Poznámka
The previous semester we work with linear equation (L-M, Fredholm, Minimizing quadratic function). This semester we will have non-linear equations like $((\partial_t u)) - \Delta u + \operatorname{arctg} u = f$ or $f = -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

We don't work with $\partial_{tt}u - \Delta_p u = f$, because nobody know how to proof it has solution (for $d \geqslant 2, p > 2$).

Poznámka (Credit)
Two homework. -10 to 10 points to exam from each. (If we hand anything we get credit.)

What we must know

Poznámka
Lebesgue spaces.

Fixed point theorem: 1) Let F be continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$. Assume that $\exists$ convex compact set in $\mathbb{R}^d$ such that $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega$ such that $F(x) = x$. 2) Let $F : X \to X$, where X is Banach space and F is continuous and compact and let $\exists\ \Omega \subseteq X$ convex and closed such that $F(\Omega) \subseteq \Omega$. Then $F(\Omega) \subseteq \Omega$. Then $\exists x \in X : F(x) = x$.

Luzin: Let Ω be a measurable set and $f \in L^1_{loc}(\Omega)$. Then $\forall \varepsilon > 0\ \exists U \in \Omega$, $\mu(U) \leqslant \varepsilon$, $f \in C(\Omega \backslash U)$.

Egorov: Let Ω be a measurable set and $f^n \to f$ in $L^1_{loc}(\Omega)$. Then $\forall \varepsilon > 0\ \exists U, \mu(U) \leqslant \varepsilon$ $f^n \to f$ in $C(\Omega \backslash U)$.

Lebesgue dominated convergence theorem.

Vitali convergence theorem: Let $\Omega \subseteq \mathbb{R}^d$ be bounded measurable, f^n a sequence of measurable functions, $f^n \to f$ almost everywhere in Ω. Then $\lim_{n\to\infty} \int_\Omega f^n = \int_\Omega f$, provided f^n is uniformly equi-integrable ($\forall \varepsilon > 0\ \exists \delta\ \forall U, \mu(U) \leqslant \varepsilon$).

Fatou lemma: $f^n \to f$ almost everywhere in Ω and $f^n \geqslant 0$, then $\liminf_{n\to\infty} \int_\Omega f^n \geqslant \int_\Omega f$.

Regularization: $\eta \in C_0^\infty(B_1(\mathbf{o}))$ non-negative, radially symmetric and $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Then $\forall f \in L^1_{loc}(\Omega)$ we extend f by „0“ to $\mathbb{R}^d$ and $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(x/\varepsilon)$. Then $f_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $\forall p \in [1,\infty)$ $f \in L^p(\Omega) \implies f_\varepsilon \to f$ in $L^p(\Omega)$. (And for $p = \infty$: $f \in L^\infty(\Omega) \implies f_\varepsilon \to f$ in $L^q(\Omega)$ $\forall q \in\in [1,\infty)$).

Reflexive and separable spaces. ($L^p(\Omega)$ is a Banach space, separable for $p \in [1,\infty)$, reflexive for $p \in (1,\infty)$.)

Nemytsky operator: (Assume that for almost all $x \in \Omega$ and , $|f(x,y)| \leqslant g(x) +$

$C\sum_{i=1}^{N}|y_1|^{p_i/p}$ for some $p_i \in [1,\infty), p \in (1,\infty), g \in L^p(\Omega)$. Then $\forall u_i \in L^{p_i}$, the function $f(\cdot,u_1,\dots,u_n)$ is measurable, $(u_1,\dots,u_n) \mapsto f(\cdot,u_1,\dots,u_n)$ is continuous $L^{p_1}(\Omega) \times \dots \times L^{p_N}(\Omega) \to L^p(\Omega)$. This mapping is called N.O.)

1 Sobolev spaces (and Bochner spaces)

Poznámka
Ω is open bounded subset of $\mathbb{R}^d$.

Věta 1.1 (Local approximation by smooth functions)

Let $f \in W^{k,p}(\Omega)$ and extend it by „0“ outside. Define $f_\varepsilon := \eta_\varepsilon * f$ and set $\Omega_\varepsilon := \{x \in \Omega | B(x,\varepsilon) \subseteq \Omega\}$. Then $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$ almost everywhere in Ω_ε $\forall \alpha, |\alpha| \leqslant k$ and $\forall \Omega' \subseteq \overline{\Omega'} \subseteq \Omega$ and $p \in [1,\infty)$ $f_\varepsilon \to f$ in $W^{k,p}(\Omega')$. (If $p=\infty$, then $f_\varepsilon \to^* f$ in $W^{1,\infty}(\Omega')$.)

