Poznámka (Literature)
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Definice 0.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka
Complete compatible metric is not unique: $\tilde{\varrho} = \min\{1, \varrho\}$.

Například
$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $2 := \{0, 1\}$, $\omega := \{0, 1, 2, \ldots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, 2^ω, ω^ω (both with product topology).

Věta 0.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X.

Důkaz
Without proof. (We should know it already.) □

Věta 0.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X, such that $\operatorname{diam}(F_n) \to 0$. *Then* $|\bigcap F_n| = 1$.

Důkaz
Without proof. (We should know it already.) □

Věta 0.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n\in\omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow H \in \mathcal{G}_\delta(X)$

Důkaz ((i))
Let d_n be CCM (complete compatible metric) on X_n, $n \in \omega$. Then

$$d(x, y) := \sum_{n=0}^{\infty} \min\left\{2^{-n}, d_n(x_n, y_n)\right\}$$

is CCM on $X = \prod_{n\in\omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „d is metric“ straightforward, „d is complete“ also easy, compatibility too). □

Důkaz ((ii))
$H = \emptyset$, $H = X$ trivial. Assume $H \neq \emptyset, X$.

„ $\implies$ “: Fix CCM ϱ on H. $V_n := \bigcup\{V \subset X | V \text{ open in } X \wedge V \cap H \neq \emptyset \wedge \operatorname{diam}_\varrho(V \cap H) < 2^{-n}\}$, $n \in \omega$. We want to show $H \overset{?}{=} \bigcap_{n\in\omega}(V_n \cap \overline{H}) \in \mathcal{G}_\delta$:

$$„\subseteq“: x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n.$$

„$\supseteq$“: $x \in V_n \cap \overline{H}$ for every $n \in \omega \implies \exists$ open sets G_n: $x \in G_n$, $G \cap H \neq \emptyset$, $\operatorname{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \overset{?}{\neq} \emptyset \impliedby x \in G_n \cap G_{n+1} \cap \overline{H} \neq \emptyset$).

$\{y\} := \bigcap_{n\in\omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O$, $G_n \cap H \subset B(y, 2^{-n})$, $n \in \omega$. $\implies \exists n \in \omega G_n \cap H \subset O$, $x \in G_n \cap (X \backslash \overline{O}) \cap \overline{H} \implies G_n \cap (X\backslash\overline{O}) \cap H \neq \emptyset$.

„ $\impliedby$ “: fix CCM d on X, $H = \bigcap_{n\in\omega} U_n$, $\emptyset = U_n \neq X$. $F_n := X \backslash U_n$, $\tilde{d}(x,y) = d(x,y) + \sum_{n=0}^{\infty} \min\left\{2^{-n}, \left|\frac{1}{\operatorname{dist}(x,F_n)} - \frac{1}{\operatorname{dist}(y,F_n)}\right|\right\}$, $x, y \in H$. Next we verified that $\tilde{d}$ is metric, that $\tilde{d}$ is equivalent with d on H (by convergence), and that $(H, \tilde{d})$ is complete metric space and separable. TODO? □

Definice 0.2 (Notation)

$A \neq 0$:

- $A^{<\omega}$:= finite sequence of elements of $A = \bigcup_{n\in\omega} A^n$;
- $s \in A^k$, $t \in A^{<\omega} \cup A^\omega$: $s^\wedge t := (s_0, s_1, \ldots, s_{k-1}, t_0, t_1, \ldots)$, where $s = (s_0, \ldots, s_{k-1})$, $t = (t_0, t_1, \ldots)$;
- $s \in A^{<\omega} \cup A^\omega$: $|s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega$, $k \in \omega$, $|s| \geqslant k$, then we denote restriction of s on first k elements as s/k;
- $s \prec t$ iff $|t| \geqslant |s|$ and $s = t/|s|$ ($s \in A^{<\omega}$, $t \in A^{<\omega} \cup A^\omega$).

1 Baire space ω^ω

Definice 1.1

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega | s \prec \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \backslash \bigcup\{\mathcal{N}(t) || t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N}|s \in \omega^{<\omega}\}$ is base of topology of ω^ω.

Věta 1.1 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz
Bez důkazu. □

Důsledek
ω^ω is homeomorphic to $\mathbb{R}\backslash\mathbb{Q}$.

Věta 1.2

Let $X \neq \emptyset$, PTS. Then X is continuous image of ω^ω.

Poznámka
$X \neq \emptyset$ PTS. Then there $\exists F \subset \omega^\omega$, F closed, and continuous injection $\varphi : F \to X$.

Důkaz
Find CCM on X such that $\operatorname{diam} X \leqslant 1$. We inductively construct closed $\emptyset \neq A_s \subset X$ for every $s \in \omega^{<\omega}$ such that 1. $A_\emptyset = X$; 2. $\operatorname{diam}(A_s) \leqslant 2^{-|s|}$; 3. $A_s = \bigcup_{i\in\omega} A_{s^\wedge i}$.

Empty set is trivial. Assume we already have A_s. Find $\{x_i|i \in \omega\} \subset A_s$ dense in A_s. $A_{s^\wedge i} := A_s \cap \overline{B(x_i, 2^{-|s|-2})} \neq \emptyset$ closed.

Fix $\forall \nu \in \omega^\omega : f(\nu) := x$, where $\{x\} = \bigcap_{k\in\omega} A_{\nu/k} \neq \emptyset$ (intersection of closed nonempty non-increasing sequence of sets). „f is surjection“: $x \in A_s \overset{3.}{\implies} \exists n \in \omega : x \in A_{s^\wedge n} \overset{1.}{\implies} \forall x \in X\ \exists \alpha \in \omega^\omega\ \forall k \in \omega : x \in A_{\alpha/k} \implies x = f(\alpha)$.

„f continuous“: $f(\mathcal{N}_{\nu/k}) \subset A_{\nu/k}$ for every $\nu \in \omega^\omega$, $k \in \omega$, $\operatorname{diam} A_{\nu/k} \leqslant 2^{-k}$. □

1.1 Cantor set 2^ω

Tvrzení 1.3

2^ω is up to homeomorphism unique nonempty nuldimensional compact metrizable space without isolated points (without isolated points is called perfect space).

Tvrzení 1.4

Let $X \neq \emptyset$ metrizable, compact. Then X is continuous image of 2^ω.

Důkaz
Without proof, but it is similar to the previous one. □

1.2 Hilbert cube $[0,1]^\omega$

Tvrzení 1.5

Let X be PTS. Then X is homeomorphic to G_δ subset of $[0,1]^\omega$.

Důkaz
X PTS, case $\emptyset$ is trivial, so assume $X \neq \emptyset$, ϱ is CCM on X, $\varrho \leqslant 1$. Let $\{x_n, n \in \omega\}$ be dense in X. Define $f : [0,1]^\omega : f(x) = (\varrho(x, x_n))_{n\in\omega}$. $\varrho \leqslant 1 \implies f(x) \in [0,1]^\omega$.

„Continuity of f“: $f^{-1}(U) = \bigcap_{i=1}^n B(x_i, b_i) \setminus \overline{B(x_i, a_i)}$ open.

„Injective“: $x \neq y \implies \exists n \in \omega : \varrho(x, x_n) < \varrho(y, x_n) \implies f(x) \neq f(y)$.

„Continuity of f^{-1}“ $f(y^n) \to f(y) \overset{?}{\implies} y^n \to y$.

$$f(y^n) \to f(y) \overset{?}{\Leftrightarrow} \forall k \in \omega : \varrho(y^n, x_k) \to \varrho(y, x_k).$$

Let $\varepsilon > 0$ be arbitrary:

$$\exists k \in \omega : \varrho(y, x_k) < \frac{\varepsilon}{3}.\ \exists n_0\ \forall n \geqslant n_0 : \varrho(y^n, x_k) < \frac{2\varepsilon}{3}.$$

Then

$$\forall n \geqslant n_0 : \varrho(y^n, y) \leqslant \varrho(y^n, x_k) + \varrho(x_k, y) < \varepsilon.$$

So $f(X)$ is homeomorphism to $X \implies f(X)$ is PTS $\implies f(X) \in \mathcal{G}_\delta([0,1]^\omega)$. □

Důsledek
Let X be compact metrizable space. Then X is homeomorphic to some closed subset of $[0,1]^\omega$.

