

1 Dynamické systémy

Definice 1.1 (Dynamický systém)

(φ, Ω) , $\Omega \subset \mathbb{R}^n$ otevřená, $\varphi : \mathbb{R} \times \Omega \rightarrow \Omega$ $\varphi(t, x)$.

- $\varphi(0, x) = x$;
- $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$
- φ je spojitý.

Definice 1.2 (Orbit)

$\gamma^+(x_0) = \{\varphi(t, x_0) | t \geq 0\}$ je pozitivní orbit.

$\gamma^-(x_0) = \{\varphi(t, x_0) | t \leq 0\}$ je negativní orbit.

$\gamma(x_0) = \{\varphi(t, x_0) | t \in \mathbb{R}\}$ je plný orbit.

Definice 1.3 (Pozitivně, negativně a úplně invariantní)

(φ, Ω) dynamický systém, $M \subset \Omega$.

M je pozitivně invariantní $\equiv \forall x \in M : \gamma^+(x) \subset M$.

M je negativně invariantní $\equiv \forall x \in M : \gamma^-(x) \subset M$.

M je úplně invariantní $\equiv \forall x \in M : \gamma(x) \subset M$.

Poznámka

$\gamma^+(x_0)$ je pozitivně invariantní, $\gamma^-(x_0)$ je negativně invariantní a $\gamma(x_0)$ je úplně invariantní.

Definice 1.4

$$\omega(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow \infty : \varphi(t_k, x_0) \rightarrow y\},$$

$$\alpha(x_0) = \{y \in \Omega | \exists \{t_k\}_{k=1}^{\infty}, t_k \rightarrow -\infty : \varphi(t_k, x_0) \rightarrow y\}.$$

Poznámka (To je ekvivalentní)

$$\omega(x_0) = \{y \in \Omega | \forall \varepsilon > 0 \forall T > 0 \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon\}.$$

Lemma 1.1

$$\omega(x_0) = \bigcap_{\tau \geq 0} \overline{\gamma^+(\tau, x_0)}.$$

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Důkaz

„ \subseteq “: $y \in \omega(x_0)$: $\forall \varepsilon > 0 \forall T \exists t \geq T : |\varphi(t, x_0) - y| < \varepsilon$. Chceme:

$$\forall \tau \geq 0 \forall \varepsilon > 0 \exists z \in \gamma^+(\tau, x_0) : |y - z| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall \tau \geq 0 \forall \varepsilon > 0 \exists s \geq \tau, z = \varphi(s, x_0) : |y - \varphi(s, x_0)| < \varepsilon.$$

$$\text{„}\supseteq\text{“: } \forall \tau \geq 0 \ y \in \overline{\gamma^+(\tau, x_0)} \implies$$

$$\implies \forall \varepsilon \exists s \geq \tau : |\varphi(s, x_0) - y| < \varepsilon.$$

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□

Věta 1.2 (Vlastnosti ω -limitní množiny)

Nechť (φ, Ω) je dynamický systém, $x_0 \in \Omega$. Potom

1. $\omega(x_0)$ je uzavřená, úplně invariantní.
2. Pokud $\gamma^+(x_0)$ je relativně kompaktní v \mathbb{R}^n , pak $\omega(x_0) \neq \emptyset$, $\omega(x_0)$ je kompaktní, souvislá.

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Důkaz

1. $\omega(x_0)$ je průnik uzavřených množin, tedy uzavřená. $y \in \omega(x_0) \exists t_k \nearrow \infty \varphi(t_k, x_0) \rightarrow y$.

$$s_k = t_k + t \quad \varphi(s_k, x_0) = \varphi(t_k + t, x_0) = \varphi(t, \varphi(t_k, x_0))$$

$$t_k \rightarrow \infty, \varphi \text{ spojitá} \quad \varphi(s_k, x_0) = \varphi(t, \varphi(t_k, x_0)) \rightarrow \varphi(t, y)$$

