Poznámka
There will be homework. We will discus it on practicals (particular solutions are good).

Poznámka (What it is about)
Functional analysis generalizes Linear Algebra. This lecture generalizes (real) Analysis in $\mathbb{R}^n$ ($Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is linear) by replacing $\mathbb{R}^n$ with Banach spaces.

Příklad (Calculus of variations)
Know things: $f : \mathbb{R} \to \mathbb{R}$, differentiable has minimizer at $x_0 \in \mathbb{R} \implies f'(x_0) = 0$ (in $\mathbb{R}^n$: $Df(x_0) = 0$). Generalize it:

Řešení
Trick: For example $F : u \mapsto \int_\Omega \frac{1}{2}|\nabla u|^2 - fudx$, $W_g^{1,2}(\Omega) \to \mathbb{R}$ (g means bounded values).
For any $\varphi \in W_0^{1,2}(\Omega)$ consider $\varepsilon \mapsto F(u + \varepsilon\varphi)$, $\mathbb{R} \to \mathbb{R}$.

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} F(u + \varepsilon\varphi) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_\Omega \frac{1}{2}|\nabla u + \varepsilon \nabla\varphi|^2 - f \cdot (u + \varepsilon\varphi) dx =$$

$$= \frac{d}{d\varepsilon}|_{\varepsilon=0} \left[\int_\Omega \frac{1}{2}|\nabla u|^2 - fudx + \varepsilon \int_\Omega \nabla u \nabla\varphi - f\varphi dx + \varepsilon^2 \int_\Omega \frac{1}{2}|\nabla\varphi|^2 dx \right] =$$

$$= \int_\Omega \nabla u \nabla \varphi - f\varphi.$$

Assume $u \in W^{2,2}(\Omega)$:

$$\overset{\text{P.I.}}{\int}_{\partial\Omega} \frac{\partial u}{\partial n}\varphi dx - \int_\Omega (\triangle u + f)\varphi dx \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

$$\overset{\text{Fundamental lemma}}{\triangle} u + f = 0.$$

Příklad (Mapping degree)
Consider $f \in \mathcal{C}([-1, 1]; \mathbb{R})$. How many zeroes does f have? Let assume $f(-1) < 0 < f(1)$.
Let assume $f \in \mathbb{C}^1$. And 0 is a regular value ($f(x_0) = 0 \implies f'(x_0) \neq 0$).

Řešení
From 0 to ∞. After assumption: by intermediate value theorem at least 1. After second assumption: odd and finitely many. Moreover, the number of zeros with positive derivative minus the number of zeros with the negative one is 1, which is called degree of f.

Observation: In one dimension $\deg(f) \in \{-1, 0, 1\}$. $\deg(f)$ is invariant under perturbations. $\deg f$ depends on boundary values. Can be extended from $\mathcal{C}^1$ to $\mathcal{C}$ (we take smooth perturbation).

Ad second observation: homotopy: $h : [0,1] \times [-1,1] \to \mathbb{R}$, $(s,x) \mapsto h_s(x)$ continuous $h_0 = f$, $h_1 = g$.. And it is admissible if $h_s(-1) \neq 0$ and $h_s(1) \neq 0$ for all s.

There is generalization to $\mathbb{R}^n$, to Manifolds, and to Banach spaces. And we get „corollaries“: Fix point theorems, topological statements,inability to comb a hedgehog,

1 Derivatives in Banach spaces

1.1 The notion of a derivative

Poznámka (In $\mathbb{R}^n$)
Partial derivative, directional derivative, total derivative.

Definice 1.1 (Directional and Gateaux derivative)

Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$. For any $x_0 \in A$, $v \in X$ if

$$\frac{\partial f}{\partial v}(x_0) := \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

exists, we call it directional derivative (at x_0, in direction v).

If $v \mapsto \frac{\partial f}{\partial v}(x_0)$ is a continuous linear operator from X to Y, we denote it by $\partial f(x_0)$ and call it the Gateaux derivative (at x_0).

Poznámka (Notation)
Some authors omit continuous and linear, i.e. for them directional $\Leftrightarrow$ Gateaux.

Some use df or Df instead of ∂f.

We will write $\frac{\partial f}{\partial v}(x_0) = \partial f(x_0) \langle v \rangle$. ($\langle \cdot \rangle$ for linear arguments.)

Například
Consider $F : L^2([0,1]) \to L^2([0,1])$, $u \mapsto F(u)$, $F(u)(x) := \sin(u(x))$. It is continuous ($\|F(u) - F(v)\|_{L^2}^2 = \int |\sin(u(x)) - \sin(v(x))|^2 \leqslant \int |u(x) - v(x)|^2$). Fix $\varphi \in L^2([0,1])$ and calculate:

$$\frac{\partial F}{\partial \varphi}(u) = \lim_{h \to 0} \frac{\sin(u(\cdot) + h\varphi(\cdot)) - \sin(u(\cdot))}{h} = \cos(u(\cdot)) \cdot \varphi(\cdot)$$

point-wise almost everywhere and by domain convergence everywhere.

$\frac{\partial F}{\partial \varphi}$ is linear in φ and bounded $\implies$ F is Gateaux differentiable. Consider $u \mapsto \frac{\partial F}{\partial \varphi}(u)$ for fixed φ. It is continuous.

Is ∂F a good linear approximation? I.e. $\|F(u+\varphi) - F(u) - \partial F(u)\langle\varphi\rangle\|_{L^2} \stackrel{?}{=} o(\|\varphi\|_{L^2})$. No: Pick $u = 0$ $\varphi_k = \pi\chi_[0, \frac{1}{k}]$, then $\|\varphi_k\|_2 = \sqrt{\frac{1}{k}\pi^2} \to 0$.

$$F(u + \varphi_k)(x) = \begin{cases} \sin(0), & x > \frac{1}{k}, \\ \sin(\pi), & x \leqslant \frac{1}{k}. \end{cases} = 0.$$

$$\|\dots\| = \|0 - 0 - \partial F(0)\langle\varphi_k\rangle\|_{L^2} = \|\varphi_k\|_{L^2} \notin o(\|\varphi_k\|_{L^2}).$$

Definice 1.2 (Fréchet derivative)

Let X, Y be Banach, $A \subset X$ open $f : A \to Y$. For any $x_0 \in A$ if there exists $Df(x_0) \in \mathcal{L}(X, Y)$ such that

$$\lim_{v \to \mathbf{o}} \frac{\|f(x_0 + v) - f(x_0)\|_Y}{\|v\|_X} = 0$$

then $Df(x_0)$ is called Fréchet derivative.

Lemma 1.1 (Fréchet $\implies$ Gateaux)

X, Y Banach spaces, $A \subset X$ open, $f : A \to Y$. If F is Fréchet differentiable at x_0, it is also Gateaux differentiable with $\partial f(x_0) = Df(x_0)$.

Důkaz
Trivial. □

Definice 1.3 (Gradient)

Let H be a Hilbert space, $A \subset H$ open $f : A \to \mathbb{R}$. If f is Gateaux differentiable at $x_0 \in A$, then the unique $\nabla f(x_0) \in H$ such that $\langle \nabla f(x_0), v\rangle_H = \partial f(x_0)\langle v\rangle \quad \forall v \in H$ is called the gradient of f at x_0.

Poznámka (Gradients in different spaces)
Derivatives are „independent“ of the space used: $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$ Banach, $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ such that $f_2|_{X_1} = f_1$. Then $Df_2(x_0)|_{X_1} = Df_1(x_0)$, if both exist.

