TODO!!!

Definice 0.1 (Dot product on the space of matrices)

$$\mathbb{A} : \mathbb{B} = \operatorname{tr}(\mathbb{A}\mathbb{B}^T).$$

Definice 0.2 (Norm of matrix)

$$|\mathbb{A}| = (\mathbb{A} : \mathbb{A})^{\frac{1}{2}}.$$

Příklad

$$(\mathbf{a} \otimes \mathbf{b})^T = \mathbf{b} \otimes \mathbf{a}.$$

Důkaz

$$\mathbf{u} \cdot (\mathbf{a} \otimes \mathbf{b})^T \mathbf{v} = (\mathbf{a} \otimes \mathbf{b})\mathbf{u} \cdot \mathbf{v} = (\mathbf{a}(\mathbf{b} \cdot \mathbf{u}))\mathbf{v} = (\mathbf{b} \cdot \mathbf{u})(\mathbf{a} \cdot \mathbf{v}) = \mathbf{u} \cdot (\mathbf{b}(\mathbf{a} \cdot \mathbf{v})) = \mathbf{u} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{v}.$$

□

Příklad

$$\det(e^{\mathbb{A}}) = e^{\operatorname{tr} \mathbb{A}}.$$

Důkaz

$$e^{\mathbb{A}} = \lim \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n.$$

$$\det e^{\mathbb{A}} = \lim_{n\to\infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)^n\right) = \lim_{n\to\infty} \left(\det \left(\mathbb{I} + \frac{\mathbb{A}}{n}\right)\right)^n =?$$

Subtask: Is there an approximation for $\det(\mathbb{I}+\mathbb{S})$, where $\mathbb{S}$ is a „small“ matrix. Yes, we did it (KontinuumDU1.pdf) for $\mathbb{S} \in \mathbb{R}^{3\times 3}$:

$$\det(\mathbb{I}+\mathbb{S}) = \det \mathbb{I} + \operatorname{tr}(\mathbb{I} \operatorname{cof} \mathbb{S}) + \operatorname{tr}(\mathbb{S}^T \operatorname{cof} \mathbb{I}) + \det \mathbb{S} \approx 1 + \operatorname{tr}(\mathbb{S}^T \operatorname{cof} \mathbb{I}) + o(\mathbb{S}^2) = 1 + \operatorname{tr}(S) + o(\mathbb{S}^2).$$

And for $\mathbb{S} \in \mathbb{R}^{n\times n}$, one can see that:

$$\det(\mathbb{I}+\mathbb{S}) = \begin{pmatrix} 1+s_{11} & s_{12} & \dots & s_{1n} \\ s_{21} & 1+s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \dots & 1+s_{nn} \end{pmatrix} = (1+s_{11})(1+s_{22})\cdot\ldots\cdot(1+s_{nn}) + o(\mathbb{S}^2) =$$

$$= 1 + s_{11} + s_{22} + \ldots + s_{nn} + o(\mathbb{S}^2) = 1 + \operatorname{tr}\mathbb{S} + o(\mathbb{S}^2).$$

$$? = \lim_{n\to\infty} \left(1 + \frac{\operatorname{tr}\mathbb{A}}{n} + \ldots\right)^n = e^{\operatorname{tr}\mathbb{A}}.$$

□

Tvrzení 0.1

$$\det(\mathbb{I}+\mathbb{S}) = 1 + \operatorname{tr}\mathbb{S} + \ldots$$

Definice 0.3 (Gateaux derivative)

$$D\mathbf{f}(\mathbf{x})[\mathbf{y}] := \frac{d}{d\tau}\mathbf{f}(\mathbf{x}+\tau\mathbf{y})|_{\tau=0}.$$

Definice 0.4 (Fréchet derivative)

$\mathbf{f} : U \to V$:

$$\lim_{\|\mathbf{y}\|_U \to 0} \frac{\|\mathbf{f}(\mathbf{x}+\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})[\mathbf{y}]\|_V}{\|\mathbf{y}\|_V} = 0.$$

Poznámka

Sometimes we write $\nabla f(\mathbf{x}) \cdot \mathbf{y}$ instead of $Df(\mathbf{x})[\mathbf{y}]$ (from Riesz representation theorem).

For matrices ($\varphi : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{R}$):

$$\frac{\|\varphi(\mathbb{A}+\mathbb{B}) - \varphi(\mathbb{A}) - D\varphi(\mathbb{A})[\mathbb{B}]\|_{\mathbb{R}}}{\|\mathbb{B}\|_{\mathbb{R}^{3\times 3}}}.$$

Poznámka
We write $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A}) : \mathbb{B}$ instead of $D\varphi(\mathbb{A})[\mathbb{B}]$, where $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A})$ is right matrix. Warning $\frac{\partial\varphi}{\partial\mathbb{A}}(\mathbb{A}) \neq D\varphi(\mathbb{A})$, because of transposition ($\mathbb{A} : \mathbb{B} = \operatorname{tr}(\mathbb{A}\mathbb{B}^T) = \operatorname{tr}(\mathbb{A}^T\mathbb{B})$).

Příklad

$$\frac{\partial \operatorname{tr}\mathbb{A}}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\operatorname{tr}(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\operatorname{tr}\mathbb{A} + \tau\operatorname{tr}\mathbb{B}\right)|_{\tau=0} = \operatorname{tr}\mathbb{B} = \mathbb{I} : \mathbb{B}.$$

So $\frac{\partial \operatorname{tr}\mathbb{A}}{\partial\mathbb{A}} = \mathbb{I}$.

Příklad

$$\frac{\partial \det\mathbb{A}}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}(\det(\mathbb{A}+\tau\mathbb{B}))|_{\tau=0} = \frac{d}{d\tau}\left(\det(\mathbb{A})\cdot\det\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left((\det\mathbb{A})\cdot\left(1+\tau\operatorname{tr}(\mathbb{A}^{-1}\mathbb{B}) + o(\tau^2)\right)\right)|_{\tau=0} = (\det\mathbb{A})\operatorname{tr}\left(\mathbb{A}^{-1}\mathbb{B}\right) =$$

$$= (\det\mathbb{A})\operatorname{tr}\left(\left(\mathbb{A}^{-T}\right)^T\mathbb{B}\right) = \left((\det\mathbb{A})\mathbb{A}^{-T}\right) : \mathbb{B}.$$

So $\frac{\partial\det\mathbb{A}}{\partial\mathbb{A}} = (\det\mathbb{A})\mathbb{A}^{-T} = \operatorname{cof}(\mathbb{A})$.

Příklad
$\mathbb{A} : \mathbb{R} \to \mathbb{R}^{3\times 3}$.

$$\frac{d}{dt}(\det\mathbb{A}(t)) = (\det\mathbb{A}(t))\operatorname{tr}\left(\mathbb{A}(t)^{-1}\frac{d\mathbb{A}(t)}{dt}\right).$$

Příklad
$\mathbb{F} : \mathbb{A} \in \mathbb{R}^{3\times 3} \to \mathbb{F}(\mathbb{A}) \in \mathbb{R}^{3\times 3}$. $\mathbb{F}(\mathbb{A}) = \mathbb{A}^{-1}$. (We know $\frac{1}{1+x} = 1 - x + \ldots$.)

$$\frac{\partial\mathbb{F}(\mathbb{A})}{\partial\mathbb{A}}(\mathbb{A})[\mathbb{B}] = \frac{d}{d\tau}\left((\mathbb{A}+\tau\mathbb{B})^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{A}\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)\right)^{-1}\right)|_{\tau=0} =$$

$$= \frac{d}{d\tau}\left(\left(\mathbb{I}+\tau\mathbb{A}^{-1}\mathbb{B}\right)^{-1}\mathbb{A}^{-1}\right)|_{\tau=0} = \frac{d}{d\tau}\left(\left(\mathbb{I}-\tau\mathbb{A}^{-1}\mathbb{B}+\ldots\right)\mathbb{A}^{-1}\right)|_{\tau=0} = -\mathbb{A}^{-1}\mathbb{B}\mathbb{A}^{-1}.$$

So we have $\frac{\partial(\mathbb{A}^{-1})_{ij}}{\partial(\mathbb{A})_{kl}}(\mathbb{B})_{kl}$.

From chain rule (but this is easily solvable by differentiating $\mathbb{A}^{-1}(t)\mathbb{A}(t) = \mathbb{I}$):

$$\frac{d}{dt}\left(\mathbb{A}^{-1}\right) = -\mathbb{A}^{-1}\frac{d\mathbb{A}}{dt}\mathbb{A}^{-1}.$$

Příklad
$\mathbb{F}(\mathbb{A}) = e^{\mathbb{A}}$

$$\frac{\partial e^{\mathbb{A}}}{\partial \mathbb{A}}[\mathbb{B}] = \frac{d}{d\tau}(e^{\mathbb{A}+\tau\mathbb{B}})|_{\tau=0} = \frac{d}{d\tau}\left(\mathbb{I} + \frac{\mathbb{A}+\tau\mathbb{B}}{1!} + \frac{(\mathbb{A}+\tau\mathbb{B})^2}{2!}\right)|_{\tau=0}.$$

Věta 0.2 (Daleckii–Krein)

$\mathbb{A}$ real symmetric matrix, $\mathbb{A} \in \mathbb{R}^{k\times k}$, $\mathbb{A} = \sum_{i=1}^{k} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$, where λ_i are eigenvalues and $\mathbf{v}_i$ are normalised orthogonal ($\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}$) eigenvectors.

f continuously differentiable real function defined on open set containing the spectrum of $\mathbb{A}$

$$\mathbb{F}(\mathbb{A}) := \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =: \sum_{i=1}^{k} f(\lambda_i)\mathbb{P}_i.$$

Then the formula for the Gateaux derivative of f at point $\mathbb{A}$ in direction $\mathbb{X}$ reads

$$D\mathbb{F}(\mathbb{A})[\mathbb{X}] = \frac{\partial \mathbb{F}}{\partial \mathbb{A}}[\mathbb{X}] = \sum_{i=1}^{k} \frac{df}{d\lambda}|_{\lambda=\lambda_i}\mathbb{P}_i\mathbb{X}\mathbb{P}_i + \sum_{i=1}^{k}\sum_{j=1,j\neq i}^{k} \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}\mathbb{P}_i\mathbb{X}\mathbb{P}_j.$$

