*Příklad* (1.)
Let $\mathbb{A} \in \mathbb{R}^{n\times n}$ be a matrix, and let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ be arbitrary vectors. Show that $\mathbb{A}(\mathbf{a}\otimes\mathbf{b}) = (\mathbb{A}\mathbf{a})\otimes\mathbf{b}$.

*Důkaz*
Podle definice tensorového součinu (a asociativity násobení):

$$\forall \mathbf{u} \in \mathbb{R}^n : \mathbb{A}((\mathbf{a}\otimes\mathbf{b})\mathbf{u}) = \mathbb{A}\left(\mathbf{a}(\mathbf{b}\cdot\mathbf{u})\right) = (\mathbb{A}\mathbf{a})(\mathbf{b}\cdot\mathbf{u}) = ((\mathbb{A}\mathbf{a})\otimes\mathbf{b})\mathbf{u}.$$

□

*Příklad* (2.)
Let $\mathbb{X} \in R^{n\times n}$ be a symmetric matrix given by the formula $\mathbb{X} = \sum_{i,j=1}^{n} X_{ij}\mathbf{v}_i\otimes\mathbf{v}_j$, where $\{v\}_{i=1}^{n}$ is an orthonormal basis in $\mathbb{R}^n$. Show that

(a) $$X_{ij} = \mathbf{v}_i\cdot\mathbb{X}\mathbf{v}_j.$$

(b) $$(\mathbf{v}_j\otimes\mathbf{v}_i)\,X_{ij} = (\mathbf{v}_j\otimes\mathbf{v}_j)\,\mathbb{X}\,(\mathbf{v}_i\otimes\mathbf{v}_i)\,.$$

Summation convention is not being used!

*Důkaz* (a)
Dosadíme do pravé strany za $\mathbb{X}$, použijeme definici tensorového součinu a aplikujeme linearitu násobení a skalárního součinu:

$$\mathbf{v}_i\cdot\mathbb{X}\mathbf{v}_j = \mathbf{v}_i\cdot\left(\sum_{k,l=1}^{n} X_{kl}\mathbf{v}_k\otimes\mathbf{v}_l\right)\mathbf{v}_j = \left(\sum_{k,l=1}^{n} X_{kl}\mathbf{v}_i\cdot\left(\mathbf{v}_k\left(\mathbf{v}_l\cdot\mathbf{v}_j\right)\right)\right) = \left(\sum_{k,l=1}^{n} X_{kl}\delta_{ki}\delta_{lj}\right) = X_{ij}.$$

□

*Důkaz* (b)
Pravou stranu upravíme pomocí příkladu 1. a definice tensorového součinu:

$$(\mathbf{v}_j\otimes\mathbf{v}_j)\,\mathbb{X}\,(\mathbf{v}_i\otimes\mathbf{v}_i) = ((\mathbf{v}_j\otimes\mathbf{v}_j)\,\mathbb{X}\mathbf{v}_i)\otimes\mathbf{v}_i = (\mathbf{v}_j\,(\mathbf{v}_j\cdot\mathbb{X}\mathbf{v}_i))\otimes\mathbf{v}_i$$

a z linearity tensorového součinu a symetričnosti $\mathbb{X}$ (tj. $\mathbf{v}_j\cdot\mathbb{X}\mathbf{v}_i = \mathbb{X}\mathbf{v}_j\cdot\mathbf{v}_i = \mathbf{v}_i\cdot\mathbb{X}\mathbf{v}_j \overset{\text{(a)}}{=} X_{ij}$)

$$(\mathbf{v}_j\,(\mathbf{v}_j\cdot\mathbb{X}\mathbf{v}_i))\otimes\mathbf{v}_i = (\mathbf{v}_j\otimes\mathbf{v}_i)\,X_{ij}$$

