1 Area formula and coarea formula

Věta 1.1

Let (P_1, ϱ_1), (P_2, ϱ_2) be metric spaces, $s > 0$, and $f : P_1 \to P_2$ be β-Lipschitz. Then $\varkappa^s(f(P_1)) \leqslant \beta^s \varkappa^s(P_1)$.

Důkaz

Choose $\delta > 0$. Let $P_1 = \bigcup_{i=1}^{\infty} A_j$, $\operatorname{diam} A_j < \delta$. Then we have $f(P_1) = \bigcup_{j=1}^{\infty} f(A_j)$, $\operatorname{diam} f(A_j) < \beta \cdot \delta$.

$$\varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \sum_{j=1}^{\infty} (\operatorname{diam} f(A_j))^s \leqslant \sum_{j=1}^{\infty} \beta^s \cdot (\operatorname{diam} A_j)^s = \beta^s \cdot \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s.$$

It holds for all possible choices of (A_j), so we can take infimum:

$$\varkappa^s(f(P_1)) \leftarrow \varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \beta^s \inf_{(A_j)} \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s = \beta^s \varkappa^s(P_1, \delta) \to \beta^s \varkappa^s(P_1).$$

□

Lemma 1.2

Let $k, n \in \mathbb{N}$, $k \leqslant n$, and $L : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear mapping. Then for every λ_k-measurable set $A \subset \mathbb{R}^k$ it holds $H^k(L(A)) = \sqrt{\det(L^T L)} \lambda_k(A)$.

Důkaz ($\dim L(\mathbb{R}^k) = k$)

We find linear isometry Q of $\mathbb{R}^k$ onto $L(\mathbb{R}^k)$, from last semester

$$H^k(L(A)) = H^k(Q^{-1} \circ L(A)) = \lambda^k(Q^{-1} \circ L(A)) = |\det(Q^{-1}L)| \cdot \lambda_k(A).$$

$$(\det(Q^{-1}L))^2 = \det((Q^{-1}L)^T) \cdot \det(Q^{-1}L) = \det((Q^{-1}L)^T \cdot (Q^{-1}L)) = \det((\langle Q^{-1}Le^i, Q^{-1}L^T e^j \rangle)_{i,j}).$$

And because Q is isometry ($\implies$ Q^{-1} is isometry), we can remove Q^{-1} from scalar product and we get $\det(L^T L)$. □

Lemma 1.3

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood V of the point x such that

- *the mapping $y \mapsto \varphi(\varphi'(x)^{-1}(y))$ is β-Lipschitz on $\varphi'(x)(V)$;*
- *the mapping $z \mapsto \varphi'(x)(\varphi^{-1}(z))$ is β-Lipschitz on $\varphi(V)$.*

Důkaz
x, β fixed. We know, that there exists $\eta > 0$ such that

$$\forall v \in \mathbb{R}^k : \|\varphi'(x)(v)\| \geqslant \eta \cdot \|v\|.$$

We find $\varepsilon \in \left(0, \frac{1}{2}\eta\right)$ such that $\frac{2\varepsilon}{\eta} + 1 < \beta$. We find a neighbourhood V of x such that $\forall y \in V : \|\varphi'(x) - \varphi'(y)\| \leqslant \varepsilon$.

We show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leqslant \varepsilon\|u - v\|.$$

Fix $v \in V$ and consider the mapping

$$g : w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v).$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$:

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| = \|g(w) - g(v)\| \leqslant \sup\{\|g'(w)\| \mid w \in V\} \cdot \|u - v\| \leqslant \varepsilon \cdot \|u - v\|.$$

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

For $u - v \in V$ we compute

$$\|\varphi(u) - \varphi(v)\| \geqslant -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \geqslant -\varepsilon\|u - v\| + \eta\|u - v\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

„First point“: TODO (řádek nebyl k přečtení)

$$\|\varphi(\varphi^{-1}(x)(a)) - \varphi(\varphi^{-1}(x)(b))\| = \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \|phi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|\varphi'(x)(y - v)\| \leqslant \varepsilon\frac{1}{\eta}\|a - b\| + \|a - b\| = \left(\frac{\varepsilon}{\eta} + 1\right)\|a - b\| \leqslant \beta \cdot \|a - b\|.$$

„Second point“: $k, q \in \varphi(V)$. We find $u, v \in V$ such that $\varphi(u) = p$ and $\varphi(v) = q$:

$$\|\varphi'(x)(\varphi^{-1}(p)) - \varphi'(x)(\varphi^{-1}(q))\| = \|\varphi'(x)(u) - \varphi'(x)(v)\| =$$

$$= \|\varphi'(x)(u - v)\| \leqslant \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|p - q\| \leqslant \frac{2\varepsilon}{\eta}\|\varphi(u) - \varphi(v)\| + \|p - q\| = \left(\frac{2\varepsilon}{\eta} + 1\right)\|p - q\| \leqslant \beta\|p - q\|.$$

$\square$

Lemma 1.4

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\alpha > 1$. Then there exists a neighbourhood of x such that for every λ^k-measurable $E \subset V$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

Důkaz

Find $\beta > 1$, $\tau > 1$ such that $\beta^k \tau < \alpha$. By previous lemma we find a neighbourhood V_1 of x such that the conclusion of the lemma holds for β. We find a neighbourhood V_2 of x such that

$$\forall t \in V_2 : \tau^{-1} \operatorname{vol} \varphi'(t) \leqslant \operatorname{vol} \varphi'(t) \leqslant \tau \operatorname{vol} \varphi'(x).$$

Set $V = V_1 \cap V_2$.