Důkaz

$$\frac{\partial}{\partial x_i}(f_\varepsilon(x)) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^d}\eta_\varepsilon(x-y)f(y)dy =$$
$$= \int_{\mathbb{R}^d}\frac{\partial}{\partial x_i}(\eta_\varepsilon(x-y))f(y)dy = -\int_{\mathbb{R}^d}\frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy =$$
$$= -\int_{B(x,\varepsilon)}\frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy = -\int_{\Omega}\frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy =$$
$$= \int_{\Omega}\eta_\varepsilon(x-y)\frac{\partial f(y)}{\partial y_i}dy = \int_{\mathbb{R}^d}\eta_\varepsilon(x-y)\frac{\partial f(y)}{\partial y_i} = \left(\frac{\partial f(y)}{\partial y_i}\right)_\varepsilon(x).$$

Now we take sufficiently small ε, such that $\Omega_\varepsilon \subseteq \Omega'$. Then $D^\alpha f_\varepsilon = (D^\alpha f)_\varepsilon \to D^\alpha f$ in $L^p(\Omega')$. □

Věta 1.2 (Composition of Lipschitz and Sobolev functions)

Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Assume that $u \in W^{1,p}(\Omega)$. Then $(f(u) - f(0)) \in W^{1,p}(\Omega)$ and $\nabla f(u) = f'(u)\nabla u \chi_{x,u(x)\notin S_f}$, where S_f are points where $f'(s)$ doesn't exists.

Moreover define $\Omega_a := \{x \in \Omega | u(x) = a\}$, then $\nabla u = 0$ almost everywhere in Ω_a.

Důkaz
We know, that $f \in C^1(\mathbb{R})$, $f_{lip} := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$.

So $|f(u(x)) - f(0)|^p \leqslant f_{lip}^p \cdot |u(x)|^p$, if $u \in L^p(\Omega) \implies f(u) - f(0) \in L^p(\Omega)$.

Next, $\frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \implies f(u) - f(0) \in W^{1,p}(\Omega)$.

We take $\eta \in C_0^\infty(\Omega)$ and $u \in W^{1,1}(u)$.

$$\int_\Omega \frac{\partial \eta}{\partial x_i} f(u) = \lim_{\varepsilon \to 0_+} \int_\Omega \frac{\partial \eta}{\partial x_i} f(u_\varepsilon) \overset{\text{IBP, both are smooth}}{-} \lim_{\varepsilon \to 0_+} \int_\Omega \eta \frac{\partial f(u_\varepsilon)}{\partial x_i} =$$

$$= - \lim_{\varepsilon \to 0_+} \int_\Omega \underbrace{\eta f'(u_\varepsilon)}_{\to \eta f(u_\varepsilon) \text{ in } L^1\text{, so weakly in } L^\infty} \cdot \underbrace{\frac{\partial u_\varepsilon}{\partial x_i}}_{\to \frac{\partial u}{\partial x_i} \text{ in } L^1} .$$

TODO? □

Věta 1.3 (Characterization of sobolev functions)

Let $\Omega \subseteq \mathbb{R}^d$ open, bounded. Define $\Omega_\delta := \{x \in \Omega | B(x,\delta) \subseteq \Omega\}$ and $u_i^h(x) := \frac{u(x+h \cdot e_i) - u(x)}{h}$, $h > 0, i \in [d]$.

- *If $u \in W^{1,p}(\Omega)$ then $\forall \delta\ \forall h < \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p} (\Omega)$.*
- *If $p \in (1, \infty]$ and $\sup_{\delta > 0} \sup_{h < \frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant k$, then $\frac{\partial u}{\partial x_i}$ exists and $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leqslant k$.*
- *If $p \in [1, \infty)$ and if $u \in W^{1,p}(\Omega)$ then $u_i^h \to \frac{\partial u}{\partial x_i}$ in $L^p_{loc}(\Omega)$.*

(If $p = 1$ and $\sup_{\delta > 0} \sup_{h < \frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant k$, then $u \in BV(\Omega)$. Moreover if $\leqslant k$ and u_i^h is equiintegrable, then $u \in W^{1,1}(\Omega)$.)*

Důkaz
„Second item“ Fix $\Omega_1 \subset\subset \Omega$. Fix δ_0, $\Omega_1 \subseteq \Omega_{\delta_0} \implies \|u_i^h\|_{L^p(\Omega_1)} \leqslant k$. $u_i^h \rightharpoonup \overline{u}$ in $L^p(\Omega_1)$ and $u_i^h \rightharpoonup^* \overline{u}$ in $L^\infty(\Omega_1)$. We want $\|\overline{u}\|_{L^p(\Omega_1)} \leqslant \liminf_{h\to 0_+} \|u_i^h\|_{L^p(\Omega_1)} \leqslant k$.

