

*Poznámka* (Exam)  
Oral, similar as in FA1.

*Poznámka* (Credit)  
Similar as in FA1.

# 1 Banach algebras

## 1.1 Basic properties

### Definice 1.1 (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$  is algebra over  $\mathbb{K}$ , if

- $(A, +, -, 0, \cdot_S)$  is vector space over  $\mathbb{K}$ ;
- $(A, +, -, 0, \cdot)$  is ring (that is we have  $a \cdot (b + c) = a \cdot b + a \cdot c$ ,  $(a + b) \cdot c = a \cdot c + b \cdot c$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ );
- $\forall \lambda \in \mathbb{K} \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$ .

### Důsledek

1)  $e \in A$  is left unit  $\equiv e \cdot a = a$ , right unit  $\equiv a \cdot e = a$ , unit  $\equiv a \cdot e = e \cdot a = a$  ( $\forall a \in A$ ).

If  $e_1$  is left unit and  $e_2$  is right unit, then  $e_1 = e_2$  is unit. ( $e_1 = e_1 \cdot e_2 = e_2$ )

2) (Algebra) homomorphism  $\varphi : A \rightarrow B \equiv \varphi$  preserves  $+, \cdot, \cdot_S$ , that is  $\varphi(x + y) = \varphi(x) + \varphi(y)$ ,  $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$  and  $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$ .

### Tvrzení 1.1

Let  $A$  be algebra over  $\mathbb{K}$ . Put  $A_e = A \times \mathbb{K}$  with operations  $A_e$  defined coordinate-wise and multiplication defined by

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \quad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

Then  $A_e$  is algebra with a unit  $(\mathbf{o}, 1)$  and  $A \equiv A \times \{0\} \subset A_e$ . Moreover, if  $A$  is commutative, then  $A_e$  is commutative.

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Důkaz

We have  $A_e$  is vector space (from linear algebra). We easy proof from definition, that  $A_e$  is algebra,  $(\mathbf{o}, 1)$  is a unit in  $A_e$  and on  $A \times \{0\}$  we have  $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$ , so  $a \mapsto (a, 0)$  is homomorphism. Commutativity is easy too.  $\square$

### Definition 1.2 (Normed algebra)

$(A, \|\cdot\|)$  is normed algebra  $\equiv A$  is algebra and  $(A, \|\cdot\|)$  is NLS and  $\|a \cdot b\| \leq \|a\| \cdot \|b\|$  ( $\forall a, b \in A$ ).

### Definition 1.3 (Banach algebra)

$(A, \|\cdot\|)$  is Banach algebra  $\equiv (A, \|\cdot\|)$  is normed algebra nad Banach space.

Například

$l_\infty(I)$  is commutative Banach algebra with a unit (all ones).

If  $T$  is Hausdorff topological space, then  $\mathcal{C}_b(T) = \{f : T \rightarrow \mathbb{K} | f \text{ is continuous and bounded}\} \subseteq l_\infty(T)$  is closed subalgebra.

If  $T$  is locally compact, Hausdorff, not compact. Then  $\mathcal{C}_0(T) = \{f : T \rightarrow \mathbb{K} \text{ continuous} | \forall \varepsilon > 0 : \{t \in T : |f(t)| > \varepsilon\} \text{ is compact}\} \subseteq \mathcal{C}_b(T)$  is closed subalgebra, which doesn't have unit.

If  $X$  is Banach,  $\dim X > 1$ , then  $\mathcal{L}(X)$ , with  $S \cdot T := S \circ T$ ,  $S, T \in \mathcal{L}(X)$ , is Banach algebra with unit (identity), which isn't commutative.

If  $X$  is Banach,  $\dim X = +\infty$ , then  $\mathcal{K}(X) \subset \mathcal{L}(X)$  is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$ , where  $*$  is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$ , where  $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$  is (commutative) Banach algebra (with unit).

### Tvrzení 1.2

If  $(A, \|\cdot\|)$  is normed algebra, then  $\cdot : A \oplus_\infty A \rightarrow A$  is Lipschitz on bounded sets.

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$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r) \quad \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) :$$

$$\|ab - cd\| \leq \|a(b-d)\| + \|(a-c) \cdot d\| \leq \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leq R \cdot (\|b-d\| + \|a-c\|) \leq 2R\|(a, b) - (c, d)\|.$$

└  $\square$

### **Tvrzení 1.3**

Let  $(A, \|\cdot\|)$  be a Banach algebra. On  $A_e$  we consider the norm

$$\|(a, \alpha)\| := \|a\| + |\alpha|, \quad (a, \alpha) \in A \times \mathbb{K} = A_e.$$

Then  $(A_e, \|\cdot\|)$  is Banach algebra.

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It is a Banach space, because  $A_e = A \oplus_1 \mathbb{K}$ . Now we need only check, that

$$\|(a, \alpha) \cdot (b, \beta)\| \leq \|(a, \alpha)\| \cdot \|(b, \beta)\|,$$

└ which is easy. □

*Poznámka*

There is more (natural) ways to define norm on  $A_e$  (unlike  $\cdot$  on  $A_e$ , which is natural).

$A$  has a unit ... we may still consider  $A_e$ .

If  $e \in A \setminus \{\mathbf{o}\}$  is a unit, then  $\|e\| \geq 1$ , because  $\|e\| = \|e^2\| \leq \|e\|^2$ .

### **Věta 1.4**

Let  $A$  be a Banach algebra, for  $a \in A$  consider  $L_a \in \mathcal{L}(A)$  defined as  $L_a(x) := a \cdot x$ ,  $x \in A$ . Then  $I : A \rightarrow \mathbb{L}(A)$ ,  $a \mapsto L_a$  is continuous algebra homomorphism,  $\|I\| \leq 1$ .

Moreover, if  $A$  has a unit  $e$ , then  $I$  is isomorphism into and  $I(e) = \text{id}$ .

If  $\|x^2\| = \|x\|^2$ ,  $x \in A$ , then  $I$  is isometry into.

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„ $L_a \in \mathcal{L}(A)$  and  $I \in \mathcal{L}(A, \mathcal{L}(A))$ ,  $\|I\| \leq 1$ “: Linearity is obvious,  $\|L_a(x)\| = \|a \cdot x\| \leq \|a\| \cdot \|x\|$ , so  $\|L_a\| \leq \|a\|$  and so  $\|I\| \leq 1$ . Since it is easily  $I$  preserves multiplication, so we are left to prove the „Moreover“ part.

„ $A$  has a unit  $e$ “: WLOG  $A \neq \{\mathbf{o}\}$ .

$$\forall a \in A : \|Ia\| = \|L_a\| \geq \|L_a\left(\frac{e}{\|e\|}\right)\| = \frac{\|a\|}{\|e\|} = \frac{1}{\|e\|} \cdot \|a\|.$$

So  $I$  is bounded from below, so  $I$  is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \text{id}.$$

Finally, if  $\|x^2\| = \|x\|^2$ ,  $x \in A$ , then  $\forall a \in A$ :

$$\|a\| \geq \|I(a)\| = \|L_a\| \geq \|L_a\left(\frac{a}{\|a\|}\right)\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So  $I$  is isometry.