2. $\exists K \subset \mathbb{R}^n$ kompaktní $\gamma^+(x_0) \subset K$. a) pokud $t_n \geq 0, t_n \rightarrow \infty \{\varphi(t_n, x_0)\}_{n=1}^\infty$ omezená posloupnost $\implies \exists \{t_{n_k}\}_{k=1}^\infty \subset \{t_n\}_{n=1}^\infty$, podposloupnost, $\exists y \in \Omega \varphi(t_{n_k}, x_0) \rightarrow y$. Pak $y \in \omega(x_0)$.

b) $\omega(x_0)$ je tedy úplná a omezená, takže kompaktní. c) ať $\omega(x_0)$ je nesouvislá, tedy $\omega(x_0) \subseteq U \cup V$, U, V otevřené disjunktní neprázdné, $U, V \subseteq K$. Vezměme $y \in \omega(x_0) \cap U$, $z \in \omega(x_0) \cap V$. Nechť t_n je posloupnost taková, že $\varphi(t_{2n}, x_0) \rightarrow y$, $\varphi(t_{2n+1}, x_0) \rightarrow z$, $t_{2n} < t_{2n+1}$, $\varphi(t_{2n}, x_0) \in U$, $\varphi(t_{2n+1}, x_0) \in V$. $F = K \setminus (U \cup V)$ uzavřená, tedy $\exists s_n \in (t_{2n}, t_{2n+1}) : \varphi(s_n, x_0) \in F$. Tedy $\{\varphi(s_n, x_0)\}$ je omezená posloupnost $\implies \exists$ podposloupnost konvergující k $w \in F$. □

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Definice 1.5 (Topologická konjugovanost)

(φ, Ω) , ψ, Θ dynamické systémy. $\exists : \Omega \rightarrow \Theta$ homeomorfismus (bijekce, spojitá, spojitá inverze):

$$\forall x \in \Omega \forall t \in \mathbb{R} \quad h(\varphi(t, x)) = \psi(t, h(x)).$$

Poznámka

Dá se zobecnit ještě zobrazováním časů.

Věta 1.3 (O rektifikaci)

$\dot{x} = f(x), f(x_0) \neq 0, (\varphi, \Omega)$ příslušný dynamický systém. $\dot{y} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, y(0) = 0$ a (ψ, Θ) je

příslušný dynamický systém. Potom $(\varphi, \Omega), (\psi, \Theta)$ jsou lokálně topologicky konjugované ($\exists U$ okolí $x_0 \in \Omega$ a V okolí $\mathbf{o} \in \mathbb{R}^n$ taková, že $\exists g : U \rightarrow V$ homeomorfismus $g(\varphi(t, x)) = \psi(t, g(x))$ $\forall x \in U, \forall t : \varphi(t, x) \in U$).

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Důkaz

BÚNO $f_1(x_0) = \alpha \neq 0$ (první souřadnice funkce f) a $x_0 = \mathbf{o}$. Buď \tilde{V} okolí $\mathbf{o} \in \mathbb{R}^n$ $G : \tilde{V} \rightarrow \mathbb{R}^n, G(y_1, \dots, y_n) = \varphi(y, (0, y_2, \dots, y_n))$. Chceme ukázat, že G je invertibilní na nějakém okolí.

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_1} \Big|_{(0, \dots, 0)} = \frac{\partial \varphi}{\partial t}(t = y_1, (0, y_2, \dots, y_n)) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(y_1(0, y_2, \dots, y_n))) \Big|_{y_1=0, \dots, y_n=0} = f(\varphi(0, y_2, \dots, y_n)) \Big|_{y_2=0, \dots, y_n=0} = f(\varphi(0, 0, \dots, 0)) = f(x_0) = \alpha \neq 0$$

$$\frac{\partial G(y_1, \dots, y_n)}{\partial y_j} \Big|_{(0, \dots, 0)} = \lim_{h \rightarrow 0} \frac{G(0, \dots, h, \dots, 0) - G(0, \dots, 0)}{h} = \lim_{h \rightarrow 0} \frac{(0, \dots, h, \dots, 0)^T - (0, \dots, 0)^T}{h} = (0, \dots, 1, \dots, 0)^T$$

Tedy $\nabla G(0, \dots, 0)$ je „jednotková matice, až na to, že a_{11} je α “, tudíž podle věty o inverzi funkce $\exists V \subseteq \tilde{V}$ okolí 0, $\exists U$ okolí bodu x_0 tak, že $G : V \rightarrow U$ je homeomorfismus. Položme $g = G^{-1}$.

