Prerequisites

0.1 Regularization

Definice 0.1 (Regularization kernel)

$\eta \in C_0^\infty(B_1(\mathbf{o}))$, non-negative, radially symmetric, $\int_{B_1(\mathbf{o})} \eta(x)dx = 1$.

Definice 0.2 (Regularization of function)

Let $f \in L^p(\Omega)$. We extend f by zero to $\mathbb{R}^d \backslash \Omega$ and define $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta(x/\varepsilon)$.

Poznámka
$f_\varepsilon \in C^\infty(\mathbb{R}^d)$, $f_\varepsilon \to f$ in $L^p(\Omega)$ if $p \in [1, \infty)$ and $f_\varepsilon \rightharpoonup^* f$ in L^∞.

Věta 0.1

$L^p(\Omega)$ is a Banach space, separable for $p \in [1, \infty)$, reflexive for $p \in (1, \infty)$.

Důsledek
f^n is a bounded sequence in $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ measurable bounded bounded. Then

1. $p \in (1, \infty)$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup f$ in $L^p(\Omega)$. ($\Leftrightarrow \forall g \in L^{p'}(\Omega) : \lim_{k\to\infty} \int_\Omega f^{n_k} g = \int_\Omega fg$, where $\frac{1}{p} + \frac{1}{p'} = 1$).
2. $p = \infty$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup^* f$ in $L^\infty(\Omega)$. ($\Leftrightarrow g \in L^1(\Omega) : \lim \int_\Omega f^{n^k} g = \int_\Omega fg$).
3. $p = 1$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup^* f$ in $M(\overline{\Omega})$ (Radon measures). ($\Leftrightarrow \forall g \in C(\overline{\Omega}) : \int_\Omega f^{n_k} g \to \langle f, g \rangle_M = \int_{\overline{\Omega}} g df$.)
4. $p = 1$: $\exists f^{n_k}, \tilde{f}$ $\exists \Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \dots, |\Omega \backslash \Omega_l| \to 0$ as $l \to \infty$: $\forall l \in \Omega : f^{n_k} \rightharpoonup \tilde{f}$ in $L^1(\Omega)$. ($\tilde{f}$ is called biting limit.)

0.2 Fixpoint theorems

Věta 0.2

$F : X \to X$, where X is a Banach space, F is continuous and compact. Let there exists closed convex non-empty set $U \subseteq X$ such that $F(U) \subset U$. Then $\exists x \in U : F(x) = x$.

Věta 0.3

$F : \mathbb{R}^d \to \mathbb{R}^d$, F is continuous. Let there exists closed, convex non-empty set $U \subseteq \mathbb{R}^d$: $F(U) \subseteq U$. Then $\exists x \in U : F(x) = x$.

0.3 Nemytskii operator

Věta 0.4

Let $\Omega \subseteq \mathbb{R}^d$ *be open and* $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ *is Carathéodory (i.e.* $\forall y \in \mathbb{R}^N : f(\cdot, y)$ *is measurable and for almost all* $x \in \Omega$*:* $f(x, \cdot)$ *is continuous). Assume that* $|f(x, y) \leqslant g(x) + c \cdot \sum_{i=1}^N |y_i|^{p_i/p}$ *for some* $p_1 \in [1, \infty)$*,* $p \in [1, \infty)$ *with* $y \in L^p(\Omega)$*.*

Then $\forall u_i \in L^{p_i}(\Omega)$*, the function* $f(x, u_1(x), \ldots, u_N(x))$ *is measurable and the mapping (named Nemytskii operator)* $(u_1, \ldots, u_N) \mapsto f(\cdot, u_1, u_2, \ldots, u_N)$ *is continuous from* $L^{p_1}(\Omega) \times \ldots \times L^{p_N}(\Omega)$ *to* $L^p(\Omega)$*.*

TODO!!!

TODO!!!

Věta 0.5

Let $\Omega \subseteq \mathbb{R}^d$ *open bounded,* $\Omega_\delta := \{x \in \Omega | B_\delta(x) \subseteq \Omega\}$*,* $u_i^h(x) := \frac{u(x+h \cdot e_i) - u(x)}{h}$*, and* $p \in [1, \infty]$*. Then*

1. *if* $u \in W^{1,p}(\Omega)$ *then* $\forall \delta > 0 \ \forall h \leqslant \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$*;*
2. *if* $p \in (1, \infty]$ *and* $\sup_{\delta > 0} \sup_{h < \delta/2} \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant K$ *then* $\frac{\partial u}{\partial x_i}$ *exists and* $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leqslant K$*;*
3. *if* $p \in [1, \infty)$ *and* $u \in W^{1,p}(\Omega)$ *then* $u_i^n \to \frac{\partial u}{\partial x_i}$ *as* $h \to 0_+$ *in* $L^p_{loc}(\Omega)$*.*