Sometimes we write $D\mathbb{F}(\mathbb{A})[\mathbb{X}] = f^{[1]}(\mathbb{A}) \bigcirc \mathbb{X}$ (Schur product of matrices, it is point-wise multiplication). Then

$$[f^{[1]}(\mathbb{A})]_{ij} = \begin{cases} \frac{df}{d\lambda}|_{\lambda=\lambda_i}, & i = j, \\ \frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, & i \neq j. \end{cases}$$

Důkaz
No summation conventions, all sums are stated explicitly!

$$\mathbb{F}(\mathbb{A}) = \sum_{i=1}^{k} f(\lambda_i)\mathbf{v}_i \otimes \mathbf{v}_i =$$

$$= \sum_{i=1}^{k} f(\lambda_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots))\mathbf{v}_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots) \otimes \mathbf{v}_i(a_{11}, a_{12}, \ldots, a_{21}, \ldots).$$

$$\frac{\partial \mathbb{F}(\mathbb{A})}{\partial \mathbb{A}} = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial \lambda_i} \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i \otimes \mathbf{v}_i + f(\lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \otimes \mathbf{v}_i + f(\lambda_i)\mathbf{v}_i \otimes \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \right) =?.$$

We derivate $\mathbb{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$:

$$\frac{\partial \mathbb{A}}{\partial \mathbb{A}} \mathbf{v}_i + \mathbb{A} \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} = \frac{\partial \lambda_i}{\partial \mathbb{A}} \mathbf{v}_i + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}}.$$

We multiply (with dot product) it by $\mathbf{v}_i$:

$$\mathbb{P}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbb{A}^T \mathbf{v}_i = \frac{\partial \lambda_i}{\partial \mathbb{A}} \cdot 1 + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \mathbb{A} \cdot \mathbf{v}_i.$$

$$\frac{\partial \lambda_i}{\partial \mathbb{A}} = \mathbb{P}_i = \mathbf{v}_i \otimes \mathbf{v}_i.$$

We again multiply derivative of $\mathbb{A}|\mathbf{v}_i = \lambda \mathbf{v}_i$, but this time by $\mathbf{v}_j$:

$$\mathbf{v}_j \otimes \mathbf{v}_i + \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \lambda_j \mathbf{v}_j = 0 + \lambda_i \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j.$$

$$(\lambda_j - \lambda_i) \frac{\partial \mathbf{v}_i}{\partial \mathbb{A}} \cdot \mathbf{v}_j = -\mathbf{v}_j \otimes \mathbf{v}_i.$$

We also need $(\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}_{ij} = \ldots = \mathbb{P}_i \mathbb{X} \mathbb{P}_j$:

$$\ldots = (\mathbf{v}_j \otimes \mathbf{v}_i)(\mathbf{v}_i \cdot \mathbb{X}\mathbf{v}_j) = (\mathbf{v}_j \otimes \mathbf{v}_i)\mathbb{X}(\mathbf{v}_j \otimes \mathbf{v}_j).$$

□

TODO!!!

1 Kinematics

Definice 1.1

We have some abstract body with point P. We can look at it in reference configuration (some point in past), where $K_0(P) = \mathbf{X}$ (K_0 = placer), $t = t_0$. Or in current configuration (how it is situated now), where $K_t(P) = \mathbf{x}$.

The change of configuration, χ in $\mathbf{x} = \chi(\mathbf{X}, t)$ is called deformation (but it contains translation and rotation too!).

Definice 1.2

Let us consider quantity θ that describes the given material point. We can describe it by:

- $\theta(P, t)$;
- $\hat{\theta}(\mathbf{X}, t)$ (referential/Lagrangian description, commonly used for solids because deformation is with respect to reference configuration);
- $\tilde{\theta}(\mathbf{x}, t)$ (spatial/Eulerian description, commonly used for fluids because velocity is time-local property).

But people write those functions without ˆ or ˜

Poznámka

$$\tilde{\theta}(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \hat{\theta}(\mathbf{X}, t).$$

Definice 1.3 (Deformation gradient)

$$d\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 = \chi(\mathbf{X}_2, t) - \chi(\mathbf{X}_1, t) =$$
$$= \chi(\mathbf{X}_1 + d\mathbf{X}, t) - \chi(\mathbf{X}_1, t) = \chi(\mathbf{X}_1, t) + \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X} + \ldots - \chi(\mathbf{X}_1, t) = \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}.$$

$$\mathbb{F}(\mathbf{X}, t) := \frac{\partial \chi}{\mathbf{X}}(\mathbf{X}_1, t)d\mathbf{X}. \qquad d\mathbf{x} = \mathbb{F}d\mathbf{X}$$

Poznámka
It can be derived by derivatives on curves (see lecture).

Důsledek
Transformation of infinitesimal line segment: $d\mathbf{x} = \mathbb{F}d\mathbf{X}$.

Transformation of infinitesimal surface elements: $d\mathbf{s} = (\det \mathbb{F})\mathbb{F}^{-T}d\mathbf{S} = \operatorname{cof} \mathbb{F}d\mathbf{S}$.

Transformation of infinitesimal volume: $dv = (\det \mathbb{F})dV$.

Důsledek (In tangent spaces)

$$F(\mathbf{X}, t_0) = f(\chi(\mathbf{X}, t), t).$$

Representation theorem:

$$(GradF)\mathbf{W} = \mathbf{U}_{GradF} \cdot \mathbf{W}$$

$$(Gradf)\mathbf{w} = \mathbf{u}_{Gradf} \cdot \mathbf{w}$$

$f(\chi(\mathbf{X}, t), t) = F(\mathbf{X}, t_0)$

$$Gradf(\mathbf{x}, t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = GradF(\mathbf{X}, t_0)$$

$$\mathbf{U}_{GradF} \cdot \mathbf{W} = (GradF)\mathbf{W} = (Gradf)\mathbb{F}\mathbf{W} = (gradf)(\mathbb{F}\mathbf{W}) = \mathbf{u}_{gradf} \cdot \mathbb{F}\mathbf{W} = \mathbb{F}^T\mathbf{u}_{Gradf} \cdot \mathbf{W}.$$

$\mathbf{u}_{gradf} = \mathbb{F}^{-T}\mathbf{U}_{GradF}$.

Příklad (Hollow cylinder)
$r = f(R)$, $\varphi = \Phi$, $z = Z$.

Řešení

$$\mathbb{F} = \frac{\partial \chi_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{E}_j$$

$$X_1 = R\cos\Phi, \qquad X_2 = R\sin\Phi, x_1 = r\cos\Phi, x_2 = r\sin\Phi.$$

$$x_1 = \chi_1(X_1, X_2, t), \qquad x_2 = \chi(x_1, x_2, t), x_i = \chi_i(X_j, t).$$

By chain rule:

$$\frac{\partial x_1}{\partial X_2} = \frac{\partial r\cos\Phi}{\partial\partial X_2} = \frac{\partial}{\partial X_2}f(R)\cos\Phi.$$

$$\mathbb{F} = F_{rR}\mathbf{e}_r \otimes \mathbf{E}_R + F_{r\Phi}\mathbf{e}_r \otimes \mathbf{E}_\Phi + \dots$$

Řešení
From image:

$$\mathbf{E}_R \xrightarrow{\mathbb{F}} F_{rR}\mathbf{e}_r.$$

$$\mathbf{E}_\Phi \xrightarrow{\mathbb{F}} F_{\varphi\Phi}\mathbf{e}_\varphi$$

So $\mathbb{F} = \begin{pmatrix} F_{rR} & 0 \\ 0 & F_{\varphi\Phi} \end{pmatrix}$

TODO? (Solution by curve)

Poznámka
How to differentiate in time tensorial quantities related to the current configuration?

Upper convected derivative:

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} = \det\mathbb{F}(\mathbf{X},t)\left[\frac{d}{dt}\left(\mathbb{F}^{-1}(\mathbf{X},t)\mathbb{A}(\chi(\mathbf{X},t),t)\mathbb{F}^{-T}(\mathbf{X},t)\right)\right]\mathbb{F}^{T}(\mathbf{X},t).$$

1.1 Derivatives

Definice 1.4 (Lagragian velocity)

$$\mathbf{V}(\mathbf{X},t) = \frac{d\chi(\mathbf{X},t)}{dt}.$$

$$\mathbf{v}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}$$

Definice 1.5 (Eulerian velocity)

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X},t)|_{\mathbf{X}=\chi^{-1}(\mathbf{x},t)}.$$

Definice 1.6 (Material time derivative)

$\frac{d}{dt}$ = keep $\mathbf{X}$ fixed, and differentiate with respect to time.

$$\psi(\mathbf{X},t)\to\frac{d}{dt}\psi(\mathbf{X},t) = \frac{\partial\psi}{\partial t}(\mathbf{X},t)$$

$$\begin{aligned}\psi(\mathbf{x},t)\to\frac{d}{dt}\psi(\chi(\mathbf{X},t),t) &= \frac{\partial\psi}{\partial t}|_{\mathbf{x}=\chi(\mathbf{X},t)} + \frac{\partial\psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\frac{d\chi_i}{dt}(\mathbf{X},t) = \\ &= \left(\frac{\partial\psi}{\partial t}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} + V_i(\mathbf{X},t)\frac{\partial\psi}{\partial x_i}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\right) = \\ &= \left(\frac{\partial\psi}{\partial t}(\mathbf{x},t) + v_i(\mathbf{x},t)\frac{\partial\psi}{\partial x_i}(\mathbf{x},t)\right)|_{\mathbf{x}=\chi(\mathbf{x},t)}\end{aligned}$$

$$\frac{d}{dt}\psi(\mathbf{x},t) = \frac{\partial\psi}{\partial t}(\mathbf{x},t) + (\mathbf{v}(\mathbf{x},t)\cdot\nabla)\psi(\mathbf{x},t).$$

Definice 1.7 (Time derivative of deformation gradient $\mathbb{F}$)

$$\frac{d}{dt}\mathbb{F}(\mathbf{X},t) = \frac{d}{dt}\left(\frac{\partial\chi(\mathbf{X},t)}{\partial\mathbf{X}}\right) = \frac{\partial}{\partial\mathbf{X}}\frac{d\chi(\mathbf{X},t)}{dt} = \frac{\partial}{\partial\mathbf{X}}\mathbf{V}(\mathbf{X},t) =$$

$$= \frac{\partial}{\partial \mathbf{X}} \mathbf{v}(\chi(\mathbf{X},t),t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \frac{\partial \chi}{\partial \mathbf{X}}(\mathbf{X},t) = \frac{\partial \mathbf{v}}{\partial x}|_{\mathbf{x}=\chi(\mathbf{X},t)} \mathbb{F}(\mathbf{X},t).$$

$$\mathbb{L}(\mathbf{x},t) := \nabla \mathbf{V}(\mathbf{x},t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\mathbf{x},t).$$

Důsledek

$$\frac{d\mathbb{F}}{dt} = \mathbb{L}\mathbb{F}$$

Důsledek

$$\overset{\triangledown}{\mathbb{A}} = \frac{d\mathbb{A}}{dt} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^T$$

TODO!!!

