Poznámka (Exam)
Oral, similar as in FA1.

Poznámka (Credit)
Similar as in FA1.

1 Banach algebras

1.1 Basic properties

Definice 1.1 (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over $\mathbb{K}$, if

- $(A, +, -, 0, \cdot_S)$ is vector space over $\mathbb{K}$;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K}\ \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

Důsledek
1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ $(\forall a \in A)$.

If e_1 is left unit and e_2 is right unit, then $e_1 = e_2$ is unit. $(e_1 = e_1 \cdot e_2 = e_2)$

2) (Algebra) homomorphism $\varphi : A \to B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

Tvrzení 1.1

Let A be algebra over $\mathbb{K}$. Put $A_e = A \times \mathbb{K}$ with operations A_e defined coordinate-wise and multiplication defined by

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \qquad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

Then A_e is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if A is commutative, then A_e is commutative.

Důkaz

We have A_e is vector space (from linear algebra). We easy proof from definition, that A_e is algebra, $(\mathbf{o}, 1)$ is a unit in A_e and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. □

Definice 1.2 (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv$ A is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ $(\forall a, b \in A)$.

Definice 1.3 (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv$ $(A, \|\cdot\|)$ is normed algebra nad Banach space.

Například

$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If T is Hausdorff topological space, then $\mathcal{C}_b(T) = \{f : T \to \mathbb{K} | f$ is continuous and bounded$\} \subseteq l_\infty(T)$ is closed subalgebra.

If T is locally compact, Hausdorff, not compact. Then $\mathcal{C}_0(T) = \{f : T \to \mathbb{K}$ continuous $|\forall \varepsilon > 0 : \{t \in T$
$\mathcal{C}_b(T)$ is closed subalgebra, which doesn't have unit.

If X is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If X is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

Tvrzení 1.2

If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \to A$ is Lipschitz on bounded sets.

Důkaz

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r) \ \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) :$$

$$\|ab - cd\| \leqslant \|a(b-d)\| + \|(a-c) \cdot d\| \leqslant \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leqslant R \cdot (\|b-d\| + \|a-c\|) \leqslant 2R \|(a, b) - (c, d)\|.$$

□

Tvrzení 1.3

Let $(A, \|\cdot\|)$ be a Banach algebra. On A_e we consider the norm

$$\|(a,\alpha)\| := \|a\| + |\alpha|, \qquad (a,\alpha) \in A \times \mathbb{K} = A_e.$$

Then $(A_e, \|\cdot\|)$ is Banach algebra.

Důkaz
It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a,\alpha)\cdot(b,\beta)\| \leqslant \|(a,\alpha)\| \cdot \|(b,\beta)\|,$$

which is easy. □

Poznámka
There is more (natural) ways to define norm on A_e (unlike $\cdot$ on A_e, which is natural).

A has a unit … we may still consider A_e.

If $e \in A \setminus \{\mathbf{o}\}$ is a unit, then $\|e\| \geqslant 1$, because $\|e\| = \|e^2\| \leqslant \|e\|^2$.

Věta 1.4

Let A be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a \cdot x$, $x \in A$. Then $I : A \to \mathbb{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leqslant 1$.

Moreover, if A has a unit e, then I is isomorphism into and $I(e) = \mathrm{id}$.

If $\|x^2\| = \|x\|^2$, $x \in A$, then I is isometry into.

Důkaz
„$L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leqslant 1$“: Linearity is obvious, $\|L_a(x)\| = \|a \cdot x\| \leqslant \|a\| \cdot \|x\|$, so $\|L_a\| \leqslant \|a\|$ and so $\|I\| \leqslant 1$. Since it is easily I preserves multiplication, so we are left to prove the „Moreover“ part.

„A has a unit e“: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geqslant \|L_a\left(\frac{e}{\|e\|}\right) = \frac{a}{\|e\|} = \frac{1}{\|e\|} \cdot a.$$

So I is bounded from below, so I is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \mathrm{id}\,.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$:

$$\|a\| \geqslant \|I(a)\| = \|L_a\| \geqslant \|L_a\left(\frac{a}{\|a\|}\right)\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So I is isometry. □

Poznámka
$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies$ $\exists$ equivalent norm $\|\cdot\|$ on A such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

Důkaz
Let $I : A \to \mathcal{L}(A)$ be as before. Put $|\|x\|| := \|I(x)\|$, $x \in A$. Since I is isomorphism, $|\|\cdot\||$ is equivalent norm. Moreover, $|\|x \cdot y\|| = \|I(x \cdot y)\| \leqslant \|I(x)\| \cdot \|I(y)\| = |\|x\|| \cdot |\|y\||$, $x, y \in A$. So $(A, |\|\cdot\||)$ is a Banach algebra. Finally

$$|\|e\|| = \|I(e)\| = \|\mathrm{id}\| = 1.$$

□

1.2 Inverse elements

Definice 1.4

$(M, \cdot, e)$ is monoid ($\cdot$ is associative, e is unit). Then invertible elements form a group ($e^{-1} = e, \exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^\times := \{x \in M \mid \exists x^{-1}\}$

Tvrzení 1.5

If $(A, \cdot, e)$ is monoid and $x_1, \ldots, x_n \in A$ commute, then $x_1 \cdot \ldots \cdot x_n \in A^x \Leftrightarrow \{x_1, \ldots, x_n\} \subset A^x$.

Důkaz

It suffices to prove it for $n = 2$ (and use induction). „If x^{-1} and y^{-1} exists, then $(xy)^{-1}$" is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so z is left inverse to y. Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so $\tilde{z}$ is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. □

Lemma 1.6

Let A be a Banach algebra with a unit.

- $\|x\| < 1 \implies \exists (e-x)^{-1} \wedge (e-x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x+e)^{-1} \wedge \|(x+h)^{-1} - x^{-1}\| \leqslant \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

Důkaz

„First item": We have $\|x^n\| \leqslant \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0} |^{\infty} x^n \in A$. Moreover,

$$(e-x) \cdot \left(\sum_{n=0}^{\infty} x^n\right) = \lim_{N\to\infty} (e-x) \cdot (e + x + \ldots + x^N) = \lim_{N\to\infty} e - x^{N+1} = e,$$

because $\lim_{N\to\infty} \|x^{n+1}\| \leqslant \lim_{N\to\infty} \|x\|^N = 0$. And similarly $\left(\sum x^n\right) \cdot (e-x) = e$.

„Second item": $x + h = x \cdot (e + x^{-1}h)$ we have x^{-1} exists and $(e + x^{-1}h)^{-1}$ exists (from first item), so from previous fact $(x+h)^{-1}$ exists. Moreover

$$(x+h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \overset{1)}{=} \sum_{n=0}^{\infty} \left(-x^{-1}h\right)^n x^{-1},$$

so

$$\|(x+h)^{-1} - x^{-1}\| = \|\sum_{n=1}^{\infty} \left(-x^{-1}h\right)^n x^{-1}\| \leqslant \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leqslant$$

$$\leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left(\|x^{-1}\| \cdot \|h\|\right)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}.$$

□

Důsledek
A Banach algebra with a unit $\implies$ $A^x \subset A$ is open and A^x is topological group.

Důkaz
$A^x \subset A$ is open by previous lemma (second item). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^x \ni x_n \to x \in A^x \overset{?}{\implies} x_n^{-1} \to x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \overset{h:=x_n-x}{\leqslant} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \to 0.$$

□

1.3 Spectral theory

Definice 1.5 (Resolvent set, spectrum and resolvent)

A Banach algebra with a unit, $x \in A$. We define resolvent set of x as $\varrho_A(x) := \{\lambda \in \mathbb{K} | \exists (\lambda \cdot e - x)^{-1}\}$. Next we define spectrum of x as $\sigma_A(x) := \mathbb{K} \setminus \varrho_A(x)$. Finally we define resolvent of x as $R_x : \varrho(x) \to A$, $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$.

If A doesn't have a unit, then notions above are defined with respect to A_e.

Tvrzení 1.7

A Banach algebra

a) $\forall x \in A : 0 \in \sigma_{A_e}(x)$ *(in particular, if A has no unit, then $0 \in \sigma_A(x)$);*

b) *A has unit* $\implies$ $\sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

Důkaz (a))

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\ldots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

□

Důkaz (b))
By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in A and second means that $((0,\lambda) - (x,0))^{-1} = (-x,\lambda)^{-1}$ exists in A. We take „$x \to -x$“.

„ $\implies$ “: find $(b,\beta) \in A_e$ such that $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$. So $(x\cdot b+\lambda\cdot b+\beta\cdot x, \lambda\cdot\beta) = (\mathbf{o},1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e+x)^{-1}\cdot x$. Similarly we find left inverse $\left(-\frac{1}{\lambda}x(x+\lambda e)^{-1}, \frac{1}{\lambda}\right)(x,\lambda)$. And next we prove that they are really inverses.

