Definice 0.1 (Category, map (arrow, morphism), composition, domain, codomain)

A category $\mathcal{A}$ consists of: a collection $\mathrm{ob}(\mathcal{A})$ of objects, and for each $A, B \in \mathcal{A}$, a collection $\mathcal{A}(A, B)$ of maps, arrows, or morphisms from A to B. Such that for each $A, B, C \in \mathrm{ob}(\mathcal{A})$ a function (named composition) $\circ : \mathcal{A}(B, C) \times \mathcal{A}(A, B) \to \mathcal{A}(A, C)$, $(g, f) \mapsto g \circ f$ meets following:

For each $f \in \mathcal{A}(A, B), g \in \mathcal{A}(B, C), h \in \mathcal{S}(C, D) : (h \circ g) \circ f = h \circ (g \circ f)$ (asociativity). For each $A \in \mathrm{ob}(\mathcal{A})$ $\exists 1_A \in \mathcal{A}(A, A)$, called the identity, such that, for each $f \in \mathcal{A}(A, B)$: $f \circ 1_A = f = 1_B \circ f$.

Poznámka (Notation)

$$A \in \mathrm{ob}(\mathcal{A}) \Leftrightarrow A \in \mathcal{A}.$$

$$f \in \mathcal{A}(A, B) \Leftrightarrow A \xrightarrow{f} B \Leftrightarrow f : A \to B.$$

For $f \in \mathcal{A}(A, B)$: $\mathrm{domain}(f) := A$ and $\mathrm{codomain}(f) := B$.

Například (of categories)
Category of:

- sets (SET): $\mathrm{ob}(SET) :=$ sets, $SET(A, B) :=$ functions from A to B, $\circ$ is composition;
- groups (GRP): $\mathrm{ob}(GRP) :=$ groups, $GRP(G, H) :=$ group homomorphisms, $\circ$ is composition;
- rings (RING): $\mathrm{ob}(RING) :=$ rings, $RING(A, B) :=$ ring homomorphisms, $\circ$ is composition;
- vector spaces ($\mathrm{VECT}_{\mathbb{K}}$): $\mathrm{ob}(VECT_{\mathbb{K}}) :=$ vector spaces over $\mathbb{K}$, $RING(A, B) := \mathbb{K}$ linear maps, $\circ$ is composition;
- topological spaces (TOP): $\mathrm{ob}(TOP) :=$ topological spaces, $RING(A, B) :=$ continuous maps, $\circ$ is composition.

Definice 0.2 (Isomorphism, inverse)

$f : A \to B$ in a category $\mathcal{A}$ is an isomorphism if exists a map $g : B \to A$ in $\mathcal{A}$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Then we call g the inberse of f.

Například
In SET isomorphisms are bijections.

Příklad
Show that inverses are unique (justifying the use of the determine article in the previous definition).

Poznámka
0-morphisms are called morphisms (between objects), 1-morphisms are called functors (between categories), 2-morphisms are called natural transformations (between functors).

Definice 0.3 (Functor)

Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A functor $F : \mathcal{A} \to \mathcal{B}$ consists of: a function $F : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$, and for each $A, A' \in \mathcal{A}$ a function $F : \mathcal{A}(A, A') \to \mathcal{B}(F(A), F(A'))$. Such that

$$F(f' \circ f) = F(f) \circ F(f'), \qquad \forall A \stackrel{f}{\ } A' \stackrel{f'}{\ } A'' \in \mathcal{A},$$

$$F(1_A) = 1_{F(A)} \qquad \forall A \in \mathcal{A}.$$

Například (Forgetful functors)
$U : GRP \to SET$, for any group $(G, \cdot)$, $U((G, \cdot)) := G$, and for any morphism f, $U(f : (G, \cdot) \to (H, *)) := f : G \to H$. (Exercise: Convince yourself that this is a well-defined functors.)

We can do the same for rings, vector spaces and topological spaces.

Například
Let $\mathcal{A}$ be the following category: $\mathrm{ob}(\mathcal{A}) = \{\cdot\}$, $\mathcal{A}(\cdot, \cdot) = 1.$, and $1. \circ 1. = 1.$. It is called discrete category with one object.

$\mathrm{ob}(\mathbb{B}) = \{\cdot, *\}$, $\mathbb{B}(\cdot, \cdot) = 1.$, $\mathbb{B}(\cdot, *) = \varnothing$

Directed transitive graph (with all loops) with concatenation of edges.

From group $(G, +)$ we construct category $\mathcal{G}$ by putting: $\mathrm{ob}(\mathcal{G}) := \cdot$, $\mathcal{G}(\cdot, \cdot) := G$ and $\circ := +$. We can generalize to a monoid $(M, +)$.

Now, let $\mathcal{A}$ be a category with one object $\{\cdot\}$ (and assume that $\mathcal{S}(\cdot, \cdot)$ is a set). Then homomorphism with composition are monoid. And isomorphisms with composition are groups (so one-object category with all homomorphism isomorphic represents group).

(Category, where $\mathcal{A}(\cdot, \cdot)$ is a set, is often called locally small.)

Let G and H be groups and $\mathcal{G}$, $\mathcal{H}$ their associated one-object categories. What is a functor from $\mathcal{G}$ to $\mathcal{H}$? For $F : \mathrm{ob}(\mathcal{G}) \to \mathrm{ob}(\mathcal{H})$ we have no other choice than $F(\cdot) := *$. For $F : \mathcal{G}(\cdot, \cdot) \to \mathcal{H}(*, *) = \mathcal{H}(F(\cdot), F(\cdot))$ we demonstrated (see lecture) that F needs to be group homomorphism (and every group homomorphism $G \to H$ is functor). (All this work for monoids too.)

Let AB be the category of $\text{ob}(AB) :=$ Abelian groups and $AB(A, B) :=$ group homomorphism. Then $U : AB \to GRP$ as „forgetful functor“ is „identity“. The same for commutative rings. Also we have forgetful functor $U : RING \to AB$, $(R, +, \cdot) \mapsto (R, +)$ and functor $U : RING \to MONOIDS$, $(R, +, \cdot) \mapsto (R, \cdot)$.

$U : SET \to VECT_{\mathbb{K}}$ we can define by $F(X) = (X \to F)$ (functions from X to F) (free vector space).