Poznámka
Topology...

Definice 0.1 (Topological vector space (TVS))

A Topological vector space over $\mathbb{F}$ is a pair (X, τ), where X is a vector space over $\mathbb{F}$ and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of $\mathbf{o}$ in (X, τ).

Definice 0.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka
Two homework (in Moodle) and one presentation.

Například
Let $(X, \|\cdot\|)$ be a normed linear space. Let τ be the topology induced by $\|\cdot\|$. The (X, τ) is HLCS.

Důkaz
$\varrho(x, y) = \|x - y\|$ metric induced by $\|\cdot\|$. τ induced by ϱ. This τ is Hausdorff. Continuity of the operations: (from Funkcionalka)

$$x_n \to x, y_n \to y, t_n \to t \implies x_n + y_n \to x + y \wedge t_n x_n \to tx.$$

So, it is a HTVS. Base of neighbourhood of $\mathbf{o}$ is e. g. $U(0, r), r > 0$, which is convex. $\square$

Let Γ be any nonempty set, $X = \mathbb{F}^\Gamma$ (= all functions $\Gamma \to \mathbb{F}$) with point-wise operations, so it is a vector space over $\mathbb{F}$. It is a HLCS.

Důkaz
„Continuity of addition:“ $x, y \in \mathbb{F}^\Gamma$, U a neighbourhood of $x+y \implies \exists F \subset \Gamma$ finite $\exists \varepsilon > 0$ such that

$$U_{\mathbf{o}} = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - (x(\gamma) + y(\gamma))| < \varepsilon \right\} \subset U$$

$$U_x = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$$U_y = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - y(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$\implies V_x$ is neighbourhood of x, and V_y is neighbourhood of y, and $U_x + U_y \subset U_0 \subset U$. Thus $z_1 \in V_x$, $z_2 \in V_y \implies z_1 + z_2 \in U_0 \subset U$.

„Continuity of multiplication“: $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^\Gamma$, U a neighbourhood of $\lambda x \implies \exists F \subset \Gamma$ finite $\exists \mu > 0$ such that

$$U_0 = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - \lambda x(\gamma)| < \varepsilon \right\} \subset U$$

$$|\mu z(\gamma) - \lambda x(\gamma)| \leqslant |\mu| \cdot |z(\gamma) - x(\gamma)| + |\mu - \lambda| \cdot |x(f)|.$$

$$M := \max_{\gamma \in F} |x(\gamma)|.$$

$$V = \left\{ \mu \in \mathbb{F} \middle| |\mu - \lambda| < \frac{\varepsilon}{2(M+1)} \right\}, \qquad W = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2(|\lambda| + \frac{\varepsilon}{2(M+1)})} \right\}.$$

$$\mu \in V, z \in W \implies \mu z \in U_0 \subset U.$$

„Local convexity“: Base of neighbourhoods of $\mathbf{o}$: $\left\{ x \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |x(\gamma)| < \varepsilon \right\}$, $F \subset \Gamma$ finite, $\varepsilon > 0$, consists of convex sets.

„Hausdorff“: $x \neq y \implies \exists \gamma \in \Gamma : x(\gamma) \neq y(\gamma)$. Take $\varepsilon = \frac{|x(\gamma) - y(\gamma)|}{2}$.

$$U = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - x(\gamma)| < \varepsilon \right\}, V = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - y(\gamma)| < \varepsilon \right\} \implies U \cap V = \varnothing.$$

□

$X = C(\mathbb{R}, \mathbb{F}) = \{f : \mathbb{R} \to \mathbb{F} \text{ continuous}\}$,

$$\varrho(f, g) = \sum_{n=1}^\infty \frac{1}{2^n} \min \left\{ 1, \max_{t \in [-n,n]} \{|f(t) - g(t)|\} \right\} =: \sum_{N=1}^\infty \frac{1}{2^N} \min\{1, p_N(f-g)\}$$

is translation invariant (that implies addition is continuous, see lecture) metric.

Důkaz
$f_n \to f$ in $\varrho \Leftrightarrow \forall N : f_n \rightrightarrows f$ on $[-N, N]$.

„$f_n \to f$, $\lambda_n \to \lambda \implies \lambda_n f_n \to \lambda f$“: Let $N \in \mathbb{N}$. We will show $\lambda_n f_n \rightrightarrows \lambda f$ in $[-N, N]$. $x \in [-N, N]$:

$$|\lambda_n f_n(x) - \lambda f(x)| \leqslant |\lambda_n| \cdot |f_n(x) - f(x)| + |\lambda_n - \lambda| \cdot |f(x)| \leqslant c \cdot p_N(f_n - f) + |\lambda_n - \lambda| \cdot p_N(f) \to 0.$$

Hence, X is HTVS. „Local convexity“: $U_{N,\varepsilon} = \{f \in X | p_N(t) < \varepsilon\}$, clearly $U_{N,\varepsilon}$ is a convex set and $U_{N,\varepsilon}$ is neighbourhood of $\mathbf{o}$. If $\varepsilon < \lambda$, then $\left\{f | \varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}\right\} \subset U_{N,\varepsilon}$, because for $\varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}$ it is $\frac{1}{2^N} p_N(f) < \frac{\varepsilon}{2^N}$. „they form a base“: $f \in U_{N,\varepsilon} \implies \varrho(f, \mathbf{o}) < \varepsilon + \frac{1}{2^N}$. Hence fix $r > 0$ and take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$. Then $U_{N,\frac{r}{2}} \subset \{f | \varrho(f, \mathbf{o}) < r\}$ □

(Ω, Σ, μ) a measure space, $p \in (0, 1)$. $L^p(\Omega, \Sigma, \mu) = \left\{f : \Omega \to \mathbb{F} \text{ measurable} | \int |f|^p d\mu < \infty\right\}$ (we identify functions equal almost everywhere). $\varrho(f, g) = \int |f - g|^p d\mu$ is a metric making $X = L^p(\Omega, \Sigma, \mu)$ a HTVS (but not locally convex).

Důkaz
„ϱ is a metric“: „$\triangle$-inequality“: $a, b \in [0, \infty) : (a + b)^p \leqslant a^p + b^p$. (Fix $a \geqslant 0$, take $\varphi_a(b) = (a + b)^p - a^p - b^p \implies \varphi_a$ is continuous on $[0, \infty)$, $\varphi_a(0) = 0$. For $b > 0$: $\varphi_a(b) = p(a + b)^{p-1} - pb^{p-1} = p \cdot ((a + b)^{p-1} - b^{p-1}) < 0$ as $p - 1 < 0 \implies \varphi_a$ decreasing on $[0, \infty)$ and $\varphi_a \leqslant 0$.)

φ is translation invariant $\implies$ addition is continuous. „Multiplication“: We can see that $\varrho(\lambda f, \mathbf{o}) = |\lambda|^p \varrho(f, \mathbf{o})$. $f_n \to f$, $\lambda_n \to \lambda$:

$$\varrho(\lambda_n f_n, \lambda f) \leqslant \varrho(\lambda_n f_n, \lambda_n f) + \varrho(\lambda_n f, \lambda f) = |\lambda_n|^p \varrho(f_n, f) + |\lambda_n - \lambda|^p \varrho(f, \mathbf{o}) \to 0.$$

Hence, we have a HTVS. □

Tvrzení 0.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x + U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 0.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1 - t)y \in A$ for $x, y \in A$, $t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}$, $|\alpha| \leqslant 1$;
- absolutely convex if it is convex and balanced;

- absorbing if $\forall x \in X\ \exists t > 0 : \{sX | s \in [0,t]\} \subset A$.

Definice 0.4

$\mathrm{co}(A)$ = convex hull, $\mathrm{b}(A)$ = balanced hull, $\mathrm{aco}(A)$ = absolutely convex hull.

Tvrzení 0.2

X is a metric space over $\mathbb{F}$, $A \subset X$. Then:

(a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow$ A is convex and symmetric.

(b) $\mathrm{co}\, A = \{t_1 x_1 + \ldots + t_k x_k | x_1 \ldots x_k \in A, t_1 \ldots t_k \geqslant 0, t_1 + \ldots + t_k = 1, k \in \mathbb{N}\}$.

(c) $\mathrm{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leqslant 1\}$.

(d) $\mathrm{aco}(A) = \mathrm{co}(\mathrm{b}(A))$.

(e) A is convex $\Leftrightarrow$ $(s+t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)
„$\implies$“: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „$\impliedby$“: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \implies \alpha \in [-1, 1].$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{(1+\alpha)}{2} x \in A$). □

Důkaz (b)
„$\subseteq$“: by induction on k:

$$t_1 x_1 + \ldots + t_{k+1} x_{k+1} = (t_1 + \ldots + t_k) \frac{t_1 x_1 + \ldots + t_k x_k}{t_1 + \ldots + t_k} + t_{k+1} x_{k+1}.$$

„$\supseteq$“: the set on the RHS is convex and contain A. □

Důkaz (c)
„$\supseteq$“: clear. „$\subseteq$“: RHS is a balanced set. □

Důkaz (d)
„$\supseteq$“: clear. „$\subseteq$“ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c): $\mathrm{co}(\mathrm{b}(A)) = \{t_1 \alpha_1 x_1 + \ldots + t_k \alpha_k x_k | x_1, \ldots, x_k \in A, |\alpha_j| \leqslant 1, t_j \geqslant 0, t_1 + \ldots + t$ is clearly balanced.) □

Důkaz (e)
„ $\Longrightarrow$ “: „$\subseteq$“: always, „$\supseteq$“: $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$.

