Poznámka (Literature)
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Definice 0.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka
Complete compatible metric is not unique: $\tilde{\varrho} = \min\{1, \varrho\}$.

Například
$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $2 := \{0, 1\}$, $\omega := \{0, 1, 2, \dots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, 2^ω, ω^ω (both with product topology).

Věta 0.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X.

Důkaz
Without proof. (We should know it already.) □

Věta 0.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X, such that $\operatorname{diam}(F_n) \to 0$. *Then* $|\bigcap F_n| = 1$.

Důkaz
Without proof. (We should know it already.) □

Věta 0.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n\in\omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow$ $H \in \mathcal{G}_\delta(X)$

Důkaz ((i))
Let d_n be CCM (complete compatible metric) on X_n, $n \in \omega$. Then

$$d(x, y) := \sum_{n=0}^{\infty} \min\left\{2^{-n}, d_n(x_n, y_n)\right\}$$

is CCM on $X = \prod_{n\in\omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „d is metric“ straightforward, „d is complete“ also easy, compatibility too). □

Důkaz ((ii))
$H = \emptyset$, $H = X$ trivial. Assume $H \neq \emptyset, X$.

„ $\implies$ “: Fix CCM ϱ on H. $V_n := \bigcup \{V \subset X | V \text{ open in } X \wedge V \cap H \neq \emptyset \wedge \operatorname{diam}_\varrho(V \cap H) < 2^{-n}\}$, $n \in \omega$. We want to show $H \stackrel{?}{=} \bigcap_{n\in\omega}(V_n \cap \overline{H}) \in \mathcal{G}_\delta$:

$$„\subseteq“\text{: } x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n.$$

„$\supseteq$“: $x \in V_n \cap \overline{H}$ for every $n \in \omega$ $\implies$ $\exists$ open sets G_n: $x \in G_n$, $G \cap H \neq \emptyset$, $\operatorname{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \stackrel{?}{\neq} \emptyset \impliedby x \in G_n \cap G_{n+1} \cap \overline{H} \neq \emptyset$).

$\{y\} := \bigcap_{n\in\omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O$, $G_n \cap H \subset B(y, 2^{-n})$, $n \in \omega$. $\implies \exists n \in \omega G_n \cap H \subset O$, $x \in G_n \cap (X \backslash \overline{O}) \cap \overline{H} \implies G_n \cap (X \backslash \overline{O}) \cap H \neq \emptyset$.

„ $\impliedby$ “: fix CCM d on X, $H = \bigcap_{n\in\omega} U_n$, $\emptyset = U_n \neq X$. $F_n := X \backslash U_n$, $\tilde{d}(x,y) = d(x,y) + \sum_{n=0}^{\infty} \min\left\{2^{-n}, \left|\frac{1}{\operatorname{dist}(x,F_n)} - \frac{1}{\operatorname{dist}(y,F_n)}\right|\right\}$, $x, y \in H$. Next we verified that $\tilde{d}$ is metric, that $\tilde{d}$ is equivalent with d on H (by convergence), and that $(H, \tilde{d})$ is complete metric space and separable. TODO? $\square$

Definice 0.2 (Notation)

$A \neq 0$:

- $A^{<\omega} :=$ finite sequence of elements of $A = \bigcup_{n\in\omega} A^n$;
- $s \in A^k$, $t \in A^{<\omega} \cup A^\omega$: $s^\wedge t := (s_0, s_1, \ldots, s_{k-1}, t_0, t_1, \ldots)$, where $s = (s_0, \ldots, s_{k-1})$, $t = (t_0, t_1, \ldots)$;
- $s \in A^{<\omega} \cup A^\omega$: $|s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega$, $k \in \omega$, $|s| \geqslant k$, then we denote restriction of s on first k elements as s/k;
- $s \prec t$ iff $|t| \geqslant |s|$ and $s = t/|s|$ ($s \in A^{<\omega}$, $t \in A^{<\omega} \cup A^\omega$).

1 Baire space ω^ω

Definice 1.1

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega | s \prec \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \backslash \bigcup \{\mathcal{N}(t) | |t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N}|s \in \omega^{<\omega}\}$ is base of topology of ω^ω.

Věta 1.1 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz

Bez důkazu. □

Důsledek

ω^ω is homeomorphic to $\mathbb{R}\backslash\mathbb{Q}$.

Věta 1.2

Let $X \neq \emptyset$, PTS. Then X is continuous image of ω^ω.

