TODO!!!

**Definice 0.1** (Dot product on the space of matrices)

$$\mathbb{A} : \mathbb{B} = \operatorname{tr}(\mathbb{A}\mathbb{B}^T).$$

**Definice 0.2** (Norm of matrix)

$$|\mathbb{A}| = (\mathbb{A} : \mathbb{A})^{\frac{1}{2}}.$$

*Příklad*

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

*Důkaz*

$$\mathbf{u} \cdot (\mathbf{a} \otimes \mathbf{b})^T \mathbf{v} = (\mathbf{a} \otimes \mathbf{b})\mathbf{u} \cdot \mathbf{v} = (\mathbf{a}(\mathbf{b} \cdot \mathbf{u}))\mathbf{v} = (\mathbf{b} \cdot \mathbf{u})(\mathbf{a} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{b}(\mathbf{a} \cdot \mathbf{v})) = \mathbf{u} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{v}.$$

□

*Příklad*

$$\det(e^{\mathbb{A}}) = e^{\operatorname{tr} \mathbb{A}}.$$

*Důkaz*

$$e^{\mathbb{A}} = \lim \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n.$$

$$\det e^{\mathbb{A}} = \lim_{n\to\infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n\right) = \lim_{n\to\infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)\right)^n =?$$

Subtask: Is there an approximation for $\det(\mathbb{I}+\mathbb{S})$, where $\mathbb{S}$ is a „small“ matrix. Yes, we did it (KontinuumDU1.pdf) for $\mathbb{S} \in \mathbb{R}^{3\times 3}$:

$$\det(\mathbb{I}+\mathbb{S}) = \det \mathbb{I} + \operatorname{tr}(\mathbb{I} \operatorname{cof} \mathbb{S}) + \operatorname{tr}(\mathbb{S}^T \operatorname{cof} \mathbb{I}) + \det \mathbb{S} \approx 1 + \operatorname{tr}(\mathbb{S}^T \operatorname{cof} \mathbb{I}) + o(\mathbb{S}^2) = 1 + \operatorname{tr}(S) + o(\mathbb{S}^2).$$

And for $\mathbb{S} \in \mathbb{R}^{n\times n}$, one can see that:

$$\det(\mathbb{I}+\mathbb{S}) = \begin{pmatrix} 1+s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & 1+s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \dots & 1+s_{nn} \end{pmatrix} = (1+s_{11})(1+s_{22})\cdot\ldots\cdot(1+s_{nn}) + o(\mathbb{S}^2) =$$

$$= 1 + s_{11} + s_{22} + \ldots + s_{nn} + o(\mathbb{S}^2) = 1 + \operatorname{tr}\mathbb{S} + o(\mathbb{S}^2).$$

$$? = \lim_{n\to\infty} \left(1 + \frac{\operatorname{tr}\mathbb{A}}{n} + \ldots\right)^n = e^{\operatorname{tr}\mathbb{A}}.$$

□

**Tvrzení 0.1**

$$\det(\mathbb{I}+\mathbb{S}) = 1 + \operatorname{tr}\mathbb{S} + \ldots$$

**Definice 0.3** (Gateaux derivative)

$$D\mathbf{f}(\mathbf{x})[\mathbf{y}] := \frac{d}{d\tau}\mathbf{f}(\mathbf{x}+\tau\mathbf{y})|_{\tau=0}.$$

**Definice 0.4** (Fréchet derivative)

$\mathbf{f}: U \to V$:

$$\lim_{\|\mathbf{y}\|_U \to 0} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})[\mathbf{y}]\|_V}{\|\mathbf{y}\|_V} = 0.$$

*Poznámka*

Sometimes we write $\nabla f(\mathbf{x}) \cdot \mathbf{y}$ instead of $Df(\mathbf{x})[\mathbf{y}]$ (from Riesz representation theorem).

