

# 1 Introduction

*Poznámka* (Literature)

„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

## 1.1 Differentiability

### Definition 1.1 (Differentiable)

A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable (also holomorphic) at a point  $z_0 \in \mathbb{C}$  if the following limit exists

$$\lim_{|h| \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call  $f'(z_0)$  the derivative of  $f$  at  $z_0$ . A function  $f$  is differentiable on a domain (open connected subset of  $\mathbb{C}$ ) if its differentiable for all points of this domain.

*Poznámka* (Writing complex numbers in cartesian coordinates)

$z = x + iy$ , for  $x, y \in \mathbb{R}$ , we can write a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  in terms of two functions  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

### Věta 1.1 (Cauchy–Riemann equations)

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function on an open subset of  $\mathbb{C}$ . Considering  $f = u + iv$ , then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

### Definition 1.2 (Orientability, orientation-preserving function)

Define an equivalence relation on the set of all bases of  $\mathbb{R}^2$  by saying that  $B_1 \sim B_2$  iff the determinant of the change of basis matrix is positive.

A function  $f : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}^2$  is said to be orientation-preserving if on an open dense subset of  $U$ , the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

*Důsledek*

Let  $f$  be a non-constant holomorphic function, then  $f$  is orientation-preserving.

*Důsledek*

Since  $f$  is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \stackrel{\text{C-R}}{=} \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \geq 0.$$

Since  $f$  is non-constant, the inequality is strict on a dense open subset of the domain of definition.

### **Věta 1.2** (Open mapping theorem)

*A non-constant holomorphic function  $f$  is open (that is if  $U$  is an open subset of  $\mathbb{C}$ , then  $f(U)$  is also open).*

## 1.2 Integration

### **Definice 1.3**

For a path  $\gamma$  (smooth function,  $\gamma : \mathbb{R} \supset [a, b] \rightarrow \mathbb{C}$ ) we define

$$\int_{\gamma} f(x) dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt$$

### **Definice 1.4** (Continuous deformation)

For  $\gamma, \mu : [a, b] \rightarrow U$  ( $U$  simply connected), paths with the same endpoints ( $\gamma(a) = \mu(a)$  and  $\gamma(b) = \mu(b)$ ). Then a continuous deformation  $\gamma$  into  $\mu$  is a continuous function  $H : [a, b] \times [0, 1] \rightarrow U \subseteq \mathbb{C}$  such that  $H(s, 0) = \gamma(s)$ ,  $H(s, 1) = \mu(s)$ ,  $H(a, t) = z_a := \gamma(a) = \mu(a)$  and  $H(b, t) = z_b := \gamma(b) = \mu(b)$ .

### **Věta 1.3**

*Suppose that  $\gamma, \mu : [a, b] \rightarrow U$  ( $U$  simply connected) are related by a continuous deformation of paths  $H$ . Then for any holomorphic function  $f$  on  $U$  we have*

$$\int_{\gamma} f(z) dz = \int_{\mu} f(z) dz.$$

┌

*Důkaz* (Partial proof assuming  $H$  admits partial derivatives)

For any  $t \in [0, 1]$  we integrate the function  $INT(t) = \int_{H(\cdot, t)} f(z)dz$ . Consider the derivative of  $INT(t)$  with respect to  $t$ :

$$\begin{aligned} \frac{d}{dt}(INT(t)) &= \frac{d}{dt} \left( \int_a^b f(H(s, t)) \frac{\partial H}{\partial s}(s, t) ds \right) \stackrel{\text{Leibniz} + \text{chain rule}}{=} \\ &= \int_a^b f'(H(s, t)) \frac{\partial H}{\partial t}(s, t) \cdot \frac{\partial H}{\partial s}(s, t) + f(H(s, t)) \frac{\partial^2 H}{\partial s \partial t}(s, t) ds = \\ &= \int_a^b \frac{d}{ds} \left[ f(H(s, t)) \frac{\partial H}{\partial t} \right] ds = \\ &= f(H(s, t)) \frac{\partial H}{\partial t} \Big|_{s=a}^{s=b} \stackrel{\text{constant endpoints}}{=} 0. \end{aligned}$$

Having derivative identically equal to 0, means that  $INT(t)$  is a constant function and  $\int_{\gamma} f(z)dz = INT(0) = INT(1) = \int_{\mu} f(z)dz$ . □

└

*Důsledek*

Let  $U$  be a simply connected subset of  $\mathbb{C}$  and  $f : U \rightarrow \mathbb{C}$  a holomorphic function. For any closed path whose image is inside  $U$ ,  $\int_{\gamma} f(z)dz = 0$ .

