subset X$ finite such that $A \subset \bigcup_{x \in F} U_\varrho(x, \varepsilon) = F + U_\varrho(0, \varepsilon)$. \square

Tvrzení 2.3

X LCS, $A \subset X$ totally bounded $\implies \text{aco } A$ is totally bounded.

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Důkaz

Let U be a neighbourhood of \mathbf{o} . Let V be an absolutely convex neighbourhood of \mathbf{o} such that $2V \subset U \implies \exists F \subset X$ finite such that $A \subset F + V$. Then clearly $\text{aco } A \subset (\text{aco } F) + V$. $\text{aco } F$ is compact,

$$F = \{x_1, \dots, x_k\} \implies \text{aco}(F) = \text{co}(\text{b}(F)) = \text{co} \{ \lambda x_j | j \in [k], |\lambda| \leq 1 \} = \left\{ t_1 \lambda_1 x_1 + \dots t_n \lambda_n x_n \mid |\lambda_j| \leq 1, t_j \right.$$

$$\left. B = \left\{ (\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mid |\lambda_j| \leq 1, t_j \geq 0, \sum t_j = 1 \right\} \right\}$$

a compact set in $\mathbb{F}^n \times \mathbb{R}^n$. $(\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mapsto t_1 \lambda_1 x_1 + \dots + t_n \lambda_n x_n$ is a continuous map and maps B onto $\text{aco } F$.

$\text{aco } F$ compact \implies totally bounded $\implies \exists H \subset X$ finite, $\text{aco } F \subset H + V$ So $\text{aco } A \subset \text{aco } F + V \subset H + V + V = H + 2V \subset H + U$. \square

Důsledek

X Fréchet space, $A \subset X$ compact $\implies \overline{\text{aco } A}$ is compact.

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Důkaz

A compact $\implies A$ is totally bounded $\implies \text{aco } A$ is totally bounded \implies (because $M \subset X$ any set $\implies \overline{M} \subset M + U$) $\overline{\text{aco } A}$ is totally bounded $\implies \overline{\text{aco } A}$ is compact.

(M totally bounded, for any $U \in \tau(\mathbf{o})$: U is neighbourhood of \mathbf{o} , $x \in \overline{M}$, U absolutely convex neighbourhood of \mathbf{o} , then $V \subseteq U$ absolutely convex such that $2V \subset U \implies (x + U) \cap M \neq \emptyset \implies x \in M + U$.)

Find F finite such that $M \subset F + V \implies \overline{M} \subset M + V \subset F + V + V \subset F + U$. \square

Věta 2.4 (Banach–Steinhaus)

Let X be a Fréchet space and let Y be LCS. Let (T_n) be a sequence of continuous linear mappings $T_n : X \rightarrow Y$ such that $\forall x \in X : \lim_{n \rightarrow \infty} T_n x$ exists in Y . Then $Tx := \lim_{n \rightarrow \infty} T_n x$

define a continuous linear map $X \rightarrow Y$.

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Důkaz

Clear: T is a linear map $X \rightarrow Y$. „Continuous“: Fix q any continuous sequence on Y .

$$A_m = \{x \in X \mid \forall n \in \mathbb{N} : q(T_n x) \leq m\}.$$

Then A_m is closed, absolutely convex and $\bigcup_{m=1}^{\infty} A_m = X$.

TODO?

Baire category theorem $\implies \exists m \in \mathbb{N} : \text{int } A_m \neq \emptyset \implies \exists x \in A_m \exists U$ an absolutely convex neighbourhood of \mathbf{o} such that $x+U \subset A_m \implies -(x+U) \subset A_m \implies (A_m \text{ convex})$
 $U \subset A_m (y \in U \implies y = \frac{1}{2}(x+y+(-x+y))) \implies q(Ty) \leq mp_U(y)$:

$$p_U(y) < c \implies \frac{y}{c} \in U \subset A_m \implies \forall n \in \mathbb{N} q(T_n \frac{y}{c}) \leq m \implies q(T \frac{y}{c}) \leq m \implies q(Ty) \leq cm.$$

└

□

Věta 2.5 (Open mapping theorem)

X, Y Fréchet, $T : X \rightarrow Y$ linear continuous surjective mapping. Then T is an open mapping

Důkaz

1. It is enough to show that $\forall U$ neighbourhood of \mathbf{o} in X : $T(U)$ is a neighbourhood of \mathbf{o} in Y .

