

Poznámka (Literature)
Kechris.

Definice 0.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka

Complete compatible metric is not unique: $\tilde{\varrho} = \min \{1, \varrho\}$.

Například

$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, 2 := \{0, 1\}, \omega := \{0, 1, 2, \dots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, $2^\omega, \omega^\omega$ (both with product topology).

Věta 0.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X .

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Důkaz

Without proof. (We should know it already.)

□

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Věta 0.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X , such that $\text{diam}(F_n) \rightarrow 0$. Then $|\bigcap F_n| = 1$.

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Důkaz

Without proof. (We should know it already.)

□

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Věta 0.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n \in \omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow H \in \mathcal{G}_\delta(X)$

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Důkaz ((i))

Let d_n be CCM (complete compatible metric) on X_n , $n \in \omega$. Then

$$d(x, y) := \sum_{n=0}^{\infty} \min \{2^{-n}, d_n(x_n, y_n)\}$$

is CCM on $X = \prod_{n \in \omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „ d is metric“ straightforward, „ d is complete“ also easy, compatibility too).

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┌ *Důkaz* ((ii))

$H = \emptyset$, $H = X$ trivial. Assume $H \neq \emptyset, X$.

„ \implies “: Fix CCM ϱ on H . $V_n := \bigcup \{V \subset X \mid V \text{ open in } X \wedge V \cap H \neq \emptyset \wedge \text{diam}_\varrho(V \cap H) < 2^{-n}\}$, $n \in \omega$. We want to show $H \stackrel{?}{=} \bigcap_{n \in \omega} (V_n \cap \overline{H}) \in \mathcal{G}_\delta$:

„ \subseteq “: $x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n$.

„ \supseteq “: $x \in V_n \cap \overline{H}$ for every $n \in \omega \implies \exists$ open sets $G_n: x \in G_n, G_n \cap H \neq \emptyset, \text{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \stackrel{?}{\neq} \emptyset \iff x \in G_n \cap G_{n+1} \cap \overline{H} \neq \emptyset$).

$\{y\} := \bigcap_{n \in \omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O, G_n \cap H \subset B(y, 2^{-n}), n \in \omega \implies \exists n \in \omega G_n \cap H \subset O, x \in G_n \cap (X \setminus \overline{O}) \cap \overline{H} \implies G_n \cap (X \setminus \overline{O}) \cap H \neq \emptyset$.

„ \Leftarrow “: fix CCM d on X , $H = \bigcap_{n \in \omega} U_n, \emptyset = U_n \neq X$. $F_n := X \setminus U_n, \tilde{d}(x, y) = d(x, y) + \sum_{n=0}^{\infty} \min \left\{ 2^{-n}, \left| \frac{1}{\text{dist}(x, F_n)} - \frac{1}{\text{dist}(y, F_n)} \right| \right\}, x, y \in H$. Next we verified that \tilde{d} is metric, that \tilde{d} is equivalent with d on H (by convergence), and that (H, \tilde{d}) is complete metric space and separable. TODO? \square

Definition 0.2 (Notation)

$A \neq \emptyset$:

- $A^{<\omega} :=$ finite sequence of elements of $A = \bigcup_{n \in \omega} A^n$;
- $s \in A^k, t \in A^{<\omega} \cup A^\omega: s \wedge t := (s_0, s_1, \dots, s_{k-1}, t_0, t_1, \dots)$, where $s = (s_0, \dots, s_{k-1}), t = (t_0, t_1, \dots)$;
- $s \in A^{<\omega} \cup A^\omega: |s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega, k \in \omega, |s| \geq k$, then we denote restriction of s on first k elements as s/k ;
- $s < t$ iff $|t| \geq |s|$ and $s = t/|s|$ ($s \in A^{<\omega}, t \in A^{<\omega} \cup A^\omega$).

1 Baire space ω^ω

Definition 1.1

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega \mid s < \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \setminus \bigcup \{\mathcal{N}(t) \mid |t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N}|s \in \omega^{<\omega}\}$ is base of topology of ω^ω .

Věta 1.1 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz

Bez důkazu. □

Důsledek

ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Věta 1.2

Let $X \neq \emptyset$, PTS. Then X is continuous image of ω^ω .

Poznámka

$X \neq \emptyset$ PTS. Then there $\exists F \subset \omega^\omega$, F closed, and continuous injection $\varphi : F \rightarrow X$.

Důkaz

Find CCM on X such that $\text{diam } X \leq 1$. We inductively construct closed $\emptyset \neq A_s \subset X$ for every $s \in \omega^{<\omega}$ such that 1. $A_\emptyset = X$; 2. $\text{diam}(A_s) \leq 2^{-|s|}$; 3. $A_s = \bigcup_{i \in \omega} A_{s^\frown i}$.

Empty set is trivial. Assume we already have A_s . Find $\{x_i | i \in \omega\} \subset A_s$ dense in A_s . $A_{s^\frown i} := A_s \cap \overline{B(x_i, 2^{-|s|-2})} \neq \emptyset$ closed.