For matrices ($\varphi : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{R}$):

$$\frac{\|\varphi(\mathbb{A}+\mathbb{B}) - \varphi(\mathbb{A}) - D\varphi(\mathbb{A})[\mathbb{B}]\|_{\mathbb{R}}}{\|\mathbb{B}\|_{\mathbb{R}^{3\times 3}}}.$$

*Poznámka*
We write $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A}) : \mathbb{B}$ instead of $D\varphi(\mathbb{A})[\mathbb{B}]$, where $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A})$ is right matrix. Warning $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A}) \neq D\varphi(\mathbb{A})$, because of transposition ($\mathbb{A} : \mathbb{B} = \operatorname{tr}(\mathbb{A}\mathbb{B}^T) = \operatorname{tr}(\mathbb{A}^T\mathbb{B})$).

*Příklad*

$$\frac{\partial \operatorname{tr}\mathbb{A}}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\operatorname{tr}(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\operatorname{tr}\mathbb{A} + \tau\operatorname{tr}\mathbb{B}\right)|_{\tau=0} = \operatorname{tr}\mathbb{B} = \mathbb{I} : \mathbb{B}.$$

So $\frac{\partial \operatorname{tr}\mathbb{A}}{\partial\mathbb{A}} = \mathbb{I}$.

*Příklad*

$$\frac{\partial \det\mathbb{A}}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\det(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\det(\mathbb{A})\cdot\det\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left((\det\mathbb{A})\cdot\left(1+\tau\operatorname{tr}(\mathbb{A}^{-1}\mathbb{B}) + o(\tau^2)\right)\right)|_{\tau=0} = (\det\mathbb{A})\operatorname{tr}\left(\mathbb{A}^{-1}\mathbb{B}\right) =$$

$$= (\det\mathbb{A})\operatorname{tr}\left(\left(\mathbb{A}^{-T}\right)^T\mathbb{B}\right) = \left((\det\mathbb{A})\mathbb{A}^{-T}\right) : \mathbb{B}.$$

So $\frac{\partial \det\mathbb{A}}{\partial\mathbb{A}} = (\det\mathbb{A})\mathbb{A}^{-T} = \operatorname{cof}(\mathbb{A})$.

*Příklad*
$\mathbb{A} : \mathbb{R} \to \mathbb{R}^{3\times 3}$.

$$\frac{d}{dt}(\det\mathbb{A}(t)) = (\det\mathbb{A}(t))\operatorname{tr}\left(\mathbb{A}(t)^{-1}\frac{d\mathbb{A}(t)}{dt}\right).$$

*Příklad*
$\mathbb{F} : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{F}(\mathbb{A}) \in \mathbb{R}^{3\times 3}$. $\mathbb{F}(\mathbb{A}) = \mathbb{A}^{-1}$. (We know $\frac{1}{1+x} = 1 - x + \ldots$.)

$$\frac{\partial\mathbb{F}(\mathbb{A})}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}\left((\mathbb{A}+\tau\mathbb{B})^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{A}\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)\right)^{-1}\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left(\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)^{-1}\mathbb{A}^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{I}-\tau\mathbb{A}^{-1}\mathbb{B}+\ldots\right)\mathbb{A}^{-1}\right)|_{\tau=0} = -\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}.$$

So we have $\frac{\partial(\mathbb{A}^{-1})_{ij}}{\partial(\mathbb{A})_{kl}}(\mathbb{B})_{kl}$.

From chain rule (but this is easily solvable by differentiating $\mathbb{A}^{-1}(t)\mathbb{A}(t) = \mathbb{I}$):

$$\frac{d}{dt}\left(\mathbb{A}^{-1}\right) = -\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\mathbb{A}^{-1}.$$

*Příklad*
$\mathbb{F}(\mathbb{A}) = e^{\mathbb{A}}$

$$\frac{\partial e^{\mathbb{A}}}{\partial \mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}(e^{\mathbb{A}+\tau\mathbb{B}})|_{\tau=0} = \frac{d}{d\tau}\left(\mathbb{I} + \frac{\mathbb{A}+\tau\mathbb{B}}{1!} + \frac{(\mathbb{A}+\tau\mathbb{B})^2}{2!}\right)|_{\tau=0}.$$

