

# 1 Dynamické systémy

## Definice 1.1 (Dynamický systém)

$(\varphi, \Omega)$ ,  $\Omega \subset \mathbb{R}^n$  otevřená,  $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$   $\varphi(t, x)$ .

- $\varphi(0, x) = x$ ;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- $\varphi$  je spojitý.

## Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geq 0\}$  je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leq 0\}$  je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$  je plný orbit.

## Definice 1.3 (Pozitivně, negativně a úplně invariantní)

$(\varphi, \Omega)$  dynamický systém,  $M \subset \Omega$ .

$M$  je pozitivně invariantní  $\equiv \forall x \in M : \gamma^+(x) \subset M$ .

$M$  je negativně invariantní  $\equiv \forall x \in M : \gamma^-(x) \subset M$ .

$M$  je úplně invariantní  $\equiv \forall x \in M : \gamma(x) \subset M$ .

*Poznámka*

$\gamma^+(x_0)$  je pozitivně invariantní,  $\gamma^-(x_0)$  je negativně invariantní a  $\gamma(x_0)$  je úplně invariantní.

## Definice 1.4

$$\omega(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow \infty : \varphi(t_k, x_0) \rightarrow y\},$$

$$\alpha(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow -\infty : \varphi(t_k, x_0) \rightarrow y\}.$$

*Poznámka* (To je ekvivalentní)

$$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0 \forall T > 0 \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon\}.$$

## Lemma 1.1

$$\omega(x_0) = \bigcap_{\tau \geq 0} \overline{\gamma^+(\tau, x_0)}.$$

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*Důkaz*„ $\subseteq$ “:  $y \in \omega(x_0)$ :  $\forall \varepsilon > 0 \forall T \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon$ . Chceme:

$$\forall \tau \geq 0 \forall \varepsilon > 0 \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geq 0 \forall \varepsilon > 0 \exists s \geq \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

$$\text{„}\supseteq\text{“: } \forall \tau \geq 0 \ y \in \overline{\gamma^+(\tau, x_0)} \implies$$

$$\implies \forall \varepsilon \exists s \geq \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

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## Věta 1.2 (Vlastnosti $\omega$ -limitní množiny)

Nechť  $(\varphi, \Omega)$  je dynamický systém,  $x_0 \in \Omega$ . Potom

1.  $\omega(x_0)$  je uzavřená, úplně invariantní.
2. Pokud  $\gamma^+(x_0)$  je relativně kompaktní v  $\mathbb{R}^n$ , pak  $\omega(x_0) \neq \emptyset$ ,  $\omega(x_0)$  je kompaktní, souvislá.

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*Důkaz*

1.  $\omega(x_0)$  je průnik uzavřených množin, tedy uzavřená.  $y \in \omega(x_0) \exists t_k \nearrow \infty \varphi(t_k, x_0) \rightarrow y$ .

$$s_k = t_k + t \quad \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0))$$

$$t_k \rightarrow \infty, \varphi \text{ spojitá} \quad \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \rightarrow \varphi(t, y)$$

2.  $\exists K \subset \mathbb{R}^n$  kompaktní  $\gamma^+(x_0) \subset K$ . a) pokud  $t_n \geq 0, t_n \rightarrow \infty \{\varphi(t_n, x_0)\}_{n=1}^\infty$  omezená posloupnost  $\implies \exists \{t_{n_k}\}_{k=1}^\infty \subset \{t_n\}_{n=1}^\infty$ , podposloupnost,  $\exists y \in \Omega \varphi(t_{n_k}, x_0) \rightarrow y$ . Pak  $y \in \omega(x_0)$ .

b)  $\omega(x_0)$  je tedy úplná a omezená, takže kompaktní. c) ať  $\omega(x_0)$  je nesouvislá, tedy  $\omega(x_0) \subseteq U \cup V$ ,  $U, V$  otevřené disjunktní neprázdné,  $U, V \subseteq K$ . Vezměme  $y \in \omega(x_0) \cap U$ ,  $z \in \omega(x_0) \cap V$ . Nechť  $t_n$  je posloupnost taková, že  $\varphi(t_{2n}, x_0) \rightarrow y$ ,  $\varphi(t_{2n+1}, x_0) \rightarrow z$ ,  $t_{2n} < t_{2n+1}$ ,  $\varphi(t_{2n}, x_0) \in U$ ,  $\varphi(t_{2n+1}, x_0) \in V$ .  $F = K \setminus (U \cup V)$  uzavřená, tedy  $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$ . Tedy  $\{\varphi(s_n, x_0)\}$  je omezená posloupnost  $\implies \exists$  podposloupnost konvergující k  $w \in F$ . □

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## Definice 1.5 (Topologická konjugovanost)

$(\varphi, \Omega)$ ,  $\psi, \Theta$  dynamické systémy.  $\exists : \Omega \rightarrow \Theta$  homeomorfismus (bijekce, spojitá, spojitá inverze):

$$\forall x \in \Omega \forall t \in \mathbb{R} \quad h(\varphi(t, x)) = \psi(t, h(x)).$$

*Poznámka*

Dá se zobecnit ještě zobrazováním časů.

### Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0, (\varphi, \Omega)$  příslušný dynamický systém.  $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, y(0) = 0$  a  $(\psi, \Theta)$  je

příslušný dynamický systém. Potom  $(\varphi, \Omega), (\psi, \Theta)$  jsou lokálně topologicky konjugované ( $\exists U$  okolí  $x_0 \in \Omega$  a  $V$  okolí  $\mathbf{o} \in \mathbb{R}^n$  taková, že  $\exists g : U \rightarrow V$  homeomorfismus  $g(\varphi(t, x)) = \psi(t, g(x))$   $\forall x \in U, \forall t : \varphi(t, x) \in U$ ).