Poznámka

$X \neq \emptyset$ PTS. Then there $\exists F \subset \omega^\omega$, F closed, and continuous injection $\varphi : F \to X$.

Důkaz

Find CCM on X such that $\operatorname{diam} X \leqslant 1$. We inductively construct closed $\emptyset \neq A_s \subset X$ for every $s \in \omega^{<\omega}$ such that 1. $A_\emptyset = X$; 2. $\operatorname{diam}(A_s) \leqslant 2^{-|s|}$; 3. $A_s = \bigcup_{i\in\omega} A_{s^\wedge i}$.

Empty set is trivial. Assume we already have A_s. Find $\{x_i|i \in \omega\} \subset A_s$ dense in A_s. $A_{s^\wedge i} := A_s \cap \overline{B(x_i, 2^{-|s|-2})} \neq \emptyset$ closed.

Fix $\forall \nu \in \omega^\omega : f(\nu) := x$, where $\{x\} = \bigcap_{k\in\omega} A_{\nu/k} \neq \emptyset$ (intersection of closed nonempty non-increasing sequence of sets). „f is surjection“: $x \in A_s \overset{3.}{\Longrightarrow} \exists n \in \omega : x \in A_{s^\wedge n} \overset{1.}{\Longrightarrow} \forall x \in X\ \exists \alpha \in \omega^\omega\ \forall k \in \omega : x \in A_{\alpha/k} \implies x = f(\alpha)$.

„f continuous“: $f(\mathcal{N}_{\nu/k}) \subset A_{\nu/k}$ for every $\nu \in \omega^\omega$, $k \in \omega$, $\operatorname{diam} A_{\nu/k} \leqslant 2^{-k}$. □

1.1 Cantor set 2^ω

Tvrzení 1.3

2^ω is up to homeomorphism unique nonempty nuldimensional compact metrizable space without isolated points (without isolated points is called perfect space).

Tvrzení 1.4

Let $X \neq \emptyset$ metrizable, compact. Then X is continuous image of 2^ω.

Důkaz
Without proof, but it is similar to the previous one. □

1.2 Hilbert cube $[0,1]^\omega$

Tvrzení 1.5

Let X be PTS. Then X is homeomorphic to G_δ subset of $[0,1]^\omega$.

Důkaz
X PTS, case $\varnothing$ is trivial, so assume $X \neq \varnothing$, ϱ is CCM on X, $\varrho \leqslant 1$. Let $\{x_n, n \in \omega\}$ be dense in X. Define $f : [0,1]^\omega : f(x) = (\varrho(x, x_n))_{n\in\omega}$. $\varrho \leqslant 1 \implies f(x) \in [0,1]^\omega$.

„Continuity of f“: $f^{-1}(U) = \bigcap_{i=1}^n B(x_i, b_i) \setminus \overline{B(x_i, a_i)}$ open.

„Injective“: $x \neq y \implies \exists n \in \omega : \varrho(x, x_n) < \varrho(y, x_n) \implies f(x) \neq f(y)$.

„Continuity of f^{-1}“ $f(y^n) \to f(y) \overset{?}{\implies} y^n \to y$.

$$f(y^n) \to f(y) \overset{?}{\Leftrightarrow} \forall k \in \omega : \varrho(y^n, x_k) \to \varrho(y, x_k).$$

Let $\varepsilon > 0$ be arbitrary:

$$\exists k \in \omega : \varrho(y, x_k) < \frac{\varepsilon}{3}. \ \exists n_0 \ \forall n \geqslant n_0 : \varrho(y^n, x_k) < \frac{2\varepsilon}{3}.$$

Then

$$\forall n \geqslant n_0 : \varrho(y^n, y) \leqslant \varrho(y^n, x_k) + \varrho(x_k, y) < \varepsilon.$$

So $f(X)$ is homeomorphism to $X \implies f(X)$ is PTS $\implies f(X) \in \mathcal{G}_\delta([0,1]^\omega)$. □

Důsledek
Let X be compact metrizable space. Then X is homeomorphic to some closed subset of $[0,1]^\omega$.

Důkaz
Compact metrizable space is Polish. And compact subset must be closed. □

1.3 $\mathcal{K}(X)$: Hyperspace of compact subsets of X

Definice 1.2

Let X be PTS, denote $\mathcal{K}(X) := \{K \subset X | K \text{ is compact}\}$. Vietoris topology on $\mathcal{K}(X)$ is generated by $\{K \in \mathcal{K}(X) | K \subset V\}$ for V open and $\{K \cap \mathcal{K}(X) | K \cap V \neq \varnothing\} = \mathcal{K}(X) \setminus \{K \in \mathbb{K}(X) | K \subset X \setminus V\}$

for V open.