2. „ $\forall U$ a neighbourhood of \mathbf{o} in X , \overline{TU} is neighbourhood of \mathbf{o} in Y “: U an neighbourhood of \mathbf{o} in X . $\exists V \subset U$ an absolutely convex neighbourhood of \mathbf{o} . V absorbing \implies

$$\implies X = \bigcup_{n=1}^{\infty} nV \implies Y = T(X) = T\left(\bigcup_{n=1}^{\infty} n \cdot V\right) = \bigcup_{n=1}^{\infty} n \cdot T(V).$$

Y Fréchet \implies by Baire category theorem

$$\exists n \in \mathbb{N} : \emptyset \neq \text{int } \overline{n \cdot T(V)} = \text{int } n \cdot \overline{T(V)} = n \cdot \text{int } \overline{T(V)} \implies \text{int } \overline{T(V)} \neq \emptyset \implies$$

$\implies \exists y \in Y \exists W$ an absolutely convex neighbourhood of \mathbf{o} in Y such that $y + W \subset \overline{T(V)}$.
 $\overline{T(V)}$ is absolutely convex $\implies -(y + w) \subset T(V) \implies W \subset T(V) \subset T(U)$.

3. „ $\forall U$ neighbourhood of \mathbf{o} in X , TU is a neighbourhood of \mathbf{o} in Y “: ϱ a translation invariant metric on X , complete, generating topology. $U_n = \{x \in X \mid \varrho(0, x) < \frac{1}{2^n}\}$. The U_n is a base of neighbourhoods of \mathbf{o} . It is enough to prove that $T(U_n)$ is a neighbourhood of \mathbf{o} for each $n \in \mathbb{N}$. We know from 2. that $\forall n : \overline{TU_n}$ is a neighbourhood of \mathbf{o} in Y . We will be done if we show that $TU_{n-1} \supset \overline{TU_n}$ for each $n \in \mathbb{N}$.

We will prove it for $n = 1$: So we will ? $TU_1 \subset TU_0$. Fix $y \in \overline{TU_1}$. Since $\overline{TU_2}$ is a neighbourhood of \mathbf{o} $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$. So there is $x_1 \in U_1$ such that $y - Tx_1 \in \overline{TU_2}$. $\overline{TU_3}$ is a neighbourhood of \mathbf{o} in $Y \implies y - Tx_1 - \overline{TU_3} \subset \text{ap}TU_2 = \emptyset$ so, there is $x_2 \in U_2$ such that $y - Tx_1 - Tx_2 \in \overline{TU_3}$.

By induction we find $x_n \in U_n$ such that

$$y - Tx_1 - Tx_2 - \dots - Tx_n \in \overline{TU_{n+1}} \quad (n \in \mathbb{N}).$$

$$x := \sum_{n=1}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n :$$

$$M > N \implies \varrho\left(\sum_{n=1}^M x_n, \sum_{n=1}^N x_n\right) = \varrho\left(\sum_{n=N+1}^M x_n, \mathbf{o}\right) \leq \underbrace{\varrho\left(\sum_{n=N+1}^M x_n, \sum_{n=N+2}^M x_n\right)}_{\varrho(x_{N+1}, \mathbf{o})} + \underbrace{\varrho\left(\sum_{n=N+2}^M x_n, \sum_{n=N+3}^M x_n\right)}_{\varrho(x_{N+2}, \mathbf{o})} + \dots$$

$$Tx = y : y - Tx = \lim_{n \rightarrow \infty} (y - Tx_1 - \dots - Tx_n)$$

$$y - Tx_1 - \dots - Tx_n \in \overline{TU_{n+1}} \subset \overline{TU_k} \quad \text{for } n+1 > k$$

so, $y - Tx \in \overline{TU_k}$ for each $k \in \mathbb{N}$, so $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k} = \{\mathbf{o}\}$. „Last equality“: $y \in Y \setminus \{\mathbf{o}\} \implies \exists V$ neighbourhood of \mathbf{o} in Y such that $y \notin \overline{V}$. T continuous $\implies \exists k \in \mathbb{N}$ such that $T(U_k) \subset V \implies \overline{T(U_1)} \subset \overline{V} \implies y \notin \overline{T(U_k)}$. \square

3 Extension and separation theorems

Definice 3.1

X LCS, X^* is the vector space of continuous linear functions on X .