**Věta 0.2** (Daleckii–Krein)

*$\mathbb{A}$ real symmetric matrix, $\mathbb{A} \in \mathbb{R}^{k\times k}$, $\mathbb{A} = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$, where $\lambda_i$ are eigenvalues and $\mathbf{v}_i$ are normalised orthogonal ($\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$) eigenvectors.*

*$f$ continuously differentiable real function defined on open set containing the spectrum of $\mathbb{A}$*

$$\mathbb{F}(\mathbb{A}) := \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =: \sum_{i=1}^{k} f(\lambda_i)\mathbb{P}_i.$$

*Then the formula for the Gateaux derivative of $f$ at point $\mathbb{A}$ in direction $\mathbb{X}$ reads*

$$D\mathbb{F}(\mathbb{A})[\mathbb{X}] = \frac{\partial \mathbb{F}}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{i=1}^{k} \frac{df}{d\lambda}|_{\lambda=\lambda_i}\mathbb{P}_i\mathbb{X}\mathbb{P}_i + \sum_{i=1}^{k}\sum_{j=1,j\neq i}^{k} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\mathbb{P}_i\mathbb{X}\mathbb{P}_j.$$

*Sometimes we write $D\mathbb{F}(\mathbb{A})[\mathbb{X}] = f^{[1]}(\mathbb{A}) \bigcirc \mathbb{X}$ (Schur product of matrices, it is point-wise multiplication). Then*

$$[f^{[1]}(\mathbb{A})]_{ij} = \begin{cases} \frac{df}{d\lambda}|_{\lambda=\lambda_i}, & i = j, \\ \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, & i \neq j. \end{cases}$$

*Důkaz*
No summation conventions, all sums are stated explicitly!

$$\mathbb{F}(\mathbb{A}) = \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =$$

$$= \sum_{i=1}^{k} f(\lambda_i(a_{11}, a_{12}, \dots, a_{21}, \dots))\mathbf{v}_i(a_{11}, a_{12}, \dots, a_{21}, \dots) \otimes \mathbf{v}_i(a_{11}, a_{12}, \dots, a_{21}, \dots).$$

$$\frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}} = \sum_{i=1}^{k} \left( \frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i \otimes \mathbf{v}_i + f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + f(\lambda_i)\mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \right) =?.$$

We derivate $\mathbb{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}.$$

We multiply (with dot product) it by $\mathbf{v}_i$:

$$\mathbb{P}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbb{A}^T \mathbf{v}_i = \frac{\partial \lambda_i}{\partial \mathbb{A}} \cdot 1 + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \mathbb{A} \cdot \mathbf{v}_i.$$

$$\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i.$$

We again multiply derivative of $\mathbb{A}|\mathbf{v}_i = \lambda \mathbf{v}_i$, but this time by $\mathbf{v}_j$:

$$\mathbf{v}_j \otimes \mathbf{v}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \lambda_j \mathbf{v}_j = 0 + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j.$$

$$(\lambda_j - \lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = -\mathbf{v}_j \otimes \mathbf{v}_i.$$

We also need $(\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}_{ij} = \ldots = \mathbb{P}_i \mathbb{X} \mathbb{P}_j$:

$$\ldots = (\mathbf{v}_j \otimes \mathbf{v}_i)(\mathbf{v}_i \cdot \mathbb{X}\mathbf{v}_j) = (\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}(\mathbf{v}_j \otimes \mathbf{v}_j).$$

□

TODO!!!

# 1 Kinematics

**Definice 1.1**

We have some abstract body with point $P$. We can look at it in reference configuration (some point in past), where $K_0(P) = \mathbf{X}$ ($K_0$ = placer), $t = t_0$. Or in current configuration (how it is situated now), where $K_t(P) = \mathbf{x}$.

The change of configuration, $\chi$ in $\mathbf{x} = \chi(\mathbf{X}, t)$ is called deformation (but it contains translation and rotation too!).

