1 Dynamické systémy

Definice 1.1 (Dynamický systém)

(φ, Ω), $\Omega \subset \mathbb{R}^n$ otevřená, $\varphi : \mathbb{R} \times \Omega \to \Omega$ $\varphi(t, x)$.

- $\varphi(0, x) = x$;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- φ je spojité.

Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geqslant 0\}$ je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leqslant 0\}$ je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$ je plný orbit.

Definice 1.3 (Pozitivně, negativně a úplně invariantní)

(φ, Ω) dynamický systém, $M \subset \Omega$.

M je pozitivně invariantní $\equiv \forall x \in M : \gamma^+(x) \subset M$.

M je negativně invariantní $\equiv \forall x \in M : \gamma^-(x) \subset M$.

M je úplně invariantní $\equiv \forall x \in M : \gamma(x) \subset M$.

Poznámka
$\gamma^+(x_0)$ je pozitivně invariantní, $\gamma^-(x_0)$ je negativně invariantní a $\gamma(x_0)$ je úplně invariantní.

Definice 1.4

$$\omega(x_0) = \left\{ y \in \Omega | \exists \left\{ t_k \right\}_{k=1}^{\infty}, t_k \to \infty : \varphi(t_k, x_0) \to y \right\},$$
$$\alpha(x_0) = \left\{ y \in \Omega | \exists \left\{ t_k \right\}_{k=1}^{\infty}, t_k \to -\infty : \varphi(t_k, x_0) \to y \right\}.$$

Poznámka (To je ekvivalentní)
$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0\ \forall T > 0\ \exists t \geqslant T : |\varphi(t_r, x_0) - y| < \varepsilon\}$.

Lemma 1.1

$\omega(x_0) = \bigcap_{\tau \geqslant 0} \overline{\gamma^+(\tau, x_0)}$.

Důkaz
„$\subseteq$“: $y \in \omega(x_0)$: $\forall \varepsilon > 0\ \forall T\ \exists t \geqslant T : |\varphi(t, x_0) - y| < \varepsilon$. Chceme:

$$\forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists s \geqslant \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

„$\supseteq$“: $\forall \tau \geqslant 0\ y \in \overline{\gamma^+(\tau, x_0)} \implies$

$$\implies \forall \varepsilon\ \exists s \geqslant \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

□

Věta 1.2 (Vlastnosti ω-limitní množiny)

Nechť (φ, Ω) je dynamický systém, $x_0 \in \Omega$. Potom

1. *$\omega(x_0)$ je uzavřená, úplně invariantní.*
2. *Pokud $\gamma^+(x_0)$ je relativně kompaktní v $\mathbb{R}^n$, pak $\omega(x_0) \neq \emptyset$, $\omega(x_0)$ je kompaktní, souvislá.*

Důkaz
1. $\omega(x_0)$ je průnik uzavřených množin, tedy uzavřená. $y \in \omega(x_0)\ \exists t_k \nearrow \infty\ \varphi(t_k, x_0) \to y$.

$$\begin{aligned} s_k = t_k + t \qquad & \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0)) \\ t_k \to \infty, \varphi \text{ spojitá} \qquad & \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \to \varphi(t, y) \end{aligned}$$

2. Víme $\exists K \subset \mathbb{R}^n$ kompaktní $\gamma^+(x_0) \subset K$. a) pokud $t_n \geqslant 0, t_n \to \infty\, \{\varphi(t_n, x_0)\}_{n=1}^{\infty}$ omezená posloupnost $\implies \exists \{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, podposloupnost, $\exists y \in \Omega \varphi(t_{n_k}, x_0) \to y$. Pak $y \in \omega(x_0)$.

b) $\omega(x_0)$ je tedy úplná a omezená, takže kompaktní. c) ať $\omega(x_0)$ je nesouvislá, tedy $\omega(x_0) \subseteq U \cup V$, U, V otevřené disjunktní neprázdné, $U, V \subseteq K$. Vezměme $y \in \omega(x_0) \cap U$, $z \in \omega(x_0) \cap V$. Nechť t_n je posloupnost taková, že $\varphi(t_{2n} x_0) \to y$, $\varphi(t_{2n+1}, x_0) \to z$, $t_{2n} < t_{2n+1}$, $\varphi(t_{2n}, x_0) \in U$, $\varphi(t_{2n+1}, x_0) \in V$. $F = K \backslash (U \cup V)$ uzavřená, tedy $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$. Tedy $\{\varphi(s_n, x_0)\}$ je omezená posloupnost $\implies \exists$ podposloupnost konvergující k $w \in F$. □

Definice 1.5 (Topologická konjugovanost)

(φ, Ω), ψ, Θ dynamické systémy. $\exists : \Omega \to \Theta$ homeomorfismus (bijekce, spojité, spojitá inverze) h:

$$\forall x \in \Omega\ \forall t \in \mathbb{R} \qquad h(\varphi(t, x)) = \psi(t, h(x)).$$

Poznámka
Dá se zobecnit ještě zobrazováním časů.

Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0$, (φ, Ω) příslušný dynamický systém. $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $y(0) = 0$ a (ψ, Θ) je příslušný dynamický systém. Potom (φ, Ω), (ψ, Θ) jsou lokálně topologicky konjugované ($\exists U$ okolí $x_0 \in \Omega$ a V okolí $\mathbf{o} \in \mathbb{R}^n$ taková, že $\exists g : U \to V$ homeomorfismus $g(\varphi(t, x)) = \psi(t, g(x))$ $\forall x \in U$, $\forall t : \varphi(t, x) \in U$).

Důkaz
BÚNO $f_1(x_0) = \alpha \neq 0$ (první souřadnice funkce f) a $x_0 = \mathbf{o}$. Buď $\tilde{V}$ okolí $\mathbf{o} \in \mathbb{R}^n$ $G : \tilde{V} \to \mathbb{R}^n$, $G(y_1, \dots, y_n) = \varphi(y_1, (0, y_2, \dots, y_n))$. Chceme ukázat, že G je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_1}|_{(0,\dots,0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \dots, y_n))|_{y_1=0,\dots,y_n=0} =$$

$$= f(\varphi(y_1(0, y_2, \dots, y_n)))|_{y_1=0,\dots,y_n=0} = f(\varphi(0, (0, \dots, 0))) = f(x_0) = \alpha.$$

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_j}|_{(0,\dots,0)} = \lim_{h\to 0} \frac{G(0, \dots, h, \dots, 0) - G(0, \dots, 0)}{h} =$$

$$= \lim_{h\to 0} \frac{(0, \dots, h, \dots, 0)^T - (0, \dots, 0)^T}{h} = (0, \dots, 1, \dots, 0)^T = e_j.$$

Tedy $\nabla G(0, \dots, 0)$ je „jednotková matice, až na to, že a_{11} je α“, tudíž podle věty o inverzi funkce $\exists V \subseteq \tilde{V}$ okolí 0, $\exists U$ okolí bodu x_0 tak, že $G : V \to U$ je homeomorfismus. Položme $g = G^{-1}$.

Nyní stačí $g(\varphi(t, x_0)) = \psi(t, g(x_0))$ $\forall x_0 \in U$ $\forall t : \varphi(t, x_0) \in U$. $\varphi(t, x_0) = G(\psi(t, g(x_0)))$

3. $x \in U = G(V)$ $\exists y \in V$ $x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

□

Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \to \Omega, f \textit{ loc. Lip.}$$

$$\exists V : \Omega \to \mathbb{R}, \textit{ bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leqslant l\} \textit{ – bounded}$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leqslant 0 \qquad \forall x \in \Omega_l.$$

$$R = \left\{ x \in \Omega_l | \dot{V}_f(x) = 0 \right\}, \quad M = \left\{ y \in R | \gamma^+(y) \subset R \right\}.$$

Then $\forall x \in \Omega_l : \omega(x) \subset M$.

Důkaz

Let $x \in \Omega_l$. $\forall y \in \omega(x)\ \exists t_k \nearrow \infty : x(t_k) \to y$. $\varphi(t, x_0) = x(t)$.

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leqslant 0.$$

$V(x(t)) \searrow$ and $\exists C : \forall x \in \Omega : V(x) > -C$ so $\exists \lim_{t\to\infty} V(x(t)) = c$.