Tvrzení 1.6

Let X be PTS, ϱ CCM on X, $\varrho \leqslant 1$. Then mapping $h : \mathcal{K}(X) \times \mathcal{K}(X) \mapsto [0, +\infty)$ defined as:

$$h(K, L) = \begin{cases} 0, & K = L = \emptyset, \\ \max\left\{\sup_{x\in K} \varrho(x, L), \sup_{y\in L} \varrho(y, K)\right\}, & K, L \neq \emptyset, \\ 1, & \text{other cases,} \end{cases}$$

is CCM on $\mathcal{K}(X)$ with Vietoris topology. h is known as Hausdorff metric.

Poznámka

$\mathcal{K}(X)$ is separable if X is PTS. X is compact metrizable $\implies$ $\mathcal{K}(X)$ is compact (totally bounded).

$$X \text{ is separable} \implies \exists D \subset X : \overline{D} = X, |D| = \omega.$$

$$M = \{K \subset D | |K| < \omega\} \implies |M| = \omega.$$

$\overline{M} = \mathcal{K}(X)$. $K \in \mathcal{K}(X)$ arbitrary, $\varepsilon > 0$ arbitrary. Then $\exists \frac{\varepsilon}{2}$ net $P \subset K$, $|P| < \omega$. We find $\{\tilde{x}_0, \dots, \tilde{x}_n\} \subset D : \varrho(x_i, \tilde{x}_i) < \frac{\varepsilon}{2} \wedge h(K, \{\tilde{x}_0, \dots, \tilde{x}_n\}) < \varepsilon$.

$$X \text{ is compact, } P \text{ is } \varepsilon\text{-net in } X, |P| < \omega \implies 2^P \text{ is finite } \varepsilon\text{-net in } \mathcal{K}(X).$$

Důkaz

($\emptyset \neq K, L, P \in \mathcal{K}(X)$.) h is metric, definition is correct, $h \geqslant 0$ trivial, $h(K, L) = h(L, K)$ trivial, $h(K, L) = 0 \implies K = L$ ($x \notin L \implies \varrho(x, L) > 0 \implies K \subset L \wedge L \subset K$).

„$\triangle$“ aka „$h(K, L) \leqslant h(K, P) + h(P, L)$“: Let $x \in K, y \in L, p \in P$. Then

$$\varrho(x, L) \leqslant \varrho(x, y) \leqslant \varrho(x, p) + \varrho(p, y) \qquad \inf y \in L$$

$$\begin{aligned} \varrho(x, L) &\leqslant \varrho(x, p) + \varrho(p, L) \qquad \sup p \in P \\ \varrho(x, L) &\leqslant \varrho(x, p) + h(P, L) \qquad \inf p \in P \\ \varrho(x, L) &\leqslant \varrho(x, P) + h(P, L) \qquad \inf p \in P \\ \sup_{x\in K} \varrho(x, L) &\leqslant h(K, P) + h(P, L). \end{aligned}$$

Similarly $\sup_{y\in L} \varrho(y, K) \leqslant h(K, P) + h(P, L)$. □

TODO!!!

Definice 1.3

X is metrizable space, $1 \leqslant \alpha < \omega_1$. We define $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, and $\Delta^0_\alpha(X)$ by induction:

$$\Sigma^0_1(X) := \{U \subset X | U \text{ open}\},$$

$$\Pi^0_\alpha(X) := \left\{A \subset X | X \backslash A \in \Sigma^0_\alpha(X)\right\},$$

$$\Sigma^0_\alpha(X) := \left\{\bigcup_{n \in \omega} A_n | A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \alpha, n \in \omega\right\},$$

$$\Delta^0_\alpha(X) := \Sigma^0_\alpha \cap \Pi^0_\alpha(X).$$

Poznámka (By induction it can be prooven)

$$\Sigma^0_\alpha(X) \subset \Sigma^0_\beta(X), \Pi^0_\alpha(X) \subseteq \Pi^0_\beta(X), \qquad 1 \leqslant \alpha < \beta < \omega_1.$$

Poznámka

$$\forall \alpha, \beta : 1 \leqslant \alpha < \beta < \omega_1 : \Sigma^0_\alpha(X) \subset \Pi^0_\beta(X).$$

Poznámka
If X contains homeomorphic copy of 2^ω then all inclusions are strict.

We denote $Borel(X)$ as σ-algebra of Borel sets (σ-algebra generated by $\Sigma^0_1(X)$).

