1 Σ_1^1 sets and trees on ω

Poznámka (Notation)

- $\mathbb{S} := \omega^{<\omega}$;
- $\nu|_k = (\nu(0), \ldots, \nu(k-1))$, $\nu \in \mathbb{S} \cup \omega^\omega$ ($\nu|_0 = \emptyset$, empty sequence);
- $t \prec s \equiv \exists s' \in \mathbb{S} \cup \mathcal{N} : s = t^\wedge s'$ ($t \in \mathbb{S}, s \in \mathbb{S} \cup \mathcal{N}$);
- $\mathcal{N} := \omega^\omega$;
- $|s|$ is the length of s, $s \in \mathbb{S}$ ($s = (s(0), \ldots, s(k-1)) \implies |s| = k$);
- $s \in \mathbb{S}$, $\nu \in \mathbb{S} \cup \mathcal{N}$: $s^\wedge\nu = (s(0), \ldots, s(|s|-1), \nu(0), \ldots)$.

Definice 1.1 (Souslin set (on TP space))

X topological space. We say $S \subset X$ be Souslin $\Leftrightarrow \exists (F_s)_{s \in S}$ Souslin scheme of closed subset of X such that $S = \mathcal{A}_s(F_s) = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n \in \omega} F_{\sigma|_n}$.

Poznámka

a) P Polish topological space, then $A \in \Sigma_1^1 \Leftrightarrow A$ Souslin in P. (We already know.)

b) P topological space, then $A \subset P$ Souslin $\Leftrightarrow \exists F \in \Pi_1^0(\mathcal{N} \times P) : A = \Pi_P(F)$. (Difficult.)

c) We will assume only regular Souslin scheme (RSS): $F_{s^\wedge t} \subset F_s$, $s, t \in \mathbb{S}$ and $F_\emptyset = P$.

1.1 Souslin operation and trees

Definice 1.2 (Trees on ω, infinite branch, ill-founded trees, well-founded trees)

We define set of trees $\mathcal{T}$ by $\mathcal{T} := \{T \in \mathcal{P}(\mathbb{S}) | \forall s \in T, t \in T : t \prec s \implies t \in T\}$.

$T \in \mathcal{T}$ has infinite branch $\equiv \exists \sigma \in \mathcal{N} \forall n \in \omega : \sigma|_n \in T$ (i.e. $\sigma \in [T]$) (i.e. $[T] \neq \emptyset$).

Trees with infinite branches are called ill-founded (IF). The set of IF trees is denoted by $\mathcal{T}_I$. Trees without infinite branches are called well-founded (WF). The set of WF trees is denoted by $\mathcal{T}_W$.

$\mathcal{T}_s := \{T \in \mathcal{T} | s \in T\}$ are all trees containing $s \in \mathbb{S}$.

Poznámka
$\mathbb{T}_I = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n \in \omega} \mathcal{T}_{\sigma|_n}$.

$\mathcal{T}^* := \mathcal{T} \setminus \{\emptyset\}$, $\mathcal{T}_W^* = \mathcal{T}_W \setminus \{\emptyset\}$.

Lemma 1.1

Let X be a topological space, $(F_s)_{s\in\mathbb{S}}$ RSS of closed subsets of X, $S := \mathcal{A}_s(F_s)$. Define $f(x) : X \to \mathcal{T}^*$ by $f(x) := \{s \in \mathbb{S} | x \in F_s\}$. Then $F_s = f^{-1}(\mathbb{T}_s)$ and $S = f^{-1}(\mathcal{T}_I)$.

Důkaz (?)

a) „$f : X \to \mathcal{T}$“: $s \in f(x) \implies x \in F_s \implies F_s \subset F_t \implies x \in F_t \implies t \in f(x)$ $(t \prec s)$.

b) $x \in F_s \Leftrightarrow s \in f(x) \Leftrightarrow f(x) \in \mathcal{T}_s \Leftrightarrow x \in f^{-1}(\mathbb{T}_s)$

c) lemma $\impliedby$ b) and the next remark. □

Poznámka

TODO!!! $\mathcal{T} \to \mathcal{T}^*$.

Důkaz

„ $\implies$ “: lemma?. „ $\impliedby$ “: $S = f^{-1}(\mathbb{T}_I) = f^{-1}\left(\bigcup\bigcap_{n\in\omega}\mathcal{T}_{\sigma|_n}\right) = \bigcup_{\sigma\in\mathcal{N}}\bigcap_{n\in\omega} f^{-1}(\mathbb{T}_{\sigma|_n})$, where $f^{-1}(\mathbb{T}_{\sigma|_n}) \in \Pi_1^0(X) \implies$ Souslin. □

1.2 Trees as PTS (compact)

Poznámka (Topology on trees)

$\mathcal{P}(\mathbb{S}) = \{A \subset \mathbb{S}\} = \{0,1\}^{\mathbb{S}}$ (product topology of product of discrete topologies) which is compact and homeomorphic to 2^ω. We assume on $\mathbb{T}$ subspace topology.

Tvrzení 1.2

$\mathbb{T}, \mathcal{T}^* \in \Pi_0^1(\{0,1\}^{\mathbb{S}})$, $\{\mathbb{T}_s, \mathbb{T}^* \setminus \mathbb{T}_s, s \in \mathbb{S}\}$ *form a subbase of topology in* $\mathbb{T}$.

Poznámka

$\mathcal{T}$, $\mathcal{T}^*$ is compact metric space, so PTS.

Důkaz
$S \in \{0,1\} \backslash \mathbb{T} \Leftrightarrow \exists s,t \in \mathbb{S}, s \prec t : t \in S \wedge s \notin s \implies \{0,1\} \backslash \mathbb{T} = \bigcup_{t \in \mathbb{S}} \bigcup_{s<t} (\{T, \chi_T(t) = 1\} \cap \{T; \chi_T(s) = 1\})$.

$\{T|\chi_T(t) = 1\}$, $\{T|\chi_T(s) = 0\}$ is subbase of product topology.

$$\mathcal{T}^* = \mathcal{T} \cap \{A \in \{0,1\} \,|\chi_A(\varnothing) = 1\} \implies \mathcal{T}^* \in \Pi_1^0(\mathcal{T}) \implies \mathcal{T}^* \text{ is compact.}$$

□

1.3 Properties of f from the lemma

Definice 1.3

$T \in \mathbb{T}$, $\sigma \in \mathcal{N}$. $h_\sigma(T) := \sup \{k \in \omega | \sigma|_k \in T\} \in \omega \cup \{\infty\}$.

Poznámka (Remind Lebesgue–H?–Banach characterization)
X, Y metric spaces, Y separable, $1 \leqslant \alpha < \omega_1$, $f : X \to Y$. Then f is Baire$_\alpha$ $\Leftrightarrow$ f is $\Sigma^0_{\alpha+1}(X)$-measurable.

Tvrzení 1.3

X metrizable (we need only $\Sigma_1^0(X) \subset \Sigma_2^0(X)$), $S \subset X$ Souslin. Then there exists $f : X \to \mathbb{T}$ such that:

1. *$f^{-1}(\mathbb{T}_I) = S$;*
2. *$f^{-1}(\mathbb{T}_s) \in \Pi_1^0(X)$, $s \in \mathbb{S}$;*
3. *$h_\sigma \circ f$ is upper semi-continuous ($h_\sigma \circ f : X \to \mathbb{R}^*$), $\sigma \in \mathcal{N}$ (i.e $\{x \in X | h_\sigma(f(x)) < n\}$ is open $\forall \sigma \in \mathcal{N}, n \in \mathbb{R}^*$);*
4. *f is Baire$_1$ (i.e. Σ_2^0-measurable).*

Důkaz
1. and 2. is from the lemma. „4.“: $\mathbb{T}$ separable metric space. So, it is enough to prove it for subbase. $f^{-1}(\mathbb{T}_s) \in \Pi_1^0 \subset \Sigma_2^0$, $f^{-1}(\mathbb{T} \backslash \mathbb{T}_s) \in \Sigma_1^0 \subset \Sigma_2^0(X)$. „3.“: $\{x \in X | h_\sigma(f(x)) < n\} = f^{-1}(\{T \in \mathbb{T} | \sigma|_n \notin T\}) = f^{-1}(\mathbb{T} \backslash \mathbb{T}_{\sigma|_n})$ is open (by the lemma). And $\{x \in X | h_\sigma(f(x)) < \infty\} = \bigcup_{n \in \omega} \{x \in X | h_\sigma(f(x)) < n\}$. □

1.4 Examples of Σ_1^1 non-Δ_1^1 sets

Poznámka

$$\Sigma^1_1(X)\backslash\Pi^1_1(X) = \Sigma^1_1(X)\backslash\Delta^1_1(X) \overset{?}{\neq} \varnothing.$$

Lemma 1.4

$\mathcal{T}_I \in \Sigma^1_1(\mathcal{T})\backslash\Delta^1_1(\mathcal{T}), \mathcal{T}_W \in \Pi^1_1(\mathcal{T})\backslash\Delta^1_1(\mathcal{T})$.

