

# Úvod

*Poznámka* (Organizační úvod)

Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (... PDE ...).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

*Poznámka* (Konvence pro PDR)

$\Omega \subseteq \mathbb{R}^d$  je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

*Poznámka*

Dále se ukazovali konkrétní parciální rovnice.

*Poznámka* (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál:  $L^p(\Omega)$ ,  $L^p_{loc}(\Omega)$ ,  $\|u\|_p$ ,  $C^k(\Omega)$ ,  $C^k(\overline{\Omega})$ ,

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) \mid \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}, \|u\|_{C^{0,\alpha}} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

- $\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$ ,  $\vec{n} = (n_1, \dots, n_d)$ .
- Funkcionální analýza 1: Banachův prostor,  $u^n \rightarrow u$  silná konvergence,  $u^n \rightharpoonup u$  slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita ( $L^p$  jsou separabilní až na  $p = \infty$ ,  $C^k(\overline{\Omega})$  je separabilní,  $C^{0,\alpha}$  není separabilní pro  $\alpha \in (0, 1]$ ).

*Poznámka* (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta u = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

# 1 Sobolevovy prostory

## Definice 1.1 (Multiindex)

$\alpha$  je multiindex  $\equiv d = (\alpha_1, \dots, \alpha_d)$ ,  $\alpha_i \in \mathbb{N}_0$ . Délka  $\alpha$  je  $|\alpha| := \alpha_1 + \dots + \alpha_d$ . Pro  $u \in C^k(\Omega)$  definujeme  $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ .

## Definice 1.2 (Slabá derivace)

Buď  $u, v_\alpha \in L^1_{loc}(\Omega)$ . Řekneme, že  $v_\alpha$  je  $\alpha$ -tá slabá derivace  $u \equiv$

$$\equiv \int_{\Omega} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} v_\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

*Příklad*

$u = \text{sign } x$  nemá slabou derivaci.

## Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

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*Důkaz*

$v_\alpha^1, v_\alpha^2$  dvě  $\alpha$ -té derivace  $u$ .

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^1 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^2 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\int_{\Omega} (v_\alpha^1 - v_\alpha^2) \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$  skoro všude v  $\Omega$ .

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají.

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□

## Definice 1.3 (Sobolevův prostor)

$\Omega \subseteq \mathbb{R}^d$  otevřená,  $k \in \mathbb{N}_0$ ,  $p \in [1, \infty]$ .

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid \forall \alpha, |\alpha| \leq k : D^\alpha u \in L^p(\Omega)\}.$$

$$\|u\|_{W^{k,p}(\Omega)} \|u\|_{k,p} := \begin{cases} \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_\infty, & p = \infty. \end{cases}$$

┌ Poznámka

Od teď  $D^\alpha$  nebo  $\frac{\partial}{\partial x_1}$  nebo  $\partial_i$  značí slabou derivaci.

### Lemma 1.2 (Základní vlastnosti slabých derivací a Sobolevových prostorů)

Nechť  $u, v \in W^{k,p}(\Omega)$ ,  $k \in \mathbb{N}$ , a  $\alpha$  multiindex s délkou  $\leq k$ .

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$  a  $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ , pro  $|\alpha| + |\beta| \leq k$ .
- $\lambda, \mu \in \mathbb{R}$ ,  $\lambda u + \mu v \in W^{k,p}(\Omega)$  a  $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$ .
- $\forall \tilde{\Omega} \subseteq \Omega$  otevřená

$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$

- $\forall \eta \in C^\infty(\Omega)$ :  $\eta u \in W^{k,p}(\Omega)$  a  $D^\alpha(\eta u) = \sum_{\beta_i \leq \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$ , kde  $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$ .

┌ Důkaz

└ Cvičení na doma. □

### Věta 1.3 (Basic properties of Sobolev spaces)

Let  $\Omega \subseteq \mathbb{R}^d$  be open set,  $k \in \mathbb{N}$  and  $p \in [1, \infty]$ . Then

- $W^{k,p}(\Omega)$  is a Banach space;
- if  $p < \infty$  it is separable space;
- if  $p \in (1, \infty)$  it is reflexive space.

Důkaz

BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness:  $u^n$  is Cauchy in  $L^p(\Omega)$  so  $\exists u \in L^p : u^n \rightarrow u$  in  $L^p$ .  $D^\alpha u^n$  is Cauchy in  $L^p(\Omega)$   $\forall |\alpha| < k$  so  $\exists v_\alpha \in L^p : D^\alpha u^n \rightarrow v_\alpha \in L^p$ . It remains prove that  $D^\alpha u = v_\alpha$ .

$$\begin{aligned} \forall \eta \in C_0^\infty(\Omega) : \int_{\Omega} v_\alpha \eta &= \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta + \int_{\Omega} D^\alpha u^n \eta = \\ &= \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta + (-1)^{|\alpha|} \int_{\Omega} D^\alpha \eta u^n = \\ &= \int_{\Omega} (v_\alpha - D^\alpha u) \eta + (-1)^{|\alpha|} \int_{\Omega} (u^n - u) D^\alpha \eta + (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \eta. \\ \left| \int_{\Omega} (v_\alpha - D^\alpha u^n) \eta \right| &\leq \|v_\alpha - D^\alpha u^n\|_p \|\eta\|_{p'} \leq C \|v_\alpha - D^\alpha u^n\| \rightarrow 0. \\ \left| \int_{\Omega} (u^n - u) D^\alpha \eta \right| &\leq \|u^n - u\|_p \|D^\alpha \eta\|_{p'} \leq C \|u^n - u\|_p \rightarrow 0. \end{aligned}$$

„2+3“:  $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \dots \times L^p(\Omega)$  ( $d+1$  times),  $X$  closed subspace from first property. Lemma: if  $X \subseteq Y$  is closed subspace then  $Y$  separable  $\implies X$  separable and  $Y$  reflexive  $\implies X$  reflexive. (From functional analysis and topology.)  $\square$

## 2 Approximation of Sobolev function

### Věta 2.1

Let  $\Omega \subseteq \mathbb{R}^d$  open,  $p \in [1, \infty)$ .

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

Pozor

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} \subsetneq W^{k,p}(\Omega).$$

Důkaz

Summer semester.  $\square$

### Věta 2.2 (Local density)

$$\begin{aligned} \forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty \\ u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} \text{ open}, \tilde{\Omega} \subseteq \Omega \\ u^n \rightarrow u \text{ in } W^{k,p}(\tilde{\Omega}) \end{aligned}$$

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*Důkaz*

$u$  is extended by 0 to  $\mathbb{R}^d \setminus \Omega$ .

$$u^\varepsilon = u * \eta^\varepsilon \quad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \quad \eta \in C_0^\infty(B_1), \eta \geq 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x) dx = 1.$$

$$u \in L^p(SET) \quad u^\varepsilon \rightarrow u \text{ in } L^p(SET).$$

We need:  $D^\alpha u^\varepsilon \rightarrow D^\alpha u$  in  $L^p(\tilde{\Omega}) \forall \alpha, |\alpha| \leq k$ . Essential step:  $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$  in  $\tilde{\Omega}$  for  $\varepsilon \leq \varepsilon_0$  (so that ball of radius  $\varepsilon_0$  and center in  $\tilde{\Omega}$  is in  $\Omega$ ):

$$\begin{aligned} (D^\alpha u)^\varepsilon(x) &= \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \int_{B_\varepsilon(x)} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \\ &= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y) dy. \\ D^\alpha u^\varepsilon &= D_x^\alpha \int_{\mathbb{R}^d} u(y) \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y) dy. \end{aligned}$$

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□

### **Tvrzení 2.3**

$\Omega$  is open connected set,  $u \in W^{1,1}(\Omega)$ , then  $u = \text{const.} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \forall i \in [d]$ .

$W^{1,1}(I) \hookrightarrow C(I)$  for  $I$  interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$ .

┌ *Důkaz*

„1.  $\implies$ “ trivial. „1.  $\Leftarrow$ “:  $\tilde{\Omega} \subseteq \Omega$  connected  $\varepsilon_0$  as before and  $\varepsilon \in (0, \varepsilon_0)$ .  $u^\varepsilon$ -modification of  $u$  is smooth, so

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial x_i} &= \left( \frac{\partial u}{\partial x_i} \right)^\varepsilon = 0 \quad \text{in } \tilde{\Omega} \\ \implies u^\varepsilon &= \text{const}(\varepsilon) \quad \text{in } \tilde{\Omega}. \end{aligned}$$

$$\begin{aligned} c(\varepsilon) &= \int_{\mathbb{R}} c(\varepsilon) \eta_\delta(x-y) dy = \int_{\mathbb{R}} u^\varepsilon(y) \eta_\delta(x-y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(z) \eta_\varepsilon(y-z) \eta_\delta(x-y) dz dy = \\ &= \iint u(z+y) \eta_\varepsilon(z) \eta_\delta(y-x) dz dy = \iint u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dz dw = \\ &= \iint u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dw dz = \int_{\mathbb{R}^d} u^\delta(z+x) \eta_\varepsilon(z) dz = \int c(\delta) \eta_\varepsilon(z) dz = c(\delta). \end{aligned}$$

„2.“: WLOG  $I = (0, 1)$ . Define  $v(x) = \int_0^x \frac{\partial u}{\partial y}(y) dy$ . We show:  $v \in W^{1,1}(I)$ ,  $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$ .

$$\begin{aligned} |v(x)| &\leq \int_0^1 \left| \frac{\partial u}{\partial x} \right| \leq \|u\|_{1,1}. \\ \varphi &\in C_0^1(0, 1) \quad \int_0^1 v(x) \frac{\partial \varphi}{\partial x}(x) dx \\ &= \int_0^1 \left( \int_0^x \frac{\partial u}{\partial y}(y) dy \right) \frac{\partial \varphi}{\partial x}(x) dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dy dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dx dy = \\ &= \int_0^1 \left( \int_y^1 \frac{\partial \varphi(x)}{\partial x} dx \right) \frac{\partial u}{\partial y}(y) dy = - \int_0^1 \varphi(y) \frac{\partial u}{\partial y}(y) dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}. \end{aligned}$$

TODO.

$$x \rightarrow y \implies \int_y^x \left| \frac{\partial u}{\partial z} \right|^\alpha \rightarrow 0 \implies |u(x) - u(y)| \rightarrow 0$$

$$\|u\|_{C(I)} \leq \|v + c\|_{C(I)} \leq \|u\|_{1,1} + |c| = \|u\|_{1,1} + |u(x) - v(x)| \forall x \in I$$

$$\|u\|_{C(I)} \leq \|u\|_{1,1} + \int_0^1 |u(x) - v(x)| dx \leq -\| \cdot \| + \int_0^1 |u| + \int_0^1 |v| \leq \|u\|_{1,1}.$$

„3.“ was shown without proof. □

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### 3 Characterization of Sobolev function

#### Věta 3.1

$\Omega \subseteq \mathbb{R}^d$ ,  $p \in [1, \infty]$ ,  $\delta > 0$ ,  $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \delta\Omega) > \delta\}$ . Then

$$\forall u \in W^{1,p}(\Omega) : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}, \quad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

$$u \in L^p \implies \forall \delta, h : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq c.$$

$$p > 1 \implies \frac{\partial u}{\partial x_i} \text{ exists and } \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq c.$$

### Definice 3.1 (Class $C^{k,\mu}$ )

Let  $\Omega \subseteq \mathbb{R}^d$  open bounded set. We say that  $\Omega \in C^{k,\mu}$  ( $\partial\Omega \in C^{k,\mu}$ ) iff:

- there exist  $M$  coordinate systems  $\mathbf{x} = (x_{r_1}, \dots, x_{r_d}) = (x'_r, x_{r_d})$  and functions  $a_r : \Delta_r \rightarrow \mathbb{R}$  where  $\Delta_r = \{x'_r \in \mathbb{R}^{d-1} \mid |x_{r_i}| \leq \alpha\}$  such that  $a_r \in C^{k,\mu}(\Delta_r)$ ,
- denoting  $\text{tr}$  the orthogonal transformation from  $(x'_r, x_{r_d})$  to  $(x', x_d)$ , then  $\forall x \in \partial\Omega \exists r \in \{1, \dots, M\}$  such that  $x = \text{tr}(x'_r, a(x_{r_d}))$ ,
- $\exists \beta > 0$ , if we define

$$V_r^+ := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\}$$

$$V_r^- := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\}$$

$$\Lambda_r := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) = x_{r_d}\}$$

Then  $\text{tr}(V_r^+) \subset \Omega$ ,  $\text{tr}(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$ ,  $\text{tr}(\Lambda_r) \subseteq \partial\Omega$  and  $\bigcup_{r=1}^M \text{tr}(\Lambda_r) = \partial\Omega$ .

### Věta 3.2 (Density of smooth functions)

Let  $\Omega \in C^0$ . Then  $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}$ ,  $p \in [1, \infty)$ .

### Věta 3.3 (Extension of Sobolev functions)

Let  $\Omega \in C^{0,1}$  ( $\Omega$  is Lipschitz) and  $k \in \mathbb{N}$ ,  $p \in [1, \infty]$ . Then there exists a continuous linear operator  $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$  such that:

- $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(\Omega)}$  ( $C$  is independent of  $u$ )
- $Eu = u$  almost everywhere in  $\Omega$ .

