Úvod

Poznámka (Organizační úvod)
K ukončení předmětu je třeba pouze udělat zkoušku: 2 příklady na definice, 2 věta-důkaz.

Literatura:

- L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.
- W. Rudin, Analýza v reálném a komplexním oboru, Academia, 2003.

1 Differentiation of measures

1.1 Covering theorems

Definice 1.1 (Vitali cover)

Let $A \subset \mathbb{R}^n$ we say that a system $\mathcal{V}$ consisting of closed balls from $\mathbb{R}^n$ forms Vitali cover of A, if

$$\forall x \in A \ \forall \varepsilon > 0 \exists B \in \mathcal{V} : x \in B \wedge \operatorname{diam} B < \varepsilon.$$

Definice 1.2 (Notation)

λ_n is Lebesgue measure on $\mathbb{R}^n$. λ_n^* is outer Lebesgue measure on $\mathbb{R}^n$. If $B \subset \mathbb{R}^n$ is a ball and $\alpha > 0$, then $\alpha \cdot B$ stands for the ball, which is concentric with B and with α-times greater radius than B.

Věta 1.1 (Vitali)

Let $A \subset \mathbb{R}^n$ and $\mathcal{V}$ be a system of closed balls forming a Vitali cover of A. Then there exists a countable disjoint subsystem $\mathcal{A} \subseteq \mathcal{V}$ such that $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$.

Důkaz
First assume that A is bounded. Take an open bounded set $G \subset \mathbb{R}^n$ with $A \subset G$. We set

$$\mathcal{V}^* = \{B \in \mathcal{V} | V \subset G\}.$$

Then $\mathcal{V}^*$ is a Vitali cover of A. If there exists a finite disjoint subsystem of $\mathcal{V}^*$ covering A, we are done. So Assume that there is no such subsystem. Mathematical induction:

First step: We set $s_1 = \sup \{\operatorname{diam} B | B \in \mathcal{V}^*\}$. We choose a ball $B_1 \in \mathcal{V}^*$ such that $B_1 > \frac{1}{2} s_1$.

k-th step: Suppose that we have already constructed balls $B_1, B_2, \ldots, B_{k-1}$. We set

$$s_k = \sup \left\{ \operatorname{diam} B | B \in \mathcal{V}^* \wedge B \cap \bigcup_{i=1}^{k-1} B_i = \emptyset \right\}.$$

We find $B_k \in \mathcal{V}^*$ such that $\operatorname{diam} B_k > \frac{1}{2} s_k > 0$, $B_k \cap \bigcup_{i=1}^{k-1} B_i = \emptyset$.

Let $\mathcal{A} = \{B_k | k \in \mathbb{N}\}$. It is disjoint, it is countable, it holds $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$:

$$\sum_{i=1}^{\infty} \lambda_n(B_i) = \lambda_n(\bigcup_{i=1}^{\infty} B_i) \leqslant \lambda_n(G) < \infty \implies$$
$$\implies \lim_{i \to \infty} = 0 \implies \lim_{i \to \infty} \operatorname{diam}(B_i) = 0 \implies \lim_{i \to \infty} s_i = 0.$$

We show that

$$\forall x \in A \setminus \bigcup \mathcal{A} \ \forall i \in \mathbb{N} \exists j \in \mathbb{N}, j > i : x \in 5 \cdot B_j$$
$$\Leftrightarrow A \setminus \bigcup \mathcal{A} \subseteq \bigcup_{j=i+1}^{\infty} 5 \cdot B_j$$

Take $x \in A \setminus \bigcup \mathcal{A}$ and $i \in \mathbb{N}$. Denote $\delta = \operatorname{dist}(x, \bigcup_{k=1}^{i} B_k) > 0$. There exists $B \in \mathcal{V}^*$ such that $x \in B$ and $\operatorname{diam} B < \delta \implies B \cap \bigcup_{k=1}^{i} B_k = \emptyset$. Then we have $\operatorname{diam} B > s_p$ for some $p \in \mathbb{N}$.

