

TODO!!!

Definition 0.1 (WLOG)

$$D := U(0, 1), \quad T = \partial D.$$

TODO!!!

Definition 0.2

$f \in \mathcal{H}(D)$. We say that the boundary T is a natural boundary of f if $R_f = \emptyset$.

Například

$f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Radius of convergence is equal to 1 and f has natural boundary.

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Důkaz

$K = \{\exp(\frac{2\pi i k}{n}) \mid k, n \in \mathbb{N}\}$ is dense in T . f is "diverges on" this set, because $f(z^{2^N}) = f(z) - \sum_{n=1}^N z^{2^n}$. For $\alpha \in (0, 1)$ we have parametrization of one "line" $\alpha \cdot \exp(\frac{2k\pi i}{2^n})$ (for k, n fixed).

$$f(\alpha^{2^N}) = f\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right) + p\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right).$$

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□

For every domain $\Omega \subseteq \mathbb{C}$, there exists $f \in \mathcal{H}(\Omega)$ such that $\partial\Omega$ is natural boundary of f .

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Důkaz

We use theorem (15.11 from Rudin or TODO from lecture).

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1 Eulerův vzorec

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 1.1

$$z \neq \frac{k}{2}, k \in \mathbb{Z} : 2\pi \cotg(2\pi z) = \pi \cotg(\pi x) + \pi \cotg(\pi(z + \frac{1}{2})).$$

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Důkaz

$$2\pi \cotg(2\pi z) = 2\pi \frac{\cos(2\pi z)}{\sin(2\pi z)} = \pi \frac{\cos^2(\pi z) - \sin^2(\pi z)}{\sin(\pi z) \cos(\pi z)} = \pi \left(\cotg(\pi z) - \frac{\sin(\pi z)}{\cos(\pi z)} \right) = \pi \left(\cotg(\pi z) + \frac{\cos \pi}{\sin \pi} \right)$$

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□

Lemma 1.2 (Herglotz)

$r > 1$, G oblast, $G \supset [0, r)$, h funkce holomorfní na G , $z, z + \frac{1}{2}, 2z \in [0, r) : 2h(2z) = h(z) + h(z + \frac{1}{2})$. Pak h je konstantní na G .

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Důkaz

Zvol $t \in (1, r)$.

$$M := \max \{ |h'(z)|, z \in [0, t] \}, \quad 4|h'(2z)| \leq |h'(z)| + |h'(z + \frac{1}{2})| \implies$$

$$\implies 4|h'(z)| \leq |h'(\frac{z}{2})| + |h'(\frac{z}{2} + \frac{1}{2})| < 2M \implies 4M \leq 2M \implies M = 0.$$

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□

Lemma 1.3

g holomorfní funkce na $\mathbb{C} \setminus \mathbb{Z}$, hlavní část Laurentovy řady g na $P(k)$, $k \in \mathbb{Z}$, je rovna $\frac{1}{z-k}$, g lichá, $2g(2z) = g(z) + g(z + \frac{1}{2})$, $z \neq \frac{k}{2}$, $k \in \mathbb{Z}$. Pak $g(z) = \pi \cotg(\pi z)$.

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Důkaz

$h(z) := g(z) - \pi \cotg(\pi z)$. h rozšíříme spojitě a holomorfně na \mathbb{C} . Z Herglotzova lemmatu je h konstantní na \mathbb{C} (obě funkce splňují podmínky). Navíc $h(0) = 0$. □

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Důsledek (Eisenstein)

$$\pi \cotg(\pi z) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}, \quad z \notin \mathbb{Z}.$$

┌ *Důkaz*

$z \mapsto \frac{2z}{z^2 - k^2}$ jsou holomorfní na $U(0, n)$, $k > n$.