„ $\impliedby$ “: Put $(b,\beta) := (x,\lambda)^{-1}$. Then $(\lambda e+x)^{-1} = b+\beta\cdot e$. We have $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$, so $\lambda\cdot\beta = 1$ and $x\cdot b+\lambda\cdot b+\beta\cdot x = \mathbf{o}$. Then

$$(\lambda e + x)\cdot(b+\beta\cdot e) = \lambda\cdot b+\lambda\cdot\beta\cdot e+x\cdot b+\beta\cdot x = e.$$

Similarly second inverse. □

Věta 1.8

$\{\mathbf{o}\} \neq A$ complex Banach algebra, $x \in A$. Then $\sigma(x) \subseteq B_{\mathbb{C}}(0,\|x\|)$ is compact, nonempty.

Důkaz
After theory. □

Definice 1.6 (Derivative)

Y Banach space, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then

$$f'(a) := \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

is the derivative of f at a.

Tvrzení 1.9 (Fact)

Y Banach, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then $f'(a)$ exists $\implies$ f is continuous at a $\wedge$ $\forall x^* \in Y^*$: $(x^* \circ f)'(a) = x^*(f'(a))$.

Důkaz
Continuity: $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}\cdot(x-a) = f'(a)\cdot 0 = 0$.

$x^* \in Y^*$ given, then

$$\lim_{x\to a} \frac{x^*(f(x)) - x^*(f(a))}{x-a} = \lim_{x\to a} x^*\left(\frac{f(x)-f(a)}{x-a}\right) = x^*(f'(a)).$$

□

Tvrzení 1.10

A Banach algebra with a unit, $x \in A$. Then

- *$\varrho(x)$ is open set;*
- *$\forall |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$;*
- *(important!) $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$ has derivative at each $\lambda \in \varrho(x)$;*
- *$\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$;*
- *$\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$.*

Důkaz

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left(e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{x}{\lambda} \right)^n .$$

For third we fix $\lambda \in \varrho(x)$ and $t \in (0, \delta)$ for δ small enough ($\lambda + t \in \varrho(x)$ and *). We shall prove that „$R_x'(\lambda) = -R_x(\lambda)^2$“:

$$0 \overset{?}{=} \left\| \frac{R_x(\lambda + t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| = \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leqslant$$

$$\overset{*\text{ for existence of the inverse}}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| (e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t \right\| =$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leqslant$$

$$\overset{\|x^n\| \leqslant \|x\|^n}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \overset{*\text{ for denominator } \leqslant 1/2}{\leqslant} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \to 0.$$

Fourth: In general $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$ (proof: $u^{-1}v^{-1} = (vu)^{-1}$). And we apply it for $u = (\mu e - x)$ and $v = (\nu e - x)$.

Fifth: In general $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$ (proof: $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$) so:

$$R_x(\mu) - R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}$$

$$= R_x(\mu)R_x(\nu) \left(R_x(\nu)^{-1} - R_x(\mu)^{-1} \right) = R_x(\mu)R_x(\nu)(\nu - \mu).$$

□

Věta 1.11 (Liouville for Banach space valued functions)

Y Banach space over $\mathbb{C}$, $f : \mathbb{C} \to Y$ has derivative at each point, f is bounded ($\equiv \|f\|$ is bounded). Then $f \equiv$ const.

Důkaz

Assume $f \not\equiv$ const, so there are $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$ (by Hahn–Banach theorem) $\exists x^* \in Y^* : x^*(f(x)) \neq x^*(f(x))$. From fact $x^* \in f : \mathbb{C} \to \mathbb{C}$ has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions. □

Důkaz (Theorem before theory)

First case: „A has a unit“: Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is closed, so $\sigma(x)$ is compact. Assume that $\varrho(x) = \mathbb{C}$. By previous tvrzeni we have $R_x : \mathbb{C} \to A$ has derivative everywhere, and it is bounded because $\lim_{|}\lambda| \to \infty R_x(\lambda) = \lim_{|\lambda|\to\infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$. From previous theorem $R_x \equiv$ const so $\lim_{|\lambda|\to\infty} R_x(\lambda) = 0 \implies R_x \equiv 0$. In particular $0 = R_x(0) = (-x)^{-1}$. ↯(If $A \neq \{0\}$ then $x^{-1} \neq 0$ for $x \in A$.)

Second case: „A hasn't a unit“, then $\sigma(x) := \sigma_{A_e}((x, 0))$ so we apply the already proven case. □

Poznámka (Convention)

If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

Věta 1.12 (Gelfand–Mazur)

$\{\mathbf{o}\} \neq A$ Banach algebra with a unit. Assume $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$. Then A is isomorphic to $\mathbb{C}$. If moreover e is a unit in A and $\|e\| = 1$, then A is isometrically isomorphic to $\mathbb{C}$.

Důkaz

Consider $\psi : \mathbb{C} \to A$ defined as $\psi(\lambda) := \lambda \cdot e$. This is algebraic homomorphism and $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$, so it is isomorphism (and isometry, if $\|e\| = 1$).

It remains „φ is surjective“: Pick $a \in A$. From previously proved theorem $\exists \lambda \in \sigma(a)$, then $(\lambda e - a) \notin A^x$. So, $\lambda \cdot e - a = 0$, then $\psi(\lambda) = a$. □

Definice 1.7 (Spectral radius)

A Banach algebra, $x \in A$. Then $r(x) := \sup\{|\lambda|, \lambda \in \sigma(x)\}$ is called spectral radius of x.

Věta 1.13 (Beurling–Gelfand)

A Banach algebra, $x \in A \implies r(x) = \inf_{n\in\mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$.

Lemma 1.14

A Banach algebra with a unit, $x \in A$. For $p(z) = \sum_{j=1}^{n} \alpha_j z^j \in \mathbb{C}$ a polynom (with complex coefficients) we put $p(x) = \sum_{j=1}^{n} \alpha_j x^j \in A$. Then $\sigma(p(x)) = p(a(x))$.

Důkaz

Fix $\lambda \in \mathbb{C}$ and write $(\lambda - p)(z) = c \cdot \prod_{i=1}^{m}(z - z_i)$, where $z_1, \dots, z_m$ are roots of $\lambda - p$. Then $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$ does not exists. $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^{m}(x - z_i \cdot e)$, so it does'nt exists if and only if $\exists i \in [m]$, such that $(x - z_i \cdot e)^{-1}$ doesn't exists $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow$ $\exists$ root ν of $\lambda - p$ such that $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$. □

Důkaz (Beurling–Gelfand)

WLOG A has a unit. Step 1, „$r(x) \leqslant \inf_n \sqrt[n]{\|x^n\|}$“: fix $\lambda \in \sigma(x)$. By previous lemma $\forall n : \lambda^n \in \sigma(x^n)$. By theorem 'Before theory' we have $\forall n : |\lambda|^n \leqslant \|x^n\|$.

Step 2, „$r(x) \geqslant \limsup_n \sqrt[n]{\|x^n\|}$“: Pick $r > r(x)$. Claim: „$\frac{x^n}{r^n} \to^w 0$“: Fix $x^* \in A^*$ and put $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$. By fact and tvrzeni after it, f has derivative at each $\lambda \in \varrho(x)$. Moreover for $|\lambda| \geqslant \|x\|$ we have $f(\lambda) = \lambda \cdot x^* \left(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$. Thus $f(\lambda) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^n}$, $\lambda \in P(0, r(x), \infty)$. From Complex analysis $f \in H(P(0, r, \infty))$ is uniquely given by Laurent series. In particular $f(r) = \sum_{n=0}^{\infty} \frac{x^*(x^n)}{r^n}$, so $x^*\left(\frac{x^n}{r^n}\right) \to 0$.

From princip of unique boundedness (last semester): $\frac{x^n}{r^n}$ if $\|\cdot\|$-bounded, so $\exists c > 0 :$ $\|x^n\| \leqslant c r^n$, $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{c} \cdot r \to r$. So $\limsup \sqrt[n]{\|x^n\|} \leqslant r$. □

Důsledek

A Banach algebra, $x \in A$ and $|\lambda| > r(x)$. Then $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ is absolutely convergent and $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

Důkaz

Fix q, such that $\frac{r(x)}{|\lambda|} < q < 1$. By previous theorem, $\exists n_0 \, \forall n \geqslant n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$, so $\frac{\|x^n\|}{|\lambda|^n} < q^n$, $n \geqslant n_0$. Thus $\sum \left\|\frac{x^n}{\lambda^n}\right\| \leqslant \infty$, so the sum is absolutely convergent.

