TODO?

Věta 0.1 (Banach–Diedinone)

X Banach space, $\varphi : X^* \to \mathbb{F}$ linear, $\varphi|_{B_{X^*}}$ w^*-continuous. Then $\varphi \in \varkappa(X)$.

Důkaz
Banach–Alaoglu $\implies$ (B_{X^*}, w^*) compact $\implies$ $\varphi(B_{X^*})$ is compact in $\mathbb{F}$. So boundedness $\implies$ $\varphi \in X^{**}$.

Assume $\mathbb{F} = \mathbb{R}$. Fix $\varepsilon > 0$ and define $A_\varepsilon := \{x^* \in B_{X^*} | \varphi(X^*) \leqslant -\varepsilon\}$ and $B_\varepsilon := \{x^* \in B_{X^*} | \varphi(x^*) \geqslant \varepsilon\}$. Then $A_\varepsilon, B_\varepsilon$ are w^*-compact, convex and disjoint. And if ε is small enough and φ nonzero (φ zero is trivially element of $\varkappa(X)$). From the Hahn–Banach separation theorem applied to (X^*, w^*): $\exists \psi \in (X^*, w^*)^* : \sup \psi(A_\varepsilon) < \inf \psi(B_\varepsilon)$. (Note: $A_\varepsilon = -B_\varepsilon$, so $\psi(A_\varepsilon) = -\psi(B_\varepsilon)$, thus $\sup \psi(A_\varepsilon) = -\inf \psi(B_\varepsilon)$, which (with previous) means $\sup \psi(A_\varepsilon) < 0 < \inf \psi(B_\varepsilon)$ and $\operatorname{Ker} \psi|_{B_{X^*}} \cap B_{X^*} \subset B_{X^*}(A_\varepsilon \cup B_\varepsilon)$.) So $\|\varphi|_{\operatorname{Ker}\psi}\| \leqslant \varepsilon$. TODO!!!

TODO!!! □

TODO!!!

1 Banach spaces and compact spaces

Poznámka
Compact := Compact Hausdorff.

Poznámka (Připomenutí)
X Banach space $\implies$ (Banach Alaoglu) (B_{X^*}, w^*) is compact.

K compact $\implies$ $(C(K), \|\cdot\|_\infty)$ is a Banach space.

Poznámka (A kind of duality)
$J : X \to C(B_{X^*}, w^*)$, $J(x)(x^*) = x^*(x)$, $x^* \in B_{X^*}$, $x \in X$. ($J(x) = \varkappa(x)|_{B_{X^*}}$)

It is well defined.

Důkaz
$J(x) \in C(B_{x^*}, w^*)$. $J(x) : x^* \mapsto x^*(x)$ is w^*-continuous. □

J is linear.

Důkaz

$$J(\alpha x+\beta y)(x^*) = x^*(\alpha x+\beta y) = \alpha x^*(x)+\beta x^*(x) = \alpha J(x)(x^*)+\beta J(y)(x^*) = (\alpha J(x)+\beta J(x))(x^*).$$

□

J is isometry.

Důkaz

$$\|J(x)\| = \sup_{x^* \in X^*} |J(x)(x^*)| = \sup_{x^* \in X^*} |x^*(x)| = \|x\|.$$

□

(Previous holds also for non-Banach space. Here, we need completeness.) $J(X)$ is $\|\cdot\|$-closed in $C(B_{X^*}, w^*)$.

Důkaz
X is complete $\implies$ $J(X)$ is complete $\implies$ $J(X)$ is closed. □

J is a homeomorphism $w \to \tau_p$.

Důkaz
„Continuity“: Fix $x^* \in B_{X^*}$. Then $x \mapsto J(x)(x^*) = x^*(x)$ is w-continuous.

„J^{-1} continuous“: Fix $x^* \in X^*$. $f(x) \ni F \mapsto x^*(J^{-1}(f))$ should be τ_p-continuous.

„$J^{-1} : J(X) \to X$“: $\exists y^* \in B_{X^*}$, $\alpha > 0$: $x^* = \alpha y^*$. Then $x^*(J^{-1}(x)) = \alpha y^*(J^{-1}(f)) = \alpha J(J^{-1}(f))(y^*) = \alpha \cdot f(y^*)$. So, $f \mapsto x^*(J^{-1}(f)) = \alpha f(y^*)$ is τ_p-continuous. □

$J(X)$ is τ_p-closed.

Důkaz
„Real case“: $J(X) = \{f \in C(B_{X^*}, w^*) | f(\mathbf{o}) = 0\}$.

RHS – set is τ_p closed. f is a t-fine (i.e. $\forall x^*, y^* \in R_{X^*}\ \forall t \in [0,1] : f(tx^* + (1-t)y^*) = t \cdot f(x^*) + (1-t)f(y^*)$).

?:?? ? : $f \in$ RHS-set. Then $\exists \tilde{f} : X^* \to \mathbb{R}$ linear. $\tilde{f}|_{B_{X^*}} = f$. TODO?

„Complex case“:

$$J(X) = \{f \in C(B_{X^*}, w^*) | f(\mathbf{o}) = 0, f \text{ is a } t\text{-fine}, f(\alpha x^*) = \alpha f(x^*) \text{ for any complex } \alpha\} =$$

$$= \{f \in C(B_{X^*}, w^*) | \forall x^*, y^* \in B_{X^*}, \forall \alpha, \beta \in \mathbb{C} : \alpha x^* + \beta y^* \in B_{X^*} \implies f(\alpha x^* + \beta y^*) = \alpha f(x^*) + \beta f(y^*)$$

□

δ is a homeomorphism of K into $(B_{C(K)^*}, w^*)$.

Důkaz
„δ is one-to-one“: $t_1, t_2 \in K$, $t_1 \neq t_2$. $\implies$ (Uryhson) $\exists f \in C(K) : f(t_1) = 1 \wedge f(t_2) = 0$. It follows the $\delta_{t_1} \neq \delta_{t_2}$.

