TODO(you should know).

TODO(motivation)

1 Sobolev spaces

Definice 1.1 (Multiindex)

α je multi-index $\equiv \alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}$. Length of multi-index α is $|\alpha| := \alpha_1 + \dots + \alpha_d$. If $u \in C^k(\Omega)$ then $D^\alpha := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$, $\alpha \leqslant k$.

Definice 1.2 (Weak derivative)

Let $u, v_\alpha \in L^1_{loc}(\Omega)$ and α be a multi-index. We say that v_α is the α-th weak derivative of u in Ω iff $\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int v_\alpha \varphi$.

Lemma 1.1

Weak derivative is unique. If the classical derivative exists then it is also the weak derivative.

Důkaz

Let v^1_α and v^2_α be two weak derivatives. Then

$$\int_\Omega (v^1_\alpha - v^2_\alpha)\varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v^1_\alpha = v^2_\alpha$ almost everywhere in Ω.

If classical $D^\alpha u$ exists, then

$$\int_\Omega \underbrace{D^\alpha u}_{v_\alpha} \varphi \overset{\text{BP}}{=} (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

□

Poznámka (Notation for this course)
D^α always means the weak derivative.

Definice 1.3 (Sobolev space)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. We define $W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}$.

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sup_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

Lemma 1.2 (Base properties of Sobolev spaces)

Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and α is multi-index. Then

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$, *if* $|\alpha| \leqslant k$;
- $\lambda u + \mu v \in W^{k,p}(\Omega)$ $\forall \lambda, \mu \in \mathbb{R}$ $(D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v)$;
- $\tilde{\Omega} \subset \Omega$ *open*, $u \in W^{k,p}(\tilde{\Omega})$;
- $\forall \eta \in C^\infty(\Omega) : \eta \cdot u \in W^{k,p}(\Omega)$.

TODO?

Věta 1.3 (Properties of Sobolev spaces)

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $k \in \mathbb{N}$:

1. $W^{k,p}(\Omega)$ *is a Banach space;*
2. *if* $p < \infty$, *then* $W^{k,p}(\Omega)$ *is separable;*
3. *if* $p \in (1, \infty)$, *then* $W^{k,p}(\Omega)$ *is reflexive.*

Důkaz (1.)
„Linear space“ is from Minkowski inequality. „Completeness“: u^n is Cauchy in $W^{k,p}(\Omega)$ $\implies \exists u \in W^{k,p}(\Omega) \implies u^n \to u$ in $L^p(\Omega)$, $D^\alpha u^n \to v_\alpha$ in $L^p(\Omega)$ $\forall |\alpha| \leqslant k$. We must check that „$v_\alpha = D^\alpha u$“:

$$\int_\Omega v_\alpha \eta dx = \int_\Omega (v_\alpha - D^\alpha u^n)\eta + \int_\Omega D^\alpha u^n \eta =$$
$$\overset{IBP}{=} \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|} \int_\Omega u^n D^\alpha \eta = TODO?$$

$$\left| \int_\Omega (v_\alpha - D^\alpha u^n)\eta \right| \leqslant \|\eta\|_\infty \cdot \|v_\alpha - D^\alpha u^n\|_{L^p} \to 0.$$

□

Důkaz (2. + 3. for $W^{1,p}(\Omega)$)
$W^{1,p}(\Omega) = X \subseteq L^p(\Omega) \times \ldots L^p(\Omega)$ ($d+1$ times) and it is closed. □

1.1 Approximation of Sobolev functions

Věta 1.4

$\Omega \subseteq \mathbb{R}^d$ open, bounded. $k \in \mathbb{N}$, $p \in [1, \infty)$. Then

$$\overline{\mathcal{C}^\infty(\Omega)}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

Důkaz
TODO!!! □

Pozor

$$\overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}} \neq W^{k,p}(\Omega).$$

Poznámka
If $\Omega \subset \mathbb{R}^d$ open, connected, then $u = \text{const} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \ \forall_i = 1, \ldots, d$.

$W^{1,1}(I)$, I interval. Then $W^{1,1}(I) \hookrightarrow C(\overline{I})$.

Důkaz

„ $\Longrightarrow$ “: easy. „ $\Longleftarrow$ “: $u_\varepsilon = u * \eta_\varepsilon$, $\Omega_\varepsilon := \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

$$x \in \Omega_\varepsilon : \frac{\partial u_\varepsilon}{\partial x_1}(x) = \left(\frac{\partial u}{\partial x_i}\right)_\varepsilon (x) = 0 \implies u_\varepsilon \equiv \text{const in } \Omega_\varepsilon.$$

Fix $\varepsilon_0 > 0$: $\varepsilon \leqslant \varepsilon_0$: $u_\varepsilon \to u$ in $W^{1,1}(\Omega_{\varepsilon_0}) \implies u \equiv \text{const}$ in Ω_{ε_0}. $u \in W^{1,1}(I)$.

