1 Dynamické systémy

Definice 1.1 (Dynamický systém)

(φ, Ω), $\Omega \subset \mathbb{R}^n$ otevřená, $\varphi : \mathbb{R} \times \Omega \to \Omega$ $\varphi(t, x)$.

- $\varphi(0, x) = x$;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- φ je spojité.

Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geqslant 0\}$ je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leqslant 0\}$ je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$ je plný orbit.

Definice 1.3 (Pozitivně, negativně a úplně invariantní)

(φ, Ω) dynamický systém, $M \subset \Omega$.

M je pozitivně invariantní $\equiv \forall x \in M : \gamma^+(x) \subset M$.

M je negativně invariantní $\equiv \forall x \in M : \gamma^-(x) \subset M$.

M je úplně invariantní $\equiv \forall x \in M : \gamma(x) \subset M$.

Poznámka

$\gamma^+(x_0)$ je pozitivně invariantní, $\gamma^-(x_0)$ je negativně invariantní a $\gamma(x_0)$ je úplně invariantní.

Definice 1.4

$$\omega(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \to \infty : \varphi(t_k, x_0) \to y\},$$
$$\alpha(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \to -\infty : \varphi(t_k, x_0) \to y\}.$$

Poznámka (To je ekvivalentní)

$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0\ \forall T > 0\ \exists t \geqslant T : |\varphi(t_r, x_0) - y| < \varepsilon\}$.

Lemma 1.1

$\omega(x_0) = \bigcap_{\tau \geqslant 0} \overline{\gamma^+(\tau, x_0)}$.

Důkaz
„$\subseteq$“: $y \in \omega(x_0)$: $\forall \varepsilon > 0\ \forall T\ \exists t \geqslant T : |\varphi(t, x_0) - y| < \varepsilon$. Chceme:

$$\forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geqslant 0\ \forall \varepsilon > 0\ \exists s \geqslant \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

„$\supseteq$“: $\forall \tau \geqslant 0\ y \in \overline{\gamma^+(\tau, x_0)} \implies$

$$\implies \forall \varepsilon\ \exists s \geqslant \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

□

Věta 1.2 (Vlastnosti ω-limitní množiny)

Nechť (φ, Ω) je dynamický systém, $x_0 \in \Omega$. Potom

1. *$\omega(x_0)$ je uzavřená, úplně invariantní.*
2. *Pokud $\gamma^+(x_0)$ je relativně kompaktní v $\mathbb{R}^n$, pak $\omega(x_0) \neq \varnothing$, $\omega(x_0)$ je kompaktní, souvislá.*

Důkaz
1. $\omega(x_0)$ je průnik uzavřených množin, tedy uzavřená. $y \in \omega(x_0)\ \exists t_k \nearrow \infty\ \varphi(t_k, x_0) \to y$.

$$\begin{aligned} s_k = t_k + t \qquad & \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0)) \\ t_k \to \infty, \varphi \text{spojitá} \qquad & \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \to \varphi(t, y) \end{aligned}$$

2. $\exists K \subset \mathbb{R}^n$ kompaktní $\gamma^+(x_0) \subset K$. a) pokud $t_n \geqslant 0, t_n \to \infty\ \{\varphi(t_n, x_0)\}_{n=1}^{\infty}$ omezená posloupnost $\implies \exists\ \{t_{n_k}\}_{k=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$, podposloupnost, $\exists y \in \Omega \varphi(t_{n_k}, x_0) \to y$. Pak $y \in \omega(x_0)$.

b) $\omega(x_0)$ je tedy úplná a omezená, takže kompaktní. c) ať $\omega(x_0)$ je nesouvislá, tedy $\omega(x_0) \subseteq U \cup V$, U, V otevřené disjunktní neprázdné, $U, V \subseteq K$. Vezměme $y \in \omega(x_0) \cap U$, $z \in \omega(x_0) \cap V$. Nechť t_n je posloupnost taková, že $\varphi(t_{2n} x_0) \to y$, $\varphi(t_{2n+1}, x_0) \to z$, $t_{2n} < t_{2n+1}$, $\varphi(t_{2n}, x_0) \in U$, $\varphi(t_{2n+1}, x_0) \in V$. $F = K \backslash (U \cup V)$ uzavřená, tedy $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$. Tedy $\{\varphi(s_n, x_0)\}$ je omezená posloupnost $\implies \exists$ podposloupnost konvergující k $w \in F$. □

