Úvod

Poznámka (Organizační úvod)
Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (… PDE …).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

Poznámka (Konvence pro PDR)
$\Omega \subseteq \mathbb{R}^d$ je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

Poznámka
Dále se ukazovali konkrétní parciální rovnice.

Poznámka (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál: $L^p(\Omega)$, $L^p_{loc}(\Omega)$, $||u||_p$, $C^k(\Omega)$, $C^k(\overline{\Omega})$,
$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) | \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha} < \infty \right\}, ||u||_{C^{0,\alpha}} = \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha}.$$
- $\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$, $\vec{n} = (n_1, \dots, n_d)$.
- Funkcionální analýza 1: Banachův prostor, $u^n \to u$ silná konvergence, $u^n \rightharpoonup u$ slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita (L^p jsou separabilní až na $p = \infty$, $C^k(\overline{\Omega})$ je separabilní, $C^{0,\alpha}$ není separabilní pro $\alpha \in (0, 1]$).

Poznámka (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

1 Sobolevovy prostory

Definice 1.1 (Multiindex)

α je multiindex $\equiv d = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$. Délka α je $|\alpha| := \alpha_1 + \ldots + \alpha_d$. Pro $u \in C^k(\Omega)$ definujeme $D^\alpha u = \frac{\partial^{|d|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$.

Definice 1.2 (Slabá derivace)

Buď $u, v_\alpha \in L^1_{loc}(\Omega)$. Řekneme, že v_α je α-tá slabá derivace $u \equiv$

$$\equiv \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Příklad

$u = \text{sign } x$ nemá slabou derivaci.

Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

Důkaz

v_α^1, v_α^2 dvě α-té derivace u.

$$(-1)^{|\alpha|} \int v_\alpha^1 \varphi = \int_\Omega u D^\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int v_\alpha^2 \varphi = \int_\Omega u D^\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$$\int_\Omega (v_\alpha^1 - v_\alpha^2) \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ skoro všude v Ω.

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají. □

Definice 1.3 (Sobolevův prostor)

$\Omega \subseteq \mathbb{R}^d$ otevřená, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}.$$

$$||u||_{W^{k,p}(\Omega)} ||u||_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} ||D^\alpha u||_p^p\right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha|\leqslant k} ||D^\alpha u||_\infty, & p = \infty. \end{cases}$$

Poznámka

Od teď D^α nebo $\frac{\partial}{\partial x_1}$ nebo ∂_i značí slabou derivaci.

Lemma 1.2 (Základní vlastnosti slabých derivací a Sobolevových prostorů)

Nechť $u, v \in W^{k,p}(\Omega), k \in \mathbb{N}$, a α multiindex s délkou $\leqslant k$.

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ *a* $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta} u$, *pro* $|\alpha| + |\beta| \leqslant k$.
- $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ *a* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
- $\forall \tilde{\Omega} \subseteq \Omega$ *otevřená*
$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$
- $\forall \eta \in C^\infty(\Omega)$: $\eta u \in W^{k,p}(\Omega)$ *a* $D^\alpha(\eta u) = \sum_{\beta_i \leqslant \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$, *kde* $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Důkaz

Cvičení na doma. □

Věta 1.3 (Basic properties of Sobolev spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then

- $W^{k,p}(\Omega)$ *is a Banach space;*
- *if $p < \infty$ it is separable space;*
- *if $p \in (1, \infty)$ it is reflexive space.*

Důkaz

BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness: u^n is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \to u$ in L^p. $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \to v_\alpha \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

$$\forall \eta \in C_0^\infty(\Omega) : \int_\Omega v_\alpha \eta = \int_\Omega (v_\alpha - D^\alpha u^n)\eta + \int_\Omega D^\alpha u^n \eta =$$

$$= \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|} \int_\Omega D^\alpha \eta u^n =$$

$$= \int_\Omega (v_\alpha - D^\alpha u)\eta + (-1)^{|\alpha|} \int_\Omega (u^n - u) D^\alpha \eta + (-1)^{|\alpha|} \int_\Omega u D^\alpha \eta.$$

$$\left| \int_\Omega (v_\alpha - D^\alpha u^n)\varphi \right| \leqslant \|v_\alpha - D^\alpha u^n\|_p \|\eta\|_{p'} \leqslant C\|v_\alpha - D^\alpha u^n\| \to 0.$$

$$\left| \int_\Omega (u^n - u) D^\alpha \eta \right| \leqslant \|u^n - u\|_p \|D^\alpha \eta\|_{p'} \leqslant C\|u^n - u\|_p \to 0.$$

„2+3“: $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \ldots \times L^P(\Omega)$ $(d+1$ times), X closed subspace from first property. Lemma: if $X \subseteq Y$ is closed subspace then Y separable $\implies$ X separable and Y reflexive $\implies$ X reflexive. (From functional analysis and topology.) □

2 Approximation of Sobolev function

Věta 2.1

Let $\Omega \subseteq \mathbb{R}^d$ open, ?. $p \in [1, \infty)$.

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} = W^{k,p}(\Omega).$$

Pozor

$$\overline{\{u \in C^\infty(\overline{\Omega})\}}^{||\cdot||_{k,p}} \subsetneq W^{k,p}(\Omega).$$

Důkaz

Summer semester. □

Věta 2.2 (Local density)

$$\forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty$$

$$u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} \; open, \overline{\tilde{\Omega}} \subseteq \Omega$$

$$u^n \to u \; in \; W^{k,p}(\tilde{\Omega})$$

Důkaz
u is extended by 0 to $\mathbb{R}^d \backslash \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \qquad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \qquad \eta \in C_0^\infty(B_1), \eta \geqslant 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

$$u \in L^P(SET) \qquad u^\varepsilon \to u \text{ in } L^p(SET).$$

We need: $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\forall \alpha, |\alpha| \leqslant k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leqslant \varepsilon_0$ (so that ball of radius ε_0 and center in $\tilde{\Omega}$ is in Ω):

$$(D^\alpha u)^\varepsilon(x) = \int_{\mathbb{R}^d} D^\alpha u(y)\eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} D^\alpha u(y)\eta_\varepsilon(x-u)dy =$$

$$= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y)dy.$$

$$D^\alpha u^\varepsilon = D_x^\alpha \int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)dy.$$

□

Tvrzení 2.3

Ω is open connected set, $u \in W^{1,1}(\Omega)$, then $u = \text{const}. \Leftrightarrow \frac{\partial u}{\partial x_i} = 0\ \forall i \in [d]$.

$W^{1,1}(I) \hookrightarrow C(I)$ for I interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$.

Důkaz
„1. $\implies$" trivial. „1. $\impliedby$": $\tilde{\Omega} \subseteq \Omega$ connected ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$. u^ε-modification of u is smooth, so

$$\frac{\partial u^\varepsilon}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^\varepsilon = 0 \quad in \tilde{\Omega}$$

$$\implies u^\varepsilon = \text{const}(\varepsilon) \quad in \tilde{\Omega}.$$

$$c(\varepsilon) = \int_{\mathbb{R}} c(\varepsilon)\eta_\delta(x-y)dy = \int_{\mathbb{R}} u^\varepsilon(y)\eta_\delta(x-y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\eta_\varepsilon(y-z)\eta_\delta(x-y)dzdy =$$

$$\iint u(z+y)\eta_\varepsilon(z)\eta_\delta(y-x)dzdy = \iint u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dzdw =$$

$$\iint u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dwdz = \int_{\mathbb{R}^d} u^\delta(z+x)\eta_\varepsilon(z)dz = \int c(\delta)\eta_\varepsilon(z)dz = c(\delta).$$

„2.": WLOG $I = (0,1)$. Define $v(x) = \int_0^x \frac{\partial u}{\partial y}(y)dy$. We show: $v \in W^{1,1}(I)$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$.

$$|v(x)| \leqslant \int_0^1 |\frac{\partial u}{\partial x}| \leqslant ||u||_{1,1}.$$

$$\varphi \in C_0^1(0,1) \qquad \int_0^1 v(x)\frac{\partial \varphi}{\partial x}(x)dx$$

$$= \int_0^1 \left(\int_0^x \frac{\partial u}{\partial y}(y)dy\right)\frac{\partial \varphi}{\partial x}(x)dx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x} x_{0<y<x} dydx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x} x_{0<y<x} dxdy =$$

$$= \int_0^1 \left(\int_y^1 \frac{\partial \varphi(x)}{\partial x}dx\right)\frac{\partial u}{\partial y}(y)dy = -\int_0^1 \varphi(y)\frac{\partial u}{\partial y}(y)dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}.$$

TODO.

$$x \to y \implies \int_y^x |\frac{\partial u}{\partial z}|^\alpha \to 0 \implies |u(x) - u(y)| \to 0$$

$$||u||_{C(I)} \leqslant ||v + c||_{C(I)} \leqslant ||u||_{1,1} + |c| = ||u||_{1,1} + |u(x) - v(x)| \forall x \in I$$

$$||u||_{C(I)} \leqslant ||u||_{1,1} + \int_0^1 |u(x) - v(x)|dx \leqslant -|| - + \int_0^1 |u| + \int_0^1 |v| \leqslant ||u||_{1,1}.$$

„3." was shown without proof. □

3 Characterization of Sobolev function

Věta 3.1

$\Omega \subseteq \mathbb{R}^d$, $p \in [1,\infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega |\, \mathrm{dist}(x, \delta\Omega) > \delta\}$. *Then*

$$\forall u \in W^{1,p}(\Omega) : ||\Delta_i^h u||_{L^p(\Omega_\delta)} \leqslant ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}, \qquad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x+he_i) - u(x)}{h}.$$

$$u \in L^P \implies \forall \delta, h : ||\Delta_i^h u||_{L^p(\Omega_\delta)} \leqslant c.$$

$p > 1 \implies \frac{\partial u}{\partial x_i}$ *exists and* $||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)} \leqslant c$.

Definice 3.1 (Class $C^{k,\mu}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \ldots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r = \left\{x'_r \in \mathbb{R}^{d-1} |\ |x_{r_i}| \leqslant \alpha\right\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting tr the orthogonal transformation from (x'_r, x_{r_d}) to (x', x_d), then $\forall x \in \partial\Omega$ $\exists r \in \{1, \ldots, M\}$ such that $x = \mathrm{tr}\left(x'_{r_1}, a(x_{r_d})\right)$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\right\}$$

$$V_r^- := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\right\}$$

$$\Lambda_r := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) = x_{r_d}\right\}$$

Then $\mathrm{tr}(V_r^+) \subset \Omega, \mathrm{tr}(V_r^-) \subset \mathbb{R}^d \backslash \overline{\Omega}, \mathrm{tr}(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M \mathrm{tr}(\Lambda_r) = \partial\Omega$.

Věta 3.2 (Density of smooth functions)

Let $\Omega \in C^0$. *Then* $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{||\cdot||_{k,p}}$, $p \in [1,\infty)$.

Věta 3.3 (Extension of Sobolev functions)

Let $\Omega \in C^{0,1}$ *(Ω is Lipschitz) and* $k \in \mathbb{N}$, $p \in [1,\infty]$. *Then there exists a continuous linear operator* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ *such that:*

- $||Eu||_{W^{k,p}(\mathbb{R}^d)} \leqslant C||Eu||_{W^{k,p}(\Omega)}$ *(C is independent of u)*
- $Eu = u$ *almost everywhere in* Ω.

Věta 3.4 (Trace theorem)

Let $\Omega \in C^{0,1}$, $p \in [1,\infty]$. *Then there exists a continuous linear operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that:*

- $|| \operatorname{tr} u||_{L^p(\partial\Omega)} \leqslant c||u||_{1,p}$,
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}$.