┌

*Důkaz* (Sketch)

The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path  $c$ .

$$\int_{\gamma} f(z)dz = \int_c f(z)dz = \int_a^b f(c(z)) \cdot c'(z)dz = \int_a^b f(c(z)) \cdot 0dz = 0$$

└

□

*Příklad*

Let  $U$  be a simple connected domain and  $f : U \rightarrow \mathbb{C}$  a holomorphic function on  $U \setminus \{z_0\}$ . For  $j = 1, 2$ , let  $\gamma_j$  be a path parametrizing a circle centered at  $z_0$  of radius  $r_j$ , oriented counterclockwise and completely contained in  $U$ . Show that  $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$ .

## 1.3 Cauchy's integral formula

**Věta 1.4** (Cauchy's integral formula)

Let  $\gamma$  be a loop around  $z \in \mathbb{C}$ , and  $f : U \rightarrow \mathbb{C}$  a holomorphic function. For  $U$  a neighbourhood of  $\gamma$ ,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} dw.$$

Důkaz

Conway 1978, Chapter IV.

□

Důsledek

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0 + z_0 - z} dw = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \cdot \left( \frac{1}{1 - \frac{z - z_0}{w - z_0}} \right) dw = \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \left( \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n} \right) dw = \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^n} \right) (z - z_0)^n. \end{aligned}$$

For sufficiently "small" (shrunk)  $\gamma$ . So  $f$  is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around  $z_0$  converges to  $f$  in a neighbourhood of  $z_0$ ).

### Definition 1.5 (Pole)

Given a positive integer  $n$ , a complex function  $f$  has pole of order  $n$  at the point  $z_0 \in \mathbb{C}$  if  $(z - z_0)^n f(z)$  is holomorphic at  $z_0$  but  $(z - z_0)^{n-1} f(z)$  is not.

Příklad

Show that if  $f$  has a pole of order  $n$  at  $z_0 \in \mathbb{C}$ . Then it admits a Laurent expansion  $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$  with  $a_{-n} \neq 0$ .

### Definition 1.6 (Residue)

Let  $f$  have a pole of order  $n$  at the point  $z_0 \in \mathbb{C}$ . Then the residue of  $f$  at  $z_0$  is the  $k = -1$  coefficient of the Laurent expansion of  $f$  at  $z_0$ .

Příklad

Show that if  $f$  has a pole of order 1 at  $z_0$ , then the residue of  $f$  at  $z_0$  can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)

Let  $\gamma : [a, b] \rightarrow U \subset \mathbb{C}$  be a simple closed path, bounding a domain  $W$  containing the points  $z_1, \dots, z_m$ . Assume that  $f$  is holomorphic on  $U \setminus \{z_1, \dots, z_m\}$  and has poles at  $\{z_1, \dots, z_m\}$ .

Show that

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{res}_{z=z_j} f(z).$$

TODO!!!

## 1.4 (Real) Projective space

*Poznámka* (Building structures)

Set  $\rightarrow$  Topology  $\rightarrow$  Differential structure (atlas)  $\rightarrow$  Riemann metric  $\rightarrow$  Connection...

### Definition 1.7 (Real projective space)

The set  $\mathbb{P}^n(\mathbb{R})$  is defined to be either of the following bijective sets: Lines through the origin in  $\mathbb{R}^{n+1}$ ; Equivalence classes of  $(n+1)$ -tuples of real numbers  $(x_0, \dots, x_n) \neq (0, \dots, 0)$ , such that for any real number  $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ :  $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$ .

*Příklad*

Confirm that the sets above are in bijection with each other.

*Poznámka* (Notation)

We will often denote a point in  $\mathbb{P}^n(\mathbb{R})$  as the equivalence class  $[x_0, \dots, x_n]$ .

### Definition 1.8 (Topology of $\mathbb{P}^n(\mathbb{R})$ )

We give a topology to  $\mathbb{P}^n(\mathbb{R})$  by endowing it with following quotient topology: consider the surjection  $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \rightarrow \mathbb{P}^n(\mathbb{R})$ ,  $(x_0, \dots, x_n) \mapsto [x_0, \dots, x_n]$ . A set  $U \subset \mathbb{P}^n(\mathbb{R})$  is defined to be open if  $\pi^{-1}(U) := \{x \in \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \mid \pi(x) \in U\}$  is open in  $\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}$ .

That is we give  $\mathbb{P}^n(\mathbb{R})$  the finest topology that makes  $\pi$  continuous.

*Příklad*

Check that for  $\mathbb{C}$  we can define  $\mathbb{P}^n(\mathbb{C})$  or  $\mathbb{CP}^n$  the same way.

*Příklad* (Projective space)

$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  is an abelian group. Let  $\mathbb{R}^*$  act on  $\mathbb{R}^{n+1}$  by component wise multiplication. When a general group  $G$  acts on a set  $X$  we have equivalence relation  $x \sim y$  if  $y = g \circ x$ . We call the equivalence classes the orbits of  $G$ . So  $\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}) / \mathbb{R}^*$ .

