

1 Introduction

Poznámka (Literature)

„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

1.1 Differentiability

Definition 1.1 (Differentiable)

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is differentiable (also holomorphic) at a point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{|h| \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call $f'(z_0)$ the derivative of f at z_0 . A function f is differentiable on a domain (open connected subset of \mathbb{C}) if its differentiable for all points of this domain.

Poznámka (Writing complex numbers in cartesian coordinates)

$z = x + iy$, for $x, y \in \mathbb{R}$, we can write a function $f : \mathbb{C} \rightarrow \mathbb{C}$ in terms of two functions $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

Věta 1.1 (Cauchy–Riemann equations)

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function on an open subset of \mathbb{C} . Considering $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definition 1.2 (Orientability, orientation-preserving function)

Define an equivalence relation on the set of all bases of \mathbb{R}^2 by saying that $B_1 \sim B_2$ iff the determinant of the change of basis matrix is positive.

A function $f : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}^2$ is said to be orientation-preserving if on an open dense subset of U , the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Důsledek

Let f be a non-constant holomorphic function, then f is orientation-preserving.

Důsledek

Since f is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \stackrel{\text{C-R}}{=} \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \geq 0.$$

Since f is non-constant, the inequality is strict on a dense open subset of the domain of definition.

Věta 1.2 (Open mapping theorem)

A non-constant holomorphic function f is open (that is if U is an open subset of \mathbb{C} , then $f(U)$ is also open).

1.2 Integration

Definice 1.3

For a path γ (smooth function, $\gamma : \mathbb{R} \supset [a, b] \rightarrow \mathbb{C}$) we define

$$\int_{\gamma} f(x) dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

Definice 1.4 (Continuous deformation)

For $\gamma, \mu : [a, b] \rightarrow U$ (U simply connected), paths with the same endpoints ($\gamma(a) = \mu(a)$ and $\gamma(b) = \mu(b)$). Then a continuous deformation γ into μ is a continuous function $H : [a, b] \times [0, 1] \rightarrow U \subseteq \mathbb{C}$ such that $H(s, 0) = \gamma(s)$, $H(s, 1) = \mu(s)$, $H(a, t) = z_a := \gamma(a) = \mu(a)$ and $H(b, t) = z_b := \gamma(b) = \mu(b)$.

Věta 1.3

Suppose that $\gamma, \mu : [a, b] \rightarrow U$ (U simply connected) are related by a continuous deformation of paths H . Then for any holomorphic function f on U we have

$$\int_{\gamma} f(z) dz = \int_{\mu} f(z) dz.$$

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Důkaz (Partial proof assuming H admits partial derivatives)

For any $t \in [0, 1]$ we integrate the function $INT(t) = \int_{H(\cdot, t)} f(z)dz$. Consider the derivative of $INT(t)$ with respect to t :

$$\begin{aligned} \frac{d}{dt}(INT(t)) &= \frac{d}{dt} \left(\int_a^b f(H(s, t)) \frac{\partial H}{\partial s}(s, t) ds \right) \stackrel{\text{Leibniz}}{=} \stackrel{\text{chain rule}}{=} \\ &= \int_a^b f'(H(s, t)) \frac{\partial H}{\partial t}(s, t) \cdot \frac{\partial H}{\partial s}(s, t) + f(H(s, t)) \frac{\partial^2 H}{\partial s \partial t}(s, t) ds = \\ &= \int_a^b \frac{d}{ds} \left[f(H(s, t)) \frac{\partial H}{\partial t} \right] ds = \\ &= f(H(s, t)) \frac{\partial H}{\partial t} \Big|_{s=a}^{s=b} \stackrel{\text{constant endpoints}}{=} 0. \end{aligned}$$

Having derivative identically equal to 0, means that $INT(t)$ is a constant function and $\int_{\gamma} f(z)dz = INT(0) = INT(1) = \int_{\mu} f(z)dz$. □

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Důsledek

Let U be a simply connected subset of \mathbb{C} and $f : U \rightarrow \mathbb{C}$ a holomorphic function. For any closed path whose image is inside U , $\int_{\gamma} f(z)dz = 0$.

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Důkaz (Sketch)

The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path c .

$$\int_{\gamma} f(z)dz = \int_c f(z)dz = \int_a^b f(c(z)) \cdot c'(z)dz = \int_a^b f(c(z)) \cdot 0dz = 0$$

└

□

Příklad

Let U be a simple connected domain and $f : U \rightarrow \mathbb{C}$ a holomorphic function on $U \setminus \{z_0\}$. For $j = 1, 2$, let γ_j be a path parametrizing a circle centered at z_0 of radius r_j , oriented counterclockwise and completely contained in U . Show that $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$.

1.3 Cauchy's integral formula

Věta 1.4 (Cauchy's integral formula)

Let γ be a loop around $z \in \mathbb{C}$, and $f : U \rightarrow \mathbb{C}$ a holomorphic function. For U a neighbourhood of γ ,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} dw.$$

Důkaz

Conway 1978, Chapter IV.

□

Důsledek

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0 + z_0 - z} dw = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \cdot \left(\frac{1}{1 - \frac{z - z_0}{w - z_0}} \right) dw = \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \left(\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} \right) dw = \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^n} \right) (z - z_0)^n. \end{aligned}$$

For sufficiently "small" (shrunk) γ . So f is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around z_0 converges to f in a neighbourhood of z_0).

Definition 1.5 (Pole)

Given a positive integer n , a complex function f has pole of order n at the point $z_0 \in \mathbb{C}$ if $(z - z_0)^n f(z)$ is holomorphic at z_0 but $(z - z_0)^{n-1} f(z)$ is not.

Příklad

Show that if f has a pole of order n at $z_0 \in \mathbb{C}$. Then it admits a Laurent expansion $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$ with $a_{-n} \neq 0$.

Definition 1.6 (Residue)

Let f have a pole of order n at the point $z_0 \in \mathbb{C}$. Then the residue of f at z_0 is the $k = -1$ coefficient of the Laurent expansion of f at z_0 .

Příklad

Show that if f has a pole of order 1 at z_0 , then the residue of f at z_0 can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)

Let $\gamma : [a, b] \rightarrow U \subset \mathbb{C}$ be a simple closed path, bounding a domain W containing the points z_1, \dots, z_m . Assume that f is holomorphic on $U \setminus \{z_1, \dots, z_m\}$ and has poles at $\{z_1, \dots, z_m\}$.

