Definice 0.1 (Weak derivative)

Let $u, v_\alpha \in L^1_{loc}(\Omega)$. We say, that v_α is α-th weak derivative of $u \equiv$

$$\equiv \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Definice 0.2 (Sobolev space ($W^{k,p}$))

$\Omega \subseteq \mathbb{R}^d$ open, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}.$$

$$||u||_{W^{k,p}(\Omega)} := ||u||_{k,p} := \begin{cases} \left(\sum_{|\alpha|\leqslant k} ||D^\alpha u||_p^p\right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha|\leqslant k} ||D^\alpha u||_\infty, & p = \infty. \end{cases}$$

Tvrzení 0.1 (Completeness of Sobolev space)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then $W^{k,p}(\Omega)$ is complete.

Tvrzení 0.2 (Separability of Sobolev spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then $W^{k,p}(\Omega)$ is separable.

Tvrzení 0.3 (Reflexivity of Sobolev spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in (1, \infty)$. Then $W^{k,p}(\Omega)$ is reflexive.

Definice 0.3 (Scalar product of $W^{k,2}$)

Let $u, v \in W^{k,2}$, we define scalar product of u and v by:

$$(u, v)_{W^{k,2}(\Omega)} := (u, v)_{k,2} := \sum_{|\alpha|\leqslant k} \int_\Omega D^\alpha u(x) \cdot D^\alpha v(x) dx.$$

Věta 0.4 (Local approximation of Sobolev functions)

$$\forall u \in W^{k,p}(\Omega)\ \exists \{u_n\}_{n=1}^\infty \subseteq C_0^\infty(\mathbb{R}^d)\ \forall \tilde{\Omega} \textit{ open}, \overline{\tilde{\Omega}} \subseteq \Omega : u^n \to u \textit{ in } W^{k,p}(\tilde{\Omega}).$$

Definice 0.4 (Domain of the class $C^{k,\mu}$)

Let $\Omega \subseteq \mathbb{R}^d$ be open bounded set and $\alpha > 0$. We say that $\Omega \in C^{k,\mu}$ iff:

- there exist M ($r \in [M]$) coordinate systems $\mathbf{x}^r = (x_1^r, \ldots, x_d^r) = (\tilde{x}^r, x_d^r)$ and functions $a^r : \Delta^r \to \mathbb{R}$, where $\Delta^r = \left\{\tilde{x}^r \in \mathbb{R}^{d-1} \mid |x_d^r| \leqslant \alpha\right\}$ such that $a^r \in C^{k,\mu}(\Delta^r)$;
- if we denote T_r the original transformation from $\mathbf{x}^r$ to $\mathbf{x} = (\tilde{x}, x_d)$, then $\forall x \in \partial\Omega$ $\exists r \in [M]$ such that $x = T_r(\tilde{x}', a(\tilde{x}_d))$;

- $\exists\beta > 0$ such that if we define

$$V_+^r := \left\{\mathbf{x}^r \in \mathbb{R}^d | \tilde{x}^r \in \Delta_r \wedge a^r(\tilde{x}^r) < x_d^r < a^r(\tilde{x}^r) + \beta\right\},$$

$$V_-^r := \left\{\mathbf{x}^r \in \mathbb{R}^d | \tilde{x}^r \in \Delta_r \wedge a^r(\tilde{x}^r) - \beta < x_d^r < a^r(\tilde{x}^r)\right\},$$

$$\Lambda^r := \left\{\mathbf{x}^r \in \mathbb{R}^d | \tilde{x}^r \in \Delta_r \wedge a^r(\tilde{x}^r) = x_d^r\right\},$$

then $t^r(V_+^r) \subseteq \Omega$, $T_r(V_-^r) \subseteq \mathbb{R}^d \backslash \Omega$, $T_r(\Lambda^r) \subseteq \partial\Omega$ and $\bigcup_{r \in [M]} T_r(\Lambda_r) = \partial\Omega$.