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*Důkaz*

BÚNO  $f_1(x_0) = \alpha \neq 0$  (první souřadnice funkce  $f$ ) a  $x_0 = \mathbf{o}$ . Buď  $\tilde{V}$  okolí  $\mathbf{o} \in \mathbb{R}^n$   $G : \tilde{V} \rightarrow \mathbb{R}^n, G(y_1, \dots, y_n) = \varphi(y, (0, y_2, \dots, y_n))$ . Chceme ukázat, že  $G$  je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_1} \Big|_{(0, \dots, 0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \dots, y_n)) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(y_1(0, y_2, \dots, y_n))) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(0, y_2, \dots, y_n)) \Big|_{y_2=0, \dots, y_n=0} = f(\varphi(0, \dots, 0)) = f(\mathbf{o}) = \alpha$$

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_j} \Big|_{(0, \dots, 0)} = \lim_{h \rightarrow 0} \frac{G(0, \dots, h, \dots, 0) - G(0, \dots, 0)}{h} = \lim_{h \rightarrow 0} \frac{(0, \dots, h, \dots, 0)^T - (0, \dots, 0)^T}{h} = (0, \dots, 1, \dots, 0)^T$$

Tedy  $\nabla G(0, \dots, 0)$  je „jednotková matice, až na to, že  $a_{11}$  je  $\alpha$ “, tudíž podle věty o inverzi funkce  $\exists V \subseteq \tilde{V}$  okolí  $0, \exists U$  okolí bodu  $x_0$  tak, že  $G : V \rightarrow U$  je homeomorfismus. Položme  $g = G^{-1}$ .

Nyní už stačí  $g(\varphi(t, x_0)) = \psi(t, g(x_0)) \forall x_0 \in U \forall t : \varphi(t, x_0) \in U. \varphi(t, x_0) = G(\psi(t, g(x_0)))$

3.  $x \in U = G(V) \exists y \in V x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

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### Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \rightarrow \Omega, f \text{loc.Lip.}$$

$$\exists V : \Omega \rightarrow \mathbb{R}, \text{ bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leq l\} - \text{bounded}$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^n \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leq 0 \quad \forall x \in \Omega_l.$$

$$R = \{x \in \Omega_l \mid \dot{V}_f(x) = 0\}, \quad M = \{y \in R \mid \gamma^+(y) \subset R\}.$$

Then  $\forall x \in \Omega_l : \omega(x) \subset M$ .

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Důkaz

Let  $x \in \Omega_l$ .  $\forall y \in \omega(x) \exists t_k \nearrow \infty : x(t_k) \rightarrow y$ .  $\varphi(t, x_0) = x(t)$ .

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leq 0.$$

$V(x(t)) \searrow$  and  $\exists C : \forall x \in \Omega : V(x) > -C$  so  $\exists \lim_{t \rightarrow \infty} V(x(t)) = c$ .

So  $\exists c \forall y \in \omega(x_0) V(y) = c$ .  $V(x(t_k)) \rightarrow V(y) = c$ .

$$\gamma^+(y) \subset \omega(x_0) \quad V(\varphi(t, y)) = c \quad \forall t \geq 0 \implies$$

$$\implies \frac{d}{dt}V(\varphi(t, y)) = 0.$$

└  $\gamma^+(y) \subset R$  in particular,  $y \in R$ . Hence  $y \in M$ . □

## 2 Poincaré-Bendixson theory

### Věta 2.1 (Poincaré-Bendixson)

Let  $p \in \Omega$ ,  $\Omega$  open connected.  $\omega(p)$  doesn't contain stat points and  $\gamma^+(p)$  is relatively compact ( $\gamma^+(p)$  is compact). Then  $\omega(p) = \Gamma$ -periodic orbit.

### Věta 2.2 (Bendixson-Dulas)

$\Omega$ -simply connected ( $\forall$  closed Jordan curve  $\gamma$  in  $\Omega$ ,  $\text{int}(\gamma) \subset \Omega$ ).  $\exists B : \Omega \rightarrow \mathbb{R} : (\text{div } Bf)(x) = \frac{\partial Bf_1}{\partial x_1}(x_1, x_2) + \frac{\partial Bf_2}{\partial x_2}(x_1, x_2) > 0$  for almost every  $x \in \Omega$ . Then  $x' = f(x)$  doesn't have nontrivial periodic solutions.

### Definice 2.1 (Transverzála)

$\Sigma$  segment on a line such that  $\forall p \in \Sigma : \Sigma \nparallel f(p)$ .

### Lemma 2.3

$\Sigma$  transversála,  $p \in \Sigma \subset \Omega$ . Then  $\exists \tilde{U}$  neighborhood of  $p$ .  $\exists \Delta > 0$  such that

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U \quad \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

Důkaz

Use Th. of rect.

□

### Lemma 2.4

Let  $p \in \Omega$  and assume that  $|\gamma^+(p) \cap \Sigma| \geq 3$ , i. e.  $\exists t_1 < t_2 < t_3 \varphi(t_j, p) \in \Sigma, j = 1, 2, 3$ . Then  $\varphi(t_2, p)$  lie between  $\varphi(t_1, p)$  and  $\varphi(t_3, p)$ .

TODO!!!

TODO!!!

## 2.1 Controllability

### Definition 2.2 (Control theory)

$$x' = f(x, u), f : \Omega \times U, \Omega \subset \mathbb{R}^n, U \subset \mathbb{R}^n,$$

$$u \in \mathcal{U} := \{u : [0, T] \rightarrow \mathbb{R}^n | \text{measurable}, \|u\|_\infty < \infty\} = L^\infty(0, T, \mathbb{R}^n).$$

( $\mathcal{U}$  is admissible functions).