Definice 3.2

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{k,p}}.$$

Věta 3.5

Let $\Omega \in C^{0,1}$ and let $p \in [1, \infty]$. Then

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$.*

Moreover

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1 - \frac{d}{p}$.*

$$X \hookrightarrow\hookrightarrow Y \Leftrightarrow X \leqslant Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$X \hookrightarrow\hookrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^\infty, \exists c : ||u^n||_{1,p} \leqslant c \implies \exists u^{n_j} : u^{n_j} \to u \text{ in } Y).$$

Důsledek (Trace theorem)
Let $\Omega \in C^{0,1}$. Then $\forall u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ we have integration by parts:

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\omega v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

Věta 3.6 (Poincaré)

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Let $\Omega_1, \Omega_2 \subseteq \Omega$, $|\Omega_i| > 0$ and $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$, $|\Gamma_i|_{d-1} > 0$. Let $\alpha_1, \alpha_2 \geqslant 0$ and $\beta_1, \beta_2 \geqslant 0$ and at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Then there exist $c_1, c_2 > 0$ such that $\forall u \in W^{1,p}(\Omega)$

$$c_1||u||_{1,p}^p \leqslant ||\nabla u||_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2| \int_{\Omega_2} u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2| \int_{\Gamma_2} u|^p \leqslant c_2||u||_{1,p}^p.$$

$$(||u||_{1,p}^p = ||u||_p^p + ||\nabla u||_p^p.)$$

Důkaz (Of the first (the only difficult) inequality)
TODO!!! □

4 Linear elliptic PDEs

Definice 4.1 (Elliptic)

Let $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$, where $\Omega \leqslant \mathbb{R}^d$ is bounded. We say that L is elliptic if $\exists c_1 > 0$ such that $\forall \zeta \in \mathbb{R}^d$ and almost all $x \in \Omega$

$$A\zeta \cdot \zeta \geqslant c_1|\zeta|^2.$$

Lemma 4.1

If u is classical solution, then $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$ on Γ_1: $B_{L,\delta}(u, \varphi) = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$.

Důkaz
TODO!!! □

Lemma 4.2

If $u \in C^2(\overline{\Omega})$ and $A, b, \mathbf{c}, \mathbf{d}$ are smooth and previous lemma holds $\forall \varphi \in C^1$, $\varphi|_{\Gamma_1} = 0$ and $u = u_0$ on Γ_1, then u is a classical solution.

Důkaz
TODO!!! □

Definice 4.2 (Weak solution)

Let $\Omega \subseteq \mathbb{R}^d$ Lipschitz, L be an elliptic operator, $u_0 \in W^{1,2}(\Omega)$, $f \in (W^{1,2}(\Omega))^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff

- $\operatorname{tr} u = \operatorname{tr} u_0$ on Γ_1 and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$, $\forall \varphi \in V$, where $V := \{\varphi \in W^{1,2}(\Omega) | \operatorname{tr} \varphi = 0 \text{ on } \Gamma_1\}$.

4.1 Existence of solution for coercive operators

Definice 4.3 (Elliptic form)

Let $B : V \times V \to \mathbb{R}$ bilinear nad V be a Hilbert space, $c_1, c_2 > 0$. We say that B is elliptic if it is

- V-bounded $\Leftrightarrow |B(u,\varphi)| \leqslant c_2 ||u||_V ||\varphi||_V$ and
- V-coercive $\Leftrightarrow B(u,u) \geqslant c_1 ||u||_V^2$.

Věta 4.3 (Lax-Milgram)

Let B be a bilinear elliptic form. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V : B(u,\varphi) =< F, \varphi > .$$

Definice 4.4

Let $B : V \to V^*$. We say that B is

- Lipschitz $\equiv \forall u, v \in V : ||B(u) - B(v)||_{V*} \leqslant c_2 ||u - v||_V$, $c_2 > 0$;
- Uniformly monotone $\equiv \forall u, v \in V :< B(u) - B(v), u - v >_V \geqslant c_1 ||u - v||_V^2$, $c_1 > 0$.

Věta 4.4 (Non-linear Lax-Milgram)

Let B be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V :< B(u), \varphi >=< F, \varphi > .$$

Důkaz
TODO!!! □

Důkaz (Lax-Milgram)
TODO!!! □

Věta 4.5

If $B_{L,\sigma}$ is bilinear, V-bounded and V-elliptic. Then there exists a unique weak solution u.

Důkaz
TODO!!! □

4.2 Existence via Fredholm alternative

TODO!!!

Věta 4.6

Let $\Omega \in C^{0,1}$, L be an elliptic operator and $\Gamma_1 = \partial\Omega$. Then

1. *Σ is at most countable and if infinite $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \to \infty$;*
2. *$(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1\ \exists! u : Lu = f + \lambda u$;*
3. *$\forall \lambda \notin \Sigma\ \exists C > 0\ \forall f \in L^2\ \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$ and $||u||_{1,2} \leqslant c||f||_2$;*

Důkaz
3) TODO improve convergence of u^{n_k} and show

$$u^{n_k} \to u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show $\{u^{n_k}\}$ is Cauchy in $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1||\nabla(u^n - u^m)||_2^2 \leqslant \int_\Omega A\nabla v^{n,m}\nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_\Omega \mathbf{c}\nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d}\nabla v^{n,m} v^{n,m} =$$

$$= \int_\Omega (f^n - f^m)v^{n,m} + \lambda(v^{n,m})^2 \pm - ||- \leqslant$$

$$\leqslant ||v^{n,m}||_2(||f^n - f^m||_2 + \lambda||v^{n,m}||_2 + ||\mathbf{c}||_\infty||\nabla v^{n,m}||_2 + ||\mathbf{d}||_\infty||\nabla v^{n,m}||_2 + ||b||_\infty||v^{n,m}||_2) \leqslant$$

$$\leqslant ||v^{n,m}||C(\lambda) \overset{u^n \text{ is Cauchy}}{\leqslant} C(\lambda)\varepsilon$$

$$\implies \nabla u^n \text{ is Cauchy sequence} \implies u^n \to u \text{ in } W_0^{1,2}(\Omega) \implies ||?||_{n_k} = 1$$

$$\int_\Omega A\nabla a u^n \nabla a\varphi + bu^n\varphi + \mathbf{c}\nabla u^n\varphi - \mathbf{d}\nabla ?u^n = \int_\Omega f^n\varphi + \lambda u^n\varphi.$$

$$n \to \infty$$

$$\int_\Omega A\nabla u\nabla\varphi + bu\varphi + \mathbf{c}\nabla u\varphi - \mathbf{d}\nabla\varphi u = \lambda\int u\varphi \Leftrightarrow Lu = \lambda u$$

But $\lambda \notin \Sigma$. □

Poznámka
Next we discussed homework.

4.3 Variational approach – minimization

Poznámka
$B_{L,\sigma}(u,v)$ must be symmetric! ($B_{L,\sigma}(u,v) = B_{L,\sigma}(v,u)$)

$$L = -\operatorname{div}(A\nabla u) + bu + \mathbf{c}\nabla u + \operatorname{div}(\mathbf{d}u)$$
$$B_{L,\sigma}(u,v) := \int_\Omega A\nabla u\cdot\nabla v + Buv + \mathbf{c}\cdot\nabla uv - \mathbf{d}\nabla vu + \int_\Gamma \sigma uv$$
$$B_{L,\sigma}(v,u) := \int_\Omega A\nabla v\cdot\nabla u + Bvu + \mathbf{c}\cdot\nabla vu - \mathbf{d}\nabla uv + \int_\Gamma \sigma vu$$
$$\implies A = A^T, \qquad \mathbf{c} = -\mathbf{d}$$

Věta 4.7

Let $B_{L,\sigma}$ be linear symmetric V-elliptic and V-bounded. $f \in V^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$, $u \in ?$. Then the following is equivalent:

- $u - u_0 \in V$ *and* $B_{L,\sigma}(u,v) =< f, \varphi > + \int_{\Gamma_2\cup\Gamma_3} g\varphi$;
- $u - u_0 \in V$ $\forall v \in W^{1,2}(\Omega)$, $v, u_0 \in V$
$$\frac{1}{2}B_{L,\sigma}(u,u) - < f, u > - \int_{\Gamma_2\cup\Gamma_3} gu \leqslant \frac{1}{2}B_{L,\sigma}(v,v) - < f, v > - \int_{\Gamma_2\cup\Gamma_3} gv.$$

Důkaz („1 $\implies$ 2“)

$$0 \overset{V-\text{elliptic}}{\leqslant} \frac{1}{2}B_{L,\sigma}(v-u,v-u) \overset{\text{linearity}}{=} \frac{1}{2}B_{L,\sigma}(v,v) + \frac{1}{2}B_{L,\sigma}(u,u) - \frac{1}{2}B_{L,\sigma}(u,v) - \frac{1}{2}B_{L,\sigma}(v,u) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u,u) - B_{L,\sigma}(u,v) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u,u-v) \overset{\text{weak formulation}}{=}$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + < f, u-v > + \int_{\Gamma_2\cup\Gamma_3} g(u-v)$$

□

Důkaz („2 $\implies$ 1")
u is minimizer, so set $v = u + \varepsilon\varphi$, $\varphi \in V$

$$\frac{1}{2}B_{L,\sigma}(u,u) - < j, u > - \int gu \leqslant \frac{1}{2}B_{L,\sigma}(u+\varepsilon\varphi, u+\varepsilon\varphi) - < j, u+\varepsilon\varphi > - \int g(u+\varepsilon\varphi) =$$

$$= \frac{1}{2}B_{L,\sigma}(u,u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi,\varphi) + \varepsilon B_{L,\sigma}(u,\varphi) - < f, u > - \varepsilon < f, \varphi > - \int ga - \varepsilon \int g\varphi$$

divide by ε and $\varepsilon \to 0_+$

$$0 \leqslant B_{L,\sigma}(u,\varphi) - < j, \varphi > - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \qquad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for $-\varphi \implies 0 = -||- \implies u$ is weak solution. □

Věta 4.8 (Dual formulation)

Let $Lu = -\operatorname{div}(A\nabla u)$ with A elliptic, bounded and symmetric, $\Gamma_1 \neq \varnothing$, $\Gamma = \varnothing$, $f \in V^*$, $g \in L^2(\Gamma_2)$, $u_0 \in W^{1,2}(\Omega)$. Then the following are equivalent:

- *u is a weak solution;*
- *$\nabla u = A^{-1}\mathbf{T}$, where $\mathbf{T}$ minimizes $\int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0 \mathbf{T}$ over the set $\tilde{V} := \Big\{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d) \mid \forall\varphi \in V : \int_\Omega \mathbf{T} \cdot$*

$$\left(\int_\Omega \mathbf{T}\cdot\nabla\varphi = < f, \varphi > + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div}\mathbf{T} = f \text{ in } \Omega, T\mathbf{u} = g \text{ on } \Gamma_2\right)$$

Důkaz („1 $\implies$ 2")
Let $\mathbf{V} \in \tilde{V}$ and $\mathbf{T} := A\nabla u \in \tilde{V}$.

$$0 \leqslant \frac{1}{2}\int_\Omega A^{-1}(\mathbf{V}-\mathbf{T})\cdot(\mathbf{V}-\mathbf{T}) = \int \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2}\int_\Omega A^{-1}\mathbf{T}\cdot\mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\left(\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0\mathbf{T}\right) + \int_\Omega \left(\nabla u_0(\mathbf{V}-\mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T}-\mathbf{V})\right) =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega \nabla(u-u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} + 0.$$

So $\mathbf{T}$ is minimizer of the formula above. □

Důkaz („2 $\implies$ 1“)
$\mathbf{T} \in \tilde{V}$ $\forall V \in \tilde{V}$: $\int_\Omega \frac{1}{2} A^{-1}\mathbf{T}\cdot\mathbf{T} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0 \mathbf{V}$. $\mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}$, $\mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$
$\forall \varphi \in V$: $\int_\Omega \mathbf{W} \cdot \nabla \varphi = 0$.

$$\int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T} + \varepsilon^2 A^{-1}\mathbf{W}\cdot\mathbf{W} + 2\varepsilon A^{-1}\mathbf{T}\cdot\mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by ε and $\varepsilon \to 0_+$:

$$0 \leqslant \int_\Omega A^{-1}\mathbf{T}\cdot\mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for $-\mathbf{W}$, co $0 = -||-$.

Now we find unique $u \in W^{1,2}$ $u - u_0 \in V$: $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega A^{-1}\mathbf{T} \cdot \nabla \varphi$ ($< F, \varphi >_V$).

$$\int_\Omega |A^{-1}\mathbf{T} - \nabla u|^2 = \int_\Omega (A^{-1}\mathbf{T} - \nabla u)(A^{-1}\mathbf{T} - \nabla u) =$$

$$= \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0) \cdot (A^{-1}\mathbf{T} - \nabla u) + \int_\Omega \nabla(u_0 - u)(A^{-1}\mathbf{T} - \nabla u) = 0 + 0 = 0$$

□

Lemma 4.9

Let X be a reflexive space and $\{u^n\}_{n=1}^\infty$ be a bounded sequence, $||u^n||_X \leqslant c < \infty$. Then $\exists u^{n_k}$, $\exists u \in x$: $u^{n_k} \rightharpoonup u$ ($\forall F \in X^* :< F, u^{n_k} > \to < F, u >$).

