# Úvod

*Poznámka* (Organizační úvod)
Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (... PDE ...).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

*Poznámka* (Konvence pro PDR)
$\Omega \subseteq \mathbb{R}^d$ je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

*Poznámka*
Dále se ukazovali konkrétní parciální rovnice.

*Poznámka* (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál: $L^p(\Omega)$, $L^p_{loc}(\Omega)$, $||u||_p$, $C^k(\Omega)$, $C^k(\overline{\Omega})$,
$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) | \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha} < \infty \right\}, ||u||_{C^{0,\alpha}} = \sup_{x \neq y} \frac{u(x) - u(y)}{|x-y|^\alpha}.$$
- $\int_\Omega \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$, $\vec{n} = (n_1, \ldots, n_d)$.
- Funkcionální analýza 1: Banachův prostor, $u^n \to u$ silná konvergence, $u^n \rightharpoonup u$ slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita ($L^p$ jsou separabilní až na $p = \infty$, $C^k(\overline{\Omega})$ je separabilní, $C^{0,\alpha}$ není separabilní pro $\alpha \in (0, 1]$).

*Poznámka* (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

# 1 Sobolevovy prostory

**Definice 1.1** (Multiindex)

$\alpha$ je multiindex $\equiv d = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$. Délka $\alpha$ je $|\alpha| := \alpha_1 + \ldots + \alpha_d$. Pro $u \in C^k(\Omega)$ definujeme $D^\alpha u = \frac{\partial^{|d|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$.

**Definice 1.2** (Slabá derivace)

Buď $u, v_\alpha \in L^1_{loc}(\Omega)$. Řekneme, že $v_\alpha$ je $\alpha$-tá slabá derivace $u \equiv$

$$\equiv \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

*Příklad*
$u = \mathrm{sign} x$ nemá slabou derivaci.

**Lemma 1.1** (O smysluplnosti)

*Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.*

*Důkaz*
$v_\alpha^1$, $v_\alpha^2$ dvě $\alpha$-té derivace $u$.

$$(-1)^{|\alpha|} \int v_\alpha^1 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int v_\alpha^2 \varphi = \int_\Omega u D^\alpha \varphi \forall \qquad \varphi \in C_0^\infty(\Omega)$$

$$\int_\Omega (v_\alpha^1 - v_\alpha^2) \varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ skoro všude v $\Omega$.

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají. □

**Definice 1.3** (Sobolevův prostor)

$\omega \subseteq \mathbb{R}^d$ otevřená, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}.$$

$$||u||_{W^{k,p}(\Omega)} ||u||_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} ||D^\alpha u||_p^p\right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha|\leqslant k} ||D^\alpha u||_\infty, & p = \infty. \end{cases}$$

*Poznámka*
Od teď $D^\alpha$ nebo $\frac{\partial}{\partial x_1}$ nebo $\partial_i$ značí slabou derivaci.

**Lemma 1.2** (Základní vlastnosti slabých derivací a Sobolevových prostorů)
*Nechť $u, v \in W^{k,p}(\Omega), k \in \mathbb{N}$, a $\alpha$ multiindex s délkou $\leqslant k$.*

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ *a* $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, *pro* $|\alpha| + |\beta| \leqslant k$.
- $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ *a* $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
- $\forall \tilde{\Omega} \subseteq \Omega$ *otevřená*
$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$
- $\forall \eta \in C^\infty(\Omega)$: $\eta u \in W^{k,p}(\Omega)$ *a* $D^\alpha(\eta u) = \sum_{\beta_i \leqslant \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$, *kde* $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

*Důkaz*
Cvičení na doma. □

**Věta 1.3** (Basic properties of Sobolev spaces)
*Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

- $W^{k,p}(\Omega)$ *is a Banach space;*
- *if* $p < \infty$ *it is separable space;*
- *if* $p \in (1, \infty)$ *it is reflexive space.*

*Důkaz*
BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness: $u^n$ is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \to u$ in $L^p$. $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \to v_a \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

*TODO*

$$|\int_\Omega (v_\alpha - D^\alpha u^n)\varphi| \leqslant ||v_\alpha - D^\alpha u^n||_p ||\varphi||_{p'} \leqslant C||v_\alpha - D^\alpha u^n|| \to 0.$$

$$|\int_\Omega (u^n - u) D^\alpha \varphi| \leqslant ||u^n - u||_p ||D^\alpha \varphi||_{p'} \leqslant C||u^n - u||_p \to 0.$$

