

# Úvod

*Poznámka* (Organizační úvod)

Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (... PDE ...).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

*Poznámka* (Konvence pro PDR)

$\Omega \subseteq \mathbb{R}^d$  je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

*Poznámka*

Dále se ukazovali konkrétní parciální rovnice.

*Poznámka* (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál:  $L^p(\Omega)$ ,  $L^p_{loc}(\Omega)$ ,  $\|u\|_p$ ,  $C^k(\Omega)$ ,  $C^k(\overline{\Omega})$ ,

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) \mid \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}, \|u\|_{C^{0,\alpha}} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

- $\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$ ,  $\vec{n} = (n_1, \dots, n_d)$ .
- Funkcionální analýza 1: Banachův prostor,  $u^n \rightarrow u$  silná konvergence,  $u^n \rightharpoonup u$  slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita ( $L^p$  jsou separabilní až na  $p = \infty$ ,  $C^k(\overline{\Omega})$  je separabilní,  $C^{0,\alpha}$  není separabilní pro  $\alpha \in (0, 1]$ ).

*Poznámka* (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta u = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

# 1 Sobolevovy prostory

## Definice 1.1 (Multiindex)

$\alpha$  je multiindex  $\equiv d = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha_i \in \mathbb{N}_0$ . Délka  $\alpha$  je  $|\alpha| := \alpha_1 + \dots + \alpha_d$ . Pro  $u \in C^k(\Omega)$  definujeme  $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ .

## Definice 1.2 (Slabá derivace)

Buď  $u, v_\alpha \in L^1_{loc}(\Omega)$ . Řekneme, že  $v_\alpha$  je  $\alpha$ -tá slabá derivace  $u \equiv$

$$\equiv \int_{\Omega} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} v_\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

*Příklad*

$u = \text{sign} x$  nemá slabou derivaci.

## Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

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*Důkaz*

$v_\alpha^1, v_\alpha^2$  dvě  $\alpha$ -té derivace  $u$ .

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^1 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^2 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\int_{\Omega} (v_\alpha^1 - v_\alpha^2) \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$  skoro všude v  $\Omega$ .

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají.

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□

## Definice 1.3 (Sobolevův prostor)

$\omega \subseteq \mathbb{R}^d$  otevřená,  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty]$ .

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid \forall \alpha, |\alpha| \leq k : D^\alpha u \in L^p(\Omega)\}.$$

$$\|u\|_{W^{k,p}(\Omega)} \|u\|_{k,p} := \begin{cases} \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_\infty, & p = \infty. \end{cases}$$

*Poznámka*

Od teď  $D^\alpha$  nebo  $\frac{\partial}{\partial x_1}$  nebo  $\partial_i$  značí slabou derivaci.

### **Lemma 1.2** (Základní vlastnosti slabých derivací a Sobolevových prostorů)

Nechť  $u, v \in W^{k,p}(\Omega)$ ,  $k \in \mathbb{N}$ , a  $\alpha$  multiindex s délkou  $\leq k$ .

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$  a  $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ , pro  $|\alpha| + |\beta| \leq k$ .
- $\lambda, \mu \in \mathbb{R}$ ,  $\lambda u + \mu v \in W^{k,p}(\Omega)$  a  $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ .
- $\forall \tilde{\Omega} \subseteq \Omega$  otevřená

$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$

- $\forall \eta \in C^\infty(\Omega)$ :  $\eta u \in W^{k,p}(\Omega)$  a  $D^\alpha(\eta u) = \sum_{\beta_i \leq \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$ , kde  $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$ .

*Důkaz*

Cvičení na doma. □

### **Věta 1.3** (Basic properties of Sobolev spaces)

Let  $\Omega \subseteq \mathbb{R}^d$  be open set,  $k \in \mathbb{N}$  and  $p \in [1, \infty]$ . Then

- $W^{k,p}(\Omega)$  is a Banach space;
- if  $p < \infty$  it is separable space;
- if  $p \in (1, \infty)$  it is reflexive space.

┌ *Důkaz*

BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness:  $u^n$  is Cauchy in  $L^p(\Omega)$  so  $\exists u \in L^p : u^n \rightarrow u$  in  $L^p$ .  $D^\alpha u^n$  is Cauchy in  $L^p(\Omega)$   $\forall |\alpha| < k$  so  $\exists v_\alpha \in L^p : D^\alpha u^n \rightarrow v_\alpha \in L^p$ . It remains prove that  $D^\alpha u = v_\alpha$ .

*TODO*

$$\left| \int_{\Omega} (v_\alpha - D^\alpha u^n) \varphi \right| \leq \|v_\alpha - D^\alpha u^n\|_p \|\varphi\|_{p'} \leq C \|v_\alpha - D^\alpha u^n\| \rightarrow 0.$$

$$\left| \int_{\Omega} (u^n - u) D^\alpha \varphi \right| \leq \|u^n - u\|_p \|D^\alpha \varphi\|_{p'} \leq C \|u^n - u\|_p \rightarrow 0.$$

„2+3“:  $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \dots \times L^p(\Omega)$  ( $d+1$  times),  $X$  closed subspace from first property. Lemma: if  $X \subseteq Y$  is closed subspace then  $Y$  separable  $\implies X$  separable and  $Y$  reflexive  $\implies X$  reflexive. (From functional analysis and topology.)  $\square$

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## 2 Approximation of Sobolev function

### Věta 2.1

Let  $\Omega \subseteq \mathbb{R}^d$  open,  $p \in [1, \infty)$ .

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

*Pozor*

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} \subsetneq W^{k,p}(\Omega).$$

┌ *Důkaz*

└ Summer semester.  $\square$

### Věta 2.2 (Local density)

$$\begin{aligned} \forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty \\ u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} \text{ open}, \bar{\tilde{\Omega}} \subseteq \Omega \\ u^n \rightarrow u \text{ in } W^{k,p}(\tilde{\Omega}) \end{aligned}$$

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*Důkaz* $u$  is extended by 0 to  $\mathbb{R}^d \setminus \Omega$ .

$$u^\varepsilon = u * \eta^\varepsilon \quad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \quad \eta \in C_0^\infty(B_1), \eta \geq 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x) dx = 1.$$

$$u \in L^p(SET) \quad u^\varepsilon \rightarrow u \text{ in } L^p(SET).$$

We need:  $D^\alpha u^\varepsilon \rightarrow D^\alpha u$  in  $L^p(\tilde{\Omega}) \forall \alpha, |\alpha| \leq k$ . Essential step:  $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$  in  $\tilde{\Omega}$  for  $\varepsilon \leq \varepsilon_0$  (so that ball of radius  $\varepsilon_0$  and center in  $\tilde{\Omega}$  is in  $\Omega$ ):

$$\begin{aligned} (D^\alpha u)^\varepsilon(x) &= \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \int_{B_\varepsilon(x)} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \\ &= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y) dy. \\ D^\alpha u^\varepsilon &= D_x^\alpha \int_{\mathbb{R}^d} u(y) \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y) dy. \end{aligned}$$

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□

### Tvrzení 2.3

$\Omega$  is open connected set,  $u \in W^{1,1}(\Omega)$ , then  $u = \text{const.} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \forall i \in [d]$ .

$W^{1,1}(I) \hookrightarrow C(I)$  for  $I$  interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$ .

