Poznámka (Note from me – autor of notes)
Bad English in this text is my fault, not lecturer's one.

Úvod

Poznámka
3 part exam: theorem -> proof; scientific paper -> understand + explain; therms + concepts -> explain

credits: homework (time demanding)

Microsoft teams

0.1 Matrix analysis / linear algebra

Poznámka
Scalar product: $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$: $\mathbf{u} \cdot \mathbf{v}$, cross product: $\mathbf{u} \times \mathbf{v}$, and more: $\mathbf{u} = u^i \mathbf{e}_i$ $\mathbf{u} \cdot \mathbf{v} = \delta_{ij}(u^i v^j)$, $(\mathbf{u} \times \mathbf{v})_i = \varepsilon_{ijk} u_j v_k$ (where ε_{ijk}, Levi-Civita symbol, does expecting thing).

Definice 0.1 (Tensor product)

$$\mathbf{u} \otimes \mathbf{v} \qquad (\mathbf{u} \otimes \mathbf{v})\mathbf{w} := \mathbf{u}(\mathbf{v} \cdot \mathbf{w})$$

Tvrzení 0.1 (Identities for Levi-Civita symbol)

$$\varepsilon_{ijk}\varepsilon_{lmn} = \det \begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix}$$

$$\varepsilon_{ijk} \cdot \delta_{lm} = \varepsilon_{jkm} \cdot \delta_{il} + \varepsilon_{klm} \cdot \delta_{jl} + \varepsilon_{ijm} \cdot \delta_{kl}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{ijm}\varepsilon_{ijn} = 2\delta_{mn}$$

Definice 0.2 (Transpose matrix)

$\mathbb{A} \in \mathbb{R}^{3\times3}$, $\mathbb{A}^T$ is defined as $\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^3 : \mathbb{A}^T\mathbf{u} \cdot \mathbf{v} := \mathbf{u} \cdot \mathbb{A}\mathbf{v}$.

Definice 0.3 (Trace of matrix)

$\mathbb{A} \in \mathbb{R}^{3\times3}$, $\operatorname{tr} \mathbb{A}$ is defined as $\operatorname{tr}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

Poznámka
Matrix, tensor and linear operator is the same.

$$\mathbb{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \qquad \mathbb{A}\mathbf{v} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(v_m\mathbf{e}_m) = A_{ij}v_m\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_m) = (A_{ij}v_j)\mathbf{e}_i.$$

Definice 0.4 (Axial vector)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$, $\mathbb{A}$ is stew-symetric ($-\mathbb{A} = \mathbb{A}^T$). Then we can prove that $\forall \mathbf{w} \in \mathbb{R}^3 : \mathbb{A}\mathbf{w} = \mathbf{v}_{\mathbb{A}} \times \mathbf{w}$. We call $\mathbf{v}$ the axial vector.

Poznámka
$\mathbf{v}_{\mathbb{A}} = (A_{23}, A_{13}, A_{12})^T$.

Tvrzení 0.2

$\mathbb{A}\mathbf{v}_{\mathbb{A}} = \mathbf{o}$ *and* $(\mathbf{u} \otimes \mathbf{v})^T = (\mathbf{v} \otimes \mathbf{u})$.

Definice 0.5 (Determinant in 3D)

$\det \mathbb{A} := \frac{\mathbb{A}\mathbf{u}\cdot(\mathbb{A}\mathbf{v}\times\mathbb{A}\mathbf{w})}{\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})}$ for three arbitrary vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$.

Poznámka (Nanson formula)

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\det \mathbb{A})^{-1}\mathbb{A}\mathbf{w} \cdot (\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) = \mathbf{w} \cdot (\det \mathbb{A})^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v}) \implies$$
$$\implies \mathbf{u} \times \mathbf{v} = (\det A)^{-1}\mathbb{A}^T(\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v})$$
$$\mathbb{A}\mathbf{u} \times \mathbb{A}\mathbf{v} = (\det \mathbb{A})\mathbb{A}^{-T}(\mathbf{u} \times \mathbf{v})$$

Definice 0.6 (Cofactor)

$$\operatorname{cof} \mathbb{A} := (\det \mathbb{A})\mathbb{A}^{-T}.$$

(Change of surface area under linear mapping $\mathbb{A}$.)

Definice 0.7 (Eigenvalues, eigenvectors)

$\mathbb{A}\mathbf{v} = \lambda\mathbf{v}$.

Characteristic polynomial: $\det(\mathbb{A} - \mu\mathbb{I}) = -\mu^3 + c_1\mu^2 - c_2\mu + c_3$.

Věta 0.3 (Cayley-Hamilton)

$$-\mathbb{A}^3 + c_1\mathbb{A}^2 - c_2\mathbb{A} + c_3\mathbb{I} = \mathbb{O}$$

Tvrzení 0.4

$$c_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det \mathbb{A}$$
$$c_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \operatorname{tr}\operatorname{cof}\mathbb{A} = \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}(\mathbb{A}^2))$$
$$c_1 = \lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}\mathbb{A}$$

Důkaz
With definition of characteristic polynomial, Cayley-Hamilton and Schur decomposition. Schur decomposition: $\mathbb{A} \in \mathbb{R}^{3\times 3}$. There exists an invertible matrix $\mathbb{U}$ and upper triangular matrix $\mathbb{T}$ such that

$$\mathbb{A} = \mathbb{U}^{-1}\mathbb{T}\mathbb{U}, \qquad \mathbb{T} = \begin{pmatrix} \lambda_1 & T_{12} & T_{13} \\ 0 & \lambda_2 & T_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

□

Tvrzení 0.5 (Useful identity from CH)

$$\mathbb{A}^{-1} = \frac{1}{c_3}\mathbb{A}^2 - \frac{c_1}{c_3}\mathbb{A} + \frac{c_2}{c_3}\mathbb{I} = \frac{1}{\det\mathbb{A}}\mathbb{A}^2 - \frac{\operatorname{tr}\mathbb{A}}{\det\mathbb{A}}\mathbb{A} + \frac{\operatorname{tr}\operatorname{cof}\mathbb{A}}{\det\mathbb{A}}\mathbb{I}$$

Poznámka (Functions of matrices)
$\exp\mathbb{A}$, $\ln\mathbb{A}$, $\sin\mathbb{A}$, ...