### Věta 3.4 (Trace theorem)

Let  $\Omega \in C^{0,1}$ ,  $p \in [1, \infty]$ . Then there exists a continuous linear operator  $\text{tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$  such that:

- $\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leq c \|u\|_{1,p},$
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}.$

### Definice 3.2

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

### Věta 3.5

Let  $\Omega \in C^{0,1}$  and let  $p \in [1, \infty]$ . Then

- if  $p < d$ , then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $1 \leq \frac{dp}{d-p}$ ,
- if  $p = d$ , then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $q < \infty$ ,
- if  $p > d$ , then  $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$ .

Moreover

- if  $p < d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$  for all  $1 \leq \frac{dp}{d-p}$ ,
- if  $p = d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$  for all  $q < \infty$ ,
- if  $p > d$ , then  $W^{1,p}(\Omega) \hookleftrightarrow C^{0,\alpha}(\overline{\Omega})$  for all  $\alpha < 1 - \frac{d}{p}$ .

$$X \hookleftrightarrow Y \Leftrightarrow X \leq Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$X \hookleftrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^\infty, \exists c : \|u^n\|_{1,p} \leq c \implies \exists u^{n_j} : u^{n_j} \rightarrow u \text{ in } Y).$$

*Důsledek* (Trace theorem)

Let  $\Omega \in C^{0,1}$ . Then  $\forall u \in W^{1,p}(\Omega)$  and  $v \in W^{1,p'}(\Omega)$  we have integration by parts:

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} dx = - \int_{\Omega} v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

### Věta 3.6 (Poincaré)

Let  $\Omega \in C^{0,1}$  and  $p \in [1, \infty]$ . Let  $\Omega_1, \Omega_2 \subseteq \Omega$ ,  $|\Omega_i| > 0$  and  $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$ ,  $|\Gamma_i|_{d-1} > 0$ . Let  $\alpha_1, \alpha_2 \geq 0$  and  $\beta_1, \beta_2 \geq 0$  and at least one of  $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ .



Then there exist  $c_1, c_2 > 0$  such that  $\forall u \in W^{1,p}(\Omega)$

$$c_1 \|u\|_{1,p}^p \leq \|\nabla u\|_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2 \int_{\Omega_2} |u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2 \int_{\Gamma_2} |u|^p \leq c_2 \|u\|_{1,p}^p.$$

$$(\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p.)$$

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Důkaz (Of the first (the only difficult) inequality)  
└ TODO!!!

□

## 4 Linear elliptic PDEs

### Definice 4.1 (Elliptic)

Let  $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$ , where  $\Omega \subseteq \mathbb{R}^d$  is bounded. We say that  $L$  is elliptic if  $\exists c_1 > 0$  such that  $\forall \zeta \in \mathbb{R}^d$  and almost all  $x \in \Omega$

$$A\zeta \cdot \zeta \geq c_1 |\zeta|^2.$$

### Lemma 4.1

If  $u$  is classical solution, then  $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$  on  $\Gamma_1$ :  $B_{L,\delta}(u, \varphi) = \int_{\Omega} f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$ .

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Důkaz  
└ TODO!!!

□

### Lemma 4.2

If  $u \in C^2(\overline{\Omega})$  and  $A, b, \mathbf{c}, \mathbf{d}$  are smooth and previous lemma holds  $\forall \varphi \in C^1, \varphi|_{\Gamma_1} = 0$  and  $u = u_0$  on  $\Gamma_1$ , then  $u$  is a classical solution.

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Důkaz  
└ TODO!!!

□

### Definice 4.2 (Weak solution)

Let  $\Omega \subseteq \mathbb{R}^d$  Lipschitz,  $L$  be an elliptic operator,  $u_0 \in W^{1,2}(\Omega)$ ,  $f \in (W^{1,2}(\Omega))^*$ ,  $g \in L^2(\Gamma_2 \cup \Gamma_3)$ . We say that  $u \in W^{1,2}(\Omega)$  is a weak solution iff

- $\text{tr } u = \text{tr } u_0$  on  $\Gamma_1$  and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \forall \varphi \in V$ , where  $V := \{\varphi \in W^{1,2}(\Omega) | \text{tr } \varphi = 0 \text{ on } \Gamma_1\}$ .

## 4.1 Existence of solution for coercive operators

### Definice 4.3 (Elliptic form)

Let  $B : V \times V \rightarrow \mathbb{R}$  bilinear nad  $V$  be a Hilbert space,  $c_1, c_2 > 0$ . We say that  $B$  is elliptic if it is

- $V$ -bounded  $\Leftrightarrow |B(u, \varphi)| \leq c_2 \|u\|_V \|\varphi\|_V$  and
- $V$ -coercive  $\Leftrightarrow B(u, u) \geq c_1 \|u\|_V^2$ .

### Věta 4.3 (Lax-Milgram)

Let  $B$  be a bilinear elliptic form. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : B(u, \varphi) = \langle F, \varphi \rangle.$$

### Definice 4.4

Let  $B : V \rightarrow V^*$ . We say that  $B$  is

- Lipschitz  $\equiv \forall u, v \in V : \|B(u) - B(v)\|_{V^*} \leq c_2 \|u - v\|_V, c_2 > 0$ ;
- Uniformly monotone  $\equiv \forall u, v \in V : \langle B(u) - B(v), u - v \rangle_V \geq c_1 \|u - v\|_V^2, c_1 > 0$ .

### Věta 4.4 (Non-linear Lax-Milgram)

Let  $B$  be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle.$$

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Důkaz  
└ TODO!!!

□

Důkaz (Lax-Milgram)  
TODO!!!

□

### Věta 4.5

If  $B_{L,\sigma}$  is bilinear,  $V$ -bounded and  $V$ -elliptic. Then there exists a unique weak solution  $u$ .

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Důkaz  
└ TODO!!!

□

## 4.2 Existence via Fredholm alternative

TODO!!!

## Věta 4.6

Let  $\Omega \in C^{0,1}$ ,  $L$  be an elliptic operator and  $\Gamma_1 = \partial\Omega$ . Then

1.  $\Sigma$  is at most countable and if infinite  $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \rightarrow \infty$ ;
2.  $(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1 \exists! u : Lu = f + \lambda u$ ;
3.  $\forall \lambda \notin \Sigma \exists C > 0 \forall f \in L^2 \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$  and  $\|u\|_{1,2} \leq C\|f\|_2$ ;

┌

*Důkaz*

3) TODO improve convergence of  $u^{n_k}$  and show

$$u^{n_k} \rightarrow u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show  $\{u^{n_k}\}$  is Cauchy in  $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1 \|\nabla(u^n - u^m)\|_2^2 \leq \int_{\Omega} A \nabla v^{n,m} \nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_{\Omega} \mathbf{c} \nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d} \nabla v^{n,m} v^{n,m} =$$

$$= \int_{\Omega} (f^n - f^m) v^{n,m} + \lambda (v^{n,m})^2 \pm \dots \leq$$

$$\leq \|v^{n,m}\|_2 (\|f^n - f^m\|_2 + \lambda \|v^{n,m}\|_2 + \|\mathbf{c}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|b\|_{\infty} \|v^{n,m}\|_2) \leq$$

$$\leq \|v^{n,m}\| C(\lambda) \stackrel{u^n \text{ is Cauchy}}{\leq} C(\lambda) \varepsilon$$

$$\implies \nabla u^n \text{ is Cauchy sequence} \implies u^n \rightarrow u \text{ in } W_0^{1,2}(\Omega) \implies \|?\|_{n_k} = 1$$

$$\int_{\Omega} A \nabla a u^n \nabla a \varphi + b u^n \varphi + \mathbf{c} \nabla u^n \varphi - \mathbf{d} \nabla u^n \varphi = \int_{\Omega} f^n \varphi + \lambda u^n \varphi.$$

$$n \rightarrow \infty$$

$$\int_{\Omega} A \nabla u \nabla \varphi + b u \varphi + \mathbf{c} \nabla u \varphi - \mathbf{d} \nabla u \varphi = \lambda \int_{\Omega} u \varphi \Leftrightarrow Lu = \lambda u$$

└ But  $\lambda \notin \Sigma$ . □

*Poznámka*

Next we discussed homework.

## 4.3 Variational approach – minimization

*Poznámka*

$B_{L,\sigma}(u, v)$  must be symmetric! ( $B_{L,\sigma}(u, v) = B_{L,\sigma}(v, u)$ )

$$L = -\operatorname{div}(A\nabla u) + bu + \mathbf{c}\nabla u + \operatorname{div}(\mathbf{d}u)$$

$$B_{L,\sigma}(u, v) := \int_{\Omega} A\nabla u \cdot \nabla v + Buv + \mathbf{c} \cdot \nabla uv - \mathbf{d}\nabla vu + \int_{\Gamma} \sigma uv$$

$$B_{L,\sigma}(v, u) := \int_{\Omega} A\nabla v \cdot \nabla u + Bvu + \mathbf{c} \cdot \nabla vu - \mathbf{d}\nabla uv + \int_{\Gamma} \sigma vu$$

$$\implies A = A^T, \quad \mathbf{c} = -\mathbf{d}$$

### Věta 4.7

Let  $B_{L,\sigma}$  be linear symmetric  $V$ -elliptic and  $V$ -bounded.  $f \in V^*$ ,  $g \in L^2(\Gamma_2 \cup \Gamma_3)$ ,  $u \in V$ . Then the following is equivalent:

- $u - u_0 \in V$  and  $B_{L,\sigma}(u, v) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$ ;
- $u - u_0 \in V \quad \forall v \in W^{1,2}(\Omega), \quad v, u_0 \in V$

$$\frac{1}{2}B_{L,\sigma}(u, u) - \langle f, u \rangle - \int_{\Gamma_2 \cup \Gamma_3} gu \leq \frac{1}{2}B_{L,\sigma}(v, v) - \langle f, v \rangle - \int_{\Gamma_2 \cup \Gamma_3} gv.$$

┌  
Důkaz („1  $\implies$  2“)

$$\begin{aligned} 0 &\stackrel{V\text{-elliptic}}{\leq} \frac{1}{2}B_{L,\sigma}(v-u, v-u) \stackrel{\text{linearity}}{=} \frac{1}{2}B_{L,\sigma}(v, v) + \frac{1}{2}B_{L,\sigma}(u, u) - \frac{1}{2}B_{L,\sigma}(u, v) - \frac{1}{2}B_{L,\sigma}(v, u) = \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u) - B_{L,\sigma}(u, v) = \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u - v) \stackrel{\text{weak formulation}}{=} \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + \langle f, u - v \rangle + \int_{\Gamma_2 \cup \Gamma_3} g(u - v) \end{aligned}$$

└

□

┌ *Důkaz („2  $\implies$  1“)*

$u$  is minimizer, so set  $v = u + \varepsilon\varphi$ ,  $\varphi \in V$

$$\begin{aligned} \frac{1}{2}B_{L,\sigma}(u, u) - \langle j, u \rangle - \int gu &\leq \frac{1}{2}B_{L,\sigma}(u + \varepsilon\varphi, u + \varepsilon\varphi) - \langle j, u + \varepsilon\varphi \rangle - \int g(u + \varepsilon\varphi) = \\ &= \frac{1}{2}B_{L,\sigma}(u, u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi, \varphi) + \varepsilon B_{L,\sigma}(u, \varphi) - \langle f, u \rangle - \varepsilon \langle f, \varphi \rangle - \int ga - \varepsilon \int g\varphi \end{aligned}$$

divide by  $\varepsilon$  and  $\varepsilon \rightarrow 0_+$

$$0 \leq B_{L,\sigma}(u, \varphi) - \langle j, \varphi \rangle - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \quad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for  $-\varphi \implies 0 = -||- \implies u$  is weak solution.  $\square$

#### Věta 4.8 (Dual formulation)

Let  $Lu = -\operatorname{div}(A\nabla u)$  with  $A$  elliptic, bounded and symmetric,  $\Gamma_1 \neq \emptyset$ ,  $\Gamma = \emptyset$ ,  $f \in V^*$ ,  $g \in L^2(\Gamma_2)$ ,  $u_0 \in W^{1,2}(\Omega)$ . Then the following are equivalent:

- $u$  is a weak solution;
- $\nabla u = A^{-1}\mathbf{T}$ , where  $\mathbf{T}$  minimizes  $\int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} = \nabla u_0 \mathbf{T}$  over the set  $\tilde{V} := \{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d)\}$ ,  $\forall \varphi \in V$ .

$$\int_{\Omega} \mathbf{T} \cdot \nabla \varphi = \langle f, \varphi \rangle + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div} \mathbf{T} = f \text{ in } \Omega, T\mathbf{u} = g \text{ on } \Gamma_2$$

┌ *Důkaz („1  $\implies$  2“)*

Let  $\mathbf{V} \in \tilde{V}$  and  $\mathbf{T} := A\nabla u \in \tilde{V}$ .

$$\begin{aligned} 0 &\leq \frac{1}{2} \int_{\Omega} A^{-1}(\mathbf{V} - \mathbf{T}) \cdot (\mathbf{V} - \mathbf{T}) = \int \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} \int_{\Omega} A^{-1}\mathbf{T} \cdot \mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} = \\ &= \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \left( \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \right) + \int_{\Omega} (\nabla u_0(\mathbf{V} - \mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T} - \mathbf{V})) = \\ &= \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} (A^{-1}\mathbf{T} - \nabla u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} \nabla(u - u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left( \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} + 0. \end{aligned}$$

┌ So  $\mathbf{T}$  is minimizer of the formula above.  $\square$

┌

Důkaz („2  $\implies$  1“)  
 $\mathbf{T} \in \tilde{V} \quad \forall V \in \tilde{V}: \int_{\Omega} \frac{1}{2} A^{-1} \mathbf{T} \cdot \mathbf{T} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V}. \quad \mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}, \quad \mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$   
 $\forall \varphi \in V: \int_{\Omega} \mathbf{W} \cdot \nabla \varphi = 0.$

$$\int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T} + \varepsilon^2 A^{-1} \mathbf{W} \cdot \mathbf{W} + 2\varepsilon A^{-1} \mathbf{T} \cdot \mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by  $\varepsilon$  and  $\varepsilon \rightarrow 0_+$ :

$$0 \leq \int_{\Omega} A^{-1} \mathbf{T} \cdot \mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for  $-\mathbf{W}$ , so  $0 = -||-$ .