Poznámka (Balance laws in Eulerian description (revision, the last lecture))

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0;$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \mathbb{T} = \mathbb{T}^T;$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q;$$

or

$$\varrho \frac{d}{dt}(e + \frac{1}{2}\mathbf{v} \cdot \mathbf{v}) = \operatorname{div}(\mathbb{T}^T \mathbf{v}) + \varrho \mathbf{b} \cdot \mathbf{v} - \operatorname{div} \mathbf{j}_q.$$

Poznámka (Balance laws in Lagrangian description)
Starting with $\frac{d}{dt}_{V(t)} \varrho(\mathbf{x},t) dv = \frac{d}{dt} m_{V(t)} = 0$ i.e. mass remains same: $m_{V(t)} = m_{V(t_0)}$). We integrate over volume:

$$\int_{V(t_0)} \varrho_R(\mathbf{X}) dV = \int_{V(t)} \varrho(\mathbf{x},t) dv = \int_{V(t_0)} \varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} dV.$$

Localization principle:

$$\varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} = \varrho_R(\mathbf{X}).$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} + \int_{V(t)} \varrho \mathbf{b} dv.$$

$$\int_{V(t)} \operatorname{div} \mathbb{T} d\mathbf{v} = \int_{\partial V(t)} \mathbb{T}\mathbf{n} \, ds = \int_{\partial V(t_0)} \mathbb{T}(\det \mathbb{F}) \mathbb{F}^{-T} \mathbf{N} \, dS \overset{\text{Stokes}}{\int}_{V(t_0)} \operatorname{div}_{\mathbf{X}}((det \mathbb{F}) \mathbb{T} \mathbb{F}^{-T}) dV =: \int_{V(t_0)} (\mathbb{T}_R) dV.$$

$$\mathbb{T}_R(\mathbf{X},t) := (\det \mathbb{F}(\mathbf{X},t))\mathbb{T}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbb{F}^{-T}(\mathbf{X},t)$$

is first Piola–Kirchhoff stress tensor. Cauchy ($\mathbb{T}$) is current → current. P–K ($\mathbb{T}_R$) is reference → current.

$$\int_{\partial V(t)} dv \to \int_{\partial V(t_0)} (\operatorname{div}_{\mathbf{X}} \mathbb{T}_R) dV.$$

$$\int_{V(t)} \varrho \mathbf{b} dv = \int_{V(t_0)} \varrho(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)}\mathbf{b}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R(\mathbf{X})\mathbf{b} dV.$$

$$\int_{V(t)} \varrho \frac{d\mathbf{v}}{dt} dv = \int_{V(t_0)} \varrho \frac{\partial^2 \chi}{\partial t^2}(\mathbf{X},t) \det \mathbb{F} dV = \int_{V(t_0)} \varrho_R \frac{\partial^2 \chi}{\partial t^2} dV.$$

Altogether:

$$\varrho_R \frac{\partial^2 \chi}{\partial t^2} = \operatorname{div}_{\mathbf{X}} \mathbb{T}_R + \varrho_R \mathbf{b} \qquad \text{(Solve for } \chi\text{)}.$$

$\mathbb{T} = \mathbb{T}^T \to \mathbb{T}_R \mathbb{F}^T = \mathbb{F}\mathbb{T}_R^T$ (P–K is not symmetric!).

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{L} - \operatorname{div} \mathbf{j}_q \to \int_{V(t)} \varrho \frac{de}{dt} dv = \int_{V(t)} \mathbb{T} : \mathbb{L} dv - \int_{V(t)} \operatorname{div} \mathbf{j}_q dv.$$

$$\int_{V(t)} \operatorname{div} \mathbf{j}_q dv = \int_{\partial V(t)} \mathbf{j}_q \cdot \mathbf{n} ds = \int_{\partial V(t_0)} \mathbf{j}_q(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} \cdot \det \mathbb{F}(\mathbf{X},t)\mathbb{F}^{-T}(\mathbf{X},t)\mathbf{N} dS =$$

$$= \int_{\partial V(t_0)} (\det \mathbb{F}(\mathbf{X},t)\mathbb{F}^{-1}(\mathbf{X},t)\mathbf{j}_q(\mathbf{x},t)) \cdot \mathbf{N} dS =$$

$$= \int_{V(t_0)} \operatorname{div}((\det \mathbb{F})\mathbb{F}^{-1}\mathbf{j}_q) dV.$$

$\mathbf{J}_q = (\det \mathbb{F})\mathbb{F}^{-1}\mathbf{j}_q$ is called referential heat flux. (It cannot be given by Fouriers law ($\mathbf{j}_q = k\nabla_{\mathbf{x}}\theta$, $\operatorname{div} \mathbf{j}_q = \operatorname{div}(k\nabla\theta)$).)

$$\int_{V(t)} \underbrace{\mathbb{T} : \overbrace{\mathbb{L}}^{\nabla_{\mathbf{x}}\mathbf{v}}}_{\operatorname{tr}(\mathbb{T}\mathbb{L}^T)=\operatorname{tr}(\mathbb{L}\mathbb{T}^T)} dv = \int_{V(t_0)} (\det \mathbb{F})\mathbb{T} : \mathbb{L} dV =$$

$$= \int_{V(t_0)} \operatorname{tr}((\det \mathbb{F})\mathbb{T}\mathbb{L}^T) dV = \int_{V(t_0)} \operatorname{tr}\left((\det \mathbb{F})\mathbb{T}\mathbb{F}^{-T}\left(\frac{d\mathbb{F}}{dt}\right)^T\right) dV = \int_{V(t_0)} \mathbb{T}_R : \dot{\mathbb{F}} dV.$$

Altogether

$$\varrho_R \frac{\partial e}{\partial t} = \mathbb{T}_R : \dot{\mathbb{F}} - \operatorname{div}_{\mathbf{X}} \mathbf{J}_q.$$

2 Entropy

Poznámka (Objective)
Find quantity that is increasing/decreasing in time.

Poznámka (With no interior)

$$\varrho\frac{d}{dt}\left(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)=\operatorname{div}\mathbb{T}+\varrho\mathbf{b}-\operatorname{div}\mathbf{j}_q=\operatorname{div}\mathbb{T}+0+\operatorname{div}(k\nabla\theta).$$

Let us work with[a] $\operatorname{div}\mathbb{T}=-p_{th}\mathbb{I}+\tilde{\lambda}(\operatorname{div}\mathbf{v})+2\mu\mathbb{D}_\delta$, and assume that $\mathbb{T}=-p_{th}(\varrho,\theta)\mathbb{I}+\tilde{\lambda}(\operatorname{div}\mathbf{v})\mathbb{I}+2\mu\mathbb{D}_\delta$ (from $\frac{pV}{T}=\text{const}$).

$$\varrho\frac{d}{dt}(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v})=\operatorname{div}(\mathbb{T}^T\mathbf{v})-\operatorname{div}\mathbf{j}_q.$$

$$\frac{d}{dt}\int_{V(t)}\varrho(e+\frac{1}{2}\mathbf{v}\cdot\mathbf{v})dv=\int_{\partial V(t)}\mathbb{T}^T\mathbf{v}\cdot\mathbf{n}ds-\int_{\partial V(t)}\mathbf{j}_q\cdot\mathbf{n}ds.$$

The first part is work and it is zero if we have boundary condition $\mathbf{v}|_{\partial V}=0$. The second part is heat exchange which is zero if we have boundary condition $\mathbf{j}_q\cdot\mathbf{n}|_{\partial V}=0$. Both boundary conditions together are math way to say system with *no interactions*.

$\varrho,\theta,\mathbf{v}\to\varrho,e,\theta.$

Assume $\eta=\eta(\varrho,e)\to e=e(\eta,\varrho)$. (We will write $e=e(\eta,\varrho)=e(\eta(\mathbf{x},t),\varrho(\mathbf{x},t))=e(\mathbf{x},t)$.)