┌ *Důkaz*

„1.  $\implies$ “ trivial. „1.  $\Leftarrow$ “:  $\tilde{\Omega} \subseteq \Omega$  connected  $\varepsilon_0$  as before and  $\varepsilon \in (0, \varepsilon_0)$ .  $u^\varepsilon$ -modification of  $u$  is smooth, so

$$\frac{\partial u^\varepsilon}{\partial x_i} = \left( \frac{\partial u}{\partial x_i} \right)^\varepsilon = 0 \quad \text{in } \tilde{\Omega}$$

$$\implies u^\varepsilon = \text{const}(\varepsilon) \quad \text{in } \tilde{\Omega}.$$

$$\begin{aligned} c(\varepsilon) &= \int_{\mathbb{R}} c(\varepsilon) \eta_\delta(x-y) dy = \int_{\mathbb{R}} u^\varepsilon(y) \eta_\delta(x-y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(z) \eta_\varepsilon(y-z) \eta_\delta(x-y) dz dy = \\ &= \int \int u(z+y) \eta_\varepsilon(z) \eta_\delta(y-x) dz dy = \int \int u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dz dw = \\ &= \int \int u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dw dz = \int_{\mathbb{R}^d} u^\delta(z+x) \eta_\varepsilon(z) dz = \int c(\delta) \eta_\varepsilon(z) dz = c(\delta). \end{aligned}$$

„2.“: WLOG  $I = (0, 1)$ . Define  $v(x) = \int_0^x \frac{\partial u}{\partial y}(y) dy$ . We show:  $v \in W^{1,1}(I)$ ,  $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$ .

$$|v(x)| \leq \int_0^1 \left| \frac{\partial u}{\partial x} \right| \leq \|u\|_{1,1}.$$

$$\begin{aligned} &\varphi \in C_0^1(0, 1) \quad \int_0^1 v(x) \frac{\partial \varphi}{\partial x}(x) dx \\ &= \int_0^1 \left( \int_0^x \frac{\partial u}{\partial y}(y) dy \right) \frac{\partial \varphi}{\partial x}(x) dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dy dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dx dy = \\ &= \int_0^1 \left( \int_y^1 \frac{\partial \varphi(x)}{\partial x} dx \right) \frac{\partial u}{\partial y}(y) dy = - \int_0^1 \varphi(y) \frac{\partial u}{\partial y}(y) dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}. \end{aligned}$$

TODO.

$$x \rightarrow y \implies \int_y^x \left| \frac{\partial u}{\partial z} \right|^\alpha \rightarrow 0 \implies |u(x) - u(y)| \rightarrow 0$$

$$\|u\|_{C(I)} \leq \|v + c\|_{C(I)} \leq \|u\|_{1,1} + |c| = \|u\|_{1,1} + |u(x) - v(x)| \forall x \in I$$

$$\|u\|_{C(I)} \leq \|u\|_{1,1} + \int_0^1 |u(x) - v(x)| dx \leq -|| - + \int_0^1 |u| + \int_0^1 |v| \leq \|u\|_{1,1}.$$

„3.“ was shown without proof. □

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### 3 Characterization of Sobolev function

#### Věta 3.1

$\Omega \subseteq \mathbb{R}^d$ ,  $p \in [1, \infty]$ ,  $\delta > 0$ ,  $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \delta\Omega) > \delta\}$ . Then

$$\forall u \in W^{1,p}(\Omega) : \|\Delta_i^h u\|_{L^p(\Omega_{\delta})} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}, \quad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

$$u \in L^p \implies \forall \delta, h : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq c.$$

$$p > 1 \implies \frac{\partial u}{\partial x_i} \text{ exists and } \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq c.$$

### Definition 3.1 (Class $C^{k,\mu}$ )

Let  $\Omega \subseteq \mathbb{R}^d$  open bounded set. We say that  $\Omega \in C^{k,\mu}$  ( $\partial\Omega \in C^{k,\mu}$ ) iff:

- there exist  $M$  coordinate systems  $\mathbf{x} = (x_{r_1}, \dots, x_{r_d}) = (x'_r, x_{r_d})$  and functions  $a_r : \Delta_r \rightarrow \mathbb{R}$  where  $\Delta_r = \{x'_r \in \mathbb{R}^{d-1} \mid |x_{r_i}| \leq \alpha\}$  such that  $a_r \in C^{k,\mu}(\Delta_r)$ ,
- denoting  $\text{tr}$  the orthogonal transformation from  $(x'_r, x_{r_d})$  to  $(x', x_d)$ , then  $\forall x \in \partial\Omega \exists r \in \{1, \dots, M\}$  such that  $x = \text{tr}(x'_r, a(x_{r_d}))$ ,
- $\exists \beta > 0$ , if we define

$$V_r^+ := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\}$$

$$V_r^- := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\}$$

$$\Lambda_r := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) = x_{r_d}\}$$

Then  $\text{tr}(V_r^+) \subset \Omega$ ,  $\text{tr}(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$ ,  $\text{tr}(\Lambda_r) \subseteq \partial\Omega$  and  $\bigcup_{r=1}^M \text{tr}(\Lambda_r) = \partial\Omega$ .

### Věta 3.2 (Density of smooth functions)

Let  $\Omega \in C^0$ . Then  $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}$ ,  $p \in [1, \infty)$ .

### Věta 3.3 (Extension of Sobolev functions)

Let  $\Omega \in C^{0,1}$  ( $\Omega$  is Lipschitz) and  $k \in \mathbb{N}$ ,  $p \in [1, \infty]$ . Then there exists a continuous linear operator  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$  such that:

- $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(\Omega)}$  ( $C$  is independent of  $u$ )
- $Eu = u$  almost everywhere in  $\Omega$ .

### Věta 3.4 (Trace theorem)

Let  $\Omega \in C^{0,1}$ ,  $p \in [1, \infty]$ . Then there exists a continuous linear operator  $\text{tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$  such that:

- $\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leq c \|u\|_{1,p},$
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}.$

### Definice 3.2

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_{k,p}}.$$

### Věta 3.5

Let  $\Omega \in C^{0,1}$  and let  $p \in [1, \infty]$ . Then

- if  $p < d$ , then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $1 \leq \frac{dp}{d-p}$ ,
- if  $p = d$ , then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $q < \infty$ ,
- if  $p > d$ , then  $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$ .

Moreover

- if  $p < d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$  for all  $1 \leq \frac{dp}{d-p}$ ,
- if  $p = d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$  for all  $q < \infty$ ,
- if  $p > d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow C^{0,\alpha}(\overline{\Omega})$  for all  $\alpha < 1 - \frac{d}{p}$ .

$$X \hookleftrightarrow Y \Leftrightarrow X \leq Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$X \hookleftrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^\infty, \exists c : \|u^n\|_{1,p} \leq c \implies \exists u^{n_j} : u^{n_j} \rightarrow u \text{ in } Y).$$

*Důsledek* (Trace theorem)

Let  $\Omega \in C^{0,1}$ . Then  $\forall u \in W^{1,p}(\Omega)$  and  $v \in W^{1,p'}(\Omega)$  we have integration by parts:

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} dx = - \int_{\Omega} v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

### Věta 3.6 (Poincaré)

Let  $\Omega \in C^{0,1}$  and  $p \in [1, \infty]$ . Let  $\Omega_1, \Omega_2 \subseteq \Omega$ ,  $|\Omega_i| > 0$  and  $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$ ,  $|\Gamma_i|_{d-1} > 0$ . Let  $\alpha_1, \alpha_2 \geq 0$  and  $\beta_1, \beta_2 \geq 0$  and at least one of  $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ .



Then there exist  $c_1, c_2 > 0$  such that  $\forall u \in W^{1,p}(\Omega)$

$$c_1 \|u\|_{1,p}^p \leq \|\nabla u\|_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2 \int_{\Omega_2} |u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2 \int_{\Gamma_2} |u|^p \leq c_2 \|u\|_{1,p}^p.$$

$$(\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p.)$$

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Důkaz (Of the first (the only difficult) inequality)  
└ TODO!!!

□

## 4 Linear elliptic PDEs

### Definice 4.1 (Elliptic)

Let  $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$ , where  $\Omega \subseteq \mathbb{R}^d$  is bounded. We say that  $L$  is elliptic if  $\exists c_1 > 0$  such that  $\forall \zeta \in \mathbb{R}^d$  and almost all  $x \in \Omega$

$$A\zeta \cdot \zeta \geq c_1 |\zeta|^2.$$

### Lemma 4.1

If  $u$  is classical solution, then  $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$  on  $\Gamma_1$ :  $B_{L,\delta}(u, \varphi) = \int_{\Omega} f \varphi + \int_{\Gamma_2 \cup \Gamma_3} g \varphi$ .

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Důkaz  
└ TODO!!!

□

### Lemma 4.2

If  $u \in C^2(\overline{\Omega})$  and  $A, b, \mathbf{c}, \mathbf{d}$  are smooth and previous lemma holds  $\forall \varphi \in C^1, \varphi|_{\Gamma_1} = 0$  and  $u = u_0$  on  $\Gamma_1$ , then  $u$  is a classical solution.

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Důkaz  
└ TODO!!!

