Definice 0.1 (Category, map (arrow, morphism), composition, domain, codomain)

A category $\mathcal{A}$ consists of: a collection $\mathrm{ob}(\mathcal{A})$ of objects, and for each $A, B \in \mathcal{A}$, a collection $\mathcal{A}(A, B)$ of maps, arrows, or morphisms from A to B. Such that for each $A, B, C \in \mathrm{ob}(\mathcal{A})$ a function (named composition) $\circ : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$, $(g, f) \mapsto g \circ f$ meets following:

For each $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{S}(C, D) : (h \circ g) \circ f = h \circ (g \circ f)$ (asociativity). For each $A \in \mathrm{ob}(\mathcal{A})$ $\exists 1_A \in \mathcal{A}(A, A)$, called the identity, such that, for each $f \in \mathcal{A}(A, B)$: $f \circ 1_A = f = 1_B \circ f$.

Poznámka (Notation)

$$A \in \mathrm{ob}(\mathcal{A}) \Leftrightarrow A \in \mathcal{A}.$$

$$f \in \mathcal{A}(A, B) \Leftrightarrow A \xrightarrow{f} B \Leftrightarrow f : A \to B.$$

For $f \in \mathcal{A}(A, B)$: $\mathrm{domain}(f) := A$ and $\mathrm{codomain}(f) := B$.

Například (of categories)
Category of:

- sets (SET): $\mathrm{ob}(SET) :=$ sets, $SET(A, B) :=$ functions from A to B, $\circ$ is composition;
- groups (GRP): $\mathrm{ob}(GRP) :=$ groups, $GRP(G, H) :=$ group homomorphisms, $\circ$ is composition;
- rings (RING): $\mathrm{ob}(RING) :=$ rings, $RING(A, B) :=$ ring homomorphisms, $\circ$ is composition;
- vector spaces ($\mathrm{VECT}_{\mathbb{K}}$): $\mathrm{ob}(VECT_{\mathbb{K}}) :=$ vector spaces over $\mathbb{K}$, $RING(A, B) := \mathbb{K}$ linear maps, $\circ$ is composition;
- topological spaces (TOP): $\mathrm{ob}(TOP) :=$ topological spaces, $RING(A, B) :=$ continuous maps, $\circ$ is composition.

Definice 0.2 (Isomorphism, inverse)

$f : A \to B$ in a category $\mathcal{A}$ is an isomorphism if exists a map $g : B \to A$ in $\mathcal{A}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Then we call g the inverse of f.

Například
In SET isomorphisms are bijections.

Příklad
Show that inverses are unique (justifying the use of the determine article in the previous definition).

Poznámka
0-morphisms are called morphisms (between objects), 1-morphisms are called functors (between categories), 2-morphisms are called natural transformations (between functors).

Definice 0.3 (Functor)

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A functor $F : \mathcal{A} \to \mathcal{B}$ consists of: a function $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$, and for each $A, A' \in \mathcal{A}$ a function $F : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$. Such that

$$F(f' \circ f) = F(f) \circ F(f'), \qquad \forall A \overset{f}{A'} \overset{f'}{A''} \in \mathcal{A},$$

$$F(1_A) = 1_{F(A)} \qquad \forall A \in \mathcal{A}.$$

Například (Forgetful functors)
$U : GRP \to SET$, for any group $(G, \cdot)$, $U((G, \cdot)) := G$, and for any morphism f, $U(f : (G, \cdot) \to (H, *)) := f : G \to H$. (Exercise: Convince yourself that this is a well-defined functors.)

We can do the same for rings, vector spaces and topological spaces.

Například
Let $\mathcal{A}$ be the following category: $\mathrm{ob}(\mathcal{A}) = \{\cdot\}$, $\mathcal{A}(\cdot, \cdot) = 1.$, and $1. \circ 1. = 1.$. It is called discrete category with one object.

$\mathrm{ob}(\mathcal{B}) = \{\cdot, *\}$, $\mathcal{B}(\cdot, \cdot) = 1.$, $\mathcal{B}(\cdot, *) = \emptyset$

Directed transitive graph (with all loops) with concatenation of edges.

From group $(G, +)$ we construct category $\mathcal{G}$ by putting: $\mathrm{ob}(\mathcal{G}) := \cdot$, $\mathcal{G}(\cdot, \cdot) := G$ and $\circ := +$. We can generalize to a monoid $(M, +)$.

Now, let $\mathcal{A}$ be a category with one object $\{\cdot\}$ (and assume that $\mathcal{S}(\cdot, \cdot)$ is a set). Then homomorphism with composition are monoid. And isomorphisms with composition are groups (so one-object category with all homomorphism isomorphic represents group).

