Poznámka
Stručný obsah: Diferencovatelnost v Banachových prostorech; Asplundovy prostory; slabé Asplundovy prostory; fragmentovanost a oddělovací spojitost; atd.

1 Diferencovatelnost

1.1 Základní pojmy

Poznámka
Většina by fungovala i pro NLP, ale my se pro jednoduchost zaměříme na Banachovy prostory.

Definice 1.1

X, Y reálné Banachovy prostory, $U \subset X$ otevřená, $f : U \to Y$, $x \in U$, $h \in X$:

$$\partial_h^+ f(x) = \lim_{t \to 0_+} \frac{f(x + t \cdot h) - f(x)}{t} \in Y, \text{ pokud existuje,}$$

$$\partial_h f(x) = \lim_{t \to 0} \frac{f(x + t \cdot h) - f(x)}{t} \in Y, \text{ pokud existuje.}$$

Poznámka
$\partial_{\mathbf{o}}^+ f(x) = \partial_{\mathbf{o}} f(x) = 0$. Pokud $\|h\| = 1$, pak je to směrová derivace.

Pokud $\alpha > 0$, pak $\partial_{\alpha h}^+ f(x) = \alpha \partial_h^+ f(x)$, má-li alespoň jedna strana smysl. Podobně pro $\alpha \in \mathbb{R} \setminus \{0\}$ je $\partial_{\alpha h} f(x) = \alpha \partial_h f(x)$, má-li alespoň jedna strana smysl (speciálně $\alpha = -1$).

$$\exists \partial_h f(x) \Leftrightarrow \exists \partial_{-h}^+ f(x) = -\partial_h^+ f(x).$$

Definice 1.2 (Gateauxova derivace)

X, Y reálné Banachovy prostory, $U \subset X$ otevřená, $f : U \to Y$, $x \in U$, $h \in X$: Pokud $\exists L \in \mathcal{L}(X, Y)$, že $\forall h \in X : L(h) = \partial_h f(x)$, značíme $f_g'(x) = L$.

Poznámka
Stačí, aby $\forall h \in X : L(h) = \partial_u^+ f(a)$. Znamená to, že $h \mapsto \partial_h^{(+)} f(x)$ je omezený lineární operátor.

Definice 1.3 (Fréchetova derivace)

f má v bodě $x \in U$ Fréchetovu derivaci, pokud $\exists L \in \mathcal{L}(X,Y)$:

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - L(h)}{\|h\|} = 0.$$

Poznámka
Pokud takové L existuje, nutně platí $L = f'_g(x)$. Fréchetovu derivaci značíme $f'_F(x)$.

Poznámka

$$\exists f'_F(x) \Leftrightarrow \exists f'_g(x) \wedge \lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = \partial_h f(x) \text{ stejnoměrně pro } h \in B_X \text{ (resp. } h \in S_X).$$

Důkaz
$f'_F(x)$ existuje $\Leftrightarrow$

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in X, \|h\| < \delta : \|f(x+h) - f(x) - \partial_h f(x)\| \leqslant \varepsilon \cdot \|h\|$$

Existenci $f'_g(x)$ máme, tedy: $\varepsilon > 0$ … najdeme to $\delta > 0$: $h \in B_x$, $t \in \mathbb{R}$, $0 < |t| < \delta$ $\implies$ $\|t \cdot h\| < \delta$:

$$\|f(x+th) - f(x) - \partial_{t\cdot h} f(x)\| \leqslant \varepsilon \|t \cdot h\| = \varepsilon \cdot |t|$$
$$\|\frac{f(x+th) - f(x)}{t} - \partial_h t(x)\| \leqslant \varepsilon$$

to dává stejnoměrnou konvergenci „ $\implies$ “.