Poznámka (Also non-trivial theorem)

$$Borel(X) = \bigcup_{1 \leqslant \alpha < \omega_1} \Sigma^0_\alpha(X) = \bigcup_{1 \leqslant \alpha < \omega_1} (X) = \bigcup_{1 \leqslant \alpha < \omega_1} \Delta^0_\alpha(X).$$

$$A_n \in \bigcup_{1 \leqslant \alpha < \omega_1} \Sigma^0_\alpha(X) \implies \exists 1 \leqslant \alpha_n < \omega_1 : A_n \in \Sigma^0_{\alpha_n}(X) \implies A_n \in \Sigma^0_{\sup\{\alpha_n | n \in \omega\}} \implies \bigcup_{n \in \omega} A_n \in \Sigma^0_{\sup\{\alpha_n, n \in \omega\}}$$

Poznámka
$F_\sigma = \Sigma^0_2$, $G_\delta = \Pi^0_2$, $F_{\sigma\delta} = \Pi^0_3$, $G_{\delta\sigma} = \Sigma^0_3$.

$\Sigma^0_\alpha(X)$ is closed under countable union and $\Pi^0_\alpha(X)$ under countable intersection.

Věta 1.7

X be metrizable, $1 \leqslant \alpha < \omega_1$. Then

1. *$\Sigma^0_\alpha(X)$ is closed under finite intersection;*
2. *$\Pi^0_\alpha(X)$ is closed under finite union.*

Důkaz

„1.“ Firstly for $\alpha = 1$, it is trivial. Then let $A, B \in \Sigma^0_\alpha(X)$, $\alpha > 1$. Then $A = \bigcup_{n\in\omega} A_n$, $A_n \in \Pi^0_{\alpha_n}(X)$, $\alpha_n < \alpha$, $B = \bigcup_{m\in\omega} B_m$, $B_m \in \Pi^0_{\beta_m}(X)$, $\beta_n < \alpha$. $A \cap B = \bigcup_{(m,n)\in\omega^2} A_n \cap B_m$, $A_n \cap B_m \in \Pi^0_{\max\{\alpha_n,\beta_n\}}(X) \implies A \cap B \in \Sigma^0_\alpha(X)$. „2.“ $\impliedby$ de Morgan and 1. □

Věta 1.8

X be metrizable, $A \subset Z \subset X$, $1 \leqslant \alpha < \omega_1$. Then $A \in \Sigma^0_\alpha(Z) \Leftrightarrow$ there exists $\tilde{A} \in \Sigma^0_\alpha(X)$: $A = \tilde{A} \cap Z$. Similarly for $\Pi^0_\alpha, \Delta^0_\alpha$.

Důkaz

Firstly $\alpha = 1$ from definition of subspace. Then assume that it is all true for all $\beta < \alpha$. We want to prove it for α. „ $\implies$ “:

$$A \in \Sigma^0_\alpha(Z) \implies A = \bigcup A_n, A_n \in \Pi^0_{\beta_n}(Z), \beta_n < \alpha \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X) : \tilde{A}_n \cap Z = A_n.$$

$$\tilde{A} = \bigcup \tilde{A}_n \in \Sigma^0_\alpha(X), \tilde{A} \cap Z = Z \cap \bigcup \tilde{A}_n = \bigcup (Z \cap \tilde{A}_n) = \bigcup A_n = A.$$

„ $\impliedby$ “:

$$\tilde{A} \in \Sigma^0_\alpha(X), A = \tilde{A} \cap Z \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X), \beta_n < \alpha, \bigcup \tilde{A}_n = \tilde{A}.$$

$$\tilde{A} \cap Z \in \Pi^0_{\beta_n}(Z) \implies A = \tilde{A} \cap Z = \left(\bigcup \tilde{A}_n\right) \cap Z = \bigcup \left(\tilde{A}_n \cap Z\right) = \bigcup A_n \in \Sigma^0_\alpha(Z).$$

□

Věta 1.9

X, Y be metric spaces, $f : X \to Y$ is continuous. If $A \in \Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$, $\Delta^0_\alpha(Y)$) then $f^{-1}(A) \in \Sigma^0_\alpha(X)$ ($\Pi^0_\alpha(X)$, $\Delta^0_\alpha(Y)$).