□

### Definice 4.2 (Weak solution)

Let  $\Omega \subseteq \mathbb{R}^d$  Lipschitz,  $L$  be an elliptic operator,  $u_0 \in W^{1,2}(\Omega)$ ,  $f \in (W^{1,2}(\Omega))^*$ ,  $g \in L^2(\Gamma_2 \cup \Gamma_3)$ . We say that  $u \in W^{1,2}(\Omega)$  is a weak solution iff

- $\text{tr } u = \text{tr } u_0$  on  $\Gamma_1$  and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g \varphi, \forall \varphi \in V$ , where  $V := \{\varphi \in W^{1,2}(\Omega) | \text{tr } \varphi = 0 \text{ on } \Gamma_1\}$ .

## 4.1 Existence of solution for coercive operators

**Definice 4.3** (Elliptic form)

Let  $B : V \times V \rightarrow \mathbb{R}$  bilinear nad  $V$  be a Hilbert space,  $c_1, c_2 > 0$ . We say that  $B$  is elliptic if it is

- $V$ -bounded  $\Leftrightarrow |B(u, \varphi)| \leq c_2 \|u\|_V \|\varphi\|_V$  and
- $V$ -coercive  $\Leftrightarrow B(u, u) \geq c_1 \|u\|_V^2$ .

**Věta 4.3** (Lax-Milgram)

Let  $B$  be a bilinear elliptic form. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : B(u, \varphi) = \langle F, \varphi \rangle.$$

**Definice 4.4**

Let  $B : V \rightarrow V^*$ . We say that  $B$  is

- Lipschitz  $\equiv \forall u, v \in V : \|B(u) - B(v)\|_{V^*} \leq c_2 \|u - v\|_V, c_2 > 0$ ;
- Uniformly monotone  $\equiv \forall u, v \in V : \langle B(u) - B(v), u - v \rangle_V \geq c_1 \|u - v\|_V^2, c_1 > 0$ .

**Věta 4.4** (Non-linear Lax-Milgram)

Let  $B$  be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle.$$

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Důkaz  
└ TODO!!!

□

Důkaz (Lax-Milgram)  
TODO!!!

□

**Věta 4.5**

If  $B_{L,\sigma}$  is bilinear,  $V$ -bounded and  $V$ -elliptic. Then there exists a unique weak solution  $u$ .

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Důkaz  
└ TODO!!!

□

## 4.2 Existence via Fredholm alternative

TODO!!!

## Věta 4.6

Let  $\Omega \in C^{0,1}$ ,  $L$  be an elliptic operator and  $\Gamma_1 = \partial\Omega$ . Then

1.  $\Sigma$  is at most countable and if infinite  $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \rightarrow \infty$ ;
2.  $(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1 \exists! u : Lu = f + \lambda u$ ;
3.  $\forall \lambda \notin \Sigma \exists C > 0 \forall f \in L^2 \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$  and  $\|u\|_{1,2} \leq C\|f\|_2$ ;

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*Důkaz*

3) TODO improve convergence of  $u^{n_k}$  and show

$$u^{n_k} \rightarrow u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show  $\{u^{n_k}\}$  is Cauchy in  $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1 \|\nabla(u^n - u^m)\|_2^2 \leq \int_{\Omega} A \nabla v^{n,m} \nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_{\Omega} \mathbf{c} \nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d} \nabla v^{n,m} v^{n,m} =$$

$$= \int_{\Omega} (f^n - f^m) v^{n,m} + \lambda (v^{n,m})^2 \pm - \leq$$

$$\leq \|v^{n,m}\|_2 (\|f^n - f^m\|_2 + \lambda \|v^{n,m}\|_2 + \|\mathbf{c}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|b\|_{\infty} \|v^{n,m}\|_2) \leq$$

$$\leq \|v^{n,m}\| C(\lambda) \stackrel{u^n \text{ is Cauchy}}{\leq} C(\lambda) \varepsilon$$

$$\implies \nabla u^n \text{ is Cauchy sequence} \implies u^n \rightarrow u \text{ in } W_0^{1,2}(\Omega) \implies \|?\|_{n_k} = 1$$

$$\int_{\Omega} A \nabla a u^n \nabla a \varphi + b u^n \varphi + \mathbf{c} \nabla u^n \varphi - \mathbf{d} \nabla u^n \varphi = \int_{\Omega} f^n \varphi + \lambda u^n \varphi.$$

$$n \rightarrow \infty$$

$$\int_{\Omega} A \nabla u \nabla \varphi + b u \varphi + \mathbf{c} \nabla u \varphi - \mathbf{d} \nabla u \varphi = \lambda \int_{\Omega} u \varphi \Leftrightarrow Lu = \lambda u$$

└ But  $\lambda \notin \Sigma$ .

□

*Poznámka*

Next we discussed homework.

## 4.3 Variational approach – minimization

*Poznámka*

$B_{L,\sigma}(u, v)$  must be symmetric! ( $B_{L,\sigma}(u, v) = B_{L,\sigma}(v, u)$ )

$$L = - \operatorname{div} (A \nabla u) + bu + \mathbf{c} \nabla u + \operatorname{div}(\mathbf{d}u)$$

$$B_{L,\sigma}(u, v) := \int_{\Omega} A \nabla u \cdot \nabla v + Buv + \mathbf{c} \cdot \nabla uv - \mathbf{d} \nabla v u + \int_{\Gamma} \sigma uv$$

$$B_{L,\sigma}(v, u) := \int_{\Omega} A \nabla v \cdot \nabla u + Bvu + \mathbf{c} \cdot \nabla vu - \mathbf{d} \nabla uv + \int_{\Gamma} \sigma vu$$

$$\implies A = A^T, \quad \mathbf{c} = -\mathbf{d}$$

### Věta 4.7

Let  $B_{L,\sigma}$  be linear symmetric  $V$ -elliptic and  $V$ -bounded.  $f \in V^*$ ,  $g \in L^2(\Gamma_2 \cup \Gamma_3)$ ,  $u \in V$ . Then the following is equivalent:

- $u - u_0 \in V$  and  $B_{L,\sigma}(u, v) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g \varphi$ ;
- $u - u_0 \in V \quad \forall v \in W^{1,2}(\Omega), \quad v, u_0 \in V$

$$\frac{1}{2} B_{L,\sigma}(u, u) - \langle f, u \rangle - \int_{\Gamma_2 \cup \Gamma_3} gu \leq \frac{1}{2} B_{L,\sigma}(v, v) - \langle f, v \rangle - \int_{\Gamma_2 \cup \Gamma_3} gv.$$

┌  
Důkaz („1  $\implies$  2“)

$$\begin{aligned} 0 &\stackrel{V\text{-elliptic}}{\leq} \frac{1}{2} B_{L,\sigma}(v-u, v-u) \stackrel{\text{linearity}}{=} \frac{1}{2} B_{L,\sigma}(v, v) + \frac{1}{2} B_{L,\sigma}(u, u) - \frac{1}{2} B_{L,\sigma}(u, v) - \frac{1}{2} B_{L,\sigma}(v, u) = \\ &= \frac{1}{2} (B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u) - B_{L,\sigma}(u, v) = \\ &= \frac{1}{2} (B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u-v) \stackrel{\text{weak formulation}}{=} \\ &= \frac{1}{2} (B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + \langle f, u-v \rangle + \int_{\Gamma_2 \cup \Gamma_3} g(u-v) \end{aligned}$$

└

□

┌ *Důkaz („2  $\implies$  1“)*

$u$  is minimizer, so set  $v = u + \varepsilon\varphi$ ,  $\varphi \in V$

$$\begin{aligned} \frac{1}{2}B_{L,\sigma}(u, u) - \langle j, u \rangle - \int gu &\leq \frac{1}{2}B_{L,\sigma}(u + \varepsilon\varphi, u + \varepsilon\varphi) - \langle j, u + \varepsilon\varphi \rangle - \int g(u + \varepsilon\varphi) = \\ &= \frac{1}{2}B_{L,\sigma}(u, u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi, \varphi) + \varepsilon B_{L,\sigma}(u, \varphi) - \langle f, u \rangle - \varepsilon \langle f, \varphi \rangle - \int ga - \varepsilon \int g\varphi \end{aligned}$$

divide by  $\varepsilon$  and  $\varepsilon \rightarrow 0_+$

$$0 \leq B_{L,\sigma}(u, \varphi) - \langle j, \varphi \rangle - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \quad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for  $-\varphi \implies 0 = -||- \implies u$  is weak solution.  $\square$