Věta 0.5 (Extension theorem for $W^{1,p}(\Omega)$)

Let $\Omega \in C^{0,1}$ and $k \in \mathbb{N}$, $p \in [1, \infty]$. Then there exists a continuous linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ such that (for C independent of u):

$$\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leqslant C \cdot \|Eu\|_{W^{k,p}(\Omega)} \wedge Eu|_\Omega = u.$$

Tvrzení 0.6 (Continuous and compact embedding of Sobolev spaces)

Let $\Omega \in C^{0,1}$ and let $p \in [1, \infty]$. Then

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$.*

Moreover

- *if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \leqslant \frac{dp}{d-p}$,*
- *if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $q < \infty$,*
- *if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1 - \frac{d}{p}$.*

$$X \hookrightarrow\hookrightarrow Y \equiv X \leqslant Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$(X \hookrightarrow\hookrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^{\infty}, \exists c : ||u^n||_{1,p} \leqslant c \implies \exists u^{n_j} : u^{n_j} \to u \text{ in } Y).)$$

Tvrzení 0.7 (Characterization of Sobolev spaces)

$$u \in W^{1,p}(\Omega) \implies \forall h, i, \delta : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leqslant \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}.$$

Also, if $\forall h, i, \delta : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leqslant c_i$ and $p > 1$ then $\frac{\partial u}{\partial x_i}$ exist $\forall i$ and $\|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)} \leqslant c_i$.

Tvrzení 0.8 (Trace theorem)

Let $\Omega \in C^{0,1}$ and $p \in [1,\infty]$. Then there exists a continuous linear operator $\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that (for c independent of u):*

$$\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leqslant c \cdot \|\operatorname{tr} u\|_{W^{1,p}(\Omega)} \wedge \forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}.$$

Věta 0.9 (Linear Lax–Milgram lemma)

Let B be a bilinear elliptic form. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V : B(u,\varphi) = \langle F, \varphi \rangle .$$

Věta 0.10 (Non-linear Lax–Milgram lemma)

Let B be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \ \exists! u \in V \ \forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle .$$

Definice 0.5 (Bochner integral)

Let $s : I \to X$ be a simple function ($|\operatorname{Im} s| = |\{x_1, \ldots, x_n\}| < \infty$) on interval. We define

$$\int_I s(t)dt := \sum_{j=1}^{n} x_j \cdot |I_j|.$$

Let $f : I \to X$ be a Bochner measurable function. We say that f is Bochner integrable if $\exists \{s^n\}_{n=1}^{\infty}$ such that $s^n(t) \to f(t)$ for almost every $t \in I$ and $\int_I \|s^n(t) - f(t)\|_X dt \to 0$ as $n \to \infty$ and we set

$$X \ni \int_I f(t)dt = \lim_{n\to\infty} \int_I s^n(t)dt.$$

Definice 0.6 (Bochner measurability, simple functions)

We say that $f : I \to X$ is measurable (strongly, Bochner) if $\exists \{s_j\}_{j=1}^{\infty}$ simple functions ($|\operatorname{Im} s_j| < \infty$), such that $||f(t) - s_n(t)||_X \to 0$ as $n \to \infty$ for almost every $t \in I$.

Definice 0.7 (The spaces $L^p(0,T;X)$)

Let X be a Banach space, then

$$L^p(0,T;X) = \left\{ f : (0,T) \to X \text{ bochner integrable} \middle| \int_I \|f(t)\|_X^p < \infty \right\}.$$

$$\|f\|_{L^p(0,T;X)} = \left(\int_I \|f(t)\|_X^p dt \right)^{1/p}.$$

Definice 0.8 (Weak time derivative for Bochner spaces)

Let $f : I \to X$ be Bochner integrable. We say that $g : I \to X$ is weak derivative of f with respect to time iff g is Bochner integrable and $\forall \tau \in C_0^\infty(I) : \int_I f(t)\tau'(t)dt = -\int_I g(t)\tau(t)dt$.