## Definice 1.2

Let us consider quantity $\theta$ that describes the given material point. We can describe it by:

- $\theta(P, t)$;
- $\hat{\theta}(\mathbf{X}, t)$ (referential/Lagrangian description, commonly used for solids because deformation is with respect to reference configuration);
- $\tilde{\theta}(\mathbf{x}, t)$ (spatial/Eulerian description, commonly used for fluids because velocity is time-local property).

But people write those functions without ˆ or ˜

*Poznámka*

$$\tilde{\theta}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \hat{\theta}(\mathbf{X}, t).$$

## Definice 1.3 (Deformation gradient)

$$d\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = \chi(\mathbf{X}_2, t) - \chi(\mathbf{X}_1, t) =$$
$$= \chi(\mathbf{X}_1 + d\mathbf{X}, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1, t) + \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X} + \ldots - \chi(\mathbf{X}_1, t) = \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}.$$

$$\mathbb{F}(\mathbf{X}, t) := \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}. \qquad d\mathbf{x} = \mathbb{F}d\mathbf{X}$$

*Poznámka*
It can be derived by derivatives on curves (see lecture).

*Důsledek*
Transformation of infinitesimal line segment: $d\mathbf{x} = \mathbb{F}d\mathbf{X}$.

Transformation of infinitesimal surface elements: $d\mathbf{s} = (\det \mathbb{F})\mathbb{F}^{-T}d\mathbf{S} = \operatorname{cof} \mathbb{F}d\mathbf{S}$.

Transformation of infinitesimal volume: $dv = (\det \mathbb{F})dV$.

*Důsledek* (In tangent spaces)

$$F(\mathbf{X}, t_0) = f(\chi(\mathbf{X}, t), t).$$

Representation theorem:

$$(GradF)\mathbf{W} = \mathbf{U}_{GradF} \cdot \mathbf{W}$$

$$(Gradf)\mathbf{w} = \mathbf{u}_{Gradf} \cdot \mathbf{w}$$

$f(\chi(\mathbf{X}, t), t) = F(\mathbf{X}, t_0)$

$$Gradf(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = GradF(\mathbf{X}, t_0)$$

$$\mathbf{U}_{GradF} \cdot \mathbf{W} = (GradF)\mathbf{W} = (Gradf)\mathbb{F}\mathbf{W} = (gradf)(\mathbb{F}\mathbf{W}) = \mathbf{u}_{gradf} \cdot \mathbb{F}\mathbf{W} = \mathbb{F}^T \mathbf{u}_{Gradf} \cdot \mathbf{W}.$$

$\mathbf{u}_{gradf} = \mathbb{F}^{-T}\mathbf{U}_{GradF}$.

*Příklad* (Hollow cylinder)
$r = f(R)$, $\varphi = \Phi$, $z = Z$.

*Řešení*

$$\mathbb{F} = \frac{\partial \chi_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{E}_j$$

$$X_1 = R\cos\Phi, \qquad X_2 = R\sin\Phi, x_1 = r\cos\Phi, x_2 = r\sin\Phi.$$

$$x_1 = \chi_1(X_1, X_2, t), \qquad x_2 = \chi(x_1, x_2, t), x_i = \chi_i(X_j, t).$$

By chain rule:

$$\frac{\partial x_1}{\partial X_2} = \frac{\partial r\cos\Phi}{\partial\partial X_2} = \frac{\partial}{\partial X_2} f(R)\cos\Phi.$$

$$\mathbb{F} = F_{rR}\mathbf{e}_r \otimes \mathbf{E}_R + F_{r\Phi}\mathbf{e}_r \otimes \mathbf{E}_\Phi + \dots$$

*Řešení*
From image:

$$\mathbf{E}_R \xrightarrow{\mathbb{F}} F_{rR}\mathbf{e}_r.$$

$$\mathbf{E}_\Phi \xrightarrow{\mathbb{F}} F_{\varphi\Phi}\mathbf{e}_\varphi$$

So $\mathbb{F} = \begin{pmatrix} F_{rR} & 0 \\ 0 & F_{\varphi\Phi} \end{pmatrix}$

TODO? (Solution by curve)

*Poznámka*
How to differentiate in time tensorial quantities related to the current configuration?