Věta 4.10 (Spectrum of symmetric operator)

V Hilbert infinity-dimensional space. Let B be linear, symmetric, V-elliptic and V-bonded operator. Then there exist $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$ and corresponding $\{u_i\}_{i=1}^\infty$ such that

- $B(u^k, \varphi) = \lambda_k \int_\Omega u^k \varphi$;
- $\lambda_k \to \infty$;
- $\{u^k\}_{k=1}^\infty$ *is basis in V and fulfils*

$$\int_\Omega u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_\Omega u u^i)$, *then* $\forall n : ||P^n u||_2 \leqslant ||u||_2$ *and* $B(P^n u, P^n u) \leqslant B(u, u)$.

Důkaz

Step 1: Construct λ_k, u^k: $\lambda_1 := \inf_{u \in V, ||u||_2 = 1} B(u,u)$ and denote u^1 function, where infimum is obtained. Then for $V^N = \left\{u \in V | \forall k \in [N] : B(u, u^k) = 0\right\}$ we do the same.

Step 2: The construction is OK:

$$0 < \lambda_1 = \lim_{n \to \infty} B(u^n, u^n), ||u^n||_2 = 1 \implies$$

$$\implies ||u^n||_V \leqslant C \implies u^{n_k} \rightharpoonup u \text{ in } V$$

$$V \hookrightarrow L^2 \implies u^{n_k} \to u \text{ in } L^2(\Omega) \implies ||u||_2 = 1$$

$$\lambda_1 = \lim_{n_k \to \infty} B(u^{n_k}, u^{n_k}) \geqslant B(u,u) \geqslant \lambda_1.$$

Step 3: λ_k, u^k eigenvalues, eigen functions: $\forall v \in V, ||v||_2 = 1$, $\lambda_1 = B(u^1, u^1) \leqslant B(v,v), \quad ||u^1||_2 = 1$

$$v = \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leqslant B(\frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2})$$

$$\lambda_1 ||u^1 + \varepsilon\psi||_2 \leqslant B(u^1 + \varepsilon\psi, u^1 + \varepsilon\psi) = B(u_1, u_1) + \varepsilon^2 B(\psi,\psi) + 2\varepsilon B(u,\psi) \leqslant \lambda_1 ||u^1||_2^2 + \lambda_1 \varepsilon^2 ||\psi||_2^2 + 2\varepsilon\lambda_1 \int_\Omega u^1$$

$$\varepsilon \to 0_+ \implies 2\lambda_1 \int_\Omega u^1\psi \leqslant 2B(u,\psi).$$

So $\lambda_1 \int_\Omega u^1\psi = B(u,\psi)$.

The same way we obtain $\lambda_k \int_\Omega u^k\psi \leqslant B(u,\psi)$ for $\psi \in V^N$.

$$u^1 : \lambda_1 \int_\Omega u^1\psi = B(u^1,\psi) \implies \psi = u^k \int_\Omega u^1 u^k = V(u_1, u^k).$$

But $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$, so $\int u^1 u^k = B(u^1, u^k) = 0$.

$$\implies \forall i \in [k-1] : \int_\Omega u^k u^1 = B(u^k, u^i) = 0.$$

Step 4: $\lambda_k \nearrow \infty$. We already know $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$. Assume a contradiction $\lambda_k \leqslant C < \infty$. $c_1 ||u^k||_V^2 \leqslant B(u^k, u^k) = \lambda_k ||u^k||_2^2 = \lambda_k < C$.

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \to u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$||u^n - y^m||_2^2 = ||u^n||_2^2 + ||u^m||_2^2 - 2\int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_? n} B(u^n, u^m) = 2 \implies \text{ not Cauchy.}$$

Step 5: λ_k are all eigenvalues (u^k is basis of V and of L^2). Assume that $\lambda \neq \lambda_j$ is also eigenvalue, so $\exists u : B(u,\varphi) = \lambda \int_\Omega u\varphi \forall\varphi$. We can find $i \in \mathbb{N}$, so $\lambda_i < \lambda < \lambda_{i+1}$.

$$B(u, u^j) = \lambda \int uu^j \wedge B(u^j, u) = \lambda_j \int u^j u \implies B(u, u_j) = 0$$

4.4 Regularity of weak solution

Poznámka
We assume that we have $u \in W^{1,2}(\Omega)$ a weak solution

$$-\operatorname{div} A\nabla u + Vu + \mathbf{c}\cdot\nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When $u \in W^{2,2}_{loc}(\Omega)$, when $u \in W^{2,2}(\Omega)$, when $u \in W^{k,2}_{loc}(\Omega)$, $u \in W^{k,2}(\Omega)$.

Simplify $-\operatorname{div} A\nabla u = f - bu - \mathbf{c}\nabla u - u\operatorname{div}\mathbf{d} - \nabla u\cdot\mathbf{d} = \tilde{f}$. If $u \in W^{1,2}$, $f \in L^2$, $b \in L^\infty$, $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$.

Problem is reduced to

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma_1,$$
$$(A\nabla u)\cdot\mathbf{v} = g \text{ on } \Gamma_2,$$
$$(A\nabla u)\cdot\mathbf{v} + \sigma u = g \text{ on } \Gamma_3.$$

Definice 4.5 (Interior regularity)

$u \in W^{2,2}_{loc}(\Omega)$; assumptions: $A \in W^{k+1,\infty}$, $f \in W^{k,2}(\Omega) \implies u \in W^{k+1,2}_{loc}(\Omega)$.

Definice 4.6 (Boundary regulariry)

$u \in W^{2,2}(\Omega)$; assumptions: on $\Omega \in C^{k+1,\infty}$, $g \in W^{\frac{1}{2}?,2}(\partial\Omega)$ and $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\varnothing\} \implies u \in W^{2,2}(\Omega)$.

Věta 4.11 (Interior regularity)

Let A be an elliptic operator and $u \in W^{1,2}$ solves

$$\int_\Omega A\nabla u\cdot\nabla\varphi = \int_\Omega f\varphi, \qquad \forall\varphi \in W^{1,2}_0(\Omega)\ \forall f \in L^2(\Omega).$$

Then if $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$, $f \in W^{k,2}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.

Moreover $\forall\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega\ \exists c(\tilde{\Omega}, A)$:

$$||u||_{W^{k+2,2}}(\tilde{\Omega}) \leqslant c(||f||_2 + ||u||_{W^{1,2}(\Omega)}).$$

Důkaz

$k=0$: Recall $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x+he_k)-v(x)|^2}{h^2} \leqslant c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2\right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left|\frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j}\right|^2}{h^2} \leqslant c,$$

$$\int_{\Omega_h} \left|\frac{\nabla u(x+he_i)-\nabla u(x)}{h}\right|^2 \leqslant c.$$

$\int_\Omega A\nabla u \nabla\varphi = \int_\Omega f\varphi$

$$h>0, \varphi \in W_0^{1,2}(\Omega), \varphi(x)=0 \text{ if } \operatorname{dist}(x,\partial\Omega) \subset h.$$

Set $\varphi(x) := \psi(x-he_k)$.

$$\implies \int_\Omega A(x)\nabla u(x)\nabla\psi(x-he_k) = \int_\Omega f(x)\psi(x-he_k) =$$

$$= \int_\Omega A(x+he_k)\nabla u(x+he_k)\cdot\nabla\psi(x)dx.$$

Set $\varphi(x) := \psi(x)$:

$$\int_\Omega A(x)\cdot\nabla u(x)\cdot\nabla\psi(x) = \int_\Omega f(x)\psi(x)dx.$$

$$\int_\Omega A(x+he_k)(\nabla u(x+he_k)-\nabla u(x))\cdot\nabla\psi(x) =$$

$$= -\int (A(x+he_k)-A(x))\nabla u(x)\cdot\nabla\psi(x) + \int_\Omega f(x)(\psi(x-he_k)-\psi(x)).$$

Set $\psi := (u(x+he_k)-u(x))\tau^2(x)$, $\tau(x)=0$, if dist $\in (x,\partial\Omega)$, $\tau \in C^1(\tilde{\Omega})$.

Evaluate all terms ($w^{h,i} = u(x+he^i)-u(x)$):

$$\int_\Omega A(x+he_i)\nabla w^{h,i}\cdot(\nabla w^{h,i}\tau^2 + 2w^{h,i}\tau\nabla\tau) \geqslant$$

$$\overset{ellip.}{\geqslant} c_1\int_\Omega |\nabla w^{hi}|^2\tau^2 - \int_\Omega \frac{2||A||_\infty |w^{h,i}| - |\nabla\tau|(|\nabla w^{hi}|\sqrt{c_1}\tau)}{\sqrt{c_1}} \geqslant$$

$$\geqslant \frac{c_1}{2}\int_\omega |\nabla w^{h,i}|^2\tau^2 - \frac{2}{c_1}||A||_\infty^2||\nabla\tau||_\infty^2 h^2 \int_{\Omega_h} \frac{|u(x+he_i)-u(x)|^2}{h^2} \geqslant$$

$$\geqslant \frac{c_1}{2}\int_\Omega |\nabla w^{h,i}|^2\tau^2 - \frac{2||A||_\infty^2||\nabla\tau||_\infty^2}{c_1}h^2 c||\nabla u||_2^2$$

TODO?

Věta 4.12 (Regularity up to the boundary)

Let u be a weak solution $-\operatorname{div}(A\nabla u) = f$ in Ω, $A\nabla u \cdot \mathbf{v} = g$ on Γ_2, $A\nabla u \cdot \mathbf{v} + \sigma u = g$ on Γ_3, $u = u_0$ on Γ_1.

Assume that $\Omega \in C^{k+1,\infty}$, $A \in W^{k,\infty}$, $f \in W^{k-1,2}$, $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega)$, $\sigma \in W^{k,\infty}(\partial\Omega)$ and Γ_1, Γ_2, Γ_3 are smooth open (in partialΩ) and $\overline{\Gamma_i} \cap \overline{\Gamma_j} = \varnothing$ $\forall i \neq j$.

Then $u \in W^{k+1,2}(\Omega)$.

Důkaz (Step 1: Flat boundary)
$\Omega = (-1,1)^{d-1} \times (0,1)$. Assume that $u \in W^{1,2}(\Omega)$ and $u = 0$ on $(x,0)$. We want that $u \in W^{2,3}((-1+\delta, 1-\delta)^{d-1} \times (0, 1-\delta)$.

1a tangential derivatives $\frac{\partial u}{\partial x_1} \in W^{1,2}(-||-)$. 1b normal derivative $\frac{\partial^2 u}{\partial x_d^2} \in L^2(-||-)$.

1a: WF $-\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi$ $\forall \varphi \in W_0^{1,2}(\Omega)$. Take continuous $\tau = 1$ in $-||-$ and $\tau = 0$ in $\Omega\setminus$ "inflated" $-||-$.

$$\varphi(x) = \psi(x - he_i)\tau, \qquad i \in [d-1], \psi \in W_0^{1,2}(\Omega\setminus \text{"inflated"} \ -||-)$$

Redefiny interior regularity

$$\int_\Omega (A(x+he_i)\nabla u(x+he_i) - A(x)\nabla u(x))\nabla\varphi(x) = \int_\Omega f(\psi(x-he_i) - \psi(x)).$$

Set $\psi = (u(x+he_i) - u(x))\tau^2 \in W_0^{1,2}$ and apply local regularity.

1b: $\varphi \in C_0^\infty(-||-)$

$$-\int_\Omega \sum_{i,j}^d \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})\varphi = -\int_\Omega \operatorname{div}(A\nabla u)\varphi = \int_\Omega A\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi$$

$$-\int_\Omega a_{dd}\frac{\partial^2 u}{\partial x_d^2}\varphi = \underbrace{\int_\Omega f\varphi}_{\in L^2(\Omega)} + \int_\Omega \varphi \left(\sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1,\neg(i=j=d}^d) \right) a_{ij}\frac{\partial u}{\partial x_i x_j}.$$

$$||a_{dd}\frac{\partial^2 u}{\partial x_d^2}||_2^2 \leqslant ||f + \sum \frac{\partial a_{ij}}{\partial x_1}\frac{\partial u}{\partial x_j} + \sum_{\neg(i=j=d)} a_{ij}\frac{\partial^2 u}{\partial x_i x_j}||_2^2 \leqslant C.$$

A is elliptic

$$c_1|\zeta|^2 \leqslant a_{ij}\zeta_i\zeta_j$$

Special choice $\zeta = (0,\dots,0,1)$, $0 < c_1 \leqslant a_{dd}(x) \implies ||\frac{\partial^2 u}{\partial x_d^2}||_{L^2}^2 \leqslant C(DATA?)$ □

Důkaz (Step 2: Transfer from flat to small parts of $\partial\Omega$)
TODO!!!