Důkaz

1. $\mathcal{T}_I \in \Sigma^1_1(\mathbb{T}) \impliedby \mathbb{T}_I = \bigcup\bigcap \mathcal{T}_{\sigma|_n}$ souslin in PTS.

2. „$\mathcal{T}_I \notin \Delta^1_1(\mathbb{T})$“: By continuity $\mathcal{T}_I \in \Delta^1_1 \implies \mathcal{T}_W \in \Delta^1_1 \implies \mathcal{T}_W \in \Sigma^1_1 \implies \mathcal{T}_W$ souslin. ϟ. □

Poznámka
f_I, f_W are mappings from the lemma for $S = \mathcal{T}_I$ and $S = \mathcal{F}_W$. Clearly $f_I = \mathrm{id}$.

Definice 1.4

$f : \mathcal{T} \to \mathcal{T}$ by $f(T) := f_I(T) \cap f_W(T) = T \cap f_w(T)$. $f(T) \in \mathcal{T} \impliedby (A, B \in \mathcal{T} \implies A \cap B \in \mathcal{T})$.

$$T \in \mathcal{T}_W \implies f(T) = T \cap f_W(T) \subset T \implies f(T) \in \mathcal{T}_W.$$

$T \in \mathcal{T}_I \implies f(T) \subset f_w(T) \in \mathcal{T}_W \impliedby$ (the lemma $\implies f^{-1}(\mathcal{T}_I) = \mathcal{T}_W \implies f^{-1}(\mathcal{T}_W) = \mathcal{T}_I) \implies f(T)$

$\implies f : \mathcal{T} \to \mathcal{T}_W \implies h_\sigma \circ f : \mathcal{T} \to \omega$. From the previous proposition $h_\sigma \circ f$ is usc, so $h_\sigma \circ f$ is usc real function on compact set. Thus $m(\sigma) := \max_{T\in\mathbb{T}} h_\sigma(f(T)) \in \omega$.

Důkaz (The previous lemma)
By contradiction $\mathcal{T}_I \in \Delta^1_1(\mathcal{T}^*) \implies \mathcal{T}^*_W \in \Sigma^1_1(\mathcal{T}^*)$. $f(T) = f_I(T) \cap f_W(T)$, $f : \mathcal{T}^* \to \mathcal{T}^*$, $f : \mathcal{T}^* \to \mathcal{T}^*_W$. $\exists m(\sigma) := \max_{T\in\mathcal{T}^*} h_\sigma(f(T)) \in \omega$.

Define $T_0 \in \mathcal{T}^* : s \in T_0 \Leftrightarrow \sigma \in \mathcal{N} : \sigma|_{m(\sigma)+1} \succ s$. $T_0 \in \mathcal{T}^*$, $\{\varnothing\} \in T_0$, $T_0 \in \mathcal{T}$ trivial. $T_0 \in \mathcal{T}^*_W$. By contradiction $\sigma \in [T_0] \implies \sigma|_{m(\sigma)+2} \in T_0 \implies \exists \nu \in \mathcal{N} : \sigma|_{m(\sigma)+2} \prec \nu|_{m(\nu)+1} \implies \nu|_{m(\sigma)+1} = \sigma|_{m(\sigma)+1}$. Definition of $m(\nu)$ gives $\exists T \in \mathcal{T}^* : m(\nu) = h_\nu(f(T)) \implies \nu|_{m(\nu)} \in f(T) \implies \sigma|_{m(\sigma)+1} \in f(T) \implies h_\sigma(f(T)) \geqslant m(\sigma) + 1$. ϟ.

Clearly

$$T_0 \supseteq \bigcup_{T\in\mathcal{T}^*} (T). T_0 \in \mathcal{T}^*_W \implies f_W(T_0) \in \mathcal{T}_I \implies \exists \sigma_0 \in [f_W(T_0)] \implies$$

$$\implies h_{\sigma_0}(f(T_0)) = \min\{k \in \omega | \sigma_0|_k \in T_0 \cap f_W(T_0)\} = \min\{k \in \omega | \sigma_0|_k \in T_0\} \supseteq m(\sigma_0) + 1. ϟ.$$

□

Věta 1.5

X PTS, $A \in \Sigma^1_1(X)$, $\mathrm{card}(A) > \mathrm{card}(\omega)$. Then there exists $B \subset A$ such that $B \in \Sigma^1_1(X) \backslash \Delta^1_1(X)$.

Důkaz

$\mathrm{card}(A) > \omega \implies \exists C \subset A$ homeomorphic copy of $2^\omega \sim 2^{\mathbb{S}}$. $2^{\mathbb{S}} \xhookrightarrow{h} A$ then $h(\mathcal{T}_I) \in \Sigma^1_1(X) \backslash \Delta^1_1(X)$. Homeomorphism of Σ^1_1, Δ^1_1 set is Σ^1_1, Δ^1_1 set. □

Poznámka

Let Γ be class of subsets of PTS and X be PTS. We say that A is $\Gamma(X)$-hard $\equiv \forall B \in \Gamma(\mathcal{N})\ \exists f \in \Delta^1_1, f : \mathcal{N} \to X : f^{-1} = B$. A is $\Gamma(X)$-complete $\Leftrightarrow A \in \Gamma$ and $A \in \Gamma$-hard.

From the previous theorem $A \in \Sigma^1_1$-complete $\implies A \in \Sigma^1_1 \backslash \Delta^1_1$ (same for Π^1_1). ($A \in \Delta^1_1 \implies f^{-1}(A) \in \Delta^1_1$, but there are $\Sigma^1_1 \backslash \Delta^1_1$ subsets of $\mathcal{N}$).

Poznámka

Σ^1_1-complete $= \Sigma^1_1 \backslash \Delta^1_1 \impliedby \Sigma^1_1$-determinacy.

Poznámka

$\mathcal{T}_I \in \Sigma^1_1$-complete, $\mathcal{T}^*_W \in \Pi^1_1$-complete.

Definice 1.5 (Universal set)

X PTS, Γ class of subsets of PTS. We say that A is $\Gamma(X)$-universal $\equiv A \in \Gamma(X \times \mathcal{N}) \wedge \Gamma(X) = \{A^s | s \in \mathcal{N}\}$.

Poznámka

X PTS. Then

1. there exists $\Sigma^0_1(X)$-universal set;
2. there exists $\Pi^0_1(X)$-universal set;
3. there exists $\Sigma^1_1(X)$-universal set.

Důkaz
„1.“: $\{B_n\}$ base of X. $G := \bigcup_{n\in\omega, s\in\omega}(B_{s(0)} \cup B_{s(1)} \cup \ldots \cup B_{s(n-1)}) \times B(s)$ $(B(s) = \{\sigma \in \mathcal{N} | s \prec \sigma\})$. $G \in \Sigma^0_1(X \times \mathcal{N})$ trivial. $\sigma \in \mathcal{N} \implies G^\sigma \in \Sigma^0_1(X)$ trivial $(G^\sigma = \bigcup_{n\in\omega}(B_{\sigma(0)} \cup B_{\sigma(1)} \cup \ldots \cup B_{\sigma(n-1)})$ open). $U \in \Sigma^0_1(X) \implies \exists \sigma \in \mathcal{N} : U = \bigcup_{n\in\omega} B_{\sigma(n)} = G^\sigma$.

„2.“: G $\Sigma^0_1(X)$-universal $\implies$ $(X \times \mathcal{N})\backslash G$ is $\Pi^0_1(X)$-universal.

„3.“: $Y = \mathcal{N} \times X$. Let $F \in \Pi^1_0(Y \times \mathcal{N})$ be $\Pi^0_1(Y)$-universal. $\Pi : \mathcal{N} \times X \times \mathcal{N} \to X \times \mathcal{N}$ be projections on 2nd and 3rd coordinate. $A := \Pi(F)$. A is $\Sigma 1^1(X)$-universal. Clearly $A \in \Sigma^1_1(X \times \mathcal{N})$, $A^\sigma \in \Sigma^1_1(X)$ for $\sigma \in \mathcal{N}$ trivial. Let $B \in \Sigma^1_1(X) \implies \exists C \in \Pi^0_1(\mathcal{N} \times X) : B = \Pi_2(C) \implies \exists \sigma \in \mathcal{N} : C = F^\sigma$.

$$A^\sigma = (\Pi_{2,3}(F))^\sigma = \Pi_2(F^\sigma) = \pi_2(C) = B.$$

□

Poznámka
Let $A \in \Sigma^1_1(\mathcal{N}^2)$ be $\Sigma^1_1(\mathcal{N})$ universal. Then

$M := \{x \in \mathcal{N} | (x,x) \notin A\} \in \Sigma^1_1(\mathcal{N}) \impliedby (M \in \Sigma^1_1 \implies \exists \sigma \in \mathcal{N} : M = A^\sigma.)(\sigma \in M? : \sigma \in M \implies (\sigma, \sigma)$

$$\{x \in \mathcal{N} | (x,x) \in \mathcal{A}\} \in \Sigma^1_1(\mathcal{N}) \impliedby \text{ diagonal is closed } \implies \{x \in \mathcal{N} | (x,x) \in A\} \in \Sigma^1_1 \backslash \Delta^1_1.$$

1.5 Derivative of trees

Definice 1.6 (Derivative)

$T \in \mathcal{T}$. $T' := \{s \in \mathbb{S} | \exists n \in \omega : s^\wedge n \in T\}$. $T^{(0)} := T$. $\sigma < \omega_1 : T^{(\alpha+1)} = (T^\alpha)'$, λ-limit ordinal: $T^{(\lambda)} := \bigcap_{\alpha<\lambda} T^{(\alpha)}$. $d_\alpha(T) := T^{(\alpha)}$, $\alpha < \omega_1$, $d_\alpha : \mathcal{T} \to \mathcal{T}$.