For Hilbert spaces $H_1 \hookrightarrow H_2$:

$$\langle a, v\rangle_{H_1} = \langle b, v\rangle_{H_2}\ \forall v \in H_1 \not\Rightarrow a = b.$$

$\implies$ ∇f depends on the space! Notation $\nabla_H f(x_0)$.

One can define „formal gradients“: Let X Banach, H Hilbert, $X \hookrightarrow H$. $f : A \subset X \to \mathbb{R}$ Gateaux differentiable. Then there might be $\nabla f(x_0) \in H$ such that

$$\langle v, \nabla f(x_0)\rangle_H = Df(x_0)(v) \qquad \forall v \in X.$$

If X is dense in H, then $\nabla f(x_0)$ is unique.

Classically gradients are associate inner product, but principle works with dual pairings, $(\langle\cdot,\cdot\rangle_{L^p\times L^q},\ \frac{1}{p}+\frac{1}{q}=1)$.

1.2 Calculation rules

Tvrzení 1.2 (Chain rule)

Let X, Y, Z be Banach, $A \subset X$, $B \subset Y$ open, $f : B \to Z$, $g : A \to B$, $x_0 \in A$, $y_0 := g(x_0)$.

1. *If f is Fréchet differentiable at y_0 and g is Gateaux differentiable at x_0, then $f \circ g$ is Gateaux differentiable at x_0 with*

$$\partial(f\circ g)(x_0)\langle v\rangle = Df(x_0)\langle \partial g(x_0)\langle v\rangle\rangle \qquad \forall v \in X.$$

2. *If g is additionally Fréchet differentiable, then so is $f \circ g$.*

Důkaz (1.)

$$\lim_{h\to 0}\left\| \frac{f(g(x_0+hv)) - f(g(x_0))}{h} - Df(y_0)\langle \partial g(x_0)\langle v\rangle\rangle \right\|_Z \leqslant$$

$$\leqslant \lim_{h\to 0}\left\| \frac{f(g(x_0+hv)+y_0-g(x_0)) - f(y_0) - Df(y_0)\langle g(x_0+hv)-g(x_0)\rangle}{h}\right\|_Z +$$

$$+\lim_{h\to 0}\underbrace{\left\| Df(y_0)\left\langle \partial g(x_0)\langle v\rangle - \frac{g(x_0+hv)-g(x_0)}{h}\right\rangle\right\|_Z}_{\to 0} =$$

$$= \lim_{h\to 0}\frac{\|f(x_0+g(x_0+hv)-g(x_0)) - f(y_0) - Df(x_0)\langle g(x_0+hv)-g(x_0)\rangle\|_Z}{\|g(x_0+hv)-g(x_0)\|_Y}\cdot\frac{\|g(x_0+hv)-g(x_0)\|_Y}{h} =$$

□

Důkaz (2.)
Last convergence in 1. is independent of v. □

Lemma 1.3 (Mean value)

Let $I \subset \mathbb{R}$ be an interval, Y Banach, $f : I \to Y$ differentiable, $a \in Y$. Then $\forall x, y \in I$, $x > y$, $\exists \xi \in [y, x]$ such that

$$\left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y \leqslant \| f'(\xi) - a \|_Y .$$

Důkaz
By Hahn–Banach $\exists \varphi \in Y^*$ such that

$$* := \left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y = \varphi \left\langle \frac{f(x) - f(y)}{x - y} - a \right\rangle \wedge \|\varphi\|_{Y*} = 1.$$

Define $\Psi : [y, x] \to \mathbb{R}$, $s \mapsto \varphi \langle f(s) - s \cdot a \rangle$. Then

$$* = \frac{\varphi\langle f(x)\rangle - \varphi\langle f(y)\rangle}{x - y} - \frac{x - y}{x - y} \varphi\langle a \rangle = \frac{\psi(x) - \psi(y)}{x - y} \overset{\text{Mean value theorem}}{=} \psi'(\xi) \overset{\text{Chain rule}}{=} \varphi\langle f'(\xi) - a\rangle \leqslant \| f'(\xi) -$$

□

Tvrzení 1.4 (Product spaces)

Let X_1, X_2, Y be Banach, $f : X_1 \times X_2 \to Y$. Let $x_1 \in X_1$, $x_2 \in X_2$, and denote by $\partial_1 f(x_1, x_2)$ the Gateaux derivative of $x \mapsto f(x, x_2)$ at x_1, by $\partial_2 f(x_1, x_2)$ the Gateaux derivative of $x \mapsto f(x_1, x)$ and similarly $D_1 f(x_1, x_2)$ and $D_2 f(x_1, x_2)$.

1. *If f is Gateaux differentiable at (x_1, x_2) then $\partial_1 f(x_1, x_2)$, $\partial_2 f(x_1, x_2)$ exists and we have*

$$\forall v_1 \in X_1, v_2 \in X_2 : \partial f(x_1, x_2) \langle (v_1, v_2) \rangle = \partial_1 f(x_1, x_2) \langle v_1 \rangle + \partial_2 f(x_1, x_2) \langle v_2 \rangle .$$

2. *If $\partial_1 f$ and $\partial_2 f$ exists at (x_1, x_2) and one of them is continuous (as a function $X_1 \times X_2 \mapsto \mathcal{L}(X_i; Y)$) then f is Gateaux differentiable.*
3. *The previous points hold also for Fréchet derivation.*

Důkaz (1.)
From definition:

$$\partial_1 f(x_1, x_2) = \partial f(x_1, x_2) \langle (v_1, 0) \rangle = \lim_{h \to 0} \frac{f(x_1 + h v_1, x_2) - f(x_1, x_2)}{h}.$$

□

Důkaz (2.)
WLOG $\partial_2 f$ is continuous.

$$\lim_{h\to 0}\left\|\frac{f(x_1+hv_1,x_2+hv_2)-f(x_1,x_2)}{h}-\partial_1 f(x_1,x_2)\langle v_1\rangle-\partial_2 f(x_1,x_2)\langle v_2\rangle\right\|_Y \leqslant$$

$$\leqslant \lim_{h\to 0}\underbrace{\left\|\frac{f(x_1+hv_1,x_2)-f(x_1,x_2)}{h}-\partial_1 f(x_1,x_2)\langle v_1\rangle\right\|_Y}_{\to 0}+$$

$$+\lim_{h\to 0}\underbrace{\left\|\frac{f(x_1+hv_1,x_2+hv_2)-f(x_1+hv_1,x_2)}{h}-\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle\right\|_Y}_{*}+$$

$$+\underbrace{\lim_{h\to 0}\|\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle-\partial_2 f(x_1,x_2)\langle v_2\rangle\|_Y}_{\to 0}=0$$

Consider $\psi : s \mapsto f(x_1+hv_1,x_2+sv_2)$.

$$* \leqslant \sup_{\xi\in[0,h]}\|\partial_2 f(x_1+hv_1,x_2+\xi v_2)\langle v_2\rangle-\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle\| \to 0$$

by continuous of $\partial_2 f$. □

Důkaz (3.)
Similarly. □

1.3 Inverse and implicit function theorem

Věta 1.5 (Inverse function theorem)