TODO!!! □

Důkaz (Step 3: Introduce a proper covering of $\partial\Omega$ and use step 2)

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$

? Ω $u \in W^{2,2}_{loc}(\Omega)$. ? of $\partial\Omega$, apply step 2.

Define $w := u - u_0 \in W_0^{1,2}(\Omega)$.

$$-\operatorname{div}(A\nabla w) = f + \operatorname{div}(A\nabla u_0)$$

if $f \in L^2$ and $\operatorname{div}(A\nabla u_0) \in L^2$, e.g. $A \in W^{1,\infty} \wedge u_0 \in W^{2,2}(\Omega)$. □

5 Bochner integral

Definice 5.1 (Measurability)

We say that $f : I \to X$ is measurable (strongly, Bochner) if $\exists \{s_j\}_{j=1}^{\infty}$ simple functions, $||f(t) - s_n(t)||_X \to 0$ as $n \to \infty$ for almost every $t \in I$.

Věta 5.1 (Measurability)

$f : I \to X$ is measurable iff

1. *f is almost separably valued;*

$$\exists E \subset I : |E| = 0, f(I \backslash E) \text{ is separable.}$$

2. *f is weakly measurable;*

$$\forall F \in X^* :< F^*, u(t) >_X \text{ is Lebesgue measure w.r.t } t \in I.$$

Definice 5.2 (Bochner integral for simple function)

Let $s : I \to X$ be a simple function on ?. We define

$$\int_I s(t)dt := \sum_{j=1}^{n} X_j |I_j|$$

Definice 5.3 (Bochner integral for measurable functions)

Let $s : I \to X$ be a Bochner measurable function. We say that f is Bochner integrable if $\exists \{s^n\}_{n=1}^{\infty}$ such that $s^n(t) \to f(t)$ a. a. t and $\int_I ||s^n(t) - f(t)||_X dt \to 0$ as $n \to \infty$ and we set

$$X \ni \int_I f(t)dt = \lim_{n\to\infty} \int_I s^n(t)dt.$$

$$\int_I s(t)dt := \sum_{j=1}^{n} X_j |I_j|$$

Definice 5.4 ($L^p(O,T,X)$ space)

Let X be a Banach space

$$L^p(O,T,X) = \left\{ f : (O,T) \to X \text{ bochner integrable} \middle| \int_I ||f(t)||_X^p < \infty \right\}$$

$$||f||_{L^p(O,T,X)} = \left(\int_I ||f(t)||_X^P dt \right)^{\frac{1}{p}}.$$

Věta 5.2 (Dual space)

Let X be a Banach space, separable and $p \in [1,\infty)$, then

$$(L^p(O,T,X))^* = L^{p'}(O,T,X^*)$$

5.1 Sobolev-Bochner spaces

Definice 5.5

Let $f : I \to X$ be Bochner integrable. We say that $g : I \to X$ is a weak derivative of f w. r. t. iff g is Bochner integrable and $\forall \tau \in C_0^\infty(I) : \int_I f(t)\tau'(t)dt = -\int_I g(t)\tau(t)dt$.

Poznámka
If $f \in L^1(I,X)$ and $\frac{\partial f}{\partial t} \in L^1(I,X)$, then $f \in C(I,X)$.

Věta 5.3

$$W^{1,p}(I,X) := \{f \in L^p(I,X), \partial_t f \in L^p(I,X)\}, \qquad ||f||_{W^{1,p}(I,X)} = \begin{cases} \left(\int_I ||f||_X^p + ||\partial_t f||_X^p\right)^{\frac{1}{p}}, & p \in [\\ \text{esssup}_{t\in I}(||f(t)||_X + ||\partial_t f||_X), & p = \end{cases}$$

Then $W^{1,p}(I,X)$ is a Banach space, is separable for $p < \infty$ and X separable and *is reflexive if $p \in (1,\infty)$ and X is reflexive and separable.*

5.2 Time derivative in heat/wave equations – Gelfand triple

Poznámka (Motivation)

$$\partial_t u - \Delta u = 0 \text{ in } (0,T)\times\Omega, u = 0 \text{ on } (0,T)\times\partial\Omega, x(0,x) = u_0(x) \text{ for } x \in \Omega, \qquad \Omega \subseteq \mathbb{R}^d$$

.

Definice 5.6 (Gelfand triple)

We say that X, H, X^* is Gelfand triple iff $X \overset{\text{dense}}{\hookrightarrow} H \cong H^* \overset{\text{dense}}{\hookrightarrow} X^*$.

Například

$$X = W_0^{1,2}(\Omega), H = L^2(\Omega), X^* = (W_0^{1,2}(\Omega))^*,$$

Neboť $W_0^{1,2}$ is dense in $C_0 \overset{\text{dense}}{\hookrightarrow} L^2(\Omega)$ and $f \in (W_0^{1,2}(\Omega))^* \implies \exists! u \in W_0^{1,2}(\Omega) : -\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$.

$$\forall \varphi \in W_0^{1,2}(\Omega) < f,\varphi >= \int_\Omega \nabla u \cdot \nabla\varphi = \lim_{n\to\infty} \int_\Omega \nabla u^n \nabla\varphi = \lim_{n\to\infty} - \int_\Omega \Delta u^n \varphi = \lim_{n\to\infty} (f^n,\varphi)_{L^2(\Omega)},$$

where $\{u^n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$, $u^n \to u$ in $W_0^{1,2}(\Omega)$,

$$(X = W_0^{1,p}(\Omega \cap L^2(\Omega)), H = L^2(\Omega))$$

Definice 5.7

Let X, H, X^* be Gelfand triple, $\varphi : H \to H^*$ is Riesz representation and define $i : X \to X^*$, such that $\forall x_0, x \in X$:

$$< i(x_0), x >_X := (\mathrm{id}(x_0), \mathrm{id}(x))_H =< \varphi\,\mathrm{id}(x_0), \mathrm{id}(x) >_H,$$

i maps X densely onto X^*.

Lemma 5.4

Let $u \in L^1(0,T,H)$, $\partial_t u \in L^1(0,T,X^*)$ and X, H, X^* be a Gelfand triple. Then $\forall w \in X$ $\forall \tau \in C_0^1(0,T)$ we have

$$\int_0^T < \partial_t u, w > \tau dt =< \int_0^T \partial_t u \tau dt, w >_X =$$

$$= - < \int_0^T u\tau' dt, w >_X = - \int_0^T < u\tau', w >_X dt =$$

$$= -\int_0^T (u\tau', w)_H dt \overset{if\ \partial_t u \in L^1(0,T)}{=} \int_0^T (\partial_t u \tau, w)_H.$$

Věta 5.5 (Integration by parts for Sobolev-Bochner function)

Let $p \in (1, \infty)$, X, H, X^* a Gelfond triple, $u, v \in L^p(0, T, X)$, $\partial_t u, \partial_t v \in L^{p'}(0, T, X^*)$. Then $u, v \in C([0, T], H)$ and $\forall 0 \leqslant t_1 < t_2 \leqslant T$.

$$\int_{t_1}^{t_2} < \partial_t u, v >_X + < \partial_t v, u >_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Důkaz

Step 1) Modify u, v in terms of the Steklov averages $u_h = \fint_t^{t+h} u(\tau) d\tau$.

Step 2) Prove for u_h, v_h from step 1).

Step 3) $h \to 0_+$. □

Důkaz („Step 1)“)
Define $u_h(t) := \frac{1}{h}\int_t^{t+h} u(\tau)d\tau$, $\forall t \in (0, T-h)$. $u_h \to h$ $L^p(0, T-h_0, X)$, $\forall h_0 \in (0, T)$. We want „$(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$“.

$$(\partial_t u)_h \to \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \qquad \forall h_0 \in (0, T).$$

$$\varphi \in C_0^\infty(0, T-h) : \int_0^{T-h} u_h(t)\varphi'(t)dt = \frac{1}{h}\int_0^{T-h} \varphi'(t)\int_t^{t+h} u(t)d\tau dt =$$
$$= \frac{1}{h}\int_0^{T-h} \varphi'(t)\left(\int_0^{t+h} u(\tau)d\tau - \int_0^t u(\tau)d\tau\right) =$$
$$= -\frac{1}{h}\int_0^{T-h} \varphi(t)(u(t+h) - u(t)) \Leftrightarrow \partial_t u_h = \frac{u(t+h) - u(t)}{h}.$$

$$\varphi \in C_0^\infty(0, T-h) : \int_0^T \varphi(t)(\partial_t u)_h(t)dt = \frac{1}{h}\int_0^{T-h} \varphi(t)\int_t^{t+h} \partial_t u(\tau)d\tau dt =$$
$$= \frac{1}{h}\int_0^{T-h} \varphi(t)\left(\int_0^{t+h} \partial_t u(\tau)d\tau - \int_0^t \partial_t u(\tau)d\tau\right) dt = (*)$$

$$\frac{1}{h}\int_0^{T-h} \varphi(t)\left(\int_0^t \partial_t u(\tau)d\tau\right) dt = \int_0^{T-h}\int_0^{T-h} \varphi(t)\partial_t u(\tau)\chi_{\tau \leqslant t} d\tau dt =$$
$$= \frac{1}{h}\int_0^{T-h} \partial_t u(\tau)\left(\int_t^{T-h} \varphi(t)dt\right) d\tau.$$
$$(*) = \frac{1}{h}\int_0^{T-h} \partial_t u(\tau)\underbrace{\left(\int_{\tau-h}^{\tau} \varphi(t)dt\right)}_{C_0^\infty(0,T)} d\tau = -\frac{1}{h}\int_0^{T-h} u(\tau)\left(\varphi(\tau) - \varphi(\tau - h)\right) d\tau dt.$$

□

Důkaz („Step 2)")
We want

$$\int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

$$\Leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

$$\int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int_{t_1}^{t} v(\tau) d\tau \right)_H =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1 - h_2}^{t} v(\tau + h_2) d\tau - \int_{t_1}^{t} v(\tau) d\tau \right)_H =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1}^{t} v(\tau + h_2) - v(\tau) d\tau \right)_H dt +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1 - h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau + h_2) - v(\tau) d\tau, \int_{t_1}^{\tau} u(t+h_1) - u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1 - h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau + h_2) - v(\tau) d\tau, \int_{t_2}^{t_2 + h_1} u(t) - \int_{t_2}^{t_2 + h_1} u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1 - h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt =$$

$$= \int_{t_1}^{t_2} \left(\frac{v(\tau + h) - v(\tau)}{h_2}, ⨍_{\tau}^{\tau + h_1} u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau + h_2) - v(\tau), \int_{t_2}^{t_2 + h} u(t) dt \right)_H + \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1 - h_2}^{t_1} v(\tau) d\tau)_H =$$

$$- \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau)) d\tau + REST$$

$$REST = \frac{1}{h_1 h_2} \left(\int_{t_2}^{t_2 + h_2} v(t) dt - \int_{t_1}^{t_1 + h_2} v(t) dt, \int_{t_2}^{t_2 + h} u(t) dt \right) + SIMILAR =$$

$$= (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \dots$$

□

Důkaz („Step 3)“)
We have

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X \, dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let $h_1 \to 0_+$ and $h_2 \to 0_+$. We have $\partial_t u_{h_1} \to \partial_t u$ in $L^{p'}(0,T,X^*)$, $\partial_t v_{h_2} \to \partial_t v$ in $L^{p'}(0,T,X^*)$, $u_{h_1} \to u$ in $L^p(0,T,X)$, $V_{h_2} \to v$ in $L^p(0,T,X)$. So for almost all t in $(0,T)$: $v_{h_2}(t) \to v(t)$ in $X \hookrightarrow H$ and $u_{h_1}(t) \to u(t)$ in $X \hookrightarrow H$.

