

1 Σ_1^1 sets and trees on ω

Poznámka (Notation)

- $\mathbb{S} := \omega^{<\omega}$;
- $\nu|_k = (\nu(0), \dots, \nu(k-1))$, $\nu \in \mathbb{S} \cup \omega^\omega$ ($\nu|_0 = \emptyset$, empty sequence);
- $t < s \equiv \exists s' \in \mathbb{S} \cup \mathcal{N} : s = t^\wedge s'$ ($t \in \mathbb{S}$, $s \in \mathbb{S} \cup \mathcal{N}$);
- $\mathcal{N} := \omega^\omega$;
- $|s|$ is the length of s , $s \in \mathbb{S}$ ($s = (s(0), \dots, s(k-1)) \implies |s| = k$);
- $s \in \mathbb{S}$, $\nu \in \mathbb{S} \cup \mathcal{N}$: $s^\wedge \nu = (s(0), \dots, s(|s| - 1), \nu(0), \dots)$.

Definition 1.1 (Souslin set (on TP space))

X topological space. We say $S \subset X$ be Souslin $\Leftrightarrow \exists (F_s)_{s \in \mathbb{S}}$ Souslin scheme of closed subset of X such that $S = \mathcal{A}_s(F_s) = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n \in \omega} F_{\sigma|_n}$.

Poznámka

- P Polish topological space, then $A \in \Sigma_1^1 \Leftrightarrow A$ Souslin in P . (We already know.)
- P topological space, then $A \subset P$ Souslin $\Leftrightarrow \exists F \in \Pi_1^0(\mathcal{N} \times P) : A = \Pi_P(F)$. (Difficult.)
- We will assume only regular Souslin scheme (RSS): $F_{s^\wedge t} \subset F_s$, $s, t \in \mathbb{S}$ and $F_\emptyset = P$.

1.1 Souslin operation and trees

Definition 1.2 (Trees on ω , infinite branch, ill-founded trees, well-founded trees)

We define set of trees \mathcal{T} by $\mathcal{T} := \{T \in \mathcal{P}(\mathbb{S}) \mid \forall s \in T, t \in T : t < s \implies t \in T\}$.

$T \in \mathcal{T}$ has infinite branch $\equiv \exists \sigma \in \mathcal{N} \forall n \in \omega : \sigma|_n \in T$ (i.e. $\sigma \in [T]$) (i.e. $[T] \neq \emptyset$).

Trees with infinite branches are called ill-founded (IF). The set of IF trees is denoted by \mathcal{T}_I . Trees without infinite branches are called well-founded (WF). The set of WF trees is denoted by \mathcal{T}_W .

$\mathcal{T}_s := \{T \in \mathcal{T} \mid s \in T\}$ are all trees containing $s \in \mathbb{S}$.

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Poznámka

$\mathbb{T}_I = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n \in \omega} \mathcal{T}_{\sigma|_n}$.

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$$\mathcal{T}^* := \mathcal{T} \setminus \{\emptyset\}, \mathcal{T}_W^* = \mathcal{T}_W \setminus \{\emptyset\}.$$

Lemma 1.1

Let X be a topological space, $(F_s)_{s \in \mathbb{S}}$ RSS of closed subsets of X , $S := \mathcal{A}_s(F_s)$. Define $f(x) : X \rightarrow \mathcal{T}^*$ by $f(x) := \{s \in \mathbb{S} \mid x \in F_s\}$. Then $F_s = f^{-1}(\mathbb{T}_s)$ and $S = f^{-1}(\mathcal{T}_I)$.

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Důkaz (?)

a) „ $f : X \rightarrow \mathcal{T}^*$ “: $s \in f(x) \implies x \in F_s \implies F_s \subset F_t \implies x \in F_t \implies t \in f(x) \ (t < s)$.

b) $x \in F_s \Leftrightarrow s \in f(x) \Leftrightarrow f(x) \in \mathcal{T}_s \Leftrightarrow x \in f^{-1}(\mathbb{T}_s)$

c) lemma \Leftarrow b) and the next remark. □

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Poznámka

TODO!!! $\mathcal{T} \rightarrow \mathcal{T}^*$.

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Důkaz

„ \implies “: lemma?. „ \Leftarrow “: $S = f^{-1}(\mathbb{T}_I) = f^{-1}(\bigcup_{n \in \omega} \mathcal{T}_{\sigma|_n}) = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n \in \omega} f^{-1}(\mathbb{T}_{\sigma|_n})$, where $f^{-1}(\mathbb{T}_{\sigma|_n}) \in \Pi_1^0(X) \implies$ Souslin. □

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1.2 Trees as PTS (compact)

Poznámka (Topology on trees)

$\mathcal{P}(\mathbb{S}) = \{A \subset \mathbb{S}\} = \{0, 1\}^{\mathbb{S}}$ (product topology of product of discrete topologies) which is compact and homeomorphic to 2^ω . We assume on \mathbb{T} subspace topology.

Tvrzení 1.2

$\mathbb{T}, \mathcal{T}^* \in \Pi_0^1(\{0, 1\}^{\mathbb{S}})$, $\{\mathbb{T}_s, \mathbb{T}^* \setminus \mathbb{T}_s, s \in \mathbb{S}\}$ form a subbase of topology in \mathbb{T} .

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Poznámka

$\mathcal{T}, \mathcal{T}^*$ is compact metric space, so PTS.

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Důkaz

$S \in \{0, 1\} \setminus \mathbb{T} \Leftrightarrow \exists s, t \in \mathbb{S}, s < t : t \in S \wedge s \notin S \implies \{0, 1\} \setminus \mathbb{T} = \bigcup_{t \in \mathbb{S}} \bigcup_{s < t} (\{T, \chi_T(t) = 1\} \cap \{T; \chi_T(s) = 1\})$.

$\{T | \chi_T(t) = 1\}, \{T | \chi_T(s) = 0\}$ is subbase of product topology.

$\mathcal{T}^* = \mathcal{T} \cap \{A \in \{0, 1\} | \chi_A(\emptyset) = 1\} \implies \mathcal{T}^* \in \Pi_1^0(\mathcal{T}) \implies \mathcal{T}^*$ is compact.

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□

1.3 Properties of f from the lemma

Definice 1.3

$T \in \mathbb{T}, \sigma \in \mathcal{N}. h_\sigma(T) := \sup \{k \in \omega | \sigma|_k \in T\} \in \omega \cup \{\infty\}$.

Poznámka (Remind Lebesgue–H?–Banach characterization)

X, Y metric spaces, Y separable, $1 \leq \alpha < \omega_1, f : X \rightarrow Y$. Then f is $\text{Baire}_\alpha \Leftrightarrow f$ is $\Sigma_{\alpha+1}^0(X)$ -measurable.