Upper convected derivative:

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \det \mathbb{F}(\mathbf{X},t) \left[ \frac{d}{dt} \left( \mathbb{F}^{-1}(\mathbf{X},t) \mathbb{A}(\chi(\mathbf{X},t),t) \mathbb{F}^{-T}(\mathbf{X},t) \right) \right] \mathbb{F}^{T}(\mathbf{X},t).$$

## 1.1 Derivatives

**Definice 1.4** (Lagragian velocity)

$$\mathbf{V}(\mathbf{X},t) = \frac{d\chi(\mathbf{X},t)}{dt}.$$

$$\mathbf{v}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}$$

**Definice 1.5** (Eulerian velocity)

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X},t)|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)}.$$

**Definice 1.6** (Material time derivative)

$\frac{d}{dt}$ = keep $\mathbf{X}$ fixed, and differentiate with respect to time.

$$\psi(\mathbf{X},t) \to \frac{d}{dt}\psi(\mathbf{X},t) = \frac{\partial \psi}{\partial t}(\mathbf{X},t)$$

$$\begin{aligned} \psi(\mathbf{x},t) \to \frac{d}{dt}\psi(\chi(\mathbf{X},t),t) &= \frac{\partial \psi}{\partial t}|_{\mathbf{x}=\chi(\mathbf{X},t)} + \frac{\partial \psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{d\chi_i}{dt}(\mathbf{X},t) = \\ &= \left( \frac{\partial \psi}{\partial t}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} + V_i(\mathbf{X},t) \frac{\partial \psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \right) = \\ &= \left( \frac{\partial \psi}{\partial t}(\mathbf{x},t) + v_i(\mathbf{x},t) \frac{\partial \psi}{\partial x_i}(\mathbf{x},t) \right) |_{\mathbf{x}=\chi(\mathbf{x},t)} \end{aligned}$$

$$\frac{d}{dt}\psi(\mathbf{x},t) = \frac{\partial \psi}{\partial t}(\mathbf{x},t) + (\mathbf{v}(\mathbf{x},t) \cdot \nabla)\psi(\mathbf{x},t).$$

**Definice 1.7** (Time derivative of deformation gradient $\mathbb{F}$)

$$\frac{d}{dt}\mathbb{F}(\mathbf{X},t) = \frac{d}{dt}\left( \frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}} \right) = \frac{\partial}{\partial \mathbf{X}} \frac{d\chi(\mathbf{X},t)}{dt} = \frac{\partial}{\partial \mathbf{X}} \mathbf{V}(\mathbf{X},t) =$$

$$= \frac{\partial}{\partial \mathbf{X}} \mathbf{v}(\chi(\mathbf{X},t),t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X},t) = \frac{\partial \mathbf{v}}{\partial x}|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbb{F}(\mathbf{X},t).$$

$$\mathbb{L}(\mathbf{x},t) := \nabla \mathbf{V}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x},t).$$

*Důsledek*

$$\frac{d\mathbb{F}}{dt} = \mathbb{L}\mathbb{F}$$

*Důsledek*

$$\overset{\triangledown}{\mathbb{A}} = \frac{d\mathbb{A}}{dt} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^T$$

TODO!!!