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Now, it is enough to show $u, v \in C([0,T); H)$. We show that u_h is Cauchy in $C([0,T]; H)$. Use IBP $u_{h_1} = u_{h^n} - u_{h^m}$, $v_{h_2} = u_{h^n} - u_{h^m}$:

$$||u_{h^n}(t_2) - u_{h^m}(t_2)||_H = ||u_{h^m}(t_1) - u_{h^m}(t_1) + 2 \int_{t_1}^{t_2} \langle \partial_t (u_h^m - u_h^n), u_{h^n} - u_{h^m} \rangle_X ||$$

$$||u_{h^n} - u_{h^m}||^2_{C\left(\left[\frac{T}{4},T\right];L^2(\Omega)\right)} = \sup_{t_2 \in \left(\frac{T}{2},T\right)} ||u_{h^n}(t_2) - u_{h^m}(t_2)||_H^2 \leqslant$$

$$\leqslant ||u_{h^m}(t_1) - u_{h^n}(t_1)||_H^2 + \int_0^T ||\partial_t(u_{h^n}) - \partial u_{h^m}||_{X*} ||u_{h^m} - u_{h^n}||_X dt.$$

Choose t_1 such that $u_h(t_1) \to u(t_1)$ in H:

$$\leqslant ||u_h(t_1) - u_{h^m}(t_1)||_H^2 + ||\partial_t u_{h^m} - \partial_2 u_{h^n}||_{L^p(X^*)} \cdot \ldots$$

$$u \in C\left(\left[\frac{T}{4}, T\right]; L^2(\Omega)\right) \wedge u \in C\left(\left[0, \frac{3T}{4}\right]; L^2(\Omega)\right) \to u \in C\left([0,T]; L^2(\Omega)\right) (u(t_1), v(t_1))_H.$$

□

6 Parabolic equations

Poznámka
Ω open set in $\mathbb{R}^d$, $T > 0$, L elliptic operator,

$$\partial_t u + Lu = f \text{ in } Q = (0,T) \times \Omega, \qquad u = 0 \text{ on } (0,T) \times \partial\Omega, \qquad u(0,x) = u_0(x) x \in \Omega.$$

$$Lu = -\operatorname{div}_x(A(t,x)\nabla_x u(t,x)) + b(t,x)u(t,x) + \mathbf{c}(t,x)\nabla u(t,x) + \operatorname{div}(\mathbf{d}(t,x)u(t,x)),$$

$$A, b, \mathbf{c}, \mathbf{d} \in L^\infty(\Omega).$$

$$A(t,x) \cdot \xi \cdot \xi \geqslant c_1 |\xi|^2, \forall \xi \in \mathbb{R}^d \text{ and almost all } (t,x) \in Q.$$

6.1 Formal a priory estimates

Poznámka
Multiply by u and $\int_\Omega dx$ and use IBP.

$$\int_\Omega \partial_t u u + \int_\Omega A\nabla u \nabla u = \int_\Omega fu - bu^2 - \mathbf{c}\nabla u u + \mathbf{d} u \nabla u.$$

Hölder's inequality:

$$\frac{d}{dt}\frac{||u||_2^2}{2} + C_1||\nabla u||_2^2 \leqslant ||f||_2||u||_2 + ||b||_\infty ||u||_2^2 + ||\mathbf{c}||_\infty ||\nabla u||_2 ||\nabla u||_2 + ||\mathbf{d}||_\infty ||\nabla u||_2 ||\nabla u||_2 \leqslant C_1 \frac{||\nabla u||_2^2}{2} + C(\mathbf{c}$$

Poincaré's inequality:

$$\frac{d}{dt}||u||_2^2 + \mathbf{c}||u||_{1,2}^2 \leqslant C(\mathbf{c}, \mathbf{d}, b)||f||_2^2 + K||u||_2^2.$$

Grönwall's inequality:

$$\sup_{t \in (0,T)} ||u(t)||_2^2 \leqslant + \int_0^T ||f||_2^2)$$

$$\int_0^T ||u||_{1,2}^2 dt \leqslant C$$

$$||\partial_t u||_{(W_0^{1,2})^*} = \sup_{||\varphi|| \leqslant 1} \langle \partial_t u, \varphi \rangle = \sup \langle f - Lu, \varphi \rangle =$$

$$= \sup_{||\varphi|| \leqslant 1} \int_\Omega (f - ? - bu - \mathbf{c}\nabla u)\varphi - \int_\Omega (A\nabla u - \mathbf{d}u)\nabla\varphi \leqslant \int_0^T ||f||_2^2 + c(||u||_2^2 + ||\nabla u||_2^2).$$

Definice 6.1

Let $\Omega \subseteq \mathbb{R}^d$ open and bounded, L be an elliptic operator, $u_0 \in L^2(\Omega)$, $f \in L^2(0, T, V^*)$ $(V = W_0^{1,2}(\Omega))$. We say that u is a weak solution to

$$\partial_t u + Lu = f \text{ in } (0, T) \times \Omega,$$

$$u = 0 \text{ on } (0, T) \times \partial\Omega,$$

$$u(0) = u_0 \text{ in } \Omega$$

iff $u \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$, $u(0) = u_0$ and for almost all $t \in (0, T)$ and $\forall \varphi \in V$:

$$\langle \partial_t u, \varphi \rangle_V + \int_\Omega A\nabla u \cdot \nabla\varphi + bu\varphi + \mathbf{c} \cdot \nabla u \varphi - \mathbf{d}\nabla\varphi u = \langle f, \varphi \rangle_V.$$

6.2 Existence and uniqueness

Věta 6.1

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^2(0,T,V^*)$, $u_0 \in L^2(\Omega)$ and L be elliptic operator. Then $\exists! u$ – weak solution.

Důkaz (Uniqueness)
u_1, u_2 are weak solutions. Define $w := u_1 - u_2 \in L^2(0,T,V) \cap W^{1,2}(0,T,V^*)$. WF for u_1 – WF for u_2:

$$\langle \partial_t w, \varphi \rangle + \int_\Omega A\nabla w \cdot \nabla\varphi = \int_\Omega -bw\varphi - \mathbf{c}\nabla w\varphi + \mathbf{d}\cdot\nabla\varphi w.$$

Follow almost everywhere, replace u by w. Set $\varphi = w \implies$

$$\langle \partial_t w, w\rangle + \mathbf{c}\|w\|_{1,2}^2 \leqslant c\|w\|_2^2.$$

Integrate in respect of time, use IBP-formula for $\langle\cdot,\cdot\rangle$:

$$\int_0^t \langle \partial_t w, w\rangle = \frac{1}{2}\|w(t)\|_2^2 - \frac{1}{2}\|w(0)\|_2^2 = \frac{1}{2}\|w(t)\|_2^2.$$

$$\implies \|w(t)\|_2^2 \leqslant c\int_0^t \|w(\tau)\|_2^2 d\tau$$

$$\frac{d}{dt}\underbrace{\int_0^t \|w(\tau)\|_2^2 d\tau}_{=:g(t)\geqslant 0} \leqslant c\int_0^t \|w(\tau)\|_2^2 d\tau$$

$$g' \leqslant c\cdot g$$

From Grönwall's inequality

$$g(t) \leqslant e^{c\cdot t}g(0) \implies \int_0^t \|w(\tau)\|_2^2 d\tau \leqslant e^{ct}\int_0^0 \|w(\tau)\|_2^2 d\tau = 0 \implies w(t) = 0.$$

□

Důkaz (Existence (via Galerkim approximation))
We know $\exists \{w_j\}_{j=1}^\infty$ basis of $\mathbf{V}$, which is ortonormal in L^2 and $\|P^N u\|_V \leqslant c\|u\|_V$, where P^N is orthogonal projection in $L^2(\Omega)$ onto $\{w_j\}_{j=1}^N$.

We solve for $u^n(t,x) = \sum_{i=1}^n a_i^n(t)w_i(x)$. We want

$$\langle \partial u^n, w_j\rangle = -\int_\Omega A\nabla u^n\nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n + \langle f, w_j\rangle$$

for $j \in [n]$ for almost all $t \in (0,T)$ (weak formulation of the problem for n, call it WFn). □

Důkaz („Existence of u^n“)
LHS of WFn:

$$\sum_{i=1}^{n} \langle \partial_t a_i^n w_i, w_j \rangle_V = \sum_{i=1}^{n} \partial_t a_i^n(t) \langle w_i, w_j \rangle_V = \sum_{i=1}^{n} \partial_t a_i^n \delta_{ij} = \partial_t a_j^n(t).$$

RHS of WFn:

$$\sum_{i=1}^{n} a_i^n(t) \left(\underbrace{-\int_\Omega A\nabla w_i \nabla w_j + b w_i w_j + \mathbf{c}\nabla w_i w_j - \mathbf{d}\nabla w_j w_i}_{G_{ij}(k) \text{ -- bounded and measurable?}} \right) + \underbrace{\langle f(t), w_j \rangle}_{g_j^{(t)} \text{ -- measurable? on } g\in L^2(0,T)} .$$

So

$$\frac{d}{dt} a_j^n(t) = \sum_{i=1}^{n} a_i^n(t) G_{ij}(t) + g_j(t), \qquad j \in [n].$$

Initial data: $u^n(0) := P^n u_0$ ($a_j^n(0) := \int_\Omega u_0 w_j$).

ODE $\implies$ $\exists \tilde{T} \leqslant T$ and $a_i^n(t) \in AC$ on $[0, \tilde{T})$ and solve for almost all $t \in (0, \tilde{T})$. Moreover either we can set $\tilde{T} = T$ or $|a^n(t)| \overset{t \to \tilde{T}}{\to} \infty$.

Now we prove $\tilde{T} = T$. We show $|a^n(t)| \leqslant c$.

Multiply WFn for j by $a_j^n(t)$ and sum it:

$$LHS = \sum_{j=1}^{n} a_j^n \langle \partial_t u_j^n, w_j \rangle = \left\langle \partial_t u^n, \sum_{j=1}^{n} a_j^n w_j \right\rangle = \langle \partial u^n, u^n \rangle.$$

$$RHS = \underbrace{\langle \partial_t u^n, u^n \rangle}_{= \frac{d}{dt}\|u^n\|_2^2} + c_1 \|u^n\|_V^2 \leqslant c(\|f\|_{V*}^2 + \|u^n\|_2^2).$$

Grönwall: $\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leqslant c(\|u^n(0)\|_2^2) + \int_0^T \|f\|_{V*}^2$.

$$\forall t < \tilde{T} : \|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leqslant c \left(\int_0^{\tilde{T}} \|f\|_{V*}^2 + \|u_0\|_2^2 \right) \leqslant \tilde{c}.$$

$$\lim_{t \to \tilde{T}_-} |a^n(t)|^2 = \lim_{t \to \tilde{T}_-} \|u^n(t)\|_2^2 \leqslant \tilde{c}.$$

□

Důkaz

$$\|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \leqslant c(f, u_0).$$

Time derivative

$$\|\partial u^n\|_{V*} = \sup_{w \in V, \|w\| \leqslant 1} \langle \partial_t u^n, w \rangle = \sup_{w \in V, \|w\| \leqslant 1} \int_\Omega \partial u^n w = \sup_{\dots} \int_\Omega \partial u^n P^n w.$$

WFn:

$$\leqslant \sup_{\dots} c(\|f\|_{V*} + \|u^n\|_V)\|P^n w\|_V \leqslant \sup_{\dots} \tilde{c}(\|f\|_{V*} + \|u^n\|_V)\|w\|_V \leqslant$$

$$\leqslant \tilde{(}\|f\|_{V*} + \|u^n\|_V).$$

$$\int_0^T \|\partial_t u^n(t)\|_{V*}^2 \leqslant \tilde{\tilde{c}} \int_0^T (\|f(t)\|_{V*}^2 + \|u^n(t)\|_V^2) \leqslant c(f, u_0).$$

u^n is a bounded sequence in $L^2(0,T,V) \cap W^{1,2}(0,T,V^*)$ so $\exists$ a subsequence u^{m_n}:

$$u^{m_n} \rightharpoonup u \text{ in } L^2(0,T,V), \qquad \partial_t u^n \rightharpoonup \partial_t u \text{ in } L^2(0,T,V^*).$$

To show u is a weak solution.

TODO? □

TODO!!!

Důkaz (Initial condition)
$\tau \in C_0^\infty(-\infty, T)$:

$$-\int_0^T \int_\Omega u^n w_j \partial_t \tau - \int_\Omega u^n(0) w_j \tau(0) + \int_0^T (\dots)\tau \dots = 0.$$

$$\to -\int_0^T \int_\Omega u w_j \partial_t \tau - \int_\Omega u_0 w_j \tau(0) + \int_0^T (\dots)\tau = 0.$$

Integration by parts in time:

$$u \in L^2(W_0^{1,2}(\Omega)) \ni \tau w_j,$$

$$\partial_u \in L^2((W_0^{1,2}(\Omega))^*) \ni \partial_t(\tau w_i) = \partial_t \tau w_i \in L^2,$$

$$-\int_0^T \int_\Omega u w_j \partial_t \tau - \int_\Omega w_j \tau(0) = -\int_0^T \langle u, \partial_t(\tau w_j)\rangle - \int_\Omega u_0 w_j \tau(0) =$$

$$= \int_0^T \langle \partial_t u, \tau w_j \rangle + \int_\Omega u(0)\tau(0) w_j - \int_\Omega u_0 w_j \tau(0).$$

□

6.3 Regularity of parabolic equations

TODO Example?