So $\exists c\ \forall y \in \omega(x_0) V(y) = c$. $V(x(t_k)) \to V(y) = c$.

$$\gamma^+(y) \subset \omega(x_0)\ V(\varphi(t, y)) = c\ \forall t \geqslant 0 \implies$$

$$\implies \frac{d}{dt} V(\varphi(t, y)) = 0.$$

$\gamma^+(y) \subset R$ in particular, $y \in R$. Hence $y \in M$. □

2 Poincaré-Bendixson theory

Věta 2.1 (Poincaré-Bendixson)

Let $p \in \Omega$, Ω *open connected.* $\omega(p)$ *doesn't contain stationary points and* $\gamma^+(p)$ *is relatively compact* ($\overline{\gamma^+(p)}$ *is compact). Then* $\omega(p) = \Gamma$*-periodic orbit.*

Věta 2.2 (Bendixson-Dulas)

Ω*-simply connected* ($\forall$ *closed Jordan curve* γ *in* Ω, $\mathrm{int}(\gamma) \subset \Omega$). $\exists B : \Omega \to \mathbb{R} : (\operatorname{div} Bf)(x) = \frac{\partial Bf_1}{\partial x_1}(x_1, x_2) + \frac{\partial Bf_2}{\partial x_2}(x_1, x_2) > 0$ *for almost every* $x \in \Omega$. *Then* $x' = f(x)$ *doesn't have nontrivial periodic solutions.*

Definice 2.1 (Transverzála)

Σ segment on a line such that $\forall p \in \Sigma : \Sigma \nparallel f(p)$.

Lemma 2.3

Σ *transverzála,* $p \in \Sigma \subset \Omega$. *Then* $\exists \tilde{U} \subset U$ *neighborhood of* p, $\exists \Delta > 0$ *such that*

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U\ \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

Důkaz
Use Th. of rect. □

Lemma 2.4

Let $p \in \Omega$ and assume that $|\gamma^+(p) \cap \Sigma| \geqslant 3$, i. e. $\exists t_1 < t_2 < t_3$ $\varphi(t_j, p) \in \Sigma$, $j = 1, 2, 3$. Then $\varphi(t_2, p)$ lies between $\varphi(t_1, p)$ and $\varphi(t_3, p)$.

Důkaz
Simple closed curve:

$$\psi := \{\varphi(t, x), t \in [t_1, t_2]\} \cup \underbrace{\operatorname{conv}\{z_1, z_2\}}_{\subset\Sigma}.$$

By uniqueness of φ and by the Jordan Lemma. □

Lemma 2.5

$\Sigma \subseteq \Omega \subseteq \mathbb{R}^2$ *transversal,* $p \in \Omega \implies |\omega(p) \cap \Sigma| \leqslant 1$.

Důkaz

$$y \neq z \in \omega(p) \cap \Sigma \implies \exists t_k \nearrow \infty : x(t_{2k}) \to y \wedge x(t_{2k+1}) \to z.$$

From lemma above: $\exists \tilde{U} \subset U$ – neighbourhoods of y and $\exists \Delta$:

$$\exists k_0 : \forall k > k_0, (x(t_{2k}) \in \tilde{U}) \implies (\exists \tilde{t}_{2k} : |\tilde{t}_{2k} - t_{2k}| < \frac{\Delta}{2} \wedge x(\tilde{t}_{2k}) \in \Sigma \cap \tilde{U}.$$

Similarly $\exists \tilde{V}$ – neighbourhood of z, $\exists \tilde{t}_{2k+1}$: $|\tilde{t}_{2k+1} - t_{2k+1}| < \frac{\Delta}{2}$ and $x(\tilde{t}_{2k+1}) \in \Sigma \cap \tilde{V}$.

WLOG $\tilde{V} \cap \tilde{U} = O$. Now continue with Lemma 2 (not monotonic). □

Důkaz (Poincaré-Bendixson theorem)
Step 1: For $q \in \omega(p)$ we want to show that q belongs to $q \in \Gamma$, where Γ is non-trivial periodic orbit.

$\exists x_0 \in \omega(q)$, $\exists t_k \nearrow \infty$: $\varphi(t_k, q) \to x_0$. x_0 is not a stationary point ($q \in \omega(p) \implies \omega(q) \subseteq \omega(p)$). So there exists a transversal $\Sigma \subseteq \Omega, x_0 \in \Sigma$.

By lemma above $\exists \tilde{t}_k$, $\exists \Delta > 0$: $|\tilde{t}_k - t_k| < \frac{\Delta}{2}$. $q \in \omega(p) \implies \varphi(\tilde{t}_k, q) \in \omega(p) \implies \varphi(\tilde{t}_k, q) \in \Sigma \cap \omega(p)$ at most 1-point set by theorem…

$$\varphi(\tilde{t}_k, q) \to x_0 \implies \varphi(\tilde{t}_k, q) = x_0.$$

Periodic orbit $\implies x_0 \in \Gamma = \left\{\varphi(t,q) | \tilde{t}_k < t < \tilde{t}_{k+1}, k \in \mathbb{N}\right\} \implies q \in \Gamma$ (uniqueness).

Now we want to show that $\omega(p) \subseteq \Gamma$. Let $M \neq \emptyset$, $M = \omega(p) \backslash \Gamma$: $\gamma^+(p)$ is bounded $\implies$ $\omega(p)$ is connected

$$\exists x_0 \in \Gamma : \exists \left\{p_n\right\}_{n \in \mathbb{N}} \subseteq M, p_n \to x_0.$$

$\exists \Sigma$ transversal: $x_0 \in \Sigma$ (because not stationary). By lemma above we have

$$\exists \tilde{p}_n \in \gamma^+(p_n) : \tilde{p}_n \in \Sigma \cap \gamma^+(p_n) \cap \tilde{U}(x_0).$$

Since $p_n \in \omega(p)$, $n \in \mathbb{N}$, then $\gamma^+(p_n) \subseteq \omega(p) \implies \tilde{p}_n \in \omega(p)$.

By previous lemma $\tilde{p}_n = x_0$ and $p_n \in \gamma^-(\tilde{p}_n) = \gamma^-(x_0) \subseteq \Gamma$. ↯. □

Důkaz (Bendixson-Dulas theorem)
Let Γ be a non-trivial periodic orbit, $\Gamma \subset \Omega$, $\Gamma = \partial M$

$$0 < \int_M \operatorname{div}[B(x) \cdot f(x)] d\lambda^2 = \int_{\partial M} \langle B(x) \cdot f(x), \nu(x) \rangle \, dS = 0.$$

□

3 Caratheodory theory

Definice 3.1 (Caratheodory theory)

f measurable, $x(t)$ absolutely continuous, Lebesgue integral.

Definice 3.2

$\Omega \subseteq \mathbb{R}^{n+1}$, $f \in Car(\Omega) \equiv \forall I \times B \subset \Omega$, $I \subseteq \mathbb{R}$ bounded interval, $B \subseteq \mathbb{R}^n$ bounded closed ball:

- $\forall x \in B$: $t \mapsto f(t, x(t))$ is measurable;
- for almost every $t \in I$: $x \mapsto f(t,x)$ is continuous;
- $\exists h \in L^1(I)$: $|f(t,x)| \leqslant |h(t)|$ for almost every $t \in I$ and $\forall x \in B$.

Definice 3.3 (*)

$x' = f(t,x)$, $x(t_0) = x_0$, $\Omega \subseteq \mathbb{R}^{n+1}$ open, $f : \Omega \to \mathbb{R}^n$, $f \in Car(\Omega)$.

Definice 3.4

$x : I \to \mathbb{R}^n$ (I interval) is a solution of * in the sense of Caratheodory, if $x \in AC_{loc}(I)$ and $graph(x) \subset \Omega$ and for almost every $t \in I : x'(t) = f(t, x(t))$ and $x(t_0) = x_0$.

Poznámka

$$\Leftrightarrow x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \qquad \text{for almost every } t.$$

Lemma 3.1

$f \in Car(\Omega)$, $x : I \to \mathbb{R}$ is continuous, $graph(f) \subseteq \Omega$, then $f(t, x(t)) \in L^1_{loc}(I)$.

Důkaz

Step 1: „f is measurable“: We approximate $x(t)$ by step function on $I = I_1 \cup \ldots \cup I_n \cup \ldots$, $\{x_n(x)\}_{n=1}^{\infty}$, piecewise constant functions, $x_n(t) \rightrightarrows x(t)$ on I_k, $I = \bigcup_j I_{j,n}$ disjoint union, $x_n(x) = \xi_{j,n}$ for $t \in I_{j,n}$. $f(t, x_n(t)) = f(t, \xi_{j,n})$ for $t \in I_{j,n}$, $f(t, \xi_{j,n})$ is measurable.

$f(t, x_n(t)) \to f(t, x(t))$ for almost every $t \in I \implies f(t, x(t))$ is measurable.

Step 2: $|f(t, x(t))| \leqslant l(t)$ for almost every $t \implies f \in L^1_{loc}(I)$. □

Lemma 3.2

$x : I \to \mathbb{R}^n$ continuous, $graph(x) \subseteq \Omega$, $f \in Car(\Omega)$ then x is solution of $*$ $\Leftrightarrow \forall t_1, t_2$: $x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s))ds$.

