Příklad (1.)
The Cauchy stress $\mathbb{T}$ tensor is related to the derivative of the Helmholtz free energy via the formula $\mathbb{T} = 2\varrho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B}$.

Since the material is isotropic, the Helmholtz free energy must be in fact a function of the invariants of $\mathbb{B}$: $\psi = \psi(\theta, I_1, I_2, I_3)$, where the invariants are given by the formulae

$$I_1 := \operatorname{tr} \mathbb{B}, \qquad I_2 := \frac{1}{2}\left((\operatorname{tr} \mathbb{B})^2 - \operatorname{tr} \mathbb{B}^2 \right), \qquad I_3 := \det \mathbb{B}.$$

Show that $\mathbb{T} = \alpha_0 \mathbb{I} + \alpha_1 \mathbb{B} + \alpha_2 \mathbb{B}^2$, where

$$\alpha_0 := 2\varrho I_3 \frac{\partial \psi}{\partial I_3}, \qquad \alpha_1 := 2\varrho \left(\frac{\partial \psi}{\partial I_1} + I_1 \frac{\partial \psi}{\partial I_2} \right), \qquad \alpha_2 := -2\varrho \frac{\partial \psi}{\partial I_2}.$$

Důkaz (Z minulého roku)
Z řetízkového pravidla:

$$\mathbb{T} = 2\varrho \frac{\partial \psi}{\partial \mathbb{B}} \mathbb{B} = 2\varrho \frac{\partial \psi}{\partial I_1} \frac{\partial I_1}{\partial \mathbb{B}} \mathbb{B} + 2\varrho \frac{\partial \psi}{\partial I_2} \frac{\partial I_2}{\partial \mathbb{B}} \mathbb{B} + 2\varrho \frac{\partial \psi}{\partial I_3} \frac{\partial I_3}{\partial \mathbb{B}} \mathbb{B}.$$

Zřejmě $\frac{\partial I_1}{\partial \mathbb{B}} = \mathbb{I}$,

$$\frac{\partial I_2}{\partial \mathbb{B}} = \frac{1}{2} \frac{\partial (\operatorname{tr} \mathbb{B})^2}{\partial \operatorname{tr} \mathbb{B}} \frac{\partial \operatorname{tr} \mathbb{B}}{\partial \mathbb{B}} - \frac{1}{2} \frac{\partial \operatorname{tr} \mathbb{B}^2}{\partial (\mathbb{B}^2)} \frac{\partial \mathbb{B}^2}{\partial \mathbb{B}} = \frac{2}{2} (\operatorname{tr} \mathbb{B}) \mathbb{I} - \frac{2}{2} \mathbb{I}\, \mathbb{B} = \mathbb{I} \operatorname{tr} \mathbb{B} - \mathbb{B}.$$

Z přednášky navíc víme $\frac{\partial I_3}{\partial \mathbb{B}} = (\det \mathbb{B}) \mathbb{B}^{-T}$. Navíc $\mathbb{B}$ je symetrické (např. z definice $\mathbb{B}^T = (\mathbb{F}\mathbb{F}^T)^T = (\mathbb{F}^T)^T \mathbb{F}^T = \mathbb{F}\mathbb{F}^T = \mathbb{B}$), tedy $\frac{\partial I_3}{\partial \mathbb{B}} = (\det \mathbb{B}) \mathbb{B}^{-1}$.

Dosazením:

$$\mathbb{T} = 2\varrho \frac{\partial \psi}{\partial I_1} \mathbb{I}\, \mathbb{B} + 2\varrho \frac{\partial \psi}{\partial I_2} (\mathbb{I}\, I_1 - \mathbb{B})\, \mathbb{B} + 2\varrho \frac{\partial \psi}{\partial I_3} \left(I_3 \mathbb{B}^{-1} \right) \mathbb{B} =$$

$$= 2\varrho I_3 \frac{\partial \psi}{\partial I_3} \mathbb{I} + 2\varrho \left(\frac{\partial \psi}{\partial I_1} + I_1 \frac{\partial \psi}{\partial I_2} \right) \mathbb{B} - 2\varrho \frac{\partial \psi}{\partial I_2} \mathbb{B}^2.$$

□

Furthermore show that an alternative form reads

$$\mathbb{T} = \beta_0 \mathbb{I} + \beta_1 \mathbb{B} + \beta_{-1} \mathbb{B}^{-1},$$

where

$$\beta_0 := 2\varrho \left(I_2 \frac{\partial \psi}{\partial I_2} + I_3 \frac{\partial \psi}{\partial I_3} \right), \qquad \beta_1 := 2\varrho \frac{\partial \psi}{\partial I_1}, \qquad \beta_{-1} := -2\varrho I_3 \frac{\partial \psi}{\partial I_2}.$$

Důkaz (Z minulého roku)
Z přednášky víme

$$\mathbb{B}^{-1} = \frac{1}{I_3}\mathbb{B}^2 - \frac{I_1}{I_3}\mathbb{B} + \frac{I_2}{I_3}\mathbb{I},$$

tedy

$$\mathbb{B}^2 = I_3\mathbb{B}^{-1} + I_1\mathbb{B} - I_2\mathbb{I}.$$

Dosadíme:

$$\begin{aligned}\mathbb{T} = 2\varrho I_3 \frac{\partial\psi}{\partial I_3}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbb{B} - 2\varrho\frac{\partial\psi}{\partial I_2}\mathbb{B}^2 = \\ = 2\varrho I_3 \frac{\partial\psi}{\partial I_3}\mathbb{I} + 2\varrho\left(\frac{\partial\psi}{\partial I_1} + I_1\frac{\partial\psi}{\partial I_2}\right)\mathbb{B} - 2\varrho\frac{\partial\psi}{\partial I_2}\left(I_3\mathbb{B}^{-1} + I_1\mathbb{B} - I_2\mathbb{I}\right) = \\ = 2\varrho\left(I_2\frac{\partial\psi}{\partial I_2} + I_3\frac{\partial\psi}{\partial I_3}\right)\mathbb{I} + 2\varrho\frac{\partial\psi}{\partial I_1}\mathbb{B} - 2\varrho I_3\frac{\partial\psi}{\partial I_2}\mathbb{B}^{-1}.\end{aligned}$$

□