1 Meromorphic functions

Definice 1.1

We say that a function f is holomorphic in a set $F \subset \mathbb{C}$ if there is an open $G \supseteq F$ such that f is holomorphic on G.

In particular, f is holomorphic at $z_0 \in \mathbb{C}$ if f is holomorphic in some neighbour ($= U(z_0) = U(z_0, \varepsilon)$) of z_0.

Definice 1.2

Function f has at ∞ a removable singularity, if $f\left(\frac{1}{z}\right)$ has a removable singularity at 0. Similarly pole and essential singularity.

Function f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Let $G \subset \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any z_0. Denote $\mathcal{H}(G) := \{f : G \to \mathbb{C} | f \text{ holomorphic}\}$.

Například
From Liouville theorem $\mathbb{H}(\mathbb{S}) = $ constant functions. So $\mathbb{H}(G)$ is interesting only for $G \subsetneq \mathbb{S}$, so WLOG $G \subset \mathbb{C}$.

Definice 1.3 (Meromorphic function)

Let $G \subset \mathbb{S}$ be open. Then a function f on G is called meromorphic if at any $z_0 \in G$ the function f is either holomorphic at z_0 or has a pole at z_0.

Denote $\mathcal{M}(G)$ the set of meromorphic functions on G.

Důsledek

- $\mathcal{H}(G) \subset \mathcal{M}(G)$.

- Denote $P_f := \{z_0 \in G | f \text{ has a pole at } z_0\}$. Then P_f has no limit points in G.

- If $f = \infty$ on P_f, then $f : G \to \mathbb{S}$ is continuous. (We always assume, that $f \in \mathcal{H}(G)$ has this property.)

Například

$$\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), \qquad e^{\frac{1}{z}} \notin \mathcal{M}(\mathbb{C}), \qquad \Gamma \in \mathcal{M}(\mathbb{C}), \qquad \zeta \in \mathcal{M}(\mathbb{C}).$$

$\mathcal{M}(\mathbb{S})$ = rational functions. (One inclusion is clear, second: Let $f \in \mathcal{M}(\mathbb{S})$, then because $\mathbb{S}$ is compact it holds that P_f is finite (has no limit point), $P_f \cap \mathbb{C} = \{z_1, \ldots, z_n\}$, so from theorem from last semester there exists $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^{n} p_j \left(\frac{1}{z - z_j}\right)$ for some polynomials p_j. f has removable singularity or pole at infinity and p_j and $\frac{1}{z-z_j}$ have removable singularity there, so $h(z)$ is polynomial, otherwise $h(z)$ has infinity Taylor polynom and $h\left(\frac{1}{z}\right)$ has essential singularity at 0.)

So $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subset \mathbb{C}$.

If $G \subset \mathbb{C}$ is domain, $f, g \in \mathbb{H}(G)$ and $g \not\equiv 0$, then $f/g \in \mathcal{M}(G)$. The inverse is also true (we will prove it) (but not for $G = \mathbb{S}$).

Lemma 1.1

Let $\mathbb{G} \subset \mathbb{C}$ be open. Then there are compacts K_n, $n \in \mathbb{N}$, in G such that $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \operatorname{int}(K_{n+1})$ and for any compact K in G, $\exists n \in \mathbb{N} : K \in K_n$.

Důkaz
Set $K_n := \left\{ z \in G \,|\, \operatorname{dist}(z, \mathbb{C} \backslash G) \geq \frac{1}{n} \right\} \cap U(0, n).$ □

Tvrzení 1.2

Let $G \subset \mathbb{S}$ be open and $M \subset G$ has no limit point in G. Then

- *$G \backslash M$ is open;*

- *if K is a compact in G, then $K \cap M$ is finite. In particular for $G = \mathbb{S}$ we have M is finite;*

- *M is at most countable. If M is infinite, then $\varnothing \neq M' \subset \partial G$;*

- *if $G \subset \mathbb{C}$ is domain (connected), then $G \backslash M$ is domain.*

Věta 1.3 (Uniqueness of meromorphic functions)

Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ and $f \not\equiv 0$. Then $N_f := \{z \in G | f(z) = 0\}$ has no limit points in G.

Věta 1.4 (Residue theorem)

Let $G \subset \mathbb{C}$ be open, φ be a closed curve (or cycle) in G and

$$\operatorname{int} \varphi := \{z_0 \in \mathbb{C} \backslash \langle \varphi \rangle \mid \operatorname{ind}_\varphi z_0 \neq 0\} \subset G.$$

Let $M \subset G \backslash \langle \varphi \rangle$ be finite and $f \in \mathcal{H}(G \backslash M)$. Then $\int_\varphi f = 2\pi i \cdot \sum_{s \in M} \operatorname{ind}_\varphi s \cdot \operatorname{res}_s f$.

> *Poznámka*
> This holds true even if instead of finiteness of M, we assume only that $M \subset G \backslash \langle \varphi \rangle$ has no limit points in G. Indeed, we have $M_0 = M \cap \operatorname{int} \varphi$ is finite, because $\langle \varphi \rangle \cup \operatorname{int} \varphi$ is compact and $G_0 := G \backslash (M \backslash M_0)$ is open and f is holomorphic on $G_0 \backslash M_0$ and by R. theorem for G_0 and M_0 we get $\int_\varphi f = 2\pi i \sum_{s \in M_0} \operatorname{res}_s f \cdot \operatorname{ind}_\varphi s$.

1.1 Logarithmic integrals

Definice 1.4 (Logarithmic integral)

Let $\varphi : [a, b] \to \mathbb{C}$ be a (regular) curve and let f be a non-zero holomorphic function on $\langle \varphi \rangle$. Then we define logarithmic integrals integral as

$$I := \frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \frac{1}{2\pi i} \int_a^b \frac{f'(\varphi(t))\varphi'(t)}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))} dt = \frac{1}{2\pi i} \int_{f \circ \varphi} \frac{dz}{z} =$$

$$= \frac{1}{2\pi i}(\Phi(b) - \Phi(a)),$$

where Φ is a branch (jednoznačná větev) of logarithm of $f \circ \varphi$. If φ is, in addition, closed, then $I = \operatorname{ind}_{f \circ \varphi} 0 = \frac{1}{2\pi}(\Theta(b) - \Theta(a)) \in \mathbb{Z}$, where Θ is a branch of argument of $f \circ \varphi$.

($\frac{f'}{f}$ is called logarithmic derivative of f, because '$(\log f)' = \frac{f'}{f}$'.)

Věta 1.5 (Argument principle)

Let $G \subseteq \mathbb{C}$ be a domain, φ be a closed curve in G and $f \in \mathcal{M}(G)$. Let $\operatorname{int} \varphi \subset G$ and $\langle \varphi \rangle \cap N_f = \varnothing$, $\langle \varphi \rangle \cap P_f = \varnothing$. Then

$$\frac{1}{2\pi i} \int_\varphi \frac{f'}{f} = \sum_{s \in \operatorname{int} \varphi, f(s) = 0} n_f(s) \cdot \operatorname{ind}_\varphi s - \sum_{s \in \operatorname{int} \varphi, f(s) = \infty} p_f(s) \cdot \operatorname{ind}_\varphi s,$$

where $n_f(s)$ is multiplicity of the zero point s of f and $p_f(s)$ is multiplicity of the pole s of f.

Důkaz

By Residua theorem, we have

$$\frac{1}{2\pi i} \int_{\varphi} \frac{f'}{f} = \sum_{s \in \text{int}\,\varphi, s \in N_f \cup P_f} \text{res}_s \left(\frac{f'}{f} \right) \cdot \text{ind}_{\varphi} s.$$

If $s \in N_f$ then on $P(s)$:

$$\frac{f'(z)}{f(z)} = \frac{p \cdot c_p (z-s)^{p-1} + \ldots}{c_p (z-s)^p + \ldots} = \frac{p}{z-s} \cdot \frac{1 + \ldots}{1 + \ldots} \implies \text{res}_s \left(\frac{f'}{f} \right) = p = n_f(s).$$

If $s \in P_f$ then on $P(s)$

$$\frac{f'(z)}{f(z)} = \frac{p \cdot c_p (z-s)^{p-1} + \ldots}{c_p (z-s)^p + \ldots} = \frac{p}{z-s} \cdot \frac{1 + \ldots}{1 + \ldots} \implies \text{res}_s \left(\frac{f'}{f} \right) = p = -p_f(s).$$

$\square$

Definice 1.5

$$\Sigma(f, \varphi) := \sum_{s \in \text{int}\,\varphi, f(s)=0} n_f(s) \cdot \text{ind}_{\varphi} s - \sum_{s \in \text{int}\,\varphi, f(s)=\infty} p_f(s) \cdot \text{ind}_{\varphi} s.$$

Lemma 1.6

Let $\varphi_1, \varphi_2 : [a, b] \to \mathbb{C}$ be closed curve and $s \in \mathbb{C} \backslash (\langle \varphi_1 \rangle \cup \langle \varphi_2 \rangle)$. Assume, for $t \in [a, b]$, $|\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - s|$. Then $\text{ind}_{\varphi_1} s = \text{ind}_{\varphi_2} s$.

Důkaz

For $t \in [a, b]$, we have $|(\varphi_1(t) - s) - (\varphi_2(t) - s)| < |\varphi_1(t) - s|$. Divide by $|\varphi_1(t) - s|$:

$$|1 - \psi(t)| < 1, \qquad \psi(t) := \frac{\varphi_2(t) - s}{\varphi_1(t) - s}.$$

Then ψ is a closed curve, $< \psi > \subset U(1, 1)$, and so

$$0 = \text{ind}_{\psi} 0 = \frac{1}{2\pi i} \int_a^b \frac{\psi'}{\psi} = \frac{1}{2\pi i} \int_a^b \frac{\frac{\varphi_2'(\varphi_1 - s) - \varphi_1'(\varphi_2 - s)}{(\varphi_1 - s)^2}}{\frac{\varphi_2 - s}{\varphi_1 - s}} = \frac{1}{2\pi i} \int_a^b \frac{\varphi_2'}{\varphi_2 - s} - \frac{1}{2\pi i} \int_a^b \frac{\varphi_1'}{\varphi_1 - s} =$$

$$= \text{ind}_{\varphi_2} s - \text{ind}_{\varphi_1} s.$$

$\square$

Věta 1.7 (Rouché)

Let $G \subset \mathbb{C}$ be a domain, $f_1, f_2 \in \mathcal{M}(G)$ and φ be closed curve in G such that $\text{int}\,\varphi \subset G$. Assume $\forall z \in \langle \varphi \rangle$:
$$|f_1(z) - f_2(z)| < |f_1(z)| < +\infty$$

Then $\Sigma(f_1, \varphi) = \Sigma(f_2, \varphi)$.

Důkaz

Set $\varphi_j = f_j \circ \varphi$. Then

$$\mathrm{ind}_{\varphi_j} 0 = \frac{1}{2\pi i} \int_\varphi \frac{f_j'}{f_j} = \Sigma(f_j, \varphi).$$

By previous lemma we have for $s = 0$: $\mathrm{ind}_{\varphi_1} 0 = \mathrm{ind}_{\varphi_2} 0$. □

Důsledek

Let f_1, f_2 be holomorphic functions on $\overline{U(z_0, r)}$ and $\forall z \in \partial U(z_0, r) : |f_1(z) - f_2(z)| < |f_1(z)|$. Then $\Sigma_1 = \Sigma_2$, where $\Sigma_j := \sum_{s \in U(z_0, r), f(s) = 0} n_{f_j}(s)$.

Důkaz

Apply Rouché's theorem to $\varphi(t) := z_0 + r \cdot e^{it}$, $t \in [0, 2\pi]$. □

Příklad

$f_2 = p$, $f_1(z) = a_0 z^n$ and big enough $U(0, r)$.

Definice 1.6 (Notation)

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. We say that $f(z_0) = w_0 \in \mathbb{C}$ p times for $p \in \mathbb{N}$ if z_0 is a zero point of $f - w_0$ of order p.

Poznámka

Following statements are equivalent to each other:

- $f(z_0) = w_0$ p times;

- $f(z_0) = w_0$, $f'(z_0) = 0 = \ldots = f^{(p-1)}(z_0)$, $f^{(p)}(z_0) \neq 0$;

- $f(z) = w_0 + \sum_{k=p}^{+\infty} c_k (z - z_0)^k$ on some neighbourhood of z_0 and $c_p \neq 0$.

We say that $f(z_0) = \infty$ p times if z_0 is a zero point of $\frac{1}{f}$ of order p. (It's the same as z_0 is pole of f of order p.) And we say that $f(\infty) = w_0 \in \mathbb{S}$ p times if $f(1/z)$ attains w_0 p times at 0.

Věta 1.8 (On a multiple value)

Let $z_0, w_0 \in \mathbb{S}$, f be a holomorphic function on a $P(z_0)$ and $f(z_0) = w_0$ p times for some $p \in \mathbb{N}$. Let $\delta_0 > 0$. Then there are $\varepsilon > 0$ and $\delta \in (0, \delta_0)$ such that, for any $w \in P(w_0, \varepsilon)$ there are just p different points $z_1, \ldots, z_p$ in $P(z_0, \delta)$ with $f(z_j) = w$. In addition, $f(z_j) = 0$ once.

Důkaz

WLOG, assume $z_0 = 0 = w_0$. Then $z_0 = 0$ is a zero point of f of order p. Choose $\delta \in (0, \delta_0)$ such that $f \neq 0$ and $f' \neq 0$ on $P(0, 2\delta)$. Set $\varepsilon := \min_{|z|=\delta} |f(z)| > 0$.

Let $w \in P(0, \varepsilon)$. Use Rouché's theorem for $f_1 := f$, $f_2 := f - w$ and $\varphi := \delta e^{it}$, $t \in [0, 2\pi]$. Of course, $|f_1 - f_2| = |w| < \varepsilon < |f_1|$ on $\langle \varphi \rangle$.