Tvrzení 1.3

X metrizable (we need only $\Sigma_1^0(X) \subset \Sigma_2^0(X)$), $S \subset X$ Souslin. Then there exists $f : X \rightarrow \mathbb{T}$ such that:

1. $f^{-1}(\mathbb{T}_I) = S$;
2. $f^{-1}(\mathbb{T}_s) \in \Pi_1^0(X), s \in \mathbb{S}$;
3. $h_\sigma \circ f$ is upper semi-continuous ($h_\sigma \circ f : X \rightarrow \mathbb{R}^*$), $\sigma \in \mathcal{N}$ (i.e. $\{x \in X | h_\sigma(f(x)) < n\}$ is open $\forall \sigma \in \mathcal{N}, n \in \mathbb{R}^*$);
4. f is Baire_1 (i.e. Σ_2^0 -measurable).

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Důkaz

1. and 2. is from the lemma. „4.“: \mathbb{T} separable metric space. So, it is enough to prove it for subbase. $f^{-1}(\mathbb{T}_s) \in \Pi_1^0 \subset \Sigma_2^0, f^{-1}(\mathbb{T} \setminus \mathbb{T}_s) \in \Sigma_1^0 \subset \Sigma_2^0(X)$. „3.“: $\{x \in X | h_\sigma(f(x)) < n\} = f^{-1}(\{T \in \mathbb{T} | \sigma|_n \notin T\}) = f^{-1}(\mathbb{T} \setminus \mathbb{T}_{\sigma|_n})$ is open (by the lemma). And $\{x \in X | h_\sigma(f(x)) < \infty\} = \bigcup_{n \in \omega} \{x \in X | h_\sigma(f(x)) < n\}$. □

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1.4 Examples of Σ_1^1 non- Δ_1^1 sets

Poznámka

$$\Sigma_1^1(X) \setminus \Pi_1^1(X) = \Sigma_1^1(X) \setminus \Delta_1^1(X) \stackrel{?}{\neq} \emptyset.$$

Lemma 1.4

$\mathcal{T}_I \in \Sigma_1^1(\mathcal{T}) \setminus \Delta_1^1(\mathcal{T}), \mathcal{T}_W \in \Pi_1^1(\mathcal{T}) \setminus \Delta_1^1(\mathcal{T})$.

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Důkaz

1. $\mathcal{T}_I \in \Sigma_1^1(\mathbb{T}) \iff \mathbb{T}_I = \bigcup \bigcap \mathcal{T}_{\sigma|_n}$ souslin in PTS.

2. „ $\mathcal{T}_I \notin \Delta_1^1(\mathbb{T})$ “: By continuity $\mathcal{T}_I \in \Delta_1^1 \implies \mathcal{T}_W \in \Delta_1^1 \implies \mathcal{T}_W \in \Sigma_1^1 \implies \mathcal{T}_W$ souslin.

└ \nexists .

□

Poznámka

f_I, f_W are mappings from the lemma for $S = \mathcal{T}_I$ and $S = \mathcal{F}_W$. Clearly $f_I = \text{id}$.

Definice 1.4

$f : \mathcal{T} \rightarrow \mathcal{T}$ by $f(T) := f_I(T) \cap f_W(T) = T \cap f_W(T)$. $f(T) \in \mathcal{T} \iff (A, B \in \mathcal{T} \implies A \cap B \in \mathcal{T})$.

$$T \in \mathcal{T}_W \implies f(T) = T \cap f_W(T) \subset T \implies f(T) \in \mathcal{T}_W.$$

$$T \in \mathcal{T}_I \implies f(T) \subset f_W(T) \in \mathcal{T}_W \iff (\text{the lemma} \implies f^{-1}(\mathcal{T}_I) = \mathcal{T}_W \implies f^{-1}(\mathcal{T}_W) = \mathcal{T}_I) \implies f(T)$$

$\implies f : \mathcal{T} \rightarrow \mathcal{T}_W \implies h_\sigma \circ f : \mathcal{T} \rightarrow \omega$. From the previous proposition $h_\sigma \circ f$ is usc, so $h_\sigma \circ f$ is usc real function on compact set. Thus $m(\sigma) := \max_{T \in \mathbb{T}} h_\sigma(f(T)) \in \omega$.

Důkaz (The previous lemma)

By contradiction $\mathcal{T}_I \in \Delta_1^1(\mathcal{T}^*) \implies \mathcal{T}_W^* \in \Sigma_1^1(\mathcal{T}^*)$. $f(T) = f_I(T) \cap f_W(T)$, $f : \mathcal{T}^* \rightarrow \mathcal{T}^*$, $f : \mathcal{T}^* \rightarrow \mathcal{T}_W^*$. $\exists m(\sigma) := \max_{T \in \mathcal{T}^*} h_\sigma(f(T)) \in \omega$.

Define $T_0 \in \mathcal{T}^* : s \in T_0 \iff \sigma \in \mathcal{N} : \sigma|_{m(\sigma)+1} > s$. $T_0 \in \mathcal{T}^*$, $\{\emptyset\} \in T_0$, $T_0 \in \mathcal{T}$ trivial. $T_0 \in \mathcal{T}_W^*$. By contradiction $\sigma \in [T_0] \implies \sigma|_{m(\sigma)+2} \in T_0 \implies \exists \nu \in \mathcal{N} : \sigma|_{m(\sigma)+2} < \nu|_{m(\nu)+1} \implies \nu|_{m(\sigma)+1} = \sigma|_{m(\sigma)+1}$. Definition of $m(\nu)$ gives $\exists T \in \mathcal{T}^* : m(\nu) = h_\nu(f(T)) \implies \nu|_{m(\nu)} \in f(T) \implies \sigma|_{m(\sigma)+1} \in f(T) \implies h_\sigma(f(T)) \geq m(\sigma) + 1$. \nexists .

Clearly

$$T_0 \supseteq \bigcup_{T \in \mathcal{T}^*} (T). T_0 \in \mathcal{T}_W^* \implies f_W(T_0) \in \mathcal{T}_I \implies \exists \sigma_0 \in [f_W(T_0)] \implies$$

$$\implies h_{\sigma_0}(f(T_0)) = \min \{k \in \omega | \sigma_0|_k \in T_0 \cap f_W(T_0)\} = \min \{k \in \omega | \sigma_0|_k \in T_0\} \supseteq m(\sigma_0) + 1. \nexists.$$

□

Věta 1.5

X PTS, $A \in \Sigma_1^1(X)$, $\text{card}(A) > \text{card}(\omega)$. Then there exists $B \subset A$ such that $B \in \Sigma_1^1(X) \setminus \Delta_1^1(X)$.

Důkaz

$\text{card}(A) > \omega \implies \exists C \subset A$ homeomorphic copy of $2^\omega \sim 2^\mathbb{S}$. $2^\mathbb{S} \xrightarrow{h} A$ then $h(\mathcal{T}_I) \in \Sigma_1^1(X) \setminus \Delta_1^1(X)$. Homeomorphism of Σ_1^1, Δ_1^1 set is Σ_1^1, Δ_1^1 set. \square

Poznámka

Let Γ be class of subsets of PTS and X be PTS. We say that A is $\Gamma(X)$ -hard $\equiv \forall B \in \Gamma(\mathcal{N}) \exists f \in \Delta_1^1, f : \mathcal{N} \rightarrow X : f^{-1} = B$. A is $\Gamma(X)$ -complete $\Leftrightarrow A \in \Gamma$ and $A \in \Gamma$ -hard.