There are several ways of define it: Analytics calculus = Taylor series, Borel calculus: $\mathbb{A} = \sum \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$ () $\implies f(\mathbb{A}) := \sum f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i$, Holomorphic calculus ($f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta$) $f(\mathbb{A}) = \frac{1}{2\pi i}\int_\gamma f(\zeta)(\zeta\mathbb{I} - \mathbb{A})^{-1} d\zeta$ (where curve ζ envelops eigenvalues of $\mathbb{A}$)

Tvrzení 0.6 (Useful identities for functions)

$$\det(\exp\mathbb{A}) = \exp(\operatorname{tr}\mathbb{A})$$
$$\exp\mathbb{A} = \lim_{n\to\infty}\left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n$$

Definice 0.8 (Invariants of matrix)

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}\mathbb{A} = I_1;$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \operatorname{tr}\operatorname{cof}\mathbb{A} = \frac{1}{2}((\operatorname{tr}\mathbb{A})^2 - \operatorname{tr}(\mathbb{A}^2)) = I_2;$$
$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det\mathbb{A} = I_3$$

0.2 Representation theorems for isotropic functions

Definice 0.9 (Isotropic function)

$\varphi : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is isotropic $\equiv \varphi(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \varphi(\mathbb{A})$ for all proper orthogonal matrices ($\mathbb{Q}\mathbb{Q}^T = \mathbb{I}$, $\det\mathbb{Q} > 0$).

$f : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is isotropic $\equiv f(\mathbb{Q}\mathbb{A}\mathbb{Q}^T) = \mathbb{Q}f(\mathbb{A})\mathbb{Q}^T$ for all proper orthogonal matrices.

Věta 0.7

A scalar function $\varphi : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{R}$ of symmetric matrices is isotropic if and only if it can be rewritten as a function of invariants of $\mathbb{A}$.

Věta 0.8

A matrix valued function $f : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ (from symmetric matrices to symmetric matrices) is isotropic if and only if it can be rewritten as

$$f(\mathbb{A}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{A} + \alpha_2\mathbb{A}^2,$$

where $\{\alpha_i\}_{i=1}^3$ are scalar function of the invariants.

Důsledek
$\mathbb{A} \mapsto \mathbb{A}^{-1}$ is isotropic function.

Poznámka (Notation)

$$\mathbb{A}, \mathbb{B} \in \mathbb{R}^{3\times 3}, \qquad \mathbb{A} : \mathbb{B} := \operatorname{tr}(\mathbb{A}\mathbb{B}^T), \qquad ||\mathbb{A}|| := (\operatorname{tr}(\mathbb{A}\mathbb{A}^T))^{1/2}$$

0.3 Calculus

Definice 0.10 (Gateaux derivative)

$$Df(x)[y] = \left(\frac{d}{d\tau} f(x + \tau y)\right)|_{\tau=0}.$$

Definice 0.11 (Fréchet derivative)

$$\lim_{||y||\to 0} \frac{||f(x+y)-f(x)-Df(x)[y]||}{||y||} = 0.$$

Poznámka
$Df(\mathbb{A})[\mathbb{B}] \sim \frac{\partial f}{\partial \mathbb{A}}(\mathbb{A})[\mathbb{B}] \sim \frac{\partial f}{\partial \mathbb{A}}(\mathbb{A}) : \mathbb{B}$.

Příklad

$$D(I_2(\mathbb{A}))[B] = D(-\frac{1}{2}\operatorname{tr}\mathbb{A}^2 + \frac{1}{2}(\operatorname{tr}\mathbb{A})^2)[B] = \frac{d}{d\tau}(-\frac{1}{2}\operatorname{tr}(\mathbb{A}+\tau\mathbb{B})^2 + \frac{1}{2}(\operatorname{tr}(\mathbb{A}+\tau\mathbb{B}))^2)|_{\tau=0} =$$
$$= -\operatorname{tr}(\mathbb{A}\mathbb{B}) + (\operatorname{tr}\mathbb{A})(\operatorname{tr}\mathbb{B}) = (\operatorname{tr}\mathbb{A})\mathbb{I} : \mathbb{B} - \mathbb{A}^T : \mathbb{B} = ((\operatorname{tr}\mathbb{A})\mathbb{I} - \mathbb{A}^T) : \mathbb{B}.$$

$$D(\det\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\det(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = (\det\mathbb{A})\frac{d}{d\tau}(\det(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}))_{\tau=0} =$$
$$= \det\mathbb{A}\frac{d}{d\tau}(1+\tau\operatorname{tr}(\mathbb{A}^{-1}\mathbb{B}) + \ldots)|_{\tau=0} = (\det\mathbb{A})\operatorname{tr}(\mathbb{A}^{-1}\mathbb{B}) = (\det\mathbb{A})\mathbb{A}^{-T} : \mathbb{B}.$$

Poznámka
Chain rule works as usual.

Příklad

$$\frac{d}{dt}(\det\mathbb{A}(t)) = (\det\mathbb{A})\operatorname{tr}\left(\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\right).$$

Příklad

$$\frac{\partial\mathbb{A}^{-1}}{\partial\mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}\left((\mathbb{A}+\tau\mathbb{B})^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left((\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B})^{-1}\mathbb{A}^{-1}\right)|_{\tau=0} =$$
$$= \frac{d}{d\tau}\left((\mathbb{I}-\tau\mathbb{A}^{-1}\mathbb{B}+\ldots)\mathbb{A}^{-1}\right)|_{\tau=0} = -\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}.$$

Příklad

$$\frac{\partial e^{\mathbb{A}}}{\partial\mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}\left(e^{\mathbb{A}+\tau\mathbb{B}}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\mathbb{I} + (\mathbb{A}+\tau\mathbb{B}) + \frac{(\mathbb{A}+\tau\mathbb{B})^2}{2!} + \ldots\right)|_{\tau=0} =$$
$$= \frac{d}{d\tau}\left(\mathbb{I} + (\mathbb{A}\tau\mathbb{B}) + \ldots + \tau(\mathbb{A}\mathbb{B}+\mathbb{B}\mathbb{A}) + \tau(\mathbb{A}\mathbb{A}\mathbb{B}+\mathbb{A}\mathbb{B}\mathbb{A}+\mathbb{B}\mathbb{A}\mathbb{A}) + \ldots\right)$$

Věta 0.9 (Daleckii-Krein theorem)

$\mathbb{A} \in \mathbb{R}^{3\times 3}$ real symmetric matrix. $\mathbb{A} = \sum_{i=1}^{3} \lambda_i \mathbb{P}_i$, $\mathbb{P}_i$-projector to i-th eigenvector, $\mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i$. f real valued function $f : \mathbb{R} \to \mathbb{R}$ differentiable.

$$f(\mathbb{A}) := \sum_{i=1}^{3} f(\lambda_i)\mathbb{P}_i = \sum_{i=1}^{3} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i.$$

$$Df(\mathbb{A})[\mathbb{B}] = \sum_{i=1}^{3} \frac{df}{d\lambda}|_{\lambda=\lambda_i}\mathbb{P}_i\mathbb{B}\mathbb{P}_i + \sum_{i=1}^{3}\sum_{j=1,j\neq i}^{3} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\mathbb{P}_i\mathbb{B}\mathbb{P}_j$$

$$(Df(\mathbb{A})[\mathbb{B}])_{ij} = \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}B_{ij}, if i \neq j, (Df(\mathbb{A})[\mathbb{B}])_{ij} = \frac{df}{d\lambda}|_{\lambda=\lambda_j}B_{ij}, if i = j.$$