Definice 1.5 (Topologická konjugovanost)

(φ, Ω), ψ, Θ dynamické systémy. $\exists : \Omega \to \Theta$ homeomorfismus (bijekce, spojité, spojitá inverze):

$$\forall x \in \Omega\ \forall t \in \mathbb{R} \qquad h(\varphi(t, x)) = \psi(t, h(x)).$$

Poznámka
Dá se zobecnit ještě zobrazováním časů.

Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0$, (φ, Ω) příslušný dynamický systém. $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $y(0) = 0$ a (ψ, Θ) je příslušný dynamický systém. Potom (φ, Ω), (ψ, Θ) jsou lokálně topologicky konjugované ($\exists U$ okolí $x_0 \in \Omega$ a V okolí $\mathbf{o} \in \mathbb{R}^n$ taková, že $\exists g : U \to V$ homeomorfismus $g(\varphi(t, x)) = \psi(t, g(x))$ $\forall x \in U$, $\forall t : \varphi(t, x) \in U$).

Důkaz
BÚNO $f_1(x_0) = \alpha \neq 0$ (první souřadnice funkce f) a $x_0 = \mathbf{o}$. Buď $\tilde{V}$ okolí $\mathbf{o} \in \mathbb{R}^n$ $G : \tilde{V} \to \mathbb{R}^n$, $G(y_1, \ldots, y_n) = \varphi(y, (0, y_2, \ldots, y_n))$. Chceme ukázat, že G je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \ldots, y_n)}{\partial y_1}|_{(0,\ldots,0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \ldots, y_n))|_{y_1=0,\ldots,y_n=0} = f(\varphi(y_1(0, y_2, \ldots, y_n)))|_{y_1=0,\ldots,y_n=0} = f(\varphi$$

$$\frac{\partial G(y_1, \ldots, y_n)}{\partial y_j}|_{(0,\ldots,0)} = \lim_{h\to 0} \frac{G(0, \ldots, h, \ldots, 0) - G(0, \ldots, 0)}{h} = \lim_{h\to 0} \frac{(0, \ldots, h, \ldots, 0)^T - (0, \ldots, 0)^T}{h} = (0$$

Tedy $\nabla G(0, \ldots, 0)$ je „jednotková matice, až na to, že a_{11} je α“, tudíž podle věty o inverzi funkce $\exists V \subseteq \tilde{V}$ okolí 0, $\exists U$ okolí bodu x_0 tak, že $G : V \to U$ je homeomorfismus. Položme $g = G^{-1}$.

Nyní už stačí $g(\varphi(t, x_0)) = \psi(t, g(x_0))$ $\forall x_0 \in U$ $\forall t : \varphi(t, x_0) \in U$. $\varphi(t, x_0) = G(\psi(t, g(x_0)))$

3. $x \in U = G(V)$ $\exists y \in V$ $x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \ldots, x_{0n} + y_n))$$

□

Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \to \Omega, floc.Lip.$$

$$\exists V : \Omega \to \mathbb{R},\ \textit{bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leqslant l\} - -\textit{bounded}$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leqslant 0 \qquad \forall x \in \Omega_l.$$

$$R = \left\{ x \in \Omega_l | \dot{V}_f(x) = 0 \right\}, \quad M = \left\{ y \in R | \gamma^+(y) \subset R \right\}.$$

Then $\forall x \in \Omega_l : \omega(x) \subset M$.

Důkaz
Let $x \in \Omega_l$. $\forall y \in \omega(x)\ \exists t_k \nearrow \infty : x(t_k) \to y$. $\varphi(t, x_0) = x(t)$.

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leqslant 0.$$

$V(x(t)) \searrow$ and $\exists C : \forall x \in \Omega : V(x) > -C$ so $\exists \lim_{t\to\infty} V(x(t)) = c$.