Let X, Y, $A\subset X$ open, $f : A\to Y$ continuously Fréchet differentiable. If $x_0\in A$ such that $Df(x_0) : X\to Y$ is an isomorphism then there exists $U\subset A$, $V\subset Y$ such that $f|_U : U\to V$ is bijection and $(f|_U)^{-1}$ is Fréchet differentiable with

$$D(f^{-1})(y_0)=(Df(x_0))^{-1}, \qquad y_0 := f(x_0).$$

Důkaz
Given $\hat{y}$ close to $f(x_0)$ find $\hat{x}$ such that $f(\hat{x}) = \hat{y}$. Idea: fix $\hat{y}$ try x: error in y is $f(x) - y$ and error in x is $(Df(x_0))^{-1} \langle f(x) - y \rangle$. Therefore try iteration:

$$F_{\hat{y}}(x) := x - (Df(x_0)) < f(x) - y > .$$

If $F_{\hat{y}}$ has fix point $\hat{x}$ then $\hat{x} = F_{\hat{y}}(\hat{x}) = \hat{x} - (Df(x_0)) \langle f(\hat{x} - y) \rangle \implies f(\hat{x}) = \hat{y}$. So we use Banach fixed point theorem: „$F_{\hat{y}}$ is contraction“: $(x_1, x_2 \in B_\delta(x_0))$

$$\|F_{\hat{y}}(x_1) - F_{\hat{y}}(x_2)\|_X = \|x_1 - x_2 - (Df(x_0))^{-1} \langle f(x_1) - f(x_2) \rangle \|_X =$$

$$= \|(Df(x_0))^{-1} \langle Df(x_0) \langle x_1, x_2 \rangle + f(x_1) - f(x_2) \rangle \|_X \leqslant$$

$$\leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \|Df(x_0) \langle x_1 - x_2 \rangle + f(x_1) - f(x_2)\|_Y = *$$

Consider $a := Df(x_0) \langle x_1 - x_2 \rangle$. $\psi : [0,1] \to Y,\ f(1-\xi)x_1 + \xi x_2)$ and apply Mennroltz? lemma.

$$* \leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \|Df(x_0) < x_1 - x_2 > -Df((1-\xi)x_1 + \xi x_2) \langle x_2 - x_1 \rangle \|_Y \leqslant$$

$$\leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \sup_{x \in B_0(x_0)} \|Df(x_0) - Df(x)\|_{\mathcal{L}(X,Y)} \cdot \|x_1 - x_2\|_X \ll 1$$

$$\|F_{\hat{y}}(x) - x_0\|_X = \|F_{\hat{y}}(x) - F_{\hat{y}}(x_j)\|_X + \|F_{\hat{y}}(x_0) - x_0\|_X \leqslant$$
$$\leqslant \frac{1}{2}\|x - x_0\|_X + \|(Df(x_0))^{-1}\| \cdot \|\hat{y} - x_0\|$$

$\|\hat{y} - x_0\|$ can chosen to be small $\implies$ $F_{\hat{y}}$ maps $\overline{B_\delta(x_0)}$ to $\overline{B_\delta(x_0)}$ $\implies$ $F_{\hat{y}}$ has unique fix point.

Next „regularity“: $(y_1 := f(x_1),\ y_2 := f(x_2))$

$$\|f^{-1}(y_1) - f^{-1}(y_2)\|_X = \|F_{y_1}(x_1) - F_{y_2}(x_2)\|_X \leqslant$$

$$\leqslant \|F_{y_1}(x_1) - F_{y_1}(x_2)\|_X + \|F_{y_1}(x_2) - F_{y_2}(x_2)\|_X \leqslant$$

$$\leqslant \frac{1}{2}\|x_1 - x_2\|_X + \|(Df(x_0))^{-1} \langle y_1 - y_2 \rangle \|_X \leqslant \frac{1}{2} \underbrace{\|x_1 + x_2\|_X}_{=\|f^{-1}(y_1) - f^{-1}(y_2)\|} + c \cdot TODO!!!$$

$$\implies \frac{1}{2}\|f^{-1}(x_1) - f^{-1}(x_2)\|_X \leqslant c \cdot \|y_1 - y_2\|_Y \implies f^{-1} \text{ is Lipschitz.}$$

Pick δ so small that

$$\|Df(x) - Df(x_0)\| \leqslant \frac{1}{2} \cdot \frac{1}{\|(Df(x_0))^{-1}\|} \qquad \forall x \in B_\delta(x_0).$$

$\implies$ $(Df(x))^{-1}$ exists and is uniformly bounded (from functional analysis).

$$\| \underbrace{f^{-1}(y+w) - f^{-1}(y)}_{=:v} - (Df(x))^{-1} \langle w \rangle \|$$

$(f(x+v) + f(x) = f(f^{-1}(y+w)) - y = w)$

$\|v - (Df(x)) \langle f(x+v) - f(x) \rangle \| = \|(Df(x))^{-1} \langle Df(x) \langle v \rangle - f(x+v) + f(x) \rangle \leqslant \|(Df(x))^{-1}\| \cdot \sigma(\|v\|) \leqslant$

because f^{-1} is Lipschitz

Věta 1.6 (Global inverse function theorem)

Let X, Y Banach, $f : X \to Y$ continuously Fréchet differentiable and $(Df(x))^{-1}$ exists, depends continuously on X and $c > 0$ such that $\|(Df(x))^{-1}\| < c\ \forall x \in X$. Then $f : X \to Y$ is a diffeomorphism.

Důkaz

Last theorem $\implies$ f is a local diffeomorphism. Left to show: f is bijective. „Surjectivity“: Fix $x_0 \in X$, $y_0 \in Y$, . Let $y \in Y$, $\varphi(t) = y_0 + t(y - y_0)$, $t \in [0,1]$. Goal: find $\psi(t)$ continuous, such that $\varphi(t) = f(\psi(t))$ (then $y = f(\varphi(t))$) (so called lifting). Local diffeomorphism implies ψ exists on $[0,\delta]$, in fact if Y is defined on $[0,t_0]$, it can be extended to $[0, t_0 + \delta]$. Similarly, if ψ is defined on $[0, t_0]$, per chain rule:

$$\|\psi'(t)\| = \|Df^{-1}(\varphi(t))\langle\varphi'(t)\rangle\| < c.$$

ψ is Lipschitz, $\lim_{t\nearrow t_0}\psi(t)$ is well defined and ψ can be extended to $[0,t_0]$. From Zorn lemma Ψ is defined on $[0,1]$.

„Injectivity“: Assume $f(x_1) = f(x_2) = y$. Pick $\psi_1(t) := x_1 + t(x_2 - x_1)$. $\varphi_1(t) = f(\psi_1(t))$. Define $\varphi_s(t) = s\varphi_1(t) + (1-s)y$ $(t, s \in [0,1])$. Similar to before (homework) $\exists\psi_s(t)$ continuous in s and t, such that $f(\psi_s(t)) = \varphi_s(t)$. But then

$$x_1 = \psi_1(0) = \psi_s(0) = \psi_0(0) = \psi_0(t) = \psi_0(1) = \psi_s(1) = \psi_1(1) = x_2.$$

□

Věta 1.7 (Implicit function theorem)

Let X_1, X_2, Y Banach, $A_1 \subset X_1$, $A_2 \subset X_2$ open, $f : A_1 \times A_2 \to Y$ continuously Fréchet differentiable and exists $\hat{x}_1 \in A_1$ and $\hat{x}_2 A_2$ with $f(x_1, x_2) = 0$. If $D_2 f(\hat{x}_1, \hat{x}_2)$ is an isomorphism (between X_2 and Y), then are neighbourhoods U_1, U_2 of x_1, x_2 such that $\forall \hat{x}_1 \in U_1\ \exists! \hat{x}_2 \in U_2$ with $f(\hat{x}_1, \hat{x}_2) = 0$.

