Příklad (1.)
Consider the energetic equation of state for the calorically perfect ideal gas $e(\eta, \varrho)$. Show that the specific Helmholtz free energy ψ for the calorically perfect ideal gas is given by the formula

$$\psi(\theta, \varrho) = -c_{V,ref}\theta \left(\ln \left(\frac{\theta}{\theta_{ref}} \right) - 1 \right) + c_{V,ref}\theta(\gamma - 1) \ln \left(\frac{\varrho}{\varrho_{ref}} \right),$$

where ϱ_{ref} and θ_{ref} are some constants. (Temperature and density at a reference state.)

Důkaz
Z pátého domácího úkolu víme

$$e(\eta, \varrho) = c_{V,ref}\theta_{ref} \left(\frac{\varrho}{\varrho_{ref}} \right)^{\gamma-1} \exp \left(\frac{\eta}{c_{V,ref}} \right).$$

Také víme (z jeho zadání), že $e(\eta, \varrho) = \theta c_{V,ref}$, což můžeme také dostat jako

$$\theta = \frac{\partial e}{\partial \eta} = \theta_{ref} \left(\frac{\varrho}{\varrho_{ref}} \right)^{\gamma-1} \frac{1}{c_{V,ref}} \exp \left(\frac{\eta}{c_{V,ref}} \right), \qquad e(\eta, \varrho) = \theta \frac{e}{\theta} = \theta c_{V,ref}.$$

Nakonec výsledkem pátého úkolu byla i entropie (vyjádřená z $e(\theta, \varrho)$), kterou můžeme upravit podle vzorců pro logaritmus:

$$\eta(\theta, \varrho) = c_{V,ref} \ln \left[\frac{\theta}{\theta_{ref}} \left(\frac{\varrho}{\varrho_{ref}} \right)^{1-\gamma} \right] = c_{V,ref} \ln \left[\frac{\theta}{\theta_{ref}} \right] + c_{V,ref} \ln \left[\frac{\varrho}{\varrho_{ref}} \right] \cdot (1 - \gamma).$$

Nyní už stačí dosadit:

$$\psi(\theta, \varrho) = e(\eta, \varrho)|_{\eta=\eta(\theta,\varrho)} - \theta\eta(\theta, \varrho) =$$

$$= \theta c_{V,ref} - \theta \left(c_{V,ref} \ln \left[\frac{\theta}{\theta_{ref}} \right] + c_{V,ref} \ln \left[\frac{\varrho}{\varrho_{ref}} \right] \cdot (1 - \gamma) \right) =$$

$$= -c_{V,ref}\theta \left(\ln \left(\frac{\theta}{\theta_{ref}} \right) - 1 \right) + c_{V,ref}\theta(\gamma - 1) \ln \left(\frac{\varrho}{\varrho_{ref}} \right).$$

□

Příklad (2.)
Consider a homogeneous isotropic elastic solid with the Helmholtz free energy in the form $\psi = \psi(\theta, \mathbb{B})$. We already know that the Cauchy stress $\mathbb{T}$ tensor is related to the derivative of the Helmholtz free energy via the formula $\mathbb{T} = 2\varrho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B}$.

Since the material is isotropic, the Helmholtz free energy must be in fact a function of the invariants of $\mathbb{B}$: $\psi = \psi(\theta, I_1, I_2, I_3)$, where the invariants are given by the formulae

$$I_1 := \operatorname{tr}\mathbb{B}, \qquad I_2 := \frac{1}{2}\left((\operatorname{tr}\mathbb{B})^2 - \operatorname{tr}\mathbb{B}^2\right), \qquad I_3 := \det\mathbb{B}.$$

Show that $\mathbb{T} = \alpha_0\mathbb{I} + \alpha_1\mathbb{B} + \alpha_2\mathbb{B}^2$, where

$$\alpha_0 := 2\varrho I_3\frac{\partial\psi}{\partial I_3}, \qquad \alpha_1 := 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right), \qquad \alpha_2 := -2\varrho\frac{\partial\psi}{\partial I_2}.$$