Důkaz
„2.“: $p \in (1,\infty)$ then L^p is reflexive. $p=\infty$ then L^∞ has separate procedure. Fix $\Omega' \subseteq \overline{\Omega'} \subseteq \Omega$. $\|u_i^h\|_{L^p(\Omega')} \leqslant K \implies$

$$p \in (1,\infty) : \exists h_n : u_i^{h_n} \rightharpoonup \overline{u_i} \text{ in } L^p(\Omega'),$$

$$p = \infty : \exists h_n : u_i^{h_n} \rightharpoonup^* \overline{u_i} in L^p(\Omega').$$

$$\implies \|\overline{u_i}\|_{L^p(\Omega')} \leqslant \lim_{h_n \to 0} \|u_i^h\|_{L^p(\Omega')} \leqslant K. \qquad \Omega' \nearrow \Omega \implies \|\overline{u_i}\|_{L^p(\Omega)} \leqslant K.$$

Remain to show: $\overline{u_i} = \frac{\partial u}{\partial x_i}$.

TODO!!!

„1.“: $u \in W^{1,p}(\Omega)$. Mollify to u_ε. $u_\varepsilon \to u$ in $W^{1,p}_{loc}(\Omega)$, for $p \neq 8$, and $u_\varepsilon \rightharpoonup^*$ in $W^{1,\infty}_{loc}(\Omega)$ for $p=\infty$. $D^\alpha u_\varepsilon = (D^\alpha u)_\varepsilon$ in Ω_ε for $p=\infty$, $D^\alpha u_\varepsilon \to D^\alpha u$ in $L^p_{loc}(\Omega)$ for $p \neq \infty$. $x \in \Omega_\varepsilon, h \leqslant \delta/2$:

$$\frac{u_\varepsilon(x+h\cdot e_i)-u_\varepsilon(x)}{h} = \frac{1}{h}\int_0^1 \frac{d}{dt} u_\varepsilon(x+h\cdot t\cdot e_i)dt = \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i}(x+h\cdot t\cdot e_i)dt.$$

$$\int_{\Omega_\delta} \left|\frac{u_\varepsilon(x+h\cdot e_i)-u_\varepsilon(x)}{h}\right|^p dx \leqslant \int_{\Omega_\delta}\left|\int_0^1 \frac{u_\varepsilon}{\partial x_i}(x+h\cdot t\cdot e_i)dt\right|^p \overset{\text{Jensen}}{\leqslant} \int_{\Omega_\delta}\int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+h\cdot t\cdot e_i)\right|^p dtdx \leqslant$$

$$\leqslant \int_0^1 \int_{\Omega_\delta} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x+h\cdot t\cdot e_i)\right|^p dxdt \leqslant \int_0^1 \int_{\Omega_{\delta/2}} \left|\frac{\partial u_\varepsilon}{\partial x_i}(x)\right|^p dxdt \leqslant \left\|\frac{\partial u}{\partial x}\right\|_{L^p(\Omega)}.$$

„3.“: It is enough to show that $u_i^{h^n}$ is Cauchy in $L^p_{loc}(\Omega)$:

$$u_i^{h^m} - u_i^{h^m} = \int_0^1 \frac{\partial u}{\partial x_i}(x+h^m\cdot t\cdot e_i) - \frac{\partial u}{\partial x_i}(x+h^n\cdot t\cdot e_i)dt.$$

$$\int_{\Omega_\delta} |u_i^{h^n} - u_i^{h^m}|^p \leqslant \int_0^1\int_\Omega \left|\frac{\partial u}{\partial x_i}(x+h^m\cdot t\cdot e_i) - \frac{\partial u}{\partial x_i}(x+h^n\cdot t\cdot e_i)\right|^p dxdt \leqslant \varepsilon \text{ provided } h^n, h^m \ll 1.$$

$\implies u_i^h$ is Cauchy. □

0.4 Properties up to the boundary

Věta 0.6

Let $\Omega \subseteq \mathbb{R}^d$ be bounded and open and $p \in [1,\infty)$. Then $\forall u \in W^{1,p}(\Omega)$:

1. *$\exists \{u^n\}_{n=1}^\infty \subseteq C^\infty(\Omega)$ such that $u^n \to u$ in $W^{1,p}(\Omega)$;*
2. *if $\Omega \in C^0$ then $\exists \{u^n\}_{n=1}^\infty \subseteq C^\infty(\overline{\Omega})$ such that $u^n \to u$ in $W^{1,p}(\Omega)$.*

Důkaz
„1.“: Prose? covering of Ω: $\Omega_i := \left\{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \frac{1}{i}\right\}$. $\Omega_i \subseteq \Omega_j$ for $i \leqslant j$. $\bigcup_{i=1}^{\infty} \Omega_i = \Omega$. Define $V_i = \Omega_{i+3} \backslash \overline{\Omega_{i+1}} = \left\{x \in \Omega | \frac{1}{i+1} > \operatorname{dist}(x, \partial\Omega) > \frac{1}{i+3}\right\}$. Find $V_0 \Subset \Omega$ such that $\bigcup_{i=0}^{\infty} V_i = \Omega$.