$$\int_{\Omega_1} \overline{u}\varphi dx = \lim_{h\to 0_+} \int_\Omega u_i^h \varphi = \lim_{h\to 0_+} \int_{\Omega_1} \frac{u(x+h\cdot e_i)-u(x)}{h}\varphi(x)dx =$$
$$= \lim_{h\to 0_+} \int_\Omega \frac{u(x+h\cdot e_i)}{h}\varphi(x) - \frac{u(x)}{h}\varphi(x)dx =$$
$$= -\lim_{h\to 0_+} \int_\Omega u(x)\frac{\varphi(x)-\varphi(x-h\cdot e_i)}{h}dx = -\int_{\Omega_1} \frac{\partial \varphi}{\partial x_i}u.$$

„First item“: $u_\varepsilon := u * \eta_\varepsilon$ (where we extend u by zero).

$$\frac{u_\varepsilon(x+h\cdot e_i)-u_\varepsilon(x)}{h} = \frac{1}{h}\int_0^1 \frac{d}{dt}u_\varepsilon(x+he_it)dt = \int_0^1 \frac{\partial u_\varepsilon(x+h\cdot e_i\cdot t)}{\partial x_i}dt.$$

$$\left|\frac{u_\varepsilon(x+h\cdot e_i)-u_\varepsilon(x)}{n}\right|^p \leqslant \left|\int_0^1 \frac{\partial u^\varepsilon}{\partial x_i}()dt\right|^p \leqslant \int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_i}()\right|^p dt.$$

$$\int_{\Omega_\delta} \left|\frac{u_\varepsilon(x+h\cdot e_i)-u_\varepsilon(x)}{h}\right|^p \leqslant \int_{\Omega_\delta}\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+h\cdot e_i\cdot t)\right|^p dtdx =$$

$$\int_0^1\int_{\Omega_\delta} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+h\cdot e_i\cdot t)\right|^p dxdt \leqslant \int_0^1\int_{\Omega_{\delta/2}} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x)\right|^p dxdt = \int_{\Omega_{\delta/2}} \left|\frac{\partial u^\varepsilon}{\partial x_i}\right|^p dx$$

$$\varepsilon \to 0_+ : \int_{\Omega_\delta} \left|\frac{u(x+h\cdot e_i)-u(x)}{h}\right|^p \leqslant \int_{\Omega_{\delta/2}} \left|\frac{\partial u}{\partial x_i}(x_i)\right|^p dx.$$

„Third item“: It is enough to show „u_i^h is Cauchy“: $\varepsilon > 0$, $u_\varepsilon := u * \eta_\varepsilon$:

$$u_{\varepsilon,i}^{h_1} - u_{\varepsilon,i}^{h_2} = \frac{u_\varepsilon(x+h_1e_i)-u_\varepsilon(x)}{h} - \frac{u_\varepsilon(x+h_2\cdot e_i)-u_\varepsilon(x)}{h} = \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i}(x+h_1\cdot e_1 t) - \frac{\partial u_\varepsilon}{\partial x_i}(x+h_2 e_i t)dt.$$

$$\int_{\Omega_\delta} |\dots|^p \leqslant \int_0^1\int_{\Omega_\delta} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+h_1\cdot e_i\cdot t) - \frac{\partial u_\varepsilon}{\partial x_i}(x+h_2\cdot e_i\cdot t)\right|^p dxdt,$$

$$\int_{\Omega_\delta} |u_i^{h_1} - u_i^{h_2}|^p \leqslant \int_0^1\int_{\Omega_\delta} |noepsilon|^p dxdt$$

□

Věta 1.4 (Approximation by smmooth function)

Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open and $p \in [1,\infty)$. Then $\forall u \in W^{k,p}(\Omega)$

- $\exists \{u^n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\Omega)$ *such that* $\|u^n - u\|_{W^{k,p}(\Omega)} \to 0$;

- *if $\Omega \in C^0$, then $\exists \{u^n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega})$ such that $\|u^n - u\|_{w^{k,p}(\Omega)} \to 0$.*

Důkaz
„First item“ at home. „Second item“: Lemma(Partition of unite): „Let $\{\Omega_r\}_{r=1}^{M+1}$ be open covering of $\overline{\Omega}$. Then $\exists \varphi_r \in C_0^\infty(\Omega_r)$, such thtat $\forall x \in \overline{\Omega} : \sum_{r=1}^{M+1} \varphi_r(x) = 1$.“ Proof at home.

Define $u_r(x) := u(x)\varphi_r(x)$. TODO!!!