„ $\Longleftarrow$ “: in particular $\forall t \in (0,1)$: $tA + (1-t)A \subset A$, it is the definition of convexity. □

Tvrzení 0.3

Let (X,τ) be a LCS, $U \in \tau(\mathbf{o})$. Then

(i) U is absorbing.

(ii) $\exists V \in T(0) : V + V \subset U$.

(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.

Důkaz (i)

$x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V$ a neighbourhood of 0 in $\mathbb{F} : V \cdot x \subset U \implies \exists t > 0 : [0,t] \subset V$

□

Důkaz (ii)

$$\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W \subset U.$$

Take $V = W_1 \cap W_2$. □

Důkaz

$$\exists U_0 \in \tau(\mathbf{o}) \text{ convex}, U_0 \subset U : \mathbf{o} \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0 \; \exists W \in \tau(\mathbf{o}) \text{ open} :$$
$$\forall \lambda, |\lambda| < c : \lambda W \subset U_0.$$

$V_1 := \bigcup_{0<|\lambda|<1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \operatorname{co} V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „V is open“:

$$V = \bigcup \{t_1 x_1 + \ldots + t_n x_n + t_{n+1} V_1 | t_1, \ldots, t_{n+1} \geqslant 0, t_1 + \ldots + t_{n+1} = 1, x_1, \ldots, x_n \in V_1\}$$

□

Věta 0.4

1. Let (X,τ) be a LCS. Then there is $\mathcal{U}$, a base of neighbourhoods of $\mathbf{o}$ with properties:

- *the elements of $\mathcal{U}$ are absorbing, open, absolutely convex;*
- $\forall U \in \mathcal{U} \; \exists V \in \mathcal{U} : 2V \subset U$.

If X is Hausdorff, then $\bigcap \mathcal{U} = \{\mathbf{o}\}$.

2. Let X be a vector space, $\mathcal{U}$ a nonempty family of subsets of X satisfying:

- *the elements of $\mathcal{U}$ are absorbing and absolutely convex;*
- $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U$*;*
- $\forall U, V \in \mathcal{U}\ \exists W \in \mathcal{U} : W \subset U \cap V$*.*

Then there is a unique topology τ on X such that (X, τ) is LCS and $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$. Further, if $\bigcap \mathcal{U} = \{\mathbf{o}\}$, the τ is Hausdorff.

Důkaz (1.)
Let $\mathcal{U}$ be the family of all open absolutely convex neighbourhoods of $\mathbf{o}$. The previous proposition (iii) gives us $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$, (1) gives us elements of $\mathcal{U}$ are absorbing, so the first item holds. (ii) gives us $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$.

Assume X i Hausdorff: $x \in X \setminus \{\mathbf{o}\} \overset{\text{Hausdorff}}{\implies} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$. □

Důkaz (2.)
Set $\tau = \{G \subset X | \forall x \in G\ \exists U \in \mathcal{U} : x + U \subset G\}$. This is a unique possibility so uniqueness is clear.

„τ is topology“: $\emptyset, X \in \tau$ and τ is closed to arbitrary union (clear). τ is closed to finite intersections by third item ($G_1, g_2 \in \tau$, $x \in G_1 \cap G_2 \ldots U_1, U_2 \in \tau$, $x + U_1 \subset G_1$, $x + U_2 \subset G_2$; $\exists V \in \mathcal{U} : V \subset U_1 \cap U_2$, then $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$).

„Elements of $\mathcal{U}$ are neighbourhoods of $\mathbf{o}$“: $U \in \mathcal{U}$. $V := \{x \in U | \exists W \in \mathcal{U} : x + W \subset U\}$. Then $V \subset U$, $0 \in V$ (take $W = U$). $V \in \tau$ ($x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$; let $\tilde{W} \in \mathcal{U}$ such that $2\tilde{W} \subset W$, then $x + \tilde{W} \subset V$, because $y \in \tilde{W} \implies X + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$).

„$\mathcal{U}$ is a base of neighbourhood of $\mathbf{o}$“: now clear.

„(X, τ) is a TVS“: $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$. Then $(x+V)+(y+V) \subset x+y+2V \subset x+y+U \subset G$. $\lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G$; $\exists V \in \mathcal{U} : 2V \subset U$; V is absorbing $\implies \exists c > 0\ \forall t \in [0, c] : tx \in V$; V balanced $\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$; assume $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1} V$,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda) y}_{(\mu-1)\cdot\left(\mu + \frac{1}{|\lambda|+1}\right)V} + \underbrace{\lambda(y - x)}_{\in \frac{\lambda}{|\lambda|+1} V \subset V}.$$

„Local convexity“: by first item: $\forall U \in \mathcal{U} : U$ is convex.

Assume $\bigcap \mathcal{U} = \{\mathbf{o}\}$. Take $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$. Take $V \in \mathcal{U} : 2V \subset U$. Then if $(x + V) \cap (y + V) = \emptyset$, $x + v_1 = y + v_2$, $x - y = v_2 - v_1 \in V + V = 2V \subset U$ ↯. □

Věta 0.5

Let X be a vector space and let $\mathcal{P}$ be a family of seminorms on X. The there is a unique topology τ on X such that (X, τ) is a LCS and $\mathcal{U} = \{\{x \in X | p_1(x) < c_1, \ldots, p_k(x) < c_k\} | p_1, \ldots, p_k \in \mathcal{P}, c_1, \ldots,$ is a base of neighbourhood of $\mathbf{o}$.

(X, τ) is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\}\ \exists p \in \mathcal{P}, p(x) > 0$.

Důkaz

Use the previous theorem (2.) on $\mathcal{U}$: The sets are absolutely convex (by properties of seminorms). „Absorbing“: $U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $x \in X$?, $j \in [k]$. Then $p_j(x) \in (0, \infty)$ as for $t > 0 : p_j(t \cdot x) = t \cdot p_j)_x$ and $\exists c > 0$ such that $c \cdot p_j(x) < c_j$ for $j \in [k]$. Now for $t \in [0, c] : tx \in U$.

$U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $V = \left\{x \in X | p_1(x) \subset \frac{c_1}{2}, \dots, p_k(x) < \frac{c_k}{2}\right\}$.

$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$ trivially.

„Hausdorffness“:

$$\bigcap U = \{x \in X | \forall p \in \mathcal{P} : p(x) = 0\}.$$

„$\supseteq$“ clear. „$\subseteq$“: Assume $y \in X$, $p \in \mathcal{P} : p(y) > 0$: $U = \{x \in X | p(x) < p(y)\} in \mathcal{U} \implies y \notin U$. □

Například

$(X, \|\cdot\|)$ is a normed space, then its topology is generated by $\mathcal{P} = \{\|\cdot\|\}$.

The topology on $\mathbb{F}^\Gamma$ is generated by seminorms $p_\gamma(f) = |f(\gamma)|$, $f \in \mathbb{F}^\Gamma$ $(\gamma \in \Gamma)$.

$C(\mathbb{R}, \mathbb{F})$ the topology is generated by this sequence of seminorms: $p_N(f) = \max_{x \in [-N,N]} |f(x)|$.

Definice 0.5 (Minkowski functional)

X vector space, $A \subset X$ convex absorbing. Then

$$p_A(x) := \inf\{\lambda > 0 | x \in \lambda \cdot A\}.$$

Lemma 0.6

Let X be LCS, $A \subset X$ convex set.

$$x \in \overline{A}, y \in \operatorname{int} A \implies \{tx + (1-t)y | t \in [0, 1)\} \subset \operatorname{int} A.$$

Důkaz

WLOG $y = 0$. $t = 0$ clear, $0 \in \operatorname{int} A$. $t \in (0, 1)$:

Fix U, an open absolutely convex neighbourhood of $\mathbf{o}$ such that $U \subset A$. Then $x + \frac{1-t}{t} U$ is a neighbourhood of $x \implies \exists$

TODO!!! □

TODO!!!

Důkaz (Continuity of multiplication? Theorem 4. TODO?)
„U is a neighbourhood of $\mathbf{o}$ in τ, $\lambda > 0 \implies \lambda U$ is neighbourhood of $\mathbf{o}$“: $\lambda \geqslant 1$: $\exists V \in \mathcal{U} : V \subset U \implies V \subset \lambda V \subset \lambda U$ (V is absolutely convex) $\implies \lambda U$ is neighbourhood of $\mathbf{o}$. $\lambda = \frac{1}{2}$: $\exists V \in \mathcal{U} : V \subset U$, then $\exists W \in \mathcal{U} : 2W \subset V$, then $W \subset \frac{1}{2}V \subset \frac{1}{2}U \implies \frac{1}{2}U$ is a neighbourhood of $\mathbf{o}$. Now by induction for $\lambda = \frac{1}{2^n}$. For $\lambda > 0$ find $n \in \mathbb{N}$ such that $\lambda > \frac{1}{2^n}$.