Věta 3.1

X LCS, $Y \subseteq X$, $f \in Y^*$. Then $\exists g \in X^*$ such that $g|_Y = f$.

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Poznámka

If topology of X is generated by \mathcal{P} a topology of seminorms TODO!!!

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Důkaz

1. Topology of Y : $U \subset Y$ is open $\Leftrightarrow \exists \tilde{U} \subset X$ open such that $U = \tilde{U} \cap Y$. U is a neighbourhood of \mathbf{o} in $Y \Leftrightarrow \exists \tilde{U}$ a neighbourhood of \mathbf{o} in X such that $U = \tilde{U} \cap Y$. Lz.pat. Y is also a LCS.

2. $f \in Y^* \implies \exists p$ a continuous seminorm on Y such that $|f(y)| \leq p(y), y \in Y$. $U = [p < 1]$ a neighbourhood of \mathbf{o} in $Y \implies \exists \tilde{U}$ a neighbourhood of \mathbf{o} in X such that $\tilde{U} \cap Y = U \implies \exists \tilde{V} \subset \tilde{U}$ an absolutely convex neighbourhood of \mathbf{o} in X . The $p_{\tilde{V}}$ is a continuous seminorm on X . Moreover, $p_{\tilde{V}}|_Y \geq p$. ($[p_{\tilde{V}}|_Y < 1] \subset \tilde{V} \cap Y \subset U = [p < 1]$). So, for $y \in Y : |f(y)| \leq p(y) \leq p_{\tilde{V}}(y) \implies$ (algebraic H-B for seminorms) $\exists g : X \rightarrow \mathbb{F}$ linear, $g|_Y = f$, $|g(x)| \leq p_{\tilde{V}}(x)$ for $x \in X \implies g$ is continuous by the proposition above. \square

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Důsledek

X LCS, $Y \subseteq X$ closed, $x \in X \setminus Y$. Then $\exists f \in X^* : f|_Y = 0, f(x) = 1$.

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Důkaz

Set $\tilde{Y} = \text{LO}(Y \cup \{x\})$. Define $g(y + \lambda x) = \lambda, y \in Y, \lambda \in \mathbb{F} \implies g$ is linear functional on \tilde{Y} , $g|_Y = 0, g(x) = 1$. $\text{Ker } g = Y$ is closed $\implies g$ is continuous $\implies g$ can be extended to $f \in X^*$. \square

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Důsledek

X LCS, $Z \subseteq Y \subseteq X$.

$$\overline{Z} \supset Y \Leftrightarrow \forall f \in X^* : f|_Z = 0 \implies f|_Y = 0.$$

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Důkaz

„ \implies “: clear. „ \Leftarrow “: $y \in Y \setminus \overline{Z} \implies \exists f \in X^* : f(y) = 1, f|_Z = 0$. \square

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Důsledek

X HLCS, $x \in X \setminus \{\mathbf{o}\} \implies \exists f \in X^* : f(x) \neq 0$.

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Důkaz

$Y = \{\mathbf{o}\}$ is closed linear subspace and use the first corollary. □

Věta 3.2 (Hahn–Banach separation theorem)

X LCS, $A, B \subset X$ nonempty convex, $A \cap B = \emptyset$.

- $\text{int } A \neq \emptyset \implies \exists f \in X^* \setminus \{0\} \exists c \in \mathbb{R} \forall a \in A \forall b \in B : \Re f(a) \leq c < \Re f(b)$.
- A compact, B closed $\implies \exists f \in X^* \exists c, d \in \mathbb{R} \forall a \in A \forall b \in B : \Re f(a) \leq c < d \leq \Re f(b)$.