1) u_{M+1} is supported in $\Omega_{M+1} \subseteq \Omega$, so it can be extened by 0. So $u_{M+1}^n = u_{M+1} * \eta_{\frac{1}{n}}$. 2) u_r for $r \in [M]$. Set $u_i^h(x) := u_i(x_1, \dots, x_{d-1}, x_d - h)$, $u_i^n := u_i * \eta_{\frac{1}{n}}$. We need h, ε edepending on n: $\|u_i^n - u_i\|_{W^{k,p}(V_1^+)} \to 0$ (because a_i is continuous). $\varphi_i \in C_0^\infty$ $\exists \delta' > 0$. φ_1 is positive on the set $x_d < a_1(x') + \beta - \delta' \wedge x_d > a_1(x') - \beta + \delta'$, $h < h_0 < \delta'$. Take $(x, \dots, x_{d-1}, x_d - h)$, where $(x_1, \dots, x_d) \in \partial\Omega$, $\mathrm{dist}((x_1, \dots, x_{d-1}, x_d - h), \partial\Omega) < \delta$. Denote this h as h_{max}, so for $h < h_{max}$ $\mathrm{dist}(\dots) < \delta$.

Give me $\delta > 0$, I find h_0, h_{max} and define $u_i^h = u_i^h * \eta^\delta$, where $h < \min(h_0, h_{max})$. Then $\|u_i - u_i^h\|_{W^{k,p}(V_i^+)} \to 0$, $\|u_i^h - (u_i^h)_\delta\|_{W^{k,p}(V_i^+)} \to 0$ □

Věta 1.5

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Then there exists a linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that

- *$Eu = u$ in Ω;*
- *$\exists B_R \subset \mathbb{R}^d$ such that $Eu \equiv 0$ in $\mathbb{R}^d \backslash B_R$;*
- *$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leqslant c(p, \Omega) \cdot \|u\|_{W^{1,p}(\Omega)}$.*

Důkaz
Focus only on V_1 (previous proof), $u = \sum u_r$ (u_{M+1} is done) and only for u_1.

TODO images!!!

□

2 Proof of $W^{1,p} \hookrightarrow C^{0,d}$

Lemma 2.1 (Morrey)

Let $u \in W^{1,1}(B_R(0))$ and 0 be the Lebesgue point of u.

$$\left| \fint_{B_R} u(x)dx - u(0) \right| \leqslant R^A c(A, d) \sup_{\varrho \leqslant R} \int_{B_\varrho} \frac{|\nabla u(x)|}{\varrho^{d-1+A}} dx \qquad A > 0.$$

Důkaz

$$|\fint_{B_R} u - u(0)| = \lim_{r\to 0_+} |\fint_{B_R} u - \fint_{B_r} u| = \lim_{r\to 0_+} |\int_r^R \frac{d}{d\varrho} \fint_{B_\varrho} u(x)dxd\varrho| = \lim_{r\to 0_+} |\int_r^R \frac{d}{d\varrho} \fint_{B_1(0)} u(\varrho x)dxd\varrho| =$$

$$\leqslant \lim_{r\to 0_+} \int_r^R \fint_{B_1(0)} |\nabla u(\varrho x)|dxd\varrho = \lim_{r\to 0_+} \int_r^R \fint_{B_\varrho} |\nabla u(x)|dxd\varrho = \lim \int_r^R \varkappa_d \int_{B_\varrho} \frac{|\nabla u(x)|dx}{\varrho^{d-1+A}} \varrho^{A-1} d\varrho \leqslant$$

$$\leqslant c(d) \sup_{\varrho<R} \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} \cdot \lim_{r\to 0_+} \int_r^R \varrho^{A-1} d\varrho = c(d,A) R^A \sup \ldots$$

□

Poznámka
Replace u by Eu.

Důsledek
x, y Lebesgue points of u.

$$|u(x) - u(y)| \leqslant |x-y|^\alpha c(\alpha, \Omega, p) \max\{I_x, I_y\},$$

where

$$I_x := \sup_{r \leqslant 2|x-y|} \int_{B_r(x)} \frac{|\nabla u|}{}$$

Důkaz
$R := |x-y|$

$$|u(x)-u(y)| \leqslant |\fint_{B_R(x)} u - u(x)| + |\fint_{B_R(y)} u - u(y)| + |\int_{B_R(x)} u - \fint_{B_R(y)} u| \leqslant c(\varrho,\alpha) r^\alpha (I_x + I_y).$$

$$?|| = |\int_0^1 \frac{d}{dt} \fint_{B_R(tx+(1-t)y)} u(z)dz| = |\int_0^1 \frac{d}{dt} \fint_{B_R(0)} u(tx+(1-t)y+z)dz| \leqslant$$