Důkaz

From chain rule:

$$\frac{\partial f(\mathbb{A})}{\partial \mathbb{A}} = \sum_{i=1}^{3} \frac{df(\lambda_i)}{d\lambda}|_{\lambda=\lambda_i}\frac{\partial \lambda_i}{\partial \mathbb{A}}\mathbf{v}_i \otimes \mathbf{v}_i + \sum_{i=1}^{3} f(\lambda_i)\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + \sum_{i=1}^{3} f(\lambda_i)\mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}$$

First derivative at right side:

$$\mathbb{A}\mathbf{v}_i = \lambda_i \mathbf{v}$$

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}}\mathbf{v}_i + \mathbb{A}\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}}\mathbf{v}_i + \lambda_i\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \qquad \cdot \mathbf{v}_i$$

$$\frac{\partial A_{mn}}{\partial A_{kl}}(\mathbf{v}_i)_n + (A_{mn})\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} = \frac{\partial \lambda_i}{\partial A_{kl}}(\mathbf{v}_i)_m + \lambda_i\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}$$

$$\delta_{mk}\delta_{nl}(\mathbf{v}_i)_n + A_{mn}\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} = \frac{\partial \lambda_i}{\partial A_{kl}}(\mathbf{v}_i)_m + \lambda_i\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} \qquad \cdot (\mathbf{v}_i)_m \sum_m$$

$$\sum_m \frac{\partial \lambda_i}{\partial A_{kl}}(\mathbf{v}_i)_m(\mathbf{v}_i)_m = \frac{\partial \lambda_i}{\partial A_{kl}}$$

From symmetry of $\mathbb{A}$ and definition of eigenvector:

$$\sum_m A_{mn}\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}(\mathbf{v}_i)_m = \lambda_i\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}(\mathbf{v}_i)_n$$

$$\sum_m \delta_{mk}\delta_{nl}(\mathbf{v}_i)_n(\mathbf{v}_i)_m = \delta_{nl}(\mathbf{v}_i)_n(\mathbf{v}_i)_k$$

So

$$\lambda_i\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}(\mathbf{v}_i)_n + \delta_{nl}(\mathbf{v}_i)_n(\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}} + \lambda_i\frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} \qquad \sum_n$$

$$\sum_n \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}} (\mathbf{v}_i)_n + (\mathbf{v}_i)_l (\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}} + \lambda_i \frac{\partial (\mathbf{v}_i)_n}{\partial A_{kl}}$$

$$(\mathbf{v}_i)_l (\mathbf{v}_i)_k = \frac{\partial \lambda_i}{\partial A_{kl}}$$

$$\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbf{v}_i \otimes \mathbf{v}_j$$

Second derivative at right side:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \qquad \cdot \mathbf{v}_j$$

$$\mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j$$

...

$$\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = \frac{\mathbf{v}_j \otimes \mathbf{v}_i}{\lambda_i - \lambda_j} i = \frac{\delta_{kj}\delta_{il}}{\lambda_i - \lambda_j}$$

$$\left(\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} [\mathbb{X}] \right)_j = \frac{\delta_{im}\delta_{jn}}{\lambda_i - \lambda_j} = \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j}.$$

$$\frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} [\mathbb{X}] = \sum_{j=1}^{3} \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j} \mathbf{v}_j.$$

□

Poznámka (V dokončení důkazu se ještě použije)

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}).$$

0.4 Differential operators

Definice 0.12

$$\operatorname{div} \mathbf{v} := \mathbf{h}(\nabla \mathbf{v})?$$

$$(\operatorname{rot} \mathbf{v}) \cdot \mathbf{w} = \operatorname{div}(\mathbf{v} \times \mathbf{w}), \qquad \forall \mathbf{w} \in \mathbb{R}^3 \text{ fixed}$$

$$(\operatorname{div} \mathbb{A}) \cdot \mathbf{w} = \operatorname{div}(\mathbb{A}^T \mathbf{w}), \qquad \forall \mathbf{w} \in \mathbb{R}^3 \text{ fixed}, \qquad (\operatorname{div} \mathbb{A})_i = \frac{\partial A_{im}}{\partial x_m}$$

$$(\operatorname{rot} \mathbb{A})^T \mathbf{w} := \operatorname{rot}(\mathbb{A}^T \mathbf{w}), \qquad \forall \mathbf{w} \in \mathbb{R}^3 \text{ fixed}, \qquad (\operatorname{rot} \mathbb{A})_{ij} = \varepsilon_{jkl} \frac{\partial A_{il}}{\partial x_k}$$

Tvrzení 0.10

$$\operatorname{rot} \nabla \varphi = \mathbf{o}, \quad \operatorname{rot}(\nabla \mathbf{v}) = \mathbb{O}, \quad \operatorname{div}(\operatorname{rot} \mathbf{v}) = 0, \quad \operatorname{div}(\operatorname{rot} \mathbb{A}) = \mathbf{o}.$$

Důkaz

$$\int_\Omega \operatorname{rot}(\nabla\varphi)\cdot d\mathbf{S} \overset{\text{Stokes}}{\int_{\partial\Omega}} \nabla\varphi\cdot d\mathbf{l} = \varphi(\text{end point}) - \varphi(\text{starting point}) \overset{\text{closed curve}}{=} 0.$$

$$\int_\Omega \operatorname{div}(\operatorname{rot}\mathbf{v})dV = \int_{\partial\Omega} \operatorname{rot}\mathbf{v}\cdot d\mathbf{S} = \int_{\partial\Omega^+} \operatorname{rot}\mathbf{v}\cdot d\mathbf{S} + \int_{\partial\Omega^-} \operatorname{rot}\mathbf{v}\cdot d\mathbf{S} = 0$$

□

Věta 0.11 (Stokes theorem)

$$\int_\Omega \operatorname{div}\mathbf{v}dV = \int_{\partial\Omega} \mathbf{v}\cdot d\mathbf{S} := \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}dS$$

$$\int_S \operatorname{rot}\mathbf{v}\cdot dS = \int_{\partial S} \mathbf{v}\cdot d\mathbf{l}$$

Using previous identities we can get integral identities using Stokes theorem:

$$\int_\Omega (\operatorname{div}\mathbb{A})\cdot\mathbf{v}dV = \int_{\partial\Omega} \mathbb{A}^T\mathbf{v}\cdot\mathbf{n}dS - \int_\Omega \mathbb{A} : \nabla\mathbf{v}dV,$$

$$\int_\Omega (\operatorname{rot}\mathbf{v})dV = -\int_{\partial\Omega} \mathbf{v}\times\mathbf{n}dS.$$

Poznámka (Kinematics)
How to describe motion (no motion, no physics, just geometry): We have starting point P, set coordinates $\implies$ we have point $\mathbf{X}$. Then we do some deformation in time: $\mathbf{x} = \chi(\mathbf{x}, t)$ describes motion and deformation.