$u_i = u\varphi_i$, where φ_i is partition of unity (from next lemma). So $\forall i\ \exists j$ such that $u_i \subset V_j$. $\forall \varepsilon$ find (by convolution) $u_i^n \in C^\infty(\mathbb{R}^d) : \|u_i - u_i^n\|_{W^{1,p}}(\Omega) \leqslant \frac{\varepsilon}{2^i}$. (Such that $u_i^n \subseteq \Omega_{i+n} \backslash \overline{\Omega_i}$)

Define $u^n := \sum_{i=0}^{\infty} u_i^n$. $K \subseteq \Omega$ compact, then

$$\|u - u^n\|_{W^{1,p}(\Omega)} = \| \sum u\varphi_i - \sum u_i^n \|_{W^{1,p}(K)} = \| \sum (u_i - u_i^n) \|_{W^{1,p}(\Omega)} \leqslant \sum \|u_i - u_i^n\|_{W^{1,p}(\Omega)} \leqslant \varepsilon \cdot \sum \frac{1}{2^i} \leqslant 2\varepsilon.$$

$$\implies \|u^n - u\|_{W^{1,p}(\Omega)} \leqslant 2\varepsilon.$$

„2.“ TODO!!!?

Start with u_{M+1}. $u_{M+1} = u \cdot \varphi_{M+1}$. $\operatorname{supp} \varphi_{M+1} \subset \Omega \implies u_{M+1} \in W^{1,p}(\mathbb{R}^d)$. $u_{M+1}^\varepsilon := u_{M+1} * \eta_{\sigma(\varepsilon)}$, where δ is taken such that $\|u_{M+1}^\varepsilon - u_{M+1}\|_{W^{1,p}(\Omega)} \leqslant \frac{\varepsilon}{M+1}$.

u_1: assume $T_r = \mathrm{id}$. $u_1 = u \cdot \varphi_1$. $u_1^h(x; x_d) := u_1(x; x_d + h)$. $h \leqslant h_0$:

$$\|u_1^n - u_1\|_{W^{1,p}(\Omega)} = \|u_1^n - u_1\|_{W^{1,p}(V^+)} \leqslant \frac{\varepsilon}{2 \cdot (M+1)}.$$

$u_1^\varepsilon = u_1^h * \eta_{\delta(\varepsilon, h, \varphi_1, a_1)} \in C^\infty(\overline{\Omega})$. $\|u_1^\varepsilon - u^h\|_{W^{1,p}(V^+)} \leqslant \frac{\varepsilon}{2(M+1)}$. Find δ: $(x; x_d) \in \Lambda$, $y \in B_\sigma(x; x_d)$, $a(y') > y_d - h$.

$$a(y') \geqslant a(x') - |a(x') - a(y')| = x_d - |a(x') - a(y')| \geqslant y_d - (|a(x') - a(y')| + |y_d - x_d|).$$

Find $\delta_0 > 0$: $\forall x, y, |x - y| \leqslant \delta_0 : |a(x') - a(y')| + |y_d - x_d| < h$. □

Lemma 0.7 (For the previous proof: Partition of unity I)

Let $\Omega \subseteq \mathbb{R}^d$ be open set. Assume that $\{V_i\}_{i \in I}$ be (uncountable) covering-?. Then there exists countable system $\{\varphi_j\}_{j=1}^{\infty}$ such that $\varphi_j \in C_0^\infty(\mathbb{R}^d)$, $\forall j \in \mathbb{N}\ \exists i \in I : \operatorname{supp} \varphi_j \subset V_i$, $0 \leqslant \varphi_j \leqslant 1$, and $\forall x \in \Omega : \sum_{j=1}^{\infty} \varphi_j(x) = 1$. Moreover, for any compact $K \subseteq \Omega$, we have that $\varphi_j(x) \neq 0$ for finitely many j's.

Lemma 0.8 (For the previous proof: Partition of unity II)

Let $\overline{\Omega}$ be a compact set and $\left\{\tilde{V}_i\right\}_{i=1}^{N}$ be its open covering ($\overline{\Omega} \subseteq \bigcup_{i=1}^{N} \tilde{V}_i$). Then $\exists \varphi_i \in C_0^\infty(\tilde{V}_i)$, $0 \leqslant \varphi_i \leqslant 1$, such that $\forall x \in \overline{\Omega} : \sum_{i=1}^{N} \varphi_i(x) = 1$.

TODO!!!?