„2+3“: $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \ldots \times L^P(\Omega)$ ($d+1$ times), $X$ closed subspace from first property. Lemma: if $X \subseteq Y$ is closed subspace then $Y$ separable $\implies$ $X$ separable and $Y$ reflexive $\implies$ $X$ reflexive. (From functional analysis and topology.) □

# 2 Approximation of Sobolev function

**Věta 2.1**

*Let $\Omega \subseteq \mathbb{R}^d$ open, ?. $p \in [1, \infty)$.*

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} = W^{k,p}(\Omega).$$

*Pozor*

$$\overline{\{u \in C^\infty(\Omega)\}}^{||\cdot||_{k,p}} \subsetneq W^{k,p}(\Omega).$$

*Důkaz*
Summer semester. □

**Věta 2.2** (Local density)

$$\forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty$$
$$u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} open, \overline{\tilde{\Omega}} \subseteq \Omega$$
$$u^n \to u in W^{k,p}(\tilde{\Omega})$$

*Důkaz*
$u$ is extended by 0 to $\mathbb{R}^d \backslash \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \qquad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \qquad \eta \in C_0^\infty(B_1), \eta \geqslant 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

$$u \in L^P(SET) \qquad u^\varepsilon \to u in L^p(SET).$$

We need: $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\forall \alpha, |\alpha| \leqslant k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leqslant \varepsilon_0$ (so that ball of radius $\varepsilon_0$ and center in $\tilde{\Omega}$ is in $\Omega$):

$$(D^\alpha u)^\varepsilon(x) = \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} D^\alpha u(y)\eta_\varepsilon(x-u)dy =$$

$$= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y)dy.$$

$$D^\alpha u^\varepsilon = D_x^\alpha \int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)dy.$$

□

## Tvrzení 2.3

*$\Omega$ is open connected set, $u \in W^{1,1}(\Omega)$, then $u = \mathrm{const}. \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \ \forall i \in [d]$.*

*$W^{1,1}(I) \hookrightarrow C(I)$ for $I$ interval.*

*$W^{d,1}(B_1) \hookrightarrow C(B_1)$.*

*Důkaz*
„1. $\implies$" trivial. „1. $\impliedby$": $\tilde{\Omega} \subseteq \Omega$ connected $\varepsilon_0$ as before and $\varepsilon \in (0, \varepsilon_0)$. $u^\varepsilon$-modification of $u$ is smooth, so

$$\frac{\partial u^\varepsilon}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^\varepsilon = 0 \quad in \tilde{\Omega}$$
$$\implies u^\varepsilon = \mathrm{const}(\varepsilon) \quad in \tilde{\Omega}.$$

$$c(\varepsilon) = \int_{\mathbb{R}} c(\varepsilon)\eta_\delta(x-y)dy = \int_{\mathbb{R}} u^\varepsilon(y)\eta_\delta(x-y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\eta_\varepsilon(y-z)\eta_\delta(x-y)dzdy =$$
$$\int\int u(z+y)\eta_\varepsilon(z)\eta_\delta(y-x)dzdy = \int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dzdw =$$
$$\int\int u(z+x+y)\eta_\varepsilon(z)\eta_\delta(u)dwdz = \int_{\mathbb{R}^d} u^\delta(z+x)\eta_\varepsilon(z)dz = \int c(\delta)\eta_\varepsilon(z)dz = c(\delta).$$

„2.": WLOG $I = (0,1)$. Define $v(x) = \int_0^x \frac{\partial u}{\partial y}(y)dy$. We show: $v \in W^{1,1}(I)$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$.

$$|v(x)| \leqslant \int_0^1 |\frac{\partial u}{\partial x}| \leqslant ||u||_{1,1}.$$
$$\varphi \in C_0^1(0,1) \qquad \int_0^1 v(x)\frac{\partial \varphi}{\partial x}(x)dx$$
$$= \int_0^1 \left(\int_0^x \frac{\partial u}{\partial y}(y)dy\right)\frac{\partial \varphi}{\partial x}(x)dx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dydx = \int_0^1\int_0^1 \frac{\partial u(y)}{\partial y}\frac{\partial \varphi(x)}{\partial x}x_{0<y<x}dxdy =$$
$$= \int_0^1 \left(\int_y^1 \frac{\partial \varphi(x)}{\partial x}dx\right)\frac{\partial u}{\partial y}(y)dy = -\int_0^1 \varphi(y)\frac{\partial u}{\partial y}(y)dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}.$$

TODO.