Důkaz
Compact metrizable space is Polish. And compact subset must be closed. □

1.3 $\mathcal{K}(X)$: Hyperspace of compact subsets of X

Definice 1.2

Let X be PTS, denote $\mathcal{K}(X) := \{K \subset X | K \text{ is compact}\}$. Vietoris topology on $\mathcal{K}(X)$ is generated by $\{K \in \mathcal{K}(X) | K \subset V\}$ for V open and $\{K \cap \mathcal{K}(X) | K \cap V \neq \emptyset\} = \mathcal{K}(X) \setminus \{K \in \mathbb{K}(X) | K \subset X \setminus V\}$

for V open.

Tvrzení 1.6

Let X be PTS, ϱ CCM on X, $\varrho \leqslant 1$. Then mapping $h : \mathcal{K}(X) \times \mathcal{K}(X) \mapsto [0, +\infty)$ defined as:

$$h(K,L) = \begin{cases} 0, & K = L = \emptyset, \\ \max\left\{\sup_{x \in K} \varrho(x,L), \sup_{y \in L} \varrho(y,K)\right\}, & K, L \neq \emptyset, \\ 1, & \textit{other cases}, \end{cases}$$

is CCM on $\mathcal{K}(X)$ with Vietoris topology. h is known as Hausdorff metric.

Poznámka

$\mathcal{K}(X)$ is separable if X is PTS. X is compact metrizable $\implies$ $\mathcal{K}(X)$ is compact (totally bounded).

X is separable $\implies \exists D \subset X : \overline{D} = X, |D| = \omega$.

$$M = \{K \subset D | |K| < \omega\} \implies |M| = \omega.$$

$\overline{M} = \mathcal{K}(X)$. $K \in \mathcal{K}(X)$ arbitrary, $\varepsilon > 0$ arbitrary. Then $\exists \frac{\varepsilon}{2}$ net $P \subset K$, $|P| < \omega$. We find $\{\tilde{x}_0, \ldots, \tilde{x}_n\} \subset D : \varrho(x_i, \tilde{x}_i) < \frac{\varepsilon}{2} \wedge h(K, \{\tilde{x}_0, \ldots, \tilde{x}_n\}) < \varepsilon$.

X is compact, P is ε-net in X, $|P| < \omega \implies 2^P$ is finite ε-net in $\mathcal{K}(X)$.

Důkaz

$(\emptyset \neq K, L, P \in \mathcal{K}(X).)$ h is metric, definition is correct, $h \geqslant 0$ trivial, $h(K,L) = h(L,K)$ trivial, $h(K,L) = 0 \implies K = L$ $(x \notin L \implies \varrho(x,L) > 0 \implies K \subset L \wedge L \subset K)$.

„$\triangle$“ aka „$h(K,L) \leqslant h(K,P) + h(P,L)$“: Let $x \in K, y \in L, p \in P$. Then

$$\varrho(x,L) \leqslant \varrho(x,y) \leqslant \varrho(x,p) + \varrho(p,y) \qquad \inf y \in L$$

$$\begin{aligned} \varrho(x,L) &\leqslant \varrho(x,p) + \varrho(p,L) && \sup p \in P \\ \varrho(x,L) &\leqslant \varrho(x,p) + h(P,L) && \inf p \in P \\ \varrho(x,L) &\leqslant \varrho(x,P) + h(P,L) && \inf p \in P \\ \sup_{x \in K} \varrho(x,L) &\leqslant h(K,P) + h(P,L). \end{aligned}$$

Similarly $\sup_{y \in L} \varrho(y,K) \leqslant h(K,P) + h(P,L)$. □

TODO!!!

Definice 1.3

X is metrizable space, $1 \leqslant \alpha < \omega_1$. We define $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, and $\Delta^0_\alpha(X)$ by induction:

$$\Sigma^0_1(X) := \{U \subset X | U \text{ open}\},$$

$$\Pi^0_\alpha(X) := \left\{A \subset X | X \setminus A \in \Sigma^0_\alpha(X)\right\},$$

$$\Sigma^0_\alpha(X) := \left\{\bigcup_{n\in\omega} A_n | A_n \in \Pi^0_{\alpha_n}(X), \alpha_n < \alpha, n \in \omega\right\},$$

$$\Delta^0_\alpha(X) := \Sigma^0_\alpha \cap \Pi^0_\alpha(X).$$

Poznámka (By induction it can be prooven)

$$\Sigma^0_\alpha(X) \subset \Sigma^0_\beta(X), \Pi^0_\alpha(X) \subseteq \Pi^0_\beta(X), \qquad 1 \leqslant \alpha < \beta < \omega_1.$$

Poznámka

$$\forall \alpha, \beta : 1 \leqslant \alpha < \beta < \omega_1 : \Sigma^0_\alpha(X) \subset \Pi^0_\beta(X).$$

Poznámka
If X contains homeomorphic copy of 2^ω then all inclusions are strict.

We denote $Borel(X)$ as σ-algebra of Borel sets (σ-algebra generated by $\Sigma^0_1(X)$).

Poznámka (Also non-trivial theorem)

$$Borel(X) = \bigcup_{1\leqslant\alpha<\omega_1} \Sigma^0_\alpha(X) = \bigcup_{1\leqslant\alpha<\omega_1} (X) = \bigcup_{1\leqslant\alpha<\omega_1} \Delta^0_\alpha(X).$$

$$A_n \in \bigcup_{1\leqslant\alpha<\omega_1} \Sigma^0_\alpha(X) \implies \exists 1 \leqslant \alpha_n < \omega_1 : A_n \in \Sigma^0_{\alpha_n}(X) \implies A_n \in \Sigma^0_{\sup\{\alpha_n | n\in\omega\}} \implies \bigcup_{n\in\omega} A_n \in \Sigma^0_{\sup\{\alpha_n, n\in\omega\}}$$

Poznámka
$F_\sigma = \Sigma^0_2$, $G_\delta = \Pi^0_2$, $F_{\sigma\delta} = \Pi^0_3$, $G_{\delta\sigma} = \Sigma^0_3$.

$\Sigma^0_\alpha(X)$ is closed under countable union and $\Pi^0_\alpha(X)$ under countable intersection.

Věta 1.7

X be metrizable, $1 \leqslant \alpha < \omega_1$. Then

1. *$\Sigma^0_\alpha(X)$ is closed under finite intersection;*
2. *$\Pi^0_\alpha(X)$ is closed under finite union.*

Důkaz

„1.“ Firstly for $\alpha = 1$, it is trivial. Then let $A, B \in \Sigma^0_\alpha(X)$, $\alpha > 1$. Then $A = \bigcup_{n\in\omega} A_n$, $A_n \in \Pi^0_{\alpha_n}(X)$, $\alpha_n < \alpha$, $B = \bigcup_{m\in\omega} B_m$, $B_m \in \Pi^0_{\beta_m}(X)$, $\beta_n < \alpha$. $A \cap B = \bigcup_{(m,n)\in\omega^2} A_n \cap B_m$, $A_n \cap B_m \in \Pi^0_{\max\{\alpha_n,\beta_n\}}(X) \implies A \cap B \in \Sigma^0_\alpha(X)$. „2.“ $\impliedby$ de Morgan and 1. □

Věta 1.8

X be metrizable, $A \subset Z \subset X$, $1 \leqslant \alpha < \omega_1$. Then $A \in \Sigma^0_\alpha(Z) \Leftrightarrow$ there exists $\tilde{A} \in \Sigma^0_\alpha(X)$: $A = \tilde{A} \cap Z$. Similarly for $\Pi^0_\alpha, \Delta^0_\alpha$.

Důkaz

Firstly $\alpha = 1$ from definition of subspace. Then assume that it is all true for all $\beta < \alpha$. We want to prove it for α. „ $\implies$ “:

$$A \in \Sigma^0_\alpha(Z) \implies A = \bigcup A_n, A_n \in \Pi^0_{\beta_n}(Z), \beta_n < \alpha \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X) : \tilde{A}_n \cap Z = A_n.$$

$$\tilde{A} = \bigcup \tilde{A}_n \in \Sigma^0_\alpha(X), \tilde{A} \cap Z = Z \cap \bigcup \tilde{A}_n = \bigcup (Z \cap \tilde{A}_n) = \bigcup A_n = A.$$

„ $\impliedby$ “:

$$\tilde{A} \in \Sigma^0_\alpha(X), A = \tilde{A} \cap Z \implies \exists \tilde{A}_n \in \Pi^0_{\beta_n}(X), \beta_n < \alpha, \bigcup \tilde{A}_n = \tilde{A}.$$

$$\tilde{A} \cap Z \in \Pi^0_{\beta_n}(Z) \implies A = \tilde{A} \cap Z = \left(\bigcup \tilde{A}_n\right) \cap Z = \bigcup \left(\tilde{A}_n \cap Z\right) = \bigcup A_n \in \Sigma^0_\alpha(Z).$$

□

Věta 1.9

X, Y be metric spaces, $f : X \to Y$ is continuous. If $A \in \Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$, $\Delta^0_\alpha(Y)$) then $f^{-1}(A) \in \Sigma^0_\alpha(X)$ ($\Pi^0_\alpha(X)$, $\Delta^0_\alpha(Y)$).