Now we find unique  $u \in W^{1,2}$   $u - u_0 \in V: \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} A^{-1} \mathbf{T} \cdot \nabla \varphi$  ( $\langle F, \varphi \rangle_V$ ).

$$\begin{aligned} \int_{\Omega} |A^{-1} \mathbf{T} - \nabla u|^2 &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u)(A^{-1} \mathbf{T} - \nabla u) = \\ &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u_0) \cdot (A^{-1} \mathbf{T} - \nabla u) + \int_{\Omega} \nabla(u_0 - u)(A^{-1} \mathbf{T} - \nabla u) = 0 + 0 = 0 \end{aligned}$$

└

□

### Lemma 4.9

Let  $X$  be a reflexive space and  $\{u^n\}_{n=1}^{\infty}$  be a bounded sequence,  $\|u^n\|_X \leq c < \infty$ . Then  $\exists u^{n_k}, \exists u \in X: u^{n_k} \rightharpoonup u$  ( $\forall F \in X^*: \langle F, u^{n_k} \rangle \rightarrow \langle F, u \rangle$ ).

### Věta 4.10 (Spectrum of symmetric operator)

$V$  Hilbert infinity-dimensional space. Let  $B$  be linear, symmetric,  $V$ -elliptic and  $V$ -bounded operator. Then there exist  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$  and corresponding  $\{u_i\}_{i=1}^{\infty}$  such that

- $B(u^k, \varphi) = \lambda_k \int_{\Omega} u^k \varphi;$
- $\lambda_k \rightarrow \infty;$
- $\{u^k\}_{k=1}^{\infty}$  is basis in  $V$  and fulfils

$$\int_{\Omega} u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_{\Omega} u u^i),$  then  $\forall n: \|P^n u\|_2 \leq \|u\|_2$  and  $B(P^n u, P^n u) \leq B(u, u).$

|

|

Dikaz

Step 1: Construct  $\lambda_k, u^k$ :  $\lambda_1 := \inf_{u \in V, \|u\|_2=1} B(u, u)$  and denote  $u^1$  function, where infimum is obtained. Then for  $V^N = \{u \in V | \forall k \in [N] : B(u, u^k) = 0\}$  we do the same.

Step 2: The construction is OK:

$$\begin{aligned} 0 < \lambda_1 &= \lim_{n \rightarrow \infty} B(u^n, u^n), \|u^n\|_2 = 1 \implies \\ &\implies \|u^n\|_V \leq C \implies u^{n_k} \rightharpoonup u \text{ in } V \\ V \hookrightarrow L^2 &\implies u^{n_k} \rightarrow u \text{ in } L^2(\Omega) \implies \|u\|_2 = 1 \\ \lambda_1 &= \lim_{n_k \rightarrow \infty} B(u^{n_k}, u^{n_k}) \geq B(u, u) \geq \lambda_1. \end{aligned}$$

Step 3:  $\lambda_k, u^k$  eigenvalues, eigen functions:  $\forall v \in V, \|v\|_2 = 1, \lambda_1 = B(u^1, u^1) \leq B(v, v), \|u^1\|_2 = 1$

$$v = \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leq B\left(\frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}\right)$$

$$\lambda_1 \|u^1 + \varepsilon \psi\|_2 \leq B(u^1 + \varepsilon \psi, u^1 + \varepsilon \psi) = B(u_1, u_1) + \varepsilon^2 B(\psi, \psi) + 2\varepsilon B(u, \psi) \leq \lambda_1 \|u^1\|_2^2 + \lambda_1 \varepsilon^2 \|\psi\|_2^2 + 2\varepsilon \lambda_1 \int_{\Omega} u^1 \psi$$

$$\varepsilon \rightarrow 0_+ \implies 2\lambda_1 \int_{\Omega} u^1 \psi \leq 2B(u, \psi).$$

So  $\lambda_1 \int_{\Omega} u^1 \psi = B(u, \psi)$ .

The same way we obtain  $\lambda_k \int_{\Omega} u^k \psi \leq B(u, \psi)$  for  $\psi \in V^N$ .

$$u^1 : \lambda_1 \int_{\Omega} u^1 \psi = B(u^1, \psi) \implies \psi = u^k \int_{\Omega} u^1 u^k = V(u_1, u^k).$$

But  $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$ , so  $\int u^1 u^k = B(u^1, u^k) = 0$ .

$$\implies \forall i \in [k-1] : \int_{\Omega} u^k u^1 = B(u^k, u^i) = 0.$$

Step 4:  $\lambda_k \nearrow \infty$ . We already know  $\lambda_1 \leq \lambda_2 \leq \dots$ . Assume a contradiction  $\lambda_k \leq C < \infty$ .  $c_1 \|u^k\|_V^2 \leq B(u^k, u^k) = \lambda_k \|u^k\|_2^2 = \lambda_k < C$ .

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \rightarrow u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$\|u^n - u^m\|_2^2 = \|u^n\|_2^2 + \|u^m\|_2^2 - 2 \int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_1 n} B(u^n, u^m) = 2 \implies \text{not Cauchy.}$$

Step 5:  $\lambda_k$  are all eigenvalues ( $u^k$  is basis of  $V$  and of  $L^2$ ). Assume that  $\lambda \neq \lambda_j$  is also eigenvalue, so  $\exists u : B(u, \varphi) = \lambda \int_{\Omega} u \varphi \forall \varphi$ . We can find  $i \in \mathbb{N}$ , so  $\lambda_i < \lambda < \lambda_{i+1}$ .

$$B(u, u^j) = \lambda \int_{\Omega} u u^j \wedge B(u^j, u) = \lambda_j \int_{\Omega} u^j u \implies B(u, u_j) = 0$$



## 4.4 Regularity of weak solution

*Poznámka*

We assume that we have  $u \in W^{1,2}(\Omega)$  a weak solution

$$-\operatorname{div} A \nabla u + Vu + \mathbf{c} \cdot \nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When  $u \in W_{loc}^{2,2}(\Omega)$ , when  $u \in W^{2,2}(\Omega)$ , when  $u \in W_{loc}^{k,2}(\Omega)$ ,  $u \in W^{k,2}(\Omega)$ .

Simplify  $-\operatorname{div} A \nabla u = f - bu - \mathbf{c} \nabla u - u \operatorname{div} \mathbf{d} - \nabla u \cdot \mathbf{d} = \tilde{f}$ . If  $u \in W^{1,2}$ ,  $f \in L^2$ ,  $b \in L^\infty$ ,  $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$ .

Problem is reduced to

$$\begin{aligned} -\operatorname{div}(A \nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_1, \\ (A \nabla u) \cdot \mathbf{v} &= g \text{ on } \Gamma_2, \\ (A \nabla u) \cdot \mathbf{v} + \sigma u &= g \text{ on } \Gamma_3. \end{aligned}$$

### Definition 4.5 (Interior regularity)

$u \in W_{loc}^{2,2}(\Omega)$ ; assumptions:  $A \in W^{k+1,\infty}$ ,  $f \in W^{k,2}(\Omega) \implies u \in W_{loc}^{k+1,2}(\Omega)$ .

### Definition 4.6 (Boundary regularity)

$u \in W^{2,2}(\Omega)$ ; assumptions: on  $\Omega \in C^{k+1,\infty}$ ,  $g \in W^{\frac{1}{2},2}(\partial\Omega)$  and  $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\emptyset\} \implies u \in W^{2,2}(\Omega)$ .

### Věta 4.11 (Interior regularity)

Let  $A$  be an elliptic operator and  $u \in W^{1,2}$  solves

$$\int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega) \quad \forall f \in L^2(\Omega).$$

Then if  $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$ ,  $f \in W^{k,2}(\Omega)$  then  $u \in W_{loc}^{k+2,2}(\Omega)$ .

Moreover  $\forall \tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega \exists c(\tilde{\Omega}, A)$ :

$$\|u\|_{W^{k+2,2}(\tilde{\Omega})} \leq c(\|f\|_2 + \|u\|_{W^{1,2}(\Omega)}).$$

Důkaz

$k = 0$ : Recall  $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x + he_k) - v(x)|^2}{h^2} \leq c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{ u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2 \right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left| \frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j} \right|^2}{h^2} \leq c,$$

$$\int_{\Omega_h} \left| \frac{\nabla u(x + he_i) - \nabla u(x)}{h} \right|^2 \leq c.$$

$$\int_{\Omega} A \nabla u \nabla \varphi = \int_{\Omega} f \varphi$$

$$h > 0, \varphi \in W_0^{1,2}(\Omega), \varphi(x) = 0 \text{ if } \text{dist}(x, \partial\Omega) \subset h.$$

Set  $\varphi(x) := \psi(x - he_k)$ .

$$\begin{aligned} \implies \int_{\Omega} A(x) \nabla u(x) \nabla \psi(x - he_k) &= \int_{\Omega} f(x) \psi(x - he_k) = \\ &= \int_{\Omega} A(x + he_k) \nabla u(x + he_k) \cdot \nabla \psi(x) dx. \end{aligned}$$

Set  $\varphi(x) := \psi(x)$ :

$$\int_{\Omega} A(x) \cdot \nabla u(x) \cdot \nabla \psi(x) = \int_{\Omega} f(x) \psi(x) dx.$$

$$\begin{aligned} &\int_{\Omega} A(x + he_k) (\nabla u(x + he_k) - \nabla u(x)) \cdot \nabla \psi(x) = \\ &= - \int_{\Omega} (A(x + he_k) - A(x)) \nabla u(x) \cdot \nabla \psi(x) + \int_{\Omega} f(x) (\psi(x - he_k) - \psi(x)). \end{aligned}$$

Set  $\psi := (u(x + he_k) - u(x)) \tau^2(x)$ ,  $\tau(x) = 0$ , if  $\text{dist} \in (x, \partial\Omega)$ ,  $\tau \in C^1(\tilde{\Omega})$ .

Evaluate all terms ( $w^{h,i} = u(x + he^i) - u(x)$ ):

$$\begin{aligned} &\int_{\Omega} A(x + he_i) \nabla w^{h,i} \cdot (\nabla w^{h,i} \tau^2 + 2w^{h,i} \tau \nabla \tau) \geq \\ &\stackrel{ellip.}{\geq} c_1 \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \int_{\Omega} \frac{2 \|A\|_{\infty} |w^{h,i}| - |\nabla \tau| (|\nabla w^{h,i}| \sqrt{c_1} \tau)}{\sqrt{c_1}} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2}{c_1} \|A\|_{\infty}^2 \frac{\|\nabla \tau\|_{\infty}^2}{18} h^2 \int_{\Omega_h} \frac{|u(x + he_i) - u(x)|^2}{h^2} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2 \|A\|_{\infty}^2 \|\nabla \tau\|_{\infty}^2}{c_1} h^2 c \|\nabla u\|_2^2 \end{aligned}$$

TODO?

### Věta 4.12 (Regularity up to the boundary)

Let  $u$  be a weak solution  $-\operatorname{div}(A\nabla u) = f$  in  $\Omega$ ,  $A\nabla u \cdot \mathbf{v} = g$  on  $\Gamma_2$ ,  $A\nabla u \cdot \mathbf{v} + \sigma u = g$  on  $\Gamma_3$ ,  $u = u_0$  on  $\Gamma_1$ .

Assume that  $\Omega \in C^{k+1,\infty}$ ,  $A \in W^{k,\infty}$ ,  $f \in W^{k-1,2}$ ,  $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega)$ ,  $\sigma \in W^{k,\infty}(\partial\Omega)$  and  $\Gamma_1, \Gamma_2, \Gamma_3$  are smooth open (in partial  $\Omega$ ) and  $\overline{\Gamma_i} \cap \overline{\Gamma_j} = \emptyset \ \forall i \neq j$ .

Then  $u \in W^{k+1,2}(\Omega)$ .

┌

Důkaz (Step 1: Flat boundary)

$\Omega = (-1, 1)^{d-1} \times (0, 1)$ . Assume that  $u \in W^{1,2}(\Omega)$  and  $u = 0$  on  $(x, 0)$ . We want that  $u \in W^{2,3}((-1 + \delta, 1 - \delta)^{d-1} \times (0, 1 - \delta))$ .

1a tangential derivatives  $\frac{\partial u}{\partial x_1} \in W^{1,2}(-||-)$ . 1b normal derivative  $\frac{\partial^2 u}{\partial x_d^2} \in L^2(-||-)$ .

1a:  $\operatorname{WF} - \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \ \forall \varphi \in W_0^{1,2}(\Omega)$ . Take continuous  $\tau = 1$  in  $-||-$  and  $\tau = 0$  in  $\Omega \setminus \text{"inflated"} -||-$ .

$$\varphi(x) = \psi(x - he_i)\tau, \quad i \in [d-1], \psi \in W_0^{1,2}(\Omega \setminus \text{"inflated"} -||-)$$

Redefine interior regularity

$$\int_{\Omega} (A(x + he_i) \nabla u(x + he_i) - A(x) \nabla u(x)) \nabla \varphi(x) = \int_{\Omega} f(\psi(x - he_i) - \psi(x)).$$

Set  $\psi = (u(x + he_i) - u(x))\tau^2 \in W_0^{1,2}$  and apply local regularity.