$$x \to y \implies \int_y^x |\frac{\partial u}{\partial z}|^\alpha \to 0 \implies |u(x) - u(y)| \to 0$$
$$||u||_{C(I)} \leqslant ||v+c||_{C(I)} \leqslant ||u||_{1,1} + |c| = ||u||_{1,1} + |u(x) - v(x)| \forall x \in I$$
$$||u||_{C(I)} \leqslant ||u||_{1,1} + \int_0^1 |u(x) - v(x)|dx \leqslant -|| - + \int_0^1 |u| + \int_0^1 |v| \leqslant ||u||_{1,1}.$$

„3." was shown without proof. □

# 3 Characterization of Sobolev function

**Věta 3.1**

$\Omega \subseteq \mathbb{R}^d$, $p \in [1,\infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega |\, \mathrm{dist}(x, \delta\Omega) > \delta\}$. *Then*

$$\forall u \in W^{1,p}(\Omega) : ||\Delta_i^h u||_{L^p(\Omega_delta)} \leqslant ||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)}, \qquad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x+he_i) - u(x)}{h}.$$

$$u \in L^P \implies \forall \delta, h : ||\Delta_i^h u||_{L^p(\Omega_\delta)} \leqslant c.$$

$p > 1 \implies \frac{\partial u}{partial x_i}$ *exists and* $||\frac{\partial u}{\partial x_i}||_{L^p(\Omega)} \leqslant c$.

**Definice 3.1** (Class $C^{k,\mu}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

- there exist $M$ coordinate systems $\mathbf{x} = (x_{r_1}, \ldots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r = \left\{x'_r \in \mathbb{R}^{d-1} |\; |x_{r_i}| \leqslant \alpha\right\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting tr the orthogonal transformation from $(x'_r, x_{r_d})$ to $(x', x_d)$, then $\forall x \in \partial\Omega$ $\exists r \in \{1, \ldots, M\}$ such that $x = \mathrm{tr}\left(x'_{r_1}, a(x_{r_d})\right)$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\right\}$$

$$V_r^- := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\right\}$$

$$\Lambda_r := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) = x_{r_d}\right\}$$

Then $\mathrm{tr}(V_r^+) \subset \Omega, \mathrm{tr}(V_r^-) \subset \mathbb{R}^d \backslash \overline{\Omega}, \mathrm{tr}(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M \mathrm{tr}(\Lambda_r) = \partial\Omega$.

**Věta 3.2** (Density of smooth functions)

*Let* $\Omega \in C^0$. *Then* $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{||\cdot||_{k,p}}$, $p \in [1,\infty)$.

**Věta 3.3** (Extension of Sobolev functions)

*Let* $\Omega \in C^{0,1}$ *(*$\Omega$ *is Lipschitz) and* $k \in \mathbb{N}$, $p \in [1,\infty]$. *Then there exists a continuous linear operator* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ *such that:*

- $||Eu||_{W^{k,p}(\mathbb{R}^d)} \leqslant C||Eu||_{W^{k,p}(\Omega)}$ *(*$C$ *is independent of* $u$*)*
- $Eu = u$ *almost everywhere in* $\Omega$.

**Věta 3.4** (Trace theorem)

*Let* $\Omega \in C^{0,1}$, $p \in [1,\infty]$. *Then there exists a continuous linear operator* tr : $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that:*

- $|| \operatorname{tr} u||_{L^p(\partial\Omega)} \leqslant c||u||_{1,p}$,
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}$.

**Definice 3.2**

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{k,p}}.$$

**Věta 3.5**

*Let $\Omega \in C^{0,1}$ and let $p \in [1, \infty]$. Then*

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$.*

*Moreover*

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1 - \frac{d}{p}$.*

$$X \hookrightarrow\hookrightarrow Y \Leftrightarrow X \leqslant Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$X \hookrightarrow\hookrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^{\infty}, \exists c : ||u^n||_{1,p} \leqslant c \implies \exists u^{n_j} : u^{n_j} \to u \text{ in } Y).$$

*Důsledek* (Trace theorem)
Let $\Omega \in C^{0,1}$. Then $\forall u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ we have integration by parts:

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = -\int_\omega v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

**Věta 3.6** (Poincaré)

*Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Let $\Omega_1, \Omega_2 \subseteq \Omega$, $|\Omega_i| > 0$ and $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$, $|\Gamma_i|_{d-1} > 0$. Let $\alpha_1, \alpha_2 \geqslant 0$ and $\beta_1, \beta_2 \geqslant 0$ and at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.*