Věta 0.9 (Extension)

Let $\Omega \in C^{0,1}$ and $p \in [1,\infty]$. Then there exists continuous linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ such that $\forall u \in W^{1,p}(\Omega)$:

1. *$Eu = u$ in Ω;*
2. *$\exists B_R \subseteq \mathbb{R}^d : Eu = 0$ in $\mathbb{R}^d \backslash B_R$;*
3. *$\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leqslant c(\Omega, p, d) \cdot \|u\|_{W^{1,p}}(\Omega)$.*

Důkaz

„1.“ By picture. „2.“: $u = \sum_{r=1}^{M+1} u_r$, where $u_r := u\varphi_r \in W^{1,p}(\Omega)$. Step 0: extension of u_{M+1} by zero is trivial. Step 1: u_1, $T_1 = \mathrm{id}$, $u_1 \in W^{1,p}(V_1^+)$. $F : V_1 \to \overline{V_1}, (x', x_d) \mapsto (y', y_d)$, $y' = x'$, $y_d = x_d - a_1(x')$. $F^{-1} : \overline{V_1} \to V_1$, $x' = y'$, $x_d = y_d + a(y')$.

TODO!!!

Proof of $(*)$: It is enough to show that $\frac{\partial Ev(y)}{\partial y_1} = \frac{\partial v(y)}{\partial y_1}$ for $y_d > 0$ and $\ldots = \frac{\partial v}{\partial y_i}(y', -y_d)$ for $y_d < 0$; and $\frac{\partial E(y)}{\partial d} = \frac{\partial v(y)}{\partial y_d}$ for $y_d > 0$ and $\ldots = -\frac{\partial v}{\partial y_d}(y'; y_d)$ for $y_d < 0$.

We know $Ev \in W^{1,p}(\overline{V_1^+})$ and $Ev \in W^{1,p}(\overline{V_1^-})$. $\|Ev\|_{W^{1,p}(V_1^-)} = \|Ev\|_{W^{1,p}(V_1^+)}$.

TODO!!!??? $\square$

1 Embeddings

Věta 1.1

Let $\Omega \in C^{0,1}$ and $p \in [1,\infty]$. Then:

- *$W^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega)$ if $p < d$;*
- *$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all q if $p = d$;*
- *$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ if $p > d$;*
- *$W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,\beta}(\overline{\Omega})$ if $p > d$ (for $\beta < 1 - \frac{d}{p}$);*
- *$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for $q < \frac{dp}{d-p}$ if $p < d$ (respectively $< \infty$ if $p = d$).*

Důkaz (Case $p > d$)
Lemma (Marey): Let $u \in W^{1,1}(B_R)$ and $\mathbf{o}$ be the Lebesgue point of u. Then

$$\left|\fint_{B_R} udx - u(\mathbf{o})\right| \leqslant c(d,A)R^A \sup_{\varrho\leqslant R}\int_{B_\varrho}\frac{|\nabla u|}{\varrho^{d-1+A}}dx \qquad \forall A \in (0,1).$$

Proof of lemma:

$$\fint_{B_R} udx - u(\mathbf{o}) = \lim_{r\to 0_+}\left(\fint_{B_R} u - \fint_{B_r} u\right) = \lim_{r\to 0_+}\int_r^R \frac{d}{d\varrho}\fint_{B_\varrho} u(x)dxd\varrho =$$

$$= \lim_{r\to 0_+}\int_r^R \frac{d}{d\varrho}\fint_{B_1} u(\varrho x)dx = \lim_{r\to 0_+}\int_r^R \fint_{B_1}\frac{\nabla u(\varrho x)\cdot x}{\sum_{i=1}^d \frac{\partial u}{\partial y_i}(\varrho x)\cdot x_i}dxd\varrho \leqslant$$

$$\leqslant \lim_{r\to 0_+}\int_r^R \fint_{B_1}|\nabla u(\varrho x)|dxd\varrho = \lim_{r\to 0_+}\int_r^R\int_{B_\varrho}\frac{|\nabla u|}{\varrho^{d-1+A}}\frac{\varrho^{d-1+A}}{\varrho^d}dxd\varrho\varkappa(\varrho)d\varrho =$$

$$= \lim_{r\to 0_+}\int_r^R \varrho^{A-1}\left(\int_{B_\varrho}\frac{|\nabla u|}{\varrho^{d-1+A}}dx\right)d\varrho \leqslant \left(\sup_{0\leqslant\varrho\leqslant R}\int_{B_\varrho}\frac{|\nabla u|}{\varrho^{d-1+A}}\right)\varkappa(d)\int_0^R \varrho^{A-1}d\varrho = \frac{\varkappa(d)}{A}R^A\sup_{r\leqslant R}\int_{B_\varrho}\frac{|\nabla u|}{\varrho^{d-1+A}}$$

Lemma (Marey II) Let $u \in W^{1,1}_{loc}(\mathbb{R}^d)$ and x, y be Lebesgue points. Then

$$|u(x) - u(y)| \leqslant c(d,A)|x-y|^A \sup_{\varrho\leqslant R, z\in[x,y]}\int_{B_\varrho(z)}\frac{|\nabla u|}{\varrho^{d-1+A}}dx.$$

