Definice 0.1 (Category, map (arrow, morphism), composition, domain, codomain)

A category $\mathcal{A}$ consists of: a collection $\mathrm{ob}(\mathcal{A})$ of objects, and for each $A, B \in \mathcal{A}$, a collection $\mathcal{A}(A, B)$ of maps, arrows, or morphisms from A to B. Such that for each $A, B, C \in \mathrm{ob}(\mathcal{A})$ a function (named composition) $\circ : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$, $(g, f) \mapsto g \circ f$ meets following:

For each $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{S}(C, D) : (h \circ g) \circ f = h \circ (g \circ f)$ (asociativity). For each $A \in \mathrm{ob}(\mathcal{A})$ $\exists 1_A \in \mathcal{A}(A, A)$, called the identity, such that, for each $f \in \mathcal{A}(A, B)$: $f \circ 1_A = f = 1_B \circ f$.

Poznámka (Notation)

$$A \in \mathrm{ob}(\mathcal{A}) \Leftrightarrow A \in \mathcal{A}.$$

$$f \in \mathcal{A}(A, B) \Leftrightarrow A \xrightarrow{f} B \Leftrightarrow f : A \to B.$$

For $f \in \mathcal{A}(A, B)$: $\mathrm{domain}(f) := A$ and $\mathrm{codomain}(f) := B$.

Například (of categories)
Category of:

- sets (SET): $\mathrm{ob}(SET)$:= sets, $SET(A, B)$:= functions from A to B, $\circ$ is composition;
- groups (GRP): $\mathrm{ob}(GRP)$:= groups, $GRP(G, H)$:= group homomorphisms, $\circ$ is composition;
- rings (RING): $\mathrm{ob}(RING)$:= rings, $RING(A, B)$:= ring homomorphisms, $\circ$ is composition;
- vector spaces ($\mathrm{VECT}_{\mathbb{K}}$): $\mathrm{ob}(VECT_{\mathbb{K}})$:= vector spaces over $\mathbb{K}$, $RING(A, B)$:= $\mathbb{K}$ linear maps, $\circ$ is composition;
- topological spaces (TOP): $\mathrm{ob}(TOP)$:= topological spaces, $RING(A, B)$:= continuous maps, $\circ$ is composition.

Definice 0.2 (Isomorphism, inverse)

$f : A \to B$ in a category $\mathcal{A}$ is an isomorphism if exists a map $g : B \to A$ in $\mathcal{A}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Then we call g the inverse of f.

Například
In SET isomorphisms are bijections.

Příklad
Show that inverses are unique (justifying the use of the determine article in the previous definition).

Poznámka
0-morphisms are called morphisms (between objects), 1-morphisms are called functors (between categories), 2-morphisms are called natural transformations (between functors).

Definice 0.3 (Functor)

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A functor $F : \mathcal{A} \to \mathcal{B}$ consists of: a function $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$, and for each $A, A' \in \mathcal{A}$ a function $F : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$. Such that

$$F(f' \circ f) = F(f) \circ F(f'), \qquad \forall A \overset{f}{A}{}' \overset{f'}{A}{}'' \in \mathcal{A},$$

$$F(1_A) = 1_{F(A)} \qquad \forall A \in \mathcal{A}.$$

Například (Forgetful functors)
$U : GRP \to SET$, for any group $(G, \cdot)$, $U((G, \cdot)) := G$, and for any morphism f, $U(f : (G, \cdot) \to (H, *)) := f : G \to H$. (Exercise: Convince yourself that this is a well-defined functors.)

We can do the same for rings, vector spaces and topological spaces.

Například
Let $\mathcal{A}$ be the following category: $\mathrm{ob}(\mathcal{A}) = \{\cdot\}$, $\mathcal{A}(\cdot, \cdot) = 1.$, and $1. \circ 1. = 1.$. It is called discrete category with one object.

$\mathrm{ob}(\mathcal{B}) = \{\cdot, *\}$, $\mathcal{B}(\cdot, \cdot) = 1.$, $\mathcal{B}(\cdot, *) = \varnothing$

Directed transitive graph (with all loops) with concatenation of edges.

From group $(G, +)$ we construct category $\mathcal{G}$ by putting: $\mathrm{ob}(\mathcal{G}) := \cdot$, $\mathcal{G}(\cdot, \cdot) := G$ and $\circ := +$. We can generalize to a monoid $(M, +)$.

Now, let $\mathcal{A}$ be a category with one object $\{\cdot\}$ (and assume that $\mathcal{S}(\cdot, \cdot)$ is a set). Then homomorphism with composition are monoid. And isomorphisms with composition are groups (so one-object category with all homomorphism isomorphic represents group).

(Category, where $\mathcal{A}(\cdot, \cdot)$ is a set, is often called locally small.)

Let G and H be groups and $\mathcal{G}, \mathcal{H}$ their associated one-object categories. What is a functor from $\mathcal{G}$ to $\mathcal{H}$? For $F : \mathrm{ob}(\mathcal{G}) \to \mathrm{ob}(\mathcal{H})$ we have no other choice than $F(\cdot) := *$. For $F : \mathcal{G}(\cdot, \cdot) \to \mathcal{H}(*, *) = \mathcal{H}(F(\cdot), F(\cdot))$ we demonstrated (see lecture) that F needs to be group homomorphism (and every group homomorphism $G \to H$ is functor). (All this work for monoids too.)