Věta 1.6

$\forall \alpha < \omega_1 : d_\alpha \in \Delta^1_1(\mathcal{T}^2)$.

Důkaz
$d_\alpha(T) \in \mathcal{T}$ $(T \in \mathcal{T})$ trivial.

a) $d_1^{-1}(\mathcal{T}_s) = \{T \in \mathcal{T} | \exists n \in \omega : s^\wedge \in T\} = \bigcup_{n\in\omega} \mathcal{T}_{s^\wedge n} \in \sum_1^0(\mathcal{T})$.

$$\implies d_1^{-1}(\mathcal{T} \backslash \mathcal{T}_s) \in \Pi_1^0(\mathcal{T}), \qquad d_1^{-1}(\varnothing) = \{\varnothing, \{\varnothing\}\} \in \Pi_1^0(\mathcal{T}) \implies$$

$$\implies (G \in \Sigma_1^0(\mathcal{T})) \implies d_1^{-1}(G) \in \Sigma_2^0(\mathcal{T}) \implies$$

$\implies$ d_1 is in the first Borel class.

b) d_0-id $\implies$ continuous.

Induction: c) $\alpha = \beta + 1$, $d_\beta \in \Delta_1^1 \implies d_\alpha = d_1 \circ d_\beta \in \Delta_1^1$.

d) λ limit ordinal, $\lambda < \omega_1$, $\forall \alpha < \lambda : d_\alpha \in \Delta_1^1$.

$$d_\lambda^{-1}(\mathcal{T}_s) = \left\{ T \in \mathcal{T} | \bigcap_{\alpha \in \lambda} d_\alpha(T) \ni s \right\} = \bigcap_{\alpha<\lambda} d_\alpha^{-1}(\mathcal{T}_s) \in \Delta_1^1 \implies$$

$$\implies d_\lambda^{-1}(\mathcal{T} \backslash \mathcal{T}_s) \in \Delta_1^1, \qquad d_\lambda^{-1}(\varnothing) = \{T \in \mathcal{T} | \exists \alpha < \lambda : d_\alpha(T) = \varnothing\} = \bigcup_{\alpha<\lambda} d_\alpha^{-1}(\varnothing) \in \Delta_1^1.$$

□

1.6 Luzin–Sierpinski index (rank, norm)

Definice 1.7

$T \in \mathcal{T}^*$, $i(T) := \min \left\{ \alpha < \omega_1 | T^{(\alpha)} = \{\varnothing\} \right\}$, if exists, otherwise ω_1.

Poznámka (Notation)
$T_s := \{t \in \mathbb{S} | s^\wedge t \in T\}, T \in \mathcal{T}^*, s \in T$.

Poznámka (Other indices)
$T_s \in \mathcal{T}^*$, $T \in \mathcal{T}^*$, $s \in T$ trivial.

Hausdorff index $:= \min \left\{ \alpha < \omega_1 | d^{(\alpha)}(T) = d^{(\alpha+1)}(T) \right\}$.

Derivation of sets: X PTS, $K \in \mathcal{K}(X)$, $K' := \{x \in K | x$ is not isolated point in $K\}$. $K^{(\alpha+1)} := (K^{(\alpha)})'$, $K^{(0)} := K$, $K^{(\lambda)} := \bigcap_{\alpha<\lambda} K^{(\alpha)}$ (λ limit ordinal).

Lemma 1.7

$T_s \in \mathcal{T}^*$, $i(T_s) = \sup \{\min \{\omega_1, i(T_{s^\wedge n})1\} | s^\wedge n \in T\}$ (sup $\varnothing := 0$).

Důkaz
$s \in T \implies T_s \neq \varnothing$, $T \in T_s$, $l < t$: $s^\wedge t \in T \implies s^\wedge l < s^\wedge t \implies s^\wedge l \in T \implies l \in T_s$.

$$i(T_s) = \omega_1 \Leftrightarrow T_s \in \mathcal{T}_I \Leftrightarrow \exists n \in \omega : T_{s^\wedge n} \in \mathcal{T}_I \Leftrightarrow \exists n \in \omega : i(T_{s^\wedge n}) = \omega_1.$$

„$i(T_s) < \omega_1 \Leftrightarrow T_s \in \mathcal{T}_W^*$“: $\alpha := \sup_{n\in\omega : s^\wedge n \in T} i(T_{s^\wedge n}) + 1$, clearly $\forall n \in \omega : s^\wedge T$, $i(T_{s^\wedge n}) \leqslant i(T_s) < \omega_1 \implies 0 < \alpha < \omega_1$. „$\alpha = i(T_s)$“:

$$T_s^{(\alpha)} = \bigcup_{s^\wedge n \in T} (\{\varnothing\} \cup n^\wedge T_{s^\wedge n})^{(\alpha)} \subseteq \bigcup_{s^\wedge n \in T} (\{\varnothing\} \cup n^\wedge T_{s^\wedge n}) = \{\varnothing\} \implies i(T_s) \leqslant \alpha.$$

Assume $\beta < \alpha \implies \exists s^\wedge n \in T : i(T_{s^\wedge n}) + 1 > \beta \implies T_s^\beta \supset (\{\varnothing\} \cup n^\wedge T_{s^\wedge n})^{(\beta)} \supsetneq \{\varnothing\} \impliedby i(\{O\} \cup n^\wedge T_{s^\wedge n}) = i(T_{s^\wedge n}) + 1. \implies \beta < i(T_s) \implies \alpha \leqslant i(T_s)$. □

Věta 1.8

a) $T \in \mathcal{T}_W^* \Leftrightarrow i(T) < \omega_1$. *b)* $i(\mathcal{T}_W^*) = \omega_1$ *(i.e.* $\{i(T) | T \in \mathcal{T}_W^*\} = \{\alpha < \omega_1\}$*).*

Důkaz
„a)“: $T \in \mathcal{T}_W^*$. $T \neq \{\varnothing\} \implies \exists s \in T : |s| \geqslant 1, \forall n \in \omega : s^\wedge n \notin T \implies s \notin T' \implies T' \subsetneq T$. And $\mathrm{card}(T) < \omega_1 \implies i(T) < \omega_1$. $i(\{\varnothing\}) = 0$. It can't happen:

$$T \neq \varnothing, \quad \{\varnothing\}, \quad T' = \varnothing$$

$$T \in \mathcal{T}_I \implies \exists \sigma \in [T] \implies \sigma \in [T'] \implies T' \in \mathcal{T}_I \implies \forall \alpha < \omega_1 : \sigma \in [T^{(\alpha)}] \implies T^{(\alpha)} \neq \{\varnothing\} \implies i(T)$$

„b)“: $i(\{\varnothing\}) = 0$. Induction $\forall \alpha < \omega_1\ \exists T_\alpha \in \mathcal{T}_W^* : i(T_\alpha) = \alpha$: First step is done; Second: $T_{\alpha+1} := 1^\wedge T_\alpha \cup \{\varnothing\} \implies i(T_{\alpha+1}) = \alpha + 1$; Assume λ is limit ordinal, $\alpha \nearrow \lambda$. $T_\lambda := \{\varnothing\} \cup \{n^\wedge T_{\alpha_n} | n \in \omega\}$. ($i(T_\lambda) = \sup\{i(T_{\alpha_n} + 1)\} = \lambda$.) □

1.7 Decomposition of $\mathcal{T}_W^*$ and cosouslin sets

Definice 1.8

$\alpha < \omega_1 : \mathcal{T}_W(\alpha) := \{T \in \mathcal{T}^* | i(T) = \alpha\}$.

Věta 1.9

$\mathcal{T}_W(\alpha) \in \Delta_1^1(\mathcal{T})$, $\alpha < \omega_1$.