Sphere quotient: Let  $S^n \subseteq \mathbb{R}^{n+1}$ . Denote the unit sphere. Then the group  $\mathbb{Z}_2 = \{+1, -1\}$  act on the sphere by  $\pm 1(x_0, \dots, x_n) = (\pm x_0, \dots, \pm x_n)$ . Then  $S^n / \mathbb{Z}_2 = \mathbb{P}^n(\mathbb{R})$ .

Disk model: Consider the  $n$ -dimensional closed unit disk  $\overline{\mathbb{D}^n} \subseteq \mathbb{R}^n$ , and the equivalence

relation on the points of the boundary:  $x \sim -x$  if  $\|x\| = 1$ . Then  $\mathbb{P}^n(\mathbb{R})$  is the quotient (collection of equivalence classes), i.e.  $\overline{D^n} \setminus \sim \simeq \mathbb{P}^n(\mathbb{R})$ .

*Příklad*

Conclude from either of these models of  $\mathbb{P}^n(\mathbb{R})$  that as a topological space,  $\mathbb{R}^n(\mathbb{P})$  is compact and Hausdorff.

*Poznámka*

Now we come to the smooth manifold structures. Let's start with  $\mathbb{P}^1(\mathbb{R})$ . Define

$$U_x := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x = 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y]) = \frac{x}{y}.$$

Similarly, we define a second chart:

$$U_y := \mathbb{P}^1(\mathbb{R}) \setminus \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | y = 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y]) = \frac{y}{x}.$$

┌ *Příklad*

Check that  $U_x, U_y$  are open and that  $\varphi_x, \varphi_y$  are homeomorphisms.

┌ *Důkaz*

Consider the transition functions:

$$U = U_x \cap U_y = \{[x, y] \in \mathbb{P}^1(\mathbb{R}) | x, y \neq 0\}, \quad \varphi_x(U) = \varphi_y(U) = \mathbb{R} \setminus \{0\}.$$

The translation function  $T_{x,y} := \varphi_y \circ (\varphi_x)^{-1}$  sends, for  $y \neq 0$ :

$$T_{x,y} : y \xrightarrow{(\varphi_x)^{-1}} [1, y] = \left[ \frac{1}{y}, 1 \right] \xrightarrow{\varphi_y} \frac{1}{y}.$$

Which is smooth on the domain  $\mathbb{R} \setminus \{0\}$ .

└ TODO smooth. Thus  $\mathbb{P}^1(\mathbb{R})$  is a smooth manifold. □

*Příklad*

Show that  $\mathbb{P}^1(\mathbb{R})$  is homomorphic to the circle  $S^1$ . We call  $\mathbb{P}^1(\mathbb{R})$  the real projective line.

*Příklad*

Try to show that  $\mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$  is a smooth manifold.

*Příklad*

For  $\mathbb{P}^2(\mathbb{R})$  the followings charts form atlas:

$$U_x := \{[x, y, z] | x \neq 0\}, \quad \varphi_x : U_x \rightarrow \mathbb{R}, \quad \varphi_x([x, y, z]) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

$$U_y := \{[x, y, z] | y \neq 0\}, \quad \varphi_y : U_y \rightarrow \mathbb{R}, \quad \varphi_y([x, y, z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$U_z := \{[x, y, z] | z \neq 0\}, \quad \varphi_z : U_z \rightarrow \mathbb{R}, \quad \varphi_z([x, y, z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Check these are open subsets and homeomorphisms, with smooth transformation functions. And extend this to  $\mathbb{P}^n(\mathbb{R})$ .

## 1.5 Compact surfaces

### Definition 1.9 (Surface)

A surface is a manifold of real dimension 2.

*Například*

$\mathbb{R}^2$ ,  $\mathbb{C}$ , and any of their open subsets are surfaces.  $S^2$  is a compact surface, as is  $\mathbb{P}^2(\mathbb{R})$ .

### Definition 1.10 (Connected surface)

Given two connected surfaces  $S_1$  and  $S_2$ , the connected surface  $S_1 \# S_2$  is the surface obtained by removing an open disc from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

*Příklad*

At the level of topological spaces, show that the operation  $\#$  is well defined up to homeomorphism, that is, show that the choice of disks in  $S_1$  and  $S_2$  does not change the definition of  $S_1 \# S_2$  / homeomorphism.

*Příklad*

Show that  $\#$  gives the set of homeomorphism classes of connected compact surfaces the structure of a monoid. (Which surface is the identity of the monoid?)