Let AB be the category of ob(AB) := Abelian groups and $AB(A,B)$:= group homomorphism. Then $U : AB \to GRP$ as „forgetful functor“ is „identity“. The same for commutative rings. Also we have forgetful functor $U : RING \to AB$, $(R,+,\cdot) \mapsto (R,+)$ and functor $U : RING \to MONOIDS$, $(R,+,\cdot) \mapsto (R,\cdot)$.

$U : SET \to VECT_{\mathbb{K}}$ we can define by $F(X) = (X \to F)$ (functions from X to F) (free vector space).

Definice 0.4 (Functor composition)

When we have functor $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{C}$. We want to $F' \circ F$ to be functor, so it has function on objects and functions on morphism classes. Function on object is simply composition $F' \circ F$. Functions on morphism classes is also composition:

$$\mathcal{A}(A,A') \xrightarrow{F} B(F(A),F(A')) \xrightarrow{F'} \mathcal{C}(F' \circ F(A), F' \circ F(A')) \implies F' \circ F : \mathcal{A}(A,A') \to \mathcal{C}(F' \circ F(A), F' \circ F(A')).$$

Důkaz
1. $(F' \circ F)(1_A) = F'(F(1_A)) = F'(1_{F(A)}) = 1_{F' \circ F(A)}$. (For $A \in \mathcal{A}$.)

2. $(F' \circ F)(f' \circ f) = F'(F(f' \circ f)) = F'((F(f')) \circ (F(f))) = (F' \circ F(f')) \circ (F' \circ F(f))$. (For $A \xrightarrow{f} A' \xrightarrow{f'} A'' \in \mathcal{A}$.) □

So $F' \circ F$ is a functor. We call it the composition of F and F'.

Definice 0.5 (CAT)

The category of categories (CAT) has categories as objects and functors as morphisms (with its composition from the previous definition).

Důkaz
We need: 1. An identity functor $1_{\mathcal{A}} \in CAT(\mathcal{A},\mathcal{A})$ (function on objects is identity, function on $CAT(\mathcal{A},\mathcal{B})$ is identity too), we can easily see that it fulfills condition from category definition.

2. Associativity of composition: composition of functions is associative, so we see this from the definition of the functor composition. □

Definice 0.6 (Dual category (opposite category))

For a category $\mathcal{A}$, its dual category (or opposite category) $\mathcal{A}^{\mathrm{op}}$ is defined by: ob$(\mathcal{A}^{\mathrm{op}})$ = ob$(\mathcal{A})$, $\mathcal{A}^{\mathrm{op}}(B,A) = \mathcal{A}(A,B)$ $(\forall A,B \in \mathrm{ob}(\mathcal{A}))$, composition in $\mathcal{A}^{\mathrm{op}}$ is the composition in $\mathcal{A}$.

Příklad (Excercise)

$$(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}.$$

Definice 0.7 (Contravariant functor)

For two cats $\mathcal{A}, \mathcal{B}$ a contravariant functor: $\mathcal{A} \to \mathcal{B}$ is a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ ($F(f' \circ f) = (F(f)) \circ (F(f')))$.

Příklad

Functor $C : TOP \to ALG_{\mathbb{K}}$ is $X \in TOP \mapsto C(X) \in ALG_{\mathbb{K}}$, where $C(X)$ is the collection of all continuous functions $X \to \mathbb{K}$ with addition, multiplication and scalar multiplication. But when we try to define C for morphisms, we find that it cannot be done this way. ($C(X \xrightarrow{f} Y) = C(X) \xrightarrow{C(f)} C(Y)$, so $C(f)(\varphi) = \varphi \circ f \implies$ this does not define a functor.)

So we „fix it“ by taking contravariant functor.

Definice 0.8 (Presheaf)

Let $\mathcal{A}$ be a category a presheaf on $\mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \to SET$.

Příklad

Let X be a topological space. Write $O(X)$ for ordered subsets of X ordered by inclusion $\to$ category $\mathcal{O}(X)$: objects are open subsets, morphisms are inclusion and $\circ$ is composition of inclusions.

Definice 0.9 (Faithful functor, full functor)

A functor $F : \mathcal{A} \to \mathcal{B}$ is faithful (resp. full) if for each $A, A' \in \mathcal{A}$ the function

$$\mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A')), \qquad f \mapsto F(f),$$

is injective (resp. surjective) $\forall A, A' \in \mathcal{A}$.

Pozor

If F is faithful, we do not have $F(f_1) \neq F(f_2)$ $\forall$ distinct morphisms f_1, f_2. ($F(A)$ still can be equal to $F(A')$, so it can be $f_1 : A \to A$, $f_2 : A' \to A'$.)

Definice 0.10 (Subcategory)

Let $\mathcal{A}$ be a category. A subcategory $\mathcal{S} \subset \mathcal{A}$ consists of a subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{A})$ together with, for $S, S' \in \mathrm{ob}(\mathcal{S})$, a subclass $\mathcal{S}(S, S') \subseteq \mathcal{A}(S, S')$ such that $\mathcal{S}$ is closed under composition.