Důkaz

„$\implies$“ $x \in AC_{loc}(I)$, $x'(t) = f(t, x)$ for almost every $t \in I$, add $\int$:

$$\int_{t_1}^{t_2} x'(t)dt = \int_{t_1}^{t_2} f(s, x(s))ds.$$

„$\impliedby$“ $t_1 = t_0$, $t_2 = t$:

$$x(t) - x_0 = \int_{t_0}^{t} f(s, x(s))ds, \qquad x(0) = x_0.$$

($f \in L^1_{loc}$, so it make sense).

$\implies$ x is AC, $graph(x) \subseteq \Omega$, $x'(t) = f(t, x)$ for almost every $t \in I$. □

Věta 3.3 (Uniform contraction theorem (generalized Banach theorem))

Λ, X metric spaces, $X \neq \varnothing$ complete, $\Phi : \Lambda \times X \to X$. $\forall x \in X : \Phi(\cdot, x)$ is continuous, $\exists \varkappa \in (0,1)$: $\varrho(\Phi(\lambda, x), \Phi(\lambda, y)) \leqslant \varkappa \cdot \varrho(x, y)$ $\forall \lambda \in \Lambda$, $\forall x, y \in X$.

Then $\forall \lambda \in \Lambda$ $\exists! x(\lambda) \in X$ such that $\Phi(\lambda, x(\lambda)) = x(\lambda)$, $\lambda \mapsto x(\lambda)$ is continuous and

$$\varrho(y, x(\lambda)) \leqslant \frac{\varrho(y, \Phi(\lambda, y))}{1 - \varkappa} \ \forall y \in X \ \forall \lambda \in \Lambda.$$

Důkaz
Let $x_0 \in X,\ x_1 = x_1(\lambda, x_0) := \Phi(\lambda, x_0),\ x_{n+1} = x_{n+1}(\lambda, x_0) := \Phi(\lambda x_n).\ \lambda \in \Lambda$ fixed:

$$\varrho(x_n(\lambda, x_0), x_m(\lambda, x_0)) \leqslant \sum_{k=n}^{m-1} \varrho(x_k(\lambda, x_0), x_{k+1}(\lambda, x_0)) =$$

$$= \sum_{k=n}^{m-1} \varrho(\Phi(\lambda, x_{k-1}(x, x_0)), \Phi(\lambda, x_k(\lambda, x_0))) \leqslant$$

$$\leqslant \sum_{k=n}^{m-1} \varkappa \varrho(x_k, x_{k-1}) \leqslant$$

$$\big(\varrho(x_k, x_{k-1}) \leqslant \varkappa \varrho(x_k, x_{k-1}) \leqslant \ldots \leqslant \varkappa^k \varrho(x_0, x_1).\big)$$

$$\leqslant \sum_{k=n}^{m-1} \varkappa^k \varrho(x_0, x_1(\lambda, x_0)) \leqslant \varrho(x_0, x_1(\lambda, x_0)) \underbrace{\sum_{k=n}^{\infty} \varkappa^k}_{\frac{\varkappa^m}{1-\varkappa}}.$$

$\implies$ sequence $\{x_n(\lambda, x_0)\}_{k=1}^{\infty}$ is Cauchy $\implies$ it has a limit:

$$\exists x(\lambda, x_0) : \lim_{n\to\infty} x_n(\lambda, x_0) = x(\lambda, x_0).$$

We want to show that $x(\lambda, x_0)$ does not depend on x_0. $\tilde{x}_0 : x_n(\lambda, \tilde{x}_0) =: \tilde{x}_n$.

$$\varrho(x_n, \tilde{x}_n) = \varrho(\Phi(\lambda x_{n-1}), \Phi(x, \tilde{x}_{n-1})) \leqslant \varkappa^n \varrho(x_0, \tilde{x}_0) \to 0 \implies x = \tilde{x}.$$

„$\Phi(\lambda, x(\lambda)) = x(\lambda)$“:

$$\varrho(\Phi(\lambda, x(\lambda)), x(\lambda)) = \lim_{n\to\infty} \varrho(\Phi(\lambda, x(\lambda)), x_n(\lambda)) =$$

$$= \lim_{n\to\infty} \varrho(\Phi(\lambda, x(\lambda)), \Phi(\lambda, x_{n-1}(\lambda))) \leqslant \varkappa \varrho(x(\lambda, x_{n-1}) = 0.$$

□

Důkaz („$\lambda \mapsto x(\lambda)$ is continuous“)

$$\begin{aligned}\varrho(x(\lambda), x(\mu)) = \varrho(\Phi(\lambda, x(\lambda)), \Phi(\mu, x(\mu))) \leqslant \\ \leqslant \varrho(\Phi(\lambda, x(\lambda)), \Phi(\lambda, x(\mu))) + \varrho(\Phi(\mu, x(\mu)), \Phi(\lambda, x(\mu))) \leqslant \\ \leqslant \varkappa\varrho(x(\lambda), x(\mu)) + \varrho(\Phi(\mu, x(\mu)), \Phi(\lambda, x(\mu))) \implies \\ \implies \varrho(x(\lambda), x(\mu)) \leqslant (1-\varkappa)^{-1}\varrho(\Phi(\lambda, x(\mu)), \Phi(\mu, x(\mu))) \to 0.\end{aligned}$$

So Φ is continuous in the first variable.

μ fixed, $\lambda = \mu_n \to \mu$. y, λ fixed, $x_1 = \Phi(\lambda, y)$, $x_n = \Phi(\lambda, x_{n-1})$, $n \geqslant 2$.

$n = 0$ $(y = x_0)$: $m \to \infty$

$$\varrho(y, \lambda(x)) \leqslant \frac{\varkappa^0}{1-\varkappa}\varrho(y, \Phi(\lambda, y)) = \frac{\varkappa^0}{1-\varkappa}\varrho(y, x_1).$$

□

Věta 3.4 (Generalized Picard theorem)

$$* : x' = f(t, x), \qquad x(t_0) = x_0, \qquad f : \Omega \to \mathbb{R}^n, \qquad \Omega \subseteq \mathbb{R}^{n+1}.$$

$I = [0, T]$, Π metric space, $f : I \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$, $f(t, x, p)$

- *$\forall p \in \Pi$ fixed $f(\cdot, \cdot, p) \in Car(I \times \mathbb{R}^n)$;*
- *$\|f(t, x, p) - f(t, y, p)\| \leqslant l(t)\|x - y\|$ for some $l(t) \in L^1(I)$, $\forall x, y \in \mathbb{R}^n$, $\forall p \in \Pi$ and almost all $t \in I$;*
- *for every $x(\cdot) \in \mathcal{L}(I)$ the map*

$$p \mapsto \int_0^t f(s, x(s), p)ds, \qquad t \in I$$

is continuous.

*Then for all $x_0 \in \mathbb{R}^n$ a $\forall p \in \Pi$ $\exists! x(\cdot) = x(x_0, p) \in AC(I)$ satisfying * in Caratheodory sense with initial condition $x(t_0) = x_0$ and $x(\cdot)$ depends continuously on x_0, p, tj.*

$$(x_0)_n \to x_0 \wedge p_n \to p \implies x_n(\cdot) \equiv x((x_0)_n, p_n) \rightrightarrows x(x_0, p).$$

Důkaz
$X := \varphi(I)$ is complete, $\|f\|_x = \sup_{t\in I}\left\{f(x)\cdot e^{-Lt}\right\}$, L will be chosen $L > \varepsilon$. $\Lambda := \mathbb{R}^n \times \Pi \ni (\lambda_0, p)$, $\int_0^t e^{L(t-s)}ds \leqslant \int_0^t e^{-Ly}dy \leqslant \int_0^\infty e^{-Ly}dy = \frac{1}{L}$.

$$\Phi(x_0, p, x(\cdot))(t) := x_0 + \int_0^t f(s, x(s))ds, \qquad t \in [0, T].$$

Φ is continuous in x_0 and p. Φ is contraction:

$$\|(\Phi(x_0, \varrho, x(\cdot)) - \Phi(x_0, p, y(\cdot)))\| = \int_0^t f(s, x(x)) - f(s, y(s)) \leqslant$$

$$\leqslant \int_0^T \|f(s, x(s), p) - f(s, y(s), p)\|ds \leqslant \int_0^T l(s)\|x - y\|ds,$$

for almost every t.