We have 1. Balance of internal energy

$$\varrho\frac{de}{dt}=\mathbb{T}:\mathbb{D}-\operatorname{div}\mathbf{j}_q;$$

2. Chain rule

$$\varrho\frac{de}{dt}=\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt}+\frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt}.$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt}=\mathbb{T}:\mathbb{D}-\operatorname{div}\mathbf{j}_q-\frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt}=\mathbb{T}:\mathbb{D}-\operatorname{div}\mathbf{j}_q+\frac{\partial e}{\partial\varrho}(\eta,\varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt}=(-p_{th}\mathbb{I}+\tilde{(}\operatorname{div}\mathbf{v})\mathbb{I}+2\mu\mathbb{D}_\delta):\mathbb{D}-\operatorname{div}\mathbf{j}_q+\frac{\partial e}{\partial\varrho}(\eta,\varrho)\varrho\operatorname{div}\mathbf{v},$$

$$\varrho\frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{d\eta}{dt}=\left(-p_{th}+\varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)\operatorname{div}\mathbf{v}+\tilde{\lambda}(\operatorname{div}\mathbf{v})^2+2\mu\mathbb{D}_\delta:\mathbb{D}_\delta-\operatorname{div}\mathbf{j}_q,$$

$$\varrho\frac{d\eta}{dt}=\frac{\left(-p_{th}+\varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial\eta}}\operatorname{div}\mathbf{v}-\frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial\eta}}+\frac{\tilde{\lambda}(\operatorname{div}\mathbf{v})^2+2\mu|\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial\eta}}.$$

There is no chance that this could be positive. (Its obvious, because the value can flow, so point-wise $\geqslant 0$ is lost case.) But we can integrate over volume. Thus instead of $\frac{d\eta}{dt}\geqslant 0$

we want just $\frac{d}{dt}\int_{V(t)} \varrho\eta dv \geqslant 0$.

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial \varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial \eta}} dv + \int_{V(t)} \frac{\tilde{\lambda}(\operatorname{div}\mathbf{v})^2 + 2\mu|\mathbb{D}_\delta|^2}{\frac{\partial e}{\partial \eta}} dv.$$

The third integral OK, if $\frac{\partial e}{\partial \eta} > 0$.

$$\operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}}\right) = \frac{\operatorname{div}\mathbf{j}_q}{\frac{\partial e}{\partial \eta}} + \nabla\left(\frac{1}{\frac{\partial e}{\partial \eta}}\right)\cdot\mathbf{j}_q.$$

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial \varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \operatorname{div}\left(\frac{\mathbf{j}_q}{\frac{\partial e}{\partial \eta}}\right) dv + \int_{V(t)} \nabla\left(\frac{1}{\frac{\partial e}{\partial \eta}}\right)\cdot\mathbf{j}_q dv + REST.$$

The second integral is zero from Stokes and boundary condition $\mathbf{j}_q\cdot\mathbf{n}|_{\partial V} = 0$. On the third integral, we can use derivative of inverse value:

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv = \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial \varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div}\mathbf{v} dv - \int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial \eta}\right)\cdot\mathbf{j}_q}{\left(\frac{\partial e}{\partial \eta}\right)^2} dv + REST =$$

$$= \int_{V(t)} \frac{\left(-p_{th} + \varrho\frac{\partial e}{\partial \varrho}(\eta,\varrho)\right)}{\frac{\partial e}{\partial \eta}} \operatorname{div}\mathbf{v} dv + k\int_{V(t)} \frac{\nabla\left(\frac{\partial e}{\partial \eta}\right)\cdot\nabla\theta}{\left(\frac{\partial e}{\partial \eta}\right)^2} dv + REST.$$

If we set $\frac{\partial e}{\partial \eta}(\eta,\varrho) = \theta$, the second integral is non-negative. Moreover, for $\theta \geqslant 0$ we satisfy the assumption for the "first third integral". Moreover if we enforce $\varrho^2\frac{\partial e}{\partial \varrho}(\varrho,\eta) = p_{th}(\theta,\varrho)$, the first integral is zero, so we win.

[a] $\mathbb{D}_\delta := \mathbb{D} - \frac{1}{3}(\operatorname{tr}\mathbb{D})\mathbb{I}$. (Traceless part of $\mathbb{D}$.)

Poznámka
Volíme si tedy $\tilde{\lambda}, \mu > 0$.

Důsledek

$$\frac{d}{dt}\int_{V(t)} \varrho\eta dv \geqslant 0$$

is granted for quantity that solves equations

$$e = e(\eta,\varrho), \qquad \frac{\partial e}{\partial \eta} = \theta, \qquad \varrho^2\frac{de}{d\varrho} = p_{th}(\theta,\varrho).$$

Příklad

$$p_{th}(\theta,\varrho) = c_V(\gamma-1)\varrho\theta, \qquad e(\theta,\varrho) = c_V\theta.$$

Poznámka (?)

1. Energy is constant.
2. Energy is function of entropy and volume.
3. Entropy increases.

Poznámka
$e = e(\eta,\varrho)$ is given $\to$ we know everything $\theta = \frac{\partial e}{\partial \eta}(\eta,\varrho)$, $p_{th} = \varrho^2 \frac{\partial e}{\partial \varrho}(\eta,\varrho)$. (Warning: $e = e(\varrho,\theta)$ is not enough!)

Poznámka
Is there a better function that will allow us to do something like this?

Definice 2.1 (Helmholtz free energy density)

$$\psi(\theta,\varrho) := e(\eta,\varrho)|_{\eta=\eta(\theta,\varrho)} - \theta\eta|_{\eta=\eta(\theta,\varrho)}.$$

Poznámka
This is le Legendre transformation of internal energy.

Důsledek

$$\frac{\partial \psi}{\partial \theta}(\theta,\varrho) = -\eta, \qquad \frac{\partial \psi}{\partial \varrho}(\theta,\varrho) = \frac{\partial e}{\partial \varrho}(\eta,\varrho)|_{\eta=\eta(\theta,\varrho)} \quad \left(= \frac{p_{th}}{\varrho^2}\right).$$

Důkaz

$$\frac{\partial \psi}{\partial \theta}(\theta,\varrho) = \frac{\partial e(\eta,\varrho)}{\partial \eta}|_{\eta=\eta(\theta,\varrho)} \frac{\partial \eta}{\partial \theta}(\theta,\varrho) - \eta|_{\eta=\eta(\theta,\varrho)} - \theta\frac{\partial \eta}{\partial \theta}(\theta,\varrho) = -\eta|_{\eta=\eta(\theta,\varrho)}.$$

$$\frac{\partial \psi}{\partial \varrho}(\theta,\varrho) = \frac{\partial e}{\partial \eta}(\eta,\theta)\frac{\partial \eta}{\partial \varrho}(\theta,\varrho) + \frac{\partial e}{\partial \varrho}(\eta,\varrho)|_{\eta=\eta(\theta,\varrho)} - \theta\frac{\partial \eta}{\partial \varrho}(\theta,\varrho) = \frac{\partial e}{\partial \varrho}(\eta,\varrho)|_{\eta=\eta(\theta,\varrho)}.$$

□

Poznámka (Why do we call $c_{V,ref}$ the specific heat at constant volume?)
(Constant volume = constant density.) $\mathbf{j}_q = -k\nabla\theta$, $\mathbb{T} \approx -p_{th}\mathbb{I}$, $\mathbb{D} = \frac{1}{2}\left((\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\right)$.

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q,$$

$$\varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt} + \varrho\frac{\partial e}{\partial\varrho}\frac{d\varrho}{dt} = -p_{th}(\underbrace{\mathbf{D}iv\mathbf{v}}_{\frac{d\varrho}{dt}+\varrho\operatorname{div}\mathbf{v}=0}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt} + 0 = 0 + \operatorname{div}(k\nabla\theta),$$

$$\int_V \varrho\frac{\partial e}{\partial\theta}(\theta,\varrho)\frac{d\theta}{dt}dv = \int_{\partial V}(k\nabla\theta)\mathbf{n}ds.$$

So in left we multiply ϱ, difference of temperature and some $c_V(\theta,\varrho) := \frac{\partial e}{\partial\theta}(\theta,\varrho)$. (On the right there is flow of heat, $\mathbf{j}_q$, through boundary).

For calorically perfect ideal gas: $e = e(\varrho,\theta) = c_{V,ref}\cdot\theta$.

Poznámka (How to get specific heat at constant pressure?)

$$e = e(\eta(\theta,p_{th}),\varrho(\theta,p_{th})), \qquad \varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div}\mathbf{j}_q = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta).$$

Chain rule:

$$\frac{de}{dt} = \frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt} + \dots\frac{dp_{th}}{dt} = \theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt}.$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \varrho\frac{\partial e}{\partial\varrho}(\eta,\varrho)\frac{d\varrho}{dt} = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} + \varrho\frac{p_{th}}{\varrho^2}(\varrho\cdot(\operatorname{div}\mathbf{v})) = -p_{th}\cdot(\operatorname{div}\mathbf{v}) + \operatorname{div}(k\nabla\theta),$$

$$\varrho\theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})\frac{d\theta}{dt} = \operatorname{div}(k\nabla\theta).$$

So $c_p(\theta,p_{th}) := \theta\frac{\partial\eta}{\partial\theta}(\theta,p_{th})$ is specific heat at constant pressure.

Poznámka (Alternative formula for the specific heat at constant volume)
Chain rule:

$$\theta\frac{\partial\eta}{\partial\theta}(\theta,\varrho) = \frac{\partial e}{\partial\eta}(\eta,\varrho)\frac{\partial\eta}{\partial\theta}(\theta,\varrho) = \frac{\partial}{\partial\theta}e(\theta,\varrho) = c_V(\theta,\varrho).$$

$$c_V(\theta,\varrho) := \theta\frac{\partial\eta}{\partial\theta}(\theta,\varrho).$$

Poznámka (Another alternative formula for the specific heat at constant volume, for usage in practice)

$$c_V(\theta, \varrho) = \theta \frac{\partial \eta}{\partial \theta}(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho),$$

because $\eta(\theta, \varrho) = -\frac{\partial \psi}{\partial \theta}(\theta, \varrho)$ (property of Helmholtz free energy).

Conclusion: If $\psi(\theta, \varrho)$ is given, then

$$c_V(\theta, \varrho) = -\theta \frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho), \qquad p_{th}(\theta, \varrho) = \varrho^2 \frac{\partial \psi}{\partial \varrho}(\theta, \varrho).$$

Poznámka (Where are my evolution equations?)
Unknowns: $\mathbf{v}, \varrho, \theta$.