□

*Příklad* (3.)
In the proof of Daleckii–Krein formula, we have already shown that

$$\frac{\partial\mathbb{F}(\mathbb{A})}{\partial\mathbb{A}} = \sum_{i=1}^{k}\left(\left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=\lambda_i}\frac{\partial\lambda_i}{\partial\mathbb{A}}\mathbf{v}_i\otimes\mathbf{v}_i + f(\lambda_i)\frac{\partial\mathbf{v}_i}{\partial\mathbb{A}}\otimes\mathbf{v}_i + f(\lambda_i)\mathbf{v}_i\otimes\frac{\partial\mathbf{v}_i}{\partial\mathbb{A}}\right),$$

which means that

$$\frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{i=1}^{k} \left( \left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=\lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}}[\mathbb{X}] \mathbf{v}_i \otimes \mathbf{v}_i + f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}[\mathbb{X}] \otimes \mathbf{v}_i + f(\lambda_i) \mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}[\mathbb{X}] \right) \quad (1)$$

(Recall that $\mathbb{X}$ is a symmetric matrix.) Furthermore, we already know that $\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbf{v}_i \otimes \mathbf{v}_i$, which implies that

$$\frac{\partial \lambda_i}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{m,n=1}^{3} (\mathbf{v}_i \otimes \mathbf{v}_i)_{mn} X_{mn} = X_{ii}. \quad (2)$$

Finally, we also know that for $i \neq j$ it holds $\frac{\partial (v_i)_j}{\partial A_{mn}} = \frac{\delta_{im}\delta_{jn}}{\lambda_i - \lambda_j}$, which implies that

$$\frac{\partial v_i}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{X_{ij}}{\lambda_i - \lambda_j} \mathbf{v}_j = \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j} \mathbf{v}_j. \quad (3)$$

(For $i = j$ it suffices to differentiate the identity $\mathbf{v}_i \cdot \mathbf{v}_i = 1$, which immediately implies that $\frac{\partial (\mathbf{v}_i)_j}{\partial A_{mn}} = 0$ for $i = j$. Consequently, we can safely ignore the identical indices.) Substitute (3) and (2) into (1) and show that the result can be rewritten as

$$D_{\mathbb{A}} \mathbb{F}(\mathbb{A})[\mathbb{X}] = \sum_{i=1}^{k} \frac{df(\lambda)}{d\lambda}|_{\lambda=\lambda_i} \mathbb{P}_i \mathbb{X} \mathbb{P}_i + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \mathbb{P}_i \mathbb{X} \mathbb{P}_j,$$

where $\{\mathbb{P}_i\}_{i=1}^{k}$ denote the projection operators to the $i$-th (normalised) eigenvector $\mathbf{v}_i$, that is $\mathbb{P}_i := \mathbf{v}_i \otimes \mathbf{v}_i$. Summation convention is not being used!

*Důkaz*

Z (2) a příkladu 2. (b) (pro $i = j$) máme

$$\frac{df(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}}[\mathbb{X}] \mathbf{v}_i \otimes \mathbf{v}_i = \frac{df(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} X_{ii} \mathbf{v}_i \otimes \mathbf{v}_i = \frac{df(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} \mathbf{v}_i \otimes \mathbf{v}_i \mathbb{X} \mathbf{v}_i \otimes \mathbf{v}_i = \frac{df(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_i} \mathbb{P}_i \mathbb{X} \mathbb{P}_i.$$

Z (3) a linearity tensorového součinu máme

$$f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}[\mathbb{X}] \otimes \mathbf{v}_i + f(\lambda_i) \mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}[\mathbb{X}] = f(\lambda_i) \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j} \mathbf{v}_j \otimes \mathbf{v}_i + f(\lambda_i) \mathbf{v}_i \otimes \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{\mathbf{v}_i \cdot \mathbb{X} \mathbf{v}_j}{\lambda_i - \lambda_j} \mathbf{v}_j.$$

V druhém členu použijeme příklad 2. (a) i (b) a dostaneme $\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{f(\lambda_i) \mathbb{P}_i \mathbb{X} \mathbb{P}_j}{\lambda_i - \lambda_j}$. První člen upravíme stejně na $\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{f(\lambda_i) \mathbb{P}_j \mathbb{X} \mathbb{P}_i}{\lambda_i - \lambda_j}$, ale u toho si ještě uvědomíme, že v součtech $\sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k}$ je tento člen i s prohozeným $i$ a $j$, čímž „dostaneme“: $\sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{f(\lambda_j) \mathbb{P}_i \mathbb{X} \mathbb{P}_j}{\lambda_j - \lambda_i}$ □