Důkaz

$\alpha = 1$ trivial. Assume it holds true for $\Sigma^0_\beta(Y)$, $\Pi^0_\beta(Y)$, $\beta < \alpha$, and we want to show for $\Sigma^0_\alpha(Y)$ ($\Pi^0_\alpha(Y)$). Let $A \in \Sigma^0_\alpha(Y)$, $\alpha > 1 \implies A = \bigcup_{n\in\omega} A_n$, $A_n \in \Pi^0_{\beta_n}(Y)$, $\beta_n < \alpha$.

$$f^{-1}(A) = f^{-1}(\bigcup A_n) = \bigcup \underbrace{f^{-1}(A_n)}_{\Pi^0_{\beta n}(X)} \in \Sigma^0_\alpha(X),$$

$$f^{-1}(Y \backslash A) = f^{-1}(Y) \backslash f^{-1}(A) = X \backslash f^{-1}(A).$$

□

Věta 1.10 (Borel classes in PTS)

X, Y be PTS, $A \in \Sigma^0_\alpha(X)$, $\alpha \geqslant 3$ (resp. $A \in \Pi^0_\alpha(X)$, $\alpha \geqslant 2$), $B \subset Y$. If B and A are homeomorphic then $B \in \Sigma^0_\alpha(Y)$ (resp. Π^0_α).

Důkaz
$f: A \to B$ is homeomorphism A onto B. The theorem above (name ?) there is extension $\tilde{f}$ of f, $\tilde{f}$ is homeomorphism $\tilde{A}$ onto $\tilde{B}$, $A \subset \tilde{A}$, $B \subset \tilde{B}$, $\tilde{A} \in \Pi_2^0(X)$, $\tilde{B} \in \Pi_2^0(Y)$. Then $B \in \Sigma_\alpha^0(\tilde{B})$ (because $B = (f^{-1})^{-1}(A)$). From the theorem above, $\exists \hat{B} \in \Sigma_\alpha^0(Y) : B = \hat{B} \cap \tilde{B} \in \Sigma_\alpha^0(Y) \impliedby \alpha \geqslant 3$. □

1.4 Analytic sets

Definice 1.4

X PTS, $A \subset X$. We say that A is analytic set in X if there exists PTS Y and continuous mapping $\varphi: Y \to X$ such that $\varphi(Y) = A$.

We denote collection of analytic subsets of X as $\Sigma_1^1(X)$. We say that A is coanalytic in X if $X \backslash A \in \Sigma_1^1(X)$ and we denote this collection as $\Pi_1^1(X)$. $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Například

$$Q = \{\alpha \in 2^\omega | \exists n \in \omega\ \forall j \geqslant n : \alpha_j = 0\} = 2^{<\omega} \in \Sigma_2^0(2^\omega) \backslash \Pi_2^0(2^\omega)$$

TODO?

Poznámka
X PTS, $F: X \to \mathcal{K}(X)$ by $F(x) = \{x\}$. Then F is continuous, $F^{-1}(\mathcal{K}(A)) = A \implies$ if $\mathcal{K}(A) \in \Sigma_\alpha^0(\mathcal{K}(X))$ (Π_α^0, Δ_α^0) then $A \in \Sigma_\alpha^0(X)$ (Π_α^0, Δ_α^0). A open $\implies \mathcal{K}(A)$ is open, A is closed $\implies \mathcal{K}(A)$ is closed. $\mathcal{K}(\bigcap A_n) = \bigcap \mathcal{K}(A_n)$. Thus for $A \in \Pi_2^0(X) : \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X))$. $A \in \Sigma_1^0(X)$ ($\Pi_1^0(X)$, $\Pi_2^0(X)$) $\Leftrightarrow \mathcal{K}(A) \in \Sigma_1^0(\mathcal{K}(X))$ ($\Pi_1^0(\mathcal{K}(X))$, $\Pi_2^0(\mathcal{K}(X))$).

Věta 1.11

X PTS, $|X| > \omega$. Assume $I \subset \mathcal{K}(X)$, I is σ-ideal ($K \in I, L \subset K \implies L \in I$; $K_n \in I, \bigcup K_n \in \mathcal{K}(X) \implies \bigcup K_n \in I$). If $I \in \Pi_2(\mathcal{K}(X))$, then $I \in \Sigma_1^1(\mathcal{K}(X))$.

Důsledek
$A \notin \Pi_2^0(X) \implies \mathcal{K}(A) \notin \Sigma_1^1(\mathcal{K}(X))$.

Poznámka
$A \in \Pi_1^1(X), \mathcal{K}(A) = \mathcal{K}(X) \backslash \{K \in \mathcal{K}(X) | \exists x \in (X \backslash A) \cap K\}$ $\{(K, x) \in \mathcal{K}(X) \times X | x \in K\}$ is closed.

Definice 1.5

$$\Sigma_1^1(X) := \{A \subset X | \exists Y \text{ PTS}, f : Y \to X \text{ continuous} : f(Y) = A\}.$$

Poznámka
- $\varnothing \in \Sigma_1^1$;
- $\Pi_2^0(X) \subset \Sigma_1^1(X)$, $f = \mathrm{id}$;
- X, Z PTS, $\psi : X \to Z$ continuous, $A \in \Sigma_1^1(X) \implies \psi(A) \in \Sigma_1^1(Z)$;
- $\Sigma_{n+1}^1(X) = \{A \subset X | \exists Y \text{ PTS}, \psi : Y \to X \text{ continuous}, B \in \Pi_n^1(X), A = \psi(B)\}, n \in \omega \backslash \{0\}$;
- $\Pi_n^1(X) = \{A \subset X | X \backslash A \in \Sigma_n^1(X)\}$, $\Delta_n^1(X) = \sum_n^1(X) \cap \Pi_n^1(X)$;
- $\bigcup_{n \in \mathbb{N}} \Sigma_n^1(X) = \bigcup_{n \in \mathbb{N}} \Pi_n^1 = \bigcup_{n \in \mathbb{N}} \Delta_n^1(x) = \mathbb{P}(X)$;
- $\#\mathbb{P}(X) \leqslant 2^\omega$, $\mathbb{P}(X)$ is closed under continuous images and inverse images;
- $\Sigma_1^1(X) = \{A \subset X | \exists \psi : \omega^\omega \to X \text{ continuous} : \psi(\omega^\omega) = A\}$; Y PTS, $f : Y \to X : f(Y) = A$, $g : \omega^\omega \to Y : g(\omega^\omega) = Y$, g, f are constant. So $\psi = f \circ g$.

Věta 1.12

X PTS, $A_n \in \Sigma_1^1(X)$, $n \in \omega$. Then $\bigcup_{n \in \omega} A_n, \bigcap_{n \in \omega} A_n \in \Sigma_1^1(X)$.

Důsledek
Similar for $\Pi_1^1(X)$.