Now we easily check that $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$. □

1.4 Subalgebra

Věta 1.15

A Banach algebra with a unit e, $B \subset A$ is closed subalgebra such that $e \in B$. Fix $x \in B$. Then

- *$C \subset \varrho_A(x)$ is component (maximum connected subset) $\implies$ $C \subseteq \sigma_B(x)$ or $C \cap \sigma_B(x) = \varnothing$;*

- $\partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$;
- $\varrho_A(x)$ *is connected* $\implies \sigma_A(x) = \sigma_B(x)$;
- $\operatorname{int}\sigma_B(x) = \varnothing \implies \sigma_A(x) = \sigma_B(x)$.

Důkaz

„$\sigma_A(x) \subseteq \sigma_B(x)$": $(\lambda e - x)^{-1}$ exists in B implies it exists (it's same) in A.

„First item": Let $C \subset \varrho_A(x)$ be component. Pick $\lambda_0 \in C \cap \sigma_B(x)$. Wanted: „$C \backslash \sigma_B(x) = \varnothing$". Pick $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$ (separate B and $R_x(\lambda) \notin B$). Then $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is holomorphic function on open (because maximum) connected set C. Which is zero[a] on $C \backslash \sigma_B(x)$.

Since $C \backslash \sigma_B(x)$ is open, if it is nonempty it contains a ball, so it has cluster point. Thus $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is such that $\{\lambda \in C | x^*(R_x(\lambda))\} = 0$ has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero. ↯with $x^*(R_x(\lambda_0)) = 1$.

„Second item": Pick $\lambda \in \sigma_B(x) \backslash \sigma_A(x)$ and let $C \subset \varrho_A(x)$ be a component containing λ. By first item, $C \subseteq \sigma_B(x)$, C is open, so $\lambda \in C \subseteq \operatorname{int}(\sigma_B(x))$.

„Third item": If $\varrho_A(x)$ is connected, we can apply first item to $C = \varrho_A(x)$, we have either $\varrho_A(x) \subseteq \sigma_B(x)$ or $\varrho_A(x) \cap \sigma_B(x) = \varnothing$. But first is not possible, because $\varrho_A(x)$ is unbounded and $\sigma_B(x)$ is bounded. Therefore $\sigma_B(x) \subseteq \sigma_A(x)$.

„Fourth item": If $\operatorname{int}(\sigma_B(x)) = \varnothing$, then (by second item) $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$. □

[a] For $\lambda \in C \backslash \sigma_B(x)$, $(\lambda e - x)^{-1}$ exists in B so $R_x(\lambda) \in B$ and therefore, $x^*(R_x(\lambda)) = 0$

Důsledek

A Banach algebra, $B \subseteq A$ closed subalgebra, $x \in B$. Then all items from previous theorem hold as well if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.

Důkaz

Without proof. (Basically same that previous; we add unit to A and B, so this unit is same $((\mathbf{o}, 1))$, etc.) □

1.5 Holomorphic calculus

Definice 1.8

X Banach, $\gamma : [a, b] \to \mathbb{C}$ path (continuous, piecewise smooth (C^1)), $f : \langle\gamma\rangle \to X$ continuous. Then

$$\int_\gamma f := \int_{[a,b]} \gamma'(t) f(\gamma(t)) dt. \qquad \text{(As Bochner integral.)}$$

If $\Gamma = \gamma_1 + \ldots + \gamma_n$ is chain in $\mathbb{C}$, $f : \langle \Gamma \rangle \to X$ continuous, then

$$\int_\Gamma f := \sum_{i=1}^n \int_{\gamma_i} f.$$

Lemma 1.16

Γ chain in $\mathbb{C}$, X Banach, $f : \langle \Gamma \rangle \to X$, $x \in X$. Then

$$\int_\Gamma f = x \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_\Gamma x^* \circ f.$$

Důkaz

„ $\Longleftarrow$ “ by Hahn–Banach theorem. „ $\Longrightarrow$ “: (by previous semester x^* and $\int$ ”commutes”)

$$x^* \left(\int_\Gamma f \right) = \sum_{i=1}^n x^* \left(\int_{\gamma_i} f \right) = \sum_{i=1}^n \int_{[a_i, b_i]} \gamma_i'(t) x^*(f(\gamma_i(t))) dt = \int_\Gamma x^* \circ f.$$

□

Poznámka (Recall)

If $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact. Then there is a cycle Γ such that $\langle \Gamma \rangle \subset \Omega \backslash K$ and $\operatorname{ind}_\Gamma z = 1$ if $z \in K$ and 0 if $z \notin \Omega$.

Then we say that Γ circulates K in Ω.

Definice 1.9

Let A be a Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open and $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $f(x) := \frac{1}{2\pi i} \int_\Gamma f \cdot R_x$, where is any cycle which circulates $\sigma(x)$ in Ω.

Poznámka

$f(x)$ exists ($f \cdot R_x$ is continuous on $\langle \Gamma \rangle$), $f(x)$ does not depend on the choice of Γ (Pick $x^* \in X^*$, then $(x^* \circ f \cdot R_x)(\lambda) = f(\lambda) \cdot x^*(R_x(\lambda))$ is holomorphic. Pick Γ_1, Γ_2 cycles circulating $\sigma(x)$ in Ω, then $\int_{\Gamma_1 - \Gamma_2} x^* \circ f \cdot R_x = 0$ from Cauchy).

Věta 1.17 (Holomorphic calculus)

A Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open such that $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $\Phi : \mathcal{H}(\Omega) \to A$ defined as $\Phi(f) = f(x)$ (from definition above) has the following properties:

- *Φ is algebra homomorphism, $\Phi(1) = e$, $\varphi(\mathrm{id}) = x$;*
- *$f_n \overset{loc.}{\rightrightarrows} f$ in $H(\Omega)$, then $f_n(x) \to f(x)$;*
- *$f(x)^{-1}$ exists $\Leftrightarrow$ $f \neq 0$ on $\sigma(x)$, in this case $f(x)^{-1} = \frac{1}{f}(x)$;*

- *$\sigma(f(x)) = f(\sigma(x))$;*
- *if Ω_1 is open and $f(\sigma(x)) \in \Omega_1$, $g \in \mathcal{H}(\Omega_1)$, then $(g \circ f)(x) = g(f(x))$;*
- *if $y \in A$ commutes with x, then y commutes with $f(x)$.*

Moreover, if $\psi : \mathcal{H}(\Omega) \to A$ satisfy first two item, then $\psi = \Phi$.

Lemma 1.18

(Ω, μ) complete measurable space, A Banach algebra, $f \in L_1(\mu, A)$. Let $x \in A$ and $E \subset \Omega$ is measurable. Then

$$x \cdot \left(\int_E f(t) d\mu(t)\right) = \int_E x \cdot f(t) d\mu(t), \qquad \left(\int_E f(t) d\mu(t)\right) \cdot x = \int_E f(t) \cdot x d\mu(t).$$

Důkaz
Easy (by commutation of integral and linear operator from last semester), skipped. □

Důkaz (Holomorphic calculus)
„1st item“: „Φ is linear“ is easy, „Φ is multipicative“: Pick $f, g \in \mathcal{H}(\Omega)$, open set U such that $\sigma(x) \subset U \subset \overline{U} \subset \Omega$. Let Γ cycle circulating $\sigma(x)$ in U, Λ cycle circulating $\overline{U}$ in Ω. Then

$$f(x) \cdot g(x) = \left(\frac{1}{2\pi i} \int_\Gamma f \cdot R_x\right) \cdot g(x) \overset{\text{lemma}}{=} \frac{1}{2\pi i} \int_\Gamma f(t) R_x(t) g(x) dt = \frac{1}{2\pi i} \int_\Gamma f(t) \cdot R_x(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(s) ds dt$$

$$= \frac{1}{2\pi i} \int_\Gamma f(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(t) \cdot R_x(s) ds dt =$$

because $\langle\Lambda\rangle \cap \langle\Gamma\rangle = \emptyset$, we can use theorem after definition of R_x:

$$= \frac{1}{(2\pi i)^2} \int_\Gamma \int_\Lambda f(t) \cdot g(s) \cdot \frac{R_x(t) - R_x(s)}{s - t} ds dt \overset{\text{Fubini to } x^*(\dots) \text{ and lemma}}{=}$$

$$= \frac{1}{(2\pi i)^2} \int_\Gamma f(t) \left(\int_\Lambda \frac{g(s)}{s - t} ds\right) R_x(t) dt - \frac{1}{(2\pi i)^2} \int_\Lambda g(s) \left(\int_\Gamma \frac{f(t)}{s - t}\right) R_x(s) ds =$$

(Now we use Cauchy theorem ($f(z) \operatorname{ind}_\Gamma z = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw$). $\forall s \in \langle\Lambda\rangle : (t \mapsto \frac{f(t)}{s-t}) \in \mathcal{H}(U) \wedge \operatorname{ind}_\Gamma z = 0, z \notin U$, so $\int_\Gamma \frac{f(t)}{s-t} dt = 0$. $\forall t \in \langle\Gamma\rangle : \operatorname{ind}_\Lambda t = 1 \wedge (s \mapsto g(s)) \in \mathcal{H}(\Omega) \implies g(t) = \frac{1}{2\pi i} \int_\Lambda \frac{g(s)}{s-t} ds$.)

$$= \frac{1}{2\pi i} \int_\Gamma f(t) g(t) R_x(t) dt - 0.$$

It remains that „if $f(z) = z^k$, $k \in \mathbb{N} \cup \{0\}$ then $f(x) = x^k$“ (we want it for $k = 0$ and

$k = 1$). Put $\Gamma(t) = r \cdot e^{it}$, $t \in [0, 2\pi]$, where $r > \|x\|$ arbitrary. By some theorem:

$$R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}, \qquad |\lambda| > \|x\|.$$

Thus (we switch integral and sum, because later we realize that sum of integral of absolute value is finite)

$$\forall x^* \in A^* : x^*(f(x)) = \frac{1}{2\pi i} \int_\Gamma \lambda^k x^*(\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}) d\lambda = \frac{1}{2\pi i} \int_\Gamma \sum_{n=0}^{\infty} \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_\Gamma \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=}$$

because Γ (is 2π periodic).