### Definition 2.3 (Linear task)

$$x' = Ax + Bu, A, B \in \mathbb{R}^{n \times m}, m < n.$$

### Definition 2.4

$$x_0 \xrightarrow[u(0)]{t} 0 \text{ iff } x(0) = x_0, x(t) = 0.$$

### Definition 2.5 (Area of controllability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in L^\infty(0, t, \mathbb{R}^n) : x_0 \xrightarrow[u(0)]{t} 0 \right\}$$

### Definition 2.6 (Kalman matrix)

$$\mathcal{K}(A, B) := (B | AB | A^2 B | \dots | A^{n-1} B)$$

### Věta 2.5

For linear problem  $\mathcal{R}(t) = \text{LO}(g_1, g_2, \dots, g_{n-m})$ , where  $\mathcal{K}(A, B) = (g_1 | g_2 | \dots | g_{n-m})$

**Tvrzení 2.6** (Observation)

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

$$x_0 \xrightarrow[u(0)]{t} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = - \int_0^t e^{-As}Bu(s)ds \quad (KO)$$

**Lemma 2.7** (1)

$$A^k \in \text{LO}(I, A, A^2, \dots, A^{n-1}), k \in \mathbb{N}_0$$

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Cayley-Hamilton.  
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Důkaz

1)  $\mathcal{R}(t)$  is vector subspace of  $\mathbb{R}^n$  from definition  $x_0 + x_1 \xrightarrow[(u_1+u_2)(0)]{t} 0, \alpha x \xrightarrow[\alpha u(0)]{t} 0$ .

2) We want  $\mathcal{R}(t)^\perp = (\text{LO}(g_1, \dots, g_n))^\perp$ . „ $\supseteq$ “:  $p \in (\text{LO}(g_1, \dots, g_n))^\perp$ .  $x_0 \in \mathcal{R}(t)$  arbitrary. From KO:

$$0 \stackrel{?}{=} p^T x_0 = - \int_0^t p^T e^{-As} Bu(s) ds = - \int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k B u(s) ds$$

We know  $(p, g_j) = 0$ ,  $p^T g_j = 0$ ,  $p^T \mathcal{K}(A, B) = 0$ ,  $p^T A^k B = 0$ ,  $k \in [n-1]$ . And from lemma 1  $k \in \mathbb{N}$ . „ $\subseteq$ “:  $p \in \mathbb{R}^n$ ,  $p \in \mathcal{R}(t)^\perp$ . We want to prove  $p \perp B, AB, A^2B, \dots, A^{n-1}B$ .  $B = (b_1 | \dots | b_m)$ .  $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$ .  $\varphi \in L^\infty(0, T, \mathbb{R})$ ,  $u(t) = \varphi(t) \cdot \mathbf{e}_j$ , where  $x_0 = - \int_0^t e^{-As} Bu(s) ds$ . We have  $x_0 \xrightarrow[u(0)]{t} 0$ , hence  $x_0 \in \mathcal{R}(t)$ .

$$0 = p^T x_0 = -p^T \int_0^t e^{-As} Bu(s) ds = - \int_0^t p^T e^{-As} b_j \varphi(s) ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have  $p^T e^{-As} b_j \equiv 0$ , we derivate it,  $p^T A^n e^{-As} b_j \equiv 0$ , and set  $s = 0$ .  
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□

Důsledek

$\mathcal{R}(t)$  doesn't depend on time.

**Definice 2.7** (Locally and globally controllable)

Linear problem is called locally controllable, iff  $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 \mid |x_0| < \delta\} \subset \mathcal{R}(t)$ . And globally if  $\mathbb{R}^n = \mathcal{R}(t)$ .

*Důsledek*

Linear problem is controllable  $\Leftrightarrow \text{rank } K(A, B) = n$ .

## 2.2 Observability

### Definition 2.8 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, y = g(x), g : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, m < n.$$

### Definition 2.9

We say that system  $x' = f(x)$  is observable through  $g(\cdot)$  on  $[0, t]$ , iff  $\forall x_1(\cdot), x_2(\cdot) : [0, T] \rightarrow \mathbb{R}^n : g(x_1(t)) = g(x_2(t)) \forall t \in [0, T] \implies x_1(0) = x_2(0)$ .

### Definition 2.10 (Linear observability)

$$x' = Ax, y = Bx, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}.$$

### Věta 2.8

$x' = Ax$  is observable on  $[0, T]$  through  $y = Bx \Leftrightarrow x' = A^T x + B^T u$  is controllable.

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*Důkaz*

(We will prove equivalence with  $\text{rank } \mathcal{K}(A^T, B^T) = n$ .) „ $\Leftarrow$ “: For contradiction

$$\exists x_1(t), x_2(t), [0, T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At} x_0, Bx(t) = B e^{At} x_0 \equiv 0 \quad \forall t \in [0, T].$$

We differentiate it, set  $t = 0$  and get  $Bx_0 = 0, BAx_0 = 0, \dots, BA^{n-1}x_0 = 0$ . So  $x_0^T B^T = 0, \dots, x_0^T (A^T)^{n-1} B^T = 0, x_0^T \mathcal{K}(A^T, B^T) = 0, x_0 \perp \mathcal{K}(A^T, B^T)$ ,  $\nexists$ .

„ $\implies$ “: For contradiction  $\text{rank}(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T \mathcal{K}(A^T, B^T) = 0$ .  
 $x_0^T (A^T)^k B^T = 0 \forall k \in [n-1]$  and from lemma 1  $\forall k \in \mathbb{N}, BA^T x_0 = 0, B e^{At} x_0 = 0$   
 $\forall t \in [0, T]$ .  $\nexists$ . □

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### Věta 2.9

$V \subset \mathbb{R}^n$  neighbourhood of 0,  $U \subset \mathbb{R}^n$  neighbourhood of 0,  $f : V \times U \rightarrow \mathbb{R}^n$   $C^1$  smooth,  $f(0, 0) = 0, \mathcal{U} = \{u : [0, T] \rightarrow U \text{ measurable}\}, A = \nabla_x f(0, 0), B = \nabla_u f(0, 0), \text{rank } \mathcal{K}(A, B) = n$ . Then

$$x' = f(x, u), x(0) = x_0 \text{ is locally controllable } \forall t \in (0, T].$$

*Důkaz*

Fix  $t > 0$ , consider  $x' = Ax + Bu$ . Since  $\text{rank}(A, B) = n$ , the linear problem is globally controllable. Take initial condition  $y_1, \dots, y_n$  linearly independent.

$$\exists u_i \in L^\infty(0, t, \mathbb{R}^n) : y_j \xrightarrow{u_\lambda(0)}^t 0$$

$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  denote by  $u_\lambda(t) = \sum_{j=1}^n \lambda_j u_j(t)$ . We know  $\sum_{j=1}^n \lambda_j y_j \xrightarrow{u_\lambda(0)}^t 0$ .