$$\|x(s) - y(s)\|e^{-L\cdot s} \leqslant \|x - y\|_X \leqslant \int_0^t l(s)e^{+L(s)}ds \cdot \|x - y\|_X?$$

$$\|\Phi(x_0, p, x(\cdot)) - \Phi(x_0, p, y(\cdot))\|_X \leqslant \|x(\cdot) - y(\cdot)\|_X \sup_t \int_0^t l(s)e^{-L(t-s)}ds \leqslant$$

$$\leqslant \int_0^T l(s)e^{-Ls}ds, \qquad l \in L^1([0, T]).$$

$$\exists l_1, l_2 \geqslant 0 : \int_0^T l_1(x)dt \leqslant \frac{1}{3}$$
$$\exists c > 0 : \|l_2\| \leqslant c \text{ for almost every } t \in I$$
$$? \leqslant \frac{1}{3} + c \cdot \frac{1}{L} \leqslant \frac{2}{3}.$$

Then $x \in \mathcal{L}(I)$ fixed point of Φ.

$$\implies x(t) = x_0 + \int_0^t f(s, x(s), p)ds \implies x \in AC(I).$$

Continuously depends on p, x_0:

$$\sup_{t\in I}\left\{(x(t, x_0, p) - y(t))e^{-Lt}\right\} \leqslant (1 - \varkappa)^{-1}((y(x) - x_0 + \int_0^t f(s, y, TODO)))TODO$$

□

4 Controllability

Definice 4.1 (Control theory)

$$x' = f(x,u), f : \Omega \times U, \Omega \subset \mathbb{R}^n, U \subset \mathbb{R}^n,$$
$$u \in \mathcal{U} := \{u : [0,T] \to \mathbb{R}^n | \text{measurable}, ||u||_\infty < \infty\} = L^\infty(0,T,\mathbb{R}^n).$$

($\mathcal{U}$ is admissible functions).

Definice 4.2 (Linear task)

$x' = Ax + Bu,\ A, B \in \mathbb{R}^{n\times m},\ m < n.$

Definice 4.3

$x_0 \underset{u(0)}{\overset{t}{\to}} 0$ iff $x(0) = x_0,\ x(t) = 0.$

Definice 4.4 (Area of controllability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n \middle| \exists u \in L^\infty(0,t,\mathbb{R}^n) : x_0 \underset{u(0)}{\overset{t}{\to}} 0 \right\}$$

Definice 4.5 (Kalman matrix)

$$\mathcal{K}(A,B) := (B|AB|A^2B|\ldots|A^{n-1}B)$$

Věta 4.1

For linear problem $\mathcal{R}(t) = \mathrm{LO}(g_1, g_2, \ldots, g_{n\cdot m})$, *where* $\mathcal{K}(A,B) = (g_1|g_2|\ldots|g_{n\cdot m})$

Tvrzení 4.2 (Observation)

$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)Bu(s)ds}.$

$$x_0 \underset{u(0)}{\overset{t}{\to}} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = -\int_0^t e^{-As}Bu(s)ds.$$

Lemma 4.3 (1)

$$A^k \in \mathrm{LO}(I, A, A^2, \ldots, A^{n-1}), k \in \mathbb{N}_0$$

Důkaz

Cayley-Hamilton. □

Důkaz
1) $\mathcal{R}(t)$ is vector subspace of $\mathbb{R}^n$ from definition $x_0 + x_1 \underset{(u_1+u_2)(0)}{\overset{t}{\to}} 0$, $\alpha x \underset{\alpha u(0)}{\overset{t}{\to}} 0$.

2) We want $\mathcal{R}(t)^\perp = (\mathrm{LO}(g_1, \dots, g_n))^\perp$. „$\supseteq$“: $p \in (\mathrm{LO}(g_1, \dots, g_n))^\perp$. $x_0 \in \mathcal{R}(t)$ arbitrary. From observation.:

$$0 \overset{?}{=} p^T x_0 = -\int_0^t p^T e^{-As} Bu(s) ds = -\int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k Bu(s) ds$$

We know $(p, g_j) = 0$, $p^T g_j = 0$, $p^T \mathcal{K}(A, B) = 0$, $p^T A^k B = 0$, $k \in [n-1]$. And from lemma 1 $k \in \mathbb{N}$. „$\subseteq$“: $p \in \mathbb{R}^n$, $p \in \mathcal{R}(t)^\perp$. We want to prove $p \perp B, AB, A^2B, \dots, A^{n-1}B$. $B = (b_1 | \dots | b_m)$. $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$. $\varphi \in L^\infty(0, T, \mathbb{R})$, $u(t) = \varphi(t) \cdot \mathbf{e}_j$, where $x_0 = -\int_0^t e^{-As} Bu(s) ds$. We have $x_0 \underset{u(0)}{\overset{t}{\to}} 0$, hence $x_0 \in \mathcal{R}(t)$.

$$0 = p^T x_0 = -p^T \int_0^t e^{-As} Bu(s) ds = -\int_0^t p^T e^{-As} b_j \varphi(s) ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have $p^T e^{-As} b_j \equiv 0$, we derivate it, $p^T A^n e^{-As} b_j \equiv 0$, and set $s = 0$. □

Důsledek
$\mathcal{R}(t)$ doesn't depend on time.

Definice 4.6 (Locally and globally controllable)

Linear problem is called locally controllable, iff $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 | \, |x_0| < \delta\} \subset \mathcal{R}(t)$. And globally if $\mathbb{R}^n = \mathcal{R}(t)$.

Důsledek
Linear problem is controllable $\Leftrightarrow \operatorname{rank} K(A, B) = n$.

4.1 Observability

Definice 4.7 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n y = g(x), g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, m < n.$$

Definice 4.8

We say that system $x' = f(x)$ is observable through $g(\cdot)$ on $[0, t]$, iff $\forall x_1(\cdot), x_2(\cdot) : [0, T] \to \mathbb{R}^n : g(x_1(t)) = g(x_2(t)) \ \forall t \in [0, T] \implies x_1(0) = x_2(0)$.

Definice 4.9 (Linear observability)

$x' = Ax$, $y = Bx$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$.

Věta 4.4

$x' = Ax$ is observable on $[0,T]$ through $y = Bx \Leftrightarrow x' = A^Tx + B^Tu$ is controllable.

Důkaz

(We will prove equivalence with $\operatorname{rank} \mathcal{K}(A^T, B^T) = n$.) „ $\Longleftarrow$ “: For contradiction

$$\exists x_1(t), x_2(t), t \in [0,T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At}x_0, Bx(t) = Be^{At}x_0 \equiv 0 \qquad \forall t \in [0,T].$$

We differentiate it, set $t = 0$ and get $Bx_0 = 0$, $BAx_0 = 0$, ..., $BA^{n-1}x_0 = 0$. So $x_0^TB^T = 0$, ..., $x_0^T \left(A^T\right)^{n-1} B^T = 0$. $x_0^T\mathcal{K}(A^T, B^T) = 0$, $x_0 \perp \mathcal{K}(A^T, B^T)$, ↯.

„ $\Longrightarrow$ “: For contradiction $\operatorname{rank}(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T\mathcal{K}(A^T, B^T) = 0$. $x_0^T \left(A^T\right)^k B^T = 0$ $\forall k \in [n-1]$ and from lemma 1 $\forall k \in \mathbb{N}$. $BA^Tx_0 = 0$. $Be^{At}x_0 = 0$ $\forall t \in [0,T]$. ↯. □

Věta 4.5 (Local controllability)

Let $V \subset \mathbb{R}^n$ neighbourhood of 0, $U \subset \mathbb{R}^n$ neighbourhood of 0, $f : V \times U \to \mathbb{R}^n$ C^1 smooth, $f(0,0) = 0$, $\mathcal{U} = \{u : [0,T] \to U \text{ measurable}\}$, $A = \nabla_x f(0,0)$, $B = \nabla_u f(0,0)$, $\operatorname{rank} \mathcal{K}(A, B) = n$. *Then*

$$x' = f(x,u), x(0) = x_0 \text{ is locally controllable } \forall t \in (0,T].$$

Důkaz
Fix $t > 0$, consider $x' = Ax + Bu$. Since $\operatorname{rank}(A, B) = n$, the linear problem is globally controllable. Take initial condition $y_1, \dots, y_n$ linearly independent.

$$\exists u_i \in L^\infty(0, t, \mathbb{R}^n) : y_j \to^t_{u_i(0)} 0$$

$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ denote by $u_{\lambda(t)} = \sum_{j=1}^n \lambda_j u_j(t)$. We know $\sum_{j=1}^n \lambda_j y_j \to^t_{u_\lambda(0)} 0$.

Step 2:

$$x'_\lambda = f(x_\lambda, u_\lambda), \qquad x_\lambda(t) = 0$$

If $\lambda = 0$, then $u_\lambda \neq 0$, then $x_\lambda \equiv 0$.

$$\psi(\lambda) := x_\lambda(0), \psi : U_\lambda(0) \subset \mathbb{R}^n \to \mathbb{R}^n.$$

We want to prove $\psi(U_\lambda(0)) \supseteq \tilde{V}$, for some $\tilde{V} \subset \mathbb{R}^n$ open, $0 \in \tilde{V}$. We will prove that ψ is C^1 smooth, and that $\nabla\varphi(0)$ is regular (if this is proved, than ψ is local diffeomorphism).