So $\exists c\ \forall y \in \omega(x_0) V(y) = c$. $V(x(t_k)) \to V(y) = c$.

$$\gamma^+(y) \subset \omega(x_0)\ V(\varphi(t, y)) = c\ \forall t \geqslant 0 \implies$$

$$\implies \frac{d}{dt} V(\varphi(t, y)) = 0.$$

$\gamma^+(y) \subset R$ in particular, $y \in R$. Hence $y \in M$. □

2 Poincaré-Bendixson theory

Věta 2.1 (Poincaré-Bendixson)

Let $p \in \Omega$, Ω *open connected.* $\omega(p)$ *doesn't contain stat points and* $\gamma^+(p)$ *is relatively compact* ($\overline{\gamma^+(p)}$ *is compact). Then* $\omega(p) = \Gamma$*-periodic orbit.*

Věta 2.2 (Bendixon-Dulas)

Ω*-simply connected (*$\forall$ *closed Jordan curve* γ *in* Ω, $\operatorname{int}(\gamma) \subset \Omega$*).* $\exists B : \Omega \to \mathbb{R} : (\operatorname{div} B)(x) = \frac{\partial B}{\partial x_1}(x_1, x_2) + \frac{\partial B}{\partial x_2}(x_1, x_2) > 0$ *for almost every* $x \in \Omega$*. Then* $x' = f(x)$ *doesn't have nontrivial periodic solutions.*

Definice 2.1 (Transverzála)

Σ segment on a line such that $\forall p \in \Sigma : \Sigma \nparallel f(p)$.

Lemma 2.3

Σ *transverzála,* $p \in \Sigma \subset \Omega$*. Then* $\exists \tilde{\subset} U$ *neighborhood of* p*.* $\exists \Delta > 0$ *such that*

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U\ \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

Důkaz

Use Th. of rect. □

Lemma 2.4

Let $p \in \Omega$ and assume that $|\gamma^+(p) \cap \Sigma| \geqslant 3$, i. e. $\exists t_1 < t_2 < t_3$ $\varphi(t_j, p) \in \Sigma$, $j = 1, 2, 3$. Then $\varphi(t_2, p)$ lie between $\varphi(t, p)$ and $\varphi(t_3, p)$.

TODO!!!

TODO!!!

2.1 Controllability

Definice 2.2 (Control theory)

$$x' = f(x, u), f : \Omega \times U, \Omega \subset \mathbb{R}^n, U \subset \mathbb{R}^n,$$

$$u \in \mathcal{U} := \{u : [0, T] \to \mathbb{R}^n | \text{measurable}, ||u||_\infty < \infty\} = L^\infty(0, T, \mathbb{R}^n).$$

($\mathcal{U}$ is admissible functions).

Definice 2.3 (Linear task)

$x' = Ax + Bu$, $A, B \in \mathbb{R}^{n \times m}$, $m < n$.

Definice 2.4

$x_0 \underset{u(0)}{\overset{t}{\to}} 0$ iff $x(0) = x_0$, $x(t) = 0$.

Definice 2.5 (Area of controlability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in L^\infty(0, t, \mathbb{R}^n) : x_0 \underset{u(0)}{\overset{t}{\to}} 0 \right\}$$

Definice 2.6 (Kalman matrix)

$$\mathcal{K}(A, B) := (B|AB|A^2B| \ldots |A^{n-1}B)$$

Věta 2.5

For linear problem $\mathcal{R}(t) = \mathrm{LO}(g_1, g_2, \ldots, g_{n \cdot m})$, where $\mathcal{K}(A, B) = (g_1|g_2| \ldots |g_{n \cdot m})$

Tvrzení 2.6 (Observation)

$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)Bu(s)ds}$.

$$x_0 \xrightarrow[u(0)]{t} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = -\int_0^t e^{-As}Bu(s)ds \qquad (KO)$$

Lemma 2.7 (1)

$$A^k \in \mathrm{LO}(I, A, A^2, \dots, A^{n-1}), k \in \mathbb{N}_0$$

Důkaz
Cayley-Hamilton. □

Důkaz
1) $\mathcal{R}(t)$ is vector subspace of $\mathbb{R}^n$ from definition $x_0 + x_1 \xrightarrow[(u_1+u_2)(0)]{t} 0$, $\alpha x \xrightarrow[\alpha u(0)]{t} 0$.

