

TODO(you should know).

TODO(motivation)

1 Sobolev spaces

Definition 1.1 (Multiindex)

α je multi-index $\equiv \alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}$. Length of multi-index α is $|\alpha| := \alpha_1 + \dots + \alpha_d$.
If $u \in C^k(\Omega)$ then $D^\alpha := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, $|\alpha| \leq k$.

Definition 1.2 (Weak derivative)

Let $u, v_\alpha \in L^1_{loc}(\Omega)$ and α be a multi-index. We say that v_α is the α -th weak derivative of u in Ω iff $\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi$.

Lemma 1.1

Weak derivative is unique. If the classical derivative exists then it is also the weak derivative.

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Důkaz

Let v_α^1 and v_α^2 be two weak derivatives. Then

$$\int_\Omega (v_\alpha^1 - v_\alpha^2) \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ almost everywhere in Ω .

If classical $D^\alpha u$ exists, then

$$\int_\Omega \underbrace{D^\alpha u}_{v_\alpha} \varphi \stackrel{\text{BP}}{=} (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

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Poznámka (Notation for this course)

D^α always means the weak derivative.

Definition 1.3 (Sobolev space)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. We define $W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leq k : D^\alpha u \in L^p(\Omega)\}$.

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{\alpha, |\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sup_{\alpha, |\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

Lemma 1.2 (Base properties of Sobolev spaces)

Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and α is multi-index. Then

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$, if $|\alpha| \leq k$;
- $\lambda u + \mu v \in W^{k,p}(\Omega) \quad \forall \lambda, \mu \in \mathbb{R} \quad (D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v)$;
- $\tilde{\Omega} \subset \Omega$ open, $u \in W^{k,p}(\tilde{\Omega})$;
- $\forall \eta \in C^\infty(\Omega) : \eta \cdot u \in W^{k,p}(\Omega)$.

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Věta 1.3 (Properties of Sobolev spaces)

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $k \in \mathbb{N}$:

1. $W^{k,p}(\Omega)$ is a Banach space;
2. if $p < \infty$, then $W^{k,p}(\Omega)$ is separable;
3. if $p \in (1, \infty)$, then $W^{k,p}(\Omega)$ is reflexive.

Důkaz (1.)

„Linear space“ is from Minkowski inequality. „Completeness“: u^n is Cauchy in $W^{k,p}(\Omega) \implies \exists u \in W^{k,p}(\Omega) \implies u^n \rightarrow u$ in $L^p(\Omega)$, $D^\alpha u^n \rightarrow v_\alpha$ in $L^p(\Omega) \quad \forall |\alpha| \leq k$. We must check that „ $v_\alpha = D^\alpha u$ “:

$$\begin{aligned} \int_{\Omega} v_\alpha \eta dx &= \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta + \int_{\Omega} D^\alpha u^n \eta = \\ &\stackrel{IBP}{=} \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta + (-1)^{|\alpha|} \int_{\Omega} u^n D^\alpha \eta = \text{TODO?} \end{aligned}$$

$$\left| \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta \right| \leq \|\eta\|_\infty \cdot \|v_\alpha - D^\alpha u^n\|_{L^p} \rightarrow 0.$$

Důkaz (2. + 3. for $W^{1,p}(\Omega)$)

$W^{1,p}(\Omega) = X \subseteq L^p(\Omega) \times \dots \times L^p(\Omega)$ ($d+1$ times) and it is closed.

1.1 Approximation of Sobolev functions

Věta 1.4

$\Omega \subseteq \mathbb{R}^d$ open, bounded. $k \in \mathbb{N}$, $p \in [1, \infty)$. Then

$$\overline{\mathcal{C}^\infty(\Omega)}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

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$$\overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}} \neq W^{k,p}(\Omega).$$

Poznámka

If $\Omega \subset \mathbb{R}^d$ open, connected, then $u = \text{const} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \ \forall_i = 1, \dots, d$.

$W^{1,1}(I)$, I interval. Then $W^{1,1}(I) \hookrightarrow C(\overline{I})$.