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \qquad \mathbb{T} = -p_{th}(\theta, \varrho)\mathbb{I} + \tilde{\lambda}(\operatorname{div} \mathbf{v})\mathbb{I} + 2\mu\mathbb{D}_\delta.$$

Poznámka (Third equation)
TODO!!!

$$\varrho\theta \frac{d\eta}{dt} = \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta.$$

$$\frac{d\eta}{dt} = -\frac{d}{dt}\left(\frac{\partial \psi}{\partial \theta}(\theta, \varrho)\right) = -\frac{\partial^2 \psi}{\partial \theta^2}(\theta, \varrho)\frac{d\theta}{dt} - \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho)\frac{d\varrho}{dt}.$$

$$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\varrho^2\theta \frac{\partial^2 \psi}{\partial \theta \partial \varrho}(\theta, \varrho) \operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta,$$

$$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta}\left(\varrho^2 \frac{\partial \psi}{\partial \varrho}\right) \operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta.$$

$$\theta c_V(\theta, \varrho)\frac{d\theta}{dt} = -\theta \frac{\partial}{\partial \theta}\left(\varrho^2 \frac{\partial \psi}{\partial \varrho}\right) \operatorname{div} \mathbf{v} + \operatorname{div}(k\nabla\theta) + \tilde{\lambda}(\operatorname{div} \mathbf{v})^2 + 2\mu\mathbb{D}_\delta : \mathbb{D}_\delta.$$

And we can get $c_V(\theta, \varrho)$ and p_{th} from $\psi = \psi(\theta, \varrho)$.

Poznámka (Why do we call γ the adiabatic exponent?)
(This applies only to the ideal gas.)

$$p_{th} = c_{V,ref}(\gamma - 1)\varrho\theta,$$

$$\varrho \frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q = \mathbb{T} : \mathbb{D}, \qquad \mathbb{T} = -p_{th}\mathbb{I},$$

$$\varrho \frac{de}{dt} = -p_{th} \operatorname{div} \mathbf{v},$$

$$e = c_{V,ref} \cdot \theta$$

$$\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt} \quad \Longleftarrow \quad \frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0.$$

$$\frac{dp_{th}}{dt} = c_{V,ref}(\gamma - 1) \frac{d\varrho}{dt} \theta + c_{V,ref}(\gamma - 1) \varrho \frac{d\theta}{dt},$$

$$\left(\varrho c_{V,ref} \frac{d\theta}{dt} = \frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref} \theta \frac{d\varrho}{dt}. \right)$$

$$\frac{1}{\gamma - 1} \frac{dp_{th}}{dt} - c_{V,ref} \theta \frac{d\varrho}{dt} = \frac{p_{th}}{\varrho} \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{c_{V,ref}\theta}{p_{th}} + \frac{1}{\varrho} \right) \frac{d\varrho}{dt},$$

$$\frac{1}{\gamma - 1} \frac{1}{p_{th}} \frac{dp_{th}}{dt} = \left(\frac{1}{\varrho} \left(\frac{1}{\gamma - 1} + 1 \right) \right) \frac{d\varrho}{dt}.$$

$$\left(\frac{c_{V,ref}\theta}{p_{th}} + \frac{1}{\varrho} =? \frac{c_{V,ref}\theta}{c_{V,ref}\theta \varrho (\gamma - 1)}. \right)$$

$$\frac{d}{dt}(L_n p_{th}) = \gamma \frac{d}{dt} L_n \varrho.$$

$$p_{th} = p_{th,ref} \left(\frac{\varrho}{\varrho_{ref}} \right)^{\gamma}.$$

Poznámka (Where is my hyperbolic equation?)
Assuming isentropic process ($\eta = \text{const}$). $\mathbb{T} = -p_{th}\mathbb{I} = -p_{th}(\varrho, \eta)\mathbb{I}$. ($\operatorname{div} -p_{th}\mathbb{I} = -\nabla p_{th}$.)

$\varrho(\mathbf{x}, t) = \hat{\varrho} + \tilde{\varrho}(\mathbf{x}, t) =$ referential density + small perturbations, similarly $\mathbf{v}(\mathbf{x}, t) = \hat{\mathbf{v}} + \tilde{\mathbf{v}}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\mathbf{x}, t)$.

$$\left(\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0, \qquad \varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta). \right)$$

$$\frac{d\varrho}{dt} = \frac{\partial \varrho}{\partial t} + (\mathbf{v} \cdot \nabla)\varrho = \frac{\partial}{\partial t}(\hat{\varrho}\tilde{\varrho}) + (\hat{\mathbf{v}} + \tilde{\mathbf{v}}) \cdot \nabla(\hat{\varrho} + \tilde{\varrho}) \approx \frac{\partial \tilde{\varrho}}{\partial t}.$$

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0 \xrightarrow{\text{Linearize}} \frac{\partial \tilde{\varrho}}{\partial t} + \hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = -\nabla p_{th}(\varrho, \eta) \xrightarrow{\text{Linearize}} \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial \varrho}(\varrho, \eta)|_{\varrho = \hat{\varrho}} \nabla \tilde{\varrho}.$$

Differentiate by time:

$$\frac{\partial^2 \tilde{\varrho}}{\partial t} - \hat{\varrho} \operatorname{div} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} \right) = 0$$

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} - \operatorname{div} \left(\frac{\partial p_{th}}{\partial \varrho} \bigg|_{\varrho=\hat{\varrho}} \nabla \tilde{\varrho} \right) = 0.$$

$$\frac{\partial^2 \tilde{\varrho}}{\partial t^2} = \left(\frac{\partial p_{th}}{\partial \varrho} \bigg|_{\varrho=\hat{\varrho}} \right) \Delta \tilde{\varrho}.$$

(Speed of sound: $c = \sqrt{\frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho}}(\varrho, \eta)}$.)

Poznámka (Stability of test state)
Set

$$\mathbf{v} = \hat{\mathbf{v}} + \tilde{\mathbf{v}}, \quad \hat{\mathbf{v}} = 0, \qquad \theta = \hat{\theta} + \tilde{\theta}, \quad \hat{\theta} \neq \hat{\theta}(\mathbf{x}, t), \qquad \varrho = \hat{\varrho} + \tilde{\varrho}, \quad \hat{\varrho} \neq \hat{\varrho}(\mathbf{x}, t).$$

Is it $\tilde{\mathbf{v}}, \tilde{\varrho}, \tilde{\theta} \to 0$?

$$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0 \longrightarrow \frac{\partial \tilde{\varrho}}{\partial t} + \hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} = 0,$$

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b} = \operatorname{div} \mathbb{T} \longrightarrow \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} = -\frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}} \nabla \tilde{\theta} + \operatorname{div} \left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right),$$

$$\varrho c_V \frac{d\theta}{dt} = -\theta \frac{\partial p_{th}}{\partial \theta}(\theta, \varrho) \operatorname{div} \mathbf{v} + \operatorname{div}(k \nabla \theta) + \tilde{\lambda} (\operatorname{div} \mathbf{v})^2 + 2\mu \mathbb{D}_\delta : \mathbb{D}_\delta = 0 \longrightarrow$$

$$\longrightarrow \hat{\varrho} c_V|_{\varrho=\hat{\varrho},\theta=\hat{\theta}} \frac{\partial \tilde{\theta}}{\partial t} = \operatorname{div}(k \nabla \tilde{\theta}) - \hat{\theta} \frac{\partial p_{th}}{\partial \varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}} \operatorname{div} \tilde{\mathbf{v}}.$$

Test by $\tilde{\mathbf{v}}$:

$$\int_V \hat{\varrho} \frac{\partial \tilde{\mathbf{v}}}{\partial t} \cdot \tilde{\mathbf{v}} = \int_V \left(-\frac{\partial p_{th}}{\partial \varrho} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta} \nabla \tilde{\theta} \right) \cdot \tilde{\mathbf{v}} dv + \int_V \operatorname{div} \left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right) \cdot \tilde{\mathbf{v}} dv =$$

$$= \ldots + \int_V \operatorname{div} \left(\left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}}) \mathbb{I} + 2\mu \tilde{\mathbb{D}}_\delta \right) \tilde{\mathbf{v}} \right) - \int_V \left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \operatorname{div} \mathbf{v} \right) = \ldots + 0 - \ldots.$$

$$\frac{d}{dt} \frac{1}{2} \int_V \hat{\varrho} (\tilde{\mathbf{v}})^2 dv = \int_V \left(-\frac{\partial p_{th}}{\partial \varrho} \nabla \tilde{\varrho} - \frac{\partial p_{th}}{\partial \theta} \nabla \tilde{\theta} \right) \cdot \tilde{\mathbf{v}} dv - \int_V \left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \right) dv.$$

See that viscosity kills kinetic energy. Now we use $\nabla \varphi \cdot \tilde{\mathbf{v}} = \operatorname{div}(\varphi \tilde{\mathbf{v}}) - \varphi \operatorname{div} \tilde{\mathbf{v}}$ and on $\operatorname{div}(\ldots)$ we use Stokes theorem.

$$\frac{d}{dt} \frac{1}{2} \int_V \hat{\varrho} (\tilde{\mathbf{v}})^2 dv = \int_V \frac{\partial p_{th}}{\partial \varrho} \tilde{\varrho} \operatorname{div} \tilde{\mathbf{v}} dv + \int_V \frac{\partial p_{th}}{\partial \theta} \tilde{\theta} \operatorname{div} \tilde{\mathbf{v}} dv - \int_V \left(\tilde{\lambda} (\operatorname{div} \tilde{\mathbf{v}})^2 + 2\mu \tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta \right) dv. (1)$$

$$\frac{\partial \tilde{\varrho}}{\partial t} = -\hat{\varrho} \operatorname{div} \tilde{\mathbf{v}} \qquad / \cdot \frac{\partial p_{th}}{\partial \varrho} \frac{\tilde{\varrho}}{\hat{\varrho}}, \int_V$$

$$\frac{d}{dt}\int_V \frac{1}{2}(\tilde{\varrho})^2 TODO!!!dv = -\int_V \frac{\partial p_{th}}{\partial \varrho}\tilde{\varrho}\operatorname{div}\tilde{\mathbf{v}}dv.(2)$$

$$\hat{\varrho}c_V|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\frac{\partial\tilde{\theta}}{\partial t} = \operatorname{div}(k\nabla\tilde{\theta}) - \hat{\theta}\frac{\partial p_{th}}{\partial\varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}\operatorname{div}\tilde{\mathbf{v}} \qquad /\cdot\frac{\tilde{\theta}}{\hat{\theta}},\int_V$$

$$\frac{d}{dt}\int_V \frac{\hat{\varrho}c_V}{\hat{\theta}}(\tilde{\theta})^2 dv = -\frac{1}{\hat{\theta}}\int_V \nabla\tilde{\theta}\cdot\nabla\tilde{\theta}dv - \int_V \frac{\partial p_{th}}{\partial\varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\operatorname{div}\tilde{\mathbf{v}})\tilde{\theta}dv.(3)$$