1b:  $\varphi \in C_0^{\infty}(-||-)$

$$\begin{aligned} - \int_{\Omega} \sum_{i,j}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) \varphi &= - \int_{\Omega} \operatorname{div}(A \nabla u) \varphi = \int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \\ - \int_{\Omega} a_{dd} \frac{\partial^2 u}{\partial x_d^2} \varphi &= \underbrace{\int_{\Omega} f \varphi}_{\in L^2(\Omega)} + \int_{\Omega} \varphi \left( \sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i,j=1, \neg(i=j=d)}^d \right) a_{ij} \frac{\partial u}{\partial x_i x_j}. \\ \|a_{dd} \frac{\partial^2 u}{\partial x_d^2}\|_2^2 &\leq \|f + \sum \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{\neg(i=j=d)} a_{ij} \frac{\partial^2 u}{\partial x_i x_j}\|_2^2 \leq C. \end{aligned}$$

$A$  is elliptic

$$c_1 |\zeta|^2 \leq a_{ij} \zeta_i \zeta_j$$

Special choice  $\zeta = (0, \dots, 0, 1)$ ,  $0 < c_1 \leq a_{dd}(x) \implies \|\frac{\partial^2 u}{\partial x_d^2}\|_{L^2}^2 \leq C(\text{DATA?})$  □

└

┌ *Důkaz* (Step 2: Transfer from flat to small parts of  $\partial\Omega$ )  
 ┌ TODO!!!

┌ TODO!!!

┌ *Důkaz* (Step 3: Introduce a proper covering of  $\partial\Omega$  and use step 2)

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

?  $\Omega$   $u \in W_{loc}^{2,2}(\Omega)$ . ? of  $\partial\Omega$ , apply step 2.

Define  $w := u - u_0 \in W_0^{1,2}(\Omega)$ .

$$-\operatorname{div}(A\nabla w) = f + \operatorname{div}(A\nabla u_0)$$

┌ if  $f \in L^2$  and  $\operatorname{div}(A\nabla u_0) \in L^2$ , e.g.  $A \in W^{1,\infty} \wedge u_0 \in W^{2,2}(\Omega)$ .  
 ┌

## 5 Bochner integral

### Definition 5.1 (Measurability)

We say that  $f : I \rightarrow X$  is measurable (strongly, Bochner) if  $\exists \{s_j\}_{j=1}^\infty$  simple functions,  $\|f(t) - s_n(t)\|_X \rightarrow 0$  as  $n \rightarrow \infty$  for almost every  $t \in I$ .

### Věta 5.1 (Measurability)

$f : I \rightarrow X$  is measurable iff

1.  $f$  is almost separably valued;

$$\exists E \subset I : |E| = 0, f(I \setminus E) \text{ is separable.}$$

2.  $f$  is weakly measurable;

$$\forall F \in X^* : \langle F^*, u(t) \rangle_X \text{ is Lebesgue measure w.r.t } t \in I.$$

### Definition 5.2 (Bochner integral for simple function)

Let  $s : I \rightarrow X$  be a simple function on  $I$ . We define

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

**Definice 5.3** (Bochner integral for measurable functions)

Let  $s : I \rightarrow X$  be a Bochner measurable function. We say that  $f$  is Bochner integrable if  $\exists \{s^n\}_{n=1}^\infty$  such that  $s^n(t) \rightarrow f(t)$  a. a. t and  $\int_I \|s^n(t) - f(t)\|_X dt \rightarrow 0$  as  $n \rightarrow \infty$  and we set

$$X \ni \int_I f(t) dt = \lim_{n \rightarrow \infty} \int_I s^n(t) dt.$$

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

**Definice 5.4** ( $L^p(O, T, X)$  space)

Let  $X$  be a Banach space

$$L^p(O, T, X) = \left\{ f : (O, T) \rightarrow X \text{ bochner integrable} \mid \int_I \|f(t)\|_X^p < \infty \right\}$$

$$\|f\|_{L^p(O, T, X)} = \left( \int_I \|f(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

**Věta 5.2** (Dual space)

Let  $X$  be a Banach space, separable and  $p \in [1, \infty)$ , then

$$(L^p(O, T, X))^* = L^{p'}(O, T, X^*)$$

## 5.1 Sobolev-Bochner spaces

**Definice 5.5**

Let  $f : I \rightarrow X$  be Bochner integrable. We say that  $g : I \rightarrow X$  is a weak derivative of  $f$  w. r. t. iff  $g$  is Bochner integrable and  $\forall \tau \in C_0^\infty(I) : \int_I f(t) \tau'(t) dt = - \int_I g(t) \tau(t) dt$ .

*Poznámka*

If  $f \in L^1(I, X)$  and  $\frac{\partial f}{\partial t} \in L^1(I, X)$ , then  $f \in C(I, X)$ .

**Věta 5.3**

$$W^{1,p}(I, X) := \{f \in L^p(I, X), \partial_t f \in L^p(I, X)\}, \quad \|f\|_{W^{1,p}(I, X)} = \begin{cases} \left( \int_I \|f\|_X^p + \|\partial_t f\|_X^p \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \text{esssup}_{t \in I} (\|f(t)\|_X + \|\partial_t f\|_X), & p = \infty \end{cases}$$

Then  $W^{1,p}(I, X)$  is a Banach space, is separable for  $p < \infty$  and  $X$  separable and

is reflexive if  $p \in (1, \infty)$  and  $X$  is reflexive and separable.

## 5.2 Time derivative in heat/wave equations – Gelfand triple

*Poznámka* (Motivation)

$$\partial_t u - \Delta u = 0 \text{ in } (0, T) \times \Omega, u = 0 \text{ on } (0, T) \times \partial\Omega, x(0, x) = u_0(x) \text{ for } x \in \Omega, \quad \Omega \subseteq \mathbb{R}^d$$

### Definition 5.6 (Gelfand triple)

We say that  $X, H, X^*$  is Gelfand triple iff  $X \xrightarrow{\text{dense}} H \cong H^* \xrightarrow{\text{dense}} X^*$ .

┌ *Například*

$$X = W_0^{1,2}(\Omega), H = L^2(\Omega), X^* = (W_0^{1,2}(\Omega))^*,$$

Neboť  $W_0^{1,2}$  is dense in  $C_0 \xrightarrow{\text{dense}} L^2(\Omega)$  and  $f \in (W_0^{1,2}(\Omega))^* \implies \exists! u \in W_0^{1,2}(\Omega) : -\Delta u = f$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ .

$$\forall \varphi \in W_0^{1,2}(\Omega) \langle f, \varphi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u^n \cdot \nabla \varphi = \lim_{n \rightarrow \infty} - \int_{\Omega} \Delta u^n \varphi = \lim_{n \rightarrow \infty} (f^n, \varphi)_{L^2(\Omega)},$$

where  $\{u^n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$ ,  $u^n \rightarrow u$  in  $W_0^{1,2}(\Omega)$ ,

$$(X = W_0^{1,p}(\Omega \cap L^2(\Omega)), H = L^2(\Omega))$$

└

### Definition 5.7

Let  $X, H, X^*$  be Gelfand triple,  $\varphi : H \rightarrow H^*$  is Riesz representation and define  $i : X \rightarrow X^*$ , such that  $\forall x_0, x \in X$ :

$$\langle i(x_0), x \rangle_X := (\text{id}(x_0), \text{id}(x))_H = \langle \varphi \text{id}(x_0), \text{id}(x) \rangle_H,$$

$i$  maps  $X$  densely onto  $X^*$ .

### Lemma 5.4

Let  $u \in L^1(0, T, H)$ ,  $\partial_t u \in L^1(0, T, X^*)$  and  $X, H, X^*$  be a Gelfand triple. Then  $\forall w \in X \forall \tau \in C_0^1(0, T)$  we have

$$\begin{aligned} \int_0^T \langle \partial_t u, w \rangle \tau dt &= \langle \int_0^T \partial_t u \tau dt, w \rangle_X = \\ &= - \langle \int_0^T u \tau' dt, w \rangle_X = - \int_0^T \langle u \tau', w \rangle_X dt = \end{aligned}$$

$$= - \int_0^T (u\tau', w)_H dt \stackrel{\text{if } \partial_t u \in L^1(0,T)}{=} \int_0^T (\partial_t u \tau, w)_H.$$

**Věta 5.5** (Integration by parts for Sobolev-Bochner function)

Let  $p \in (1, \infty)$ ,  $X, H, X^*$  a Gelfond triple,  $u, v \in L^p(0, T, X)$ ,  $\partial_t u, \partial_t v \in L^{p'}(0, T, X^*)$ . Then  $u, v \in C([0, T], H)$  and  $\forall 0 \leq t_1 < t_2 \leq T$ .

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

┌  
Důkaz

Step 1) Modify  $u, v$  in terms of the Steklov averages  $u_h = \int_t^{t+h} u(\tau) d\tau$ .

Step 2) Prove for  $u_h, v_h$  from step 1).

Step 3)  $h \rightarrow 0_+$ .

□

└

┌ *Důkaz* („Step 1“)

Define  $u_h(t) := \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$ ,  $\forall t \in (0, T-h)$ .  $u_h \rightarrow h$   $L^p(0, T-h_0, X)$ ,  $\forall h_0 \in (0, T)$ . We want „ $(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$ “.

$$(\partial_t u)_h \rightarrow \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \quad \forall h_0 \in (0, T).$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^{T-h} u_h(t) \varphi'(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \int_t^{t+h} u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \left( \int_0^{t+h} u(\tau) d\tau - \int_0^t u(\tau) d\tau \right) = \\ &= -\frac{1}{h} \int_0^{T-h} \varphi(t) (u(t+h) - u(t)) dt \Leftrightarrow \partial_t u_h = \frac{u(t+h) - u(t)}{h}. \end{aligned}$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^T \varphi(t) (\partial_t u)_h(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi(t) \int_t^{t+h} \partial_t u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi(t) \left( \int_0^{t+h} \partial_t u(\tau) d\tau - \int_0^t \partial_t u(\tau) d\tau \right) dt = (*) \end{aligned}$$

$$\begin{aligned} \frac{1}{h} \int_0^{T-h} \varphi(t) \left( \int_0^t \partial_t u(\tau) d\tau \right) dt &= \int_0^{T-h} \int_0^{T-h} \varphi(t) \partial_t u(\tau) \chi_{\tau \leq t} d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \left( \int_t^{T-h} \varphi(t) dt \right) d\tau. \end{aligned}$$

$$(*) = \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \underbrace{\left( \int_{\tau-h}^\tau \varphi(t) dt \right)}_{C_0^\infty(0, T)} d\tau = -\frac{1}{h} \int_0^{T-h} u(\tau) (\varphi(\tau) - \varphi(\tau-h)) d\tau dt.$$

└

□



┌ *Dikaz* („Step 2“)

We want

$$\begin{aligned}
& \int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \\
& \Leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \\
& \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau \right)_H dt = \\
& = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int_{t_1}^t v(\tau) d\tau \right)_H = \\
& = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1-h_2}^t v(\tau + h_2) d\tau - \int_{t_1}^t v(\tau) d\tau \right)_H = \\
& = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1}^t v(\tau + h_2) - v(\tau) d\tau \right)_H dt + \\
& + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt = \\
& = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( v(\tau + h_2) - v(\tau) d\tau, \int_{t_1}^\tau u(t + h_1) - u(t) dt \right)_H d\tau + \\
& + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt = \\
& = \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( v(\tau + h_2) - v(\tau) d\tau, \int_{t_2}^{t_2+h_1} u(t) - \int_{t_2}^{t_2+h_1} u(t) dt \right)_H d\tau + \\
& + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( u(t + h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau + h_2) d\tau \right)_H dt = \\
& = \int_{t_1}^{t_2} \left( \frac{v(\tau + h) - v(\tau)}{h_2}, \int_\tau^{\tau+h_1} u(t) dt \right)_H d\tau + \\
& + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} \left( v(\tau + h_2) - v(\tau), \int_{t_2}^{t_2+h} u(t) dt \right)_H + \int_{t_1}^{t_2} (u(t + h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau) d\tau)_H = \\
& - \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau)) d\tau + REST \\
& REST = \frac{1}{h_1 h_2} \left( \int_{t_2}^{t_2+h_2} v(t) dt - \int_{t_1}^{t_1+h_2} v(t) dt, \int_{t_2}^{t_2+h} u(t) dt \right) + SIMILAR = \\
& = (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \dots
\end{aligned}$$

└

□

┌ *Důkaz („Step 3“)*

We have

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let  $h_1 \rightarrow 0_+$  and  $h_2 \rightarrow 0_+$ . We have  $\partial_t u_{h_1} \rightarrow \partial_t u$  in  $L^{p'}(0, T, X^*)$ ,  $\partial_t v_{h_2} \rightarrow \partial_t v$  in  $L^{p'}(0, T, X^*)$ ,  $u_{h_1} \rightarrow u$  in  $L^p(0, T, X)$ ,  $v_{h_2} \rightarrow v$  in  $L^p(0, T, X)$ . So for almost all  $t$  in  $(0, T)$ :  $v_{h_2}(t) \rightarrow v(t)$  in  $X \hookrightarrow H$  and  $u_{h_1}(t) \rightarrow u(t)$  in  $X \hookrightarrow H$ .