$$\left| \frac{2z}{z^2 - k^2} \right| \leq \frac{2n}{k^2 - n^2} \quad \wedge \quad \sum_{k=n+1}^{\infty} \frac{2n}{k^2 - n^2} K \implies \sum \in \mathcal{M}.$$

f je holomorfní na $\mathbb{C} \setminus \mathbb{Z}$. Také je lichá. Nakonec

$$s_n(z) = \frac{1}{2} + \sum_{k=1}^n \frac{2z}{z^2 - k^2} = \frac{1}{2} + \sum_{k=1}^n \frac{1}{z + k} + \sum_{k=1}^n \frac{1}{z - k} = \sum_{-n}^n \frac{1}{z + k},$$

$$s_n\left(\frac{z}{2}\right) + s_n\left(\frac{z+1}{2}\right) = \sum_{k=-n}^n \frac{2}{z + 2k} + \frac{2}{z + 2k + 1} = 2 \sum_{k=-2n}^{2n} \frac{1}{z + k} + 2 \cdot \frac{1}{z + 2n + 1} = 2s_{2n}(k) + \frac{2}{z + 2n + 1},$$

$$n \rightarrow \infty : f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right) = 2f(z) + 0.$$

└ Z předchozího lemmatu vyplývá důkaz. □

Věta 1.4 (Euler)

$$\underbrace{\sin(\pi z)}_g = \underbrace{\pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)}_f.$$

┌ *Důkaz*

$\forall z \in \mathbb{C} \exists$ neighbourhood U : $\sum_{k=1}^{\infty} \left\| \left(z \mapsto \frac{z^2}{k^2} \right) \right\|_{\infty}$ is convergent $\implies \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$ is holomorphic.

$$f_k(z) := 1 - \frac{z^2}{k^2}, \quad \frac{f'_k(z)}{f_k(z)} = \frac{\frac{-2z}{k^2}}{\frac{k^2 - z^2}{k^2}} = \frac{2z}{z^2 - k^2}.$$

$$\frac{f'(z)}{f(z)} = \frac{1}{2\pi \prod (1 - \frac{z^2}{k^2})} \left(\pi \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) + \pi z \left(\prod_{k=1}^{\infty} \dots \right)' \right) = \frac{1}{2} = \frac{\Pi'}{\Pi} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{f'_k(z)}{f_k(z)} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}$$

$$\left(\frac{g'(z)}{g(z)} \right) = \frac{\pi \cos(\pi z)}{\sin \pi z} = \pi \cot g(\pi z), \quad \left(\frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} = 0 \implies \frac{f}{g} \text{ is constant}.$$

$$\lim_{z \rightarrow 0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow 0} \frac{\pi \sin(z\pi)}{TODO} TODO(online).$$

└ □

2 Gamma function

Definice 2.1

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x \in (0, \infty).$$

Poznámka (Notion)

$$I_n = \int_0^1 1 \cdot (-\log x)^n dx \stackrel{\text{IBP}}{=} [x(-\log x)^n]_0^1 - \int_0^1 xn(-\log x)^{n-1} \left(-\frac{1}{x}\right) dx = n \cdot I_{n-1}.$$

So $I_n = n!$. Set $\log x = t$, $e^t = x$:

$$I_n = \int_{-\infty}^0 (-t)^n \cdot e^t dt = \int_0^\infty t^n e^{-t} dt.$$

Lemma 2.1

$\forall n \in \mathbb{N}$ we define $\Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt$, then $\Gamma_n(x) \rightarrow \Gamma(x)$.

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Důkaz

$$„0 \leq e^t - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{n} e^{-t}“:$$

$$0 \leq e^{-t} \cdot \left(1 - \frac{t}{n}\right)^n = e^{-t} \left(1 - e^t \left(1 - \frac{t}{n}\right)^n\right) \leq e^{-t} \left(1 - \left(1 - \frac{t}{n}\right)^n \cdot \left(1 + \frac{t}{n}\right)^n\right) = e^{-t} \left(1 - \left(1 - \frac{t^2}{n^2}\right)^n\right)$$

(Last inequality from Bernoulli: $(1+x)^n \geq 1+n \cdot x$, $x \geq -1$.)

$$\begin{aligned} |\Gamma(x) - \Gamma_n(x)| &\leq \left| \int_0^n \left(e^{-t} \left(1 - \frac{t}{n}\right)^n\right) \cdot t^{x-1} dt \right| + \left| \int_n^\infty e^{-t} t^{x-1} dt \right| \leq \\ &\leq \frac{1}{n} \int_0^n e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \leq \frac{1}{n} \int_0^\infty e^{-t} t^{x+1} dt + \int_n^\infty e^{-t} t^{x-1} dt \rightarrow 0. \end{aligned}$$