(Category, where $\mathcal{A}(\cdot, \cdot)$ is a set, is often called locally small.)

Let G and H be groups and $\mathcal{G}$, $\mathcal{H}$ their associated one-object categories. What is a functor from $\mathcal{G}$ to $\mathcal{H}$? For $F : \mathrm{ob}(\mathcal{G}) \to \mathrm{ob}(\mathcal{H})$ we have no other choice than $F(\cdot) := *$. For $F : \mathcal{G}(\cdot, \cdot) \to \mathcal{H}(*, *) = \mathcal{H}(F(\cdot), F(\cdot))$ we demonstrated (see lecture) that F needs to be group homomorphism (and every group homomorphism $G \to H$ is functor). (All this work for monoids too.)

Let AB be the category of $\text{ob}(AB) :=$ Abelian groups and $AB(A, B) :=$ group homomorphism. Then $U : AB \to GRP$ as „forgetful functor" is „identity". The same for commutative rings. Also we have forgetful functor $U : RING \to AB$, $(R, +, \cdot) \mapsto (R, +)$ and functor $U : RING \to MONOIDS$, $(R, +, \cdot) \mapsto (R, \cdot)$.

$U : SET \to VECT_{\mathbb{K}}$ we can define by $F(X) = (X \to F)$ (functions from X to F) (free vector space).

Definice 0.4 (Functor composition)

When we have functor $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{C}$. We want to $F' \circ F$ to be functor, so it has function on objects and functions on morphism classes. Function on object is simply composition $F' \circ F$. Functions on morphism classes is also composition:

$$\mathcal{A}(A, A') \xrightarrow{F} B(F(A), F(A')) \xrightarrow{F'} \mathcal{C}(F' \circ F(A), F' \circ F(A')) \implies F' \circ F : \mathcal{A}(A, A') \to \mathcal{C}(F' \circ F(A), F' \circ F(A')).$$

Důkaz

1. $(F' \circ F)(1_A) = F'(F(1_A)) = F'(1_{F(A)}) = 1_{F' \circ F(A)}$. (For $A \in \mathcal{A}$.)

2. $(F' \circ F)(f' \circ f) = F'(F(f' \circ f)) = F'((F(f')) \circ (F(f))) = (F' \circ F(f')) \circ (F' \circ F(f))$. (For $A \xrightarrow{f} A' \xrightarrow{f'} A'' \in \mathcal{A}$.) □

So $F' \circ F$ is a functor. We call it the composition of F and F'.

Definice 0.5 (CAT)

The category of categories (CAT) has categories as objects and functors as morphisms (with its composition from the previous definition).

Důkaz

We need: 1. An identity functor $1_{\mathcal{A}} \in CAT(\mathcal{A}, \mathcal{A})$ (function on objects is identity, function on $CAT(\mathcal{A}, \mathcal{B})$ is identity too), we can easily see that it fulfills condition from category definition.

2. Associativity of composition: composition of functions is associative, so we see this from the definition of the functor composition. □

Definice 0.6 (Dual category (opposite category))

For a category $\mathcal{A}$, its dual category (or opposite category) $\mathcal{A}^{\text{op}}$ is defined by: $\text{ob}(\mathcal{A}^{\text{op}}) = \text{ob}(\mathcal{A})$, $\mathcal{A}^{\text{op}}(B, A) = \mathcal{A}(A, B)$ $(\forall A, B \in \text{ob}(\mathcal{A}))$, composition in $\mathcal{A}^{\text{op}}$ is the composition in $\mathcal{A}$.

Příklad (Excercise)

$$(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{A}.$$

Definice 0.7 (Contravariant functor)

For two cats $\mathcal{A}, \mathcal{B}$ a contravariant functor: $\mathcal{A} \to \mathcal{B}$ is a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{B}$ ($F(f' \circ f) = (F(f)) \circ (F(f'))$).

Příklad
Functor $C : TOP \to ALG_{\mathbb{K}}$ is $X \in TOP \mapsto C(X) \in ALG_{\mathbb{K}}$, where $C(X)$ is the collection of all continuous functions $X \to \mathbb{K}$ with addition, multiplication and scalar multiplication. But when we try to define C for morphisms, we find that it cannot be done this way. ($C(X \xrightarrow{f} Y) = C(X) \xrightarrow{C(f)} C(Y)$, so $C(f)(\varphi) = \varphi \circ f \implies$ this does not define a functor.)

So we „fix it“ by taking contravariant functor.

Definice 0.8 (Presheaf)

Let $\mathcal{A}$ be a category a presheaf on $\mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \to SET$.