#### Věta 4.8 (Duel formulation)

Let  $Lu = -\operatorname{div}(A\nabla u)$  with  $A$  elliptic, bounded and symmetric,  $\Gamma_1 \neq \emptyset$ ,  $\Gamma = \emptyset$ ,  $f \in V^*$ ,  $g \in L^2(\Gamma_2)$ ,  $u_0 \in W^{1,2}(\Omega)$ . Then the following are equivalent:

- $u$  is a weak solution;
- $\nabla u = A^{-1}\mathbf{T}$ , where  $\mathbf{T}$  minimizes  $\int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} = \nabla u_0 \mathbf{T}$  over the set  $\tilde{V} := \{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d)\}$ ,  $\forall \varphi \in V$ .

$$\int_{\Omega} \mathbf{T} \cdot \nabla \varphi = \langle f, \varphi \rangle + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div} \mathbf{T} = f \text{ in } \Omega, T\mathbf{u} = g \text{ on } \Gamma_2$$

┌ *Důkaz („1  $\implies$  2“)*

Let  $\mathbf{V} \in \tilde{V}$  and  $\mathbf{T} := A\nabla u \in \tilde{V}$ .

$$\begin{aligned} 0 &\leq \frac{1}{2} \int_{\Omega} A^{-1}(\mathbf{V} - \mathbf{T}) \cdot (\mathbf{V} - \mathbf{T}) = \int \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} \int_{\Omega} A^{-1}\mathbf{T} \cdot \mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} = \\ &= \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \left( \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \right) + \int_{\Omega} (\nabla u_0(\mathbf{V} - \mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T} - \mathbf{V})) = \\ &= \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} (A^{-1}\mathbf{T} - \nabla u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} \nabla(u - u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} + 0. \end{aligned}$$

┌ So  $\mathbf{T}$  is minimizer of the formula above.  $\square$

┌

Důkaz („2  $\implies$  1“)  
 $\mathbf{T} \in \tilde{V} \quad \forall V \in \tilde{V}: \int_{\Omega} \frac{1}{2} A^{-1} \mathbf{T} \cdot \mathbf{T} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V}, \quad \mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}, \quad \mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$   
 $\forall \varphi \in V: \int_{\Omega} \mathbf{W} \cdot \nabla \varphi = 0.$

$$\int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T} + \varepsilon^2 A^{-1} \mathbf{W} \cdot \mathbf{W} + 2\varepsilon A^{-1} \mathbf{T} \cdot \mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by  $\varepsilon$  and  $\varepsilon \rightarrow 0_+$ :

$$0 \leq \int_{\Omega} A^{-1} \mathbf{T} \cdot \mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for  $-\mathbf{W}$ , so  $0 = -||-$ .

Now we find unique  $u \in W^{1,2}$   $u - u_0 \in V: \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} A^{-1} \mathbf{T} \cdot \nabla \varphi$  ( $\langle F, \varphi \rangle_V$ ).

$$\begin{aligned} \int_{\Omega} |A^{-1} \mathbf{T} - \nabla u|^2 &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u)(A^{-1} \mathbf{T} - \nabla u) = \\ &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u_0) \cdot (A^{-1} \mathbf{T} - \nabla u) + \int_{\Omega} \nabla(u_0 - u)(A^{-1} \mathbf{T} - \nabla u) = 0 + 0 = 0 \end{aligned}$$

└

□

### Lemma 4.9

Let  $X$  be a reflexive space and  $\{u^n\}_{n=1}^{\infty}$  be a bounded sequence,  $\|u^n\|_X \leq c < \infty$ . Then  $\exists u^{n_k}$ ,  $\exists u \in X: u^{n_k} \rightharpoonup u$  ( $\forall F \in X^*: \langle F, u^{n_k} \rangle \rightarrow \langle F, u \rangle$ ).

### Věta 4.10 (Spectrum of symmetric operator)

$V$  Hilbert infinity-dimensional space. Let  $B$  be linear, symmetric,  $V$ -elliptic and  $V$ -bounded operator. Then there exist  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$  and corresponding  $\{u_i\}_{i=1}^{\infty}$  such that

- $B(u^k, \varphi) = \lambda_k \int_{\Omega} u^k \varphi;$
- $\lambda_k \rightarrow \infty;$
- $\{u^k\}_{k=1}^{\infty}$  is basis in  $V$  and fulfils

$$\int_{\Omega} u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_{\Omega} u u^i),$  then  $\forall n: \|P^n u\|_2 \leq \|u\|_2$  and  $B(P^n u, P^n u) \leq B(u, u).$

|

|

Dikaz

Step 1: Construct  $\lambda_k, u^k$ :  $\lambda_1 := \inf_{u \in V, \|u\|_2=1} B(u, u)$  and denote  $u^1$  function, where infimum is obtained. Then for  $V^N = \{u \in V | \forall k \in [N] : B(u, u^k) = 0\}$  we do the same.

Step 2: The construction is OK:

$$\begin{aligned} 0 < \lambda_1 &= \lim_{n \rightarrow \infty} B(u^n, u^n), \|u^n\|_2 = 1 \implies \\ &\implies \|u^n\|_V \leq C \implies u^{n_k} \rightharpoonup u \text{ in } V \\ V \hookrightarrow L^2 &\implies u^{n_k} \rightarrow u \text{ in } L^2(\Omega) \implies \|u\|_2 = 1 \\ \lambda_1 &= \lim_{n_k \rightarrow \infty} B(u^{n_k}, u^{n_k}) \geq B(u, u) \geq \lambda_1. \end{aligned}$$

Step 3:  $\lambda_k, u^k$  eigenvalues, eigen functions:  $\forall v \in V, \|v\|_2 = 1, \lambda_1 = B(u^1, u^1) \leq B(v, v), \|u^1\|_2 = 1$

$$v = \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leq B\left(\frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}\right)$$

$$\lambda_1 \|u^1 + \varepsilon \psi\|_2 \leq B(u^1 + \varepsilon \psi, u^1 + \varepsilon \psi) = B(u_1, u_1) + \varepsilon^2 B(\psi, \psi) + 2\varepsilon B(u, \psi) \leq \lambda_1 \|u^1\|_2^2 + \lambda_1 \varepsilon^2 \|\psi\|_2^2 + 2\varepsilon \lambda_1 \int_{\Omega} u^1 \psi$$

$$\varepsilon \rightarrow 0_+ \implies 2\lambda_1 \int_{\Omega} u^1 \psi \leq 2B(u, \psi).$$

So  $\lambda_1 \int_{\Omega} u^1 \psi = B(u, \psi)$ .

The same way we obtain  $\lambda_k \int_{\Omega} u^k \psi \leq B(u, \psi)$  for  $\psi \in V^N$ .

$$u^1 : \lambda_1 \int_{\Omega} u^1 \psi = B(u^1, \psi) \implies \psi = u^k \int_{\Omega} u^1 u^k = V(u_1, u^k).$$

But  $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$ , so  $\int u^1 u^k = B(u^1, u^k) = 0$ .

$$\implies \forall i \in [k-1] : \int_{\Omega} u^k u^1 = B(u^k, u^i) = 0.$$

Step 4:  $\lambda_k \nearrow \infty$ . We already know  $\lambda_1 \leq \lambda_2 \leq \dots$ . Assume a contradiction  $\lambda_k \leq C < \infty$ .  $c_1 \|u^k\|_V^2 \leq B(u^k, u^k) = \lambda_k \|u^k\|_2^2 = \lambda_k < C$ .