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□

Lemma 2.2

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)}.$$

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Důkaz

$$\Gamma_n(x) = \frac{n!n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)} \because \Gamma_n(x) = \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = \frac{1}{n^n} \cdot \int_0^n t^{x-1} (n - A^t) dt \stackrel{\text{IBP}}{=} \frac{1}{n^n} \cdot \int_0^n t^{x-1} (n - A^t) dt$$

$$= \left(\frac{1}{n}\right)^n \left(\left[\frac{1}{x} \cdot t \cdot (n-t)^n \right]_0^n + \int_0^n \frac{n}{x} t^x (n-t)^{n-1} dt \right) = \frac{1}{n^n} \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{x \cdot (x+1) \cdot (x+2) \cdot \dots \cdot (x+n-1)} \cdot \int_0^n t^{x-1} dt$$

└

□

3 Weierstrass function

Definice 3.1

$$H(z) := z \cdot \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}.$$

Lemma 3.1

$H \in \mathcal{H}(\mathbb{C})$ has simple zero points just in \mathbb{N}_0 .

$$H(z) \cdot H(-z) = -\frac{z}{\pi} \sin(\pi z).$$

$H(1) = e^{-\gamma}$, where $\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n\right)$ is the Euler–Mascheroni constant. It is known $\gamma \doteq 0,577$, but it isn't known if it is even irrational (much less transcendent).

Důkaz

$\left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}$ is $E_1\left(\frac{z}{k}\right)$ (first Weierstrass factor). So from WF, we know that 1. hold because $\sum_{k=1}^{\infty} \frac{|z|^2}{k^2}$ converges locally uniformly on \mathbb{C} .

$$H(z) \cdot H(-z) = -z^2 \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right) = -\frac{z}{\pi} \sin(\pi z).$$

$$H(1) = \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{1}{k}\right) e^{-\frac{1}{k}} = e^{-\gamma},$$

because

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \dots \cdot \frac{n+1}{n} \exp\left(-\sum_{k=1}^n \frac{1}{k}\right) = \exp\left(\log(n+1) - \sum_{k=1}^n \frac{1}{k}\right) \rightarrow e^{-\gamma}.$$

└

□

Definice 3.2 (Weierstrass)

$$\Delta(z) := e^{\gamma z} H(z).$$

Lemma 3.2

$\Delta \in \mathcal{H}(\mathbb{C})$ has simple zero points just in \mathbb{N}_0 .

$$\Delta(z) = \lim_{n \rightarrow \infty} \frac{z \cdot (z+1) \cdot \dots \cdot (z+n)}{n! n^z}, \quad z \in \mathbb{C}.$$

$$\Delta(1) = 1, \quad z \cdot \Delta(z+1) = \Delta(z) \text{ for } z \in \mathbb{C}.$$

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Důkaz

First is similar to previous lemma.

$$\begin{aligned} \Delta(z) &= e^{\gamma z} \cdot \lim_{n \rightarrow \infty} z \cdot \prod_{k=1}^n \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} = e^{\gamma z} \cdot \lim_{n \rightarrow \infty} z \cdot \frac{(z+1) \cdot (z+2) \cdot \dots \cdot (z+n)}{1 \cdot 2 \cdot \dots \cdot n} e^{-z \cdot \sum_{k=1}^n \frac{1}{k}} = \\ &= e^{\gamma z} \cdot \lim_{n \rightarrow \infty} \frac{z \cdot (z+1) \cdot \dots \cdot (z+n)}{n! \cdot n^z} \cdot e^{-z \cdot \sum_{k=1}^n \frac{1}{k} - \log n} = \lim_{n \rightarrow \infty} \frac{z \cdot (z+1) \cdot \dots \cdot (z+n)}{n! \cdot n^z}. \end{aligned}$$

$$\Delta(1) = 1 \text{ is obvious. } z \cdot \Delta(z+1) = \Delta(z) \cdot \lim_{n \rightarrow \infty} \frac{z+n+1}{n} \text{ previous.}$$