How to describe properties of continuously distributed matter? We have 3 possibilities (e. g. for θ):

- $\theta(P, t)$,
- $\theta(\mathbf{X}, t)$ – Lagrangian description,
- $\theta(\mathbf{x}, t)$ – Eulerian description.

But we don't know what χ is and how to work with it → we linearize (use derivations), write down PDEs and hopefully form local solutions. So

Problem: What is the relation between $d\mathbf{x}$ and $d\mathbf{X}$?

$$d\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = \chi(\mathbf{X}_2, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1 + d\mathbf{X}, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1, t) + \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}_1, t) d\mathbf{X} + \dots$$

$$\implies d\mathbf{x} = \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t) d\mathbf{X}.$$

Definice 0.13 (Deformation gradient)

$$\mathbb{F}(\mathbf{X}, t) := \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X}, t).$$

Důsledek

$$dv = (\det \mathbb{F}) dV$$
$$d\mathbf{s} = (\det \mathbb{F}) \mathbb{F}^{-T} d\mathbf{S}$$
$$\mathbf{n} = \mathbb{F}^{-T} \mathbf{N}$$

TODO?

Poznámka (How to characterise deformation/strain?)

$$d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} = \mathbb{F} d\mathbf{X} \cdot \mathbb{F} d\mathbf{X} - d\mathbf{X} \cdot d\mathbf{X} = (\mathbb{F}^T \mathbb{F} - \mathbb{I}) d\mathbf{X} \cdot d\mathbf{X}$$

Definice 0.14

Green-Stieland? strain tensor:

$$\mathbb{E}(\mathbf{X}, t) := \frac{1}{2}(\mathbb{F}^T \mathbb{F} - \mathbb{I})$$

Right Cauchy-Green tensor:

$$\mathbb{C}(\mathbf{X}, t) := \mathbb{F}^T(\mathbf{X}, t) \cdot \mathbb{F}(\mathbf{X}, t)$$

Euler-Almans strain tensor:

$$@e(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X}, t)} = \frac{1}{2}\left(\mathbb{I} - F^{-T}(\mathbf{X}, t) \mathbb{F}^{-1}(\mathbf{X}, t)\right)$$

Left Cauchy-Green tensor

$$\mathbb{B}(\mathbf{x}, t) = \mathbb{F}(\mathbf{X}, t) \mathbb{F}^T(\mathbf{X}, t)$$

TODO?

Poznámka (Velocity)

$$\mathbf{V}(\mathbf{X},t) := \frac{d\chi(\mathbf{X},t)}{dt}$$
$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\chi^{-1}(\mathbf{x},t),t)$$

or

$$\mathbf{v}(\chi(\mathbf{X},t),t) = \mathbf{V}(\mathbf{X},t)$$

(Euler velocity field.)

Definice 0.15

$$\frac{d\mathbf{v}}{dt}(\mathbf{x},t) := \frac{\partial \mathbf{v}}{\partial t}(\mathbf{x},t) + (\mathbf{v}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}})\mathbf{v}(\mathbf{x},t),$$
$$\frac{d\varphi(\mathbf{x},t)}{dt} := \frac{\partial\varphi}{\partial t} + (\mathbf{v}\cdot\nabla)\varphi.$$

Definice 0.16 (Time derivative of the deformation gradient)

$$\frac{d\mathbb{F}}{dt}(\mathbf{X},t) = \frac{d}{dt}\frac{\partial\chi(\mathbf{X},t)}{\partial\mathbf{X}} = \frac{\partial}{\partial\mathbf{X}}\frac{d\chi(\mathbf{X},t)}{dt} = \frac{\partial}{\partial\mathbf{X}}\mathbf{V}(\mathbf{X},t) = \frac{\partial}{\partial\mathbf{X}}(\mathbf{v}(\chi(\mathbf{X},t),t)) =$$
$$= \frac{\partial\mathbf{V}}{\partial\mathbf{x}}|_{\mathbf{x}=\chi(\mathbf{X},t)}\frac{\partial\chi}{\partial\mathbf{X}}(\mathbf{X},t) \implies \frac{d\mathbb{F}}{dt}(\mathbf{X},t) = \mathbb{L}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbb{F}(\mathbf{X},t),$$

where $\mathbb{L}(\mathbf{x},t) := \frac{\partial\mathbf{v}}{\partial\mathbf{x}}(\mathbf{x},t) = \nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x},t)$.

Tvrzení 0.12 (Time derivateives of $d\mathbf{x}, d\mathbf{s}, dv$)

$$\frac{d}{dt}d\mathbf{x} = \frac{d}{dt}\mathbb{F}d\mathbf{X} = \left(\frac{d}{dt}\mathbb{F}(\mathbf{X},t)\right)d\mathbf{X} = \mathbb{L}(\mathbf{x},t)\mathbb{F}(\mathbf{X},t)d\mathbf{X} = \mathbb{L}(\mathbf{x},t)d\mathbf{x}$$

Tvrzení 0.13 (Rate(s) of strain)

$$\frac{d\mathbb{E}}{dt}(\mathbf{X},t) = \frac{d}{dt}\left(\frac{1}{2}(\mathbb{F}^T\mathbb{F}-\mathbb{I})\right) = \frac{1}{2}\left(\frac{d\mathbb{F}^T}{dt}\mathbb{F} + \mathbb{F}^T\frac{d\mathbb{F}}{dt}\right) = \frac{1}{2}\left(\mathbb{F}^T\mathbb{L}^T\mathbb{F} + \mathbb{F}^T\mathbb{L}\mathbb{F}\right) =$$
$$= \mathbb{F}^T\left(\frac{1}{2}(\mathbb{L}^T+\mathbb{L})\right)\mathbb{F} =: \mathbb{F}\mathbb{D}\mathbb{F},$$
$$\mathbb{D}(\mathbf{x},t) := \frac{1}{2}(\mathbb{L}+\mathbb{L}^T).$$

$$\frac{d@e}{dt}(\mathbf{x},t) = \frac{1}{2}\left(\frac{d}{dt}\left(\mathbb{I}-\mathbb{F}^{-T}\mathbb{F}^{-1}\right)\right) = \frac{1}{2}\left(-\left(\frac{d\mathbb{F}^{-1}}{dt}\right)^T\mathbb{F}^{-1} - \mathbb{F}^{-T}\frac{d\mathbb{F}^{-1}}{dt}\right) =$$