Proof of lemma: $(R = |x-y|)$

$$|u(x)-u(y)| \leqslant \left|\fint_{B_R(x)} u(z)dz - u(x)\right| + \left|\fint_{B_R(y)} u(z)dz - u(y)\right| + \left|\fint_{B_R(x)} u(z)dz - \fint_{B_R(y)} u(z)dz\right| \leqslant$$

$$\leqslant c(d,A)R^A\left(\sup_{\varrho\leqslant R}\int_{B_\varrho(x)}\frac{|\nabla u|}{\varrho^{d-1+A}} + \sup_{\varrho\leqslant R}\int_{B_\varrho(y)}\frac{|\nabla u|}{\varrho^{d-1+A}}\right) + \left|\int_0^1\frac{d}{dt}\fint_{B_R(tx+(1-t)y)} u(z)dzdt\right| =$$

$$= \ldots + \left|\int_0^1 \frac{d}{dt}\int_{B_R(\mathbf{o})} u(tx+(1-t)y+z)dz\right| \leqslant \ldots + \left|\int_0^1\fint_{B_R(\mathbf{o})}\nabla u(tx+(1-t)y+z)\cdot(x-y)dz\right| \leqslant$$

$$\leqslant \ldots + \int R^A\int_0^1 \varkappa^{-1}()\int_{B_R(tx+(1-t)y)}\frac{|\nabla u|}{R^{d-1+A}}dzdt \leqslant$$

$$\leqslant \tilde{c}(d,A)R^A\sup_{\varrho\leqslant R}\sup_{z\in[x,y]}\int_{B_\varrho(z)}\frac{|\nabla u|}{\varrho^{d-1+A}}.$$

Proof of theorem: We have $\|u\|_{C^{0,\alpha}} \leqslant c\cdot\|u\|_{1,p}$ for $u \in C^1(\overline{\Omega})$

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|Eu\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant \|Eu\|_{C^{0,\alpha}(B_R)} \overset{1.}{\leqslant} c(\overline{B_R},p,d)\cdot\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \overset{\text{Extension}}{\leqslant} C(\overline{B_R},p,d,\Omega)\|u\|_{W^{1,p}(\Omega)},$$

where $\overline{B_R}$ is support of E.

$u \in C^1_0(\overline{B_R})$

$$\sup_{x\neq y}\frac{u(x)-u(y)}{|x-y|^A} \leqslant \sup_{x\neq y} c(d,A)\sup_{\varrho\leqslant|x-y|, z\in[x,y]}\int_{B_\varrho(z)}\frac{|\nabla u|}{\varrho^{d-1+A}}dx \leqslant$$

Důkaz (Case $d > p$ ($d = p$), only for $u \in C_0^\infty(\mathbb{R}^d)$)

$$(*): \quad \|u\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \leqslant c(d,p)\|\nabla u\|_{L^p(\mathbb{R}^d)} \overset{\text{At home}}{W}^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{d-p}}(\Omega) \qquad p < d, \Omega \in C^{0,1}.$$

„Step 1: If $(*)$ is true for $p = 1$, then $(*)$ is true for $p \in [1,d)$“: (Set $v := |u|^q$)

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{q\cdot d}{d-1}}\right)^{\frac{d-1}{d}} = \|v\|_{L^{\frac{d}{d-1}}} \leqslant c(d)\cdot\|\nabla v\|_{L^1} \leqslant c(d)\int_{\mathbb{R}^d} q\cdot|u|^{q-1}\cdot|\nabla u| \leqslant c(d,q)\|\nabla u\|_{L^p}\cdot\|u\|_{L^{p'(q-1)}}^{q-1}.$$

Set $q := \frac{p\cdot(d-1)}{d-p}$:

$$\left(\int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}}\right)^{\frac{d-1}{d}} \leqslant c(d,p)\|\nabla u\|_p \cdot \|u\|_{\frac{dp}{d-p}}^{\frac{p\cdot(d-1)}{d-p}-1}.$$

$$\left(\frac{p}{p-1}\cdot\left(\frac{p\cdot(d-1)}{d-p} - 1\right) = \frac{dp}{d-p}\right).$$

Lemma (Gagliardo): Let $u_i \in C_0^\infty(\mathbb{R}^{d-1})$, $i \in [d]$. Define $v_i(x_1,\dots,x_d) := u_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d)$ Then

$$\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)|dx \leqslant \prod_{i=1}^d \|u_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