Show that

$$\oint_{\gamma} f(z)dz = 2\pi i \sum_{j=1}^m \text{res}_{z=z_j} f(z).$$

TODO!!!

1.4 (Real) Projective space

Poznámka (Building structures)

Set \rightarrow Topology \rightarrow Differential structure (atlas) \rightarrow Riemann metric \rightarrow Connection...

Definition 1.7 (Real projective space)

The set $\mathbb{P}^n(\mathbb{R})$ is defined to be either of the following bijective sets: Lines through the origin in \mathbb{R}^{n+1} ; Equivalence classes of $(n+1)$ -tuples of real numbers $(x_0, \dots, x_n) \neq (0, \dots, 0)$, such that for any real number $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$: $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$.

Příklad

Confirm that the sets above are in bijection with each other.

Poznámka (Notation)

We will often denote a point in $\mathbb{P}^n(\mathbb{R})$ as the equivalence class $[x_0, \dots, x_n]$.

Definition 1.8 (Topology of $\mathbb{P}^n(\mathbb{R})$)

We give a topology to $\mathbb{P}^n(\mathbb{R})$ by endowing it with following quotient topology: consider the surjection $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \rightarrow \mathbb{P}^n(\mathbb{R})$, $(x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]$. A set $U \subset \mathbb{P}^n(\mathbb{R})$ is defined to be open if $\pi^{-1}(U) := \{x \in \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \mid \pi(x) \in U\}$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}$.

That is we give $\mathbb{P}^n(\mathbb{R})$ the finest topology that makes π continuous.

Příklad

Check that for \mathbb{C} we can define $\mathbb{P}^n(\mathbb{C})$ or \mathbb{CP}^n the same way.

Příklad (Projective space)

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is an abelian group. Let \mathbb{R}^* act on \mathbb{R}^{n+1} by component wise multiplication. When a general group G acts on a set X we have equivalence relation $x \sim y$ if $y = g \circ x$. We call the equivalence classes the orbits of G . So $\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}) / \mathbb{R}^*$.

Sphere quotient: Let $S^n \subseteq \mathbb{R}^{n+1}$. Denote the unit sphere. Then the group $\mathbb{Z}_2 = \{+1, -1\}$ act on the sphere by $\pm 1(x_0, \dots, x_n) = (\pm x_0, \dots, \pm x_n)$. Then $S^n / \mathbb{Z}_2 = \mathbb{P}^n(\mathbb{R})$.

Disk model: Consider the n -dimensional closed unit disk $\overline{\mathbb{D}^n} \subseteq \mathbb{R}^n$, and the equivalence

relation on the points of the boundary: $x \sim -x$ if $\|x\| = 1$. Then $\mathbb{P}^n(\mathbb{R})$ is the quotient (collection of equivalence classes), i.e. $\overline{D^n} \setminus \sim \simeq \mathbb{P}^n(\mathbb{R})$.

Příklad

Conclude from either of these models of $\mathbb{P}^n(\mathbb{R})$ that as a topological space, $\mathbb{R}^n(\mathbb{P})$ is compact and Hausdorff.

Poznámka

Now we come to the smooth manifold structures. Let's start with $\mathbb{P}^1(\mathbb{R})$. Define

$$U_x := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x = 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y]) = \frac{x}{y}.$$

Similarly, we define a second chart:

$$U_y := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | y = 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y]) = \frac{y}{x}.$$

┌ *Příklad*

Check that U_x, U_y are open and that φ_x, φ_y are homeomorphisms.

┌ *Důkaz*

Consider the transition functions:

$$U = U_x \cap U_y = \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x, y \neq 0\}, \quad \varphi_x(U) = \varphi_y(U) = \mathbb{R} \setminus \{0\}.$$

The translation function $T_{x,y} := \varphi_y \circ (\varphi_x)^{-1}$ sends, for $y \neq 0$:

$$T_{x,y} : y \xrightarrow{(\varphi_x)^{-1}} [1, y] = \left[\frac{1}{y}, 1 \right] \xrightarrow{\varphi_y} \frac{1}{y}.$$

Which is smooth on the domain $\mathbb{R} \setminus \{0\}$.

└ TODO smooth. Thus $\mathbb{P}^1(\mathbb{R})$ is a smooth manifold. □

Příklad

Show that $\mathbb{P}^1(\mathbb{R})$ is homomorphic to the circle S^1 . We call $\mathbb{P}^1(\mathbb{R})$ the real projective line.

Příklad

Try to show that $\mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$ is a smooth manifold.

Příklad

For $\mathbb{P}^2(\mathbb{R})$ the followings charts form atlas:

$$U_x := \{[x, y, z] | x \neq 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y, z]) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

$$U_y := \{[x, y, z] | y \neq 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y, z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$U_z := \{[x, y, z] | z \neq 0\}, \quad \varphi_z : U_z \rightarrow \mathbb{R}, \quad \varphi_z([x, y, z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Check these are open subsets and homeomorphisms, with smooth transformation functions. And extend this to $\mathbb{P}^n(\mathbb{R})$.

1.5 Compact surfaces

Definition 1.9 (Surface)

A surface is a manifold of real dimension 2.

Například

\mathbb{R}^2 , \mathbb{C} , and any of their open subsets are surfaces. S^2 is a compact surface, as is $\mathbb{P}^2(\mathbb{R})$.

Definition 1.10 (Connected surface)

Given two connected surfaces S_1 and S_2 , the connected surface $S_1 \# S_2$ is the surface obtained by removing an open disc from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

Příklad

At the level of topological spaces, show that the operation $\#$ is well defined up to homeomorphism, that is, show that the choice of disks in S_1 and S_2 does not change the definition of $S_1 \# S_2$ / homeomorphism.

Příklad

Show that $\#$ gives the set of homeomorphism classes of connected compact surfaces the structure of a monoid. (Which surface is the identity of the monoid?)

Věta 1.5 (Classification of compact surfaces)

Any connected, compact surfaces is homeomorphic to exactly one surface in the following list:

- S^2 ;
- $T^{\#g} := T \# \dots \# T$, $g \in \mathbb{N}_0$;

- $\mathbb{P}^2(\mathbb{R})^{\#n} := \mathbb{P}^2(\mathbb{R}) \# \dots \# \mathbb{P}^2(\mathbb{R}), n \in \mathbb{N}_0.$

Poznámka (Deep fact)

For $d \leq 3$, if two d -dimensional manifolds are homeomorphic, then they are diffeomorphic.