Důkaz
„Union“: Assume $A_n \neq \varnothing$, $n \in \omega \implies \varphi_n : \omega^\omega \to X : \varphi_n(\omega^\omega) = A_n$ continuous. Define $\varphi : \omega^\omega \to X$ by $\varphi(\nu_0, \nu_1, \ldots) = \varphi_{\nu_0}(\nu_1, \nu_2, \ldots)$. „$\varphi$ is continuous“: $\nu^j \to \nu \implies \exists n_0 \in \omega \ \forall j \geqslant n_9 : \nu_0^j = \nu_0$.

$$\lim_{j \to \infty} \varphi(\nu^j) = \lim_{j \to \infty} \varphi_{\nu_0^j}(\nu_1^j, \nu_2^j, \ldots) = \lim_{j \to \infty} \varphi_{\nu_0}(\nu_1^j, \ldots) = \varphi_{\nu_0}(\nu_1, \ldots) = \varphi(\nu).$$

„$\varphi(\omega^\omega) = \bigcup_{n \in \omega} A_n$“:

$$x \in \bigcup A_n \implies \exists n \in \omega : x \in A_n \implies \exists \nu \in \omega^\omega : \varphi_n(\nu) = x \implies \varphi(n^\wedge \nu) = x.$$

$$x \in \varphi(\omega^\omega) \implies \exists \tilde{\nu} \in \omega^\omega : \varphi(\tilde{\nu}) = x \implies x = \varphi_{\tilde{\nu}_0}(\tilde{\nu}_1, \ldots) \implies z \in A_{\tilde{\nu}_0} \implies x \in \bigcup A_n.$$

□

Poznámka (Intersection)
WLOG: $A_n \neq \varnothing$, $n \in \omega$. $Y := (\omega^\omega)^\omega$, Y PTS by the theorem above (first item). $\varphi_n : \omega^\omega \to$

X, meh that $\varphi_n(\omega^\omega) = A_n$.

$F := \{y = (y_0, y_1, \dots) \in Y | \forall n, m \in \omega : \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap_{n,m\in\omega} \{y \in Y | \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap))n, m \in$

intersection of closed, so F is closed and is PTS.

„$\varphi_0 \circ \pi_0(F) = \bigcap_{n\in\omega} A_n$“:

$x \in \varphi_0 \circ \pi_0(F) \implies \exists y \in F : x = \varphi_0(y_0) = \varphi_1(y_1) = \varphi_2(y_2) = \dots \implies x \in \bigcap_{n\in\omega} A_n.$

$x \in \bigcap A_n \implies \exists y_0, y_1, \dots \in \omega^\omega : \varphi_0(y_0) = x, \varphi_1(y_1) = x, \dots \implies y = (y_0, y_1, \dots) \in F, \varphi_0 \circ \pi_0(y) = x \implies$

Poznámka
$\Sigma^1_1(X)$ is not closed under complement: $\sigma(\Sigma^1_1(X)) \supset \Sigma^1_1(X) \cup \Pi^1_1(x)$.

$Borel(X) \subset \Sigma^1_1(X) \cap \Pi^1_1(X) = \Delta^1_1(X)$.

Věta 1.13

X, Y PTS, $A \in \Sigma^1_1(X)$ (respective $\Pi^1_1(X)$), $B \subset Y$, A and B are homeomorphism. Then $B \in \Sigma^1_1(Y)$ (resp. $\Pi^1_1(Y)$).

Důkaz
For Σ^1_1 trivial. $A \in \Pi^1_1(X)$, $\varphi : A \to B$ homeomorphism. Then from the theorem above, $\exists \tilde{A} \in \Pi^0_2(X), \tilde{B} \in \Pi^0_2(Y)$ and $\tilde{\varphi} : \tilde{A} \to \tilde{B}$ homeomorphism extending φ, $A \subset \tilde{A}$, $B \subset \tilde{B}$. Then $\tilde{A} \backslash A = (X \backslash A) \cap \tilde{A} \in \Sigma^1_1(X) \implies \tilde{B} \backslash B \in \Sigma^1_1(Y)$. $B = Y \backslash (\tilde{B} \backslash B \cup Y \backslash \tilde{B}) \in \Pi^1_1(Y)$. □

Věta 1.14

X PTS. Then $Borel(X) \subset \Delta^1_1(X)$.

Důkaz
Trivial. □

1.5 Luzin theorem

Věta 1.15 (Luzin)

X PTS, $A_1, A_2 \in \Sigma^1_1(X)$, $A_1 \cap A_2 = \emptyset$. Then there exists $B \in Borel(X)$, such that $A_1 \subset B \subset X \backslash A_2$.

Důsledek
X PTS. $\Delta_1^1(X) = Borel(X)$.

Důkaz
$\Delta_1^1(X) \subseteq Borel(X)$ we already have.

$$A \in \Delta_1^1(X) \implies A \in \Sigma_1^1(X), X \backslash A \in \Sigma_1^1 \implies \exists B \in Borel(X) : A \subset B \subset X \backslash (X \backslash A) = A \implies A = B =$$

□

Lemma 1.16

$C_n, D_n \subset X$, $n, m \in \omega$ and $\forall n, m \in \omega$ we can separate C_n, D_m by some Borel set. Then we can separate $\bigcup_{n\in\omega} C_n$ and $\bigcup_{m\in\omega} D_m$ by Borel set.

Důkaz
Let $B_{n,m} \in Borel(X)$ separating C_n from D_m ($C_n \subset B_{n,m} \subset X \backslash D_m$). Put $B := \bigcup_{n\in\omega} \bigcap_{m\in\omega} B_{n,m}$. □

Důkaz (Luzin theorem)
Assume $A_1, A_2 \neq \emptyset$. Then exists $\varphi_1, \varphi_2 : \omega^\omega \to X$ $\varphi_i(\omega^\omega) = A_i$. We assume A_1 can't be separated from A_2 by any Borel set.

$$A_i = \varphi_i(\omega^\omega) \implies A_i = \bigcup_{n\in\omega} \varphi_i(\mathcal{N}(n)) \implies \exists \nu_0, \mu_0 \in \omega : \varphi_i(\mathcal{N}(\mu_0)) \text{ can't be separated from } \varphi_2(\mathcal{N}(\nu_0)).$$

We use lemma again and obtain $\mu, \nu \in \omega^\omega$ such that $\forall k \in \omega : \varphi_1(\mathcal{N}(\mu/k))$ can't be separated from $\varphi_2(\mathcal{N}(\nu/k))$

$$\varphi_1(\mu) \in A_1, \varphi_2(\nu) \in A_2 \implies \varphi_1(\mu) \neq \varphi_2(\nu) \implies \exists G_1, G_2 \text{ open }, G_1 \cap G_2 = \emptyset$$

such that $\varphi_1(\mu) \in G_1$, $\varphi_2(\nu) \in G_2$, φ_1, φ_2 are continuous $\implies \exists k \in \omega : \varphi_1(\mathcal{N}(\mu/k)) \subset G_1$, $\varphi_2(\mathcal{N}(\nu/k)) \subset G_2$ which is continuous. □

Například
$\{f \in C([0,1]) | \forall x \in [0,1] : f'(x) \in \mathbb{R}\} \in \Pi_1^1 \backslash \Delta_1^1$.

$$\{f \in C([0, 2\pi) | \text{ Fourier series converges to } f \text{ for every } x \in [0, 2\pi]\} \in \Pi_1^1 \backslash \Delta_1^1.$$

$$\{K \in \mathcal{K}([0,1]) | |K| \leqslant \omega\}, \{K \in \mathcal{K}(\mathbb{R}) | K \subset \mathbb{Q}\} \in \Pi_1^1 \backslash \Delta_1^1.$$

Například
$\{x \in X | \exists y \in Y : (x, y) \in B\} \in \Sigma_1^1(X)$.

TODO!!!

Lemma 1.17

(X,τ) PTS, $F \in \Pi^0_1(X)$. Let τ_F be topology generated by $\tau \cup \{F\}$. Then τ_F is Polish, $F \in \Delta^0_1(X,\tau_F)$, $\Delta^1_1(X,\tau_F) = \Delta^1_1(X,\tau)$.

Důkaz

(X,τ_F) is homeomorphic with $((X\backslash F)\times\{0\}) \cup (F\times\{1\}) \subset X\times\{0,1\}$ which is PTS and those two subsets are G_δ in $X\times\{0,1\}$, so, (X,τ_F) is Polish.

$$\Delta^1_1(((X\backslash F)\times\{0\})\cup(F\times\{1\})) \leftrightarrow \Delta^1_1(\tau_F) = \left\{A\cup B | A \in \Delta^1_1(X\backslash F,\tau), B\in\Delta^1_1(F,\tau)\right\} \subset \Delta^1_1(\tau) \subset \Delta^1_1(\tau_F).$$

□

Lemma 1.18

(X,τ) PTS, $(\tau_n)_{n\in\omega}$ Polish topology, $\tau\subset\tau_n$, $n\in\omega$. Then topology τ_∞ generated by $\bigcup_{n\in\omega}\tau_n$ is polish. If $\forall n\in\omega : \tau_n \subset \Delta^1_1(\tau)$, then $\Delta^1_1(\tau) = \Delta^1_1(\tau_\infty)$.