Důkaz
$\mathcal{T}_W(\alpha) = d_\alpha^{-1}(\{\varnothing\})$, $d_\alpha \in \Delta_1^1$. □

Poznámka
C cosouslin in X ($X \backslash C = S$, which is souslin). $\exists \Delta^1_1 f : X \to \mathcal{T}^*$: $f^{-1}(\mathcal{T}_I) = S = f^{-1}(\mathcal{T}^*_W) = C$. Define $C_\alpha = f^{-1}(\mathcal{T}_W(\alpha))$, $\alpha < \omega_1$. It is called a decomposition of C on Δ^1_1 subsets. If $\{\alpha | C_\alpha \neq \emptyset\}$ is countable $\implies C \in \Delta^1_1$. „Inverse implication“ is going to be in some weeks (Theorem 15).

Poznámka

$$A \in \Pi^1_1(X) \backslash \Pi^0_2(x) \implies \mathcal{K}(A) \in \Pi^1_1 - \text{complete}.$$
$$A \in \Pi^0_2(X) \Leftrightarrow \mathcal{K}(A) \in \Pi^0_2(\mathcal{K}(X)).$$

1.8 Luzin–Sierpinski index as partial ordering

Poznámka (Goal)
Study $\{(T_1, T_2) \in (\mathcal{T}^*_W)^2 | i(T_1) \leqslant i(T_2)\}$.

Definice 1.9

$f : \mathbb{S} \to \mathbb{S}$ is strategy $\equiv \forall s \in \mathbb{S} : |f(s)| = |s|$ (respect length) and $\forall s, t \in \mathbb{S} : s < t \implies f(s) < f(t)$ (monotone.)

Poznámka
a) f strategy. We define $\overline{f} : \omega^\omega \to \omega^\omega$ by $f(\sigma) = \mathbb{T} \Leftrightarrow \forall n \in \omega : T|_n = f(\sigma|_n)$.

b) For first $|s|$ steps of player I describes f first $|s|$ steps of player II (strategy for II player).

c) $T \in \mathcal{T}^* : f(T), f^{-1}(T) \in \mathcal{T}^*$.

d) $\alpha < \omega_1 : (f^{-1}(T))^{(\alpha)} \subset f^{-1}(T^{(\alpha)})$.

Důkaz
„a)“, „b)“ trivial. „c)“: $s \in f(T), t < s \implies \exists x \in T : f(x) = s \implies |x| = |s| \geqslant |t| \implies x|_{|t|} \in T \implies f(x|_{|t|}) \in f(T), f(x|_{|t|}) < f(x) = s, |f(x|_{|t|})| = |t| \implies f(x|_{|t|}) = t \implies f(T) \in \mathcal{T}^*$. $f^{-1}(T) \in \mathcal{T}^*$ similar.

„d)“: By induction: First step ($\alpha = 0$) is trivial. For $\alpha = 1$: $s \in (f^{-1}(T))' \implies \exists n \in \omega : s^\wedge n \in f^{-1}(T) \implies f(s^\wedge n) \subset f(s), f(s^\wedge n) \in T \implies f(s) \in T \implies f(s) \in T'$ $(\exists m \in \omega : f(s^\wedge m) = f(s)^\wedge m)$. For successor ordinal: $(f^{-1})^{(\beta+1)} = \big((f^{-1}(T))^{(\beta)}\big)' \subset (f^{-1}(T^{(\beta)})) \subset f^{-1}(T^{(\beta+1)})$. For limit ordinal $\lambda < \omega_1$: $(f^{-1}(T))^{(\lambda)} = \bigcap_{\alpha<\lambda} (f^{-1}(T))^{(\alpha)} \subseteq \bigcap_{\alpha<\lambda} f^{-1}(T^{(\alpha)}) = f^{-1}(\bigcap_{\alpha<\lambda} T^{(\alpha)}) = f^{-1}(T^{(\lambda)})$. □

Lemma 1.10

$T_1, T_2 \in \mathcal{T}^*_W$. $i(T_1) \leqslant i(T_2) \Leftrightarrow \exists f : \mathbb{S} \to \mathbb{S}$ *strategy such that* $T_1 \subset f^{-1}(T_2)$ *(*$f(T_1) \subset T_2$*).*

Důkaz
„ $\Longleftarrow$ “: $T_1 \subset f^{-1}(T_2) \implies i(T_1) \leqslant i(f^{-1}(T_2)) \leqslant i(T_2)$ (second equation holds, because: $(f^{-1}(T_2))^{(\alpha)} \subset f^{-1}(T_2^{(\alpha)})$, put $\alpha = i(T_2) \implies (f^{-1}(T_2))^{(\alpha)} \subseteq \{\varnothing\} \implies i(f^{-1}(T_2)) \leqslant \alpha$).

„ $\implies$ “: a) $i(T_2) = \omega_1 \implies T_2 \in \mathcal{T}_I \implies \sigma \in [T_2]$. Define $f(s)$ by $f(s) = \sigma|_s$. Clearly f is strategy and $f(T_1) \subset \{\sigma|_k, k \in \omega\} \subset T_2$.

b) $i(T_2) < \omega_1 \implies T_2 \in \mathcal{T}^*_W$. We will construct f by induction on $|s|$, $s \in \mathbb{S}$, and we also want $(+_n) : i_{T_1}(s) \leqslant i_{T_2}(f(s))$, $s \in T_1$, $|s| \leqslant n \implies f(s) \in T_2$, $s \in T_1$ (where $i_T(s) = i(T_s)$, $T \in \mathcal{T}^*$, $s \in T$).

Firstly $f(\{\varnothing\}) = \{\varnothing\}$. f monotone, respect length and $(+_0) : i_{T_1}(\{\varnothing\}) = i(T_1) \leqslant i(T_2) = i_{T_2}(\{\varnothing\})$. Let f be defined for $s \in \mathbb{S}$, $|s| \leqslant n$, $n \in \omega$, f respect length and be monotone and satisfy $(+_n)$. Let $s \in \omega^n$. i) $s_0 \notin T_1$ or $i_{T_1}(s_0) = 0$ TODO!!!

ii) $i_{T_1}(s_0) > 0$ TODO!!!

TODO!!! $\square$

1.9 Luzin–Sierpinski index as Π^1_1 rank

Věta 1.11

$$A := \{(T_1, T_2) \in (\mathcal{T}^*)^2 | i(T_1) \leqslant i(T_2)\} \in \Sigma^1_1((\mathcal{T}^*)^2).$$
$$C := \{(T_1, T_2) \in (\mathcal{T}^*)^2 | T_1 \in \mathcal{T}^*_W, i(T_1) \leqslant i(T_2)\} \in \Pi^1_1((\mathcal{T}^*)^2).$$
$$B := \{(T_1, T_2) \in (\mathcal{T}^*)^2 | i(T_1) < i(T_2)\} \in \Pi^1_1((\mathcal{T}^*)^2).$$
$$D := \{(T_1, T_2) \in (\mathcal{T}^*_W)^2 | i(T_1) \leqslant i(T_2)\} \text{ bisouslin in } (\mathcal{T}^*_W)^2.$$

Důkaz
„$A \implies C$“: Define $h : (\mathcal{T}^*)^2 \to (\mathcal{T}^*)^2$ homeomorphism by $h(T_1, T_2) = (T_2, T_1)$. Then $(\mathcal{T}^*)^2 \backslash A = h(B) \implies B \in \Pi^1_1((\mathcal{T}^*)^2)$.

„$C \implies B$“: $E := \{(T,T) \in (\mathcal{T}^*)^2 | T \in \mathcal{T}^*_W\} \simeq \mathcal{T}^*_W \implies E \in \Pi^1_1$. $C = B \cup E \in \Pi^1_1((\mathcal{T}^*)^2)$.

„D“: $A \cap (\mathcal{T}^*_W)^2$ Souslin, $D = C \cap \left((\mathcal{T}^*_W)^2\right) \in \Pi^1_1\left((\mathcal{T}^*)^2\right)$ cosouslin.