Definice 0.11 (Full subcategory)

We say that subcategory $\mathcal{S}$ is full if $\mathcal{S}(S, S') = \mathcal{A}(S, S')$, $\forall S, S' \in \mathrm{ob}(\mathcal{S})$.

Poznámka
A full subcategory is identified by its objects.

Například
AB is the full subcategory of GRP.

Příklad
For any subcategory $\mathcal{S} \subset \mathcal{A}$, we have an inclusion functor $I : \mathcal{S} \to \mathcal{A}$.

I is faithful, and it is full $\Leftrightarrow$ $\mathcal{S}$ is full.

Definice 0.12

$F : \mathcal{A} \to \mathcal{B}$, $\mathrm{Im}(F)$ has objects $F(A)$ and morphisms $F(f)$.

Pozor
$\mathrm{Im}(F)$ nemusí být kategorie. (Mohou vzniknout „možnosti složení“, které v původní kategorii nebyly.)

0.1 2-morphism and natural transformations

Definice 0.13 (Natural transformation)

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ two functors. A natural transformation between F and G ia s family of morphisms in $\mathcal{B}$: $(F(A) \overset{\alpha_A}{\to} G(A))_{A \in \mathcal{A}}$ such that $F(f) \circ \alpha_B = \alpha_A G(f)$ for every $A \overset{f}{\to} B \in \mathcal{A}$.

We call the morphisms α_A the components of the natural transformation.

Příklad
Define a composition of natural transformations and use it to define the functor category of $\mathcal{A}$ and $\mathcal{B}$ (objects functors $F : \mathcal{A} \to \mathcal{B}$ and morphisms natural transformations α).

Příklad
For two graphs H, K, functors between their 1-object cats $\leftrightarrow$ group homomorphism. What is a natural transformations between two functors?

0.2 Free functors

Poznámka
Recall forgetfull functors. What about functors in the other direction?

Například
$F : SET \to VECT_{\mathbb{K}}$, $X \mapsto F(X)$. $F(X)$ (the free $\mathbb{K}$-vector space) is is functions $f : X \to \mathbb{K}$ endowed with the vector space structure (addition and scalar multiplication). (Alternatively $F(X)$ is the vector space with a basis $\left\{e_x^X | x \in X\right\}$).

Morphisms: $F(f)(e_x^X) := e_{f(x)}^X$.

Například
$U : GRP \to SET$, so free functor should look like $F : SET \to GRP$. $S \mapsto F(S)$, where $F(S)$ (the free group) is a sets for which $\exists i : S \to F(S)$ inclusion of sets to $F(S)$, that for every $f : S \to \mathcal{G}$ function between sets and groups, $\exists! \varphi_i$ such that $i \circ \varphi_i$ commutes.

Think about / look up: this defines $F(S)$ uniquely up to group isomorphism.

Příklad
Take the set $\mathcal{S}^{-1} = \{S^{-1} | S \in \mathcal{S}\}$. Take all words in the alphabet $\mathcal{S} \cup \mathcal{S}^{-1}$ that are reduced, i.e. we remove pairs of the form SS^{-1}, $S^{-1}S$ and ? is concatenation of words with reduction.

Příklad
How does act on morphisms.

TODO!!!

1 Adjunction

Definice 1.1

Let $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$ be categories and functors. We say that F is left adjoint to G, and G is right adjoint to F, and write $F - | G$ if $B(F(A), B) \cong \mathcal{A}(A, G(B))$ „naturally“ in $A \in \mathcal{A}$, and $B \in \mathcal{B}$.

Poznámka
Naturally: $\overline{-} : \mathcal{B}(F(A), B) \to \mathcal{A}(A, G(B))$ and $\overline{-} : \mathcal{A}(A, G(B)) \to \mathcal{B}(F(A), B)$.

1. $\overline{F(A) \xrightarrow{g} B \xrightarrow{q} B'} = A \xrightarrow{\bar{g}} G(B) \xrightarrow{F(q)} G(B') \in \mathcal{A}$. 2. $\overline{A' \xrightarrow{p} A \xrightarrow{f} G(B)} = F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\bar{f}} B \in \mathcal{B}$.

An adjunction between F and G is a choice of such isomorphism in $B(F(A), B) \cong \mathcal{A}(A, G(B))$.

Příklad (Think about this)
Adjoints may not exist. But if an adjunction does exist, then it is unique up to unique isomorphism.

Definice 1.2 (Initial, terminal and zero object)

Let $\mathcal{A}$ be a category. An object $I \in \mathcal{A}$ is initial if for every $A \in \mathcal{A}$, $\exists!$ map $I \to A$. An object $T \in \mathcal{A}$ is terminal if for every $A \in \mathcal{A}$ $\exists!$ map $A \to T$. If object is both initial and terminal, we say that it is a zero object.

Například
In SET, we have an initial object. It is empty set.

In GRP we have an initial object $\{e\}$. And it is also a terminal object.

What object is a terminal object in SET? $T =$ the set with one element.

The terminal object in CAT is 1, the discrete category with one object.

