TODO!!!

Definice 0.1 (WLOG)

$$D := U(0,1), \qquad T = \partial D.$$

TODO!!!

Definice 0.2

$f \in \mathcal{H}(D)$. We say that the boundary T is a natural boundary of f if $R_f = \emptyset$.

Například
$f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Radius of convergence is equal to 1 and f has natural boundary.

Důkaz
$K = \left\{\exp\left(\frac{2\pi ik}{n}\right) | k, n \in \mathbb{N}\right\}$ is dense in T. f is "diverges on" this set, because $f(z^{2^N}) = f(z) - \sum_{n=1}^{N}$. For $\alpha \in (0,1)$ we have parametrization of one "line" $\alpha \cdot \exp\left(\frac{2k\pi i}{2^n}\right)$ (for k, n fixed).

$$f\left(\alpha^{2^N}\right) = f\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right) + p\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right).$$

□

For every domain $\Omega \subseteq \mathbb{C}$, there exists $f \in \mathcal{H}(\Omega)$ such that $\partial\Omega$ is natural boundary of f.

Důkaz
We use theorem (15.11 from Rudin or TODO from lecture). □

TODO!!!

TODO!!!

1 Eulerův vzorec

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 1.1

$$z \neq \frac{k}{2}, k \in \mathbb{Z} : 2\pi \operatorname{cotg}(2\pi z) = \pi \operatorname{cotg}(\pi x) + \pi \operatorname{cotg}(\pi(z + \frac{1}{2})).$$

Důkaz

$$2\pi \operatorname{cotg}(2\pi z) = 2\pi \frac{\cos(2\pi z)}{\sin(2\pi z)} = \pi \frac{\cos^2(\pi z) - \sin^2(\pi z)}{\sin(\pi z)\cos(\pi z)} = \pi \left(\operatorname{cotg}(\pi z) - \frac{\sin(\pi z)}{\cos(\pi z)} \right) = \pi \left(\operatorname{cotg}(\pi z) + \frac{\cos \pi}{\sin \pi} \right.$$

$\square$

Lemma 1.2 (Herglotz)

$r > 1$, G oblast, $G \supset [0, r)$, h funkce holomorfní na G, $z, z + \frac{1}{2}, 2z \in [0, r) : 2h(2z) = h(z) + h(z + \frac{1}{2})$. Pak h je konstantní na G.

Důkaz
Zvol $t \in (1, r)$.

$$M := \max \{|h'(z)|, z \in [0, t]\}, \qquad 4|h'(2z)| \leqslant |h'(z)| + |h'(z + \frac{1}{2})| \implies$$

$$\implies 4|h'(z)| \leqslant |h'(\frac{z}{2})| + |h'(\frac{z}{2} + \frac{1}{2})| < 2M \implies 4M \leqslant 2M \implies M = 0.$$

$\square$

Lemma 1.3

g holomorfní funkce na $\mathbb{C} \backslash \mathbb{Z}$, hlavní část Laurenotvy řady g na $P(k)$, $k \in \mathbb{Z}$, je rovna $\frac{1}{z-k}$, g lichá, $2g(2z) = g(z) + g(z + \frac{1}{2})$, $z \neq \frac{k}{2}$, $k \in \mathbb{Z}$. Pak $g(z) = \pi \operatorname{cotg}(\pi z)$.

Důkaz
$h(z) := g(z) - \pi \operatorname{cotg}(\pi z)$. h rozšíříme spojitě a holomorfně na $\mathbb{C}$. Z Herglotzova lemmatu je h konstantní na $\mathbb{C}$ (obě funkce splňují podmínky). Navíc $h(0) = 0$. $\square$

Důsledek (Eisenstein)
$\pi \operatorname{cotg}(\pi z) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}$, $z \notin \mathbb{Z}$.

Důkaz
$z \mapsto \frac{2z}{z^2-k^2}$ jsou holomorfní na $U(0,n)$, $k > n$.

$$\left|\frac{2z}{z^2-k^2}\right| \leqslant \frac{2n}{k^2-n^2} \qquad \wedge \qquad \sum_{k=n+1}^{\infty} \frac{2n}{k^2-n^2}K \implies \sum \in \mathcal{M}.$$

f je holomorfní na $\mathbb{C}\backslash\mathbb{Z}$. Také je lichá. Nakonec

$$s_n(z) = \frac{1}{2} + \sum_{k=1}^{n} \frac{2z}{z^2-k^2} = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{z+k} + \sum_{k=1}^{n} \frac{1}{z-k} = \sum_{-n}^{n} \frac{1}{z+k},$$

$$s_n\left(\frac{z}{2}\right) + s_n\left(\frac{z+1}{2}\right) = \sum_{k=-n}^{n} \frac{2}{z+2k} + \frac{2}{z+2k+1} = 2\sum_{k=-2n}^{2n} \frac{1}{z+k} + 2\cdot\frac{1}{z+2n+1} = 2s_{2n}(k) + \frac{2}{z+2n+1},$$

$$n \to \infty : f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right) = 2f(z) + 0.$$