2 Riemann surfaces

Definice 2.1 (Riemann surface)

A Riemann surface is a complex analytic manifold of dimension 1:

- X is a Hausdorff, connected topological space;
- for all $x \in X$, there is a homeomorphism $\varphi_x : U_x \rightarrow V_x$, where U_x is an open neighbourhood of $x \in X$, V_x is an open set in \mathbb{C} ;
- for any U_x, U_y such that $U_x \cap U_y \neq \emptyset$, the transition function $T_{x,y} := \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \rightarrow \varphi_y(U_x \cap U_y)$ is holomorphic.

Poznámka

We saw in the first lecture that a holomorphic preserves orientation when thought of as a function from the real plane to itself. Since our transition functions are holomorphic, any Riemann surface is orientable.

Příklad (The complex projective line)

Just as for $\mathbb{P}^1(\mathbb{R})$, we define $\mathbb{P}^1(\mathbb{C})$ to be the set whose elements are complex 1-dimensional subspaces of \mathbb{C}^2 .

Let $U_1 = U_2 := \mathbb{C}$ and define $g : U_1 \setminus \{0\} \rightarrow U_2 \setminus \{0\}, z \mapsto \frac{1}{z}$. We define $\mathbb{P}^1(\mathbb{C})$ to be the quotient $\mathbb{P}^1(\mathbb{C}) := U_1 \coprod U_2 / (z \sim g(z))$.

Příklad (Show that)

As a set $\mathbb{P}^1(\mathbb{C})$ is \mathbb{C} plus a point.

As a topological space $\mathbb{P}^1(\mathbb{C})$ is the one point compactification of \mathbb{C} .

Conclude from the previous sentence that $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the two sphere.

Poznámka

In complex analysis $\mathbb{P}^1(\mathbb{C})$ is known as the Riemann sphere.

Poznámka

For $i = 1, 2$, we denote the image of U_i in the quotient $U_1 \coprod U_2 / (z \sim g(z))$ by $[U_i]$. Note that U_i define the local coordinate functions: $\varphi_i : [U_i] \rightarrow U_i, p \mapsto z_i$, where z_i is the complex numbers in U_i such that $[z_i] = p_i$. Both φ_1, φ_2 are homeomorphisms.

We now consider the transition functions: the intersection

$$[U_1] \cap [U_2] = [U_1 \setminus \{0\}] = [U_2 \setminus \{0\}].$$

The image of the intersection under φ_1 is $\varphi_1([U_1] \cap [U_2]) = \mathbb{C} \setminus \{0\}$ (*). Thus (*) is the domain of our single transition function $T = \varphi_2 \circ \varphi_1^{-1}$. For $z_1 \neq 0$, we have $T : z_1 \xrightarrow{\varphi_1^{-1}} [z_1] = [z_2 := g(z_1) = 1/z_1] \xrightarrow{\varphi_2} z_2 = 1/z_1$. Thus

$$T : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \quad z \mapsto \frac{1}{z}.$$

Since T has a pole only at $z_1 = 0$, we see that it is holomorphic on $\mathbb{C} \setminus \{0\}$. A symmetric (exchange 1 for 2) calculation shows that T^{-1} is also holomorphic. So $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

Příklad (Hopf fibration)

Consider the 3-dimensional real sphere $S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4$. Given a point $p \in S^3$, \exists a unique line l_p through the origin and p . Thus we got a function $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C}), p \mapsto l_p$.

Check that H is continuous and surjective. Since S^3 is closed and bounded, it is compact. Moreover, since the image of a compact set under a continuous function is compact, $\mathbb{P}^1(\mathbb{C})$ is compact.

What is the fiber of the surjective map $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C})$, i.e. what is $H^{-1}(p)$, for any point $p \in \mathbb{P}^1(\mathbb{C})$. (Hint: It is S^1 .)

This gives us $S^2 \times S^1 = S^3$ as set. (Not as topological space!)

Příklad (Complex tori)

Definition: Let τ_1 and τ_2 be two complex numbers, that are linearly independent. The set of all integral linear combinations of τ_1 and τ_2 :

$$\Lambda := \{n\tau_1, m\tau_2 | n, m \in \mathbb{Z}\} \subseteq \mathbb{C}$$

is called the lattice of τ_1 and τ_2 .

Observe that we can assume that $\tau_1 = 1$, and $\Im(\tau_2) > 0$, allowing to make simplifying assumption: that our lattice has the form $\Lambda = \{n + m\tau | n, m \in \mathbb{Z}, \tau \in \mathbb{H}\}$, where \mathbb{H} is the upper half plane.

Consider the quotient space $T = \mathbb{C}/\Lambda$. That is the quotient space with respect to the equivalence relation $z_2 \sim z_1 \Leftrightarrow z_2 = z_1 + w$ for $w \in \Lambda$.

The canonical projection map $\pi : \mathbb{C} \rightarrow T$ (i.e. $\pi(z) = [z]$) induces a quotient topology on T (i.e. $V \subseteq T$ is open iff $p^{-1}(V)$ is open in \mathbb{C}).

Příklad

For P the closed parallelogram with vertices $0, 1, \tau, 1 + \tau$, show that for any $z \in \mathbb{C} \exists z' \in P$ that $\pi|_P \rightarrow T$ is surjective. Hence we can restrict our attention to p .

Poznámka

By considering the identification points in p , we see that T is topologically a torus.

Příklad

Prove that π (from previous exercise) is an open map, i.e. that V an open subset of \mathbb{C} implies that $\pi(V)$ is open in T .

Poznámka

Now to the complex structure: from the previous exercise, we see that if π restricted to a subset $V \subseteq \mathbb{C}$ is bijective, then it is a homeomorphism onto its image in T . In this case, $(\pi|_V)^{-1}$ is also a homeomorphism from the image of $\pi|_V$ to V . Hence we can use $(\pi|_V)^{-1}$ as a chart for T .

Příklad

Find a real number r (depends on t) such that for any $z \in \mathbb{C}$: π restricted to $B_r(z)$ is a bijective map.