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \rightarrow u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$\|u^n - y^m\|_2^2 = \|u^n\|_2^2 + \|u^m\|_2^2 - 2 \int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_1 n} B(u^n, u^m) = 2 \implies \text{not Cauchy.}$$

Step 5:  $\lambda_k$  are all eigenvalues ( $u^k$  is basis of  $V$  and of  $L^2$ ). Assume that  $\lambda \neq \lambda_j$  is also eigenvalue, so  $\exists u : B(u, \varphi) = \lambda \int_{\Omega} u \varphi \forall \varphi$ . We can find  $i \in \mathbb{N}$ , so  $\lambda_i < \lambda < \lambda_{i+1}$ .

$$B(u, u^j) = \lambda \int_{\Omega} u u^j \wedge B(u^j, u) = \lambda_j \int_{\Omega} u^j u \implies B(u, u_j) = 0$$



## 4.4 Regularity of weak solution

*Poznámka*

We assume that we have  $u \in W^{1,2}(\Omega)$  a weak solution

$$-\operatorname{div} A \nabla u + Vu + \mathbf{c} \cdot \nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When  $u \in W_{loc}^{2,2}(\Omega)$ , when  $u \in W^{2,2}(\Omega)$ , when  $u \in W_{loc}^{k,2}(\Omega)$ ,  $u \in W^{k,2}(\Omega)$ .

Simplify  $-\operatorname{div} A \nabla u = f - bu - \mathbf{c} \nabla u - u \operatorname{div} \mathbf{d} - \nabla u \cdot \mathbf{d} = \tilde{f}$ . If  $u \in W^{1,2}$ ,  $f \in L^2$ ,  $b \in L^\infty$ ,  $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$ .

Problem is reduced to

$$\begin{aligned} -\operatorname{div}(A \nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_1, \\ (A \nabla u) \cdot \mathbf{v} &= g \text{ on } \Gamma_2, \\ (A \nabla u) \cdot \mathbf{v} + \sigma u &= g \text{ on } \Gamma_3. \end{aligned}$$

### Definition 4.5 (Interior regularity)

$u \in W_{loc}^{2,2}(\Omega)$ ; assumptions:  $A \in W^{k+1,\infty}$ ,  $f \in W^{k,2}(\Omega) \implies u \in W_{loc}^{k+1,2}(\Omega)$ .

### Definition 4.6 (Boundary regularity)

$u \in W^{2,2}(\Omega)$ ; assumptions: on  $\Omega \in C^{k+1,\infty}$ ,  $g \in W^{\frac{1}{2},2}(\partial\Omega)$  and  $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\emptyset\} \implies u \in W^{2,2}(\Omega)$ .

### Věta 4.11 (Interior regularity)

Let  $A$  be an elliptic operator and  $u \in W^{1,2}$  solves

$$\int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega) \quad \forall f \in L^2(\Omega).$$

Then if  $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$ ,  $f \in W^{k,2}(\Omega)$  then  $u \in W_{loc}^{k+2,2}(\Omega)$ .

Moreover  $\forall \tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega \exists c(\tilde{\Omega}, A)$ :

$$\|u\|_{W^{k+2,2}(\tilde{\Omega})} \leq c(\|f\|_2 + \|u\|_{W^{1,2}(\Omega)}).$$

Důkaz

$k = 0$ : Recall  $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x + he_k) - v(x)|^2}{h^2} \leq c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{ u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2 \right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left| \frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j} \right|^2}{h^2} \leq c,$$

$$\int_{\Omega_h} \left| \frac{\nabla u(x + he_i) - \nabla u(x)}{h} \right|^2 \leq c.$$

$$\int_{\Omega} A \nabla u \nabla \varphi = \int_{\Omega} f \varphi$$

$$h > 0, \varphi \in W_0^{1,2}(\Omega), \varphi(x) = 0 \text{ if } \text{dist}(x, \partial\Omega) \subset h.$$

Set  $\varphi(x) := \psi(x - he_k)$ .

$$\begin{aligned} \implies \int_{\Omega} A(x) \nabla u(x) \nabla \psi(x - he_k) &= \int_{\Omega} f(x) \psi(x - he_k) = \\ &= \int_{\Omega} A(x + he_k) \nabla u(x + he_k) \cdot \nabla \psi(x) dx. \end{aligned}$$

Set  $\varphi(x) := \psi(x)$ :

$$\int_{\Omega} A(x) \cdot \nabla u(x) \cdot \nabla \psi(x) = \int_{\Omega} f(x) \psi(x) dx.$$

$$\begin{aligned} &\int_{\Omega} A(x + he_k) (\nabla u(x + he_k) - \nabla u(x)) \cdot \nabla \psi(x) = \\ &= - \int_{\Omega} (A(x + he_k) - A(x)) \nabla u(x) \cdot \nabla \psi(x) + \int_{\Omega} f(x) (\psi(x - he_k) - \psi(x)). \end{aligned}$$

Set  $\psi := (u(x + he_k) - u(x)) \tau^2(x)$ ,  $\tau(x) = 0$ , if  $\text{dist} \in (x, \partial\Omega)$ ,  $\tau \in C^1(\tilde{\Omega})$ .

Evaluate all terms ( $w^{h,i} = u(x + he^i) - u(x)$ ):

$$\begin{aligned} &\int_{\Omega} A(x + he_i) \nabla w^{h,i} \cdot (\nabla w^{h,i} \tau^2 + 2w^{h,i} \tau \nabla \tau) \geq \\ &\stackrel{ellip.}{\geq} c_1 \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \int_{\Omega} \frac{2 \|A\|_{\infty} |w^{h,i}| - |\nabla \tau| (|\nabla w^{h,i}| \sqrt{c_1} \tau)}{\sqrt{c_1}} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2}{c_1} \|A\|_{\infty}^2 \|\nabla \tau\|_{\infty}^2 h^2 \int_{\Omega_h} \frac{|u(x + he_i) - u(x)|^2}{h^2} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2 \|A\|_{\infty}^2 \|\nabla \tau\|_{\infty}^2 h^2 c_1 \|\nabla u\|_2^2}{c_1} \end{aligned}$$

TODO?

### Věta 4.12 (Regularity up to the boundary)

Let  $u$  be a weak solution  $-\operatorname{div}(A\nabla u) = f$  in  $\Omega$ ,  $A\nabla u \cdot \mathbf{v} = g$  on  $\Gamma_2$ ,  $A\nabla u \cdot \mathbf{v} + \sigma u = g$  on  $\Gamma_3$ ,  $u = u_0$  on  $\Gamma_1$ .

Assume that  $\Omega \in C^{k+1,\infty}$ ,  $A \in W^{k,\infty}$ ,  $f \in W^{k-1,2}$ ,  $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega)$ ,  $\sigma \in W^{k,\infty}(\partial\Omega)$  and  $\Gamma_1, \Gamma_2, \Gamma_3$  are smooth open (in partial  $\Omega$ ) and  $\overline{\Gamma_i} \cap \overline{\Gamma_j} = \emptyset \ \forall i \neq j$ .

Then  $u \in W^{k+1,2}(\Omega)$ .

┌

Důkaz (Step 1: Flat boundary)

$\Omega = (-1, 1)^{d-1} \times (0, 1)$ . Assume that  $u \in W^{1,2}(\Omega)$  and  $u = 0$  on  $(x, 0)$ . We want that  $u \in W^{2,3}((-1 + \delta, 1 - \delta)^{d-1} \times (0, 1 - \delta))$ .

1a tangential derivatives  $\frac{\partial u}{\partial x_1} \in W^{1,2}(-||-)$ . 1b normal derivative  $\frac{\partial^2 u}{\partial x_d^2} \in L^2(-||-)$ .

1a: WF  $-\int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \ \forall \varphi \in W_0^{1,2}(\Omega)$ . Take continuous  $\tau = 1$  in  $-||-$  and  $\tau = 0$  in  $\Omega \setminus \text{"inflated"} -||-$ .

$$\varphi(x) = \psi(x - he_i)\tau, \quad i \in [d-1], \psi \in W_0^{1,2}(\Omega \setminus \text{"inflated"} -||-)$$

Redefine interior regularity

$$\int_{\Omega} (A(x + he_i) \nabla u(x + he_i) - A(x) \nabla u(x)) \nabla \varphi(x) = \int_{\Omega} f(\psi(x - he_i) - \psi(x)).$$

Set  $\psi = (u(x + he_i) - u(x))\tau^2 \in W_0^{1,2}$  and apply local regularity.