„2nd item“: For $\Gamma = \gamma_1 + \ldots + \gamma_N$:

$$\|f_n(x) - f(x)\| = \frac{1}{2\pi i} \| \int_\Gamma (f_n(\lambda) - f(\lambda)) R_x(\lambda) d\lambda \| \leqslant \frac{1}{2\pi} \int_\Gamma |f_n(\lambda) - f(\lambda)| \cdot \|R_x(\lambda)\| d\lambda \leqslant \frac{1}{2\pi} \sum_{i=1}^{N} \int_{a_i}^{b_i} |\gamma_i'(t)| \sup_{z \in \langle \Gamma}$$

„Moreover part“: By Runge theorem (and second item) it is enough prove it for rational functions. If R was polynom, then $\Phi(R) = \Psi(R)$ by second item. So it suffices „$\forall p$ polynom: $\frac{1}{p} \in \mathcal{H}(\Omega) \implies \Phi(\frac{1}{p}) = \psi(\frac{1}{p})$“. Pick p polynom. Then $e = \psi(1) = \psi(p \cdot \frac{1}{p}) = \psi(p) \cdot \psi(\frac{1}{p}) = \Phi(p) \cdot \psi(\frac{1}{p})$ (similarly for $\frac{1}{p} \cdot p$). So $\psi(\frac{1}{p}) = \Phi(p)^{-1} = \Phi(\frac{1}{p})$.

„3rd item“: „ $\implies$ “ Let $f(z) = 0$ for some $z \in \sigma(x)$. Then exists $g \in H(\Omega) : f(u) = (z - u)g(z)$. By item one, we have $(ze - x)g(x) = f(x) = g(x)(ze - x)$. But $(ze - x)^{-1}$ does not exist, so $f(x)^{-1}$ does not exists.

„ $\impliedby$ “ Suppose $f \neq 0$ on $\sigma(x)$ by compactness. $\exists \Omega_1 \subset \Omega$ open: $\sigma(x) \subset \Omega_1$ and $f \neq 0$ on Ω_1. Then $\frac{1}{f} \in H(\Omega_1)$ and by first item we have $e = (f \cdot \frac{1}{f})(x) = f(x)\frac{1}{f}(x) = \ldots = \frac{1}{f}(x) \cdot f(x) \implies f(x)^{-1} = \frac{1}{f}(x)$. □

Poznámka
$f = g$ on a neighbourhood of $\sigma(x) \implies f(x) = g(x)$ (from definition), other implication doesn't hold!

1.6 Multiplicative functionals

Definice 1.10 (Multiplicative functional)

Let A be a Banach algebra. We say $\varphi : A \to \mathbb{C}$ is multiplicative linear functional $\equiv \varphi$ preserves $+, \cdot, \cdot_S$.

$$\Delta(A) := \{\varphi : A \to \mathbb{C} | \varphi \text{ multiplicative linear functional }, \varphi \not\equiv 0\}.$$

Tvrzení 1.19

A Banach algebra, $\varphi \in \Delta(A) \cup \{0\}$. Then

- $\exists! \tilde{\varphi} \in \Delta(A_e) : \tilde{\varphi}((x,0)) = \varphi(x), \forall x \in A$. *It is given by*

  $$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda.$$

  Moreover, $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$.

- $\forall x \in A : \varphi(x) \in \sigma(x)$ *whenever* $\varphi \not\equiv 0$.

- $\Delta(A) \subseteq B_{A^*}$.

- *A has a unit,* $\varphi \not\equiv 0 \implies \|\varphi\| \geqslant \frac{1}{\|e\|}$. *In particular if* $\|e\| = 1$*, then* $\|\varphi\| = 1$.

Důkaz

„1. uniqueness“: For $\tilde{\varphi} \in \Delta(A_e)$ such that $\tilde{\varphi}((x,0)) = \varphi(x)$, $x \in A$:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda\tilde{\varphi}((\mathbf{o},1)) = \varphi(x) + \lambda,$$

second equality by $\varphi \in \Delta(A) \implies \varphi(e) = \varphi(e^2) = \varphi^2(e)$. „1. existence“ is proven by check that defined $\tilde{\varphi}$ is multiplicative linear functional (and it is nonzero, but $\tilde{\varphi}((0,1)) = 1 \neq 0$). This is easy (omitted).

„$\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$“: „$\subseteq$“: $\varphi \in LHS$, put $\varphi(x) := \psi((x,0))$. Then $\varphi \in \Delta(A) \cup \{0\}$ and $\tilde{\varphi} = \psi$ became:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda = \psi((x,0)) + \lambda = \psi((x,\lambda)).$$

„$\supseteq$“: We know already that $\tilde{\varphi} \in \Delta(A_e)$.

„2. with A has unit e“: $\varphi \neq 0$, $\varphi \in \Delta(A)$: If $\lambda \in \varrho(x)$, then $\varphi(\lambda e - x) \neq 0$ ($\varphi(x) \neq 0$ if x^{-1} exists). $0 \neq \varphi(\lambda e - x) = \lambda - \varphi(x) \implies \lambda \neq \varphi(x)$. Thus $\varphi(x) \notin \varrho(x)$, so $\varphi(x) \in \sigma(x)$. „2. with A hasn't unit“, then $\varphi(x) = \tilde{\varphi}((x,0)) \in \sigma_{A_e}((x,0)) = \sigma_A(x)$.

„3.“: $\varphi \in \Delta(A)$. Then $\forall x \in A : \varphi(x) \in \sigma(x) \subseteq B(\mathbf{o}, \|x\|)$, so $|\varphi(x)| \leqslant \|x\|$.

„4.“: A has a unit e, then $\|\varphi\| \geqslant \left|\varphi\left(\frac{e}{\|e\|}\right)\right| = \frac{1}{\|e\|}$. □

Věta 1.20

A Banach algebra, $M := \Delta(A) \cup \{0\}$. Then $M \subset (B_{A^*}, w^*)$ is compact, $\Delta(A)$ is locally compact and if A has u unit, then $\Delta(A)$ is compact. The mapping $\Phi : M \to \Delta(A_e)$, $\Phi(\varphi) = \tilde{\varphi}$ is w^*–w^* homeomorphism.

Důkaz
By previous proposition, $M \subset (B_{A^*}, w^*)$ ((B_{A^*}, w^*) is compact by previous semester). So, it suffices to check that M is w^*-closed.

$$M = \bigcap_{x,y \in A} \{\varphi \in A^* | \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}.$$

Sets from RHS is closed by previous semester, so, M is closed. Thus M is compact.

$\Delta \subset M$ is open, so $\Delta(A)$ is locally compact (and M is 1-point compactification of $\Delta(A)$). If Δ has a unit, then $\Delta(A) = \{\varphi \in M | \varphi(e) = 1\}$ is w^*-closed, so $\Delta(A)$ is compact (and 0 is isolated in M).