Nyní už stačí $g(\varphi(t, x_0)) = \psi(t, g(x_0)) \forall x_0 \in U \forall t : \varphi(t, x_0) \in U. \varphi(t, x_0) = G(\psi(t, g(x_0)))$

3. $x \in U = G(V) \exists y \in V x = G(y)$

$$x = \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

$$\varphi(t, x) = \varphi(t, \varphi(y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))) = \varphi(t + y, (x_{01}, x_{02} + y_2, \dots, x_{0n} + y_n))$$

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□

Věta 1.4 (La Salle invariance principle)

$$x' = f(x), (\varphi, \Omega) \quad \varphi : \mathbb{R} \rightarrow \Omega, f \text{ loc.Lip.}$$

$$\exists V : \Omega \rightarrow \mathbb{R}, \text{ bounded from below.}$$

$$\exists l \in \mathbb{R} : \Omega_l = \{x \in \Omega | V(x) \leq l\} - \text{bounded}$$

$$\dot{V}_f(x) := \nabla V(x) \cdot f(x) = \sum_{j=1}^n \frac{\partial V(x)}{\partial x_j} \cdot f_j(x) \leq 0 \quad \forall x \in \Omega_l.$$

$$R = \{x \in \Omega_l \mid \dot{V}_f(x) = 0\}, \quad M = \{y \in R \mid \gamma^+(y) \subset R\}.$$

Then $\forall x \in \Omega_l : \omega(x) \subset M$.

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Důkaz

Let $x \in \Omega_l$. $\forall y \in \omega(x) \exists t_k \nearrow \infty : x(t_k) \rightarrow y$. $\varphi(t, x_0) = x(t)$.

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)) \cdot x'(t) = \dot{V}_f(x(t)) \leq 0.$$

$V(x(t)) \searrow$ and $\exists C : \forall x \in \Omega : V(x) > -C$ so $\exists \lim_{t \rightarrow \infty} V(x(t)) = c$.

So $\exists c \forall y \in \omega(x_0) V(y) = c$. $V(x(t_k)) \rightarrow V(y) = c$.

$$\gamma^+(y) \subset \omega(x_0) \quad V(\varphi(t, y)) = c \quad \forall t \geq 0 \implies$$

$$\implies \frac{d}{dt}V(\varphi(t, y)) = 0.$$

└ $\gamma^+(y) \subset R$ in particular, $y \in R$. Hence $y \in M$. □

2 Poincaré-Bendixson theory

Věta 2.1 (Poincaré-Bendixson)

Let $p \in \Omega$, Ω open connected. $\omega(p)$ doesn't contain stat points and $\gamma^+(p)$ is relatively compact ($\gamma^+(p)$ is compact). Then $\omega(p) = \Gamma$ -periodic orbit.

Věta 2.2 (Bendixon-Dulas)

Ω -simply connected (\forall closed Jordan curve γ in Ω , $\text{int}(\gamma) \subset \Omega$). $\exists B : \Omega \rightarrow \mathbb{R} : (\text{div } B)(x) = \frac{\partial B}{\partial x_1}(x_1, x_2) + \frac{\partial B}{\partial x_2}(x_1, x_2) > 0$ for almost every $x \in \Omega$. Then $x' = f(x)$ doesn't have nontrivial periodic solutions.

Definice 2.1 (Transverzála)

Σ segment on a line such that $\forall p \in \Sigma : \Sigma \nparallel f(p)$.