Since in $U(0, \delta)$ the function $f = f_1$ has the only zero point of order p at origin, $f - w = f_2$ has just p simple zero points in $P(0, \delta)$. □

Důsledek

Let $G \subset \mathbb{S}$ be a domain, $f \in \mathcal{M}(G)$ and f be not constant on G. Then $f : G \to \mathbb{S}$ is an open map (for any open $\Omega \subset G$, $f(\Omega)$ is open).

Důkaz

Let $\Omega \subset G$ be open and $w_0 \in f(\Omega)$. Then there is a $z_0 \in \Omega$ and $p \in \mathbb{N}$ such that $f(z_0) = w_0$ p times. Choose $\delta_0 > 0$ such that $U(z_0, \delta_0) \subset \Omega$. By the previous theorem, there is $\varepsilon > 0$, $\delta \in (0, \delta_0)$ such that $P(w_0, \varepsilon) \subset f(P(z_0, \delta))$, so $U(w_0, \varepsilon) \subset f(U(z_0, \delta)) \subset f(\Omega)$. □

Poznámka

This is true for $\mathcal{H}(G)$ too.

Důsledek

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. Then $f'(z_0) \neq 0$ if and only if there is $U(z_0)$ such that $f|_{U(z_0)}$ is one-to-one.

Důkaz

„$\Longrightarrow$": Let $f'(z_0) \neq 0$. Then $f(z_0) = w_0$ once, so we choose $\delta_0 > 0$ such that $f \neq w_0$ on a $P(z_0, \delta_0)$. By the previous theorem choose $\varepsilon > 0$, $\delta \in (0, \delta_0)$. Moreover, due to the continuity of f at z_0 choose $\delta_1 \in (0, \delta)$ such that $f(U(z_0, \delta_1)) \subset U(w_0, \varepsilon)$. Then $f|_{U(z_0, \delta_1)}$ is one-to-one.

„$\Longleftarrow$": Let $f'(z_0) = 0$ and let f be not constant on any neighbourhood of z_0. Then $f(z_0) = w_0$ p times ($p \in \mathbb{N} \setminus \{1\}$). By the previous theorem f is not one-to-one on any neighbourhood of z_0. □

Věta 1.9 (On holomorphic inverse)

Let $G \subset \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be a one-to-one holomorphic[a] function, then $f' \neq 0$ on G, $\Omega := f(G)$ is open and $f_{-1} : \Omega \overset{onto}{\to} G$ is holomorphic.

In addition, $(f_{-1})' = \frac{1}{f' \circ f_{-1}}$ on Ω.

Důkaz

WLOG, $G \subset \mathbb{C}$ is a domain. By first „dusledek" of previous theorem f is an open map, so $\Omega := f(G)$ is open and $f_{-1} : \Omega \to G$ is continuous. Let $z_0 \in G$ and $w_0 = f(z_0)$. By second „dusledek" we have $f'(z_0) \neq 0$, and

$$\frac{1}{f'(z_0)} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} \stackrel{*}{=} \lim_{w \to w_0} \frac{f_{-1}(w) - f_{-1}(w_0)}{w - w_0} = f'_{-1}(w_0).$$

The equality * follows from theorem on limits of composite functions because f_{-1} is continuous and $f_{-1}(w) \neq f_{-1}(w_0)$ for $w \neq w_0$. □

---
^{a}One-to-one holomorphic function is sometimes called conformal.

Věta 1.10 (Hurwitz)

Let $G \subset \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(G)$, $f_n \stackrel{loc.}{\rightrightarrows} f$ on G and $f \not\equiv 0$. Let $z_0 \in G$ be a zero point of f. Then $\exists \{z_n\}_{n=1}^{\infty} \subset G$ and a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $z_n \to 0$ and $f_{k_n}(z_n) = 0$.

Poznámka

Not true in $\mathbb{R}$! The assumption $f \not\equiv 0$ is important! ($f_n(z) := z/n$)

Důsledek

Let $G \subset \mathbb{C}$ be a domain, f_n be one-to-one holomorphic functions on G and $f_n \stackrel{loc}{\rightrightarrows} f$ on G. Then f is either one-to-one and holomorphic, or constant.

Důkaz (Hurwitz theorem)

Choose $\delta > 0$ such that $U(z_0, \delta) \subset G$ and $f \neq 0$ on $P(z_0, \delta)$. For $n \in \mathbb{N}$ put $\varrho_n := \frac{\delta}{n+1}$ and $\varphi_n(t) := z_0 + \varrho_n e^{it}$, $t \in [0, 2\pi]$. Of course, $\tau_n := \min_{\langle \varphi_n \rangle} |f| > 0$. For a given n, there is (from uniformly convergence) $k_n \in \mathbb{N}$ such that $\forall z \in \langle \varphi_n \rangle : |f_{k_n}(z) - f(z)| < \tau_n \leqslant |f|$.

By Rouché's theorem there is $z_n \in U(z_0, \varrho_n)$ such that $f_{k_n}(z_n) = 0$. Of course, we can choose $\{k_n\}$ to be increasing. □

Důkaz (Corollary)

Assume that there is $w_0 \in \mathbb{C}$ such that $f \neq w_0$ but, for different $z', z'' \in G$ we have $f(z') = w_0 = f(z'')$. WLOG $w_0 = 0$. Choose $\delta > 0$ such that $U(z', \delta) \cap U(z'', \delta) = \varnothing$. By Hurwitz, there are $\{z'_n\} \subset U(z', \delta)$ and $\{f_{k'_n}\}$ of $\{f_n\}$ such that $z'_n \to z'$ and $f_{k'_n}(z'_n) = 0$. By Hurwitz, there are also $\{z''_n\} \subset U(z'', \delta)$ and $\{f_{k''_n}\} \subset \{f_{k'_n}\}$ such that $z''_n \to z''$ and $f_{k''_n}(z''_n) = 0$.

Every $f_{k''_n}$ has at least two different zero points which is contradiction. □

Věta 1.11 (Mittag-Leffler)

Let $\{s_j\} \subset \mathbb{C}$ be one-to-one, $s_j \to \infty$ and $s_0 := 0 < |s_1| \leqslant |s_2| \leqslant |s_3| \leqslant \ldots \leqslant |s_j| \leqslant \ldots$

Let $P_0, P_1, \ldots, P_j, \ldots$ be polynomials such that $P_j(0) = 0$. Then the function

$$f(z) := P_0\left(\frac{1}{z}\right) + \sum_{j=1}^{\infty} \left(P_j\left(\frac{1}{z - s_j}\right) - Q_j(z) \right)$$

for some polynomials Q_j satisfies:

1. *series in definition converges locally uniformly on $\mathbb{C}$, i. e., on any compact $K \subset \mathbb{C}$, the series converges uniformly if we omit finitely many terms which have poles.*

2. *$f \in \mathcal{M}(\mathbb{C})$ and f has poles just at $s_0, s_1, \ldots, s_j, \ldots$, while at s_j the function f has its principal part equal to $P_j\left(\frac{1}{z - s_j}\right)$.*

3. *If $g \in \mathcal{M}(\mathbb{C})$ satisfies previous property, then there is $h \in \mathcal{H}(\mathbb{C})$ such that $g = f + h$ on G.*

Důkaz
Let $k \in \mathbb{N}$. Then $H_k(z) := P_k\left(\frac{1}{z - s_k}\right) \in \mathcal{H}(U(0, |s_k|))$, $H_k(z) = \sum_{n=0}^{\infty} c_n^k z^n$ for $|z| < |s_k|$. There is $n_k \in \mathbb{N}$ such that $Q_k(z) = \sum_{n=1}^{n_k} c_n^k z^n$ satisfies $|H_k(z) - Q_k(z)| < \frac{1}{2^k}$, $|z| \leqslant \frac{|s_k|}{2}$ (*).

Let $K \subset \mathbb{C}$ be a compact. Choose $k_0 \in \mathbb{N}$ such that $K \subset \overline{U(0, |s_{k_0}|/2)}$. If $k > k_0$, (*) holds on K which implies 1. obviously, 2. is valid.

3. follow from the fact that $g - f \in \mathcal{M}(\mathbb{C})$ has all isolated singularities removable. □

2 Zero points of holomorphic functions

Tvrzení 2.1

Let f be non-zero holomorphic function on a simply connected domain (G is domain, and $\mathbb{S} \backslash G$ is connected) $G \subset \mathbb{C}$. Then there is $L \in \mathcal{H}(G)$ such that $f = e^L$ on G.

Důkaz
1) Let $L \in \mathcal{H}(G)$ and $f = e^L$ on G. Then $f' = L' \cdot e^L$ and $f'/f = L'$.

2) Since G is a simply connected domain and $f'/f \in \mathcal{H}(G)$, by Cauchy theorem, there is $L_0 \in \mathcal{H}(G)$ such that $L_0' = f'/f$.

3) On G we have $\left(f \cdot e^{-L_0}\right)' = e^{-L_0} \cdot (f' - L_0' \cdot f) = 0$ on G, hence $f \cdot e^{-L_0} = e^c$ is constant, i. e. $c \in \mathbb{C}$. Put $L := L_0 + c$. □

Poznámka

Polynomial $f(z) = \prod_{j=1}^{n}(z - z_j)$ has zero points just at $z_1, \dots, z_n$ and their multiplicity corresponds to their occurrence.

Let $g \in \mathcal{H}(\mathbb{C})$ have the same zero points including multiplicity as f. Then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on $\mathbb{C}$. (Proof: use previous tvrzeni for g/f.)

Poznámka (Notation)

Let $\{a_j\} \subset \mathbb{C}$. Then we define

$$\prod_{j=1}^{\infty} a_j := \lim_{n \to \infty} \prod_{j=1}^{n} a_j,$$

if the limit on the right-hand side exists.

Tvrzení 2.2

Let $0 \neq z_j \to \infty$ and $k \in \mathbb{N}_0$ (multiplicity of 0 as zero point). Then consider

$$f(z) := z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right).$$

It sometimes converges and then f has zero points in z_j with right multiplicities.

Věta 2.3 (On infinite product)

Let M be a set (in $\mathbb{C}$), $u_j : M \to \mathbb{C}$ be bounded and $\sum_{j=1}^{\infty} |u_j|$ converges uniformly on M. Then $p_n := \prod_{j=1}^{n}(1 + u_j)$ converge uniformly to a function $f : M \to \mathbb{C}$, and it holds that $f = \prod_{j=1}^{\infty}(1 + u_{n(j)})$ on M, where n is bijection onto $\mathbb{N}$.

If $z_0 \in M$, then $f(z_0) = 0$ if and only if $u_{j_0}(z_0) = -1$ for some $j_0 \in \mathbb{N}$.

Důkaz

Denote $p_n^* := \prod_{j=1}^n (1 + |u_j|)$. Then $p_n^* \leqslant \exp\left(\sum_{j=1}^n |u_j|\right)$ and $|p_n - 1| \leqslant p_n^* - 1$ (from $1 + x \leqslant e^x$ and the second inequality by induction on n: $n = 1$ yes, $p_{n+1} - 1 = p_n(1 + u_{n+1}) - 1 = (p_n - 1) \cdot (1 + u_{n+1}) + u_{n+1}$ so $|p_{n+1} - 1| \leqslant (p_n^* - 1) \cdot (1 + |u_{n+1}|) + |u_{n+1}| = p_{n+1}^* - 1$).

$\sum_{j=1}^\infty |u_j|$ is bounded on M, because there is $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^\infty |u_j| < 1$. By inequalities there is $C \in (0, +\infty)$ such that $|p_n| \leqslant C \ \forall n \in \mathbb{N}$.

Let $0 < \varepsilon < \frac{1}{2}$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty |u_n| < \varepsilon$ on M. Let $\{n_1, n_2, \ldots\}$ be a permutation of $\mathbb{N}$ and $q_m := \prod_{j=1}^m (1 + u_{n_j})$, $m \in \mathbb{N}$. Let $n \geqslant n_0$ and $m \in \mathbb{N}$ be such that $\{n_1, \ldots, n_m\} \supseteq [n]$. Then

$$|q_m - p_n| = \left|p_n \cdot \left(\prod_{n_j > n, j \in [m]} (1 + u_{n_j}) - 1\right)\right| \leqslant |p_n| \left(\prod_{\ldots} (1 + |u_{n_j}|) - 1\right) \leqslant$$

$$\leqslant |p_n| \cdot \left(e^{\sum_{\ldots} |u_{n_j}|} - 1\right) \leqslant |p_n| \cdot (e^\varepsilon - 1) \leqslant |p_n| \cdot 2\varepsilon \leqslant 2C\varepsilon.$$

If $n_j = j \ \forall j \in \mathbb{N}$, then $q_m = p_m$ and we get $\forall m > n : |q_m - p_n| < 2C\varepsilon$, so $p_n \rightrightarrows f$ on M. Moreover we have, for $n \geqslant n_0$, $|p_n - p_{n_0}| \leqslant 2\varepsilon |p_{n_0}|$, so $|p_n| \geqslant |p_{n_0}| - |p_n - p_{n_0}| \geqslant (1 - 2\varepsilon)|p_{n_0}|$. For $n \to \infty$: $|f| \geqslant (1 - 2\varepsilon)|p_{n_0}|$, hence $f(z_0) = 0 \Leftrightarrow p_{n_0}(z_0) = 0$.

If n_j is any, then $q_m \rightrightarrows f$ on M. □

Důsledek

Let $G \subset \mathbb{C}$ be open, $f_n \in \mathcal{H}(G)$ and $f_n \not\equiv 0$ on any component of G. We assume $\sum_{n=1}^\infty |1 - f_n|$ converges locally uniformly on G. Then $f = \prod_{n=1}^\infty f_n$ converges locally uniformly on G, $f \in \mathcal{H}(G)$ and the resulting infinite product f does not depend on the order of functions f_n. Moreover, we have

$$n_f(s) = \sum_{k=1}^\infty n_{f_k}(s), \qquad s \in G$$

where $n_f(s)$ is multiplicity of a zero point s of f. Here we put $n_f(s) = 0$ if $f(s) \neq 0$.

Poznámka

Moreover the ? in previous sum contains only finitely many non-zero terms for any $s \in G$.