Finally, by previous proposition, Φ is bijection. Φ is w^*-continuous:

$$\varphi_i \xrightarrow{w^*} \varphi \implies \forall (x, \lambda) : \tilde{\varphi}_i((x, \lambda)) = \varphi_i(x) + \lambda \to \varphi(x) + \lambda = \tilde{\varphi}((x, \lambda)) \implies \tilde{\varphi}_i \xrightarrow{w^*} \tilde{\varphi}$$

So, Φ is homeomorphism (continuous bijection on compact, last semester?). □

Například
$\Delta(\mathcal{C}(K)) = \{\delta_x | x \in K\}$. ($f \mapsto f(x)$ is multiplicative. Suppose $\varphi \in \Delta(\mathcal{C}(K))$, $\varphi \notin \{\delta_x | x \in K\}$. So for $x \in K$ there is $g_x \in C(x) : \varphi(g_x) \neq g_x(x)$. Consider $f_x = g_x - \varphi(g_x)$. Then $\varphi(f_x) = 0$, $f_x(x) \neq 0$. So there is U_x open neighbourhood of x such that $f_x \neq 0$ on U_x. Compactness implies $\exists x_1, \ldots, x_n \in K : K \subset \bigcup_{i=1}^n U_{x_i}$. Consider $h := \sum_{i=1}^n |f_{x_i}|^2$. Then $h > 0$ on K, so h^{-1} exists and therefore $\varphi(h) \neq 0$. But $\varphi(h) = \sum_{i=1}^n \varphi(f_{x_i}) \overline{\varphi_{x_i}} = 0$.)

$\Delta \{M_n\} = \emptyset$, $n \geqslant 2$, where M_n is (non-commutative) algebra of $n \times n$ matrices. ($M_n = \mathrm{LO}\{E^{i,j}\}$. $E^{ij} \cdot E^{kl} = E^{il}$ if $j = k$, else 0. So $\varphi(E^{ij}) \cdot \varphi(E^{ij}) = \varphi(E^{ij} \cdot E^{ij}) = 0$ if $i \neq j$. $\varphi(E^{ii}) = \varphi(E^{in} E^{ni}) = \varphi(E^{in}) \varphi(E^{in}) = 0$. $\varphi(E^{nn}) = \varphi(E^{n1} E^{1n}) = 0$.)

Definice 1.11 (Ideal, maximal ideal)

A Banach algebra. Ideal in A is a subspace $I \subset A$ if $\forall x \in I\ \forall y \in A : x \cdot y \in I \wedge y \cdot x \in I$.

Maximal ideal $\equiv$ proper ($I \neq A$) ideal and it is maximal proper ideal with respect to inclusion.

Například (2021, Johnson-Schetman, Acta mathematica)
$\mathcal{L}(L_p)$ has 2^{2^ω} non-isomorphic closed ideals.

Tvrzení 1.21

A Banach algebra with a unit. Then:

- *Any proper ideal is contained in a maximum ideal. (From Zorn's lemma. And $I \subset A$ ideal is proper $\Leftrightarrow e \notin I$.)*

- *$I \subset A$ proper ideal $\implies \overline{I} \in A$ is proper ideal. In particular, maximal ideals are closed. (Easy: $\overline{I}$ is ideal. Moreover, $I \cap A^* = \varnothing$ (if $x \in I$ was invertible thus $e = x \cdot x^{-1} \in I$, but $e \notin I$). So (A^* is open) $\overline{I} \cap A^* = \varnothing$ and therefore $e \notin \overline{I}$.)*

Tvrzení 1.22

A Banach algebra, $I \subseteq A$ closed ideal $\implies A/I$ is Banach algebra ($[x] \cdot [y] := [x \cdot y]$).

Důkaz

Straightforward from definition. (Omitted.) □

Poznámka

From now on, A will be commutative.

Step 1: „Hahn-Banach“: $I \subset A$ closed ideal $\implies \exists \varphi \in \Delta(A) : \varphi/I \equiv \ldots$.

Věta 1.23

A commutative Banach algebra with a unit. Then $\Phi : \Delta(A) \to \{$maximal ideals in $A\}$, $\Phi(\varphi) := \operatorname{Ker} \varphi$, is bijection.

Důkaz

Pick $\varphi \in \Delta(A)$. Then „$\operatorname{Ker}\varphi$ is maximal ideal“: ideal: $y \in \operatorname{Ker}\varphi, x \in A : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \ldots \cdot 0 = 0$, proper: $\varphi \not\equiv 0$, maximal: $\operatorname{codim} \operatorname{Ker}\varphi = 1$: pick $x_0 : \varphi(x_0) \neq 0$, $a = a - \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} + \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} \in \operatorname{Ker}\varphi \oplus \mathbb{R}$.

„Φ is one-to-one“: Pick $\varphi, \psi \in \Delta(A)$: $\operatorname{Ker}\varphi = \operatorname{Ker}\psi$. Then (by lemma from previous semester) $\varphi = c \cdot \psi$ for some $c \in \mathbb{K}$. But $\varphi(e) = 1 = \psi(1)$ so $\varphi = \psi$.

„Φ is surjective“: Let $I \subset A$ be maximal ideal ($\implies$ closed). Step 1 „Any nonzero element in A/I is invertible“: For contradiction assume $\exists q(x) \in A/I$ ($q(x) = [x]$), $q(x) \neq 0 \wedge q(x)^{-1}$ does not exist. By next lemma $q(x)(A/I)$ is proper ideal. Then $q^{-1}\left(q(x)(A/I)\right)$ is an ideal in A which is proper and $I \subsetneq q^{-1}\left(q(x)(A/I)\right)$, which contradicts maximality of I. It follows from: ideal: follows from the fact that q is algebra homomorphism; proper: $q(e) = [e] \notin q(x)A/I$; $I \subseteq q^{-1}(\ldots)$: $0 \in q(x)A/I$; $I \neq q^{-1}(\ldots)$: $q(x) \neq 0 \implies x \notin I$, but $q(x) = q(x)q(e) \in q(x)(A/I)$, so $x \in q^{-1}(\ldots)$.

From Gelfand–Mazur theorem $\exists$ surjective isomorphism $j : A/I \to \mathbb{C}$. Then $\varphi := j \circ q \in \Delta(A)$. It remains „$I = \operatorname{Ker}\varphi$“: $x \in \operatorname{Ker}\varphi \Leftrightarrow j(q(x)) = 0 \Leftrightarrow q(x) = 0 \Leftrightarrow x \in I$. □

Lemma 1.24

A commutative *Banach algebra with a unit, $x \in A$ does not have inverse $\implies xA$ is proper ideal.*

Důkaz
xA is ideal, because A is commutative. Then xA is proper ($e \notin xA$). □

Důsledek (Hahn–Banach like theorem)
A is commutative Banach algebra with a unit, $I \subset A$ proper ideal. Then $\exists \varphi \in \Delta(A) : \varphi / I \equiv 0$.

Důkaz
Let $\tilde{I} \supseteq I$ be maximal ideal. By previous theorem there is $\varphi \in \Delta(A) : \tilde{I} = \operatorname{Ker} \varphi$. □

Tvrzení 1.25

A, B Banach algebras, $\Phi : A \to B$ algebraic isomorphism. Then $\Phi^\# : \Delta(B) \to \Delta(A)$ defined as $\Phi^\#(\varphi) := \varphi \circ \Phi$ is homeomorphism.

Důkaz
„$\Phi^\#(\varphi) \in \Delta(A)$“: $\Phi^\#(\varphi) = \varphi \circ \Phi \in \Delta(A) \cup \{0\}$ and $\varphi \not\equiv 0 \wedge \Phi$ is onto $\implies \varphi \circ \Phi \neq 0$.

„$\Phi^\#$ is w^*-W^* continuous“: $\varphi_i \overset{w^*}{\to} \varphi \implies \varphi_i \circ \Phi \overset{w^*}{\to} \varphi \circ \Phi$.

Apply the proven part to Φ^{-1}, obtain that $(\Phi^{-1})^\# : \Delta(A) \to \Delta(B)$ is w^*-W^* continuous. Moreover we have $\Phi^\# \circ (\Phi^{-1})^\# = \mathrm{id} \wedge (\Phi^{-1})^\# \circ \Phi^\#$. □

Tvrzení 1.26

L locally compact T_2. Then $\delta : L \to \Delta(C_0(L)), x \mapsto \delta_x$ is homeomorphism onto.

Důkaz
„Case 1: L is compact“: By example δ is onto. Of course, δ is one-to-one, continuous. So δ is homeomorphism.

„Case 2: L is not compact“: Then there is $K = L \cup \{\infty\}$, one-point compactification, and $\{f \in \mathcal{C}(K) | f(\infty) = 0\} \ni f \mapsto f|_L \in C_0(L)$ is isometric isomorphism. Moreover $\Phi : \mathcal{C}_0(L)_e \to \mathcal{C}(K)$, $\Phi(f, \lambda) := f + \lambda$, is algebraic isomorphism.

So, we have $K \overset{\eta}{\to} \Delta(C(K)) \overset{\Phi^\#}{\to} \Delta(C_0(L)_e) \overset{\psi}{\to} \Delta(C_0(L)) \cup \{0\}$, where η is homeomorphism from case 1 and $\psi(\varphi) := \varphi|_{C_0(L)}$.