$$= \frac{1}{2}(\mathbb{L}^T\mathbb{F}^{-T}\mathbb{F}^{-1} + \mathbb{F}^{-T}\mathbb{F}^{-1}\mathbb{L}) = \frac{1}{2}(-\mathbb{L}^T(\mathbb{I} - \mathbb{F}^{-T}\mathbb{F}^{-1}) + \mathbb{L}^T - (\mathbb{I} - \mathbb{F}^{-T}\mathbb{F}^{-1})\mathbb{L} + \mathbb{L}) =$$

$$= -\mathbb{L}^T @e - @e\mathbb{L} + \mathbb{D} \implies$$

$$\frac{d\mathbb{E}}{dt}(\mathbf{x},t) + \mathbb{L}^T(\mathbf{x},t)@e(\mathbf{x},t) + @e(\mathbf{x},t)\mathbb{L}(\mathbf{x},t) = \mathbb{D}(\mathbf{x},t)$$

$$\frac{d\mathbb{B}}{dt}(\mathbf{x},t) = \mathbb{L}\mathbb{F}\mathbb{F}^T + \mathbb{F}\mathbb{F}^T\mathbb{L}^T = \mathbb{L}\mathbb{B} + \mathbb{B}\mathbb{L}^T \implies$$

$$\implies \overset{\triangledown}{\mathbb{B}} := \frac{d\mathbb{B}}{dt} - \mathbb{L}\mathbb{B} - \mathbb{B}\mathbb{L}^T = \mathbb{O}$$

Tvrzení 0.14

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t) := \mathbb{F}(\mathbf{X},t)\left[\frac{d}{dt}(\mathbb{F}^{-1}(\mathbf{X},t)\mathbb{A}(\chi(\mathbf{X},t),t)\mathbb{F}^{-T}(\mathbf{X},t))\right]\mathbb{F}^T(\mathbf{X},t)$$

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t) = -\mathbb{L}\mathbb{A} + \frac{d\mathbb{A}}{dt} - \mathbb{L}\mathbb{A}^T$$

Poznámka
A reasonable definition of time derivative for a tensor field of type "normal vector $\to$ tangent vector" is previous.

Poznámka (Material surface)
$f(\mathbf{x},t) = 0$ ...implicit declaration of surface.

$$\frac{\partial f}{\partial t}(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t)\cdot\nabla_{\mathbf{x}} f(\mathbf{x},t) = 0 \qquad \forall \mathbf{x}, f(\mathbf{x},t) = 0$$

$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = 0$ for $z = g(x,y,t)$ and the velocities $\mathbf{v}$ are computed through this equation.

$\nabla_{\mathbf{x}} f(\mathbf{x},t)ds$ a normal to the surface $\implies \frac{1}{|\nabla_{\mathbf{x}} f|}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\nabla_{\mathbf{x}} f}{|\nabla_{\mathbf{x}} f|} = 0 \implies \mathbf{v}\cdot\mathbf{n} = -\frac{\frac{\partial f}{\partial t}}{|\nabla_{\mathbf{x}} f|}$.

Věta 0.15 (Reynolds transport theorem)

$$\frac{d}{dt}\int_{V(t)} \varphi(\mathbf{x},t)dv = \int_{V(t)}\left(\frac{d\varphi}{dt}(\mathbf{x},t) + \varphi(\mathbf{x},t)\operatorname{div}\mathbf{v}(\mathbf{x},t)\right)dv$$

Důkaz

$$\frac{d}{dt}\int_{V(t_0)} \varphi(\mathbf{x},t)dv = \frac{d}{dt}\int_{V(t_0)} \varphi(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F}(\mathbf{X},t)dV =$$

$$= \int_{V(t_0)} \frac{d}{dt}\left(\varphi(\chi(\mathbf{X},t),t)\det \mathbb{F}(\mathbf{X},t)\right)dV =$$

$$= \int_{V(t_0)} \left[\frac{\partial \varphi}{\partial t}(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t)\cdot \nabla_{\mathbf{x}}\varphi(\mathbf{x},t)|_{\dots} \det \mathbb{F}(\mathbf{X},t) + \varphi(\mathbf{x},t)|_{\dots}\frac{d}{dt}\det \mathbb{F}(\mathbf{X},t)\right] dV =$$

$$= \int_{V(t_0)} \left(\frac{d\varphi}{dt}(\mathbf{x},t) + \varphi(\mathbf{x},t)\operatorname{div}\mathbf{v}(\mathbf{x},t)\right)|_{\mathbf{x}=\chi(\mathbf{X},t)}(\det \mathbb{F})dV =$$

$$= \int_{V(t_0)} \left(\frac{d\varphi}{dt}(\mathbf{x},t) + \varphi(\mathbf{x},t)\operatorname{div}\mathbf{v}(\mathbf{x},t)\right) dv$$

□

TODO?

Definice 0.17 (Displacement)

$$\mathbf{U}(\mathbf{X},t) := \chi(\mathbf{X},t) - \mathbf{X}$$

1 Dynamics

Poznámka (Outline)
Newtons 2nd law for single particle: $m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}$. We want to find a counterpart for continuously distributed matter…

$$\frac{d}{dt}(m\frac{d\mathbf{x}}{dt}) = \mathbf{F}$$

We have $\frac{dm}{dt} = 0$ ($m = \text{const}$). In continuum mechanics we have $\frac{d}{dt}m_{V(t)} = 0$, $m_{V(t)} := \int_{V(t)} \varrho(\mathbf{x},t)dv$ (ϱ = density). Than (from Reynolds transport theorem)

$$0 = \frac{d}{dt}m_{V(t)} = \frac{d}{dt}\int_{V(t)} \varrho(\mathbf{x},t)dv = \int_{V(t)} \left(\frac{d\varrho}{dt}(\mathbf{x},t) + \varrho(\mathbf{x},t) \div_{\mathbf{x}} \mathbf{v}(\mathbf{x},t)\right) dv$$

From localization lemma ($V(t)$ is arbitrary material volume) we get Balance of mass:

$$\frac{d\varrho}{dt}(\mathbf{x},t) + \varrho(\mathbf{x},t) \div_{\mathbf{x}} \mathbf{v}(\mathbf{x},t) = 0.$$

Spring: $m\frac{d^2\mathbf{x}}{dt^2} = -k\mathbf{x} - b\frac{d\mathbf{x}}{dt}$, k is spring stiffness, $-b\dots$ is damping (proportional to the velocity).

$$m\frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d\mathbf{x}}{dt} = -k\mathbf{x}\cdot\frac{d\mathbf{x}}{dt} - b\frac{d\mathbf{x}}{dt}\cdot\frac{d\mathbf{x}}{dt},$$