Given this r , define $U_z := \pi(B_r(z)) \subseteq T$ and $\varphi_z := (\pi|_{B_r(z)})^{-1}$. We claim that the collection $\mathcal{A} = \{U_z, \varphi_z | z \in \mathbb{C}\}$ forms an atlas for T . It is clear that \mathcal{A} gives a cover for T . Moreover, by definition the maps φ_z are homeomorphic to their images. Assume that $U_{z_1} \cap U_{z_2} \neq \emptyset$. For $j \in [2]$ denote by (α_j, β_j) the unique pair of real numbers such that $z_j = \alpha_j + t\beta_j$. We have that $T_{21}(z) = (\varphi_{z_1} \circ \varphi_{z_2}^{-1})(z) = z + k$, where $k = ([\alpha_2] - [\alpha_1]) + ([\beta_1] - [\beta_2])t$ is just a constant depending on z_1 and z_2 . Therefore the transition function T_{21} is holomorphic $\implies T$ is a Riemann surface.

3 Graph of complex functions

Definice 3.1 (Graph)

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function. The graph of f is the set

$$\Gamma_f := \{(z, f(z)) | z \in \mathbb{C}\} \subseteq \mathbb{C} \times \mathbb{C},$$

given the subset topology.

Poznámka

Note that Γ_f is Hausdorff since $\mathbb{C} \times \mathbb{C}$ is Hausdorff. The graph of f is naturally given the structure of a Riemann surface by an atlas, with one chart, namely Γ_f : the local coordinate function is the projection map $\varphi := \pi_1|_{\Gamma_f}$, i.e. $(z, f(z)) \mapsto z$. TODO!!!(Jedna celá tabule)

Definice 3.2 (Affine plane curve)

For any polynomial $p(x, y) \in \mathbb{C}[x, y]$, the set $V(p) := \{(x, y) | p(x, y) = 0\} \subseteq \mathbb{C}^2$, is called an affine plane curve. We say that $V(p)$ is smooth if $\nexists (x_0, y_0) \in V(p)$ such that $\frac{\partial p}{\partial x}(x_0, y_0) = 0 = \frac{\partial p}{\partial y}(x_0, y_0)$.

Věta 3.1

A smooth affine plane curve is a Riemann surface.

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Důkaz

Let $(x_0, y_0) \in V(p)$. Since $V(p)$ is smooth, then for at least one of $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$ is non-zero at (x_0, y_0) . Assume (WLOG) that $\frac{\partial p}{\partial y}(x_0, y_0) \neq 0$. Then by the implicit function theorem, exists a neighbourhood $U_{(x_0, y_0)} \subseteq \mathbb{C}^2$, and a neighbourhood $V_{x_0} \subseteq \mathbb{C}$ and a holomorphic function $f : V_{x_0} \rightarrow \mathbb{C}$ such that $V(p) \cap U_{(x_0, y_0)} = \{(x, f(x)) | x \in V_{x_0}\}$. We call this the graph of f .

We get a local chart on $V(p)$ around (x_0, y_0) (as in the previous example) by setting $\varphi_{(x_0, y_0)} : V(p) \cap U_{(x_0, y_0)} \rightarrow V_{x_0}, (x, f(x)) \mapsto x$. Finally, we show that the transition functions are holomorphic: for $U_{(x_0, y_0)} \cap U_{(x, y)} \cap V(p) \neq \emptyset$, if $\varphi_{(x_0, y_0)}$ and $\varphi_{(x, y)}$ are both projections to the same axis, the transition function $\varphi_{(x_0, y_0)} \circ \varphi_{(x, y)}^{-1}$ is the identity function restricted to the appropriate domain in \mathbb{C} . Assume now that $\varphi_{(x_0, y_0)}$ is projection onto the x -axis and that $\varphi_{(x, y)}$ is projection to the axis y . Then set $U_{(x_0, y_0)} \cap U_{(x, y)} \cap V(p)$ is simultaneously on the graph of a holomorphic function f_0 and of a holomorphic function f_1 . Then functions all $\varphi_{(x, y)} \circ \varphi_{(x_0, y_0)}^{-1} = f_0(x)$ and $\varphi_{(x_0, y_0)} \circ \varphi_{(x, y)}^{-1} = f_1(x)$ restricted to the appropriate domains, which are holomorphic. □

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4 Projective curves

Příklad

Consider the polynomial $p(x, y, z) = x^2 + y + z + 1$. Note that $p(1, 1, 1) = 4 \neq 7 = p(2, 2, 2)$ since $[1, 1, 1] = [2, 2, 2]$ in $\mathbb{P}^2(\mathbb{C})$ p does not restrict to $\mathbb{P}^2(\mathbb{C})$.

Definice 4.1 (Homogeneous polynomial)

A polynomial $p \in \mathbb{C}[x, y, z]$ is said to be homogeneous of degree l , if the following equivalent conditions hold

- every monomial of p has degree l ;
- for each $t \in \mathbb{C}$: $p(tx, ty, tz) = t^l p(x, y, z)$;
- $x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} + z \frac{\partial p}{\partial z} = lp$.

┌ *Důsledek*

If p is homogeneous, $V(p) \subset \mathbb{P}^2(\mathbb{C})$ is well-defined.

Příklad

Confirm that these three conditions are equivalent.

Příklad

Show that if $p \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial, then the set of points $[x, y, z] \in \mathbb{P}^2(\mathbb{C})$ satisfying $p(x, y, z) = 0$ is well-defined.

Definice 4.2

We call

$$V(p) := \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid p(x, y, z) = 0\}$$

the vanishing locus of p . Moreover, we call $V(p)$ a (plane) projective curve of degree l .

If

$$\left\{ (x, y, z) \in \mathbb{C}^3 \mid \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \right\}$$

the $V(p)$ is said to be smooth.

Tvrzení 4.1

A smooth projective plane curve $V(p)$ is a compact Riemann surface.

Důkaz

We first show that $V(p)$ is compact by showing that $V(p)$ is closed set in $\mathbb{P}^2(\mathbb{C})$ which is a compact space.

Consider the diagram $\mathbb{P}^2(\mathbb{C}) \xleftarrow{\pi} \mathbb{C}^3 \setminus \{(0, 0, 0)\} \xrightarrow{p} \mathbb{C}$, where π is the natural projection function and p is the continuous function defined by the homogeneous polynomial $p : \mathbb{C}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{C}$, $(x, y, z) \mapsto p(x, y, z)$ by definition $V(p)$ is a closed subset of $\mathbb{P}^2(\mathbb{C})$ if $\pi^{-1}(V(p))$ is closed in $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$. But $\pi^{-1}(V(p)) = p^{-1}(0)$ is the inverse image of the closed set $\{0\} \subseteq \mathbb{C}$. Thus since p is continuous, $p^{-1}(0)$ is closed, in other words, $\pi^{-1}(V(p))$ is closed. Thus $V(p)$ is compact.