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*Poznámka*

$A \neq \{\mathbf{o}\}$  Banach algebra with a unit  $\implies \exists$  equivalent norm  $\|\cdot\|$  on  $A$  such that  $(A, \|\cdot\|)$  is Banach algebra and  $\|e\| = 1$ .

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Let  $I : A \rightarrow \mathcal{L}(A)$  be as before. Put  $\|x\| := \|I(x)\|$ ,  $x \in A$ . Since  $I$  is isomorphism,  $\|\cdot\|$  is equivalent norm. Moreover,  $\|x \cdot y\| = \|I(x \cdot y)\| \leq \|I(x)\| \cdot \|I(y)\| = \|x\| \cdot \|y\|$ ,  $x, y \in A$ . So  $(A, \|\cdot\|)$  is a Banach algebra. Finally

$$\|e\| = \|I(e)\| = \|\text{id}\| = 1.$$

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□

## 1.2 Inverse elements

### Definice 1.4

$(M, \cdot, e)$  is monoid ( $\cdot$  is associative,  $e$  is unit). Then invertible elements form a group ( $e^{-1} = e$ ,  $\exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ ); if  $x \in M$ , and  $y \in M$  is its left inverse and  $z \in M$  is its right inverse, then  $y = z$  is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote  $M^\times := \{x \in M \mid \exists x^{-1}\}$

### Tvrzení 1.5

If  $(A, \cdot, e)$  is monoid and  $x_1, \dots, x_n \in A$  commute, then  $x_1 \dots x_n \in A^\times \Leftrightarrow \{x_1, \dots, x_n\} \subset A^\times$ .

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It suffices to prove it for  $n = 2$  (and use induction). „If  $x^{-1}$  and  $y^{-1}$  exists, then  $(xy)^{-1}$ “ is easy from asociativity.

If we have  $(xy)^{-1}$ . Put  $z := (xy)^{-1}x$ . Then  $zy = (xy)^{-1}(xy) = e$ , so  $z$  is left inverse to  $y$ . Next we show that there is also right inverse: Put  $\tilde{z} := x(xy)^{-1}$ :  $y\tilde{z} = (xy)(xy)^{-1} = e$ , so  $\tilde{z}$  is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse.  $\square$

### Lemma 1.6

Let  $A$  be a Banach algebra with a unit.

- $\|x\| < 1 \implies \exists (e - x)^{-1} \wedge (e - x)^{-1} = \sum_{n=0}^{\infty} x^n$ ;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x + h)^{-1} \wedge \|(x + h)^{-1} - x^{-1}\| \leq \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$ .

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„First item“: We have  $\|x^n\| \leq \|x\|^n$ , so  $\sum_{n=0}^{\infty} x^n$  is absolute convergent series, so  $\sum_{n=0}^{\infty} x^n \in A$ . Moreover,

$$(e - x) \cdot \left( \sum_{n=0}^{\infty} x^n \right) = \lim_{N \rightarrow \infty} (e - x) \cdot (e + x + \dots + x^N) = \lim_{N \rightarrow \infty} e - x^{N+1} = e,$$

because  $\lim_{N \rightarrow \infty} \|x^{N+1}\| \leq \lim_{N \rightarrow \infty} \|x\|^{N+1} = 0$ . And similarly  $(\sum x^n) \cdot (e - x) = e$ .

„Second item“:  $x + h = x \cdot (e + x^{-1}h)$  we have  $x^{-1}$  exists and  $(e + x^{-1}h)^{-1}$  exists (from first item), so from previous fact  $(x + h)^{-1}$  exists. Moreover

$$(x + h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \stackrel{1)}{=} \sum_{n=0}^{\infty} (-x^{-1}h)^n x^{-1},$$

so

$$\begin{aligned} \|(x + h)^{-1} - x^{-1}\| &= \left\| \sum_{n=1}^{\infty} (-x^{-1}h)^n x^{-1} \right\| \leq \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leq \\ &\leq \|x^{-1}\| \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} (\|x^{-1}\| \cdot \|h\|)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}. \end{aligned}$$

*Důsledek*

A Banach algebra with a unit  $\implies A^x \subset A$  is open and  $A^x$  is topological group.

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$A^x \subset A$  is open by previous lemma (second item). So it remains to prove  $x \mapsto x^{-1}$  is continuous:

$$A^x \ni x_n \rightarrow x \in A^x \stackrel{?}{\implies} x_n^{-1} \rightarrow x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \stackrel{h:=x_n-x}{\leq} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \rightarrow 0.$$

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□

## 1.3 Spectral theory

### Definice 1.5 (Resolvent set, spectrum and resolvent)

A Banach algebra with a unit,  $x \in A$ . We define resolvent set of  $x$  as  $\varrho_A(x) := \{\lambda \in \mathbb{K} \mid \exists (\lambda \cdot e - x)^{-1}\}$ . Next we define spectrum of  $x$  as  $\sigma_A(x) := \mathbb{K} \setminus \varrho_A(x)$ . Finally we define resolvent of  $x$  as  $R_x : \varrho(x) \rightarrow A$ ,  $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$ .

If  $A$  doesn't have a unit, then notions above are defined with respect to  $A_e$ .

### Tvrzení 1.7

*A Banach algebra*

- a)  $\forall x \in A : 0 \in \sigma_{A_e}(x)$  (in particular, if  $A$  has no unit, then  $0 \in \sigma_A(x)$ );
- b)  $A$  has unit  $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$ .

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*Důkaz (a)*

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\dots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

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Důkaz (b))

By a) we have  $0 \in \sigma_{A_e}(x)$ . So it suffices:  $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$ . First means  $(\lambda \cdot e - x)^{-1}$  exists in  $A$  and second means that  $((0, \lambda) - (x, 0))^{-1} = (-x, \lambda)^{-1}$  exists in  $A$ . We take „ $x \rightarrow -x$ “.

„ $\Rightarrow$ “: find  $(b, \beta) \in A_e$  such that  $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$ . So  $(x \cdot b + \lambda \cdot b + \beta \cdot x, \lambda \cdot \beta) = (\mathbf{o}, 1)$ . So  $\beta = \frac{1}{\lambda}$  and  $b = -\frac{1}{\lambda}(\lambda e + x)^{-1} \cdot x$ . Similarly we find left inverse  $(-\frac{1}{\lambda}x(x + \lambda e)^{-1}, \frac{1}{\lambda}) (x, \lambda)$ . And next we prove that they are really inverses.