Lemma 2.3

Σ transversála, $p \in \Sigma \subset \Omega$. Then $\exists \tilde{U}$ neighborhood of p . $\exists \Delta > 0$ such that

$$\forall y \in \tilde{U} : \varphi(t, y) \subset U \quad \forall t : |t| < \Delta \wedge \exists \tau : |\tau| < \frac{\Delta}{2} : \varphi(\tau, y) \in \Sigma \cap \tilde{U}.$$

Důkaz

Use Th. of rect.

□

Lemma 2.4

Let $p \in \Omega$ and assume that $|\gamma^+(p) \cap \Sigma| \geq 3$, i. e. $\exists t_1 < t_2 < t_3 \varphi(t_j, p) \in \Sigma, j = 1, 2, 3$. Then $\varphi(t_2, p)$ lie between $\varphi(t_1, p)$ and $\varphi(t_3, p)$.

TODO!!!

TODO!!!

2.1 Controllability

Definition 2.2 (Control theory)

$$x' = f(x, u), f : \Omega \times U, \Omega \subset \mathbb{R}^n, U \subset \mathbb{R}^n,$$

$$u \in \mathcal{U} := \{u : [0, T] \rightarrow \mathbb{R}^n | \text{measurable}, \|u\|_\infty < \infty\} = L^\infty(0, T, \mathbb{R}^n).$$

(\mathcal{U} is admissible functions).

Definition 2.3 (Linear task)

$$x' = Ax + Bu, A, B \in \mathbb{R}^{n \times m}, m < n.$$

Definition 2.4

$$x_0 \xrightarrow[u(0)]{t} 0 \text{ iff } x(0) = x_0, x(t) = 0.$$

Definition 2.5 (Area of controllability)

$$\mathcal{R}(t) = \left\{ x_0 \in \mathbb{R}^n | \exists u \in L^\infty(0, t, \mathbb{R}^n) : x_0 \xrightarrow[u(0)]{t} 0 \right\}$$

Definition 2.6 (Kalman matrix)

$$\mathcal{K}(A, B) := (B | AB | A^2 B | \dots | A^{n-1} B)$$

Věta 2.5

For linear problem $\mathcal{R}(t) = \text{LO}(g_1, g_2, \dots, g_{n-m})$, where $\mathcal{K}(A, B) = (g_1 | g_2 | \dots | g_{n-m})$

Tvrzení 2.6 (Observation)

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

$$x_0 \xrightarrow[u(0)]{t} 0 \Leftrightarrow x(t) = 0 \Leftrightarrow x_0 = - \int_0^t e^{-As}Bu(s)ds \quad (KO)$$

Lemma 2.7 (1)

$$A^k \in \text{LO}(I, A, A^2, \dots, A^{n-1}), k \in \mathbb{N}_0$$

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Důkaz
Cayley-Hamilton.
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Důkaz

1) $\mathcal{R}(t)$ is vector subspace of \mathbb{R}^n from definition $x_0 + x_1 \xrightarrow[(u_1+u_2)(0)]{t} 0, \alpha x \xrightarrow[\alpha u(0)]{t} 0$.

2) We want $\mathcal{R}(t)^\perp = (\text{LO}(g_1, \dots, g_n))^\perp$. „ \supseteq “: $p \in (\text{LO}(g_1, \dots, g_n))^\perp$. $x_0 \in \mathcal{R}(t)$ arbitrary. From KO:

$$0 \stackrel{?}{=} p^T x_0 = - \int_0^t p^T e^{-As} Bu(s) ds = - \int_0^t \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} p^T A^k B u(s) ds$$

We know $(p, g_j) = 0$, $p^T g_j = 0$, $p^T \mathcal{K}(A, B) = 0$, $p^T A^k B = 0$, $k \in [n-1]$. And from lemma 1 $k \in \mathbb{N}$. „ \subseteq “: $p \in \mathbb{R}^n$, $p \in \mathcal{R}(t)^\perp$. We want to prove $p \perp B, AB, A^2B, \dots, A^{n-1}B$. $B = (b_1 | \dots | b_m)$. $\forall j \in [n] : p \perp b_j, Ab_j, \dots, A^{n-1}b_j$. $\varphi \in L^\infty(0, T, \mathbb{R})$, $u(t) = \varphi(t) \cdot \mathbf{e}_j$, where $x_0 = - \int_0^t e^{-As} Bu(s) ds$. We have $x_0 \xrightarrow[u(0)]{t} 0$, hence $x_0 \in \mathcal{R}(t)$.