From the previous theorem $A \in \Sigma_1^1$ -complete $\implies A \in \Sigma_1^1 \setminus \Delta_1^1$ (same for Π_1^1). ($A \in \Delta_1^1 \implies f^{-1}(A) \in \Delta_1^1$, but there are $\Sigma_1^1 \setminus \Delta_1^1$ subsets of \mathcal{N}).

Poznámka

Σ_1^1 -complete $= \Sigma_1^1 \setminus \Delta_1^1 \iff \Sigma_1^1$ -determinacy.

Poznámka

$\mathcal{T}_I \in \Sigma_1^1$ -complete, $\mathcal{T}_W^* \in \Pi_1^1$ -complete.

Definice 1.5 (Universal set)

X PTS, Γ class of subsets of PTS. We say that A is $\Gamma(X)$ -universal $\equiv A \in \Gamma(X \times \mathcal{N}) \wedge \Gamma(X) = \{A^s | s \in \mathcal{N}\}$.

Poznámka

X PTS. Then

1. there exists $\Sigma_1^0(X)$ -universal set;
2. there exists $\Pi_1^0(X)$ -universal set;
3. there exists $\Sigma_1^1(X)$ -universal set.

⌈

Důkaz

„1.“: $\{B_n\}$ base of X . $G := \bigcup_{n \in \omega, s \in \omega} (B_{s(0)} \cup B_{s(1)} \cup \dots \cup B_{s(n-1)}) \times B(s)$ ($B(s) = \{\sigma \in \mathcal{N} \mid s < \sigma\}$). $G \in \Sigma_1^0(X \times \mathcal{N})$ trivial. $\sigma \in \mathcal{N} \implies G^\sigma \in \Sigma_1^0(X)$ trivial ($G^\sigma = \bigcup_{n \in \omega} (B_{\sigma(0)} \cup B_{\sigma(1)} \cup \dots \cup B_{\sigma(n-1)})$ open). $U \in \Sigma_1^0(X) \implies \exists \sigma \in \mathcal{N} : U = \bigcup_{n \in \omega} B_{\sigma(n)} = G^\sigma$.

„2.“: $G \in \Sigma_1^0(X)$ -universal $\implies (X \times \mathcal{N}) \setminus G$ is $\Pi_1^0(X)$ -universal.

„3.“: $Y = \mathcal{N} \times X$. Let $F \in \Pi_0^1(Y \times \mathcal{N})$ be $\Pi_1^0(Y)$ -universal. $\Pi : \mathcal{N} \times X \times \mathcal{N} \rightarrow X \times \mathcal{N}$ be projections on 2nd and 3rd coordinate. $A := \Pi(F)$. A is $\Sigma_1^1(X)$ -universal. Clearly $A \in \Sigma_1^1(X \times \mathcal{N})$, $A^\sigma \in \Sigma_1^1(X)$ for $\sigma \in \mathcal{N}$ trivial. Let $B \in \Sigma_1^1(X) \implies \exists C \in \Pi_1^0(\mathcal{N} \times X) : B = \Pi_2(C) \implies \exists \sigma \in \mathcal{N} : C = F^\sigma$.

$$A^\sigma = (\Pi_{2,3}(F))^\sigma = \Pi_2(F^\sigma) = \pi_2(C) = B.$$

⌋

□

Poznámka

Let $A \in \Sigma_1^1(\mathcal{N}^2)$ be $\Sigma_1^1(\mathcal{N})$ universal. Then

$$M := \{x \in \mathcal{N} \mid (x, x) \notin A\} \in \Sigma_1^1(\mathcal{N}) \iff (M \in \Sigma_1^1 \implies \exists \sigma \in \mathcal{N} : M = A^\sigma) \implies (\sigma \in M? : \sigma \in M \implies (\sigma, \sigma) \in A)$$

$$\{x \in \mathcal{N} \mid (x, x) \in A\} \in \Sigma_1^1(\mathcal{N}) \iff \text{diagonal is closed} \implies \{x \in \mathcal{N} \mid (x, x) \in A\} \in \Sigma_1^1 \setminus \Delta_1^1.$$

1.5 Derivative of trees

Definice 1.6 (Derivative)

$T \in \mathcal{T}$. $T' := \{s \in \mathbb{S} \mid \exists n \in \omega : s \wedge n \in T\}$. $T^{(0)} := T$. $\sigma < \omega_1 : T^{(\alpha+1)} = (T^\alpha)'$, λ -limit ordinal: $T^{(\lambda)} := \bigcap_{\alpha < \lambda} T^{(\alpha)}$. $d_\alpha(T) := T^{(\alpha)}$, $\alpha < \omega_1$, $d_\alpha : \mathcal{T} \rightarrow \mathcal{T}$.

Věta 1.6

$\forall \alpha < \omega_1 : d_\alpha \in \Delta_1^1(\mathcal{T}^2)$.

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Důkaz

$d_\alpha(T) \in \mathcal{T}$ ($T \in \mathcal{T}$) trivial.

$$\text{a) } d_1^{-1}(\mathcal{T}_s) = \{T \in \mathcal{T} \mid \exists n \in \omega : s^\wedge \in T\} = \bigcup_{n \in \omega} \mathcal{T}_{s^\wedge n} \in \Sigma_1^0(\mathcal{T}).$$

$$\implies d_1^{-1}(\mathcal{T} \setminus \mathcal{T}_s) \in \Pi_1^0(\mathcal{T}), \quad d_1^{-1}(\emptyset) = \{\emptyset, \{\emptyset\}\} \in \Pi_1^0(\mathcal{T}) \implies$$

$$\implies (G \in \Sigma_1^0(\mathcal{T})) \implies d_1^{-1}(G) \in \Sigma_2^0(\mathcal{T}) \implies$$

$\implies d_1$ is in the first Borel class.

b) $d_0\text{-id} \implies$ continuous.

Induction: c) $\alpha = \beta + 1$, $d_\beta \in \Delta_1^1 \implies d_\alpha = d_1 \circ d_\beta \in \Delta_1^1$.

d) λ limit ordinal, $\lambda < \omega_1$, $\forall \alpha < \lambda : d_\alpha \in \Delta_1^1$.

$$d_\lambda^{-1}(\mathcal{T}_s) = \left\{ T \in \mathcal{T} \mid \bigcap_{\alpha \in \lambda} d_\alpha(T) \ni s \right\} = \bigcap_{\alpha < \lambda} d_\alpha^{-1}(\mathcal{T}_s) \in \Delta_1^1 \implies$$

$$\implies d_\lambda^{-1}(\mathcal{T} \setminus \mathcal{T}_s) \in \Delta_1^1, \quad d_\lambda^{-1}(\emptyset) = \{T \in \mathcal{T} \mid \exists \alpha < \lambda : d_\alpha(T) = \emptyset\} = \bigcup_{\alpha < \lambda} d_\alpha^{-1}(\emptyset) \in \Delta_1^1.$$

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□

1.6 Luzin–Sierpinski index (rank, norm)

Definice 1.7

$T \in \mathcal{T}^*$, $i(T) := \min \{\alpha < \omega_1 \mid T^{(\alpha)} = \{\emptyset\}\}$, if exists, otherwise ω_1 .