„δ is continuous“: Fix $f \in C(K)$. Then $t \mapsto \delta_t(f) = f(t)$ is continuous.

„δ is a homeomorphism“: $F \subset K$ closed $\implies$ F is compact $\implies$ $\delta(F)$ is compact $\implies$ $\delta(F)$ is closed. □

TODO(Examples)

TODO!!!

Věta 1.1

X vector space, $M \subset X^{\#}$ separating points $\implies$ $(X, \sigma(X, M))$ has ccc.

Lemma 1.2

Let $B \subset \operatorname{span}(M)$ be an algebraic basis of $\operatorname{span}(M)$. Consider $\Phi : X \to \mathbb{F}^B$ defined by $\Phi(x)(b) = b(x)$, $b \in B$, $x \in X$. Then Φ is a homeomorphism of $(X, \sigma(X, M))$ onto $\Phi(X)$ and $\Phi(X)$ is dense in $\mathbb{F}^B$.

Důkaz
„Φ is one-to-one“: $x, y \in X$, $\Phi(x) = \Phi(y) \implies \forall b \in B : b(x) = b(y) \implies \forall m \in M : m(x) = m(y) \implies x = y$ (M separates points).

„Φ is homeomorphism“: observe $\sigma(X, M) = \sigma(X, B)$. ($b \in B : X \mapsto \Phi(x)(b) = b(x)$ is $\sigma(X, B)$ continuous). $f \in \Phi(x) \mapsto b(\Phi^{-1}(f)) = f(b)$ is continuous.

„density“: $b_1, \dots, b_n \in B$ discrete $\alpha_1, \dots, \alpha_n \in \mathbb{F} \implies x \in X$ such that $\Phi(x)(b_j) = \alpha_j$, $j \in [n]$. $\exists x_1, \dots, x_n \in X$ such that $b_i(x_j) = 1$ if $i = j$ and $b_i(x_j) = 0$ if $i \neq j$. (Assume not, WLOG x_1 does not exists $\implies \bigcap_{j=2}^n \operatorname{Ker} b_j \subset \operatorname{Ker} b_1 \implies b_1 \in \operatorname{span}(b_2, \dots, b_n)$ ↯.) $x \coloneqq \sum \alpha_j x_j$. □

Lemma 1.3

T topological space, $A \subset T$ dense. Then T has ccc $\Leftrightarrow$ A has ccc.

Důkaz
„ $\Longleftarrow$ “ trivial. „ $\implies$ “: $\mathcal{U}$ a disjoint-family of open sets in A. $U \in \mathcal{U} \implies \exists \tilde{U}$ open in T such that $\tilde{U} \cap A = U$. $\tilde{\mathcal{U}} = \left\{ \tilde{U} | U \in \mathcal{U} \right\}$. $\tilde{\mathcal{U}}$ disjoint: $\tilde{U} \cap \tilde{V} \neq \varnothing$. $\tilde{U} \cap \tilde{V}$ open $\implies$ $\tilde{U} \cap \tilde{V} \cap A \neq \varnothing \implies U \cap V \neq \varnothing$. □

Lemma 1.4

$\mathbb{F}^\Gamma$ has ccc for each Γ.

Důkaz
Since $\mathbb{C} = \mathbb{R}^2$ WLOG $\mathbb{F} = \mathbb{R}$. $\mathbb{R}^\mathbb{R}$ is separable (has its own ccc). Assume $a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$ are rational numbers, $r_1, \ldots, r_n \in \mathbb{Q}$. Define $f_{a_1,b_1,\ldots,a_n,b_n,r_1,\ldots,r_n}(x) = r_i$ if $x \in [a_i, b_i]$ and 0 elsewhere. There are countably such functions. They form a dense set ($f \in \mathbb{R}^\mathbb{R}$, $x_1 < \ldots < x_n$, $\varepsilon > 0 \implies$ find $a_1, b_1, \ldots, a_n, b_n, r_1, \ldots, r_n \in \mathbb{Q}$). TODO!!! $\square$

Věta 1.5

X vector space, $M \subset X^\#$ separating points, $f : X \to \mathbb{F}$ $\sigma(X, M)$-continuous $\implies \exists S \subset M$ countable, such that $p_S : X \to \mathbb{F}^S$ is the evaluation mapping ($p_S(x)(m) = m(x), m \in S, x \in X$), then $\exists h : p_S(X) \to \mathbb{F}$ continuous such that $f = h \circ p_S$.

Tvrzení 1.6

Let T be a T_3-topological space, $|T| \geqslant 2$, Γ countable set. Then following assertions are equivalent:

1. *T^Γ has ccc;*
2. *$\forall U \subset T^\Gamma$ open $\exists S \subset \Gamma$ countable such that $\overline{U} = \pi_S^{-1}(\pi_S(\overline{U}))$;*
3. *$\forall U \subset T^\Gamma$ open $\forall X \subset U$ dense $\forall f; X \to \mathbb{F}$ continuous $\exists S \subset \Gamma$ countable $\exists h : \pi_S(X) \to \mathbb{F}$ continuous such that $f = h \circ \pi_S|_X$.*

Důkaz (tvrzeni $\implies$ veta)
Take $B \subset M$, a basis of $\operatorname{span} M$. Let Φ be as in the lemma above $\implies$ Φ is a homeomorphism $(X, \sigma(X, M))$ onto $\Phi(X)$, $\Phi(X)$ is dense in $\mathbb{F}^B$.