So, to show that $V(p)$ is Riemann surface, we need to show that its intersection with any of the coordinate open sets of $\mathbb{P}^2(\mathbb{C})$ is a Riemann surface. So let us consider (WLOG) the chart $U_z = \{[x, y, z] | z \neq 0\} \subseteq \mathbb{P}^2(\mathbb{C})$ with affine coordinates $\varphi_z(x, y, z) = (\frac{x}{z}, \frac{y}{z})$. The set $\varphi_z(V(p) \cap U_z)$ is equal to $V(\tilde{p})$ where $\tilde{p}(x, y) = p(x, y, 1)$.

Now for any $(x, y) \in \mathbb{C}^2$:

$$(*) : \frac{\partial \tilde{p}}{\partial x}(x, y) = \frac{\partial p}{\partial x}(x, y, 1), \quad (**) : \frac{\partial \tilde{p}}{\partial y}(x, y) = \frac{\partial p}{\partial y}(x, y, 1).$$

We claim \nexists an $(\hat{x}, \hat{y}) \in \mathbb{C}^2$ such that

$$\tilde{p}(\hat{x}, \hat{y}) = \frac{\partial \tilde{p}}{\partial x}(\hat{x}, \hat{y}) = \frac{\partial \tilde{p}}{\partial y}(\hat{x}, \hat{y}) = 0.$$

This claim implies that $V(\tilde{p})$ is a smooth affine plane curve and hence a Riemann surface. Since the restriction of $V(p)$ with any affine chart is a Riemann surface, then so is $V(p)$.

So it remains to prove the claim: Assume $\exists(\hat{x}, \hat{y}) \in \mathbb{C}^2$ satisfying condition above. By $(*)$ and $(**)$, together smoothness of $V(p)$, which would imply that $\frac{\partial p}{\partial z}(\hat{x}, \hat{y}, 1) \neq 0$. But now Euler's identity implies $0 \neq \frac{\partial p}{\partial x}(\hat{x}, \hat{y}, 1) + \frac{\partial p}{\partial y}(\hat{x}, \hat{y}, 1) + \frac{\partial p}{\partial z}(\hat{x}, \hat{y}, 1) = lp(\hat{x}, \hat{y}, 1) = 0 \implies$ contradiction \implies we are done. \square

Příklad

Confirm that $V(p)$ is Hausdorff.

Například (Elliptic curves)

Consider a polynomial p of the form $p(x, y, z) = y^2z - (x - \alpha_1z)(x - \alpha_2z)(x - \alpha_3z)$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ are distinct complex numbers. Note that the partial derivative with respect to y satisfies $\frac{\partial p}{\partial y} = 2yz$, which is zero only if $y = 0$ or $z = 0$. We show that $V(p)$ is a smooth projective curve by considering the case $z = 0$, $y = 0$, and finding in each case a non-vanishing partial derivative.

„Case $z = 0$ “: Then the only part in $\mathbb{P}^2(\mathbb{C})$ belonging to $V(p)$ is $[0, 1, 0]$. But we have $\frac{\partial p}{\partial z} = y^2 + Q(x, z) = 1 + 0 \neq 0$.

„Case $y = 0$ “: Then the parts belonging to $V(p)$ are $[\alpha_1, 0, 1]$, $[\alpha_2, 0, 1]$, $[\alpha_3, 0, 1]$. For $j \in [3]$: $\frac{\partial p}{\partial x}(\alpha_j, 0, 1) \neq 0$, follows from the fact that the $\alpha_1, \alpha_2, \alpha_3$ are distinct.

So $V(p)$ is a smooth projective curve of degree 3.

Například

Consider the function $\varphi : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^3(\mathbb{C})$ defined in homogeneous coordinates by $\varphi[s, t] = [s^3, s^2t, st^2, t^3]$. We call the image of φ the twisted cubic in $\mathbb{P}^3(\mathbb{C})$.

5 Holomorphic maps of Riemann surfaces

Definition 5.1

Let X, Y be two Riemann surfaces, and $f : X \rightarrow Y$ a function (of sets).

- We say that f is holomorphic at $x \in X$ if for every choice of charts $\varphi_x, \varphi_{f(x)}$, the function $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic.
- If $U \subseteq X$ is open, we say that f is holomorphic on U if f is holomorphic at each $x \in U$.
- If f is holomorphic on $U = X$, then we say that f is a holomorphic map.

The function $F := \varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is called a local expression for f .

Příklad

Show that a map of Riemann surfaces $f : X \rightarrow Y$ is holomorphic at $x \in X \Leftrightarrow \exists$ a choice of charts $\varphi_x, \varphi_{f(x)}$ such that $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic at x .

Příklad (Eg)

Let X, Y be two Riemann surfaces and choose a point $y \in Y$. Denote associated constraint map $c : X \rightarrow Y, x \mapsto y$. We see that c is a holomorphic map.

Příklad (Eg)

Let X be a Riemann surface. The identity map on X is the function $I_X : X \rightarrow X, x \mapsto x$. We see that I_x is a holomorphic map.

Příklad (Eg)

Consider an arbitrary rational function $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z) \in \mathbb{C}[z]$ are two polynomials with distinct roots. We can extend f to a function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$ as follows:

- $f(\alpha) := \infty$ (point at infinity of the Riemann sphere) where α is a root of q ;
- $f(\infty) := \lim_{z \rightarrow \infty} \frac{p(z)}{q(z)}$.

Show that f is a holomorphic function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$.

Recall that $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{C} \cup \infty$ with a topology. Identifying \mathbb{C} with image of the first affine chart $\varphi_1([U_1])$, the additional chart point corresponds to the image of zero in the second affine chart. We denote this point by ∞ .

This we have that $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Using this identification, define the function $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$, $z \mapsto z^2 =: \omega, \infty \mapsto \infty$. We will now show that f is a holomorphic map.

We denote by z the local coordinate on the source and by ω the local coordinate on the range. Then since $\omega = f(z) = z^2$ is a holomorphic function on all of \mathbb{C} , f is holomorphic on the image of U_1 .

It remains to show that f is holomorphic at ∞ . We consider the local expression for f using the chart U_2 whose image contains ∞ . We denote by $\tilde{z} := \frac{1}{z}$ the corresponding local coordinate for the source and $\tilde{\omega} := \frac{1}{\omega}$ for the range.