$$\leqslant \int_0^1 \fint_{B_R(0)} |\nabla u(\dots)\cdot(x-y)|dzdt \leqslant c(d) \int_0^1 \int_{B_R(0)} \frac{|\nabla u(\dots)|}{R^{d-1}} dzdt \leqslant$$

$$\leqslant c(d) \int_0^1 \int_{B_{2R}(x)} \frac{|\nabla u(z)|dz}{R^{d-1}} dt = c(d) \int_{B_{2R}(x)} \frac{|\nabla u(z)|dz}{(2R)^{d-1+\alpha}} (2R)^\alpha \leqslant c(d,\alpha) I_x (2R)^\alpha.$$

$$\sup_{R>0, x\in\mathbb{R}^d} I_R(x) = \sup_{R,x} \int_{B_\varrho(x)} \frac{|\nabla Eu(z)|}{R^{d-1+\alpha}} dz \leqslant \sup \left(\int_{B_R} |\nabla Eu|^p\right)^{1/p} \left(\int_{B_R} R^{(1-d-\alpha)p'}\right)^{1/p'} \leqslant c\|u\|_{W^{1,p}(\Omega)} \Big(R^{(}$$

It remains $\|u\|_{L^\infty(\Omega)}$:

$$|u(x)| \leqslant |u(x)-u(y)| + |u(y)|$$

$$|u(x)| = \fint_\Omega |u(x)|dy \leqslant \fint_\Omega |u(y)|dy + K\|u\|_{W^{1,p}(\Omega)} \leqslant C(\Omega)\|u\|_{1,p},$$

$$\|\frac{u(x)-u(y)}{|x-y|^\alpha}\| \leqslant c\|u\|_{1,p}.$$

x, y Lebesgue points, so estimates TODO? □

Poznámka
$W^{1,d}(\Omega) \not\hookrightarrow C^0(\overline{\Omega})$, but $W^{1,d}(\Omega) \hookrightarrow BMO(\Omega)(VMO)$.

$W^{d,1}(\Omega) \hookrightarrow W^{1,d} \not\hookrightarrow C^0(\overline{\Omega})$, but $W^{d,1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$.

Věta 2.2

Let $\Omega \in C^{0,1}$, $p \in [1,\infty)$. Then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ if

- *for $p \in [1,d)$ and $q \leqslant \frac{dp}{d-p}$;*
- *for $p = d$ and $q \in [1,\infty)$;*
- *for $p > d$ and $q \in [1,\infty]$.*

And $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *if previous holds, except for* $p \in [1,d)$ *and* $q = \frac{dp}{d-p}$.

Důkaz (Scheme of the proof)
We use extension $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ compactly supported. Mollification $W^{1,p}(\mathbb{R}^d) \to \mathcal{C}_0^\infty(\mathbb{R}^d)$. Show all estimates only for smooth functions. $\square$

Lemma 2.3 (Gagliardo–Nirenberg inequality)

$\exists C(d), C(d,p)$ *such that* $\forall u \in C_0^\infty(\mathbb{R}^d)$*:*

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leqslant C(d)\|\nabla u\|_{L^1(\mathbb{R}^d)},$$

$$\|u\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \leqslant C(d,p)\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

Důkaz (Proof of lemma)
Firstly we show that first inequality implies second. Define $v := |u|^q$ for some $q > 1$. Then from first inequality $\|v\|_{\frac{d}{d-1}} \leqslant c \cdot \|\nabla v\|_1$:

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{qd}{d-1}}\right)^{\frac{d-1}{d}} \leqslant C(d) \int_{\mathbb{R}^d} |\nabla |u|^q| \leqslant c(d,q) \int_{\mathbb{R}^d} |u|^{q-1} |\nabla u| \overset{\text{Hölder}}{\leqslant}$$

$$\leqslant c(d,q)\|\nabla u\|_p \cdot \||u|^{q-1}\|_{p'} = c(d,q)\|\nabla u\|_p \left(\int_{\mathbb{R}^d} |u|^{\frac{p(q-1)}{p-1}}\right)^{\frac{p-1}{p}}.$$

Choose q such that $\frac{qd}{d-1} = \frac{p(q-1)}{p-1}$, i. e. $q = \frac{p(d-1)}{d-p}$.