$f : X \to F$ $\sigma(X, M)$-continuous $\implies \tilde{f} \coloneqq f \circ \Phi^{-1} : \Phi(X) \to \mathbb{F}$ is continuous. From the previous lemma $\exists S \subset B$ countable, $h : \pi_S(\Phi(X)) \to \mathbb{F}$ continuous, $h \circ \pi_S|_{\Phi(X)} = \tilde{f} \implies h \circ \pi_S \circ \Phi = \hat{f} \circ \varphi = f$. $\pi_S \circ \Phi = p_S$. That's it. $\square$

Důkaz (tvrzeni)
„3 $\implies$ 2“: Let $U \subset T^\Gamma$ be open $\implies X \coloneqq U \cup (T^\Gamma \backslash \overline{U}) \implies X$ is open, $\overline{X} = T^\Gamma$. $f \coloneqq \psi_U$ continuous $X \to \mathbb{R} \overset{3}{\implies} \exists S \subset P$ countable, $h : P_S(X) \to \mathbb{R}$ continuous such that $f = h \circ \pi_S|_X$.

Then $\overline{U} = \pi_S^{-1}(\pi_S(\overline{U}))$: „$\subset$“ always, „$\supset$“: $\pi_S(U) \cap \pi_S(T^\Gamma \backslash \overline{U}) = \varnothing$ ($h|_{\pi_S(U)} = 1$, $h|_{\pi_S(T^\Gamma \backslash \overline{U})} = 0$). π_S is an open mapping $\implies \overline{\pi_S(U)} \cap \pi_S(T^\Gamma \backslash \overline{U}) = \varnothing \implies \pi_S(\overline{U}) \cap \pi_S(T^\Gamma \backslash \overline{U}) = \varnothing \implies \pi_S^{-1}(\pi_S(\overline{U})) \subset \overline{U}$.

„2 $\implies$ 1": Assume T^Γ fails ccc. $\implies$ $\exists U_\alpha, \alpha < \omega_1$ disjoin nonempty open sets. WLOG

$$U_\alpha = \pi_{F_\alpha}^{-1}(\prod_{j\in F_a} O_j^\alpha), \qquad F_\alpha \subset \Gamma \text{ finite}, O_j^a \subset \pi \text{ open}.$$

$\alpha < \omega_1$ find $S_\alpha \in \Gamma\backslash F_\alpha$, $\alpha \neq \beta \implies S_\alpha \neq S_\beta$ $(S_\alpha \in \Gamma\backslash(F_\alpha \cup \{S_\beta|\beta < \alpha\}))$. $V_\alpha \subset U_\alpha$ open such that $\overline{\pi_{S_\alpha}(V_\alpha)} \neq T$. $H \coloneqq \overline{\bigcup_\alpha V_\alpha} \implies \exists S$ countable $H = \pi_s^{-1}(\pi_S(H))$. $\exists \alpha : S_\alpha \notin S$ $\implies \exists x \in H : x \in U_\alpha\backslash\overline{V_\alpha} \implies x \notin \overline{\bigcup_{b\neq\alpha} V_b}$ ↯. □

TODO!!!

Tvrzení 1.7

X NLS then following assertions are equivalent:

1. (X, w) *has countable base;*
2. (X, w) *is metrizable;*
3. (X, w) *has countable character;*
4. $\dim X < \infty$.

Důkaz
„(4. $\implies$ 1. $\wedge$ 2.)": $\dim X < \infty \implies w = \|\cdot\|$ on X. ($X = \mathbb{F}^n$.)

„1. $\implies$ 3." and „2. $\implies$ 3.": obvious.

„3. $\implies$ 4.": Assume $(U_n)_{n\in\mathbb{N}}$ is a base of neighbourhood of $\mathbf{o}$. WLOG $U_n = \{x \in X \big| |f_1^n(x)| < \varepsilon_1^n, \ldots,$
for $\varepsilon_j^n > 0$, $f_j^n \in X^*$.

Claim: „$\operatorname{span}\{f_j^n | 1 \leqslant j \leqslant k_n, n \in \mathbb{N}\} = X^*$". Let $f \in X^*$ be as the $V = \{x \in X | |f(x)| < 1\}$ is a weak neighbourhood of 0. $\implies \exists n \in \mathbb{N} : U_n \subset V \implies \bigcap_{j=1}^{k_n} \operatorname{Ker} f_j^n \subset \operatorname{Ker} f$. ($x \in \bigcap_{j=1}^{k_n} \operatorname{Ker} f_j^n \implies f_j^n(x) = 0$, for all $i \in [k_n] \implies \forall m \in \mathbb{N} : mx \in U_n \subset V \implies$ $|f(x)| < 1 \implies |f(x)| < \frac{1}{m} \implies f(x) = 0$.) Hence $f \in \operatorname{span}\{f_1^n, \ldots, f_{k_n}^n\}$.

$\implies \exists (y_n) \subset X^*$ such that $\operatorname{span}\{g_n; n \in \mathbb{N}\} = X^*$. $F_n \coloneqq \operatorname{span}\{g_1, \ldots, g_n\}$. Then $F_n \Subset X^*$, closed, $\bigcup_n F_n = X^*$. X^* is complete $\implies$ (Baire) $\exists n : \operatorname{int} F_n \neq \emptyset$. $\implies$ $F_n = X^* \implies \dim X^* < \infty \implies$ *char*$X < \infty$. □

Věta 1.8

X measure space, $A \subset X$. Then following assertions are equivalent:

1. $(A, \|\cdot\|)$ *is separable;*
2. (A, w) *is separable;*
3. (A, w) *has countable network.*

Důkaz
„1. $\implies$ 3.“: $(A, \|\cdot\|)$ is separable $\implies$ $(A, \|\cdot\|)$ has countable base and this base is a network for (A, w).

„3. $\implies$ 2.“: obvious.

„2. $\implies$ 1.“: $C \subset (A, w)$ countable dense. $\tilde{C} \coloneqq$ rational convex combinations of elements of C. $\implies$ $\tilde{C}$ is countable, the $\overline{\tilde{C}}^{\|\cdot\|}$ is a convex set, $\|\cdot\|$-closed, hence w-closed $\implies$ $A \subset \overline{C}^{w} \subset \overline{\tilde{C}}^{w} \subset \overline{\tilde{C}}^{\|\cdot\|}$ (from Mazur's theorem). $(\overline{\tilde{C}}^{\|\cdot\|}, \|\cdot\|)$ is separable $\implies$ $(A, \|\cdot\|)$ is separable. □

Důsledek
X Banach space. Then X is separable $\Leftrightarrow$ (X, w) is separable $\Leftrightarrow$ (X, w) has countable network.