Step 3:

$$x_\lambda(s) = x_\lambda(t) + \int_t^s f(x_\lambda(s), u_\lambda(s))ds.$$

Formally differentiate:

$$\frac{\partial x_\lambda(s)}{\partial \lambda_j} = \int_t^s (\nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot \frac{\partial x_\lambda(s)}{\partial \lambda_j} + \nabla_u f(x_\lambda(s), u_j(s)))ds.$$

Denote $y_{\lambda,j}(s) = \frac{\partial x_\lambda(s)}{\partial \lambda_j}$.

$$y'_{\lambda,j}(s) = \nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot y_{\lambda,j}(s) + \nabla_u f(x_\lambda(s), u_\lambda(s)) \cdot u_j(s).$$

$$y_{\lambda,j}(t) = 0.$$

Consider $(LPy) \to y_{\lambda,j}(\cdot)$.

$$x_{\lambda+\Delta\lambda}(s) - x_\lambda(s) - \Delta\lambda \cdot y_{\lambda,j}(s) = 0$$

(as in Thn? of differentiability w. r. t. initial condition)

$$\frac{\partial \psi}{\partial \lambda_j}(\lambda = 0) = \frac{\partial x_\lambda(s = 0)}{\partial \lambda_j}|_{\lambda=0} = y_{\lambda,j}(s = 0)|_{\lambda=0} = y_{\lambda,j}(s = 0)|_{\lambda=0} = y_j.$$

If $\lambda = 0$, then (LPy): $y'_{0,j}(s) = Ay_{0,t}(s) + Bu_j(s)$, $y_{0,j}(t) = 0$. From uniq.: $y_{0,j}(0) = y_{j,0}$.

$$\nabla\psi(0) = \left(\frac{\partial \psi}{\partial \lambda_1}(0) \dots \frac{\partial \psi}{\partial \lambda_n}(0) \right) = (y_1, \dots, y_n)$$

regular matrix. □

Poznámka

$$x' = Ax + Bu, u \in \mathcal{U} = \{u : [0,T] \to [-1,1] \text{ measurable}\}, x(0) = x_0.$$

Definice 4.10

$$\mathcal{R}(t) = \left\{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U} \wedge x_0 \to^t_{u(0)} 0\right\}.$$

Definice 4.11

$u_n \in \mathcal{U}_0$: $u_n \rightharpoonup^* u \in \mathcal{U} \equiv \forall f \in L(0,T,\mathbb{R}^n) : \int_0^T f(s)u_n(s)ds \to \int_0^T f(s)u^*(s)ds.$

Věta 4.6 (Alaoglu)

$\mathcal{U}$ is weak- sequentially compact (i. e. $\forall \{u_n\}_{n=1}^\infty \in \mathcal{U}\ \exists \{u_{n_k}\}$ weekly-* convergent).*

Věta 4.7

$\mathcal{R}(t)$ convex, symmetric, closed $0 < t_1 < t_2 \implies \mathcal{R}(t_1) \subset \mathcal{R}(t_2)$.

Důkaz

Convex: $x_{01}, x_{02} \in \mathcal{R}(t)$, $\alpha \in [0,1] \implies \alpha x_{01} + (1-\alpha)x_{02} \in \mathcal{R}(t)$.

$$x(t) = e^{At}x_0 + \int_0^t e^{As}Bu(s)ds. x_{01} \to^t_{u_{01}} 0 \wedge x_{02} \to^t_{u_{02}} 0 \Leftrightarrow x_{0i} = -\int_0^t e^{(s-t)A}Bu_{0i}(s)ds.$$

Symmetry: $x_0 \in \mathcal{R}(t) \implies -x_0 \in \mathcal{R}(t)$, $x_0 \to^t_u 0 \implies -x_0 \to^t_{-u} 0$.

Closedness: $x_{0n} \in \mathbb{R}(t), x_{0n} \to x_0$. $x_0 \in \mathcal{R}(t)$? $\exists u_n(0) \in \mathcal{U}, x_{0n} = -\int_0^t e^{(s-t)A}Bu_n(s)ds \to -\int_0^t e^{(s-t)A}Bu(s)ds$. WLOG $u_n \rightharpoonup^* u \in \mathcal{U}$. Then $x_0 \to^t_u 0$.

$$\mathcal{R}(t_1) \subset \mathcal{R}(t_2), \qquad 0 < t_1 < t_2 < T$$
$$\exists u_1 \in \mathcal{U} \qquad x_0 = -\int_0^t e^{(s-t)A}Bu_1(s)ds.$$

Define $u_2(s) = u_1(s)$ if $0 \leqslant s \leqslant t$, else 0. □

Definice 4.12 (Area of controllability)

$$\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t).$$

Věta 4.8

$$\operatorname{rank}\mathcal{K}(A,B) = n \Leftrightarrow \forall t > 0 : \mathcal{R}(t) \supseteq U(0),$$

where $U(0) \subset \mathbb{R}^n$ is some neighbourhood of 0.

Důkaz

„$\Longleftarrow$“: If $\exists t > 0$ $\mathcal{R}(t) \supset U(0)$ open, $0 \in U(0)$. $\tilde{\mathcal{R}} : u \in L^\infty, \mathcal{R} : ||u||_\infty \leqslant 1$, then $\tilde{\mathcal{R}}(t) \supset \mathcal{R}(t) \supset U(0) \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n. \implies \operatorname{rank}\mathcal{K}(A,B) = n$.

„$\Longrightarrow$“: $\operatorname{rank}\mathcal{K}(A,B) = n \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n$. From theorem of local controllability. □

Věta 4.9 (Minimal time)

$$x' = Ax + Bu$$
$$\forall x_0 \in \mathcal{R} = \bigcup_{s>0} \mathcal{R}(s)$$
$$\exists t > 0\ \exists u(0) \in \mathcal{U} : x_0 \to_u^t 0$$
$$t = \inf\{s > 0 | x_0 \in \mathcal{R}(s)\}.$$

Důkaz

$$t > 0, \exists t_n \searrow t, t_n \in (0,T], \exists u_n \in U, x_0 = -\int_0^{t_n} e^{(t_n-s)A} V u_n(s).$$

Alaoglu: WLOG $u_n \xrightarrow{*} u \in U$.

$$x_0 = -\int_0^{t_n} e^{(t-s)A} Bu_n(s)ds - \int_0^{t_n} \left[e^{(t-s)A} - e^{(t_n-s)A}\right] Bu_n(s)ds$$

$$x_0 = -\underbrace{\int_0^t e^{(t-s)A} Bu_n(s)ds}_{\xrightarrow{*} \int_0^t e^{(t-s)A} Bu(s)ds} - \underbrace{\int_t^{t_n} e^{(t-s)A} Bu_n(s)ds}_{\to 0} - \underbrace{\int_0^t \left[e^{(t-s)A} - e^{(t_n-s)A}\right] Bu_n(s)ds}_{\to 0 (DLCT)}.$$

□

Definice 4.13 (Bang-bang)

We say that a regulation $u \in U(0)$ is of type bang-bang, if for almost every $t \in [0,T]$: $u(t) = \pm 1$.

Věta 4.10 (Bang-bang)

If $x_0 \in \mathcal{R}(t) \implies \exists \tilde{u}(0)$ of type bang-bang $x_0 \to_{\tilde{u}}^t 0$.

Definice 4.14 (Extremal point)

X vector space, $K \subset X$. $x \in K$ is called an extremal point, if it cannot be written as $x = \frac{y+z}{2}$, $y, z \in K$, $y \neq z$. We denote $ex(K)$ the set of extremal points.

Tvrzení 4.11 (Krein-Milman theorem)

X locally convex vector space, $K \subset X$: $K \neq \varnothing$, K convex and compact. Then $ex(K) \cap K \neq \varnothing$.

Důkaz (Bang-bang)

$$K = \left\{u \in \mathcal{U} | x_0 \rightarrow^t_{u(0)} 0\right\}, \qquad X = L^\infty(0,T,\mathbb{R}^n).$$

$K \neq \varnothing$ ($u \in \mathcal{R}(t)$), K convex, K is compact (sequential compactness: Alaoglu theorem? $L'(0,T,\mathbb{R}^n)$ separable $\implies$ $L^\infty(0,T,\mathbb{R}^n)$ with locale $*$ topology is metrizable $\implies$ sequential compactness $\implies$ compactness.