Důkaz

Sufficient to prove previous equality. Let $s \in G$. There is a neighbourhood V of s such that $f_n \rightrightarrows 1$ on V. Choose $n_0 \in \mathbb{N}$ such that $f_n \neq 0$ on V for $n > n_0$. By previous theorem, we get $\prod_{n=n_0+1}^\infty f_n \neq 0$ on V. Since $f = \left(\prod_{n=1}^{n_0} f_n\right) \cdot \left(\prod_{n=n_0+1}^\infty f_n\right)$ we get $n_f(s) = \sum_{k=1}^{n_0} n_{f_k}(s) = \sum_{k=1}^\infty n_{f_k}(s)$. □

Příklad (Homework)

Under the assumption of previous corollary prove that

$$\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f_n'}{f_n} \text{ on } G\backslash N_f.$$

Například (Euler formula)

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 2.4 (Weierstrass's factor)

Let $E_0(z) := (1-z)$ and $E_m(z) := (1-z) \cdot e^{z+\dots+\frac{z^m}{m}}$, $z \in \mathbb{C}$, $m \in \mathbb{N}$. Then $|1 - E_m(z)| \leqslant |z|^{m+1}$, $|z| \leqslant 1$.

$\ulcorner$

  Důkaz

$$E_m'(z) = e^{z+\dots+\frac{z^m}{m}} \cdot (-1 + (1-z) \cdot (1 + \dots + z^m)) = -z^m \cdot e^{z+\dots+\frac{z^m}{m}} = -z^m \cdot \sum_{k=0}^{\infty} b_k z^k,$$

where $b_0 = 1$, $b_k \geqslant 0$, $k \in \mathbb{N}$. Hence

$$E_m(0) - E_m(z) = 1 - E_m(z) = -\int_{[0,z]} E_m'(w)dw = +\sum_{k=0}^{\infty} c_k z^{k+m+1}$$

with $c_k = \frac{b_k}{m+k+1} \geqslant 0$.

  By this, if $|z| \leqslant 1$, $z \neq 0$, then $\left|\frac{1-E_m(z)}{z^m}\right| \leqslant \sum_{k=0}^{\infty} c_k = 1 - E_m(1) = 1.$ $\square$

$\llcorner$

Věta 2.5 (Weierstrass factorization in $\mathbb{C}$)

Let $k \in \mathbb{N}_0$ and $0 \neq z_i \to \infty$. Then there is $\{m_j\} \subset \mathbb{N}_0$ such that

$$f(z) = z^k \cdot \prod_{j=1}^{\infty} E_{m_j}\left(\frac{z}{z_j}\right)$$

converges locally uniformly on $\mathbb{C}$, $f \in \mathcal{H}(\mathbb{C})$ and f has at 0 zero point of multiplicity K and 'non-zero' zero points just at $z_1, z_2, \dots, z_j, \dots$, and their multiplicity corresponds to their occurrence in $\{z_j\}$. We can always take $m_j := j - 1$, $j \in \mathbb{N}$.

If $g \in \mathcal{H}(\mathbb{C})$ has the same zero points as f including multiplicities, then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on $\mathbb{C}$.

By the previous corollary, we know the product converges locally uniformly in $\mathbb{C}$ if

$$\sum_{j=1}^{\infty} \left| 1 - E_{m_j} \left(\frac{z}{z_j} \right) \right|$$

converges locally uniformly on $\mathbb{C}$. By lemma, this is true if $\sum_{j=1}^{\infty} \left| \frac{z}{z_j} \right|^{m_j+1}$ converges locally uniformly on $\mathbb{C}$.

Let $r > 0$ and $|z| \leqslant r$. Choose $j_0 \in \mathbb{N}$ such that $\frac{r}{|z_j|} < \frac{1}{2}$ for $j \geqslant j_0$. If $m_j := j - 1$, then $\left| \frac{z}{z_j} \right|^j \leqslant \frac{1}{2^j}$, $j \geqslant j_0$ and $|z| \leqslant r$. So, for $m_j := j - 1$, sum converges uniformly on $|z| \leqslant r$. □

Poznámka
If $\sum_{j=1}^{\infty} \frac{1}{|z_j|} < +\infty$, take $m_j = 0$. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|^2} < +\infty$, take $m_j = 1$. Etc.

Věta 2.6 (Weierstrass factorization in a general open set)

Let $G \subsetneq \mathbb{S}$ be open, $N \subset G$ have no limit points in G and $n : N \to \mathbb{N}$. Then there is $f \in \mathcal{H}(G)$ such that $N_f = N$ and $n_f(s) = n(s)$, $s \in N_f$.

WLOG $\infty \in G \backslash N$. Then $K := \mathbb{S} \backslash G = \mathbb{C} \backslash G$ is compact in $\mathbb{C}$. For a finite N it is obvious. Assume that N is (infinite) countable. We put points of N into the sequence $s_1, s_2, \ldots, s_n$ such that any $s \in N$ occurs in $\{s_n\}$ just $n(s)$ times. For any n, take $t_n \in K$ such that $|s_n - t_n| = \text{dist}(s_n, K)$, $n \in \mathbb{N}$.

Then „$|s_n - t_n| \to 0$": Let $\varepsilon > 0$ and $\{n_k\} \subset \mathbb{N}$ such that $|s_{n_k} - t_{n_k}| \geqslant \varepsilon$, i. e., $\text{dist}(s_{n_k}, K) \geqslant \varepsilon$. If s_∞ is a limit point of s_{n_k}, then $\text{dist}(s_\infty, K) \geqslant \varepsilon$. Hence $s_\infty \in G$, a contradiction.

Put $f(z) := \prod_{n=1}^{\infty} E_n \left(\frac{s_n - t_n}{z - t_n} \right)$, $z \in G$. The infinite product converges locally uniformly on G. In fact, let L be a compact in G. Put $r_n := 2 \cdot |s_n - t_n|$. Since $\text{dist}(L, K) > 0$, there is $n_0 \in \mathbb{N}$ such that $|z - t_n| > r_n$, $\forall z \in L$, $\forall n \geqslant n_0$. So

$$\left| \frac{s_n - t_n}{z - t_n} \right| < \frac{1}{2} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

By lemma on Weierstrass factors, we get

$$\left| 1 - E_n \left(\frac{s_n - t_n}{z - t_n} \right) \right| < \frac{1}{2^n} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

Now use theorem on infinite product. □

Lemma 2.7

If $G \subseteq \mathbb{C}$ is open and $f \in \mathcal{M}(G)$, then there are $g, h \in \mathcal{H}(G)$ such that $f = \frac{g}{h}$ on G.

Důkaz
Let P_f be the set of poles of f. By Weierstrass factorization, we construct $h \in \mathcal{H}(G)$ such that $N_h = P_f$ and $n_h = p_f$ on P_f. Put $g := f \cdot h$. Then $g \in \mathcal{H}(G)$ because at the points of P_f g has a removable singularities. □

3 The space H(G)

Poznámka (Arzela–Ascoli theorem)
Let $\mathcal{F} \subset \mathcal{C}(K)$ and let the functions of $\mathcal{F}$ be equibounded (i.e. $\exists M \in (0, +\infty) \ \forall f \in \mathcal{F} : |f| \leqslant M$ on K) and equicontinuous (i.e. $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall f \in \mathcal{F} \ \forall x, y \in K : \varrho(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$, where ϱ is metric on K). Then every $\{f_n\} \subset \mathcal{F}$ has $\{f_{n_k}\}$ which is uniformly convergent on K.

3.1 The space C(G)

Definice 3.1

Let $G \subseteq \mathbb{C}$, then $\mathcal{C}(G) := \{f : G \to \mathbb{C} \mid f \text{ continuous}\}$.

Tvrzení 3.1

For $f_n, f \in \mathcal{C}(G)$ and K_m compact in G such that $\bigcup_{m=1}^{\infty} K_m = G$ and $\forall m \in \mathbb{N} : K_m \subseteq \text{int } K_{m+1}$, TSAE:

- *$f_n \overset{loc.}{\rightrightarrows}$ on G;*

- *for any compact K in G, $\|f_n - f\| \to 0$, where $\|f\|_K := \sup_K |f|$ is a seminorm on $\mathcal{C}(G)$;*

- *$\forall m \in \mathbb{N} : \|f_n - f\|_{K_m} \to 0$ for $u \to \infty$;*

- *$\varrho(f_n, f) \to 0$, where $\varrho(f_n, f) := \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|f_n - f\|_{K_m}}{1 + \|f_n - f\|_{K_m}}$.*

Důkaz
„1 ⇔ 2 $\implies$ 3" is obvious. „2 $\impliedby$ 3": Let K be a compact in G. Then $K \subset K_{m_0}$ for come $m_0 \in \mathbb{N}$. Then $\|f_n - f\|_K \leqslant \|f_n - f\|_{K_{m_0}}$. „3 ⇔ 4" homework. □

Věta 3.2 (Moore–Osgood, Montöl)

Let $G \subset \mathbb{C}$ be open and let $\{f_n\} \subset \mathcal{H}(G)$ be locally equibounded (i.e. on every compact K in G $\{f_n\}$ is equibounded). Then there is $\{f_{n_k}\}$ which converges locally uniformly on G.

Důkaz

First step: Let $\overline{U(z_0, 2r)} \subset G$ and $\varphi(t) := z_0 + 2re^{it}$, $t \in [0, 2\pi]$. Let $z_1, z_2 \in \overline{U(z_0, r)}$. Then by the Cauchy formula we get $f_n(z_j) = \frac{1}{2\pi i} \int_\varphi \frac{f_n(z)}{z - z_j} dz$. There is $M \in (0, +\infty)$ such that $\forall n \in \mathbb{N}$ $|f_n| \leqslant M$ on $\langle \varphi \rangle$. Then we have

$$|f_n(z_1) - f_n(z_2)| = \frac{1}{2\pi} \left| \int_\varphi f_n(z) \cdot \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz \right| \leqslant$$

$$\leqslant \frac{2\pi \cdot 2r}{2\pi} \cdot M \cdot \frac{|z_1 - z_2|}{r^2}$$

$(\left| \frac{1}{z - z_1} - \frac{1}{z - z_2} \right| = \left| \frac{z_1 - z_2}{(z - z_1) \cdot (z - z_2)} \right| \leqslant \frac{|z_1 - z_2|}{r^2})$.

By this $\{f_n\}$ are equicontinuous on $\overline{U(z_0, r)}$, and by Arzela–Ascoli, there is $\{f_{n_k}\}$ which is uniformly convergent on $\overline{U(z_0, r)}$.

Second step: Let us cover the set G by $U_j = U(z_j, r_j)$, $j \in \mathbb{N}$, such that $\overline{U(z_j, 2r_j)} \subset G$. Then use a diagonal choice: 1. By first step choose $\left\{ f_{n_k^1} \right\}$ of $\{f_n\}$ such that $\left\{ f_{n_k^1} \right\}$ converges uniformly on $\overline{U_1}$. 2. By first step choose $\left\{ f_{n_k^2} \right\}$ subsequence of $\left\{ f_{n_k^1} \right\}$ such that $\left\{ f_{n_k^2} \right\}$ converges uniformly on $\overline{U_2}$ and so on.

Then $\left\{ f_{n_k^k} \right\}_{k=1}^\infty$ converges uniformly on any $\overline{U_j}$, i.e., locally uniformly on G. □

Definice 3.2

Let E be a (complex) linear space and let $\mathcal{P}$ be a system of seminorms on E. Then $(E, \mathcal{P})$ is called locally convex space (LCS). In $(E, \mathcal{P})$ we define:

- convergence: $f_n \to f \Leftrightarrow \forall p \in \mathcal{P} : p(f_n - f) \to 0$;

- topology τ is the weakest topology on E for which all $p \in \mathcal{P}$ are continuous;

- $\mathcal{F} \subset E$ is bounded if $\mathcal{F}$ is bounded with respect to any $p \in \mathcal{P}$, i.e.,

$$\forall p \in \mathcal{P} \; \exists C \in (0, +\infty) : p(f) \leqslant C \; \forall f \in \mathcal{F};$$

- the dual space to $(E, \mathbb{P})$ is defined as

$$E^* := \{L : E \to \mathbb{C} \mid L \text{ linear and continuous}\}.$$

Poznámka

$\mathcal{C}(G)$ is the so-called Fréchet space, i.e., completely metrizable LCS. So is $\mathcal{H}(G)$ because $\mathcal{H}(G)$ is closed subspace of $\mathcal{C}(G)$.

Topology τ on $\mathcal{C}(G)$ is generated by the system of seminorms

$$\mathcal{P} := \{\|\cdot\|_K \mid K \text{ is compact in } G\}.$$

$U \subset \mathcal{C}(G)$ is neighbourhood of $f \in \mathcal{C}(G)$ iff there are a compact $K \in G$ and $\varepsilon > 0$ such that

$$U \supset U_{K,\varepsilon}(f) := \{g \in \mathcal{C}(G) \mid \|g - f\|_K < \varepsilon\}.$$

Důkaz

„$\Longleftarrow$": obvious. „$\Longrightarrow$": There are $m \in \mathbb{N}$, compact, $K_1, \ldots, K_m$ in G and $\varepsilon_1, \ldots, \varepsilon_m > 0$ such that

$$U \supset \bigcap_{j=1}^{m} U_{K_j, \varepsilon_j}(f) \supset U_{K,\varepsilon}(f),$$

where $K := K_1 \cup \ldots \cup K_m$ and $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_m\} > 0$. □

Poznámka

Let $X = \mathcal{H}(G)$. Then in the sense of (LCS) $\mathcal{F} \subset \mathcal{H}(G)$ is bounded iff in the functions of $\mathcal{F}$ are locally equibounded on G. By the Montal theorem, we get $\overline{\mathcal{F}}$ is a compact in $\mathcal{H}(G)$. Easily we get that $\mathcal{F} \subset X$ is compact iff $\mathcal{F}$ is closed and bounded in X.