(1) + (2) + (3):

$$\frac{d}{dt}\int_V \frac{1}{2}\hat{\varrho}(\tilde{\mathbf{v}})^2 + \frac{1}{2}\frac{1}{\hat{\varrho}}\frac{\partial p_{th}}{\partial\varrho}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\tilde{\varrho})^2 + \frac{\hat{\varrho}c_V}{\hat{\theta}}|_{\varrho=\hat{\varrho},\theta=\hat{\theta}}(\tilde{\theta})^2 =$$

$$= -\frac{1}{\hat{\theta}}\int_V \nabla\tilde{\theta}\nabla\tilde{\theta}dv - \int_V \tilde{\lambda}(\operatorname{div}\tilde{\mathbf{v}})^2 + 2\mu\tilde{\mathbb{D}}_\delta : \tilde{\mathbb{D}}_\delta dv.$$

o

When we rightly choose Helmholtz free energy, $\frac{\partial p_{th}}{\varrho,\theta} > 0$ and c_V will be positive and we win (perturbations go to zero).

Z $c_V > 0$ máme $\frac{\partial^2\psi}{\partial\theta^2} < 0$.

TODO!!!

Poznámka (For Homework)
$\frac{d\mathbb{H}}{dt} \neq \frac{1}{2}\mathbb{B}^{-1}\frac{d\mathbb{B}}{dt}$, because there is no commutativity. We must use $\mathbb{T} : \ldots$.

Poznámka (*)
$\mathbb{T} = \underbrace{-p_{th}(\varrho,\tau)I}_{} + \underbrace{\tilde{\lambda}(\operatorname{div}\mathbf{v}) + 2\mu\mathbb{D}_\delta}_{\text{Why this?}}.$

$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\mathbf{v}).$$

Now we use Galilei principle of relativity „$\mathbf{x} = \mathbf{x} + \mathbf{w}t$“. We see that this don't work. So we use $\nabla\mathbf{v}$ ($\nabla(\mathbf{v}+\mathbf{w}) = \nabla\mathbf{v} + \nabla\mathbf{w} = \nabla\mathbf{v}$):

$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\nabla\mathbf{v}).$$

$\mathbb{T}$ is symmetric matrix, so:

$$\mathbb{T} = -p_{th}(\varrho,\theta)\mathbb{I} + \mathbb{S}(\mathbb{D}).$$

2.1 Representation theorems for isotropic functions

Definice 2.2

We say that $\varphi : \mathbb{R}^{3\times 3} \to \mathbb{R}$ is isotropic $\equiv \varphi(\mathbb{QAQ}^T) = \varphi(\mathbb{A})$ holds for any $\mathbb{Q} \in Orth^+$ (= proper orthogonal matrices, i.e. $\mathbb{QQ}^T = \mathbb{I}$, $\det \mathbb{Q} > 0$).

Například
tr is isotropic, $\mathbb{A} \mapsto a_{11}$ is not isotropic.

We say that $\mathbb{F} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is isotropic $\equiv \mathcal{F}(\mathbb{QAQ}^T) = \mathbb{QF}(\mathbb{A})\mathbb{Q}^T$ holds for any $\mathbb{Q} \in Orth^+$.

Například
id, $^{-1}$, exp, ln, … is isotropic. $\mathbb{A} \mapsto a_{ii}\mathbb{I}$ is not isotropic.

Věta 2.1

$\varphi : Sym(\mathbb{R}^{3\times 3}) \to \mathbb{R}$, φ isotropic $\implies \varphi(\mathbb{A}) = \varphi(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

$f : Sym(\mathbb{R}^{3\times 3}) \to Sym(\mathbb{R}^{3\times 3})$, f isotropic $\implies f(\mathbb{A}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{A} + \alpha_2\mathbb{A}$, where $\alpha_i = \alpha_i(I_1(\mathbb{A}), I_2(\mathbb{A}), I_3(\mathbb{A}))$.

Poznámka
The second one goes from: Assume $f(\mathbb{A}) = \sum_{i=0}^{+\infty} f_i\mathbb{A}^i$. Then by Cayley–Hamilton $f(\mathbb{A}) = \sum_{i=0}^{2} f_i\mathbb{A}^i$. Furthermore, $f_i(\mathbb{A})$ must be isotropic…

Poznámka (Continuation of *)
Isotropic fluid: $\mathbb{S}(\mathbb{D})$ is isotropic function, $\mathbb{S}(\mathbb{D}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{D} + \alpha_2\mathbb{D}^2$. What if we want Linear relation (i.e. $\mathbb{S}(\mathbb{D})$ is linear function of $\mathbb{D}$)? Then $\mathbb{S}(\mathbb{D}) = c_0(\operatorname{tr}\mathbb{D})\mathbb{I} + c_1\mathbb{D}$, where $c_0, c_1 = \text{const}$.

Pozor
This remarks (not theorem) works only for fluids, where 0 velocity means 0 stress.

Poznámka (Governing equations for incompressible isotropic fluids)
TODO?

$$\operatorname{div}\mathbf{v} = 0$$

$$\varrho\frac{d\mathbf{v}}{dt} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}) + \varrho\mathbf{b}$$

$$\varrho c_V\frac{d\theta}{dt} = \dots$$

So

$$\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}, \qquad \operatorname{div}\mathbb{T} = -\nabla p + \operatorname{div}(2\mu\mathbb{D}).$$

The kind of this p is other than p_{th}. This is (artificial) pressure maintaining the incompressibility, and we solve! for it. (p_{th} is function of ϱ and θ). $\mathbb{T}$ is not obtained by simply substitution, but from equations above!

Příklad (Archimedes law)
Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

Poznámka
Fluid = incompressible fluid. So $\mathbb{T} = -p\mathbb{I} + 2\mu\mathbb{D}$.

Poznámka
Weight = body force = gravitation force

Poznámka
It talks about floating, so nothing moves!

Poznámka (Governing equations)

$$\begin{aligned}
\operatorname{div}\mathbf{v} &= 0,\\
\varrho\frac{d\mathbf{v}}{dt} &= \operatorname{div}\mathbb{T} + \varrho\mathbf{b}, \qquad \mathbf{b} = -g\mathbf{e}_{\hat{z}},\\
\mathbb{T} &= -p\mathbb{I} + 2\mu\mathbb{D},\\
\mathbf{F} &= \int_{\partial\mathcal{B}} \mathbb{T}\mathbf{n}\,ds.
\end{aligned}$$

Řešení
1. Solve for $\mathbf{v}$ and p.

2. Evaluate $\mathbf{F}$.

„1." is easy, we have static problem, so $\mathbf{v} = 0$.

$$\varrho \frac{d\mathbf{v}}{dt} = \operatorname{div} \mathbb{T} + \varrho \mathbf{b} \wedge \operatorname{div} \mathbb{T} = -\nabla p + \operatorname{div}(2\mu \mathbb{D}) \implies \implies 0 = -\nabla p - \varrho_{fluid} g \mathbf{e}_{\hat{z}} \implies$$

$$\implies \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right) = (0, 0, -\varrho g) \implies p = -\varrho g z + p_0.$$

„2.":

$$\int_{\partial \mathcal{B}} \mathbb{T} \mathbf{n} ds = \int_{\partial \mathcal{B}} -p \mathbb{I} \mathbf{n} ds = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z - p_0) = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{n}) ds - p_0 \int_{\partial \mathcal{B}} \mathbf{n} ds =: *.$$

By stokes (for constant $\mathbf{w}$)

$$\mathbf{w} \cdot \int_{\partial \mathcal{B}} \mathbf{n} ds = \int_{\partial \mathcal{B}} \mathbf{w} \cdot \mathbf{n} = \int_{\mathcal{B}} \operatorname{div} \mathbf{w} dv = 0.$$

So $\int_{\partial \mathcal{B}} \mathbf{n} ds = 0$. Continue:

$$* = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{n}) ds,$$

for constant $\mathbf{w}$:

$$\mathbf{F} \cdot \mathbf{w} = \int_{\partial \mathcal{B}} (\varrho_{fluid} g z \mathbf{w}) \cdot \mathbf{n} ds = \int_{\mathcal{B}} \varrho_{fluid} g \operatorname{div}(z \mathbf{w}) dv + \int_{\mathcal{B}} \varrho_{fluid} g ((\nabla z) \cdot \mathbf{w} + z \operatorname{div} \mathbf{w} dv = \varrho_{fluid} g \int_{\mathcal{B}} 1 dv \mathbf{e}_{\hat{z}} \cdot \mathbf{w}.$$

So $\mathbf{F} = \varrho_{fluid} g V \mathbf{e}_{\hat{z}}$.