Poznámka (Notation)

$T_s := \{t \in \mathbb{S} \mid s^\wedge t \in T\}$, $T \in \mathcal{T}^*$, $s \in T$.

Poznámka (Other indices)

$T_s \in \mathcal{T}^*$, $T \in \mathcal{T}^*$, $s \in T$ trivial.

Hausdorff index $:= \min \{\alpha < \omega_1 \mid d^{(\alpha)}(T) = d^{(\alpha+1)}(T)\}$.

Derivation of sets: X PTS, $K \in \mathcal{K}(X)$, $K' := \{x \in K \mid x \text{ is not isolated point in } K\}$.
 $K^{(\alpha+1)} := (K^{(\alpha)})'$, $K^{(0)} := K$, $K^{(\lambda)} := \bigcap_{\alpha < \lambda} K^{(\alpha)}$ (λ limit ordinal).

Lemma 1.7

$T_s \in \mathcal{T}^*$, $i(T_s) = \sup \{\min \{\omega_1, i(T_{s^\wedge n})\} \mid s^\wedge n \in T\}$ ($\sup \emptyset := 0$).

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Důkaz

$s \in T \implies T_s \neq \emptyset, T \in \mathcal{T}_s, l < t: s^\wedge t \in T \implies s^\wedge l < s^\wedge t \implies s^\wedge l \in T \implies l \in T_s.$

$$i(T_s) = \omega_1 \Leftrightarrow T_s \in \mathcal{T}_I \Leftrightarrow \exists n \in \omega : T_{s^\wedge n} \in \mathcal{T}_I \Leftrightarrow \exists n \in \omega : i(T_{s^\wedge n}) = \omega_1.$$

„ $i(T_s) < \omega_1 \Leftrightarrow T_s \in \mathcal{T}_W^*$ “: $\alpha := \sup_{n \in \omega: s^\wedge n \in T} i(T_{s^\wedge n}) + 1$, clearly $\forall n \in \omega : s^\wedge T, i(T_{s^\wedge n}) \leq i(T_s) < \omega_1 \implies 0 < \alpha < \omega_1$. „ $\alpha = i(T_s)$ “:

$$T_s^{(\alpha)} = \bigcup_{s^\wedge n \in T} (\{\emptyset\} \cup n^\wedge T_{s^\wedge n})^{(\alpha)} \subseteq \bigcup_{s^\wedge n \in T} (\{\emptyset\} \cup n^\wedge T_{s^\wedge n}) = \{\emptyset\} \implies i(T_s) \leq \alpha.$$

Assume $\beta < \alpha \implies \exists s^\wedge n \in T : i(T_{s^\wedge n}) + 1 > \beta \implies T_s^\beta \supset (\{\emptyset\} \cup n^\wedge T_{s^\wedge n})^{(\beta)} \supsetneq \{\emptyset\} \Leftarrow i(\{\emptyset\} \cup n^\wedge T_{s^\wedge n}) = i(T_{s^\wedge n}) + 1. \implies \beta < i(T_s) \implies \alpha \leq i(T_s).$ \square

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Věta 1.8

a) $T \in \mathcal{T}_W^* \Leftrightarrow i(T) < \omega_1$. b) $i(\mathcal{T}_W^*) = \omega_1$ (i.e. $\{i(T) | T \in \mathcal{T}_W^*\} = \{\alpha < \omega_1\}$).

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Důkaz

„a“: $T \in \mathcal{T}_W^*, T \neq \{\emptyset\} \implies \exists s \in T : |s| \geq 1, \forall n \in \omega : s^\wedge n \notin T \implies s \notin T' \implies T' \subsetneq T$. And $\text{card}(T) < \omega_1 \implies i(T) < \omega_1$. $i(\{\emptyset\}) = 0$. It can't happen:

$$T \neq \emptyset, \quad \{\emptyset\}, \quad T' = \emptyset$$

$$T \in \mathcal{T}_I \implies \exists \sigma \in [T] \implies \sigma \in [T'] \implies T' \in \mathcal{T}_I \implies \forall \alpha < \omega_1 : \sigma \in [T^{(\alpha)}] \implies T^{(\alpha)} \neq \{\emptyset\} \implies i(T)$$

„b“: $i(\{\emptyset\}) = 0$. Induction $\forall \alpha < \omega_1 \exists T_\alpha \in \mathcal{T}_W^* : i(T_\alpha) = \alpha$: First step is done; Second: $T_{\alpha+1} := 1^\wedge T_\alpha \cup \{\emptyset\} \implies i(T_{\alpha+1}) = \alpha + 1$; Assume λ is limit ordinal, $\alpha \nearrow \lambda$. $T_\lambda := \{\emptyset\} \cup \{n^\wedge T_{\alpha_n} | n \in \omega\}$. $(i(T_\lambda) = \sup \{i(T_{\alpha_n}) + 1\} = \lambda)$ \square

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1.7 Decomposition of \mathcal{T}_W^* and cosouslin sets

Definice 1.8

$\alpha < \omega_1 : \mathcal{T}_W(\alpha) := \{T \in \mathcal{T}^* | i(T) = \alpha\}.$

Věta 1.9

$\mathcal{T}_W(\alpha) \in \Delta_1^1(\mathcal{T}), \alpha < \omega_1.$

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Důkaz

$$\mathcal{T}_W(\alpha) = d_\alpha^{-1}(\{\emptyset\}), d_\alpha \in \Delta_1^1.$$

\square

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Poznámka

C cosouslin in X ($X \setminus C = S$, which is souslin). $\exists \Delta_1^1 f : X \rightarrow \mathcal{T}^* : f^{-1}(\mathcal{T}_I) = S = f^{-1}(\mathcal{T}_W^*) = C$. Define $C_\alpha = f^{-1}(\mathcal{T}_W(\alpha))$, $\alpha < \omega_1$. It is called a decomposition of C on Δ_1^1 subsets. If $\{\alpha | C_\alpha \neq \emptyset\}$ is countable $\implies C \in \Delta_1^1$. „Inverse implication“ is going to be in some weeks (Theorem 15).

Poznámka

$$A \in \Pi_1^1(X) \setminus \Pi_2^0(x) \implies \mathcal{K}(A) \in \Pi_1^1 - \text{complete.}$$

$$A \in \Pi_2^0(X) \Leftrightarrow \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X)).$$

1.8 Luzin–Sierpinski index as partial ordering

Poznámka (Goal)

Study $\{(T_1, T_2) \in (\mathcal{T}_W^*)^2 | i(T_1) \leq i(T_2)\}$.

Definice 1.9

$f : \mathbb{S} \rightarrow \mathbb{S}$ is strategy $\equiv \forall s \in \mathbb{S} : |f(s)| = |s|$ (respect length) and $\forall s, t \in \mathbb{S} : s < t \implies f(s) < f(t)$ (monotone.)