Lemma 1.1

Let I and I' be two initial objects in a category $\mathcal{A}$. Then there is a unique isomorphism $I \to I'$, i.e. $I \cong I'$.

Důkaz
Since I and I' are both initial objects, $\exists!$ morphisms $\mathrm{id}_I : I \to I$, $f : I \to I'$, $g : I' \to I$ and $\mathrm{id}_{I'} : I' \to I'$. Because $g \circ f = \mathrm{id}_I$ and $f \circ g = \mathrm{id}_{I'}$, f and g give an isomorphism between I and I'. Moreover we see that it is unique. $\square$

Například
$VECT_{\mathbb{K}}$: initial object and terminal object is zero vector space (this is part of the „abelian category structure“ of $VECT_{\mathbb{K}}$).

Let R be a ring. Then we denote by MOD_R the category of R-modules with R-linear maps. This has zero object 0 – the zero module.

Příklad
Initial and terminal objects can be described via adjunctions: Let $\mathcal{A}$ be a category, then $\exists!$ functor $\mathcal{A} \to 1$ (the discrete category with one element). What bout a functor $1 \to \mathcal{A}$? We see that such functor $F \leftrightarrow$ objects $A \in \mathcal{A}$.

TODO?

TODO!!!

TODO!!!

TODO!!!

Věta 1.2

Take cats and functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{A}$. There is a bijective correspondence between

1. *(hom-class) adjunctions $F - | G$;*
2. *pairs $(1 \xrightarrow{?} GF,\ FG \xrightarrow{\varepsilon} 1_B)$ of natural transformations, satisfying the triangle identities;*
3. *"initial objects in certain comma categories".*

TODO!!!

Lemma 1.3

Take an adjunction $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{B}$, $F - | G$, and $A \in \mathcal{A}$. Then $(F(A), \eta_A : A \to GF(A))$ is an initial object in the category $(A \Rightarrow G)$

TODO!!!

TODO!!!

2 Representable functors

Poznámka
From now on all categories are assumed to be locally small (i.e. $\mathcal{A}(A, B)$ is a set).

Definice 2.1

Let $\mathcal{A}$ be a locally small category and let $A \in \mathcal{A}$. We define a functor

$$H^A(\cdot) := \mathcal{A}(A, \cdot) : \mathcal{A} \to SET$$

as follows:

- objects: $B \in \mathcal{A}$, $H^A(B) := \mathcal{A}(A, B)$;
- morphisms: for $B \xrightarrow{g} B' \in \mathcal{A}$ the map $H^A(g) := \mathcal{A}(A, g) : \mathcal{A}(A, B) \to \mathcal{A}(A, B')$ is defined by $p \mapsto g \circ p$.

Poznámka
Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Define $f_V := \langle v, \cdot \rangle : V \to \mathbb{K}$, $w \mapsto \langle v, w \rangle$. For some $v \in V$.

Definice 2.2

Let $\mathcal{A}$ be a locally small category, a functor $X : \mathcal{A} \to SET$ is called representable if, for some $A \in \mathcal{A}$, we have $X \simeq H^A$. A representation is a choice of isomorphism: $X \to H^A$.

Například
Let G be a group and let $\mathcal{G}$ be the associated one object category. Recall that (functors $\mathcal{G} \to SET \Leftrightarrow G$-sets.)

Since a representable functor is a functor, it must correspond to a G-set. The corresponding G-set is G itself, i.e. the left regular representation. (Since we only have one object, we only have one representable functor $H^{\cdot} : \mathcal{G} \to SET, \cdot \mapsto \mathcal{G}(\cdot, \cdot)$.)

Tvrzení 2.1

Any SET valued with a left adjoint is representable.

Důkaz
$G : \mathcal{A} \to SET \implies \mathcal{G}(A) \simeq SET(1, G(A))$, where 1 is 1-element set. □

Příklad
The forgetful functor $U : VECT_{\mathbb{K}} \to SET$ is representable, since it admits a left adjoint, i.e. the free functor.

$(f_v : V \to \mathbb{K}$, $w \mapsto \langle v, w \rangle$, $v \in V$, $\implies V \to V^* = LIN_{\mathbb{K}}(V, \mathbb{K})$, $v \mapsto f_v = \langle v, \cdot \rangle$.)

Poznámka
A morphism $A' \xrightarrow{f} A$ induces a natural transformation $H^A \overset{H^f}{\Rightarrow} H^{A'}$, defined by $H^A(B) = \mathcal{A}(A, B) \xrightarrow{H^f_B} H^{A'}(B) = \mathcal{A}(A', B)$, $p \mapsto p \circ f$.

Definice 2.3

Let $\mathcal{A}$ be a locally small cat, the functor $H^{\cdot} : \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, SET]$ (functor category: objects are $F : \mathcal{A} \to SET$, morphisms are natural transformations) is defined on objects $H^{\cdot}(A) = H^A$ and on morphisms $H^{\cdot}(f) = H^f$.

Poznámka (Moral)
This is a „representation“ of $\mathcal{A}^{\mathrm{op}}$ in $[\mathcal{A}, SET]$. (Functor categories „nicer“ than general categories.)