Fix $\forall \nu \in \omega^\omega : f(\nu) := x$, where $\{x\} = \bigcap_{k \in \omega} A_{\nu/k} \neq \emptyset$ (intersection of closed nonempty non-increasing sequence of sets). „ f is surjection“: $x \in A_s \xrightarrow{3.} \exists n \in \omega : x \in A_{s^\frown n} \xrightarrow{1.} \forall x \in X \exists \alpha \in \omega^\omega \forall k \in \omega : x \in A_{\alpha/k} \implies x = f(\alpha)$.

„ f continuous“: $f(\mathcal{N}_{\nu/k}) \subset A_{\nu/k}$ for every $\nu \in \omega^\omega$, $k \in \omega$, $\text{diam } A_{\nu/k} \leq 2^{-k}$. □

1.1 Cantor set 2^ω

Tvrzení 1.3

2^ω is up to homeomorphism unique nonempty nuldimensional compact metrizable space without isolated points (without isolated points is called perfect space).

Tvrzení 1.4

Let $X \neq \emptyset$ metrizable, compact. Then X is continuous image of 2^ω .

┌ *Důkaz*

Without proof, but it is similar to the previous one. □

1.2 Hilbert cube $[0, 1]^\omega$

Tvrzení 1.5

Let X be PTS. Then X is homeomorphic to G_δ subset of $[0, 1]^\omega$.

┌ *Důkaz*

X PTS, case \emptyset is trivial, so assume $X \neq \emptyset$, ϱ is CCM on X , $\varrho \leq 1$. Let $\{x_n, n \in \omega\}$ be dense in X . Define $f : [0, 1]^\omega : f(x) = (\varrho(x, x_n))_{n \in \omega}$. $\varrho \leq 1 \implies f(x) \in [0, 1]^\omega$.

„Continuity of f “: $f^{-1}(U) = \bigcap_{i=1}^n B(x_i, b_i) \setminus \overline{B(x_i, a_i)}$ open.

„Injective“: $x \neq y \implies \exists n \in \omega : \varrho(x, x_n) < \varrho(y, x_n) \implies f(x) \neq f(y)$.

„Continuity of f^{-1} “: $f(y^n) \rightarrow f(y) \stackrel{?}{\implies} y^n \rightarrow y$.

$$f(y^n) \rightarrow f(y) \stackrel{?}{\iff} \forall k \in \omega : \varrho(y^n, x_k) \rightarrow \varrho(y, x_k).$$

Let $\varepsilon > 0$ be arbitrary:

$$\exists k \in \omega : \varrho(y, x_k) < \frac{\varepsilon}{3}. \exists n_0 \forall n \geq n_0 : \varrho(y^n, x_k) < \frac{2\varepsilon}{3}.$$

Then

$$\forall n \geq n_0 : \varrho(y^n, y) \leq \varrho(y^n, x_k) + \varrho(x_k, y) < \varepsilon.$$

So $f(X)$ is homeomorphism to $X \implies f(X)$ is PTS $\implies f(X) \in \mathcal{G}_\delta([0, 1]^\omega)$. □

Důsledek

Let X be compact metrizable space. Then X is homeomorphic to some closed subset of $[0, 1]^\omega$.

┌ *Důkaz*

Compact metrizable space is Polish. And compact subset must be closed. □

1.3 $\mathcal{K}(X)$: Hyperspace of compact subsets of X

Definice 1.2

Let X be PTS, denote $\mathcal{K}(X) := \{K \subset X \mid K \text{ is compact}\}$. Vietoris topology on $\mathcal{K}(X)$ is generated by $\{K \in \mathcal{K}(X) \mid K \subset V\}$ for V open and $\{K \in \mathcal{K}(X) \mid K \cap V \neq \emptyset\} = \mathcal{K}(X) \setminus \{K \in \mathcal{K}(X) \mid K \subset X \setminus V\}$

for V open.

Tvrzení 1.6

Let X be PTS, ϱ CCM on X , $\varrho \leq 1$. Then mapping $h : \mathcal{K}(X) \times \mathcal{K}(X) \mapsto [0, +\infty)$ defined as:

$$h(K, L) = \begin{cases} 0, & K = L = \emptyset, \\ \max \left\{ \sup_{x \in K} \varrho(x, L), \sup_{y \in L} \varrho(y, K) \right\}, & K, L \neq \emptyset, \\ 1, & \text{other cases,} \end{cases}$$

is CCM on $\mathcal{K}(X)$ with Vietoris topology. h is known as Hausdorff metric.