„ $\Leftarrow$ “: Put  $(b, \beta) := (x, \lambda)^{-1}$ . Then  $(\lambda e + x)^{-1} = b + \beta \cdot e$ . We have  $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$ , so  $\lambda \cdot \beta = 1$  and  $x \cdot b + \lambda \cdot b + \beta \cdot x = \mathbf{o}$ . Then

$$(\lambda e + x) \cdot (b + \beta \cdot e) = \lambda \cdot b + \lambda \cdot \beta \cdot e + x \cdot b + \beta \cdot x = e.$$

└ Similarly second inverse. □

## Věta 1.8

$\{\mathbf{o}\} \neq A$  complex Banach algebra,  $x \in A$ . Then  $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$  is compact, nonempty.

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└ After theory. □

## Definice 1.6 (Derivative)

$Y$  Banach space,  $\Omega \subset \mathbb{K}$ ,  $f : \Omega \rightarrow Y$ ,  $a \in \Omega$ . Then

$$f'(a) := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of  $f$  at  $a$ .

## Tvrzení 1.9 (Fact)

$Y$  Banach,  $\Omega \subset \mathbb{K}$ ,  $f : \Omega \rightarrow Y$ ,  $a \in \Omega$ . Then  $f'(a)$  exists  $\implies f$  is continuous at  $a \wedge \forall x^* \in Y^* : (x^* \circ f)'(a) = x^*(f'(a))$ .

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Continuity:  $\lim_{x \rightarrow a} f(x) - f(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = f'(a) \cdot 0 = 0$ .

$x^* \in Y^*$  given, then

$$\lim_{x \rightarrow a} \frac{x^*(f(x)) - x^*(f(a))}{x - a} = \lim_{x \rightarrow a} x^* \left( \frac{f(x) - f(a)}{x - a} \right) = x^*(f'(a)).$$

└ □

## Tvrzení 1.10

A Banach algebra with a unit,  $x \in A$ . Then

- $\varrho(x)$  is open set;
- $\forall |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$ ;
- (important!)  $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$  has derivative at each  $\lambda \in \varrho(x)$ ;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$ ;
- $\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$ .

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Důkaz

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left( e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{\lambda} \right)^n.$$

For third we fix  $\lambda \in \varrho(x)$  and  $t \in (0, \delta)$  for  $\delta$  small enough ( $\lambda + t \in \varrho(x)$  and \*). We shall prove that „ $R'_x(\lambda) = -R_x(\lambda)^2$ “:

$$0 \stackrel{?}{=} \left\| \frac{R_x(\lambda + t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| = \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leq$$

$$\begin{aligned} & \text{* for existence of the inverse} \\ & \leq \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \|(e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t\| = \end{aligned}$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leq$$

$$\stackrel{\|x^n\| \leq \|x\|^n}{\leq} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \stackrel{\text{* for denominator} \leq 1/2}{\leq} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \rightarrow 0.$$

Fourth: In general  $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$  (proof:  $u^{-1}v^{-1} = (vu)^{-1}$ ). And we apply it for  $u = (\mu e - x)$  and  $v = (\nu e - x)$ .

Fifth: In general  $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$  (proof:  $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$ ) so:

$$\begin{aligned} R_x(\mu) - R_x(\nu) &= R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) \\ &= R_x(\mu)R_x(\nu) (R_x(\nu)^{-1} - R_x(\mu)^{-1}) = R_x(\mu)R_x(\nu)(\nu - \mu). \end{aligned}$$

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□



**Věta 1.11** (Liouville for Banach space valued functions)

$Y$  Banach space over  $\mathbb{C}$ ,  $f : \mathbb{C} \rightarrow Y$  has derivative at each point,  $f$  is bounded ( $\equiv \|f\|$  is bounded). Then  $f \equiv \text{const}$ .

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Assume  $f \not\equiv \text{const}$ , so there are  $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$  (by Hahn–Banach theorem)  $\exists x^* \in Y^* : x^*(f(a)) \neq x^*(f(b))$ . From fact  $x^* \in f : \mathbb{C} \rightarrow \mathbb{C}$  has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions.  $\square$

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*Důkaz* (Theorem before theory)

First case: „ $A$  has a unit“: Then  $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$  is closed, so  $\sigma(x)$  is compact. Assume that  $\varrho(x) = \mathbb{C}$ . By previous tvrzení we have  $R_x : \mathbb{C} \rightarrow A$  has derivative everywhere, and it is bounded because  $\lim_{|\lambda| \rightarrow \infty} \lambda \rightarrow \infty R_x(\lambda) = \lim_{|\lambda| \rightarrow \infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$ . From previous theorem  $R_x \equiv \text{const}$  so  $\lim_{|\lambda| \rightarrow \infty} R_x(\lambda) = 0 \implies R_x \equiv 0$ . In particular  $0 = R_x(0) = (-x)^{-1}$ .  $\nexists$  (If  $A \neq \{0\}$  then  $x^{-1} \neq 0$  for  $x \in A$ .)

Second case: „ $A$  hasn't a unit“, then  $\sigma(x) := \sigma_{A_e}((x, 0))$  so we apply the already proven case.  $\square$

*Poznámka* (Convention)

If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

**Věta 1.12** (Gelfand–Mazur)

$\{\mathbf{o}\} \neq A$  Banach algebra with a unit. Assume  $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$ . Then  $A$  is isomorphic to  $\mathbb{C}$ . If moreover  $e$  is a unit in  $A$  and  $\|e\| = 1$ , then  $A$  is isometrically isomorphic to  $\mathbb{C}$ .

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Consider  $\psi : \mathbb{C} \rightarrow A$  defined as  $\psi(\lambda) := \lambda \cdot e$ . This is algebraic homomorphism and  $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$ , so it is isomorphism (and isometry, if  $\|e\| = 1$ ).

It remains „ $\varphi$  is surjective“: Pick  $a \in A$ . From previously proved theorem  $\exists \lambda \in \sigma(a)$ , then  $(\lambda e - a) \notin A^x$ . So,  $\lambda \cdot e - a = 0$ , then  $\psi(\lambda) = a$ .  $\square$

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**Definice 1.7** (Spectral radius)

A Banach algebra,  $x \in A$ . Then  $r(x) := \sup \{|\lambda|, \lambda \in \sigma(x)\}$  is called spectral radius of  $x$ .

**Věta 1.13** (Beurling–Gelfand)

A Banach algebra,  $x \in A \implies r(x) = \inf_{n \in \mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$ .

### Lemma 1.14

A Banach algebra with a unit,  $x \in A$ . For  $p(z) = \sum_{j=1}^n \alpha_j z^j \in \mathbb{C}$  a polynom (with complex coefficients) we put  $p(x) = \sum_{j=1}^n \alpha_j x^j \in A$ . Then  $\sigma(p(x)) = p(\sigma(x))$ .