Důkaz
Z řetízkového pravidla:

$$\mathbb{T} = 2\varrho\frac{\partial\psi}{\partial\mathbb{B}}\mathbb{B} = 2\varrho\frac{\partial\psi}{\partial I_1}\frac{\partial I_1}{\partial\mathbb{B}}\mathbb{B} + 2\varrho\frac{\partial\psi}{\partial I_2}\frac{\partial I_2}{\partial\mathbb{B}}\mathbb{B} + 2\varrho\frac{\partial\psi}{\partial I_3}\frac{\partial I_3}{\partial\mathbb{B}}\mathbb{B}.$$

Zřejmě $\frac{\partial I_1}{\partial\mathbb{B}} = \mathbb{I}$,

$$\frac{\partial I_2}{\partial\mathbb{B}} = \frac{1}{2}\frac{\partial(\operatorname{tr}\mathbb{B})^2}{\partial\operatorname{tr}\mathbb{B}}\frac{\partial\operatorname{tr}\mathbb{B}}{\partial\mathbb{B}} - \frac{1}{2}\frac{\partial\operatorname{tr}\mathbb{B}^2}{\partial(\mathbb{B}^2)}\frac{\partial\mathbb{B}^2}{\partial\mathbb{B}} = \frac{2}{2}(\operatorname{tr}\mathbb{B})\mathbb{I} - \frac{2}{2}\mathbb{I}\,\mathbb{B} = \mathbb{I}\operatorname{tr}\mathbb{B} - \mathbb{B}.$$

Z přednášky navíc víme $\frac{\partial I_3}{\partial\mathbb{B}} = (\det\mathbb{B})\mathbb{B}^{-T}$. Navíc $\mathbb{B}$ je symetrické (např. z definice $\mathbb{B}^T = (\mathbb{F}\mathbb{F}^T)^T = (\mathbb{F}^T)^T\mathbb{F}^T = \mathbb{F}\mathbb{F}^T = \mathbb{B}$), tedy $\frac{\partial I_3}{\partial\mathbb{B}} = (\det\mathbb{B})\mathbb{B}^{-1}$.

Dosazením:

$$\mathbb{T} = 2\varrho\frac{\partial\psi}{\partial I_1}\mathbb{I}\,\mathbb{B} + 2\varrho\frac{\partial\psi}{\partial I_2}\left(\mathbb{I}\,I_1 - \mathbb{B}\right)\mathbb{B} + 2\varrho\frac{\partial\psi}{\partial I_3}\left(I_3\mathbb{B}^{-1}\right)\mathbb{B} =$$

$$= 2\varrho I_3\frac{\partial\psi}{\partial I_3}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbb{B} - 2\varrho\frac{\partial\psi}{\partial I_2}\mathbb{B}^2.$$

□

Some people also use the formulae

$$\mathbb{T} = \beta_0\mathbb{I} + \beta_1\mathbb{B} + \beta_{-1}\mathbb{B}^{-1},$$

where

$$\beta_0 := 2\varrho\left(I_2\frac{\partial\psi}{\partial I_2} + I_3\frac{\partial\psi}{\partial I_3}\right), \qquad \beta_1 := 2\varrho\frac{\partial\psi}{\partial I_1}, \qquad \beta_{-1} := -2\varrho I_3\frac{\partial\psi}{\partial I_2}.$$

Show that they are equivalent to the previous ones.

Důkaz
Z přednášky víme
$$\mathbb{B}^{-1} = \frac{1}{I_3}\mathbb{B}^2 - \frac{I_1}{I_3}\mathbb{B} + \frac{I_2}{I_3}\mathbb{I},$$
tedy
$$\mathbb{B}^2 = I_3\mathbb{B}^{-1} + I_1\mathbb{B} - I_2\mathbb{I}.$$

Dosadíme:
$$\begin{aligned}\mathbb{T} &= 2\varrho I_3\frac{\partial\psi}{\partial I_3}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbb{B} - 2\varrho\frac{\partial\psi}{\partial I_2}\mathbb{B}^2 = \\ &= 2\varrho I_3\frac{\partial\psi}{\partial I_3}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbb{B} - 2\varrho\frac{\partial\psi}{\partial I_2}\left(I_3\mathbb{B}^{-1} + I_1\mathbb{B} - I_2\mathbb{I}\right) = \\ &= 2\varrho\left(I_2\frac{\partial\psi}{\partial I_2} + I_3\frac{\partial\psi}{\partial I_3}\right)\mathbb{I} + 2\varrho\frac{\partial\psi}{\partial I_1}\mathbb{B} - 2\varrho I_3\frac{\partial\psi}{\partial I_2}\mathbb{B}^{-1}.\end{aligned}$$

□

Příklad (3.)
Some people prefer to write $\psi = \psi(\theta, \mathbb{B})$ as $\psi = \psi(\theta, J, \overline{\mathbb{B}})$, where $J := \det\mathbb{F}$ and $\overline{\mathbb{B}} := \frac{\mathbb{B}}{J^{\frac{2}{3}}}$.