Důkaz

$\alpha = 1$ trivial. Assume it holds true for $\Sigma^0_\beta(Y)$, $\Pi^0_\beta(Y)$, $\beta < \alpha$, and we want to show for $\Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$). Let $A \in \Sigma^0_\alpha(Y)$, $\alpha > 1 \implies A = \bigcup_{n\in\omega} A_n$, $A_n \in \Pi^0_{\beta_n}(Y)$, $\beta_n < \alpha$.

$$f^{-1}(A) = f^{-1}(\bigcup A_n) = \bigcup \underbrace{f^{-1}(A_n)}_{\Pi^0_{\beta n}(X)} \in \Sigma^0_\alpha(X),$$

$$f^{-1}(Y \backslash A) = f^{-1}(Y) \backslash f^{-1}(A) = X \backslash f^{-1}(A).$$

□

Věta 1.10 (Borel classes in PTS)

X,Y be PTS, $A \in \Sigma^0_\alpha(X)$, $\alpha \geqslant 3$ (resp. $A \in \Pi^0_\alpha(X)$, $\alpha \geqslant 2$), $B \subset Y$. If B and A are homeomorphic then $B \in \Sigma^0_\alpha(Y)$ (resp. Π^0_α).

Důkaz
$f : A \to B$ is homeomorphism A onto B. The theorem above (name ?) there is extension $\tilde{f}$ of f, $\tilde{f}$ is homeomorphism $\tilde{A}$ onto $\tilde{B}$, $A \subset \tilde{A}$, $B \subset \tilde{B}$, $\tilde{A} \in \Pi_2^0(X)$, $\tilde{B} \in \Pi_2^0(Y)$. Then $B \in \Sigma_\alpha^0(\tilde{B})$ (because $B = (f^{-1})^{-1}(A)$). From the theorem above, $\exists \hat{B} \in \Sigma_\alpha^0(Y) : B = \hat{B} \cap \tilde{B} \in \Sigma_\alpha^0(Y) \impliedby \alpha \geqslant 3$. $\square$

1.4 Analytic sets

Definice 1.4

X PTS, $A \subset X$. We say that A is analytic set in X if there exists PTS Y and continuous mapping $\varphi : Y \to X$ such that $\varphi(Y) = A$.

We denote collection of analytic subsets of X as $\Sigma_1^1(X)$. We say that A is coanalytic in X if $X \backslash A \in \Sigma_1^1(X)$ and we denote this collection as $\Pi_1^1(X)$. $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Například

$$Q = \{\alpha \in 2^\omega | \exists n \in \omega \; \forall j \geqslant n : \alpha_j = 0\} = 2^{<\omega} \in \Sigma_2^0(2^\omega) \backslash \Pi_2^0(2^\omega)$$

TODO?

Poznámka
X PTS, $F : X \to \mathcal{K}(X)$ by $F(x) = \{x\}$. Then F is continuous, $F^{-1}(\mathcal{K}(A)) = A \implies$ if $\mathcal{K}(A) \in \Sigma_\alpha^0(\mathcal{K}(X))$ (Π_α^0, Δ_α^0) then $A \in \Sigma_\alpha^0(X)$ (Π_α^0, Δ_α^0). A open $\implies \mathcal{K}(A)$ is open, A is closed $\implies \mathcal{K}(A)$ is closed. $\mathcal{K}(\bigcap A_n) = \bigcap \mathcal{K}(A_n)$. Thus for $A \in \Pi_2^0(X) : \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X))$. $A \in \Sigma_1^0(X)$ ($\Pi_1^0(X)$, $\Pi_2^0(X)$) $\Leftrightarrow \mathcal{K}(A) \in \Sigma_1^0(\mathcal{K}(X))$ ($\Pi_1^0(\mathcal{K}(X))$, $\Pi_2^0(\mathcal{K}(X))$).

Věta 1.11

X PTS, $|X| > \omega$. Assume $I \subset \mathcal{K}(X)$, I is σ-ideal ($K \in I, L \subset K \implies L \in I$; $K_n \in I, \bigcup K_n \in \mathcal{K}(X) \implies \bigcup K_n \in I$). If $I \in \Pi_2(\mathcal{K}(X))$, then $I \in \Sigma_1^1(\mathcal{K}(X))$.

Důsledek
$A \notin \Pi_2^0(X) \implies \mathcal{K}(A) \notin \Sigma_1^1(\mathcal{K}(X))$.

Poznámka
$A \in \Pi_1^1(X), \mathcal{K}(A) = \mathcal{K}(X) \backslash \{K \in \mathcal{K}(X) | \exists x \in (X \backslash A) \cap K\}$ $\{(K, x) \in \mathcal{K}(X) \times X | x \in K\}$ is closed.