If we call $\hat{x}_2 = g(x_1)$, then g is continuously Fréchet differentiable with $Dg(x) = -(D_2 f(x, g(x)))^{-1} \circ D_1 f(x, g(x))$.

Důkaz

Apply the inverse function theorem to

$$F(x_1, x_2) := (x_1, (D(f(\hat{x}_1, \hat{x}_2)))^{-1})\langle f(x_1, x_2)\rangle$$

□

TODO!!!

Tvrzení 1.8 (Noether–type theorem)
Let $\Omega \subset \mathbb{R}^n$, $F(u) := \int_\Omega f(x,u,Du)$ with $f \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{m\times n})$ and $(\psi_s)_{s\in\mathbb{R}} \subset C^2(\mathbb{R}^n,\mathbb{R}^n)$ is a smooth family with $\psi_0 = \mathrm{id}$, such that

$$f(x,\psi_s\circ u, D(\psi_s \circ u)) = f(x,u,Du).$$

Then there exists a conservation ? $0 \neq Q : \Omega\times\mathbb{R}^n\times\mathbb{R}^{m\times n} \to \mathbb{R}^n$ *such that* $\operatorname{div}(Q(x,u,Du)) = 0$ $\forall$ *critical points of* u.

Důkaz

$$0 = \frac{d}{ds}|_{s=0} f(x,\psi_s\circ u, TODO!!! \circ u) =$$

$$\sum_i \frac{\partial \psi^i}{\partial s}|_{s=0}\frac{\partial f}{\partial z^i}(x,u,D_u) + \sum_{ij}\frac{\partial^2\psi_j}{\partial s\partial y^j}\frac{\partial u^i}{\partial x_j}\frac{\partial f}{\partial p^{ij}}(x,u,Du) =$$

$$= \sum_i \frac{\partial\psi^i}{\partial s}|_{s=0}\sum_j\frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial p_{ij}}(x,u,Du)\right) + \sum_{ij}\frac{\partial^2\psi_s}{\partial s\partial y^j}\frac{\partial u^j}{\partial x_i}\frac{\partial f}{\partial p_{ij}}(x,y,Du) =$$

$$= \sum_j \frac{\partial}{\partial x^j}\left(\sum_i \left.\frac{\partial(\psi^i\circ u)}{\partial s}\right|_{s=0}\frac{\partial f}{\partial p^{ij}}(x,u,Du)\right).$$

□

Příklad (Particle in potential well)
$y : I \to \mathbb{R}^n$ position of a particle, $V : \mathbb{R}^n \to \mathbb{R}$ a physical potential. $F(u) := \int_I \frac{m}{2}|\dot y|^2 - V(y)dt$ (Physics: critical points are behaviour of a ion particle). El eg: $\frac{\partial V}{\partial x_i} + \frac{d}{dt}\left(m\dot y^i\right) = 0 \implies m\ddot y = -\nabla V(y)$.

Assume that V is invariant under rotations, i.e. $V(R(\theta)y) = V(y)$, where $R(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & I\end{pmatrix}$. And always $|\frac{d}{dt}R(\theta)y|^2 = y^T R(\theta)^T R(\theta)$. $\implies$ (Noether)

$$0 = \frac{d}{dt}\left(\frac{dR(\theta)}{d\theta}|_{\theta=0}\frac{\partial f}{\partial p}(y,\dot y)\right) =$$

$$= \frac{d}{dt}\cdot\left(\left(\begin{pmatrix}0 & -1 & \dots\\ 1 & 0 & \dots\\ \dots & \dots & 0\end{pmatrix} y\right)\cdot m\dot y = m\left(y_1\dot y_2 - y_2\dot y_1\right).\right.$$

(Which is angular momentum.)

Poznámka (Conservation law in $n+1$ dimensions)
If we single out one direction as time, e.g.

$$(t,x) = (t,x_1,\dots,x_n),$$

then the conservation law reads as

$$\frac{\partial}{\partial t}Q_0 + \operatorname{div}_x(\overline{Q}) = 0.$$

(Q_0 – conserved quantity, $\overline{Q}$ – conservation current.) And

$$\frac{d}{dt}\int_\Omega Q_0 = \int_\Omega \operatorname{div}_x \overline{Q}.$$

Tvrzení 1.9 (2nd Variation)

Let X be Banach space $A \subset X$ open, $F : A \to \mathbb{R}$.

1. *If $x_0 \in A$ is local minimizer of F and F is twice Gateaux differentiable in x_0, then $\partial^2 F(x)\langle v, v\rangle \geqslant 0\ \forall v \in X$;*
2. *If x_0 is critical point of F and F is twice Fréchet differentiable and $D^2F(x_0)\langle v, v\rangle \geqslant c \cdot \|v\|^2\ \forall v \in X$ with c independent of v, then x_0 is a local minimum.*

Důkaz
„1.“: Consider $\varphi : \varepsilon \mapsto F(x_0 + \varepsilon \cdot v)$, if x_0 is local minimum of F, then 0 is local minimum of $\varphi \implies$

$$\implies 0 \leqslant \varphi''(0) = \frac{d^2}{d\varepsilon^2}|_\varepsilon F(x_0 + \varepsilon v) = \partial^2 F(x_0)\langle v, v\rangle.$$

„2.“: By continuity $\exists \delta > 0$ such that $D^2F(x)\langle v, v\rangle \geqslant \frac{c}{2}\|v\|^2\ \forall v \in X\ \forall x \in B_\delta(x_0)$. Pick $x \in B_\delta(x_0)$, define $\psi(t) := x_0 + t(x - x_0)$, $H(t) := J(\psi(t))$.

$$H(j) - H(0) = \int_0^1 1 \cdot H'(t)dt \overset{BP'}{H}(0) + \int_0^1 (1-t)H''(t)dt = (*).$$

$$H'(t) = DF(\psi(t))\langle x - x_0\rangle \implies H'(0) = 0.$$

$$H''(t) = D^2F(\psi(t))\langle x - x_0, x - x_0\rangle \geqslant 0.$$

$$\implies (*) \geqslant 0 \implies F(x) \geqslant F(x_0)\forall x \in B_\delta(x_0).$$

□

Poznámka (Lebesgue–Hadamard)
If $F(u) = \int_\Omega f(x, u, Du)$, then $D^2F(u)\langle \varphi, \varphi\rangle$ includes

$$\int_\Omega \sum_{ijkl} \frac{\partial}{\partial p_{ij}} \frac{\partial f}{\partial p_{kl}}(x, u, Du) \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_k}{\partial x_l} ds.$$

This is the dominant term. Even more, its enough:

$$\sum_{ijkl} \frac{\partial}{\partial p_{ij}} \frac{\partial}{\partial p_{kl}} f(x,u,Du) \xi^i \xi^j \eta^k \eta^l \geqslant c \cdot ?.$$

1.4 Lagrange multipliers

Tvrzení 1.10 (Lagrange multipliers)

Let X Banach, $A \subset X$ open $F, G : A \to \mathbb{R}$ continuous Fréchet differentiable. Let x_0 be a local minimizer of $F|_{\{G=0\}}$ with $DG(x) \neq 0$. Then $\exists \lambda \in \mathbb{R}$ such that $DF(x_0) + \lambda DG(x_0) = 0$.

λ is called the Lagrange multiplier, any x_0 that satisfies this equation is called critical point.

TODO!!!