Choose $\tilde{u}_j \in ex(K)$ (from Krein-Milman). It remains to check that $\tilde{u}_j(s) = \pm 1$, $\forall j \in [n]$ for almost every $s \in (0,t)$. For contradiction: for some $j \in [n]$ $\exists E \subset (0,t)$, $\lambda(E) > 0$ $\forall s \in E$ $|\tilde{u}_j(s)| < 1$. WLOG

$$\exists \varepsilon > 0 \ \forall s \in E |\tilde{u}_j(s)| < 1 - \varepsilon \qquad \left[E = \bigcup_{n \in \mathbb{N}} \left\{ s \in (0,t) \middle| |\tilde{u}_j(s)| \leqslant 1 - \frac{1}{n} \right\}\right].$$

$$x_0 = -\int_0^t e^{-sA} B \tilde{u}(s) ds$$

We find (from ortogonality to $B_i e^{-sA}$) $\varphi \in L^\infty(0,T,\mathbb{R})$ such that:

1. $\operatorname{supp} \varphi \subset E$;
2. $\int_E e^{-sA} B(0,\dots,0,\varphi(s),0,\dots,0)^T ds = 0$;
3. $\forall s \in E |\varphi(s)| < \varepsilon$.

Define $u_1(s) = \tilde{u}(s) + (0,\dots,0,\varphi(s),0,\dots,0)^T$ and $u_2(s) = \tilde{u}(s) - (0,\dots,0,\varphi(s),0,\dots,0)^T$. Then $x_0 \rightarrow^t_{u_{1,2}(0)} 0$, and $u_1, u_2 \in K$. □

Věta 4.12 (Global controllability)

We have (LTP) $x' = Ax + Bu$, $x(0) = x_0$, $u \in \mathcal{U}$.

1. $\operatorname{rank} \mathcal{K}(A,B) = n \implies$ *(LTP) is locally controllable.*

2. $\operatorname{rank} \mathcal{K}(A,B) = n$ *and* $\Re\lambda \leqslant 0$ $\forall \lambda$*-eigenvalues of A. Then (LTP) is globally controllable $\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t) = \mathbb{R}^n$.*

Důkaz
1) folows from „In theorem of local controllability for the problem $x' = f(x,u)$ we could take $u \in \mathcal{U}$."

2a) If $\forall \lambda$ eigenvalue of A we have $\Re\lambda < 0 \implies$ theorem follows from text above: first, set $u = 0$. Then we arrive at a neighbourhood of zero.

2b) For contradiction $x_0 \in \mathbb{R}^n \setminus \mathcal{R}$. $\mathcal{R}$ convex $\exists z_0 \in \partial\mathcal{R}$, n normal vector. $\forall x_1 \in \mathcal{R}$: $n^T(x_1 - x_0) \leqslant 0$, $n^T x_1 \leqslant n^T x_0 =: M$.

$$x_1 = -\int_0^t e^{-sA} Bu(s) ds$$

$$n^T x_1 = -\int_0^t \underbrace{n^T e^{-sA} B}_{v(s)} u(s) ds$$

$$\tilde{u}(s) := \begin{cases} 0, & v(s) = 0, \\ \frac{-v(s)}{||v(s)||_2}, & v(s) \neq 0. \end{cases}$$

If $v(s) \equiv 0$, then apply $\frac{d^p}{(ds)^p}$, $n^T A^p e^{-sA} B \equiv 0$, then $n^T \mathcal{K}(A,B) = 0$. ↯

$$\int_0^\infty ||v(s)||_2 ds = \infty.$$

If this is true, then $t_k \nearrow \infty$, $u_k = \tilde{u}|_{[0,t_k]}$, $x_{1,k} = -\int_0^{t_k} e^{-sA} Bu_k(s) ds$.

$$n^T x_{1,k} = -\int_0^{t_k} v^t(s) \cdot \tilde{u}(s) ds = \int_0^{t_k} ||v(s)||_2 ds \to \infty. ↯$$

$v(s)$ is linear combination of $s^j e^{-s\lambda_p}, \Re\lambda_p \leqslant 0$. Then $\int_0^\infty |v(s)| ds = \infty$. □

Věta 4.13 (Pontrjagin maximum)

$$x' = Ax + Bu, ||u||_\infty \leqslant 1, x(0) = x_0.$$

Let $x_0 \to^{t^*}_{u^*(0)} 0$, t^* *is the minimal. Then* $\exists h \in \mathbb{R}^n \setminus \{\mathbf{o}\}$*:*

$$h^T \cdot e^{-sA} Bu^*(s) = \max_{\eta \in [-1,1]^m} h^t e^{-sA} B\eta$$

for almost every $s \in (0, t^*)$.

Důkaz
$x_0 \in \partial\mathcal{R}(t^*)$.

Step 2 – contradiction: $\exists E \subset (0,t^*), \lambda(E) > 0$, $\forall s \in E\ \exists \eta_s \in [-1,1]^m\ h^T e^{-sA} B u^*(s) < h^T e^{-sA} B \eta_s$. But $x_j(\delta) \in \mathcal{R}(t^* - \delta)$, hence $x_0 \in \mathcal{R}(t^* - \delta)$ and t^* is not minimal.

Step 1: $x_0 \in \partial\mathcal{R}(t^*)$. For contradiction $x_0 \in \operatorname{int}\mathcal{R}(t^*)$.

$$\exists x_1, \dots, x_{n+1} \in \mathcal{R}(t^*), x_0 \in CO(x_1, \dots, x_{n+1}).$$

$$\exists u_1, \dots, u_{n+1} \in U, x_j \to^{t^*}_{u_j(\cdot)} 0\ \forall j \in [n+1].$$

Let $\tilde{u}_j(t)$ are the corresponding solutions

TODO!!! □

Věta 4.14 (Pontrjagin)

$x'(f,u)$, $x(0) = x_0$, $u \in \mathcal{U} = \{u : (0,T) \to U \subset \mathbb{R}^n\}$, T fixed,

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s)) ds \to \textit{maximum}.$$

$f, g, r, \nabla_x f, \nabla_x g, \nabla_x r$ are continuous.

Let u is a local maximum of this problem (it maximizes P), then for p solving:

$$H(x,p,u) := p^T f(x,u) + r(x,u),$$

$$p' = -\nabla_x H(x,p,u),$$
$$p(T) = \nabla_x g(x(T)),$$

we have
$$H(x,p,u) = \max_{\eta \in U} H(x,p,\eta) \textit{ for almost every } t \in (0,T).$$

Důkaz
Step one „WLOG $r = 0$“: We set

$$x' = f(x,u), \qquad x'_{n+1} = r(x,u), x_{n+1}(0) = 0, P[u(\cdot)] = \hat{g}(\hat{x}(T)) = g(x(T)) + x_{n+1}(T).$$

Step 2: Fix $\tau \in (0,T)$, $\eta \in U$, $u_\varepsilon(T) = \begin{cases} \eta, & t \in (\tau - \varepsilon, \tau), \\ u(t), & \text{elsewhere}, \end{cases}$ and corresponding $x_\varepsilon(t)$.

$$u \text{ "best"} \implies P[u_\varepsilon(0)] \leqslant P[u(0)] \implies g(x_\varepsilon(T)) \leqslant g(x(t)).$$

$$D := \frac{d}{d\varepsilon}|_{\varepsilon=0^+} \qquad Dg(x_\varepsilon(T))|_{\varepsilon=0^+} \leqslant 0$$
$$\nabla_x g(x(T)) \cdot Dx_\varepsilon(T)|_{\varepsilon=0^+} \leqslant 0.$$

Step 2.2: $x_\varepsilon(t) = x_0 + \int_0^t f(x_\varepsilon(s), u_\varepsilon(s))ds$. If $t < \tau$, then $u_\varepsilon \equiv u$, $x_\varepsilon \equiv x$, $Dx_\varepsilon(t) \equiv 0$ on $[0,t]$. If $t > \tau$, then $x_\varepsilon(t) =: y(t)$, $y'(t) = f(y(t), u(t))$, $u(\tau) = x_\varepsilon(\tau)$,

$$Dx_\varepsilon(t) \equiv z(t) : z' = \nabla_x f(y(t), u(t))z, z(\tau) = Dx_\varepsilon(\tau), \qquad \text{variational equation.}$$

Statement: $z' = A(t)z, p' = -A^T(t)p \implies p^T z = const$. Proof: $(p^T z)' = (p^T)'z + p^T z' = -p^T Az + p^T Az = 0$.

Step 2.3: $p' = -(\nabla_x f(y(t), u(t)))^T p$, $p(T) = (\nabla_x g(x(T)))^T$. Then $p^T(t)z(t)$ constant on (τ, T), $p^T(\tau)z(\tau) \leqslant 0$.