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Now, it is enough to show  $u, v \in C([0, T]; H)$ . We show that  $u_h$  is Cauchy in  $C([0, T]; H)$ .

Use IBP  $u_{h_1} = u_{h^n} - u_{h^m}$ ,  $v_{h_2} = u_{h^n} - u_{h^m}$ :

$$\|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H = \|u_{h^m}(t_1) - u_{h^n}(t_1) + 2 \int_{t_1}^{t_2} \langle \partial_t (u_h^m - u_h^n), u_{h^n} - u_{h^m} \rangle_X \|$$

$$\|u_{h^n} - u_{h^m}\|_{C([\frac{T}{4}, T]; L^2(\Omega))}^2 = \sup_{t_2 \in (\frac{T}{2}, T)} \|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H^2 \leq$$

$$\leq \|u_{h^m}(t_1) - u_{h^n}(t_1)\|_H^2 + \int_0^T \|\partial_t (u_{h^n}) - \partial_t u_{h^m}\|_{X^*} \|u_{h^m} - u_{h^n}\|_X dt.$$

Choose  $t_1$  such that  $u_h(t_1) \rightarrow u(t_1)$  in  $H$ :

$$\leq \|u_h(t_1) - u_{h^m}(t_1)\|_H^2 + \|\partial_t u_{h^m} - \partial_t u_{h^n}\|_{L^p(X^*)} \cdot \dots$$

$$u \in C\left(\left[\frac{T}{4}, T\right]; L^2(\Omega)\right) \wedge u \in C\left(\left[0, \frac{3T}{4}\right]; L^2(\Omega)\right) \rightarrow u \in C([0, T]; L^2(\Omega)) \quad (u(t_1), v(t_1))_H.$$

└

□

## 6 Parabolic equations

*Poznámka*

$\Omega$  open set in  $\mathbb{R}^d$ ,  $T > 0$ ,  $L$  elliptic operator,

$$\partial_t u + Lu = f \text{ in } Q = (0, T) \times \Omega, \quad u = 0 \text{ on } (0, T) \times \partial\Omega, \quad u(0, x) = u_0(x) \text{ } x \in \Omega.$$

$$Lu = -\operatorname{div}_x(A(t, x)\nabla_x u(t, x)) + b(t, x)u(t, x) + \mathbf{c}(t, x)\nabla u(t, x) + \operatorname{div}(\mathbf{d}(t, x)u(t, x)),$$

$$A, b, \mathbf{c}, \mathbf{d} \in L^\infty(\Omega).$$

$$A(t, x) \cdot \xi \cdot \xi \geq c_1 |\xi|^2, \forall \xi \in \mathbb{R}^d \text{ and almost all } (t, x) \in Q.$$

## 6.1 Formal a priory estimates

*Poznámka*

Multiply by  $u$  and  $\int_{\Omega} dx$  and use IBP.

$$\int_{\Omega} \partial_t u u + \int_{\Omega} A \nabla u \nabla u = \int_{\Omega} f u - b u^2 - \mathbf{c} \nabla u u + \mathbf{d} u \nabla u.$$

Hölder's inequality:

$$\frac{d}{dt} \frac{\|u\|_2^2}{2} + C_1 \|\nabla u\|_2^2 \leq \|f\|_2 \|u\|_2 + \|b\|_{\infty} \|u\|_2^2 + \|\mathbf{c}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 \leq C_1 \frac{\|\nabla u\|_2^2}{2} + C(\mathbf{c})$$

Poincaré's inequality:

$$\frac{d}{dt} \|u\|_2^2 + \mathbf{c} \|u\|_{1,2}^2 \leq C(\mathbf{c}, \mathbf{d}, b) \|f\|_2^2 + K \|u\|_2^2.$$

Grönwall's inequality:

$$\sup_{t \in (0, T)} \|u(t)\|_2^2 \leq + \int_0^T \|f\|_2^2$$

$$\int_0^T \|u\|_{1,2}^2 dt \leq C$$

$$\|\partial_t u\|_{(W_0^{1,2})^*} = \sup_{\|\varphi\| \leq 1} \langle \partial_t u, \varphi \rangle = \sup \langle f - Lu, \varphi \rangle =$$

$$= \sup_{\|\varphi\| \leq 1} \int_{\Omega} (f - ? - bu - \mathbf{c} \nabla u) \varphi - \int_{\Omega} (A \nabla u - \mathbf{d} u) \nabla \varphi \leq \int_0^T \|f\|_2^2 + c(\|u\|_2^2 + \|\nabla u\|_2^2).$$

### Definice 6.1

Let  $\Omega \subseteq \mathbb{R}^d$  open and bounded,  $L$  be an elliptic operator,  $u_0 \in L^2(\Omega)$ ,  $f \in L^2(0, T, V^*)$  ( $V = W_0^{1,2}(\Omega)$ ). We say that  $u$  is a weak solution to

$$\partial_t u + Lu = f \text{ in } (0, T) \times \Omega,$$

$$u = 0 \text{ on } (0, T) \times \partial\Omega,$$

$$u(0) = u_0 \text{ in } \Omega$$

iff  $u \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$ ,  $u(0) = u_0$  and for almost all  $t \in (0, T)$  and  $\forall \varphi \in V$ :

$$\langle \partial_t u, \varphi \rangle_V + \int_{\Omega} A \nabla u \cdot \nabla \varphi + bu\varphi + \mathbf{c} \cdot \nabla u \varphi - \mathbf{d} \nabla \varphi u = \langle f, \varphi \rangle_V.$$

## 6.2 Existence and uniqueness

## Věta 6.1

Let  $\Omega \subseteq \mathbb{R}^d$  be open and bounded,  $f \in L^2(0, T, V^*)$ ,  $u_0 \in L^2(\Omega)$  and  $L$  be elliptic operator. Then  $\exists! u$  – weak solution.

┌

*Důkaz (Uniqueness)*

$u_1, u_2$  are weak solutions. Define  $w := u_1 - u_2 \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$ . WF for  $u_1$  – WF for  $u_2$ :

$$\langle \partial_t w, \varphi \rangle + \int_{\Omega} A \nabla w \cdot \nabla \varphi = \int_{\Omega} -b w \varphi - \mathbf{c} \nabla w \varphi + \mathbf{d} \cdot \nabla \varphi w.$$

Follow almost everywhere, replace  $u$  by  $w$ . Set  $\varphi = w \implies$

$$\langle \partial_t w, w \rangle + \mathbf{c} \|w\|_{1,2}^2 \leq c \|w\|_2^2.$$

Integrate in respect of time, use IBP-formula for  $\langle \cdot, \cdot \rangle$ :

$$\begin{aligned} \int_0^t \langle \partial_t w, w \rangle &= \frac{1}{2} \|w(t)\|_2^2 - \frac{1}{2} \|w(0)\|_2^2 = \frac{1}{2} \|w(t)\|_2^2. \\ \implies \|w(t)\|_2^2 &\leq c \int_0^t \|w(\tau)\|_2^2 d\tau \\ \frac{d}{dt} \underbrace{\int_0^t \|w(\tau)\|_2^2 d\tau}_{=: g(t) \geq 0} &\leq c \int_0^t \|w(\tau)\|_2^2 d\tau \\ g' &\leq c \cdot g \end{aligned}$$

From Grönwall's inequality

$$g(t) \leq e^{c \cdot t} g(0) \implies \int_0^t \|w(\tau)\|_2^2 d\tau \leq e^{ct} \int_0^0 \|w(\tau)\|_2^2 d\tau = 0 \implies w(t) = 0.$$

└

□

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*Důkaz (Existence (via Galerkin approximation))*

We know  $\exists \{w_j\}_{j=1}^{\infty}$  basis of  $\mathbf{V}$ , which is orthonormal in  $L^2$  and  $\|P^N u\|_V \leq c \|u\|_V$ , where  $P^N$  is orthogonal projection in  $L^2(\Omega)$  onto  $\{w_j\}_{j=1}^N$ .

We solve for  $u^n(t, x) = \sum_{i=1}^n a_i^n(t) w_i(x)$ . We want

$$\langle \partial_t u^n, w_j \rangle = - \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n + \langle f, w_j \rangle$$

for  $j \in [n]$  for almost all  $t \in (0, T)$  (weak formulation of the problem for  $n$ , call it WF $_n$ ). □

└

┌ *Dukaz* („Existence of  $u^n$ “)

LHS of WFn:

$$\sum_{i=1}^n \langle \partial_t a_i^n w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n(t) \langle w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n \delta_{ij} = \partial_t a_j^n(t).$$

RHS of WFn:

$$\sum_{i=1}^n a_i^n(t) \left( \underbrace{- \int_{\Omega} A \nabla w_i \nabla w_j + b w_i w_j + \mathbf{c} \nabla w_i w_j - \mathbf{d} \nabla w_j w_i}_{G_{ij}(k) - \text{bounded and measurable?}} \right) + \underbrace{\langle f(t), w_j \rangle}_{g_j^{(t)} - \text{measurable? on } g \in L^2(0,T)}.$$

So

$$\frac{d}{dt} a_j^n(t) = \sum_{i=1}^n a_i^n(t) G_{ij}(t) + g_j(t), \quad j \in [n].$$

Initial data:  $u^n(0) := P^n u_0$  ( $a_j^n(0) := \int_{\Omega} u_0 w_j$ ).

ODE  $\implies \exists \tilde{T} \leq T$  and  $a_i^n(t) \in AC$  on  $[0, \tilde{T})$  and solve for almost all  $t \in (0, \tilde{T})$ .  
Moreover either we can set  $\tilde{T} = T$  or  $|a^n(t)| \xrightarrow{t \rightarrow \tilde{T}} \infty$ .

Now we prove  $\tilde{T} = T$ . We show  $|a^n(t)| \leq c$ .

Multiply WFn for  $j$  by  $a_j^n(t)$  and sum it:

$$LHS = \sum_{j=1}^n a_j^n \langle \partial_t u_j^n, w_j \rangle = \left\langle \partial_t u^n, \sum_{j=1}^n a_j^n w_j \right\rangle = \langle \partial_t u^n, u^n \rangle.$$

$$RHS = \underbrace{\langle \partial_t u^n, u^n \rangle}_{= \frac{d}{dt} \|u^n\|_2^2} + c_1 \|u^n\|_V^2 \leq c(\|f\|_{V^*}^2 + \|u^n\|_2^2).$$

Grönwall:  $\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c(\|u^n(0)\|_2^2) + \int_0^T \|f\|_{V^*}^2$ .

$$\forall t < \tilde{T} : \|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c \left( \int_0^{\tilde{T}} \|f\|_{V^*}^2 + \|u_0\|_2^2 \right) \leq \tilde{c}.$$

$$\lim_{t \rightarrow \tilde{T}_-} |a^n(t)|^2 = \lim_{t \rightarrow \tilde{T}_-} \|u^n(t)\|_2^2 \leq \tilde{c}.$$

└

□

┌ *Důkaz*

$$\|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \leq c(f, u_0).$$

Time derivative

$$\|\partial_t u^n\|_{V^*} = \sup_{w \in V, \|w\| \leq 1} \langle \partial_t u^n, w \rangle = \sup_{w \in V, \|w\| \leq 1} \int_{\Omega} \partial_t u^n w = \sup_{\dots} \int_{\Omega} \partial_t u^n P^n w.$$

WFn:

$$\begin{aligned} &\leq \sup_{\dots} c(\|f\|_{V^*} + \|u^n\|_V) \|P^n w\|_V \leq \sup_{\dots} \tilde{c}(\|f\|_{V^*} + \|u^n\|_V) \|w\|_V \leq \\ &\leq (\tilde{c}(\|f\|_{V^*} + \|u^n\|_V)). \\ &\int_0^T \|\partial_t u^n(t)\|_{V^*}^2 \leq \tilde{c} \int_0^T (\|f(t)\|_{V^*}^2 + \|u^n(t)\|_V^2) \leq c(f, u_0). \end{aligned}$$

$u^n$  is a bounded sequence in  $L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$  so  $\exists$  a subsequence  $u^{m_n}$ :

$$u^{m_n} \rightharpoonup u \text{ in } L^2(0, T, V), \quad \partial_t u^{m_n} \rightharpoonup \partial_t u \text{ in } L^2(0, T, V^*).$$

To show  $u$  is a weak solution.

└ TODO?

□

*TODO!!!*

┌ *Důkaz* (Initial condition)

$\tau \in C_0^\infty(-\infty, T)$ :

$$\begin{aligned} &-\int_0^T \int_{\Omega} u^n w_j \partial_t \tau - \int_{\Omega} u^n(0) w_j \tau(0) + \int_0^T (\dots) \tau \dots = 0. \\ &\rightarrow -\int_0^T \int_{\Omega} u w_j \partial_t \tau - \int_{\Omega} u_0 w_j \tau(0) + \int_0^T (\dots) \tau = 0. \end{aligned}$$

Integration by parts in time:

$$\begin{aligned} &u \in L^2(W_0^{1,2}(\Omega)) \ni \tau w_j, \\ &\partial_t u \in L^2((W_0^{1,2}(\Omega))^*) \ni \partial_t(\tau w_j) = \partial_t \tau w_j \in L^2, \\ &-\int_0^T \int_{\Omega} u w_j \partial_t \tau - \int_{\Omega} w_j \tau(0) = -\int_0^T \langle u, \partial_t(\tau w_j) \rangle - \int_{\Omega} u_0 w_j \tau(0) = \\ &= \int_0^T \langle \partial_t u, \tau w_j \rangle + \int_{\Omega} u(0) \tau(0) w_j - \int_{\Omega} u_0 w_j \tau(0). \end{aligned}$$

└

□

## 6.3 Regularity of parabolic equations

TODO Example?