Věta 6.2

Let $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in L^\infty$, $\operatorname{div} \mathbf{d} \in L^\infty$, A, ∇A, $\partial_t A \in L^\infty$, $f \in L^2(0, T, L^2(\Omega))$, *then* $\forall \delta > 0$*:*

$$\int_\delta^T \|\partial_t u\|_2^2 + \sup_{t \geqslant \delta} \|\nabla u(t)\|_2^2 \leqslant \frac{c}{\delta}$$

.

Moreover if $u_0 \in W_0^{1,2}(\Omega)$*, then* $\partial_t u \in L^2(0, T, L^2(\Omega))$*,* $u \in L^\infty(0, T, W_0^{1,2}(\Omega))$*.*

Moreover $u \in L^2(0, T, W_{loc}^{1,2}(\Omega))$ *and if* $\Omega \in C^{1,1}$*, then* $u \in L^2(0, T, W^{2,2}(\Omega))$*.*

Důkaz
Consider u^n–Galeikin approximation

$$u^n(t,x)=\sum_{i=1}^n a_i^n w_i : \int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n \nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n = \langle f, w_j\rangle.$$

Multiply by $\partial_t a_j^n(t)$ and

$$\sum_{i=1}^n \int_\Omega \partial_t u^n \partial_k a_i^n w_j = \int_\Omega \partial_t u^n \left(\sum \partial_t a_i^n w_j\right) = \int_\Omega \partial_t u^n(\partial_t u^n).$$

$$\int_\Omega \partial_t u^n \partial_t u^n + \int_\Omega A\nabla u^n\cdot\nabla\partial_t u^n + bu^n\partial_t u^n + \mathbf{c}\cdot\nabla u^n \partial_t u^n - \mathbf{d}\nabla\partial_t u^n u^n = \langle f, \partial_t u^n\rangle.$$

Good guy: $\int_\Omega \partial_t u^n \partial_t u^n = \|\partial_t u^n\|_2^2$.

First half of other guy: $\int_\Omega A\nabla u\nabla\partial_t u =$

$$\int_\Omega \frac{(A+A^T)}{2}\nabla u\nabla\partial_t u + \int_\Omega \frac{A-A^T}{2}\nabla u\nabla\partial_t u =$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \frac{A+A^T}{2}\nabla u\nabla u - \frac{1}{2}\int_\Omega \frac{\partial_t(A-A^T)}{2}\nabla u\nabla u - \int_\Omega \sum_{i,j}\frac{\partial}{\partial x_i}\frac{(A_{ij}-A_{ji})}{2}\frac{\partial u}{\partial x_j}\partial_t u - \underbrace{\sum_{i,j}\frac{A_{ij}-A_{ji}}{2}\frac{\partial^2 u}{\partial x_i \partial}}_{\text{sum with symmetric dot antis}}$$

Worst guy: $\int_\Omega \mathbf{d}\cdot\nabla\partial_t u^n u^n =$

$$= -\int_\Omega \partial_t u^n \operatorname{div}(\mathbf{d}u^n) = -\int_\Omega \partial_t u^n(\operatorname{div}\mathbf{d}u^n + \mathbf{d}\cdot\nabla u^n).$$

$$\|\partial_t\|_2^2 + \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \frac{A+A^T}{2}\nabla u\nabla u \leqslant \int_\Omega |\partial_t u^n|(|f|+\ldots) + |\nabla u|^2|\partial_t A|.$$

From Young's inequality:

$$\leqslant \frac{1}{2}\int_\Omega |\partial_t u^n|^2 + C\int_\Omega |b|^2|u^n|^2 + |\mathbf{c}|^2|\nabla u^n|^2 + |\operatorname{div}\mathbf{d}|^2|u^n|^2 + |\mathbf{d}|^2|\nabla u^n|^2 + |f|^2 + \left|\nabla\frac{A-A^T}{2}\right|^2|\nabla u^n|^2 + \left|\partial_t\frac{A+A}{2}\right.$$

$$\leqslant \frac{1}{2}\|\partial_t u^n\|_2^2 \leqslant c(b,\mathbf{c},\mathbf{d})(\|f\|_2^2 + \|u^n\|_{1,2}^2).$$

$$\implies \|\partial u^n\|_2^2 + \frac{d}{dt}\int_\Omega A\nabla u^n\cdot\nabla u^n \leqslant c(\ldots)\cdot(\|f\|_2^2 + \|u^n\|_{1,2}^2).$$

We want to know, if right hand side is integrable in time:

$$\int_\tau^t \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t)\nabla u^n(t) \leqslant \int_\Omega A\nabla u^n(\tau)\nabla u^n(\tau) + c\cdot\int_\tau^t \|f\|_2^2 + \|u^n\|_{1,2}^2.$$

With $\tau\leqslant\delta$ we add $\fint_0^\delta \cdot d\tau$:

$$\int_\delta^t \|\partial_t u^n\|_2^2 + \int_\Omega A\nabla u^n(t)\nabla u^n(t) \leqslant \fint_0^\delta \int_\Omega A\nabla u^n(\tau)\nabla u^n(\tau)d\tau + C(DATA)$$

Věta 6.3

Let $\partial_t f \in L^2(0,T,L^2(\Omega))$, $\partial_t A, \partial_t b, \partial_t \mathbf{c}, \partial_t d \in L^\infty$. Then $\forall \delta > 0 : \partial_{tt} u \in L^2(\delta, T, V^*), \partial_t u \in L^2(\delta, T, W_0^{1,2}(\Omega))$. If $-Lu_0 + f(0) \in L^2(\Omega)$, then

$$\partial_{tt} u \in L^2(0,T,V^*), \qquad \partial_t u \in L^2(0,T,W_0^{1,2}(\Omega)).$$

Důkaz (Sketch)
Take u^n – Galerkin approximation. Apply ∂_t to it:

$$\int_\Omega \partial_t u^n w_j + \int_\Omega A\nabla u^n \nabla u^n w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j - \mathbf{d}\nabla w_j u^n = \int_\Omega f w_j, \qquad \forall j \in [n] \text{ and almost every } t \in (0,$$

$$\int_\Omega \partial_{tt} u^n w_j + \int_\Omega A\nabla \partial_t u^n \nabla w_j = \int_\Omega -\partial_t A \nabla u^n \nabla w_j + (\partial_t b u^n + b\partial_t u^n) w_j + \partial_t \mathbf{c} \nabla u^n + \mathbf{c}\nabla \partial_t u^n w_j.$$

Similar as before we replace w_j, b_j, $\partial_t u^n$:

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u^n\|_2^2 + c_1\|\nabla \partial_t u^n\|_2^2 \leqslant \int_\Omega \|\nabla \partial_t u^n\|(SOMETHING).$$

$$\implies \frac{d}{dt}\|\partial_t u^n\|_2^2 + \|\nabla \partial_t u^n\|_2^2 \leqslant C(\|\partial_t u^n\|_2^2 + \ldots).$$

$$t \geqslant 2\delta : \|\partial_t u(t)\|_2^2 + \int_\tau^t \|\partial u^n\|_2^2 \leqslant C(1 + \int_\tau^t \|\partial_t u^n\|_2^2) + \|\partial u^n(\tau)\|_2^2.$$

Add $\fint_\delta^2 \delta d\tau$:

$$\|\partial_t u(t)\|_2^2 + \int_{2\delta}^T \|\nabla u\|_2^2 \leqslant X(\int_\delta^T \|\partial_t u^n\|_2^2 + 1 + \fint_\delta^{2\delta} \|\partial_t u^n(\tau)\|_2^2) \leqslant C(1 + \frac{c}{\delta} + \frac{c}{\delta^2}).$$

$$\to C(\int_0^T \|\partial_t u^n\|_2^2 + \|\partial_t u^n(0)\|_2^2 + 1) \leqslant$$

$$\leqslant C + C\|\partial_t u^n(0)\|_2^2 = C + C\| - Lu_0^n + f(0)\|_2^2 \leqslant \text{const}.$$

□

7 Linear hyperbolic equations

Poznámka (Prototype)

$$\frac{\partial u^2}{\partial t^2} - \Delta u = 0 \text{ in } (0,T)\times\Omega, \qquad u = 0 \text{ on } (0,T)\times\partial\Omega.$$
$$u(0,x) = u_0(x) \in W_0^{1,2}(\Omega), \qquad \partial_t u(0,x) = u_1(x) \in L^2(\Omega).$$

Poznámka (Formal a priory estimate)
Test by $\partial_t u$:

$$\int_\Omega \partial_{tt} u \partial_t u - \Delta u \partial_t u = 0$$
$$\frac{1}{2}\frac{d}{dt}\|\partial_t u\|_2^2 + \int_\Omega \underbrace{\nabla u \nabla \partial_t u}_{\frac{1}{2}\partial_t\|\nabla u\|^2} = 0$$

$$\frac{d}{dt}\left(\|\partial_t u\|_2^2 + \|\nabla u\|_2^2\right) = 0$$
$$\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leqslant \|\partial_t u(0)\|_2^2 + \|\nabla u(0)\|_2^2 = \|u_1\|_2^2 + \|\nabla u_0\|_2^2.$$

$$\|\partial_{tt}^2 u\|_{(W_0^{1,2}(\Omega))^*} = \sup_{\|\varphi\|\leqslant 1}\left\langle \partial_{tt}^2 u, \varphi\right\rangle \sim \sup \int_\Omega \partial_{tt}^2 u\varphi = \sup\int_\Omega \nabla u \varphi.$$

Věta 7.1

L be an elliptic operator such that $\int_0^T (\|\partial_t u\|_\infty + \|A\|_{1,\infty} + \|b\|_\infty + \|\mathbf{c}\|_\infty + \|\mathbf{u}\|_{1,\infty}) < \infty$ and $f \in L^2(0,T,L^2(\Omega))$. Assume that $u_0 \in W_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there $\exists! u \in L^2(0,T,W_0^{1,2}(\Omega)) \cap W^{1,2}(0,T,L^2(\Omega)) \cap W^{2,2}(0,T < V^*)$.

And $u(t) \to u_0$ in $L^2(\Omega)$, $\partial_t u(t) \to u_1$ in V^*.

Důkaz (Existence)
Step one: Galleikin approximation. Step two: Uniform estimates. Step three: $n \to \infty$. □

Důkaz (Step one)
$\{w_j\}_{j=1}^\infty$ base of $W_0^{1,2}$ ($\|P^n u\|_{1,2} \leqslant c\|u\|_{1,2}$). ? for $u^n(t,x) = \sum_{j=1}^n a_j^n(t) w_j(x)$.

$$\int_\Omega \partial_{tt} u^n w_j + \int_\Omega A\nabla u^n \nabla w_j + bu^n w_j + \mathbf{c}\nabla u^n w_j + \mathbf{d}\nabla w_j u^n = \int_\Omega f w_j.$$

Weak formulation for n-th coord. (WFn). $\partial_t u^n(0) = P^n u_1$ and $u^n(0) = P^n u_0$.

$$(a_j^n)'(0) = \int u_1 w_j, \qquad a_j^n(0) = \int u_0 w_j, (a_j^n)''(t) = F_j(a^n,t) + b_j(t).$$

Assume there exists u^n a solution to (WFn). □

Důkaz (Step two)
Uniform (N-independent) estimates: Multiply WFn by $(e^n_j)'(t)$ and $\sum_{j=1}^n$:

$$\int_\Omega \partial_{tt}u^n \partial_t u^n + \int_\Omega A\nabla u^n \partial\nabla u^n = \int_\Omega f\partial_t u^n + \mathbf{d}\nabla\partial_t u^n u^n - \mathbf{c}\nabla u^n \partial_t u^n - bu^n\partial_t u^n.$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\partial_t u^n|^2 + \frac{A+A^T}{2}\nabla u^n\nabla u^n\right) = -||-\int_\Omega \partial_t\left(\frac{A+A^T}{2}\nabla u^n\nabla u^n\right) - \underbrace{\int_\Omega \frac{A-A^T}{2}\nabla u^n\partial\nabla u^n}_{\text{by part}} \overset{\text{Hölder}}{\leqslant}$$

$$\leqslant c(\|f\|_2^2 + \|\partial_t u^n\|_2^2 + \|\nabla u^n\|_2^2) \leqslant \tilde{c}\left(\|f\|_2^2 + \int_\Omega |\partial_t u^n|^2 + \int_\Omega \frac{A+A^T}{2}\nabla u^n\nabla u^n\right).$$