---

4 The dual space $\mathcal{H}^*(G)$

Poznámka

1. Let $G = \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. Let $L \in \mathcal{H}^*(\mathbb{D})$. Let $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, and $R := \frac{1}{\limsup_{n \to +\infty} \sqrt[n]{|a_n|}} \geqslant 1$. Then

$$L(f) = L(\sum_{n=0}^{\infty} a_n z^n) = L\left(\lim_{n \to \infty} \sum_{k=0}^{n} a_k z^k\right) = \lim_{n \to \infty} \sum_{k=0}^{n} a_k L(z^k) = \sum_{n=0}^{\infty} a_n \cdot b_n,$$

where $b_n := L(z^n) \in \mathbb{C}$. We show $r := \limsup_{n \to \infty} \sqrt[n]{|b_n|} < 1$:

If $r > 1$, then for $a_n := 1$, $n \in \mathbb{N}_0$, we get $\sum_{n=0}^{\infty} a_n \cdot b_n$ is divergent. If $r = 1$, then there is $\{n_k\}$ such that such that $0 \neq \sqrt[n_k]{|b_{n_k}|} \to 1$. Putting $a_n = \frac{1}{b_{n_k}}$, $n = n_k$, we get $\sum_{n=0}^{\infty} a_n b_n$ is divergent.

Conclusion: $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\{b_n\} \subset \mathbb{C}$ such that $\limsup_{n \to \infty} \sqrt[n]{|b_n|} < 1$ and $L(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$. In addition, $b_n = L(z^n)$, $n \in \mathbb{N}_0$. ($\Longleftarrow$ obvious, HW.)

Poznámka (Integral form of L)

Let $\{b_n\} \subset \mathbb{C}$ and $r := \limsup_{n \to \infty} \sqrt[n]{|b_n|} < 1$. Define

$$\lambda(z) := \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}, \qquad |z| > r.$$

Of course, $\lambda \in \mathcal{H}(\mathbb{S} \setminus \overline{U(0,r)})$, $\lambda(\infty) = 0$ and $b_n = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$. Here $\lambda^{(k)}(\infty) := \left(\lambda \left(\frac{1}{z} \right) \right)^{(k)}(0)$.

Let $R \in (r,1)$ and $\varphi(t) := Re^{it}$, $t \in [0, 2\pi]$. Let $f \in \mathcal{H}(\mathbb{D})$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Then

$$\frac{1}{2\pi i} \int_\varphi f(z) \cdot \lambda(z) dz = \frac{1}{2\pi i} \int_\varphi \left(\sum_{n=0}^{\infty} a_n \cdot z^n \right) \cdot \left(\sum_{m=0}^{\infty} \frac{b_m}{z^{m+1}} \right) dz =$$

$$= \frac{1}{2\pi i} \int_\varphi \sum_{n,m=0}^{\infty} a_n b_m z^{n-m-1} dz = \sum_{n,m=0}^{\infty} a_n \cdot b_m \cdot \frac{1}{2\pi i} \int_\varphi z^{n-m-1} dz = \sum_{n=0^{\infty}} a_n \cdot b_n = L(f).$$

Definice 4.1 (Notation)

Let $A \subset \mathbb{S}$. Then a function f is holomorphic on A if f is holomorphic on some open superset $U \supset A$. Let f_1, f_2 be holomorphic function on A. We say that $f_1 \sim f_2$ if there are open $U_1, U_2 \subset \mathbb{S}$ such that $A \subset U_1 \cap U_2$, $f_1 \in \mathcal{H}(U_1)$, $f_2 \in \mathcal{H}(U_2)$ and $f_1 = f_2$ on $U_1 \cap U_2$. Denote $\mathcal{H}(A) := \{[f] \mid f \text{ is holomorphic on } A\}$, where $[f]$ is an equivalence class for $\sim$. As usual, we do not often distinguish between $[f]$ and f.

We have that $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus \mathbb{D}) := \{\mu \in \mathcal{H}(\mathbb{S} \setminus \mathbb{D}) \mid \mu(\infty) = 0\}$. Moreover, we have

$$(*) L(f) = \frac{1}{2\pi i} \int_\varphi f(z) \cdot \lambda(z) dz, \qquad f \in \mathcal{H}(\mathbb{D});$$

$$L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, \qquad n \in \mathbb{N}_0;$$

$$\lambda(w) = L\left(\frac{1}{w-z} \right), \qquad |w| \geq 1.$$

Důkaz

In fact, we have

$$L\left(\frac{1}{w-z}\right) = L\left(\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{b_n}{w^{n+1}} = \lambda(w),$$

because $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{1}{w}} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$, $z \in \mathbb{D}$. $\quad\square$

Poznámka (Conclusion)

$$\mathcal{H}^*(\mathbb{D}) = \mathcal{H}_0(\mathbb{S}\backslash\mathbb{D}).$$

In particular, $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash\mathbb{D})$ such that $(*)$ hold true.

Příklad (Birkhoff)

There is a universal entire function, i.e., $f \in \mathcal{H}(\mathbb{C})$ such that $\overline{\{\tau_\gamma(f)|\gamma \in \mathbb{C}\}} = \mathcal{H}(\mathbb{C})$, where $\tau_\gamma(f) := f(z - \gamma)$, $z, \gamma \in \mathbb{C}$.

Řešení

HW.

Poznámka

2. Let $G = \bigcup_{j=1}^{n} D_j$ with $D_j = U(z_j, r_j)$ and $D_j \cap D_k = \varnothing$ for $j = k$.

Let $L \in \mathcal{H}^*(G)$. For $j \in [n]$, put $L_j(d) := L(\tilde{f})$ for $f \in \mathcal{H}(D_j)$ and $\tilde{f} := f$ on D_j and $\tilde{f} := 0$ on D_k, $k \neq j$. Then

$$L(f) = \sum_{j=1}^{n} L_j(f|_{D_j}), \qquad f \in \mathcal{H}(G).$$

By 1., for each $j \in [n]$, there are $\tilde{r}_j \in (0, r_j)$ and $\lambda_j \in \mathcal{H}_0(\mathbb{S}\backslash\overline{U(z_j, \tilde{r}_j)})$ such that

$$L_j(f) = \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz, \qquad f \in \mathcal{H}(D_j),$$

where $\varphi_j(t) := z_j + R_j e^{it}$, $t \in [0, 2\pi]$ for some $R_j \in (\tilde{r}_j, r_j)$.

In addition, we have

$$L_j(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, \qquad n \in \mathbb{N}_0.$$

If $f \in \mathcal{H}(G)$, then $L(f) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz$.

$\overset{?}{\implies}$ $L(f) = \frac{1}{2\pi i} \int_\Gamma f(z) \cdot \lambda(z) dz$, where $\Gamma := \{\varphi_1, \ldots, \varphi_n\}$ and $\lambda := \sum_{j=1}^n \lambda_j$.

? holds true because $\int_{\varphi_j} f(z) \cdot \lambda_k(z) dz = 0$ for $k \neq j$ by Cauchy $(f(z) \cdot \lambda_k(z) \in \mathcal{H}(D_j))$.

We have $L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$.

Poznámka (Conclusion)
$(G = \bigcup_{j=0}^n D_j.)$ $\mathcal{H}^*(G) = \mathcal{H}_0(\mathbb{S} \setminus G)$. Indeed, $L \in \mathcal{H}^*(G)$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S} \setminus G)$ such that last 2 equation hold true.

5 Hahn–Banach theorem

Lemma 5.1

Let $L : E \to \mathbb{C}$ be linear. Then $L \in E^*$ iff there is a compact K in G and $M \in [0, +\infty)$ such that $|L(f)| \leqslant M \cdot \|f\|_K$, $f \in E$.

Důkaz
„ $\impliedby$ “ from continuity of $\| \cdot \|_K$. „ $\implies$ “: Since $U := L^{-1}(\mathbb{D})$ is a neighbourhood of $\mathbf{o}$ in E, there are a compact K in G and $\varepsilon > 0$ such that $U \supseteq U_{K,\varepsilon}(0) := \{f \in E | \|f\|_K < \varepsilon\}$. Let $f \in E$.

1. Let $\|f\|_K \neq 0$. Then
$$\left| L\left(\frac{f}{\|f\|_K} \cdot \frac{\varepsilon}{2}\right)\right| < 1,$$
hence $|L(f)| < \frac{2}{\varepsilon}\|f\|_K$. Put $M := \frac{2}{\varepsilon}$.

2. Let $\|f\|_K = 0$. Then for any $n \in \mathbb{N}$, we have $\|nf\|_K = 0$, so $|L(n \cdot f)| < 1$, $|L(f)| < \frac{1}{n} \to 0$, $L(f) = 0$. $\quad\square$

Věta 5.2 (Hahn–Banach)

Let A be a linear subspace of E. Then

- *if $L \in A^*$, then there is $\tilde{L} \in E^*$ such that $\tilde{L}|_A = L$;*

- *if A is closed and $0 \neq b \in E \setminus A$, then there is $L \in E^*$ such that $L(b) = 1$ and $L = 0$ on A;*

- *$\overline{A} = E$ iff ($L \in E^*$, $L = 0$ on A $\implies$ $L = 0$ on E).*

Věta 5.3 (Runge (special))

Let $G \subset \mathbb{C}$ be a finite union of pairwise open discs as in above "poznamka"s. Then for each $f \in \mathcal{H}(G)$ there are polynomials P_n, $n \in \mathbb{N}$, such that $P_n \overset{loc.}{\rightrightarrows} f$ on G.

Důkaz

Let $\mathcal{P} := \mathrm{LO}\,\{1, z, \ldots\}$ be the space of polynomials. Then $\mathcal{P} \subset \mathcal{H}(G)$. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathcal{P}$. We know that there is $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash G)$ such that ? is valid. So, $\lambda^{(n)}(\infty) = 0$, $n \in \mathbb{N}_0$. By the uniqueness theorem, we get $\lambda \equiv 0$, so $L = 0$ on $\mathcal{H}(G)$ (because $L = 0$ fits and is uniquely determined by λ). By HB theorem, $\overline{\mathcal{P}} = \mathcal{H}(G)$.   □

Věta 5.4 (Cauchy formula for compact)

Let $G \subset \mathbb{C}$ be open, $K \subset G$ compact. Then there is a cycle $\Gamma \subset G$, $K \subseteq \mathrm{int}\,\Gamma \subseteq G$ and $\forall a \in \mathrm{int}\,\Gamma : \mathrm{ind}_\Gamma\,a = 1$.

In addition

$$\forall f \in \mathcal{H}(G) : \int_\Gamma f = 0 \wedge \forall a \in \mathrm{int}\,\Gamma : f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} dz.$$

Poznámka

„In addition“ follows from the properties of Γ and residue's theorem for cycles, but we prove it directly.

Důkaz (Cauchy formula for compact)

Choose $0 < \delta < \frac{1}{2} \mathrm{dist}(K, \mathbb{C} \backslash G)$, if $G \subsetneq \mathbb{C}$, otherwise, if $G = \mathbb{C}$, take $\delta := 1$. For $m, n \in \mathbb{Z}$ let $Q_{m,n}$ be the closed square with edges (parallel to the axes) with length δ, and such that $m\delta + in\delta$ is the lower left vertex of $Q_{m,n}$.

Denote $Q^* := \{Q_{n,m} | Q_{n,m} \cap K \neq \varnothing\}$, $U := \int(\bigcup Q^*)$. Q^* is finite because of compactness of K. Of course, $K \subseteq U \subseteq \bigcup Q^* \subseteq G$ (by choice of δ).

We understand $\partial Q_{m,n}$ as a positively oriented curve (piece-wise linear curve). Let Γ be the system of all edges $\Gamma_1, \ldots, \Gamma_k$ of squares of Q^* when we omit those edges which occur twice ($\pm$). Of course, $U = \bigcup Q^* \backslash \mathrm{Im}\,\Gamma$.

a) Let $f \in \mathcal{H}(G)$. Then $\int_\Gamma f := \sum_{j=1}^k \int_{\Gamma_j} f = \sum_{Q_{m,n} \subset Q^*} \int_{\partial Q_{m,n}} f = 0$.

b) Γ can be viewed as a cycle. In fact the edges $\Gamma_1, \ldots, \Gamma_k$ form finitely many closed simple piece-wise linear curves.

For $j \in [k]$ put $\Gamma_j =: [a_j, b_j]$.

(*) „Every point $c \in \mathbb{C}$ is the starting point of some edge of Γ as many times as it is the ending point of some edge in Γ":

Take a polynomial P such that $p(c) = 1$ and $p(a) = 0$, if $a \neq c$ and $[a, b] \in \Gamma$ for some b. $p(b) = 0$, if $b \neq c$ and $[a, b] \in \Gamma$ for some a. By a):

$$0 = \int_\Gamma p' = \sum_{j=1}^k \int_{\Gamma_j} p' = \sum_{j=1}^k (p(b_j) - p(a_j)) = \sum_{j=1}^k p(b_1) - \sum_{j=1}^k p(a_j) =$$

$$= \# \; c \text{ is the ending point} - \# \; c \text{ is the starting point.}$$

„Γ can be viewed as a cycle": Let L be longest (one of the longest) simple piecewise linear curve consisting of edges of Γ which begins with $\Gamma 1$, i. e.,

- $L = [c_1, c_2, \ldots, c_l] := [c_1, c_2] + [c_2, c_3] + \ldots + [c_{l-1}, c_l]$;

- $\Gamma 1 = [c_1, c_2]$;

- $c_i \neq c_j$ for $i \neq j$ (simple curve);

- l is the biggest.