Důkaz

Analogous to the theorem above. Assume X is a real space ($\mathbb{F} = \mathbb{R}$). „First item“: $\text{int } A \neq \emptyset \implies \text{int}(B - A) \neq \emptyset$ and $- \notin B - A$. Fix $z \in \text{int}(B - A)$, set $U := z - (B - A)$. The U is a convex neighbourhood of \mathbf{o} , $z \notin U \implies p_U(z) \geq 1$. Define $g_0 : \text{LO}\{z\} \rightarrow \mathbb{R}$ by $g_0(t \cdot z) = t \cdot p_U(z) \implies g_0$ is a linear functional, $g_0 \leq p_U$ on $\text{LO}\{z\}$ ($t \geq 0 \implies g_0(t \cdot z) = t \cdot p_U(z) = p_U(t \cdot z)$, $t < 0 \implies g_0(t \cdot z) = t \cdot p_U(z) < 0 \leq p_U(t \cdot z)$).

From algebraic Hahn–Banach $\exists g : X \rightarrow \mathbb{R}$ linear, $g|_{\text{LO}\{z\}} = g_0$, $g \leq p_U$ on X . g is continuous ($g \leq 1$ on $U \implies g \geq -1$ on $-U$, so $|g| \leq 1$ on $U \cap (-U)$, a neighbourhood of \mathbf{o}). $a \in A$, $b \in B \implies$

$$\implies g(z) - g(b) + g(a) = g(z - (b - a)) \leq p_U(z - (b - a)) \leq 1,$$

$$g(a) \leq g(b) + \underbrace{1 - \overbrace{g(z)}^{=p_U(z) \geq 1}}_{\leq 0}.$$

So, $\forall a \in A \forall b \in B : g(a) \leq g(b)$, $c := \sup g(A)$.

„Second item“: A compact, B closed. For $x \in A$ $\exists U_x$ an absolutely convex open neighbourhood of \mathbf{o} such that $(x + U_x) \cap B = \emptyset$. The $(x + \frac{1}{2}U_x)_{x \in A}$, is an open cover of A . A is compact $\implies \exists x_1, \dots, x_n \in A : A \subset (x_1 + \frac{1}{2}U_{x_1}) \cup \dots \cup (x_n + \frac{1}{2}U_{x_n})$. Set $V := \frac{1}{2}U_{x_1} \cap \dots \cap \frac{1}{2}U_{x_n}$ an absolutely convex open neighbourhood of \mathbf{o} . Then $(A + V) \cap B = \emptyset$

$$\left(a \in A \implies \exists j : a \in x_j + \frac{1}{2}U_{x_j} \implies a + V \subset x_j + \frac{1}{2}U_{x_j} + V \subset x_j + U_{x_j} \right).$$

Apply first item to $A + V$ (open convex), B (convex) $\implies \exists f \in X^* \setminus \{0\}$ such that

$$\sup f(A) + \sup f(V) = \sup(f(A) + f(V)) = \sup f(A + V) \leq \inf f(B),$$

observe that $\sup f(V) > 0$ ($f \neq 0$, V is neighbourhood of \mathbf{o} , hence absorbing).

$$c := \sup f(A), \quad d := \sup f(A) + \sup f(V).$$

„ X complex“: look at X as a real space, $f : X \rightarrow \mathbb{R}$ real-linear such that. Define $f_c(x) = f(x) - if(ix)$, $x \in X$. □

Důsledek

X LCS, $\emptyset \neq A \subset X$, $x \in X$.

- $x \in X \setminus \overline{\text{co}}A \Leftrightarrow \exists f \in X^* : \Re f(x) > \sup \{\Re f(a) | a \in A\}$. („ \Leftarrow “: Clear as $\{y \in X, \Re f(y) \leq \sup \{\Re f(a) | a \in A\}\}$ is closed convex set containing A . „ \Rightarrow “: Apply the previous theorem to $\{x\}$ and $\overline{\text{co}}A$, get f and take $-f$.)
- $x \in X \setminus \overline{\text{aco}}A \Leftrightarrow \exists f \in X^* : |f(x)| > \sup \{|f(a)| | a \in A\}$ („ \Leftarrow “: Clear. „ \Rightarrow “: Apply the previous theorem to $\{x\}$ and $\overline{\text{aco}}A$ (and multiply by -1), $f \in X^*$:

$$|f(x)| \geq \Re f(x) > \sup \{ \Re f(y) | y \in \overline{\text{aco}} A \} = \sup \{ |f(y)| | y \in \overline{\text{aco}} A \}. \text{ „}\leq\text{“ clear, „}\geq\text{“:}$$

$$y \in \overline{\text{aco}} A \implies \exists \alpha \in \mathbb{F}, |\alpha| = 1 : |f(y)| = \alpha f(y), \text{ then } |f(y)| = \lambda f(y) = \Re \alpha f(y) = \Re f(\alpha y), \alpha y \in \overline{\text{aco}} A.$$