Důkaz

Set $X_n := (X,\tau_n)$, $\varphi : X \to \prod_{n\in\omega} X_n$, $\varphi(x) = (x,x,x,x,\ldots)$. φ is homomorphism (X,τ_∞) on $\varphi(X)$. ($U\in$ base of $\tau_\infty \implies \exists n\in\omega : U\in\tau_n, \varphi(U) = x_1\times x_2\times\ldots\times x_{n-1}\times U\times x_{n+1}\times\ldots\cap\varphi(X)$ is open. $\varphi(X)\in\Pi^0_1(\prod X_n) \implies \varphi(X)$ PTS $\implies (X,\tau_\infty)$ PTS.)

$$\Delta^1_1(\tau) = \Delta^1_1(\tau_\infty) \impliedby \sigma(\sigma(M)) = \sigma(M).\ (\tau_\infty\subset\Delta^1_1(\tau) = \Delta^1_1(\tau_n).)\ \tau_\infty \subset \bigcup\Delta^1_1(\tau_n).$$ □

Věta 1.19

(X,τ) PTS, $A\in\Delta^1_1(X,\tau)$. There exists polish topology τ_A such that $\tau\subset\tau_A$, $\Delta^1_1(\tau_A) = \Delta^1_1(\tau)$ and $A\in\Delta^0_1(X,\tau_A)$.

Důkaz

$\mathcal{S} := \{D\in\Delta^1_1(X)|$ exists polish topology $\tau_D\supset\tau$ and $\Delta^1_1(\tau_D) = \Delta^1_1(\tau), D\in\Delta^0_1(X,\tau_D)\}$. We know that $\tau\subset\mathcal{S}$ and that $\mathcal{S}$ is closed under complements. Moreover, $\mathcal{S}$ is closed under countable union ($A_n\in\mathcal{S}\to\tau_{A_n}\to\tau_\infty = \tau_{\bigcup A_n}$). So $\mathcal{S} = \Delta^1_1(X,\tau)$. □

Lemma 1.20

X, Y PTS. $f : X\to Y$ Borel. Then graph$(f)\in\Delta^1_1(X\times Y)$.

Důkaz

Fix compatible complete metric ϱ on Y. U_n, $n \in \omega$, countable colection of open balls with diam $< 2^{-n}$ covering Y.

$$\operatorname{graph} f \stackrel{?}{=} \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U \in \Delta^1_1(X \times Y).$$

„$\subseteq$“: $(x, y) \in \operatorname{graph}(f) \Leftrightarrow f(x) = y \implies \forall n \in \omega\ \exists U \in U_n : y \in U \wedge x \in f^{-1}(U) \implies (x, y) \in \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U$.

„$\supseteq$“: $(x, y) \notin \operatorname{graph}(f) \Leftrightarrow f(x) \neq y \implies \exists n \in \omega : \varrho(f(x), y) > \frac{1}{n} \implies \exists n \in \omega \forall U \in U_n \neg(x \in f^{-1}(U) \cap y \in U) \implies (x, y) \notin \bigcap_{n\in\omega} \bigcup_{U\in U_n} f^{-1}(U) \times U$. □

Poznámka (Notation)
If f is Borel, we write $f \in \Delta^1_1$.

Věta 1.21

X, Y PTS, $f \in \Delta^1_1(X \times Y)$. If $A \in \Delta^1_1(X)$ and $f|_A$ is injective, then $f(A) \in \Delta^1_1(Y)$.

Důkaz
If $f : X \to Y$ is injective, then $f(A) = \prod_Y(\text{graph}(f) \cap A \times Y) \in \Sigma^1_1(Y)$.

$$Y \backslash F(A) = \prod_Y(\text{graph}(f) \cap (X \backslash A) \times Y) \in \Sigma^1_1(Y) \implies f(A) \in \Delta^1_1(Y).$$

Assume f is continuous, $A \in \Pi^0_1(X)$. From the theorem above $A \subset \omega^\omega$, $B_s := f(\mathcal{N}(s) capA)$. $\forall s \in \omega^{<\omega}\ \forall i, j, i \neq j : B_{s^\wedge i} \cap B_{s^\wedge j} = \varnothing \impliedby f$ is injection. $\forall s \in \omega^{<\omega} : B_s = \bigcup_{i\in\omega} B_{s^\wedge i}$.

From Luzin separation theorem, there exists (by induction) $(B'_s)_{s\in\omega^{<\omega}}$ of Borel sets:

$$\forall s \in \omega^{<\omega}\ \forall i, j \in \omega, i \neq j B'_{s^\wedge i} \cap B'_{s^\wedge j} = \varnothing.$$

(separation $B_{s^\wedge i}$, $\bigcup_{j<i} B_{s^\wedge j} \cup \bigcup_{l>i} B_{s^\wedge l}$) $\forall s \in \omega^{<\omega} : B_s \subset B'_s$.

Put: $B^*_\varnothing = Y$, $B^*_{s^\wedge j} = B_{s^\wedge j} \cap \overline{B_{s^\wedge j}} \cap B^*_s$. $\forall s \in \omega^{<\omega} : B^*_s \in \Delta^1_1(Y), B_s \subset B^*_s \subset \overline{B_s}$, $B^*_{s^\wedge j} \subset B^*_s$, $B^*_{s^\wedge j} \cap B^*_{s^\wedge i} = \varnothing$, $s \in \omega^{<\omega}$, $i, j \in \omega, i \neq j$. We proof: $f(A) \stackrel{?}{=} \bigcup_{s\in\omega^{<\omega}} \bigcap_{k\in\omega} B^*_{s/k} = \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s \in \Delta^1_1(Y)$.

$B^*_s, s \in \omega^{<\omega}, B^*_s \in \Delta^1_1(Y)$. $f(A) = \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s$:

„$\subseteq$“: $x \in f(A) \implies \exists \nu \in A : f(\nu) = x$. Then $x \in f(\mathcal{N}_{\nu/k} \cap A) = B_{\nu/k} \subset B^*_{\nu/k}$, $k \in \omega$ $\implies x \in \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s$.

„$\supseteq$“: $x \in \bigcap_{k\in\omega} \bigcup_{s\in\omega^k} B^*_s \implies \forall k \in \omega\ \exists \nu^k \in \omega^\omega : x \in B^*_{\nu/k}$. $\exists \nu \in \omega^\omega : \nu^k = \nu$, $k \in \omega$. $\implies \forall k \in \omega\ \exists \nu \in \omega^\omega : f(\mathcal{N}(\nu/k) \cap A) \neq \varnothing$. $\implies \exists \nu^k \in \mathcal{N}(\nu/k) \cap A, \nu^k \to \nu \implies \nu \in A$ (A is closed). $f(\nu) = x$? Assume $f(\nu) \neq x \implies \exists U$ neighbourhood of $f(\nu)$, such that $x \notin \overline{U} \implies$ (f is continuous) $\exists k_0 \in \omega : x \in B^*_{\nu/k_0} \subset \overline{f(\mathcal{N}_{\nu/k_0}) \cap A} = \overline{B_{\nu/k_0}} \subset \overline{U}$ which is contradiction.

a) Let f is continuous and $A \in \Delta^1_1(X)$. On X we find Polish topology τ_A such that $A \in \Delta^0_1(\tau_A)$, $\tau \subset \tau_A$ (so f is continuous with respect to τ_A), $\Delta^1_1(\tau) = \Delta^1_1(\tau_A)$.

b) Let $f \in \Delta^1_1$. Then $f(A) = \pi_Y(\text{graph}(f) \cap A \times Y)$. Observe that π_Y is injective on (graph$(f) \cap A \times Y$ if f is injective on A. $\square$

Věta 1.22

X, Y PTS, $f \in \Delta^1_1(X \times Y)$.

1. $A \in \Sigma^1_1(X) \implies f(A) \in \Sigma^1_1(Y)$;
2. $B \in \Sigma^1_1(Y) \implies f^{-1}(B) \in \Sigma^1_1(X)$;
3. $B \in \Pi^1_1(Y) \implies f^{-1}(B) \in \Pi^1_1(X)$.

Důkaz
„1.“: $f(A) = \pi_Y((\mathrm{graph}(f) \cap A \times Y)$ is continuous image of Σ^1_1 set.