*Then there exist $c_1, c_2 > 0$ such that $\forall u \in W^{1,p}(\Omega)$*

$$c_1||u||_{1,p}^p \leqslant ||\nabla u||_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2| \int_{\Omega_2} u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2| \int_{\Gamma_2} u|^p \leqslant c_2||u||_{1,p}^p.$$

$$(||u||_{1,p}^p = ||u||_p^p + ||\nabla u||_p^p.)$$

*Důkaz* (Of the first (the only difficult) inequality)
TODO!!! □

# 4 Linear elliptic PDEs

**Definice 4.1** (Elliptic)

Let $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$, where $\Omega \leqslant \mathbb{R}^d$ is bounded. We say that $L$ is elliptic if $\exists c_1 > 0$ such that $\forall \zeta \in \mathbb{R}^d$ and almost all $x \in \Omega$

$$A\zeta \cdot \zeta \geqslant c_1 |\zeta|^2.$$

**Lemma 4.1**

*If $u$ is classical solution, then $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$ on $\Gamma_1$: $B_{L,\delta}(u, \varphi) = \int_\Omega f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$.*

*Důkaz*
TODO!!! □

**Lemma 4.2**

*If $u \in C^2(\overline{\Omega})$ and $A, b, \mathbf{c}, \mathbf{d}$ are smooth and previous lemma holds $\forall \varphi \in C^1$, $\varphi|_{\Gamma_1} = 0$ and $u = u_0$ on $\Gamma_1$, then $u$ is a classical solution.*

*Důkaz*
TODO!!! □

**Definice 4.2** (Weak solution)

Let $\Omega \subseteq \mathbb{R}^d$ Lipschitz, $L$ be an elliptic operator, $u_0 \in W^{1,2}(\Omega)$, $f \in (W^{1,2}(\Omega))^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff

- $\operatorname{tr} u = \operatorname{tr} u_0$ on $\Gamma_1$ and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$, $\forall \varphi \in V$, where $V := \{\varphi \in W^{1,2}(\Omega) | \operatorname{tr} \varphi = 0 \text{ on } \Gamma_1\}$.

## 4.1 Existence of solution for coercive operators

**Definice 4.3** (Elliptic form)

Let $B : V \times V \to \mathbb{R}$ bilinear nad $V$ be a Hilbert space, $c_1, c_2 > 0$. We say that $B$ is elliptic if it is

- $V$-bounded $\Leftrightarrow |B(u,\varphi)| \leqslant c_2||u||_V||\varphi||_V$ and
- $V$-coercive $\Leftrightarrow B(u,u) \geqslant c_1||u||_V^2$.

**Věta 4.3** (Lax-Milgram)

*Let $B$ be a bilinear elliptic form. Then*

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V : B(u,\varphi) =< F, \varphi > .$$

**Definice 4.4**

Let $B : V \to V^*$. We say that $B$ is

- Lipschitz $\equiv \forall u, v \in V : ||B(u) - B(v)||_{V*} \leqslant c_2||u - v||_V$, $c_2 > 0$;
- Uniformly monotone $\equiv \forall u, v \in V :< B(u) - B(v), u - v >_V \geqslant c_1||u - v||_V^2$, $c_1 > 0$.

**Věta 4.4** (Non-linear Lax-Milgram)

*Let $B$ be Lipschitz continuous and uniformly monotone. Then*

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V :< B(u), \varphi >=< F, \varphi > .$$

*Důkaz*
TODO!!! □

*Důkaz* (Lax-Milgram)
TODO!!! □

**Věta 4.5**

*If $B_{L,\sigma}$ is bilinear, $V$-bounded and $V$-elliptic. Then there exists a unique weak solution $u$.*

*Důkaz*
TODO!!! □

## 4.2 Existence via Fredholm alternative

TODO!!!

**Věta 4.6**

*Let $\Omega \in C^{0,1}$, $L$ be an elliptic operator and $\Gamma_1 = \partial\Omega$. Then*

1. *$\Sigma$ is at most countable and if infinite $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \to \infty$;*
2. *$(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1\ \exists! u : Lu = f + \lambda u$;*
3. *$\forall \lambda \notin \Sigma\ \exists C > 0\ \forall f \in L^2\ \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$ and $||u||_{1,2} \leqslant c||f||_2$;*

*Důkaz*

3) TODO improve convergence of $u^{n_k}$ and show

$$u^{n_k} \to u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show $\{u^{n_k}\}$ is Cauchy in $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1||\nabla(u^n - u^m)||_2^2 \leqslant \int_\Omega A\nabla v^{n,m}\nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_\Omega \mathbf{c}\nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d}\nabla v^{n,m} v^{n,m} =$$

$$= \int_\Omega (f^n - f^m)v^{n,m} + \lambda(v^{n,m})^2 \pm - ||- \leqslant$$

$$\leqslant ||v^{n,m}||_2(||f^n - f^m||_2 + \lambda||v^{n,m}||_2 + ||\mathbf{c}||_\infty||\nabla v^{n,m}||_2 + ||\mathbf{d}||_\infty||\nabla v^{n,m}||_2 + ||b||_\infty||v^{n,m}||_2) \leqslant$$

$$\leqslant ||v^{n,m}||C(\lambda) \overset{u^n \text{ is Cauchy}}{\leqslant} C(\lambda)\varepsilon$$