$$0 = p^T x_0 = -p^T \int_0^t e^{-As} Bu(s) ds = - \int_0^t p^T e^{-As} b_j \varphi(s) ds \implies y(s) := p^T e^{-As} b_j \equiv 0$$

So we have $p^T e^{-As} b_j \equiv 0$, we derivate it, $p^T A^n e^{-As} b_j \equiv 0$, and set $s = 0$.
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□

Důsledek

$\mathcal{R}(t)$ doesn't depend on time.

Definice 2.7 (Locally and globally controllable)

Linear problem is called locally controllable, iff $\exists \delta > 0 : \{x_0 \in \mathbb{R}^2 \mid |x_0| < \delta\} \subset \mathcal{R}(t)$. And globally if $\mathbb{R}^n = \mathcal{R}(t)$.

Důsledek

Linear problem is controllable $\Leftrightarrow \text{rank } K(A, B) = n$.

2.2 Observability

Definition 2.8 (System for observability)

$$x' = f(x), x(0) = x_0, f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, y = g(x), g : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, m < n.$$

Definition 2.9

We say that system $x' = f(x)$ is observable through $g(\cdot)$ on $[0, t]$, iff $\forall x_1(\cdot), x_2(\cdot) : [0, T] \rightarrow \mathbb{R}^n : g(x_1(t)) = g(x_2(t)) \forall t \in [0, T] \implies x_1(0) = x_2(0)$.

Definition 2.10 (Linear observability)

$$x' = Ax, y = Bx, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}.$$

Věta 2.8

$x' = Ax$ is observable on $[0, T]$ through $y = Bx \Leftrightarrow x' = A^T x + B^T u$ is controllable.

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Důkaz

(We will prove equivalence with $\text{rank } \mathcal{K}(A^T, B^T) = n$.) „ \Leftarrow “: For contradiction

$$\exists x_1(t), x_2(t), [0, T], Bx_1(t) \equiv Bx_2(t) : x(t) = x_1(t) - x_2(t), x(0) = x_0 \neq 0, Bx(t) \equiv 0.$$

$$x(t) = e^{At} x_0, Bx(t) = B e^{At} x_0 \equiv 0 \quad \forall t \in [0, T].$$

We differentiate it, set $t = 0$ and get $Bx_0 = 0, BAx_0 = 0, \dots, BA^{n-1}x_0 = 0$. So $x_0^T B^T = 0, \dots, x_0^T (A^T)^{n-1} B^T = 0, x_0^T \mathcal{K}(A^T, B^T) = 0, x_0 \perp \mathcal{K}(A^T, B^T)$, \nexists .

„ \implies “: For contradiction $\text{rank}(A^T, B^T) < n \implies \exists x_0 \neq 0 : x_0^T \mathcal{K}(A^T, B^T) = 0$.
 $x_0^T (A^T)^k B^T = 0 \forall k \in [n-1]$ and from lemma 1 $\forall k \in \mathbb{N}, BA^T x_0 = 0, B e^{At} x_0 = 0$
 $\forall t \in [0, T]$. \nexists . □

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Věta 2.9

$V \subset \mathbb{R}^n$ neighbourhood of 0, $U \subset \mathbb{R}^n$ neighbourhood of 0, $f : V \times U \rightarrow \mathbb{R}^n$ C^1 smooth, $f(0, 0) = 0, \mathcal{U} = \{u : [0, T] \rightarrow U \text{ measurable}\}, A = \nabla_x f(0, 0), B = \nabla_u f(0, 0), \text{rank } \mathcal{K}(A, B) = n$. Then

$$x' = f(x, u), x(0) = x_0 \text{ is locally controllable } \forall t \in (0, T].$$