Proof of lemma: By induction (with respect to d):

$$d = 2: \qquad \int_{\mathbb{R}^d} |v_1(x)|\cdot|v_2(x)|dx = \int_{\mathbb{R}^2} |u_1(x_2)|\cdot|u_2(x_1)|dx_1dx_2 = \|u_1\|_{L^1(\mathbb{R})}\cdot\|u_2\|_{L^1(\mathbb{R})}.$$

$$d \implies d+1: \qquad \int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1}|v_i(x)|dx = \int_{\mathbb{R}^d} |v_{d+1}(x)|\cdot\left(\int_{\mathbb{R}}\prod_{i=1}^d |v_i(x)|dx_{d+1}\right)dx_1\dots dx_d \overset{\text{Hölder}}{\leqslant}$$

$$\leqslant \left(\int_{\mathbb{R}^d} |v_{d+1}(x)|^d dx_1\dots dx_d\right)^{1/d}\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\prod_{i=1}^d |v_i|dx_{d+1}\right)^{d'} dx_1\dots dx_d\right)^{1/d'} \overset{\text{Hölder}}{\leqslant}$$

$$\leqslant \|u_{d+1}\|_{L^d(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\left(\prod_{i=1}^d\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{1/d}\right)^{d'} dx_1\dots dx_d\right)^{1/d'} \leqslant$$

$$= \|u_{d+1}\|_{L^d(\mathbb{R}^d)}\cdot\left(\int_{\mathbb{R}^d}\prod_{i=1}^d\underbrace{\left(\int_{\mathbb{R}}|v_i|^d dx_{d+1}\right)^{\frac{1}{d-1}}}_{=:z_i} dx_1\dots dx_d\right)^{1/d'} \leqslant$$

$$\leqslant \|u_{d+1}\|_{L^d(\mathbb{R}^d)}\cdot\left(\int_{\mathbb{R}^d}\prod_{i=1}^d |z_i|dz\right)^{1/d'} \overset{\text{Induction hypothesis}}{\leqslant} \|u_{d+1}\|_{L^d(\mathbb{R}^d)}\prod_{i=1}^d \|z_i\|_{L^{d-1}}^{\frac{d-1}{d}} =$$

$$= \|u_{d+1}\|_{L^d(\mathbb{R}^d)}\prod_{i=1}^d TODO = \prod_{i=1}^{d+1}\|u_i\|_{L^d}.$$

Proof of theorem: We want $\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \overset{\text{Gagliardo–Nirenberg inequality}}{\leqslant} c(d)\|\nabla u\|_{L^1(\mathbb{R}^d)}$ $\forall u \in C_0^\infty(\mathbb{R}^d)$.

Důkaz (Compact embeddings)
Step 1: $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$. Step 2: $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $q < \frac{dp}{d-p}$.

„Step 1 $\implies$ Step 2“: $p \leqslant q \leqslant z$:

$$\|u\|_{L^q} \leqslant \|u\|_{L^p}^{\alpha} \cdot \|u\|_z^{1-\alpha}, \qquad \frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{z}.$$

S bounded set in $W^{1,p}(\Omega)$. Goal: $\forall \varepsilon > 0\ \exists \{u_i\}_{i=1}^N \subset L^q(\Omega)\ \forall u \in S : \min_i \|u - u_i\| < \varepsilon$. $W^{1,p}(\Omega) \hookrightarrow W^{1,1}(\Omega) \overset{\text{Step 1}}{\implies} \forall \tilde{\varepsilon} > 0\ \exists \{u_i\}_{i=1}^{N(\tilde{\varepsilon},S)} : \min_i \|u - u_i\|_{L^1(\Omega)} \leqslant \tilde{\varepsilon}$.

$$\|u - u_i\|_{L^q(\Omega)} \leqslant \|u - u_i\|_{L^1(\Omega)}^{\alpha} \cdot \|u - u_i\|_{\frac{dp}{d-p}}^{1-\alpha}, \qquad \frac{1}{q} = \frac{\alpha}{1} + \frac{(1-\alpha)\cdot(d-p)}{dp} \leqslant$$

$$\leqslant c(\Omega, p)\|u - u_i\|_{L^1(\Omega)}^{\alpha} \cdot \|u - u_i\|_{W^{1,p}(\Omega)}^{1-\alpha} \leqslant c(\Omega, p, S) \cdot \|u - u_i\|_{L^1(\Omega)}^{\alpha}.$$

$$\min \|u - u_i\|_{L^q(\Omega)} \leqslant c(\Omega, p, S)\tilde{\varepsilon}^{\alpha}.$$

Given $\varepsilon > 0$. $\tilde{\varepsilon} := \frac{\varepsilon^{1/\alpha}}{c(\Omega,p,S)^{1/\alpha}}$, find $\{u_i\}$ from Step 1 $\implies$ $\min_i \|u - u_i\|_{L^q} \leqslant \varepsilon$.