Definice 2.4

Let $\mathcal{A}$ be a locally small category and $A \in \mathcal{A}$. We define a functor $H_A : \mathcal{A}(\cdot, A) : \mathcal{A} \to SET$, as following:

- objects: $H_A(B) = \mathcal{A}(B, A)$, $B \in \mathcal{A}$;
- morphism: $B' \xrightarrow{g} B$ define $H_A(g) := \mathcal{A}(g, A) : \mathcal{A}(B, A) \to \mathcal{A}(B', A)$, $p \mapsto p \circ g$.

Poznámka
This now gives the definition of representable functor for functors $X : \mathcal{A}^{\mathrm{op}} \to SET$.

Definice 2.5 (Recall)

The functor category $[\mathcal{A}^{\mathrm{op}}, SET]$ is called the category of pre-sheaves on $\mathcal{A}$.

Definice 2.6

Let $\mathcal{A}$ be a locally small category. The Yoneda embedding of $\mathcal{A}$ is the functor $H_{\cdot} : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, SET]$ (defined in analogy with $H^{\cdot}$).

Poznámka
Embedding for categories is defined at the level of homomorphism sets $\mathcal{A}(A, B) \xrightarrow{F} \mathcal{B}(F(A), F(B))$ is injective, $\forall A, B \in \mathcal{A}$, i.e. F is a faithfull functor.

Například
Recall the functor $C : TOP^{\mathrm{op}} \to RING$, $X \mapsto C(X)$ (continuous functions from X to $\mathbb{C}$ or $\mathbb{R}$, ring with respect to point-wise operations). The functor $TOP^{\mathrm{op}} \xrightarrow{C} RING \underset{U}{\to} SET$ is representable since

$$U(C(X)) = TOP(X, \mathbb{R}) \text{ or } TOP(X, \mathbb{C}) = H_{\mathbb{R}}(X) \text{ or } H_{\mathbb{C}}(X).$$

Věta 2.2 (Yoneda lemma)

Let $\mathcal{A}$ be locally small small category. Then $[\mathcal{A}^{op}, SET](H_A, X) \simeq X(A)$. Naturally in $A \in \mathcal{A}$, and $X \in [\mathcal{A}^{op}, SET]$ (pre-sheaf), where naturality means that the composite functor

$$\mathcal{A}^{op} \times [\mathcal{A}^{op}, SET] \overset{H^{op}\times 1}{\to} [\mathcal{A}^{op}, SET]^{op} \times [\mathcal{A}^{op}, SET] \overset{Hom_{[\mathcal{A}^{op},SET]}}{S} ET,$$

$$(A, X) \mapsto (H_A, X) \mapsto [\mathcal{A}^{op}, SET](H_A, X)$$

is naturally isomorphic to the evaluation functor

$$\mathcal{A}^{op} \times [\mathcal{A}^{op}, SET] \overset{en}{\to} SET, \qquad (A, X) \mapsto X(A).$$

Příklad
Confirm how the two functors act on morphisms.

Důkaz (Yoneda)
Strategy for the proof: we want a natural isomorphism between our two functors:

- components are isomorphisms in SET labelled by the objects of $\mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, SET]$; (This we need to look up.)
- naturality conditions labelled by morphisms in $\mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, SET]$. (This is what we need to check.)

Let's focus this week on the first point. So for every $A \in \mathcal{A}^{\text{op}}$, and every $X \in [\mathcal{A}^{\text{op}}, SET]$ we want an isomorphism of sets:

$$[\mathcal{A}^{\text{op}}, SET](H_A, X) \overset{\hat{}\,(A,X)}{\to} X(A) \overset{\tilde{}\,(A,X)}{\to} [\mathcal{A}^{\text{op}}, SET](H_A, X).$$

$$F(A) \overset{\alpha_A}{\to} G(A) \overset{\alpha_A^{-1}}{F} (A).$$

Lemma 2.3 (Observation)

A function is defined by $[\mathcal{A}^{op}, SET](H_A, X) \overset{\hat{}}{\to} X(A)$, $(\alpha : H_A \to X) \mapsto \hat{\alpha} := \alpha_A(1_A)$.

Rough work: $\alpha : H_A \to X \Leftrightarrow \alpha_B : \mathcal{A}(B, A) \to X(B)$ $(B \in \mathcal{A})$. Let's look at the case $B = A : \alpha_A : \mathcal{A}(A, A) \to X(A)$, $1_A \mapsto \alpha_A(1_A)$.

Lemma 2.4

A function is defined by

$$[\mathcal{A}^{op}, SET](H_A, X) \overset{\tilde{}}{\leftarrow} X(A), \qquad \tilde{x} \leftarrow | x.$$

Důkaz (The previous lemma)
$x \in X(A)$, we need natural transformation $\tilde{X} : H_A \to X$, that is, for each $B \in \mathcal{A}^{\mathrm{op}}$ a function $\tilde{x}_B : H_A(B) = \mathcal{A}(B, A) \to X(V)$, which is natural in B.

TODO!!! □

TODO!!! □

TODO!!!

3 Limits and colimits

Poznámka
Change of viewpoint. We will look at the zero level, i.e. back inside categories.

Například
In $VECT_{\mathbb{K}}$, the category of all vector spaces, take V, W two objects. Consider $V \oplus W$ as a way to produce a new object from V and W.