Gronwall's lemma:

$$\int_\Omega |\partial_t u(t)|^2 + \frac{A+A^T}{2}\nabla u^n(t)\nabla u^n(t) \leqslant C(T)\cdot\left(\int_0^T \|f\|_2^2 + \int_\Omega |\partial_t u^n(0)|^2 + \frac{A+A^T}{2}\nabla u^n(0)\nabla u^n(0)\right) \leqslant C_t\cdot$$

$$\leqslant \tilde{C}_T(1 + \|v_0\|_2^2 + \|u_0\|_{1,2}^2).$$

$$\sup_{t\in(0,T)} \|\partial_t u^n(t)\|_2^2 + \|u^n(t)\|_{1,2}^2 \leqslant C(DATA).$$

$$\|\partial_{tt}u^n\|_{(W_0^{1,2}(\Omega))^*} := \sup_{\varphi\in W_0^{1,2}(\Omega)} \langle \partial_{tt}u^n, \varphi\rangle \overset{\text{Gelfand}}{=}$$

$$= \sup_\varphi \int_\Omega \partial_{tt}u^n\varphi \overset{\text{basis}}{=} \sup_\varphi \int_\Omega \partial_{tt}u^n P^n(\varphi) \overset{\text{WFn}}{=}$$

$$= -\int_\Omega A\nabla u^n\nabla P^n(\varphi)\ldots \overset{\text{Hölder}}{\leqslant} C\cdot\|P^n(\varphi)\|_{1,2}\,(\|f\|_2 + \|u^n\|_{1,2}) \leqslant$$

$$\leqslant \tilde{c}\|\varphi\|_{1,2}(\|f\|_2 + \|u^n\|_{1,2}).$$

$$\int_0^T \|\partial_{tt}u^n\|^2_{(W_0^{1,2}(\Omega))^*} \leqslant \tilde{c}\cdot 2\cdot\int_0^T (\|f\|_2^2 + \|u^n\|_{1,2}^2) \leqslant C(DATA).$$

□

Důkaz (Step three)
$u^n \rightharpoonup^* u$ in $W^{1,\infty}(0,T,L^2(\Omega)) \cap L^\infty(0,T,W_0^{1,2}(\Omega))$. $u^n \rightharpoonup u$ in $W^{2,2}(0,T,(W_0^{1,2}(\Omega))^*)$.

Limits:

$$\lim_{n\to\infty} \int_0^T \int_\Omega \partial_{tt} u^n w_j \tau dx dt = \int_0^T \langle \partial_{tt} u^n, w_j \tau \rangle dt = \langle \partial_{tt} u^n, w_j \tau \rangle_{L^2(0,T,W_0^{1,2}(\Omega))}.$$

$$\lim_{n\to\infty} \int_0^T \int_\Omega A \nabla u^n \nabla w_j \tau = \left\langle \nabla u^n, A^T \nabla w_j \tau \right\rangle_{L^2(0,T,L^2(\Omega))} \to \left\langle \nabla u, A^T \nabla w_j \tau \right\rangle = \int_0^T \int_\Omega A \nabla u \nabla w_j \tau.$$

$$WF: \int_0^T \langle \partial_{tt} u, w_j \rangle \tau + \int_\Omega A \nabla u \nabla w_j \tau + b u w_j \tau + \mathbf{c} \nabla u w_j \tau - \mathbf{d} \nabla w_j \tau = \int_0^T \int_\Omega f w_j \tau.$$

TODO?

TODO!!! □

8 Semigrupy

Definice 8.1 (Značení)

$$\mathcal{L}(x) := \{L : X \to X | L \text{ lineární omezený operátor}\}, \qquad \|L\|_{\mathcal{L}} = \sup_{\|x\|<1} \|Lx\|_X.$$

Dvojice $(A, D(A))$ je neomezený operátor, kde $D(A) \subset X$ je definiční obor A – podprostor X. $A : D(A) \to X$ je lineární.

Definice 8.2 (Semigrupa (jednoparametrická lineární semigrupa))

$S(t) : [0,\infty] \to \mathbb{L}(x)$ se nazývá semigrupa $\equiv$

- $S(0) = \text{id}$, neboli $S(0)x = x \ \forall x \in X$;
- $\forall t, s \geqslant 0 : S(t)S(s) = S(t+s)$.
- Pokud navíc $S(t)x \to x$ pro $t \to 0_+$, pak $S(t)$ nazýváme C_0-semigrupa.

Lemma 8.1

Nechť $S(t)$ je C_0-semigrupa. Potom

1. $\exists M \geqslant 1 \ \exists \omega \geqslant 0 \ \forall t \in [0,\infty) : \|S(t)\|_{\mathcal{L}(X)} \leqslant M e^{\omega t}$;

2. $\forall x$ *pevné* $t \mapsto S(t)x$ *je spojité zobrazení z* $[0,\infty)$ *do* X.

Důkaz
Krok 1): „$\exists M\ \exists \delta > 0\ \forall t \in [0,\delta] : \|S(t)\|_{\mathcal{L}(X)} \leqslant M$“: Pro spor nechť toto neplatí. Tedy $\exists t_n \to 0_+ : \|S(t_n)\|_{\mathcal{L}(x)} \to \infty$. Víme, že $\forall x : S(t_n)x \to x$. To implikuje $\forall x \sup_{t_n} \|S(t_n)x\|_X < \infty$. Z Principu stejnoměrné omezenosti (Věta Banach-Steinhaus, z úvodu do funkcionály) $\exists M > 0 \|S(t_n)\|_{\mathcal{L}(X)} \leqslant M$. $\lightning$.

„1.“: Definujeme $\omega = \frac{1}{\delta} \ln M$. $t \geqslant 0\ \exists \varepsilon \in (0,\delta) : t = n\delta + \varepsilon$.

$$\|S(t)\|_{\mathcal{L}(X)} = \|S(\delta) \cdot \ldots \cdot S(\delta) \cdot S(\varepsilon)\|_{\mathcal{L}(X)} \leqslant \|S(\delta)\|^n_{\mathcal{L}(X)} \cdot \|S(\varepsilon)\|_{\mathcal{L}(x)} \leqslant Me^{\omega t}.$$

„2.“: Spojitost v 0_+ plyne z třetího bodu definice semigrupy. Pro $t_0 > 0$, $t \to (t_0)_+$:

$$\lim_{h\to 0_+} S(t_0+h)x = \lim_{h\to 0_+} S(t_0)S(h)x = S(t_0) \lim_{h\to 0} S(h)x = S(t_0)x.$$

Pro $t \to (t_0)_-$ (bereme h dostatečně malé, aby $t_0 - h > 0$):

$$\|S(t_0-h)x - S(t_0)x\|_X = \|S(t_0-h)(x - S(h)x)\|_X \leqslant \|S(t-h)\|_{\mathcal{L}} \cdot \|x - S(h)x\|_X \to 0.$$

$\square$

Definice 8.3 (Generator of semigroup)

An unbounded operator $(A, D(A))$ is called a generator of $S(t)$ iff

$$Ax := \lim_{h\to 0_+} \frac{S(h)x - x}{h}, \qquad D(A) := \left\{ x \in X |\ \lim_{h\to 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\}.$$

Věta 8.2 (Properties of generator)

Let $(A, D(A))$ *be a generator of a* C_0*-semigroup* $S(t)$. *Then*

1. $x \in D(A) \implies S(t)x \in D(a)\ \forall t \geqslant 0$;
2. $x \in D(A) \implies AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x)\ \ \forall t \geqslant 0$;
3. $x \in X$, $t > 0 \implies x_t := \int_0^t S(s)x ds \in D(A)$, $A(x_t) = S(t)x - x$.

Poznámka (Použití)
$u_0 \in D(A) \subseteq X$, $u(t) := S(t)u_0$, 2. $\implies \frac{d}{dt}u(t) = \frac{d}{dt}(S(t)u_0) = AS(t)u_0 = Au(t)$. E. g. if $S(t)$ corresponds to the solution operator of $\partial_t u = \Delta u$, then generator $S(t)$ is Laplace.

Důkaz

$$A(S(t)x) \overset{\text{if exists}}{=} \lim_{h\to 0_+} \frac{S(h)(S(t)x) - S(t)x}{h} = \lim_{h\to 0_+}$$

TODO!!!

$$\frac{d}{dt}(S(t)x)_{t\to t_{0+}} = \lim_{h\to 0_+} \frac{S(t_0+h)x - S(t_0)x}{h} = S(t_0)\lim \frac{S(h)x - x}{h} = S(t_0)Ax.$$

$$\frac{d}{dt}(S(t)x)_{t\to t_{0-}} = \lim_{h\to 0_+} \frac{S(t_0-h)x - S(t_0)x}{h} = \lim \left(S(t_0-h)\left[\frac{x - S(h)x}{h} - S(h)Ax\right]\right) + \lim_{h\to 0_+} S(t_0-h)S$$

because

$$\|S(t_0-h)\left(\frac{x-S(h)x}{h} - S(h)Ax\right)\|_X \leqslant \|S(t_0-h)\|_{\mathcal{L}(X)} \cdot \|\frac{x-S(h)x}{-h} - Ax + Ax - S(h)Ax\|_X \leqslant$$

$$\leqslant Me^{\omega t_0}\left(\|\frac{S(h)x-x}{h} - Ax\|_X + \|Ax - S(h)Ax\|_X\right) \to 0.$$

3.)

$$\frac{S(h)x_t - x_t}{h} = \frac{S(h)\int_0^t S(s)xds - \int_0^t S(s)xds}{h} = \frac{\int_0^t S(h+s)xds - \int_0^t S(s)xds}{h} =$$

$$= \frac{\int_h^{t+h} S(s)xds - \int_0^t S(s)xds}{h} = \frac{1}{h}\int_t^{t+h} S(s)xds - \frac{1}{h}\int_0^h S(s)xds.$$

□

Definice 8.4 (Closed operator)

We say that $(A, D(A))$ is closed iff

$$(u_n \in D(A), \qquad u_n \to u \text{ in } X, \qquad A(u_n) \to v \text{ in } X)$$

$$\implies u \in D(A), \qquad Au = v.$$

Věta 8.3 (Density and closedness of generator)

Let $(A, D(A))$ be a generator to a C_0-semigroup $S(t)$ in X. Then $D(A)$ is dense in X and $(A, D(A))$ is closed.

Důkaz

$x \in X$ arbitrary $\implies x_t \in D(A)$.

$$\left[\frac{x_t}{t}\right] = \frac{1}{t}\int_0^t S(s)xds \to x \implies D(A) \text{ is dense in } X.$$

$x_n \in D(A)$, $x_n \to x$ in X, $Ax_n \to y$ in X. We want $x \in D(A)$ and $Ax = y$.

$$S(h)x_n - S(0)x_n \int_0^t \frac{d}{dt}(S(t)x_n)dt = \int_0^h S(t)(Ax_n)dt$$

$$n \to \infty : \frac{S(h)x - x}{h} = \fint_0^h S(t)ydt.$$

$$A(x) = \lim_{h \to 0_+} \frac{S(h)x - x}{h} = \lim_{n \to \infty} \fint_0^h S(t)ydt = S(0)y = y.$$

□

Poznámka

We have $A(\int_0^t S(s)xds) = S(t)x - x = \int_0^t AS(t)xdt$.

Lemma 8.4 (Uniqueness of $S(t)$)

Let $S(t)$ and $\tilde{S}(t)$ be C_0 semigroup with the same generator. Then $S(t) = tildeS(t)$ $\forall t \geqslant 0$.

Důkaz

$$y(t) = S(T-t)\tilde{S}(t)x, \qquad x \in D(A), \quad T > 0 \text{ fixed.}$$

$$\frac{d}{dt}y(t) = -S(T-t)A\tilde{S}(t)x + S(T-t)A\tilde{S}(t)x = 0.$$

$$y(t) = y(0) = y(T) \implies S(T)x = \tilde{S}(T)x.$$

An $D(A)$ is dense, so $S = \tilde{S}$ on X. □

Definice 8.5 (Resolvent)

Let $(A, D(A))$ be unbounded operator. We define resolvent set $\varrho(A) := \{\lambda \in \mathbb{R} | (\lambda I - A) : D(A) \to X$ is or
$\forall \lambda \in \varrho(A)$ we define the resolvent operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \to D(A).$$

Poznámka

$(A, D(A))$ closed $\implies R(\lambda, A)$ is continuous $\implies R(\lambda, A) \in \mathcal{L}(X)$.

Lemma 8.5 (Properties of $R(\lambda, A)$)

Let $(A, D(A))$ be a generator of C_0-semigroup $S(t)$ and let $\|S(t)\| \leqslant Me^{\omega t}$.

1. $AR(\lambda, A)x = \lambda R(\lambda, A)x - x \ \forall x \in X$.
2. $R(\lambda, A)Ax = \lambda R(\lambda, A)x - x \ \forall x \in D(A)$.
3. $R(\lambda, A)x - R(\mu, A)x = (\mu - \lambda)R(\lambda, A)R(\mu, A)x$.
4. $\forall \lambda > \omega \ \lambda \in \varrho(A)$*:* $R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)xdt$ *adn* $\|R(\lambda, A)\| \leqslant \frac{M}{\lambda \omega}$.