Thus $\delta := \psi \circ \Phi^\# \circ \eta$ is homeomorphism between $L \cup \{\infty\}$ and $\Delta(C_0(L)) \cup \{0\}$. Finally, for $x \in K$ and $f \in \mathcal{C}_0(L)$:

$$\Phi^\# \circ \eta(x)(f) = (\eta(x) \circ \Phi)(f) = f(x),$$

so $\psi \circ \Phi^\# \circ \eta(x) = \Phi^\# \circ \eta(x)|_{C_0(L)} = \delta_x|_{C_0(L)}$. □

Věta 1.27

K, L locally compact T_2. Then following is ekvivalent

- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as Banach algebra;*
- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as algebras;*
- *$K \approx L$ as topological spaces.*

Důkaz

„1 $\implies$ 2" trivial. „2 $\implies$ 3": $K \approx \Delta(\mathcal{C}_0(K)) \approx \Delta(\mathcal{C}_0(L)) \approx L$ from previous two tvrzeni. „3 $\implies$ 1": Given $h : K \to L$ homeomorphism, $f \mapsto f \circ h$ is isometry between Banach algebras. □

Definice 1.12 (Semi-simple Banach algebra)

A commutative Banach algebra. It is semi-simple $\equiv$ $\Delta(A)$ separates points of A. ($\Leftrightarrow$ $\bigcap \{\operatorname{Ker}\varphi | \varphi \in \Delta(A)\} = \{\mathbf{o}\}$.)

Poznámka

Semi-simple $\implies$ commutative. (Semi-simple and $x \cdot y \neq y \cdot x \implies \exists \varphi \in \Delta(A) : \varphi(x) \cdot \varphi(y) = \varphi(x \cdot y) \neq \varphi(y \cdot x) = \varphi(y) \cdot \varphi(x)$ ↯.)

Věta 1.28

A, B Banach algebras, B is semi-simple, then every (algebra) homomorphism $\Phi : A \to B$ is continuous.

Důkaz

Use Closed graph theorem. Pick $x_n \to x$, $\varphi(x_n) \to y$. Wanted „$\Phi(x) = y$" ($\Leftrightarrow \forall \varphi \in \Delta(B) : \varphi(\Phi(x)) = \varphi(y)$). For $\varphi \in \Delta(B)$ we have $\varphi(y) = \lim_n \varphi(\Phi(x_n)) \overset{\varphi\circ\Phi\in\Delta(A)\subseteq A^*}{=} \varphi \circ \Phi(\lim_n x_n) = \varphi(\Phi(x))$. □

Důsledek

$(A, \|\cdot\|)$ semi-simple Banach algebra and $(A, |||\cdot|||)$ is Banach algebra (with other norm), then $\|\cdot\|$ and $|||\cdot|||$ are equivalent.

Důkaz

We have that $\mathrm{id} : (A, |||\cdot|||) \to (A, \|\cdot\|)$ is algebra homomorphism, so continuous by previous theorem. Similarly inverse is continuous (semi-simplicity doesn't depend on norm). So, id is isomorphism. □

2 Gelfand transformation

Definice 2.1 (Gelfand transformation)

A Banach algebra. For $x \in A$ we define $\hat{x} : \Delta(A) \to \mathbb{C}$, $\hat{x}(\varphi) := \varphi(x)$. We say that $\hat{x}$ is Gelfand transformation of x.

Poznámka
$\hat{x} \in \mathcal{C}_0(\Delta(A))$.

$A = \mathcal{C}_0(L) \implies \Delta(A) = \{\delta_x | x \in L\} \implies \forall f \in A : \hat{f}(\delta_x) = f(x), x \in L$. So, $\hat{f} = f$.

$A = L_1(\mathbb{R}^d) \implies \Delta(A) = \{e^{it\cdot x} | x \in \mathbb{R}\} \subseteq L_\infty(\mathbb{R}^d) = A^*$ and $\hat{f}$ is Fourier transformation.

Věta 2.1

A commutative Banach algebra, $x \in A$. Then

- *A has a unit* $\implies \sigma(x) = \operatorname{Rng} \hat{x}$*;*
- *A doesn't have a unit* $\implies \sigma(x) = \operatorname{Rng} \hat{x} \cup \{0\}$*;*
- $\|\hat{x}\|_\infty = r(x) = \sup\{|\lambda| | \lambda \in \sigma(x)\}$.

Důkaz
„a) $\subseteq$“: $\lambda \in \sigma(x) \Leftrightarrow (\lambda \cdot e - x)^{-1}$ does not exists $\implies$ (Lemma above) $(\lambda e - x)A$ is proper ideal $\implies \exists \varphi \in \Delta(A) : \varphi|_{(\lambda e - x)A} \equiv 0 \implies \exists \varphi \in \Delta(A) : 0 = \varphi(\lambda e - x) = \lambda - \varphi(x) = \lambda - \hat{x}(\varphi) \implies \lambda \in \operatorname{Rng} \hat{x}$.

„$\supseteq$“ follows from Tvrzeni above, $\varphi(x) \in \sigma(x)$ for $\varphi \in \Delta(A)$.

„b)“ For $x \in A$:

$$\sigma(x) = \sigma_{A_e}((x,0)) \overset{\text{a)}}{=} \operatorname{Rng} \widehat{(x,0)} = (\{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}) =$$
$$= \{\varphi(x) | \varphi \in \Delta(A) \cup \{0\}\} = \operatorname{Rng} \hat{x} \cup \{0\}.$$

„c)“ $\|\hat{x}\|_\infty = \sup\{|\lambda| | \lambda \in \operatorname{Rng} \hat{x}\} = \sup\{|\lambda| | \lambda \in \operatorname{Rng} \hat{x} \cup \{0\}\} = \sup\{|\lambda| | \lambda \in \sigma(x)\} = r(x)$. $\square$

Definice 2.2 (Gelfand transformation of algebra)

A Banach algebra, then $\Gamma : A \to \mathcal{C}_0(\Delta(A))$, $\Gamma(x) := \hat{x}$ is the Gelfand transformation of A.

Věta 2.2

A commutative Banach algebra, Γ Gelfand transformation. Then

- *Γ is algebra transformation, continuous, $\|\Gamma\| \leqslant 1$;*
- *$\Gamma(A)$ separates the points of $\Delta(A)$;*
- *Γ is one-to-one $\Leftrightarrow$ A is semi-simple;*
- *Γ is an isomorphism into $\Leftrightarrow \exists K > 0 : \|x^2\| \geqslant K \cdot \|x\|^2$, $x \in A$; ($\Leftrightarrow$ Γ is one-to-one and $\Gamma(A)$ is closed;)*
- *Γ is an isometry into $\Leftrightarrow \|x^2\| = \|x\|^2$, $x \in A$.*

Důkaz

„a)“: Γ is linear (obvious), Γ preserves multiplication (obvious). Finally, $\|\Gamma(x)\|_\infty = \|\hat{x}\|_\infty = r(x) \leqslant \|x\|$. So $\|\Gamma\| \leqslant 1$.

„b)“: Let $\varphi \neq \psi \in \Delta(A)$ and $x \in A : \hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$.

„c)“: $\Gamma(x) = 0 \Leftrightarrow \hat{x}(\varphi) = 0, \varphi \in \Delta(A) \Leftrightarrow \varphi(x) = 0, \varphi \in \Delta(A)$. So, Γ is one-to-one $\Leftrightarrow \forall x \neq 0\ \exists \varphi \in \Delta(A) : \varphi(x) \neq 0 \Leftrightarrow A$ is semi-simple.

„d) second“: Γ is isomorphism into $\Leftrightarrow$ Γ is bijection between A and $\Gamma(A) \wedge \Gamma(A)$ is closed. ($\Gamma(A)$ is closed, then we use Open mapping theorem; if Γ is isomorphism, $\Gamma(A)$ is a Banach space.).

„d) + e), $\implies$ “: Suppose $\exists c > 0$: $\|\Gamma(x)\| \geqslant c \cdot \|x\|$, $x \in A$. Then $\forall x \in A : \|x^2\| \overset{\text{a)}}{\geqslant} \|\Gamma(x^2)\| = \|\Gamma(x)\|^2 \geqslant c^2 \cdot \|x\|^2$.