Příklad (Stability of flow in a container)
$\mathbf{v}|_{\partial \Omega} = \mathbf{o}$, $\mathbb{T} = -p\mathbb{I} + 2\mu \mathbb{D}$.

$$t \to +\infty \implies \mathbf{v} \to \mathbf{o}.$$

Příklad

$$\operatorname{div} \mathbb{T} = -\nabla p + \mu \Delta \mathbf{v}.$$

Poznámka (Governing equations)

$$
\begin{aligned}
\operatorname{div} \mathbf{v} &= 0,\\
\varrho \frac{d\mathbf{v}}{dt} &= \operatorname{div} \mathbb{T} = -\nabla p + \mu \Delta \mathbf{v},\\
\mathbf{v}|_{\partial\Omega} &= \mathbf{o},\\
\mathbf{v}|_{t_0} &= \mathbf{v}_0.
\end{aligned}
$$

Řešení
Test equation by solution:

$$\int_\Omega \varrho \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \int_\Omega (-(\nabla p)\cdot \mathbf{v} + \mu(\Delta \mathbf{v})\cdot \mathbf{v}) dv.$$

$$\varrho \int_\Omega ((\mathbf{v}\cdot\nabla)\mathbf{v})\cdot \mathbf{v} dv = \varrho \int_\Omega v_i \frac{\partial v_j}{\partial x_i} v_j dv = \frac{1}{2}\varrho \int_\Omega v_i \frac{\partial}{\partial x_i}(v_j)^2 dv =$$

$$= \varrho \in_\Omega \mathbf{v}\cdot\nabla\left(\frac{|\mathbf{v}|^2}{2}\right) dv = \varrho \int_\Omega \operatorname{div}(\mathbf{v}\frac{|\mathbf{v}|^2}{2}) dv = \varrho \int_{\partial\Omega} \frac{|\mathbf{v}|^2}{2}(\mathbf{v}\cdot\mathbf{n}) ds = 0.$$

$$\frac{d}{dt}\int_{\mathcal{B}} \frac{1}{2}\varrho(\mathbf{v})^2 dv = \ldots - \mu \int_{\mathcal{B}} (\nabla \mathbf{v}) : \nabla \mathbf{v} dv.$$

Poincaré inequality (we live in Dirichlet zero $\Longleftarrow$ boundary conditions):

$$\int |\nabla \mathbf{v}|^2 \leqslant C_p^2 \int |\nabla \mathbf{v}|^2.$$

$$\frac{1}{2}\varrho \frac{d}{dt}\int_\Omega \leqslant -\frac{\mu}{C_p}\int_\Omega |\mathbf{v}|^2 dv$$

$$\frac{d}{dt}(\|\mathbf{v}\|^2_{L^2(\Omega)}) \leqslant \frac{-2\mu}{\varrho C_p}\|\mathbf{v}\|^2_{L^2(\Omega)}.$$

Příklad

$$\operatorname{div} \mathbf{v} = 0 \qquad \varrho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \Delta \mathbf{v}.$$

Poznámka (Dimensionless form)
l_{char} = Characteristic length = arbitrary chosen length. v_{char} = characteristic velocity. $t_{char} = l_{char}/v_{char}$.

Dimensionless variables: $\mathbf{x}^* = \mathbf{x}/l_{char}$, $t^* = t/t_{char}$, $\mathbf{v}^* = \mathbf{v}/v_{char}$.

$$\operatorname{div}^* \mathbf{v}^* = 0.$$
$$\varrho \frac{v_{char}}{t_{char}} \frac{d\mathbf{v}^*}{dt^*} = -\frac{1}{l_{char}}(\nabla^* p^*) p_{char} + \frac{\mu}{l_{char}^2}(\Delta^* \mathbf{v}^*) v_{char},$$
$$\frac{d\mathbf{v}^*}{dt^*} = \frac{t_{char}}{\varrho v_{char} l_{char}}(\nabla^* p^*) p_{char} + \frac{\mu v_{char}}{l_{char}^2}\frac{t_{char}}{\varrho v_{char}} \Delta^* \mathbf{v}^*,$$
$$\frac{dv^*}{dt^*} = -\frac{p_{char}}{\varrho v_{char}^2}(\nabla^* p^*) + \frac{\mu}{\varrho l_{char} v_{char}} \Delta^* \mathbf{v}^*.$$

$p_{char} := \varrho v_{char}^2$
$$\implies \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* p^* + \frac{1}{Re}\Delta^* \mathbf{v}^*,$$
where $\frac{1}{Re} := \frac{\mu}{\varrho l_{char} v_{char}}$ is Reinold's number.

3 Solids

TODO!!!

Příklad
TODO!!!

Řešení
TODO!!!

$$0 = \operatorname{div}\boldsymbol{\tau} = \begin{pmatrix} \frac{\partial \tau_{\hat r\hat r}}{\partial r} + \frac{1}{r}\left(\frac{\partial \tau_{\hat r\hat\varphi}}{\partial\varphi} - \tau_{\hat\varphi\hat\varphi} + \tau_{\hat r\hat r}\right) + \frac{\partial \tau_{\hat r\hat z}}{\partial z} \\ \frac{\partial \tau_{\hat\varphi\hat r}}{\partial r} + \frac{1}{r}\left(\frac{\partial \tau_{\hat\varphi\hat\varphi}}{\partial\varphi} + \tau_{\hat r\hat\varphi} + \tau_{\hat\varphi\hat r}\right) + \frac{\partial \tau_{\hat\varphi\hat z}}{\partial z} \\ \frac{\partial \tau_{\hat r\hat r}}{\partial r} + \frac{1}{r}\left(\frac{\partial \tau_{\hat z\hat\varphi}}{\partial\varphi}\tau_{\hat z\hat r}\right) + \frac{\partial \tau_{\hat z\hat z}}{\partial z} \end{pmatrix},$$

where $\boldsymbol{\tau} = \begin{pmatrix} \tau_{\hat r\hat r} & \tau_{\hat r\hat\varphi} & \tau_{\hat r\hat z} \\ & \tau_{\hat\varphi\hat\varphi} & \tau_{\hat\varphi\hat z} \\ & & \tau_{\hat z\hat z} \end{pmatrix}$. So $\boldsymbol{\tau} = \begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ & & \tau_{\hat z\hat z} \end{pmatrix} =: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix}$.

Thus $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{pmatrix}\Bigg|_{z=L} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{S}\begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}$, so $T = \frac{F}{S}$.

$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\mathfrak{e})\mathbb{I} + 2\mu\mathfrak{e}. \operatorname{tr}\boldsymbol{\tau} = (3\lambda + 2\mu)\operatorname{tr}\mathfrak{e} \implies \operatorname{tr}\mathfrak{e} = \frac{\operatorname{tr}\boldsymbol{\tau}}{3\lambda + 2\mu}.$$

$$\mathfrak{e} = f(\boldsymbol{\tau}). \qquad \mathfrak{e} = \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right).$$

$$RHS = \frac{1}{2\mu}(\mathfrak{e} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}) = \begin{pmatrix} -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{F}{S} & 0 & 0 \\ 0 & -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{F}{S} & 0 \\ 0 & 0 & \frac{\lambda+\mu}{\mu(3\lambda+2\mu)}\frac{F}{S} \end{pmatrix}.$$

$$LHS = \frac{1}{2}(\nabla\mathbf{U} + (\nabla\mathbf{U})^T).$$

Symmetric gradient in cylindrical coordinate system is another long formula, so expect: $\mathbf{U} = (U_{\hat r}(r), 0, U_{\hat z}(z))^T$.

$$\frac{1}{2}\nabla\mathbf{U} + \frac{1}{2}(\nabla\mathbf{U})^T = \begin{pmatrix} \frac{dU_{\hat r}}{dr} & 0 & 0 \\ 0 & \frac{U_{\hat r}}{r} & 0 \\ 0 & 0 & \frac{dU_{\hat z}}{dz} \end{pmatrix}.$$

Together: $U_{\hat r} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot r$, $U_{\hat z} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot z$.

Length of cylinder: $\varepsilon := \frac{\Delta L}{L} = \frac{\mathbf{U}|_{z=L}}{L} = \frac{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot L}{L} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} \cdot \frac{F}{S} =: \frac{1}{E}\sigma$, where $\sigma = E\varepsilon$ is Hooke law, where E is Young modulus, σ is „applied force“ and ε is change of length.

Change of radius:

$$-\frac{\frac{\Delta R}{R}}{\frac{\Delta L}{L}} = -\frac{\frac{-\frac{\lambda}{2\mu(3\lambda+2\mu)} \cdot \frac{F}{S} \cdot R}{R}}{\frac{\Delta L}{L}} = \frac{\frac{\lambda}{2\mu(3\lambda+2\mu)}}{\frac{\lambda+\mu}{\mu(3\lambda+2\mu)}} = \frac{\lambda}{2(\lambda + \mu)} =: \nu.$$

Poznámka
This is easy to measure -> we can measure Young modulus $E := \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and Poisson ration $\nu := \frac{\lambda}{2(\lambda+\mu)}$.

Poznámka
Young modulus is certainly positive. However, Poisson ration can be negative. (Move Vs horizontally in VΛVΛVΛVΛVΛVΛ.)

Poznámka (Fixing negativity of constants (Poisson ration))

$$\boldsymbol{\tau} = \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right), \qquad \boldsymbol{\tau} = \lambda(\operatorname{tr} < BS > \boldsymbol{\varepsilon})\mathbb{I} + 2\mu\boldsymbol{\varepsilon}.$$

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left(\left(\boldsymbol{\tau} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right) + \left(\frac{1}{3} - \frac{\lambda}{3\lambda+2\mu}\right)(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right).$$

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\boldsymbol{\tau}_\delta + \frac{1}{9(\lambda+\frac{2}{3}\mu)}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}.$$

Experiment: compress material. Then we see, that $K := \lambda + \frac{2}{3}\mu$ must be positive. It is called bulk modulus (related to change of volume). $G := \mu > 0$ is shear modulus.

Důsledek

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)},$$

$$\mu = \frac{E}{2(1+\nu)}, \qquad E = \frac{9K\mu}{\mu+3K}, \qquad \nu = \frac{3K-2\mu}{2(3K+\mu)}.$$

$$\implies -1 < \nu < \frac{1}{2}.$$

$\nu = \frac{1}{2}$ means incompressible (solid) material (= no $\boldsymbol{\varepsilon}$).

Definice 3.1

Spherical stress: $\boldsymbol{\tau} = \begin{pmatrix} \tau & 0 & 0 \\ 0 & \tau & 0 \\ 0 & 0 & \tau \end{pmatrix}$. Sheer stress: $\boldsymbol{\tau} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3.1 Elastic materials

Definice 3.2

Elastic material is a material that does not produce entropy. No energy loss/gain in cyclic mechanical processes.