Příklad
Let X be a topological space. Write $O(X)$ for ordered subsets of X ordered by inclusion $\to$ category $\mathcal{O}(X)$: objects are open subsets, morphisms are inclusion and $\circ$ is composition of inclusions.

Definice 0.9 (Faithful functor, full functor)

A functor $F : \mathcal{A} \to \mathcal{B}$ is faithful (resp. full) if for each $A, A' \in \mathcal{A}$ the function

$$\mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A')), \qquad f \mapsto F(f),$$

is injective (resp. surjective) $\forall A, A' \in \mathcal{A}$.

Pozor
If F is faithful, we do not have $F(f_1) \neq F(f_2)$ $\forall$ distinct morphisms f_1, f_2. ($F(A)$ still can be equal to $F(A')$, so it can be $f_1 : A \to A$, $f_2 : A' \to A'$.)

Definice 0.10 (Subcategory)

Let $\mathcal{A}$ be a category. A subcategory $\mathcal{S} \subset \mathcal{A}$ consists of a subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{A})$ together with, for $S, S' \in \mathrm{ob}(\mathcal{S})$, a subclass $\mathcal{S}(S, S') \subseteq \mathcal{A}(S, S')$ such that $\mathcal{S}$ is closed under composition.

Definice 0.11 (Full subcategory)

We say that subcategory $\mathcal{S}$ is full if $\mathcal{S}(S, S') = \mathcal{A}(S, S')$, $\forall S, S' \in \mathrm{ob}(\mathcal{S})$.

Poznámka
A full subcategory is identified by its objects.

Například
AB is the full subcategory of GRP.

Příklad
For any subcategory $\mathcal{S} \subset \mathcal{A}$, we have an inclusion functor $I : \mathcal{S} \to \mathcal{A}$.

I is faithful, and it is full $\Leftrightarrow$ $\mathcal{S}$ is full.

Definice 0.12

$F : \mathcal{A} \to \mathcal{B}$, $\mathrm{Im}(F)$ has objects $F(A)$ and morphisms $F(f)$.

Pozor
$\mathrm{Im}(F)$ nemusí být kategorie. (Mohou vzniknout „možnosti složení“, které v původní kategorii nebyly.)

0.1 2-morphism and natural transformations

Definice 0.13 (Natural transformation)

Let $\mathcal{A}$ and $\mathcal{B}$ be categories and $\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B}$ two functors. A natural transformation between F and G ia s family of morphisms in $\mathcal{B}$: $(F(A) \xrightarrow{\alpha_A} G(A))_{A \in \mathcal{A}}$ such that $F(f) \circ \alpha_B = \alpha_A G(f)$ for every $A \xrightarrow{f} B \in \mathcal{A}$.

We call the morphisms α_A the components of the natural transformation.

Příklad
Define a composition of natural transformations and use it to define the functor category of $\mathcal{A}$ and $\mathcal{B}$ (objects functors $F : \mathcal{A} \to \mathcal{B}$ and morphisms natural transformations α).

Příklad
For two graphs H, K, functors between their 1-object cats $\leftrightarrow$ group homomorphism. What is a natural transformations between two functors?

0.2 Free functors

Poznámka
Recall forgetfull functors. What about functors in the other direction?

Například
$F : SET \to VECT_{\mathbb{K}}$, $X \mapsto F(X)$. $F(X)$ (the free $\mathbb{K}$-vector space) is is functions $f : X \to \mathbb{K}$ endowed with the vector space structure (addition and scalar multiplication). (Alternatively $F(X)$ is the vector space with a basis $\left\{e_x^X \mid x \in X\right\}$).

Morphisms: $F(f)(e_x^X) := e_{f(x)}^X$.

Například
$U : GRP \to SET$, so free functor should look like $F : SET \to GRP$. $S \mapsto F(S)$, where $F(S)$ (the free group) is a sets for which $\exists i : S \to F(S)$ inclusion of sets to $F(S)$, that for every $f : S \to \mathcal{G}$ function between sets and groups, $\exists! \varphi_i$ such that $i \circ \varphi_i$ commutes.

Think about / look up: this defines $F(S)$ uniquely up to group isomorphism.

Příklad
Take the set $\mathcal{S}^{-1} = \{S^{-1} | S \in \mathcal{S}\}$. Take all words in the alphabet $\mathcal{S} \cup \mathcal{S}^{-1}$ that are reduced, i.e. we remove pairs of the form SS^{-1}, $S^{-1}S$ and ? is concatenation of words with reduction.

Příklad
How does act on morphisms.