„Step 1“: Enough $W_0^{1,1}(B_R) \hookrightarrow\hookrightarrow L^1(B_R)$. $u \in W_0^{1,1}(B_R)$, $u_\delta := u * \eta_\delta$.

$$\int_{\mathbb{R}^d} |u - u_\delta| dx = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (u(y) - u(x))\eta_\delta(x - y) dx \right| dy =$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x+y) - u(x)|}{|y|} \eta_\delta(y)|y| dx dy,$$

$$\leqslant \|\nabla u\|_{L^1(\mathbb{R}^d)} \cdot \int_{\mathbb{R}^d} |y|\eta_\delta(y) dy \leqslant \delta \|\nabla u\|_{L^1(\mathbb{R}^d)}.$$

Recall: $C^1(\overline{B_R}) \hookrightarrow\hookrightarrow C^0(\overline{B_R}) \hookrightarrow\hookrightarrow L^1(\overline{B_R})$ (Arzela–Ascoli + Hölder).

S set bounded in $W^{1,1}(\overline{B_R})$. $S_\delta := \{u_\delta, u \in S\}$. $\|u_\delta\|_{C^1(\overline{\overline{B_R}})} \leqslant \frac{C(B_R)\cdot\|u\|_{W^{1,1}(B_R)}}{\delta}$ (for δ small).

Given $\varepsilon > 0$ $\exists \delta$: $\|u - u_\delta\|_{L^1} \leqslant \frac{\varepsilon}{2}$ $\forall u \in S$. $u_\delta \in S_\delta$ (bounded set in $C^1(\overline{B_R})$), $\|u_\delta\|_{C^1} \leqslant \frac{c}{\delta} = c(\varepsilon)$. Find $\{u_\delta^i\}_{i=1}^{N(\varepsilon)}$: $\min \|u_\delta - u_\delta^i\|_{L^1} \leqslant \frac{\varepsilon}{2}$.

$$\|u - u_\delta^i\|_{L^1} \leqslant \|u - u_\delta\|_{L^1} + \|u_\delta - u_\delta^i\| \leqslant \varepsilon.$$

□

1.1 Traces

Poznámka (C^1 functions on cube)
$\Omega = (-1,1)^{d-1} \times (0,1)$, $x' = (x_1, \ldots, x_{d-1})$. $u \in C^1(\overline{\Omega})$, $u(x', 1) = 0$.

Optimal q such that $\int_{(-1,1)^{d-1}} |u(x', \mathbf{o})|^q dx_1 \dots dx_{d-1} \leqslant c \cdot \|\nabla u\|_{L^p(\Omega)}^q$?

$$\int_{(-1,1)^{d-1}} |u(x',0)|^q dx' = \int_{(-1,1)^{d-1}} -\int_0^1 \frac{\partial}{\partial x_d} |u(x', x_d)|^q dx_d dx' \leqslant q \cdot \int_\Omega |u|^{q-1} \cdot |\nabla u| dx \leqslant$$

$$\leqslant q \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \left\| |u|^{q-1} \right\|_{L^{p'}(\Omega)} \overset{?}{\leqslant} q \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \|u\|^{q-1}_{L^{\frac{dp}{d-p}}(\Omega)}.$$

Set q: $(q-1)p' = \frac{dp}{d-p} \implies q = \frac{d(p-1)}{d-p} + 1 = \frac{dp-p}{d-p} = \frac{p \cdot (d-1)}{d-p}$.

$$\|u\|_{L^{\frac{p(d-1)}{d-p}}((-1,1)^{d-1})} \leqslant C(\Omega, p) \cdot \|u\|_{W^{1,p}(\Omega)}.$$

Poznámka (Integral on boundary for $\Omega \in C^{0,1}$)

$$\int_{\partial\Omega} f ds := \int_{\partial\Omega} \sum_{i=1}^N f\varphi_i = \sum_{i=1}^N \int_{(-1,1)^{d-1}} f(T_i(y)) \varphi_i(T_i(y)) \sqrt{1 + |\nabla y|^2} dy',$$

where φ_i is partition of unity corresponding to $C^{0,1}$ and T_i.

We should show independence on φ_i, V_i. Also we can show $\int_\Omega \frac{\partial f}{\partial x_i} dx = \int_{\partial\Omega} f n_i dS$. ($\forall f \in C^1(\overline{\Omega})$.)

TODO!!!

Poznámka (On spaces with non-integer derivative)
tr is not onto $L^{\frac{(d-1)p}{d-p}}(\partial\Omega)$.

Věta 1.2 (Inverse trace theorem)

$\Omega \in C^{0,1}$, $p \in (1, \infty]$, $s \in (1/p, 1]$. *Then* tr *is bounded linear from* $W^{s,p}(\Omega)$ *to* $W^{s-\frac{1}{p},p}(\partial\Omega)$. *Moreover* $\exists\, \mathrm{tr}^{-1} : W^{s-\frac{1}{p},p}(\partial\Omega) \to W^{s,p}(\Omega)$ *linear bounded, such that* $\mathrm{tr}(\mathrm{tr}^{-1}) = u$ *on* $\partial\Omega$.