$\implies \nabla u^n$ is Cauchy sequence $\implies u^n \to u$ in $W_0^{1,2}(\Omega) \implies ||?||_{n_k} = 1$

$$\int_\Omega A\nabla a u^n \nabla a\varphi + bu^n\varphi + \mathbf{c}\nabla u^n\varphi - \mathbf{d}\nabla ?u^n = \int_\Omega f^n\varphi + \lambda u^n\varphi.$$

$$n \to \infty$$

$$\int_\Omega A\nabla u\nabla\varphi + bu\varphi + \mathbf{c}\nabla u\varphi - \mathbf{d}\nabla\varphi u = \lambda\int u\varphi \Leftrightarrow Lu = \lambda u$$

But $\lambda \notin \Sigma$. □

*Poznámka*

Next we discussed homework.

## 4.3 Variational approach – minimization

*Poznámka*
$B_{L,\sigma}(u,v)$ must be symmetric! ($B_{L,\sigma}(u,v) = B_{L,\sigma}(v,u)$)

$$L = -\div(A\nabla u) + bu + \mathbf{c}\nabla u + \div(\mathbf{d}u)$$
$$B_{L,\sigma}(u,v) := \int_\Omega A\nabla u \cdot \nabla v + Buv + \mathbf{c}\cdot\nabla uv - \mathbf{d}\nabla vu + \int_\Gamma \sigma uv$$
$$B_{L,\sigma}(v,u) := \int_\Omega A\nabla v \cdot \nabla u + Bvu + \mathbf{c}\cdot\nabla vu - \mathbf{d}\nabla uv + \int_\Gamma \sigma vu$$
$$\implies A = A^T, \qquad \mathbf{c} = -\mathbf{d}$$

## Věta 4.7

*Let $B_{L,\sigma}$ be linear symmetric $V$-elliptic and $V$-bounded. $f \in V^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$, $u \in ?$. Then the following is equivalent:*

- $u - u_0 \in V$ *and* $B_{L,\sigma}(u,v) =< f,\varphi > + \int_{\Gamma_2\cup\Gamma_3} g\varphi$;
- $u - u_0 \in V\ \forall v \in W^{1,2}(\Omega),\ v, u_0 \in V$
$$\frac{1}{2}B_{L,\sigma}(u,u) - < f,u > - \int_{\Gamma_2\cup\Gamma_3} gu \leqslant \frac{1}{2}B_{L,\sigma}(v,v) - < f,v > - \int_{\Gamma_2\cup\Gamma_3} gv.$$

*Důkaz* („1 $\implies$ 2“)

$$0 \overset{V-\text{elliptic}}{\leqslant} \frac{1}{2}B_{L,\sigma}(v-u,v-u) \overset{\text{linearity}}{=} \frac{1}{2}B_{L,\sigma}(v,v) + \frac{1}{2}B_{L,\sigma}(u,u) - \frac{1}{2}B_{L,\sigma}(u,v) - \frac{1}{2}B_{L,\sigma}(v,u) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u,u) - B_{L,\sigma}(u,v) =$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + B_{L,\sigma}(u,u-v) \overset{\text{weak formulation}}{=}$$
$$= \frac{1}{2}\left(B_{L,\sigma}(v,v) - B_{L,\sigma}(u,u)\right) + < f, u-v > + \int_{\Gamma_2\cup\Gamma_3} g(u-v)$$

□

*Důkaz* („2 $\implies$ 1")
$u$ is minimizer, so set $v = u + \varepsilon\varphi$, $\varphi \in V$

$$\frac{1}{2}B_{L,\sigma}(u,u) - < j, u > - \int gu \leqslant \frac{1}{2}B_{L,\sigma}(u+\varepsilon\varphi, u+\varepsilon\varphi) - < j, u+\varepsilon\varphi > - \int g(u+\varepsilon\varphi) =$$

$$= \frac{1}{2}B_{L,\sigma}(u,u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi,\varphi) + \varepsilon B_{L,\sigma}(u,\varphi) - < f, u > -\varepsilon < f, \varphi > - \int ga - \varepsilon \int g\varphi$$

divide by $\varepsilon$ and $\varepsilon \to 0_+$

$$0 \leqslant B_{L,\sigma}(u,\varphi) - < j, \varphi > - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \qquad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for $-\varphi \implies 0 = -||- \implies u$ is weak solution. □