Step 2:

$$x'_\lambda = f(x_\lambda, u_\lambda), \quad x_\lambda(t) = 0$$

If  $\lambda = 0$ , then  $u_\lambda \equiv 0$ , then  $x_\lambda \equiv 0$ .

$$\psi(\lambda) := x_\lambda(0), \psi : U_\lambda(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We want to prove  $\psi(U_\lambda(0)) \supseteq \tilde{V}$ , for some  $\tilde{V} \subset \mathbb{R}^n$  open,  $0 \in \tilde{V}$ . We will prove that  $\psi$  is  $C^1$  smooth, and that  $\nabla \psi(0)$  is regular (if this is proved, then  $\psi$  is local diffeomorphism).

Step 3:

$$x_\lambda(s) = x_\lambda(t) + \int_t^s f(x_\lambda(s), u_\lambda(s)) ds.$$

Formally differentiate:

$$\frac{\partial x_\lambda(s)}{\partial \lambda_j} = \int_t^s (\nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot \frac{\partial x_\lambda(s)}{\partial \lambda_j} + \nabla_u f(x_\lambda(s), u_\lambda(s))) ds.$$

Denote  $y_{\lambda,j}(s) = \frac{\partial x_\lambda(s)}{\partial \lambda_j}$ .

$$y'_{\lambda,j}(s) = \nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot y_{\lambda,j}(s) + \nabla_u f(x_\lambda(s), u_\lambda(s)) \cdot u_j(s).$$

$$y_{\lambda,j}(t) = 0.$$

Consider  $(LPy) \rightarrow y_{\lambda,j}(\cdot)$ .

$$x_{\lambda+\Delta\lambda}(s) - x_\lambda(s) - \Delta\lambda \cdot y_{\lambda,j}(s) = 0$$

(as in Thm? of differentiability w. r. t. initial condition)

$$\frac{\partial \psi}{\partial \lambda_j}(\lambda = 0) = \frac{\partial x_\lambda(s=0)}{\partial \lambda_j}|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_j.$$

If  $\lambda = 0$ , then  $(LPy)$ :  $y'_{0,j}(s) = Ay_{0,t}(s) + Bu_j(s)$ ,  $y_{0,j}(t) = 0$ . From uniq.:  $y_{0,j}(0) = y_{j,0}$ .

$$\nabla \psi(0) = \left( \frac{\partial \psi}{\partial \lambda_1}(0) \dots \frac{\partial \psi}{\partial \lambda_n}(0) \right) = (y_1, \dots, y_n)$$

regular matrix. □



*Poznámka*

$$x' = Ax + Bu, u \in \mathcal{U} = \{u : [0, T] \rightarrow [-1, 1] \text{ measurable}\}, x(0) = x_0.$$

### Definice 2.11

$$\mathcal{R}(t) = \{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U} \wedge x_0 \rightarrow_{u(0)}^t 0\}.$$

### Definice 2.12

$$u_n \in \mathcal{U}_0: u_n \rightarrow^* u \in \mathcal{U} \equiv \forall f \in L(0, T, \mathbb{R}^n) : \int_0^T f(s)u_n(s)ds \rightarrow \int_0^T f(s)u^*(s)ds.$$

### Věta 2.10 (Alaoglu)

$\mathcal{U}$  is weak-\* sequentially compact (i. e.  $\forall \{u_n\}_{n=1}^\infty \in \mathcal{U} \exists \{u_{n_k}\}$  weakly-\* convergent).

### Věta 2.11

$\mathcal{R}(t)$  convex, symmetric, closed

$$0 < t_1 < t_2 \implies \mathcal{R}(t_1) \subset \mathcal{R}(t_2).$$

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*Důkaz*

Convex:  $x_{01}, x_{02} \in \mathcal{H}(t), \alpha \in [0, 1] \implies \alpha x_{01} + (1 - \alpha)x_{02} \in \mathcal{R}(t).$

$$x(t) = e^{At}x_0 + \int_0^t e^{As}Bu(s)ds. x_{01} \rightarrow_u^t 0 \wedge x_{02} \rightarrow_u^t 0 \Leftrightarrow x_1 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

$$\text{Symmetry: } x_0 \in \mathcal{R}(t) \implies -x_0 \in \mathcal{R}(t), x_0 \rightarrow_u^t 0 \implies -x_0 \rightarrow_u^t 0.$$

Closedness:  $x_{0n} \in \mathcal{R}(t), x_{0n} \rightarrow x_0. x_0 \in \mathcal{R}(t)? \exists u_n(0) \in \mathcal{U}, x_{0n} = - \int_0^t e^{(s-t)A}Bu_n(s)ds \rightarrow - \int_0^t e^{(s-t)A}Bu(s)ds. \text{ WLOG } u_n \rightarrow^* u \in \mathcal{U}. \text{ Then } x_0 \rightarrow_u^t 0.$

$$\mathcal{R}(t_1) \subset \mathcal{R}(t_2) 0 < t_1 < t_2 < T x_0$$

$$\exists u_1 \in \mathcal{U} \quad x_0 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

Define  $u_2(s) = u_1(s)$  if  $0 \leq s \leq t$ , else 0.