The local expression \tilde{F} for f in these coordinates is then obtained by composing $F(z)$ with the transition functions for the local coordinate:

$$\tilde{F}(\tilde{z}) = \tilde{\omega} = \frac{1}{\omega} = \frac{1}{z^2} = (\tilde{z})^2.$$

Since the point at ω corresponding to $\tilde{z} = \tilde{\omega} = 0$, and we have that $f(\infty) = \infty$ and $\tilde{F}(0) = 0$, the local expression extends to the whole chart, and in particular, it is a holomorphic function at 0. So f is holomorphic at every point of $\mathbb{P}^1(\mathbb{C})$.

Definition 5.2

Two Riemann surfaces X, Y are called isomorphic (or bi-holomorphic) if \exists holomorphic maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. In this case we write $X \simeq Y$. We call f and g isomorphisms (or bi-holomorphisms). An isomorphism $h : X \rightarrow X$ from a Riemann surface to itself is called an automorphism.

Příklad

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function and $\Gamma_f \subseteq \mathbb{C}^2$ its graph. Show that $\Gamma_f \simeq \mathbb{C}$.

Definition 5.3

We say that a chart (U_x, φ_x) for a Riemann surface is centered at x if $\varphi_x(x) = 0$.

Věta 5.1

Let $f : X \rightarrow Y$ be a non-constant holomorphic map of Riemann surfaces. For any $x \in X$ \exists charts centered at x such that the local expression of f in terms of these charts is $z \mapsto z^h$, for some $h \geq 1$.

┌

Důkaz

Let φ and ψ be two charts centered at x , and $f(x)$, respectively. This gives the local expression $F = \psi \circ f \circ \varphi^{-1}$. Consider the Taylor expansion of F at 0 and let k be the smallest positive integer such that the coefficient of z^k does not vanish. Since $F(0) = 0$, $k > 1$, and $F(z) = z^k (\sum_{n=0}^{\infty} a_{k+n} z^n)$. Denote by $G(z) = a_k + a_{k+1}z + \dots$ the second factor in that equation. The function $G(z)$ is holomorphic at 0 and $G(0) = a_k \neq 0$. Thus we can make a choice of root $\sqrt[k]{G(z)}$ such that the associated map is well-defined and holomorphic around $0 = \varphi(z)$.

Defining $h(z) = z \cdot \sqrt[k]{G(z)}$ we have that $F = h^k$. The function h is holomorphic in a neighbourhood U of $0 = \varphi(x)$, $h(0) = 0$, and $h'(0) \neq 0$. The inverse function theorem now implies that h is bi-holomorphic on a neighbourhood $U' \subseteq U$ of $\varphi(x)$, and therefore the composition $\tilde{\varphi} = h \circ \varphi$ gives local chart for x centered at x . The local coordinate \tilde{z} coming from $\tilde{\varphi}$ is related to z with $\tilde{z} = h(z)$. The local expression for f allows x is now obtained by changing coordinates from z to \tilde{z} in F :

$$\tilde{F}(\tilde{z}) = F(z(\tilde{z})) = h(z)^k = (\tilde{z})^k.$$

└

□

Definice 5.4

Let $f : X \rightarrow Y$ be a non-constant holomorphic map of Riemann surfaces

- Given a point $x \in X$, the integer k_x such that \exists a local expression centered at x of the form $F(z) = z^{k_x}$ is called the Ramification index of f at x .
- The quantity $V_x = k_x - 1$ the differential length of f at x .
- A point x with Ramification index $k_x = 1$ is called unramified.
- A point for which $k_x \geq 2$ is ramified (ramification point).
- If x is a ramification point, then $f(x) \in Y$ is called a branch point.
- The ramification locus is the subset of X consisting of all ramification points of f .
- The branch locus is $f(R)$.

Příklad

Let $f : X \rightarrow Y$ be a non-constant holomorphic map of two Riemann surfaces, and let $x \in X$. Suppose that f has two local expressions around x of the form $F(z) = z^k$, $\tilde{F}(\tilde{z}) = \tilde{z}^{\tilde{k}}$, then $k = \tilde{k}$.

Řešení (Hint)

Let $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$ be charts. Observe that the change of coordinates near x must be of the form $\tilde{z} = z\alpha(z)$, where $\alpha(z)$ is holomorphic in a neighbourhood of 0, and $\alpha(0) \neq 0$. (Try to draw this in picture.)

Similarly (but going in the opposite direction) $\omega = \tilde{\omega}\beta(\tilde{\omega})$ for some holomorphic function β with $\beta(0) \neq 0$. Write down the Taylor expansions of α and β and use the fact that F is obtained from \tilde{F} via the above changes of coordinates.

Lemma 5.2

The ramification locus R is a discrete subset of open subsets (i.e. $\exists U_i \subset X$, such that U_i contains exactly one element of R).

Poznámka

A discrete subset of a compact set is finite (an infinite set has a limit point, thus a discrete subset, which has no limit points, must be finite).

Lemma 5.3

If X is a compact Riemann surface and $f : X \rightarrow Y$ is a non-constant holomorphic map of two Riemann surfaces, the ramification locus R is a finite set. Moreover, the branch locus is also finite.

6 Holomorphic maps between compact Riemann surfaces

Věta 6.1

Let $f : X \rightarrow Y$ be a holomorphic of Riemann surfaces. If X is compact and f is non-constant, then f is onto / surjective.

Důkaz

Consider the image $f(X) \subset Y$. By Liouville's theorem (from complex analysis) $f(x)$ is open. On the other hand, since X is compact and f is continuous, $f(X)$ is compact. Since Y is Hausdorff, we must have that $f(X)$ is closed.

Finally, since $f(X)$ is an open, closed, and non-empty subset of a connected topological space, we must have that $f(X) = Y$. \square

Důsledek

Hence Y is also compact.

Důsledek

If X is a compact Riemann surface, the only holomorphic functions $f : X \rightarrow \mathbb{C}$ are the constant functions.

Poznámka (Discussion)

Let $f : X \rightarrow Y$ be a non-constant map of compact Riemann surfaces. Consider a point $x \in f^{-1}(y)$. We know that \exists a neighbourhood U_x and coordinate charts centered at x such that we have local expression $z \rightarrow z^{k_x}$.