*Poznámka* (Balance laws in Eulerian description (revision, the last lecture))

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0;$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \mathbb{T} = \mathbb{T}^T;$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q;$$

or

$$\varrho \frac{d}{dt}(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = \operatorname{div}(\mathbb{T}^T \mathbf{v}) + \varrho \mathbf{b} \cdot \mathbf{v} - \operatorname{div} \mathbf{j}_q.$$

*Poznámka* (Balance laws in Lagrangian description)
Starting with $\frac{d}{dt}{}_{V(t)}\varrho(\mathbf{x},t)dv = \frac{d}{dt} m_{V(t)} = 0$ i.e. mass remains same: $m_{V(t)} = m_{V(t_0)}$). We integrate over volume:

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) dV = \int_{V(t)} \varrho(\mathbf{x},t) dv = \int_{V(t_0)} \varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} dV.$$

Localization principle:

$$\varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} = \varrho_R(\mathbf{X}).$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} + \int_{V(t)} \varrho \mathbf{b} dv.$$

$$\int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} = \int_{\partial V(t)} \mathbb{T}\mathbf{n}\, ds = \int_{\partial V(t_0)} \mathbb{T}(\det \mathbb{F})\mathbb{F}^{-T}\mathbf{N}\, dS \overset{\text{Stokes}}{\int}_{V(t_0)} \operatorname{div}_{\mathbf{X}}((det\mathbb{F})\mathbb{T}\mathbb{F}^{-T}) dV =: \int_{V(t_0)} (\mathbb{T}_R) dV.$$

$$\mathbb{T}_R(\mathbf{X},t) := (\det \mathbb{F}(\mathbf{X},t))\mathbb{T}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbb{F}^{-T}(\mathbf{X},t)$$

is first Piola–Kirchhoff stress tensor. Cauchy ($\mathbb{T}$) is current → current. P–K ($\mathbb{T}_R$) is reference → current.

$$\int_{\partial V(t)} dv \to \int_{\partial V(t_0)} (\operatorname{div}_{\mathbf{X}} \mathbb{T}_R) dV.$$

$$\int_{V(t)} \varrho \mathbf{b} dv = \int_{V(t_0)} \varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbf{b}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R(\mathbf{X}) \mathbf{b} dV.$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t_0)} \varrho \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R \frac{\partial^2 \chi}{\partial t^2} dV.$$

Altogether:

$$\varrho_R \frac{\partial^2 \chi}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbb{T}_R + \varrho_R \mathbf{b} \qquad \text{(Solve for } \chi).$$

$\mathbb{T} = \mathbb{T}^T \to \mathbb{T}_R \mathbb{F}^T = \mathbb{F}\mathbb{T}_R^T$ (P–K is not symmetric!).

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q \to \int_{V(t)} \varrho \frac{de}{dt} dv = \int_{V(t)} \mathbb{T} : \mathbb{L} dv - \int_{V(t)} \operatorname{div} \mathbf{j}_q dv.$$

$$\int_{V(t)} \operatorname{div} \mathbf{j}_q dv = \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds = \int_{\partial V(t_0)} \mathbf{j}_q(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \cdot \det \mathbb{F}(\mathbf{X},t)\mathbb{F}^{-T}(\mathbf{X},t)\mathbf{N} dS =$$

$$= \int_{\partial V(t_0)} (\det \mathbb{F}(\mathbf{X},t)\mathbb{F}^{-1}(\mathbf{X},t)\mathbf{j}_q(\mathbf{x},t)) \cdot \mathbf{N} dS =$$

$$= \int_{V(t_0)} \operatorname{div}((\det \mathbb{F})\mathbb{F}^{-1}\mathbf{j}_q) dV.$$

$\mathbf{J}_q = (\det \mathbb{F})\mathbb{F}^{-1}\mathbf{j}_q$ is called referential heat flux. (It cannot be given by Fouriers law ($\mathbf{j}_q = k\nabla_{\mathbf{x}}\theta$, $\operatorname{div} \mathbf{j}_q = \operatorname{div}(k\nabla\theta)$).)

$$\int_{V(t)} \underbrace{\mathbb{T} : \overbrace{\mathbb{L}}^{\nabla_{\mathbf{x}}\mathbf{v}}}_{\operatorname{tr}(\mathbb{T}\mathbb{L}^T)=\operatorname{tr}(\mathbb{L}\mathbb{T}^T)} dv = \int_{V(t_0)} (\det \mathbb{F})\mathbb{T} : \mathbb{L} dV =$$

$$= \int_{V(t_0)} \operatorname{tr}((\det \mathbb{F})\mathbb{T}\mathbb{L}^T) dV = \int_{V(t_0)} \operatorname{tr}\left((\det \mathbb{F})\mathbb{T}\mathbb{F}^{-T}\left(\frac{d\mathbb{F}}{dt}\right)^T\right) dV = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV.$$