## **Věta 4.8** (Duel formulation)

*Let $Lu = -\operatorname{div}(A\nabla u)$ with $A$ elliptic, bounded and symmetric, $\Gamma_1 \neq \varnothing$, $\Gamma = \varnothing$, $f \in V^*$, $g \in L^2(\Gamma_2)$, $u_0 \in W^{1,2}(\Omega)$. Then the $f$ following are equivalent:*

- *$u$ is a weak solution;*
- *$\nabla u = A^{-1}\mathbf{T}$, where $\mathbf{T}$ minimizes $\int \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} = \nabla u_0 \mathbf{T}$ over the set $\tilde{V} := \{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d)\}$, $\forall \varphi \in V$.*

$$\int_\Omega \mathbf{T}\cdot\nabla\varphi =< f, \varphi > + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div}\mathbf{T} = f \ \text{in } \Omega, T\mathbf{u} = g \ \text{on } \Gamma_2$$

*Důkaz* („1 $\implies$ 2")
Let $\mathbf{V} \in \tilde{V}$ and $\mathbf{T} := A\nabla u \in \tilde{V}$.

$$0 \leqslant \frac{1}{2}\int_\Omega A^{-1}(\mathbf{V}-\mathbf{T})\cdot(\mathbf{V}-\mathbf{T}) = \int \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2}\int_\Omega A^{-1}\mathbf{T}\cdot\mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\left(\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0\mathbf{T}\right) + \int_\Omega \left(\nabla u_0(\mathbf{V}-\mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T}-\mathbf{V})\right) =$$

$$= \int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \int_\Omega \nabla(u-u_0)\cdot(\mathbf{V}-\mathbf{T}) =$$

$$\int\left(\frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0\mathbf{V}\right) - \int\frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} + 0.$$

So $\mathbf{T}$ is minimizer of the formula above. □

*Důkaz* („2 $\implies$ 1")
$\mathbf{T} \in \tilde{V}$ $\forall V \in \tilde{V}$: $\int_\Omega \frac{1}{2} A^{-1}\mathbf{T} \cdot \mathbf{T} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{V}\cdot\mathbf{V}}{2} - \nabla u_0 \mathbf{V}$. $\mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}$, $\mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$ $\forall \varphi \in V$: $\int_\Omega \mathbf{W} \cdot \nabla \varphi = 0$.

$$\int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leqslant \int_\Omega \frac{A^{-1}\mathbf{T}\cdot\mathbf{T} + \varepsilon^2 A^{-1}\mathbf{W}\cdot\mathbf{W} + 2\varepsilon A^{-1}\mathbf{T}\cdot\mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by $\varepsilon$ and $\varepsilon \to 0_+$:

$$0 \leqslant \int_\Omega A^{-1}\mathbf{T} \cdot \mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for $-\mathbf{W}$, co $0 = -||-$.

Now we find unique $u \in W^{1,2}$ $u - u_0 \in V$: $\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega A^{-1}\mathbf{T} \cdot \nabla \varphi$ $(< F, \varphi >_V)$.

$$\int_\Omega |A^{-1}\mathbf{T} - \nabla u|^2 = \int_\Omega (A^{-1}\mathbf{T} - \nabla u)(A^{-1}\mathbf{T} - \nabla u) =$$

$$= \int_\Omega (A^{-1}\mathbf{T} - \nabla u_0) \cdot (A^{-1}\mathbf{T} - \nabla u) + \int_\Omega \nabla(u_0 - u)(A^{-1}\mathbf{T} - \nabla u) = 0 + 0 = 0$$

□

## Lemma 4.9

*Let $X$ be a reflexive space and $\{u^n\}_{n=1}^\infty$ be a bounded sequence, $||u^n||_X \leqslant c < \infty$. Then $\exists u^{n_k}$, $\exists u \in x$: $u^{n_k} \rightharpoonup u$ ($\forall F \in X^* :< F, u^{n_k} > \to < F, u >$).*

## Věta 4.10 (Spectrum of symmetric operator)

*$V$ Hilbert infinity-dimensional space. Let $B$ be linear, symmetric, $V$-elliptic and $V$-bonded operator. Then there exist $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_m$ and corresponding $\{u_i\}_{i=1}^\infty$ such that*

- $B(u^k, \varphi) = \lambda_k \int_\Omega u^k \varphi$;
- $\lambda_k \to \infty$;
- $\{u^k\}_{k=1}^\infty$ *is basis in $V$ and fulfils*

$$\int_\Omega u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_\Omega u u^i)$, *then* $\forall n : ||P^n u||_2 \leqslant ||u||_2$ *and* $B(P^n u, P^n u) \leqslant B(u, u)$.