└

□

### Definice 2.13 (Area of controllability)

$$\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t).$$

## Věta 2.12

$$\text{rank } \mathcal{K}(A, B) = n \Leftrightarrow \forall t > 0 \mathcal{R}(t) \supseteq U(0),$$

where  $U(0) \subset \mathbb{R}^n$  is some neighbourhood of 0.

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*Důkaz*

„ $\Leftarrow$ “: If  $\exists t > 0 \mathcal{R}(t) \supseteq U(0)$  open,  $0 \in U(0)$ .  $\tilde{\mathcal{R}} : u \in L^\infty, \mathcal{R} : \|u\|_\infty \leq 1$ , then  $\tilde{\mathcal{R}}(t) \supseteq \mathcal{R}(t) \supseteq U(0) \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n \implies \text{rank } \mathcal{K}(A, B) = n$ .

└

„ $\implies$ “:  $\text{rank}(A, B) = n \implies \tilde{\mathcal{R}}(t) = \mathbb{R}^n$ . TODO? □

## Věta 2.13 (Minimal time)

$$x' = Ax + Bu$$

$$\forall x_0 \in \mathcal{R} = \bigcup_{s>0} \mathcal{R}(s)$$

$$\exists t > 0 \exists u(0) \in \mathcal{U} : x_0 \xrightarrow{u}^t 0$$

$$t = \inf \{s > 0 | x_0 \in \mathcal{R}(s)\}.$$

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*Důkaz*

$$t > 0, \exists t_n \searrow t, t_n \in (0, T], \exists u_k \in U, x_0 = - \int_0^{t_n} e^{(t_n-s)A} B u_n(s) ds.$$

Alaoglu: WLOG  $u_n \xrightarrow{*} u \in U$ .

$$x_0 = - \int_0^{t_n} e^{(t-s)A} B u_n(s) ds - \int_0^{t_n} [e^{(t-s)A} - e^{(t_n-s)A}] B u_n(s) ds$$

$$x_0 = - \underbrace{\int_0^t e^{(t-s)A} B u_n(s) ds}_{\xrightarrow{*} \int_0^t e^{(t-s)A} B u(s) ds} - \underbrace{\int_t^{t_n} e^{(t-s)A} B u_n(s) ds}_{\rightarrow 0} - \underbrace{\int_0^T [e^{(t-s)A} - e^{(t_n-s)A}] B u_n(s) ds}_{\rightarrow 0(DLCT)}.$$

└

□

## Definice 2.14 (Bang-bang)

We say that a regulation  $u \in U(0)$  is of type bang-bang, if for almost every  $t \in [0, T]$ :  $u(t) = \pm 1$ .

## Věta 2.14 (Bang-bang)

If  $x_0 \in \mathcal{R}(t) \implies \exists \tilde{u}(0)$  of type bang-bang  $x_0 \xrightarrow{\tilde{u}}^t 0$ .

**Definice 2.15** (Extremal point)

$X$  vector space,  $K \subset X$ .  $x \in K$  is called an extremal point, if it cannot be written as  $x = \frac{y+z}{2}$ ,  $y, z \in K$ ,  $y \neq z$ . We denote  $ex(K)$  the set of extremal points.

**Tvrzení 2.15** (Krein-Milman theorem)

$X$  locally convex vector space,  $K \subset X$ :  $K \neq \emptyset$ ,  $K$  convex and compact. Then  $ex(K) \cap K \neq \emptyset$ .

*Důkaz* (Bang-bang)

$$K = \{u \in \mathcal{U} | x_0 \rightarrow_{u(0)}^t 0\}, \quad X = L^\infty(0, T, \mathbb{R}^n).$$

$K \neq \emptyset$  ( $u \in \mathcal{R}(t)$ ),  $K$  convex,  $K$  is compact (sequential compactness: Alaoglu theorem?  $L'(0, T, \mathbb{R}^n)$  separable  $\implies L^\infty(0, T, \mathbb{R}^n)$  with locale  $*$  topology is metrizable  $\implies$  sequential compactness  $\implies$  compactness.

It remains to check that  $\tilde{u}_j(s) = \pm 1$ ,  $\forall j \in [n]$  for almost every  $s \in (0, t)$ . For contradiction: for some  $j \in [n]$   $\exists E \subset (0, t)$ ,  $\lambda(E) > 0$   $\forall s \in E$   $|\tilde{u}_j(s)| < 1$ . WLOG

$$\exists \varepsilon > 0 \quad \forall s \in E |\tilde{u}_j(s)| < 1 - \varepsilon \cdot \left[ E = \bigcup_{n \in \mathbb{N}} \left\{ s \in (0, t) | |\tilde{u}_j(s)| \leq 1 - \frac{1}{n} \right\} \right].$$

$$x_0 = - \int_0^t e^{-sA} B \tilde{u}(s) ds$$

We want to find  $\varphi \in L^\infty(0, T, \mathbb{R})$  such that:

1.  $\text{supp } \varphi \subset E$ ;
2.  $\int_E e^{-sA} B(0, \dots, 0, \varphi(s), 0, \dots, 0)^T ds = 0$ ;
3.  $\forall s \in E |\varphi(s)| < \varepsilon$ .

Define  $u_1(s) = \tilde{u}(s) + (0, \dots, 0, \varphi(s), 0, \dots, 0)^T$  and  $u_2(s) = \tilde{u}(s) - (0, \dots, 0, \varphi(s), 0, \dots, 0)^T$ . Then  $x_0 \rightarrow_{u_{1,2}(0)}^t 0$ , and  $u_1, u_2 \in \mathcal{K}$ . □

**Věta 2.16** (Global controllability)

We have (LTP)  $x' = Ax + Bu$ ,  $x(0) = x_0$ ,  $u \in \mathcal{U}$ .

1.  $\text{rank } \mathcal{K}(A, B) = n \implies$  (LTP) is locally controllable.
2.  $\text{rank } \mathcal{K}(A, B) = n$  and  $\Re \lambda \leq 0 \quad \forall \lambda\text{-eigenvalues of } A$ . Then (LTP) is globally controllable  $\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t) = \mathbb{R}^n$ .