$\lambda x \in G$ ($\lambda \in \mathbb{F}, x \in X, G \in \tau$) $\implies \exists U \in \mathcal{U} : \lambda x + U \in G$. Find $V \in \mathcal{U} : 2V \subset U$ such that V is absorbing ($\implies \exists c > 0\ \forall t \in [0, c] : tx \in V$) and V is balanced ($\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$). Let $\mu \in F, y \in X$ such that

$$|\mu - \lambda| < c \wedge y \in x + \frac{1}{|\lambda| + c} V \text{ (a neighbourhood of } \mathbf{o})$$

$$\implies \mu y - \lambda x = \mu(y - x) + (\mu - \lambda)x \in V + V = 2V \subset U \implies \mu y \in \lambda x + U \subset G.$$

□

Tvrzení 0.7 (8. see notes of lecturer)

Let X be LCS, $A \subset X$ a convex neighbourhood of $\mathbf{o}$.

Clearly: $[p_A \subset 1] \subset A \subset [p_A \leqslant 1]$.

Důkaz
„$[p_a < 1] = \operatorname{int} A$“: „$\subseteq$“: $p_A(x) < 1 \implies \exists c > 1$ such that $cx \in A \implies x = \frac{1}{c}cx \in \operatorname{int} A$. „$\supseteq$“: $x \in \operatorname{int} A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$. U absorbing $\implies \exists \alpha > 0 : \alpha x \in U$. Then $(1 + \alpha)x \in A \implies p(x) \leqslant \frac{1}{1+\alpha} < 1$.

„$[p_A \leqslant 1] = \overline{A}$“: „$\subseteq$“: $p_A(x) \leqslant 1 \implies \forall n \in \mathbb{N} : p_x\left(\left(1 - \frac{1}{n}\right)x\right) = \left(1 - \frac{1}{n}\right)p_A(x) \leqslant 1$. $\left(1 - \frac{1}{n}\right)x \in \operatorname{int} A \implies x \in \overline{\operatorname{int} A} \subset \overline{A}$. „$\supseteq$“: $x \in \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in \operatorname{int} A$, so, $p_A\left(\left(1 - \frac{1}{n}\right)x\right) < 1 \overset{n\to\infty}{\implies} p_A(x) \leqslant 1$. □

p_A is continuous on X.

Důkaz
$[p_A < c] = \emptyset$ if $c \leqslant 0$ and $c \cdot \operatorname{int} A$ if $c > 0$. $[p_A > c] = X$ if $c < 0$, $X \backslash (c \cdot \overline{A})$ if $c > 0$, and $\bigcup_{t>0} X \backslash t\overline{A}$ if $c = 0$. All these sets are open. □

$p_A = p_{\overline{A}} = p_{\operatorname{int} A}$.

Důkaz
$\operatorname{int} A \subset A \subset \overline{A} \implies p_{\overline{A}} \leqslant p_A \leqslant p_{\operatorname{int} A}$. „Conversely“: Assume that $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in d \operatorname{int} A \implies \left(1 - \frac{1}{n}\right)p_{\operatorname{int} A}(x) \leqslant d \implies p_{\operatorname{int} A}(x) \leqslant d < c$. □

Důsledek
Any LCS (X) is completely regular.

Důkaz
$x \in X$, U an open neighbourhood of x. Take V a convex neighbourhood of $\mathbf{o}$ such that $x + V \in U$. $f(y) := \min\{1, p_V(y-x)\}$. The f is continuous by the previous proposition, $f(x) = 0$.

$$y \in X \backslash U \implies y - x \notin V \implies p_V(y-x) \geqslant 1 \implies f(y) = 1.$$

□

Věta 0.8

TODO!!! The topology generated by $\mathcal{P}_\tau$ coincides with τ.

Důkaz
Let τ_1 be topology induced by $\mathcal{P}_\tau$. $\tau_1 \subset \tau$ (seminorms from $\mathcal{P}_\tau$ are τ-continuous, hence the sets from theorem 5? are τ-open). „$\tau \subset \tau_1$“: Let $U \in \tau(\mathbf{o}) \implies \exists V$ a neighbourhood of $\mathbf{o}$ such that $V \subset U$. The $p_V \in \mathcal{P}_\tau$ (from the previous proposition is continuous) $\implies$ $[p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$. □

Tvrzení 0.9

X a vector space.

1. *p is seminorm $\implies$ $[p < 1]$ is absolutely convex, absorbing, and $p_{[p<1]} = p$.*
2. *p, q are seminorms, then $p \leqslant q \Leftrightarrow [p < 1] \supset [q < 1]$.*
3. *$\mathcal{P}$ a set of seminorms generated by a topology τ. p a seminorm on X. Then p is τ-continuous $\Leftrightarrow \exists p_1, \ldots, p_k \in \mathcal{P}\ \exists c > 0 : p \leqslant c \cdot \max\{p_1, \ldots, p_k\}$.*

Důkaz (1.)
Absolutely convex and absorbing is clear.

$$p_{[p<1]}(x) = \inf\{\lambda > 0 | x \in \lambda[p < 1]\} = \inf\{\lambda > 0 | x \in [p < \lambda]\} = p(x).$$

□

Důkaz (2.)
„ $\implies$ “ trivial. „ $\impliedby$ “: $[p < 1] \supset [q < 1] \implies p = p_{[p<1]} \leqslant p_{[q<1]} = q$. □

Důkaz (3.)
„ $\Longleftarrow$ “: $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = \left[p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}\right]$, which is a τ-open set $\implies A$ is a neighbourhood of $\mathbf{o} \implies p = p_A$ is continuous (by 1. and the previous proposition).

„ $\implies$ “: p is continuous $\implies [p < 1]$ is neighbourhood of $\mathbf{o}$ ($p(\mathbf{o}) = 0$) $\implies \exists p_1, \dots, p_k \in \mathcal{P}\ \exists c_1, \dots, c_k > 0$ such that $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c}\max\{p_1, \dots, p_k\} < 1]$ ($c = \min\{c_1, \dots, c_k\}$). Use 2. for seminorms $p, \frac{1}{2\max\{p_1,\dots,p_k\}}$ and get $p \leqslant \frac{1}{c}\max\{p_1, \dots, p_k\}$. □

1 Continuous and bounded linear mapping

Tvrzení 1.1

$(X, \tau), (Y, \mathcal{U})$ LCS, $L : X \to Y$ linear. Then the following assertions are equivalent:

1. *L is continuous;*
2. *L is continuous at $\mathbf{o}$;*
3. *L is uniformly continuous.*

Důkaz
„1. $\implies$ 2.“ trivial, „2. $\implies$ 3.“ assume L continuous at $\mathbf{o}$. Then, given $U \in \mathcal{U}(\mathbf{o})$, there is $V \in \tau(\mathbf{o})$ such that $L(V) \subset U$. Take $x, y \in X$ such that $x - y \in V$. Then $L(x) - L(y) = L(x - y) \in U$ and that's continuous. „3. $\implies$ 1.“ trivial. □

Tvrzení 1.2

$L : X \to Y$ linear. L is continuous $\Leftrightarrow \forall q$ a continuous seminorm on Y $\exists p$ a continuous seminorm on X: $\forall x \in X : q(L(x)) \leqslant p(x)$.

Důkaz
„ $\implies$ “: L continuous, q a continuous seminorm on Y, the $p(x) = q(L(x))$ is a continuous seminorm on X. „ $\Longleftarrow$ “: By the previous proposition it is enough „L is continuous at $\mathbf{o}$“: U neighbourhood of $\mathbf{o}$ in Y, $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. $q := p_V$ is a continuous seminorm. Let p be a continuous seminorm on X such that $q \circ L \leqslant p$. $W := [p < 1]$ a neighbourhood of $\mathbf{o}$ in X and $L(W) \subset V \subset U$. $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$. □

TODO!!!

TODO!!!

Věta 1.3

TODO[Theorem 22]!!!

Důkaz
„2. $\implies$ 1." trivial. „1. $\implies$ 3." if ϱ a metric generating τ, then $U_n = \left\{x \in X | \varrho(x, 0) < \frac{1}{n}\right\}$ $\implies$ $(U_n)_n$ is a base of neighbourhoods of $\mathbf{o}$. „3. $\implies$ 4.": (see the proof of the previous proposition, 1.) (U_n) base of neighbourhood of $\mathbf{o}$, take $V_n \subset U_n$ absolutely convex neighbourhood of $\mathbf{o}$, $p_n = p_{V_n}$ $\implies$ (p_n) generate τ. „4. $\implies$ 2.": the previous proposition 2. □

Věta 1.4

(X, τ) is HLCS. X is normable $\Leftrightarrow$ $\exists U$, a bounded neighbourhood of $\mathbf{o}$.

Důkaz
„ $\implies$ ": τ generated by $\|\cdot\|$, $U := \{x \in X | \|x\| < 1\}$ is a bounded neighbourhood of $\mathbf{o}$.