Poznámka

a) f strategy. We define $\bar{f} : \omega^\omega \rightarrow \omega^\omega$ by $f(\sigma) = \mathbb{T} \Leftrightarrow \forall n \in \omega : T|_n = f(\sigma|_n)$.

b) For first $|s|$ steps of player I describes f first $|s|$ steps of player II (strategy for II player).

c) $T \in \mathcal{T}^* : f(T), f^{-1}(T) \in \mathcal{T}^*$.

d) $\alpha < \omega_1 : (f^{-1}(T))^{(\alpha)} \subset f^{-1}(T^{(\alpha)})$.

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Důkaz

„a“, „b“ trivial. „c“: $s \in f(T), t < s \implies \exists x \in T : f(x) = s \implies |x| = |s| \geq |t| \implies x|_{|t|} \in T \implies f(x|_{|t|}) \in f(T), f(x|_{|t|}) < f(x) = s, |f(x|_{|t|})| = |t| \implies f(x|_{|t|}) = t \implies f(T) \in \mathcal{T}^*. f^{-1}(T) \in \mathcal{T}^*$ similar.

„d“: By induction: First step ($\alpha = 0$) is trivial. For $\alpha = 1$: $s \in (f^{-1}(T))' \implies \exists n \in \omega : s^{\wedge} n \in f^{-1}(T) \implies f(s^{\wedge} n) \subset f(s), f(s^{\wedge} n) \in T \implies f(s) \in T \implies f(s) \in T'$ ($\exists m \in \omega : f(s^{\wedge} m) = f(s)^{\wedge} m$). For successor ordinal: $(f^{-1})^{(\beta+1)} = ((f^{-1}(T))^{(\beta)})' \subset (f^{-1}(T^{(\beta)})) \subset f^{-1}(T^{(\beta+1)})$. For limit ordinal $\lambda < \omega_1$: $(f^{-1}(T))^{(\lambda)} = \bigcap_{\alpha < \lambda} (f^{-1}(T))^{(\alpha)} \subseteq \bigcap_{\alpha < \lambda} f^{-1}(T^{(\alpha)}) = f^{-1}(\bigcap_{\alpha < \lambda} T^{(\alpha)}) = f^{-1}(T^{(\lambda)})$. \square

└

Lemma 1.10

$T_1, T_2 \in \mathcal{T}_W^*$. $i(T_1) \leq i(T_2) \Leftrightarrow \exists f : \mathbb{S} \rightarrow \mathbb{S}$ strategy such that $T_1 \subset f^{-1}(T_2)$ ($f(T_1) \subset T_2$).

┌

Důkaz

„ \Leftarrow “: $T_1 \subset f^{-1}(T_2) \implies i(T_1) \leq i(f^{-1}(T_2)) \leq i(T_2)$ (second equation holds, because: $(f^{-1}(T_2))^{(\alpha)} \subset f^{-1}(T_2^{(\alpha)})$, put $\alpha = i(T_2) \implies (f^{-1}(T_2))^{(\alpha)} \subseteq \{\emptyset\} \implies i(f^{-1}(T_2)) \leq \alpha$).

„ \implies “: a) $i(T_2) = \omega_1 \implies T_2 \in \mathcal{T}_I \implies \sigma \in [T_2]$. Define $f(s)$ by $f(s) = \sigma|_s$. Clearly f is strategy and $f(T_1) \subset \{\sigma|_k, k \in \omega\} \subset T_2$.

b) $i(T_2) < \omega_1 \implies T_2 \in \mathcal{T}_W^*$. We will construct f by induction on $|s|$, $s \in \mathbb{S}$, and we also want $(+_n) : i_{T_1}(s) \leq i_{T_2}(f(s))$, $s \in T_1$, $|s| \leq n \implies f(s) \in T_2$, $s \in T_1$ (where $i_T(s) = i(T_s)$, $T \in \mathcal{T}^*$, $s \in T$).

Firstly $f(\{\emptyset\}) = \{\emptyset\}$. f monotone, respect length and $(+_0) : i_{T_1}(\{\emptyset\}) = i(T_1) \leq i(T_2) = i_{T_2}(\{\emptyset\})$. Let f be defined for $s \in \mathbb{S}$, $|s| \leq n$, $n \in \omega$, f respect length and be monotone and satisfy $(+_n)$. Let $s \in \omega^n$. i) $s_0 \notin T_1$ or $i_{T_1}(s_0) = 0$ TODO!!!

ii) $i_{T_1}(s_0) > 0$ TODO!!!

└

TODO!!!

□

1.9 Luzin–Sierpinski index as Π_1^1 rank**Věta 1.11**

$$\begin{aligned} A &:= \{(T_1, T_2) \in (\mathcal{T}^*)^2 \mid i(T_1) \leq i(T_2)\} \in \Sigma_1^1((\mathcal{T}^*)^2). \\ C &:= \{(T_1, T_2) \in (\mathcal{T}^*)^2 \mid T_1 \in \mathcal{T}_W^*, i(T_1) \leq i(T_2)\} \in \Pi_1^1((\mathcal{T}^*)^2). \\ B &:= \{(T_1, T_2) \in (\mathcal{T}^*)^2 \mid i(T_1) < i(T_2)\} \in \Pi_1^1((\mathcal{T}^*)^2). \\ D &:= \{(T_1, T_2) \in (\mathcal{T}_W^*)^2 \mid i(T_1) \leq i(T_2)\} \text{ bisouslin in } (\mathcal{T}_W^*)^2. \end{aligned}$$

┌

Důkaz

„ $A \implies C$ “: Define $h : (\mathcal{T}^*)^2 \rightarrow (\mathcal{T}^*)^2$ homeomorphism by $h(T_1, T_2) = (T_2, T_1)$. Then $(\mathcal{T}^*)^2 \setminus A = h(B) \implies B \in \Pi_1^1((\mathcal{T}^*)^2)$.

„ $C \implies B$ “: $E := \{(T, T) \in (\mathcal{T}^*)^2 \mid T \in \mathcal{T}_W^*\} \simeq \mathcal{T}_W^* \implies E \in \Pi_1^1$. $C = B \cup E \in \Pi_1^1((\mathcal{T}^*)^2)$.

„ D “: $A \cap (\mathcal{T}_W^*)^2$ Souslin, $D = C \cap ((\mathcal{T}_W^*)^2) \in \Pi_1^1((\mathcal{T}^*)^2)$ cosouslin.