„2.“: $f^{-1}(B) = \pi_X((\mathrm{graph}(f) \cap X \times B)$ is continuous image of Σ^1_1 set.

„3.“: $f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(Y \backslash B)$. □

1.6 Standard Borel spaces (SBS)

Definice 1.6 (Standard Borel space (SBS))

Measurable space $(X, \mathcal{S})$ is called standard Borel space (SBS) if there exists Polish topology τ on X such that $\Delta^1_1(X, \tau) = \mathcal{S}$.

Definice 1.7 (Effros Borel space)

Let X be PTS and $\mathcal{F}(X) := \Pi^0_1(X)$. Let $\mathcal{S}$ be σ-algebra generated by sets of form $\{F \in \mathcal{F}(X) | F \cap U \neq \varnothing\} =: M_U$, where $U \in \Sigma^0_1(X)$. $(\mathcal{F}(X), \mathcal{S})$ is called Effros Borel space.

Věta 1.23

X PTS. Then $(\mathcal{F}(X), \mathcal{S})$ is SBS.

Důkaz
Without proof. □

Poznámka
X be measurable compact. Then $\mathcal{F}(X)$ can be equipped by Vietoris topology.

Příklad
$SB := \{Y \in \mathcal{F}(C([0,1])) | Y$ is Banach subspace of $C([0,1])\}$. If we restrict Effros σ-algebra on SB then SB is SBS.

$$SD = \{Y \in SB | Y \text{ has separable dual}\},$$
$$NU = \{Y \in SB | Y \text{ is not universal}\},$$
$$REFL = \{Y \in SB | Y \text{ is reflexive}\},$$
$$NL_1 = \{Y \in SB | Y \text{ does not contain } l_1\}.$$

2 Regularity of Σ^1_1 sets

2.1 Sets with Baire property (BP)

Definice 2.1 (Baire property (BP))

X TS, $A \subset X$ has Baire property (BP) in X if there exists open $U \subset X$ and set of 1. category $M \subset X$ such that $A = U \triangle M := (U \backslash M) \cup (M \backslash U)$. Collection of all sets with BP we denote as $Baire(X)$.

Věta 2.1

X TS. Then $Baire(X)$ is σ-algebra and $Baire(X) \supset \Delta^1_1(X)$.

Důkaz

1. „$Baire(X) \supset \Sigma^0_1(X)$“ trivial. 2. „$Baire(X)$ is σ-algebra“: a) „$A \in Baire(X) \overset{?}{\implies} X \backslash A \in Baire(X)$“: $A \in Baire(X) \implies \exists G \in \Sigma^0_1(X)$ and M meagre such that $A = G \triangle M$.

$$X \backslash A = X \backslash (G \triangle M) = (X \backslash G) \triangle M = (\operatorname{int}(X \backslash G) \cup (X \backslash G) \backslash \operatorname{int}(X \backslash G)) \triangle M =$$

$$= (V \cup M_1) \triangle M_2 = V \triangle M \qquad (M = M_1 \triangle M_2).$$

b) „$A_n \in Baire(X) \overset{?}{\implies} \bigcup A_n \in Baire(X)$“: $A_n = G_n \triangle M_n$, $G_n \in \Sigma^0_1(X)$, M_n meager. $M_n' = G_n \cap M_n$ (meager), $M_n'' = M_n \backslash G_n$ (meager).

$$\bigcup A_n = \bigcup ((G_n \backslash M_n') \cup M_n'') = ((\bigcup G_n) \backslash M''') \cup \bigcup M_n'',$$

where $M''' \subset \bigcup_{n \in \omega} M_n'$. □

Lemma 2.2

X TS, $A \subset X$. Then A is meager iff $\forall x \in A\ \exists V \in \Sigma^0_1(X)$ such that $x \in V$ and $A \cap V$ is meager.

Důkaz
„ $\Longrightarrow$ “ trivial. „ $\Longleftarrow$ “ $\mathcal{U}$ denote as maximal collection of disjoint Σ_1^0 sets such that $U \cap A$ is meager for $U \in \mathcal{U}$. We show that $A \cap \bigcup \mathcal{U}$ is meager, $X \backslash \bigcup \mathcal{U}$ is nowhere dense, so meager.

„$X \backslash \bigcup \mathcal{U}$ is nowhere dense“: By contradiction we assume that there exists $\emptyset \neq V \in \Sigma_1^0(X)$, $V \subset X \backslash \bigcup \mathcal{U}$. Now we have 2 cases: $A \cap V = \emptyset \implies V \in \mathcal{U}$ contradiction, or $A \cap V \neq \emptyset \implies \exists x \in A \cap V \implies \exists W \in \Sigma_1^0(X) : x \in W, W \cap A$ is meager $\implies x \in W \cap V \neq \emptyset, W \cap V \cap A$ is meager $\implies W \cap V \in \mathcal{U}$ contradiction.

„$\bigcup \mathcal{U} \cap A$ is meager“: $\mathcal{U} := \{U_\alpha | \alpha \in I\}$, $U_\alpha \cap A$ meager $\implies$ exist? $F_n^\alpha \in \Pi_1^0(X)$ nowhere dense: $U_\alpha \cap A \subset \bigcup F_n^\alpha \subset \overline{U_\alpha}$. We show that $\bigcup_{\alpha \in I} F_n^\alpha$ is nowhere dense:

$$a) \bigcup_{\alpha \in I} U_\alpha \backslash F_n^\alpha \in \Sigma_1^0(X), \quad (\bigcup_{\alpha \in I} U_\alpha \backslash F_n^\alpha) \cap (\bigcup_{\alpha \in I} F_n^\alpha) = \emptyset \impliedby F_n^\alpha \subset \overline{U_\alpha}, \quad \overline{U_\alpha} \cap U_\beta = \emptyset, \alpha \neq \beta$$

So $\mathcal{U}$ is disjoint collection, so $\bigcup_{\alpha \in I} U_\alpha F_n^\alpha \cap \overline{\bigcup_{\alpha \in I} F_n^\alpha} = \emptyset$.

$$\implies \overline{\bigcup_{\alpha \in I} F_n^\alpha} \subset (\bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha)) \cup (X \backslash \bigcup \mathcal{U}).$$

b) We assume $\exists V \in \Sigma_1^0(X)$, $V \neq \emptyset$, $V \subset \overline{\bigcup_{\alpha \in I} F_n^\alpha}$.

$$? \implies V \not\subset X \backslash \bigcup \mathcal{U} \overset{a)}{\implies} V \cap \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \neq \emptyset \implies \exists \alpha \in I : V \cap U_\alpha \neq \emptyset.$$

$$a) \implies V \cap U_\alpha \subset \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \overset{\mathcal{U} \text{ disjoint}}{\implies} V \cap U_\alpha \subset F_n^\alpha \ \lightning.$$

□

TODO!!!

2.2 Solecky theorem

Poznámka (Notation)
X PTS, $\mathcal{I} \subset \Pi_1^0(X)$.

$$\mathcal{I}^{ext} := \left\{ A \subset X | \exists \mathcal{F} \subset \mathcal{I}, |\mathcal{F}| = \omega, A \subset \bigcup \mathcal{F} \right\}.$$

Například
$\mathcal{I} = \{A \subset X || A| < \omega\}$, $\mathcal{I} = \{A \subset X | A \text{ nowhere dense}\}$.

$$\mathcal{I}^{perf} = \left\{A \subset X | A \neq \varnothing, \forall U \in \Sigma_1^0(X) : U \cap A \neq \varnothing \implies U \cap A \notin \mathcal{I}^{ext}\right\}.$$

$$\operatorname{Ker} A := A \backslash \bigcup \left\{U \subset X | U \in \Sigma_1^0(X), U \cap A \in \mathcal{I}^{ext}\right\} =$$

$=$ max perfect subset of $A \impliedby X$ has countable base.

$$MGR(A) = \{Z \subset A | Z \text{ be meager in } A\}, \qquad A \subset X.$$

Věta 2.3 (Solecki)

X PTS, $A \in \Sigma_1^1(A)$, $\mathcal{I} \subset \Pi_1^0(X)$. $A \notin \mathcal{I}^{ext} \implies \exists H \in \Pi_2^0(X), H \subset A, H \notin \mathcal{I}^{ext}$

Lemma 2.4 (For proof of Solecki)

$A \in \Sigma_1^1(X) \backslash \mathcal{I}^{ext}$. Then there exists Suslin scheme $(A_s)_{s \in \omega^{<\omega}}$ of closed subsets of X such that:

$$A_\varnothing = \varnothing, \qquad a_s A_s \subset A, \qquad A_s \neq \varnothing \implies A \cap A_s \in \mathcal{I}^{perf}, \overline{A \cap A_s} = A_s, \qquad \overline{\bigcup_{n \in \omega} A_{s^\wedge n}} = A_s.$$

Důkaz

$(H_s)_{s \in \omega^{<\omega}}$ closed subsets of X, decreasing ($H_s \supset H_{s^\wedge n}, n \in \omega$), $A = a_s H_s \impliedby A \in \Sigma_1^1(X)$.
For $s \in \omega^{<\omega} : L_s := a_t H_{s^\wedge t}, A_s := \overline{\operatorname{Ker}(L_s)}$.