4 Weak topologies

4.1 General weak topologies and duality

Definice 4.1 (Algebraic dual, general weak topology)

X vector space. $X^\#$ is the algebraic dual of X (it means set of all linear functionals on X). $\emptyset \neq M \subset X^\#$, then $\sigma(X, M)$ is the topology on X generated by seminorms $X \mapsto |f(x)|$, $f \in M$.

Tvrzení 4.1

Properties:

1. $(X, \sigma(X, M))$ is LCS (by the theorem above).
2. $(X, \sigma(X, M))$ is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{0\} \exists f \in M : f(x) \neq 0$ (i.e. M separates points) (by the theorem above).
3. $f \in M \implies f$ is continuous on $(X, \sigma(X, M))$ (fix $f \in M$, $p(x) = |f(x)|$, $x \in X$ is a continuous seminorm and $|f(x)| = p(x) \leq p(x)$).
4. $\sigma(X, M)$ is the weakest topology on X making all $f \in M$ continuous.
5. $\sigma(X, M) = \sigma(X, \text{LO}(M))$.
6. T a topological space, $F : T \rightarrow X$ mapping. Then F is continuous $T \rightarrow \sigma(X, M) \Leftrightarrow \forall f \in M : f \circ F$ is continuous ($T \rightarrow \mathbb{F}$).

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Důkaz (4.)

Assume τ is any topology on X such that all $f \in M$ are τ -continuous \implies

$$\implies \forall x \in X \forall f_1, \dots, f_n \in M \forall c_1, \dots, c_n > 0 : \{y \in X | |f_j(y) - f_j(x)| < c_j \forall j \in [n]\} \text{ is } \tau\text{-open}$$

but these sets form a base of $\sigma(X, M) \implies \sigma(X, M) \subset \tau$. □

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┌ *Důkaz (5.)*

„ \subseteq “: Clear. „ \supseteq “: $f \in \text{LO } M \implies f$ is $\sigma(X, M)$ -continuous (the linear combination of continuous linear functionals is continuous) $f = \alpha_1 f_1 + \dots + \alpha_n f_n$, $f_1, \dots, f_n \in M$, $x_1, \dots, x_n \in \mathbb{F}$.

$$|f(x)| \leq |\alpha_1| \cdot |f_1(x)| + \dots + |\alpha_n| \cdot |f_n(x)| \leq (|\alpha_1| + \dots + |\alpha_n|) \cdot \max \{|f_1(x)|, \dots, |f_n(x)|\}.$$

So by the previous point we get $\sigma(X, \text{LO } M) \subset \sigma(X, M)$. □

┌ *Důkaz (6.)*

„ \implies “: $f \in M \implies f$ is $\sigma(X, M)$ -continuous, so $f \circ F$ is continuous. „ \impliedby “: $t \in T$, U neighbourhood of $F(t)$ in $\sigma(X, M) \implies \exists f_1, \dots, f_n \in M \exists c_1, \dots, c_n > 0$ such that

$$F(t) \in \{y \in X \mid \forall j \in [n] |f_j(y) - f_j(F(t))| < c_j\} \subset U.$$

Let $G = \{u \in T \mid \forall j \in [n] : |(f_j \circ F)(u) - (f_j \circ F)(t)| < c_j\}$. Then G is an open neighbourhood of t (by continuity of $f_j \circ F$ and $F(G) \subset U$). □

Příklad

X LCS. Then $X^* \subseteq X^\#$. So, we may consider $\sigma(X, X^*)$ „the weak topology of X “. $\sigma(X, X^*)$ is Hausdorff when X is HLCS.