Step 2.4: $Dx_\varepsilon(\tau)|_{\varepsilon=0^+} \stackrel{?}{=} f(x(\tau), \eta) - f(x(\tau), u(\tau))$. Then

$$p^T(\tau)\left(f(x(\tau), \eta) - f(x(\tau), u(\tau))\right) \leqslant 0$$

$$\frac{1}{\varepsilon}(x_\varepsilon(\tau) - x(\tau)) = \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \left[f(x_\varepsilon(s), \eta) - f(x(s), u(s))\right] ds =$$

$$\frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau} \left[f(x_\varepsilon(s), \eta) - f(x(s), \eta)\right] ds + \int_{\tau-\varepsilon}^{\tau} \left[f(x(s), \eta) - f(x(s), u(s))\right] ds.$$

Fist converge to zero from Lebesgue theorem about average value. Second to $f(x(\tau), \eta) - f(x(\tau), u(\tau)) \to 0$. □

Věta 4.15 (Potrjagin for fixed point („fixed finish“))

Same as previous, but T is not fixed, $x(T)$ is fixed $\implies$ $g \equiv 0$ (we don't "rate" final point, because it's the same for all u).

5 Bifurcation

Definice 5.1

$x' = \mu - x^2$ is saddle-node bifurcation, $x' = \mu x - x^2 = x(\mu - x)$ is transcritical bifurcation, $x' = \mu x - x^3 = x(\mu - x^2)$ is fork bifurcation, in $x' = x - \sin\mu$ there is no bifurcation.

Pozorování

$f(x_0, \mu_0) \neq 0 \implies$ no bifurcation. (From lemma of rect.) (Bifurcation $\implies f = 0$.)

Pozorování

$$f(x_0, \mu_0) = 0, \sigma(\nabla_x f(x_0, \mu_0)) = \{\lambda_j | \Re\lambda_j \neq 0\}.$$

Definice 5.2

Point from previous observation is called hyperbolic stationary point.

Věta 5.1

$f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be C^1, (x_0, μ_0) is a hyperbolic stationary point. Then $\exists\Delta > 0$ $\exists\delta > 0$ $\forall\mu \in U_\delta(\mu_0)$ $\exists x = x(\mu) \in U_\Delta(x_0)$, stationary point $x(\mu)$ is a hyperbolic stationary point of $\mu \mapsto x(\mu)$, which is C^1.

Důkaz

IFT (Implicit function theorem):

$$f(x_0, \mu_0) = 0 \wedge \nabla_x f(x_0, \mu_0)\text{regular?} \wedge f \in C^1 \implies x = x(\mu), f(x(\mu), \mu) = 0.$$

Hyperbolic? Eigenvalues of $A = \nabla_x f(x(\mu), \mu)$, $\det(\lambda I - A(\mu))$ – polynomial in λ, deg $= n$. □

Věta 5.2

$f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 on neighborhood $(0, 0) \in \mathbb{R}^2$.

$$f(0, 0) = 0, \qquad f_\mu(0, 0) \neq 0, \qquad f_x(0, 0) = 0, \qquad f_{xx}(0, 0) \neq 0.$$

Then f has bifurcation at $(0, 0)$ of the type saddle-node.

Důkaz

Without proof. □

Věta 5.3

$f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 on neighborhood $(0, 0) \in \mathbb{R}^2$.

$$f(0, 0) = 0, \quad f_x(0, 0) = 0, \quad f(0, \mu) = 0 \ \forall\mu \in U_\delta(0), \quad f_{xx}(0, 0) \neq 0, \quad f_{x\mu}(0, 0) \neq 0.$$

Then f has bifurcation at $(0, 0)$ of the type transcritical. ($f(0, \mu) = 0 \implies f_\mu(0, 0) = 0$.)

Důkaz
Without proof. □

Lemma 5.4 (About division)

$h : U(0,0) \to \mathbb{R}$ be C^k for some $k \in \mathbb{N}$. $h(0,\lambda) = 0\ \forall \lambda \in U_\delta(0)$. Then

$$h(x,\lambda) = xH(x,\lambda), H \in C^{k-1}(U(0,0),\mathbb{R}).$$

$$H(0,0) = h_x(0,0), \qquad H_x(0,0) = \frac{1}{2}h_{xx}(0,0), \qquad H_\lambda(0,0) = h_{x\lambda}(0,0),$$
$$H_{xx}(0,0) = \frac{1}{3}h_{xxx}(0,0),$$

if k is sufficiently large.

Důkaz

$$H(x,\lambda) := \int_0^1 \partial_x h(\sigma x, \lambda) d\sigma.$$

□

Důkaz (Theorem of transcritical bifurcation)
$f(x,\mu) = xF(x,\mu)$. $F_\mu(0,0) \neq 0$?

$$F(x,\mu(x)) = 0? \to \tfrac{d}{dt} : \mu'(x) = \tfrac{-\partial_x F(x,\mu(x))}{\partial_\mu(F(x,\mu(x)))}.$$

$$f_{x\mu}(x,\mu) = F_\mu(x,\mu) + xF_{x\mu}(x,\mu) \implies F_\mu(0,0) = f_{x\mu}(0,0) \neq 0.$$

□

Věta 5.5 (Fork)

$$f \in C^3(U), \qquad f(0,0) = f_x(0,0) = f_{xx}(0,0) = 0, \qquad f_{xxx}(0,0) \neq 0,$$
$$f(0,\mu) = 0\ \forall (0,\mu) \in U, \qquad (f_\mu(0,0) = 0), \qquad f_{x\mu}(0,0) \neq 0.$$

Then f has bifurcation at $(0,0)$ of type fork.

Důkaz
$\mu'(0) = 0$, $\mu''(0) = \frac{\cdots}{-\partial_\mu F(x,\mu(x))}$, $\partial_{x,x}F(0,0) = \frac{1}{3}f_{xxx}(0,0) \neq 0$.

$$\mu''(0) \neq 0 \implies \mu''(x) \text{ doesn't change sign } \implies \mu(x) \text{ has a local extreme at } (0,0).$$

□

Věta 5.6 (?)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f(x,y,\mu), f \in C^2, f(0,0,\mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \sigma\left(\nabla f(0,0,\mu)\right) = \left\{\alpha(\mu) \pm i\omega(\mu)\right\}.$$

$$\alpha(0) = 0, \alpha'(0) \neq 0, \omega(0) \neq 0, \quad \alpha, \omega \in C^1.$$

Then $\exists \delta > 0, \varepsilon > 0 : \mu = \mu(a), a \in (0,\varepsilon) \mapsto \mu(a) \in (-\delta, \delta)$.

$\forall a \in (0,\varepsilon)$ $\exists$ *nontrivial periodic solution passing through* $(a, 0)$.

Důkaz
Rotation: $x' = \alpha(\mu)x - \omega(\mu)y + f_1(x,y,\mu)$, $y' = \omega(\mu)x + \alpha(\mu)y + f_2(x,y,\mu)$, $f_2(x,y,\mu) = O(x^2+y^2)$.

Polar coords:

$$x = r\cos\theta, y = r\sin\theta, x' = g_1(x,y), y' = g_2(x,y),$$

$$r'\cos\theta - r\sin\theta\cdot\theta' = g_1, \qquad r'\sin\theta + r\cos\theta\cdot\theta' = g_2.$$

$$r' = g_i\cdot\cos\theta + g_2\sin\theta, \qquad r\cdot\theta' = -g_1\sin\theta + g_2\cos\theta.$$

$$r; = \alpha\cdot r + \underbrace{f_1\cdot\cos\theta + f_2\cdot\sin\theta}_{=R}, \qquad r\theta' = \omega(\mu)\cdot r + \underbrace{(-f_1\cdot\sin\theta + f_2\cdot\cos\theta)}_{=r\cdot Q}.$$

$$r' = \alpha(\mu)r + R(r,\theta,\mu), R = O(r^2), \quad \theta' = \Omega(\mu) + Q(r,\theta,\mu), Q = O(r).$$

WLOG $\omega(0) > 0$. $\exists\varepsilon,\delta > 0$ $\forall r \leqslant \varepsilon$ $\forall\mu \in [-\delta,\delta]$, $\theta'(t) > 0$. $r(t) \mapsto \hat{r}(\theta) := r(t(\theta))$. $t \mapsto \theta(t)$ is simple $\implies$ $\exists t = t(\theta)$.