┌  
Důkaz

1) follows from „In theorem of local controllability for the problem  $x' = f(x, u)$  we could take  $u \in \mathcal{U}$ .“

2a) If  $\forall \lambda$  eigenvalue of  $A$  we have  $\Re \lambda < 0 \implies$  theorem follows from text above: first, set  $u = 0$ . Then we arrive at a neighbourhood of zero.

2b) For contradiction  $x_0 \in \mathbb{R}^n \setminus \mathcal{R}$ .  $\mathcal{R}$  convex  $\exists z_0 \in \partial \mathcal{R}$ ,  $n$  normal vector.  $\forall x_1 \in \mathcal{R} : n^T(x_1 - x_0) \leq 0$ ,  $n^T x_1 \leq n^T x_0 =: M$ .

$$x_1 = - \int_0^t e^{-sA} B u(s) ds$$

$$n^T x_1 = - \int_0^t \underbrace{n^T e^{-sA} B}_{v(s)} u(s) ds$$

$$\tilde{u}(s) := \begin{cases} 0, & v(s) = 0, \\ \frac{-v(s)}{\|v(s)\|_2}, & v(s) \neq 0. \end{cases}$$

If  $v(s) \equiv -$ , then apply  $\frac{d^p}{(ds)^p}$ ,  $n^T A^p e^{-sA} B \equiv 0$ , then  $n^T \mathcal{K}(A, B) = 0$ .  $\nmid$

$$\int_0^\infty \|v(s)\|_2 ds = \infty.$$

If this is true, then  $t_k \nearrow \infty$ ,  $u_k = \tilde{u}|_{[0, t_k]}$ ,  $x_{1,k} = - \int_0^{t_k} e^{-sA} B u_k(s) ds$ .

$$n^T x_{1,k} = - \int_0^{t_k} v^t(s) \cdot \tilde{u}(s) ds = \int_0^{t_k} \|v(s)\|_2 ds \rightarrow \infty. \nmid$$

$v(s)$  is linear combination of  $s^j e^{-s\lambda_p}$ ,  $\Re \lambda_p \leq 0$ . Then  $\int_0^\infty |v(s)| ds = \infty$ . □

## Věta 2.17 (Pontrjagin maximum)

$$x' = Ax + Bu, \|u\|_\infty \leq 1, x(0) = x_0.$$

Let  $x_0 \rightarrow_{u^*(0)}^* 0$ ,  $t^*$  is the minimal. Then  $\exists h \in \mathbb{R}^n \setminus \{\mathbf{0}\} : h^T \cdot e^{-sA} B u^*(s) = \max_{\eta \in [-1, 1]^m} h^T e^{-sA} B \eta$  for almost every  $s \in (0, t^*)$ .

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*Důkaz*

$x_0 \in \partial \mathcal{R}(t^*)$ .

Step 2 – contradiction:  $\exists E \subset (0, t^*), \lambda(E) > 0, \forall s \in E \exists \eta_s \in [-1, 1]^m h^T e^{-sA} B u^*(s) < h^T e^{-sA} B \eta_s$ . But  $x_j(\delta) \in \mathcal{R}(t^* - \delta)$ , hence  $x_0 \in \mathcal{R}(t^* - \delta)$  and  $t^*$  is not minimal.

Step 1:  $x_0 \in \partial \mathcal{R}(t^*)$ . For contradiction  $x_0 \in \text{int } \mathcal{R}(t^*)$ .

$$\exists x_1, \dots, x_{n+1} \in \mathcal{R}(t^*), x_0 \in CO(x_1, \dots, x_{n+1}).$$

$$\exists u_1, \dots, u_{n+1} \in U, x_j \xrightarrow{u_j(\cdot)}^{t^*} 0 \quad \forall j \in [n+1].$$

Let  $\tilde{u}_j(t)$  are the corresponding solutions

TODO!!!

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□

### Věta 2.18 (Pontrjagin)

$x'(f, u), x(0) = x_0, u \in \mathcal{U} = \{u : (0, T) \rightarrow U \subset \mathbb{R}^n\}, T \text{ fixed},$

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s), u(s)) ds \rightarrow \text{maximum}.$$

$f, g, r, \nabla_x f, \nabla_x g, \nabla_x r$  are continuous.

Let  $u$  is a local maximum of this problem (it maximizes  $P$ ), then for  $p$  solving  $(H(x, p, u) := p^T f(x, u) + r(x, u))$ :

$$p' = -\nabla_x H(x, p, u),$$

$$p(T) = \nabla_x g(x(T)),$$

we have

$$H(x, p, u) = \max_{\eta \in U} H(x, p, \eta) \text{ for almost every } t \in (0, T).$$

┌ *Důkaz*

Step one „WLOG  $r = 0$ “: We set

$$x' = f(x, u), \quad x'_{n+1} = r(x, u), \quad x_{n+1}(0) = 0, \quad P[u(\cdot)] = \hat{g}(\hat{x}(T)) = g(x(T)) + x_{n+1}(T).$$

Step 2: Fix  $\tau \in (0, T)$ ,  $\eta \in U$ ,  $u_\varepsilon(T) = \begin{cases} \eta, & t \in (\tau - \varepsilon, \tau), \\ u(t), & \text{elsewhere,} \end{cases}$  and corresponding  $x_\varepsilon(t)$ .