Příklad (Principal eigenvalue of Δ)
Consider $\Omega \subset \mathbb{R}^n$ domain, bounded. Minimize $F(u) := \int_\Omega \frac{1}{2}|Du|^2$, $u \in W_0^{1,2}(\Omega)$, under constraint $\frac{1}{2}\int_\Omega |u|^2 = 1$, i.e. $G(u) = \frac{1}{2}\int |u|^2 dx - 1 = 0$.

Řešení
We are looking for $u_1 \in W_0^{1,2}(\Omega)$ such that

$$\forall \varphi \in W_0^{2,2}(\Omega) : 0 = DF(u_1)\langle \varphi \rangle + \lambda_1 DG(u_1) \langle \varphi \rangle =$$

$$= \langle \nabla u_1, \nabla \varphi \rangle_{L^2} + \lambda_1 \langle u_1, \varphi \rangle .$$

I.e. a weak solution to $\Delta u_1 = \lambda_1 u_1$ in Ω and $u_1 = 0$ on $\partial\Omega$. Additionally take $\varphi = u_1 \implies \lambda_1 = -\frac{\int_\Omega |\nabla u_1|^2}{\int_\Omega |u_1|^2} \implies \lambda_1$ is largest eigenvalue.

Příklad (Stokes problem)
Minimize $F(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 - f u dx$ in $W_0^{1,2}(\Omega, \mathbb{R}^3)$ under the constant $\mathrm{div}(u) = 0$.

TODO!!!!

2 The direct method on convex integrands

2.1 Direct method

Tvrzení 2.1 (Direct method in the calculus of variations)

Let X be topological space, $f : X \to \mathbb{R}$ such that

1. *All sublevel sets ($\{x \in X | F(x) \leqslant c\}$) are sequentially precompact;*
2. *F is sequentially lower-semi-continuous ($x_k \to x_0 \implies \liminf_{k\to\infty} F(x_k) \geqslant F(x_0)$.)*

Then F has a minimizer in X.

Důkaz
Let $s := \inf_X F$. Pick sequence $(x_k)_k \subset X$ such that $F(x_n) \to s$. For k_0 large enough $(x_i)_{i\geqslant k_0} \subset x \in X : F(x) \leqslant s+1.$ $\overset{1.}{\implies}$ $\exists$ subsequence (not relabeled) and $x_0 \in X$ such that $x_k \to x_0$. $s = \inf F \leqslant F(x_0) \leqslant \liminf_{k\to\infty} F(x_k) = s$. □

Poznámka (The three c's of the direct method)
Equivalent conditions: Coercivity (sublebel sets are bounded with respect to metric), Compactness (bounded sets are compact with respect to some topology) and lower-semi-Continuity (As before.)

Sometimes also Convexity (if F is strictly convex, then the minimum is unique).

TODO!!!

TODO!!!

Důkaz (of tonelli, sketch)
Weak convergence "averages" functions, convex functions decrease when taking averages.

Reminder (Mazur's Lemma): If $u_k \rightharpoonup u$ then $\exists v_k \in \operatorname{conv}\{u_k, \dots, u_{N(k)}\}$ such that $v_k \to u$.

First step: If $f(x, z, p) = f(x, p)$ and $v_k = \sum_{i=k}^{N(k)} \alpha_{i,k} u_k$ with $\sum_{i=k}^{N(k)} \alpha_{i,k} = 1$. Then Nemitsky:

$$F(u) = \lim_{k\to\infty} F(v_k) = \lim_{k\to\infty} \int_\Omega f(x, \sum \alpha_{i,k} Du_k) \overset{\text{Jensen}}{\leqslant} \lim_{k\to\infty} \sum \alpha_{i,k} \int_\Omega f(x, Du_k) = \lim_{k\to\infty} \sum_{i=k}^{N(k)} \alpha_{i,k} F(u_i) \leqslant \lim_{k\to\infty}$$

Second step: Replace f by $\tilde{f}(x, z, p) = f(x, zp) - a(x) \cdot p - b(x) - c|z|^q$. Then $\tilde{f}$ has the same mean, continuous, and ? condition and

$$u \mapsto \int \tilde{f}(x, u, Du) - f(x, u, Du)$$

is weakly continuous. So we can assume $f(\dots) \geqslant 0$.

TODO!!! By first step:

$$\liminf_{k\to\infty} \int f(x,u,Du_k) \geqslant \int_\Omega f(x,u,Du).$$

Now need to estimate $|f(x,u,Du_k) - f(x,u_k,Du_k)| =: *$. Similarly to the proof of Nemitzky:

$$\forall \varepsilon > 0\ \exists K_\varepsilon \subset \Omega, |?_\varepsilon| < \varepsilon : f|_{\Omega\setminus K_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^{m\cdot n}} \text{ is continuous and } \int_{K_\varepsilon} * \xrightarrow{\varepsilon\to 0} 0.$$

As last time, $u_k = \overline{u}_k + \tilde{u}_k =$ uniformly convergent + small support.

$$\int_{\operatorname{supp}\tilde{u}_k} * \to 0.$$

$$\int_{\Omega\setminus(K_\varepsilon\cup\operatorname{supp}\tilde{u}_k)} |f(x,u,Du_k) - f(x,u_k,Du_k)| \to 0.$$

□

Poznámka (Convexity v.s. convexity)
$F(u)$ convex $\not\Leftrightarrow$ $f(x,z,p)$ convex. (For example $\int_\Omega \det Du dx$ is convex for fixed boundary, but $\det p$ is not convex. For example $\int \frac{1}{4}(1-u^2)^2 + \frac{1}{2}|u'|^2$ not convex, but $(1-z^2) + p^2$ is convex in p.)

3 The mapping degree in finite dimensions

Definice 3.1 (Axioms of mapping degree)

The degree $\deg_{\mathbb{R}^n}(u,\Omega,y_0)$ should be an integer defined for all continuous functions all domains Ω and all $y_0 \notin u(\partial\Omega)$ and it should satisfy

D1 Unity of identity

$$\deg(id,\Omega,y_0) = \begin{cases} 1, & \text{if } y_0 \in \Omega \\ 0, & \text{if } y_0 \notin \overline{\Omega}. \end{cases}$$

D2 Additivity of domains: If $(\Omega_i)_{i\in[k]}$ are disjoint domains such that $\overline{\Omega} = \overline{\bigcup_{i=1}^k \Omega_i}$, then $\forall y_0 \notin u(\partial\Omega) \cup \bigcup_i u(\partial\Omega_i)$, then

$$\deg_{\mathbb{R}^n}(u,\Omega,y_0) = \sum_{i=1}^k \deg(u,\Omega_i,y_0).$$

D3 Base point invariance: $y \mapsto \deg(u,\Omega,y)$ is continuous in $\mathbb{R}^n\setminus u(\partial\Omega) \implies$ if y_1, y_2 are in the same connected component, then $\deg(u,\Omega,y_1) = \deg(u,\Omega,y_2)$.

D4 Homotopy invariance: If $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous such that $y_0 \notin h(s, \partial\Omega)$ $\forall s \in [0,1]$ then $s \mapsto \deg_{\mathbb{R}^n}(h_s, \Omega, y_0)$ is constant.

Věta 3.1

There exists a unique function $\deg_{\mathbb{R}^n}$ *satisfying these axioms.*

Poznámka (Notation)
When clear; $y_0 = \mathbf{o}$; if Ω is clear:

$$\deg_{\mathbb{R}^n}(u, \Omega, y_0) = \deg(u, \Omega, y_0) = \deg(u, \Omega) = \deg(u).$$

Lemma 3.2

TODO!!!