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*Důkaz*

Fix  $\lambda \in \mathbb{C}$  and write  $(\lambda - p)(z) = c \cdot \prod_{i=1}^m (z - z_i)$ , where  $z_1, \dots, z_m$  are roots of  $\lambda - p$ . Then  $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$  does not exist.  $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^m (x - z_i \cdot e)$ , so it doesn't exist if and only if  $\exists i \in [m]$ , such that  $(x - z_i \cdot e)^{-1}$  doesn't exist  $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow \exists$  root  $\nu$  of  $\lambda - p$  such that  $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$ .  $\square$

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*Důkaz (Beurling–Gelfand)*

WLOG  $A$  has a unit. Step 1, „ $r(x) \leq \inf_n \sqrt[n]{\|x^n\|}$ “: fix  $\lambda \in \sigma(x)$ . By previous lemma  $\forall n : \lambda^n \in \sigma(x^n)$ . By theorem 'Before theory' we have  $\forall n : |\lambda|^n \leq \|x^n\|$ .

Step 2, „ $r(x) \geq \limsup_n \sqrt[n]{\|x^n\|}$ “: Pick  $r > r(x)$ . Claim: „ $\frac{x^n}{r^n} \rightarrow^w 0$ “: Fix  $x^* \in A^*$  and put  $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$ . By fact and tvrzení after it,  $f$  has derivative at each  $\lambda \in \rho(x)$ . Moreover for  $|\lambda| \geq \|x\|$  we have  $f(\lambda) = \lambda \cdot x^* \left( \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$ . Thus  $f(\lambda) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$ ,  $\lambda \in P(0, r(x), \infty)$ . From Complex analysis  $f \in H(P(0, r, \infty))$  is uniquely given by Laurent series. In particular  $f(r) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{r^n}$ , so  $x^* \left( \frac{x^n}{r^n} \right) \rightarrow 0$ .

From principle of unique boundedness (last semester):  $\frac{x^n}{r^n}$  if  $\|\cdot\|$ -bounded, so  $\exists c > 0 : \|x^n\| \leq cr^n$ ,  $\sqrt[n]{\|x^n\|} \leq \sqrt[n]{c} \cdot r \rightarrow r$ . So  $\limsup \sqrt[n]{\|x^n\|} \leq r$ .  $\square$

*Důsledek*

A Banach algebra,  $x \in A$  and  $|\lambda| > r(x)$ . Then  $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$  is absolutely convergent and  $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$ .

┌

*Důkaz*

Fix  $q$ , such that  $\frac{r(x)}{|\lambda|} < q < 1$ . By previous theorem,  $\exists n_0 \forall n \geq n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$ , so  $\frac{\|x^n\|}{|\lambda|^n} < q^n$ ,  $n \geq n_0$ . Thus  $\sum \left\| \frac{x^n}{\lambda^n} \right\| \leq \infty$ , so the sum is absolutely convergent.

└

Now we easily check that  $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$ .  $\square$

## 1.4 Subalgebra

### Věta 1.15

A Banach algebra with a unit  $e$ ,  $B \subset A$  is closed subalgebra such that  $e \in B$ . Fix  $x \in B$ . Then

- $C \subset \rho_A(x)$  is component (maximum connected subset)  $\implies C \subseteq \sigma_B(x)$  or  $C \cap \sigma_B(x) = \emptyset$ ;

- $\partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$ ;
- $\varrho_A(x)$  is connected  $\implies \sigma_A(x) = \sigma_B(x)$ ;
- $\text{int } \sigma_B(x) = \emptyset \implies \sigma_A(x) = \sigma_B(x)$ .

*Důkaz*

„ $\sigma_A(x) \subseteq \sigma_B(x)$ “:  $(\lambda e - x)^{-1}$  exists in  $B$  implies it exists (it's same) in  $A$ .

„First item“: Let  $C \subset \varrho_A(x)$  be component. Pick  $\lambda_0 \in C \cap \sigma_B(x)$ . Wanted: „ $C \setminus \sigma_B(x) = \emptyset$ “. Pick  $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$  (separate  $B$  and  $R_x(\lambda) \notin B$ ). Then  $C \ni \lambda \mapsto x^*(R_x(\lambda))$  is holomorphic function on open (because maximum) connected set  $C$ . Which is zero<sup>a</sup> on  $C \setminus \sigma_B(x)$ .

Since  $C \setminus \sigma_B(x)$  is open, if it is nonempty it contains a ball, so it has cluster point. Thus  $C \ni \lambda \mapsto x^*(R_x(\lambda))$  is such that  $\{\lambda \in C \mid x^*(R_x(\lambda))\} = 0$  has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero.  $\nLeftarrow$ with  $x^*(R_x(\lambda_0)) = 1$ .

„Second item“: Pick  $\lambda \in \sigma_B(x) \setminus \sigma_A(x)$  and let  $C \subset \varrho_A(x)$  be a component containing  $\lambda$ . By first item,  $C \subseteq \sigma_B(x)$ ,  $C$  is open, so  $\lambda \in C \subseteq \text{int}(\sigma_B(x))$ .

„Third item“: If  $\varrho_A(x)$  is connected, we can apply first item to  $C = \varrho_A(x)$ , we have either  $\varrho_A(x) \subseteq \sigma_B(x)$  or  $\varrho_A(x) \cap \sigma_B(x) = \emptyset$ . But first is not possible, because  $\varrho_A(x)$  is unbounded and  $\sigma_B(x)$  is bounded. Therefore  $\sigma_B(x) \subseteq \sigma_A(x)$ .

„Fourth item“: If  $\text{int}(\sigma_B(x)) = \emptyset$ , then (by second item)  $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$ . □

<sup>a</sup>For  $\lambda \in C \setminus \sigma_B(x)$ ,  $(\lambda e - x)^{-1}$  exists in  $B$  so  $R_x(\lambda) \in B$  and therefore,  $x^*(R_x(\lambda)) = 0$

*Důsledek*

A Banach algebra,  $B \subseteq A$  closed subalgebra,  $x \in B$ . Then all items from previous theorem hold as well if we replace  $\sigma_A(x)$  and  $\sigma_B(x)$  by  $\sigma_A(x) \cup \{0\}$  and  $\sigma_B(x) \cup \{0\}$ .