Altogether

$$\varrho_R \frac{\partial e}{\partial t} = \mathbb{T}_R : \dot{\mathbb{F}} - \operatorname{div}_{\mathbf{X}} \mathbf{J}_q.$$

# 2 Entropy

*Poznámka* (Objective)
Find quantity that is increasing/decreasing in time.

*Poznámka* (With no interior)

$$\varrho \frac{d}{dt}\left(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \operatorname{div}\mathbb{T} + \varrho\mathbf{b} - \operatorname{div}\mathbf{j}_q = \operatorname{div}\mathbb{T} + 0 + \operatorname{div}(k\nabla\theta).$$

Let us work with[a] $\operatorname{div}\mathbb{T} = -p_{th}\mathbb{I} + \tilde{\lambda}(\operatorname{div}\mathbf{v}) + 2\mu\mathbb{D}_\delta$, and assume that $\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \tilde{\lambda}(\operatorname{div}\mathbf{v})\mathbb{I} + 2\mu\mathbb{D}_\delta$ (from $\frac{pV}{T} = \text{const}$).

$$\varrho \frac{d}{dt}(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}) = \operatorname{div}(\mathbb{T}^T\mathbf{v}) - \operatorname{div}\mathbf{j}_q.$$

$$\frac{d}{dt}\int_{V(t)} \varrho(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v})dv = \int_{\partial V(t)} \mathbb{T}^T\mathbf{v}\cdot\mathbf{n}ds - \int_{\partial V(t)} \mathbf{j}_q\cdot\mathbf{n}ds.$$

The first part is work and it is zero if we have boundary condition $\mathbf{v}|_{\partial V} = 0$. The second part is heat exchange which is zero if we have boundary condition $\mathbf{j}_q\cdot\mathbf{n}|_{\partial V} = 0$. Both boundary conditions together are math way to say system with *no interactions*.

$\varrho, \theta, \mathbf{v} \to \varrho, e, \theta.$

Assume $\eta = \eta(\varrho, e) \to e = e(\eta,\varrho)$. (We will write $e = e(\eta,\varrho) = e(\eta(\mathbf{x},t),\varrho(\mathbf{x},t)) = e(\mathbf{x},t)$.)

We have 1. Balance of internal energy

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q;$$

2. Chain rule

$$\varrho\frac{de}{dt} = \frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt} + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt}.$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q - \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt} = (-p_{th}\mathbb{I} + \tilde{(}\operatorname{div}\mathbf{v})\mathbb{I} + 2\mu\mathbb{D}_\delta) : \mathbb{D} - \operatorname{div}\mathbf{j}_q + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt} = \left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)\operatorname{div}\mathbf{v} + \tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta - \operatorname{div}\mathbf{j}_q,$$

$$\varrho\frac{d\eta}{dt} = \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}}\operatorname{div}\mathbf{v} - \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial\eta}} + \frac{\tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu|\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial\eta}}.$$

There is no chance that this could be positive. (Its obvious, because the value can flow, so point-wise $\geqslant 0$ is lost case.) But we can integrate over volume. Thus instead of $\frac{d\eta}{dt} \geqslant 0$

we want just $\frac{d}{dt}\int_{V(t)} \varrho\eta dv \geqslant 0$.