Definice 0.9 (Sobolev space $W^{1,p}(I; X)$)

$$W^{1,p}(I; X) := \{f \in L^p(I; X) | \partial_t f \in L^p(I; X)\};$$

$$||f||_{W^{1,p}(I;X)} = \begin{cases} \left(\int_I ||f||_X^p + ||\partial_t f||_X^p\right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \operatorname{esssup}_{t \in I}(||f(t)||_X + ||\partial_t f||_X), & p = \infty. \end{cases}$$

Tvrzení 0.11 (Completeness of $W^{1,p}(I; X)$)

$W^{1,p}(I; X)$ is complete.

Tvrzení 0.12 (Reflexivity, separability of $L^p(0, T; X)$)

$W^{1,p}(I; X)$ is separable for $p < \infty$ and X separable. $W^{1,p}(I; X)$ is reflexive if $p \in (1, \infty)$ and X is reflexive and also separable.

Definice 0.10 (Scalar product of $W^{1,2}(I; H)$)

If H is Hilbert space and $u, v \in$, then

$$(u, v)_{H^1(I;X)} := (u, v)_{L^2(I;X)} + (u', v')_{L^2(I;X)},$$

where

$$(u, v)_{L^2(I;X)} := \int_I (u(t), v(t))dt.$$

Definice 0.11 (Gelfand triple)

We say that X, H, X^* is Gelfand triple iff $X \overset{\text{dense}}{\hookrightarrow} H \cong H^* \overset{\text{dense}}{\hookrightarrow} X^*$.

Věta 0.13 (Integration by parts for Sobolev-Bochner functions)

Let $p \in (1, \infty)$, X, H, X^* a Gelfond triple, $u, v \in L^p(0, T; X)$, $\partial_t u, \partial_t v \in L^{p'}(0, T; X^*)$. Then $u, v \in C([0, T]; H)$ and $\forall 0 \leqslant t_1 < t_2 \leqslant T$:

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Důkaz (Completeness of Sobolev space)
u^n is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \to u$ in L^p. $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \to v_\alpha \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

$$\forall \eta \in C_0^\infty(\Omega) : \int_\Omega v_\alpha \eta = \int_\Omega (v_\alpha - D^\alpha u^n)\eta + \int_\Omega D^\alpha u^n \eta =$$

$$= \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|} \int_\Omega D^\alpha \eta u^n =$$

$$= \int_\Omega (v_\alpha - D^\alpha u)\eta + (-1)^{|\alpha|} \int_\Omega (u^n - u) D^\alpha \eta + (-1)^{|\alpha|} \int_\Omega u D^\alpha \eta.$$

$$\left| \int_\Omega (v_\alpha - D^\alpha u^n)\varphi \right| \leqslant \|v_\alpha - D^\alpha u^n\|_p \|\eta\|_{p'} \leqslant C\|v_\alpha - D^\alpha u^n\| \to 0.$$

$$\left| \int_\Omega (u^n - u) D^\alpha \eta \right| \leqslant \|u^n - u\|_p \|D^\alpha \eta\|_{p'} \leqslant C\|u^n - u\|_p \to 0.$$

□

Důkaz (Separability and reflexivity of Sobolev spaces)
$W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \ldots \times L^P(\Omega)$ $(d+1$ times), X closed subspace from previous.

Lemma: if $X \subseteq Y$ is closed subspace then Y separable $\implies$ X separable and Y reflexive $\implies$ X reflexive. (From functional analysis and topology.) □

Důkaz (Local approximation of Sobolev functions)
u is extended by 0 to $\mathbb{R}^d \backslash \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \qquad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \qquad \eta \in C_0^\infty(B_1), \eta \geqslant 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x)dx = 1.$$

$$u \in L^P(SET) \qquad u^\varepsilon \to u \text{ in } L^p(SET).$$

We need: $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(\tilde{\Omega})$ $\forall \alpha, |\alpha| \leqslant k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leqslant \varepsilon_0$ (so that ball of radius ε_0 and center in $\tilde{\Omega}$ is in Ω):

$$(D^\alpha u)^\varepsilon(x) = \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y)dy = \int_{B_\varepsilon(x)} D^\alpha u(y) \eta_\varepsilon(x-u)dy =$$

$$= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y)dy.$$

$$D^\alpha u^\varepsilon = D_x^\alpha \int_{\mathbb{R}^d} u(y)\eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y)dy.$$