*Důkaz*

Without proof. (Basically same that previous; we add unit to  $A$  and  $B$ , so this unit is same  $((\mathbf{o}, 1))$ , etc.) □

## 1.5 Holomorphic calculus

### Definice 1.8

$X$  Banach,  $\gamma : [a, b] \rightarrow \mathbb{C}$  path (continuous, piecewise smooth ( $C^1$ )),  $f : \langle \gamma \rangle \rightarrow X$  continuous. Then

$$\int_{\gamma} f := \int_{[a,b]} \gamma'(t) f(\gamma(t)) dt. \quad (\text{As Bochner integral.})$$

If  $\Gamma = \gamma_1 + \dots + \gamma_n$  is chain in  $\mathbb{C}$ ,  $f : \langle \Gamma \rangle \rightarrow X$  continuous, then

$$\int_{\Gamma} f := \sum_{i=1}^n \int_{\gamma_i} f.$$

### Lemma 1.16

$\Gamma$  chain in  $\mathbb{C}$ ,  $X$  Banach,  $f : \langle \Gamma \rangle \rightarrow X$ ,  $x \in X$ . Then

$$\int_{\Gamma} f = x \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_{\Gamma} x^* \circ f.$$

┌

*Důkaz*

„ $\Leftarrow$ “ by Hahn–Banach theorem. „ $\Rightarrow$ “: (by previous semester  $x^*$  and  $\int$  “commutes”)

$$x^* \left( \int_{\Gamma} f \right) = \sum_{i=1}^n x^* \left( \int_{\gamma_i} f \right) = \sum_{i=1}^n \int_{[a_i, b_i]} \gamma_i'(t) x^*(f(\gamma_i(t))) dt = \int_{\Gamma} x^* \circ f.$$

└

□

*Poznámka* (Recall)

If  $\Omega \subset \mathbb{C}$  open,  $K \subset \Omega$  compact. Then there is a cycle  $\Gamma$  such that  $\langle \Gamma \rangle \subset \Omega \setminus K$  and  $\text{ind}_{\Gamma} z = 1$  if  $z \in K$  and 0 if  $z \notin \Omega$ .

Then we say that  $\Gamma$  circulates  $K$  in  $\Omega$ .

### Definice 1.9

Let  $A$  be a Banach algebra with unit,  $x \in A$ ,  $\Omega \subset \mathbb{C}$  open and  $\sigma(x) \subset \Omega$ ,  $f \in \mathcal{H}(\Omega)$ . Then  $f(x) := \frac{1}{2\pi i} \int_{\Gamma} f \cdot R_x$ , where is any cycle which circulates  $\sigma(x)$  in  $\Omega$ .

*Poznámka*

$f(x)$  exists ( $f \cdot R_x$  is continuous on  $\langle \Gamma \rangle$ ),  $f(x)$  does not depend on the choice of  $\Gamma$  (Pick  $x^* \in X^*$ , then  $(x^* \circ f \cdot R_x)(\lambda) = f(\lambda) \cdot x^*(R_x(\lambda))$  is holomorphic. Pick  $\Gamma_1, \Gamma_2$  cycles circulating  $\sigma(x)$  in  $\Omega$ , then  $\int_{\Gamma_1 - \Gamma_2} x^* \circ f \cdot R_x = 0$  from Cauchy).

### Věta 1.17 (Holomorphic calculus)

A Banach algebra with unit,  $x \in A$ ,  $\Omega \subset \mathbb{C}$  open such that  $\sigma(x) \subset \Omega$ ,  $f \in \mathcal{H}(\Omega)$ . Then  $\Phi : \mathcal{H}(\Omega) \rightarrow A$  defined as  $\Phi(f) = f(x)$  (from definition above) has the following properties:

- $\Phi$  is algebra homomorphism,  $\Phi(1) = e$ ,  $\Phi(\text{id}) = x$ ;
- $f_n \xrightarrow{\text{loc.}} f$  in  $H(\Omega)$ , then  $f_n(x) \rightarrow f(x)$ ;
- $f(x)^{-1}$  exists  $\Leftrightarrow f \neq 0$  on  $\sigma(x)$ , in this case  $f(x)^{-1} = \frac{1}{f}(x)$ ;

- $\sigma(f(x)) = f(\sigma(x))$ ;
- if  $\Omega_1$  is open and  $f(\sigma(x)) \in \Omega_1$ ,  $g \in \mathcal{H}(\Omega_1)$ , then  $(g \circ f)(x) = g(f(x))$ ;
- if  $y \in A$  commutes with  $x$ , then  $y$  commutes with  $f(x)$ .

Moreover, if  $\psi : \mathcal{H}(\Omega) \rightarrow A$  satisfy first two item, then  $\psi = \Phi$ .

### Lemma 1.18

$(\Omega, \mu)$  complete measurable space,  $A$  Banach algebra,  $f \in L_1(\mu, A)$ . Let  $x \in A$  and  $E \subset \Omega$  is measurable. Then

$$x \cdot \left( \int_E f(t) d\mu(t) \right) = \int_E x \cdot f(t) d\mu(t), \quad \left( \int_E f(t) d\mu(t) \right) \cdot x = \int_E f(t) \cdot x d\mu(t).$$

┌ *Důkaz*

Easy (by commutation of integral and linear operator from last semester), skipped. □

*Důkaz* (Holomorphic calculus)

„1st item“: „ $\Phi$  is linear“ is easy, „ $\Phi$  is multiplicative“: Pick  $f, g \in \mathcal{H}(\Omega)$ , open set  $U$  such that  $\sigma(x) \subset U \subset \overline{U} \subset \Omega$ . Let  $\Gamma$  cycle circulating  $\sigma(x)$  in  $U$ ,  $\Lambda$  cycle circulating  $\overline{U}$  in  $\Omega$ . Then

$$\begin{aligned} f(x) \cdot g(x) &= \left( \frac{1}{2\pi i} \int_{\Gamma} f \cdot R_x \right) \cdot g(x) \stackrel{\text{lemma}}{=} \frac{1}{2\pi i} \int_{\Gamma} f(t) R_x(t) g(x) dt = \frac{1}{2\pi i} \int_{\Gamma} f(t) \cdot R_x(t) \cdot \frac{1}{2\pi i} \int_{\Lambda} g(s) \cdot R_x(s) ds \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(t) \cdot \frac{1}{2\pi i} \int_{\Lambda} g(s) \cdot R_x(t) \cdot R_x(s) ds dt = \end{aligned}$$

because  $\langle \Lambda \rangle \cap \langle \Gamma \rangle = \emptyset$ , we can use theorem after definition of  $R_x$ :

$$\begin{aligned} &= \frac{1}{(2\pi i)^2} \int_{\Gamma} \int_{\Lambda} f(t) \cdot g(s) \cdot \frac{R_x(t) - R_x(s)}{s - t} ds dt \stackrel{\text{Fubini to } x^*(\dots) \text{ and lemma}}{=} \\ &= \frac{1}{(2\pi i)^2} \int_{\Gamma} f(t) \left( \int_{\Lambda} \frac{g(s)}{s - t} ds \right) R_x(t) dt - \frac{1}{(2\pi i)^2} \int_{\Lambda} g(s) \left( \int_{\Gamma} \frac{f(t)}{s - t} dt \right) R_x(s) ds = \end{aligned}$$