„ $\impliedby$ ": U bounded neighbourhood of $\mathbf{o}$. WLOG U is absolutely convex. Then $\frac{1}{n}U$, $n \in \mathbb{N}$ is a base of neighbourhoods of $\mathbf{o}$ (V neighbourhood of $\mathbf{o}$, $W \subset V$ an absolutely convex neighbourhood of $\mathbf{o}$ $\implies$ $\exists \lambda > 0 : U \subset \lambda W$ Take $n \in \mathbb{N}$ such that $n > \lambda$. Then $U \subset n \cdot W$ so $\frac{1}{n}U \subset W \subset V$). Finally, p_U is a norm generating the topology (U absolutely convex neighbourhood of $\mathbf{o}$ $\implies$ p_U is a continuous seminorm. $\frac{1}{n}U = [p_U < \frac{1}{n}], n \in \mathbb{N}$ is a base of neighbourhood of $\mathbf{o}$ $\implies$ p_U generated topology of X. From X Hausdorff, p_U is a norm.) □

2 Fréchet spaces

Definice 2.1 (Fréchet space)

A LCS whose topology is generated by a complete translation invariant metric is called Fréchet space.

Například
X Banach space $\implies$ X Fréchet space. $\mathbb{F}^{\mathbb{N}}, C(\mathbb{R}, \mathbb{F}), H(\Omega)$ are Fréchet spaces.

Důkaz ($\mathbb{F}^{\mathbb{N}}$)

$$p_n((x_k)) = \max\{|x_k||k \in [n]\}$$

seminorms generating the topology, $p_1 \leqslant p_2 \leqslant \dots$

$$\varrho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x-y)\}$$

is translation invariant metric generating the topology. It is complete: $((x_k^m)_k)_{m=1}^{\infty}$ a ϱ-Cauchy sequence $\implies \forall n \in \mathbb{N} : ((x_k^m))_m$ is p_n-Cauchy $\implies$ it is $\|\cdot\|_\infty$-Cauchy in $\mathbb{F}^{\mathbb{N}}$ $\implies$ (because $\mathbb{F}^{\mathbb{N}}$ is complete) $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \overset{m\to\infty}{\longrightarrow} (y_1^n, \dots, y_n^n) \in \mathbb{F}^n$.

Moreover, if $i \leqslant n_1 \leqslant n_2$, then $y_i^{n_1} = y_i^{n_2}$. So, we have $y = (y_k)_{k=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$, such that $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \overset{m}{\to} (y_k)_{k=1}^n$

$$\implies \forall n \in \mathbb{N} : p_n(x^n - y) \overset{m}{\to} 0 \implies \varrho(x^n, y) \to 0, \text{ i.e. } x^n \to y \text{ in } X.$$

□

Důkaz ($\mathbb{C}(\mathbb{R}, \mathbb{F})$)

$$p_n(f) = \max_{x \in [-n,n]} |f(x)|.$$

(f_k) ϱ-Cauchy $\implies \forall n : (f_k)$ is p_n-Cauchy $\implies \forall n : (f_k|_{[-n,n]})$ is $\|\cdot\|_\infty$-Cauchy in $C([-n,n) \implies \forall n\, \exists g_n \in C([-n,n])$ such that $f_k|_{[-n,n]} \overset{k}{\to} g_n$ in $C([-n,n])$.

$\forall n_1 \leqslant n_2 : g_{n_2}|_{[-n_1,n_1]} = g_{n_1}$ so, we have one function $g : \mathbb{R} \to \mathbb{F}$ such that $\forall n \in \mathbb{N} : g|_{[-n,n]=g_n}$. Then g is continuous, i.e. $g \in C(\mathbb{R}, \mathbb{F})$ and $\forall n \in \mathbb{N} : p_n(f_k - g) \overset{k}{\to} 0$. So $p_n(f_k, g) \to 0 \implies f_n \to g$. □

Tvrzení 2.1

(X, τ) is a Fréchet space, ϱ any translation invariant metric on X generating $\tau \implies \varrho$ is complete.

Důkaz
ϱ, d two translation invariant metrics generating by τ. Idea: convergent sequences with respect to ϱ and d coincide, Cauchy sequences with respect to ϱ and d coincide. (x_n) ϱ-Cauchy: $\varepsilon > 0 \implies \{x | d(x, \mathbf{o}) < \varepsilon\}$ is a neighbourhood of $\mathbf{o} \implies \exists \delta > 0 : \{x | \varrho(x, \mathbf{o}) < \delta\} \subset \{x | d(x, \mathbf{o}) < \varepsilon\}$.

$\exists n_0\, \forall m,n > n_0 : \varrho(x_m - x_n, \mathbf{o}) = \varrho(x_m, x_n) < \delta \implies d(x_m - x_n, 0) = d(x_m, x_n) < \varepsilon \implies (x_n)$ is d-Cauch

□

Tvrzení 2.2

X Fréchet, $A \subset X$. A is compact $\Leftrightarrow$ A is closed and totally bounded.

Důkaz

Let ϱ be a complete translation invariant metric generating the topology. A is compact $\Leftrightarrow$ A is closed and ϱ-totally bounded. But ϱ-totally boundedness = total boundedness in X. A is totally bounded in X $\Leftrightarrow$ $\forall U$ neighbourhood of $\mathbf{o}$ $\exists F \subset X$ finite $A \subset F+U$. A is totally bounded in $\varrho \Leftrightarrow \forall \varepsilon > 0\ \exists F \subset X$ finite such that $A \subset \bigcup_{x\in F} U_\varrho(x,\varepsilon) = F + U_\varrho(0,\varepsilon)$. □

Tvrzení 2.3

X LCS, $A \subset X$ totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded.

Důkaz

Let U be a neighbourhood of $\mathbf{o}$. Let V be an absolutely convex neighbourhood of $\mathbf{o}$ such that $2V \subset U \implies \exists F \subset X$ finite such that $A \subset F + V$. Then clearly $\operatorname{aco} A \subset (\operatorname{aco} F) + V$. $\operatorname{aco} F$ is compact,

$$F = \{x_1,\dots,x_k\} \implies \operatorname{aco}(F) = \operatorname{co}(\mathrm{b}(F)) = \operatorname{co}\{\lambda x_j | j \in [k], |\lambda| \leqslant 1\} = \left\{t_1\lambda_1x_1 + \dots t_n\lambda_nx_n \middle| |\lambda_j| \leqslant 1, t_j\right.$$

$$B = \left\{(\lambda_1,\dots,\lambda_n,t_1,\dots,t_n) \middle| |\lambda_j| \leqslant 1, t_j \geqslant 0, \sum t_j = 1\right\}$$

a compact set in $\mathbb{F}^n \times \mathbb{R}^n$. $(\lambda_1,\dots,\lambda_n,t_1,\dots,t_n) \mapsto t_1\lambda_1x_1 + \dots + t_n\lambda_nx_n$ is a continuous map and maps B onto $\operatorname{aco} F$.

$\operatorname{aco} F$ compact $\implies$ totally bounded $\implies$ $\exists H \subset X$ finite, $\operatorname{aco} F \subset H + V$ So $\operatorname{aco} A \subset \operatorname{aco} F + V \subset H + V + V = H + 2V < H + U$. □

Důsledek

X Fréchet space, $A \subset X$ compact $\implies$ $\overline{\operatorname{aco} A}$ is compact.

Důkaz

A compact $\implies$ A is totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded $\implies$ (because $M \subset X$ any set $\implies$ $\overline{M} \subset M + U$) $\overline{\operatorname{aco} A}$ is totally bounded $\implies$ $\overline{\operatorname{aco} A}$ is compact.

(M totally bounded, for any $U \in \tau(\mathbf{o})$: U is neighbourhood of $\mathbf{o}$, $x \in \overline{M}$, U absolutely convex convex neighbourhood of $\mathbf{o}$, then $V \subseteq U$ absolutely convex such that $2V \subset U$ $\implies$ $(x+U) \cap M \neq 0$ $\implies$ $x \in M + U$.)

Find F finite such that $M \subset F + V$ $\implies$ $\overline{M} \subset M + V \subset F + V + V \subset F + U$. □

Věta 2.4 (Banach–Steinhaus)

Let X be a Fréchet space and let Y be LCS. Let (T_n) be a sequence of continuous linear mappings $T_n : X \to Y$ such that $\forall x \in X : \lim_{n\to\infty} T_n x$ exists in Y. Then $Tx := \lim_{n\to\infty} T_n x$

define a continuous linear map $X \to Y$.

Důkaz

Clear: T is a linear map $X \to Y$. „Continuous“: Fix q any continuous sequence on Y.

$$A_m = \{x \in X | \forall n \in \mathbb{N} : q(T_n x) \leqslant m\}.$$

Then A_m is closed, absolutely convex and $\bigcup_{m=1}^{\infty} A_m = X$.

TODO?

Baire category theorem $\implies \exists m \in \mathbb{N} : \operatorname{int} A_m \neq \varnothing \implies \exists x \in A_m \; \exists U$ an absolutely convex neighbourhood of $\mathbf{o}$ such that $x + U \subset A_m \implies -(x+U) \subset A_m \implies$ (A_m convex) $U \subset A_m$ ($y \in U \implies y = \frac{1}{2}(x + y + (-x + y))$) „$\implies q(Ty) \leqslant m p_U(y)$“:

$$p_U(y) < c \implies \frac{y}{c} \in U \subset A_m \implies \forall n \in \mathbb{N} q(T_n \frac{y}{c}) \leqslant m \implies q(T\frac{y}{c}) \leqslant m \implies q(Ty) \leqslant cm.$$

□

Věta 2.5 (Open mapping theorem)

X, Y Fréchet, $T : X \to Y$ linear continuous surjective mapping. Then T is an open mapping

Důkaz
1. It is enough to show that $\forall U$ neighbourhood of $\mathbf{o}$ in X: $T(U)$ is a neighbourhood of $\mathbf{o}$ in Y.