Therefore there exists $j > i$ with $B_j \cap B \neq \emptyset$. Let j be the smallest number with this property. Then we have $s_j \geqslant \operatorname{diam} B$ since $B \cap \bigcup_{l=1}^{j-1} B_l = \emptyset$. Further we have $\operatorname{diam} B_j > \frac{1}{2} s_j \geqslant \frac{1}{2} \operatorname{diam} B \implies 2 \operatorname{diam} B_j \geqslant \operatorname{diam} B$ This implies that $x \in B \subset 5 \cdot B_j$.

$$\lambda_n^*(A \setminus \bigcup A) \leqslant \lambda_n \left(\bigcup_{j=i+1}^{\infty} 5 \cdot B_j \right) \leqslant \sum_{j=i+1}^{\infty} \lambda_n(5 \cdot B_j) = \sum_{j=i+1}^{\infty} 5^n \lambda_n(B_j) = 5^n \cdot \sum_{j=i+1}^{\infty} \lambda_n(B_j) \to 0 \implies \lambda_n(A \setminus \bigcup$$

General case (A not bounded): Let $(G_j)_{j=1}^{\infty}$ be a sequence of disjoint open sets such that $\lambda_n(\mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} G_j) = 0$. We define $\mathcal{V}_j = \{B \in \mathcal{V}_i, B \subseteq G_j\}$. $\mathcal{V}_j$ is a Vitali cover of $A \cap G_j$ $\implies \exists \mathcal{A}_j \subseteq \mathcal{V}_j$ countable disjoint and $\lambda_n(A \cap G_j \setminus \bigcup A_j) = 0$. We set $\mathcal{A} = \bigcup_{j=1}^{\infty} \mathcal{A}_j$. $\mathcal{A}$ is countable, disjoint and $\lambda_n(A \setminus \bigcup \mathcal{A}) = 0$. □

Definice 1.3

We say that a measure μ on $\mathbb{R}^n$ satisfies Vitali theorem, if for every Vitaly cover $\mathcal{V}$ of $M \subseteq \mathbb{R}^n$ there exists a disjoint countable $\mathcal{A} \subset \mathcal{V}$ with $\mu(M \setminus \bigcup \mathcal{A}) = 0$.

Poznámka
If μ satisfies Vitali theorem and $\nu \ll \mu$, then ν satisfies Vitali theroem.

Věta 1.2

Set $E \subset \mathbb{R}^n$ be Lebesgue measurable and $\mathcal{S}$ be a finite system of closed balls covering E. Then there exists a disjoint system $\mathcal{L} \subset \mathbb{S}$ such that $\lambda_n(E) \leqslant 3^n \cdot \sum_{B \in \mathcal{L}} \lambda_n(B)$.

Důkaz
WLOG $\mathcal{S} \neq \varnothing$. SUppose $B_1 \in \mathcal{S}$ with maximal radius among balls from $\mathcal{S}$.

Suppose that we have already constructed $B_1, \ldots, B_{k-1} \in \mathcal{S}$. If possible, choose $B_k \in \mathcal{S}$ disjoint with $\bigcup_{j<k} B_j$ and with maximal radius among balls satisfying this property.

We set $\mathcal{L} = \{B_1, \ldots, B_N\}$. We show $E \subseteq \bigcup_{B \in \mathcal{L}} 3 * B = \bigcup_{i=1}^N 3 * B_i$. $x \in E$. Find $B \in \mathcal{S}$ with $x \in B$. Find smallest k with $B \cap B_k \neq \varnothing$. This means $\operatorname{rad}(B) \leqslant \operatorname{rad}(B_k) \implies x \in B \subseteq 3 * B_k$.

Now $\lambda_n(E) \leqslant \lambda_n \left(\bigcup_{i=1}^N 3 * B_i \right) \leqslant \sum_{i=1}^N \lambda_n(3 * B_i) = 3^n \sum_{i=1}^N \lambda_n(B_i)$. □

Věta 1.3 (Besicovitch theorem)

For each $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ with the following property:

If $A \subset \mathbb{R}^n$ and $\Delta : A \to (0, \infty)$ is a bounded function, then there exist sets A_1, …, A_N $\subseteq A$ such that

- $\left\{ \overline{B}(x, \Delta x) | x \in A_j \right\}$ *is disjoint for every $j \in [N]$;*
- $A \subset \bigcup \left\{ \overline{B}(x, \Delta x) | x \in \bigcup_{i=1}^N A_i \right\}$.