### Věta 1.5 (Classification of compact surfaces)

Any connected, compact surfaces is homeomorphic to exactly one surface in the following list:

- $S^2$ ;
- $T^{\#g} := T \# \dots \# T$ ,  $g \in \mathbb{N}_0$ ;

- $\mathbb{P}^2(\mathbb{R})^{\#n} := \mathbb{P}^2(\mathbb{R}) \# \dots \# \mathbb{P}^2(\mathbb{R}), n \in \mathbb{N}_0.$

*Poznámka* (Deep fact)

For  $d \leq 3$ , if two  $d$ -dimensional manifolds are homeomorphic, then they are diffeomorphic.

## 2 Riemann surfaces

### Definice 2.1 (Riemann surface)

A Riemann surface is a complex analytic manifold of dimension 1:

- $X$  is a Hausdorff, connected topological space;
- for all  $x \in X$ , there is a homeomorphism  $\varphi_x : U_x \rightarrow V_x$ , where  $U_x$  is an open neighbourhood of  $x \in X$ ,  $V_x$  is an open set in  $\mathbb{C}$ ;
- for any  $U_x, U_y$  such that  $U_x \cap U_y \neq \emptyset$ , the transition function  $T_{x,y} := \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \rightarrow \varphi_y(U_x \cap U_y)$  is holomorphic.

*Poznámka*

We saw in the first lecture that a holomorphic preserves orientation when thought of as a function from the real plane to itself. Since our transition functions are holomorphic, any Riemann surface is orientable.

*Příklad* (The complex projective line)

Just as for  $\mathbb{P}^1(\mathbb{R})$ , we define  $\mathbb{P}^1(\mathbb{C})$  to be the set whose elements are complex 1-dimensional subspaces of  $\mathbb{C}^2$ .

Let  $U_1 = U_2 := \mathbb{C}$  and define  $g : U_1 \setminus \{0\} \rightarrow U_2 \setminus \{0\}, z \mapsto \frac{1}{z}$ . We define  $\mathbb{P}^1(\mathbb{C})$  to be the quotient  $\mathbb{P}^1(\mathbb{C}) := U_1 \coprod U_2 / (z \sim g(z))$ .

*Příklad* (Show that)

As a set  $\mathbb{P}^1(\mathbb{C})$  is  $\mathbb{C}$  plus a point.

As a topological space  $\mathbb{P}^1(\mathbb{C})$  is the one point compactification of  $\mathbb{C}$ .

Conclude from the previous sentence that  $\mathbb{P}^1(\mathbb{C})$  is homeomorphic to the two sphere.

*Poznámka*

In complex analysis  $\mathbb{P}^1(\mathbb{C})$  is known as the Riemann sphere.



### *Poznámka*

For  $i = 1, 2$ , we denote the image of  $U_i$  in the quotient  $U_1 \coprod U_2 / (z \sim g(z))$  by  $[U_i]$ . Note that  $U_i$  define the local coordinate functions:  $\varphi_i : [U_i] \rightarrow U_i, p \mapsto z_i$ , where  $z_i$  is the complex numbers in  $U_i$  such that  $[z_i] = p_i$ . Both  $\varphi_1, \varphi_2$  are homeomorphisms.

We now consider the transition functions: the intersection

$$[U_1] \cap [U_2] = [U_1 \setminus \{0\}] = [U_2 \setminus \{0\}].$$

The image of the intersection under  $\varphi_1$  is  $\varphi_1([U_1] \cap [U_2]) = \mathbb{C} \setminus \{0\}$  (\*). Thus (\*) is the domain of our single transition function  $T = \varphi_2 \circ \varphi_1^{-1}$ . For  $z_1 \neq 0$ , we have  $T : z_1 \xrightarrow{\varphi_1^{-1}} [z_1] = [z_2 := g(z_1) = 1/z_1] \xrightarrow{\varphi_2} z_2 = 1/z_1$ . Thus

$$T : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \quad z \mapsto \frac{1}{z}.$$

Since  $T$  has a pole only at  $z_1 = 0$ , we see that it is holomorphic on  $\mathbb{C} \setminus \{0\}$ . A symmetric (exchange 1 for 2) calculation shows that  $T^{-1}$  is also holomorphic. So  $\mathbb{P}^1(\mathbb{C})$  is a Riemann surface.

### *Příklad (Hopf fibration)*

Consider the 3-dimensional real sphere  $S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4$ . Given a point  $p \in S^3$ ,  $\exists$  a unique line  $l_p$  through the origin and  $p$ . Thus we got a function  $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C}), p \mapsto l_p$ .

Check that  $H$  is continuous and surjective. Since  $S^3$  is closed and bounded, it is compact. Moreover, since the image of a compact set under a continuous function is compact,  $\mathbb{P}^1(\mathbb{C})$  is compact.

What is the fiber of the surjective map  $H : S^3 \rightarrow \mathbb{P}^1(\mathbb{C})$ , i.e. what is  $H^{-1}(p)$ , for any point  $p \in \mathbb{P}^1(\mathbb{C})$ . (Hint: It is  $S^1$ .)