„ $\impliedby$ “: Nechť $\forall \varepsilon > 0\ \exists \delta > 0\ \forall h \in \{x | \forall t \in P(\mathbf{o}, \delta)\}$:

$$\|\frac{f(x+t\cdot h) - f(x)}{t} - \partial_h f(x)\| \leqslant \varepsilon.$$

$\varepsilon > 0$ … najdeme to $\delta > 0$: Zvolíme $h \in X$, $0 < \|h\| < \delta \implies \frac{h}{\|h\|} \in S_X \implies$

$$\implies \|\frac{f(x+h) - f(h)}{\|h\|} - \frac{\partial_h f(x)}{\|h\|}\| \leqslant \varepsilon \implies$$

$$\implies \frac{\|f(x+h) - f(x) - \partial_h f(x)\|}{\|h\|} < \varepsilon.$$

□

Poznámka

1. $X = \mathbb{R}$, pak je F. derivace, G. derivace a běžná derivace to samé.

2. TODO?

3. TODO?

Tvrzení 1.1

$\dim X < \infty$, $U \subset X$ otevřená; $f : U \to Y$ lipschitzovská, $x \in U$, $f'_g(x)$ existuje $\implies$ $f'_F(x)$ existuje.

Důkaz

f lipschitzovská $\implies$ existuje $L > 0 : \|f(x) - f(y)\| \leqslant L \cdot \|x - y\|$ $(x, y \in U)$. Nechť existuje $f'_g(x)$. Potom $\forall \varepsilon > 0$ existuje $h_1, \dots, h_N \in S_X$ ε-síť. Nechť $\delta > 0$ je takové, že $B(x, \delta) \subset U$ a $0 < |t| < \delta \implies \|\frac{f(x+th_i)-f(x)}{t} - f'_g(x)(h_i)\| < \varepsilon$.

Vezmeme $h \in S_X$ libovolné, $0 < |t| < \delta$. Existuje i, že $\|h - h_i\| < \varepsilon$:

$$\|\frac{f(x+t\cdot h) - f(x)}{t} - f'_g(x)(h)\| \leqslant \|\frac{f(x+t\cdot h) - f(x+t\cdot h_i)}{t}\| + \|\frac{f(x+t\cdot h_i) - f(x)}{t} - f'_g(x)(h_i)\| + \|f'_g(x$$

□

Poznámka

Stačí lokálně lipschitzovská.

Tvrzení 1.2

$f : (a, b) \to \mathbb{R}$ konvexní $\implies$ $f'(x)$ existuje v každém bodě (a, b) až na spočetně mnoho.

Důkaz

1) $\forall x \in (a, b)$ existuje vlastní $f'_+(x)$, neboť $f'_+(x) = \lim_{y\to x+} \frac{f(y)-f(x)}{y-x}$, což je neklesající funkce v $y \in (x, b)$ a zdola omezená hodnotou $\frac{f(z)-f(x)}{z-x}$ pro $z \in (a, x)$.

2) $x \mapsto f'_+(x)$ je neklesající na (a, b). 3) Podobně pro f'_-. Tedy f je spojitá na (a, b). 4) $f'(x)$ neexistuje $\Leftrightarrow$ f'_+ má v bodě x skok. (f'_+ je spojitá v x $\implies$ $f'_x(x) = \lim_{y\to x-} f'_+(y) = \lim_{y\to x-} f'_-(y)$, $f'_-(y) \leqslant f'_+(y) \leqslant f'_-(z)$ pro $z > y$). □

Tvrzení 1.3

f convex and bounded from above on $B(x, r)$, $x \in X, r > 0$ $\implies$ f is Lipschitz on $B\left(x, \frac{1}{2}\right)$.

Důkaz
1) „$f \leqslant M$ on $B(x,r) \implies f \geqslant 2f(x) - M$ on $B(x,r)$“: $y \in B(x,r)$, $z := x + (x-y)$ $\implies z \in B(x,r)$, $x = \frac{1}{2}(y+z)$. $f(x) \leqslant \frac{1}{2}(f(y)+f(z))$, $f(y) \geqslant 2f(x) - f(z) \geqslant 2f(x) - M$.