Důkaz (Case A is bounded)
Let $R := \sup_A \Delta$. Choose $B_1 := \overline{B}(a_1, \Delta(a_1))$ such that $a_1 \in A$ and $r_1 := \Delta(a_1) > \frac{3}{4}R$.

Assume that we already constructed $B_1, \dots, B_{j-1}$, $j \geqslant 2$. $B_{j-1} = \overline{B}(a_{j-1}, \Delta(a_{j-1})) = \overline{B}(a_{j-1}, r_{j-1})$. Let $F_j := A \setminus \bigcup_{i=1}^{j-1} B_i$. If $F_j = \varnothing$ we set $J := j$. If not $B_j := \overline{B}(a_j, \Delta(a_j)) = \overline{B}(a_j, r_j)$, $a_j \in F_j$ and $r_j > \frac{3}{4} \sup_{F_j} \Delta$.

If $F_j \neq \varnothing$ for every $j \in \mathbb{N}$, then we set $J := \infty$. So we have $(B_j)_{j<J}$. If $J < \infty$, then we covered A. „If $J = \infty$, then $A \subset \bigcup_{j<J} B_j$“:

„$\lim_{j\to\infty} r_j = 0$“: because A is bounded

$$\|a_i - a_j\| \geqslant r_i = \frac{1}{3}r_i + \frac{2}{3}r_i > \frac{1}{3}r_i + \frac{1}{2}r_j > \frac{1}{3}r_i + \frac{1}{3}r_j = \frac{1}{3}(r_i + r_j) \implies \frac{1}{3} * B_i \cap \frac{1}{3} * B_j = \varnothing.$$

$\left\{\frac{1}{3}B_j | j < J\right\}$ is a disjoint family $\implies \sum_{j=1}^{\infty} \lambda_n(\frac{1}{3} * B_j) < \infty$.

If $A \in A \setminus \bigcup_{j=1}^{\infty} B_j$, then $a \in \bigcap_{j=1}^{\infty} F_j$. We find $j_0 \in \mathbb{N}$ with $r_{j_0} \leqslant \frac{3}{4}\Delta(a)$. ↯.

Fix $k < J$. We set $I = \{i < k | B_i \cap B_k \neq \varnothing\}$, $I_1 = \{i < k_i | B_i \cap B_k \neq \varnothing \wedge r_i < 10 r_k\}$, $I_2 = \{i < k_i | B_i \cap B_k \wedge r_i \geqslant 10 r_k\}$. The estimate of I_1: „We have $\frac{1}{3}B_i \subseteq 15 * B_k$ for every $i \in I_1$“: Take $x \in \frac{1}{3} * B_i$. Then

$$\|x - a_k\| \leqslant \|x - a_j\| + \|a_i - a_k\| \leqslant \frac{1}{3}r_i + r_i + r_k \leqslant \frac{10}{3}r_k + 10r_k + r_k \leqslant 15 r_k$$

$$\lambda_n(\frac{1}{3} * B_i) = \lambda(\overline{B}(0,1)) \cdot (\frac{1}{3}r_i)^n \geqslant \lambda_n(\overline{B}(0,1)) \cdot (\frac{1}{3} \cdot \frac{3}{4}r_k)^n = \lambda_n(\overline{B}(0,1)) \cdot \frac{1}{4^n}r_k^n =$$

$$= \frac{1}{60^n}\lambda_n(15 * B_k) \implies |I_1| \leqslant 60^n.$$

Denote $b_i = a_i - a_k$, vector between centers of balls. Take a family $\{Q_m | 1 \leqslant m \leqslant (22n)^n\}$ of closed cubes with edge length $\frac{1}{11n}$ which cover $[-1,1]^n$. We claim that „for each $1 \leqslant m \leqslant (22n)^n$ there is at most one $i \in I_2$ with $\frac{b_i}{\|b_i\|} \in Q_m$“:

$$i, j \in I_2, i < j, \left\| \frac{b_i}{\|b_i\|} - \frac{b_j}{\|b_j\|} \right\| \leqslant \frac{1}{11}.$$