Poznámka

$\mathcal{K}(X)$ is separable if X is PTS. X is compact metrizable $\implies \mathcal{K}(X)$ is compact (totally bounded).

$$X \text{ is separable} \implies \exists D \subset X : \overline{D} = X, |D| = \omega.$$

$$M = \{K \subset D \mid |K| < \omega\} \implies |M| = \omega.$$

$\overline{M} = \mathcal{K}(X)$. $K \in \mathcal{K}(X)$ arbitrary, $\varepsilon > 0$ arbitrary. Then $\exists \frac{\varepsilon}{2}$ net $P \subset K$, $|P| < \omega$. We find $\{\tilde{x}_0, \dots, \tilde{x}_n\} \subset D : \varrho(x_i, \tilde{x}_i) < \frac{\varepsilon}{2} \wedge h(K, \{\tilde{x}_0, \dots, \tilde{x}_n\}) < \varepsilon$.

$$X \text{ is compact, } P \text{ is } \varepsilon\text{-net in } X, |P| < \omega \implies 2^P \text{ is finite } \varepsilon\text{-net in } \mathcal{K}(X).$$

Důkaz

($\emptyset \neq K, L, P \in \mathcal{K}(X)$.) h is metric, definition is correct, $h \geq 0$ trivial, $h(K, L) = h(L, K)$ trivial, $h(K, L) = 0 \implies K = L$ ($x \notin L \implies \varrho(x, L) > 0 \implies K \subset L \wedge L \subset K$).

„ Δ “ aka „ $h(K, L) \leq h(K, P) + h(P, L)$ “: Let $x \in K, y \in L, p \in P$. Then

$$\varrho(x, L) \leq \varrho(x, y) \leq \varrho(x, p) + \varrho(p, y) \quad \inf y \in L$$

$$\varrho(x, L) \leq \varrho(x, p) + \varrho(p, L) \quad \sup p \in P$$

$$\varrho(x, L) \leq \varrho(x, p) + h(P, L) \quad \inf p \in P$$

$$\varrho(x, L) \leq \varrho(x, P) + h(P, L) \quad \inf p \in P$$

$$\sup_{x \in K} \varrho(x, L) \leq h(K, P) + h(P, L).$$

Similarly $\sup_{y \in L} \varrho(y, K) \leq h(K, P) + h(P, L)$. □

TODO!!!

Definice 1.3

X is metrizable space, $1 \leq \alpha < \omega_1$. We define $\Sigma_\alpha^0(X)$, $\Pi_\alpha^0(X)$, and $\Delta_\alpha^0(X)$ by induction:

$$\Sigma_1^0(X) := \{U \subset X \mid U \text{ open}\},$$

$$\Pi_\alpha^0(X) := \{A \subset X \mid X \setminus A \in \Sigma_\alpha^0(X)\},$$

$$\Sigma_\alpha^0(X) := \left\{ \bigcup_{n \in \omega} A_n \mid A_n \in \Pi_{\alpha_n}^0(X), \alpha_n < \alpha, n \in \omega \right\},$$

$$\Delta_\alpha^0(X) := \Sigma_\alpha^0 \cap \Pi_\alpha^0(X).$$

Poznámka (By induction it can be proven)

$$\Sigma_\alpha^0(X) \subset \Sigma_\beta^0(X), \Pi_\alpha^0(X) \subseteq \Pi_\beta^0(X), \quad 1 \leq \alpha < \beta < \omega_1.$$

Poznámka

$$\forall \alpha, \beta : 1 \leq \alpha < \beta < \omega_1 : \Sigma_\alpha^0(X) \subset \Pi_\beta^0(X).$$

Poznámka

If X contains homeomorphic copy of 2^ω then all inclusions are strict.

We denote $Borel(X)$ as σ -algebra of Borel sets (σ -algebra generated by $\Sigma_1^0(X)$).

Poznámka (Also non-trivial theorem)

$$Borel(X) = \bigcup_{1 \leq \alpha < \omega_1} \Sigma_\alpha^0(X) = \bigcup_{1 \leq \alpha < \omega_1} (X) = \bigcup_{1 \leq \alpha < \omega_1} \Delta_\alpha^0(X).$$

$$A_n \in \bigcup_{1 \leq \alpha < \omega_1} \Sigma_\alpha^0(X) \implies \exists 1 \leq \alpha_n < \omega_1 : A_n \in \Sigma_{\alpha_n}^0(X) \implies A_n \in \Sigma_{\sup\{\alpha_n \mid n \in \omega\}}^0 \implies \bigcup_{n \in \omega} A_n \in \Sigma_{\sup\{\alpha_n, n \in \omega\}}^0$$

Poznámka

$$F_\sigma = \Sigma_2^0, G_\delta = \Pi_2^0, F_{\sigma\delta} = \Pi_3^0, G_{\delta\sigma} = \Sigma_3^0.$$

$\Sigma_\alpha^0(X)$ is closed under countable union and $\Pi_\alpha^0(X)$ under countable intersection.

Věta 1.7

X be metrizable, $1 \leq \alpha < \omega_1$. Then

1. $\Sigma_\alpha^0(X)$ is closed under finite intersection;
2. $\Pi_\alpha^0(X)$ is closed under finite union.