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}}\right)^{\frac{d-1}{d}} \leqslant C(d,p)\|\nabla u\|_p \left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}}\right)^{\frac{p-1}{p}} \implies \|u\|_{\frac{dp}{d-p}} \leqslant C(d,p)\|\nabla u\|_p.$$

Then we proof first inequality by next lemma: (u is smooth, compactly supported)

$$u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d) ds,$$

$$|u(x)| \leqslant \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds.$$

$$|u(x)|^d dx \leqslant \prod_{i=1}^d \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds,$$

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx \leqslant \int_{\mathbb{R}^d} \prod_{i=1}^d \left(\int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds\right) dx =: \int_{\mathbb{R}^d} \prod_{i=1}^d v_i \overset{\text{Gagliardo}}{\leqslant}$$

$$\leqslant \prod_{i=1}^d \left(\int_{\mathbb{R}^{d-1}} \left[\left(\int_{-\infty}^{\infty} |\nabla u(x_1, \dots, s, \dots, x_d)| ds\right)^{\frac{1}{d-1}}\right]^{d-1} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_d\right)^{\frac{1}{d-1}} \leqslant$$

$$\leqslant \prod_{i=1}^d \|\nabla u\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}} = \|\nabla u\|_1^{\frac{d}{d-1}}.$$

□

Lemma 2.4 (Gagliardo)

Let $u_i \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1})$, $i \in [d]$. Define $v_i(x_1, \dots, x_d) := u_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$. Then

$$\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)| dx \leqslant \prod_{i=1}^d \|u_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

Důkaz (Proof of lemma)
By induction: 1) „$d=2$“:

$$\int_{\mathbb{R}^d}\prod_{i=1}^{2}|v_i(x)|dx=\int_{\mathbb{R}^2}|u_1(x_2)|\cdot|u_2(x_1)|dx_1dx_2=\|u_1\|_{L^1(\mathbb{R})}\cdot\|u_2\|_{L^1(\mathbb{R})}.$$

2) „$d \implies d+1$“:

$$\int_{\mathbb{R}^{d+1}}\prod_{i=1}^{d+1}|v_i(x)|dx=\int_{\mathbb{R}^d}|v_{d+1}(x)|\cdot\left(\int_{\mathbb{R}}\prod_{i=1}^{d}|v_i(x)|dx_{d+1}\right)dx_1\dots dx_d\leqslant$$

$$\leqslant\|v_{d+1}\|_{L^d(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\prod_{i=1}^{d}|v_i(x)|dx_{d+1}\right)^{d'}dx_1\dots dx_d\right)^{\frac{1}{d}}=RHS$$

$$(\dots)^{d'}=\left(\int_{\mathbb{R}}|v_1|\cdot\ldots\cdot|v_d|dx_{d+1}\right)^{d'}\overset{\text{Hölder}}{\leqslant}\left(\prod_{i=1}^{d}\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{\frac{1}{d}}\right)^{d'}.$$

$$RHS\leqslant\|v_{d+1}\|_{L^d}\left(\int_{\mathbb{R}^d}\left(\prod_{i=1}^{d}\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{\frac{1}{d}}\right)^{\frac{d}{d-1}}dx_1\dots dx_d\right)^{\frac{d-1}{d}}\leqslant$$

$$\leqslant\|u_{d+1}\|_d\left(\int_{\mathbb{R}^d}\prod_{i=1}^{d}\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{\frac{1}{d-1}}dx_1\dots dx_d\right)^{\frac{d-1}{d}}\overset{\text{Induction step}}{\leqslant}$$

$$\leqslant\|u_{d+1}\|_d\cdot\prod_{i=1}^{d}\|\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{\frac{1}{d-1}}\|_{L^{d-1}(\mathbb{R}^{d-1})}^{\frac{d-1}{d}}=\prod_{i=1}^{d}\|u_i\|_{L^d}$$

□

Důkaz
If $p < d$ Galiardo–Nirenberg finishes $W^{1,p} \hookrightarrow L^{\frac{dp}{d-p}}$. If $p = d$, $W^{1,d} \hookrightarrow W^{1,d-\varepsilon} \hookrightarrow L^{\frac{d(d-\varepsilon)}{d-(d-\varepsilon)}} = L^{\frac{d(d-\varepsilon)}{\varepsilon}}$. If $p > d$ forget G–N and use $W^{1,p} \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q(\Omega)$.

„Compact embeddings:“ 1. step: $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$. 2. step $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$.