„A“: $i(T_1) \leqslant i(T_2) \Leftrightarrow \exists f$ strategy : $f^{-1}(T_2) \supset T_1$. So $A = \Pi(F)$, $F := \big\{(T_1, T_2, f) \in (\mathcal{T}^*)^2 \times \mathbb{S}^{\mathbb{S}} | T_1 \subseteq$
where $\mathcal{S}$ is set of strategies. We show $F \in \Pi^0_1$. Clearly $\mathbb{S}^{\mathbb{S}}$ is PTS.

a) „$\mathcal{S} \subset \Pi^0_1(\mathbb{S}^{\mathbb{S}})$“: $f_n \in \mathcal{S}$, $f_n \to f$, $f \in \mathcal{S}$? Set $s < t$, $s, t \in \mathbb{S} \implies \forall n \in \omega$: $f_n(s) < f_n(t)$ ($f_n \in \mathcal{S}$). (Convergence in product space is point-wise) $\implies \exists n_0 \in \omega\ \forall n \geqslant n_0 : f_n(s) = f(s)$, $f_n(t) = f(t) \implies f(s) < f(t)$. Similarly $\exists n_1\ \forall n \geqslant n_1 : f_n(s) = f(s) \implies |f(s)| = |f_n(s)| = |s| \implies f \in \mathcal{S}$.

b) $f^{-1}(T_2) \supset T_1$ is Π^0_1 cond? $T^n_1 \to T_1$, $T^n_2 \to T_2$, $f_n \to f$ such that $f^{-1}_n(T^n_2) \supset T^n_1$. By contradiction: $\exists v \in T_1 \backslash f^{-1}(T_2)$. $\exists n_0\ \forall n \geqslant n_0 : f_n(v) = f(v)$, $v \in T^n_1$, $f(v) \notin T^n_2 \implies v \in T^n_1 \backslash f^{-1}_n(T^n_2)$. ↯. □

Definice 1.10

$\mathcal{S} : L \to \omega_1$ is Π^1_1-rank $\equiv L \in \Pi^1_1(X)$, X PTS and $\exists C \in \Pi^1_1(X^2)$, $A \in \Sigma^1_1(X^2)$: $\{(x,y) \in L^2 | S(x) \subseteq \mathcal{S}(y)\} = C \cap (X \times L) = A \cap (X \times L)$.

Poznámka
TODO!!!

Důsledek
TODO!!!

1.10 Boundedness of Π^1_1-rank

Lemma 1.12

X PTS, $L \subset X$. Let $\mathcal{S} : L \to \omega_1$ be Π^1_1-rank, $L \notin \Sigma^1_1(X)$ and $B \subset L$, $B \in \Sigma^1_1(X)$. Then $\sup\{\mathcal{S}(x), x \in B\} < \omega_1$.

Důkaz
Define $\mathcal{S}(x) = \omega_1$, $x \in X \backslash L$. A as in definition of Π^1_1-rank. By contradiction: $\sup \mathcal{S}(B) = \omega_1$. Then

$$L = \{x \in X | \exists y \in B : \mathcal{S}(x) \leqslant \mathcal{S}(y)\} = \{x \in X | \exists y \in X : (x,y) \in A \cap (X \times B)\} = \Pi_1(A \cap (X \times B)) \in \Sigma^1_1. ↯$$

□

Věta 1.13

Let $B \subset \mathcal{T}_W^*$*,* $B \in \Sigma_1^1(\mathcal{T}^*)$*. Then* $\sup \{i(T) | T \in B\} < \omega_1$*.*

Důkaz

Trivial. □

Poznámka

$B \subset X$ PTS, $B \in \Delta_1^1(X) \implies B \in \Pi_1^1 \implies \exists f \in \Delta_1^1, f : X \to \mathcal{T}^* : f^{-1}(\mathcal{T}_W^*)B, f(B) \subset \mathcal{T}_W^*, f(B) \in \Sigma_1^1 \implies \{\alpha | f^{-1}(\mathcal{T}_W^*(\alpha)) \neq \varnothing\}$ is countable.

$$\implies \exists \alpha < \omega_1 : B \subset f^{-1}(\textstyle\bigcup_{\beta<\alpha} \mathcal{T}_W^*(\beta)),\ X \backslash B = f^{-1}(\mathcal{T}_I^*).$$

1.11 Luzin first separation principle

Věta 1.14

Assume M *is metric space,* $S \subset M$ *souslin,* $A \in \Sigma_1^1(M)$*,* $A \cap S = \varnothing$*. Then there exists* $B \in \Delta_1^1(M)$ *such that* $A \subset B \subset M \backslash S$*.*

Důkaz

S Souslin $\implies S = f^{-1}(\mathcal{T}_I)$, $f \in \Delta_1^1$, $f : M \to \mathcal{T}^*$. Define $\mathcal{S}(x) := i(f(x))$.

$$f(A) \in \Sigma_1^1(\mathcal{T}^*), f(A) \subset \mathcal{T}_W^* \impliedby A \cap S = \varnothing \implies \sup \mathcal{S}(A) = \alpha < \omega_1 \implies$$

$$A \subset B = f^{-1}(\bigcup_{\beta \leqslant \alpha} \mathcal{T}_W^*(\beta)) \in \Delta_1^1, B \cap S = \varnothing.$$

□

Příklad

$\exists C_1, C_2 \in \Pi_1^1(\mathbb{R})$, $C_1 \cap C_2 = \varnothing$, C_1 cannot be Δ_1^1-separated from C_2. (C_1, C_2 are bisouslin in $C_1 \cup C_2$ and cannot be separated by $\Delta_1^1(C_1 \cup C_2)$ set.)

Důkaz
$C_1 = \{(S,T) \in (\mathcal{T}^*)^2 | i(s) < i(T)\} \in \Pi^1_1 \impliedby$ the theorem above. $C_2 = \{(S,T) \in (\mathcal{T}^*)^2 | i(T) < i(S)\} \in \Pi^1_1$. $C_1 \cap C_2 = \emptyset$. $M := C_1 \cup C_2 \implies C_1$ and C_2 are bisouslin in M.

For contradiction $\exists H \in \Delta^1_1((\mathcal{T}^*)^2)$. $C_1 \subset H \subset (\mathcal{T}^*)^2 \setminus C_2 \implies \exists \alpha < \omega_1 : H \in \Sigma^0_\alpha((\mathcal{T}^*)^2)$. Find $B \in \Delta^1_1 \setminus \Sigma^0_{\alpha+1}((\mathcal{T}^*)^2) \impliedby$ use Σ^0_j universal sets $\impliedby$ Kechris.

Find f_{B^C} from the lemma, $f_{B^C} : (\mathcal{T}^*)^2 \to \mathcal{T}^*$, $f^{-1}_{B^C}(\mathcal{T}_I) = (\mathcal{T}^*)^2 \setminus B$, $B = f^{-1}_{B^C}(\mathcal{T}^*_W)$ $\implies \Sigma^1_1 \ni f_{B^C}(B) \subset \mathcal{T}^*_W$, $f_{B^C} \in B_{\sigma_1}$ ($f^{-1}_{B^C}(\Sigma^0_1) \subset \Sigma^0_2$).

From the theorem above $\sup_{x \in B} i(f(x)) = \alpha_B < \omega_1$. From the other theorem $\exists T \in \mathcal{T}^*_W : i(T) > \alpha_B$. Define $F(x) = (f(x), T) \in (\mathcal{T}^*)^2$, $x \in (\mathcal{T}^*)^2$. $F \in B_{\sigma_1}$.

Then $F^{-1}(C_1) = B \impliedby x \in B \implies i(f(x)) \leqslant \alpha_B < i(T)$, $x \in B \implies f(x) \in \mathcal{T}_I \implies$ $(f(x), T) \notin C_1, \in C_2$.

$F^{-1}(C_1) = F^{-1}(H) \impliedby x \in (\mathcal{T}^*)^2 \implies F(x) \subset C_1 \cup C_2 . H \in \Sigma^0_\alpha, F \in B_{\sigma_1} \implies B = F^{-1}(H) \in \Sigma^0_{\alpha+1}(($

$\square$

1.12 Luzin second separation principle and reduction theorem

Věta 1.15 (Reduction theorem)

C_1, C_2 cosouslin in metric space M. Then there exists cosouslin $D_1, D_2 \subset M$ such that

$$\forall i = 1,2 : \quad D_i \subset C_i, \qquad D_1 \cap D_2 = \emptyset, \qquad D_1 \cup D_2 = C_1 \cup C_2.$$

Důkaz
From the lemma $\exists f_i : M \to \mathcal{T}^*$, $f_i \in \Delta^1_1$, $f_i^{-1}(\mathcal{T}^*_W) = C_i$.

$$D_1 := \{x \in M | i(f_1(x)) < \omega_1, i(f_1(x)) \leqslant i(f_2(x))\} \implies D_1 \subset C_1 \quad (i(f_1(x)) \leqslant \omega_1).$$

$$D_1 := \{x \in M | i(f_2(x)) < i(f_1(x))\} \implies D_2 \subset C_2 \quad (i(f_2(x)) \leqslant \omega_1).$$

$D_1 \cup D_2 = C_1 \cup C_2$ ($x \in C_1 \cup C_2 \implies i(f_1(x)) < \omega_1 \vee i(f_2(x)) < \omega$, if $i(f_1(x)) \leqslant i(f_2(x))$ then $x \in D_1$ otherwise $x \in D_2$).

