1 Introduction

Poznámka (Literature)
„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

1.1 Differentiability

Definice 1.1 (Differentiable)

A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable (also holomorphic) at a point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{|h| \to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call $f'(z_0)$ the derivative of f at z_0. A function f is differentiable on a domain (open connected subset of $\mathbb{C}$) if its differentiable for all points of this domain.

Poznámka (Writing complex numbers in cartesian cooridnates)
$z = x + iy$, for $x, y \in \mathbb{R}$, we can write a function $f : \mathbb{C} \to \mathbb{C}$ in terms of two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

Věta 1.1 (Cauchy–Riemann equations)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}$. Considering $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definice 1.2 (Orientability, orientation-preserving function)

Define and equivalence relation on the set of all bases of $\mathbb{R}^2$ by saying that $B_1 \sim B_2$ iff the determinant of the change of basis matrix is positive.

A function $f : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ is said to be orientation-preserving if on an open dense subset of U, the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Důsledek
Let f be a non-constant holomorphic function, then f is orientation-preserving.

Důsledek
Since f is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x}\frac{\partial v}{y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \overset{\text{C–R}}{=} \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geqslant 0.$$

Since f is non-constant, the inequality is strict on a dense open subset of the domain of definition.

Věta 1.2 (Open mapping theorem)

A non-constant holomorphic function f is open (that is if U is an open subset of $\mathbb{C}$, then $f(U)$ is also open).

1.2 Integration

Definice 1.3

For a path γ (smooth function, $\gamma : \mathbb{R} \supset [a, b] \to \mathbb{C}$) we define

$$\int_\gamma f(x)dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

Definice 1.4 (Continuous deformation)

For $\gamma, \mu : [a, b] \to U$ (U simply connected), paths with the same endpoints ($\gamma(a) = \mu(a)$ and $\gamma(b) = \mu(b)$). Then a continuous deformation γ into μ is a continuous function $H : [a, b] \times [0, 1] \to U \subseteq \mathbb{C}$ such that $H(s, 0) = \gamma(s)$, $H(s, 1) = \mu(s)$, $H(a, t) = z_a := \gamma(a) = \mu(a)$ and $H(b, t) = z_b := \gamma(b) = \mu(b)$.

Věta 1.3

Suppose that $\gamma, \mu : [a, b] \to U$ (U simply connected) are related by a continuous deformation of paths H. Then for any holomorphic function f on U we have

$$\int_\gamma f(z)dz = \int_\mu f(z)dz.$$

Důkaz (Partial proof assuming H admits partial derivatives)
For any $t \in [0,1]$ we integrate the function $INT(t) = \int_{H(\cdot,t)} f(z)dz$. Consider the derivative of $INT(t)$ with respect to t:

$$\frac{d}{dt}(INT(t)) = \frac{d}{dt}\left(\int_a^b f(H(s,t))\frac{\partial H}{\partial s}(s,t)ds\right) \overset{\text{Leibniz + chain rule}}{=}$$

$$= \int_a^b f'(H(s,t))\frac{\partial H}{\partial t}(s,t)\cdot\frac{\partial H}{\partial s}(s,t) + f(H(s,t))\frac{\partial^2 H}{\partial s \partial t}(s,t)ds =$$

$$= \int_a^b \frac{d}{ds}\left[f(H(s,t))\frac{\partial H}{\partial t}\right]ds =$$

$$= f(H(s,t))\frac{\partial H}{\partial t}|_{s=a}^{s=b} \overset{\text{constant endpoints}}{=} 0.$$

Having derivative identically equal to 0, means that $INT(t)$ is a constant function and $\int_\gamma f(z)dz = INT(0) = INT(1) = \int_\mu f(z)dz$. □

Důsledek
Let U be a simply connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. For any closed path whose image is inside U, $\int_\gamma f(z)dz = 0$.

Důkaz (Sketch)
The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path c.

$$\int_\gamma f(z)dz = \int_c f(z)dz = \int_a^b f(c(z))\cdot c'(z)dz = \int_a^b f(c(z))\cdot 0dz = 0$$

□

Příklad
Let U be a simple connected domain and $f : U \to \mathbb{C}$ a holomorphic function on $U\backslash\{z_0\}$. For $j = 1,2$, let γ_j be a path parametrizing a circle centered at z_0 of radius r_j, oriented counterclockwise and completely contained in U. Show that $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$.

1.3 Cauchy's integral formula

Věta 1.4 (Cauchy's integral formula)

Let γ be a loop around $z \in \mathbb{C}$, and $f : U \to \mathbb{C}$ a holomorphic function. For U a neighbourhood of γ,

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw.$$

Důkaz
Conway 1978, Chapter IV. ◻

Důsledek

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z_0 + z_0 - z} dw = \frac{1}{2\pi i} \oint \frac{f(w)}{w - z_0} \cdot \left(\frac{1}{1 - \frac{z-z_0}{w-z_0}} \right) dw =$$
$$= \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z_0} \left(\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} \right) dw =$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^n} \right) (z - z_0)^n.$$

For sufficiently ”small” (shrunken) γ. So f is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around z_0 converges to f in a neighbourhood of z_0).

Definice 1.5 (Pole)

Given a positive integer n, a complex function f has pole of order n at the point $z_0 \in \mathbb{C}$ if $(z - z_0)^n f(z)$ is holomorphic at z_0 but $(z - z_0)^{n-1} f(z)$ is not.

Příklad
Show that if f has a pole of order n at $z_0 \in \mathbb{C}$. Then it admits a Laurient expansion $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$ with $a_{-n} \neq 0$.

Definice 1.6 (Residue)

Let f have a pole of order n at the point $z_0 \in \mathbb{C}$. Then the residue of f at z_0 is the $k = -1$ coefficient of the Laurent expansion of f at z_0.

Příklad
Show that if f has a pole of order 1 at z_0, then the residue of f at z_0 can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)
Let $\gamma : [a, b] \to U \subset \mathbb{C}$ be a simple closed path, bounding a domain W containing the points $z_1, \ldots, z_m$. Assume that f is holomorphic on $U \setminus \{z_1, \ldots, z_m\}$ and has poles at $\{z, \ldots, z_m\}$.

Show that

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname{res}_{z=z_j} f(z).$$