2) Assume $|f| \leqslant M$ on $B(x,r)$. Take $v, w \in B(x, \frac{r}{2})$, $v \neq w$, $z := w + \frac{z}{2}\frac{w-v}{\|w-v\|} \implies$ $z \in B(x,r)$. $w(1 + \frac{z}{2\|w-v\|}) = z + \frac{z}{2\|w-v\|}v$,

$$f(w) \leqslant \frac{f(z) + \frac{z}{2\|w-v\|}f(v)}{1 + \frac{z}{2\|w-v\|}}$$

$$f(w) - f(v) \leqslant \frac{f(z) + f(v)}{1 + \frac{z}{2\|w-v\|}}$$

$$\frac{f(w) - f(v)}{\|w-v\|} \leqslant \frac{f(z) - f(v)}{\|w-v\| + 1/2} \leqslant \frac{2M}{\frac{r}{2}} = \frac{4M}{r}$$

$\implies f$ is $\frac{4M}{r}$-lipschitz on $B(x, \frac{y}{2})$. □

Důsledek

- $\dim X < \infty$, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex $\implies f$ is locally lipschitz on U. (WLOG: $X = (\mathbb{R}^n, \|\cdot\|_1)$. $x \in U \implies \exists r > 0 \overline{B_{\|\cdot\|_1}(x,r)} \subset U$. $\overline{B_{\|\cdot\|_1}(x,r)} = \overline{\operatorname{conv}\{x \pm re_i | i \in [n]\}}$. $f \leqslant \max_{i \in [n]} f(x \pm r \cdot e_i)$ on $\overline{B_{\|\cdot\|_1}(x,r)} \implies f$ is Lipschitz on $\overline{B_{\|\cdot\|_1}(x, \frac{r}{2})}$)
- $\dim X < \infty$, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex, $x \in U \implies f'_F(x)$ exists if and only if f'_g („ $\implies$ “ always, „ $\impliedby$ “ from first item and tvrzeni above).
- X Banach space, $U \subset X$ open convex, $f : U \to \mathbb{R}$ continuous convex, then f is locally Lipschitz on U (f continuous $\implies f$ is locally bounded $\implies f$ is locally Lipschitz).

Věta 1.4

$X = l_1$, $f : X \to \mathbb{R}$, $f(x) = \|x\| = \sum_{n=1}^{\infty} |x_n|$.

$$\exists f'_g(x) \Leftrightarrow \forall n \in \mathbb{N} : x_n \neq 0. \qquad \implies f'_g(x) = (\operatorname{sgn} x_n)_{n=1}^{\infty} \in l_\infty,$$

$$\forall x \in l_1 \ \nexists f'_F(x).$$

Důkaz
1) $x \in l_1$, $n \in \mathbb{N}$, $x_n = 0$. Take $h = e_n \sum_{k \neq n} |x_k| + |t|$. $\partial_h f(x) = \lim_{t \to 0} \frac{\|x + t \cdot e_n\| - \|x\|}{t} = \lim_{t \to} \frac{|t|}{t}$ doesn't exist. This prove „ $\implies$ “.

„ $\impliedby$ “: Assume $\forall n \in \mathbb{N}$: $x_n \neq 0$, $h \in l_1$, $h \neq 0$, $\varepsilon > 0$:

$$\left| \frac{f(x + t \cdot h) - f(x)}{t} - \sum_{n=1}^{\infty} h_n \cdot \operatorname{sgn} x_n \right| = \left| \frac{1}{t} \sum_{n=1}^{\infty} (|x_n + t \cdot h_n| - |x_n| - t h_n \operatorname{sgn} x_n) \right| \leqslant \left| \frac{1}{t} \sum_{n=1}^{N} (\ldots) \right| + \left| \frac{1}{t} \sum_{n>N} (\right.$$

□

TODO?