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Důkaz

„1.“ Firstly for $\alpha = 1$, it is trivial. Then let $A, B \in \Sigma_\alpha^0(X)$, $\alpha > 1$. Then $A = \bigcup_{n \in \omega} A_n$, $A_n \in \Pi_{\alpha_n}^0(X)$, $\alpha_n < \alpha$, $B = \bigcup_{m \in \omega} B_m$, $B_m \in \Pi_{\beta_m}^0(X)$, $\beta_m < \alpha$. $A \cap B = \bigcup_{(m,n) \in \omega^2} A_n \cap B_m$, $A_n \cap B_m \in \Pi_{\max\{\alpha_n, \beta_n\}}^0(X) \implies A \cap B \in \Sigma_\alpha^0(X)$. „2.“ \iff de Morgan and 1. \square

Věta 1.8

X be metrizable, $A \subset Z \subset X$, $1 \leq \alpha < \omega_1$. Then $A \in \Sigma_\alpha^0(Z) \iff$ there exists $\tilde{A} \in \Sigma_\alpha^0(X) : A = \tilde{A} \cap Z$. Similarly for $\Pi_\alpha^0, \Delta_\alpha^0$.

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Důkaz

Firstly $\alpha = 1$ from definition of subspace. Then assume that it is all true for all $\beta < \alpha$. We want to prove it for α . „ \implies “:

$$A \in \Sigma_\alpha^0(Z) \implies A = \bigcup A_n, A_n \in \Pi_{\beta_n}^0(Z), \beta_n < \alpha \implies \exists \tilde{A}_n \in \Pi_{\beta_n}^0(X) : \tilde{A}_n \cap Z = A_n.$$

$$\tilde{A} = \bigcup \tilde{A}_n \in \Sigma_\alpha^0(X), \tilde{A} \cap Z = Z \cap \bigcup \tilde{A}_n = \bigcup (Z \cap \tilde{A}_n) = \bigcup A_n = A.$$

„ \impliedby “:

$$\tilde{A} \in \Sigma_\alpha^0(X), A = \tilde{A} \cap Z \implies \exists \tilde{A}_n \in \Pi_{\beta_n}^0(X), \beta_n < \alpha, \bigcup \tilde{A}_n = \tilde{A}.$$

$$\tilde{A} \cap Z \in \Pi_{\beta_n}^0(Z) \implies A = \tilde{A} \cap Z = \left(\bigcup \tilde{A}_n \right) \cap Z = \bigcup (\tilde{A}_n \cap Z) = \bigcup A_n \in \Sigma_\alpha^0(Z).$$

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Věta 1.9

X, Y be metric spaces, $f : X \rightarrow Y$ is continuous. If $A \in \Sigma_\alpha^0(Y)$ ($\Pi_\alpha^0(Y)$, $\Delta_\alpha^0(Y)$) then $f^{-1}(A) \in \Sigma_\alpha^0(X)$ ($\Pi_\alpha^0(X)$, $\Delta_\alpha^0(X)$).

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Důkaz

$\alpha = 1$ trivial. Assume it holds true for $\Sigma_\beta^0(Y)$, $\Pi_\beta^0(Y)$, $\beta < \alpha$, and we want to show for $\Sigma_\alpha^0(Y)$ ($\Pi_\alpha^0(Y)$). Let $A \in \Sigma_\alpha^0(Y)$, $\alpha > 1 \implies A = \bigcup_{n \in \omega} A_n$, $A_n \in \Pi_{\beta_n}^0(Y)$, $\beta_n < \alpha$.

$$f^{-1}(A) = f^{-1}\left(\bigcup A_n\right) = \bigcup \underbrace{f^{-1}(A_n)}_{\Pi_{\beta_n}^0(X)} \in \Sigma_\alpha^0(X),$$

$$f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A).$$

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Věta 1.10 (Borel classes in PTS)

X, Y be PTS, $A \in \Sigma_\alpha^0(X)$, $\alpha \geq 3$ (resp. $A \in \Pi_\alpha^0(X)$, $\alpha \geq 2$), $B \subset Y$. If B and A are homeomorphic then $B \in \Sigma_\alpha^0(Y)$ (resp. $\Pi_\alpha^0(Y)$).

Důkaz

$f : A \rightarrow B$ is homeomorphism A onto B . The theorem above (name ?) there is extension \tilde{f} of f , \tilde{f} is homeomorphism \tilde{A} onto \tilde{B} , $A \subset \tilde{A}$, $B \subset \tilde{B}$, $\tilde{A} \in \Pi_2^0(X)$, $\tilde{B} \in \Pi_2^0(Y)$. Then $B \in \Sigma_\alpha^0(\tilde{B})$ (because $B = (f^{-1})^{-1}(A)$). From the theorem above, $\exists \hat{B} \in \Sigma_\alpha^0(Y) : B = \hat{B} \cap \tilde{B} \in \Sigma_\alpha^0(Y) \iff \alpha \geq 3$. \square

1.4 Analytic sets

Definice 1.4

X PTS, $A \subset X$. We say that A is analytic set in X if there exists PTS Y and continuous mapping $\varphi : Y \rightarrow X$ such that $\varphi(Y) = A$.