Důkaz

„1.“: $AR(\lambda, A)x = [(A - \lambda I) + \lambda I]\, R(\lambda, A)x = -x + \lambda R(\lambda, A)x$. „2. + 3.“: obdobně.

„4.“: Rescale and define $\tilde{S}(t) = e^{-\omega t}S(t)$. We will prove the result for $\tilde{S}(t)$ and transform it to $S(t)$:

$$\|\tilde{S}(t)\| \leqslant e^{-\omega t}\|S(t)\| \leqslant Me^{0 \cdot t} = M.$$

□

Věta 8.6 (Hille-Yosida)

Nechť $(A, D(A))$ je neomezený operátor. Pak následující je ekvivalentní:

- $\exists C_0$*-semigupa, jejíž generátor je $(A, D(A))$ a je neexpanzivní;*
- $(A, D(A))$ *je uzavřený,* $D(A)$ *je husté v* X*,* $(0, \infty) \subseteq \varrho(A)$*,* $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leqslant \frac{1}{\lambda}$.

Důkaz

„ $\Longrightarrow$ “ máme. „ $\Longleftarrow$ “: (Myšlenka: $A \to A_n$ aproximace pomocí omezených operátorů, $S_n(x) \sim e^{tA_n}$, $n \to \infty$). □

Důkaz (Krok 1)
A_n aproximace (Hille-Yosida):

$$A_n := nAR(n, A) \qquad \forall n \in \mathbb{N}.$$

„A_n je omezený operátor“:

$$A_n = nAR(n, A) = n^2 R(n, A) - nI \in \mathcal{L}(X)$$

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leqslant \|R(n, A)\|_{\mathcal{L}(X)} \|Ax\|_X \leqslant \frac{\|Ax\|_X}{n} \to 0.$$

$D(A)$ je husté v X. $\|nR(n, A) - I\|_{\mathcal{L}(X)} \leqslant n\|R(n, A)\|_{\mathcal{L}(x)} + 1 \leqslant 1 + 1 = 2$. Z tohoto a principu stejnoměrné omezenosti (někdy také Banach-Steinhaus) $nR(n, A)x \to x \ \forall x \in X$. Spolu s lemmatem bod 3:

$$A_n x = nAR(n, A)x = nR(n, A)Ax \to Ax, \qquad \forall x \in D(A).$$

□

Důkaz (Krok 2)
Definujeme $S_n(t)$ jako:

$$S_n(t) := e^{tA_n} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_n^k \quad \in \mathcal{L}(x).$$

Za domácí úkol si ověříme, že $S_n(t)$ je semigrupa.

$$S_n(t) = e^{tA_n} = e^{-ntI + n^2 tR(n,A)} = e^{-nt} e^{n^2 tR(n,A)}$$

$$\|S_n(t)\|_{\mathcal{L}(X)} \leqslant e^{-n} \|e^{n^2 tR(n,A)}\|_{\mathcal{L}(x)} \leqslant e^{-nt} \| \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} (nR(n, A))^k \|_{\mathcal{L}(X)} \leqslant$$

$$\leqslant e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n, A)\|_{\mathcal{L}(X)} \leqslant e^{-nt} \sum \frac{(nt)^k}{k!} = e^{-nt} e^{nt} = 1.$$

□

Důkaz (Krok 3)
Ukážeme „$\exists \lim S_n(t)x$“:

$$(S_n(t) - S_m(t))x = \int_0^t \frac{d}{ds}(S_m(t-s)S_n(s)x)ds = \int_0^t \frac{d}{ds}(e^{(t-s)A_m}A^{sA_n}x)ds =$$

$$= \int_0^t (-A_n e^{(t-s)A_m}e^{sA_n}x + A_n e^{(t-s)A_m}e^{sA_n}x)ds = \int_0^t e^{(t-s)A_m}e^{sA_n}(A_n x - A_m x),$$

$$\|S_n(t)x - S_m(t)x\|_X \leqslant \left\| \int_0^t e^{(t-s)A_m}e^{sA_n}(A_n x - A_m x) \right\|_X \leqslant$$

$$\leqslant \left\| \int_0^t S_m(t-s)S_n(s)(A_n x - A_m x) \right\|_X \leqslant t\|A_n x - A_m x\|_X$$

$\forall x \in D(A)$ $S_n(t)x$ je cauchyovská posloupnost $\implies \exists S(t)x$, $S_n(t)x \to S(t)x$. $D(A)$ je hustý v x a $\|S_n(t)\|_{\mathcal{L}(x)} \leqslant 1 \implies \forall x \in X\ \exists \lim_{n\to\infty} S_n(t)x =: S(t)x$. Z linearity plyne, že $S(t)$ je semigrupa. □

Důkaz (Krok 4)
„$(A, D(A))$ je generátor $S(t)$“: Označíme $(\tilde{A}, D(\tilde{A}))$ generátor $S(t)$.

$$\left(\frac{S(t)x - x}{t}\right) \qquad S_n(t)x - x = \int_0^t \frac{d}{ds}S_n(s)xds = \int_0^t S_n(s)A_n xds$$

$$n \to \infty, x \in D(A) : S(t)x - x = \lim_{n\to\infty} \int_0^t S_n(s)A_n xds = \lim_{n\to\infty} \int_0^t S_n(s)(A_n x - Ax) + (S_n(s) - S(s)) + S(s)Ax$$

$$\frac{S(t)x - x}{t} = \fint S(s)Axds \to Ax \qquad \forall x \in D(A).$$

$$\implies D(A) \subseteq D(\tilde{A}) \implies A = \tilde{A} \text{ na } D(A).$$

Tedy zbývá ukázat $D(\tilde{A}) \subseteq D(A)$.

Víme $\forall \lambda > 0 : \lambda I - A$ je prostý a na ($\|R(\lambda, A)\| \leqslant \frac{1}{\lambda}$). Také víme, že $S(t)$ je neexpanzivní a $\|S(t)\|_{\mathcal{L}(X)} \leqslant 1 \implies R(\lambda, \tilde{A}) \leqslant \frac{1}{\lambda}$ $\forall \lambda > 0$. Nakonec víme $\forall \lambda > 0 : \lambda I - \tilde{A}$ je prostý a na. To nám dává $D(A) = D(\tilde{A})$. □

Věta 8.7 (Obecná Hille-Yosida)

$(A, D(A))$ generuje C_0-semigrupu splňující $\|S(t)\|_{\mathcal{L}(x)} \leqslant Me^{\omega t} \Leftrightarrow$ ($(A, D(A))$ je uzavřený a $\forall \lambda > \omega$, $\lambda \leqslant \varrho(A)$ a $\|R^n(\lambda, A)\|_{\mathcal{L}(x)} \leqslant \frac{M}{(\lambda-\omega)^n}$ $\forall n$).

Důkaz
Podobný předchozímu, jen se musí udělat lepší odhady. Nebyl na přednášce. □

8.1 Aplikace na lineární evoluční PDR

Příklad
$\partial_t u - \Delta u = 0$ v $(0,T)\times\Omega$, $u(0) = u_0$ v Ω a $u = 0$ na $\partial\Omega \times (0,T)$.

Přepis do semigrup: $\partial_t u = \Delta u = Au$. A ptáme se $\exists S(t)$ tak, že A je uzavřený, $D(A)$ je husté v $L^2(\Omega)$ a $(0,\infty) \subseteq \varrho(A)$ a $R(\lambda, A) \leqslant \frac{1}{\lambda}$.

$D(A) = L^2(\Omega) \cap W_0^{1,2}(\Omega) \cap \{u|\Delta u \in L^2(\Omega)\}$ a $X = L^2(\Omega)$. $\overline{D(A)} = X$, neboť $C_0^\infty(\Omega) \subseteq D(A)$ a $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$.

$(A, D(A))$ je uzavřený:

$$(u^n \to u \wedge Au^n \to f) \implies u \in D(A) \wedge Au = f :$$

$$u^n \in D(A) \wedge \Delta u^n =: f^n \in L^2(\Omega) \wedge f^n \to f \implies$$

$$\implies \forall \varphi \in W_0^{1,2} \int_\Omega \nabla u^n \nabla \varphi = \int_\Omega f^n \varphi$$

Vezmeme $\varphi = u^n \implies \|\nabla u^n\|_2 \leqslant c\|f^n\|_2 \overset{\text{Poisson}}{\implies} \|u^n\|_{1,2} \leqslant C$:

$$u^n \rightharpoonup u \implies \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

TODO!!!

TODO!!!

Příklad (RVT s pravou strano)
$\partial_t u - \Delta u = f$ v $(0,T)\times\Omega$, $u = 0$ na $(0,T)\times\partial\Omega$, $u(0) = u_0$ v Ω.

$Au = \Delta u$, $D(A) = \left\{u \in W_0^{1,2}, \Delta u \in L^2(\Omega)\right\}$. To má semigrupu $S(t)$ pro $f \in L^1(0,T,L^2(\Omega))$. Volíme

$$u(t) := S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau.$$

Co řeší $u(t)$?

$$\partial_t u = \partial_t(S(t)u_0) + \partial_t \int_0^t S(t-\tau)f(\tau)d\tau = AS(t)u_0 + S(0)f(t) + \int_0^t \partial_t S(t-\tau)f(\tau)d\tau =$$

$$= f(t) + AS(t)u_0 + \int_0^t AS(t-\tau)f(\tau)d\tau = f(t) + A(S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau) = f(t) + Au(t).$$

Příklad
$\partial_t u - \Delta u = 0$ v $(0,T)\times\Omega$, $u = 0$ na $(0,T)\times\partial\Omega$, $u(0) = u_0 \in L^p(\Omega)$, $\infty > p \geqslant 2$ (lze i pro $p \in (1,2)$).

Řešení

Chceme $X = L^p(\Omega)$, $Au = \Delta u$, $D(A) = \left\{u \in W_0^{1,2}(\Omega), \Delta u \in L^p(\Omega)\right\}$. Pak $A : D(A) \to X$. Hustota je zadarmo. „$(A, D(A))$ uzavřený“: rozmyslet doma (důkaz stejný jako výše).

„$\lambda u - \Delta u = f \implies \|u\|_p \leqslant \frac{\|f\|_p}{\lambda}$“:

$$\forall f \in L^p(\Omega)\ \exists! u \in W_0^{1,2}(\Omega) : \int_\Omega \lambda u\varphi + \nabla u \nabla \varphi = \int_\Omega f\varphi \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

Kdyby $\varphi = |u|^{p-2}u$ (pozor, nemáme, že toto $\varphi \in W_0^{1,2}$, pouze zkoušíme), pak

$$\lambda \|u\|_p^p + \int_\Omega \nabla u \nabla(|u|^{p-2}u) = \int_\Omega f|u|^{p-2}u \overset{\text{Hölder}}{\leqslant} \|f\|_p \cdot \|u^{p-1}\|_{p'} = \|f\|_p \cdot \|u\|_p^{p-1},$$

což je kýžený odhad, jelikož $(p-1)|u|^{p-2}\nabla u \geqslant 0$. Volme tedy $\varphi := \left(\frac{|u|}{1+\varepsilon|u|}\right)^{p-2} u \in W_0^{1,2}(\Omega)$. Zopakujme postup (TODO?) a máme to.

Příklad

$\partial_t u + Lu = 0$ v $(0,T) \times \Omega$, $Lu = \operatorname{div} A\nabla u + b \cdot u + \mathbf{c} \cdot \nabla u$, kde A, b, c nezávisí na čase t.

Řešení

$Au = -Lu$. $D(A) := \left\{u \in W_0^{1,2}(\Omega) \middle| \sup_{\varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^2(\Omega)}=1} \int_\Omega A\nabla u \nabla \varphi + bu\varphi + \mathbf{c} \cdot \nabla u \varphi < \infty\right\}$.

$(A, D(A))$ je uzavřený (doma). $\exists \omega > 0\ \forall \lambda \geqslant \omega : \lambda u + Lu = f\ \exists! u$. Odhad $\|u\|_2 \leqslant \frac{\|f\|_2}{\lambda - \omega}$ už jsme dělali:

$$(\lambda - u)u + \omega u + Lu = f, TODO?$$

Poznámka

Zkouška: První příklad musíme vysypat z rukávu (5–10 minut max) jinak jdeme domu. Na zbytek spoustu času. Příklad nemusí být ukázaný na přednášce, ale stačí ji následovat. Druhý příklad bude čistá teorie (může to být např. list ANO-NE).