2 Subdiferential

Definice 2.1

X Banach, $U \subset X$ open + convex, $f : U \to \mathbb{R}$ convex + continuous ($\implies$ locally Lipschitz). $x \in U$,

$$\partial f(x) := \{x^* \in X^* | \forall y \in U : x^*(y - x) \leqslant f(y) - f(x)\}.$$

Poznámka

$$\forall h \in X \ \exists \partial_h^+ f(x)$$

$$x^* \in \partial f(x) \Leftrightarrow \forall h \in X : x^*(h) \leqslant \partial_h^+ f(x)$$

(„ $\implies$ “: Fix $h \in X$, find $\delta > 0$: $\forall |t| < \delta : x + t \cdot h \in U$. Then $\forall t \in (0, \delta) : x^*(x + t \cdot h - x) \leqslant f(x + t \cdot h) - f(x)$, $x^*(h) \leqslant \frac{f(x+t \cdot h) - f(x)}{t} \to \partial_h^+ f(x)$. „ $\impliedby$ “: Fix $y \in X$, put $h := y - x$. Then $x^*(y - x) = x^*(h) \leqslant \partial_h^+ f(x) \leqslant \frac{f(x+h) - f(x)}{1} = f(y) - f(x)$.)

$$U = X, f(x) = \|x\| \implies \partial f(x) = \{x^* \in B_{X^*} | x^*(x) = \|x\|\}.$$

(„$\subseteq$“ Let $x^* \in \partial f(x)$. Then $x^*(x) \leqslant \|x + x\| - \|x\| = \|x\|$, $x^*(-x) \leqslant \|0\| - \|x\| = -\|x\|$. Thus $x^*(x) = \|x\|$. And for $h \in X : x^*(h) \leqslant \|x + h\| - \|x\| \leqslant \|h\|$, therefore $\|x^*\| \leqslant 1$. „$\supseteq$“: Let $x^* \in B_{X^*}$, $\|x\| = x^*(x)$. Then $\forall y \in X : x^*(y - x) = x^*(y) - x^*(x) \leqslant \|y\| - \|x\|$.)

Tvrzení 2.1

$\forall x \in U : \partial f(x) \neq \emptyset$, convex, w^*-compact.

Důkaz
$h \mapsto \partial_h^+ f(x)$ is sublinear functional ($t \cdot \partial_h^+ f(x) = \partial_{t\cdot h}^+ f(x)$, $t > 0$, and

$$\partial_{h_1+h_2}^+ f(x) = \lim_{t\to 0_+} \frac{f(x+t\cdot(h_1+h_2)) - f(x)}{t} \leqslant \lim_{t\to 0_+} \left(\frac{f(x+2\cdot t\cdot h_1) - f(x)}{2t} + \frac{f(x+2\cdot t\cdot h_2) - f(x)}{2t} \right.$$

so it is sublinear functional).

By Hahn–Banach theorem, $\exists x^* \in X^\# : x^*(h) \leqslant \partial_h^+ f(x)$, $h \in X$. Moreover x^* is continuous ($x^* \in X^*$), because f is locally Lipschitz, so $\exists r > 0\ \exists L > 0 : f|_{B(x,r)}$ is L–Lipschitz, so $\left|\frac{f(x+t\cdot h)-f(x)}{t}\right| \leqslant L \cdot \|h\|$ and so $x^*(h) \leqslant \partial_h^+ f(x) \leqslant L \cdot \|h\|$, $h \in X$.

So by remark $x^* \in \partial f(x)$. Thus $\partial f(x) \neq \varnothing$. And also $\forall y^* \in \partial f(x)$. $\|?x\| \leqslant L$. Thus $\partial f(x)$ is bounded, so $\subseteq R(B_{X^*}, w^*)$ for some $R > 0$, which is w^*-compact. So since $\partial f(x)$ is w^*-closed, it is w^*-compact. (It is closed, because $\partial f(x) = \bigcap_{y\in U} \{x^* \in X^* | x^*(y-x) \leqslant f(y) - f(x)\}$).

Finally „$\partial f(x)$ is convex“: For $x^*, y^* \in \partial f(x)$, $\lambda \in (0,1)$:

$$\forall y \in U : (\lambda x^* + (1-\lambda)y^*)(y-x) \leqslant \lambda(f(y)-f(x)) + (1-\lambda)(f(y)-f(x)) = f(y) - f(x).$$

□

Tvrzení 2.2

$x \in U$. Then following is equivalent:

- $\exists f'_G(x)$;
- $|\partial f(x)| = 1$;
- $\forall h \in X : \partial_h^+ f(x) = -\partial_{-h}^+ f(x)$.