(Now we use Cauchy theorem ( $f(z) \text{ ind}_{\Gamma} z = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw$ ).  $\forall s \in \langle \Lambda \rangle : (t \mapsto \frac{f(t)}{s - t}) \in \mathcal{H}(U) \wedge \text{ind}_{\Gamma} z = 0, z \notin U$ , so  $\int_{\Gamma} \frac{f(t)}{s - t} dt = 0$ .  $\forall t \in \langle \Gamma \rangle : \text{ind}_{\Lambda} t = 1 \wedge (s \mapsto g(s)) \in \mathcal{H}(\Omega) \implies g(t) = \frac{1}{2\pi i} \int_{\Lambda} \frac{g(s)}{s - t} ds$ .)

$$= \frac{1}{2\pi i} \int_{\Gamma} f(t) g(t) R_x(t) dt - 0.$$

It remains that „if  $f(z) = z^k$ ,  $k \in \mathbb{N} \cup \{0\}$  then  $f(x) = x^k$ “ (we want it for  $k = 0$  and

$k = 1$ ). Put  $\Gamma(t) = r \cdot e^{it}$ ,  $t \in [0, 2\pi]$ , where  $r > \|x\|$  arbitrary. By some theorem:

$$R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}, \quad |\lambda| > \|x\|.$$

Thus (we switch integral and sum, because later we realize that sum of integral of absolute value is finite)

$$\forall x^* \in A^* : x^*(f(x)) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k x^* \left( \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} \right) d\lambda = \frac{1}{2\pi i} \int_{\Gamma} \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=k-1}^{\infty} \int_{\Gamma} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda$$

because  $\Gamma$  (is  $2\pi$  periodic).

„2nd item“: For  $\Gamma = \gamma_1 + \dots + \gamma_N$ :

$$\|f_n(x) - f(x)\| = \frac{1}{2\pi i} \left\| \int_{\Gamma} (f_n(\lambda) - f(\lambda)) R_x(\lambda) d\lambda \right\| \leq \frac{1}{2\pi} \int_{\Gamma} |f_n(\lambda) - f(\lambda)| \cdot \|R_x(\lambda)\| d\lambda \leq \frac{1}{2\pi} \sum_{i=1}^N \int_{a_i}^{b_i} |\gamma_i'(t)| \sup_{z \in \langle \Gamma \rangle} \|R_x(z)\| dt$$

„Moreover part“: By Runge theorem (and second item) it is enough prove it for rational functions. If  $R$  was polynomial, then  $\Phi(R) = \Psi(R)$  by second item. So it suffices „ $\forall p$  polynomial:  $\frac{1}{p} \in \mathcal{H}(\Omega) \implies \Phi(\frac{1}{p}) = \psi(\frac{1}{p})$ “. Pick  $p$  polynomial. Then  $e = \psi(1) = \psi(p \cdot \frac{1}{p}) = \psi(p) \cdot \psi(\frac{1}{p}) = \Phi(p) \cdot \psi(\frac{1}{p})$  (similarly for  $\frac{1}{p} \cdot p$ ). So  $\psi(\frac{1}{p}) = \Phi(p)^{-1} = \Phi(\frac{1}{p})$ .

„3rd item“: „ $\implies$ “ Let  $f(z) = 0$  for some  $z \in \sigma(x)$ . Then exists  $g \in H(\Omega) : f(u) = (z - u)g(z)$ . By item one, we have  $(ze - x)g(x) = f(x) = g(x)(ze - x)$ . But  $(ze - x)^{-1}$  does not exist, so  $f(x)^{-1}$  does not exist.

„  $\Leftarrow$  “ Suppose  $f \neq 0$  on  $\sigma(x)$  by compactness.  $\exists \Omega_1 \subset \Omega$  open:  $\sigma(x) \subset \Omega_1$  and  $f \neq 0$  on  $\Omega_1$ . Then  $\frac{1}{f} \in H(\Omega_1)$  and by first item we have  $e = (f \cdot \frac{1}{f})(x) = f(x)\frac{1}{f}(x) = \dots = \frac{1}{f}(x) \cdot f(x) \implies f(x)^{-1} = \frac{1}{f}(x)$ .  $\square$

### Poznámka

$f = g$  on a neighbourhood of  $\sigma(x) \implies f(x) = g(x)$  (from definition), other implication doesn't hold!

## 1.6 Multiplicative functionals

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**Definice 1.10** (Multiplicative functional)

Let  $A$  be a Banach algebra. We say  $\varphi : A \rightarrow \mathbb{C}$  is multiplicative linear functional  $\equiv \varphi$  preserves  $+, \cdot, \cdot^*$ .

$$\Delta(A) := \{\varphi : A \rightarrow \mathbb{C} \mid \varphi \text{ multiplicative linear functional, } \varphi \not\equiv 0\}.$$

### Tvrzení 1.19

$A$  Banach algebra,  $\varphi \in \Delta(A) \cup \{0\}$ . Then

- $\exists! \tilde{\varphi} \in \Delta(A_e) : \tilde{\varphi}((x, 0)) = \varphi(x), \forall x \in A$ . It is given by

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda.$$

Moreover,  $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$ .

- $\forall x \in A : \varphi(x) \in \sigma(x)$  whenever  $\varphi \neq 0$ .
- $\Delta(A) \subseteq B_{A^*}$ .
- $A$  has a unit,  $\varphi \neq 0 \implies \|\varphi\| \geq \frac{1}{\|e\|}$ . In particular if  $\|e\| = 1$ , then  $\|\varphi\| = 1$ .

┌

*Důkaz*

„1. uniqueness“: For  $\tilde{\varphi} \in \Delta(A_e)$  such that  $\tilde{\varphi}((x, 0)) = \varphi(x), x \in A$ :

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda \tilde{\varphi}((\mathbf{o}, 1)) = \varphi(x) + \lambda,$$

second equality by  $\varphi \in \Delta(A) \implies \varphi(e) = \varphi(e^2) = \varphi^2(e)$ . „1. existence“ is proven by check that defined  $\tilde{\varphi}$  is multiplicative linear functional (and it is nonzero, but  $\tilde{\varphi}((0, 1)) = 1 \neq 0$ ). This is easy (omitted).

„ $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$ “: „ $\subseteq$ “:  $\varphi \in LHS$ , put  $\varphi(x) := \psi((x, 0))$ . Then  $\varphi \in \Delta(A) \cup \{0\}$  and  $\tilde{\varphi} = \psi$  became:

$$\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda = \psi((x, 0)) + \lambda = \psi((x, \lambda)).$$

„ $\supseteq$ “: We know already that  $\tilde{\varphi} \in \Delta(A_e)$ .