This gives us  $S^2 \times S^1 = S^3$  as set. (Not as topological space!)

### *Příklad (Complex tori)*

Definition: Let  $\tau_1$  and  $\tau_2$  be two complex numbers, that are linearly independent. The set of all integral linear combinations of  $\tau_1$  and  $\tau_2$ :

$$\Lambda := \{n\tau_1, m\tau_2 | n, m \in \mathbb{Z}\} \subseteq \mathbb{C}$$

is called the lattice of  $\tau_1$  and  $\tau_2$ .

Observe that we can assume that  $\tau_1 = 1$ , and  $\Im(\tau_2) > 0$ , allowing to make simplifying assumption: that our lattice has the form  $\Lambda = \{n + m\tau | n, m \in \mathbb{Z}, \tau \in \mathbb{H}\}$ , where  $\mathbb{H}$  is the upper half plane.

Consider the quotient space  $T = \mathbb{C}/\Lambda$ . That is the quotient space with respect to the equivalence relation  $z_2 \sim z_1 \Leftrightarrow z_2 = z_1 + w$  for  $w \in \Lambda$ .

The canonical projection map  $\pi : \mathbb{C} \rightarrow T$  (i.e.  $\pi(z) = [z]$ ) induces a quotient topology on  $T$  (i.e.  $V \subseteq T$  is open iff  $p^{-1}(V)$  is open in  $\mathbb{C}$ ).

*Příklad*

For  $P$  the closed parallelogram with vertices  $0, 1, \tau, 1 + \tau$ , show that for any  $z \in \mathbb{C} \exists z' \in P$  that  $\pi|_P \rightarrow T$  is surjective. Hence we can restrict our attention to  $p$ .

*Poznámka*

By considering the identification points in  $p$ , we see that  $T$  is topologically a torus.

*Příklad*

Prove that  $\pi$  (from previous exercise) is an open map, i.e. that  $V$  an open subset of  $\mathbb{C}$  implies that  $\pi(V)$  is open in  $T$ .

*Poznámka*

Now to the complex structure: from the previous exercise, we see that if  $\pi$  restricted to a subset  $V \subseteq \mathbb{C}$  is bijective, then it is a homeomorphism onto its image in  $T$ . In this case,  $(\pi|_V)^{-1}$  is also a homeomorphism from the image of  $\pi|_V$  to  $V$ . Hence we can use  $(\pi|_V)^{-1}$  as a chart for  $T$ .

*Příklad*

Find a real number  $r$  (depends on  $t$ ) such that for any  $z \in \mathbb{C}$ :  $\pi$  restricted to  $B_r(z)$  is a bijective map.

Given this  $r$ , define  $U_z := \pi(B_r(z)) \subseteq T$  and  $\varphi_z := (\pi|_{B_r(z)})^{-1}$ . We claim that the collection  $\mathcal{A} = \{U_z, \varphi_z | z \in \mathbb{C}\}$  forms an atlas for  $T$ . It is clear that  $\mathcal{A}$  gives a cover for  $T$ . Moreover, by definition the maps  $\varphi_z$  are homeomorphic to their images. Assume that  $U_{z_1} \cap U_{z_2} \neq \emptyset$ . For  $j \in [2]$  denote by  $(\alpha_j, \beta_j)$  the unique pair of real numbers such that  $z_j = \alpha_j + t\beta_j$ . We have that  $T_{21}(z) = (\varphi_{z_1} \circ \varphi_{z_2}^{-1})(z) = z + k$ , where  $k = ([\alpha_2] - [\alpha_1]) + ([\beta_1] - [\beta_2])t$  is just a constant depending on  $z_1$  and  $z_2$ . Therefore the transition function  $T_{21}$  is holomorphic  $\implies T$  is a Riemann surface.

### 3 Graph of complex functions

**Definice 3.1** (Graph)

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a continuous function. The graph of  $f$  is the set

$$\Gamma_f := \{(z, f(z)) | z \in \mathbb{C}\} \subseteq \mathbb{C} \times \mathbb{C},$$

given the subset topology.

### Poznámka

Note that  $\Gamma_f$  is Hausdorff since  $\mathbb{C} \times \mathbb{C}$  is Hausdorff. The graph of  $f$  is naturally given the structure of a Riemann surface by an atlas, with one chart, namely  $\Gamma_f$ : the local coordinate function is the projection map  $\varphi := \pi_1|_{\Gamma_f}$ , i.e.  $(z, f(z)) \mapsto z$ . TODO!!!(Jedna celá tabule)

### Definice 3.2 (Affine plane curve)

For any polynomial  $p(x, y) \in \mathbb{C}[x, y]$ , the set  $V(p) := \{(x, y) | p(x, y) = 0\} \subseteq \mathbb{C}^2$ , is called an affine plane curve. We say that  $V(p)$  is smooth if  $\nexists (x_0, y_0) \in V(p)$  such that  $\frac{\partial p}{\partial x}(x_0, y_0) = 0 = \frac{\partial p}{\partial y}(x_0, y_0)$ .