┌ *Důkaz*

Fix $t > 0$, consider $x' = Ax + Bu$. Since $\text{rank}(A, B) = n$, the linear problem is globally controllable. Take initial condition y_1, \dots, y_n linearly independent.

$$\exists u_i \in L^\infty(0, t, \mathbb{R}^n) : y_j \xrightarrow{u_\lambda(0)}^t 0$$

$\forall \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ denote by $u_\lambda(t) = \sum_{j=1}^n \lambda_j u_j(t)$. We know $\sum_{j=1}^n \lambda_j y_j \xrightarrow{u_\lambda(0)}^t 0$.

Step 2:

$$x'_\lambda = f(x_\lambda, u_\lambda), \quad x_\lambda(t) = 0$$

If $\lambda = 0$, then $u_\lambda \equiv 0$, then $x_\lambda \equiv 0$.

$$\psi(\lambda) := x_\lambda(0), \psi : U_\lambda(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

We want to prove $\psi(U_\lambda(0)) \supseteq \tilde{V}$, for some $\tilde{V} \subset \mathbb{R}^n$ open, $0 \in \tilde{V}$. We will prove that ψ is C^1 smooth, and that $\nabla \psi(0)$ is regular (if this is proved, then ψ is local diffeomorphism).

Step 3:

$$x_\lambda(s) = x_\lambda(t) + \int_t^s f(x_\lambda(s), u_\lambda(s)) ds.$$

Formally differentiate:

$$\frac{\partial x_\lambda(s)}{\partial \lambda_j} = \int_t^s (\nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot \frac{\partial x_\lambda(s)}{\partial \lambda_j} + \nabla_u f(x_\lambda(s), u_\lambda(s))) ds.$$

Denote $y_{\lambda,j}(s) = \frac{\partial x_\lambda(s)}{\partial \lambda_j}$.

$$y'_{\lambda,j}(s) = \nabla_x f(x_\lambda(s), u_\lambda(s)) \cdot y_{\lambda,j}(s) + \nabla_u f(x_\lambda(s), u_\lambda(s)) \cdot u_j(s).$$

$$y_{\lambda,j}(t) = 0.$$

Consider $(LPy) \rightarrow y_{\lambda,j}(\cdot)$.

$$x_{\lambda+\Delta\lambda}(s) - x_\lambda(s) - \Delta\lambda \cdot y_{\lambda,j}(s) = 0$$

(as in Thm? of differentiability w. r. t. initial condition)

$$\frac{\partial \psi}{\partial \lambda_j}(\lambda = 0) = \frac{\partial x_\lambda(s=0)}{\partial \lambda_j}|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_{\lambda,j}(s=0)|_{\lambda=0} = y_j.$$

If $\lambda = 0$, then (LPy) : $y'_{0,j}(s) = Ay_{0,t}(s) + Bu_j(s)$, $y_{0,j}(t) = 0$. From uniq.: $y_{0,j}(0) = y_{j,0}$.

$$\nabla \psi(0) = \left(\frac{\partial \psi}{\partial \lambda_1}(0) \dots \frac{\partial \psi}{\partial \lambda_n}(0) \right) = (y_1, \dots, y_n)$$

└ regular matrix. □

Poznámka

$$x' = Ax + Bu, u \in \mathcal{U} = \{u : [0, T] \rightarrow [-1, 1] \text{ measurable}\}, x(0) = x_0.$$

Definice 2.11

$$\mathcal{R}(t) = \{x_0 \in \mathbb{R}^n | \exists u \in \mathcal{U} \wedge x_0 \rightarrow_{u(0)}^t 0\}.$$

Definice 2.12

$$u_n \in \mathcal{U}_0: u_n \rightarrow^* u \in \mathcal{U} \equiv \forall f \in L(0, T, \mathbb{R}^n) : \int_0^T f(s)u_n(s)ds \rightarrow \int_0^T f(s)u^*(s)ds.$$

