Consider a linearised homogeneous isotropic elastic solid, that is a continuous medium where the stress tensor is given by the formula $\boldsymbol{\tau} = \lambda(\operatorname{tr}\mathfrak{e})\mathbb{I} + 2\mu\mathfrak{e}$, where $\mathfrak{e} := \frac{1}{2}\left(\nabla U + (\nabla U)^T\right)$ denotes the linearised strain tensor.

Příklad (1.)
Show that dynamic governing equation $\varrho_R \frac{\partial^2 \mathbf{U}}{\partial t^2} = \operatorname{div}\boldsymbol{\tau}$ in this case reduces to

$$\varrho_R \frac{\partial^2 \mathbf{U}}{\partial t^2} = (\lambda + \mu)\nabla(\operatorname{div}\mathbf{U}) + \mu\Delta\mathbf{U}.$$

Důkaz

$$\begin{aligned}
(\operatorname{div}\boldsymbol{\tau})_j &= (\operatorname{div}(\lambda(\operatorname{tr}\mathfrak{e})\mathbb{I} + 2\mu\mathfrak{e}))_j = \\
&= \lambda\partial_k\left(\mathbb{I}_{jk}\sum_i\left(\frac{1}{2}\partial_i U_i + \frac{1}{2}\partial_i U_i\right)\right) + 2\mu\left(\frac{1}{2}\sum_i \partial_i(\partial_i U_j) + \frac{1}{2}\sum_i \partial_i(\partial_j U_i)\right) = \\
&= \lambda\partial_j\left(\sum_i(\partial_i U_i)\right) + \mu\left(\sum_i \partial_i\partial_i U_j + \partial_j\left(\sum_i \partial_i U_i\right)\right) = \\
&= \lambda(\nabla(\operatorname{div}\mathbf{U}))_j + \mu\Delta U_j + \mu(\nabla(\operatorname{div}\mathbf{U}))_j = ((\lambda+\mu)\nabla(\operatorname{div}\mathbf{U}) + \mu\Delta\mathbf{U})_j
\end{aligned}$$

□

Příklad (2.)
Assume that solution to previous equation has the form of a travelling plane wave, that is $\mathbf{U} = \mathbf{A}\sin(\mathbf{K}\cdot\mathbf{X} - \omega t)$. Substitute and show that the result can be rewritten in the form

$$\varrho_R c^2 \mathbf{A} = \left[\mu\left(\mathbb{I} - \frac{\mathbf{K}}{K}\otimes\frac{\mathbf{K}}{K}\right) + (\lambda + 2\mu)\frac{\mathbf{K}}{K}\otimes\frac{\mathbf{K}}{K}\right]\mathbf{A}.$$

Důkaz
Levá strana:

$$\varrho_R \frac{\partial^2 \mathbf{U}}{\partial t^2} = \varrho_R \mathbf{A} \frac{\partial \sin(\mathbf{K} \cdot \mathbf{X} - \omega t)}{\partial t^2} = \varrho_R \mathbf{A} \frac{\partial(-\omega \cos(K \cdot X - \omega t))}{\partial t} =$$

$$= \varrho_R \mathbf{A} - \omega^2 \sin(\mathbf{K} \cdot \mathbf{X} - \omega t) = \varrho_R c^2 \mathbf{A}(-K^2 \sin(\ldots)).$$

První člen pravé strany:

$$((\lambda + \mu)\nabla(\operatorname{div} \mathbf{U}))_j = (\lambda + \mu)\partial_j \left(\sum_i \partial_i A_i \sin(\mathbf{K} \cdot \mathbf{X} - \omega t) \right) =$$

$$= (\lambda + \mu) \sum_i \partial_j A_i K_i \cos(\mathbf{K} \cdot \mathbf{X} - \omega t) = -(\lambda + \mu) \sum_i A_i K_i K_j \sin(\mathbf{K} \cdot \mathbf{X} - \omega t) =$$

$$= -(\lambda + \mu)(\mathbf{K} \otimes \mathbf{K})\mathbf{A} \sin(\mathbf{K} \cdot \mathbf{X} - \omega t) = (\lambda + \mu) \left(\frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K} \right) \mathbf{A}(-K^2 \sin(\ldots)).$$

Druhý člen pravé strany:

$$\mu \Delta \mathbf{U} = \mu \Delta \left(\mathbf{A} \sin(\mathbf{K} \cdot \mathbf{X} - \omega t) \right) = \mu \mathbf{A} \Delta \left(\sin(\mathbf{K} \cdot \mathbf{X} - \omega t) \right) = \mu \mathbf{A} \sum_i K_i^2 (-\sin(\mathbf{K} \cdot \mathbf{X} - \omega t)) =$$

$$= \mu \mathbf{A}(-K^2 \cdot \sin(\ldots)).$$

Předpokládáme, že $K \neq 0$, tedy můžeme dělit rovnici K^2 a navíc $\sin(\ldots)$ bude nulový pouze na množině s prázdným vnitřkem, takže rovnici můžeme dodefinovat ze spojitosti a můžeme dělit i $-\sin(\ldots)$:

$$\varrho_R c^2 \mathbf{A} = (\lambda + \mu) \left(\frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K} \right) \mathbf{A} + \mu \mathbf{A} = \left[\mu \left(\mathbb{I} - \frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K} \right) + (\lambda + 2\mu) \frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K} \right] \mathbf{A}$$

□

This is an eigenvalue problem of the form $c^2 \mathbf{A} = \mathbb{A}\mathbf{A}$, where

$$\mathbb{A} := \frac{\mu}{\varrho} \left(\mathbb{I} - \frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K} \right) + \frac{\lambda + 2\mu}{\varrho} \frac{\mathbf{K}}{K} \otimes \frac{\mathbf{K}}{K}.$$