□

Důkaz (Extension theorem for $W^{1,p}(\Omega)$)
Without proof. □

Důkaz (Continuous and compact embedding of Sobolev spaces)
Without proof. □

Důkaz (Characterization of Sobolev spaces)
Without proof. □

Důkaz (Trace theorem)
Without proof. □

Důkaz (Linear Lax–Milgram lemma by non-linear version)
We define $B(u) : V \to V^*$ by $\langle B(u), \varphi \rangle := B(u, \varphi)$. Then $B(u)$ is Lipschitz and uniformly monotone.

Důkaz (Lipschitz)

$$\|B(u) - B(v)\|_{V*} = \sup_{\varphi \in V, \|\varphi\|_V \leqslant 1} \langle B(u) - B(v), \varphi \rangle = \sup_{\varphi} (B(u, \varphi) - B(v, \varphi)) =$$

$$= \sup_{\varphi} B(u - v, \varphi) \leqslant \sup_{\varphi} c_2 \cdot \|u - v\|_V \cdot \|\varphi\|_V = c_2 \cdot \|u - v\|_V .$$

□

Důkaz (Uniformly monotone)

$$\langle B(u) - B(v), u - v \rangle = B(u - v, u - v) \geqslant c_1 \cdot \|u - v\|_V^2 .$$

□

So it satisfies assumptions of Non-linear Lax–Milgram lemma. □

Důkaz (Non-linear Lax–Milgram lemma)
„Uniqueness“: $u, v \in V$, $\forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle = \langle B(v), \varphi \rangle$. Then

$$\forall \varphi \in V : \langle B(u) - B(v), \varphi \rangle = 0 \overset{(\varphi := u - v)}{\Longrightarrow} \langle B(u) - B(v), u - v \rangle = 0 \geqslant c_1 \|u - v\|_V^2 \implies u = v.$$

„Existence“: $\forall \varphi : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle \Leftrightarrow$

$$\Leftrightarrow \forall \varepsilon > 0 \ \forall \varphi : (u, \varphi)_V = (u, \varphi)_V - \varepsilon \cdot (\langle B(u), \varphi \rangle - \langle F, \varphi \rangle).$$

Define a problem for $v \in V$: Find $u \in V$ such that

$$\forall \varphi : (u, \varphi)_V = (v, \varphi)_V - \varepsilon \cdot (\langle B(v), \varphi\rangle - \langle F, \varphi\rangle).$$

Define $M : V \to V$, $v \mapsto u$. If M has a fixed point, then we find a solution to the original problem.

1. „M is well-defined“: For given $v \in V$, define $\tilde{F} \in V^*$: $\forall \varphi : \left\langle \tilde{F}, \varphi \right\rangle := (v, \varphi)_V - \varepsilon(\langle B(v), \varphi\rangle - \langle F, \varphi\rangle)$. $\left\langle \tilde{F}, \varphi \right\rangle$ linear in φ. Riesz tells us that $\forall \tilde{F} \in V^*\ \exists! u \in V\ \forall \varphi \in V : (u, \varphi)_V = \left\langle \tilde{F}, \varphi \right\rangle$.

2. „M has a fixed point“: We show that

$$\exists \delta > 0\ \forall u, v \in V : \|M(u) - M(v)\|_V \leqslant (1 - \delta)\|u - v\|_V.$$

Then from Banach theorem M has a fixed point. From linearity (and definition of M):

$$(\overline{u} - \overline{v}, \varphi)_V = (u - v, \varphi)_V - \varepsilon \cdot (\langle B(u) - B(v), \varphi\rangle + 0).$$

From Rietsz theorem there exists w_1, w_2 such that $\forall \varphi : (w_1, \varphi)_V = \langle B(u), \varphi\rangle \wedge (w_2, \varphi)_V = \langle B(v), \varphi\rangle \implies$

$$\implies \|M(u) - M(v)\|_V^2 = \|u - v - \varepsilon(w_1 - w_2)\|_V^2 = \|u - v\|_V^2 - 2\varepsilon(u - v, w_1 - w_2) + \varepsilon^2 \cdot \|w_1 - w_2\|_V^2.$$