We denote collection of analytic subsets of X as $\Sigma_1^1(X)$. We say that A is coanalytic in X if $X \setminus A \in \Sigma_1^1(X)$ and we denote this collection as $\Pi_1^1(X)$. $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Například

$$Q = \{\alpha \in 2^\omega \mid \exists n \in \omega \forall j \geq n : \alpha_j = 0\} = 2^{<\omega} \in \Sigma_2^0(2^\omega) \setminus \Pi_2^0(2^\omega)$$

TODO?

Poznámka

X PTS, $F : X \rightarrow \mathcal{K}(X)$ by $F(x) = \{x\}$. Then F is continuous, $F^{-1}(\mathcal{K}(A)) = A \implies$ if $\mathcal{K}(A) \in \Sigma_\alpha^0(\mathcal{K}(X))$ ($\Pi_\alpha^0, \Delta_\alpha^0$) then $A \in \Sigma_\alpha^0(X)$ ($\Pi_\alpha^0, \Delta_\alpha^0$). A open $\implies \mathcal{K}(A)$ is open, A is closed $\implies \mathcal{K}(A)$ is closed. $\mathcal{K}(\bigcap A_n) = \bigcap \mathcal{K}(A_n)$. Thus for $A \in \Pi_2^0(X) : \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X))$. $A \in \Sigma_1^0(X)$ ($\Pi_1^0(X), \Pi_2^0(X)$) $\iff \mathcal{K}(A) \in \Sigma_1^0(\mathcal{K}(X))$ ($\Pi_1^0(\mathcal{K}(X)), \Pi_2^0(\mathcal{K}(X))$).

Věta 1.11

X PTS, $|X| > \omega$. Assume $I \subset \mathcal{K}(X)$, I is σ -ideal ($K \in I, L \subset K \implies L \in I$; $K_n \in I, \bigcup K_n \in \mathcal{K}(X) \implies \bigcup K_n \in I$). If $I \in \Pi_2(\mathcal{K}(X))$, then $I \in \Sigma_1^1(\mathcal{K}(X))$.

Důsledek

$A \notin \Pi_2^0(X) \implies \mathcal{K}(A) \notin \Sigma_1^1(\mathcal{K}(X))$.

Poznámka

$A \in \Pi_1^1(X)$, $\mathcal{K}(A) = \mathcal{K}(X) \setminus \{K \in \mathcal{K}(X) \mid \exists x \in (X \setminus A) \cap K\}$ $\{(K, x) \in \mathcal{K}(X) \times X \mid x \in K\}$ is closed.

Definice 1.5

$$\Sigma_1^1(X) := \{A \subset X \mid \exists Y \text{ PTS}, f : Y \rightarrow X \text{ continuous} : f(Y) = A\}.$$

Poznámka • $\emptyset \in \Sigma_1^1$;

- $\Pi_2^0(X) \subset \Sigma_1^1(X)$, $f = \text{id}$;
- $X, Z \text{ PTS}$, $\psi : X \rightarrow Z$ continuous, $A \in \Sigma_1^1(X) \implies \psi(A) \in \Sigma_1^1(Z)$;
- $\Sigma_{n+1}^1(X) = \{A \subset X \mid \exists Y \text{ PTS}, \psi : Y \rightarrow X \text{ continuous}, B \in \Pi_n^1(X), A = \psi(B)\}$, $n \in \omega \setminus \{0\}$;
- $\Pi_n^1(X) = \{A \subset X \mid X \setminus A \in \Sigma_n^1(X)\}$, $\Delta_n^1(X) = \Sigma_n^1(X) \cap \Pi_n^1(X)$;
- $\bigcup_{n \in \mathbb{N}} \Sigma_n^1(X) = \bigcup_{n \in \mathbb{N}} \Pi_n^1 = \bigcup_{n \in \mathbb{N}} \Delta_n^1(x) = \mathbb{P}(X)$;
- $\#\mathbb{P}(X) \leq 2^\omega$, $\mathbb{P}(X)$ is closed under continuous images and inverse images;
- $\Sigma_1^1(X) = \{A \subset X \mid \exists \psi : \omega^\omega \rightarrow X \text{ continuous} : \psi(\omega^\omega) = A\}$; $Y \text{ PTS}$, $f : Y \rightarrow X$: $f(Y) = A$, $g : \omega^\omega \rightarrow Y : g(\omega^\omega) = Y$, g, f are constant. So $\psi = f \circ g$.

Věta 1.12

$X \text{ PTS}$, $A_n \in \Sigma_1^1(X)$, $n \in \omega$. Then $\bigcup_{n \in \omega} A_n, \bigcap_{n \in \omega} A_n \in \Sigma_1^1(X)$.

