TODO(you should know).

TODO(motivation)

1 Sobolev spaces

Definice 1.1 (Multiindex)

α je multi-index $\equiv \alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}$. Length of multi-index α is $|\alpha| := \alpha_1 + \ldots + \alpha_d$. If $u \in C^k(\Omega)$ then $D^\alpha := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$, $\alpha \leqslant k$.

Definice 1.2 (Weak derivative)

Let $u, v_\alpha \in L^1_{loc}(\Omega)$ and α be a multi-index. We say that v_α is the α-th weak derivative of u in Ω iff $\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int v_\alpha \varphi$.

Lemma 1.1

Weak derivative is unique. If the classical derivative exists then it is also the weak derivative.

Důkaz
Let v_α^1 and v_α^2 be two weak derivatives. Then

$$\int_\Omega (v_\alpha^1 - v_\alpha^2)\varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ almost everywhere in Ω.

If classical $D^\alpha u$ exists, then

$$\int_\Omega \underbrace{D^\alpha u}_{v_\alpha} \varphi \overset{\text{BP}}{=} (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

□

Poznámka (Notation for this course)
D^α always means the weak derivative.

Definice 1.3 (Sobolev space)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. We define $W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}$.

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sup_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

Lemma 1.2 (Base properties of Sobolev spaces)

Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and α is multi-index. Then

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$, *if* $|\alpha| \leqslant k$;
- $\lambda u + \mu v \in W^{k,p}(\Omega)$ $\forall \lambda, \mu \in \mathbb{R}$ $(D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v)$;
- $\tilde{\Omega} \subset \Omega$ *open*, $u \in W^{k,p}(\tilde{\Omega})$;
- $\forall \eta \in C^\infty(\Omega) : \eta \cdot u \in W^{k,p}(\Omega)$.