„d) + e), $\impliedby$ “: Let d) hold with K (this holds in every algebra). Then (proven by induction)

$$\forall x \in A : \|x^{2^n}\| \geqslant K^{2^{n-1}} \|x\|^{2^n}, \qquad n \in \mathbb{N}.$$

$$\implies \sqrt[2^n]{\|x^{2^n}\|} \geqslant K^{1-2^{-n}} \|x\|,$$

where left side converges (by Beurling) to $r(x)$ and right side converges to $\|x\|$. So $r(x) \geqslant K \cdot \|x\|$ and from previous theorem $r(x) \geqslant \|\hat{x}\|_\infty = \|\Gamma(x)\|$. □

2.1 C^*-algebras

Definice 2.3 (Involution)

A is a Banach algebra. Involution is a mapping $* : A \to A$ such that

$$\forall x, y \in A\ \forall \lambda \in \mathbb{C} :$$

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x^*, \qquad (xy)^* = y^* \cdot x^*, \qquad (x^*)^* = x.$$

Definice 2.4 (C^*-algebra)

Banach algebra with involution $*$ is a C^*-algebra, if

$$\forall x \in A : \|x \cdot x^*\| = \|x\|^2, x \in A.$$

Definice 2.5 (Self-adjoint element, normal element)

For A with involution $*$ and $x \in A$ we say that x is self-adjoint $\equiv x = x^*$, and x is normal $\equiv x \cdot x^* = x^* \cdot x$.

Tvrzení 2.3 (Properties)

A Banach algebra with involution, $x \in A$. Then

- *e is left/right unit $\implies$ e is unit and $e = e^*$. (e is left unit $\Leftrightarrow$ e^* is right unit. So there is unit.)*
- *A is C^*-algebra $\Leftrightarrow \|x \cdot x^*\| \geqslant \|x\|^2$, $x \in A$. Then $\|x^*\| = \|x\|$, $x \in A$. („ $\implies$ “: clear, „ $\impliedby$ “: Then $\forall x \in A : \|x\|^2 \leqslant \|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\|$, so $\|x\| \leqslant \|x^*\|$, and applying to x^* we get $\|x^*\| \leqslant \|x\|$. But then we have $\|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\| = \|x\|^2$.)*
- *Let A has a unit. then x^{-1} exists $\Leftrightarrow (x^*)^{-1}$ exists. Then $(x^*)^{-1} = (x^{-1})^*$. („ $\implies$ “: $x^* \cdot (x^{-1})^* = (x^{-1}x)^* = e^* = e$, analogically $(x^{-1})^* x^* = e$. „ $\impliedby$ “: Apply the proven part to x^*.)*
- *$\lambda \in \sigma(x) \Leftrightarrow \overline{\lambda} \in \sigma(x^*)$. ($A$ has a unit: $\lambda \notin \sigma(x) \Leftrightarrow \exists (\lambda e - x)^{-1} \Leftrightarrow \exists \left((\lambda e - x)^*\right)^{-1} \Leftrightarrow \overline{\lambda} \notin \sigma(x^*)$. If A has not a unit, then we use previous sentence and next theorem?)*
- *$x + x^*$, $x^* \cdot x$, $x \cdot x^*$, $i \cdot (x - x^*)$ are self-adjoint. (Easy, omitted.)*
- *$\exists! u, v \in A$ self-adjoint: $x = u + i \cdot v$. Then $x^* = u - i \cdot v$, and x is normal $\Leftrightarrow uv = vu$. („Existence“: $u := \frac{1}{2}(x + x^*), v := \frac{1}{2i}(x - x^*)$. Then $x = u + iv$. „Formulas“: $(u + i \cdot v)^* = u^* + \overline{i} v^*$. „Uniqueness“: Pick $a, b \in A_{sa} : x = a + i \cdot b$. Then $a + i \cdot b = x = u + i \cdot v$, $a - i \cdot b = x^* = u - i \cdot v$. By subtracting or summing equation we get $a = u$ and $b = v$. „Normality“: x normal $\Leftrightarrow (u + i \cdot v)(u - i \cdot v) = (u - i \cdot v)(u + i \cdot v) \Leftrightarrow -i \cdot u \cdot v + i \cdot v \cdot u = i \cdot u \cdot v - i \cdot v \cdot u \Leftrightarrow u \cdot v = v \cdot u$.)*

Věta 2.4

A is C^*-algebra, $x \in A$ is normal. Then $r(x) = \|x\|$.

Důkaz

„Step 1: $\|x^2\| = \|x\|^2$“:

$$\|x\|^4 = \|x^*x\|^2 = \|(x^*x)^*(x^*x)\| = \|x^*xx^*x\| = \|x^*x^*xx\| = \|(xx)^*xx\| = \|xx\|^2 = \|x^2\|^2.$$

Thus inductively, we obtain $\|x^{2^k}\| = \|x\|^{2^k}$, $k \in \mathbb{N}$. Thus, Beurling gives $r(x) = \lim_k \sqrt[2^k]{\|x^{2^k}\|} = \|x\|$. □

Důsledek

A (Banach) algebra with involution. Then there is at most one norm $\|\cdot\|$ on A, such that $(A, \|\cdot\|)$ is C^*-algebra.

Důkaz

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on A such that $(A, \|\cdot\|)$ is C^*-algebra, then by previous theorem

$$\forall x \in A : \|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \|x^*x\|_2 = \|x\|_2^2.$$

□

Věta 2.5

$(A, \|\cdot\|)$ Banach algebra.

- *$(a, \lambda)^* = (a^*, \overline{\lambda})$, $(a, \lambda) \in A_e$ defines an involution on A_e. (Trivial.)*
- *If A is C^*-algebra, then on A_e there exists a norm $|\|\cdot\||$ (equivalent to the norm from $A \oplus_1 \mathbb{K}$) such that $(A_e, |\|\cdot\||)$ is C^*-algebra and $|\|(a, 0)\|| = \|a\|$, $a \in A$.*

Věta 2.6

A is C^*-algebra, $x \in A$. Then

- *$x = x^* \implies \sigma(x) \subseteq \mathbb{R}$;*
- *A has a unit and $x^* = x^{-1}$ (that is, x is unitary) $\implies \sigma(x) \subseteq \{\lambda | |\lambda| = 1\}$.*

Důkaz

By previous theorem, WLOG A has a unit.

„a)": Let $\alpha + i\beta \in \sigma(x)$, $\alpha, \beta \in \mathbb{R}$. We want $\beta = 0$. Trick: $x_t := x + i \cdot t \cdot e$, $t \in \mathbb{R}$. Then

$$\alpha + i \cdot (\beta + t) \in \sigma(x_t)(\Longleftarrow (\alpha + i(\beta + t))e - x_t = (\alpha + i \cdot \beta)e - x),$$

$$\alpha^2 + (\beta+t)^2 = |\alpha + i(\beta+t)|^2 \leqslant \|x_t\|^2 = \|x_t^* x_t\| = \|(x - i \cdot t \cdot e) \cdot (x + i \cdot t \cdot e)\| = \|x^2 + (t \cdot e)^2\| \leqslant \|x^2\| + t^2.$$

So, $\alpha^2 + (\beta + t)^2 - t^2 \leqslant \|x^2\|$, $t \in \mathbb{R} \implies \beta = 0$ (Otherwise $LHS \to +\infty$ for $t \to \pm\infty$.)

„b)": ($\|e\| = \|e^2\| = \|e\|^2$.) $1 = \|e\| = \|x^* x\| = \|x\|^2$, so $\|x\| = 1$. Then, for $\lambda \in \sigma(x)$, we have $|\lambda| \leqslant \|x\| = 1$. On the other hand $\frac{1}{\lambda} \in \sigma(x^{-1})$ (because if not, then $\frac{1}{\lambda}e - x^{-1}$ ha inverse $\implies \lambda e - x = (\lambda e - x)x^{-1}x = (\lambda x^{-1} - e)x = -\lambda(\frac{1}{\lambda}e - x^{-1})x \implies \lambda e - x$ has inverse.) So

$$\left|\frac{1}{\lambda}\right| \leqslant \|x^{-1}\| = \|x^*\| = \|x\| = 1.$$

□

Definice 2.6

A, B are C^*-algebras, then $\Phi : A \to B$ is $*$-homomorphism if Φ is homomorphism preserving $*$ (that is, $\Phi(x^*) = (\Phi(x))^*$).

Důsledek

Let A be a C^*-algebra and $\Phi \in \Delta_A$. Then Φ is $*$-homomorphism.

Důkaz

„If $x = x^*$", then $\Phi(x) \in \sigma(x) \subseteq \mathbb{R}$, so $\Phi(x^*) = \Phi(x) = \overline{\Phi(x)}$.

„In general", if $x = u + i \cdot v$ ($u = u^*$, $v = v^*$), then $\Phi(x^*) = \Phi(u - i \cdot v) = \Phi(u) - i \cdot \Phi(v) = \overline{\Phi(u) + i \cdot \Phi(v)} = \overline{\Phi(x)}$. □

Tvrzení 2.7 (Automatical continuous)

Let A, B be C^*-algebras, $\Phi : A \to B$ is $*$-homomorphism. Then Φ is continuous and $\|\Phi\| \leqslant 1$.

Důkaz

$\forall x \in A : \|\Phi(x)\|^2 = \|\Phi(x)^* \cdot \Phi(x)\| = r(\Phi(x^*) \cdot \Phi(x)) = r(\Phi(x^* x)) \overset{*}{=} r(x^* x) = \|x^* x\| = \|x\|^2$.