Assume that $E \subset V$ is a λ^k-measurable set. We have

$$\tau^{-1} \operatorname{vol} \varphi'(x) \cdot \lambda^k(E) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau \operatorname{vol} \varphi'(t) \lambda^k(E).$$

By lemma above we have $\operatorname{vol} \varphi'(t) \lambda^k(E) = H^k(\varphi'(x)(E))$:

$$\tau^{-1} H^k(\varphi'(x)(E)) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau H^k(\varphi'(x)(E)).$$

By previous lemma we get

$$H^k(\varphi(E)) = H^k \left(\left(\varphi \circ (\varphi'(x))^{-1} \circ \varphi'(x) \right) (E) \right) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant$$

$$\leqslant \beta^k \tau \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

By lemma above we get

$$H^k(\varphi(E)) \geqslant \beta^{-k} H^k \left(\left(\varphi'(x) \circ \varphi^{-1} \circ \varphi \right) (E) \right) = \beta^{-k} H^k(\varphi'(x)(E)) \geqslant$$

$$\geqslant \beta^{-k} \tau^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \geqslant \alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

□

Věta 1.5

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping and $f : \varphi(G) \to \mathbb{R}$ be H^k-measurable. Then we have

$$\int_{\varphi(G)} f(x) dH^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t),$$

if the integral at the right side converges.

Důkaz

„φ^{-1} is well defined“: If $H \subset G$ is open, then we can write $H = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact for every $n \in \mathbb{N}$. Then we have $\varphi(H) = \bigcup_{n=1}^{\infty} \underbrace{\varphi(K_n)}_{\text{compact}}$ is F_σ. This implies that φ^{-1} is Borel. The mappings φ, φ^{-1} are locally Lipschitz by lemma above. ($\varphi(G)$ is Borel.) $\varphi(G)$ is H^k-σ-finite.

1. „$f = \chi_L$, $L \subset \varphi(G)$ is H^k-measurable“: We show $H^k(L) = \int_{\varphi^{-1}(L)} \varphi'(t) d\lambda^k(t)$. Choose $\alpha > 1$. By previous lemma we find for every $y \in G$ neighbourhood $V_y \subset G$ of the point y such that for every λ^k-measurable set $E \subset V_y$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

We have $\subset \{V_y |\ y \in G\} = G$. There exists a sequence $\{y_j\}_{j=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} V_{y_j} = G$. Using lemma from previous semester we find Borel sets $B, N \subset \varphi(G)$ such that $B \subset L \subset B \cup N$, $H^k(N) = 0$.

$\lambda^k(\varphi^{-1}(N)) = 0$. $\varphi^{-1}(B) \subset \varphi^{-1}(L) \subset \varphi^{-1}(B) \cup \varphi^{-1}(N) \implies \varphi^{-1}(L)$ is λ^k-measurable. We set

$$A_j = \varphi^{-1}(L) \cap \left(V_{y_j} \setminus \bigcup_{i=1}^{j-1} V_{y_j} \right).$$

Then we have

- A_j is λ^k-measurable;
- $A_j \subset V_{y_j}$ for every $j \in \mathbb{N}$;
- $\forall j, j' \in \mathbb{N}, j \neq j' : A_j \cap A_{j'} = \varnothing$;
- $\bigcup_{j=1}^{\infty} A_j = \varphi^{-1}(L)$;
- for every $j \in \mathbb{N}$ we have

$$\alpha^{-1} \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(A_j)) \leqslant \alpha \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

From all except for second point we have

$$\alpha^{-1} \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(L) \leqslant \underbrace{\sum_{j=1}^{\infty} H^k(\varphi(A_j))}_{=H^k\left(\bigcup_{j=1}^{\infty} \varphi(A_j)\right) = H^k(L)} \leqslant \alpha \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

2. „$f \geqslant 0$ simple H^k-measurable“: From linearity of integrals. 3. „$f \geqslant 0$ H^k-measurable“: we approximate f by $0 \leqslant f_j \leqslant f_{j+1}$ simple functions and from Levi

$$\lim_{j\to\infty} \int_{\varphi(G)} f_j(x) dH^k(x) = \int_{\varphi(G)} f(x) dH^k(x), \qquad \lim_{j\to\infty} \int_G f_j(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda$$

3. „f H^k-measurable“: We add positive and negative part. □