1. $A_\varnothing = \overline{\operatorname{Ker}(L_\varnothing)} = \overline{\operatorname{Ker}(A)} \neq \varnothing \impliedby A \notin \mathcal{I}^{ext}$ (X has countable base).

2. $H_s \searrow \implies L_s \subset H_s \implies \operatorname{Ker}(L_s) \subset H_s \overset{H_s \in \Pi_1^0(X)}{\implies} A_s \subset H_s \implies a_s A_s \subset a_s H_s = A$.

3. $\operatorname{Ker}(L_s) \subset A_s, L_s \subset A : (A = \bigcup_{|s|=k} L_s, k \in \omega \impliedby H_s \searrow) \implies \operatorname{Ker}(L_s) \subset A_s \cap A$, $\overline{\operatorname{Ker}(L_s)} = A_s$.
$$A_s = \overline{\operatorname{Ker}(L_s)} \subset \overline{A_s \cap A} \subset \overline{A_s} = A_s.$$
Assume $A_s \neq \varnothing \implies A \cap A_s \neq \varnothing$. $U \in \Sigma_1^0(X)$, $U \cap A \cap A_s \neq \varnothing \implies U \cap \operatorname{Ker}(L_s) \neq \varnothing \implies U \cap \operatorname{Ker}(L_s) \notin \mathcal{I}^{ext}. \implies U \cap A \cap A_s \notin \mathcal{I}^{ext}$.

4. $\bigcup_{n \in \omega} A_{s^\wedge n} \subset A_s \impliedby (H_s \searrow \implies L_s \searrow \implies A_s \searrow)$. Let $U \in \Sigma_1^0(X)$, $U \cap A_s \neq \varnothing \implies U \cap \operatorname{Ker}(L_s) \neq \varnothing \implies U \cap L_s \notin \mathcal{I}^{ext}$.
$$L_s = \bigcup_{n \in \omega} L_{s^\wedge n} \implies \exists n_0 \in \omega : U \cap L_{s^\wedge n_0} \notin \mathcal{I}^{ext} \implies U \cap \operatorname{Ker}(L_{s^\wedge n_0}) \notin \mathcal{I}^{ext} \implies U \cap A_{s^\wedge n_0} \neq \varnothing.$$

□

Důkaz (Solecki theorem, not in exam)
$A \in \Sigma_1^1(X) \backslash \mathcal{I}^{ext}$, $(A_s)_{s \in \omega^{<\omega}}$ from the previous lemma. There are 2 cases:

„1st case $\exists s \in \omega^{<\omega}\ \exists U \in \Sigma_1^0(X) : A_s \cap U \neq \varnothing \wedge MGR(A_s \cap U) \subset \mathcal{I}^{ext}$“: Put $\tilde{A} := A \cap A_s \cap U$. Then from the third item of the previous lemma $\tilde{A} \in \mathcal{I}^{perf}$, $\tilde{A} \in \Sigma_1^1(X)$. $A_s \neq \varnothing$,

$A \cap A_s \in \mathcal{I}^{perf}$, $U \cap A_s \neq \varnothing \implies U \cap A \cap A_s \neq \varnothing \impliedby \overline{A \cap A_s} = A_s$.

$$\implies \tilde{A} \in Baire(A_s \cap U) \impliedby (A_s \cap U \in \Pi^0_2(X)), A_s \cap U \text{ PTS.}$$

$$\tilde{A} = H \cup M, H \in \Pi^0_2(A_s \cup U), M \in MGR(A_s \cap U) \subset \mathcal{I}^{ext} \implies H \notin \mathcal{I}^{ext}, H \subset A.$$

„2nd case $\forall s \in \omega^{<\omega}\ \forall U \in \Sigma^0_1(X), U \cap A_s \neq \varnothing : MGR(A_s \cap U) \backslash \mathcal{I}^{ext} \neq \varnothing$“: Notation: $\mathcal{F} \subset 2^X : \mathcal{F}^d := \overline{\bigcup \mathcal{F}} \backslash \bigcup \left\{ \overline{F} | F \in \mathcal{F} \right\}$. Choose CCM $\leqslant 1$ on X. We will inductively construct $\varphi : \omega^{<\omega} \to \omega^{<\omega}$, $U_s \subset X$, $s \in \omega^{<\omega}$ such that:

1. $|\varphi(s)| = |s|$ TODO
2. $U_s \in \Sigma^0_1(X)$;
3. $\operatorname{diam} U_s \leqslant 2^{-|s|}$;
4. $\lim_{n\to\infty} \operatorname{diam}(U_{s^\wedge n}) = 0$;
5. $\forall t, s \in \omega^{<\omega}, t < s, t \neq s : \overline{U_s} \subset U_t$;
6. $\forall s \in \omega^{<\omega}\ \forall m, n \in \omega, m \neq n : U_{s^\wedge m \cap U_{s^\wedge n}} = \varnothing$;
7. $U_a \cap A_{\varphi(s)} \neq \varnothing$;
8. $\{U_{s^\wedge n} | n \in \omega\}^d \notin \mathcal{I}^{ext}$;
9. $\{U_{s^\wedge n} | n \in \omega\}^d \subset U_s$;
10. (9. + 5.) $\overline{\bigcup_{n\in\omega} U_{s^\wedge n}} \subset U_s$.

Construction: $\varphi(\varnothing) = \varnothing$, $U_\varnothing$ be arbitrary open subset of X: $U_\varnothing \cap A_\varnothing \neq \varnothing$. Then all items are satisfied. We assume that U_s, φ_s are constructed for all $s \in \omega^{<\omega}$, $|s| \leqslant N \in \omega$. Let $s \in \omega^{<\omega}$, $|s| \leqslant N$ be arbitrary. From 7th item $U_s \cap A_{\varphi(s)} \neq \varnothing$, $MGR(A_{\varphi(s)} \cap U_s) \notin \mathcal{I}^{ext}$ $\implies \exists K \subset A_{\varphi(s)} \cap U_s, K \in \Pi^0_1(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s$, $K \notin \mathcal{I}^{ext}$. Because

$$\exists L \in MGR(A_{\varphi(s)} \cap U_s) \backslash \mathcal{I}^{ext} \implies \exists H \in \Sigma^0_2(X), H \supset L, H \in \Sigma^0_2(A_{\varphi(s)} \cap U_s), H \notin \mathcal{I}^{ext},$$

so $H = \bigcup F_n$, $F_n \in \Pi^0_1(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s \implies \exists n_0 \in \omega : F_{n_0} = K \notin \mathcal{I}^{ext}$.

Find $D \subset A_{\varphi(s)} \cap U_s$: D is discrete in $X \backslash K$. $D \cap K = \varnothing$. $\overline{D} = K \cup D$. Let $\{y_n\} \subset K$, $\overline{\{y_n\}} = K$, and every element of $\{y_n\}$ repeats infinitely many times. Find $x_n \in (A_{\varphi(s)} \cap U_s) \backslash K$ such that $\varrho(x_n, y_n) < \frac{1}{n}$ (it exists $\impliedby$ K is nowhere dense in $A_{\varphi(s)} \cap U_s$). Then $D = \{x_n | n \in \omega\}$, $D \cap K = \varnothing$, $\overline{D} \supset \overline{D \cup \{y_n | n \in \omega\}} \supset D \cup K$, $x \notin K \cup D \implies \exists n \in \omega \backslash \{0\}$: $\varrho(x, K) > \frac{1}{n} \implies \#(B(x, 1/2n) \cap D) \leqslant 2n \implies x \notin \overline{D}$. $\implies \overline{D} = D \cup K$, D is discrete in $X \backslash K$. Assume $x_n \neq x_m$, $n \neq m$.