Moreover $\partial f(x) = \{f'_G(x)\}$, if one of item is true.

Důkaz

„1. $\implies$ 2.“: We have $\forall h \in X : f'_G(x)(h) = \partial_h^+ f(x) \implies f'_G(x) \in \partial f(x)$. Moreover

$$\forall x^* \in \partial f(x)\ \forall h \in X : x^*(h) \leqslant \partial_h^+ f(x) = f'_G(x)(h) \wedge -x^*(h) = x^*(-h) \leqslant f'_G(x)(-h) = -f'_G(x)(h) \implies x$$

„2. $\implies$ 3.“: Let $\exists h \in X : \partial_h^+ f(x) \neq -\partial_{-h}^+ f(x)$. Always holds $\partial_h^+ f(x) \geqslant -\partial_{-h}^+ f(x)$ ($\varphi(t) = f(x + t \cdot h)$ is convex, then $-\partial_{-h}^+ f(x) = \partial'_-(0) \leqslant \partial'_+(0) = \partial_h^+ f(x)$). So $\partial_h^+ f(x) > -\partial_{-h}^+ f(x)$.

Define $x_1^*(t \cdot h) := t \cdot \partial_h^+ f(x)$ and $x_2^*(t \cdot h) := -t \partial_{-h}^+ f(x)$, $t \in \mathbb{R}$. Then $x_1^*, x_2^* \in (\mathrm{LO}(h))^*$. And for $j = 1, 2$:

$$x_j^*(t \cdot h) \leqslant \partial_{t \cdot h}^+ f(x), \qquad t \in \mathbb{R}.$$

For $t \geqslant 0 : x_1^*(t \cdot h) = t \partial_h^+ f(x) = \partial_{t \cdot h}^+ f(x)$. For $t < 0 : x_1^*(t \cdot h) = t \cdot x_1^*(h) = t \cdot \partial_h^+ f(x) < -t \cdot \partial_{-h}^+ f(x) = \partial_{t \cdot h} f(x)$. Same for x_2^*. By Hahn–Banach theorem, we extend x_j^*, $j \in \{1, 2\}$ to $x_j^* \in X^\#$ satisfying $x_j^*(z) \leqslant \partial_z^+ f(x)$, $z \in X$. And because f is locally Lipschitz, similarly as before we have $x_1^*, x_2^* \in X^*$. Thus $x_1^*, x_2^* \in \partial f(x)$ and $x_1^* \neq x_2^*$.

„3. $\implies$ 2.“: We know $\varphi : h \mapsto \partial_h^+ f(x)$ is sublinear and we know $\varphi(h) = -\varphi(-h)$. This implies, that φ is linear ($\varphi(t \cdot h) = t \cdot \varphi(h), t \in \mathbb{R}$ arbitrary, $\varphi(h_1 + h_2) \leqslant \varphi(h_1) + \varphi(h_2)$, $\varphi(h_1 + h_2) = -\varphi(-h_1 - h_2) \geqslant -(\varphi(-h_1) + \varphi(-h_2)) = \varphi(h_1) + \varphi(h_2)$). Moreover, φ is continuous, because $\varphi(h) \leqslant \varphi_h^+ f(x)$ and f is Lipschitz. □

Důsledek
$f(x) = \|x\|$, $x \in X$. Then $f'_G(x)$ exists $\Leftrightarrow \exists! x^* \in Bx^* : x^*(x) = \|x\|$.

TODO?

TODO?

Důsledek
$X = \mathbb{R}^n$, $U \subset X$ open, $f : U \to \mathbb{R}$ convex, $x \in U$. Then $f'_F(x)$ exists $\Leftrightarrow \forall i \in [n] : \frac{\partial f}{\partial x_i}(x)$ exists.

