Poznámka
Topology...

Definice 0.1 (Topological vector space (TVS))

A Topological vector space over $\mathbb{F}$ is a pair (X, τ), where X is a vector space over $\mathbb{F}$ and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of $\mathbf{o}$ in (X, τ).

Definice 0.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka
Two homework (in Moodle) and one presentation.

Například
Let $(X, \|\cdot\|)$ be a normed linear space. Let τ be the topology induced by $\|\cdot\|$. The (X, τ) is HLCS.

Důkaz
$\varrho(x, y) = \|x - y\|$ metric induced by $\|\cdot\|$. τ induced by ϱ. This τ is Hausdorff. Continuity of the operations: (from Funkcionalka)

$$x_n \to x, y_n \to y, t_n \to t \implies x_n + y_n \to x + y \wedge t_n x_n \to tx.$$

So, it is a HTVS. Base of neighbourhood of $\mathbf{o}$ is e. g. $U(0, r), r > 0$, which is convex. □

Let Γ be any nonempty set, $X = \mathbb{F}^\Gamma$ (= all functions $\Gamma \to \mathbb{F}$) with point-wise operations, so it is a vector space over $\mathbb{F}$. It is a HLCS.

Důkaz

„Continuity of addition:“ $x, y \in \mathbb{F}^\Gamma$, U a neighbourhood of $x+y \implies \exists F \subset \Gamma$ finite $\exists \varepsilon > 0$ such that

$$U_{\mathbf{o}} = \left\{ z \in \mathbb{F}^\Gamma \middle| \; \forall \gamma \in F : |z(\gamma) - (x(\gamma) + y(\gamma))| < \varepsilon \right\} \subset U$$

$$U_x = \left\{ z \in \mathbb{F}^\Gamma \middle| \; \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$$U_y = \left\{ z \in \mathbb{F}^\Gamma \middle| \; \forall \gamma \in F : |z(\gamma) - y(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$\implies V_x$ is neighbourhood of x, and V_y is neighbourhood of y, and $U_x + U_y \subset U_0 \subset U$. Thus $z_1 \in V_x$, $z_2 \in V_y \implies z_1 + z_2 \in U_0 \subset U$.

„Continuity of multiplication“: $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^\Gamma$, U a neighbourhood of $\lambda x \implies \exists F \subset \Gamma$ finite $\exists \mu > 0$ such that

$$U_0 = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - \lambda x(\gamma)| < \varepsilon \right\} \subset U$$

$$|\mu z(\gamma) - \lambda x(\gamma)| \leqslant |\mu| \cdot |z(\gamma) - x(\gamma)| + |\mu - \lambda| \cdot |x(f)|.$$

$$M := \max_{\gamma \in F} |x(\gamma)|.$$

$$V = \left\{ \mu \in \mathbb{F} \middle| |\mu - \lambda| < \frac{\varepsilon}{2(M+1)} \right\}, \qquad W = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2(|\lambda| + \frac{\varepsilon}{2(M+1)})} \right\}.$$

$$\mu \in V, z \in W \implies \mu z \in U_0 \subset U.$$

„Local convexity“: Base of neighbourhoods of $\mathbf{o}$: $\left\{ x \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |x(\gamma)| < \varepsilon \right\}$, $F \subset \Gamma$ finite, $\varepsilon > 0$, consists of convex sets.

„Hausdorff“: $x \neq y \implies \exists \gamma \in \Gamma : x(\gamma) \neq y(\gamma)$. Take $\varepsilon = \frac{|x(\gamma) - y(\gamma)|}{2}$.

$$U = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - x(\gamma)| < \varepsilon \right\}, V = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - y(\gamma)| < \varepsilon \right\} \implies U \cap V = \emptyset.$$

□

$$X = C(\mathbb{R}, \mathbb{F}) = \{ f : \mathbb{R} \to \mathbb{F} \text{ continuous} \},$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \max_{t \in [-n,n]} \{ |f(t) - g(t)| \} \right\} =: \sum_{N=1}^{\infty} \frac{1}{2^N} \min \{ 1, p_N(f - g) \}$$

is translation invariant (that implies addition is continuous, see lecture) metric.

Důkaz
$f_n \to f$ in $\varrho \Leftrightarrow \forall N : f_n \rightrightarrows f$ on $[-N, N]$.

„$f_n \to f$, $\lambda_n \to \lambda \implies \lambda_n f_n \to \lambda f$“: Let $N \in \mathbb{N}$. We will show $\lambda_n f_n \rightrightarrows \lambda f$ in $[-N, N]$. $x \in [-N, N]$:

$$|\lambda_n f_n(x) - \lambda f(x)| \leqslant |\lambda_n| \cdot |f_n(x) - f(x)| + |\lambda_n - \lambda| \cdot |f(x)| \leqslant c \cdot p_N(f_n - f) + |\lambda_n - \lambda| \cdot p_N(f) \to 0.$$

Hence, X is HTVS. „Local convexity“: $U_{N,\varepsilon} = \{f \in X | p_N(t) < \varepsilon\}$, clearly $U_{N,\varepsilon}$ is a convex set and $U_{N,\varepsilon}$ is neighbourhood of $\mathbf{o}$. If $\varepsilon < \lambda$, then $\left\{f | \varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}\right\} \subset U_{N,\varepsilon}$, because for $\varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}$ it is $\frac{1}{2^N} p_N(f) < \frac{\varepsilon}{2^N}$. „they form a base“: $f \in U_{N,\varepsilon} \implies \varrho(f, \mathbf{o}) < \varepsilon + \frac{1}{2^N}$. Hence fix $r > 0$ and take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$. Then $U_{N,\frac{r}{2}} \subset \{f | \varrho(f, \mathbf{o}) < r\}$ □

(Ω, Σ, μ) a measure space, $p \in (0, 1)$. $L^p(\Omega, \Sigma, \mu) = \left\{f : \Omega \to \mathbb{F} \text{ measurable} | \int |f|^p d\mu < \infty\right\}$ (we identify functions equal almost everywhere). $\varrho(f, g) = \int |f - g|^p d\mu$ is a metric making $X = L^p(\Omega, \Sigma, \mu)$ a HTVS (but not locally convex).