### Věta 6.2

Let  $\mathbf{b}, \mathbf{c}, \mathbf{d} \in L^\infty$ ,  $\operatorname{div} \mathbf{d} \in L^\infty$ ,  $A, \nabla A, \partial_t A \in L^\infty$ ,  $f \in L^2(0, T, L^2(\Omega))$ , then  $\forall \delta > 0$ :

$$\int_\delta^T \|\partial_t u\|_2^2 + \sup_{t \geq \delta} \|\nabla u(t)\|_2^2 \leq \frac{c}{\delta}$$

.

Moreover if  $u_0 \in W_0^{1,2}(\Omega)$ , then  $\partial_t u \in L^2(0, T, L^2(\Omega))$ ,  $u \in L^\infty(0, T, W_0^{1,2}(\Omega))$ .

Moreover  $u \in L^2(0, T, W_{loc}^{1,2}(\Omega))$  and if  $\Omega \in C^{1,1}$ , then  $u \in L^2(0, T, W^{2,2}(\Omega))$ .

Dukaz

Consider  $u^n$ -Galeikin approximation

$$u^n(t, x) = \sum_{i=1}^n a_i^n w_i : \int_{\Omega} \partial_t u^n w_j + \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n = \langle f, w_j \rangle.$$

Multiply by  $\partial_t a_j^n(t)$  and

$$\sum_{i=1}^n \int_{\Omega} \partial_t u^n \partial_t a_i^n w_j = \int_{\Omega} \partial_t u^n \left( \sum \partial_t a_i^n w_j \right) = \int_{\Omega} \partial_t u^n (\partial_t u^n).$$

$$\int_{\Omega} \partial_t u^n \partial_t u^n + \int_{\Omega} A \nabla u^n \cdot \nabla \partial_t u^n + b u^n \partial_t u^n + \mathbf{c} \cdot \nabla u^n \partial_t u^n - \mathbf{d} \nabla \partial_t u^n u^n = \langle f, \partial_t u^n \rangle.$$

$$\text{Good guy: } \int_{\Omega} \partial_t u^n \partial_t u^n = \|\partial_t u^n\|_2^2.$$

$$\text{First half of other guy: } \int_{\Omega} A \nabla u \nabla \partial_t u =$$

$$\begin{aligned} & \int_{\Omega} \frac{(A + A^T)}{2} \nabla u \nabla \partial_t u + \int_{\Omega} \frac{A - A^T}{2} \nabla u \nabla \partial_t u = \\ &= \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \frac{A + A^T}{2} \nabla u \nabla u - \frac{1}{2} \int_{\Omega} \frac{\partial_t (A - A^T)}{2} \nabla u \nabla u - \int_{\Omega} \sum_{i,j} \frac{\partial}{\partial x_i} \frac{(A_{ij} - A_{ji})}{2} \frac{\partial u}{\partial x_j} \partial_t u - \underbrace{\sum_{i,j} \frac{A_{ij} - A_{ji}}{2} \frac{\partial^2 u}{\partial x_i \partial x_j}}_{\text{sum with symmetric dot antisymmetric}} \end{aligned}$$

$$\text{Worst guy: } \int_{\Omega} \mathbf{d} \cdot \nabla \partial_t u^n u^n =$$

$$= - \int_{\Omega} \partial_t u^n \operatorname{div}(\mathbf{d} u^n) = - \int_{\Omega} \partial_t u^n (\operatorname{div} \mathbf{d} u^n + \mathbf{d} \cdot \nabla u^n).$$

$$\|\partial_t\|_2^2 + \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \frac{A + A^T}{2} \nabla u \nabla u \leq \int_{\Omega} |\partial_t u^n| (|f| + \dots) + |\nabla u|^2 |\partial_t A|.$$

From Young's inequality:

$$\begin{aligned} & \leq \frac{1}{2} \int_{\Omega} |\partial_t u^n|^2 + C \int_{\Omega} |b|^2 |u^n|^2 + |\mathbf{c}|^2 |\nabla u^n|^2 + |\operatorname{div} \mathbf{d}|^2 |u^n|^2 + |\mathbf{d}|^2 |\nabla u^n|^2 + |f|^2 + \left| \nabla \frac{A - A^T}{2} \right|^2 |\nabla u^n|^2 + \left| \partial_t \frac{A + A^T}{2} \right|^2 |\nabla u^n|^2 \\ & \leq \frac{1}{2} \|\partial_t u^n\|_2^2 \leq c(b, \mathbf{c}, \mathbf{d}) (\|f\|_2^2 + \|u^n\|_{1,2}^2). \\ & \implies \|\partial_t u^n\|_2^2 + \frac{d}{dt} \int_{\Omega} A \nabla u^n \cdot \nabla u^n \leq c(\dots) \cdot (\|f\|_2^2 + \|u^n\|_{1,2}^2). \end{aligned}$$

We want to know, if right hand side is integrable in time:

$$\int_{\tau}^t \|\partial_t u^n\|_2^2 + \int_{\Omega} A \nabla u^n(t) \nabla u^n(t) \leq \int_{\Omega} A \nabla u^n(\tau) \nabla u^n(\tau) + c \cdot \int_{\tau}^t \|f\|_2^2 + \|u^n\|_{1,2}^2.$$

With  $\tau \leq \delta$  we add  $\int_0^{\delta} \cdot d\tau$ :

$$\int_{\delta}^t \|\partial_t u^n\|_2^2 + \int_{\Omega} A \nabla u^n(t) \nabla u^n(t) \leq \int_0^{\delta} \int_{\Omega} A \nabla u^n(\tau) \nabla u^n(\tau) d\tau + C(\text{DATA})$$



### Věta 6.3

Let  $\partial_t f \in L^2(0, T, L^2(\Omega))$ ,  $\partial_t A, \partial_t b, \partial_t \mathbf{c}, \partial_t d \in L^\infty$ . Then  $\forall \delta > 0 : \partial_{tt} u \in L^2(\delta, T, V^*), \partial_t u \in L^2(\delta, T, W_0^{1,2}(\Omega))$ . If  $-Lu_0 + f(0) \in L^2(\Omega)$ , then

$$\partial_{tt} u \in L^2(0, T, V^*), \quad \partial_t u \in L^2(0, T, W_0^{1,2}(\Omega)).$$

┌

*Důkaz* (Sketch)

Take  $u^n$  – Galerkin approximation. Apply  $\partial_t$  to it:

$$\int_{\Omega} \partial_t u^n w_j + \int_{\Omega} A \nabla u^n \nabla u^n w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n = \int_{\Omega} f w_j, \quad \forall j \in [n] \text{ and almost every } t \in (0, T)$$

$$\int_{\Omega} \partial_{tt} u^n w_j + \int_{\Omega} A \nabla \partial_t u^n \nabla w_j = \int_{\Omega} -\partial_t A \nabla u^n \nabla w_j + (\partial_t b u^n + b \partial_t u^n) w_j + \partial_t \mathbf{c} \nabla u^n + \mathbf{c} \nabla \partial_t u^n w_j.$$

Similar as before we replace  $w_j, b_j, \partial_t u^n$ :

$$\frac{1}{2} \frac{d}{dt} \|\partial_t u^n\|_2^2 + c_1 \|\nabla \partial_t u^n\|_2^2 \leq \int_{\Omega} \|\nabla \partial_t u^n\| (SOMETHING).$$

$$\implies \frac{d}{dt} \|\partial_t u^n\|_2^2 + \|\nabla \partial_t u^n\|_2^2 \leq C(\|\partial_t u^n\|_2^2 + \dots).$$

$$t \geq 2\delta : \|\partial_t u(t)\|_2^2 + \int_{\tau}^t \|\partial_t u^n\|_2^2 \leq C(1 + \int_{\tau}^t \|\partial_t u^n\|_2^2) + \|\partial_t u^n(\tau)\|_2^2.$$

Add  $\int_{\delta}^2 \delta d\tau$ :

$$\|\partial_t u(t)\|_2^2 + \int_{2\delta}^T \|\nabla u\|_2^2 \leq X \left( \int_{\delta}^T \|\partial_t u^n\|_2^2 + 1 + \int_{\delta}^{2\delta} \|\partial_t u^n(\tau)\|_2^2 \right) \leq C(1 + \frac{c}{\delta} + \frac{c}{\delta^2}).$$

$$\begin{aligned} & \rightarrow C \left( \int_0^T \|\partial_t u^n\|_2^2 + \|\partial_t u^n(0)\|_2^2 + 1 \right) \leq \\ & \leq C + C \|\partial_t u^n(0)\|_2^2 = C + C \|-Lu_0 + f(0)\|_2^2 \leq \text{const}. \end{aligned}$$

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□

## 7 Linear hyperbolic equations

*Poznámka* (Prototype)

$$\begin{aligned} \frac{\partial u^2}{\partial t^2} - \Delta u &= 0 \text{ in } (0, T) \times \Omega, & u &= 0 \text{ on } (0, T) \times \partial\Omega. \\ u(0, x) &= u_0(x) \in W_0^{1,2}(\Omega), & \partial_t u(0, x) &= u_1(x) \in L^2(\Omega). \end{aligned}$$

*Poznámka* (Formal a priory estimate)

Test by  $\partial_t u$ :

$$\begin{aligned} \int_{\Omega} \partial_{tt} u \partial_t u - \Delta u \partial_t u &= 0 \\ \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_2^2 + \int_{\Omega} \underbrace{\nabla u \nabla \partial_t u}_{\frac{1}{2} \partial_t \|\nabla u\|^2} &= 0 \end{aligned}$$

$$\frac{d}{dt} (\|\partial_t u\|_2^2 + \|\nabla u\|_2^2) = 0$$

$$\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq \|\partial_t u(0)\|_2^2 + \|\nabla u(0)\|_2^2 = \|u_1\|_2^2 + \|\nabla u_0\|_2^2.$$

$$\|\partial_{tt}^2 u\|_{(W_0^{1,2}(\Omega))^*} = \sup_{\|\varphi\| \leq 1} \langle \partial_{tt}^2 u, \varphi \rangle \sim \sup \int_{\Omega} \partial_{tt}^2 u \varphi = \sup \int_{\Omega} \nabla u \varphi.$$

## Věta 7.1

$L$  be an elliptic operator such that  $\int_0^T (\|\partial_t u\|_{\infty} + \|A\|_{1,\infty} + \|b\|_{\infty} + \|\mathbf{c}\|_{\infty} + \|\mathbf{u}\|_{1,\infty}) < \infty$  and  $f \in L^2(0, T, L^2(\Omega))$ . Assume that  $u_0 \in W_0^{1,2}(\Omega)$  and  $u_1 \in L^2(\Omega)$ . Then there  $\exists! u \in L^2(0, T, W_0^{1,2}(\Omega)) \cap W^{1,2}(0, T, L^2(\Omega)) \cap W^{2,2}(0, T, V^*)$ .

And  $u(t) \rightarrow u_0$  in  $L^2(\Omega)$ ,  $\partial_t u(t) \rightarrow u_1$  in  $V^*$ .

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*Důkaz* (Existence)

Step one: Galleikin approximation. Step two: Uniform estimates. Step three:  $n \rightarrow \infty$ .  $\square$

┌

*Důkaz* (Step one)

$\{w_j\}_{j=1}^{\infty}$  base of  $W_0^{1,2}$  ( $\|P^n u\|_{1,2} \leq c \|u\|_{1,2}$ ). ? for  $u^n(t, x) = \sum_{j=1}^n a_j^n(t) w_j(x)$ .

$$\int_{\Omega} \partial_{tt} u^n w_j + \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j + \mathbf{d} \nabla w_j u^n = \int_{\Omega} f w_j.$$

Weak formulation for  $n$ -th coord. (WFn).  $\partial_t u^n(0) = P^n u_1$  and  $u^n(0) = P^n u_0$ .

$$(a_j^n)'(0) = \int u_1 w_j, \quad a_j^n(0) = \int u_0 w_j, \quad (a_j^n)''(t) = F_j(a^n, t) + b_j(t).$$

┌

Assume there exists  $u^n$  a solution to (WFn).  $\square$

┌ *Důkaz* (Step two)

Uniform ( $N$ -independent) estimates: Multiply WFn by  $(e_j^n)'(t)$  and  $\sum_{j=1}^n$ :

$$\int_{\Omega} \partial_{tt} u^n \partial_t u^n + \int_{\Omega} A \nabla u^n \partial \nabla u^n = \int_{\Omega} f \partial_t u^n + \mathbf{d} \nabla \partial_t u^n u^n - \mathbf{c} \nabla u^n \partial_t u^n - b u^n \partial_t u^n.$$

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} |\partial_t u^n|^2 + \frac{A + A^T}{2} \nabla u^n \nabla u^n \right) = - \underbrace{\int_{\Omega} \partial_t \left( \frac{A + A^T}{2} \nabla u^n \nabla u^n \right) - \int_{\Omega} \frac{A - A^T}{2} \nabla u^n \partial \nabla u^n}_{\text{by part}} \stackrel{\text{Hölder}}{\leq}$$

$$\leq c(\|f\|_2^2 + \|\partial_t u^n\|_2^2 + \|\nabla u^n\|_2^2) \leq \tilde{c} \left( \|f\|_2^2 + \int_{\Omega} |\partial_t u^n|^2 + \int_{\Omega} \frac{A + A^T}{2} \nabla u^n \nabla u^n \right).$$