$$\frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{\alpha(\mu)r + R}{\omega(\mu) + Q} = \frac{\alpha(\mu)}{\omega(\mu)}r + T(r,\theta,\mu).$$

$$\lambda(\mu) := \frac{\alpha(\mu)}{\omega(\mu)} : r'(\theta) = \lambda(\mu)r(\theta) + T(r,\theta,\mu),$$

$$(r(\theta)e^{-\lambda(\mu)\theta})' = T(r,\theta,\mu)e^{-\lambda(\mu)\theta}$$

$$r(\theta)\cdot e^{-\lambda(\mu)\theta} = r'(\theta_0)\cdot e^{-\lambda(\mu)\theta_0} + \int_{\theta_0}^{\theta} T(r(\psi),\psi,\mu)\cdot e^{-\lambda(\mu)\psi}d\psi.$$

We get $r(\theta_0) = r(\theta_0 + 2\pi)$ – periodicity. If we denote $r(\theta_0) = a$, we get

$$\left(e^{2\pi\lambda(\mu)} - 1\right)a + \int_{\theta_0}^{\theta_0+2\pi} T(r(\psi),\psi,\mu)\cdot e^{-\lambda(\mu)(\psi-\theta_0)}d\psi.$$

$h_\mu(0,0) \neq 0$? $h''(a,\mu)$, $a = 0 \implies r = 0$, $T = 0$, $h(0,\mu) = 0$.

$$h(a,\mu) =: a\cdot H(a,\mu).$$

$$H(0,0) = 0? \quad H_\mu(0,0) \neq 0? \quad H \in C^1 \quad H(0,0) = \partial_? h(0,0).$$

□

6 Central manifolds

Poznámka

$$x' = Ax + f(x,y), \qquad y' = By + g(x,y),$$
$$\sigma(B) \subset \{z \in \mathbb{C} | \Re z < -\beta\}, \quad \beta > 0, \quad \sigma(A) \subset \{z \in \mathbb{C} | \Re z \geqslant 0\},$$
$$x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad f, g \in C^1(\mathbb{R}^{n+m}),$$
$$f(0) = 0, \quad g(0) = 0, \quad \nabla f(0) = 0, \quad \nabla g(0) = 0,$$
$$|f| \leqslant \varrho, \quad |g| \leqslant \varrho, \quad |\nabla f| \leqslant \sigma, \quad |\nabla g| \leqslant \sigma.$$

Goal: $\exists \varphi : \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $\varphi(0) = 0$, $\nabla\varphi(0) = 0$:

INV: if $(x(t), y(t))$ solution to previous and $y(0) = \varphi(x(0))$, then $\forall t : y(t) = \varphi(x(t))$.

Definice 6.1 (Reduced equation)

$$p'(t) = A \cdot p(t) + f(p(t), \varphi(p(t))), p(t) \in \mathbb{R}^n.$$

Lemma 6.1

φ satisfies INV iff φ satisfies RED: (if $p(t)$ satisfies reduced equation then $(x(t), y(t)) := (p(t), \varphi(p(t)))$ satisfies INV equation)

Důkaz

Straightforward. □

Definice 6.2 (RED')

If $p(t)$ satisfies reduced equation, then $y(t) := \varphi(p(t))$ satisfies

$$y'(t) = By(t) + g(p(t), \varphi(p(t))).$$

Lemma 6.2

$\gamma(t)$ is bounded on $(-\infty, 0]$. Then $\exists! y(t) : y'(t) = By(t) + \gamma(t)$, such that $y(t)$ is bounded on $(-\infty, 0]$. For this y, $y(0) = \int_{-\infty}^{0} e^{-Bs}\gamma(s)ds$.

Důkaz

$$e^{-Bt}y'(t) - Be^{-Bt}y(t) = e^{-Bt}\gamma(t).(e^{-Bt}y(t))' = e^{-Bt}\gamma(t).$$

$$e^{-Bt}y(t) = y(0) + \int_0^t e^{-Bs}\gamma(s)ds.$$

If y is bounded on $(-\infty, 0]$, then $y(0) + \int_0^{-\infty} e^{-Bs}\gamma(s)ds = 0$, $y(0) = \int_{-\infty}^0 e^{-Bs}\gamma(s)ds$.

Take y with (i. c.). Then

$$y(t) = e^{Bt}\left(\int_{-\infty}^t e^{-Bs}\gamma(s)ds\right) = \int_{-\infty}^t e^{B(t-s)}\gamma(s)ds.$$

$$|e^{s\cdot B}| \leqslant c_0 e^{-\beta s}, \quad c_0 > 0, \forall s$$

$$|y(t)| \leqslant \int_{-\infty}^t |e^{B(t-s)}| \cdot |\gamma(s)|ds \leqslant \|\gamma\|_\infty \int_{-\infty}^t c_0 e^{-\beta(t-s)}ds = \frac{\|\gamma\|_\infty c_0}{\beta}.$$

□

Lemma 6.3

φ satisfies INV $\Leftrightarrow$ φ satisfies (RED) $\Leftrightarrow$ φ satisfies (RED') $\Leftrightarrow$ φ satisfies FP:

$$\forall p_0 \in \mathbb{R}^n : \varphi(p_0) = \int_{-\infty}^0 e^{-Bs}g(p(s), \varphi(p(s)))ds,$$

where p satisfies reduced equation with $p(0) = p_0$.

Důkaz

„ $\Longrightarrow$ “: φ RED' $\Longrightarrow$ y satisfies $y'(t) = By(t) + g(p(t), \varphi(p(t)))$ and y is bounded. Then previous lemma:

$$\varphi(p_0) = \varphi(p(0)) = y(0) = \int_{-\infty}^0 e^{-B\cdot s}g(p(s), \varphi(p(s)))ds.$$

„ $\Longleftarrow$ “: Let y satisfy $y'(t) = By(t) + g(p(t), \varphi(p(t)))$, $y(0) = \varphi(p(0))$. $\forall t_1 : y(t_1) = \varphi(p(t_1))$?

t_1 arbitrary $y_1 := y(t+t_1), p_1(t) := p(t+t_1).p_1'(t) = Ap_1(t)+f(p_1(t), \varphi(p,t)), p_1(0) = p(t_1)$,

$$y_1'(t) = B \cdot y_1(t) + g(p_1(t), \varphi(p,t)), y_1(0) = y(t_1).$$

$y(0) = \varphi(p(0)) = \int_{-\infty}^0 e^{-B\cdot s}g(p(s), \varphi(p(s)))ds \implies y$ is bounded on $(-\infty, 0] \implies y_1$ is bounded. From lemma:

$$y(t_1) = y_1(0) = \int_{-\infty}^0 e^{-B\cdot s}g(p(s), \varphi(p,s))ds = \varphi(p(0)) = \varphi(p(t_1)).$$

□

Věta 6.4 (Existence of central manifold)

$$\forall \beta \ \exists \varrho > 0, \sigma > 0, b > 0, l > 0 : \exists ! \varphi \in \mathcal{X} \ \textit{satisfying INV.}$$

Důkaz

$$\mathcal{X} \subset C(\mathbb{R}^n, \mathbb{R}^n), \|\varphi\|_{\mathcal{X}} = \sup_{x \in \mathbb{R}^n} |\varphi(x)|,$$

$$T : \mathcal{X} \to \mathcal{X}, \varphi \mapsto T\varphi, \quad (T\varphi)(p_0) = \int_{-\infty}^{0} e^{-B \cdot s} g(p(s), \varphi(p(s))) ds.$$

Step 1: T is well-defined $\forall \varphi \in \mathcal{X}$: $T\varphi \in \mathcal{X}$.

Step 2: T is contraction.

Step 1 + 2 $\implies$ (Banach) $\exists ! \varphi \in \mathcal{X}$: $T\varphi = \varphi$.

- $(T\varphi)(0) = 0$? Take $p(0)$ satisfying reduced equation, $p(0) = 0 \implies p(t) \equiv 0$.
- $|(T\varphi)(p_0)| \leqslant \int_{-\infty}^{0} |e^{-B \cdot s}| \cdot |g(p(s), \varphi(p(s)))| ds \leqslant \frac{\varrho c_0}{\beta} \overset{?}{\leqslant} f$ (true for sufficiently small ϱ).
- $(T\varphi)(p_0) - (T\varphi)(q_0) = \int_{-\infty}^{0} e^{-B \cdot s} \left[g(p(s), \varphi(p(s))) - g(q(s), \varphi(q(s))) \right] ds$.

□

Definice 6.3 (Central manifold)

$\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is called a central manifold if $\varphi(0) = 0$, $\nabla \varphi(0) = 0$, $\varphi \in C^1$, it satisfies INV.

Definice 6.4

$$M[\varphi](x) = \nabla \varphi(x)(Ax + f(x, \varphi(x))) - B\varphi(x) - g(x, \varphi(x)).$$

Důsledek
φ is a central manifold $\Leftrightarrow M[\varphi] = 0$.

Poznámka
Dělal se podrobně důkaz Existence centrální variety.

Poznámka
Dodělával se důkaz Hopfovy bifurkace, Pontrjagina.