This decomposition is motivated by the fact that J is related to the volume-changing part of the deformation, while $\overline{\mathbb{B}}$ characterises the volume-preserving part of the deformation. (Check that $\det\overline{\mathbb{B}} = 1$.)

Důkaz
Ve třech dimenzích $\det c\mathbb{A} = c^3\det\mathbb{A}$, tedy (víme $\mathbb{B} = \mathbb{F}\mathbb{F}^T$, $\det\mathbb{A}\mathbb{B} = (\det\mathbb{A})(\det\mathbb{B})$, $\det\mathbb{A}^T = \det\mathbb{A}$):
$$\det\overline{\mathbb{B}} = \det\frac{\mathbb{B}}{J^{\frac{2}{3}}} = \det\frac{\mathbb{F}\mathbb{F}^T}{J^{\frac{2}{3}}} = \frac{1}{J^2}(\det\mathbb{F})(\det\mathbb{F}^T) = \frac{1}{J^2}\cdot J\cdot J = 1.$$

□

Show that in this case the counterpart of $\mathbb{T} = 2\varrho\frac{\partial\psi}{\partial\mathbb{B}}\mathbb{B}$ is
$$\mathbb{T} = \varrho J\frac{\partial\psi}{\partial J}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial\overline{\mathbb{B}}}\overline{\mathbb{B}}\right)_\delta, \qquad \mathbb{A}_\delta := \mathbb{A} - \frac{1}{3}(\operatorname{tr}\mathbb{A})\,\mathbb{I}.$$

Důkaz

Podle řetízkového pravidla

$$\mathbb{T} = 2\varrho \frac{\partial \psi(\theta, J, \overline{\mathbb{B}})}{\partial \mathbb{B}} \mathbb{B} = 0 + 2\varrho \frac{\partial \psi}{\partial J} \frac{\partial J}{\partial \mathbb{B}} \mathbb{B} + 2\varrho \frac{\partial \psi}{\partial \overline{\mathbb{B}}} \frac{\partial \overline{\mathbb{B}}}{\partial \mathbb{B}} \mathbb{B}.$$

První nás tedy zajímá $\frac{\partial J}{\partial \mathbb{B}} = \frac{\partial \det \mathbb{F}}{\partial \mathbb{F}\mathbb{F}^T}$. Použijeme toho, že víme, kolik je $\frac{\partial \det(\mathbb{F}\mathbb{F}^T)}{\partial \mathbb{F}\mathbb{F}^T}$ a že umíme derivovat součin funkcí ($\det \mathbb{A}\mathbb{B} = (\det \mathbb{A})(\det \mathbb{B})$, $\det \mathbb{A}^T = \det \mathbb{A}^T$):

$$J^2 \mathbb{B}^{-1} = (\det \mathbb{F})^2 \mathbb{B}^{-1} = (\det \mathbb{F})(\det \mathbb{F}^T)\mathbb{B}^{-1} = (\det \mathbb{F}\mathbb{F}^T)\mathbb{B}^{-1} = (\det \mathbb{B})\mathbb{B}^{-1} = LHS$$

$$LHS = \frac{\partial \det \mathbb{B}}{\partial \mathbb{B}} = \frac{\partial \det(\mathbb{F}\mathbb{F}^T)}{\partial \mathbb{F}\mathbb{F}^T} = \frac{\partial \det \mathbb{F}^T}{\partial \mathbb{F}\mathbb{F}^T} \det \mathbb{F} + \frac{\partial \det \mathbb{F}}{\partial \mathbb{F}\mathbb{F}^T} \det \mathbb{F}^T = RHS$$

$$RHS = \frac{\partial \det \mathbb{F}}{\partial \mathbb{F}\mathbb{F}^T} \det \mathbb{F} + \frac{\partial \det \mathbb{F}}{\partial \mathbb{F}\mathbb{F}^T} \det \mathbb{F} = \frac{\partial J}{\partial \mathbb{B}} J + \frac{\partial J}{\partial \mathbb{B}} J = 2\frac{\partial J}{\partial \mathbb{B}} J.$$