1b:  $\varphi \in C_0^{\infty}(-||-)$

$$\begin{aligned} - \int_{\Omega} \sum_{i,j}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) \varphi &= - \int_{\Omega} \operatorname{div}(A \nabla u) \varphi = \int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \\ - \int_{\Omega} a_{dd} \frac{\partial^2 u}{\partial x_d^2} \varphi &= \underbrace{\int_{\Omega} f \varphi}_{\in L^2(\Omega)} + \int_{\Omega} \varphi \left( \sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i,j=1, \neg(i=j=d)}^d \right) a_{ij} \frac{\partial u}{\partial x_i x_j}. \\ \|a_{dd} \frac{\partial^2 u}{\partial x_d^2}\|_2^2 &\leq \|f + \sum \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{\neg(i=j=d)} a_{ij} \frac{\partial^2 u}{\partial x_i x_j}\|_2^2 \leq C. \end{aligned}$$

$A$  is elliptic

$$c_1 |\zeta|^2 \leq a_{ij} \zeta_i \zeta_j$$

Special choice  $\zeta = (0, \dots, 0, 1)$ ,  $0 < c_1 \leq a_{dd}(x) \implies \|\frac{\partial^2 u}{\partial x_d^2}\|_{L^2}^2 \leq C(DATA?)$  □

└

┌ *Důkaz* (Step 2: Transfer from flat to small parts of  $\partial\Omega$ )  
 TODO!!!

└ TODO!!! □

┌ *Důkaz* (Step 3: Introduce a proper covering of  $\partial\Omega$  and use step 2)

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

?  $\Omega \ u \in W_{loc}^{2,2}(\Omega)$ . ? of  $\partial\Omega$ , apply step 2.

Define  $w := u - u_0 \in W_0^{1,2}(\Omega)$ .

$$-\operatorname{div}(A\nabla w) = f + \operatorname{div}(A\nabla u_0)$$

└ if  $f \in L^2$  and  $\operatorname{div}(A\nabla u_0) \in L^2$ , e.g.  $A \in W^{1,\infty} \wedge u_0 \in W^{2,2}(\Omega)$ . □

## 5 Bochner integral

### Definition 5.1 (Measurability)

We say that  $f : I \rightarrow X$  is measurable (strongly, Bochner) if  $\exists \{s_j\}_{j=1}^\infty$  simple functions,  $\|f(t) - s_n(t)\|_X \rightarrow 0$  as  $n \rightarrow \infty$  for almost every  $t \in I$ .

### Věta 5.1 (Measurability)

$f : I \rightarrow X$  is measurable iff

1.  $f$  is almost separably valued;

$$\exists E \subset I : |E| = 0, f(I \setminus E) \text{ is separable.}$$

2.  $f$  is weakly measurable;

$$\forall F \in X^* : \langle F^*, u(t) \rangle_X \text{ is Lebesgue measure w.r.t } t \in I.$$

### Definition 5.2 (Bochner integral for simple function)

Let  $s : I \rightarrow X$  be a simple function on  $I$ . We define

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

**Definice 5.3** (Bochner integral for measurable functions)

Let  $s : I \rightarrow X$  be a Bochner measurable function. We say that  $f$  is Bochner integrable if  $\exists \{s^n\}_{n=1}^\infty$  such that  $s^n(t) \rightarrow f(t)$  a. a. t and  $\int_I \|s^n(t) - f(t)\|_X dt \rightarrow 0$  as  $n \rightarrow \infty$  and we set

$$X \ni \int_I f(t) dt = \lim_{n \rightarrow \infty} \int_I s^n(t) dt.$$

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

**Definice 5.4** ( $L^p(O, T, X)$  space)

Let  $X$  be a Banach space

$$L^p(O, T, X) = \left\{ f : (O, T) \rightarrow X \text{ bochner integrable} \mid \int_I \|f(t)\|_X^p < \infty \right\}$$

$$\|f\|_{L^p(O, T, X)} = \left( \int_I \|f(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

**Věta 5.2** (Dual space)

Let  $X$  be a Banach space, separable and  $p \in [1, \infty)$ , then

$$(L^p(O, T, X))^* = L^{p'}(O, T, X^*)$$

## 5.1 Sobolev-Bochner spaces

**Definice 5.5**

Let  $f : I \rightarrow X$  be Bochner integrable. We say that  $g : I \rightarrow X$  is a weak derivative of  $f$  w. r. t. iff  $g$  is Bochner integrable and  $\forall \tau \in C_0^\infty(I) : \int_I f(t) \tau'(t) dt = - \int_I g(t) \tau(t) dt$ .

*Poznámka*

If  $f \in L^1(I, x)$  and  $\frac{\partial f}{\partial t} \in L^1(I, x)$ , then  $f \in C(I, x)$ .

**Věta 5.3**

$$W^{1,p}(I, X) := \{f \in L^p(I, x), \partial_t f \in L^p(I, X)\}, \quad \|f\|_{W^{1,p}(I, X)} = \begin{cases} (\int_I \|f\|_X^p + \|\partial_t f\|_X^p)^{\frac{1}{p}}, & p \in [1, \infty) \\ \text{esssup}_{t \in I} (\|f(t)\|_X + \|\partial_t f\|_X), & p = \infty \end{cases}$$

Then  $W^{1,p}(I, X)$  is a Banach space, is separable for  $p < \infty$  and  $X$  separable and

is reflexive if  $p \in (1, \infty)$  and  $X$  is reflexive and separable.

## 5.2 Time derivative in heat/wave equations – Gelfand triple

*Poznámka* (Motivation)

$$\partial_t u - \Delta u = 0 \text{ in } (0, T) \times \Omega, u = 0 \text{ on } (0, T) \times \partial\Omega, x(0, x) = u_0(x) \text{ for } x \in \Omega, \quad \Omega \subseteq \mathbb{R}^d$$

### Definition 5.6 (Gelfand triple)

We say that  $X, H, X^*$  is Gelfand triple iff  $X \xrightarrow{\text{dense}} H \cong H^* \xrightarrow{\text{dense}} X^*$ .

┌ *Například*

$$X = W_0^{1,2}(\Omega), H = L^2(\Omega), X^* = (W_0^{1,2}(\Omega))^*,$$

Neboť  $W_0^{1,2}$  is dense in  $C_0 \xrightarrow{\text{dense}} L^2(\Omega)$  and  $f \in (W_0^{1,2}(\Omega))^* \implies \exists! u \in W_0^{1,2}(\Omega) : -\Delta u = f$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ .

$$\forall \varphi \in W_0^{1,2}(\Omega) \langle f, \varphi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u^n \cdot \nabla \varphi = \lim_{n \rightarrow \infty} - \int_{\Omega} \Delta u^n \varphi = \lim_{n \rightarrow \infty} (f^n, \varphi)_{L^2(\Omega)},$$

where  $\{u^n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ ,  $u^n \rightarrow u$  in  $W_0^{1,2}(\Omega)$ ,

$$(X = W_0^{1,p}(\Omega) \cap L^2(\Omega), H = L^2(\Omega))$$

### Definition 5.7

Let  $X, H, X^*$  be Gelfand triple,  $\varphi : H \rightarrow H^*$  is Riesz representation and define  $i : X \rightarrow X^*$ , such that  $\forall x_0, x \in X$ :

$$\langle i(x_0), x \rangle_X := (\text{id}(x_0), \text{id}(x))_H = \langle \varphi \text{id}(x_0), \text{id}(x) \rangle_{H^*},$$

$i$  maps  $X$  densely onto  $X^*$ .

### Lemma 5.4

Let  $u \in L^1(0, T, H)$ ,  $\partial_t u \in L^1(0, T, X^*)$  and  $X, H, X^*$  be a Gelfand triple. Then  $\forall w \in X \forall \tau \in C_0^1(0, T)$  we have

$$\begin{aligned} \int_0^T \langle \partial_t u, w \rangle \tau dt &= \langle \int_0^T \partial_t u \tau dt, w \rangle_X = \\ &= - \langle \int_0^T u \tau' dt, w \rangle_X = - \int_0^T \langle u \tau', w \rangle_X dt = \end{aligned}$$

$$= - \int_0^T (u\tau', w)_H dt \stackrel{\text{if } \partial_t u \in L^1(0,T)}{=} \int_0^T (\partial_t u \tau, w)_H.$$

**Věta 5.5** (Integration by parts for Sobolev-Bochner function)

Let  $p \in (1, \infty)$ ,  $X, H, X^*$  a Gelfond triple,  $u, v \in L^p(0, T, X)$ ,  $\partial_t u, \partial_t v \in L^{p'}(0, T, X^*)$ . Then  $u, v \in C([0, T], H)$  and  $\forall 0 \leq t_1 < t_2 \leq T$ .