We have $\mathbb{T} = \mathbb{T}(\mathbb{B}) = \alpha_0\mathbb{I} + \alpha_1\mathbb{B} + \alpha_2\mathbb{B}^2$ (for solids, isotropic solids) (hence $\mathbb{T}\mathbb{B} = \mathbb{B}\mathbb{T}$), $e(\eta, \varrho)$, $\psi(\theta, \varrho)$, and from $\varrho \det \mathbb{F} = \varrho_R$ ($\varrho(\det \mathbb{B})^{1/2} = \varrho_R$), we get $\psi(\theta, \det \mathbb{B})$. So $\psi = \psi(\theta, \mathbb{B}) = \psi(\theta, I_1(\mathbb{B}), I_2(\mathbb{B}), I_3(\mathbb{B}))$.

From $\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q$ we get $\psi(\theta, \mathbb{B}) = (e(\eta, \mathbb{B}) - \theta\eta)|_{\eta=\eta(\theta,\mathbb{B})}$.

$$\frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt} + \frac{\partial \psi}{\partial \theta}\frac{d\theta}{dt} = \frac{de}{dt} - \frac{d\theta}{dt}\eta - \theta \cdot \frac{d\eta}{dt}.$$

Důsledek ($\overset{\triangledown}{\mathbb{B}} = \mathbb{O}$)

$$\frac{de}{dt} = \frac{\partial \psi}{\partial \mathbb{B}}(\theta, \mathbb{B}) : \frac{d\mathbb{B}}{dt} + \theta\frac{d\eta}{dt},$$

$$\varrho\theta\frac{d\eta}{dt} - \varrho\frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt}.$$

$$\varrho\frac{de}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q \to \varrho\theta\frac{d\eta}{dt} = \mathbb{T} : \mathbb{D} - \operatorname{div} \mathbf{j}_q - \varrho\frac{\partial \psi}{\partial \mathbb{B}} : \frac{d\mathbb{B}}{dt} =$$

$$= \left(\mathbb{T} : \mathbb{D} - \varrho\frac{\partial \psi}{\partial \mathbb{B}} : (\mathbb{L}\mathbb{B} + \mathbb{B}\mathbb{L}^T)\right) - \operatorname{div} \mathbf{j}_q = \mathbb{T} : \mathbb{D} - 2\varrho\mathbb{B}\frac{\partial \psi}{\partial \mathbb{B}} : \mathbb{D} - \operatorname{div} \mathbf{j}_q.$$

$$\frac{\partial \psi}{\partial \mathbb{B}}\mathbb{B} = \mathbb{B}\frac{\partial \psi}{\partial \mathbb{B}},$$

because ψ is isotropic function of $\mathbb{B}$. Thus we get evolution equation for entropy.

$$\varrho\theta\frac{d\eta}{dt} = \left(\mathbb{T} - 2\varrho\mathbb{B}\frac{\partial \psi}{\partial \mathbb{B}}\right) : \mathbb{D} - \operatorname{div} \mathbf{j}_q.$$

Důsledek

No entropy production can be done (for all processes at once) ”only” by setting $\mathbb{T} = 2\varrho\mathbb{B}\frac{\partial \psi}{\partial \mathbb{B}}$.

Důsledek

$$0 = \left(\int_V \varrho\psi(\theta, \mathbb{B})\right)|_{t_{end}} - \left(\int_V \varrho\psi(\theta, \mathbb{B})\right)|_{t_{start}} =$$

$$= \int_{t_{start}}^{t_{end}} \left(\int_V \mathbb{T} : \mathbb{D} dv\right) dt = \int_{t_{start}}^{t_{end}} \int_V 2\varrho\mathbb{B}\frac{\partial \psi}{\partial \mathbb{B}} : \mathbb{D} dv dt =$$

$$= \int_{t_{start}}^{t_{end}} \int_V \left(\varrho\frac{\partial \psi}{\partial \mathbb{B}}\frac{d\mathbb{B}}{dt} dv\right) dt = \int_{t_{start}}^{t_{end}} \frac{d}{dt}\int_V \varrho\psi dv dt.$$

TODO!!!

TODO? (Torsion of right circular cylinder)

Poznámka
Where is my biharmonic equation $\Delta(\Delta\varphi) = g$?

And related question: What are the compatibility conditions good for?

Poznámka
$\mathbf{o} = \operatorname{div}\boldsymbol{\tau} + \varrho_R\mathbf{b}$ (where $\varrho_R\mathbf{b}$ is force $\mathbf{f}$ and $\boldsymbol{\tau}\mathbf{n}|_{\partial\Omega} = \mathbf{g}$) is only 3 equations (2 in 2D) for ($\boldsymbol{\tau}$ is symmetric) 6 equations (3 in 3D). Here comes the compatibility condition $\operatorname{rot}\left((\operatorname{rot}\mathfrak{e})^T\right) = \mathbb{O}$ ($\mathfrak{e} := \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right)$).

Poznámka (Going from equations for $\mathfrak{e}$ to equations for $\boldsymbol{\tau}$)
Using $\operatorname{rot}(\operatorname{rot}\mathbf{v}) = \nabla(\operatorname{div}\mathbf{v}) - \Delta\mathbf{v}$.

$$\mathbb{O} = \nabla(\operatorname{div}\mathfrak{e}) + (\nabla(\operatorname{div}\mathfrak{e}))^T - \nabla(\nabla\operatorname{tr}\mathfrak{e}) - \Delta\mathfrak{e}.$$

Doing algebra (and using $\mathbf{o} = \operatorname{div}\boldsymbol{\tau} + \mathbf{f}$, problem for exam), we get (Beltram–Michell equation):

$$\Delta\boldsymbol{\tau} + \frac{1}{1+\nu}\nabla(\nabla\operatorname{tr}\boldsymbol{\tau}) = -(\nabla\mathbf{f}) + (\nabla\mathbf{f})^T - \frac{\nu}{1-\nu}(\operatorname{div}\mathbf{f})\mathbb{I}.$$

Definice 3.3 (Plane strain problems)

$$\boldsymbol{\tau} = \begin{pmatrix} \varepsilon_{xx}(x,y) & \varepsilon_{xy}(x,y) & 0 \\ \varepsilon_{yx}(x,y) & \varepsilon_{yy}(x,y) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Důsledek

$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\mathfrak{e})\mathbb{I} + 2\mu\mathfrak{e} \implies \boldsymbol{\tau} = \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & \tau_{\hat{z}\hat{z}} \end{pmatrix}.$$

$$\mathfrak{e} := \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\boldsymbol{\tau})\mathbb{I}\right) \implies 0 = \varepsilon_{\hat{z}\hat{z}} = \frac{1}{2\mu}\left(\tau_{\hat{z}\hat{z}} - \frac{\lambda}{3\lambda+2\mu}\left(\tau_{xx} + \tau_{yy} + \tau_{zz}\right)\right) \implies$$

$$\tau_{\hat{z}\hat{z}} = \frac{\lambda}{2(\lambda+\mu)}(\tau_{\hat{x}\hat{x}} + \tau_{\hat{y}\hat{y}}) = \frac{\lambda}{2(\lambda+\mu)}\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D}.$$

Důsledek (ε_{2D} vs. $\boldsymbol{\tau}_{2D}$)

$$\operatorname{tr}\boldsymbol{\tau} = \tau_{\hat{z}\hat{z}} + \operatorname{tr}_{2D}\boldsymbol{\tau}_{2D} = \left(\frac{\lambda}{2(\lambda+\mu)} + 1\right)\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D}.$$

$$\varepsilon_{2D} = \frac{1}{2\mu}\left(\boldsymbol{\tau}_{2D} - \frac{\lambda}{2(\lambda+\mu)}(\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D})\mathbb{I}_{2D}\right).$$

It is different from the 2D theory!

$$\mathbf{o} = \operatorname{div}_{2D}\boldsymbol{\tau}_{2D} + \mathbf{f}_{2D}. \qquad \boldsymbol{\tau}_{2D}\mathbf{n}|_{\partial\Omega} = \mathbf{g}. \qquad \varepsilon_{2D} = \frac{1}{2}\left(\nabla_{2D}\mathbf{u}_{2D} + (\nabla_{2D}\mathbf{u}_{2D})\right).$$

Důsledek

We need just one additional equation. We take trace of Beltrami–Michelle:

$$\Delta(\operatorname{tr}\boldsymbol{\tau}) + \frac{1}{1+\nu}\Delta(\operatorname{tr}\boldsymbol{\tau}) = -2(\operatorname{div}\mathbf{f}) - \frac{3\nu}{1-\nu}(\operatorname{div}\mathbf{f}).$$

Using $\operatorname{tr}\boldsymbol{\tau} = \left(\frac{\lambda}{2(\lambda+\mu)} + 1\right)\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D}$:

$$\Delta_{2D}(\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D}) = -\frac{1}{1-\nu}\operatorname{div}_{2D}\mathbf{f}_{2D}.$$

Poznámka (Airy stress function)

$$\mathbf{o} = \operatorname{div}_{2D}\boldsymbol{\tau}_{2D} - \nabla_{2D}\varphi.$$

Let us think $\boldsymbol{\tau}$ is given by this ”strange” potential:

$$\boldsymbol{\tau}_{2D} = \begin{pmatrix} \frac{\partial^2\varphi}{\partial y^2} + \varphi & -\frac{\partial^2\varphi}{\partial x\partial y} \\ -\frac{\partial^2\varphi}{\partial x\partial y} & \frac{\partial^2\varphi}{\partial x^2} + \varphi \end{pmatrix}.$$

$\implies \mathbf{i} = \operatorname{div}_{2D}\boldsymbol{\tau}_{2D} - \nabla_{2D}\varphi.$

$$\Delta_{2D}(\operatorname{tr}_{2D}\boldsymbol{\tau}_{2D}) = \frac{1}{1-\nu}\operatorname{div}_{2D}(\nabla_{2D}\varphi) \implies \Delta_{2D}(\Delta_{2D}\varphi) + \frac{1-2\nu}{1-\nu}\Delta_{2D}\varphi = 0.$$

This is nice equation, but boundary conditions are awful.