Z předchozího lemmatu vyplývá důkaz. □

Věta 1.4 (Euler)

$$\underbrace{\sin(\pi z)}_{g} = \underbrace{\pi z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)}_{f}.$$

Důkaz
$\forall z \in \mathbb{C}\ \exists$ neighbourhood U: $\sum_{k=1}^{\infty} \| \left(z \mapsto \frac{z^2}{k^2}\right) \|_\infty$ is convergent $\implies \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right)$ is holomorphic.

$$f_k(z) := 1 - \frac{z^2}{k^2}, \qquad \frac{f_k'(z)}{f_k(z)} = \frac{\frac{-2z}{k^2}}{\frac{k^2-z^2}{k^2}} = \frac{2z}{z^2-k^2}.$$

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi \prod \left(1 - \frac{z^2}{k^2}\right)} \left(\pi \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{k^2}\right) + \pi z \left(\prod_{k=1}^{\infty} \dots\right)'\right) = \frac{1}{2} = \frac{\prod'}{\prod} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{f_k'(z)}{f_k(z)} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2}$$

$\left(\frac{g'(z)}{g(z)} = \frac{\pi\cos(\pi z)}{\sin \pi z} = \pi \operatorname{cotg}(\pi z),\ \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = 0 \implies \frac{f}{g}\right.$ is constant).

$$\lim_{z\to 0} \frac{f(z)}{g(z)} = \lim_{z\to 0} \frac{\pi \sin(z\pi)}{TODO} TODO(oneline).$$

□

2 Gamma function

Definice 2.1

$$\Gamma(x) = \int_0^\infty e^{-t} t^{\lambda - 1} dt, \qquad x \in (0, \infty).$$

Poznámka (Notion)

$$I_n = \int_0^1 1 \cdot (-\log x)^n dx \overset{\text{IBP}}{=} [x(-\log x)^n]_0^1 - \int_0^1 xn(-\log x)^{n-1} \left(-\frac{1}{x}\right) dx = n \cdot I_{n-1}.$$

So $I_n = n!$. Set $\log x = t$, $e^t = x$:

$$I_n = \int_{-\infty}^0 (-t)^n \cdot e^t dt = \int_0^\infty t^n e^{-t} dt.$$

Lemma 2.1

$\forall n \in \mathbb{N}$ we define $\Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt$, then $\Gamma_n(x) \to \Gamma(x)$.

Důkaz

„$0 \leqslant e^t - \left(1 - \frac{t}{n}\right)^n \leqslant \frac{t^2}{n} e^{-t}$“:

$$0 \leqslant e^{-t} \cdot \left(1 - \frac{t}{n}\right)^n = e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) \leqslant e^{-t} \left(1 - \left(1 - \frac{t}{n}\right)^n \cdot \left(1 + \frac{t}{n}\right)^n\right) = e^{-t} \left(1 - \left(1 - \frac{t^2}{n^2}\right.\right.$$

(Last inequality from Bernoulli: $(1 + x)^n \geqslant 1 + n \cdot x$, $x \geqslant -1$.)

$$|\Gamma(x) - \Gamma_n(x)| \leqslant \left| \int_0^n \left(e^{-t} \left(1 - \frac{t}{n}\right)^n\right) \cdot t^{x-1} \right| + \left| \int_n^\infty e^{-t} t^{x-1} dt \right| \leqslant$$

$$\leqslant \frac{1}{n} \int_0^n e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \leqslant \frac{1}{n} \int_0^\infty e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \to 0.$$

□

Lemma 2.2

$$\Gamma(x) = \lim_{n \to \infty} \frac{n! \cdot n^x}{x \cdot (x + 1) \cdot \ldots \cdot (x + n)}.$$

Důkaz
„$\Gamma_n(x) = \frac{n! n^x}{x \cdot (x+1) \cdot \ldots \cdot (x+n)}$“: $\Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{1}{n^n} \cdot \int_0^n t^{x-1} (n - A^t) dt \overset{\text{IBP}}{=}$

$$= \left(\frac{1}{n}\right)^n \left(\left[\frac{1}{x} \cdot t \cdot (n-t)^n \right]_0^n + \int_0^n \frac{n}{x} t^x (n-t)^{n-1} dt \right) = \frac{1}{n^n} \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{x \cdot (x+1) \cdot (x+2) \cdot \ldots \cdot (x+n-1)} \cdot \int_0^n t^{x+$$

□

3 Weierstrass function

Definice 3.1

$$H(z) := z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Lemma 3.1

$H \in \mathcal{H}(\mathbb{C})$ has simple zero points just in $\mathbb{N}_0$.