Since $F^{-1}(0) = 0$, there are no other pre-images of y in U_x . Thus the pre-image set $f^{-1}(y)$ is discrete, and since X is compact, the pre-image is a finite set.

Věta 6.2

Let $f : X \rightarrow Y$ be a non-constant holomorphic map of compact Riemann surfaces. If y_0, y_1 are not in the branch locus of f then $|f^{-1}(y_0)| = |f^{-1}(y_1)|$.

┌

Důkaz

└

Next week.

□

Definice 6.1

For $f : X \rightarrow Y$ a non-constant map of compact Riemann surfaces, the degree of f is the cardinality of the fiber $f^{-1}(y)$, for any $y \notin$ branch locus of f . If f is constant, then we say that its degree is 0.

Důkaz (next week proof)

Let B be the branch locus of f . Since B is finite, $Y \setminus B$ is a connected topological space (Y is a surface). Hence it cannot have a proper subset which is both closed and open. Let $y_0 \in Y \setminus B$ and set $d := |f^{-1}(y_0)|$.

The set $A := \{y \in Y \setminus B \mid |f^{-1}(y)| = d\}$ is non-empty since it contains y_0 . We claim that „ A is open in $Y \setminus B$ “: For any $y \in A$ we show that y is an interior point of A . Denote by $\{x_1, \dots, x_d\}$ the inverse images of y and by U_{x_1}, \dots, U_{x_d} pairwise disjoint charts around each inverse such that f admits the local expression $F(z) = z$ on each chart. We can choose (see exercise below) $V_y \subseteq F(U_{x_1}) \cap \dots \cap f(U_{x_d})$ a connected neighbourhood of y homeomorphic to a disk, such that the inverse image $f^{-1}(V_y)$ consists of d connected components \tilde{U}_{x_i} (each contains one of the x_i and is contained in U_{x_i}). Then f restricted to each \tilde{U}_{x_i} is bijective onto V_y . This implies that every point of V_y has d inverse images, and hence that y is an interior point for A .

Now consider $A^c = \{y \in Y \setminus B \mid |f^{-1}(y)| \neq d\}$. With essentially the same argument, one shows that A^c is also open in $Y \setminus B$. Thus we can conclude that A is closed. Since A is a non-empty, open and closed subset of a connected topological space, $A = Y \setminus B$. □

Příklad

Show that a V_y (as in the previous proof) exists by showing that otherwise one may construct a sequence $\{\xi_n\}$ of points of X such that $\xi_n \rightarrow \xi$, $f(\xi_n) \rightarrow y$ but $f(\xi) \neq y$ violating continuity.

Definition 6.2 (Degree)

For $f : X \rightarrow Y$ a non-constant holomorphic map of compact Riemann surfaces, the degree of f is the cardinality of the fiber $|f^{-1}(y)|$, for any $y \in Y$ such that $y \notin \text{branch locus}$. If f is constant then we say that it has degree 0.

Příklad

Let $f : X \rightarrow Y$ be a holomorphic map of compact Riemann surfaces, of degree $d > 0$, $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_n\}$. Prove that $\sum_{i=1}^n k_{x_i} = d$.

Poznámka (Recall the classification of compact Riemann surfaces)

Any compact, connected surface is homeomorphic to one surface in the following list: S^2 (two sphere, also \mathbb{C} -projective plane $\mathbb{P}\mathbb{C}$), $T^{\#g}$, $\mathbb{P}^2(\mathbb{R})^{\#n}$.

The first two are Riemann surfaces (Ex: the connected sum of Riemann surfaces is again a Riemann surface). The third are not orientable, so not Riemann surfaces.

Definition 6.3 (Genus)

Define the genus of a compact connected Riemann surface as 0 for 2 sphere and g for connected sum of g tori.

┌ *Poznámka* (Alternative definition of genus)

A vector field X on a differential manifold is a derivation on $C^\infty(m)$ i.e. a \mathbb{R} linear map $X : C^\infty(m) \rightarrow C^\infty(m)$ such that $X(f \cdot g) = X(f) \cdot g + f \cdot X(g) \forall f, g \in C^\infty(m)$. The space of vector fields is denoted by $\mathcal{X}(m)$. Note that $\mathcal{X}(m)$ is a $C^\infty(m)$ -module. $(fX)(g) = f(X(g))$. $((fX)(g \cdot h) = f(X(g \cdot h)) = f(X(g) \cdot h + g \cdot X(h)) = f(X(g)) \cdot h + g \cdot f(X(h))$.)

Denote $\Omega^1(m) = \mathcal{X}(m)^* = \text{Mor}_{C^\infty(m)}(\mathcal{X}(m), C^\infty(m))$. This is again $C^\infty(m)$ -module (just as for any dual module)

$$(f\omega)(X) := f(\omega(X)) \quad ((f\omega)(gX) = f(\omega(gX)) = g(f\omega(X)) = g((f\omega)(X))).$$

Consider the map $d : C^\infty(m) \rightarrow \Omega^1(m)$, $f \mapsto df$, where $df : \mathcal{X}(m) \rightarrow C^\infty(m)$, $X \mapsto X(f)$. (Note that $d(fg)(X) = X(fg) = X(f) \cdot g + f \cdot X(g) = df(X) \cdot g + f dg(X) = (g \cdot df)(X) + (f \cdot dg)(X) \implies d(f \cdot g) = g \cdot df + f \cdot dg$.)

Now let's restrict to surfaces:

$$\Omega^2(m) := \frac{\Omega^1(m) \otimes_{C^\infty(m)} \Omega^1(m)}{\{\omega \otimes \nu + \nu \otimes \omega\}}.$$

We denote the product in $\Omega^2(m)$ by \wedge . $\implies \omega \wedge \nu = -\nu \wedge \omega$.

└ TODO example?

Věta 6.3 (Riemann–Hurwitz)

For $f : X \rightarrow Y$ be non-constant, degree d , holomorphic map of compact Riemann surfaces. Denote by g_x , resp. g_y , the genus of X , and Y . Then $2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} v_x$, where $v_x = h_x - 1$ is the differential length of f at x .

Příklad

Observe that $\sum_{x \in X} v_x = \sum_{x \in \text{Ram. locus}} v_x$ and hence conclude that this sum is well-defined.