Definice 1.1 (Sobolev–Slobodeckij spaces)

We say that $u \in W^{s,p}(\Omega)$, $s \in (0,1)$, iff

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+ps}} dx dy < \infty \qquad \wedge\ u \in L^p(\Omega).$$

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)}.$$

Definice 1.2 (Nikólskii spaces)

We say that $u \in N^{s,p}(\Omega)$, $p \in [1, \infty]$, $s \in (0, 1]$, iff

$$\sup_{h,i} \int_{\Omega_h} \frac{|u(x + he_i) - u(x)|^p}{h^{p \cdot s}} dx < \infty.$$

Lemma 1.3

$$W^{s,p}(\Omega) \hookrightarrow N^{s,p}(\Omega) \hookrightarrow W^{s-\varepsilon,p}(\Omega), \qquad \forall 0 < \varepsilon < s.$$

Důkaz
At home. □

2 Nonlinear elliptic equations as a „compact“ perturbation of linear PDE

Poznámka
$-\Delta u = f(x, u, \nabla u)$ in Ω, $u = 0$ on $\partial\Omega$, $f : \Omega \times \mathbb{R} \times \mathbb{R}^d$ is Caratheódory, $|f(x, u, \xi)| \leqslant c$ ($\forall u \in \mathbb{R}, \xi \in \mathbb{R}^d$, and almost all $x \in \Omega$).

Lemma 2.1

$$\exists u \in W_0^{1,2}(\Omega) : \int_\Omega \nabla u \nabla \varphi = \int_\Omega f(\cdot, u, \nabla u)\varphi \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

Důkaz
Consider $M : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega), v \mapsto u$:

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f(x, v, \nabla v)\varphi \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

Linear theory $\implies \forall v \in W_0^{1,2}(\Omega)\ \exists! u \in W_0^{1,2}(\Omega)$ satisfying this equation.

Does M have a fixpoint? Schauder fixpoint theorem assumptions: (A1) $\exists K$ convex set in $W_0^{1,2}(\Omega)$ such that $M(K) \subseteq K$, (A2) M compact, continuous.

Ad (A1): Set $\varphi := u$.

$$C_1\|u\|_{1,2}^2 \leqslant \|\nabla u\|_2^2 = \int_\Omega \nabla u \nabla u = \int_\Omega f(x, v, \nabla v)u \leqslant C \int_\Omega |u| \leqslant \tilde{C}(\Omega)\|u\|_{1,2} \implies \|u\|_{1,2} \leqslant \tilde{\tilde{C}}.$$

$K \subseteq W_0^{1,2}(\Omega)$, $K = \left\{ u \in W_0^{1,2}(\Omega) \middle| \|u\|_{1,2} \leqslant \tilde{\tilde{C}} \right\}$.

Ad (A2): $v^n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$, $u^n = M(v^n)$. $\exists u^{n_k} \to u$ strongly in $W_0^{1,2}(\Omega)$?

$$\|u^n\|_{1,2} \leqslant \tilde{\tilde{C}}, u^{n_k} \rightharpoonup u \text{ in } W_0^{1,2}(\Omega), u^{n_k} \to u \text{ in } L^2(\Omega).$$

Set $\varphi := u^n - u \in W_0^{1,2}(\Omega)$.

$$\lim_{n_k \to 0} \|u^{n_k} - u\|_{1,2}^2 \leqslant C \cdot \lim \|\nabla u^{n_k} - \nabla u\|_2^2 = C \cdot \lim \int_\Omega \nabla u^{n_k} \cdot (\nabla u^{n_k}, \nabla u) - C \cdot \lim \int_\Omega \nabla u \cdot (\nabla u^{n_k} - \nabla u) =$$

$$= C \cdot \lim \int_\Omega f(\cdot, v^n, \nabla v^n)(u^{n_k} - u) - C \cdot \lim \int_\Omega \nabla u \cdot (\nabla u^{n_k} - \nabla u),$$

$$TOOD!!!$$

$$TODO!!!$$

□

Poznámka
Works also if $|f(x, u, \xi)| \leqslant C(g(x) + |u|^\alpha + |\xi|^\alpha)$, where $0 \leqslant \alpha < 1$ and $g \in L^2(\Omega)$.

No estimates available (in principle): $-\Delta u = l|u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$. Does the problem have nontrivial solution? TODO!!!

$$-\Delta v = \lambda A^{2-p} \cdot |v|^{p-2} v$$

if $\exists \lambda > 0\ u \not\equiv 0$ solving $-\Delta u = \lambda |u|^{p-2}u$, then $\forall B > 0\ \exists v : -\Delta v = B|v|^{p-2}v$.

Lemma 2.2

Let $p < \frac{2d}{d-2}$. Then $\exists \lambda > 0\ \exists u \in W_0^{1,2}(\Omega)$ such that $-\Delta u = \lambda |u|^{p-2}u$ in Ω.x TODO!!!

Důkaz
TODO!!! $\square$

TOOD!!!

TODO!!!

TODO!!!

TODO!!!

TODO!!!

TODO!!!

TODO!!!

TODO!!!

TODO!!!