Důkaz
Assume $u_0, u_1 : \mathbb{R}^n \to \mathbb{R}^n$ continuous such that $u_0|_{\overline{\Omega}} = u_1|_{\overline{\Omega}}$. Consider: $h_s(x) := (1 - s)u_0(x) + su_1(x)$. $h_s(\partial\Omega) = u_0(\partial\Omega) = u_1(\partial\Omega)$. $\implies$ $\deg(u_0, \Omega, y_0) = \deg(h_0, \Omega, y_0) = \deg(h_1, \Omega, y_0) = \deg(u_1, \Omega, y_0)$. □

Tvrzení 3.3 (Degree as existence criterion)

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ *continuous,* $\Omega \subset \mathbb{R}^n$ *bounded domain,* $y_0 \in \mathbb{R}^n \backslash u(\partial\Omega)$. *If* $y_0 \notin u(\Omega)$, *then* $\deg(u, \Omega, y_0) = 0$. *Conversely if* $\deg(u, \Omega, y_0) \neq 0$ *then* $\exists x_0 \in \Omega$ *such that* $u(x_0) = y_0$.

Důkaz
Assume $y_0 \in u(\Omega)$. Split Ω into finitely many disjoint subdomains Ω_i (with $\overline{\Omega} = \overline{\bigcup \Omega_i}$) such that $u(\Omega_i) \subset B_\varepsilon(y_i)$, where ε is such that $B_\varepsilon(y_0) \subset \mathbb{R}^n \backslash u(\Omega)$. Pick $\tilde{y}_0$ such that $|\tilde{y}_0| \geqslant \sup_{x \in u(\Omega)} |y| + \sup_{x \in \Omega} |x|$.

$$\deg(u, \Omega, y_0) \overset{\text{D2}}{=} \sum_{i=1}^{k} \deg_{\mathbb{R}^n}(u, \Omega_i, y_0) \overset{\text{D3}}{\sum}{}_{i=1}^{k} \deg_{\mathbb{R}^n}(u, \Omega_i, \tilde{y}_0) =: *.$$

$h_s(x) := (1 - s)u(x) + sx$.

$$* = \sum_{i=1}^{k} \deg_{\mathbb{R}^n}(\mathrm{id}, \Omega_i, \tilde{y}_0) = 0.$$

□

TODO!!!

Tvrzení 3.4

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ *continuous,* $\Omega \subset \mathbb{R}^n$ *bounded domain* $y_0 \in \mathbb{R} \backslash u(\partial\Omega)$. *If* $u|_\Omega \in \mathcal{C}^1$ *and* y_0 *is a regular value of* $u|_\Omega$, *then* $\deg_{\mathbb{R}^n}(u, \Omega, y_0) = \sum_{x \in u^{-1}(y_0)} \operatorname{sgn} \det Du(x)$.

Důkaz

Split Ω into $\Omega_0, \Omega_1, \dots, \Omega_k$, where $k = \# u^{-1}(y_0)$, such that $\Omega_0 \cap u^{-1}(y_0) = \emptyset$, $u|_{\Omega_i}$ diffeomorphism? $\Omega_i \cap u^{-1}(y_0) = \{x_i\}$. Then $\deg(u, \Omega, y_0) = \sum_{i=1}^n \deg(u, \Omega_i, y_0) + \deg(u, \Omega_0, y_0) = \sum_{i=1}^n \operatorname{sgn} \det Du(x_i) + 0$. □

Věta 3.5 (Sard)

Let $\Omega \subset \mathbb{R}^n$ open, $u \in \mathcal{C}^1(\Omega, \mathbb{R}^n)$. Then the set of singular (i.e. not regular) values is a Lebesgue zero set.

Důkaz (Idea)

If $\det Du(x_0) = 0$, then exists v such that $\frac{\partial u}{\partial v} = 0$. □

Tvrzení 3.6 (Integral formula)

Let $u \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, Ω bounded, $y_0 \in \mathbb{R}^n \backslash u(\partial\Omega)$. If $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ is any function such that $\operatorname{supp} f$ is in the connected component of y_0 in $\mathbb{R}^n \backslash u(\partial\Omega)$, then

$$\deg(u, \Omega, y_0) \int_{\mathbb{R}^n} f dy = \int_\Omega f(u(x)) \det Du dx.$$

Důkaz

By Sard and inverse of degree y_0 is regular. Pick $\varepsilon > 0$ such that $u^{-1}(B_\varepsilon(y_0))$ consists of neighbourhoods of $\{x_i\}_i = u^{-1}(y_0) \cap \Omega$, where u is a diffeomorphism. This means that $\operatorname{sgn} \det Du$ is constant in each connected component of $u^{-1}(B_\varepsilon(y_0))$. Assume f such that $\operatorname{supp} f \subset B_\varepsilon(y_0)$.

$$\deg(u, \Omega, y_0) \int_{\mathbb{R}^n} f dy = \sum_{x_i \in u^{-1}(y_0)} \operatorname{sgn} \det Du(x_i) \cdot \int_{\mathbb{R}^n} f dy =$$

$$\overset{\text{Tento}}{=} \sum_{i=1}^k \operatorname{sgn} \det Du(x_i) \int_{U_i} f(u(x)) |\det Du| dx = \sum_{i=1}^k \int_{U_i} f(u(x)) \det Du dx = \int_\Omega f(u) \det Du dx.$$

Now let $\tilde{f}$ arbitrary, but $\int_{\mathbb{R}^n} \tilde{f} = 0$. Then $LHS = 0$, we need to prove

$$\int_\Omega \tilde{f}(u(x)) \det Du(x) dx = 0. \qquad \text{(Homework.)}$$

$$(f_0, \quad \int f_0 \neq 0, \quad \operatorname{supp} f_0 \subset B_\varepsilon(y_0), \qquad \tilde{f} = f - \frac{\int f}{\int f_0} f_0.)$$

Now generic f can be written as sum of both cases and equation is linear in f. □

Důsledek (Integral definition of degree)
For any $u \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ $\deg_{\mathbb{R}^n}(u, \Omega, y_0)$ is uniquely defined by

$$\deg_{\mathbb{R}^n}(u, \Omega, y_0) = \frac{\int_\Omega f((u(x))) \det Du dx}{\int_{\mathbb{R}^n} f dy},$$

where f is as in the last theorem and $\int_{\mathbb{R}^n} f \neq 0$.

Důkaz
(D1) $u = \mathrm{id} \implies \deg = 1$ if $x_0 \in \Omega$ and 0 otherwise.

(D2) Additivity of domains is trivial.

(D3) Base point invariance: proof of last theorem independence choice of f.

(D4) $s \mapsto \int_\Omega f(h_s) \det Dh_s(x) dx$ is continuous. □

Důkaz (Theorem above (C^0-degree?))
If $u, \tilde{u} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\|u - \tilde{u}\|_{C^0} < \varepsilon$, where $\varepsilon < \mathrm{dist}(y_0, u(\partial\Omega))$. By homotopy invariance $\deg(u, \Omega, y_0) = \deg(u, \Omega, y_0)$. Let $u_0 \in C^0(\mathbb{R}^n, \mathbb{R}^n)$ by convolution arg $\exists u \in C^\infty$ such that $\|u_0 - u\|_{C^0} < \frac{\varepsilon}{2}$.

$\deg(u_0, \Omega, y_0) := \deg(u, \Omega, y_0)$. Well defined (independent of u). Axioms can be derived easily. □

Tvrzení 3.7 (Odd maps have odd degree)

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ continuous and odd ($u(x) = -u(-x)$ $\forall x \in \mathbb{R}^n$). $0 \in \Omega, 0 \notin u(\partial\Omega)$, $\Omega = -\Omega$. Then $\deg(u, \Omega, 0)$ is odd.