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 + \frac{k}{2}(\mathbf{x})^2\right) = -b\left(\frac{d\mathbf{x}}{dt}\right)^2,$$

where $(\mathbf{x})^2 = \mathbf{x}\cdot\mathbf{x}$. With $b = 0$ we have conservation of energy, with $b > 0$ we have energy loss through $-b\left(\frac{d\mathbf{x}}{dt}\right)^2$ = dissipation = conversion of mechanical energy into heat.

$$dU = dQ + dW, \qquad dS = \frac{dQ}{T} \qquad \text{(classical equilibrium thermodynamic)}$$

Lemma 1.1

$$\frac{d}{dt}\int_{V(t)}\varrho(\mathbf{x},t)\varphi(\mathbf{x},t)dv = \int_{V(t)}\varrho(\mathbf{x},t)\frac{d\varphi}{dt}(\mathbf{x},t)dv.$$

Důkaz

$$\frac{d}{dt}\int_{V(t)}\varrho\varphi dv \overset{\text{Reynolds}}{=} \int_{V(t)}\left(\frac{d}{dt}(\varrho\varphi) + \varrho\varphi\operatorname{div}\mathbf{v}\right)dv =$$

$$= \int_{V(t)}\left(\frac{d\varrho}{dt}\varphi + \varrho\frac{d\varphi}{dt} + \varrho\varphi\operatorname{div}\mathbf{v}\right)dv \overset{\text{Balance of mass}}{=} \int_{V(t)}\varrho(\mathbf{x},t)\frac{d\varphi}{dt}(\mathbf{x},t)dv.$$

□

Poznámka (Newton second law)

$$\mathbf{p}_{V(t)} := \int_{V(t)}\varrho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv$$

$$LHS = \frac{d}{dt}\mathbf{p}_{V(t)} = \frac{d}{dt}\int_{V(t)}\varrho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv$$

$$RHS = \mathbf{F}_{V(t)} = \underbrace{\mathbf{F}_{body}}_{:=\int_{V(t)}\varrho(\mathbf{x},t)\mathbf{b}(\mathbf{x},t)dv} + \underbrace{\mathbf{F}_{surface}}_{:=\int_{\partial V(t)}\mathbf{t}(t,\mathbf{x})ds}$$

$\mathbf{F}_{surface}$ is called surface faces or contact forces. $\mathbf{t}$ is traction vector.

Poznámka
We will assume that $\mathbf{t} = \mathbf{t}(\mathbf{x}, t, \mathbf{n}(\mathbf{x}, \mathbf{t}))$.

From Newton's 3rd law (action + reaction): $\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = -\mathbf{t}(\mathbf{x}, t, -\mathbf{n})$. Simplest formula that guarantees this is linear, so we will assume (it can be proved with Cauchy tetrahedron argument, which isn't part of this lecture, but proof can be in exam) that $\mathbf{t}$ is linear with respect to $\mathbf{n}$:

$$\mathbf{t}(\mathbf{x}, t, \mathbf{n}) = \mathbb{T}(\mathbf{x}, t)\mathbf{n},$$

$\mathbb{T}$ is tensor matrix $\mathbb{R}^{3\times 3}$ and is called Cauchy stress tensor.

Například

$$\mathbb{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbb{I}, \qquad \text{where } p \text{ is pressure.}$$

With this, we can rewrite Newton's second law:

$$\frac{d}{dt}\int_{V(t)} \varrho \mathbf{v} dx = \int_{V(t)} \varrho \mathbf{b} dv + \int_{\partial V(t)} \mathbb{T}\mathbf{n} ds \overset{\text{Stokes}}{=} \int_{V(t)} \varrho \mathbf{b} dv + \int_{V(t)} (\operatorname{div}_{\mathbf{x}} \mathbb{T}) dv.$$

With Reynolds and Localization lemma we get balance of (linear) momentum (and it's counterpart of N. 2nd law):

$$\varrho \frac{d\mathbf{v}}{dt} = \varrho \mathbf{b} dv + \operatorname{div}_{\mathbf{x}} \mathbb{T}.$$

Poznámka

$$\frac{d}{dt}\left(\frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 + \frac{k}{2}(\mathbf{x})^2\right) = -b\left(\frac{d\mathbf{x}}{dt}\right)^2,$$

$$\underbrace{\frac{d}{dt}\int_{V(t)}\left(\frac{1}{2}\varrho \cdot (\mathbf{v})^2 + \underbrace{\varrho e}_{\text{internal energy density}}\right) dv}_{\text{net total energy}} =$$

$$= \int_{V(t)} \varrho \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial V(t)} \mathbb{T}\mathbf{n} \cdot \mathbf{v} ds - \int_{\partial V(t)} \underbrace{\mathbf{J}_q}_{\text{heat flux vector}} \cdot \mathbf{n} ds$$

$(\mathbf{J}_q := -k\nabla\theta)$

dU is internal energy, dW is work done by internal + surface forces (first two expressions on RHS), dQ is exchange of heat on surface (the third expression)

Stokes theorem, Reynolds transport theorem, Localization lemma $\implies$ First law of thermodynamics:

$$\varrho \frac{d}{dt}\left(\frac{1}{2}(\mathbf{v})^2 + e\right) = \varrho \mathbf{b} \cdot \mathbf{v} - \operatorname{div} \mathbf{J}_q + \operatorname{div}(\mathbb{T}^T \mathbf{v})$$

With $\varrho \frac{d}{dt}(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = (\operatorname{div} \mathbb{T}) \cdot \mathbf{v} + \varrho \mathbf{b} \cdot \mathbf{v}$

$$\varrho \frac{de}{dt} = \operatorname{div}(\mathbb{T}^T \mathbf{v}) - (\operatorname{div} \mathbb{T}) \cdot \mathbf{v} - \operatorname{div} \mathbf{J}_q$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q, \qquad \mathbb{L} := \nabla_{\mathbf{x}} \mathbf{v}.$$

Definice 1.1 (Entropy)

Basic feat: Continuous medium in an isolated vessel: $\mathbf{v} \to 0$, $\theta \to$ const (θ = temperature field).