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

┌  
Důkaz

Step 1) Modify  $u, v$  in terms of the Steklov ar?  $u_h = \int_t^{t+h} u(\tau) d\tau$ .

Step 2) Prove for  $u_h, v_h$  from step 1).

Step 3)  $h \rightarrow 0_+$ .

□

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Důkaz („Step 1“)

Define  $u_h(t) := \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$ ,  $\forall t \in (0, T-h)$ .  $u_h \rightarrow u$  in  $L^p(0, T-h, X)$ ,  $\forall h_0 \in (0, T)$ . We want „ $(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$ “.

$$(\partial_t u)_h \rightarrow \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \quad \forall h_0 \in (0, T).$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^{T-h} u_h(t) \varphi'(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \int_t^{t+h} u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \left( \int_0^{t+h} u(\tau) d\tau - \int_0^t u(\tau) d\tau \right) = \\ &= -\frac{1}{h} \int_0^{T-h} \varphi(t) (u(t+h) - u(t)) dt \Leftrightarrow \partial_t u_h = \frac{u(t+h) - u(t)}{h}. \end{aligned}$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^T \varphi(t) (\partial_t u)_h(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi(t) \int_t^{t+h} \partial_t u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi(t) \left( \int_0^{t+h} \partial_t u(\tau) d\tau - \int_0^t \partial_t u(\tau) d\tau \right) dt = (*) \end{aligned}$$

$$\frac{1}{h} \int_0^{T-h} \varphi(t) \left( \int_0^t \partial_t u(\tau) d\tau \right) dt = \int_0^{T-h} \int_0^{T-h} \varphi(t) \partial_t u(\tau) \chi_{\tau \leq t} d\tau dt = \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \left( \int_t^{T-h} \varphi(t) dt \right) d\tau.$$

$$(*) = \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \underbrace{\left( \int_{\tau-h}^\tau \varphi(t) dt \right)}_{C_0^\infty(0, T)} d\tau = -\frac{1}{h} \int_0^{T-h} u(\tau) (\varphi(\tau) - \varphi(\tau-h)) d\tau.$$

└

□



┌ *Dikaz* („Step 2“)

We want

$$\begin{aligned} \int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \Leftrightarrow \\ \Leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \end{aligned}$$

$$\begin{aligned} \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau)_H dt = \frac{1}{h_1 h_2} \int_{t_1}^t (u(t+h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int_{t_1}^t v(\tau) d\tau)_H dt \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^t v(\tau+h_2) d\tau - \int_{t_1}^t v(\tau) d\tau)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau)_H dt + \int_{t_1}^{t_2} \left( u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_1}^{t_2} u(t+h_1) - u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left( u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h_1} u(t) - \int_{t_2}^{t_2+h_1} u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left( u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt \\ &\quad + \int_{t_1}^{t_2} \left( \frac{v(\tau+h_2) - v(\tau)}{h_2}, \int_{\tau}^{\tau+h_1} u(t) dt \right)_H d\tau + \\ &\quad + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h_1} u(t) dt)_H d\tau + \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau) d\tau)_H dt = \\ &\quad - \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau)) d\tau + REST \\ REST &= \frac{1}{h_1 h_2} \left( \int_{t_2}^{t_2+h_2} v(t) dt - \int_{t_1}^{t_1+h_2} v(t) dt, \int_{t_2}^{t_2+h_2} u(t) dt \right) + SIMILAR = \\ &= (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \dots \end{aligned}$$

└

□

┌ *Důkaz („Step 3“)*

We have

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let  $h_1 \rightarrow 0_+$  and  $h_2 \rightarrow 0_+$ . We have  $\partial_t u_{h_1} \rightarrow \partial_t u$  in  $L^{p'}(0, T, X^*)$ ,  $\partial_t v_{h_2} \rightarrow \partial_t v$  in  $L^{p'}(0, T, X^*)$ ,  $u_{h_1} \rightarrow u$  in  $L^p(0, T, X)$ ,  $v_{h_2} \rightarrow v$  in  $L^p(0, T, X)$ . So for almost all  $t$  in  $(0, T)$ :  $v_{h_2}(t) \rightarrow v(t)$  in  $X \hookrightarrow H$  and  $u_{h_1}(t) \rightarrow u(t)$  in  $X \hookrightarrow H$ .

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Now, it is enough to show  $u, v \in C([0, T], H)$ . We show that  $u_h$  is Cauchy in  $C([0, T], H)$ .

Use IBP  $u_{h_1} = u_{h^n} - u_{h^m}$ ,  $v_{h_2} = u_{h^n} - u_{h^m}$ :

$$\|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H = \|u_{h^m}(t_1) - u_{h^n}(t_1) + 2 \int_{t_1}^{t_2} \langle \partial_t (u_{h^n} - u_{h^m}), u_{h^n} - u_{h^m} \rangle_X \|$$

$$\begin{aligned} \|u_{h^n} - u_{h^m}\|_{C([\frac{T}{4}, T], L^2(\Omega))}^2 &= \sup_{t_2 \in (\frac{T}{2}, T)} \|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H^2 \leq \\ &\leq \|u_{h^m}(t_1) - u_{h^n}(t_1)\|_H^2 + \int_0^T \|\partial_t(u_{h^n}) - \partial_t(u_{h^m})\|_{X^*} \|u_{h^n} - u_{h^m}\|_X dt. \end{aligned}$$

Choose  $t_1$  such that  $u_h(t_1) \rightarrow u(t_1)$  in  $H$ :

$$\leq \|u_h(t_1) - u_{h^m}(t_1)\|_H^2 + \|\partial_t u_{h^m} - \partial_t u_{h^n}\|_{L^p(X^*)} \cdot \dots$$

$$u \in C([\frac{T}{4}, T], L^2(\Omega)) \wedge u \in C([0, \frac{3T}{4}], L^2(\Omega)) \rightarrow u \in C([0, T], L^2(\Omega)) (u(t_1), v(t_1))_H$$

└

□

## 6 Parabolic equations

*Poznámka*

$\Omega$  open set in  $\mathbb{R}^d$ ,  $T > 0$ ,  $L$  elliptic operator,

$$\partial_t u + Lu = f \text{ in } Q = (0, T) \times \Omega, \quad u = 0 \text{ on } (0, T) \times \partial\Omega, \quad u(0, x) = u_0(x) \text{ } x \in \Omega.$$

$$Lu = -\operatorname{div}_x(A(t, x)\nabla_x u(t, x)) + b(t, x)u(t, x) + \mathbf{c}(t, x)\nabla u(t, x) + \operatorname{div}(\mathbf{d}(t, x)u(t, x)),$$

$$A, b, \mathbf{c}, \mathbf{d} \in L^\infty(\Omega).$$

$$A(t, x) \cdot \xi \cdot \xi \geq c_1 |\xi|^2, \forall \xi \in \mathbb{R}^d \text{ and almost all } (t, x) \in Q.$$

## 6.1 Formal a priory estimates

*Poznámka*

Multiply by  $u$  and  $\int_{\Omega} dx$  and use IBP.

$$\int_{\Omega} \partial_t u u + \int_{\Omega} A \nabla u \nabla u = \int_{\Omega} f u - b u^2 - \mathbf{c} \nabla u u + \mathbf{d} u \nabla u.$$

Hölder's inequality:

$$\frac{d}{dt} \frac{\|u\|_2^2}{2} + C_1 \|\nabla u\|_2^2 \leq \|f\|_2 \|u\|_2 + \|b\|_{\infty} \|u\|_2^2 + \|\mathbf{c}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 \leq C_1 \frac{\|\nabla u\|_2^2}{2} + C(\mathbf{c})$$

Poincaré's inequality:

$$\frac{d}{dt} \|u\|_2^2 + \mathbf{c} \|u\|_{1,2}^2 \leq C(\mathbf{c}, \mathbf{d}, b) \|f\|_2^2 + K \|u\|_2^2.$$