### Věta 3.1

*A smooth affine plane curve is a Riemann surface.*

┌

#### Důkaz

Let  $(x_0, y_0) \in V(p)$ . Since  $V(p)$  is smooth, then for at least one of  $\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}$  is non-zero at  $(x_0, y_0)$ . Assume (WLOG) that  $\frac{\partial p}{\partial y}(x_0, y_0) \neq 0$ . Then by the implicit function theorem, exists a neighbourhood  $U_{(x_0, y_0)} \subseteq \mathbb{C}^2$ , and a neighbourhood  $V_{x_0} \subseteq \mathbb{C}$  and a holomorphic function  $f : V_{x_0} \rightarrow \mathbb{C}$  such that  $V(p) \cap U_{(x_0, y_0)} = \{(x, f(x)) | x \in V_{x_0}\}$ . We call this the graph of  $f$ .

We get a local chart on  $V(p)$  around  $(x_0, y_0)$  (as in the previous example) by setting  $\varphi_{(x_0, y_0)} : V(p) \cap U_{(x_0, y_0)} \rightarrow V_{x_0}, (x, f(x)) \mapsto x$ . Finally, we show that the transition functions are holomorphic: for  $U_{(x_0, y_0)} \cap U_{(x, y)} \cap V(p) \neq \emptyset$ , if  $\varphi_{(x_0, y_0)}$  and  $\varphi_{(x, y)}$  are both projections to the same axis, the transition function  $\varphi_{(x_0, y_0)} \circ \varphi_{(x, y)}^{-1}$  is the identity function restricted to the appropriate domain in  $\mathbb{C}$ . Assume now that  $\varphi_{(x_0, y_0)}$  is projection onto the  $x$ -axis and that  $\varphi_{(x, y)}$  is projection to the axis  $y$ . Then set  $U_{(x_0, y_0)} \cap U_{(x, y)} \cap V(p)$  is simultaneously on the graph of a holomorphic function  $f_0$  and of a holomorphic function  $f_1$ . Then functions all  $\varphi_{(x, y)} \circ \varphi_{(x_0, y_0)}^{-1} = f_0(x)$  and  $\varphi_{(x_0, y_0)} \circ \varphi_{(x, y)}^{-1} = f_1(x)$  restricted to the appropriate domains, which are holomorphic. □

└

## 4 Projective curves

### Příklad

Consider the polynomial  $p(x, y, z) = x^2 + y + z + 1$ . Note that  $p(1, 1, 1) = 4 \neq 7 = p(2, 2, 2)$  since  $[1, 1, 1] = [2, 2, 2]$  in  $\mathbb{P}^2(\mathbb{C})$   $p$  does not restrict to  $\mathbb{P}^2(\mathbb{C})$ .

### Definice 4.1 (Homogeneous polynomial)

A polynomial  $p \in \mathbb{C}[x, y, z]$  is said to be homogeneous of degree  $l$ , if the following equivalent conditions hold

- every monomial of  $p$  has degree  $l$ ;
- for each  $t \in \mathbb{C}$ :  $p(tx, ty, tz) = t^l p(x, y, z)$ ;
- $x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} + z \frac{\partial p}{\partial z} = lp$ .

*Důsledek*

If  $p$  is homogeneous,  $V(p) \subset \mathbb{P}^2(\mathbb{C})$  is well-defined.

*Příklad*

Confirm that these three conditions are equivalent.

*Příklad*

Show that if  $p \in \mathbb{C}[x, y, z]$  is a homogeneous polynomial, then the set of points  $[x, y, z] \in \mathbb{P}^2(\mathbb{C})$  satisfying  $p(x, y, z) = 0$  is well-defined.

## Definice 4.2

We call

$$V(p) := \{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) \mid p(x, y, z) = 0\}$$

the vanishing locus of  $p$ . Moreover, we call  $V(p)$  a (plane) projective curve of degree  $l$ .

If

$$\left\{ (x, y, z) \in \mathbb{C}^3 \mid \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \right\}$$

the  $V(p)$  is said to be smooth.

## Tvrzení 4.1

A smooth projective plane curve  $V(p)$  is a compact Riemann surface.

*Důkaz*

We first show that  $V(p)$  is compact by showing that  $V(p)$  is closed set in  $\mathbb{P}^2(\mathbb{C})$  which is a compact space.