„$1 \implies 2$“: $\|u\|_q \leqslant \|u\|^\alpha_{\frac{dp}{d-p}} \|u\|_1^{1-\alpha} \leqslant c\|u\|^\alpha_{1,p}\|u\|_1^{1-\alpha}$. Let B be a bounded set in $W^{1,p}(\Omega)$. Use 1. step: (for $\frac{1}{q} = 1 - \alpha + \frac{\alpha(d-p)}{dp}$, i.e. $1 \leqslant q < \frac{dp}{d-p}$)

$$\exists \{u_i\}_{i=1}^N \subseteq W^{1,p}(\Omega)\ \forall u \in B : \min_{i\in[N]} \|u - u_i\|_{L^1} \leqslant \tilde{\varepsilon}.$$

$$\|u - u_i\|_q \leqslant c \cdot \|u - u_i\|^\alpha_{1,p} \cdot \|u - u_i\|_1^{1-\alpha} \leqslant c(\alpha, B)(\tilde{\varepsilon})^{1-\alpha}.$$

„1. step“: Let B be a bounded set in $W^{1,1}(\Omega)$, EB be bounded set in $W^{1,1}(\mathbb{R}^d)$ and compactly supported in B_Ω?.

$$\forall u \in EB : u_\delta := u * \eta_\delta :$$

$$\int_{\mathbb{R}^d} |u(x) - u_\delta(x)| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} u(x)\eta_\delta(y) - u(x+y)\eta_\delta(y) dy \right| dx \leqslant$$
$$\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x) - u(x+y)|}{|y|} \eta_\delta(y)|y| dx dy \overset{\text{we already had it}}{\leqslant}$$
$$\leqslant \int_{\mathbb{R}^d} \|\nabla u\|_1 \eta_\delta(y)|y| dy \leqslant \|\nabla u\|_1 \delta \int_{\mathbb{R}^d} \eta_\delta(y) dy = \|\nabla u\|_1 \delta \leqslant C(B)\delta.$$

Given $\varepsilon > 0$ choose $\delta > 0$, $C(B)\delta < \frac{\varepsilon}{2}$ and use Arsela–Ascoli. □

Poznámka (Lipschitz domain is necessary)

$$u = \frac{1}{(1 + |x|)^{3/2}}.$$

2.1 Compact embedding in Bochner spaces

Lemma 2.5 (Aubin–Lions)

$V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$ Banach spaces. $p \in [1, \infty)$. Then $U := \{u \in L^p(0, T, V_1) | \partial_t u \in L^1(0, T, V_3)\} \hookrightarrow\hookrightarrow L^p(0, T, V_2)$.

Lemma 2.6 (Ehrling (start of proof of Aubin–Lions))

Let $V_1 \hookrightarrow\hookrightarrow V_2 \hookrightarrow V_3$ be Banach spaces. Then

$$\forall \varepsilon > 0\ \exists c > 0\ \forall u \in V_1 : \|u\|_{V_2} \leqslant \varepsilon \|u\|_{V_1} + c \cdot \|u\|_{V_3}.$$

Důkaz (By contradiction)

$$\exists u^n \in V_1 : \|u^n\|_{V_2} > \varepsilon \|u^n\|_{V_1} + n\|u^n\|_{V_3}.$$
$$u^n \neq 0 \implies v^n := \frac{u^n}{\|u^n\|_{V_2}} \implies 1 = \|v^n\|_{V_2} > \varepsilon \cdot \|v^n\|_{V_1} + n \cdot \|v^n\|_{V_3}.$$
$$v^n \to \text{ in } V_3 \implies v^n \text{ bounded in } V_1 \hookrightarrow\hookrightarrow V_2 \implies V^n \to v \text{ in } V_2 \implies$$
$$\implies \|v\|_{V_2} = 1 \implies v \neq 0. \lightning$$

□

Důkaz (Aubin–Lions)
$M \subseteq U$ bounded set: $\exists c^* \ \forall u \in M : \int_0^T \|u\|_{V_1}^p + \|\partial_t u\|_{V_3} \leqslant c^*$.

We want M is precompact in $L^p(0, T < V_2) \Leftrightarrow$

$$\forall \varepsilon > 0 \ \exists \{w_i\}_{i=k}^N \ \ \forall u \in M \ \exists i \in [N] : \int_0^T \|u - w_i\|_{V_2}^p \leqslant \varepsilon.$$

1. Mollify with respect to time and use Arsela–Ascoli for $C^1(0, T; V_1) \hookrightarrow\hookrightarrow C^0(0, T < V_2)$.

2. Mollification is „not far“ from u in $L^1(0, T; V_3)$.

3, Ehrling interpolation to $L^p(0, T; V_2)$.