Since we have (*) there is an index $j \in [l-2]$ such that $[c_l, c_j] \in \Gamma$ (otherwise we would have a longer curve).

$$L' := [c_j, c_{j+1}] + \ldots + [c_{l-2}, c_l] + [c_l, c_j] \subseteq L$$

$\implies L'$ is simple closed piece-wise linear curve. The proper subset Γ', which we get from Γ_k by omitting the edges of L' has again (*). We can process in this fashion for Γ', by finitely many steps we get what we want.

c) Let $f \in \mathcal{H}(G)$ and $a \in U = \mathrm{int}(\bigcup Q^*)$. c1) $a \in \mathrm{int}\left(\tilde{Q}\right)$ for some $\tilde{Q} \in Q^*$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz = \sum_{Q_{m,n} \in Q^*} \frac{1}{2\pi i} \int_{\partial Q_{m,n}} \frac{f(z)}{z-a} dz = f(a)$$

(residue's theorem for $Q_{m,n} = \tilde{Q}$, Cauchy for $Q_{m,n} \neq \tilde{Q}$).

c2) $a \in \partial \tilde{Q}$, $\tilde{Q} \in Q^*$ ($a \notin \partial Q^* \implies a \notin \mathrm{Im}(\Gamma)$). Take $a_j \in \mathrm{int}\,\tilde{Q}$, $a_j \to a$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a_j} dt \to f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - a} dz.$$

d) „$U = \mathrm{int}(\Gamma)$": Let $a \in \mathbb{C} \backslash (U \cup \mathrm{Im}\,\Gamma) \implies a \in \mathbb{C} \backslash \bigcup Q^*$. As in a), we show easily that $\mathrm{ind}_\Gamma a = 0$. If $a \in U$, then by c) we have that $\mathrm{ind}_p(1) = 1$. $\qquad \square$

Věta 5.5 (Description of $\mathcal{H}^*(G)$)

Let $G \subset \mathbb{C}$ be open subset. Then $\mathcal{H}^*(G) \simeq \mathcal{H}_0(\mathbb{S}\backslash G)$.

In more detail, let $L \in \mathcal{H}^*(G)$. Then there are a compact $K \subset G$ and $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash K)$ such that

$$L(f) = \frac{1}{2\pi i} \int_\Gamma f(z)\lambda(z)dz, \qquad f \in \mathcal{H}(G),$$

where Γ is a cycle in $G\backslash K$ with $K \subset \operatorname{int}\Gamma \subset G$ and $\forall z_0 \in \operatorname{int}\Gamma : \operatorname{ind}_\Gamma z_0 = 1$.

In addition, as an element of $\mathcal{H}_0(\mathbb{S}\backslash G)$, λ is uniquely determined by

$$\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k), k \in \mathbb{N}_0, \frac{\lambda^{(k)}(z_0)}{k!} = -L\left(\frac{1}{(z-z_0)^{k+1}}\right), z_0 \in \mathbb{C}\backslash G, k \in \mathbb{N}_0.$$

Důkaz (Step 1)

Let $L \in \mathcal{H}^*(G)$.

Step 1: There are a compact $K \subset G$ and $L_1 \in (\mathcal{C}(K))^* =: \mathcal{C}^*(K)$ such that $L(f) = L_1(f|_K), f \in \mathcal{H}(G)$.

We know that there are a compact $K \subseteq G$ and $C \in (0, +\infty)$ such that $\forall f \in \mathcal{H}(G) :$ $|L(f)| \leqslant \|f\|_K \cdot C$.

By the Hahn–Banach theorem we can extend L (from $\mathcal{H}^*(G)$ to $\mathcal{C}^*(G)$) to $\tilde{L} \Leftrightarrow \tilde{L} \in$ $\mathcal{C}^*(G)$ such that $\tilde{L}_2|_{\mathcal{H}}(G) = L$ and $|L(f)| \leqslant \|f\|_K \cdot C, f \in \mathbb{C}(G)$.

For each $f \in \mathcal{C}(K)$ put $L_1(f) := \tilde{L}_1(\tilde{f})$, where $\tilde{f} \in \mathcal{C}(G)$ and $\tilde{f}|_K = f$.

Is definition of L_1 correct?

i) by Tietze theorem: $f \in \mathcal{C}(K)$ can be extended to $f \in \mathcal{C}(G)$,

$$\forall f \in \mathcal{C}(K) \, \exists \tilde{f} \in \mathcal{C}(\mathbb{C}) \, (\mathcal{C}(G)) : \tilde{f}|_K = f;$$

ii) for any extension we want to get the same result. $\tilde{f}_1, \tilde{f}_2 \in \mathcal{C}(G), \tilde{f}_i|_U = f, i = 1, 2$.

$$\implies |\tilde{L}_1(\tilde{f}_1) - \tilde{L}_1(\tilde{f}_2)| = |\tilde{L}_1(\tilde{f}_1 - \tilde{f}_2)| \leqslant C \cdot \|\tilde{f}_1 - \tilde{f}_2\|_K = C\|f - f\|_K = 0.$$

$\square$

Poznámka ($\mathcal{C}^*(K)$)

By the Riesz representation theorem, for each $L_1 \in \mathcal{C}^*(K)$ there is a unique complex Borel measure μ on K such that

$$L_1(f) = \int_K f d\mu, \qquad \forall f \in \mathcal{C}(K).$$

Důkaz

Step 2: By the Cauchy formula for compact, there is a cycle $\Gamma \subset G$ such that $K \subset \operatorname{int}\Gamma \subset G$, $\forall a \in \operatorname{int}\Gamma : \operatorname{ind}_\Gamma a = 1$ and we have, $\forall f \in \mathcal{H}(G)$:

$$f(z_1) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z_2)dz_2}{z_2 - z_1}, \qquad z_1 \in K.$$

Denote

$$L_2(f) := \frac{1}{2\pi i}\int_\Gamma f(z_2)dz_2, f \in \mathcal{C}(\langle\Gamma\rangle), \qquad F(z_1, z_2) := \frac{f(z_2)}{z_2 - z_1}.$$

Of course $L_2 \in \mathcal{C}^*(\langle\Gamma\rangle)$ and $f(z_1) = L_2(F(z_1, z_3))$, $z_1 \in K$.

Step 3: For a given $f \in \mathcal{H}(G)$,

$$L(f) = L_1(f(z_1)) = L_1(L_2(F(z_1, z_2))) \overset{\text{Fubini}}{L} {}_2(L_1(F(z_1, z_2))),$$

hence

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z_2)\cdot\lambda(z_2)dz_2,$$

where

$$\lambda(z_2) := L_1\left(\frac{1}{z_2 - z_1}\right), \qquad z_2 \in \mathbb{C}\backslash K.$$

Step 4: $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash K)$ satisfies „in addition": Let $U(\infty, \varepsilon) \subset \mathbb{S}\backslash K$. For $u \in P(0, \varepsilon)$, we have

$$\lambda\left(\frac{1}{u}\right) = L_1(\frac{u}{1 - u\cdot z_1}) = L_1\left(\sum_{k=0}^\infty z_1^k u^{k+1}\right) = \sum_{k=0}^\infty L_1(z_1^k)u^{k+1},$$

hence $\lambda(\infty) = 0$ and

$$\forall k \in \mathbb{N}_0 : \frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L_1\left(z_1^k\right).$$

Let $U(z_0, \varepsilon) \subset \mathbb{C}\backslash K$. Then $\forall z_2 \in U(z_0, \varepsilon)$:

$$\lambda(z_2) = L_1\left(\frac{1}{z_2 - z_1}\right) = -L_1\left(\sum_{k=0}^\infty \frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}\right) = -\sum_{k=0}^\infty L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right)(z_2 - z_0)^k;$$

$$\forall z_1 \in K : \frac{1}{z_2 - z_1} = \frac{1}{(z_2 - z_0) - (z_1 - z_0)} = -\frac{1}{z_1 - z_0}\cdot\frac{1}{1 - \frac{z_2 - z_0}{z_1 - z_0}} = -\sum_{k=0}^\infty \frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}.$$

Hence $\frac{\lambda^{(k)}(z_0)}{k!} = -L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right)$, $k \in \mathbb{N}_0$.

Step 5: As an element of $\mathcal{H}_0(\mathbb{S}\backslash G)$, λ is uniquely determined by „in addition". (Proof below.) □

Lemma 5.6

Let $G \subset \mathbb{C}$ be open and K be a compact in G. There is a compact K_1 such that $K \subset K_1 \subset G$ and each component of $\mathbb{S}\backslash K_1$ contains some component of $\mathbb{S}\backslash G$.

Důkaz

Take $n \in \mathbb{N}$ such that $K_1 := \left\{ z \in G \,|\, \mathrm{dist}(z, \mathbb{C}\backslash G) \geqslant \frac{1}{n} \right\} \cap \overline{U(0, n)} \supset K$. In addition, we have

$$\mathbb{S}\backslash K_1 = \bigcup_{z_0 \in \mathbb{S}\backslash G} U(z_0, \frac{1}{n}).$$

Let V be a component of $\mathbb{S}\backslash K_1$. There is $z_0 \in \mathbb{S}\backslash G$ such that $U\left(z_0, \frac{1}{n}\right) \subset V$. If W is a component of $\mathbb{S}\backslash G$ containing z_0, then $W \subset V$. □

Důkaz (Step 5)

Let $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\backslash G)$ satisfying „in addition". Then there is a compact $K \subset G$ such that $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\backslash K)$.

By the previous lemma, WLOG we assume that each component V of $\mathbb{S}\backslash K$ intersect $\mathbb{S}\backslash G$. We show $\lambda_1 = \lambda_2$ on $\mathbb{S}\backslash K$.

Let V be any component of $\mathbb{S}\backslash K$ and $z_0 \in V \cap (\mathbb{S}\backslash G) \neq 0$. By „in addition" we have $\lambda_1^{(k)}(z_0) = \lambda_2^{(k)}(z_0) \;\forall k \in \mathbb{N}_0$. By the uniqueness theorem $\lambda_1 = \lambda_2$ on the domain B, so $\lambda_1 = \lambda_2$ on $\mathbb{S}\backslash K$. □

Lemma 5.7 (Fubini)

Let $K_1, K_2 \subset \mathbb{C}$ be compact, $L_j \in \mathcal{C}^*(K_j)$ for $j = 1, 2$ and $F \in \mathcal{C}(K_1 \times K_2)$. Then we have

$$L_1(L_2(F(z_1, z_2))) = L_2(L_1(F(z_1, z_2))).$$

Důkaz (Sketch)

Obviously it holds true for the functions of the following form: $F(z_1, z_2) = f(z_1) \cdot g(z_2)$ for $f \in \mathcal{C}(K_1)$, $G \in \mathbb{C}(K_2)$.

Now we can use the Stone–Weierstrass theorem which show that the linear span of the functions of this form is dense in $\mathcal{C}(K_1 \times K_2)$. □

6 Runge's theorem

Definice 6.1 (Notation)

Let $E \subset \mathbb{C}$ and $m : E \to \mathbb{N} \cup \{\infty\}$. We call $m(e)$ the multiplicity of $e \in E$. We say that (E, m) has a limit point $e \in \mathbb{S}$ if e is a limit point of E, or $e \in E$ with $m(e) = \infty$.

Denote by $\mathcal{F}(E, m)$ system of functions which consists of

- $\frac{1}{z-e}$ if $e \in E \cap \mathbb{C}$, $m(e) < \infty$;

- $\frac{1}{(z-e)^k}$, $k \in \mathbb{N}$ if $e \in E \cap \mathbb{C}$, $m(e) = \infty$;

- z^k, $k \in \mathbb{N}_0$ if $\infty \in E$, $m(\infty) = \infty$.

Věta 6.1 (Runge)

Let $G \subset \mathbb{C}$ be open, $E \subset \mathbb{S} \backslash G$ and $m : E \to N \cup \{\infty\}$. If (E, m) has a limit point in every component of $\mathbb{S} \backslash G$, then the linear span of $\mathbb{F}(E, m)$ is dense in $\mathcal{H}(G)$.

Důkaz

We shall use Hahn–Banach theorem. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathbb{F}(E, m)$. We need to show $L = 0$ on $\mathcal{H}(G)$. Let $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash G)$ which represents L in the sense of theorem describing $\mathcal{H}^*(G)$.

If $e \in E \cap \mathbb{C}$, $m(e) < \infty$, then $\lambda(e) = -L\left(\frac{1}{z-e}\right) = 0$. If $e \in E \cap \mathbb{C}$, $m(e) = \infty$, then $\frac{\lambda^{(k)}(e)}{k!} = -L\left(\frac{1}{(z-e)^k}\right) = 0$ $\forall k \in \mathbb{N}_0$. If $\infty \in E$, $m(\infty) = \infty$, then $\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k) = 0$ $\forall k \in \mathbb{N}_0$.

We show that $\lambda = 0$ in $\mathcal{H}_0(\mathbb{S} \backslash G)$. There is a compact $K \subset G$ such that $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash K)$ and every component of $\mathbb{S} \backslash K$ contains some component of $\mathbb{S} \backslash G$.

Let V be any component of $\mathbb{S} \backslash K$. Then V is domain and V contains a limit point e of (E, m). By the uniqueness theorem, we get $\lambda = 0$ on V, so on $\mathbb{S} \backslash K$. □

Věta 6.2 (Runge, classical version)

Let $G \subset \mathbb{C}$ be open and $f \in \mathcal{H}(G)$. Then there are rational functions R_n, $n \in \mathbb{N}$ with poles outside G such that $R_n \overset{Loc.}{\to} f$ on G.

If, in addition, $\mathbb{S} \backslash G$ is connected, then there are polynomials P_n, $n \in \mathbb{N}$, such that $P_n \overset{Loc.}{\to} f$ on G.

Důkaz

„Second part": Let $E = \{\infty\}$ and put $m(\infty) = \infty$. Then

$$\mathbb{F}(E, m) = \left\{1, z, \ldots, z^k, \ldots\right\}$$

and by the previous theorem, the polynomials are dense in $\mathcal{H}(G)$.

„First part": Let $E \subset \mathbb{S} \backslash G$ containing at least one point of every component of $\mathbb{S} \backslash G$. Put $m = \infty$ on E. Then $\mathrm{LO}(\mathcal{F}(E, m))$ is dense in $\mathcal{H}(G)$ and it is a subspace of rational functions with poles outside G. □

Důsledek (Cauchy's theorem for simply connected domains)

Let $G \subset \mathbb{C}$ be open a nd $\mathbb{S}\backslash G$ be connected. If $f \in \mathcal{H}(G)$ and φ is a closed curve in G, then $\int_\varphi f = 0$.