Gronwall's lemma:

$$\begin{aligned} \int_{\Omega} |\partial_t u(t)|^2 + \frac{A + A^T}{2} \nabla u^n(t) \nabla u^n(t) &\leq C(T) \cdot \left( \int_0^T \|f\|_2^2 + \int_{\Omega} |\partial_t u^n(0)|^2 + \frac{A + A^T}{2} \nabla u^n(0) \nabla u^n(0) \right) \leq C_t. \\ &\leq \tilde{C}_T (1 + \|v_0\|_2^2 + \|u_0\|_{1,2}^2). \end{aligned}$$

$$\sup_{t \in (0, T)} \|\partial_t u^n(t)\|_2^2 + \|u^n(t)\|_{1,2}^2 \leq C(\text{DATA}).$$

$$\|\partial_{tt} u^n\|_{(W_0^{1,2}(\Omega))^*} := \sup_{\varphi \in W_0^{1,2}(\Omega)} \langle \partial_{tt} u^n, \varphi \rangle \stackrel{\text{Gelfand}}{=}$$

$$= \sup_{\varphi} \int_{\Omega} \partial_{tt} u^n \varphi \stackrel{\text{basis}}{=} \sup_{\varphi} \int_{\Omega} \partial_{tt} u^n P^n(\varphi) \stackrel{\text{WFn}}{=}$$

$$\begin{aligned} &= - \int_{\Omega} A \nabla u^n \nabla P^n(\varphi) \dots \stackrel{\text{Hölder}}{\leq} C \cdot \|P^n(\varphi)\|_{1,2} (\|f\|_2 + \|u^n\|_{1,2}) \leq \\ &\leq \tilde{c} \|\varphi\|_{1,2} (\|f\|_2 + \|u^n\|_{1,2}). \end{aligned}$$

$$\int_0^T \|\partial_{tt} u^n\|_{(W_0^{1,2}(\Omega))^*}^2 \leq \tilde{c} \cdot 2 \cdot \int_0^T (\|f\|_2^2 + \|u^n\|_{1,2}^2) \leq C(\text{DATA}).$$

└

□

┌ *Důkaz* (Step three)

$u^n \rightharpoonup^* u$  in  $W^{1,\infty}(0, T, L^2(\Omega)) \cap L^\infty(0, T, W_0^{1,2}(\Omega))$ .  $u^n \rightharpoonup u$  in  $W^{2,2}(0, T, (W_0^{1,2}(\Omega))^*)$ .

Limits:

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \partial_{tt} u^n w_j \tau dx dt = \int_0^T \langle \partial_{tt} u^n, w_j \tau \rangle dt = \langle \partial_{tt} u^n, w_j \tau \rangle_{L^2(0, T, W_0^{1,2}(\Omega))}.$$

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} A \nabla u^n \nabla w_j \tau = \langle \nabla u^n, A^T \nabla w_j \tau \rangle_{L^2(0, T, L^2(\Omega))} \rightarrow \langle \nabla u, A^T \nabla w_j \tau \rangle = \int_0^T \int_{\Omega} A \nabla u \nabla w_j \tau.$$

$$WF : \int_0^T \langle \partial_{tt} u, w_j \rangle \tau + \int_{\Omega} A \nabla u \nabla w_j \tau + buw_j \tau + \mathbf{c} \nabla u w_j \tau - \mathbf{d} \nabla w_j \tau = \int_0^T \int_{\Omega} f w_j \tau.$$

TODO?

└ TODO!!! □

## 8 Semigrupy

### Definice 8.1 (Značení)

$$\mathcal{L}(X) := \{L : X \rightarrow X \mid L \text{ lineární omezený operátor}\}, \quad \|L\|_{\mathcal{L}} = \sup_{\|x\| < 1} \|Lx\|_X.$$

Dvojice  $(A, D(A))$  je neomezený operátor, kde  $D(A) \subset X$  je definiční obor  $A$  – podprostor  $X$ .  $A : D(A) \rightarrow X$  je lineární.

### Definice 8.2 (Semigrupa (jednparametrická lineární semigrupa))

$S(t) : [0, \infty] \rightarrow \mathcal{L}(X)$  se nazývá semigrupa  $\equiv$

- $S(0) = \text{id}$ , neboli  $S(0)x = x \ \forall x \in X$ ;
- $\forall t, s \geq 0 : S(t)S(s) = S(t+s)$ .
- Pokud navíc  $S(t)x \rightarrow x$  pro  $t \rightarrow 0_+$ , pak  $S(t)$  nazýváme  $C_0$ -semigrupa.

### Lemma 8.1

*Nechť  $S(t)$  je  $C_0$ -semigrupa. Potom*

1.  $\exists M \geq 1 \ \exists \omega \geq 0 \ \forall t \in [0, \infty) : \|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$ ;

2.  $\forall x$  pevné  $t \mapsto S(t)x$  je spojité zobrazení z  $[0, \infty)$  do  $X$ .

┌

*Důkaz*

Krok 1): „ $\exists M \exists \delta > 0 \forall t \in [0, \delta] : \|S(t)\|_{\mathcal{L}(X)} \leq M$ “: Pro spor nechť toto neplatí. Tedy  $\exists t_n \rightarrow 0_+ : \|S(t_n)\|_{\mathcal{L}(X)} \rightarrow \infty$ . Víme, že  $\forall x : S(t_n)x \rightarrow x$ . To implikuje  $\forall x \sup_{t_n} \|S(t_n)x\|_X < \infty$ . Z Principu stejnoměrné omezenosti (Věta Banach-Steinhaus, z úvodu do funkcí)  $\exists M > 0 \|S(t_n)\|_{\mathcal{L}(X)} \leq M$ .  $\nmid$ .

„1.“: Definujeme  $\omega = \frac{1}{\delta} \ln M$ .  $t \geq 0 \exists \varepsilon \in (0, \delta) : t = n\delta + \varepsilon$ .

$$\|S(t)\|_{\mathcal{L}(X)} = \|S(\delta) \cdot \dots \cdot S(\delta) \cdot S(\varepsilon)\|_{\mathcal{L}(X)} \leq \|S(\delta)\|_{\mathcal{L}(X)}^n \cdot \|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M e^{\omega t}.$$

„2.“: Spojitost v  $0_+$  plyne z třetího bodu definice semigrupy. Pro  $t_0 > 0, t \rightarrow (t_0)_+$ :

$$\lim_{h \rightarrow 0_+} S(t_0 + h)x = \lim_{h \rightarrow 0_+} S(t_0)S(h)x = S(t_0) \lim_{h \rightarrow 0} S(h)x = S(t_0)x.$$

Pro  $t \rightarrow (t_0)_-$  (bereme  $h$  dostatečně malé, aby  $t_0 - h > 0$ ):

$$\|S(t_0 - h)x - S(t_0)x\|_X = \|S(t_0 - h)(x - S(h)x)\|_X \leq \|S(t_0 - h)\|_{\mathcal{L}} \cdot \|x - S(h)x\|_X \rightarrow 0.$$

└

□

### Definice 8.3 (Generator of semigroup)

An unbounded operator  $(A, D(A))$  is called a generator of  $S(t)$  iff

$$Ax := \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h}, \quad D(A) := \left\{ x \in X \mid \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\}.$$

### Věta 8.2 (Properties of generator)

Let  $(A, D(A))$  be a generator of a  $C_0$ -semigroup  $S(t)$ . Then

1.  $x \in D(A) \implies S(t)x \in D(A) \forall t \geq 0$ ;
2.  $x \in D(A) \implies AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x) \forall t \geq 0$ ;
3.  $x \in X, t > 0 \implies x_t := \int_0^t S(s)x ds \in D(A), A(x_t) = S(t)x - x$ .

┌

*Poznámka (Použití)*

$u_0 \in D(A) \subseteq X, u(t) := S(t)u_0, 2. \implies \frac{d}{dt}u(t) = \frac{d}{dt}(S(t)u_0) = AS(t)u_0 = Au(t)$ . E. g. if  $S(t)$  corresponds to the solution operator of  $\partial_t u = \Delta u$ , then generator  $S(t)$  is Laplace.

└

┌  
Důkaz

$$A(S(t)x) \stackrel{\text{if exists}}{=} \lim_{h \rightarrow 0_+} \frac{S(h)(S(t)x) - S(t)x}{h} = \lim_{h \rightarrow 0_+}$$

TODO!!!

$$\frac{d}{dt}(S(t)x)_{t \rightarrow t_0+} = \lim_{h \rightarrow 0_+} \frac{S(t_0+h)x - S(t_0)x}{h} = S(t_0) \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} = S(t_0)Ax.$$

$$\frac{d}{dt}(S(t)x)_{t \rightarrow t_0-} = \lim_{h \rightarrow 0_+} \frac{S(t_0-h)x - S(t_0)x}{h} = \lim_{h \rightarrow 0_+} \left( S(t_0-h) \left[ \frac{x - S(h)x}{h} - S(h)Ax \right] \right) + \lim_{h \rightarrow 0_+} S(t_0-h)S$$

because

$$\begin{aligned} \|S(t_0-h) \left( \frac{x - S(h)x}{h} - S(h)Ax \right)\|_X &\leq \|S(t_0-h)\|_{\mathcal{L}(X)} \cdot \left\| \frac{x - S(h)x}{h} - Ax + Ax - S(h)Ax \right\|_X \leq \\ &\leq Me^{\omega t_0} \left( \left\| \frac{S(h)x - x}{h} - Ax \right\|_X + \|Ax - S(h)Ax\|_X \right) \rightarrow 0. \end{aligned}$$

3.)

$$\begin{aligned} \frac{S(h)x_t - x_t}{h} &= \frac{S(h) \int_0^t S(s)x ds - \int_0^t S(s)x ds}{h} = \frac{\int_0^t S(h+s)x ds - \int_0^t S(s)x ds}{h} = \\ &= \frac{\int_h^{t+h} S(s)x ds - \int_0^t S(s)x ds}{h} = \frac{1}{h} \int_t^{t+h} S(s)x ds - \frac{1}{h} \int_0^h S(s)x ds. \end{aligned}$$

└

□

### Definice 8.4 (Closed operator)

We say that  $(A, D(A))$  is closed iff

$$\begin{aligned} (u_n \in D(A), \quad u_n \rightarrow u \text{ in } X, \quad A(u_n) \rightarrow v \text{ in } X) \\ \implies u \in D(A), \quad Au = v. \end{aligned}$$

### Věta 8.3 (Density and closedness of generator)

Let  $(A, D(A))$  be a generator to a  $C_0$ -semigroup  $S(t)$  in  $X$ . Then  $D(A)$  is dense in  $X$  and  $(A, D(A))$  is closed.

┌ *Důkaz*

$x \in X$  arbitrary  $\implies x_t \in D(A)$ .

$$\left[\frac{x_t}{t}\right] = \frac{1}{t} \int_0^t S(s)x ds \rightarrow x \implies D(A) \text{ is dense in } X.$$

$x_n \in D(A)$ ,  $x_n \rightarrow x$  in  $X$ ,  $Ax_n \rightarrow y$  in  $X$ . We want  $x \in D(A)$  and  $Ax = y$ .

$$S(h)x_n - S(0)x_n \int_0^t \frac{d}{dt}(S(t)x_n)dt = \int_0^h S(t)(Ax_n)dt$$

$$n \rightarrow \infty : \frac{S(h)x - x}{h} = \int_0^h S(t)y dt.$$

$$A(x) = \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} = \lim_{n \rightarrow \infty} \int_0^h S(t)y dt = S(0)y = y.$$

└

□

*Poznámka*

We have  $A(\int_0^t S(s)x ds) = S(t)x - x = \int_0^t AS(t)x dt$ .

### **Lemma 8.4** (Uniqueness of $S(t)$ )

Let  $S(t)$  and  $\tilde{S}(t)$  be  $C_0$  semigroup with the same generator. Then  $S(t) = \tilde{S}(t) \forall t \geq 0$ .

┌ *Důkaz*

$$y(t) = S(T-t)\tilde{S}(t)x, \quad x \in D(A), \quad T > 0 \text{ fixed.}$$

$$\frac{d}{dt}y(t) = -S(T-t)A\tilde{S}(t)x + S(T-t)A\tilde{S}(t)x = 0.$$

$$y(t) = y(0) = y(T) \implies S(T)x = \tilde{S}(T)x.$$

An  $D(A)$  is dense, so  $S = \tilde{S}$  on  $X$ .

└

□

### **Definice 8.5** (Resolvent)

Let  $(A, D(A))$  be unbounded operator. We define resolvent set  $\varrho(A) := \{\lambda \in \mathbb{R} | (\lambda I - A) : D(A) \rightarrow X \text{ is on}\}$   
 $\forall \lambda \in \varrho(A)$  we define the resolvent operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \rightarrow D(A).$$

┌ *Poznámka*

$(A, D(A))$  closed  $\implies R(\lambda, A)$  is continuous  $\implies R(\lambda, A) \in \mathcal{L}(X)$ .

└

**Lemma 8.5** (Properties of  $R(\lambda, A)$ )

Let  $(A, D(A))$  be a generator of  $C_0$ -semigroup  $S(t)$  and let  $\|S(t)\| \leq Me^{\omega t}$ .