2. „$\forall U$ a neighbourhood of $\mathbf{o}$ in X, $\overline{TU}$ is neighbourhood of $\mathbf{o}$ in Y": U an neighbourhood of $\mathbf{o}$ in X. $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. V absorbing $\implies$

$$\implies X = \bigcup_{n=1}^{\infty} nV \implies Y = T(X) = T\left(\bigcup_{n=1}^{\infty} n \cdot V\right) = \bigcup_{n=1}^{\infty} n \cdot T(V).$$

Y Fréchet $\implies$ by Baire category theorem

$$\exists n \in \mathbb{N} : \emptyset \neq \operatorname{int} \overline{n \cdot T(V)} = \operatorname{int} n \cdot \overline{T(V)} = n \cdot \operatorname{int} \overline{T(V)} \implies \operatorname{int} \overline{T(V)} \neq \emptyset \implies$$

$\implies \exists y \in Y\ \exists W$ an absolutely convex neighbourhood of $\mathbf{o}$ in Y such that $y + W \subset \overline{T(V)}$. $\overline{T(V)}$ is absolutely convex $\implies -(y+w) \subset \overline{T(V)} \implies W \subset \overline{T(V)} \subset \overline{T(U)}$.

3. „$\forall U$ neighbourhood of $\mathbf{o}$ in X, TU is a neighbourhood of $\mathbf{o}$ in Y": ϱ a translation invariant metric on X, complete, generating topology. $U_n = \left\{x \in X | \varrho(0,x) < \frac{1}{2^n}\right\}$. The U_n is a base of neighbourhoods of $\mathbf{o}$. It is enough to prove that $T(U_n)$ is a neighbourhood of $\mathbf{o}$ for each $n \in \mathbb{N}$. We know from 2. that $\forall n : \overline{TU_n}$ is a neighbourhood of $\mathbf{o}$ in Y. We will be done if we show that $TU_{n-1} \supset \overline{TU_n}$ for each $n \in \mathbb{N}$.

We will prove it for $n = 1$: So we will ? $TU_1 \subset TU_0$. Fix $y \in \overline{TU_1}$. Since $\overline{TU_2}$ is a neighbourhood of $\mathbf{o}$ $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$. So there is $x_1 \in U_1$ such that $y - Tx_1 \in \overline{TU_2}$. $\overline{TU_3}$ is a neighbourhood of $\mathbf{o}$ in $Y \implies y - Tx_1 - \overline{TU_3} \subset apTU_2 = \emptyset$ so, there is $x_2 \in U_2$ such that $y - Tx_1 - Tx_2 \in \overline{TU_3}$.

By induction we find $x_n \in U_n$ such that

$$y - Tx_1 - Tx_2 - \ldots - Tx_n \in \overline{TU_{n+1}} \quad (n \in \mathbb{N}).$$

$$x := \sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n :$$

$$M > N \implies \varrho\left(\sum_{n=1}^{M} x_n, \sum_{n=1}^{N} x_n\right) = \varrho\left(\sum_{n=N+1}^{M} x_n, \mathbf{o}\right) \leqslant \underbrace{\varrho\left(\sum_{n=N+1}^{M} x_n, \sum_{n=N+2}^{M}\right)}_{\varrho(x_{N+1},\mathbf{o})} + \underbrace{\varrho\left(\sum_{n=N+2}^{M} x_n, \sum_{n=N+3}^{M}\right)}_{\varrho(x_{N+2},\mathbf{o})} + \ldots$$

$$Tx = y : y - Tx = \lim_{n\to\infty} (y - Tx_1 - \ldots - Tx_n)$$

$$y - Tx_1 - \ldots - Tx_n \in \overline{TU_{N+1}} \subset \overline{TU_k} \qquad \text{for } n+1 > k$$

so, $y - Tx \in \overline{TU_k}$ for each $k \in \mathbb{N}$, so $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k} = \{\mathbf{o}\}$. „Last equality": $y \in Y \setminus \{\mathbf{o}\}$ $\implies \exists V$ neighbourhood of $\mathbf{o}$ in Y such that $y \notin \overline{B}$. T continuous $\implies \exists k \in \mathbb{N}$ such that $T(U_k) \subset V \implies \overline{T(U_1)} \subset \overline{V} \implies y \notin \overline{T(U_k)}$. □

3 Extension and separation theorems

Definice 3.1

X LCS, X^* is the vector space of continuous linear functions on X.

Věta 3.1

X LCS, $Y \Subset X$, $f \in Y^*$. Then $\exists g \in X^*$ such that $g|_Y = f$.

Poznámka

If topology of X is generated by $\mathcal{P}$ a topology of seminorms TODO!!!

Důkaz

1. Topology of Y: $U \subset Y$ is open $\Leftrightarrow \exists \tilde{U} \subset X$ open such that $U = \tilde{U} \cap Y$. U is a neighbourhood of $\mathbf{o}$ in $Y \Leftrightarrow \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $U = \tilde{U} \cap Y$. Lz.pat. Y is also a LSC.

2. $f \in Y^* \implies \exists p$ a continuous seminorm on Y such that $|f(y)| \subseteq p(y), y \in Y$. $U = [p < 1]$ a neighbourhood of $\mathbf{o}$ in $Y \implies \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $\tilde{U} \cap Y = U \implies \exists \tilde{V} \subset \tilde{U}$ an absolutely convex neighbourhood of $\mathbf{o}$ in X. The $p_{\tilde{V}}$ is a continuous seminorm on X. Moreover, $p_{\tilde{V}}|_Y \geqslant p$. ($[p_{\tilde{V}}|_Y < 1] \subset \tilde{V} \cap Y \subset U = [p < 1]$). So, for $y \in Y : |f(y)| \leqslant p(y) \leqslant p_{\tilde{V}}(y) \implies$ (algebraic H–B for seminorms) $\exists g : X \to \mathbb{F}$ linear, $g|_Y = f$, $|g(x)| \leqslant p_{\tilde{V}}(x)$ for $x \in X \implies g$ is continuous by the proposition above. □

Důsledek

X LCS, $Y \Subset X$ closed, $x \in X \setminus Y$. Then $\exists f \in X^* : f|_Y = 0, f(x) = 1$.

Důkaz

Set $\tilde{Y} = \mathrm{LO}(Y \cup \{x\})$. Define $g(y + \lambda x) = \lambda$, $y \in Y$, $\lambda \in \mathbb{F} \implies g$ is linear functional on $\tilde{Y}$, $g|_Y = 0$, $g(x) = 1$. $\operatorname{Ker} g = Y$ is closed $\implies g$ is continuous $\implies g$ can be extended to $f \in X^*$. □

Důsledek

X LCS, $Z \Subset Y \Subset X$.

$$\overline{Z} \supset Y \Leftrightarrow \forall f \in X^* : f|_Z = 0 \implies f|_Y = 0.$$

Důkaz

„ $\implies$ “: clear. „ $\impliedby$ “: $y \in Y \setminus \overline{Z} \implies \exists f \in X^* : f(y) = 1, f|_Z = 0$. □

Důsledek
X HLCS, $x \in X \setminus \{\mathbf{o}\} \implies \exists f \in X^* : f(x) \neq 0.$

Důkaz
$Y = \{\mathbf{o}\}$ is closed linear subspace and use the first corollary. □

Věta 3.2 (Hahn–Banach separation theorem)

X LCS, $A, B \subset X$ nonempty convex, $A \cap B = \emptyset$.

- $\operatorname{int} A \neq \emptyset \implies \exists f \in X^* \setminus \{0\}\ \exists c \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < \Re f(s).$
- *A compact, B closed* $\implies \exists f \in X^*\ \exists c, d \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < d \leqslant \Re f(b).$

Důkaz
Analogous to the theorem above. Assume X is a real space ($\mathbb{F}=\mathbb{R}$). „First item“: $\operatorname{int} A \neq \emptyset \implies \operatorname{int}(B-A) \neq \emptyset$ and $- \notin B-A$. Fix $z \in \operatorname{int}(B-A)$, set $U := z-(B-A)$. The U is a convex neighbourhood of $\mathbf{o}$, $z \notin U \implies p_U(z) \geqslant 1$. Define $g_0 : \mathrm{LO}\{z\} \to \mathbb{R}$ by $g_0(t\cdot z) = t\cdot p_U(z) \implies g_0$ is a linear functional, $g_0 \leqslant p_U$ on $\mathrm{LO}\{z\}$ ($t \geqslant 0 \implies g_0(t\cdot z) = t\cdot p_U(z) = p_U(t\cdot z)$, $t<0 \implies g_0(t\cdot z) = t\cdot p_U(z) < 0 \leqslant p_U(t\cdot z)$).