Příklad
In SET let X, Y be two objects. How can we form a new object from X and Y.

Řešení
Cartesian product $X \times Y$.

Například
In $GROUP$, we can also take Cartesian product of two groups $G \times H$.

Definice 3.1

Let $\mathcal{A}$ be a category. A product of X and Y consists of object P and morphisms $X \overset{p_1}{\leftarrow} P \overset{p_2}{\to} Y$ with the property that for all objects and morphisms $X \overset{f_1}{\leftarrow} A \overset{f_2}{\to} Y$ $\exists!$ map $\overline{f} : A \to P$ such that $f_2 = \overline{f} \circ p_2$ and $f_1 = \overline{f} \circ p_1$.

We call p_1 and p_2 the projections of the product.

Například
For $V, W \in VECT_{\mathbb{K}}$ we define $P = V \oplus W$, $p_1 : V \oplus W \to V, (v, w) \mapsto v$ and $p_2 : V \oplus W \to W, (v, w) \mapsto w$.

Například
For $V, W \in VECT_{\mathbb{K}}$, we have maps

$$p_1 : V \oplus W \to V, \quad (v, w) \mapsto v, \qquad p_2 : V \oplus W \to W, \quad (v, w) \mapsto w.$$

Assume $\exists A \in VECT_{\mathbb{K}}$ + maps $f_1 : A \to V$, $f_2 : A \to W$. Then for $\overline{f}$ we take the linear map

$$\overline{f}(a) = (\overline{f}_V(a), \overline{f}_w(a)) = (f_1(a), f_2(a)).$$

$\implies$ by definition the diagram commutes and $\hat{f}$ is the unique linear such that this happens.
$\implies$ $V \oplus W$ is the product of V and W.

Příklad
Verify that products are unique up to unique isomorphism.

Příklad
For $X, Y \in SET$ consider the Cartesian product $X \times Y$, and the maps $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$. Consider another set Z and functions $f_1 : Z \to X$, $f_2 : Z \to Y$. Consider the diagram of the product. What is $\hat{f}$?

Řešení
Take $\hat{f}(z) := (f_1(z), f_2(z))$ and we are done.

Například
$X, Y \in TOP$. Then the product of X and Y is the Cartesian product of X and Y endowed with the product topology.

Take $2 = (XY)$ (two-objects category with only identities). There is no cone, objects that maps to both X and Y, hence the product of X and Y does not exist.

Příklad
Show that for $1 = ()$ the product of X with itself is X.

Příklad
$V_1, \ldots, V_n \in VECT_{\mathbb{K}}$ then we can consider $V_1 \oplus V_2 \oplus \ldots \oplus V_k \in VECT_{\mathbb{K}}$.

Poznámka
This is a generalised product over $I := \{1, \ldots, k\}$.

Definice 3.2 (Generalised product)

Let $\mathcal{A}$ be a category, I a set and $(X_i)_{i\in I}$ a family of objects of $\mathcal{A}$. A product of $(X_i)_{i\in I}$ consists of an object P and a family of maps $(P \xrightarrow{p_i} X_i)_{i\in I}$ such that $\forall A \in \mathcal{A}$ and maps $(A\overset{f_i}{X}_i)_{i\in I}$ ('cone') $\exists!$ map $\overline{f} : A \to P$ such that $p_i \circ \overline{f} = f_i$, $\forall i \in I$.

Poznámka
Generalised products do not need to exist, but if they do exist, then they are unique.

Důsledek
When $|I| = 2$, generalised products reduce to products (binary products).

Definice 3.3 (Diagram of shape, cone, limit)

Let $\mathcal{A}$ be a category, and let $\mathcal{I}$ be a small category. A diagram of shape $\mathcal{I}$ is a functor $D : \mathcal{I} \to \mathcal{A}$.

Například
A binary product is a diagram of shape $(A \leftarrow B \to C)$.

- A cone on D is an object $A \in \mathcal{A}$ (the vertex) together with a family $(A \xrightarrow{f_I} D(I))_{I\in\mathcal{I}}$ ('cone') of maps in $\mathcal{A}$ such that for all $I \xrightarrow{U} J$ in $\mathcal{I}$ the triangle $A \xrightarrow{f_I} D(I) \xrightarrow{D(U)} D(J) \xleftarrow{f_J} A$ commutes
- A limit of D is a cone $(L\overset{p_i}{D}(I))_{I\in\mathcal{I}}$ such that for any other cone L on D, $\exists!$ map $\overline{f} : A \to L$ such that $p_I \circ \overline{f} = f_I$, $\forall I \in \mathcal{I}$. (Factorisation in terms of $\overline{f}$.)

Například
A binary product is a limit of shape $I = 2 := (XY)$. A (generalised) product is a limit of shape $\mathcal{I}$, where $\mathcal{I}$ is a discrete category.

Například (Equaliser)
For a diagram $E = (X \rightrightarrows Y)$. A limit in a category $\mathcal{A}$ is called the equaliser.

Například (Pullback)
For a diagram of shape $P = (A \to B \leftarrow C)$, the limit is called a pullback.