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Definice 3.3

$$\Gamma := \frac{1}{\Delta}.$$

Lemma 3.3

$\Gamma \in \mathcal{M}(\mathbb{C})$ has simple poles just in $(-\mathbb{N}_0) =: \mathbb{N}_0^-$, $\Gamma \neq 0$ on \mathbb{C} .

$$\text{Gauss formula: } \Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z \cdot (z+1) \cdot \dots \cdot (z+n)}, \quad z \in \mathbb{C} \setminus \mathbb{N}_0^-.$$

$$\Gamma(1) = 1, \quad \Gamma(z+1) = z \cdot \Gamma(z).$$

$$\text{res}_{-n} \Gamma = \frac{(-1)^n}{n!}, \quad n \in \mathbb{N}_0.$$

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Důkaz (Only last proposition)

We know that (with $z \notin \mathbb{N}_0^-$ in limits)

$$\text{res}_{-n} \Gamma = \lim_{z \rightarrow -n} (z+n) \Gamma(z) = \lim_{z \rightarrow -n} \frac{\Gamma(z+n+1)}{z \cdot (z-1) \cdot \dots \cdot (z+n-1)} \frac{\Gamma(1)}{(-1)^n \cdot n!} = \frac{(-1)^n}{n!}.$$

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□

4 ?

Poznámka

Ω open, bounded, $f \in \mathcal{C}(\overline{\Omega})$, $f \in \mathcal{H}(\Omega) \implies \sup_{\Omega} |f| = \max_{\partial\Omega} |f|$.

Věta 4.1

$\Omega = \{x + iy | x \in (a, b) \wedge y \in \mathbb{R}\}$, $f \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{H}(\Omega)$, $|f| < B < \infty$ on Ω . $M(x) := \sup_{y \in \mathbb{R}} |f(x + iy)|$. Then $M(x)^{b-a} \leq M(a)^{b-x} \cdot M(b)^{x-a}$.

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Důkaz

$M(a) = M(b) = 1$, $|f| \leq 1$ on Ω . Let $\varepsilon > 0$, $h_{\varepsilon}(z) = \frac{1}{1+\varepsilon(z-a)}$. $|h_{\varepsilon}(z)| \leq 1$ on $\overline{\Omega}$.

$\Re\{1 + \varepsilon(z-a)\} = 1 + \varepsilon(x-a) \geq 1$. $|1 + \varepsilon(z-a)| \geq \varepsilon|y|$. $|f(z)h_{\varepsilon}(z)| \leq \frac{B}{\varepsilon} \cdot \frac{1}{y}$, $y \neq 0$.

└ $|fh_{\varepsilon}| \leq 1$ on ∂R . □

5 Riemann zeta function

Poznámka

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \quad \Re z > 1.$$

Definice 5.1 (Riemann zeta function)

Riemann zeta function is defined on $\{\Re z > 1\}$ by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ and

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Poznámka

$$\begin{aligned} \Gamma(z) &= \int_0^{\infty} e^{-t} t^{z-1} dt = n^z \int_0^{\infty} e^{-nt} t^{z-1} dt \implies \\ &\implies n^{-z} \Gamma(z) = \int_0^{\infty} e^{-nt} t^{z-1} dt. \end{aligned}$$

$$\zeta(z) \cdot \Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} t^{z-1} dt.$$

└

Lemma 5.1

Let $S = \{\Re z \geq a\}$, $a > 1$. If $\varepsilon > 0$, there is $\delta \in (0, 1)$ such that $\forall z \in S$:

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon, \quad \delta > \beta > \alpha > 0.$$

Let $S = \{\Re z \leq A\}$, $A \in \mathbb{R}$. If $\varepsilon > 0$, there is $\varkappa > 1$ such that $\forall z \in S$:

$$\left| \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt \right| < \varepsilon, \quad \beta > \alpha > \varkappa.$$