„$D_1 \cap D_2 = \emptyset$“: Define $F = (f_1, f_2) \in \Delta^1_1$, $F : M \to ((\mathcal{T}^*)^2) \impliedby F^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$. ($(\mathcal{T}^*)^2$ has countable base.)

$$C = \left\{(T_1, T_2) \in (\mathcal{T}^*)^2 | i(T_1) < \omega_1, i(T_1) \leqslant i(T_2)\right\} \in \Pi^1_1,$$

$$B = \left\{(T_1, T_2) \in (\mathcal{T}^*)^2 | i(T_2) < i(T_1)\right\} \in \Pi^1_1,$$

$$F^{-1}(C) = D_1 \wedge F^{-1}(B) = D_2 \implies D_1, D_2 \in \Pi^1_1 \implies \text{cosouslin}.$$

□

Důsledek (Luzin second separation principle)
Let M be metric space, A_1, A_2 Souslin in M. Then there exists cosouslin B_1, B_2 such that $A_2 \backslash A_1 \subset B_1$, $A_1 \backslash A_2 \subset B_2$, $B_1 \cap B_2 = \emptyset$. Moreover, it is possible to manage $B_1 \cup B_2 = M \backslash (A_1 \cap A_2) \implies$ if $A_1 \cap A_2 = \emptyset$, then B_i are bisouslin.

Důkaz
$C_i = M \backslash A_i$, B_i reduction of C_i. $B_1 \cup B_2 = C_1 \cup C_2 = M \backslash (A_1 \cap A_2)$, $B_1 \cap B_2 = \emptyset$, $B_i \supset C_i \backslash C_j = A_j \backslash A_i$ $(i \neq j)$. $A_1 \cap A_2 = \emptyset \implies B_1 = M \backslash B_2$. □

2 Kuratowski–Ulam theorem

Poznámka (Notation)
$A \subset X \times Y$, X, Y sets. $A_X := \{y \in Y | [x, y] \in A\}$. $A^y := \{x \in X | [x, y] \in A\}$.

X topological space, $T(x)$ statement. $\forall^* x : T(x) \Leftrightarrow \{x \in X | T(x)\}$ is co-meager. $\exists^* x : T(x) \Leftrightarrow \{x \in X | T(x)\}$ is non-meager.

Věta 2.1 (Kuratowski–Ulam)
X, Y be topological spaces with countable base, $A \subset X \times Y$ has Baire property in $X \times Y$. Then

1. *$\forall^* x : A_x$ has Baire property in Y, $\forall^* y : A^y$ has Baire property in X;*

2. *A is meager $\Leftrightarrow \forall^* x : A_x$ is meager $\Leftrightarrow \forall^* y : A^y$ is meager;*

3. *A is co-meager $\Leftrightarrow \forall^* x : A_x$ is co-meager $\Leftrightarrow \forall^* y : A^y$ is co-meager.*

Lemma 2.2

X, Y topological spaces, Y has countable base, $F \subset X \times Y$ nowhere dense. Then $\forall^* x : F_x$ is nowhere dense.

Důkaz
WLOG $Y \neq \varnothing$. $F \in \Pi^0_1(X \times Y)$ (otherwise for $\overline{F}$). Let $U := (X \times Y) \setminus F$. It is open and dense. We want $\forall^* x : \overline{U_x} = Y$.

$\{V_n\}$ base of Y, $V_n \neq \varnothing$. $U_n := \Pi_X(U \cap X \times V_n)$ dense open in X. (Open trivial. Dense $\Longleftarrow$ $G \in \Sigma^0_1(X), G \neq \varnothing \implies (G \times V_n) \cap U \neq \varnothing \implies [x, y] \in U \cap (X \times V_n)$.)

$x \in \bigcap U_n \implies x \in U_n \implies U_x \cap V_n \neq \varnothing \implies U_x$ is dense in Y. □

Důkaz (Kuratowski–Ulam)
$F \subset X \times Y$ meager $\implies F \subset \bigcup F_n$, $F_n \in \Pi^0_1$, nowhere dense. By the previous lemma $\exists M_n \subset X$ co-meager: $\forall x \in M_n$: $(F_n)_x$ is nowhere dense. $M := \bigcap M_n$ co-meager $\implies \forall x \in M\ \forall n \in \omega$: $(F_n)_x$ is nowhere dense $\implies F_x \subset \bigcup (F_n)_x$ is meager.

Let $A \subset X \times Y$ has Baire property $\implies A = U \triangle M$, $U \in \Sigma^0_1$, M meager. $A_x = U_x \triangle M_x$ (open $\triangle$ meager for co-meager many x) $\implies \forall^* x : A_x$ has Baire property. This implies 1.

Clearly 2. $\Leftrightarrow$ 3. using complements. It remains to show 2. $\Longleftarrow$. □

Lemma 2.3

X, Y topological spaces with countable base, $A \subset X$, $B \subset Y$. Then $A \times B$ is meager $\Leftrightarrow$ A or B is meager.

Důkaz
„ $\implies$ “: $A \times B$ meager, A non-meager. Then by the previous lemma $\exists x \in A : (A \times B)_x = B$ meager.

„ $\Longleftarrow$ “: A is meager, $A \subseteq \bigcup F_n$, $F_n \in \Pi^0_1$, nowhere dense. Then $A \times B \subset \bigcup (F_n \times B)$. We need to show that $F_n \times B$ is nowhere dense. $X \setminus F_n$ open dense $\implies (X \setminus F_n) \times Y$ open dense in $X \times Y \implies F_n \times Y$ is nowhere dense $\implies F_n \times B$ is nowhere dense. □

Důkaz (Kuratowski–Ulam remaining 2. $\Longleftarrow$)
$A \subset X \times Y$ has Baire property, $\forall^* x : A_x$ is meager. $A = U \triangle M$ (open $\triangle$ meager). For contradiction we assume that A is not meager (U is not meager). $\implies \exists G \in \Sigma^0_1(X), H \in \Sigma^0_1(Y) : G \times H \subset U$, $G \times H$ is not meager ($\Longleftarrow$ X, Y have countable base).

$\overset{lemma}{\Longrightarrow}$ G, H non-meager $\implies$ $\exists x \in G : A_x$ is meager, M_x is meager ($\impliedby$ $\forall^* x : M_x$ is meager). Clearly non-meager $H \backslash M_x \subseteq U_x \backslash M_x \subset U_x \triangle M_x = A_x$ meager. ↯. □

Například
$\exists A \subset [0,1]^2$, A non-meager and there are no three points in A on a straight line.

Důkaz
$\{F_\alpha, \alpha < 2^\omega\}$ meager F_σ sets. We will construct $\{x_\alpha, \alpha < 2^\omega\}$ such that $x_\alpha \notin F_\alpha$ and there are no 3 points on the same line. By induction: 1) $\alpha = 0 : x_0 \in [0,1]^2 \backslash F_0$.

2) We already have $\{x_\beta, \beta < \alpha\} \subset [0,1]^2$, $\alpha < 2^\omega$ such that $\forall \beta < \alpha : x_\beta \notin F_\beta$ and there are no 3 points on the same line. $\mathcal{M} := \{p \text{ line} | \exists \beta, \gamma < \alpha : x_\beta \neq x_\gamma \wedge x_\beta, x_\gamma \in p\}$. Clearly $\# < 2^\omega$. From Kuratowski–Ulam: $\forall^* t \in [0,1]$: $(F_\alpha)_t$ is meager. We find $t \in [0,1] \backslash \Pi_1(\{x_\beta, \beta < \alpha\})$ such that $(F_\alpha)_t$ is meager.

$\implies$ line $\{[l, y], y \in \mathbb{R}\} \notin \mathcal{M} \implies \forall p \in \mathcal{M} : \#\{y \in [0,1] | [t,y] \in p\} \leqslant 1$. So $\exists y \in [0,1] : [t,y] \notin \bigcup \mathcal{M} \cup F_\alpha$. $(F_\alpha)_t$ is meager and $\#(\bigcup \mathcal{M}) \cap \{[t,y], y \in \mathbb{R}\} \leqslant \#\mathcal{M} < 2^\omega$. Put $x_\alpha := [t,y]$. □

3 Measurable selections

Definice 3.1 (Uniformization, selection)

Let X, Y sets, $C \subset X \times Y$ and $F : x \mapsto C_x$ is mapping from X to $\mathcal{P}(Y)$. $U \subset C$ is uniformization of C if $|U_x| = 1$ for $C_x \neq \emptyset$ ($x \in \Pi_X(C)$), (U is a graph of mapping $X \to Y$).

Mapping $f : D_f \to Y$ ($D_f = \Pi_x(C) = \{x \in X | F(x) \neq \emptyset\}$) is selection of F, if $f(x) \in F(x)$, $x \in D_F$.

Poznámka (Kondo–Norikov)
X, Y Polish topological spaces, $C \in \Pi^1_1(X \times Y)$. Then there exists $B \in \Pi^1_1(X \times Y)$ unifomization of C.

Poznámka
The theorem above implies if $A \subset M \times \{0,1\}$ (M metric space) is cosouslin then there exists cosouslin uniformization.