Definice 3.4 (Has all limits)

A category has all limits, if it has limits of shape $\mathcal{I}$, $\forall$ small categories $\mathcal{I}$.

Tvrzení 3.1

Let $\mathcal{A}$ be a category.

- *If $\mathcal{A}$ has all products and equaliser $\implies$ it has all limits.*
- *If $\mathcal{A}$ has binary products, a terminal objects and equalisers. Then $\mathcal{A}$ has finite limits ($\mathcal{I}$ is a finite category).*

Definice 3.5 (Monics)

Let $\mathcal{A}$ be a category, a map $X \xrightarrow{f} Y$ in $\mathcal{A}$ is monic, if for all objects A and maps $A \underset{x'}{\overset{x}{\rightrightarrows}} X$,

$$f \circ x = f \circ x' \implies x = x'.$$

Příklad
In SET a map is monic if it is injective: f injective $\implies$ monic (easy), f monic $\implies$ injective $1 \underset{x'}{\overset{x}{\rightrightarrows}} X \xrightarrow{f} Y$, tj. $x \neq x' \implies f(x(1)) \neq f(x'(1))$.

Příklad
Similarly, for GRP, $VEC_{\mathbb{K}}$, $RING$, ... monics $\Leftrightarrow$ injective.

Lemma 3.2

$X \xrightarrow{f} Y$ is monic iff $X \xrightarrow{1_X} X \xrightarrow{f} Y \xleftarrow{f} X \xleftarrow{1_X} X$ is a pullback.

4 Colimits

Definice 4.1 (Cocone, colimit)

Let $\mathcal{A}$ be a category and $\mathcal{I}$ a small category in $\mathcal{A}$, and write $D^{\mathrm{op}} : \mathcal{I}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. A cocone on D is a cone on D^{op} and a colimit of D is a limit of D^{op}.

Explicitly a cocone on D is an object $A \in \mathcal{A}$ (vertex) plus a family $(D(I) \xrightarrow{f_I} A)_{I \in \mathcal{I}}$ (cocone) of morphisms such that $\forall (I \overset{u}{J}) \in \mathcal{I}$: $D(I) \xrightarrow{D(u)} D(J) \xrightarrow{f_J} A \xleftarrow{f_I} D(I)$.

A colimit is a universal cocone $(D(I) \overset{p_I}{L})$, i.e. it satisfies TODO (diagram with commutation).

Definice 4.2 (Coproduct / sum)

Coproduct (or sum) is a colimit of a discrete category.

Například
SET coproducts/sums are disjoint unions.

$GROUP$. The coproduct of two groups G and H is $G * H$, the free product of G and H.

$V, W \in VECT_{\mathbb{K}}$, what is the coproduct of V and W? $V \oplus W$.

Důsledek
Sometimes products and coproducts are same. (This is a general fact about Abelian categories.)

Například (Coequaliser)
A colimit of shape $E := (X \rightrightarrows Y)$ is called a coequaliser.

Například (Pushout)
A colimit of shape $(A \leftarrow B \rightarrow C)$ is called pushout (i.e. a pushback in $\mathcal{A}^{\mathrm{op}}$).

Například
Take $\mathcal{A} = AB$ (the category of Abelian groups). The coequaliser of $A \overset{s}{\underset{d}{\rightrightarrows}} B$ is $B \overset{proj}{\rightarrow} B/(\Im(t) \cup \Im(s))$. For $s = 0$ we get the cokernel of $t : A \rightarrow B$.

Definice 4.3 (Epics)

A map $X \overset{f}{\rightarrow} Y$ in $\mathcal{A}$ is epic, if $\forall Y \overset{g}{\underset{g'}{\rightrightarrows}} Z$:

$$g \circ f = g' \circ f \implies g = g'.$$

Například
SET epics $\Leftrightarrow$ surjective maps.

$HAUSDTOP$. Any morphism $f : X \rightarrow Y$ with dense image is an epic.

Příklad
Express epics as a pushout.

Definice 4.4 (Preservation of limits)

Let $\mathcal{I}$ be a small category, a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ preserves limits of shape $\mathcal{I}$ if for all diagrams $D : \mathcal{I} \rightarrow \mathcal{A}$ and all cones $(A \overset{p_I}{\rightarrow} D(I))$ on D, $(A \overset{p_I}{\rightarrow} D(I))_{I \in \mathcal{I}}$ is a limit on D in A. $\implies$ $(F(A) \overset{F(p_1)}{\rightarrow} F(D(I)))_{I \in \mathcal{I}}$, cone on $F \circ D$ in B, is a limit cone.

Například
Take $I = \varnothing$, empty category, and $D : \varnothing \to \mathcal{A}$. What is any object $A \in \mathcal{A}$? L is a limit if for any other cone A we have a unique morphism $A \xrightarrow{\hat{f}} L$. So a limit of shape $\varnothing$ is a terminal object. Dually a colimit over $\varnothing$ is an initial object.

Například (Equalisers)
A fork in a category consists of objects and maps $A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ such that $s \circ f = t \circ f$.

Definice 4.5

Let $\mathbb{A}$ be a category, and $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be objects and morphisms. An equaliser of s and t is $E \xrightarrow{i} X$ such that $E \xrightarrow{i} X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a fork such that for any other fork $\exists!$ map $\hat{f} : A \to E$ such that $X \leftarrow A \xrightarrow{\hat{f}} E \xrightarrow{i} X$.