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Důkaz

„First part“: $e^t - 1 \geq t$, $t \geq 0$, so for $0 < t \leq 1$:

$$\begin{aligned} z \in S : |(e^t - 1)^{-1} t^{z-1}| &\leq |t^{z-2}| \implies \\ \implies \int_0^1 |(e^t - 1)^{-1} t^{z-1}| dt &\leq \int_0^1 |t^{z-2}| dt < \infty. \end{aligned}$$

„Second part“: $t \geq 1$, $z \in S$:

$$\begin{aligned} |(e^t - 1)^{-1} t^{z-1}| &\leq (e^t - 1)^{-1} \cdot t^{A-1} < C \cdot e^{\frac{1}{2}t} (e^t - 1)^{-1} \implies \\ \implies \int_1^{\infty} |(e^t - 1)^{-1} t^{z-1}| dt &\leq C \cdot \int_1^{\infty} e^{\frac{1}{2}t} (e^t - 1)^{-1} dt < \infty. \end{aligned}$$

└

□

Důsledek

If $S = \{a \leq \Re z \leq A\}$, $1 < a < A < \infty$, then $\int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt$ converges uniformly on S .

If $S = \{\Re z \leq A\}$, $A \in \mathbb{R}$, then $\int_1^{\infty} (e^t - 1)^{-1} t^{z-1} dt$ converges uniformly on S .

Tvrzení 5.2

For $\Re z > 1$

$$\zeta(z) \cdot \Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt.$$

┌ *Důkaz*

By the previous lemma for $\varepsilon > 0$ there exist $0 < \alpha < \beta < \infty$ such that

$$\int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt < \frac{3}{4},$$

$$\int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{3}{4}.$$

$$\sum_{k=1}^{\infty} e^{-kt} \leq (e^t - 1)^{-1} \quad \forall n \geq 1 :$$

$$\sum_{n=1}^{\infty} \int_0^\alpha e^{-nt} t^{x-1} dt < \frac{3}{4},$$

$$\sum_{n=1}^{\infty} \int_\beta^\infty e^{-nt} t^{x-1} dt < \frac{3}{4}.$$

$$\left| \zeta(x) \cdot \Gamma(x) - \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt \right| = \left| \sum \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\infty \right) - \left(\int_0^\alpha + \int_\alpha^\beta + \int_\beta^\infty \right) \right| \leq ? + \left| \sum_{n=1}^{\infty} \int_\alpha^\beta e^{-nt} t^{x-1} dt \right|$$

└ since on $[\alpha, \beta]$ $\sum e^{-nt}$ converges uniformly to $(e^t - 1)^{-1}$. □

Poznámka

Extend to $\{\Re z > -1\}$: Laurent expansion (in 0):

$$(e^z - 1)^{-1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n.$$

TODO?

6 Interpolační problém

Poznámka (Zadání)

Nechť $A \subseteq \Omega$, A nemá hromadný bod v Ω . Můžeme najít funkci $f \in \mathcal{H}(\Omega)$ tak, že f má předepsané hodnoty v A ?

Poznámka (Odpověď)

Ano, dokonce můžeme předepsat konečně mnoho derivací.

Věta 6.1

Nechť $\Omega \subseteq \mathbb{C}$ otevřená, $A \subset \Omega$ nemá v Ω hromadný bod, $m : A \rightarrow \mathbb{N}$, $w_{n,a} \in \mathbb{C}$, $0 \leq n \leq m(a)$, $a \in A$. Pak existuje $f \in \mathcal{H}(\Omega)$ tak, že $f^{(n)}(a) = n!w_{n,a}$.

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Důkaz

Z Věty Weierstrass factorization existuje $g \in \mathcal{H}(\Omega)$ tak že její nulové body jsou právě body A a řád nulového bodu $a \in A$ je $(m(a) + 1)$. Ke každému bodu $a \in A$ najdeme funkci $P_a(z)$ tvaru

$$P_a(z) = \sum_{j=1}^{1+m(a)} c_{j,a}(z-a)^{-j}$$

tak, aby gP_z měla hodnoty $y(z)P_n(z) = w_{0,a} + w_{1,a}(z-a) + \dots + w_{m(a),a}(za)^{m(a)}$ na nějakém okolí a . Pro jednoduchost $a = 0$, $m = m(a)$. Pro z v nějakém okolí 0 máme

$$g(z) = b_1 z^{m+1} + b_2 z^{m+2} + \dots, \quad b_1 \neq 0.$$

Pokud $P(z) = c_1 z^{-1} + \dots + c_{m+1} z^{-m-1}$, pak $g(z)P(z) = (c_{m+1} + c_m z + \dots + c_1 z^m)(b_1 + b_2 z + b_2 z^2 + \dots)$.