$$\tilde{u}(x) := \int_0^x \frac{\partial u(y)}{\partial y} dy, \qquad \|\tilde{u}(x)\|_\infty \leqslant \int_0^1 |\nabla u| dx.$$

Aim $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}$. $\eta \in C_0^\infty(0,1)$:

$$\int_0^1 \tilde{u}(x) \frac{\partial \eta}{\partial x}(x) dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \eta(x)}{\partial x} \chi_{\{0 \leqslant y \leqslant x\}} dx dy =$$

$$= \int_0^1 \int_y^1 \frac{\partial u}{\partial y}(y) \frac{\partial \eta(x)}{\partial x_i} dx dy =$$

$$= -\int_0^1 \frac{\partial u(y)}{\partial y} \eta(y) dy.$$

$\implies \tilde{u} - u = \text{const} =: c$.

$$|u(x_1) - u(x_2)| = |\tilde{(}x_1) - c - \tilde{u}(x_2) + c| = |\tilde{u}(x_1) - \tilde{u}(x_2)| \leqslant \int_{x_1}^{x_2} \left|\frac{\partial u}{\partial y}\right| dy \to 0.$$

$\implies C(I)$.

„$\|u\|_\infty \leqslant K \cdot \|u\|_1$“:

$$|c| = \int_0^1 |c| = \int_0^1 |\tilde{u}(x) - u(x)| \leqslant \|\tilde{u}\|_\infty + \|u\|_1 \leqslant \|u\|_{1,1}.$$

□

$W^{d,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ (for Lipschitz domain Ω).

1.2 Characterization of Sobolev functions

Věta 1.5

Let $\Omega \subset \mathbb{R}^d$, $p \in [1, \infty]$, $\Omega_d := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \delta\}$. *1. Then*

$$u \in W^{1,p}(\Omega) : \|\triangle_i^n u\|_{L^p(\Omega_\delta)} \leqslant \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)},$$

where $\triangle_i^n u(x) = \frac{u(x+he_i) - u(x)}{h}$.

2. If $\forall h, i, \delta : \|\triangle_i^h u\|_{L^p(\Omega_\delta)} \leqslant c_i$ $(p > 1)$. Then

$$\exists \frac{\partial u}{\partial x_i}, \qquad \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)} \leqslant c_i.$$

Pozor
This works only for $(p > 1)$.

Definice 1.4 (Domains of class $C^{k,\alpha}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ $(\partial\Omega \in C^{k,\mu})$ iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \ldots, x_{r_d}) = (x_r', x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r = \left\{x_r' \in \mathbb{R}^{d-1} | \, |x_{r_i}| \leqslant \alpha\right\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting Tr the orthogonal transformation from (x_r', x_{r_d}) to (x', x_d), then $\forall x \in \partial\Omega$ $\exists r \in \{1, \ldots, M\}$ such that $x = Tr\left(x_{r_1}', a(x_{r_d})\right)$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \left\{(x_r', x_{r_d}) \in \mathbb{R}^d | x_r' \in \Delta_r, a(x_r') < x_{r_d} < a(x_r') + \beta\right\}$$

$$V_r^- := \left\{(x_r', x_{r_d}) \in \mathbb{R}^d | x_r' \in \Delta_r, a(x_r') - \beta < x_{r_d} < a(x_r')\right\}$$

$$\Lambda_r := \left\{(x_r', x_{r_d}) \in \mathbb{R}^d | x_r' \in \Delta_r, a(x_r') = x_{r_d}\right\}$$

  Then $Tr(V_r^+) \subset \Omega, Tr(V_r^-) \subset \mathbb{R}^d \backslash \overline{\Omega}, Tr(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M Tr(\Lambda_r) = \partial\Omega$.

Věta 1.6 (Density)

Let $\Omega \in C^{0,1}$ and $p \neq \infty$, then $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}$.

Věta 1.7 (Extension)

Let $\Omega \in C^{0,1}$, $k \in \mathbb{N}$, $p \in [1, \infty]$. Then $\exists$ continuous bounded operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ such that

1. $\|Eu\|_{W^{k,p}}(\mathbb{R}^d) \leqslant c \cdot \|u\|_{W^{k,p}(\Omega)}$ *(Eu has compact support);*
2. $Eu = u$ *almost everywhere in* Ω.

Věta 1.8 (Trace)

Let $\Omega \in C^{0,1}$, $p \in [1, \infty]$. Then $\exists$ continuous bounded operator $\mathrm{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that:*

1. $\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leqslant c \cdot \|u\|_{W^{1,p}(\Omega)}$;

2. $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \implies \operatorname{tr} u = u|_{\partial\Omega}$.

Definice 1.5

$W_0^{k,p}(\Omega) = \overline{u \in C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}$.

Poznámka

$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) | \operatorname{tr} u = 0\}$.