Příklad
X, Y Banach spaces, (X, w) and (Y, w) are homeomorphic. Are X, Y isomorphic?

Věta 1.9

X Banach space. Then following assertions are equivalent:

1. *X is seprable;*
2. *$\exists T : X^* \to \mathbb{F}^{\mathbb{N}}$ linear, one-to-one, w^*-continuous;*
3. *$\exists T : X^* \to c_0$ linear, one-to-one, $w^* - \tau_p$ continuous, $\|T\| \leqslant 1$;*
4. *$\exists T : X^* \to c_0$ linear, one-to-one, $w^* - w$ continuous, $\|T\| \leqslant 1$.*

Důkaz
„4. $\implies$ 3. $\implies$ 2.“: obvious.

„2. $\implies$ 1.“: Let T be as in 2. Define $\varphi_n : X^* \to \mathbb{F}$ by $\varphi_n(x^*) \coloneqq$ the n-th coordinate of $T(x^*)$. Then φ_n is linear, w^*-continuous. $\implies \exists x_n \in X$ such that $\varphi_n(x^*) = x^*(x_n)$, $x^* \in X^*$. Then $\{x_n; n \in \mathbb{N}\}$ separates points of X^*. ($x^* \in X^* \setminus \{\mathbf{o}\} \overset{\text{one-to-one}}{\implies} Tx^* \neq \mathbf{o} \implies \exists n \in \mathbb{N} : x^*(x_n) = \varphi_n(x^*) = (Tx^*)(n) \neq 0.$) $\implies \overline{\operatorname{span}\{x_n n \in \mathbb{N}\}}^{\|\cdot\|} = X \implies X$ is separable.

„1. $\implies$ 4.“: X separable $\implies$ B_X is separable. Let $\{x_n\}_{n=1}^\infty$ be base in B_X. Define $T : X^* \to c_0$ by $Tx^* = \left(\frac{1}{n} x^*(x_n)\right)_{n=1}^\infty$. Then $Tx^* \in c_0$. ($|\frac{1}{n} x^*(x_n)| \leqslant \frac{1}{n} \|x^*\| \cdot \|x_n\| \leqslant \frac{1}{n\|x^*\| \to 0}$.)

„T is linear, $\|T\| \leqslant 1$“: $w^* - w$ continuity: Let $f \in c_0^*$, we will show that $f \circ T$ is w^*-continuous. So, fix $f \in c_0^* \implies \exists (y_n)_{n=1}^\infty \in l_1$ representing f. Then

$$(f \circ T)(x^*) = f(T(x^*)) = f\left(\left(\frac{1}{n} x^*\left(x_n\right)\right)_{n=1}^\infty\right) = \sum_{n=1}^\infty y_n \cdot \frac{1}{n} x^*\left(x_n\right) = x^*\left(\sum_{n=1}^\infty \frac{y_n}{n} x_n\right),$$

so it is w^*-continuous.

$$\sum_{n=1}^\infty \left\| \frac{y_n}{n} x_n \right\| = \sum_{n=1}^\infty \frac{|y_n|}{n} \|x_n\| \leqslant \sum_{n=1}^\infty |y_n \leqslant \|f\| < \infty.$$

□

Věta 1.10

X separable Banach space.

1. *Any $\|\cdot\|$-open set is weakly F_σ. In particular non-Borel and weakly Borel sets coincide.*
2. *X is $F_{\sigma\delta}$ in (X^{**}, w^*).*

Důkaz
„1.“: $U \subset X$ $\|\cdot\|$-open $\implies \forall x \in U\ \exists \delta_x > 0 : \overline{U(x,\delta_x)} \subset U \implies \bigcup_{x\in U} U(x,\delta_x) = U \implies$ (X separable) $\bigcup_{n=1}^{\infty} U(x_n,\delta_{x_n}) = U$. Then $U = \bigcup_{n=1}^{\infty} \overline{U(x_n,\delta_{x_n})} \implies U$ is weakly F_σ.

„2.“: Let $\{x_n\} \subset X$ be $\|\cdot\|$-dense. Then

$$X = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (x_n + \frac{1}{k} B_{X^{**}}).$$

RHS is $F_{\sigma\delta}$ in w^*.

„$\subset$“: $x \in X$, $k \in \mathbb{N} \implies \exists n \in \mathbb{N} : \|x - x_n\| < \frac{1}{k} \implies x \in x_n + \frac{1}{k} B_X \subset x_n + \frac{1}{k} B_{X^{**}}$. So $x \in RHS$.

„$\supset$“: $x^{**} \in RHS$. Fix $k \in \mathbb{N} \implies \exists n : x^{**} \in x_n + \frac{1}{k} B_{X^{**}}$. $\implies \|x^{**} - x_n\| \leqslant \frac{1}{k} \implies \operatorname{dist}(x^{**}, X) \leqslant \frac{1}{k}$. This holds for each $k \in \mathbb{N}$, hence $\operatorname{dist}(x^{**}, X) = 0$ and from X is closed, we get $x^{**} \in X$. □

Věta 1.11

1. X Banach space. Then X is separable $\Leftrightarrow (B_{X^*}, w^*)$ is metrizable.

2. K compact T_2. Then K is metrizable $\Leftrightarrow C(K)$ is separable.

Důkaz
„1., $\implies$“: X separable $\implies$ (from the theorem above) $\exists T: X^* \to c_0$ linear, one-to--one, $\|T\| \leqslant 1$, $w^* - \tau_p$ continuous. Then $T|_{B_{X^*}}$ is a homeomorphism of (B_{X^*}, w^*) into (B_{c_0}, τ_p) (use compactness of (B_{X^*}, w^*)). $T(B_{X^*}) \subset B_{c_0} \subset \{t \in \mathbb{F} | |t| \leqslant 1\}^{\mathbb{N}}$. The last one is metrizable compact, so (B_{X^*}, w^*) is metrizable.