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Důsledek

Similar for $\Pi_1^1(X)$.

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Důkaz

„Union“: Assume $A_n \neq \emptyset$, $n \in \omega \implies \varphi_n : \omega^\omega \rightarrow X : \varphi_n(\omega^\omega) = A_n$ continuous. Define $\varphi : \omega^\omega \rightarrow X$ by $\varphi(\nu_0, \nu_1, \dots) = \varphi_{\nu_0}(\nu_1, \nu_2, \dots)$. „ φ is continuous“: $\nu^j \rightarrow \nu \implies \exists n_0 \in \omega \forall j \geq n_0 : \nu_0^j = \nu_0$.

$$\lim_{j \rightarrow \infty} \varphi(\nu^j) = \lim_{j \rightarrow \infty} \varphi_{\nu_0^j}(\nu_1^j, \nu_2^j, \dots) = \lim_{j \rightarrow \infty} \varphi_{\nu_0}(\nu_1^j, \dots) = \varphi_{\nu_0}(\nu_1, \dots) = \varphi(\nu).$$

„ $\varphi(\omega^\omega) = \bigcup_{n \in \omega} A_n$ “:

$$x \in \bigcup A_n \implies \exists n \in \omega : x \in A_n \implies \exists \nu \in \omega^\omega : \varphi_n(\nu) = x \implies \varphi(n^\wedge \nu) = x.$$

$$x \in \varphi(\omega^\omega) \implies \exists \tilde{\nu} \in \omega^\omega : \varphi(\tilde{\nu}) = x \implies x = \varphi_{\tilde{\nu}_0}(\tilde{\nu}_1, \dots) \implies z \in A_{\tilde{\nu}_0} \implies x \in \bigcup A_n.$$

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□

Poznámka (Intersection)

WLOG: $A_n \neq \emptyset$, $n \in \omega$. $Y := (\omega^\omega)^\omega$, $Y \text{ PTS}$ by the theorem above (first item). $\varphi_n : \omega^\omega \rightarrow$

X , meh that $\varphi_n(\omega^\omega) = A_n$.

$$F := \{y = (y_0, y_1, \dots) \in Y \mid \forall n, m \in \omega : \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap_{n, m \in \omega} \{y \in Y \mid \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap_{n, m \in \omega} F_{n, m}$$

intersection of closed, so F is closed and is PTS.

$$„\varphi_0 \circ \pi_0(F) = \bigcap_{n \in \omega} A_n“:$$

$$x \in \varphi_0 \circ \pi_0(F) \implies \exists y \in F : x = \varphi_0(y_0) = \varphi_1(y_1) = \varphi_2(y_2) = \dots \implies x \in \bigcap_{n \in \omega} A_n.$$

$$x \in \bigcap_{n \in \omega} A_n \implies \exists y_0, y_1, \dots \in \omega^\omega : \varphi_0(y_0) = x, \varphi_1(y_1) = x, \dots \implies y = (y_0, y_1, \dots) \in F, \varphi_0 \circ \pi_0(y) = x \implies x \in \varphi_0 \circ \pi_0(F).$$

Poznámka

$\Sigma_1^1(X)$ is not closed under complement: $\sigma(\Sigma_1^1(X)) \supset \Sigma_1^1(X) \cup \Pi_1^1(X)$.

$$\text{Borel}(X) \subset \Sigma_1^1(X) \cap \Pi_1^1(X) = \Delta_1^1(X).$$

Věta 1.13

X, Y PTS, $A \in \Sigma_1^1(X)$ (respective $\Pi_1^1(X)$), $B \subset Y$, A and B are homeomorphism. Then $B \in \Sigma_1^1(Y)$ (resp. $\Pi_1^1(Y)$).

┌

Důkaz

For Σ_1^1 trivial. $A \in \Pi_1^1(X)$, $\varphi : A \rightarrow B$ homeomorphism. Then from the theorem above, $\exists \tilde{A} \in \Pi_2^0(X), \tilde{B} \in \Pi_2^0(Y)$ and $\tilde{\varphi} : \tilde{A} \rightarrow \tilde{B}$ homeomorphism extending φ , $A \subset \tilde{A}$, $B \subset \tilde{B}$.

Then $\tilde{A} \setminus A = (X \setminus A) \cap \tilde{A} \in \Sigma_1^1(X) \implies \tilde{B} \setminus B \in \Sigma_1^1(Y)$. $B = Y \setminus (\tilde{B} \setminus B \cup Y \setminus \tilde{B}) \in \Pi_1^1(Y)$. \square

└

Věta 1.14

X PTS. Then $\text{Borel}(X) \subset \Delta_1^1(X)$.

┌

Důkaz

└ Trivial. \square

1.5 Luzin theorem

Věta 1.15 (Luzin)

X PTS, $A_1, A_2 \in \Sigma_1^1(X)$, $A_1 \cap A_2 = \emptyset$. Then there exists $B \in \text{Borel}(X)$, such that $A_1 \subset B \subset X \setminus A_2$.