Isolated = no mechanical energy exchange with outside environment ($\mathbf{b} = \mathbf{o}$, $\mathbf{v}|_{\partial\Omega} = \mathbf{o}$) and no heat flux through boundary ($\mathbf{J}_q \cdot \mathbf{n}|_{\partial\Omega} = 0$)

We want a quantity that increases in time! Let denote it η. Assume $\eta = \eta(e, \varrho)$ and $e = e(\eta, \varrho)$. Chain rule:

$$\frac{de}{dt} = \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} + \frac{\partial e}{\partial \varrho} \frac{d\varrho}{dt}$$

We know balance of internal energy $\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q$. So

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} + \varrho \frac{\partial e}{\partial \varrho} \frac{d\varrho}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q - \varrho \frac{\partial e}{\partial \varrho} \frac{d\varrho}{dt}$$

$$\varrho \frac{\partial e}{\partial \eta}(\eta, \varrho) \frac{d\eta}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q - \varrho^2 \frac{\partial e}{\partial \varrho} \operatorname{div} \mathbf{v}$$

$$\varrho \frac{d\eta}{dt} = \frac{\mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{J}_q - \varrho^2 \frac{\partial e}{\partial \varrho} \operatorname{div} \mathbf{v}}{\frac{\partial e}{\partial \eta}(\eta, \varrho)}$$

This is point-wise equation and it is too demanding. So let us work with ret quantities:

$$\frac{d}{dt} \int_{V(t)} \varrho \eta = \int_{V(t)} \frac{1}{\frac{\partial e}{\partial \eta}(\eta, \varrho)} \left(-p + \varrho^2 \frac{\partial e}{\partial \varrho} \right) \operatorname{div} \mathbf{v} dv + \int_{V(t)} \frac{\operatorname{div}(k \nabla \theta)}{\frac{\partial e}{\partial \eta}(\eta, \varrho)} dv$$

Let us try to fix $\varrho^2 \frac{\partial e}{\partial \varrho}(\eta, \varrho) = p$:

$$\frac{d}{dt} \int_{V(t)} \varrho \eta = 0 + \int_{V(t)} \frac{\operatorname{div}(k \nabla \theta)}{\frac{\partial e}{\partial \eta}(\eta, \varrho)} dv$$

$$\frac{d}{dt} \int_{V(t)} \varrho \eta = \int_{V(t)} \operatorname{div} \left(\frac{k \nabla \theta}{\frac{\partial e}{\partial \eta}(\eta, \varrho)} \right) dv - \int_{V(t)} k \nabla \theta \cdot \nabla \left(\frac{1}{\frac{\partial e}{\partial \eta}} \right) dv$$

From boundary condition $k \nabla \theta \cdot \mathbf{n}|_{\partial\Omega} = 0$ and stokes theorem

$$\frac{d}{dt} \int_{V(t)} \varrho \eta = 0 - \int_{V(t)} k \nabla \theta \cdot \nabla \left(\frac{1}{\frac{\partial e}{\partial \eta}} \right) dv$$

$$\frac{d}{dt}\int_{V(t)} \varrho\eta = -\int_{V(t)} \frac{k\nabla\theta\cdot\nabla\left(\frac{\partial e}{\partial\eta}(\eta,\varrho)\right)}{\left(\frac{\partial e}{\partial\eta}(\eta,\varrho)\right)^2} dv.$$

Let us try to fix $\frac{\partial e}{\partial\eta}(\eta,\varrho) = \theta \implies \frac{d}{dt}\int_{V(t)} \varrho\eta \geqslant 0$.

Summary: Solve $\frac{\partial e}{\partial\eta}(\eta,\varrho) = \theta$ and $\varrho^2\frac{\partial e}{\partial\varrho}(\eta,\varrho) = p$ and we will get $\eta = \eta(e,\varrho)$ such that $\frac{d}{dt}\int_{V(t)} \varrho\eta \geqslant 0$. Function η is called the entropy density.

Poznámka (Experimental facts)

$$p(\varrho,\theta) = c_V(\gamma-1)\varrho\theta \qquad \text{(something like } \frac{PV}{T} = \text{const)}$$

$$e(\varrho,\theta) = c)V\theta$$

$$c_V = \text{const}, \qquad \gamma = \text{const}$$

c_V = specific heat at constant volume.

Poznámka (c_V)

$$\frac{de}{dt} = \frac{\partial e}{\partial\varrho}\frac{d\varrho}{dt} + \frac{\partial e}{\partial\theta}\frac{d\theta}{dt}$$

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div}\mathbf{J}_q = -p\operatorname{div}\mathbf{v} + \operatorname{div}(\varkappa\nabla\theta)$$

Constant volume $\implies \frac{d\varrho}{dt} = 0$ and $\operatorname{div}\mathbf{v} = 0$

$$\text{change of temperature } = \int_{V(t)} \varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt} = \int_{V(t)} \operatorname{div}(\varkappa\nabla\theta)dv = \text{ heat exchange}$$

Poznámka (Balance of angular momentum)

$$(m\frac{d\mathbf{x}^2}{dt^2} = \mathbf{F} \qquad \times\,\mathbf{x} \qquad m\frac{d\mathbf{x}^2}{d\mathbf{t}}\times\mathbf{x} = \mathbf{F}\times\mathbf{x}.)$$

Primitive equation:

$$\int_{V(t)} \varrho\frac{d\mathbf{v}}{dt}\times\mathbf{x}dv = \int_{V(t)} \varrho\mathbf{b}\times\mathbf{x} + \underline{\int_{\partial V(t)} \mathbb{T}\mathbf{n}\times\mathbf{x}ds}.$$

We already know that

$$\varrho\frac{d\mathbf{v}}{dt} = \varrho\mathbf{b} + \operatorname{div}\mathbb{T}$$

$$\int_{V(t)} \varrho\frac{d\mathbf{v}}{dt}\times\mathbf{x}dv = \int_{V(t)} \varrho\mathbf{b}\times\mathbf{x}dv + \underline{\int_{V(t)} \operatorname{div}\mathbb{T}\times\mathbf{x}dv}$$

Is underlined parts equivalent?

$$\int_{\partial V(t)} \mathbb{T}\mathbf{n} \times \mathbf{x} ds = \int_{V(t)} \operatorname{div} \mathbb{T} \times \mathbf{x} dv.$$

$$LHS = \int_{\partial V(t)} \varepsilon_{ijk} T_{jl} n_l x_k ds = \int_{\partial V(t)} (\varepsilon_{ijk} T_{jl} x_k) \cdot n_l ds \overset{\text{Stokes}}{=} \int_{V(t)} \frac{\partial}{\partial x_l} (\varepsilon_{ijk} T_{jl} x_k) dv =$$

$$= \int_{V(t)} \varepsilon_{ijk} \frac{\partial T_{jl}}{\partial x_l} x_k + \int_{V(t)} \varepsilon_{ijk} T_{jl} \underbrace{\frac{\partial x_k}{\partial x_l}}_{\delta_{kl}} dv = \int_{V(t)} (\operatorname{div} \mathbb{T} \times \mathbf{x}) dv + \int_{V(t)} \varepsilon_{ijl} T_{jl} dv.$$

LHS is equal to RHS (balance of angular momentum holds) provided that

$$\int_{V(t)} \varepsilon_{ijl} \mathbb{T}_{jl} dv = 0.$$

This holds provided that $\mathbb{T}$ is symmetric matrix.