Důkaz

By Runge, there are polynomials P_n such that $P_n \overset{\text{Loc.}}{\rightrightarrows} f$ on G. Then (P_n has a primitive function in $\mathbb{C}$) $0 = \int_\varphi P_n \to \int_\varphi f$. □

Důsledek (Cauchy's theorem for cycles)

Let $G \subset \mathbb{C}$ be open and Γ be a cycle in G (i.e., $\langle \Gamma \rangle \subset G$). Then $\left(\forall f \in \mathcal{H}(G) : \int_\Gamma f = 0 \right) \Leftrightarrow \operatorname{int} \Gamma \subset G$.

Důkaz

„$\Longrightarrow$": If $z_0 \in \mathbb{C}\backslash G$, then $f(z) := \frac{1}{z - z_0} \in \mathcal{H}(G)$ and $\operatorname{ind}_\Gamma z_0 = \frac{1}{2\pi i} \int_\Gamma f = 0$.

„$\Longleftarrow$": Let $f \in \mathcal{H}(G)$. By Runge, there are rational R_n with poles outside G such that $R_n \overset{\text{Loc.}}{\rightrightarrows} f$. Then $0 = \int_\Gamma R_n \to \int_\Gamma f$. (First equality is from: Let $\Gamma = \{\varphi_1, \ldots, \varphi_m\}$, where φ_j are closed curves in G. Then $\int_\Gamma R_n = \sum_{j=1}^m \int_{\varphi_j} R_n = \sum_{j=1}^m 2\pi i \sum_{R_n(s)=\infty} \operatorname{res}_s R_n \operatorname{ind}_{\varphi_j} s = 2\pi i \cdot \sum_{R_n(s)=\infty} \operatorname{res}_s R_n \cdot \operatorname{ind}_\Gamma s$, but s lies outside of G, so it is equal to 0.) □

Věta 6.3 (Runge, for compacts)

Let K be a compact in $\mathbb{C}$ and let $S \subset \mathbb{S}\backslash K$ contain at least one point of any component of $\mathbb{S}\backslash K$. Let f be a holomorphic function on K. Then there are rational functions R_n with poles in S such that $R_n \rightrightarrows f$ on K.

Poznámka (Technique: pushing poles)

Each rational function R can be uniquely expressed in the form (rational function has $n \in \mathbb{N}$ poles, and we will write the principal part of Laurent expansion around the pole z_k):

$$R(z) = \sum_{k=1}^n \sum_{j=1}^{n_k} \frac{A_j^k}{(z - z_k)^j} + C_0 + C_1 z + \ldots + C_m z^m,$$

where $n, m, n_k \in \mathbb{N}$, $z_k \in \mathbb{C}$ and $A_{n_k}^k \neq 0$, $C_m \neq 0$. Then z_k is a pole of R of multiplicity n_k and ∞ is a pole of R of multiplicity m. A rational function R is a polynomial iff R has a pole at most at ∞.

Notation: Let K be a compact in $\mathbb{C}$, $U \subset \mathbb{S}$ and $U \cap K = \varnothing$. Put $B(K, U) = \overline{\{R|_K | R \text{ is rational with poles in } U\}}^{\mathcal{C}(K)}$. ($B(K, U)$ is a closed subalgebra of $\mathcal{C}(K)$.)

Theorem (pushing poles): Let K be a compact in $\mathbb{C}$, $U \subset \mathbb{S}$ be a domain, $K \cap U = \varnothing$ and $z_0 \in U$. If R is rational function with poles in U, then $R \in R(K, \{z_0\})$.

Corollary: By theorem, we have $B(K, U) = B(K, z_0)$.

25

Poznámka (Technique: pushing poles)

Proof: Put $V := \left\{ \xi \in U | \frac{1}{z-\xi} \in B(K, x_0), \text{ for } \xi \in \mathbb{C} \text{ and } z \in B(K, z_0) \text{ for } \xi = \infty \right\}$. Of course $B(K, z_0) = B(K, V)$. Indeed, if $\xi \in V$, then $\frac{1}{(z-\xi)^k} \in B(K, z_0)$, for $\xi \in \mathbb{C}$ and $k \in \mathbb{N}$, and $z^k \in B(K, z_0)$ for $\xi = \infty$, $k \in \mathbb{N}$.

Then each rational R with poles in V is contained in $B(K, z_0)$. Hence $B(K, V) \subset B(K, z_0)$. Since $z_0 \in V$, we have $B(K, z_0) \subset B(K, V)$.

„V is closed in U": Let $\xi_n \in V$, $\xi_n \to \xi_0$ and $\xi_0 \in U$. We need to show that $\xi_0 \in V$. WLOG $\forall n \in \mathbb{N} : \xi_n \in \mathbb{C}$.

„$\xi_0 \in \mathbb{C}$". Then put $\delta := \text{dist}(\xi_0, K) > 0$. Choose $n_0 \in \mathbb{N}$ such that $\text{dist}(\xi_n, K) \geqslant \frac{\delta}{2}$ for $n > n_0$. Then $\frac{1}{z-\xi_n} \rightrightarrows \frac{1}{z-\xi_0}$, $z \in K$,

$$\Longleftarrow \left| \frac{1}{z-\xi_n} - \frac{1}{z-\xi_0} \right| = \frac{|\xi_n - \xi_0|}{|z-\xi_n| \cdot |z-\xi_0|} \leqslant \frac{2}{\delta^2} \cdot |\xi_n - \xi_0| \to 0,$$

if $n > n_0$ and $z \in K$. Hence $\frac{1}{z-\xi_n} \in B(K, z_0)$, so $\xi_0 \in V$.

„$\xi_0 = \infty$". Then

$$\frac{\xi_n z}{\xi_n - z} = -\xi_n \left(\frac{\xi_n}{z-\xi_n} + 1 \right) \in B(K, z_0).$$

Take $C > 0$ with $\forall z \in K : |z| \leqslant C$. Take $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : |\xi_n| > C$. Then $\forall z \in K : \frac{\xi_n z}{\xi_n - z} \rightrightarrows z$, because

$$\left| \frac{\xi_n z}{\xi_n - z} - z \right| = \frac{|z|^2}{|\xi_n - z|} \leqslant \frac{C^2}{|\xi_n| - C} \to 0.$$

if $n > n_0$ and $z \in K$. Hence $z \in B(K, z_0)$, so $\infty \in V$.

„V is open (so $V = U$)": Let $\xi_0 \in V$.

„$\xi \in \mathbb{C}$": Put $\delta := \text{dist}(\xi_0, K) > 0$. Let $\xi \in U(\xi \in U(\xi_0, \delta/2))$. Then

$$\frac{1}{z-\xi} = \frac{1}{(z-\xi_0) - (\xi-\xi_0)} = \frac{1}{z-\xi_0} \cdot \frac{1}{1 - \frac{\xi-\xi_0}{z-\xi_0}} = \sum_{k=0}^{\infty} \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}}$$

converges uniformly for $z \in K$ because $\left| \frac{(\xi-\xi_0)^k}{(z-\xi_0)^{k+1}} \right| \leqslant \frac{\left(\frac{\delta}{2}\right)^k}{\delta^{k+1}} = \frac{1}{\delta \cdot 2^k}$, for $z \in K$ and $k \in \mathbb{N}_0$. Hence $\frac{1}{z-\xi} \in B(K, \xi_0) \subset B(K, V) = B(K, z_0)$. So $U(\xi_0, \delta/2) \subset V$.

„$\xi_0 = \infty$": Take $C > 0$ with $\forall z \in K : |z| \leqslant C$. Let $\xi \in \mathbb{C}$ with $|\xi| > 2C$. Then $\frac{1}{z-\xi} = -\frac{1}{\xi} \cdot \frac{1}{1 - \frac{z}{\xi}} = -\sum_{k=0}^{\infty} \frac{z^k}{\xi^{k+1}}$ converges uniformly for $z \in K$, because $|z^k/\xi^{k+1}| \leqslant C^k/(2C)^{k+1} = \frac{1}{C \cdot 2^{k+1}}$ for $z \in K$ and $k \in \mathbb{N}_0$. Hence $\frac{1}{z-\xi} \in B(K, \infty) \subset B(K, z_0)$. So $U\left(\infty, \frac{1}{2C}\right) \subset V$.

7 Characterization of simple connectedness

Tvrzení 7.1

Let $G \subset \mathbb{C}$ be open. FSAE:

SC1 If φ is closed (regular) curve in G, then $\operatorname{int} \varphi \subset G$;

SC2 $\mathbb{S} \backslash G$ is connected;

SC3 $\forall f \in \mathcal{H}(G) \; \exists$ polynomials P_n: $P_n \stackrel{loc.}{\rightrightarrows} f$ on G;

SC4 $\forall f \in \mathcal{H}(G) : \int_\varphi f = 0$ for any closed regular curve φ in G;

SC5 $\forall f \in \mathcal{H}(G) \; \exists F \in \mathcal{H}(G) : F' = f$ on G;

SC6 $\forall f \in \mathcal{H}(G)$, $f \neq 0$ on G, $\exists g \in \mathcal{H}(G)$: $f = e^g$ on G;

SC7 $\forall f \in \mathcal{H}(G)$, $f \neq 0$ on G, $\exists h \in \mathcal{H}(G)$: $h^2 = f$ on G.

Důkaz (SC1 $\implies$ SC2)

Assume that $\mathbb{S} \backslash G$ is not connected. Then there are disjoint closed sets $\emptyset \neq K, L \subset \mathbb{S}$ such that $\mathbb{S} \backslash G = K \cup L$. WLOG $\infty \notin K$. Then K is compact in $\mathbb{C}$, $G_0 := G \cup K$ is an open set in $\mathbb{C}$ and, by theorem Cauchy formula for compact we know, there is cycle Γ in G_0 such that $K \subset \operatorname{int} \Gamma \subset G_0$.

Let $z_0 \in K$. Since $\operatorname{ind}_\Gamma z_0 \neq 0$, there is $\varphi \in \Gamma$ with $\operatorname{ind}_\varphi z_0 \neq 0$. Of course, $z_0 \in (\mathbb{C} \backslash G) \cap \operatorname{int} \varphi$. □

Důkaz (SC2 $\implies$ SC3, SC3 $\implies$ SC4)

See Runge's theorem (classical version).

See the proof of the Cauchy theorem for simply connected domains. □

Důkaz (SC4 ⇔ SC5, SC5 $\implies$ SC6, SC6 $\implies$ SC7)
We know from introduction to complex analysis.

    See the proof of proposition about non-zero holomorphic function.

    Put $h := e^{\frac{1}{2}g}$.   □

7.1   The right topological definition

Definice 7.1 (Loop)

Let $G \subset \mathbb{C}$ be open. WLOG: We assume that all curves are defined on $[0, 1]$ (otherwise we can make linear reparametrization). A continuous closed curve $\varphi : [0, 1] \to G$ is called a loop in G.

Definice 7.2 (Homotopic loops)

We say that two loops φ and ψ are homotopic (in G) provided that there is a continuous map $H : [0, 1] \times [0, 1] \to G$ such that $\varphi_0(t) = \varphi(t)$ and $\varphi_1(t) = \psi(t)$ and $\varphi_s(0) = \varphi_s(1)$, where $\varphi_s(t) := H(s, t)$.

Tvrzení 7.2 (Continuation of the previous tvrzeni)

SC8: Every loop φ in G is homotopic in G to a constant loop.

Důkaz (SC7 $\implies$ SC8)
Let φ be a loop in G. Let G_0 be a component of G containing $\langle\varphi\rangle$. If $G_0 = \mathbb{C}$, then (all star-like domain has the property SC8) φ is homotopic to a constant loop. So assume $G_0 \subsetneq \mathbb{C}$. Then $\varnothing \neq G_0 \subsetneq \mathbb{C}$ is a domain with the property SC7. By Riemann (next) theorem, G_0 is homeomorphic to $\mathbb{D}$ and hence G_0 has SC8 property (all star-like domains have SC8 and homomorphism preserve homotopic loops).   □

Důkaz (SC8 $\implies$ SC1)
Of course, every constant loop ψ has int $\psi = \varnothing$. Hence this implication follows from the theorem after Riemann theorem.   □

Věta 7.3 (Riemann)

Let $\varnothing \neq G_0 \subsetneq \mathbb{C}$ be a domain with SC7. Then there is a one-to-one holomorphic function $h : G_0 \overset{onto}{\to} \mathbb{D} := U(0, 1)$.

Poznámka (Recall)
Proposition: Let $\varphi_1, \varphi_2 : [0, 1] \to \mathbb{C}$ be closed (regular) curves and $z_0 \in \mathbb{C}\backslash(\langle\varphi_1\rangle \cup \langle\varphi_2\rangle)$. If

$\forall t \in [0,1]$:
$$|\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - z_0|,$$

then $\operatorname{ind}_{\varphi_1} z_0 = \operatorname{ind}_{\varphi_2} z_0$.

Věta 7.4

Let φ, ψ be two loops homotopic in an open set $G \subset \mathbb{C}$. Then $\operatorname{ind}_\varphi z_0 = \operatorname{ind}_\psi z_0 \;\forall z_0 \in \mathbb{C} \backslash G$.

Definice 7.3 (Index of (non-regular) loop)

Let $\varphi : [0,1] \to \mathbb{C}$ be a loop and $z_0 \in \mathbb{C} \backslash \langle \varphi \rangle$. There are regular closed curves $\varphi_n : [0,1] \to \mathbb{C}$ such that $\varphi_n \rightrightarrows \varphi$. Indeed using the uniform continuity of φ, φ can be uniformly approximated by piecewise linear closed curves with vertices on φ given by sufficiently fine partitions of $[0,1]$.

Define $\operatorname{ind}_\varphi z_0 := \lim_{n \to \infty} \operatorname{ind}_{\varphi_n} z_0$.

By recalled proposition, the definition is correct because there is $n_0 \in \mathbb{N}$ such that $\operatorname{ind}_{\varphi_n} z_0$, $n \geqslant n_0$, are constant and $\operatorname{ind}_\varphi z_0$ does not depend of the choice of $\{\varphi_n\}$.