From algebraic Hahn–Banach $\exists g : X \to \mathbb{R}$ linear, $g|_{\mathrm{LO}\{z\}} = g_0$, $g \leqslant p_U$ on X. g is continuous ($g \leqslant 1$ on $U \implies g \geqslant -1$ on $-U$, so $|g| \leqslant 1$ on $U\cap(-U)$, a neighbourhood of $\mathbf{o}$). $a \in A$, $b \in B \implies$

$$\implies g(z)-g(b)+g(a) = g(z-(b-a)) \leqslant p_U(z-(b-a)) \leqslant 1,$$

$$g(a) \leqslant g(b) + \underbrace{1 - \overbrace{g(z)}^{=p_U(z)\geqslant 1}}_{\leqslant 0}.$$

So, $\forall a \in A\ \forall b \in B : g(a) \leqslant g(b)$, $c := \sup g(A)$.

„Second item“: A compact, B closed. For $x \in A\ \exists U_x$ an absolutely convex open neighbourhood of $\mathbf{o}$ such that $(x+U_x)\cap B = \emptyset$. The $(x+\frac{1}{2}U_x)_{x\in A}$, is an open cover of A. A is compact $\implies \exists x_1,\dots,x_n \in A : A \subset \left(x_1+\frac{1}{2}U_{x_1}\right)\cup\dots\cup\left(x_n+\frac{1}{2}U_{x_n}\right)$. Set $V := \frac{1}{2}U_{x_1}\cap\dots\cap\frac{1}{2}U_{x_n}$ an absolutely convex open neighbourhood of $\mathbf{o}$. Then $(A+V)\cap B = \emptyset$

$$\left(a \in A \implies \exists j : a \in x_j + \frac{1}{2}U_{x_j} \implies a+V \subset x_j+\frac{1}{2}U_{x_j}+V \subset x_j+U_{x_j}\right).$$

Apply first item to $A+V$ (open convex), B (convex) $\implies \exists f \in X^*\setminus\{0\}$ such that

$$\sup f(A) + \sup f(V) = \sup(f(A)+f(V)) = \sup f(A+V) \leqslant \inf f(B),$$

observe that $\sup f(V) > 0$ ($f \neq 0$, V is neighbourhood of $\mathbf{o}$, hence absorbing).

$$c := \sup f(A), \qquad d := \sup f(A) + \sup f(V).$$

„X complex“: look at X as a real space, $f : X \to \mathbb{R}$ real-linear such that. Define $f_c(x) = f(x) - i f(ix), x \in X$. □

Důsledek
X LCS, $\emptyset \neq A \subset X$, $x \in X$.

- $x \in X\setminus\overline{\mathrm{co}}A \Leftrightarrow \exists f \in X^* : \Re f(x) > \sup\{\Re f(a) | a \in A\}$. („ $\impliedby$ “: Clear as $\{y \in X, \Re f(y) \leqslant \sup\{\Re f(x$ is closed convex set containing A. „ $\implies$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{co}}A$, get f and take $-f$.)
- $x \in X\setminus\overline{\mathrm{aco}}A \Leftrightarrow \exists f \in X^* : |f(x)| > \sup\{|f(a)| | a \in A\}$ („ $\impliedby$ “: Clear. „ $\implies$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{aco}}A$ (and multiply by -1), $f \in X^*$:

$|f(x)| \geqslant \Re f(x) > \sup\{\Re f(y)|y \in \overline{\text{aco}}A\} = \sup\{|f(y)||y \in \overline{\text{aco}}A\}$. „$\leqslant$“ clear, „$\geqslant$“: $y \in \overline{\text{aco}\,A} \implies \exists \alpha \in \mathbb{F}, |\alpha| = 1 : |f(y)| = \alpha f(y)$, then $|f(y)| = \lambda f(y) = \Re \alpha f(y) = \Re f(\alpha y)$, $\alpha y \in \overline{\text{aco}\,A}$).

4 Weak topologies

4.1 General weak topologies and duality

Definice 4.1 (Algebraic dual, general weak topology)

X vector space. $X^{\#}$ is the algebraic dual of X (it means set of all linear functionals on X). $\emptyset \neq M \subset X^{\#}$, then $\sigma(X, M)$ is the topology on X generated by seminorms $X \mapsto |f(x)|$, $f \in M$.

Tvrzení 4.1

Properties:

1. *$(X, \sigma(X, M))$ is LCS (by the theorem above).*
2. *$(X, \sigma(X, M))$ is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{0\}\ \exists f \in M : f(x) \neq 0$ (i.e. M separates points) (by the theorem above).*
3. *$f \in M \implies f$ is continuous on $(X, \sigma(X, M))$ (fix $f \in M$, $p(x) = |f(x)|$, $x \in X$ is a continuous seminorm and $|f(x)| = p(x) \leqslant p(x)$).*
4. *$\sigma(X, M)$ is the weakest topology on X making all $f \in M$ continuous.*
5. *$\sigma(X, M) = \sigma(X, \mathrm{LO}(M))$.*
6. *T a topological space, $F : T \to X$ mapping. Than F is continuous $T \to \sigma(X, M) \Leftrightarrow \forall f \in M : f \circ F$ is continuous ($T \to \mathbb{F}$).*

Důkaz (4.)
Assume τ is any topology on X such that all $f \in M$ are τ-continuous $\implies$

$$\implies \forall x \in X\ \forall f_1, \ldots, f_n \in M\ \forall c_1, \ldots, c_n > 0 : \{y \in X||f_j(y) - f_j(x)| < c_j\ \forall j \in [n]\} \text{ is } \tau\text{-open}$$

but these sets form a base of $\sigma(X, M) \implies \sigma(X, M) \subset \tau$. $\square$

Důkaz (5.)
„$\subseteq$“: Clear. „$\supseteq$“: $f \in \operatorname{LO} M \implies f$ is $\sigma(X, M)$-continuous (the linear combination of continuous linear functionals is continuous) $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$, $f_1, \ldots, f_n \in M$, $x_1, \ldots, x_n \in \mathbb{F}$.

$$|f(x)| \leqslant |\alpha_1| \cdot |f_1(x)| + \ldots + |\alpha_n| \cdot |f_n(x)| \leqslant (|\alpha_1| + \ldots + |\alpha_n|) \cdot \max\{|f_1(x)|, \ldots, |f_n(x)|\}.$$

So by the previous point we get $\sigma(X, \operatorname{LO} M) \subset \sigma(X, M)$. □

Důkaz (6.)
„ $\implies$ “: $f \in M \implies f$ is $\sigma(X, M)$-continuous, so $f \circ F$ is continuous. „ $\impliedby$ “: $t \in T$, U neighbourhood of $F(t)$ in $\sigma(X, M) \implies \exists f_1, \ldots, f_n \in M\ \exists c_1, \ldots, c_n > 0$ such that

$$F(t) \in \{y \in X | \forall j \in [n] | f_j(y) - f_j(F(t)) < c_j\} \subset U.$$

Let $G = \{u \in T | \forall j \in [n] : |(f_j \circ F)(u) - (f_j \circ F)(t)| < c_j\}$. Then G is an open neighbourhood of t (by continuity of $f_j \circ F$ and $F(G) \subset U$). □

Příklad
X LCS. Then $X^* \Subset X^{\#}$. So, we may consider $\sigma(X, X^*)$ „the weak topology of X“. $\sigma(X, X^*)$ is Hausdorff when X is HLCS.

TODO!!!

TODO!!!

Lemma 4.2

TODO

a) $\|\cdot\|_N$ *is a norm on* $\mathcal{D}(\Omega)$*;*

b) $\mathcal{D}_K(\Omega)$ *is a Fréchet space when equipped with* $(\|\cdot\|_N)_{N \in \mathbb{N}_0}$.

Důkaz (a))
TODO!!! □

Důkaz (b))
$\|\cdot\|_0 \leqslant \|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \dots \implies \mathcal{D}_K(\Omega)$ is a metrizable LCS (by translation invarinat metric ϱ from the proposition above).

$(\varphi_n) \subset \mathcal{D}_k(\Omega)$ ϱ-cauchy, then $\forall N \in \mathbb{N}_0$: (φ_n) is $\|\cdot\|_N$-cauchy $\implies \forall\alpha$: $(D^\alpha\varphi_n)$ is $\|\cdot\|_\infty$-cauchy $\implies \forall\alpha\ \exists\psi_n$ such that $D^\alpha\varphi_n \rightrightarrows \psi_\alpha$ on Ω. The ψ_α is continuous, $\varphi_\alpha = 0$ on $\Omega\backslash K$. Fix $\alpha \in \mathbb{N}_0^d$ and $j \in [d]$. Then

$$D^\alpha\varphi_n \rightrightarrows \psi_\alpha \wedge \frac{\partial}{\partial x_j} D^\alpha\varphi_n = D^{\alpha+e_j}\varphi_n \rightrightarrows \psi_{\alpha+e_j} \implies \psi_{\alpha+e_j} = \frac{\partial}{\partial x_j}\psi_\alpha.$$

$\implies \psi_\alpha = D^\alpha\psi_0.$

TODO!!! □

Tvrzení 4.3

$\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ linear then following assertions are equivalent:

1. *$\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to \Lambda(\varphi)$;*
2. *$\varphi_n \to 0$ in $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to 0$;*
3. *$\forall K \subset \Omega$ compact and $\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous;*
4. *$\forall K \subset \Omega$ compact $\exists N \in \mathbb{N}_0$ $\exists C > 0$ such that*

$$|\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \qquad \varphi \in \mathcal{D}_K(\Omega).$$

Důkaz
„1. $\implies$ 2.“ is trivial. „2. $\implies$ 3.“: Fix $K \subset \Omega$ compact. $\varphi_n \to 0$ on $\mathcal{D}_K(\Omega) \implies \varphi_n \to 0$ in $\mathcal{D}(\Omega) \overset{2.}{\implies} \Lambda(\varphi_n) \to 0$. Thus $\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous at $\mathbf{o}$, so it is continuous.