1.  $AR(\lambda, A)x = \lambda R(\lambda, A)x - x \quad \forall x \in X$ .
2.  $R(\lambda, A)Ax = \lambda R(\lambda, A)x - x \quad \forall x \in D(A)$ .
3.  $R(\lambda, A)x - R(\mu, A)x = (\mu - \lambda)R(\lambda, A)R(\mu, A)x$ .
4.  $\forall \lambda > \omega \quad \lambda \in \varrho(A): R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}$ .

┌

*Důkaz*

„1.“:  $AR(\lambda, A)x = [(A - \lambda I) + \lambda I] R(\lambda, A)x = -x + \lambda R(\lambda, A)x$ . „2. + 3.“: obdobně.

„4.“: Rescale and define  $\tilde{S}(t) = e^{-\omega t} S(t)$ . We will prove the result for  $\tilde{S}(t)$  and transform it to  $S(t)$ :

$$\|\tilde{S}(t)\| \leq e^{-\omega t} \|S(t)\| \leq Me^{0 \cdot t} = M.$$

└

□

**Věta 8.6** (Hille-Yosida)

Nechť  $(A, D(A))$  je neomezený operátor. Pak následující je ekvivalentní:

- $\exists C_0$ -semigupa, jejíž generátor je  $(A, D(A))$  a je neexpanzivní;
- $(A, D(A))$  je uzavřený,  $D(A)$  je husté v  $X$ ,  $(0, \infty) \subseteq \varrho(A)$ ,  $\|R(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}$ .

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*Důkaz*

„ $\implies$ “ máme. „ $\impliedby$ “: (Myšlenka:  $A \rightarrow A_n$  aproximace pomocí omezených operátorů,  $S_n(x) \sim e^{tA_n}$ ,  $n \rightarrow \infty$ ). □

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*Důkaz (Krok 1)* $A_n$  aproximace (Hille-Yosida):

$$A_n := nAR(n, A) \quad \forall n \in \mathbb{N}.$$

„ $A_n$  je omezený operátor“:

$$A_n = nAR(n, A) = n^2R(n, A) - nI \in \mathcal{L}(X)$$

$$\|nR(n, A)x - x\|_X = \|R(n, A)Ax\|_X \leq \|R(n, A)\|_{\mathcal{L}(X)}\|Ax\|_X \leq \frac{\|Ax\|_X}{n} \rightarrow 0.$$

$D(A)$  je husté v  $X$ .  $\|nR(n, A) - I\|_{\mathcal{L}(X)} \leq n\|R(n, A)\|_{\mathcal{L}(X)} + 1 \leq 1 + 1 = 2$ . Z tohoto a principu stejnoměrné omezenosti (někdy také Banach-Steinhaus)  $nR(n, A)x \rightarrow x \quad \forall x \in X$ . Spolu s lemmatem bod 3:

$$A_n x = nAR(n, A)x = nR(n, A)Ax \rightarrow Ax, \quad \forall x \in D(A).$$

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□

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*Důkaz (Krok 2)*Definujeme  $S_n(t)$  jako:

$$S_n(t) := e^{tA_n} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_n^k \in \mathcal{L}(X).$$

Za domácí úkol si ověříme, že  $S_n(t)$  je semigrupa.

$$S_n(t) = e^{tA_n} = e^{-ntI + n^2tR(n, A)} = e^{-nt} e^{n^2tR(n, A)}$$

$$\begin{aligned} \|S_n(t)\|_{\mathcal{L}(X)} &\leq e^{-n} \|e^{n^2tR(n, A)}\|_{\mathcal{L}(X)} \leq e^{-nt} \left\| \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} (nR(n, A))^k \right\|_{\mathcal{L}(X)} \leq \\ &\leq e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \|nR(n, A)\|_{\mathcal{L}(X)}^k \leq e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} = e^{-nt} e^{nt} = 1. \end{aligned}$$

└

□

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Důkaz (Krok 3)

Ukážeme „ $\exists \lim S_n(t)x$ “:

$$\begin{aligned} (S_n(t) - S_m(t))x &= \int_0^t \frac{d}{ds} (S_m(t-s)S_n(s)x) ds = \int_0^t \frac{d}{ds} (e^{(t-s)A_m} A^{sA_n} x) ds = \\ &= \int_0^t (-A_n e^{(t-s)A_m} e^{sA_n} x + A_n e^{(t-s)A_m} e^{sA_n} x) ds = \int_0^t e^{(t-s)A_m} e^{sA_n} (A_n x - A_m x), \\ \|S_n(t)x - S_m(t)x\|_X &\leq \left\| \int_0^t e^{(t-s)A_m} e^{sA_n} (A_n x - A_m x) \right\|_X \leq \\ &\leq \left\| \int_0^t S_m(t-s)S_n(s)(A_n x - A_m x) \right\|_X \leq t \|A_n x - A_m x\|_X \end{aligned}$$

$\forall x \in D(A)$   $S_n(t)x$  je cauchyovská posloupnost  $\implies \exists S(t)x, S_n(t)x \rightarrow S(t)x$ .  $D(A)$  je hustý v  $x$  a  $\|S_n(t)\|_{\mathcal{L}(X)} \leq 1 \implies \forall x \in X \exists \lim_{n \rightarrow \infty} S_n(t)x =: S(t)x$ . Z linearit y plyne, že  $S(t)$  je semigrupa.  $\square$

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Důkaz (Krok 4)

„ $(A, D(A))$  je generátor  $S(t)$ “: Označíme  $(\tilde{A}, D(\tilde{A}))$  generátor  $S(t)$ .

$$\begin{aligned} \left( \frac{S(t)x - x}{t} \right) \quad S_n(t)x - x &= \int_0^t \frac{d}{ds} S_n(s)x ds = \int_0^t S_n(s)A_n x ds \\ n \rightarrow \infty, x \in D(A) : S(t)x - x &= \lim_{n \rightarrow \infty} \int_0^t S_n(s)A_n x ds = \lim_{n \rightarrow \infty} \int_0^t S_n(s)(A_n x - Ax) + (S_n(s) - S(s)) + S(s)Ax \end{aligned}$$

$$\begin{aligned} \frac{S(t)x - x}{t} &= \int_0^t S(s)Ax ds \rightarrow Ax \quad \forall x \in D(A). \\ \implies D(A) &\subseteq D(\tilde{A}) \implies A = \tilde{A} \text{ na } D(A). \end{aligned}$$

Tedy zbývá ukázat  $D(\tilde{A}) \subseteq D(A)$ .

Víme  $\forall \lambda > 0 : \lambda I - A$  je prostý a na ( $\|R(\lambda, A)\| \leq \frac{1}{\lambda}$ ). Také víme, že  $S(t)$  je neexpanzivní a  $\|S(t)\|_{\mathcal{L}(X)} \leq 1 \implies R(\lambda, \tilde{A}) \leq \frac{1}{\lambda} \forall \lambda > 0$ . Nakonec víme  $\forall \lambda > 0 : \lambda I - \tilde{A}$  je prostý a na. To nám dává  $D(A) = D(\tilde{A})$ .  $\square$

## Věta 8.7 (Obecná Hille-Yosida)

$(A, D(A))$  generuje  $C_0$ -semigrupu splňující  $\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t} \Leftrightarrow ((A, D(A)) \text{ je uzavřený a } \forall \lambda > \omega, \lambda \leq \varrho(A) \text{ a } \|R^n(\lambda, A)\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n} \forall n)$ .

┌  
Důkaz

Podobný předchozímu, jen se musí udělat lepší odhady. Nebyl na přednášce.  $\square$

## 8.1 Aplikace na lineární evoluční PDR

*Příklad*

$\partial_t u - \Delta u = 0$  v  $(0, T) \times \Omega$ ,  $u(0) = u_0$  v  $\Omega$  a  $u = 0$  na  $\partial\Omega \times (0, T)$ .

Přepis do semigrup:  $\partial_t u = \Delta u = Au$ . A ptáme se  $\exists S(t)$  tak, že  $A$  je uzavřený,  $D(A)$  je husté v  $L^2(\Omega)$  a  $(0, \infty) \subseteq \varrho(A)$  a  $R(\lambda, A) \leq \frac{1}{\lambda}$ .

$D(A) = \underline{L^2(\Omega)} \cap W_0^{1,2}(\Omega) \cap \{u | \Delta u \in L^2(\Omega)\}$  a  $X = L^2(\Omega)$ .  $\overline{D(A)} = X$ , neboť  $C_0^\infty(\Omega) \subseteq D(A)$  a  $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$ .

$(A, D(A))$  je uzavřený:

$$(u^n \rightarrow u \wedge Au^n \rightarrow f) \implies u \in D(A) \wedge Au = f :$$

$$u^n \in D(A) \wedge \Delta u^n =: f^n \in L^2(\Omega) \wedge f^n \rightarrow f \implies$$

$$\implies \forall \varphi \in W_0^{1,2} \int_{\Omega} \nabla u^n \nabla \varphi = \int_{\Omega} f^n \varphi$$

Vezmeme  $\varphi = u^n \implies \|\nabla u^n\|_2 \leq c \|f^n\|_2 \xrightarrow{\text{Poisson}} \|u^n\|_{1,2} \leq C:$

$$u^n \rightarrow u \implies \int_{\Omega} \nabla u \nabla \varphi = \int_{\Omega} f \varphi \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

TODO!!!

TODO!!!

*Příklad* (RVT s pravou stranou)

$\partial_t u - \Delta u = f$  v  $(0, T) \times \Omega$ ,  $u = 0$  na  $(0, T) \times \partial\Omega$ ,  $u(0) = u_0$  v  $\Omega$ .

$Au = \Delta u$ ,  $D(A) = \{u \in W_0^{1,2}, \Delta u \in L^2(\Omega)\}$ . To má semigrupu  $S(t)$  pro  $f \in L^1(0, T, L^2(\Omega))$ .  
Volíme

$$u(t) := S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau.$$

Co řeší  $u(t)$ ?

$$\begin{aligned} \partial_t u &= \partial_t(S(t)u_0) + \partial_t \int_0^t S(t-\tau)f(\tau)d\tau = AS(t)u_0 + S(0)f(t) + \int_0^t \partial_t S(t-\tau)f(\tau)d\tau = \\ &= f(t) + AS(t)u_0 + \int_0^t AS(t-\tau)f(\tau)d\tau = f(t) + A(S(t)u_0 + \int_0^t S(t-\tau)f(\tau)d\tau) = f(t) + Au(t). \end{aligned}$$

*Příklad*

$\partial_t u - \Delta u = 0$  v  $(0, T) \times \Omega$ ,  $u = 0$  na  $(0, T) \times \partial\Omega$ ,  $u(0) = u_0 \in L^p(\Omega)$ ,  $\infty > p \geq 2$  (lze i pro  $p \in (1, 2)$ ).

Řešení

Chceme  $X = L^p(\Omega)$ ,  $Au = \Delta u$ ,  $D(A) = \{u \in W_0^{1,2}(\Omega), \Delta u \in L^p(\Omega)\}$ . Pak  $A : D(A) \rightarrow X$ . Hustota je zadarmo. „ $(A, D(A))$  uzavřený“: rozmyslet doma (důkaz stejný jako výše).

$$\lambda u - \Delta u = f \implies \|u\|_p \leq \frac{\|f\|_p}{\lambda}:$$

$$\forall f \in L^p(\Omega) \exists! u \in W_0^{1,2}(\Omega) : \int_{\Omega} \lambda u \varphi + \nabla u \nabla \varphi = \int_{\Omega} f \varphi \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Kdyby  $\varphi = |u|^{p-2}u$  (pozor, nemáme, že toto  $\varphi \in W_0^{1,2}$ , pouze zkusíme), pak

$$\lambda \|u\|_p^p + \int_{\Omega} \nabla u \nabla (|u|^{p-2}u) = \int_{\Omega} f |u|^{p-2}u \stackrel{\text{Hölder}}{\leq} \|f\|_p \cdot \|u^{p-1}\|_{p'} = \|f\|_p \cdot \|u\|_p^{p-1},$$

což je kýžený odhad, jelikož  $(p-1)|u|^{p-2}\nabla u \geq 0$ . Volme tedy  $\varphi := \left(\frac{|u|}{1+\varepsilon|u|}\right)^{p-2} u \in W_0^{1,2}(\Omega)$ . Zopakujeme postup (TODO?) a máme to.

Příklad

$\partial_t u + Lu = 0$  v  $(0, T) \times \Omega$ ,  $Lu = \operatorname{div} A \nabla u + b \cdot u + \mathbf{c} \cdot \nabla u$ , kde  $A, b, c$  nezávisí na čase  $t$ .

Řešení

$$Au = -Lu. D(A) := \left\{ u \in W_0^{1,2}(\Omega) \mid \sup_{\varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^2(\Omega)}=1} \int_{\Omega} A \nabla u \nabla \varphi + bu\varphi + \mathbf{c} \cdot \nabla u \varphi < \infty \right\}$$

$(A, D(A))$  je uzavřený (doma).  $\exists \omega > 0 \forall \lambda \geq \omega : \lambda u + Lu = f \exists! u$ . Odhad  $\|u\|_2 \leq \frac{\|f\|_2}{\lambda - \omega}$  už jsme dělali:

$$(\lambda - u)u + \omega u + Lu = f, \text{ TODO?}$$

Poznámka

Zkouška: První příklad musíme vyspat z rukávu (5–10 minut max) jinak jdeme domu. Na zbytek spoustu času. Příklad nemusí být ukázaný na přednášce, ale stačí ji následovat. Druhý příklad bude čistá teorie (může to být např. list ANO-NE).