Grönwall's inequality:

$$\sup_{t \in (0, T)} \|u(t)\|_2^2 \leq + \int_0^T \|f\|_2^2$$

$$\int_0^T \|u\|_{1,2}^2 dt \leq C$$

$$\|\partial_t u\|_{(W_0^{1,2})^*} = \sup_{\|\varphi\| \leq 1} \langle \partial_t u, \varphi \rangle = \sup \langle f - Lu, \varphi \rangle =$$

$$= \sup_{\|\varphi\| \leq 1} \int_{\Omega} (f - ? - bu - \mathbf{c} \nabla u) \varphi - \int_{\Omega} (A \nabla u - \mathbf{d} u) \nabla \varphi \leq \int_0^T \|f\|_2^2 + c(\|u\|_2^2 + \|\nabla u\|_2^2).$$

### Definice 6.1

Let  $\Omega \subseteq \mathbb{R}^d$  open and bounded,  $L$  be an elliptic operator,  $u_0 \in L^2(\Omega)$ ,  $f \in L^2(0, T, V^*)$  ( $V = W_0^{1,2}(\Omega)$ ). We say that  $u$  is a weak solution to

$$\partial_t u + Lu = f \text{ in } (0, T) \times \Omega,$$

$$u = 0 \text{ on } (0, T) \times \partial\Omega,$$

$$u(0) = u_0 \text{ in } \Omega$$

iff  $u \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$ ,  $u(0) = u_0$  and for almost all  $t \in (0, T)$  and  $\forall \varphi \in V$ :

$$\langle \partial_t u, \varphi \rangle_V + \int_{\Omega} A \nabla u \cdot \nabla \varphi + bu \varphi + \mathbf{c} \cdot \nabla u \varphi - \mathbf{d} \nabla \varphi u = \langle f, \varphi \rangle_V.$$

## 6.2 Existence and uniqueness

## Věta 6.1

Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded,  $f \in L^2(0, T, V^*)$ ,  $u_0 \in L^2(\Omega)$  and  $L$  be elliptic operator. Then  $\exists! u$  – weak solution.

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*Důkaz* (Uniqueness)

$u_1, u_2$  are weak solutions. Define  $w := u_1 - u_2 \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$ . WF for  $u_1$  – WF for  $u_2$ :

$$\langle \partial_t w, \varphi \rangle + \int_{\Omega} A \nabla w \cdot \nabla \varphi = \int_{\Omega} -b w \varphi - \mathbf{c} \nabla w \varphi + \mathbf{d} \cdot \nabla \varphi w.$$

Follow almost everywhere, replace  $u$  by  $w$ . Set  $\varphi = w \implies$

$$\langle \partial_t w, w \rangle + \mathbf{c} \|w\|_{1,2}^2 \leq c \|w\|_2^2.$$

Integrate in respect of time, use IBP-formula for  $\langle \cdot, \cdot \rangle$ :

$$\begin{aligned} \int_0^t \langle \partial_t w, w \rangle &= \frac{1}{2} \|w(t)\|_2^2 - \frac{1}{2} \|w(0)\|_2^2 = \frac{1}{2} \|w(t)\|_2^2. \\ \implies \|w(t)\|_2^2 &\leq c \int_0^t \|w(\tau)\|_2^2 d\tau \\ \frac{d}{dt} \underbrace{\int_0^t \|w(\tau)\|_2^2 d\tau}_{=: g(t) \geq 0} &\leq c \int_0^t \|w(\tau)\|_2^2 d\tau \\ g' &\leq c \cdot g \end{aligned}$$

From Grönwall's inequality

$$g(t) \leq e^{c \cdot t} g(0) \implies \int_0^t \|w(\tau)\|_2^2 d\tau \leq e^{ct} \int_0^0 \|w(\tau)\|_2^2 d\tau = 0 \implies w(t) = 0.$$

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□

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*Důkaz* (Existence (via Galerkin approximation))

We know  $\exists \{w_j\}_{j=1}^{\infty}$  basis of  $\mathbf{V}$ , which is ortonormal in  $L^2$  and  $\|P^N u\|_V \leq c \|u\|_V$ , where  $P^N$  is orthogonal projection in  $L^2(\Omega)$  onto  $\{w_j\}_{j=1}^N$ .

We solve for  $u^n(t, x) = \sum_{i=1}^n a_i^n(t) w_i(x)$ . We want

$$\langle \partial_t u^n, w_j \rangle = - \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n + \langle f, w_j \rangle$$

for  $j \in [n]$  for almost all  $t \in (0, T)$  (weak formulation of the problem for n, call it WF<sub>n</sub>). □

└

┌ *Důkaz* („Existence of  $u^n$ “)

LHS of WFn:

$$\sum_{i=1}^n \langle \partial_t a_i^n w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n(t) \langle w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n \delta_{ij} = \partial_t a_j^n(t).$$

RHS of WFn:

$$\sum_{i=1}^n a_i^n(t) \left( \underbrace{- \int_{\Omega} A \nabla w_i \nabla w_j + b w_i w_j + \mathbf{c} \nabla w_i w_j - \mathbf{d} \nabla w_j w_i}_{G_{ij}(k) - \text{bounded and measurable?}} \right) + \underbrace{\langle f(t), w_j \rangle}_{g_j^{(t)} - \text{measurable? on } g \in L^2(0,T)}.$$

So

$$\frac{d}{dt} a_j^n(t) = \sum_{i=1}^n a_i^n(t) G_{ij}(t) + g_j(t), \quad j \in [n].$$

Initial data:  $u^n(0) := P^n u_0$  ( $a_j^n(0) := \int_{\Omega} u_0 w_j$ ).

ODE  $\implies \exists \tilde{T} \leq T$  and  $a_i^n(t) \in AC$  on  $[0, \tilde{T})$  and solve for almost all  $t \in (0, \tilde{T})$ .  
Moreover either we can set  $\tilde{T} = T$  or  $|a^n(t)| \xrightarrow{t \rightarrow \tilde{T}} \infty$ .

Now we prove  $\tilde{T} = T$ . We show  $|a^n(t)| \leq c$ .

Multiply WFn for  $j$  by  $a_j^n(t)$  and sum it:

$$LHS = \sum_{j=1}^n a_j^n \langle \partial_t u_j^n, w_j \rangle = \langle \partial_t u^n, \sum_{j=1}^n a_j^n w_j \rangle = \langle \partial_t u^n, u^n \rangle.$$

$$RHS = \underbrace{\langle \partial_t u^n, u^n \rangle}_{= \frac{d}{dt} \|u^n\|_2^2} + c_1 \|u^n\|_V^2 \leq c(\|f\|_{V^*}^2 + \|u^n\|_2^2).$$

Grönwall:  $\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c(\|u^n(0)\|_2^2) + \int_0^T \|f\|_{V^*}^2$ .

$$\forall t < \tilde{T} : \|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c \left( \int_0^{\tilde{T}} \|f\|_{V^*}^2 + \|u_0\|_2^2 \right) \leq \tilde{c}.$$

$$\lim_{t \rightarrow \tilde{T}_-} |a^n(t)|^2 = \lim_{t \rightarrow \tilde{T}_-} \|u^n(t)\|_2^2 \leq \tilde{c}.$$

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$$\|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \leq c(f, u_0).$$

Time derivative

$$\|\partial_t u^n\|_{V^*} = \sup_{w \in V, \|w\| \leq 1} \langle \partial_t u^n, w \rangle = \sup_{w \in V, \|w\| \leq 1} \int_{\Omega} \partial_t u^n w = \sup_{\dots} \int_{\Omega} \partial_t u^n P^n w.$$

WFn:

$$\begin{aligned} &\leq \sup_{\dots} c(\|f\|_{V^*} + \|u^n\|_V) \|P^n w\|_V \leq \sup_{\dots} \tilde{c}(\|f\|_{V^*} + \|u^n\|_V) \|w\|_V \leq \\ &\leq (\tilde{\|f\|_{V^*}} + \|u^n\|_V). \end{aligned}$$

$$\int_0^T \|\partial_t u^n(t)\|_{V^*}^2 \leq \tilde{c} \int_0^T (\|f(t)\|_{V^*}^2 + \|u^n(t)\|_V^2) \leq c(f, u_0).$$

$u^n$  is a bounded sequence in  $L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$  so  $\exists$  a subsequence  $u^{m_n}$ :

$$u^{m_n} \rightharpoonup u \text{ in } L^2(0, T, V), \quad \partial_t u^{m_n} \rightharpoonup \partial_t u \text{ in } L^2(0, T, V^*).$$

To show  $u$  is a weak solution.

└ TODO?

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