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Důkaz

„ \implies “: easy. „ \impliedby “: $u_\varepsilon = u * \eta_\varepsilon$, $\Omega_\varepsilon := \{x \in \Omega, \text{dist}(x, \partial\Omega) > \varepsilon\}$.

$$x \in \Omega_\varepsilon : \frac{\partial u_\varepsilon}{\partial x_1}(x) = \left(\frac{\partial u}{\partial x_i} \right)_\varepsilon(x) = 0 \implies u_\varepsilon \equiv \text{const in } \Omega_\varepsilon.$$

Fix $\varepsilon_0 > 0$: $\varepsilon \leq \varepsilon_0$: $u_\varepsilon \rightarrow u$ in $W^{1,1}(\Omega_{\varepsilon_0}) \implies u \equiv \text{const in } \Omega_{\varepsilon_0}$. $u \in W^{1,1}(I)$.

$$\tilde{u}(x) := \int_0^x \frac{\partial u(y)}{\partial y} dy, \quad \|\tilde{u}(x)\|_\infty \leq \int_0^1 |\nabla u| dx.$$

Aim $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}$. $\eta \in C_0^\infty(0, 1)$:

$$\begin{aligned} \int_0^1 \tilde{u}(x) \frac{\partial \eta}{\partial x}(x) dx &= \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \eta(x)}{\partial x} \chi_{\{0 \leq y \leq x\}} dx dy = \\ &= \int_0^1 \int_y^1 \frac{\partial u}{\partial y}(y) \frac{\partial \eta(x)}{\partial x_i} dx dy = \\ &= - \int_0^1 \frac{\partial u(y)}{\partial y} \eta(y) dy. \end{aligned}$$

$\implies \tilde{u} - u = \text{const} =: c$.

$$|u(x_1) - u(x_2)| = |\tilde{u}(x_1) - c - \tilde{u}(x_2) + c| = |\tilde{u}(x_1) - \tilde{u}(x_2)| \leq \int_{x_1}^{x_2} \left| \frac{\partial u}{\partial y} \right| dy \rightarrow 0.$$

$\implies C(I)$.

„ $\|u\|_\infty \leq K \cdot \|u\|_1$ “:

$$|c| = \int_0^1 |c| = \int_0^1 |\tilde{u}(x) - u(x)| \leq \|\tilde{u}\|_\infty + \|u\|_1 \leq \|u\|_{1,1}.$$

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$W^{d,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ (for Lipschitz domain Ω).

1.2 Characterization of Sobolev functions

Věta 1.5

Let $\Omega \subset \mathbb{R}^d$, $p \in [1, \infty]$, $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$. 1. Then

$$u \in W^{1,p}(\Omega) : \|\Delta_i^n u\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)},$$

where $\Delta_i^n u(x) = \frac{u(x+he_i) - u(x)}{h}$.

2. If $\forall h, i, \delta : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq c_i$ ($p > 1$). Then

$$\exists \frac{\partial u}{\partial x_i}, \quad \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq c_i.$$

Pozor

This works only for ($p > 1$).

Definice 1.4 (Domains of class $C^{k,\alpha}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \dots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \rightarrow \mathbb{R}$ where $\Delta_r = \{x'_r \in \mathbb{R}^{d-1} \mid |x_{r_i}| \leq \alpha\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting Tr the orthogonal transformation from (x'_r, x_{r_d}) to (x', x_d) , then $\forall x \in \partial\Omega$ $\exists r \in \{1, \dots, M\}$ such that $x = Tr(x'_r, a(x_{r_d}))$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\}$$

$$V_r^- := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\}$$

$$\Lambda_r := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) = x_{r_d}\}$$

Then $Tr(V_r^+) \subset \Omega$, $Tr(V_r^-) \subset \mathbb{R}^d \setminus \bar{\Omega}$, $Tr(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M Tr(\Lambda_r) = \partial\Omega$.

Věta 1.6 (Density)

Let $\Omega \in C^{0,1}$ and $p \neq \infty$, then $W^{k,p}(\Omega) = \overline{C^\infty(\bar{\Omega})}^{\|\cdot\|_{k,p}}$.

Věta 1.7 (Extension)

Let $\Omega \in C^{0,1}$, $k \in \mathbb{N}$, $p \in [1, \infty]$. Then \exists continuous bounded operator $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$ such that

1. $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leq c \cdot \|u\|_{W^{k,p}(\Omega)}$ (Eu has compact support);

2. $Eu = u$ almost everywhere in Ω .