2) We want $\mathcal{R}(t)^\perp = (\mathrm{LO}(g_1, \dots, g_n))^\perp$. „$\supseteq$“: $p \in (\mathrm{LO}(g_1, \dots, g_n))^\perp$. $x_0 \in \mathcal{R}(t)$ arbitrary. From KO:

$$0 \stackrel{?}{=} p^T x_0 = -\int_0^t p^T e^{-As}Bu(s)ds = -\int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k Bu(s)ds$$

We know $(p, g_j) = 0$, $p^T g_j = 0$, $p^T \mathcal{K}(A, B) = 0$, $p^T A^k B = 0$, $k \in [n-1]$. And from lemma 1 $k \in \mathbb{N}$. „$\subseteq$“: $p \in \mathbb{R}^n$, $p \in \mathcal{R}(t)^\perp$. We want to prove $p \perp B, AB, A^2B, \dots, A^{n-1}B$. $B = (b_1|\dots|b_m)$. $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$. $\varphi \in L^\infty(0, T, \mathbb{R})$, $u(t) = \varphi(t) \cdot \mathbf{e}_j$, where $x_0 = -\int_0^t e^{-As}Bu(s)ds$. We have $x_0 \xrightarrow[u(0)]{t} 0$, hence $x_0 \in \mathcal{R}(t)$.

$$0 = p^T x_0 = -p^T \int_0^t e^{-As}Bu(s)ds = -\int_0^t p^T e^{-As} b_j \varphi(s)ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have $p^T e^{-As} b_j \equiv 0$, we derivate it, $p^T A^n e^{-As} b_j \equiv 0$, and set $s = 0$. □

Důsledek
$\mathcal{R}(t)$ doesn't depend on time.

Definice 2.7 (Locally and globally controllable)

Linear problem is called locally controllable, iff $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 | \, |x_0| < \delta\} \subset \mathcal{R}(t)$. And globally if $\mathbb{R}^n = \mathcal{R}(t)$.

Důsledek
Linear problem is controllable $\Leftrightarrow$ rank $K(A, B) = n$.

2.2 Observability

Definice 2.8 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n y = g(x), g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, m < n.$$

Definice 2.9

We say that system $x' = f(x)$ is observable through $g(\cdot)$ on $[0, t]$, iff $\forall x_1(\cdot), x_2(\cdot) : [0, T] \to \mathbb{R}^n : g(x_1(t)) = g(x_2(t))\ \forall t \in [0, T] \implies x_1(0) = x_2(0)$.

Definice 2.10 (Linear observability)

$x' = Ax$, $y = Bx$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$.

Věta 2.8

$x' = Ax$ is observable on $[0, T]$ through $y = Bx \Leftrightarrow x' = A^T x + B^T u$ is controllable.

Důkaz
(We will prove equivalence with rank $\mathcal{K}(A^T, B^T) = n$.) „ $\Longleftarrow$ “: For contradiction

$$\exists x_1(t), x_2(t), [0, T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At}x_0, Bx(t) = B^{At}x_0 \equiv 0 \qquad \forall t \in [0, T].$$

We differentiate it, set $t = 0$ and get $Bx_0 = 0$, $BAx_0 = 0$, ..., $BA^{n-1}x_0 = 0$. So $x_0^T B^T = 0$, ..., $x_0^T \left(A^T\right)^{n-1} B^T = 0$. $x_0^T \mathcal{K}(A^T, B^T) = 0$, $x_0 \perp \mathcal{K}(A^T, B^T)$, ↯.

„ $\Longrightarrow$ “: For contradiction rank$(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T \mathcal{K}(A^T, B^T) = 0$. $x_0^T \left(A^T\right)^k B^T = 0\ \forall k \in [n-1]$ and from lemma 1 $\forall k \in \mathbb{N}$. $BA^T x_0 = 0$. $Be^{At}x_0 = 0$ $\forall t \in [0, T]$. ↯. □