„2., $\impliedby$“: $C(K)$ is separable, so (from first part) $(B_{C(K)^*}, w^*)$ is metrizable. Sic $K \hookrightarrow (B_{C(K)^*}, w^*)$, K is metrizable.

„2., $\implies$“: Let K be metrizable. Then $\exists (f_n) \subset C(K, \mathbb{R})$ separating points of K (K metrizable compact $\implies$ it has countable base $(U_n)_{n \in \mathbb{N}}$. Let $I \coloneqq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} \middle| \varnothing \neq \overline{U_m} \subset U_n\right\}$, then I is a countable set. $(m,n) \in I \implies \exists f_{m,n}: K \to [0,1]$ continuous,

$$f_{m,n}(x) = \frac{\operatorname{dist}(x, K \backslash U_m)}{\operatorname{dist}(x, K \backslash U_n) + \operatorname{dist}(x, K \backslash U_m)}, \qquad f_{m,n}|_{U_m} = 1, \quad f_{m,n}|_{U_n} = 0.$$

$(f_{m,n})_{(m,n) \in I}$ separates points of K. $x \neq y \in K \implies \exists n \in \mathbb{N}: x \in U_n, y \notin U_n$ because $K \backslash \{y\}$ is open and $x \in K \backslash \{y\}$. $\implies \exists V$ open: $x \in V \subset \overline{V} \subset U_n \implies \exists m \in \mathbb{N}: x \in U_m \subset V$. Then $(n, m \in I)$, $f_{m,n}(x) = 1$, $f_{m,n}(y) = 0$.)

Let $\mathcal{A} \coloneqq \operatorname{span}\{1, \text{ finite products of } (f_n)\}$. Then $\mathcal{A}$ is separable, separates points of K, contains constants, $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$, so $\mathcal{A}$ is an algebra $\implies$ (Stone–Weierstrass) $\overline{\mathcal{A}}^{\|\cdot\|} = C(K)$, so $C(K)$ is separable.

„1., $\impliedby$“: (B_{X^*}, w^*) is metrizable, so (from the third part) $C(B_{X^*}, w^*)$ is separable and $X \subset C(B_{X^*}, w^*)$ is also separable. □

Poznámka
The first part of the proof provides a formula for a metric on B_{X^*}:

$$(x_n) \subset B_X \text{ dense} \qquad \implies \qquad \varrho(x^*, y^*) = \sum_{n=1}^{\infty} \frac{1}{2^n} |x^*(x_n) - y^*(x_n)|.$$

Kelloc's theorem $\implies$ (X is separable, $\dim X = \infty \implies (B_{X^*}, w^*)$ is homeomorphic to $[0,1]^{\mathbb{N}}$).

In particular, X separable, reflexive, $\dim X = \infty \implies (B_X, w)$ is homeomorphic to $[0,1]^{\mathbb{N}}$.

Příklad
$T: l_p \to l_q$, $T((x_n)_n) = ((\operatorname{sgn} f_n)|f_n|^{\frac{p}{q}})_n \implies T$ is a bijection l_p onto l_q. $\|Tx\|_q^q = \|x\|_p^p$. In particular $T(B_{l_p}) = B_{l_q}$ and T is $\tau_p \to \tau_q$ homeomorphism ($\implies T$ is homeomorphism $(B_{l_p}, w) \to (B_{l_q}, w)$).

Definice 1.1

Let X be a Banach space. A Markuševič basis of X is a system $(x_\alpha, x_\alpha^*)_{\alpha\in A} \subset X \times X^*$, such that:

- $x_\alpha^*(x_\beta) = 1$ if $\alpha = \beta$ and 0 if $\alpha \neq \beta$;
- $\overline{\operatorname{span}}\{x_\alpha, \alpha \in A\} = X$;
- $\forall x \in X \setminus \{\mathbf{o}\}\ \exists \alpha \in A : x_\alpha^*(x) \neq 0$ (i.e. $(x_\alpha^*)_{\alpha\in A}$ separate points of $X \Leftrightarrow \overline{\operatorname{span}}^{w^*}\{x_\alpha^*, \alpha \in A = X^*\}$).

Poznámka
X separable Banach space, $(x_\alpha, x_\alpha^*)_{\alpha\in A}$ is an Markuševič basis. Then A is countable.

Důkaz
WLOG $\|x_\alpha^*\| = 1$ for each α.

$\alpha \neq \beta \implies \|x_\alpha - x_\beta\| \geqslant |x_\alpha^*(x_\alpha - x_\beta)| = 1 \implies (x_\alpha)_{\alpha\in A}$ 1-discrete $\implies A$ is countable. □

Věta 1.12 (Markuševič)

X separable Banach space ($\dim X = \infty$*), $(z_n) \subset X$ such that $\overline{\operatorname{span}}\{z_n, n \in \mathbb{N}\} = X$, $(z_n^*) \subset X^*$ separates points of X. Then $\exists$ an Markuševič basis $(x_n, x_n^*)_{n\in\mathbb{N}}$ such that* $\operatorname{span}\{x_n, n \in \mathbb{N}\} = \operatorname{span}\{z_n, n \in \mathbb{N}\}$ *and* $\operatorname{span}\{x_n^*, n \in \mathbb{N}\} = \operatorname{span}\{z_n^*, n \in \mathbb{N}\}$.

Důkaz
$k_1 :=$ the first index such that $z_{k_1} \neq 0$. $x_1 := z_{k_1}$. Find $l_1 \in \mathbb{N}$ such that $z_{l_1}^*(x_1) \neq 0$. $x_1^* := \frac{z_{l_1}^*}{z_{l_1}^*(x_1)}$.