„2. with  $A$  has unit  $e$ “:  $\varphi \neq 0, \varphi \in \Delta(A)$ : If  $\lambda \in \varrho(x)$ , then  $\varphi(\lambda e - x) \neq 0$  ( $\varphi(x) \neq 0$  if  $x^{-1}$  exists).  $0 \neq \varphi(\lambda e - x) = \lambda - \varphi(x) \implies \lambda \neq \varphi(x)$ . Thus  $\varphi(x) \notin \varrho(x)$ , so  $\varphi(x) \in \sigma(x)$ . „2. with  $A$  hasn't unit“, then  $\varphi(x) = \tilde{\varphi}((x, 0)) \in \sigma_{A_e}((x, 0)) = \sigma_A(x)$ .

„3.“:  $\varphi \in \Delta(A)$ . Then  $\forall x \in A : \varphi(x) \in \sigma(x) \subseteq B(\mathbf{o}, \|x\|)$ , so  $|\varphi(x)| \leq \|x\|$ .

„4.“:  $A$  has a unit  $e$ , then  $\|\varphi\| \geq \left| \varphi\left(\frac{e}{\|e\|}\right) \right| = \frac{1}{\|e\|}$ . □

└

### Věta 1.20

$A$  Banach algebra,  $M := \Delta(A) \cup \{0\}$ . Then  $M \subset (B_{A^*}, w^*)$  is compact,  $\Delta(A)$  is locally compact and if  $A$  has a unit, then  $\Delta(A)$  is compact. The mapping  $\Phi : M \rightarrow \Delta(A_e)$ ,  $\Phi(\varphi) = \tilde{\varphi}$  is  $w^*-w^*$  homeomorphism.

*Důkaz*

By previous proposition,  $M \subset (B_{A^*}, w^*)$  ( $(B_{A^*}, w^*)$  is compact by previous semester). So, it suffices to check that  $M$  is  $w^*$ -closed.

$$M = \bigcap_{x, y \in A} \{\varphi \in A^* \mid \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}.$$

Sets from RHS is closed by previous semester, so,  $M$  is closed. Thus  $M$  is compact.

$\Delta \subset M$  is open, so  $\Delta(A)$  is locally compact (and  $M$  is 1-point compactification of  $\Delta(A)$ ). If  $\Delta$  has a unit, then  $\Delta(A) = \{\varphi \in M \mid \varphi(e) = 1\}$  is  $w^*$ -closed, so  $\Delta(A)$  is compact (and 0 is isolated in  $M$ ).

Finally, by previous proposition,  $\Phi$  is bijection.  $\Phi$  is  $w^*$ -continuous:

$$\varphi_i \xrightarrow{w^*} \varphi \implies \forall (x, \lambda) : \tilde{\varphi}_i((x, \lambda)) = \varphi_i(x) + \lambda \rightarrow \varphi(x) + \lambda = \tilde{\varphi}((x, \lambda)) \implies \tilde{\varphi}_i \xrightarrow{w^*} \tilde{\varphi}$$

So,  $\Phi$  is homeomorphism (continuous bijection on compact, last semester?).  $\square$

*Například*

$\Delta(\mathcal{C}(K)) = \{\delta_x \mid x \in K\}$ . ( $f \mapsto f(x)$  is multiplicative. Suppose  $\varphi \in \Delta(\mathcal{C}(K))$ ,  $\varphi \notin \{\delta_x \mid x \in K\}$ . So for  $x \in K$  there is  $g_x \in \mathcal{C}(x) : \varphi(g_x) \neq g_x(x)$ . Consider  $f_x = g_x - \varphi(g_x)$ . Then  $\varphi(f_x) = 0$ ,  $f_x(x) \neq 0$ . So there is  $U_x$  open neighbourhood of  $x$  such that  $f_x \neq 0$  on  $U_x$ . Compactness implies  $\exists x_1, \dots, x_n \in K : K \subset \bigcup_{i=1}^n U_{x_i}$ . Consider  $h := \sum_{i=1}^n |f_{x_i}|^2$ . Then  $h > 0$  on  $K$ , so  $h^{-1}$  exists and therefore  $\varphi(h) \neq 0$ . But  $\varphi(h) = \sum_{i=1}^n \varphi(f_{x_i}) \overline{\varphi(f_{x_i})} = 0$ .)

$\Delta\{M_n\} = \emptyset$ ,  $n \geq 2$ , where  $M_n$  is (non-commutative) algebra of  $n \times n$  matrices. ( $M_n = \text{LO}\{E^{i,j}\}$ .  $E^{ij} \cdot E^{kl} = E^{il}$  if  $j = k$ , else 0. So  $\varphi(E^{ij}) \cdot \varphi(E^{ij}) = \varphi(E^{ij} \cdot E^{ij}) = 0$  if  $i \neq j$ .  $\varphi(E^{ii}) = \varphi(E^{in} E^{ni}) = \varphi(E^{in}) \varphi(E^{ni}) = 0$ .  $\varphi(E^{nn}) = \varphi(E^{n1} E^{1n}) = 0$ .)

### Definition 1.11 (Ideal, maximal ideal)

A Banach algebra. Ideal in  $A$  is a subspace  $I \subset A$  if  $\forall x \in I \forall y \in A : x \cdot y \in I \wedge y \cdot x \in I$ .

Maximal ideal  $\equiv$  proper ( $I \neq A$ ) ideal and it is maximal proper ideal with respect to inclusion.

*Například* (2021, Johnson-Schetman, Acta mathematica)

$\mathcal{L}(L_p)$  has  $2^{2^\omega}$  non-isomorphic closed ideals.

### Tvrzení 1.21

A Banach algebra with a unit. Then:

- Any proper ideal is contained in a maximum ideal. (From Zorn's lemma. And  $I \subset A$  ideal is proper  $\Leftrightarrow e \notin I$ .)



- $I \subset A$  proper ideal  $\implies \bar{I} \in A$  is proper ideal. In particular, maximal ideals are closed. (Easy:  $\bar{I}$  is ideal. Moreover,  $I \cap A^* = \emptyset$  (if  $x \in I$  was invertible thus  $e = x \cdot x^{-1} \in I$ , but  $e \notin I$ ). So ( $A^*$  is open)  $\bar{I} \cap A^* = \emptyset$  and therefore  $e \notin \bar{I}$ .)

### Tvrzení 1.22

$A$  Banach algebra,  $I \subseteq A$  closed ideal  $\implies A/I$  is Banach algebra ( $[x] \cdot [y] := [x \cdot y]$ ).

*Důkaz*

Straightforward from definition. (Omitted.) □

*Poznámka*

From now on,  $A$  will be commutative.

Step 1: „Hahn-Banach“:  $I \subset A$  closed ideal  $\implies \exists \varphi \in \Delta(A) : \varphi|_I \equiv \dots$

### Věta 1.23

$A$  commutative Banach algebra with a unit. Then  $\Phi : \Delta(A) \rightarrow \{\text{maximal ideals in } A\}$ ,  $\Phi(\varphi) := \text{Ker } \varphi$ , is bijection.