$$u \text{ "best"} \implies P[u_\varepsilon(0)] \leq P[u(0)] \implies g(x_\varepsilon(T)) \leq g(x(T)).$$

$$D = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0+} Dg(x_\varepsilon(T)) \Big|_{\varepsilon=0+} \leq 0$$

$$\nabla_x g(x(T)) \cdot Dx_\varepsilon(T) \Big|_{\varepsilon=0+} \leq 0.$$

Step 2.2:  $x_\varepsilon(t) = x_0 + \int_0^t f(x_\varepsilon(s), u_\varepsilon(s)) ds$ . If  $t < \tau$ , then  $u_\varepsilon \equiv u$ ,  $x_\varepsilon \equiv x$ ,  $Dx_\varepsilon(t) \equiv 0$  on  $[0, t]$ . If  $t > \tau$ , then  $x_\varepsilon(t) =: y(t)$ ,  $y'(t) = f(y(t), u(t))$ ,  $u(\tau) = x_\varepsilon(\tau)$ ,

$$Dx_\varepsilon(t) \equiv z(t) : z' = \nabla_x f(y(t), u(t))z, \quad z(\tau) = Dx_\varepsilon(\tau), \quad \text{variational equation.}$$

Statement:  $z' = A(t)z$ ,  $p' = -A^T(t)p \implies p^T z = \text{const.}$  Proof:  $(p^T z)' = (p^T)'z + p^T z' = -p^T A z + p^T A z = 0$ .

Step 2.3:  $p' = -(\nabla_x f(y(t), u(t)))^T p$ ,  $p(T) = (\nabla_x g(x(T)))^T$ . Then  $p^T(t)z(t)$  constant on  $(\tau, T)$ ,  $p^T(\tau)z(\tau) \leq 0$ .

Step 2.4:  $Dx_\varepsilon(\tau) \Big|_{\varepsilon=0+} \stackrel{?}{=} f(x(\tau), \eta) - f(x(\tau), u(\tau))$ . Then

$$p^T(\tau) (f(x(\tau), \eta) - f(x(\tau), u(\tau))) \leq 0$$

$$\frac{1}{\varepsilon} (x_\varepsilon(\tau) - x(\tau)) = \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), u(s))] ds =$$

$$\frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} [f(x_\varepsilon(s), \eta) - f(x(s), \eta)] ds + \int_{\tau-\varepsilon}^{\tau} [f(x(s), \eta) - f(x(s), u(s))] ds.$$

First converge to zero from Lebesgue theorem about average value. Second to  $f(x(\tau), \eta) - f(x(\tau), u(\tau)) \rightarrow 0$ . □

### Věta 2.19 (Potrjagin for fixed point („fixed finish“))

Same as previous, but  $T$  is not fixed,  $x(T)$  is fixed  $\implies g \equiv 0$  (we don't "rate" final point, because it's the same for all  $u$ ).

## 3 Bifurcation

### Definice 3.1

$x' = \mu - x^2$  is saddle-node bifurcation,  $x' = \mu x - x^2 = x(\mu - x)$  is transcritical bifurcation,  $x' = \mu x - x^3 = x(\mu - x^2)$  is fork bifurcation, in  $x' = x - \sin \mu$  there is no bifurcation.

*Pozorování*

$f(x_0, \mu_0) \neq 0 \implies$  no bifurcation. (From lemma of rect.) (Bifurcation  $\implies f = 0$ .)

*Pozorování*

$$f(x_0, \mu_0) = 0, \sigma(\nabla_x f(x_0, \mu_0)) = \{\lambda_j | \Re \lambda_j \neq 0\}.$$

### Definice 3.2

Point from previous observation is called hyperbolic stationary point.

### Věta 3.1

$f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  be  $C^1$ ,  $(x_0, \mu_0)$  is a hyperbolic stationary point. Then  $\exists \Delta > 0 \exists \delta > 0 \forall \mu \in U_\delta(\mu_0) \exists x = x(\mu) \in U_\Delta(x_0)$ , stationary point  $x(\mu)$  is a hyperbolic stationary point of  $\mu \mapsto x(\mu)$ , which is  $C^1$ .

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*Důkaz*

IFT:

$$f(x_0, \mu_0) = 0 \wedge \nabla_x f(x_0, \mu_0) \text{ regular?} \wedge f \in C^1 \implies x = x(\mu), f(x(\mu), \mu) = 0.$$

Hyperbolic? Eigenvalues of  $A = \nabla_x f(x(\mu), \mu)$ ,  $\det(\lambda I - A(\mu))$  – polynomial in  $\lambda$ ,  $\deg = n$ . □

└

### Věta 3.2

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be  $C^2$  on neighborhood  $(0, 0) \in \mathbb{R}^2$ .

$$f(0, 0) = 0, \quad f_\mu(0, 0) \neq 0, \quad f_x(0, 0) = 0, \quad f_{xx}(0, 0) \neq 0.$$

Then  $f$  has bifurcation at  $(0, 0)$  of the type saddle-node.

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*Důkaz*

Without proof. □

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### Věta 3.3

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be  $C^2$  on neighborhood  $(0, 0) \in \mathbb{R}^2$ .

$$f(0, 0) = 0, \quad f_x(0, 0) = 0, \quad f(0, \mu) = 0 \forall \mu \in U_\delta(0), \quad f_{xx}(0, 0) \neq 0, \quad f_{x\mu}(0, 0) \neq 0.$$

Then  $f$  has bifurcation at  $(0, 0)$  of the type transcritical.

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Důkaz

└ Without proof. □

### Věta 3.4

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be  $C^2$  on neighborhood  $(0, 0) \in \mathbb{R}^2$ .

$$f(0, 0) = 0, \quad f_x(0, 0) \neq 0, \quad f_{xx}(0, 0) = 0, \quad f_{xxx}(0, 0) \neq 0,$$

$$f(0, \mu) = 0 \quad \forall \mu \in U_\delta(0), \quad f_{x\mu}(0, 0) \neq 0.$$

Then  $f$  has bifurcation at  $(0, 0)$  of the type fork.

