TODO!!!

Definice 0.1 (WLOG)

$$D := U(0,1), \qquad T = \partial D.$$

TODO!!!

Definice 0.2

$f \in \mathcal{H}(D)$. We say that the boundary T is a natural boundary of f if $R_f = \varnothing$.

Například
$f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Radius of convergence is equal to 1 and f has natural boundary.

Důkaz
$K = \left\{\exp\left(\frac{2\pi ik}{n}\right) | k, n \in \mathbb{N}\right\}$ is dense in T. f is "diverges on" this set, because $f(z^{2^N}) = f(z) - \sum_{n=1}^{N}$. For $\alpha \in (0,1)$ we have parametrization of one "line" $\alpha \cdot \exp\left(\frac{2k\pi i}{2^n}\right)$ (for k, n fixed).

$$f\left(\alpha^{2^N}\right) = f\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right) + p\left(\alpha \exp\left(\frac{2k\pi i}{2^N}\right)\right).$$

□

For every domain $\Omega \subseteq \mathbb{C}$, there exists $f \in \mathcal{H}(\Omega)$ such that $\partial\Omega$ is natural boundary of f.

Důkaz
We use theorem (15.11 from Rudin or TODO from lecture). □

TODO!!!

TODO!!!

1 Eulerův vzorec

$$\sin \pi z = \pi z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 1.1

$$z \neq \frac{k}{2}, k \in \mathbb{Z} : 2\pi \operatorname{cotg}(2\pi z) = \pi \operatorname{cotg}(\pi x) + \pi \operatorname{cotg}(\pi(z + \frac{1}{2})).$$

Důkaz

$$2\pi \operatorname{cotg}(2\pi z) = 2\pi \frac{\cos(2\pi z)}{\sin(2\pi z)} = \pi \frac{\cos^2(\pi z) - \sin^2(\pi z)}{\sin(\pi z)\cos(\pi z)} = \pi \left(\operatorname{cotg}(\pi z) - \frac{\sin(\pi z)}{\cos(\pi z)} \right) = \pi \left(\operatorname{cotg}(\pi z) + \frac{\cos \pi}{\sin \pi} \right.$$

□

Lemma 1.2 (Herglotz)

$r > 1$, G oblast, $G \supset [0, r)$, h funkce holomorfní na G, $z, z + \frac{1}{2}, 2z \in [0, r) : 2h(2z) = h(z) + h(z + \frac{1}{2})$. Pak h je konstantní na G.

Důkaz

Zvol $t \in (1, r)$.

$$M := \max \{|h'(z)|, z \in [0, t]\}, \qquad 4|h'(2z)| \leqslant |h'(z)| + |h'(z + \frac{1}{2})| \implies$$

$$\implies 4|h'(z)| \leqslant |h'(\frac{z}{2})| + |h'(\frac{z}{2} + \frac{1}{2})| < 2M \implies 4M \leqslant 2M \implies M = 0.$$

□

Lemma 1.3

g holomorfní funkce na $\mathbb{C} \backslash \mathbb{Z}$, hlavní část Laurenotvy řady g na $P(k)$, $k \in \mathbb{Z}$, je rovna $\frac{1}{z-k}$, g lichá, $2g(2z) = g(z) + g(z + \frac{1}{2})$, $z \neq \frac{k}{2}$, $k \in \mathbb{Z}$. Pak $g(z) = \pi \operatorname{cotg}(\pi z)$.

Důkaz

$h(z) := g(z) - \pi \operatorname{cotg}(\pi z)$. h rozšíříme spojitě a holomorfně na $\mathbb{C}$. Z Herglotzova lemmatu je h konstantní na $\mathbb{C}$ (obě funkce splňují podmínky). Navíc $h(0) = 0$. □

Důsledek (Eisenstein)

$\pi \operatorname{cotg}(\pi z) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2}$, $z \notin \mathbb{Z}$.

Důkaz
$z \mapsto \frac{2z}{z^2-k^2}$ jsou holomorfní na $U(0,n)$, $k>n$.

$$\left|\frac{2z}{z^2-k^2}\right| \leqslant \frac{2n}{k^2-n^2} \qquad \wedge \qquad \sum_{k=n+1}^{\infty} \frac{2n}{k^2-n^2} K \implies \sum \in \mathcal{M}.$$

f je holomorfní na $\mathbb{C} \backslash \mathbb{Z}$. Také je lichá. Nakonec

$$s_n(z) = \frac{1}{2} + \sum_{k=1}^{n} \frac{2z}{z^2-k^2} = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{z+k} + \sum_{k=1}^{n} \frac{1}{z-k} = \sum_{-n}^{n} \frac{1}{z+k},$$

$$s_n\left(\frac{z}{2}\right) + s_n\left(\frac{z+1}{2}\right) = \sum_{k=-n}^{n} \frac{2}{z+2k} + \frac{2}{z+2k+1} = 2\sum_{k=-2n}^{2n} \frac{1}{z+k} + 2\cdot\frac{1}{z+2n+1} = 2s_{2n}(k) + \frac{2}{z+2n+1},$$

$$n \to \infty : f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right) = 2f(z) + 0.$$

Z předchozího lemmatu vyplývá důkaz. $\square$