*Důkaz*

Pick  $\varphi \in \Delta(A)$ . Then „Ker  $\varphi$  is maximal ideal“: ideal:  $y \in \text{Ker } \varphi, x \in A : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \dots \cdot 0 = 0$ , proper:  $\varphi \not\equiv 0$ , maximal:  $\text{codim Ker } \varphi = 1$ : pick  $x_0 : \varphi(x_0) \neq 0$ ,  $a = a - \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} + \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} \in \text{Ker } \varphi \oplus \mathbb{R}$ .

„ $\Phi$  is one-to-one“: Pick  $\varphi, \psi \in \Delta(A) : \text{Ker } \varphi = \text{Ker } \psi$ . Then (by lemma from previous semester)  $\varphi = c \cdot \psi$  for some  $c \in \mathbb{K}$ . But  $\varphi(e) = 1 = \psi(1)$  so  $\varphi = \psi$ .

„ $\Phi$  is surjective“: Let  $I \subset A$  be maximal ideal ( $\implies$  closed). Step 1 „Any nonzero element in  $A/I$  is invertible“: For contradiction assume  $\exists q(x) \in A/I$  ( $q(x) = [x]$ ),  $q(x) \neq 0 \wedge q(x)^{-1}$  does not exist. By next lemma  $q(x)(A/I)$  is proper ideal. Then  $q^{-1}(q(x)(A/I))$  is an ideal in  $A$  which is proper and  $I \subsetneq q^{-1}(q(x)(A/I))$ , which contradicts maximality of  $I$ . It follows from: ideal: follows from the fact that  $q$  is algebra homomorphism; proper:  $q(e) = [e] \notin q(x)A/I$ ;  $I \subseteq q^{-1}(\dots) : 0 \in q(x)A/I$ ;  $I \neq q^{-1}(\dots) : q(x) \neq 0 \implies x \notin I$ , but  $q(x) = q(x)q(e) \in q(x)(A/I)$ , so  $x \in q^{-1}(\dots)$ .

From Gelfand–Mazur theorem  $\exists$  surjective isomorphism  $j : A/I \rightarrow \mathbb{C}$ . Then  $\varphi := j \circ q \in \Delta(A)$ . It remains „ $I = \text{Ker } \varphi$ “:  $x \in \text{Ker } \varphi \Leftrightarrow j(q(x)) = 0 \Leftrightarrow q(x) = 0 \Leftrightarrow x \in I$ . □

### Lemma 1.24

$A$  commutative Banach algebra with a unit,  $x \in A$  does not have inverse  $\implies xA$  is proper ideal.

*Důkaz*

$xA$  is ideal, because  $A$  is commutative. Then  $xA$  is proper ( $e \notin xA$ ).  $\square$

*Důsledek* (Hahn–Banach like theorem)

$A$  is commutative Banach algebra with a unit,  $I \subset A$  proper ideal. Then  $\exists \varphi \in \Delta(A) : \varphi/I \equiv 0$ .

*Důkaz*

Let  $\tilde{I} \supseteq I$  be maximal ideal. By previous theorem there is  $\varphi \in \Delta(A) : \tilde{I} = \text{Ker } \varphi$ .  $\square$

### Tvrzení 1.25

$A, B$  Banach algebras,  $\Phi : A \rightarrow B$  algebraic isomorphism. Then  $\Phi^\# : \Delta(B) \rightarrow \Delta(A)$  defined as  $\Phi^\#(\varphi) := \varphi \circ \Phi$  is homeomorphism.

*Důkaz*

„ $\Phi^\#(\varphi) \in \Delta(A)$ “:  $\Phi^\#(\varphi) = \varphi \circ \Phi \in \Delta(A) \cup \{0\}$  and  $\varphi \not\equiv 0 \wedge \Phi$  is onto  $\implies \varphi \circ \Phi \neq 0$ .

„ $\Phi^\#$  is  $w^*$ - $W^*$  continuous“:  $\varphi_i \xrightarrow{w^*} \varphi \implies \varphi_i \circ \Phi \xrightarrow{w^*} \varphi \circ \Phi$ .

Apply the proven part to  $\Phi^{-1}$ , obtain that  $(\Phi^{-1})^\# : \Delta(A) \rightarrow \Delta(B)$  is  $w^*$ - $W^*$  continuous. Moreover we have  $\Phi^\# \circ (\Phi^{-1})^\# = \text{id} \wedge (\Phi^{-1})^\# \circ \Phi^\#$ .  $\square$

### Tvrzení 1.26

$L$  locally compact  $T_2$ . Then  $\delta : L \rightarrow \Delta(C_0(L)), x \mapsto \delta_x$  is homeomorphism onto.

*Důkaz*

„Case 1:  $L$  is compact“: By example  $\delta$  is onto. Of course,  $\delta$  is one-to-one, continuous. So  $\delta$  is homeomorphism.

„Case 2:  $L$  is not compact“: Then there is  $K = L \cup \{\infty\}$ , one-point compactification, and  $\{f \in \mathcal{C}(K) | f(\infty) = 0\} \ni f \mapsto f|_L \in C_0(L)$  is isometric isomorphism. Moreover  $\Phi : C_0(L)_e \rightarrow \mathcal{C}(K)$ ,  $\Phi(f, \lambda) := f + \lambda$ , is algebraic isomorphism.

So, we have  $K \xrightarrow{\eta} \Delta(C(K)) \xrightarrow{\Phi^\#} \Delta(C_0(L)_e) \xrightarrow{\psi} \Delta(C_0(L)) \cup \{0\}$ , where  $\eta$  is homeomorphism from case 1 and  $\psi(\varphi) := \varphi|_{C_0(L)}$ .

Thus  $\delta := \psi \circ \Phi^\# \circ \eta$  is homeomorphism between  $L \cup \{\infty\}$  and  $\Delta(C_0(L)) \cup \{0\}$ . Finally, for  $x \in K$  and  $f \in C_0(L)$ :

$$\Phi^\# \circ \eta(x)(f) = (\eta(x) \circ \Phi)(f) = f(x),$$

so  $\psi \circ \Phi^\# \circ \eta(x) = \Phi^\# \circ \eta(x)|_{C_0(L)} = \delta_x|_{C_0(L)}$ .  $\square$

### Věta 1.27

$K, L$  locally compact  $T_2$ . Then following is equivalent

- $\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$  as Banach algebra;
- $\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$  as algebras;
- $K \approx L$  as topological spaces.

┌

*Důkaz*

„1  $\implies$  2“ trivial. „2  $\implies$  3“:  $K \approx \Delta(\mathcal{C}_0(K)) \approx \Delta(\mathcal{C}_0(L)) \approx L$  from previous two tvrzení.

„3  $\implies$  1“: Given  $h : K \rightarrow L$  homeomorphism,  $f \mapsto f \circ h$  is isometry between Banach algebras. □

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