$A_i := \Pi_x(A \cap M \times \{i\})$. $B_0 \cup B_1 = M \implies B_0 \times \{0\} \cup B_1 \{1\}$ is uniformization. $B_0 \subset A_0$, $B_1 \subset A_1$, $B_i \in \Pi^1_1$, $B_0 \cap B_1 = \emptyset$. Similarly, we can do reduction for countable collections.

TODO!!! (Kondo–Norikov is not true in Σ^1_1.)

Příklad
There exists $F \in \Pi_1()$

TODO!!!

TODO!!!

TODO? (Example)

TODO!!!

TODO!!!

4 Continuous selections

Poznámka
It is enough: $F^{-1}(\Sigma_1^0) \subset \Sigma_1^0$ (yes, in 0-dim spaces $\Sigma_1^0 = (\Delta_0^1)_\sigma$, generally no).

Příklad ($F(x)$ is connected is not enought)
$A(t) = \begin{cases} S(0,1) : t = 0 \\ S(0,1) \setminus B(e^{\frac{i}{t}}, t) \end{cases}$. There is no continuous selection.

Poznámka (Notation)
Y Baire space, $\mathcal{F}_c(Y) :=$ convex non-empty closed subsets of Y.

$F : X \to \mathcal{P}(Y)$ lower semi continuous $\equiv F^{-1}(\Sigma_1^0) \subset \Sigma_1^0$. ($X$, Y topological spaces.)

Poznámka (E. Michael)
Let X be T_1 topological space. Then following assertions are equivalent:

- if Y is Baire space, then every lower semi continuous $F : X \to \mathcal{F}_c(Y)$ admits continuous selection;
- X is paracompact.

Poznámka
Let X be T_1 topological space. Then following assertions are equivalent:

- X paracompact and T_2;

- $\forall$ open cover of X admits partition of unity.

Definice 4.1

M be open cover of topological space X. Then M admits partition of unity $\equiv \exists \{u_j\}_{j\in I}$, $u_j : X \to [0,1]$ be continuous and $\forall j \in I\ \exists G \in M$: $\overline{\{u_j > 0\}} \subset G$, $\left\{\overline{\{u_j > 0\}}, j \in I\right\}$ is locally finite and $\forall x \in X : \sum_{j\in I} u_j(x) = 1$.

Poznámka (Stone)
X metric space, then X is paracompact and T_2.

Věta 4.1

X be T_2 topological space such that every open cover admits partition of unity. Y Baire space, $F : X \to \mathcal{F}_c(Y)$ be lower semi continuous. Then there exists continuous selection.

Věta 4.2 (Tietze)

If $A \to \mathbb{R}$ continuous, $A \in \Pi^0_1(X)$, X normal topological space. Then there exists continuous extension.

Důsledek
Let X be T_2, paracompact, Y be Baire space, $A \in \Pi^0_1(X)$ and $f : A \to Y$ be continuous. Then there is continuous extension.

$F(x) = f(x)$, $x \in A$, and $F(x) = Y$, $x \notin A$ TODO!!!

Důsledek
TODO!!!

Důsledek
X be T_2, precompact $\implies$ X normal.

Důsledek
TODO!!!

Lemma 4.3 (Approximation)

Let X be like in the previous theorem, Y normed linear space, $G : X \to \mathcal{F}_c(Y)$ be lower semi--continuous, and W be convex open neighbourhood of $\mathbf{o}$ in Y. Then there exists continuous $g : X \to Y$ such that $g(x) \in G(x) + W$.

Důkaz
$\{y_\alpha\}_{\alpha\in I}$ dense in Y, $U_\alpha := G^{-1}(y_\alpha - W)$ is open cover of X. $x \in X$ be arbitrary. $\emptyset \neq G(x) \subset Y$, $\{y_\alpha - W, \alpha \in I\}$ covers $Y \implies \exists \alpha \in I : G(x) \cap (y_\alpha - W) \neq \emptyset \implies G^{-1}(y_\alpha - W) \ni x$. Find $\{g_\alpha, \alpha \in I\}$ locally finite partition of unity subordinate to cover $\{U_\alpha, \alpha \in I\}$. Put $g(x) := \sum_{\alpha\in I} g_\alpha(x) \cdot y_\alpha$, it is clearly continuous ($\impliedby$ locally finiteness).

$$g(x) \in G(x)+W \impliedby g_\alpha(x) > 0 \implies x \in U_\alpha = G^{-1}(y_\alpha - W) \implies G(x) \cap y_\alpha - W \neq \emptyset \implies y_\alpha \in G(x)+$$

$$\implies g(x) \text{ is convex combination of elements of } G(x)+W \implies g(x) \in G(x)+W.$$ □

Lemma 4.4 (Lower semi-continuity and intersection)

X topological space, Y normed linear space, $F, G : X \to \mathcal{P}(Y)$ lower semi-continuous and W neighbourhood of $\mathbf{o}$ in Y. Let $H(x) := F(x) \cap (G(x)+W) \neq \emptyset$, $x \in X$. Then H is lower semi continuous.

Důkaz
$U \in \Sigma^0_1(Y)$:

$$H^{-1}(U) = \{x \in X | H(x) \cap U \neq \emptyset\} = \{x \in X | G(x) \cap (G(x)+W) \cap U \neq \emptyset\} = \{TODO!!!\} = TODO!!!$$

TODO!!! □

Důkaz (of the previous theorem)
$W_n := B(0, 2^{-n}) \subset Y$. We will inductively construct continuous f_n such that $f_n(x) \in F(x) + W_n$ and $f_n(x) \in f_{n-1}(x) + 2W_{n-1}$. Then $f_n \rightrightarrows f \impliedby$ completeness of Y and second condition on f_n. First condition $\implies f(x) \in F(x) \impliedby F(x) \in \Pi^0_1$. f continuous $\impliedby$ continuous convergence.

Take f_0 from approximation lemma for W_0. Assume we already have continuous $f_0, \dots, f_n$ satisfying those two conditions. Put $F_{n+1}(x) = F(x) \cap (f_n(x) + W_n) \neq \emptyset$ for $x \in X \impliedby$ first condition for n. F_{n+1} lower semi-continuous $\impliedby$ lower semi-continuity of F and intersection lemma. Take continuous $f_{n+1}(x) \in F_{n+1}(x) + W_{n+1}$ from approximation lemma.

First condition $\impliedby F_{n+1}(x) \subset F(x)$ and the definition of f_{n+1}. Second condition $\impliedby$ $f_{n+1}(x) \in F_{n+1} + W_{n+1} \subset (f_n(x) + W_n) + W_{n+1} \subset f_n(x) + 2W_n$. □

5 Borel selections for sets with large sections

Věta 5.1

$(X, \mathcal{M})$-measurable space, Y Polish topological space, η Δ^1_1 probability on Y, $B \in \mathcal{M} \otimes \Delta^1_1(Y)$. Then $\{x \in X, \eta(B_x > 0)\} \in \mathcal{M}$.

Důkaz
For $B \in \mathcal{M} \otimes \Delta_1^1(Y)$ we define $B(r) := \{x \in X, \eta(B_x) > r\}$, $r \geqslant 0$. (Our set is $B(0)$.)

$$\mathcal{A} := \left\{A \in \mathcal{M} \otimes \Delta_1^1(Y) | \forall r > 0 : A(r) \in \mathcal{M}\right\}.$$

We want $\mathcal{A} = \mathcal{M} \otimes \Delta_1^1(Y)$ ($\forall B \in \mathcal{A} : B(1/n) \in \mathcal{M}$, $B(0) = \bigcup_{n=1}^{\infty} B(1/n) \in \mathcal{M}$).

- $M \in \mathcal{M}$, $W \in \Delta_1^1(Y) \implies (M \times W)(i) = M$ if $\eta(W) > r$, else $= \varnothing$. Both in $\mathcal{M}$ $\implies M \times W \in \mathcal{A}$.
- „$B \in \mathcal{A} \implies B^C := (X \times Y) \backslash B \in \mathcal{A}$“:

  $$B^C(r) = \{x \in X, \eta(B_x) < 1 - r\} = \bigcup_{n=1}^{\infty} \left\{x \in X, \eta(B_x) \leqslant 1 - r - \frac{1}{n}\right\} = \bigcup_{n=1}^{\infty} \left(X \backslash \left\{x \in X | \eta(B_x) >\right.\right.$$

- „$B_n \in \mathcal{A}$ disjoint $\implies \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$“:

  $$\left(\bigcup_{n=1}^{\infty} B_n\right)(r) = \left\{x \in X | \sum_{n=1}^{\infty} \mu((B_n)_x) > r\right\} = \bigcup_{n_0=1}^{\infty} \left\{x \in X | \sum_{n=1}^{n_0} \mu((B_n)_x) > r\right\} = \bigcup_{n_0=1}^{\infty} \bigcup_{\{q_1,\dots,q_{n_0}\} \in \mathbb{Q}}$$

We have $\mathcal{M} \times \Delta_1^1(Y) \subset \mathcal{A}$, $\mathcal{A}$ is closed under complements and continuous disjoint union $\implies \mathcal{A}$ is σ-algebra $\implies \mathcal{A} = \mathcal{M} \otimes \Delta_1^1(Y)$. $\square$

Poznámka
$(X, \mathcal{M})$ measurable space, Y Polish topological space, $B \in \mathcal{M} \otimes \Delta_1^1(Y)$.