Z Mittagovy–Lessenovy věty existuje h meromorfní v Ω tak že její hlavní části jsou právě P_a . Zbývá položit $f = gh$. □

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Definice 6.1

Ideál $[g_1, \dots, g_n]$ generovaný funkcemi $g_1, \dots, g_n \in \mathcal{H}(\Omega)$ je množina všech funkcí tvaru $\sum f_i g_i$, kde $f_i \in \mathcal{H}(\Omega)$, $1 \leq i \leq n$. Hlavní ideál je ideál generovaný jedinou funkcí.

Definice 6.2 (Ideal, principal ideal, $m(f, \alpha)$)

The ideal $[g_1, \dots, g_n]$ generated by $g_1, \dots, g_n \in \mathcal{H}(\Omega)$ is the set of all functions of the form $\sum f_i g_i$ for some $f_i \in \mathcal{H}(\Omega)$.

A principal ideal is ideal generated by a single function.

By $m(f, \alpha)$ denote multiplicity of zero of f at α .

Věta 6.2

Every finitely generated ideal is principal.

Důkaz

WLOG, we may assume that Ω is a domain. Let us consider the proposition $p(n)$: If $g_1, \dots, g_n \in \mathcal{H}(\Omega)$, no g_i is identically zero and no point of Ω is a zero point of every g_i , then $[g_1, \dots, g_n] = [1] = \mathcal{H}(\Omega)$.

$p(1)$ is trivial. Let $n > 1$ and assume $p(n-1)$ holds. Let $g_1, \dots, g_n \in \mathcal{H}(\Omega)$ be without common zero point. By Weierstrass theorem, we find $\varphi \in \mathcal{H}(\Omega)$ such that $m(\varphi, \alpha) = \min \{m(g_i, \alpha) | i \in [n-1]\}$, $\alpha \in \Omega$. For $i \in \{1, \dots, n-1\}$ we define $f_i = \frac{g_i}{\varphi} \in \mathcal{H}(\Omega)$ and these functions have no common zero point in Ω . From induction: $[f_1, \dots, f_{n-1}] = [1]$. This implies $[g_1, \dots, g_n] = [\varphi, g_n]$. By the choice of φ , we have that $g_n(z)$ is nonzero whenever $\varphi(z) = 0$. By theorem above we find $h \in \mathcal{H}(\Omega)$ such that $m(1-h \cdot g_n, \alpha) \geq m(\varphi, \alpha)$, $\alpha \in \Omega$ by prescribing $h^{(n)}(\alpha)$ for $\alpha \in \{z \in \mathbb{C} | \varphi(z) = 0\}$. $0 \leq k \leq m(\varphi, \alpha)$.

$$h = \sum a_n z^n, \quad g_n = \sum b_n z^n, \quad hg_n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_k a_{n-k} z^n.$$

This means $\frac{1-hg_n}{\varphi}$ has removable singularities \implies there exists $f \in \mathcal{H}(\Omega)$ such that $1 = f\varphi + hg_n \implies 1 \in [g_1, \dots, g_n]$.

Let $G_1, \dots, G_n \in \mathcal{H}(\Omega)$, no G_i is identically zero. By Weierstrass theorem we find $\varphi \in \mathcal{H}(\Omega)$, such that $m(\varphi, \alpha) = \min_{1 \leq i \leq n} (G_i, \alpha)$, $\alpha \in \Omega$. Put $g_i := \frac{G_i}{\varphi} \in \mathcal{H}(\Omega) \implies [g_1, \dots, g_n] = [1] \implies [G_1, \dots, G_n] = [\varphi]$. \square