Věta 1.8 (Trace)

Let $\Omega \in C^{0,1}$, $p \in [1, \infty]$. Then \exists continuous bounded operator $\text{tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ such that:

1. $\|\text{tr } u\|_{L^p(\partial\Omega)} \leq c \cdot \|u\|_{W^{1,p}(\Omega)}$;

$$2. \ u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \implies \operatorname{tr} u = u|_{\partial\Omega}.$$

Definice 1.5

$$W_0^{k,p}(\Omega) = \overline{u \in C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

Poznámka

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \mid \operatorname{tr} u = 0\}.$$

TODO!!!

1.3 Existence theory via Lax–Milgram

Definice 1.6 (Elliptic forms)

Let $B : V \times V \rightarrow \mathbb{R}$ a linear form and V be a Hilbert space. We say that B is elliptic iff

1. B is V -bounded, it is $\exists c_2 \ \forall u, \varphi \in V : |B(u, \varphi)| \leq c_2 \|u\|_V \cdot \|\varphi\|_V$;
2. B is V -coercive, it is $\exists c_1 > 0 \ \forall u \in V : B(u, u) \geq c_1 \|u\|_V^2$.

Věta 1.9 (Lax–Milgram)

Let B be linear and satisfying two previous properties. Then $\forall F \in V^* \ \exists! u \in V : \forall \varphi \in V : B(u, \varphi) = \langle F, \varphi \rangle$.

Definice 1.7 (Lipschitz, unifomly monototne)

Let $B : V \rightarrow V^*$. We say that B is

1. Lipschitz iff $\forall u, v \in V : \|B(u) - B(v)\|_{V^*} \leq \overline{c}_2 \|u - v\|_V$;
2. uniformly monotone iff $\forall u, v \in V : \langle B(u) - B(v), u - v \rangle \geq \overline{c}_1 \|u - v\|_V^2$.

Věta 1.10 (Non-linear Lax–Milgram)

Let V be a Hilbert space, $B : V \rightarrow V^*$ be Lipschitz and uniformly monotone. Then $\forall F \in V^* \ \exists! u \in V : \forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle$. ($B(u) = F$.)

Důkaz (Lax–Milgram by using non-linear version)

Define $B : V \rightarrow V^* : \langle B(u), \varphi \rangle =: B(u, \varphi)$. We show that B is Lipschitz and uniformly monotone:

$$\|B(u) - B(v)\|_{V^*} = \sup_{\varphi \in V, \|\varphi\| \leq 1} \langle B(u) - B(v), \varphi \rangle = \sup_{\varphi} (B(u, \varphi) - B(v, \varphi)) \stackrel{\text{linear}}{=} \sup_{\varphi} B(u - v, \varphi) \stackrel{\text{bounded}}{\leq} \sup_{\varphi} c_2 \|u - v\| \|\varphi\|$$

$$\langle B(u) - B(v), u - v \rangle = B(u, u - v) - B(v, u - v) = B(u - v, u - v) \geq c_1 \|u - v\|_V^2.$$

$$\text{So } \forall F^* \exists! u \in V : B(u, \varphi) = \langle B(u), \varphi \rangle = \langle F, \varphi \rangle \ (\forall \varphi \in V). \quad \square$$

┌ *Důkaz* (Non-linear Lax–Milgram)

„Uniqueness“: $u_1 \neq u_2$:

$$\forall \varphi : \langle B(u_1), \varphi \rangle = \langle B(u_2), \varphi \rangle = \langle F, \varphi \rangle \implies \forall \varphi \in V : \langle B(u_1) - B(u_2), \varphi \rangle = 0.$$

$$\varphi = u_1 - u_2 \implies 0 = \langle B(u_1) - B(u_2), u_1 - u_2 \rangle \geq \bar{c}_1 \|u_1 - u_2\|_V^2 \implies u_1 = u_2.$$

„Existence“: $\forall \langle B(u), \varphi \rangle = \langle F, \varphi \rangle \Leftrightarrow \exists \varepsilon > 0 : (u, \varphi)_V = (u, \varphi)_V - \varepsilon (\langle B(u), \varphi \rangle - \langle F, \varphi \rangle)$.
Desire $M : V \rightarrow V, v \mapsto u$:

$$(u, v)_V = (v, \varphi)_V - \varepsilon (\langle B(v), \varphi \rangle - \langle F, \varphi \rangle).$$

If M is well-defined and if it has a fixed point then we find solution.