Find the smallest l_2 such that $z_{l_2}^* \notin \operatorname{span}\{x_1^*\}$. $x_2^* := z_{l_2}^* - z_{l2}^*(x_1) \cdot x_1^*$ ($\implies x_2^*(x_1) = 0, x_2^* \neq 0$). Find k_2 such that $x_2^*(z_{k_2}) \neq 0$. $x_2 := \frac{z_{k_2} - x_1^*(z_{k_2}) \cdot x_1}{x_2^*(z_{k_2})}$. ($\implies x_1^*(x_2) = 0$, $x_2^*(x_2) = 0$.)

Find the smallest k_3 such that $z_{k_3} \notin \operatorname{span}\{x_1, x_2\}$. $x_3 := z_{k_3} - x_1^*(z_{k_3})x_1 - x_2^*(z_{k_3}) \cdot x_2$. Find l_3 such that $z_{l_3}^*(x_3) \neq 0$. $x_3^* := \frac{z_{l_3}^* - z_{l_3}^*(x_1)x_1^* - z_{l_3}^*(x_2)x_2^*}{z_{l_3}^*(x_3)}$.

Etc. Then $(x_n, x_n^*)_{n\in\mathbb{N}}$ satisfies first. $\operatorname{span}\{x_n, n \in \mathbb{N}\} = \operatorname{span}\{z_n, n \in \mathbb{N}\}$ („$\subset$“ clear, „$\supset$“: k_{2n-1} is always the smallest such that $z_{k_{2n-1}}$ is not covered by $\operatorname{span}\{x_1, \dots, x_{2n-1}\}$ and $z_{k_{2n-1}} \in \operatorname{span}\{x_1, \dots, x_{2n-1}\}$). $\operatorname{span}\{x_n^*, n \in \mathbb{N}\} = \operatorname{span}\{z_n^*, n \in \mathbb{N}\}$ (analogous). □

Věta 1.13

Let X be a Banach space. Then following assertions are equivalent:

1. X^* is separable;

2. (B_X, w) is metrizable;

3. $X = \bigcup_n F_n$ for F_n weakly closed and (F_n, w) metrizable;

Důkaz (1. $\implies$ 2.)
X^* separable $\implies$ (by the theorem above) $(B_{X^{**}}, w^*)$ is metrizable. And note that $(B_X, w) \subset (B_{X^{**}}, w^*)$. □

Důkaz (2. $\implies$ 3.)
Take $F_n = n \cdot B_X$. □

Důkaz (3. $\implies$ 2.)
$X = \bigcup_n F_n$ as in 3. F_n is weakly closed $\implies$ F_n are $\|\cdot\|$ closed $\implies$ (Baire) $\exists n : \operatorname{int}_{\|\cdot\|} F_n \neq \emptyset \implies \exists x \in X\ \exists r > 0 : x + r \cdot B_X \in F_n \implies (x + rB_X, w)$ is metrizable (homeomorphic to (B_x, w), so (B_x, w) is metrizable). □

Důkaz (2. $\implies$ 1.)
Let ϱ be a metric on B_X inducing w. Then $U_n = \left\{x \in B_x \middle| \varrho(0, x) < \frac{1}{n}\right\}$ is w-open $\implies$ $\exists x^*_{n,1}, \dots, x^*_{n,k_n} \in X^*$ such that $\left\{x \in B_x \middle| |x^*_{n,j}(x)| < 1, j \in [k_n]\right\} \subset U_n$. Define $Y \coloneqq \overline{\operatorname{span}} \{x^*_{n,j}, j \in [k_n], n \in$
We will show $Y = X^*$.

Assume $X^* \setminus Y \neq \emptyset$. Fix $x^* \in X^* \setminus Y$. Then $d \coloneqq \operatorname{dist}(x^*, Y) > 0$. From Hahn--Banach theorem: $\exists x^{**} \in X^{**} : \|x^{**}\| = \frac{1}{d}$, $x^{**}|_Y = 0$, $x^{**}(x^*) = 1$. Define $V \coloneqq \left\{x \in B_X \middle| |x^*(x)| < \frac{d}{2}\right\}$. Then V is w-open neighbourhood of $\mathbf{o}$ in $B_X \implies \exists n : U_n \subset V$.

Goldstine $\implies \exists x_1 \in B_X$ such that

$$|x^*(x_1) - d| = |x^*(x_1) - dx^{**}(x^*)| < \frac{d}{2} \wedge |x^*_{n,j}(x_1) - dx^{**}(x_{n,j})| = |x^*_{n,j}(x_1)| < 1, \quad \text{for } j \in [k_n].$$

Then $x_1 \in U_n \subset V \implies |x^*(x_1)| < \frac{d}{2}$, $|x^*(x_1)| \geqslant d - |x^*(x_1) - d| > \frac{d}{2}$. ↯. □

Poznámka
1. $\implies$ X is separable. (X^* separable $\implies$ $(B_{X^{**}}, w^*)$ is a metrizable compact, so, it is a separable metrizable space. So $(B_X, w) \subset (B_{X^{**}}, w^*)$ is also separable metrizable. Thus from the theorem above $(B_X, \|\cdot\|)$ is separable $\implies$ X is separable.)

If, moreover, X is separable, also next ones are equivalent to 1.-3. (The remark means that 1. $\implies$ 4. holds and that 4. $\implies$ 1. needs separability of X.) (TODO count from 4.)

1. X has a shrinking (i.e., $\overline{\operatorname{span}\{x^*_\beta, \beta \in A\}}^{\|\cdot\|} = X^*$) Markuševič basis.

2. (X^*, w) is Lindelöf.

3. $Borel(X^*, \|\cdot\|) = Borel(X^*, w^*)$.

4. $Borel(X^*, w) = Borel(X^*, w^*)$.