„ A “: $i(T_1) \leq i(T_2) \Leftrightarrow \exists f$ strategy : $f^{-1}(T_2) \supset T_1$. So $A = \Pi(F)$, $F := \{(T_1, T_2, f) \in (\mathcal{T}^*)^2 \times \mathbb{S} \mid T_1 \subseteq \text{where } \mathcal{S} \text{ is set of strategies. We show } F \in \Pi_1^0$. Clearly \mathbb{S} is PTS.

a) „ $\mathcal{S} \subset \Pi_1^0(\mathbb{S})$ “: $f_n \in \mathcal{S}$, $f_n \rightarrow f$, $f \in \mathcal{S}$? Set $s < t$, $s, t \in \mathbb{S} \implies \forall n \in \omega: f_n(s) < f_n(t)$ ($f_n \in \mathcal{S}$). (Convergence in product space is point-wise) $\implies \exists n_0 \in \omega \forall n \geq n_0 : f_n(s) = f(s)$, $f_n(t) = f(t) \implies f(s) < f(t)$. Similarly $\exists n_1 \forall n \geq n_1 : f_n(s) = f(s) \implies |f(s)| = |f_n(s)| = |s| \implies f \in \mathcal{S}$.

b) $f^{-1}(T_2) \supset T_1$ is Π_1^0 cond? $T_1^n \rightarrow T_1$, $T_2^n \rightarrow T_2$, $f_n \rightarrow f$ such that $f_n^{-1}(T_2^n) \supset T_1^n$. By contradiction: $\exists v \in T_1 \setminus f^{-1}(T_2)$. $\exists n_0 \forall n \geq n_0 : f_n(v) = f(v)$, $v \in T_1^n$, $f(v) \notin T_2^n \implies v \in T_1^n \setminus f_n^{-1}(T_2^n)$. \nexists . □

Definice 1.10

$\mathcal{S} : L \rightarrow \omega_1$ is Π_1^1 -rank $\equiv L \in \Pi_1^1(X)$, X PTS and $\exists C \in \Pi_1^1(X^2)$, $A \in \Sigma_1^1(X^2) : \{(x, y) \in L^2 \mid \mathcal{S}(x) \leq \mathcal{S}(y)\} = C \cap (X \times L) = A \cap (X \times L)$.

┌

Poznámka

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TODO!!!

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Důsledek

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TODO!!!

1.10 Boundedness of Π_1^1 -rank

Lemma 1.12

X PTS, $L \subset X$. Let $\mathcal{S} : L \rightarrow \omega_1$ be Π_1^1 -rank, $L \notin \Sigma_1^1(X)$ and $B \subset L$, $B \in \Sigma_1^1(X)$. Then $\sup \{\mathcal{S}(x), x \in B\} < \omega_1$.

┌

Důkaz

Define $\mathcal{S}(x) = \omega_1$, $x \in X \setminus L$. A as in definition of Π_1^1 -rank. By contradiction: $\sup \mathcal{S}(B) = \omega_1$. Then

$L = \{x \in X \mid \exists y \in B : \mathcal{S}(x) \leq \mathcal{S}(y)\} = \{x \in X \mid \exists y \in X : (x, y) \in A \cap (X \times B)\} = \Pi_1(A \cap (X \times B)) \in \Sigma_1^1$. \nexists .

┌

□

Věta 1.13

Let $B \subset \mathcal{T}_W^*$, $B \in \Sigma_1^1(\mathcal{T}^*)$. Then $\sup \{i(T) | T \in B\} < \omega_1$.

┌

Důkaz

└ Trivial. □

┌

Poznámka

$B \subset X$ PTS, $B \in \Delta_1^1(X) \implies B \in \Pi_1^1 \implies \exists f \in \Delta_1^1, f : X \rightarrow \mathcal{T}^* : f^{-1}(\mathcal{T}_W^*)B, f(B) \subset \mathcal{T}_W^*, f(B) \in \Sigma_1^1 \implies \{\alpha | f^{-1}(\mathcal{T}_W^*(\alpha)) \neq \emptyset\}$ is countable.

$\implies \exists \alpha < \omega_1 : B \subset f^{-1}(\bigcup_{\beta < \alpha} \mathcal{T}_W^*(\beta)), X \setminus B = f^{-1}(\mathcal{T}_I^*)$.

└

1.11 Luzin first separation principle

Věta 1.14

Assume M is metric space, $S \subset M$ souslin, $A \in \Sigma_1^1(M)$, $A \cap S = \emptyset$. Then there exists $B \in \Delta_1^1(M)$ such that $A \subset B \subset M \setminus S$.

┌

Důkaz

S Souslin $\implies S = f^{-1}(\mathcal{T}_I)$, $f \in \Delta_1^1$, $f : M \rightarrow \mathcal{T}^*$. Define $\mathcal{S}(x) := i(f(x))$.

$f(A) \in \Sigma_1^1(\mathcal{T}^*), f(A) \subset \mathcal{T}_W^* \iff A \cap S = \emptyset \implies \sup \mathcal{S}(A) = \alpha < \omega_1 \implies$

$A \subset B = f^{-1}(\bigcup_{\beta \leq \alpha} \mathcal{T}_W^*(\beta)) \in \Delta_1^1, B \cap S = \emptyset$.

└

□

Příklad

$\exists C_1, C_2 \in \Pi_1^1(\mathbb{R})$, $C_1 \cap C_2 = \emptyset$, C_1 cannot be Δ_1^1 -separated from C_2 . (C_1, C_2 are bisouslin in $C_1 \cup C_2$ and cannot be separated by $\Delta_1^1(C_1 \cup C_2)$ set.)

┌

Důkaz

$C_1 = \{(S, T) \in (\mathcal{T}^*)^2 \mid i(s) < i(T)\} \in \Pi_1^1 \iff$ the theorem above. $C_2 = \{(S, T) \in (\mathcal{T}^*)^2 \mid i(T) < i(S)\} \in \Pi_1^1$. $C_1 \cap C_2 = \emptyset$. $M := C_1 \cup C_2 \implies C_1$ and C_2 are bisouslin in M .

For contradiction $\exists H \in \Delta_1^1((\mathcal{T}^*)^2)$. $C_1 \subset H \subset (\mathcal{T}^*)^2 \setminus C_2 \implies \exists \alpha < \omega_1 : H \in \Sigma_\alpha^0((\mathcal{T}^*)^2)$. Find $B \in \Delta_1^1 \setminus \Sigma_{\alpha+1}^0((\mathcal{T}^*)^2) \iff$ use Σ_j^0 universal sets \iff Kechris.

Find f_{B^C} from the lemma, $f_{B^C} : (\mathcal{T}^*)^2 \rightarrow \mathcal{T}^*$, $f_{B^C}^{-1}(\mathcal{T}_I) = (\mathcal{T}^*)^2 \setminus B$, $B = f_{B^C}^{-1}(\mathcal{T}_W^*) \implies \Sigma_1^1 \ni f_{B^C}(B) \subset \mathcal{T}_W^*$, $f_{B^C} \in B_{\sigma_1}$ ($f_{B^C}^{-1}(\Sigma_1^0) \subset \Sigma_2^0$).

From the theorem above $\sup_{x \in B} i(f(x)) = \alpha_B < \omega_1$. From the other theorem $\exists T \in \mathcal{T}_W^* : i(T) > \alpha_B$. Define $F(x) = (f(x), T) \in (\mathcal{T}^*)^2$, $x \in (\mathcal{T}^*)^2$. $F \in B_{\sigma_1}$.

Then $F^{-1}(C_1) = B \iff x \in B \implies i(f(x)) \leq \alpha_B < i(T)$, $x \in B \implies f(x) \in \mathcal{T}_I \implies (f(x), T) \notin C_1, \in C_2$.