Důkaz
„ϱ is a metric“: „$\triangle$-inequality“: $a, b \in [0, \infty) : (a + b)^p \leqslant a^p + b^p$. (Fix $a \geqslant 0$, take $\varphi_a(b) = (a + b)^p - a^p - b^p \implies \varphi_a$ is continuous on $[0, \infty)$, $\varphi_a(0) = 0$. For $b > 0$: $\varphi_a(b) = p(a + b)^{p-1} - pb^{p-1} = p \cdot ((a + b)^{p-1} - b^{p-1}) < 0$ as $p - 1 < 0 \implies \varphi_a$ decreasing on $[0, \infty)$ and $\varphi_a \leqslant 0$.)

φ is translation invariant $\implies$ addition is continuous. „Multiplication“: We can see that $\varrho(\lambda f, \mathbf{o}) = |\lambda|^p \varrho(f, \mathbf{o})$. $f_n \to f$, $\lambda_n \to \lambda$:

$$\varrho(\lambda_n f_n, \lambda f) \leqslant \varrho(\lambda_n f_n, \lambda_n f) + \varrho(\lambda_n f, \lambda f) = |\lambda_n|^p \varrho(f_n, f) + |\lambda_n - \lambda|^p \varrho(f, \mathbf{o}) \to 0.$$

Hence, we have a HTVS. □

Tvrzení 0.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x + U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 0.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1 - t)y \in A$ for $x, y \in A$, $t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}$, $|\alpha| \leqslant 1$;
- absolutely convex if it is convex and balanced;

- absorbing if $\forall x \in X\ \exists t > 0 : \{sX | s \in [0, t]\} \subset A$.

Definice 0.4

$\operatorname{co}(A)$ = convex hull, $\operatorname{b}(A)$ = balanced hull, $\operatorname{aco}(A)$ = absolutely convex hull.

Tvrzení 0.2

X is a metric space over $\mathbb{F}$, $A \subset X$. Then:

(a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow$ A is convex and symmetric.

(b) $\operatorname{co} A = \{t_1 x_1 + \ldots + t_k x_k | x_1 \ldots x_k \in A, t_1 \ldots t_k \geqslant 0, t_1 + \ldots + t_k = 1, k \in \mathbb{N}\}$.

(c) $\operatorname{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leqslant 1\}$.

(d) $\operatorname{aco}(A) = \operatorname{co}(\operatorname{b}(A))$.

(e) A is convex $\Leftrightarrow$ $(s+t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)
„ $\implies$ “: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „ $\impliedby$ “: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \implies \alpha \in [-1, 1].$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{(1+\alpha)}{2} x \in A$). □

Důkaz (b)
„$\subseteq$“: by induction on k:

$$t_1 x_1 + \ldots + t_{k+1} x_{k+1} = (t_1 + \ldots + t_k) \frac{t_1 x_1 + \ldots + t_k x_k}{t_1 + \ldots + t_k} + t_{k+1} x_{k+1}.$$

„$\supseteq$“: the set on the RHS is convex and contain A. □

Důkaz (c)
„$\supseteq$“: clear. „$\subseteq$“: RHS is a balanced set. □

Důkaz (d)
„$\supseteq$“: clear. „$\subseteq$“ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c): $\operatorname{co}(\operatorname{b}(A)) = \{t_1 \alpha_1 x_1 + \ldots + t_k \alpha_k x_k | x_1, \ldots, x_k \in A, |\alpha_j| \leqslant 1, t_j \geqslant 0, t_1 + \ldots + t$ is clearly balanced.) □

Důkaz (e)
„ $\implies$ “: „$\subseteq$“: always, „$\supseteq$“: $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t} a_1 + \frac{t}{s+t} a_2\right)$.

„ $\impliedby$ “: in particular $\forall t \in (0,1)$: $tA + (1-t)A \subset A$, it is the definition of convexity. □

Tvrzení 0.3

Let (X, τ) be a LCS, $U \in \tau(\mathbf{o})$. Then

(i) U is absorbing.

(ii) $\exists V \in T(0) : V + V \subset U$.

(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.

Důkaz (i)

$x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V$ a neighbourhood of 0 in $\mathbb{F} : V \cdot x \subset U \implies \exists t > 0 : [0, t] \subset V$

□

Důkaz (ii)

$$\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W \subset U.$$

Take $V = W_1 \cap W_2$. □

Důkaz

$$\exists U_0 \in \tau(\mathbf{o}) \text{ convex}, U_0 \subset U : \mathbf{o} \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0\ \exists W \in \tau(\mathbf{o}) \text{ open} :$$
$$\forall \lambda, |\lambda| < c : \lambda W \subset U_0.$$

$V_1 := \bigcup_{0<|\lambda|<1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \operatorname{co} V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „V is open“:

$$V = \bigcup \{t_1 x_1 + \ldots + t_n x_n + t_{n+1} V_1 | t_1, \ldots, t_{n+1} \geqslant 0, t_1 + \ldots + t_{n+1} = 1, x_1, \ldots, x_n \in V_1\}$$

□