Tedy za předpokladu, že $J \neq 0$, což je rozumný předpoklad, protože jinak bychom nemohli definovat $\overline{\mathbb{B}}$, máme $\frac{\partial J}{\partial \mathbb{B}} = \frac{1}{2} J \mathbb{B}^{-1}$. První člen je tedy (a tak přesně vychází do vzorce ze zadání)

$$2\varrho \frac{\partial \psi}{\partial J} \frac{\partial J}{\partial \mathbb{B}} \mathbb{B} = 2\varrho \frac{\partial \psi}{\partial J} \frac{1}{2} J \mathbb{B}^{-1} \mathbb{B} = \varrho J \frac{\partial \psi}{\partial J} \mathbb{I}.$$

Z druhého členu potřebujeme spočítat (použil jsem derivaci součinu):

$$\frac{\partial \overline{\mathbb{B}}}{\partial \mathbb{B}} = \frac{\partial \left(\frac{\mathbb{B}}{J^{-\frac{2}{3}}}\right)}{\partial \mathbb{B}} = \frac{\partial \mathbb{B} \frac{\mathbb{I}}{J^{-\frac{2}{3}}}}{\partial \mathbb{B}} = \frac{\partial \mathbb{B}}{\partial \mathbb{B}} \frac{\mathbb{I}}{J^{-\frac{2}{3}}} + \mathbb{B} \frac{\partial \left(\frac{\mathbb{I}}{J^{-\frac{2}{3}}}\right)}{\partial \mathbb{B}}.$$

Jistě $\frac{\partial \mathbb{B}_{ik}}{\partial \mathbb{B}_{mn}} \mathbb{I}_{kj} = \delta_{im}\delta_{kn}\delta_{kj}$. Tedy při posčítání přes i, k a j dostaneme dostaneme 1 pro všechny m a n, tedy vychází

$$2\varrho \frac{\partial \psi}{\partial \overline{\mathbb{B}}} \frac{\partial \mathbb{B}}{\partial \mathbb{B}} \frac{\mathbb{I}}{J^{-\frac{2}{3}}} \mathbb{B} = 2\varrho \frac{\partial \psi}{\partial \overline{\mathbb{B}}} \frac{\mathbb{B}}{J^{-\frac{2}{3}}} = 2\varrho \frac{\partial \psi}{\partial \overline{\mathbb{B}}} \overline{\mathbb{B}}.$$

Což je druhý „člen“ ve vzorci ze zadání. Zbývá už jen $\mathbb{B}\frac{\partial \mathbb{I} \cdot J^{-\frac{2}{3}}}{\partial \mathbb{B}}$. To můžeme derivovat jako složenou funkci a dosadit výsledek prvního členu:

$$\mathbb{B}\frac{\partial \mathbb{I} \cdot J^{-\frac{2}{3}}}{\partial \mathbb{B}} = \mathbb{B}\left(-\frac{2}{3} J^{-\frac{5}{3}} \mathbb{I}\right) \frac{\partial J}{\partial \mathbb{B}} = \mathbb{B}\left(-\frac{2}{3} J^{-\frac{5}{3}} \mathbb{I}\right) \frac{1}{2} J \mathbb{B}^{-1} = -\frac{1}{3} \overline{\mathbb{B}}\, \mathbb{I}\, \mathbb{B}^{-1}.$$

Tedy dostáváme („třetí člen“ výrazu ze zadání, tedy máme hotovo):

$$-\frac{1}{3} 2\varrho \frac{\partial \psi}{\partial \overline{\mathbb{B}}} : \overline{\mathbb{B}} \cdot \mathbb{I} \cdot \mathbb{B}^{-1} \cdot \mathbb{B} = -\frac{1}{3} 2\varrho \operatorname{tr}\left(\frac{\partial \psi}{\partial \overline{\mathbb{B}}} \cdot \overline{\mathbb{B}}\right) \mathbb{I}.$$

□