We have $r_i < \|b_i\| \leqslant r_i + r_k$ and $r_j < \|b_j\| \leqslant r_j + r_k$. So $|\|b_i\| - \|b_j\|| \leqslant |r_i - r_j| + r_k$. $\|b_j\| \leqslant r_j + r_k \leqslant r_j + \frac{1}{10}r_j = \frac{11}{10}r_j$.

$$\|a_i - a_j\| = \|b_i - b_j\| \leqslant \left\| b_i - \frac{\|b_j\|}{\|b_i\|}b_i \right\| + \left\| \frac{\|b_j\|}{\|b_i\|}b_i - b_j \right\| \leqslant |\|b_i\| - \|b_j\|| + \frac{1}{11}\|b_j\| \leqslant$$

$$\leqslant |r_i - r_j| + r_k + \frac{1}{11} \cdot \frac{11}{10}r_j \leqslant |r_i - r_j| + \frac{1}{5}r_j.$$

We distinguish two cases:

$$(1) r_i > r_j : \|a_i - a_j\| \leqslant r_i - \frac{4}{5}r_j < r_i;$$
$$4$$
$$(2) r_i \leqslant r_j : \|a_i - a_j\| \leqslant -r_i + r_j + \frac{1}{5}r_j = -r_i + \frac{6}{5}r_j \leqslant -r_i + \frac{8}{5}r_i < r_i \implies a_j \in \overline{B}(a_i, r_i) = B_i, ↯.$$

Důkaz (Case A is not bounded)
Let $A^l := A \cap \{x \in \mathbb{R}^n | 3(l-1)R \leqslant \|x\| < 3lR\}$, $l \in \mathbb{N}$. We get A_i^l, $i \in [M]$ by the previous. $A_i = \bigcup_{l=2k+1} A_i^l$, $A_{M+i} = \bigcup_{l=2k} A_i^l$. □

Definice 1.4 (Radon measure)

Let P be a locally compact Hausdorff space and $\mathcal{S}$ a σ-algebra of subsets of P. We say that μ is a Radon measure if

- $\mathcal{S}$ contains all Borel sets,
- $\mu(K) < \infty$ for every compact $K \in P$,
- $\mu(G) = \sup\{\mu(K) | K \subset G \text{ is compact}\}$ for every $G \subset P$ open,
- $\mu(A) = \inf\{\mu(K) | A \subset G \text{ is open}\}$ for every $A \in \mathcal{S}$,
- μ is complete.

Lemma 1.4

Let μ be a measure on X and $\{A_j\}_{j=1}^{\infty}$ be an increasing sequence of subsets of X. Then $\lim \mu^*(A_j) = \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right)$.

Věta 1.5

Let μ be a Radon measure on $\mathbb{R}^n$ and $\mathcal{F}$ be a collection of closed balls in $\mathbb{R}^n$. Let A denote the set of centers of balls in $\mathcal{F}$. Assume $\inf\{r | B(a,r) \in \mathcal{F}\} = 0$ *for each $a \in A$. Then there exists a countable disjoint system $\mathcal{G} \subset \mathcal{F}$ such that $\mu(A \setminus \bigcup \mathcal{G}) = 0$.*

Důkaz (The case $\mu^*(A) < \infty$)
Let $N \in \mathbb{N}$ be the constant from Besicovitch theorem. We find Θ such that $1 - \frac{1}{N} < \Theta < 1$. Claim: „Let $U \subset \mathbb{R}^n$ be an open set. Then there exists a disjoint finite system $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset U$ and

$$\mu^*((A \cap U) \setminus \bigcup \mathcal{H}) \leqslant \Theta \cdot \mu^*(A \cap U).$$

“

$$\mathcal{F}_1 \subset \mathbb{F}, \mathbb{F}_1 = \{B \in \mathbb{F}, \operatorname{diam} B < 1 \wedge B \subset U\}$$

□