$H(z) \cdot H(-z) = -\frac{z}{\pi} \sin(\pi z)$.

$H(1) = e^{-\gamma}$, where $\gamma := \lim_{n \to \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$ is the Euler–Mascheroni constant. It is known $\gamma \doteq 0,577$, but it isn't known if it is even irrational (much less transcendent).

Důkaz
$\left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$ is $E_1(\frac{z}{k})$ (first Weierstrass factor). So from WF, we know that 1. hold because $\sum_{k=1}^{\infty} \frac{|z|^2}{k^2}$ converges locally uniformly on $\mathbb{C}$.

$$H(z) \cdot H(-z) = -z^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = -\frac{z}{\pi} \sin(\pi z).$$

$$H(1) = \lim_{n \to \infty} \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = e^{-\gamma},$$

because

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{n+1}{n} \exp\left(-\sum_{k=1}^{n} \frac{1}{k}\right) = \exp\left(\log(n+1) - \sum_{k=1}^{n} \frac{1}{k}\right) \to e^{-\gamma}.$$

□

Definice 3.2 (Weierstrass)

$\Delta(z) := e^{\gamma z} H(z)$.

Lemma 3.2

$\Delta \in \mathcal{H}(\mathbb{C})$ *has simple zero points just in* $\mathbb{N}_0$.

$$\Delta(z) = \lim_{n\to\infty} \frac{z\cdot(z+1)\cdot\ldots\cdot(z+n)}{n!n^z}, \qquad z\in\mathbb{C}.$$

$\Delta(1) = 1,\ z\cdot\Delta(z+1) = \Delta(z)$ *for* $z\in\mathbb{C}$.

Důkaz
First is similar to previous lemma.

$$\Delta(z) = e^{\gamma z}\cdot \lim_{n\to\infty} z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = e^{\gamma z}\cdot\lim_{n\to\infty} z\cdot\frac{(z+1)\cdot(z+2)\cdot\ldots\cdot(z+n)}{1\cdot2\cdot\ldots\cdot n}e^{-z\cdot\sum_{k=1}^{n}\frac{1}{k}} =$$

$$= e^{\gamma z}\cdot\lim_{n\to\infty}\frac{z\cdot(z+1)\cdot\ldots\cdot(z+n)}{n!\cdot n^z}\cdot e^{-z\cdot\sum_{k=1}^{n}\frac{1}{k}-\log n} = \lim_{n\to\infty}\frac{z\cdot(z+1)\cdot\ldots\cdot(z+n)}{n!\cdot n^z}.$$

$\Delta(1) = 1$ is obvious. $z\cdot\Delta(z+1) = \Delta(z)\cdot\lim_{n\to\infty}\frac{z+n+1}{n}$ previous. □

Definice 3.3

$\Gamma := \frac{1}{\Delta}$.

Lemma 3.3

$\Gamma \in \mathcal{M}(\mathbb{C})$ *has simple poles just in* $(-\mathbb{N}_0) =: \mathbb{N}_0^-$, $\Gamma \neq 0$ *on* $\mathbb{C}$.

Gauss formula: $\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z\cdot(z+1)\cdot\ldots\cdot(z+n)}$, $z\in\mathbb{C}\backslash\mathbb{N}_0^-$.

$$\Gamma(1) = 1, \qquad \Gamma(z+1) = z\cdot\Gamma(z).$$

$$\operatorname{res}_{-n}\Gamma = \frac{(-1)^n}{n!}, n\in\mathbb{N}_0.$$

Důkaz (Only last proposition)
We know that (with $z\notin\mathbb{N}_0^-$ in limits)

$$\operatorname{res}_{-n}\Gamma = \lim_{z\to-n}(z+n)\Gamma(z) = \lim_{z\to-n}\frac{\Gamma(z+n+1)}{z\cdot(z-1)\cdot\ldots\cdot(z+n-1)}\frac{\Gamma(1)}{(-1)^n\cdot n!} = \frac{(-1)^n}{n!}.$$

□

4 ?

Poznámka
Ω open, bounded, $f \in \mathcal{C}(\overline{\Omega})$, $f \in \mathcal{H}(\Omega) \implies \sup_\Omega |f| = \max_{\partial\Omega} |f|$.

Věta 4.1

$\Omega = \{x + iy | x \in (a,b) \wedge y \in \mathbb{R}\}$, $f \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{H}(\Omega)$, $|f| < B < \infty$ *on* Ω. $M(x) := \sup_{y\in\mathbb{R}} |f(x+iy)|$. *Then* $M(x)^{b-a} \leqslant M(a)^{b-x} \cdot M(b)^{x-a}$.