Věta 2.10 (Alanglu)

\mathcal{U} is weak- * sequentially compact ($\forall \{u_n\}_{n=1}^\infty \in \mathcal{U} \exists \{u_{n_k}\}$ weakly- * convergent)

Věta 2.11

$\mathcal{H}(t)$ convex, symmetric, closed

$$0 < t_1 < t_2 \implies \mathcal{H}(t_1) \subset \mathcal{R}(t_2).$$

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Důkaz

Convex: $x_{01}, x_{02} \in \mathcal{H}(t), \alpha \in [0, 1] \implies \alpha x_{01} + (1 - \alpha)x_{02} \in \mathcal{R}(t).$

$$x(t) = e^{At}x_0 + \int_0^t e^{As}Bu(s)ds. x_{01} \rightarrow_u^t 0 \wedge x_{02} \rightarrow_u^t 0 \Leftrightarrow x_1 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

$$\text{Symmetry: } x_0 \in \mathcal{R}(t) \implies -x_0 \in \mathcal{R}(t), x_0 \rightarrow_u^t 0 \implies -x_0 \rightarrow_u^t 0.$$

Closedness: $x_{0n} \in \mathcal{R}(t), x_{0n} \rightarrow x_0. x_0 \in \mathcal{R}(t)? \exists u_n(0) \in \mathcal{U}, x_{0n} = - \int_0^t e^{(s-t)A}Bu_n(s)ds \rightarrow - \int_0^t e^{(s-t)A}Bu(s)ds. \text{ WLOG } u_n \rightarrow^* u \in \mathcal{U}. \text{ Then } x_0 \rightarrow_u^t 0.$

$$\mathcal{R}(t_1) \subset \mathcal{R}(t_2) 0 < t_1 < t_2 < T x_0$$

$$\exists u_1 \in \mathcal{U} \quad x_0 = - \int_0^t e^{(s-t)A}Bu_1(s)ds.$$

Define $u_2(s) = u_1(s)$ if $0 \leq s \leq t$, else 0.

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□

Definice 2.13 (Area of controllability)

$$\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t).$$

Věta 2.12

$$\text{rank } \mathcal{K}(A, B) = n \Leftrightarrow \forall t > 0 \mathcal{R}(t) \supseteq U(0),$$

where $U(0) \subset \mathbb{R}^n$ is some neighbourhood of 0.

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Důkaz

„ \Leftarrow “: If $\exists t > 0 \mathcal{R}(t) \supset U(0)$ open, $0 \in U(0)$. $\tilde{\mathcal{R}} : u \in L^\infty, \mathcal{R} : \|u\|_\infty \leq 1$, then $\tilde{\mathcal{R}}(t) \supset \mathcal{R}(t) \supset U(0) \Rightarrow \tilde{\mathcal{R}}(t) = \mathbb{R}^n. \Rightarrow \text{rank } \mathcal{K}(A, B) = n$.

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„ \Rightarrow “: $\text{rank}(A, B) = n \Rightarrow \tilde{\mathcal{R}}(t) = \mathbb{R}^n$. TODO?

□

Věta 2.13

$$x' = Ax + Bu$$

$$\forall x_0 \in \mathcal{R} = \bigcup_{s>0} \mathcal{R}(s)$$

$$\exists t > 0 \exists u(0) \in \mathcal{U} : x_0 \xrightarrow{t}_u 0$$

$$t = \inf \{s > 0 | x_0 \in \mathcal{R}(s)\}.$$

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Důkaz

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TODO!!!

□

Definice 2.14 (Bang-bang)

We say that a regulation $u \in U(0)$ is of type bang-bang, if for almost every $t \in [0, T]$: $u(t) = \pm 1$.

Věta 2.14 (Bang-bang)

If $x_0 \in \mathcal{R}(t) \Rightarrow \exists \tilde{u}(0)$ of type bang-bang $x_0 \xrightarrow{t}_{\tilde{u}} 0$.