But from Lipschitz and uniformly monotone:

$$(u - v, w_1 - w_2) = \langle B(u) - B(v), u - v\rangle \geqslant c_1 \cdot \|u - v\|_V^2,$$

$$\|w_1 - w_2\|_V^2 = (w_1 - w_2, w_1 - w_2)_V = \langle B(u) - B(v), w_1 - w_2\rangle \leqslant \|B(u) - B(v)\|_V + \|w_1 - w_2\|$$

$$\implies \|w_1 - w_2\|_V^2 \leqslant \|B(u) - B(v)\|_{V*}^2 \leqslant c_2 \cdot \|u - v\|_V^2.$$

So (for sufficiently small ε $\exists d > 0$)

$$\|M(u) - M(v)\|_V^2 \leqslant \|u - v\|_V^2 - 2\varepsilon \cdot c_1 \cdot \|u - v\|_V^2 + \varepsilon^2 c_2 \cdot \|u - v\|_V^2 = (1 - 2\varepsilon \cdot c_1 + \varepsilon^2 \cdot c_2)\|u - v\|_V^2 \leqslant$$

$$\leqslant (1 - \delta)\|u - v\|_V^2.$$

□

Důkaz (Completeness of $W^{1,p}(I; X)$)
Without proof. □

Důkaz (Reflexivity, separability of $L^p(0, T; X)$)
Without proof. □

Důkaz (Integration by parts for Sobolev-Bochner functions)

- Step 1: Modify u, v in terms of the Steklov averages $u_h = \fint_t^{t+h} u(\tau)d\tau$.
- Step 2: Prove for u_h, v_h from step 1).
- Step 3: $h \to 0_+$.

Důkaz (Step 1)
Define $u_h(t) := \frac{1}{h}\int_t^{t+h} u(\tau)d\tau$, $\forall t \in (0, T-h)$. $u_h \to h$ $L^p(0, T-h_0, X)$, $\forall h_0 \in (0,T)$. We want „$(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$“.

$$(\partial_t u)_h \to \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \qquad \forall h_0 \in (0,T).$$

$$\begin{aligned}\varphi \in C_0^\infty(0,T-h) : \int_0^{T-h} u_h(t)\varphi'(t)dt &= \frac{1}{h}\int_0^{T-h}\varphi'(t)\int_t^{t+h} u(t)d\tau dt = \\ &= \frac{1}{h}\int_0^{T-h}\varphi'(t)\left(\int_0^{t+h} u(\tau)d\tau - \int_0^t u(\tau)d\tau\right) = \\ &= -\frac{1}{h}\int_0^{T-h}\varphi(t)(u(t+h)-u(t)) \Leftrightarrow \partial_t u_h = \frac{u(t+h)-u(t)}{h}.\end{aligned}$$

$$\begin{aligned}\varphi \in C_0^\infty(0,T-h) : \int_0^T \varphi(t)(\partial_t u)_h(t)dt &= \frac{1}{h}\int_0^{T-h}\varphi(t)\int_t^{t+h}\partial_t u(\tau)d\tau dt = \\ &= \frac{1}{h}\int_0^{T-h}\varphi(t)\left(\int_0^{t+h}\partial_t u(\tau)d\tau - \int_0^t \partial_t u(\tau)d\tau\right)dt = (*)\end{aligned}$$

$$\begin{aligned}\frac{1}{h}\int_0^{T-h}\varphi(t)\left(\int_0^t \partial_t u(\tau)d\tau\right)dt &= \int_0^{T-h}\int_0^{T-h}\varphi(t)\partial_t u(\tau)\chi_{\tau\leqslant t}d\tau dt = \\ &= \frac{1}{h}\int_0^{T-h}\partial_t u(\tau)\left(\int_t^{T-h}\varphi(t)dt\right)d\tau.\end{aligned}$$

$$(*) = \frac{1}{h}\int_0^{T-h}\partial_t u(\tau)\underbrace{\left(\int_{\tau-h}^{\tau}\varphi(t)dt\right)}_{C_0^\infty(0,T)}d\tau = -\frac{1}{h}\int_0^{T-h}u(\tau)\left(\varphi(\tau)-\varphi(\tau-h)\right)d\tau dt.$$