Consider the diagram  $\mathbb{P}^2(\mathbb{C}) \xleftarrow{\pi} \mathbb{C}^3 \setminus \{(0, 0, 0)\} \xrightarrow{p} \mathbb{C}$ , where  $\pi$  is the natural projection function and  $p$  is the continuous function defined by the homogeneous polynomial  $p : \mathbb{C}^3 \setminus \{(0, 0, 0)\} \rightarrow \mathbb{C}$ ,  $(x, y, z) \mapsto p(x, y, z)$  by definition  $V(p)$  is a closed subset of  $\mathbb{P}^2(\mathbb{C})$  if  $\pi^{-1}(V(p))$  is closed in  $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ . But  $\pi^{-1}(V(p)) = p^{-1}(0)$  is the inverse image of the closed set  $\{0\} \subseteq \mathbb{C}$ . Thus since  $p$  is continuous,  $p^{-1}(0)$  is closed, in other words,  $\pi^{-1}(V(p))$  is closed. Thus  $V(p)$  is compact.

So, to show that  $V(p)$  is Riemann surface, we need to show that its intersection with any of the coordinate open sets of  $\mathbb{P}^2(\mathbb{C})$  is a Riemann surface. So let us consider (WLOG) the chart  $U_z = \{[x, y, z] | z \neq 0\} \subseteq \mathbb{P}^2(\mathbb{C})$  with affine coordinates  $\varphi_z(x, y, z) = (\frac{x}{z}, \frac{y}{z})$ . The set  $\varphi_z(V(p) \cap U_z)$  is equal to  $V(\tilde{p})$  where  $\tilde{p}(x, y) = p(x, y, 1)$ .

Now for any  $(x, y) \in \mathbb{C}^2$ :

$$(*) : \frac{\partial \tilde{p}}{\partial x}(x, y) = \frac{\partial p}{\partial x}(x, y, 1), \quad (**) : \frac{\partial \tilde{p}}{\partial y}(x, y) = \frac{\partial p}{\partial y}(x, y, 1).$$

We claim  $\nexists$  an  $(\hat{x}, \hat{y}) \in \mathbb{C}^2$  such that

$$\tilde{p}(\hat{x}, \hat{y}) = \frac{\partial \tilde{p}}{\partial x}(\hat{x}, \hat{y}) = \frac{\partial \tilde{p}}{\partial y}(\hat{x}, \hat{y}) = 0.$$

This claim implies that  $V(\tilde{p})$  is a smooth affine plane curve and hence a Riemann surface. Since the restriction of  $V(p)$  with any affine chart is a Riemann surface, then so is  $V(p)$ .

So it remains to prove the claim: Assume  $\exists(\hat{x}, \hat{y}) \in \mathbb{C}^2$  satisfying condition above. By  $(*)$  and  $(**)$ , together smoothness of  $V(p)$ , which would imply that  $\frac{\partial p}{\partial z}(\hat{x}, \hat{y}, 1) \neq 0$ . But now Euler's identity implies  $0 \neq \frac{\partial p}{\partial x}(\hat{x}, \hat{y}, 1) + \frac{\partial p}{\partial y}(\hat{x}, \hat{y}, 1) + \frac{\partial p}{\partial z}(\hat{x}, \hat{y}, 1) = lp(\hat{x}, \hat{y}, 1) = 0 \implies$  contradiction  $\implies$  we are done.  $\square$

*Příklad*

Confirm that  $V(p)$  is Hausdorff.

*Například (Elliptic curves)*

Consider a polynomial  $p$  of the form  $p(x, y, z) = y^2z - (x - \alpha_1z)(x - \alpha_2z)(x - \alpha_3z)$  where  $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$  are distinct complex numbers. Note that the partial derivative with respect to  $y$  satisfies  $\frac{\partial p}{\partial y} = 2yz$ , which is zero only if  $y = 0$  or  $z = 0$ . We show that  $V(p)$  is a smooth projective curve by considering the case  $z = 0$ ,  $y = 0$ , and finding in each case a non-vanishing partial derivative.

„Case  $z = 0$ “: Then the only part in  $\mathbb{P}^2(\mathbb{C})$  belonging to  $V(p)$  is  $[0, 1, 0]$ . But we have  $\frac{\partial p}{\partial z} = y^2 + Q(x, z) = 1 + 0 \neq 0$ .

„Case  $y = 0$ “: Then the parts belonging to  $V(p)$  are  $[\alpha_1, 0, 1]$ ,  $[\alpha_2, 0, 1]$ ,  $[\alpha_3, 0, 1]$ . For  $j \in [3]$ :  $\frac{\partial p}{\partial x}(\alpha_j, 0, 1) \neq 0$ , follows from the fact that the  $\alpha_1, \alpha_2, \alpha_3$  are distinct.

So  $V(p)$  is a smooth projective curve of degree 3.

*Například*

Consider the function  $\varphi : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^3(\mathbb{C})$  defined in homogeneous coordinates by  $\varphi[s, t] = [s^3, s^2t, st^2, t^3]$ . We call the image of  $\varphi$  the twisted cubic in  $\mathbb{P}^3(\mathbb{C})$ .