Důkaz (1. $\implies$ 4.)
X^* and X are separable, so $\exists (z_n) \subset X$ and $\exists (z_n^*) \subset X^*$ $\|\cdot\|$-dense. Apply the theorem above. □

Důkaz (4. $\implies$ 1.)
X separable $\implies$ any Markuševič basis is countable. So, there is a shrinking Markuševič basis $(X_n, X_n^*)_{n\in\mathbb{N}}$. Then $X^* = \overline{\text{span}}^{\|\cdot\|}\{x_n^*, n \in \mathbb{N}\} \implies X^*$ is separable. □

Důkaz (1. $\implies$ 5.)
Trivial. (X^* separable $\implies$ $(X^*, \|\cdot\|)$ is Lindelöf $\implies$ (X^*, w) is Lindelöf.) □

Důkaz (1. $\implies$ 6.)
X^* separable $\implies$ any $\|\cdot\|$-open set in X^* is w^*-F_σ. See the proof above. (U is $\|\cdot\|$-open. $x^* \in U$ $\exists r_{x^*}$: $\overline{U(x^*, r_{x^*})} \subset U$. $U(x^*, r_{x^*}), x^* \in U$ is an open cover of U $\implies$ $\exists (x_n^*) : U = \bigcup_n U(x^*, r_{x^*}) = \bigcup_n \overline{U(x^*, r_{x^*})}$ which is w^* compact.) □

Důkaz (6. $\implies$ 7.)
Clear, as $w^* \subset w \subset \|\cdot\|$. Hence $Borel(X^*, w^*) \subset Borel(X^*, w) \subset Borel(X, \|\cdot\|)$. □

Důkaz (5. $\implies$ 1. $\wedge$ 7. $\implies$ 1.)
Claim: X separable, X^* non-separable $\implies \exists \Delta \subset S_{X^*}$ such that (Δ, w^*) is homeomorphic to $\{0,1\}^{\mathbb{N}}$ (= Cantor set) and (Δ, w) is discrete.

„If we prove this claim, we are done“: Δ is w^*-closed (is homeomorphic to $\{0,1\}^{\mathbb{N}}$), hence w-closed. w-closed and discrete $\implies$ (Δ, w) is not Lindelöf, hence (X^*, w) is not Lindelöf. $\lightning$ with 5. (Δ, w) is closed and discrete, hence each subset is w-closed. But (Δ, w^*) is homeomorphic to $\{0,1\}^{\mathbb{N}}$ and there are non-Borel sets in $\{0,1\}^{\mathbb{N}}$. $\lightning$ with 7.

„Claim“: Assume X is separable, X^* is non-separable.

Step 1: Given $\varepsilon > 0$ $\exists (x^*_\alpha, x^{**}_\alpha)_{\alpha<\omega_1} \subset X^* \times X^{**}$ such that $\|x^*_\alpha\| = 1$, $\|x^{**}_\alpha\| < 1+\varepsilon$ and $x^{**}_\alpha(x^*_\beta) = 1$ if $\beta = \alpha$ and $= 0$ if $\beta < \alpha$. In sketch: Fix x^*_0, x^{**}_0 such that $\|x^*_0\| = 1 = \|x^{**}_0\| = x^{**}_0(x^*_0)$. Now assume $1 \leqslant \alpha < \omega_1$ and that we already have $(x^*_\beta, x^{**}_\beta)$ for $\beta < \alpha$. $Z \coloneqq \overline{\operatorname{span}}\left\{x^*_\beta, \beta < \alpha\right\} \implies Z$ is separable, so $Z \subsetneq X^* \implies \exists x^{**}_\alpha \in X^{**}$ such that $x^{**}_\alpha|_Z = 0$, $\|x^{**}_\alpha\| = 1 + \frac{\varepsilon}{2}$. Then find $x^*_\alpha \in X^*$, $\|x^*_\alpha\| = 1$ and $x^{**}_\alpha(x^*_\alpha) = 1$.

Step 2: WLOG $\{x^*_\alpha, \alpha < \omega_1\}$ is locally uncountable in (X^*, w^*). ($\{x^*_\alpha, \alpha < \omega_1\} \eqqcolon A$ is an uncountable subset of (B_{X^*}, w^*). (B_{X^*}, w^*) is a separable metrizable space. $\mathcal{U} = \{U \subset B_{X^*} | w^*\text{-open } \wedge U \cap A \text{ is countable}\}$. $\implies$ $\exists \mathcal{U}' \subset \mathcal{U}$ countable such that $\bigcup \mathcal{U} = \bigcup U'$. Thus, $A \cap \bigcup \mathcal{U}$ is countable. $A' \coloneqq A \setminus \bigcup \mathcal{U}$. Then A' is uncountable (in fact $A \setminus A'$ is countable) and $\forall U \subset B_{X^*}$ w^*-open $U \cap A' \neq \varnothing \implies U \cap A'$ is uncountable.)

Step 3: Fix ϱ a metric generating w^* on B_{X^*}. We will construct w^*-open sets $U_s \subset B_{X^*}$, $s \in \bigcup_{n=0}^{\infty} \{0,1\}^n$ and $X_s \in X$, $\|x_s\| < 1 + \varepsilon$ such that:

- $U_\varnothing = B_{X^*}$;
- $\operatorname{diam} U_s < \frac{1}{|s|+2}$ if $|s| \geqslant 1$;
- $\overline{U_s}^{w^*} \cap \left(1 - \frac{1}{|s|+2}\right) \cdot B_{X^*} = \varnothing$, for $|s| \geqslant 1$;
- $\overline{U_{s^\wedge 0}}^{w^*} \cup \overline{U_{s^\wedge 1}}^{w^*} \subset U_s$, $\overline{U_{s^\wedge 0}}^{w^*} \cap \overline{U_{s^\wedge 1}}^{w^*} = \varnothing$;
- $U_s \cap A \neq \varnothing$;
- $\forall x^* \in U_s : (x^*(x_s) - 1) < \frac{1}{|s|+1}$, for $|s| \geqslant 1$;
- $\forall x^* \in \bigcup \{U_t | |t| = |s|, t \neq s\} : |x^*(x_s)| < \frac{1}{|s|+1}$, for $|s| \geqslant 1$.