Důsledek

X PTS. $\Delta_1^1(X) = \text{Borel}(X)$.

┌

Důkaz

$\Delta_1^1(X) \subseteq \text{Borel}(X)$ we already have.

$A \in \Delta_1^1(X) \implies A \in \Sigma_1^1(X), X \setminus A \in \Sigma_1^1 \implies \exists B \in \text{Borel}(X) : A \subset B \subset X \setminus (X \setminus A) = A \implies A = B$

└

□

Lemma 1.16

$C_n, D_n \subset X$, $n, m \in \omega$ and $\forall n, m \in \omega$ we can separate C_n, D_m by some Borel set. Then we can separate $\bigcup_{n \in \omega} C_n$ and $\bigcup_{m \in \omega} D_m$ by Borel set.

┌

Důkaz

Let $B_{n,m} \in \text{Borel}(X)$ separating C_n from D_m ($C_n \subset B_{n,m} \subset X \setminus D_m$). Put $B := \bigcup_{n \in \omega} \bigcap_{m \in \omega} B_{n,m}$.

└

□

Důkaz (Luzin theorem)

Assume $A_1, A_2 \neq \emptyset$. Then exists $\varphi_1, \varphi_2 : \omega^\omega \rightarrow X$ $\varphi_i(\omega^\omega) = A_i$. We assume A_1 can't be separated from A_2 by any Borel set.

$A_i = \varphi_i(\omega^\omega) \implies A_i = \bigcup_{n \in \omega} \varphi_i(\mathcal{N}(n)) \implies \exists \nu_0, \mu_0 \in \omega : \varphi_i(\mathcal{N}(\mu_0))$ can't be separated from $\varphi_2(\mathcal{N}(\nu_0))$.

We use lemma again and obtain $\mu, \nu \in \omega^\omega$ such that $\forall k \in \omega : \varphi_1(\mathcal{N}(\mu/k))$ can't be separated from $\varphi_2(\mathcal{N}(\nu/k))$

$\varphi_1(\mu) \in A_1, \varphi_2(\nu) \in A_2 \implies \varphi_1(\mu) \neq \varphi_2(\nu) \implies \exists G_1, G_2$ open, $G_1 \cap G_2 = \emptyset$

such that $\varphi_1(\mu) \in G_1, \varphi_2(\nu) \in G_2, \varphi_1, \varphi_2$ are continuous $\implies \exists k \in \omega : \varphi_1(\mathcal{N}(\mu/k)) \subset G_1, \varphi_2(\mathcal{N}(\nu/k)) \subset G_2$ which is continuous. □

Například

$\{f \in C([0, 1]) \mid \forall x \in [0, 1] : f'(x) \in \mathbb{R}\} \in \Pi_1^1 \setminus \Delta_1^1$.

$\{f \in C([0, 2\pi]) \mid \text{Fourier series converges to } f \text{ for every } x \in [0, 2\pi]\} \in \Pi_1^1 \setminus \Delta_1^1$.

$\{K \in \mathcal{K}([0, 1]) \mid |K| \leq \omega\}, \{K \in \mathcal{K}(\mathbb{R}) \mid K \subset \mathbb{Q}\} \in \Pi_1^1 \setminus \Delta_1^1$.

Například

$\{x \in X \mid \exists y \in Y : (x, y) \in B\} \in \Sigma_1^1(X)$.

TODO!!!

Lemma 1.17

(X, τ) PTS, $F \in \Pi_1^0(X)$. Let τ_F be topology generated by $\tau \cup \{F\}$. Then τ_F is Polish, $F \in \Delta_1^0(X, \tau_F)$, $\Delta_1^1(X, \tau_F) = \Delta_1^1(X, \tau)$.

┌

Důkaz

(X, τ_F) is homeomorphic with $((X \setminus F) \times \{0\}) \cup (F \times \{1\}) \subset X \times \{0, 1\}$ which is PTS and those two subsets are G_δ in $X \times \{0, 1\}$, so, (X, τ_F) is Polish.

$$\Delta_1^1(((X \setminus F) \times \{0\}) \cup (F \times \{1\})) \leftrightarrow \Delta_1^1(\tau_F) = \{A \cup B \mid A \in \Delta_1^1(X \setminus F, \tau), B \in \Delta_1^1(F, \tau)\} \subset \Delta_1^1(\tau) \subset \Delta_1^1(\tau_F).$$

└

□

Lemma 1.18

(X, τ) PTS, $(\tau_n)_{n \in \omega}$ Polish topology, $\tau \subset \tau_n$, $n \in \omega$. Then topology τ_∞ generated by $\bigcup_{n \in \omega} \tau_n$ is polish. If $\forall n \in \omega : \tau_n \subset \Delta_1^1(\tau)$, then $\Delta_1^1(\tau) = \Delta_1^1(\tau_\infty)$.