□

Důkaz (Step 2)
We want

$$\int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

$$\Leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

$$\int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int_{t_1}^{t} v(\tau) d\tau \right)_H =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t} v(\tau+h_2) d\tau - \int_{t_1}^{t} v(\tau) d\tau \right)_H =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1}^{t} v(\tau+h_2) - v(\tau) d\tau \right)_H dt +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau+h_2) - v(\tau) d\tau, \int_{t_1}^{\tau} u(t+h_1) - u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt =$$

$$= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau+h_2) - v(\tau) d\tau, \int_{t_2}^{t_2+h_1} u(t) - \int_{t_2}^{t_2+h_1} u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt =$$

$$= \int_{t_1}^{t_2} \left(\frac{v(\tau+h) - v(\tau)}{h_2}, \fint_{\tau}^{\tau+h_1} u(t) dt \right)_H d\tau +$$

$$+ \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left(v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h} u(t) dt \right)_H + \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau) d\tau)_H =$$

$$- \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau)) d\tau + REST$$

$$REST = \frac{1}{h_1 h_2} \left(\int_{t_2}^{t_2+h_2} v(t) dt - \int_{t_1}^{t_1+h_2} v(t) dt, \int_{t_2}^{t_2+h} u(t) dt \right) + SIMILAR =$$

$$= (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \ldots$$

□

Důkaz (Step 3)
We have

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X \, dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let $h_1 \to 0_+$ and $h_2 \to 0_+$. We have $\partial_t u_{h_1} \to \partial_t u$ in $L^{p'}(0,T,X^*)$, $\partial_t v_{h_2} \to \partial_t v$ in $L^{p'}(0,T,X^*)$, $u_{h_1} \to u$ in $L^p(0,T,X)$, $V_{h_2} \to v$ in $L^p(0,T,X)$. So for almost all t in $(0,T)$: $v_{h_2}(t) \to v(t)$ in $X \hookrightarrow H$ and $u_{h_1}(t) \to u(t)$ in $X \hookrightarrow H$.

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Now, it is enough to show $u, v \in C([0,T);H)$. We show that u_h is Cauchy in $C([0,T];H)$. Use IBP $u_{h_1} = u_{h^n} - u_{h^m}$, $v_{h_2} = u_{h^n} - u_{h^m}$:

$$||u_{h^n}(t_2) - u_{h^m}(t_2)||_H = ||u_{h^m}(t_1) - u_{h^m}(t_1) + 2\int_{t_1}^{t_2} \langle \partial_t(u_h^m - u_h^n), u_{h^n} - u_{h^m} \rangle_X \, ||$$

$$||u_{h^n} - u_{h^m}||^2_{C\left(\left[\frac{T}{4},T\right];L^2(\Omega)\right)} = \sup_{t_2 \in \left(\frac{T}{2},T\right)} ||u_{h^n}(t_2) - u_{h^m}(t_2)||_H^2 \leqslant$$

$$\leqslant ||u_{h^m}(t_1) - u_{h^n}(t_1)||_H^2 + \int_0^T ||\partial_t(u_{h^n}) - \partial u_{h^m}||_{X^*} ||u_{h^m} - u_{h^n}||_X dt.$$

Choose t_1 such that $u_h(t_1) \to u(t_1)$ in H:

$$\leqslant ||u_h(t_1) - u_{h^m}(t_1)||_H^2 + ||\partial_t u_{h^m} - \partial_2 u_{h^n}||_{L^p(X^*)} \cdot \ldots$$

$$u \in C\left(\left[\frac{T}{4}, T\right]; L^2(\Omega)\right) \wedge u \in C\left(\left[0, \frac{3T}{4}\right]; L^2(\Omega)\right) \to u \in C\left([0,T]; L^2(\Omega)\right) (u(t_1), v(t_1))_H.$$

□

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