Balance of linear momentum + symmetry of $\mathbb{T} \implies$ balance of angular momentum holds. So from now we work with symmetric $\mathbb{T}$. (There are parts of continuum mechanics, where this doesn't hold (especially some electric continuum), but we continue in our part.)

Poznámka (Eulerian description)
So far we have ($\varrho(\mathbf{x}, t)$, $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$)

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \qquad \varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q, \qquad \mathbb{T} = \mathbb{T}^T.$$

Poznámka (Lagragian description)
($\varrho(\mathbf{X}, t)$) Balance of mass:

$$\frac{d}{dt} \int_{V(t)} \varrho(\mathbf{x}, t) dv = 0 \implies \int_{V(t)} \varrho(\mathbf{x}, t) dv = \int_{V(t_0)} \varrho_R(\mathbf{X}) dV$$

$$\int_{V(t)} \varrho(\chi(\mathbf{X}, t), t) \det \mathbb{F}(\mathbf{X}, t) dV = \int_{V(t_0)} \varrho_R(\mathbf{X}) dV$$

$$\overset{\text{Localization lemma}}{\varrho} (\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F}(\mathbf{X}, t) = \varrho_R(\mathbf{X}).$$

Balance of momentum:

$$\int_{V(t)} \varrho(\mathbf{x}, t) \frac{d\mathbf{v}}{dt}(\mathbf{x}, t) dv = \int_{V(t)} \operatorname{div}_{\mathbf{x}} \mathbb{T}(\mathbf{x}, t) dv + \int_{V(t)} \varrho(\mathbf{x}, t) \mathbf{b}(\mathbf{x}, t) dv$$

$$\int_{V(t_0)} \varrho(\chi(\mathbf{X}, t), t) \frac{d\mathbf{v}}{dt}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F}(\mathbf{X}, t) dv = \int_{\partial V(t)} \mathbb{T}(\mathbf{x}, t) \mathbf{n}(\mathbf{x}, t) ds + \int_{V(t_0)} \varrho(\chi(\mathbf{x}, t), t) \mathbf{b}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathrm{d}$$

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) dV = \int_{\partial V(t_0)} (\det \mathbb{F}(\mathbf{X},t)) \mathbb{T}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbb{F}^{-T}(\mathbf{X},t)\mathbf{N}(\mathbf{X}) dS + \int_{V(t_0)} \varrho_R(\mathbf{X}) \mathbf{b}(\chi(\mathbf{X},t),t) d$$

First Piola-Kirchhoff stress tensor: $\mathbb{T}_R(\mathbf{X},t) := \det \mathbb{F}(\mathbf{X},t)\mathbb{T}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbb{F}^{-T}(\mathbf{X},t)$.

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) dV = \int_{\partial V(t_0)} \mathbb{T}_R(\mathbf{X},t)\mathbf{N} dS + \int_{V(t_0)} \varrho_R(\mathbf{X})\mathbf{b}(\chi(\mathbf{X},t),t) dV$$

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) dV = \int_{V(t_0)} \operatorname{div}_{\mathbf{X}} \mathbb{T}_R(\mathbf{X},t) dV + \int_{V(t_0)} \varrho_R(\mathbf{X})\mathbf{b}(\chi(\mathbf{X},t),t) dV$$

$$\varrho_R(\mathbf{X}) \frac{\partial^2 \chi}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbb{T}_R dV + \varrho_R \mathbf{b}.$$

Internal energy:

$$\int_{V(t)} \varrho \frac{de}{dt} dV = \int_{V(t)} \mathbb{T} : \mathbb{L} dv - \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds$$

$\mathbb{T} : \mathbb{L} = \operatorname{tr}(\mathbb{T}\mathbb{L}^T) = \operatorname{tr}((\frac{1}{\det \mathbb{F}}\mathbb{T}_R\mathbb{F}^T)(\mathbb{F}^{-T}\frac{d\mathbb{F}^T}{dt})) = \frac{1}{\det \mathbb{F}}\mathbb{T}_R : \frac{d\mathbb{F}}{dt}$.

$$\int_{V(t_0)} \varrho_R \frac{de}{dt} dv = \int_{V(t)} \frac{1}{\det \mathbb{F}} \mathbb{T}_R : \frac{d\mathbb{F}}{dt} dv - \int_{\partial V(t_0)} \mathbf{j}_q \cdot (\det \mathbb{F})\mathbb{F}^{-T}\mathbf{N} dS$$

$$\int_{V(t_0)} \varrho_R \frac{de}{dt} dv = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV - \int_{V(t_0)} \operatorname{div}_{\mathbf{X}}((\det \mathbb{F})\mathbb{F}^{-1}\mathbf{j}_q) dV$$

Refferential heat flux: $\mathbf{J}_q(\mathbf{X},t) := (\det \mathbb{F}(\mathbf{X},t))\mathbb{F}^{-1}(\mathbf{X},t)\mathbf{j}_q(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}$

$$\int_{V(t_0)} \varrho_R \frac{de}{dt} dv = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV - \int_{V(t_0)} \operatorname{div}_{\mathbf{X}} \mathbf{J}_q dV$$

$$\varrho_R \frac{de}{dt} = \mathbb{T}_R : \dot{\mathbb{F}} - \operatorname{div}_{\mathbf{X}} \mathbf{J}_q.$$

Angular momentum:

$$\mathbb{T}_R\mathbb{F}^T = \mathbb{F}\mathbb{T}_R^T$$

Poznámka (How to find a formula for the Cauchy stress tensor?)
Solids: $\mathbb{T} = f(\text{deformation})$, but χ contains also rotations or translations. $\mathbb{T} = f(\nabla\chi)$, but it still contains rotations. $\mathbb{T} = f(\mathbb{F}\mathbb{F}^T - \mathbb{I}) = f(\mathbb{B} - \mathbb{I})$. $\mathbf{x} = \mathbb{Q}\mathbf{X} \implies \mathbb{F} = \mathbb{Q} \implies \mathbb{F}^T\mathbb{F} = \mathbb{I}$. So

$$\mathbb{T} = f(\mathbb{B}).$$

Can we formulate an evolution equation for $\mathbb{B}$? Yes, we did it long time ago:

$$\overset{\triangledown}{\mathbb{B}} = \mathbb{O}, \qquad \overset{\triangledown}{\mathbb{B}} := \frac{d\mathbb{B}}{dt} - \mathbb{L}\mathbb{B} - \mathbb{B}\mathbb{L}^T.$$

We add this to equations we have for Eulerian description and solve them for $\varrho, \mathbf{v}, \mathbb{B}, e$ at points $\mathbf{x} \in V(t) = \chi(V(t_0))$, so we must find χ, but we may find it a posteriori after solving equations.