Důkaz

„ $\implies$ “ ?. „ $\impliedby$ “:

$$x^* \in \partial f(x) \implies x^*(e_i) \leqslant \partial_{e_i}^+ f(x) = \frac{\partial f}{\partial x_i}(x) \wedge x^*(-e_i) \leqslant \partial_{-e_i}^+ f(x) = -\frac{\partial f}{\partial x_i}(x) \implies x^*(e_i) = \frac{\partial f}{\partial x_i}(x) \implies$$

$\implies \partial f(x)$ contains at most one point $\implies f$ contains exactly one point $\implies f'_G(x)$ exists $\implies$ (locally Lipschitz, $\dim \mathbb{R}^n < \infty$) $f'_F(x)$ exists. □

Definice 2.2 (Monotone, upper semi-continuous (usc))

X Banach space, $D \subset X$, $T : D \to 2^{x^*}$ is monotone if $\forall x \in D : Tx \subset X^*$, $Tx \neq \emptyset$ and $\forall x, y \in D\ \forall x^* \in Tx\ \forall y^* \in Ty : \langle x^* - y^*, x - y \rangle \geqslant 0$.

Poznámka
$f : (a, b) \to \mathbb{R}$ is non-decreasing $\Leftrightarrow \forall x, y \in (a, b) : (f(x) - f(y))(x - y) \geqslant 0$

Let S and T be topological spaces. Then $\varphi : S \to 2^T$ is usc (upper semi-continuous) $\equiv$ $\forall U \subset T$ open: $\{x \in S | \varphi(x) \subset U\}$ is open in S.

Poznámka (This we will not use)
lsc $\equiv \forall U \subset T$ open $\{x \in S | \varphi(x) \cap U \neq \emptyset\}$ is open.

Tvrzení 2.3

X Banach space, $U \subset X$ open convex, $f : U \to \mathbb{R}$ convex continuous. Then $\partial f : U \to 2^{X^*}$ is

a) monotone;

b) locally bounded;

c) usc from $\|\cdot\|$ to w^*.

Důkaz
„a)“: $x, y \in U$, $x^* \in \partial f(x)$, $y^* \in \partial f(y)$. Then $x^*(y-x) \leqslant f(y)-f(x)$, $y^*(x-y) \leqslant f(x)-f(y)$.

$$x^*(y-x) + y^*(x-y) \leqslant 0, \qquad (x^*-y^*)(x-y) \geqslant 0.$$

„b)“: f is locally Lipschitz:

$$x \in U \implies \exists z > 0, L > 0, B(x,z) \subset U,$$

f is L-Lipschitz on $B(x,z) \implies$

$$\forall y \in B(x,z) : \partial f(y) \subset L \cdot B_{X^*}.$$

„c)“ $G \subset X^*$ w^*-open, $x \in U$, $\partial f(x) \subset G$. We want: „$\exists z > 0 : B(x,r) \subset I$ and $\forall y \in B(x,z) : \partial f(y) \subset G$“. It's enough to show „$\forall (x_n) \subset U, x_n \to x, \exists n_0\ \forall n \geqslant n_0 : \partial f(x_n) \subset G$“.

We show it by contradiction: Assume not, i.e., $\exists (y_n) \subset U, y_n \to x$, $\forall n : \partial f(y_n) \neq 0$. Fix $y_n^* \in \partial f(y_n) \backslash G$. By b) we know ? $\implies \exists R > 0 : \forall n : y_n^* \in \overline{B(0,R)}$ (in X^*).

Let y^* be a w^* cluster point of (y_n^*). Thus „$y^* \in \partial f(x)$“: If not, $\exists y \in U : y^*(y-x) > f(y) - f(x) \implies \varepsilon > 0 : y^*(y-x) \geqslant f(y) - f(x) + \varepsilon$, now $y_n^*(y - x + y_n - y_n) \leqslant f(y - x + y_n) - f(y_n)$ ($y - x + y_n \in U$ for large n).

So, for n large enough:

$$y_n^*(y-x) \leqslant f(y-x+y_n) - f(y_n)$$

with $n \to \infty$ LHS has cluster point $y^*(y-x)$ and $RHS \to f(y) - f(x)$

$$\implies y^*(y-x) \leqslant f(y) - f(x). \ ↯$$

(But $y^* \in X^* \backslash S \wedge y^* \in \partial f(y) \subset S$.) □