Construction: Set $U_\varnothing = B_{X^*}$. Assume that for $n \in \mathbb{N}$, we have the construction for $|s| < n$. For $|s| = n-1$ we find $V_{s^\wedge 0}$ and $V_{s^\wedge 1}$ (w^*-open in B_{X^*}) such that „II to V“ are satisfied.

We have V_s, $s \in \{0,1\}^n$, order that by $V_0, V_1, \dots, V_{2^n-1}$. Find $\alpha_1, \alpha_2, \dots, \alpha_{2^n-1} < \omega_1$ such that $x^*_{\alpha_i} \in V_i$, $i \in [2^n - 1]$. Next find $\alpha_0 > \max\{\alpha_1, \dots, \alpha_{2^n-1}\}$ such that $x^*_{\alpha_0} \in V_0$. Then $x^{**}_{\alpha_0}(x^*_{\alpha_0}) = 1$, $x^{**}_{\alpha_0}(x^*_{\alpha_i}) = 0$, $i \in [2^n - 1]$ $\implies$ (Goldstine) $\exists x_0 \in X$, $\|x_0\| < 1 + \varepsilon$, $|x^*_{\alpha_0}(x_0) - 1| < \frac{1}{n+1}$, $|x^*_{\alpha_i}(x_0)| < \frac{1}{n+1}$, $i \in [2^n - 1]$.

$$V_{0,0} \coloneqq \left\{x^* \in V_0 \middle| |x^*(x_0) - 1| < \frac{1}{n+1}\right\},$$

$$V_{i,0} \coloneqq \left\{x^* \in V_i \middle| |x^*(x_0)| < \frac{1}{n+1}\right\}, \quad i \in [2^n - 1].$$

2 Reflexive spaces

TODO?

Tvrzení 2.1

1. X is reflexive $\Leftrightarrow$ X^* is reflexive.

2. X is reflexive and separable $\implies$ X^* is separable.

Důkaz

„2.“: X reflexive and separable $\implies$ X^{**} separable $\implies$ X^* separable. „1.“: Intro to FA. □

Tvrzení 2.2

X Banach space, then following assertions are equivalent:

1. *X is reflexive;*
2. *(B_X, w) is compact;*
3. *(X, w) is σ-compact.*

Důkaz

„1. $\implies$ 2.“: X reflexive $\implies$ $\varkappa(X) = X^{**}$ $\implies$ $\varkappa(B_X) = B_{X^{**}}$. $\varkappa$ is w-w^* homeomorphism and $B_{X^{**}}$ is w^* compact, so B_X is w-compact.

„2. $\implies$ 1.“: Assume B_X is w-compact, then $\varkappa(B_X)$ is w^*-compact, hence w^*-closed and by Goldstine it is w^*-dense in $B_{X^{**}}$ $\implies$ $\varkappa(B_X) = B_{X^{**}}$ $\implies$ $\varkappa(X) = X^{**}$.

„2. $\implies$ 3.“: $X = \bigcup_{n=1}^{\infty} n \cdot B_X$. „3. $\implies$ 2.“: $X = \bigcup_{n=1}^{\infty} K_n$, (K_n, w) compact $\implies$ K_n are w-closed, so $\|\cdot\|$-closed. Baire $\implies$ $\exists n : \operatorname{int}_{\|\cdot\|} K_n \neq \emptyset$ $\implies$ $\exists x \in X\ \exists r > 0$ such that $x + r \cdot B_X \subset K_n$ $\implies$ $x + r \cdot B_X$ is w-compact $\implies$ B_X is w-compact. □

Tvrzení 2.3

X reflexive $\Leftrightarrow$ (B_{X^*}, w) is compact.

Důkaz
„Method 1“: Use the previous propositions.

„Method 2“: „ $\implies$ “ X reflexive $\implies$ on X^* we have $w = w^*$. (B_{X^*}, w^*) is compact (Banach-Alaoglu) $\implies$ (B_{X^*}, w) is compact.

„ $\impliedby$ “ (B_{X^*}, w) is compact $\implies$ $w = w^*$ on B_{X^*}. $x^{**} \in X^{**}$ $\implies$ x^{**} is w-continuous $\implies$ $x^{**}|_{B_{X^*}}$ is w^*-continuous $\implies$ (Banach-Diledin) x^{**} is w^*-continuous $\implies$ $x^{**} \in \varkappa(X)$. □

Tvrzení 2.4

$C(K)$ is reflexive $\Leftrightarrow$ K is finite.

Důkaz
„ $\impliedby$ “: K finite $\implies$ $\dim C(K) < \infty$ $\implies$ $C(K)$ is reflexive.

„ $\implies$ “: K is infinite $\implies$ $C(K)$ not reflexive:

„Method 1“: Using Riesz theorem: $C(K)^* \approx M(K)$, $\mu \in M(K)$, $\mu(f) = \int f d\mu$. K is infinite $\implies$ $\exists x_0 \in K$ non-isolated. Define $\varphi \in M(K)^*$ by $\varphi(\mu) = \mu(\{x_0\})$. Then $\|\varphi\| = 1$ and $\varphi \notin \varkappa(C(K))$. (Assume $f \in C(K), \varphi = \varkappa(f)$. Then for $x \in K : f(x) = \delta_x(f) = \varkappa(f)(\Gamma_x) = \varphi(\delta_x) = \varphi_x(\{x_0\}) = 1$ if $x = x_0$ and 0 for $x \neq x_0$. So, $f = \chi_{x_0}$, which is not a continuous function. ↯.) □