Prerequisites

0.1 Regularization

Definice 0.1 (Regularization kernel)

$\eta \in C_0^\infty(B_1(\mathbf{o}))$, non-negative, radially symmetric, $\int_{B_1(\mathbf{o})} \eta(x) dx = 1$.

Definice 0.2 (Regularization of function)

Let $f \in L^p(\Omega)$. We extend f by zero to $\mathbb{R}^d \backslash \Omega$ and define $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta(x/\varepsilon)$.

Poznámka

$f_\varepsilon \in C^\infty(\mathbb{R}^d)$, $f_\varepsilon \to f$ in $L^p(\Omega)$ if $p \in [1, \infty)$ and $f_\varepsilon \rightharpoonup^* f$ in L^∞.

Věta 0.1

$L^p(\Omega)$ is a Banach space, separable for $p \in [1, \infty)$, reflexive for $p \in (1, \infty)$.

Důsledek

f^n is a bounded sequence in $L^p(\Omega)$, $\Omega \subseteq \mathbb{R}^d$ measurable bounded bounded. Then

1. $p \in (1, \infty)$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup f$ in $L^p(\Omega)$. ($\Leftrightarrow \forall g \in L^{p'}(\Omega) : \lim_{k\to\infty} \int_\Omega f^{n_k} g = \int_\Omega fg$, where $\frac{1}{p} + \frac{1}{p'} = 1$).
2. $p = \infty$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup^* f$ in $L^\infty(\Omega)$. ($\Leftrightarrow g \in L^1(\Omega) : \lim \int_\Omega f^{n^k} g = \int_\Omega fg$).
3. $p = 1$: $\exists f^{n_k}, f$: $f^{n_k} \rightharpoonup^* f$ in $M(\overline{\Omega})$ (Radon measures). ($\Leftrightarrow \forall g \in C(\overline{\Omega}) : \int_\Omega f^{n_k} g \to \langle f, g \rangle_M = \int_{\overline{\Omega}} g df$.)
4. $p = 1$: $\exists f^{n_k}, \tilde{f}$ $\exists \Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \ldots, |\Omega \backslash \Omega_l| \to 0$ as $l \to \infty$: $\forall l \in \Omega : f^{n_k} \rightharpoonup \tilde{f}$ in $L^1(\Omega)$. ($\tilde{f}$ is called biting limit.)

0.2 Fixpoint theorems

Věta 0.2

$F : X \to X$, where X is a Banach space, F is continuous and compact. Let there exists closed convex non-empty set $U \subseteq X$ such that $F(U) \subset U$. Then $\exists x \in U : F(x) = x$.

Věta 0.3

$F : \mathbb{R}^d \to \mathbb{R}^d$, F is continuous. Let there exists closed, convex non-empty set $U \subseteq \mathbb{R}^d$: $F(U) \subseteq U$. Then $\exists x \in U : F(x) = x$.

0.3 Nemytskii operator

Věta 0.4

Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is Carathéodory (i.e. $\forall y \in \mathbb{R}^N : f(\cdot, y)$ is measurable and for almost all $x \in \Omega$: $f(x, \cdot)$ is continuous). Assume that $|f(x, y) \leqslant g(x) + c \cdot \sum_{i=1}^{N} |y_i|^{p_i/p}$ for some $p_1 \in [1, \infty)$, $p \in [1, \infty)$ with $y \in L^p(\Omega)$.

Then $\forall u_i \in L^{p_i}(\Omega)$, the function $f(x, u_1(x), \dots, u_N(x))$ is measurable and the mapping (named Nemytskii operator) $(u_1, \dots, u_N) \mapsto f(\cdot, u_1, u_2, \dots, u_N)$ is continuous from $L^{p_1}(\Omega) \times \dots \times L^{p_N}(\Omega)$ to $L^p(\Omega)$.