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Důkaz

└ Without proof. □

### Lemma 3.5 (About division)

$h : U(0, 0) \rightarrow \mathbb{R}$  be  $C^k$  for some  $k \in \mathbb{N}$ .  $h(0, \lambda) = 0 \quad \forall \lambda \in U_\delta(0)$ . Then

$$h(x, \lambda) = xH(x, \lambda), \quad H \in C^{k-1}(U(0, 0), \mathbb{R}).$$

$$H(0, 0) = h_x(0, 0), \quad H_x(0, 0) = \frac{1}{2}h_{xx}(0, 0), \quad H_\lambda(0, 0) = h_{x\lambda}(0, 0),$$

$$H_{xx}(0, 0) = \frac{1}{3}h_{xxx}(0, 0),$$

if  $k$  is sufficiently large.

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Důkaz

$$H(x, \lambda) := \int_0^1 \partial_x h(\sigma x, \lambda) d\sigma.$$

└ □

TODO?

Důkaz (Theorem of transcritical bifurcation)

$f(x, \mu) = xF(x, \mu)$ .  $F_\mu(0, 0) \neq 0$ ?

$$F(x, \mu(x)) = 0? \rightarrow \frac{d}{dt} : \mu'(x) = \frac{-\partial_x F(x, \mu(x))}{\partial_\mu (F(x, \mu(x)))}.$$

$$f_{x\mu}(x, \mu) = F_\mu(x, \mu) + xF_{x\mu}(x, \mu) \implies F_\mu(0, 0) = f_{x\mu}(0, 0) \neq 0.$$

□



### Věta 3.6 (Fork)

$$f \in C^3(U), \quad f(0,0) = f_x(0,0) = f_{xx}(0,0) = 0, \quad f_{xxx}(0,0) \neq 0, \quad f(0,\mu) = 0 \quad \forall (0,\mu) \in U,$$

$$(f_\mu(0,0) = 0), \quad f_{x\mu}(0,0) \neq 0.$$

Then ?

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*Důkaz*

$$\mu'(0) = 0, \quad \mu''(0) = \frac{\dots}{-\partial_\mu F(x, \mu(x))}, \quad \partial_{x,x} F(0,0) = \frac{1}{3} f_{xxx}(0,0) \neq 0.$$

$$\mu''(0) \neq 0 \implies \mu''(x) \text{ doesn't change sign} \implies \mu(x) \text{ has a local extreme at } (0,0).$$

└

□

### Věta 3.7 (?)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f(x, y, \mu), \quad f \in C^2, \quad f(0,0,\mu) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \sigma(\nabla f(0,0,\mu)) = \{\alpha(\mu) \pm i\omega(\mu)\}.$$

$$\alpha(0) = 0, \quad \alpha'(0) \neq 0, \quad \omega(0) \neq 0, \quad \alpha, \omega \in C^1.$$

$$\text{Then } \exists \delta > 0, \varepsilon > 0 : \mu = \mu(a), \quad a \in (0, \varepsilon) \mapsto \mu(a) \in (-\delta, \delta).$$

$$\forall a \in (0, \varepsilon) \exists \text{ nontrivial periodic solution passing through } (a, 0).$$

Důkaz

Rotation:  $x' = d(\mu)x - \omega(\mu)y + f_1(x, y, \mu)$ ,  $y' = \omega(\mu)x + \alpha(\mu)y + f_2(x, y, \mu)$ ,  $f_2(x, y, \mu) = O(x^2 + y^2)$ .

Polar coords:

$$\begin{aligned} x &= r \cos \theta, y = r \sin \theta, x' = g_1(x, y), y' = g_2(x, y), \\ r' \cos \theta - r \sin \theta \cdot \theta' &= g_1, \quad r' \sin \theta + r \cos \theta \cdot \theta' = g_2. \\ r' &= g_1 \cdot \cos \theta + g_2 \sin \theta, \quad r \cdot \theta' = -g_1 \sin \theta + g_2 \cos \theta. \\ r; &= \alpha \cdot r + \underbrace{f_1 \cdot \cos \theta + f_2 \cdot \sin \theta}_{=R}, \quad r\theta' = \omega(\mu) \cdot r + \underbrace{(-f_1 \cdot \sin \theta + f_2 \cdot \cos \theta)}_{=r \cdot Q}. \end{aligned}$$

$$r' = \alpha(\mu)r + R(r, \theta, \mu), R = O(r^2), \quad \theta' = \Omega(\mu) + Q(r, \theta, \mu), Q = O(r).$$

WLOG  $\omega(0) > 0$ .  $\exists \varepsilon, \delta > 0 \forall r \leq \varepsilon \forall \mu \in [-\delta, \delta], \theta'(t) > 0$ .  $r(t) \mapsto \hat{r}(\theta) := r(t(\theta))$ .  $t \mapsto \theta(t)$  is simple  $\implies \exists t = t(\theta)$ .

$$\frac{dr}{d\theta} = \frac{\frac{dr}{dt}}{\frac{d\theta}{dt}} = \frac{\alpha(\mu)r + R}{\omega(\mu) + Q} = \frac{\alpha(\mu)}{\omega(\mu)}r + T(r, \theta, \mu).$$

$$\lambda(\mu) := \frac{\alpha(\mu)}{\omega(\mu)} : r'(\theta) = \lambda(\mu)r(\theta) + T(r, \theta, \mu),$$

$$(r(\theta)e^{-\lambda(\mu)\theta})' = T(r, \theta, \mu)e^{-\lambda(\mu)\theta}$$

$$r(\theta) \cdot e^{-\lambda(\mu)\theta} = r'(\theta_0) \cdot e^{-\lambda(\mu)\theta_0} + \int_{\theta_0}^{\theta} T(r(\psi), \psi, \mu) \cdot e^{-\lambda(\mu)\psi} d\psi.$$

We get  $r(\theta_0) = r(\theta_0 + 2\pi)$  – periodicity. If we denote  $r(\theta_0) = a$ , we get

$$a = a \cdot e^{2\pi\lambda(\mu)} + \int_{\theta_0}^{\theta_0+2\pi} T(r(\psi), \psi, \mu) \cdot e^{-\lambda(\mu)(\psi-\theta_0)} d\psi.$$

□

□