Důkaz
$M(a) = M(b) = 1$, $|f| \leqslant 1$ on Ω. Let $\varepsilon > 0$, $h_\varepsilon(z) = \frac{1}{1+\varepsilon(z-a)}$. $|h_\varepsilon(z)| \leqslant 1$ on $\overline{\Omega}$.

$\Re\{1 + \varepsilon(z-a)\} = 1 + \varepsilon(x-a) \geqslant 1$. $|1 + \varepsilon(z-a)| \geqslant \varepsilon|y|$. $|f(z)h_\varepsilon(z)| \leqslant \frac{B}{\varepsilon} \cdot \frac{1}{y}$, $y \neq 0$. $|fh_\varepsilon| \leqslant 1$ on ∂R. □

5 Riemann zeta function

Poznámka

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \qquad \Re z > 1.$$

Definice 5.1 (Riemann zeta function)

Riemann zeta function is defined on $\{\Re z > 1\}$ by $\zeta(z) = \sum_{n=1}^\infty n^{-z}$ and

Poznámka

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = n^z \int_0^\infty e^{-nt} t^{z-1} dt \implies$$
$$\implies n^{-t}\Gamma(z) = \int_0^\infty e^{-nt} t^{z-1} dt.$$

$$\zeta(z) \cdot \Gamma(z) = \sum_{n=1}^{\infty} \int_0^\infty e^{-nt} t^{z-1} dt.$$

Lemma 5.1

Let $S = \{\Re z \geqslant a\}$, $a > 1$. *If* $\varepsilon > 0$, *there is* $\delta \in (0,1)$ *such that* $\forall z \in S$*:*

$$\left| \int_\alpha^\beta (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon, \qquad \delta > \beta > \alpha > 0.$$

Let $S = \{\Re z \leqslant A\}$, $A \in \mathbb{R}$. If $\varepsilon > 0$, there is $\varkappa > 1$ such that $\forall z \in S$:

$$\left|\int_\alpha^\beta (e^t - 1)^{-1} t^{z-1} dt\right| < \varepsilon, \qquad \beta > \alpha > \varkappa.$$

Důkaz
„First part“: $e^t - 1 \geqslant t$, $t \geqslant 0$, so for $0 < t \leqslant 1$:

$$z \in S : \left|(e^t - 1)^{-1} t^{z-1}\right| \leqslant |t^{z-2}| \implies$$

$$\implies \int_0^1 \left|(e^t - 1)^{-1} t^{z-1}\right| dt \leqslant \int_0^1 |t^{z-2}| dt < \infty.$$

„Second part“: $t \geqslant 1$, $z \in S$:

$$\left|(e^t - 1)^{-1} t^{z-1}\right| \leqslant (e^t - 1)^{-1} \cdot t^{A-1} < C \cdot e^{\frac{1}{2}t}(e^t - 1)^{-1} \implies$$

$$\implies \int_1^\infty \left|(e^t - 1)^{-1} t^{z-1}\right| dt \leqslant C \cdot \int_1^\infty e^{\frac{1}{2}t}(e^t - 1)^{-1} dt < \infty.$$

□

Důsledek
If $S = \{a \leqslant \Re z \leqslant A\}$, $1 < a < A < \infty$, then $\int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$ converges uniformly on S.

If $S = \{\Re z \leqslant A\}$, $A \in \mathbb{R}$, then $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ converges uniformly on S.

Tvrzení 5.2

For $\Re z > 1$

$$\zeta(z) \cdot \Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt.$$

Důkaz
By the previous lemma for $\varepsilon > 0$ there exist $0 < \alpha < \beta < \infty$ such that

$$\int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt < \frac{3}{4},$$

$$\int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{3}{4}.$$

$$\sum_{k=1}^{\infty} e^{-kt} \leqslant (e^t - 1)^{-1} \qquad \forall n \geqslant 1 :$$

$$\sum_{n=1}^{\infty} \int_0^\alpha e^{-nt} t^{x-1} dt < \frac{3}{4},$$

$$\sum_{n=1}^{\infty} \int_\beta^\infty e^{-nt} t^{x-1} dt < \frac{3}{4}.$$

$$\left|\zeta(x) \cdot \Gamma(x) - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt\right| = \left|\sum \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\infty\right) - \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\infty\right)\right| \leqslant ?+ \left|\sum_{n=1}^{\infty} \int_\alpha^\beta e^{-nt} t\right.$$

since on $[\alpha, \beta]$ $\sum e^{-nt}$ converges uniformly to $(e^t - 1)^{-1}$. □

Poznámka
Extend to $\{\Re z > -1\}$: Laurent expansion (in 0):

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n.$$

TODO?