Důkaz
WLOG assume that $u \in C^\infty$ and 0 is regular value. $u(0) = -u(0) = 0$. Other zeros occur in pairs such that $(-1)^n \det(Du)(-x) = \det D(u(-x)) = \det D(-u(x)) = (-1)^n \det(Du)(x)$ $\implies$ sign is related. □

3.1 Degrees on manifolds

Poznámka
Let M, N be n-dimensional oriented manifolds.

Definice 3.2 (C^1 degree on manifolds)

Let $u \in C^1(M, N)$, $\Omega \subset M$ open, such that $\overline{\Omega}$ is compact and $y_0 \in N \backslash u(\partial\Omega)$ be regular value (in the sense $Du(x) : T_x M \to T_{u(x)} N$ is an isomorphism $\forall x \in u^{-1}(y_0)$). Then define

$\deg_{M\to N}(u,\Omega,y_0) := \sum_{x\in u^{-1}\cap\Omega} \sigma(Du)$, where

$$\sigma(Du) := \begin{cases} +1, & \text{if } Du \text{ is orientation preserving,} \\ -1, & \text{if not.} \end{cases}$$

Tvrzení 3.8

$\deg_{M\to N}$ *fulfills (D2), (D3) and (D4).*

Důkaz

Domain additivity from definition $\implies$ We can pick domains small enough to fit in coordinate chart. Then

$$\deg_N(u,\Omega,y_0) = \deg_{\mathbb{R}^n}(\psi^{-1}\circ u\circ\varphi, \varphi^{-1}(\Omega), \psi^{-1}(y_0))$$

implies the rest. □

Poznámka

(D1) only makes sense if $M = N$, otherwise id is not well defined.

If M is compact, then $\deg_{M\to N}(u,M,y_0) = \deg_{M\to N}(u)$.

There are cases where $\deg_{M\to N}(u) = 0\ \forall u$.

Příklad ($\mathbb{S}^n$ degree)

Consider $M = N = \mathbb{S}^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$. $\mathbb{S}^n$ is compact $\implies$ choose $\Omega = \mathbb{S}^n$. $\mathrm{id}\,\mathbb{S}^n \to \mathbb{S}^n$ is well defined and $\deg_{\mathbb{S}^n}(\mathrm{id}) = 1$. Pick $f = 1$ in the integral formulation:

$$\deg_{\mathbb{S}^n}(u) = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} \det(u|Du)dx,$$

where u is normal vector at u and Du as matrix for orthonormal basis of $Tx\mathbb{S}^n$.

In parametrization: stereoscopic projection $\Phi : \mathbb{R}^n \to \mathbb{S}^n\backslash\{N\}$; Φ is angle-preserving, then

$$\deg_{\mathbb{S}^n}(u) = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{R}^n} \det(u|\partial_1 u|\dots|\partial_n u)dx.$$

We have Hopf's theorem: $C^0(\mathbb{S}^n,\mathbb{S}^n)/\sim_{\text{Homotopy}} \overset{\deg_{\mathbb{S}^n}}{\simeq} \mathbb{Z}$.

Tvrzení 3.9 (Relation between $\mathbb{R}^{n+1}$ and $\mathbb{S}^n$ degree)

Let $u : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *continuous differentiable and* $0 \notin u(\mathbb{S}^n)$ *(where* $\mathbb{S}^n \subset \mathbb{R}^n$*). Then*

$$\deg_{\mathbb{S}^n}\left(\frac{u}{|u|}\bigg|_{\mathbb{S}^n}\right) = \deg_{\mathbb{R}^{n+1}}(u, B_1(\mathbf{o}), \mathbf{o}).$$

Důkaz
Let $\varrho : [0,\infty) \to [0,1]$ smooth such that $\varrho(0) = 0$, $\varrho(s) = 1$ for $s > r$, and $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $y \mapsto \varrho(|y|) \cdot \frac{y}{|y|^{n+1}}$. Then

$$\operatorname{div}\varphi(y) = \varrho'(|y|)\frac{y}{|y|}\cdot\frac{y}{|y|^{n+1}}+\varrho(|y|)\left(\frac{y}{|y|^{n+1}} - n\frac{y}{|y|}\cdot\frac{y}{|y|^{n+2}}\right) = \frac{\varrho'(|y|)}{|y|^n} \implies \operatorname{supp}\operatorname{div}\varphi \subset B_r(\mathbf{o}), \qquad r <$$

$$\implies \int_{B_1} \operatorname{div}\varphi dy = \int_{\partial B_1} \varphi \cdot \nu dy = \int_{\partial B_1} \frac{y \cdot \nu}{|y|^{n+1}} dy = |\mathbb{S}^n|.$$

$$\deg_{\mathbb{R}^n}(u, B_1(\mathbf{o}), \mathbf{o}) = \frac{1}{|\mathbb{S}^n|}\int_{B_1(\mathbf{o})} (\operatorname{div}\varphi)\circ u \det Du dx =$$
$$= \frac{1}{|\mathbb{S}^n|}\int_{B_1(\mathbf{o})} \operatorname{div}(\varphi\circ u \operatorname{cof} Du)dx = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} \varphi\circ u \operatorname{cof} Du\cdot\nu dx =$$
$$= \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u\cdot\operatorname{cof} Du\cdot\nu dx.$$

(Last equation WLOF from homotopy, $|u| = 1$, $u \in \mathbb{S}^n$). It equals to

$$\frac{1}{|\mathbb{S}^n|}\int \det(u|Du)dx = \deg_{\mathbb{S}^n}(u).$$

□

3.2 Brouwer's fixed-point theorem and other consequences

Věta 3.10 (No interaction)

There is no continuous map $u : \overline{B_1(\mathbf{o})} \subset \mathbb{R}^{n+1} \to \mathbb{S}^n$ such that $u|_{\partial B_1(\mathbf{o})} = \mathrm{id}$.

Důkaz
Assume u is such a map. Define $h_s : [0,1]\times\mathbb{S}^n \to \mathbb{S}^n$, $(s,x) \mapsto u(s\cdot x)$. h_s is homotopy. So $\deg_{\mathbb{S}^n}(\mathrm{const}) = \deg_{\mathbb{S}^n}(h_0) = \deg_{\mathbb{S}^n}(h_1) = \deg_{\mathbb{S}^n}(\mathrm{id}) = 1$ □

Věta 3.11 (Brouwer's fixed-point theorem)

Let $u : \overline{B_1(\mathbf{o})} \to \overline{B_1(\mathbf{o})}$ continuous. Then u has a fixed-point, i.e. $\exists x_0 \in \overline{B_1(\mathbf{o})}$ such that $u(x_0) = x_0$.

Důkaz
Assume u has no fixed-point. Let $g(x) \in \mathbb{S}^n$ such that $u(x)$, x, $g(x)$ are on a line (in that order). $f : \overline{B_1(\mathbf{o})} \to \mathbb{S}^n$ is continuous, $x \in \mathbb{S}^n \implies g(x) = x$, ↯. □

Důsledek
Let $\Omega \subset \mathbb{R}^n$ compact and convex, $u : \Omega \to \Omega$ continuous, then u has a fixed point.

Důkaz
If Ω has interior, then Ω is homeomorphic to a ball, so apply the previous theorem. If not, restrict to lower dimenzional subspace. □

Věta 3.12 (Borsak–Ulam)

If $u : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there is a pair of antipodal points with the same value, i..e. $\exists x_0 \in \mathbb{S}^n$ such that $u(x_0) = u(-x_0)$.