- Mapping (X, μ): $X \times$ prob.$(Y) \mapsto \mu(B_x)$ is $\mathcal{M} \otimes \Delta_1^1(Y)$ measurable.
- If η_x is Δ_1^1 and $\mathcal{M} = \Delta_1^1(X)$. Then $\{x \in X | \eta_x(B_x) > 0\} \in \Delta_1^1$.

Lemma 5.2

Y Polish topological space (it is sufficient Baire space), $B \in BP(T)$, $\{U_n\}$ base of Y consisting of non-empty sets. Then $Y \backslash B$ is non-meager $\Leftrightarrow \exists n \in \omega : B \cap U_n$ is meager.

Poznámka
$A \in BP \implies$ (A non-meager $\Leftrightarrow A$ is nowhere co-meager).

Důkaz

„ $\Longleftarrow$ “ $B \cap U_n$ meager $\implies$ (nonempty subsets of PTS are non-meager) $U_n \backslash B$ is non--meager $\implies$ $Y \backslash B$ is non-meager.

$$„ \implies “: Y\backslash B = G \triangle E \implies (Y\backslash B)\triangle G = E, G \neq \varnothing \implies \exists n \in \omega : U_n \subset G. \text{ Then}$$

$$U_n \cap B \subset G \cap B = G\backslash(Y\backslash B) \subset (Y\backslash B)\triangle G = E \implies U_n \cap B \text{ is meager.}$$

□

Věta 5.3 (Montgomery, Novikov)

$(X, \mathcal{M})$ measurable space, Y Polish topological space, $B \in \mathcal{M} \otimes \Delta^1_1(Y)$. Then $\{x \in X | B_x$ non-meager$\}$, $\{x \in$ $\mathcal{M}$.

TODO!!!

TODO!!!

Důsledek

X, Y Polish topological spaces, $B \in \Delta^1_1(X \times Y)$, B_x monmeager, $x \in \Pi_X(B)$. Then there exists Δ^1_1 selection of $x \mapsto B_x$, Δ^1_1 unif. of B, $\Pi_X(B) \in \Delta^1_1$.

Důkaz

Trivial. □

Věta 5.4 (Srivastava)

X, Y Polish topological spaces, $F : X \to \Pi^2_0(Y)$, $\mathrm{graph}(f) \in \Delta^1_1$, $F^{-1}(\Sigma^0_1(Y)) \in \Delta^1_1(X)$. Then F has Δ^1_1 selection.

Důkaz

$G(x) : \overline{F(x)}$ is Δ^1_1 meager ($G^{-1}(U) = \{x \in X | G(x) \cap U \neq \varnothing\} = \{x \in X | F(x) \cap U \neq \varnothing\} = F^{-1}(U)$). $\mathcal{I}_x = \{E | E \text{ is meager in } G(x)\}$. By the previous theorem $x \mapsto \mathcal{I}_x$ is (BB), $\mathrm{graph}(F) = B$, $B_x \notin \mathcal{I}_x$, $x \in \Pi(B) \Longleftarrow B_x \in \Pi^0_2(\overline{B_x})$, B_x dense in $\overline{B_x}$. □

Příklad

$\{f_x | x \in [0,1]\}$ be set of Δ^1_1 functions $[0,1] \to [0,1]$. Put $G(x) := [0,1] \backslash \{f_x(x)\} \in \Pi^0_2$, $U \neq \varnothing$, $U \in \Sigma^0_1([0,1]) \implies G^{-1}(U) = [0,1] \in \Delta^1_1$. G is Δ^1_1, but there is no Δ^1_1 selection ($\mathrm{graph}(f) \in \Delta^1_1$). (By diagonal argument.)

Důsledek

X, Y Polish topological spaces, $B \in \Delta^1_1(X \times Y)$, $\forall x \in \Pi_X(B) : \mu(B_x) > 0$ (μ is some Δ^1_1 probability measure). Then there exists Δ^1_1 selection of $x \mapsto B_x$.

We can also assume that there is Δ^1_1 map $x \mapsto \mu_x$, $\forall x \in \Pi_X(B) : \mu_x(B_x) > 0$ instead of

$\mu(B_x) > 0$.

6 Small sections

6.1 Compact selections

Věta 6.1 (Norikov separation principle)

X Polish topological space, $A_n \in \Sigma^1_1(X), \bigcap A_n = \emptyset$. Then there exists $B_n \supset A_n$, $B_n \in \Delta^1_1(X)$ such that $\bigcap B_n = \emptyset$.

Definice 6.1

(E_n) can't be approximated, if there does not exists $B_n \supset E_n$, $B_n \in \Delta^1_1$, $\bigcap B_n = \emptyset$.

Lemma 6.2

$E_n \subset X$, (E_n) can't be approximated, $k \in \omega$, $E_n = \bigcup_i E_{n,i}$, $n \leqslant k$. Then there exists $i_1, \dots, i_k$: $E_{1,i_1}, E_{2,i_2}, \dots, E_{k,i_k}, E_{k+1}, \dots$ can't be approximated.

Důkaz

We will find $i_1, i_2, \dots, i_k$ by induction on k. „$k = 0$“ is trivial. „$k > 0$ and we already found E_{1,i_1}, E_{2,i_2}, ..., $E_{k-1,i_{k-1}}$ such that E_{1,i_1}, E_{2,i_2}, ..., $E_{k-1,i_{k-1}}$ cannot be approximated“: By contradiction $\forall i \in$: $E_{i,i_1}, \dots, E_{k-1,i_{k-1}}, E_{k,i}, E_{k+1}, \dots$ can be approximated by B^i_l: $B^i_l \supset E_{l,i_l}$, $l < k$, $B^i_k \supset E_{k,i}$, $B^i_l \supset E_l$, $l > k$.

Put $B_l := \bigcap_{i\in\omega} B^i_l$, $l \neq \ k$. $B_k := \bigcup_{i\in\omega} B^i_k \implies B_k \in \Delta^1_1$, $B_l \supset E_{l,i_l}$, $l < k$, $B_l \supset E_l$, $l > \ k$. $\bigcap_{l\in\omega} B_l = \emptyset \impliedby (x \in B_k \implies \exists i : x \in B^i_k \implies x \notin \bigcap_{l\neq\ k} B^i_l \implies x \in \bigcap_{l\neq k} B_l)$ which is contradiction. □

Důkaz (Norikov separation principle)

If $A_n = \emptyset$ then put $B_n = \emptyset$, $B_k = X$, $k \neq n$. Se we can assume $A_n \neq \emptyset$. Set $f_n : \mathcal{N} \to A_n$ continuous surjection. By contradiction, let (A_n) can't be approximated. From the previous lemma $\exists n^1_1, n^1_2, n^1_3, n^2_1, n^2_2, n^3_1 \in \omega$:

$$f_1(\mathcal{N}(n^1_1, n^1_2, n^1_3)), f_2(\mathcal{N}(n^2_1, n^2_2)), f_3(\mathcal{N}(n^3_1)), A_4, A_5, \dots$$

can't be approximated.

By lemma it holds also for $\sigma_k = (n^k_1, n^k_2, \dots) \in \mathcal{N}$:

$$\forall k \in \omega : f_1(\mathcal{N}(\sigma_1|_{k-1})), f_2(\mathcal{N}(\sigma_2|_{k-2})), \dots, f_k(\mathcal{N}(\sigma_k|_0)), A_{k+1}, \dots$$

can't be approximated.

$\bigcap A_n \neq \emptyset \implies \exists i < j : f_i(\sigma_i) \neq f_j(\sigma_j)$ (otherwise $\forall k, l \in \omega : f_k(\sigma_k) = f_l(\sigma_l) \implies f_k(\sigma_k) \in \bigcap A_l$). $\implies U_i, U_j \in \Sigma^0_1(X)$: $U_i \cap U_j = \emptyset$, $f_k(\sigma_k) \in U_k$, $k \in \{i, j\}$.

f_i, f_j continuous $\exists k \in \omega : f_i(\mathcal{N}_{\sigma_i|_{k-i}}) \subset U_i, f_j(\mathcal{N}_{\sigma_j|_{k-j}}) \subset U_j \implies f_1(\mathcal{N}_{\sigma_1|_{k-1}}), \ldots, f_k(\mathcal{N}_{\sigma_{k-1}|_1}), A_k, \ldots$ can be approximated by $X, X, \ldots, U_i, X, X, \ldots, U_j, X, X, \ldots$, which is contradiction. $\square$