$F^{-1}(C_1) = F^{-1}(H) \iff x \in (\mathcal{T}^*)^2 \implies F(x) \subset C_1 \cup C_2$. $H \in \Sigma_\alpha^0$, $F \in B_{\sigma_1} \implies B = F^{-1}(H) \in \Sigma_{\alpha+1}^0((\mathcal{T}^*)^2)$.

└

□

1.12 Luzin second separation principle and reduction theorem

Věta 1.15 (Reduction theorem)

C_1, C_2 cosouslin in metric space M . Then there exists cosouslin $D_1, D_2 \subset M$ such that

$$\forall i = 1, 2 : \quad D_i \subset C_i, \quad D_1 \cap D_2 = \emptyset, \quad D_1 \cup D_2 = C_1 \cup C_2.$$

┌

Důkaz

From the lemma $\exists f_i : M \rightarrow \mathcal{T}^*, f_i \in \Delta_1^1, f_i^{-1}(\mathcal{T}_W^*) = C_i$.

$$D_1 := \{x \in M \mid i(f_1(x)) < \omega_1, i(f_1(x)) \leq i(f_2(x))\} \implies D_1 \subset C_1 \quad (i(f_1(x)) \leq \omega_1).$$

$$D_1 := \{x \in M \mid i(f_2(x)) < i(f_1(x))\} \implies D_2 \subset C_2 \quad (i(f_2(x)) \leq \omega_1).$$

$D_1 \cup D_2 = C_1 \cup C_2$ ($x \in C_1 \cup C_2 \implies i(f_1(x)) < \omega_1 \vee i(f_2(x)) < \omega$, if $i(f_1(x)) \leq i(f_2(x))$ then $x \in D_1$ otherwise $x \in D_2$).

„ $D_1 \cap D_2 = \emptyset$ “: Define $F = (f_1, f_2) \in \Delta_1^1, F : M \rightarrow ((\mathcal{T}^*)^2) \iff F^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2)$. ($(\mathcal{T}^*)^2$ has countable base.)

$$C = \{(T_1, T_2) \in (\mathcal{T}^*)^2 \mid i(T_1) < \omega_1, i(T_1) \leq i(T_2)\} \in \Pi_1^1,$$

$$B = \{(T_1, T_2) \in (\mathcal{T}^*)^2 \mid i(T_2) < i(T_1)\} \in \Pi_1^1,$$

$$F^{-1}(C) = D_1 \wedge F^{-1}(B) = D_2 \implies D_1, D_2 \in \Pi_1^1 \implies \text{cosouslin}.$$

└

□

Důsledek (Luzin second separation principle)

Let M be metric space, A_1, A_2 Souslin in M . Then there exists cosouslin B_1, B_2 such that $A_2 \setminus A_1 \subset B_1, A_1 \setminus A_2 \subset B_2, B_1 \cap B_2 = \emptyset$. Moreover, it is possible to manage $B_1 \cup B_2 = M \setminus (A_1 \cap A_2) \implies$ if $A_1 \cap A_2 = \emptyset$, then B_i are bisouslin.

┌

Důkaz

$C_i = M \setminus A_i, B_i$ reduction of C_i . $B_1 \cup B_2 = C_1 \cup C_2 = M \setminus (A_1 \cap A_2), B_1 \cap B_2 = \emptyset, B_i \supset C_i \setminus C_j = A_j \setminus A_i$ ($i \neq j$). $A_1 \cap A_2 = \emptyset \implies B_1 = M \setminus B_2$. □

└

2 Kuratowski–Ulam theorem

Poznámka (Notation)

$A \subset X \times Y, X, Y$ sets. $A_X := \{y \in Y \mid [x, y] \in A\}$. $A^y := \{x \in X \mid [x, y] \in A\}$.

X topological space, $T(x)$ statement. $\forall^* x : T(x) \iff \{x \in X \mid T(x)\}$ is co-meager. $\exists^* x : T(x) \iff \{x \in X \mid T(x)\}$ is non-meager.

Věta 2.1 (Kuratowski–Ulam)

X, Y be topological spaces with countable base, $A \subset X \times Y$ has Baire property in $X \times Y$. Then

1. $\forall^* x : A_x$ has Baire property in $Y, \forall^* y : A^y$ has Baire property in X ;

2. A is meager $\Leftrightarrow \forall^* x : A_x$ is meager $\Leftrightarrow \forall^* y : A^y$ is meager;
3. A is co-meager $\Leftrightarrow \forall^* x : A_x$ is co-meager $\Leftrightarrow \forall^* y : A^y$ is co-meager.

Lemma 2.2

X, Y topological spaces, Y has countable base, $F \subset X \times Y$ nowhere dense. Then $\forall^* x : F_x$ is nowhere dense.

Důkaz

WLOG $Y \neq \emptyset$. $F \in \Pi_1^0(X \times Y)$ (otherwise for \overline{F}). Let $U := (X \times Y) \setminus F$. It is open and dense. We want $\forall^* x : \overline{U}_x = Y$.

$\{V_n\}$ base of Y , $V_n \neq \emptyset$. $U_n := \Pi_X(U \cap X \times V_n)$ dense open in X . (Open trivial. Dense $\Leftarrow G \in \Sigma_1^0(X), G \neq \emptyset \Rightarrow (G \times V_n) \cap U \neq \emptyset \Rightarrow [x, y] \in U \cap (X \times V_n)$.)

$x \in \bigcap U_n \Rightarrow x \in U_n \Rightarrow U_x \cap V_n \neq \emptyset \Rightarrow U_x$ is dense in Y . □

Důkaz (Kuratowski–Ulam)

$F \subset X \times Y$ meager $\Rightarrow F \subset \bigcup F_n, F_n \in \Pi_1^0$, nowhere dense. By the previous lemma $\exists M_n \subset X$ co-meager: $\forall x \in M_n : (F_n)_x$ is nowhere dense. $M := \bigcap M_n$ co-meager $\Rightarrow \forall x \in M \forall n \in \omega : (F_n)_x$ is nowhere dense $\Rightarrow F_x \subset \bigcup (F_n)_x$ is meager.

Let $A \subset X \times Y$ has Baire property $\Rightarrow A = U \Delta M, U \in \Sigma_1^0, M$ meager. $A_x = U_x \Delta M_x$ (open Δ meager for co-meager many x) $\Rightarrow \forall^* x : A_x$ has Baire property. This implies 1.

Clearly 2. \Leftrightarrow 3. using complements. It remains to show 2. \Leftarrow . □

Lemma 2.3

X, Y topological spaces with countable base, $A \subset X, B \subset Y$. Then $A \times B$ is meager $\Leftrightarrow A$ or B is meager.

Důkaz

„ \Rightarrow “: $A \times B$ meager, A non-meager. Then by the previous lemma $\exists x \in A : (A \times B)_x = B$ meager.

