Příklad (1.)
Let $\mathbb{A}$ be a sufficiently smooth tensor/matrix field, and let $\mathbb{A}$ be (at every point $\mathbf{x}$) a *symmetric matrix*. Show that

$$\operatorname{rot}\left((\operatorname{rot}\mathbb{A})^T\right) = [\Delta \operatorname{tr}\mathbb{A} - \operatorname{div}(\operatorname{div}\mathbb{A})]\mathbb{I} + \nabla(\operatorname{div}\mathbb{A}) + [\nabla(\operatorname{div}\mathbb{A})]^T - \nabla(\nabla \operatorname{tr}\mathbb{A}) - \Delta\mathbb{A}.$$

Důkaz
Z definice:

$$\operatorname{rot}\left((\operatorname{rot}\mathbb{A})^T\right) = \operatorname{rot}\left(\varepsilon_{jkl}\frac{\partial A_{il}}{\partial x_k}\cdot \mathbf{e}_j \otimes \mathbf{e}_i\right) = \varepsilon_{nop}\varepsilon_{mkl}\frac{\partial^2 A_{pl}}{\partial x_k \partial x_o}\mathbf{e}_m \otimes \mathbf{e}_n$$

Z rovnosti

$$\varepsilon_{nop}\varepsilon_{mkl} = \det\begin{pmatrix}\delta_{nm} & \delta_{nk} & \delta_{nl}\\ \delta_{om} & \delta_{ok} & \delta_{ol}\\ \delta_{pm} & \delta_{pk} & \delta_{pl}\end{pmatrix}$$

dostáváme na pravé straně 6 členů:

- $\frac{\partial^2 A_{pp}}{\partial x_k \partial x_o}\mathbf{e}_m \otimes \mathbf{e}_n = (\Delta \operatorname{tr}\mathbb{A})\mathbb{I}$, neboť

$$A_{pp} = \operatorname{tr}\mathbb{A}, \qquad \frac{\partial^2}{\partial x_o^2} = \nabla\cdot\nabla = \Delta, \qquad \mathbf{e}_n \otimes \mathbf{e}_n = \mathbb{I};$$

- $-\frac{\partial^2 A_{po}}{\partial x_p \partial x_o}\mathbf{e}_n \otimes \mathbf{e}_n = -\operatorname{div}(\operatorname{div}\mathbb{A})$;
- $\frac{\partial^2 A_{po}}{\partial x_n \partial x_o}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A}^T)_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A})_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \frac{\partial(\operatorname{div}\mathbb{A})_p}{\partial x_n}\mathbf{e}_p \otimes \mathbf{e}_n = \nabla(\operatorname{div}\mathbb{A})$;
- $\frac{\partial^2 A_{pn}}{\partial x_p \partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = \frac{\partial(\nabla\cdot\mathbb{A})_n}{\partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = \frac{\partial(\operatorname{div}\mathbb{A})_n}{\partial x_o}\mathbf{e}_o \otimes \mathbf{e}_n = [\nabla(\operatorname{div}\mathbb{A})]^T$;
- $-\frac{\partial^2 A_{pp}}{\partial x_n \partial x_o}e_n \otimes e_o = -\frac{\partial^2 \operatorname{tr}\mathbb{A}}{\partial x_n \partial x_o}e_n \otimes e_o = -\nabla(\nabla \operatorname{tr}\mathbb{A})$;
- $-\frac{\partial^2}{\partial x_o^2}A_{pm}\mathbf{e}_p \otimes \mathbf{e}_n = -\frac{\partial^2}{\partial x_o^2}\mathbb{A} = -\Delta\mathbb{A}$.

Tím jsme dokázali rovnost. □

Příklad (2.)
Prove the following. Let u, v be smooth scalar valued functions, $v : \mathbb{R}^3 \to \mathbb{R}$, $u : \mathbb{R}^3 \to \mathbb{R}$, and let $\mathbb{A}$ be a smooth tensor valued function $\mathbb{A} : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary, then

$$\int_\Omega u(\nabla v) dv = \int_{\partial\Omega} uv d\mathbf{s} - \int_\Omega (\nabla u) v dv$$

Důkaz
Stokesova věta říká:

$$\int_{\partial\Omega} (uv\mathbf{w}) \cdot d\mathbf{s} = \int_\omega \nabla \cdot (uv\mathbf{w}) dv \quad \left(= \int_\Omega \operatorname{div}(uv\mathbf{w}) dv \right)$$

Z předchozího domácího úkolu víme, že $\operatorname{div}(a\mathbf{x}) = \mathbf{x} \cdot \nabla a + a \operatorname{div} \mathbf{x}$. To použijeme dvakrát:

$$= \int_\Omega \left(v\mathbf{w} \cdot \nabla u + u \operatorname{div} v\mathbf{w} \right) dv = \int_\Omega \left(v\mathbf{w} \cdot \nabla u + u\mathbf{w} \cdot \nabla v + uv \operatorname{div} \mathbf{w} \right) dv =$$

Zvolili jsme si konstantní vektor, tedy jeho derivace jsou nulové:

$$\int_\Omega \left(v\mathbf{w} \cdot \nabla u + u\mathbf{w} \cdot \nabla v \right) dv.$$

Jelikož toto platí pro libovolný vektor $\mathbf{w}$, tak platí

$$\int_{\partial\Omega} uv d\mathbf{s} = \int_\Omega \left(v(\nabla u) + u(\nabla v) \right) dv = \int_\Omega v(\nabla u) dv + \int_\Omega u(\nabla v) dv,$$

$$\int_\Omega u(\nabla v) dv = \int_{\partial\Omega} uv d\mathbf{s} - \int_\Omega v(\nabla u) dv.$$

□

$$\int_\Omega (\operatorname{div} \mathbb{A}) \cdot \mathbf{v} \, dv = \int_{\partial\Omega} (\mathbb{A}^T \mathbf{v}) \cdot d\mathbf{s} - \int_\Omega \mathbb{A} : \nabla \mathbf{v} \, dv.$$

Důkaz
Zase podle Stokese platí:

$$\int_{\partial\Omega} (\mathbb{A}^T \mathbf{v}) \cdot d\mathbf{s} = \int_\Omega \operatorname{div}(\mathbb{A}^T \mathbf{v}) dv.$$

Takže nám stačí dokázat $\operatorname{div}(\mathbb{A}^T \mathbf{v}) = \operatorname{div}(\mathbb{A}) \cdot \mathbf{v} + \mathbb{A} : \nabla \mathbf{v}$. Potom už dostaneme chtěnou rovnost pouze z linearity integrálu a „přehozením“ jednoho integrálu „na druhou stranu“.

$$\operatorname{div}(\mathbb{A}^t \mathbf{v}) = \frac{\partial A_{ji} v_j}{\partial x_i} = \frac{\partial A_{ji}}{\partial x_i} v_j + A_{ji} \frac{\partial v_j}{\partial x_i} = (\operatorname{div} \mathbb{A})_j v_j + A_{ji} (\nabla \mathbf{v})_{ji} = (\operatorname{div} \mathbb{A}) \cdot \mathbf{v} + \mathbb{A} : \nabla \mathbf{v}.$$

□