Poznámka

Surface	$\chi = 2 - 2g$	genus g
\mathbb{S}^2	2	0
$T^{\#m}$	$2 - 2m$	$2m$
$\mathbb{P}^2(\mathbb{R})^{\#m}$	$2 - m$	m

Příklad

Let $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ be a degree d holomorphic map. Assume two points $x_1, x_2 \in \mathbb{P}^1(\mathbb{C})$ have full ramification, that is $k_{x_1} = k_{x_2} = d$. We will use Riemann–Hurwitz to show that there are no more ramification points for f .

Řešení

Since $g_x = g_y = 0$, we have $\text{TODO? } (-2 = d(-2) + (d-1) + (d-1) + \sum_{x \neq x_1, x_2} \nu_x \Leftrightarrow 0 = \sum_{x \neq x_1, x_2} \nu_x)$.

Since differential lengths are non-negative integers, for $x \neq x_1, x_2$ we must have $\nu_x = 0$.

Příklad

Consider $f : E \rightarrow \mathbb{P}^1(\mathbb{C})$ with E a Riemann surface with $g_E = 1$. As in the previous example, assume \exists two points $x_1, x_2 \in E$ where f is fully ramified. How many more ramification points can f have, and what ramification indices are possible?

Příklad

Suppose that $f : X \rightarrow Y$ is a non-constant holomorphic map of connected compact Riemann surfaces. Prove that $\sum_{x \in X} \nu_x$ is even, $g_X \geq g_Y$ (a Riemann surface X can never map (non-trivial) to a Riemann surface of higher genus), if $\sum_{x \in X} \nu_x = 0$, then $g_X = dg_Y - d$.

Příklad (Recall)

Consider an arbitrary rational function $f = \frac{p}{q}$, for $p, q \in \mathbb{C}[z]$ two polynomials with distinct roots. We can extend f to a function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$ by defining $f(\alpha) = \infty$ for roots of q and $f(\infty) = \lim_{z \rightarrow \infty} \frac{p(z)}{q(z)}$.

This extension is a holomorphic map.

Věta 6.4

If $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is a holomorphic map of Riemann surfaces, then f is a rational function $f = \frac{p}{q}$, for $p, q \in \mathbb{C}[z]$.

Důkaz

If f is a constant function, then it is rational. Let f be a non-constant holomorphic function and denote by z_1, \dots, z_n the inverse images of 0. Moreover, let p_1, \dots, p_m be the inverse images of ∞ . Define the rational function

$$\varphi(z) := \frac{\prod_{i=1}^n (z - z_i)^{k_{z_i}}}{\prod_{j=1}^m (z - p_j)^{k_{p_j}}},$$

where the natural numbers k_{z_i} and k_{p_j} are the ramification indices of f at these points. In the expression of φ we adopt the notation $z - \infty = 1$.

The function $\frac{f}{\varphi}$ does not take the values 0 or ∞ . In particular, $\frac{f}{\varphi}$ is not surjective. Hence, since any non-constant holomorphic $h : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is surjective, we can conclude that $\frac{f}{\varphi}$ is constant, and hence $f = c \cdot \varphi$ for some constant c . \square

Poznámka

If one prefers to describe points of $\mathbb{P}^1(\mathbb{C})$ with their homogeneous coordinates, then a holomorphic function $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is given by $f(x, y) = (p(x, y), q(x, y))$, where p and q are homogeneous polynomials of the same degree.

7 Maps of elliptic curves

Definice 7.1

Let $p(x, y, z) = y^2z - (x - a_1z) \cdot (x - a_2z) \cdot (x - a_3z)$ for distinct a_1, a_2, a_3 , and $E = V(p)$ the associated elliptic curve.

Věta 7.1

An elliptic curve is a Riemann surface of genus 1.

┌

Důkaz

Consider the affine chart $U_z = \{z \neq \mathbf{o}\} \subseteq \mathbb{P}^1(\mathbb{C})$, with coordinates $(x, y) = (x/z, y/z)$, the restriction of E to this chart is the affine curve E_z determined by the equation $y^2 = (x - a_1) \cdot (x - a_2) \cdot (x - a_3)$. The vertical projection map $\pi : (x, y) \mapsto x$ restricts to a holomorphic map $\pi : E_z \rightarrow \mathbb{C}$. One can verify that, for every point $x \in \mathbb{C} \setminus \{a_i\}_{i=1}^3$, π^{-1} consists of two points, i.e. the degree of π is 2.

Using next fact, the result is map of degree 2 of connected compact Riemann surfaces. The Brach locus for $\tilde{\pi}$ is $B = \{a_1, a_2, a_3, a_4 = \infty\}$. Denote $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ the corresponding ramification points. Since for a map of degree 2, the only non-trivial ramification has differential length equal to 1, the Riemann–Hurwitz formula now reads:

$$2g_E - 2 = 2(-2) + \sum_{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4} 1$$

┌ which is to say $g_E = 1$. □

Poznámka (Fact)

It can be shown (As the consequence of Riemann existence theorem), that π extends to a holomorphic map $\tilde{\pi} : E \rightarrow \mathbb{P}^1(\mathbb{C})$.

Příklad

A non-constant holomorphic map between complex tori $\tilde{f} : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ is called an isogeny. An isogeny is defined by a holomorphic map $f : \mathbb{C} \rightarrow \mathbb{C}$ which is well-defined, when the mod? out by the lattices Λ, Λ' , that is $f(z + l) = f(z) + l', \forall z \in \mathbb{C}, l \in \Lambda$ and some $l' \in \Lambda'$.

Use the Riemann–Hurwitz formula to show that any isogeny is unramified.

Show that any isogeny is induced by a map $f(z) = az$, $f(\Lambda) \subseteq \Lambda'$.

Consider the isogeny $\tilde{f} : \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda'$ induced by the function $f(z) = z + 1$, where $\Lambda = \{n + m(1 + i), n, m \in \mathbb{Z}\}$, $\Lambda' = \{n(1/2) + m(1/2 + i/2), n, m \in \mathbb{Z}\}$, find the degree of \tilde{f} .

Příklad

Consider a lattice Λ generated by the vectors $\varrho_1 e^{i\theta_1}$, $\varrho_2 e^{i\theta_2}$, where $0 \leq \theta_1 < \theta_2 < 2\pi$. Denote by $\tau := \frac{\varrho_2}{\varrho_1} e^{i\theta_2 - i\theta_1}$, and by Λ' the lattice generated by the vectors 1 and τ . Prove that the complex tori \mathbb{C}/Λ and \mathbb{C}/Λ' are isomorphic.