Define $U_{s^\wedge n}$ as open ball with center x_n: $\overline{U_{s^\wedge n}} \subset U_s$. $U_{s^\wedge n} \cap U_{s^\wedge m} = \varnothing$ (D is discrete), $\operatorname{diam} U_{s^\wedge n} \leqslant 2^{-|s|-1}$, $\lim_{n\to\infty} \operatorname{diam} U_{s^\wedge n} = 0$, $\overline{\bigcup_{n\in\omega} U_{s^\wedge n}} \backslash \bigcup_{n\in\omega} \overline{U_{s^\wedge n}} = \underline{\{U_{s^\wedge n} | n \in \omega\}} =$ $K \impliedby \overline{U_{s^\wedge n}} \cap K = \varnothing$, $\overline{D} = K \cup D$. $x_n \in A_{\varphi(s)} \implies U_{s^\wedge n} \cap A_{\varphi(s)} \neq \varnothing$, $\overline{\bigcup_{k\in\omega} A_{\varphi(s)^\wedge k}} =$ $A_{\varphi(s)} \implies \exists k \in \omega : U_{s^\wedge n} \cap A_{\varphi(s)^\wedge k} \neq \varnothing$.

Put $\varphi(s^\frown n) = \varphi(s)^\frown k$. And then all items are satisfied. $H = \bigcap_{n\in\omega} \bigcup_{|s|=n, s\in\omega^{<\omega}} U_s \in \Pi^0_2(X)$, $H \subset A$, $H \notin \mathcal{I}^{ext}$.

$$H := \bigcap_{n\in\omega} \bigcup \{U_s | s \in \omega^{<\omega}, |s| = n\} \in \Pi^0_2(\Longleftarrow 2.).$$

$H \subset A$?, $H \notin \mathcal{I}^{ext}$. „$H \subset A$“: 5. and 6. $\implies H = \bigcup_{\nu\in\omega^\omega} \bigcap_{n\in\omega} \bigcup_{\nu/m} = a_s U_s$?

$$x \in H \Leftrightarrow \forall n \in \omega\ \exists \omega^{<\omega}, |s| = n : x \in U_s \overset{5.\wedge 6.}{\Leftrightarrow} \exists s \in \omega^\omega\ \forall n \in \omega x \in U_{s/n} \Leftrightarrow a_s U_s.$$

$$\big(3. \implies \operatorname{diam}(U_{\nu/n}) \leqslant 2^{-n}, \big(7. \implies U_{\nu/m} \cap A_{\varphi(\nu/n)} \neq \emptyset\big)\big) \implies \bigcap_{n\in\omega} \overline{U_{\nu/n}} \subset \bigcap_{n\in\omega} A_{\varphi(\nu/n)} \subset A \implies H \subset A$$

„$H \notin \mathcal{I}^{ext}$“: $\forall \nu \in \omega^\omega : \bigcap_{n\in\omega} U_{\nu/n} \neq \emptyset \Longleftarrow 3. \wedge 5.$, so $\forall s \in \omega^{<\omega} : U_s \cap H \neq \emptyset$. Assume $H \subset \bigcup_{m\in\omega} F_m$, $F_m \in \mathcal{I}$. $H \in G_\delta \implies \exists n_0 \in \omega : F_{n_0}$ is not meager in $H \implies \exists U \neq \emptyset$ open in X: $\emptyset \neq U \cap H \subset F_{n_0}$. Let $x \in U \cap H$. Then there exist $\nu \in \omega^\omega\ \forall n \in \omega : x \in U_{\nu/n}$ $\implies \exists m_0 \in \omega : U_{\nu/m_0} \subset U \implies \emptyset \neq U_{\nu/m_0} \cap H \subset F_{n_0}$. Denote $\nu/m_0 =: s$. For contradiction assume 9. $\wedge$ 3. $\overset{?}{\implies} \{U_{s^\frown n}, n \in \omega\}^d \subset F_{n_0}$ which contradicts 8. So 9. $\implies$ $\{U_{s^\frown n}, n \in \omega\}^d \subset U_s$,

$$x \in \{U_{s^\frown n}, n \in \omega\}^d \implies \exists n_k \nearrow, x_k \in U_{s^\frown n_k} : x_k \to x, y_k \in U_{s^\frown n_k} \cap H \subset F_{n_0} \implies$$

$$\implies 4. \implies y_k \to x \implies x \in F_{n_0}.$$ □

TODO!!!

3 Infinite games

3.1 Baire definitions

Definice 3.1

Assume $A \neq \emptyset$, $X \subset A^\omega$. In game $G(X)$, there are 2 players I, II and those players choose $a_i, i \in \omega, a_i \in A$:

$$I : a_0, a_2, a_4, \ldots$$

$$II : a_1, a_3, a_5, \ldots$$

Player I wins $\equiv (a_i) \in X$, otherwise player II wins.

Strategy for I is mapping $\mathcal{S} : A^{<\omega} \to A^{<\omega}$ such that $\forall s \in A^{<\omega} : |\mathcal{S}(s)| = |s| + 1$ and $\forall s, t \in A^{<\omega}$, $s < t$: $\mathcal{S}(s) < \mathcal{S}(t)$.

Definice 3.2 (Notation)

$\sigma \subset A^{<\omega}$ be tree iff $\forall s, t \in A^{<\omega}$, $s < t : t \in \sigma \implies s \in \sigma$

Let σ be tree, $s \in \sigma$. Then s is leaf iff $\forall n \in A : s^\frown n \notin \sigma$.

Let σ be tree. Then σ is prunned $\equiv$ σ does not have leaves ($\forall s \in \sigma : \exists a \in A : s^\frown a \in \sigma$).

$$[\sigma] = \{\nu \in A^\omega | \forall n \in \omega : \nu/n \in \sigma\}, \qquad [\sigma] \in \Pi_1^0(A^\omega).$$

$\forall F \in \Pi_1^0(A^\omega)$ $\exists!$ prunned tree $\sigma \subset A^{<\omega}$: $[\sigma] = F$.

Strategy for I is prunned tree $\sigma \subset A^{<\omega}$ such that $\sigma \neq \emptyset$, $(a_0, a_1, \dots, a_{2j}) \in \sigma \implies \forall a \in A : (a_0, \dots, a_{2j}, a) \in \sigma$, and $(a_0, \dots, a_{2j+1}) \in \sigma \implies \exists! a \in A : (a_0, \dots, a_{2j+1}, a) \in \sigma$.

Definice 3.3

Strategy σ for I is winning $\Leftrightarrow$ I wins if the follows this strategy $[\sigma] \subset X$.

Poznámka
In Game $G(A, X)$ at most one player has winning strategy. It can happen (ZFC) that nobody has winning strategy.

Definice 3.4 (Game with rulse)

$T \subset A^{<\omega}$ be prunned tree, $X \subset [T]$.

$$I : a_0, a_2, a_4, \dots$$

$$II : a_1, a_3, a_5, \dots$$

such that $\forall n \in \omega : (a_0, \dots, a_n) \in T$ (T is tree of rules). Other notions are similar.

Poznámka
Assume $X = \{x \in A^\omega | x \in X \cap [T] \cup (\exists n \in \omega : x/n \notin T$ and the least $n \in \omega : x/n \notin T$ is odd$)\}$ then I (resp. II) has wining strategy in $G(X')$ $\Leftrightarrow$ I (resp. II) has winning strategy in $G(T, X)$.

Například
$SG(A, B_0, B_1)$. Let S, T be nonempty prunned trees on ω, $A \subset [S]$, $B_0, B_1 \subset [T]$.

$$I : x(0), x(1), x(2), \dots$$

$$II : y(0), y(1), y(2), \dots$$

$x(i), y(i) \in \omega, x/n \in S, y/n \in T$. Player II wins $\Leftrightarrow$ $(x \in A \implies y \in B_0) \wedge (x \notin A \implies y \in B_1)$.

I has winning strategy $\Leftrightarrow$ $\exists f : [T] \to [S]$ continuous: $f(B_0) \cap A = \emptyset$, $f(B_1) \subset A$ $\Leftrightarrow$ $f^{-1}(A)$ separates B_1 from B_2.

II has winning strategy $\Leftrightarrow \exists g : [S] \to [T]$ continuous: $g(A) \subset B_0$, $g(A^c) \subset B_1$.