Poznámka
„Proposition from recall holds true for (non-regular) loops φ_1, φ_2.": Indeed, let loops φ_1 and φ_2 satisfy
$$\forall t \in [0,1] : |\varphi_1(t) - \varphi_2(t)| < |\varphi_1(t) - z_0|.$$

Then, by definition, there are approximations $\tilde{\varphi}_1, \tilde{\varphi}_2$ which are regular, satisfy the assumptions of proposition from recall and $\operatorname{ind}_{\varphi_j} z_0 = \operatorname{ind}_{\tilde{\varphi}_j}$, $j = 1, 2$.

Důkaz
Let $H : [0,1] \times [0,1] \to G$ be continuous, $\varphi_0 = \varphi$, $\varphi_1 = \psi$ and $\varphi_s(0) = \varphi_s(1)$, $\forall s \in [0,1]$, where $\varphi_s(t) = H(s,t)$. Put $\varepsilon := \operatorname{dist}(z_0, H([0,1]^2)) > 0$ ($H([0,1]^2)$ is compact).

Since H is uniformly continuous, there is $n \in \mathbb{N}$ such that for each $k \in [n-1]$ and $t \in [0,1]$ we have

$$\left| \varphi_{\frac{k}{n}}(t) - \varphi_{\frac{k+1}{n}}(t) \right| = \left| H(\frac{k}{n}, t) - H(\frac{k+1}{n}, t) \right| < \varepsilon.$$

In particular, $\varphi_{\frac{k}{n}}$ and $\varphi_{\frac{k+1}{n}}$ satisfy the assumptions of proposition. Hence

$$\operatorname{ind}_{\varphi_0} z_0 = \operatorname{ind}_{\varphi_{\frac{1}{n}}} z_0 = \operatorname{ind}_{\varphi_{\frac{2}{n}}} z_0 = \ldots = \operatorname{ind}_{\varphi_1} z_0.$$

$\square$

Věta 7.5 (The Schwarz lemma)

Let $f \in \mathcal{H}(\mathbb{D})$, $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Then $|f(z)| \leqslant |z|$, $z \in \mathbb{D}$ and $|f'(0)| \leqslant 1$. If the equality occurs in first inequality got some $z \in \mathbb{D} \backslash \{0\}$ or in second inequality, then f is a rotation, i.e., $f(z) = \lambda z$, $z \in \mathbb{D}$ for some $\lambda \in \mathbb{C}, |\lambda| = 1$.

Důkaz
Put $g(z) := \frac{f(z)}{z}$, for $z \in \mathbb{D}\setminus\{0\}$, and $f'(0)$, $z = 0$. Then $g \in \mathcal{H}(\mathbb{D})$. Let $0 < r < 1$. Then $|g(r)| \leqslant \frac{1}{r}$, $|z| = r$. By the maximum modulus theorem, we get $|g(z)| \leqslant \frac{1}{z}$, $|z| \leqslant r$.

Let $z \in \mathbb{D}$. Then, for r close enough to $1-$, we have this inequality and letting $r \to 1-$ we get $|g(z)| \leqslant 1$. If $|g(z)| = 1$ for some $z \in \mathbb{D}$, then by maximum modulus theorem g is constant on $\mathbb{D}$. □

Lemma 7.6

For $\alpha \in \mathbb{D}$ put $\varphi_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$. Then

- $\varphi_\alpha \in \mathcal{H}\left(\mathbb{C}\setminus\left\{\frac{1}{\overline{\alpha}}\right\}\right)$, φ_α *is one-to-one*, $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$, $\varphi_\alpha(\mathbb{T}) = \mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$;

- $(\varphi_\alpha)_{-1} = \varphi_{-\alpha}$;

- $\varphi_\alpha(\alpha) = 0$, $\varphi'_\alpha(\alpha) = \frac{1}{1-|\alpha|^2}$, $\varphi'_\alpha(0) = 1 - |\alpha|^2$.

Důkaz (ii))

$$w = \frac{z-\alpha}{1-\overline{\alpha}z} : w - \overline{\alpha}wz = z - \alpha, \qquad w + \alpha = z \cdot (1 + \overline{\alpha}w), \qquad z = \frac{w+\alpha}{1+\overline{\alpha}w} = \varphi_{-\alpha}(w).$$

□

Důkaz (i))
If $z \in \mathbb{T}$, then
$$\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right| = \frac{|z-\alpha|}{|\overline{z}-\overline{\alpha}| \cdot |z|} = 1.$$
Hence $\varphi_\alpha(\mathbb{T}) \subseteq \mathbb{T}$. The same is true for $(\varphi_\alpha)_{-1} = \varphi_{-\alpha}$, so $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$. By the fact $\varphi_\alpha(\mathbb{T}) = \mathbb{T}$ and maximum modulus theorem, we get $\varphi_\alpha(\mathbb{D}), \varphi_{-\alpha}(\mathbb{D}) \subset \mathbb{D}$, so $\varphi_\alpha(\mathbb{D}) = \mathbb{D}$. □

Důkaz (iii))

$$\varphi'_\alpha(\alpha) = \lim_{z \to \alpha} \frac{\varphi_\alpha(z)}{z-\alpha} = \frac{1}{1-|\alpha|^2}.$$

$$\varphi'_\alpha(0) = \left.\frac{1 - \overline{\alpha}z + (z-\alpha)\overline{\alpha}}{(1-\overline{\alpha}z)^2}\right|_{z=0} = 1 - |\alpha|^2.$$

□

Věta 7.7 (Conformal transformations of $\mathbb{D}$)

A function f is one-to-one holomorphic map of $\mathbb{D}$ onto $\mathbb{D}$, iff there are $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that
$$f(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}(= rot_\theta \cdot \varphi_\alpha), \qquad z \in \mathbb{D}.$$

Důkaz

„ $\Longleftarrow$ ": by the previous lemma. „ $\Longrightarrow$ ": Let $\alpha \in \mathbb{D}$ and $f(\alpha) = 0$. Then $g := f \circ \varphi_{-\alpha}$ is a one-to-one holomorphic map of $\mathbb{D}$ onto $\mathbb{D}$ and $g(0) = 0$. By the Schwarz lemma, for $z \in \mathbb{D}$, $|g(z)| \leqslant |z|$, $|g_{-1}(z)| \leqslant |z|$, so $|g(z)| = |z|$. By Schwarz lemma, g is a rotation. $\quad\square$

Lemma 7.8 (Schwarz–Pick)

Let $F \in \mathcal{H}(\mathbb{D})$, $F(\mathbb{D}) \subset \mathbb{D}$ and $F(\alpha) = \beta$. Then $|F'(\alpha)| \leqslant \frac{1-|\beta|^2}{1-|\alpha|^2}$. If the equality occurs, then $F(z) = \varphi_{-b}(\lambda \varphi_\alpha(z))$, $z \in \mathbb{D}$, for $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

In particular $F'(0) < 1$ unless F is a rotation.

Důkaz

Use the Schwarz lemma for $f := \varphi_\beta \circ F \circ \varphi_{-\alpha}$. Then $|f'(0)| \leqslant 1$ and

$$f'(0) = \varphi'_\beta(\beta) \circ F'(\alpha) \circ \varphi'_{-\alpha}(0) = \frac{1}{1-|\beta|^2} \circ F'(\alpha) \circ (1 - |\alpha|^2)$$

If $\alpha = 0 = \beta$ and F is not rotation, then $|F'(0)| < 1$. $\quad\square$

Věta 7.9 (Riemann)

Let $\varnothing \neq G \subsetneq \mathbb{C}$ be a simply connected domain. Then there is a one-to-one holomorphic map $f : G \stackrel{onto}{\to} \mathbb{D}$.

Poznámka

By this, we finish our proof that conditions (SC1)-(SC8) are equivalent with each other. (In proof we use SC7.)

Důkaz

Let $\varnothing \neq G \subsetneq \mathbb{C}$ be a domain with (SC7). Take a point $z_0 \in G$. Denote by Σ the set of all one-to-one holomorphic maps $\psi : G \to \mathbb{D}$. Then we have (we will prove it later)

- $\Sigma \neq \varnothing$;

- If $\psi \in \Sigma$ and $\psi(G) \neq \mathbb{D}$, then there is $\tilde{\psi} \in \Sigma$ such that $|\tilde{\psi}'(z_0)| > |\psi'(z_0)|$.

Put $\eta := \sup_{\psi \in \Sigma} |\psi'(z_0)|$. Take $\psi \in \Sigma \neq \varnothing$. Since ψ is one-to-one we have $\psi'(z_0) \neq 0$ and hence $\eta > 0$.

By the definition of η, there are $\psi_n \in \Sigma$, $n \in \mathbb{N}$, such that $|\psi_n'(z_0)| \to \eta$. Since φ_n, $n \in \mathbb{N}$, are uniformly bounded, by the Montol theorem, there is a subsequence $\{\psi_{n_k}\}$ such that $\psi_{n_k} \overset{\text{loc.}}{\rightrightarrows} f$ on G. By the Weierstrass theorem, $f \in \mathcal{H}(G)$ and $|f'(z_0)| = \eta > 0$ thus f is not constant (but it is limit of one-to-one functions), so by Hurwitz theorem f is one-to-one.

Of course, $f(G) \subset \overline{\mathbb{D}}$ but by openness of f we have $f(G) \subset \mathbb{D}$. Hence $f \in \Sigma$ and $f(G) = \mathbb{D}$. □

Důkaz (First property of Σ)

Let $w_0 \in \mathbb{C} \backslash G$. Then by (SC7), there is $\varphi \in \mathcal{H}(G)$ such that $z - w_0 = \varphi^2(z)$, $z \in G$. „If $\varphi(z_1) = \pm \varphi(z_2)$, then $z_1 = z_2$.": Indeed, $z_1 - w_0 = \varphi^2(z_1) = \varphi^2(z_2) = z_2 - w_0$.

By this, φ is one-to-one and $0 \neq w \in \varphi(G) \implies -w \notin \varphi(G)$. Since $\varnothing \neq \varphi(G)$ is open, there is $0 \notin U(a, r) \subset \varphi(G)$. By previous implication, we have $U(-a, r) \cap \varphi(G) = \varnothing$, i.e., $|\varphi(z) + a| \geqslant r$, $\forall z \in G$.

Put $\psi := \frac{r}{2(\varphi(z)+a)}, z \in G$. Then $|\psi| \leqslant \frac{1}{2}$ on G, so $\psi \in \Sigma$. □

Důkaz (Second property of Σ)

Let $\psi \in \Sigma$ and $\alpha \in \mathbb{D} \backslash \psi(G)$. Consider the map $\varphi_\alpha(z) := \frac{z-\alpha}{1-\bar{\alpha}z}$, $z \in \mathbb{D}$. Then $\varphi_\alpha \circ \psi \in \Sigma$ and $\varphi_\alpha \circ \psi \neq 0$ on G.

By (SC7), there is $g \in \mathcal{H}(G)$ such that $\varphi_\alpha \circ \psi = g^2$ on G. Then g is one-to-one, because

$$g(z_1) = g(z_2) \implies g^2(z_1) = g^2(z_2) \implies \varphi_\alpha \circ \psi(z_1) = \varphi_\alpha \circ \psi(z_2) \implies z_1 = z_2.$$

Hence $g \in \Sigma$. If $\beta := g(z_0)$, then put $\tilde{\psi} := \varphi$. Of course, $\tilde{\psi} \in \Sigma$ and $\tilde{\psi}(z_0) = 0$.

Denoting $s(w) := w^2$, $w \in \mathbb{C}$, we have that $\Psi = (\varphi_{-\alpha} \circ s \circ \varphi_{-\beta}) \circ z\tilde{\psi} = F \circ \tilde{\psi}$, where $F := \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. We have $F \in \mathcal{H}(\mathbb{D})$, $F(\mathbb{D}) \subset \mathbb{D}$ and F is not a rotation (because F is not one-to-one). By the Schwarz–Pick lemma, we have $|F'(0)| < 1$. Since $\psi'(z_0) = F'(0) \cdot \tilde{\Psi}'(z_0)$, we have $0 < |\psi'(z_0)| < |\tilde{\psi}'(z_0)|$. □

32

Definice 7.4

Let $G \subset \mathbb{S}$ be open. We say that $f : G \to \mathbb{S}$ is a conformal map if f is one-to-one meromorphic on G.

Definice 7.5

Let $\Omega, G \subset \mathbb{S}$ be open. We say that G and Ω are conformally equivalent (we write $G \sim \Omega$) if there is a conformal map $f : G \overset{\text{onto}}{\Omega}$.

Příklad
Show that there are just 4 classes of conform equivalent simply connected domains in $\mathbb{S}$, namely:
$$\varnothing, \qquad \mathbb{S}, \qquad [\mathbb{C}] := \{\mathbb{S} \setminus \{z_0\} \,|\, z_0 \in \mathbb{S}\}, \qquad [\mathbb{D}].$$

8 Preservation of angles

Definice 8.1

For $z \in \mathbb{C} \setminus \{0\}$, put $A(z) := \frac{z}{|z|}$.

Definice 8.2 (Map preserving angles)

Let $G \subseteq \mathbb{C}$ be open, $f : G \to \mathbb{C}$, $z_0 \in G$ have $P(z_0) \subset G$ such that $f(z) \neq f(z_0) \ \forall z \in P(z_0)$. Then we say that f preserves angles (with orientation) at z_0 if $\forall \theta \in \mathbb{R}$:

$$\lim_{r \to 0_+} e^{-k*\theta} \cdot A[f(z_0 + re^{i\theta}) - f(z_0)]$$

exists and is independent of θ.