„ M well defined“:

$$\forall v \in V \exists \tilde{F} \in V^* \left\langle \tilde{F}, \varphi \right\rangle_V = (v, \varphi)_V - \varepsilon (\langle B(v), \varphi \rangle - \langle F, \varphi \rangle).$$

$$\text{Riesz} \implies \exists! u \in V (u, \varphi) = \left\langle \tilde{F}, \varphi \right\rangle.$$

„ M contraction“: We want bound $\|M(u) - M(v)\|_V^2$.

$$M(u) = \bar{u}, \quad (\bar{u}, \varphi) = (u, \varphi) - \varepsilon (\langle B(u), \varphi \rangle - \langle F, \varphi \rangle).$$

$$M(v) = \bar{v}, \quad (\bar{v}, \varphi) = (v, \varphi) - \varepsilon (\langle B(v), \varphi \rangle - \langle F, \varphi \rangle).$$

$$(\bar{u} - \bar{v}, \varphi) = (u - v, \varphi) - \varepsilon \langle B(u) - B(v), \varphi \rangle.$$

$$\text{Riesz: } \exists! w_1 : \langle B(u), \varphi \rangle = (w_1, \varphi) \text{ and } \exists! w_2 : \langle B(v), \varphi \rangle = (w_2, \varphi).$$

$$(*) \|\bar{u} - \bar{v}\|_V^2 = \|u - v - \varepsilon(w_1 - w_2)\|_V^2 = \|u - v\|_V^2 + \varepsilon^2 \|w_1 - w_2\|_V^2 - 2\varepsilon(w_1 - w_2, u - v).$$

$$(w_1 - w_2, w_1 - w_2) = \|w_1 - w_2\|_V^2 = \langle B(u) - B(v), w_1 - w_2 \rangle \leq \|B(u) - B(v)\|_{V^*} \|w_1 - w_2\|_V \implies \|w_1 - w_2\|_V \leq$$

$$(w_1 - w_2, u - v) = \langle B(u) - B(v), u - v \rangle \stackrel{\text{Uniformly monotone}}{\geq} c_1 \|u - v\|_V^2$$

By (*):

$$\begin{aligned} \|M(u) - M(v)\|_V^2 &\leq \|u - v\|_V^2 + \varepsilon^2 \bar{c}_2^2 \|u - v\|_V^2 - 2\varepsilon \bar{c}_1 \|u - v\|_V^2 = \\ &= (1 + \varepsilon^2 \bar{c}_2^2 - 2\varepsilon \bar{c}_1) \|u - v\|_V^2. \end{aligned}$$

$\implies M$ is contraction and has a fixed point (for ε such that this constant is less than 1, so $\varepsilon(\varepsilon \bar{c}_2^2 - 2\bar{c}_1) < 0$, so $0 < \varepsilon < \frac{2\bar{c}_1}{\bar{c}_2^2}$). \square

Věta 1.11

Let $B_{L,\sigma}$ be bilinear, V -bounded and V -elliptic. Then $\exists! u$ weak solution:

$$u_0 \in W^{1,2}, \quad u - 0 \in V, \quad B_{L,\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g \varphi.$$

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Důkaz

„Uniqueness“: $u_1, u_2 \implies u_1 - u_0 \in V, u_2 - u_0 \in V$ TODO!!!

„Existence“: $w \in V$ ($u - u_0 = w$):

$$B_{L,\sigma}(w, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g \varphi - B_{L,\sigma}(u_0, \varphi) := \langle \overline{F}, \varphi \rangle.$$

Find $w \in V$.

$$\langle \overline{F}, \varphi \rangle := \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g \varphi + \int_{\Omega} -A \nabla u \cdot \nabla \varphi - b u_0 \varphi - \mathbf{c} \cdot \nabla u_0 \varphi + \mathbf{d} \cdot \nabla \varphi u_0 - \int_{\Gamma_2} \sigma u_0 \varphi.$$

Is \overline{F} in V^* ? ($\varphi \in V \subseteq W^{1,2}(\Omega)$)

$$|\langle \overline{F}, \varphi \rangle| \leq \|\varphi\|_{V^*} (\|f\|_{V^*} + \|g\|_{L^2(\partial\Omega)} + \|A\|_{\infty} \cdot \|u_0\|_{1,2} + \|b\|_{\infty} \|u_0\|_2 + \|c\|_{\infty} \|u_0\|_{1,2} + \dots).$$

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TODO!!!

1.4 Existence theory via Fredholm alternative

Lemma 1.12 (Fredholm alternative)

Let H be a Hilbert space and $K : H \rightarrow H$ be linear compact.