„3. $\implies$ 1.“ $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \implies \exists K \subset \Omega$ compact such that $\operatorname{supp}\varphi_n \subset K$ for each n. Then $(\varphi_n) \subset \mathcal{D}_K(\Omega) \implies \varphi_n \to \varphi$ in $\mathcal{D}_K(\Omega) \overset{3.}{\implies} \Lambda(\varphi_n) \to \varphi(\varphi)$.

„3. $\Leftrightarrow$ 4.“. By the proposition above. □

Definice 4.2 (Distribution, finite order)

A distribution on Ω is a linear functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ satisfying assertions from the previous proposition. We will denote distributions on Ω by $\mathcal{D}'(\Omega)$.

$\Lambda \in \mathcal{D}'(\Omega)$ is of finite order, if $N \in \mathbb{N}_0$ in 4. of the previous proposition can be chosen independently on K.

Například
$f \in L^1_{loc}(\Omega)$. $\Lambda_f(\varphi) = \int_\Omega f \cdot \varphi$ ($\varphi \in \mathcal{D}(\Omega)$) $\implies$ Λ_f is a distribution of order 0. Because $K \subset \Omega$ compact $\implies \int_K |f| < \infty$, $\varphi \in D_K(\Omega)$:

$$|\Lambda_f(\varphi)| = |\int_\Omega f \cdot \varphi| = |\int_K f \cdot \varphi| \leqslant \int_K |f\varphi| \leqslant \|\varphi\|_\infty \cdot \int_K |f| = \|\varphi\|_0 \cdot \int_K |f|.$$

$\mu \geqslant 0$ regular Borel measure, finite on compacts. $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution on Ω of order 0. Because if $K \subset \Omega$, $\varphi \in \mathcal{D}_K(\Omega)$, then

$$|\Lambda_\mu(\varphi)| = |\int_\Omega \varphi d\mu| = |\int_K \varphi d\mu| \leqslant \|\varphi\|_\infty \mu(K).$$

μ is a signed (or complex) finite measure $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution of order 0:

$$|\int_K \varphi d\mu| \leqslant \int_K |\varphi| d|\mu| \leqslant |\mu|(K) \cdot \|\varphi\|_\infty \leqslant \|\mu\| \cdot \|\varphi\|_\infty.$$

$\Lambda(\varphi) = \varphi'(0)$, $\varphi \in \mathcal{D}(\mathbb{R})$ is a distribution of order 1. (Clearly $|\Lambda(\varphi)| \leqslant \|\varphi'\|_\infty \leqslant \|\varphi\|_1$.) Λ not of order 0: Find $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi'(0) = 1$, $\operatorname{supp} \varphi \subset [-c, c]$ for some $c > 0$. $\varphi_n(x) = \varphi(nx)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\implies$ $\varphi_n \in \mathcal{D}(\mathbb{R})$. $\operatorname{supp} \varphi_n \subset [-c/n, c/n] \subset [-c, c]$. $\|\varphi_n\|_0 = \|\varphi\|_0$. $\Lambda(\varphi_n) = \varphi_n'(0) = \varphi'(0) \cdot n = n$.

$\Lambda(\varphi) = \sum_{n=0}^\infty \varphi^{(n)}(n)$, $\varphi \in \mathcal{D}(\mathbb{R})$ $\implies$ Λ is a distribution on $\mathbb{R}$, not of finite order ($\operatorname{supp} \varphi \subset [-k, k], k \in \mathbb{N}, \implies |\Lambda(\varphi)| \leqslant (k+1)\|\varphi\|_K$.)

Poznámka
If $f, g \in L^1_{loc}(\Omega)$, $\Lambda_f = \Lambda_g$, then $f = g$ almost everywhere. If μ, ν measures, $\Lambda_\mu = \Lambda_\nu$, then $\mu = \nu$.

If $f \in L^1(\Omega)$, μ finite measure, $\Lambda_f = \Lambda_\mu$, then $\mu(A) = \int_A f$, for each $A \subset \Omega$ Borel.

Definice 4.3

$\Lambda \in \mathcal{D}'(\Omega)$.

- For $\alpha \in \mathbb{N}_0^d$ define $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)
- For $f \in C^\infty(\Omega)$ define $(f\Lambda)(\varphi) = \Lambda(f\varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)

Tvrzení 4.4

a) $\Lambda \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}_0^d \implies D^\alpha \Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
Clear: $D^\alpha\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ linear, $K \subset \Omega$ compact $\implies \exists N \in \mathbb{N}_0, C > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega)$. Then $\forall \varphi \in \mathcal{D}_k(\Omega)$:

$$|D^\alpha\Lambda(\varphi)| = |\Lambda(D^\alpha\varphi)| \leqslant C \cdot \|D^\alpha\varphi\|_N \leqslant C \cdot \|\varphi\|_{|\alpha|+N}$$

. □

b) $f \in C^\infty(\Omega) \implies D^\alpha\Lambda_f = \Lambda_{D^\alpha f}$

Důkaz (For $\partial/\partial x_1$)

$$\frac{\partial}{\partial x_1}\Lambda_f(\varphi) = -\Lambda_f\left(\frac{\partial\varphi}{\partial x_1}\right) =? = -\int_\Omega f \cdot \frac{\partial\varphi}{\partial x_1}$$

TODO □

c) $d = 1$, $\Omega = (a, b)$, $f \in L^1_{loc}(\Omega)$. Then $(\Lambda_f)' = \Lambda_g \Leftrightarrow g$ is the weak derivative of f. And $(\Lambda_f)' = \Lambda\mu \Leftrightarrow \mu$ is the weak derivative of f.

Důkaz
By definitions. □

d) $\Lambda \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega) \implies f\Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
clear: $f\Lambda : \mathcal{D}(\Omega) \implies$ IF linear □

Tvrzení 4.5

a) $\Lambda \in \mathcal{D}'((a, b)), \Lambda' = 0 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
We will prove $\operatorname{Ker}\Lambda_1 \subset \operatorname{Ker}\Lambda$. Then $\exists c : \Lambda = c \cdot \Lambda_1 = \Lambda c$.

$$\varphi \in \operatorname{Ker}\Lambda_1 \implies \Lambda_1(\varphi) = 0, i.e. \int_a^b \varphi = 0.$$

Define $\varphi(t) = \int_a^t \varphi$, $t \in (a, b)$. Then $\psi \in \mathcal{D}((a, b))$, $\psi' = \varphi$ ($\psi' = \varphi$ … differentiation of indefinite integral $\implies \psi \in C^\infty((a, b))$, $\psi = 0$ on $(a, \min\operatorname{supp}\varphi)$ and $(\max\operatorname{supp}\varphi, b)$ $\implies \psi \in \mathcal{D}((a, b))$). Hence $\Lambda(\varphi) = \Lambda(\psi') = -\Lambda'(\psi) = 0$, so $\varphi \in \operatorname{Ker}\Lambda$. □

b) $\Omega \subset \mathbb{R}^d$ open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha\Lambda = 0$ for $|\alpha| = 1 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
„Step 1: $\Omega = \prod_{j=1}^{d}(a_j, b_j)$“: Induction on d. For $d = 1$ use a). Assume it holds for $d-1$, denote $\Omega' = \prod_{j=1}^{d-1}(a_j, b_j)$, $x \in \Omega \implies x = (x', x_d)$ ($x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$), $\alpha \in N_0^d \implies \alpha = (\alpha', \alpha_d)$.

$\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$ for $|\alpha| = 1$. It means: $\forall \varphi \in \mathcal{D}(\Omega)\ \forall j \in [d] : \Lambda\left(\frac{\partial \varphi}{\partial x_j}\right) = 0$.

Claim: $\psi \in \mathcal{D}(\Omega)$. Then $\exists \varphi \in \mathcal{D}(\Omega) : \frac{\partial \varphi}{\partial x_d} = \psi \Leftrightarrow \forall x' \in \Omega' : \int_{a_d}^{b_d} \psi(x', x_d) dx_d = 0$. („ $\implies$ “ clear, „ $\impliedby$ “: define $\varphi(x', x_d) = \int_{a_d}^{x_d} \psi(x', t) dt$). Define

$$T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega'), \qquad T\varphi(x') = \int_{a_d}^{b_d} \varphi(x', x_d) dx_d, \quad \varphi \in \mathcal{D}(\Omega).$$

T is linear, $\operatorname{Ker} T \subset \operatorname{Ker} \Lambda$ ($T\varphi = 0 \implies \exists \psi \in \mathcal{D}(\Omega) : \varphi = \frac{\partial \psi}{\partial x_d}$, thus $\Lambda(\varphi) = 0$). Fix $\eta \in \mathcal{D}((a_d, b_d))$, $\int_{a_d}^{b_d} \eta = 1$. For $\varphi \in \mathcal{D}(\Omega')$ define $(\varphi\eta)(x) = \varphi(x')\eta(x_d)$. Then $\varphi\eta \in \mathcal{D}(\Omega)$. $\tilde{\Lambda}(\varphi) = \Lambda(\varphi\eta)$, $\varphi \in \mathcal{D}(\Omega')$. Then $\tilde{\Lambda} \in \mathcal{D}'(\Omega')$.