Důkaz
Assume the opposite. Define $v : \mathbb{S}^n \to \mathbb{S}^{n-1}$, $x \mapsto \frac{u(x)-u(-x)}{|u(x)-u(-x)|}$. Consider

$$h_s : [0,1] \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \qquad h_s(x) = v(sx, \sqrt{1-s^2}),$$

then $h_0 = \text{const} \implies \deg_{\mathbb{S}^{n-1}}(h_0) = 0$, h_1 is odd $\implies \deg_{\mathbb{S}^{n-1}}(h_1) = \textit{odd}$. ↯. □

Důsledek (?Lyusternik–Shnirelmann?)
Let $A_1, \dots, A_{n+1} \subset \mathbb{S}^n$ open cover of $\mathbb{S}^n$. Then there is a set A_i that contains an antipodal pair of points.

Důkaz
for $i \in [n]$ define $u_i := \operatorname{dist}(x, \mathbb{S}^n \setminus A_i)$. Then $u : \mathbb{S}^n \to \mathbb{R}^n$ is continuous $\implies$ (by Borsak–Ulam) $\exists x_0 \in \mathbb{S}^n$: $u(x_0) = u(-x_0)$. Either $u_i(x_0) > 0$ for some $i \implies x_0, -x_0 \in A_i$ or $u(x_0) = 0 \implies x_0, -x_0 \in A_{n+1}$. □

TODO!!!

TODO!!!

Věta 3.13 (Peano)

Let $Q = [0,T] \times \overline{B_R(y_0)} \subset \mathbb{R} \times \mathbb{R}^n$, $f : Q \to \mathbb{R}^n$ bounded and continuous. Then the ODE $\dot{y}(t) = f(t,y)$, $y(0) = y_0$ has a solution in the interval $\left[0, \min(T, \frac{R}{\sup f})\right] =: [0, T^*]$.

Důkaz
Consider

$$y(t) = F(y)(t) := y_0 + \int_0^t f(s, y(s))ds,$$

TODO!!!

„F is continuous“: $\|y - \hat{y}\|_{\sup} < \delta \implies \|F(y) - F(\hat{y})\|_{\sup} \leqslant \sup_{t\in(0,T^*)} \int_0^t |f(s,y) - f(s,\hat{y})| < T^*\varepsilon$.

„F is compact“: All functions in $F(\mathcal{C}([0,T^*], B_R(y_0)))$ are equibounded and equicontinuous, so by Arzela–Ascoli $\exists$ converging subsequence $\implies$ precompact. □

Poznámka
Consider $\dot{y}(t) = |y|^{1/3}$ (continuous and bounded for small y), $y(0) = 0$. It has many solutions (0, $(2/3)^{3/2}(t-a)^{3/2}$ for $t \geqslant a$ and 0 otherwise, …).

4 The Leray–Schauder degree

Definice 4.1

$$\deg_X(\mathrm{id}, \Omega, y_0) = \begin{cases} 0, & \text{if } x \in \Omega, \\ 1, & \text{if } x \in X\backslash\Omega. \end{cases}$$

Věta 4.1 (Leray–Schauder degree)
Let X be Banach, $T : X \to X$ compact and $(P_n)_n$ be a finite dimension approximation with $X_n \subset X$ finite dimensional, such that $P_n(X) \subset X_n$. Let $\Omega \subset X$ open, bounded, $0 \notin (\mathrm{id} - T)(\partial\Omega)$ then $\deg_X(\mathrm{id} - T, \Omega, 0) := \lim_{n\to\infty} \deg_{X_n}((\mathrm{id} - P_n)|_{X_n}, \Omega \cap X_n, 0)$ is well defined (actually RHS is constant for n large enough). We'll call this the Leray–Schauder degree.

Důkaz

1. Make sense of $\deg_{X_n}((\mathrm{id} - R)|_{X_n}, \Omega \cap X_n, 0)$. Assume $\exists (x_n)_n$ such that $x_n \in \partial(X_n \cap \Omega)$ such that $x_n - P_n x_n = 0$. x_n bounded and T compact $\implies \exists$ subsequence $Tx_n \to x$:

$$\|Tx_n - P_n x_n\| < \frac{1}{n} \implies P_n x \to x.$$

$x_n \to x \implies Tx = x$. ↯.

$$\operatorname{dist}((\mathrm{id} - T)(\partial\Omega), 0) =: r > 0.$$

2. Let P_n, P_m be such that $\frac{1}{n} < \frac{r}{2}$, $\frac{1}{m} < \frac{r}{2}$. Denote by $\tilde{X} := X_n + X_m$ the smallest linear subspace of X including X_n and X_m.

$$\deg_{X_n}((\mathrm{id}\, P_n)|_{X_n}, \Omega \cap X_n, 0) = \deg_{\tilde{X}}((\mathrm{id} - P_n)|_{\tilde{X}}, \Omega \cap \tilde{X}, 0),$$

since $(\mathrm{id} - P_n)(x) = 0 \implies x - P_n x = 0 \implies x \in X_n$. WLOG for all such x $\det((I - DP_n))(x) \neq 0$. TODO!!!

3. TODO!!! □

Důsledek (Leray–Schauder degree as existencial criterion)
Let X be Banach space and $\Omega \subset X$ open, bounded, $T : X \to X$ compact and $0 \notin (\mathrm{id} - T)(\partial\Omega)$. If $\deg_X(\mathrm{id} - T, \Omega, 0) \neq 0$, then there is $x \in \Omega$ such that $x = Tx$.

Důkaz
Approx T by P_n as before. Then $\deg_{X_n}((\mathrm{id} - P_n)|_{X_n}, \Omega \cap X_n, 0) \neq 0$ for n large enough $\implies \exists (x_n)$ such that $x_n = P_n x_n$. $\exists$ subsequence $Tx_n \to x$. As before $x = Tx$. □

Věta 4.2 (Homotopies for the Leray-Schauder degree)

Let X Banach, $T_s : X \to X$ for $s \in [0,1]$ a family of compact operators, uniformly continuous in the sense

$$\exists \varepsilon > 0, \Omega \subset X \text{ bounded } \exists \delta > 0\ \forall x \in \Omega\ \forall |s_1 - s_2| < \delta : \|T_{s_1}(x) - T_{s_2}(x)\| < \varepsilon.$$

If Ω is open and bounded such that $0 \notin (\mathrm{id} - T_s)(\partial\Omega)$ $\forall s \in [0,1]$, then $s \mapsto \deg_X(\mathrm{id} - T_n, \Omega, 0)$ is constant.

Důkaz
Similar to before we show $\operatorname{dist}((\operatorname{id} - T_s)(\partial\Omega), 0) \geqslant r > 0$ independently of s. Assume $\exists (s_n)_n \subset [0,1], (x_n)_n \subset \partial\Omega$ such that $\|x_n - T_{s_n} x_n\| \to 0$. By compactness $\exists$ subsequence $s_n \to s$ and $T_s x_n \to x$. Now

$$\|x_n - T_s x_n\| \leqslant \underbrace{\|x_n - T_{s_n} x_n\|}_{\to 0 \text{ by assumption}} + \underbrace{\|T_s x_n - T_{s_n} x_n\|}_{\to 0 \text{ by uniform continuity}} \implies$$

$\implies x_n \to x \in \Omega \wedge x - Tx = 0.$ ↯.

TODO!!! □