Důkaz (1.)
$u \in M$ extend to $(0, 2T)$ as $\tilde{u}(t) = u(t)$ if $t < T$ and $\tilde{u}(t) = u(2T - t)$ if $t > \ T$.

$$\int_0^{2T} \|\tilde{u}(t)\|_{V_1}^p + \|\partial_t \tilde{u}(t)\|_{V_3} = 2 \int_0^T \|u\|_{V_1} + \|\partial_t u\|_{V_3} \leqslant 2C^*.$$

$\forall 0 < \delta < T$ and $t \in (0, T)$, $u_\delta(t) = \int_0^\delta \tilde{u}(t + s)\varphi_\delta(s) ds = \int_{\mathbb{R}} \tilde{u}(s)\varphi_\delta(s - t)$, where $\varphi \in C_0^\infty(0, 1)$, $\varphi \geqslant 0$.

$$\|u_\delta(t)\|_{V_1} \leqslant \frac{c}{\delta} \int_0^{2T} \|\tilde{u}\|_{V_1} \leqslant \frac{c \cdot c^*}{\delta}.$$
$$\|\partial_t u_\delta(t)\|_{V_1} \leqslant \int_{\mathbb{R}} \|\tilde{u}\|_{V_1} |\varphi_\delta'| \leqslant c(\delta) \cdot c^*.$$

$M_\delta := \{u_\delta, u \in M\} \implies M_\delta$ is bounded in $C^1(0, T; V_1)$. $\forall \tilde{\varepsilon} > 0 \ \exists \{w_k\}_{k=1}^N \subseteq L^p(0, T; V_1)$ such that $\forall u_\delta \in M_\delta \ \exists k : \int_0^T \|w_\delta - w_k\|_{V_2}^p < \tilde{\varepsilon}$. □

Důkaz (2.)

$$u(t)-u_\delta(t)=u(t)-\int_0^\delta \tilde{u}(t+s)\varphi_\delta(s)ds=$$

$$=\int_0^\delta (u(t)-\tilde{u}(t+s))\varphi_\delta(s)ds=-\int_0^\delta (u(t)-\tilde{u}(t+s))\frac{d}{ds}\left(\int_s^\delta \varphi_\delta(\tau)d\tau\right)ds=$$

$$=-\int_0^\delta \partial_t\tilde{u}(t+s)\int_s^\delta \varphi_\delta(\tau)d\tau ds=-\int_0^\delta\int_0^\tau \partial_t\tilde{u}(t+s)\varphi_\delta(\tau)dsd\tau.$$

$$\int_0^T \|u(t)-u_0(t)\|_{V_3}dt\leqslant \int_0^T\int_0^\delta\int_0^\tau \|\partial_t\tilde{u}(t+s)\|_{V_3}\varphi_\delta(\tau)dsd\tau dt\leqslant \int_0^T\int_0^\delta\int_0^\delta \|\partial_t\tilde{u}(t+s)\|_{V_3}\varphi_\delta(\tau)dsd\tau dt\leqslant c$$

$$\|u(t)-u_\delta(t)\|_{V_3}\leqslant \int_0^\delta\int_0^\tau \|\partial_t u(t+s)\|_{V_3}\varphi_\delta(\tau)dsd\tau\leqslant c^*.$$

□

Důkaz (3.)

$$\int_0^T \|u-w_k\|_{V_2}^p \overset{\text{Ehrling}}{\leqslant} \tilde{\tilde{\varepsilon}}\int_0^T \|u-w_k\|_{V_1}^p+c(\tilde{\tilde{\varepsilon}})\int_0^T \|u-w_k\|_{V_3}^p\leqslant k(c^*)\tilde{\tilde{\varepsilon}}+c(\tilde{\tilde{\varepsilon}})\int_0^T \|u-w_k\|_{V^3}^p\leqslant$$

$$\leqslant k(c^*)\tilde{\tilde{\varepsilon}}+c(\tilde{\tilde{\varepsilon}},p)\int_0^T \|u-u_\delta\|_{V_3}^p\leqslant$$

$$\leqslant k(c^*)\tilde{\tilde{\varepsilon}}+c(\tilde{\tilde{\varepsilon}},p)\sup_{t\in(0,T)}\left\{\|u(t)-u_\delta(t)\|_{V_3}^{p-1}\right\}\int_0^T \|u-u_\delta\|_{V_3}+-||-\leqslant$$

$$\leqslant k(c^*)\tilde{\tilde{\varepsilon}}+c(c^*,p,\tilde{\tilde{\varepsilon}})\delta+c(\tilde{\tilde{\varepsilon}},p,c^*)\int_0^T \|u_\delta-w_k\|_{V_3}^p.$$

Find $\tilde{\tilde{\varepsilon}}$ such that $k(c^*)\tilde{\tilde{\varepsilon}}\leqslant\frac{\varepsilon}{3}$. Find $\delta>0$ such that $c(c^*,p,\varepsilon)<\frac{\varepsilon}{3}$. Find N-covering $\{U_i\}_{i=1}^N$ such that $\min_k c(\tilde{\tilde{\varepsilon}},p,c^*)\int_0^T \|u_\delta-w_k\|\leqslant\frac{\varepsilon}{3}$. □

□