1 Introduction

Poznámka (Literature)
„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

1.1 Differentiability

Definice 1.1 (Differentiable)

A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable (also holomorphic) at a point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{|h| \to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call $f'(z_0)$ the derivative of f at z_0. A function f is differentiable on a domain (open connected subset of $\mathbb{C}$) if its differentiable for all points of this domain.

Poznámka (Writing complex numbers in cartesian cooridnates)
$z = x + iy$, for $x, y \in \mathbb{R}$, we can write a function $f : \mathbb{C} \to \mathbb{C}$ in terms of two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

Věta 1.1 (Cauchy–Riemann equations)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}$. Considering $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definice 1.2 (Orientability, orientation-preserving function)

Define and equivalence relation on the set of all bases of $\mathbb{R}^2$ by saying that $B_1 \sim B_2$ iff the determinant of the change of basis matrix is positive.

A function $f : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ is said to be orientation-preserving if on an open dense subset of U, the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Důsledek
Let f be a non-constant holomorphic function, then f is orientation-preserving.

Důsledek
Since f is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x}\frac{\partial v}{y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \overset{\text{C–R}}{=} \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geqslant 0.$$

Since f is non-constant, the inequality is strict on a dense open subset of the domain of definition.

Věta 1.2 (Open mapping theorem)

A non-constant holomorphic function f is open (that is if U is an open subset of $\mathbb{C}$, then $f(U)$ is also open).

1.2 Integration

Definice 1.3

For a path γ (smooth function, $\gamma : \mathbb{R} \supset [a, b] \to \mathbb{C}$) we define

$$\int_\gamma f(x)dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$