Moreover, $\forall \alpha'$ with $|\alpha'| = 1 : D^{\alpha'}\tilde{\Lambda} = 0$.

$$\left(\forall j \in [d-1] : \frac{\partial}{\partial x_j}\tilde{\Lambda}(\varphi) = -\tilde{\Lambda}\left(\frac{\partial \varphi}{\partial x_j}\right) = -\Lambda\left(\frac{\partial \varphi}{\partial x_j}\eta\right) = -\Lambda\left(\frac{\partial}{\partial x_j}(\varphi\eta)\right) = 0.\right)$$

$\implies \exists c \in \mathbb{F} : \tilde{\Lambda} = \Lambda_c$ in $\mathcal{D}'(\Omega')$. Then $\Lambda = \Lambda_c$ (in $\mathcal{D}(\Omega)$) cause

$$\varphi \in \mathcal{D}(\Omega) \implies \varphi - (T\varphi)\eta \in \mathcal{D}(\Omega), \varphi - (T\varphi)\eta \in \operatorname{Ker} T \subset \operatorname{Ker}\Lambda, \text{ so,}$$

$$\Lambda(\varphi) = \Lambda((T\varphi)\eta) = \tilde{\Lambda}(T\varphi) = \Lambda_c(T\varphi) = \int_{\Omega'} c \cdot T\varphi = \int_{\Omega'} c \cdot \int_{a_d}^{b_d} \varphi(x', x_d) dx_d dx' \overset{\text{FUBINI}}{=} \int_{\Omega} c \cdot \varphi = \Lambda_c(\varphi).$$

„Step 2: Ω is open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$, $|\alpha| = 1$.“: Step 1 $\implies \forall Q \subset \Omega$ cuboid $\exists c : \Lambda|_{\mathcal{D}(Q)} = \Lambda_c$. Fix one cuboid $Q_0 \subset \Omega$ and the respective c.

$$A := \left\{x \in \Omega | \exists Q \subset \Omega \text{ cuboid}, x \in Q, \Lambda|_{\mathcal{D}(Q)} = \Lambda_c\right\}.$$

Fix $A \neq \emptyset$ ($Q_0 \subset A$), A is open, A is closed in Ω ($x \in \overline{A} \cap \Omega$, $Q \cap A \neq \emptyset$, $\Lambda|_{\mathcal{D}(Q)} = \Lambda_d$, $y \in Q \cap A \implies \Lambda|_{\mathcal{D}(Q_y)} = \Lambda_c \implies$ on $\mathcal{D}(Q \cap Q_y) : \Lambda = \Lambda_c = \Lambda_d \implies c = d \implies x \in A$.). So $A = \Omega$ as Ω is connected. The $\Lambda = \Lambda_c$ in $\mathcal{D}'(\Omega)$. (Proof of this was skipped, it remains that for every $\varphi \in \mathcal{D}(\Omega)$, not only for every $\varphi \in \mathcal{D}(Q)$, it holds $\Lambda(\varphi) = \Lambda_c(\varphi)$.) □

4.2 A bit more on distributions

Definice 4.4

$\Lambda_n \to \Lambda$ in $\mathcal{D}(\Omega) \equiv \forall \varphi \in \mathcal{D}(\Omega) : \Lambda_n(\varphi) = \Lambda(\varphi)$.

Tvrzení 4.6

a) $\Lambda_n \to \Lambda$ *in* $\mathcal{D}(\Omega)$*, then:*

- $\forall \alpha : D^\alpha \Lambda_n \to D^\alpha \Lambda$;

  Důkaz

  $$D^\alpha \Lambda_n(\varphi) = (-1)^{|\alpha|}\Lambda_n(D^\alpha \varphi) \to (-1)^{|\alpha|}\Lambda(D^\alpha \varphi) = D^\alpha \Lambda(\varphi).$$

  □

- $f \in C^\infty(\Omega) : f\Lambda_n \to f\Lambda$.

  Důkaz

  $$f\Lambda_n(\varphi) = \Lambda_n(f\varphi) \to \Lambda(f\varphi) = f\Lambda(\varphi).$$

  □

b) $f_n \to f$ *in* $L^1_{loc}(\Omega)$ *(*$\forall K \subset \Omega$ *compact:* $\int_K |f_n - f| \to 0$*). Then* $\Lambda_{f_n} \to \Lambda_f$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi \in \mathcal{D}(\Omega) : |\Lambda_{f_n}(\varphi) - \Lambda_f(\varphi)| = \left| \int_\Omega f_n \varphi - \int_\Omega f\varphi \right| \leqslant \int_\Omega |f_n - f| \cdot |\varphi| = \int_{\operatorname{supp}\varphi} |f_n - f| \cdot |\varphi| \leqslant \|\varphi\|_\infty \int_{\operatorname{supp}\varphi} |f_n -$$

□

c) $f_n \to f$ *in* $L^p(\Omega)$ *for some* $p \in [1, \infty]$*. Then* $\Lambda_{f_n} \to \Lambda_f$*.*

Důkaz

Let $K \subset \Omega$ be compact, q the dual exponent. Then use b) with

$$\int_K |f_n - f| \leqslant \|f_n - f\|_{L^p(K)} \cdot \|1\|_{L^q(K)} \to 0.$$

□

d) $\varphi_n \to \varphi$ *in* $\mathcal{D}(\Omega)$*. Then* $\Lambda_{\varphi_n} \to \Lambda_\varphi$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) \implies \varphi_n \to \varphi \text{ in } C^\infty(\Omega), \text{ and use c).}$$

□

Věta 4.7

$(\Lambda_n) \subset \mathcal{D}'(\Omega)$ and $\forall \varphi \in \mathcal{D}(\Omega) : (\Lambda_n(\varphi))$ converges in $\mathbb{F}$. Then $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$ is a distribution on Ω.

Důkaz

Clearly Λ is a linear functional on $\mathcal{D}(\Omega)$. Further: $K \subset \Omega$ compact $\implies \forall n : \Lambda_n|_{\mathcal{D}_K(\Omega)}$ is continuous. $\mathcal{D}_K(\Omega)$ is a Fréchet space $\overset{\text{the lemma above, b)}}{\implies} \Lambda|_{\mathcal{D}_K(\Omega)}$ continuous $\implies \Lambda \in \mathcal{D}'(\Omega)$. □

Definice 4.5

$\Lambda \in \mathcal{D}'(\Omega)$.

- $G \subset \Omega$ open. Λ vanishes on G if $\Lambda(\varphi) = 0$ whenever $\varphi \in \mathcal{D}(\Omega), \operatorname{supp} \varphi \subset G$.
- $\operatorname{supp} \Lambda = \Omega \setminus \{G \subset \Omega$ open $|\Lambda$ vanishes on $G\} = \{x \in \Omega | \forall \varepsilon > 0 \exists \varphi \in \mathcal{D}(\Omega) : \operatorname{supp} \varphi \subset U(x, \varepsilon) \wedge \Lambda(\varphi$
- Λ has compact support if $\operatorname{supp} \Lambda$ is a compact subset of Ω.

Tvrzení 4.8

a) $\Lambda = \Lambda_f$ for some $f \in L^1_{loc}(\Omega)$. Then

$$\operatorname{supp} \Lambda_f = \operatorname{supp} f = \left\{x \in \Omega | \forall \varepsilon > 0 : \lambda^d \left(\{y \in U(x, \varepsilon) \cap \Omega | f(y) \neq 0\}\right) > 0\right\}$$

Důkaz

„$\subseteq$“: $X \notin \operatorname{supp} f \implies \exists \varepsilon > 0 : f = 0$ almost everywhere on $U(x, \varepsilon) \cap \Omega \implies \Lambda_f$ vanishes on $U(x, \varepsilon) \cap \Omega \implies x \notin \operatorname{supp} \Lambda_f$.

„$\supseteq$“: $x \in$ supp. Let $\varepsilon > 0$. Then f is not 0 almost everywhere on $U(x, \varepsilon) \cap \Omega \implies \exists \varphi \in \mathcal{D}(U(x, \varepsilon) \cap \Omega)$ □

b) $\Lambda = \Lambda_\mu$. Then $\operatorname{supp} \Lambda = \operatorname{supp} \mu = \Omega \setminus \bigcup \{G \subset \Omega$ *open* $|\forall B \subset G$ *Borel* $\mu(B) = 0\}$.

Důkaz

$G \subset \Omega$ open the $\forall B \subset G$ Borel $\mu(B) = 0 \Leftrightarrow \forall \varphi \in \mathcal{D}(G) : \int \varphi d\mu = 0 \Leftrightarrow \Lambda_\mu$ vanishes on G. □

Poznámka

f is continuous $\implies \operatorname{supp} f = \overline{\{x | f(x) \neq 0\}} \cap \Omega$.