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial\eta}} dv + \int_{V(t)} \frac{\tilde\lambda(\operatorname{div}\mathbf{v})^2 + 2\mu|\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial\eta}} dv.$$

The third integral OK, if $\frac{\partial e}{\partial\eta} > 0$.

$$\operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial\eta}}\right) = \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial\eta}} + \nabla\left(\frac{1}{\frac{\partial e}{\partial\eta}}\right)\cdot\mathbf{j}_q.$$

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial\eta}}\right) dv + \int_{V(t)} \nabla\left(\frac{1}{\frac{\partial e}{\partial\eta}}\right)\cdot\mathbf{j}_q dv + REST.$$

The second integral is zero from Stokes and boundary condition $\mathbf{j}_q\cdot\mathbf{n}|_{\partial V} = 0$. On the third integral, we can use derivative of inverse value:

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial\eta}\right)\cdot\mathbf{j}_q}{\left(\frac{\partial e}{\partial\eta}\right)^2} dv + REST =$$

$$= \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}} \operatorname{div}\mathbf{v} dv + k\int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial\eta}\right)\cdot\nabla\theta}{\left(\frac{\partial e}{\partial\eta}\right)^2} dv + REST.$$

If we set $\frac{\partial e}{\partial\eta}(\eta,\varrho) = \theta$, the second integral is non-negative. Moreover, for $\theta \geqslant 0$ we satisfy the assumption for the "first third integral". Moreover if we enforce $\varrho^2\frac{\partial e}{\partial\varrho}(\varrho,\eta) = p_{th}(\theta,\varrho)$, the first integral is zero, so we win.

[a] $\mathbb{D}_\delta := \mathbb{D} - \frac{1}{3}(\operatorname{tr}\mathbb{D})\mathbb{I}$. (Traceless part of $\mathbb{D}$.)

*Poznámka*
Volíme si tedy $\tilde\lambda, \mu > 0$.

*Důsledek*

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv \geqslant 0$$

is granted for quantity that solves equations

$$e = e(\eta,\varrho), \qquad \frac{\partial e}{\partial\eta} = \theta, \qquad \varrho^2\frac{de}{d\varrho} = p_{th}(\theta,\varrho).$$

*Příklad*

$$p_{th}(\theta, \varrho) = c_V(\gamma - 1)\varrho\theta, \qquad e(\theta, \varrho) = c_V\theta.$$

*Poznámka* (?)

1. Energy is constant.
2. Energy is function of entropy and volume.
3. Entropy increases.

*Poznámka*
$e = e(\eta, \varrho)$ is given $\to$ we know everything $\theta = \frac{\partial e}{\partial \eta}(\eta, \varrho)$, $p_{th} = \varrho^2 \frac{\partial e}{\partial \varrho}(\eta, \varrho)$. (Warning: $e = e(\varrho, \theta)$ is not enough!)

*Poznámka*
Is there a better function that will allow us to do something like this?

**Definice 2.1** (Helmholtz free energy density)

$$\psi(\theta, \varrho) := e(\eta, \varrho)|_{\eta=\eta(\theta,\varrho)} - \theta\eta|_{\eta=\eta(\theta,\varrho)}.$$

*Poznámka*
This is le Legrende transformation of internal energy.

*Důsledek*

$$\frac{\partial \psi}{\partial \theta}(\theta, \varrho) = -\eta, \qquad \frac{\partial \psi}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta,\varrho)} \quad \left(= \frac{p_{th}}{\varrho^2}\right).$$

*Důkaz*

$$\frac{\partial \psi}{\partial \theta}(\theta, \varrho) = \frac{\partial e(\eta, \varrho)}{\partial \eta}|_{\eta=\eta(\theta,\varrho)} \frac{\partial \eta}{\partial \theta}(\theta, \varrho) - \eta|_{\eta=\eta(\theta,\varrho)} - \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\eta|_{\eta=\eta(\theta,\varrho)}.$$

$$\frac{\partial \psi}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \eta}(\eta, \theta)\frac{\partial \eta}{\partial \varrho}(\theta, \varrho) + \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta,\varrho)} - \theta\frac{\partial \eta}{\partial \varrho}(\theta, \varrho) = \frac{\partial e}{\partial \varrho}(\eta, \varrho)|_{\eta=\eta(\theta,\varrho)}.$$

□