Thus it suffices to show that (by following lemma)

$$\sigma(\Phi(x^* x)) \subseteq \sigma(x^* x) \cup \{0\} .$$

□

Lemma 2.8

Let A, B be Banach algebras, $\Phi : A \to B$ algebra homomorphism. Then $\forall x \in A : \sigma_B(\varphi(x)) \subseteq \sigma_A(x) \cup \{0\}$.

Důkaz

Consider $\tilde{\Phi} : A_e \to B_e$ defined as $\tilde{\Phi}(a, \lambda) := (\Phi(a), \lambda)$. Then $\tilde{\Phi}$ is algebra homomorphism preserving unit. Moreover $\sigma_B(\Phi(x)) \subseteq \sigma_{B_e}((\Phi(x), 0)) \cup \{0\}$ and $\sigma_{A_e}((x, 0)) \subseteq \sigma_A(x) \cup \{0\}$. Thus, WLOG A, B have units and $\Phi(e_A) = e_B$.

But then for $\lambda \neq 0$ and $x \in A : \lambda e - x$ has inverse in A, then $\Phi(\lambda e - x) = \lambda \Phi(e) - \Phi(x)$ has inverse in B. So, $\lambda \notin \sigma_A(x) \cup \{0\} \implies \lambda \notin \sigma_B(\Phi(x))$. □

Věta 2.9 (Gelfand–Naimark)

A commutative C^*-algebra. Then the Gelfand transformation $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ is isometric $*$-isomorphism onto.

Důkaz

By proposition above, Γ is algebra homomorphism, $\|\Gamma\| \leqslant 1$ and from theorem above $\|\Gamma(x)\|_\infty = r(x)$, $x \in A$. „Γ is $*$-homomorphism“:

$$\forall a \in A \ \forall \varphi \in \Delta(A) : \Gamma(a^*)(\varphi) = \varphi(a^*) = \overline{(\varphi(a))} = \overline{\Gamma(a)}(\varphi).$$

„Γ is isometry“:

$$\forall x \in A : \|\Gamma(x)\|^2 = \|\overline{\Gamma(x)} \cdot \Gamma(x)\| = \|\Gamma(x^* x)\| = r(x^* x) = \|x^x\| = \|x\|^2.$$

„Γ is onto“: $\Gamma(A)$ is Banach space so $\Gamma(A) \subseteq \mathcal{C}_0(\Delta(A))$ is closed and $*$-subalgebra. And $\Gamma(A)$ separates points of $\Delta(A)$. So from Stone–Weierstrass theorem ($A \subset \mathcal{C}_0(K)$ is $*$-subalgebra separating the points, then $\overline{A}^{\|\cdot\|} = \mathcal{C}_0(K)$) $\Gamma(A) = \mathcal{C}_0(\Delta(A))$. □

Důsledek

A, B commutative C^*-algebras. Then the following items are equivalent:

- A and B are isometrically $*$-isomorphic;

- A and B are algebraically isomorphic;
- $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

Důkaz
„2. $\Leftrightarrow$ 3.“ follows from theorem above (where it is proved for $\mathcal{C}_0(K)$-spaces). „1. $\implies$ 2.“: trivial.

„3. $\implies$ 1.“: easy for $\mathcal{C}_0(K)$-spaces, because if $h : K \to L$ is homeomorphism, then $f \mapsto f \circ h$ is isometrical $*$-isomorphism. □

Definice 2.7

A Banach algebra, $M \subset A$. Then $alg(M) = \bigcap\{B \supseteq M | B$ is subalgebra of $A\}$.

Poznámka (Easy)

$$= \left\{\sum_{i=1}^{n} \alpha_i \prod_{j=1}^{m} x_{ij} | n, m \in \mathbb{N}, \alpha_i \in \mathbb{C}, x_{ij} \in M\right\}.$$

Moreover $\overline{alg} M = \bigcap\{B \supseteq M | B$ is closed subalgebra of $A\}$.

Poznámka (Easy)

$$= \overline{algM}^{\|\cdot\|}.$$

Tvrzení 2.10 (Fact)

A is C^*-algebra, $M \subset A$ is commutative and closed under $*$, then $\overline{alg} M$ is commutative C^*-subalgebra of A.

Věta 2.11

A, B are C^*-algebras, $h : A \to B$ is $*$-homomorphism, one-to-one. Then h is isometry.

Důkaz
WLOG A and B have units and $h(e)$ is a unit ($(a,\lambda) \mapsto (h(a),\lambda)$ is one-to-one $*$-homomorphism). Suffices: $\forall x \in A$ self-adjoint: $\|x\| = \|h(x)\|$ ($\forall y \in A : \|h(y)\|^2 = \|h(y^*y)\| = \|y^*y\| = \|y\|^2$). Let $x \in A$ be self-adjoint. Put $A_0 := \overline{alg}\{e,x\} = \overline{\mathrm{LO}}\{e,x,x^2,x^3,\dots\}$ is commutative and C^*-subalgebra.

$$B_y := \overline{alg}\{e,h(x)\} = \overline{\mathrm{LO}}\left\{e,h(x),h(x^2),\dots\right\}$$

is commutative and C^*-subalgebra. So, we have $A_0 \xrightarrow{h} B_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(B_0))$, $A_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(A_0))$. So there is $\tilde{h} : \mathcal{C}(\Delta(A_0)) \to \mathcal{C}(\Delta(B_0))$ one-to-one $*$-homeomorphism, $\tilde{h}(1) = 1$. So, it suffices to prove the following lemma. □

Lemma 2.12

Let K, L be T_2 compact spaces, $\varphi : \mathcal{C}(K) \to \mathcal{C}(L)$ $*$-homomorphism, $\varphi(1) = 1$. Then $\exists \alpha : L \to K$ continuous mapping such that $\varphi(f) := f \circ \alpha$, $f \in \mathcal{C}(K)$.

Moreover, if φ is one-to-one, then α is onto and so φ is isometry.

Důkaz
By proposition above $\|\varphi\| \leqslant 1$ and φ is continuous. Consider $\varphi^* : \mathcal{M}(L) \to \mathcal{M}(K)$. Then „$\varphi^*(\Delta(\mathcal{C}(L)) \subseteq \Delta(\mathcal{C}(K))$“:

$$\forall h \in \Delta(\mathcal{C}(L))\ \forall f,g : \varphi^* h(fg) = h(\varphi(fg)) = h(\varphi(f))h(\varphi(g)) = \varphi^* h(f)\varphi^* h(g).$$

So, we have: $L \xrightarrow{\delta} \Delta(\mathcal{C}(L)) \xrightarrow{\varphi^*} \Delta(\mathcal{C}(K)) \xrightarrow{\delta^{-1}} K$. So, $\alpha(x) := \delta^{-1}(\varphi^*(\delta(x)))$, $x \in L$ is continuous from L to K.

For this α we have:

$$\forall x \in L\ \forall f \in \mathcal{C}(K) : \varphi(f)(x) = \delta_x(\varphi(f)) = (\varphi^* \circ \delta_x)(f) = f(\delta^{-1}\varphi^*\delta_x) = f(\alpha(x)).$$

Moreover, „if φ is one-to-one, then α is onto“: Suppose $\alpha(L) \subsetneq K \implies \exists f \in C(K)\setminus\{0\} : f|_{\alpha(L)} \equiv 0$. But then $\varphi(f) \equiv 0$, but $f \neq 0$. ↯(φ should be one-to-one.) Thus φ is isometry. □

Poznámka (GNS construction)
A is C^*-algebra $\implies \exists H$ Hilbert $\exists \varphi : A \to B(H)$ $*$-isomorphism into.

Důkaz (Sketch)
$f \geqslant 0$ $(\sigma(f) \geqslant 0)$ on $A|_{\{a|f(a*a)=0\}}$ constructs inner product $\langle [x],[y]\rangle := f(y^*x)$. Put $H := \overline{A|_{\{a|f(a*a)=0\}}}$. Then $\varphi(a)([x]) = [ax]$. □

3 Continuous calculus for formal elements of C^*-algebras

Poznámka
Idea: $\varphi(\sigma(x)) \ni f \mapsto f(x) \in A$.

For $A = C(K)$:

$$g \in \mathcal{C}(K), \varphi(\sigma(x)) \ni f \implies g \circ f \in C(K).$$

Let A be C^*-algebra with a unit, $x \in A$ normal. Consider

$$B = \overline{alg}\{e, x, x^*\} \in A \implies \Gamma_B : B \to \mathcal{C}(\Delta(B)) \wedge f(x) := \Gamma_B^{-1}(f \circ \Gamma_B(x)), f \in \mathcal{C}(\sigma_A(x)).$$

Problem is when $\Gamma_B(x) \subseteq \sigma_A(x)$.