┌

Důkaz

Set $X_n := (X, \tau_n)$, $\varphi : X \rightarrow \prod_{n \in \omega} X_n$, $\varphi(x) = (x, x, x, x, \dots)$. φ is homomorphism (X, τ_∞) on $\varphi(X)$. ($U \in \text{base of } \tau_\infty \implies \exists n \in \omega : U \in \tau_n, \varphi(U) = x_1 \times x_2 \times \dots \times x_{n-1} \times U \times x_{n+1} \times \dots \cap \varphi(X)$ is open. $\varphi(X) \in \Pi_1^0(\prod X_n) \implies \varphi(X)$ PTS $\implies (X, \tau_\infty)$ PTS.)

$$\Delta_1^1(\tau) = \Delta_1^1(\tau_\infty) \iff \sigma(\sigma(M)) = \sigma(M). (\tau_\infty \subset \Delta_1^1(\tau) = \Delta_1^1(\tau_n).) \tau_\infty \subset \bigcup \Delta_1^1(\tau_n). \quad \square$$

└

Věta 1.19

(X, τ) PTS, $A \in \Delta_1^1(X, \tau)$. There exists polish topology τ_A such that $\tau \subset \tau_A$, $\Delta_1^1(\tau_A) = \Delta_1^1(\tau)$ and $A \in \Delta_1^0(X, \tau_A)$.

┌

Důkaz

$\mathcal{S} := \{D \in \Delta_1^1(X) \mid \text{exists polish topology } \tau_D \supset \tau \text{ and } \Delta_1^1(\tau_D) = \Delta_1^1(\tau), D \in \Delta_1^0(X, \tau_D)\}$. We know that $\tau \subset \mathcal{S}$ and that \mathcal{S} is closed under complements. Moreover, \mathcal{S} is closed under countable union ($A_n \in \mathcal{S} \rightarrow \tau_{A_n} \rightarrow \tau_\infty = \tau_{\bigcup A_n}$). So $\mathcal{S} = \Delta_1^1(X, \tau)$. □

└

Lemma 1.20

X, Y PTS. $f : X \rightarrow Y$ Borel. Then $\text{graph}(f) \in \Delta_1^1(X \times Y)$.

┌ *Důkaz*

Fix compatible complete metric ϱ on Y . U_n , $n \in \omega$, countable collection of open balls with $\text{diam} < 2^{-n}$ covering Y .

$$\text{graph } f \stackrel{?}{=} \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U \in \Delta_1^1(X \times Y).$$

$$„\subseteq“: (x, y) \in \text{graph}(f) \Leftrightarrow f(x) = y \implies \forall n \in \omega \exists U \in U_n : y \in U \wedge x \in f^{-1}(U) \implies (x, y) \in \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U.$$

$$„\supseteq“: (x, y) \notin \text{graph}(f) \Leftrightarrow f(x) \neq y \implies \exists n \in \omega : \varrho(f(x), y) > \frac{1}{n} \implies \exists n \in \omega \forall U \in U_n \neg (x \in f^{-1}(U) \cap y \in U) \implies (x, y) \notin \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U. \quad \square$$

└

Poznámka (Notation)

If f is Borel, we write $f \in \Delta_1^1$.

Věta 1.21

X, Y PTS, $f \in \Delta_1^1(X \times Y)$. If $A \in \Delta_1^1(X)$ and $f|_A$ is injective, then $f(A) \in \Delta_1^1(Y)$.

┌ *Důkaz*

If $f : X \rightarrow Y$ is injective, then $f(A) = \prod_Y(\text{graph}(f) \cap A \times Y) \in \Sigma_1^1(Y)$.

$$Y \setminus F(A) = \prod_Y(\text{graph}(f) \cap (X \setminus A) \times Y) \in \Sigma_1^1(Y) \implies f(A) \in \Delta_1^1(Y).$$

Assume f is continuous, $A \in \Pi_1^0(X)$. From the theorem above $A \subset \omega^\omega$, $B_s := f(\mathcal{N}(s) \cap A)$. $\forall s \in \omega^{<\omega} \forall i, j, i \neq j : B_{s^\wedge i} \cap B_{s^\wedge j} = \emptyset \iff f$ is injection. $\forall s \in \omega^{<\omega} : B_s = \bigcup_{i \in \omega} B_{s^\wedge i}$.

From Luzin separation theorem, there exists (by induction) $(B'_s)_{s \in \omega^{<\omega}}$ of Borel sets:

$$\forall s \in \omega^{<\omega} \forall i, j \in \omega, i \neq j : B'_{s^\wedge i} \cap B'_{s^\wedge j} = \emptyset.$$

(separation $B_{s^\wedge i}, \bigcup_{j < i} B_{s^\wedge j} \cup \bigcup_{l > i} B_{s^\wedge l}$) $\forall s \in \omega^{<\omega} : B_s \subset B'_s$.

Put: $B_\emptyset^* = Y$, $B_{s^\wedge j}^* = B_{s^\wedge j}$ TODO!!! □

└