*Důkaz*
Step 1: Construct $\lambda_k, u^k$: $\lambda_1 := \inf_{u \in V, ||u||_2 = 1} B(u,u)$ and denote $u^1$ function, where infimum is obtained. Then for $V^N = \left\{ u \in V | \forall k \in [N] : B(u, u^k) = 0 \right\}$ we do the same.

Step 2: The construction is OK:

$$0 < \lambda_1 = \lim_{n \to \infty} B(u^n, u^n), ||u^n||_2 = 1 \implies$$

$$\implies ||u^n||_V \leqslant C \implies u^{n_k} \rightharpoonup u \text{ in } V$$

$$V \hookrightarrow L^2 \implies u^{n_k} \to u \text{ in } L^2(\Omega) \implies ||u||_2 = 1$$

$$\lambda_1 = \lim_{n_k \to \infty} B(u^{n_k}, u^{n_k}) \geqslant B(u,u) \geqslant \lambda_1.$$

Step 3: $\lambda_k$, $u^k$ eigenvalues, eigen functions: $\forall v \in V, ||v||_2 = 1$, $\lambda_1 = B(u^1, u^1) \leqslant B(v,v)$, $\quad ||u^1||_2 = 1$

$$v = \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leqslant B(\frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2}, \frac{u^1 + \varepsilon\psi}{||u^1 + \varepsilon\psi||_2})$$

$$\lambda_1 ||u^1 + \varepsilon\psi||_2 \leqslant B(u^1 + \varepsilon\psi, u^1 + \varepsilon\psi) = B(u_1, u_1) + \varepsilon^2 B(\psi,\psi) + 2\varepsilon B(u,\psi) \leqslant \lambda_1 ||u^1||_2^2 + \lambda_1 \varepsilon^2 ||\psi||_2^2 + 2\varepsilon\lambda_1 \int_\Omega u^1$$

$$\varepsilon \to 0_+ \implies 2\lambda_1 \int_\Omega u^1 \psi \leqslant 2B(u,\psi).$$

So $\lambda_1 \int_\Omega u^1\psi = B(u,\psi)$.

The same way we obtain $\lambda_k \int_\Omega u^k \psi \leqslant B(u,\psi)$ for $\psi \in V^N$.

$$u^1 : \lambda_1 \int_\Omega u^1 \psi = B(u^1, \psi) \implies \psi = u^k \int_\Omega u^1 u^k = V(u_1, u^k).$$

But $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$, so $\int u^1 u^k = B(u^1, u^k) = 0$.

$$\implies \forall i \in [k-1] : \int_\Omega u^k u^1 = B(u^k, u^i) = 0.$$

Step 4: $\lambda_k \nearrow \infty$. We already know $\lambda_1 \leqslant \lambda_2 \leqslant \dots$. Assume a contradiction $\lambda_k \leqslant C < \infty$. $c_1 ||u^k||_V^2 \leqslant B(u^k, u^k) = \lambda_k ||u^k||_2^2 = \lambda_k < C$.

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \to u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$||u^n - y^m||_2^2 = ||u^n||_2^2 + ||u^m||_2^2 - 2\int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_? n} B(u^n, u^m) = 2 \implies \text{ not Cauchy.}$$

Step 5: $\lambda_k$ are all eigenvalues ($u^k$ is basis of $V$ and of $L^2$). Assume that $\lambda \neq \lambda_j$ is also eigenvalue, so $\exists u : B(u,\varphi) = \lambda \int_\Omega u\varphi \forall\varphi$. We can find $i \in \mathbb{N}$, so $\lambda_i < \lambda < \lambda_{i+1}$.



$$B(u, u^j) = \lambda \int uu^j \wedge B(u^j, u) = \lambda_j \int u^j u \implies B(u, u_j) = 0$$

## 4.4 Regularity of weak solution

*Poznámka*
We assume that we have $u \in W^{1,2}(\Omega)$ a weak solution

$$-\operatorname{div} A\nabla u + Vu + \mathbf{c}\cdot\nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When $u \in W^{2,2}_{loc}(\Omega)$, when $u \in W^{2,2}(\Omega)$, when $u \in W^{k,2}_{loc}(\Omega)$, $u \in W^{k,2}(\Omega)$.

Simplify $-\operatorname{div} A\nabla u = f - bu - \mathbf{c}\nabla u - u \operatorname{div}\mathbf{d} - \nabla u\cdot\mathbf{d} = \tilde{f}$. If $u \in W^{1,2}$, $f \in L^2$, $b \in L^\infty$, $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$.