Definice 8.3 (Notation)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, where $\mathbb{R}^2 = \mathbb{C}$, have the total differential $df(z_0)$ at $z_0 \in \mathbb{R}^2 = \mathbb{C}$, i.e.,

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0) - df(z_0)h}{|h|} = 0.$$

Then

$$df(z_0)h = \frac{\partial f}{\partial x}(z_0)h_1 + \frac{\partial f}{\partial y}(z_0)h_2,$$

$$h = (h_1, h_2) = h_1 + ih_2 \in \mathbb{R}^2 = \mathbb{C}.$$

We have $h_1 = \frac{h + \overline{h}}{2}$, $h_2 = \frac{h - \overline{h}}{2i}$ and

$$df(z_0)h = \partial f(z_0)h + \overline{\partial} f(z_0)\overline{h},$$

where

$$\partial f(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i \cdot \frac{\partial f}{\partial y}(z_0)\right),$$

$$\overline{\partial} f(z_0) := \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i \cdot \frac{\partial f}{\partial y}(z_0)\right).$$

Poznámka
We know $f'(z_0)$ exists iff $df(z_0)$ exists and $\overline{\partial} f(z_0) = 0$, in this case $f'(z_0) = \partial f(z_0)$.

Věta 8.1

Let $G, \Omega \subset \mathbb{C}$ be open. Then $f : G \to \Omega$ is conformal iff f is a diffeomorphism of G onto Ω preserving angles at any point of G.

TODO!!!

9 Harmonic functions

Poznámka
We study $f : \mathbb{C} \to \mathbb{C}$. Since $\mathbb{C} = \mathbb{R}^2$ we have $z = x + iy$, $x = \Re z$, $y = \Im z$ and $f = u + iv$ with $u = \Re f$, $v = \Im f$.

Poznámka (Observation)
If $G \subseteq \mathbb{C}$ is a domain, $f, g \in \mathcal{H}(G)$ and $\Re f = \Re g$ on G, then there is $c \in \mathbb{C}$ such that $\Im f = \Im g + c$ on G.

Důkaz
It follows from CR condition:

$$\frac{\partial v}{\partial y} = +\frac{\partial u}{\partial x}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

$\square$

What are the real parts of holomorphic functions?

Lemma 9.1

If $f \in \mathbb{C}^2(G)$, $G \subseteq \mathbb{C}$ open, then $\partial \overline{\partial} f = \overline{\partial} \partial f = \frac{1}{4}\Delta f$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

$$\partial\overline{\partial}f = \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f = \frac{1}{4}\Delta f.$$

□

Definice 9.1

If $G \subseteq \mathbb{C}$ is open, we say that $u \in \mathbb{C}^2(G)$ is harmonic if $\Delta u = 0$ on G.

TODO example?

Příklad
If $G \subset \mathbb{C}$ is a simply connected domain, $f \in \mathcal{H}(G)$ and $f \neq 0$ on G, then $\log|f| = \Re F$ for some $F \in \mathcal{H}(G)$. In particular, $\log|f|$ is harmonic.

Důkaz
We know that there is a holomorphic branch F of $\log f$, i.e., $F \in \mathcal{H}(G)$ and $f = e^F$. Then $|f| = e^{\Re F}$.

□

Důsledek (From examples)
If $G \subset \mathbb{C}$ is open, $0 \neq f \in \mathcal{H}(G)$, then $\log|f|$ is harmonic.

Důkaz
By the previous exercise, $\log|f|$ is harmonic on any open ball U in G.

□

TODO example?

Věta 9.2

If $G \subset \mathbb{C}$ is a simply connected domain and $u : G \to \mathbb{R}$ is harmonic, then there is $f \in \mathcal{H}(G)$ such that $\Re f = u$.

Poznámka
Every harmonic function is locally (but not globally) the real part of a holomorphic function.

If $f \in \mathcal{H}(G)$, then $f' = \partial f = \partial(f + \overline{f}) = 2\partial(\Re f)$, because $\partial\overline{f} = \overline{(\overline{\partial}f)} = 0$.

Důkaz

We have $\partial u \in \mathcal{H}(G)$, because $\overline{\partial}(\partial u) = \frac{1}{4}\Delta u = 0$. Then there is $f_0 \in \mathcal{H}(G)$ such that $f_0' = 2\partial u$.

By remark, $2\partial(\Re f_0) = f_0' = 2\partial u$, $\partial(\Re f_0 - u) = 0$. $\implies \frac{\partial}{\partial x}(\Re f_0 - u) = 0$, $\frac{\partial}{\partial y}(\Re f_0 - u) = 0$ on the domain G. $\qquad \square$

Důsledek

Let $G \subset \mathbb{C}$ be open and $u : G \to \mathbb{R}$ be harmonic. Then $u \in \mathcal{C}^\infty(G)$ and u satisfies the mean value property:

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt = u(z_0),$$

whenever $\overline{U(z_0, r)} \subset G$.

Důkaz

Let $\overline{U(z_0, r)} \subseteq G$. Take $R \in (r, +\infty)$ such that $U(z_0, R) \subseteq G$. Then $u = \Re F$ for some $F \in \mathcal{H}(U(z_0, R))$. So $u \in \mathcal{C}^\infty(U(z_0, R))$ and, by the Cauchy integral formula,

$$F(z_0) = \frac{1}{2\pi i} \int_\varphi \frac{F(z)}{z - z_0}dz = \frac{1}{2\pi i} \int_0^{2\pi} F(z_0 + re^{it}) \cdot \frac{ire^{it}}{re^{it}}dt,$$

where $\varphi(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. This implies mean value property. $\qquad \square$

Věta 9.3 (The maximum principle)

Let $G \subset \mathbb{C}$ be a domain and $u : G \to \mathbb{R}$ be continuous and u satisfies mean value property. If u is not constant, then u does not attain an extreme in G.

Důkaz

Let us assume that $z_0 \in G$ and, say, $u(z_0) \geqslant u$ on G. Put $M := \{z \in G | u(z) = u(z_0)\}$. Obviously, $\varnothing \neq M$ is closed in G. If we show that M is open, then $M = G$. Let $z_1 \in M$ and $U(z_1, r) \subset G$. We show that $U(z_1, r) \subset M$. Assume that there is $z_2 \in U(z_1, r) \backslash M$.

By mean value property, $u(z_0) = u(z_1) = \frac{1}{2\pi} \int_0^{2\pi} u(z_1 + \varrho e^{it})dt < u(z_0)$, where $\varrho := |z_1 - z_2|$. The Part inequality is strict because $u \leqslant u(z_0)$ on G and $u < u(z_0)$ in a neighbourhood of z_2. $\qquad \square$

Důsledek

Let $G \subset \mathbb{C}$ be bounded and open, $u \in \mathcal{C}(\overline{G})$ and u be harmonic on G. Then

$$\min_{\partial G} u \leqslant u \leqslant \max_{\partial G} u \text{ on } G.$$

Poznámka

The assumption of harmonicity can be replaced with mean value property.

Důkaz
Let $z_0 \in G$ and $u(z_0) = \max_{\overline{G}} u$. Then u is constant on the component G_0 of G containing z_0. So u attains the maximum at the boundary. □

10 Poisson integral

Poznámka
Let $u : \overline{\mathbb{D}} \to \mathbb{R}$ be harmonic, i.e. u is harmonic on $U(0, r)$ for some $r > 1$. Then there is $f \in \mathcal{H}(\overline{\mathbb{D}})$ such that $u = \Re f$ on $\overline{\mathbb{D}}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $|z| \leqslant 1$.

For $|z| \leqslant 1$, $z = re^{i\theta}$, we have

$$u(z) = \Re f(z) = \Re a_0 + \sum_{n=0}^{+\infty} \frac{1}{2}(a_n r^n e^{in\theta} + \overline{a_n} r^n e^{-in\theta}).$$

Hence $u(z) = \sum_{n=-\infty}^{\infty} b_n \cdot r^{|n|} e^{in\theta}$, where $b_n = \Re a_0$ for $n = 0$, $b_n = \frac{1}{2} a_n$ for $n > 0$ and $b_n = \frac{1}{2} \overline{a_{-n}}$ for $n < 0$.

In addition, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{-imt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} b_n e^{int} \right) e^{-imt} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} b_n e^{i(n-m)t} dt =$$

$$= \sum_{n=-\infty}^{\infty} b_n \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t} dt = b_m.$$

Putting this equations together, we get, for $z = re^{i\theta}$ with $r \in [0, 1)$,

$$u(z) = \sum_{n=-\infty}^{\infty} r^{|n|} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}) e^{in(\theta-t)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)} \right) u(e^{it}) dt.$$

Definice 10.1 (Poisonn kernel, poisson integral)

For $0 \leqslant r < 1$, $\theta \in \mathbb{R}$, put

$$P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

$$[Pu](re^{i\theta}) := \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it}) dt$$

for $0 \leqslant r < 1$, $\theta \in \mathbb{R}$.

$$P_r(\theta) = \Re\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) = \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}.$$

Důkaz
TODO?

□

Věta 10.1

If u is harmonic on $\overline{\mathbb{D}}$, then $u = Pu$ on $\mathbb{D}$.

Poznámka
The Poisson formula for harmonic functions is an analogue of the Cauchy integral formula.

$P1 = 1.$

Věta 10.2 (Properties of PI)

Let $g \in L^1(\mathbb{T})$. Then Pg is harmonic on $\mathbb{D}$ and if g is continuous at $z_0 \in \mathbb{T}$, then $\lim_{z \to z_0, z \in \mathbb{D}}[Pg](z) = g(z_0)$. In particular, if $g \in \mathcal{C}(\mathbb{T})$, then the function $u := Pg$ in $\mathbb{D}$ and $u := g$ in $\mathbb{T}$ is continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$.

Důkaz

WLOG, we assume that g is real-valued. For $|z| < 1$, $z = re^{i\theta}$, we have

$$Pg(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(\frac{1 + re^{i(\theta - t)}}{1 - re^{i(\theta - t)}}\right) g(e^{it}) dt =$$

$$= \Re\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} g(e^{it})\right) dt.$$

So we have $f \in \mathcal{H}(\mathbb{D})$ and $Pg \Re f$ is harmonic on $\mathbb{D}$.

WLOG, assume $g(z_0) = 0$ (otherwise, consider $g - g(z_0)$ and use the fact that $P1|_{\mathbb{T}} = 1|_{\mathbb{D}}$). Let $\varepsilon > 0$. Take $\delta \in (0, \pi)$ such that $|g(e^{it})| < \varepsilon$ $\forall t \in (\theta_0 - \delta, \theta_0 + \delta)$. Let $z \in \mathbb{D}$, $z = r \cdot e^{i\theta}$ and $|\theta - \theta_0| < \frac{\delta}{2}$.

Then

$$Pg(z) = \frac{1}{2\pi} \int_{\theta_0 - \delta}^{\theta_0 + \delta} P_r(\theta - t) g(e^{it}) dt + \frac{1}{2\pi} \int_A P_r(\theta - t) g(e^{it}) dt = I_1 + I_2$$

with $A := (\theta_0 - \pi, \theta_0 - \delta) \cup (\theta_0 + \delta, \theta_0 + \pi)$. Of course, $P_r > 0$ and

$$P1(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) dt = 1$$

for $0 \leqslant r < 1$, $\theta \in \mathbb{R}$. Hence $|I_1| < \varepsilon$.

Next we have

$$|I_2| \leqslant \frac{1}{2\pi} \int_A P_r(\theta - t) |g(e^{it})| dt \leqslant P_r\left(\frac{\delta}{2}\right) \|g\|_{L^1(\mathbb{T})} \to 0,$$

because

$$0 < P_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \overset{*}{\leqslant} P_r\left(\frac{\delta}{2}\right),$$

here $\|g\|_{L^1} := \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{it})| dt$.

„*": For $t \in A$, $\frac{\delta}{2} \leqslant |\theta - t| \leqslant \frac{3\pi}{2}$. Indeed, if, say, $t \in (\theta_0 + \delta, \theta_0 + \pi)$, then

$$\theta_0 + \delta < t < \theta_0 + \pi,$$

$$-\frac{\delta}{2} < \theta_0 - \theta < \frac{\delta}{2},$$

$$\delta < t - \theta_0 < \pi.$$

By $|I_2| \to 0$, take $r_0 \in (0, 1)$ such that $|I_2| < \varepsilon$ $\forall r \in (r_0, 1)$. Then for all $z = re^{i\theta}$ with $|\theta - \theta_0| < \frac{\delta}{2}$ and $r \in (r_0, 1)$, we have $|Pg(z)| \leqslant |I_1| + |I_2| \leqslant 2\varepsilon$. □

Definice 10.2 (The Dirichlet problem)

Let $G \subset I$ be open and bounded. Let $g \in \mathcal{C}(\partial G)$. The Dirichlet problem (DP) is to find $u \in \mathcal{C}(\overline{G})$ such that u is harmonic on G and $u = g$ on ∂G.

Poznámka

DP compounds to many problems from physics, e.g., a ? distribution g on ∂G gives an electric potential u on G.

By maximum principle (DP) has at most one solution (indeed, let u_1, u_2 be solutions of DP, then, for $u := u_1 - u_2$ we have $0 = \min_{\partial G} u \leqslant u \leqslant \max_{\partial G} u = 0$ on G).

DP has not always (classical) solution, e.g. for $G := \mathbb{D} \setminus \{0\}$ and $g := 0$ on $\mathbb{T}$, 1 at 0.

DP has a unique solution on "nice" domains, e.g., on domains with Lipschitz boundaries.

In particular, DP has a unique solution on any open disc in $\mathbb{C}$. (For $G = \mathbb{D}$ we use the previous theorem. For general disc, use a simple transformations of the form $z \mapsto rz + b$ for $r > 0$ and $b \in \mathbb{C}$.)

Věta 10.3

Let $G \subseteq \mathbb{C}$ be open and $u : G \to \mathbb{C}$. Then u is harmonic on G iff u is continuous on G and satisfies the mean value property.

Důkaz

„$\Longrightarrow$ ": we already know. „$\Longleftarrow$ " Let $u \in \mathcal{C}(G)$ satisfy mean value property on G. Let $U := U(z_0, r)$ with $\overline{U} \subset G$. Let h be a unique solution of DP on U with the boundary delta $u|_{\partial u}$. Then $v := u - h$ is continuous on $\overline{U}$, satisfies mean value property on U and $v = 0$ on ∂U.

By the maximum principle we get $v = 0$ on U, $u = h$ on U. In particular, u is harmonic in U, so in G. □