Například
In SET $X \underset{t}{\overset{s}{\rightrightarrows}} Y$. $E := \{x \in X | s(x) = t(x)\}$.

Let $o : G \to H$ be a morphism in $GROUP$. This gives us a fork $\mathrm{Ker}(o) \hookrightarrow G \underset{e}{\overset{o}{\rightrightarrows}} H$, where $e(g) = e_H$ (identity of H) $\implies$ Kernels are equalisers.

$V \underset{t}{\overset{s}{\rightrightarrows}} W \in VECT_{\mathbb{K}}$ then the equaliser of this diagram is $\mathrm{Ker}(t - s) \hookrightarrow V$.

Poznámka (Pullbacks)
Recall that a pullback was a limit of shape $\mathcal{I} := (A \to B \leftarrow C)$. For a category $\mathcal{A}$, $D : \mathcal{I} \to \mathcal{A}$ we get $(X \xrightarrow{s} Z \xleftarrow{t} Y) \in \mathcal{A}$.

A pullback is an object $P \in \mathcal{A}$ together with maps $p_1 : P \to X$, $p_2 : P \to Y$ such that $P \xrightarrow{p_1} Y \xrightarrow{t} Z \xleftarrow{s} X \xleftarrow{p_1} P$ commutes and such that for any other cone ... we have ... commutes.

Například (Pullbacks in SET)
The pullback of a diagram $(X \to Z \leftarrow Y)$ in SET is $P = \{(x, y) \in X \times Y | s(x) = t(y)\}$. $P \xrightarrow{p_X} X \xrightarrow{t} Z \xleftarrow{s} Y \xleftarrow{p_Y} P$.

Inverse image (on SET): $f^{-1}(Y') \xrightarrow{p_y} Y \overset{j}{\hookrightarrow} Y \xleftarrow{f} X \xleftarrow{p_X} f^{-1}(Y')$, where j is inclusion, $f^{-1}(Y')$ are (x, y) such that $f(x) = y \in Y'$. $\Leftrightarrow$ the pullback $f^{-1}(Y)$ since y is determined by x.

Intersection: Let $X, Y \subseteq Z$. Then $P \xrightarrow{p_y} Y \overset{\text{inclusion}}{\hookrightarrow} Z \overset{\text{inclusion}}{\hookleftarrow} X \xleftarrow{p_X} P$, $(x, y) \overset{p_y}{\mapsto} y \mapsto y =$

$x \leftarrow\!\shortmid x \overset{p_X}{\leftarrow\!\shortmid} (x, y) \implies P \simeq X \cap Y.$

TODO!!!

Tvrzení 4.1

A morphism $f : X \to Y$ of objects in an additive category $\mathcal{C}$ is monic (resp. epic) iff $\mathrm{Ker}(f) = (0, 0 \to m)$*, respective (resp. coker*$(f) = (0, n \to 0)$*).*

Důkaz
Exercise. □

Definice 4.6 (Pre-abelian category)

A pre-abelian category $\mathcal{C}$ is an additive category such that every morphism in $\mathcal{C}$ has a kernel and a cokernel.

Poznámka
Now suppose we have a morphism $f : X \to Y$ in a pre-abelian category. Then consider following diagram:

$$0 \to \mathrm{Ker}(f) \to X \xrightarrow{f} Y \to \mathit{coker}(f) \to 0,$$

$$X \to \mathit{coker}\,\mathrm{ker}(f) \xrightarrow{h} \mathrm{ker}\,\mathit{coker}(f) \leftarrow Y, \qquad \mathit{coker}\,\mathrm{ker}(f) \xrightarrow{g?} Y.$$

The arrow g is induced by the universal property of $\mathit{coker}\,\mathrm{Ker}(f)$ applied to the composite $\mathrm{ker}(f) \to X \to Y$ and the same for h.

So we see that in a pre-abelian category, an morphism factors through its coimage $\mathit{coim}(f) := \mathit{coker}\,\mathrm{Ker}(f)$ and its image $\Im(f) := \mathrm{Ker}\,\mathit{ckoer}(f)$.

Definice 4.7 (Abelian category)

A pre-abelian category $\mathcal{A}$ is abelian if for every morphism $f \in \mathcal{A}$, the canonical morphism $\mathit{coim}(f) \to \Im(f)$ is an isomorphism.

Příklad
Show that this is true in MOD_R. (Use the fundamental 3-theorems of modules).

Věta 4.2 (Frey–Mitchel embedding theorem)

Let $\mathcal{A}$ be a small abelian category. Then exists a ring R and a full-faithful functor $F : \mathcal{A} \to MOD_R$.

Poznámka
How can we tell when $\mathcal{A} \simeq MOD_R$?

Věta 4.3 (Morita's theorem)

Let $\mathcal{A}$ be a complete abelian category, with „pregenerator" $P \in \mathcal{A}$, and let $R = \mathcal{A}(P, P)$. Then the functor $\mathcal{A}(P, -) : \mathcal{A} \to MOD_R$ is an equivalence of categories.