F1 $\text{Ker}(I - K)$ has finite dimension ($u \in \text{Ker}(I - K) \Leftrightarrow (I - K)(u) = 0$);

F2 $\text{Rng}(I - K)$ is closed ($u \in \text{Rng}(I - K) \Leftrightarrow \exists w \in H (I - K)w = u$);

F3 $\text{Rng}(I - K) = (\text{Ker}(I - K^*))^{\perp}$ ($u \in \text{Rng}(I - K), w \in \text{Ker}(I - K^*) \Leftrightarrow (u, w) = 0$);

F4! $\text{Ker}(I - K) = \{\mathbf{0}\} \Leftrightarrow \text{Rng}(I - K) = H$;

F5 $\dim(\text{Ker}(I - K)) = \dim(\text{Ker}(I - K^*)) < \infty$;

F6 spectrum of K is at most countable and if it is infinite then zero is the only attracting point.

Věta 1.13 (Fredholm alternative for PDR)

Let $\Omega \in C^{0,1}$, $u_0 = 0$ and $\Gamma_1 = \partial\Omega$ and L be an elliptic operator.

1. Either $\forall f \in L^2(\Omega) \exists! u \in W_0^{1,2}(\Omega) : Lu = f$ in Ω and $u = 0$ on $\partial\Omega$, or $\exists u \neq 0 : Lu = 0$ in Ω and $u = 0$ on $\partial\Omega$.
2. $N_L := \{u \in V | Lu = \mathbf{0}\} : B_L(u, \varphi) = 0 \ \forall \varphi \in W_0^{1,2}(\Omega)$, $N_{L^*} := \{\varphi \in V | L^*\varphi = \mathbf{0}\}$. Then N_L and N_{L^*} are closed subspaces of $W_0^{1,2}(\Omega)$, $\dim N_L = \dim N_{L^*} < \infty$.
3. For $f \in L^2(\Omega) : (\exists u \in W_0^{1,2} : Lu = f) \Leftrightarrow (\forall \varphi \in N_{L^*} : \int_{\Omega} f\varphi = 0)$.

Where

$$Lu = -\operatorname{div}(A\nabla u) + \mathbf{c} \cdot \nabla u + \operatorname{div}(\mathbf{d}u) + bu,$$

$$L^*\varphi = -\operatorname{div}(A^T\nabla\varphi) - \mathbf{d} \cdot \nabla\varphi - \operatorname{div}(\mathbf{c}\varphi) + b\varphi.$$

$$Lu = f \Leftrightarrow \forall \varphi : B_L(u, \varphi) = \int_{\Omega} f\varphi, \quad L^*\varphi = g \Leftrightarrow \forall u : B_L(\varphi, u) = \int_{\Omega} gu.$$

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Důkaz

From Lax–Milgram $\exists p > 0$:

$$\forall f \in L^2 \exists! u \in W_0^{1,2}(\Omega) L_p u := Lu + pu = f.$$

$$B_{L,p}(u, \varphi) = B_L(u, \varphi) + p \cdot \int_{\Omega} u\varphi.$$

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TODO!!! (First lecture)

Definice 1.8 (Real spectrum of L)

$\Gamma_1 = \partial\Omega$. We say that $\lambda \in \mathbb{R}$ belongs to the spectrum of L iff $\exists u \neq 0, Lu = \lambda u$ in Ω , $u = 0$ on $\partial\Omega$. Denote $\Sigma := \{\lambda \in \mathbb{R} | \exists u \neq \mathbf{0}, Lu = \lambda u\}$.

Věta 1.14

Let $\Omega \in C^{0,1}$, L elliptic operator. Then

1. Σ is at most countable, and if it is infinite, $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ then $\lambda_k \rightarrow \infty$;
2. $(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^2 \exists! u \in W_0^{1,2} : Lu = f + \lambda u$;
3. $\forall \lambda \notin \Sigma \exists c > 0 \forall f \in L^2 \exists! u \in W_0^{1,2} : Lu = f + \lambda u, \|u\|_{1,2} \leq c\|f\|_2$.

$$L_\gamma u = Lu + \gamma u, \quad K = \gamma L_\gamma^{-1}, \quad Lu + \gamma u = \gamma u + \lambda u = -?u$$

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$$\begin{aligned} Lu + \gamma u &= \lambda u + \gamma u = (\lambda + \gamma)u \\ L_\gamma u + (-\lambda - \gamma)u &= 0 \implies u \equiv 0. \end{aligned}$$

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