Problem is reduced to

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma_1,$$
$$(A\nabla u)\cdot\mathbf{v} = g \text{ on } \Gamma_2,$$
$$(A\nabla u)\cdot\mathbf{v} + \sigma u = g \text{ on } \Gamma_3.$$

**Definice 4.5** (Interior regularity)

$u \in W^{2,2}_{loc}(\Omega)$; assumptions: $A \in W^{k+1,\infty}$, $f \in W^{k,2}(\Omega) \implies u \in W^{k+1,2}_{loc}(\Omega)$.

**Definice 4.6** (Boundary regulariry)

$u \in W^{2,2}(\Omega)$; assumptions: on $\Omega \in C^{k+1,\infty}$, $g \in W^{\frac{1}{2}?,2}(\partial\Omega)$ and $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\varnothing\} \implies u \in W^{2,2}(\Omega)$.

**Věta 4.11** (Interior regularity)

*Let $A$ be an elliptic operator and $u \in W^{1,2}$ solves*

$$\int_\Omega A\nabla u\cdot\nabla\varphi = \int_\Omega f\varphi, \qquad \forall\varphi \in W^{1,2}_0(\Omega)\ \forall f \in L^2(\Omega).$$

*Then if $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$, $f \in W^{k,2}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.*

*Moreover $\forall\tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega\ \exists c(\tilde{\Omega}, A)$:*

$$||u||_{W^{k+2,2}}(\tilde{\Omega}) \leqslant c(||f||_2 + ||u||_{W^{1,2}(\Omega)}).$$

*Důkaz*
$k=0$: Recall $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x+he_k)-v(x)|^2}{h^2} \leqslant c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{ u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2 \right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left|\frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j}\right|^2}{h^2} \leqslant c,$$

$$\int_{\Omega_h} \left| \frac{\nabla u(x+he_i) - \nabla u(x)}{h} \right|^2 \leqslant c.$$

$\int_\Omega A \nabla u \nabla \varphi = \int_\Omega f \varphi$

$$h > 0, \varphi \in W_0^{1,2}(\Omega), \varphi(x) = 0 \text{ if } \operatorname{dist}(x, \partial\Omega) \subset h.$$

Set $\varphi(x) := \psi(x - he_k)$.

$$\implies \int_\Omega A(x) \nabla u(x) \nabla \psi(x - he_k) = \int_\Omega f(x) \psi(x - he_k) =$$

$$= \int_\Omega A(x+he_k) \nabla u(x+he_k) \cdot \nabla \psi(x) dx.$$

Set $\varphi(x) := \psi(x)$:

$$\int_\Omega A(x) \cdot \nabla u(x) \cdot \nabla \psi(x) = \int_\Omega f(x) \psi(x) dx.$$

$$\int_\Omega A(x+he_k)(\nabla u(x+he_k) - \nabla u(x)) \cdot \nabla \psi(x) =$$

$$= -\int (A(x+he_k) - A(x)) \nabla u(x) \cdot \nabla \psi(x) + \int_\Omega f(x)(\psi(x-he_k) - \psi(x)).$$

Set $\psi := (u(x+he_k) - u(x))\tau^2(x)$, $\tau(x) = 0$, if dist $\in (x, \partial\Omega)$, $\tau \in C^1(\tilde{\Omega})$.

Evaluate all terms ($w^{h,i} = u(x+he^i) - u(x)$):

$$\int_\Omega A(x+he_i) \nabla w^{h,i} \cdot (\nabla w^{h,i} \tau^2 + 2 w^{h,i} \tau \nabla \tau) \geqslant$$

$$\overset{ellip.}{\geqslant} c_1 \int_\Omega |\nabla w^{hi}|^2 \tau^2 - \int_\Omega \frac{2||A||_\infty |w^{h,i}| - |\nabla \tau|(|\nabla w^{hi}| \sqrt{c_1} \tau)}{\sqrt{c_1}} \geqslant$$

$$\geqslant \frac{c_1}{2} \int_\omega |\nabla w^{h,i}|^2 \tau^2 - \frac{2}{c_1} ||A||_\infty^2 ||\nabla \tau||_\infty^2 h^2 \int_{\Omega_h} \frac{|u(x+he_i) - u(x)|^2}{h^2} \geqslant$$



$$\geqslant \frac{c_1}{2} \int_\Omega |\nabla w^{h,i}|^2 \tau^2 - \frac{2||A||_\infty^2 ||\nabla \tau||_\infty^2}{c_1} h^2 c ||\nabla u||_2^2$$