„ \Leftarrow “: A is meager, $A \subseteq \bigcup F_n, F_n \in \Pi_1^0$, nowhere dense. Then $A \times B \subset \bigcup (F_n \times B)$. We need to show that $F_n \times B$ is nowhere dense. $X \setminus F_n$ open dense $\Rightarrow (X \setminus F_n) \times Y$ open dense in $X \times Y \Rightarrow F_n \times Y$ is nowhere dense $\Rightarrow F_n \times B$ is nowhere dense. □

Důkaz (Kuratowski–Ulam remaining 2. \Leftarrow)

$A \subset X \times Y$ has Baire property, $\forall^* x : A_x$ is meager. $A = U \Delta M$ (open Δ meager). For contradiction we assume that A is not meager (U is not meager). $\Rightarrow \exists G \in \Sigma_1^0(X), H \in \Sigma_1^0(Y) : G \times H \subset U, G \times H$ is not meager ($\Leftarrow X, Y$ have countable base).

$\xRightarrow{\text{lemma}} G, H \text{ non-meager} \implies \exists x \in G : A_x \text{ is meager, } M_x \text{ is meager (} \iff \forall^* x : M_x \text{ is meager)}$. Clearly non-meager $H \setminus M_x \subseteq U_x \setminus M_x \subset U_x \triangle M_x = A_x$ meager. ζ . \square

Například

$\exists A \subset [0, 1]^2$, A non-meager and there are no three points in A on a straight line.

Důkaz

$\{F_\alpha, \alpha < 2^\omega\}$ meager F_σ sets. We will construct $\{x_\alpha, \alpha < 2^\omega\}$ such that $x_\alpha \notin F_\alpha$ and there are no 3 points on the same line. By induction: 1) $\alpha = 0 : x_0 \in [0, 1]^2 \setminus F_0$.

2) We already have $\{x_\beta, \beta < \alpha\} \subset [0, 1]^2$, $\alpha < 2^\omega$ such that $\forall \beta < \alpha : x_\beta \notin F_\beta$ and there are no 3 points on the same line. $\mathcal{M} := \{p \text{ line} \mid \exists \beta, \gamma < \alpha : x_\beta \neq x_\gamma \wedge x_\beta, x_\gamma \in p\}$. Clearly $\# < 2^\omega$. From Kuratowski–Ulam: $\forall^* t \in [0, 1] : (F_\alpha)_t$ is meager. We find $t \in [0, 1] \setminus \Pi_1(\{x_\beta, \beta < \alpha\})$ such that $(F_\alpha)_t$ is meager.

$\implies \text{line } \{[t, y], y \in \mathbb{R}\} \notin \mathcal{M} \implies \forall p \in \mathcal{M} : \#\{y \in [0, 1] \mid [t, y] \in p\} \leq 1$. So $\exists y \in [0, 1] : [t, y] \notin \bigcup \mathcal{M} \cup F_\alpha$. $(F_\alpha)_t$ is meager and $\#(\bigcup \mathcal{M}) \cap \{[t, y], y \in \mathbb{R}\} \leq \#\mathcal{M} < 2^\omega$. Put $x_\alpha := [t, y]$. \square

3 Measurable selections

Definition 3.1 (Uniformization, selection)

Let X, Y sets, $C \subset X \times Y$ and $F : x \mapsto C_x$ is mapping from X to $\mathcal{P}(Y)$. $U \subset C$ is uniformization of C if $|U_x| = 1$ for $C_x \neq \emptyset$ ($x \in \Pi_X(C)$), (U is a graph of mapping $X \rightarrow Y$).

Mapping $f : D_f \rightarrow Y$ ($D_f = \Pi_x(C) = \{x \in X \mid F(x) \neq \emptyset\}$) is selection of F , if $f(x) \in F(x)$, $x \in D_f$.

Poznámka (Kondo–Norikov)

X, Y Polish topological spaces, $C \in \Pi_1^1(X \times Y)$. Then there exists $B \in \Pi_1^1(X \times Y)$ uniformization of C .

Poznámka

The theorem above implies if $A \subset M \times \{0, 1\}$ (M metric space) is cosouslin then there exists cosouslin uniformization.

$A_i := \Pi_x(A \cap M \times \{i\})$. $B_0 \cup B_1 = M \implies B_0 \times \{0\} \cup B_1 \times \{1\}$ is uniformization. $B_0 \subset A_0$, $B_1 \subset A_1$, $B_i \in \Pi_1^1$, $B_0 \cap B_1 = \emptyset$. Similarly, we can do reduction for countable collections.

TODO!!! (Kondo–Norikov is not true in Σ_1^1 .)

Příklad

There exists $F \in \Pi_1()$

TODO!!!

TODO!!!

TODO? (Example)

TODO!!!

TODO!!!

4 Continuous selections

Poznámka

It is enough: $F^{-1}(\Sigma_1^0) \subset \Sigma_1^0$ (yes, in 0-dim spaces $\Sigma_1^0 = (\Delta_0^1)_\sigma$, generally no).

Příklad ($F(x)$ is connected is not enough)

$A(t) = \begin{cases} S(0, 1) : t = 0 \\ S(0, 1) \setminus B(e^i, t) \end{cases}$. There is no continuous selection.

Poznámka (Notation)

Y Baire space, $\mathcal{F}_c(Y) :=$ convex non-empty closed subsets of Y .

$F : X \rightarrow \mathcal{P}(Y)$ lower semi continuous $\equiv F^{-1}(\Sigma_1^0) \subset \Sigma_1^0$. (X, Y topological spaces.)

Poznámka (E. Michael)

Let X be T_1 topological space. Then following assertions are equivalent:

- if Y is Baire space, then every lower semi continuous $F : X \rightarrow \mathcal{F}_c(Y)$ admits continuous selection;
- X is paracompact.

Poznámka

Let X be T_1 topological space. Then following assertions are equivalent:

- X paracompact and T_2 ;

- \forall open cover of X admits partition of unity.

Definice 4.1

M be open cover of topological space X . Then M admits partition of unity $\equiv \exists \{u_j\}_{j \in I}$, $u_j : X \rightarrow [0, 1]$ be continuous and $\forall j \in I \exists G \in M : \overline{\{u_j > 0\}} \subset G$, $\{\overline{\{u_j > 0\}}, j \in I\}$ is locally finite and $\forall x \in X : \sum_{j \in I} u_j(x) = 1$.

Poznámka (Stone)

X metric space, then X is paracompact and T_2 .

Věta 4.1

X be T_2 topological space such that every open cover admits partition of unity. Y Baire space, $F : X \rightarrow \mathcal{F}_c(Y)$ be lower semi continuous. Then there exists continuous selection.

Věta 4.2 (Tietze)

If $A \rightarrow \mathbb{R}$ continuous, $A \in \Pi_1^0(X)$, X normal topological space. Then there exists continuous extension.

Důsledek

Let X be T_2 , paracompact, Y be Baire space, $A \in \Pi_1^0(X)$ and $f : A \rightarrow Y$ be continuous. Then there is continuous extension.

$$F(x) = f(x), x \in A, \text{ and } F(x) = Y, x \notin A \text{ TODO!!!}$$

Důsledek

TODO!!!

Důsledek

X be T_2 , precompact $\implies X$ normal.

Důsledek

TODO!!!