## 5 Holomorphic maps of Riemann surfaces

### Definition 5.1

Let  $X, Y$  be two Riemann surfaces, and  $f : X \rightarrow Y$  a function (of sets).

- We say that  $f$  is holomorphic at  $x \in X$  if for every choice of charts  $\varphi_x, \varphi_{f(x)}$ , the function  $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$  is holomorphic.
- If  $U \subseteq X$  is open, we say that  $f$  is holomorphic on  $U$  if  $f$  is holomorphic at each  $x \in U$ .
- If  $f$  is holomorphic on  $U = X$ , then we say that  $f$  is a holomorphic map.

The function  $F := \varphi_{f(x)} \circ f \circ \varphi_x^{-1}$  is called a local expression for  $f$ .

*Příklad*

Show that a map of Riemann surfaces  $f : X \rightarrow Y$  is holomorphic at  $x \in X \Leftrightarrow \exists$  a choice of charts  $\varphi_x, \varphi_{f(x)}$  such that  $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$  is holomorphic at  $x$ .

*Příklad (Eg)*

Let  $X, Y$  be two Riemann surfaces and choose a point  $y \in Y$ . Denote associated constraint map  $c : X \rightarrow Y, x \mapsto y$ . We see that  $c$  is a holomorphic map.

*Příklad (Eg)*

Let  $X$  be a Riemann surface. The identity map on  $X$  is the function  $I_X : X \rightarrow X, x \mapsto x$ . We see that  $I_x$  is a holomorphic map.

*Příklad (Eg)*

Consider an arbitrary rational function  $f(z) = \frac{p(z)}{q(z)}$ , where  $p(z), q(z) \in \mathbb{C}[z]$  are two polynomials with distinct roots. We can extend  $f$  to a function from  $\mathbb{P}^1(\mathbb{C})$  to  $\mathbb{P}^1(\mathbb{C})$  as follows:

- $f(\alpha) := \infty$  (point at infinity of the Riemann sphere) where  $\alpha$  is a root of  $q$ ;
- $f(\infty) := \lim_{z \rightarrow \infty} \frac{p(z)}{q(z)}$ .

Show that  $f$  is a holomorphic function from  $\mathbb{P}^1(\mathbb{C})$  to  $\mathbb{P}^1(\mathbb{C})$ .

Recall that  $\mathbb{P}^1(\mathbb{C})$  is  $\mathbb{C} \cup \infty$  with a topology. Identifying  $\mathbb{C}$  with image of the first affine chart  $\varphi_1([U_1])$ , the additional chart point corresponds to the image of zero in the second affine chart. We denote this point by  $\infty$ .

This we have that  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ . Using this identification, define the function  $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ ,  $z \mapsto z^2 =: \omega, \infty \mapsto \infty$ . We will now show that  $f$  is a holomorphic map.

We denote by  $z$  the local coordinate on the source and by  $\omega$  the local coordinate on the range. Then since  $\omega = f(z) = z^2$  is a holomorphic function on all of  $\mathbb{C}$ ,  $f$  is holomorphic on the image of  $U_1$ .

It remains to show that  $f$  is holomorphic at  $\infty$ . We consider the local expression for  $f$  using the chart  $U_2$  whose image contains  $\infty$ . We denote by  $\tilde{z} := \frac{1}{z}$  the corresponding local coordinate for the source and  $\tilde{\omega} := \frac{1}{\omega}$  for the range.

The local expression  $\tilde{F}$  for  $f$  in these coordinates is then obtained by composing  $F(z)$  with the transition functions for the local coordinate:

$$\tilde{F}(\tilde{z}) = \tilde{\omega} = \frac{1}{\omega} = \frac{1}{z^2} = (\tilde{z})^2.$$

Since the point at  $\omega$  corresponding to  $\tilde{z} = \tilde{\omega} = 0$ , and we have that  $f(\infty) = \infty$  and  $\tilde{F}(0) = 0$ , the local expression extends to the whole chart, and in particular, it is a holomorphic function at 0. So  $f$  is holomorphic at every point of  $\mathbb{P}^1(\mathbb{C})$ .

## Definition 5.2

Two Riemann surfaces  $X, Y$  are called isomorphic (or bi-holomorphic) if  $\exists$  holomorphic maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $g \circ f = I_X$  and  $f \circ g = I_Y$ . In this case we write  $X \simeq Y$ . We call  $f$  and  $g$  isomorphisms (or bi-holomorphisms). An isomorphism  $h : X \rightarrow X$  from a Riemann surface to itself is called an automorphism.

### Příklad

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function and  $\Gamma_f \subseteq \mathbb{C}^2$  its graph. Show that  $\Gamma_f \simeq \mathbb{C}$ .

## Definition 5.3

We say that a chart  $(U_x, \varphi_x)$  for a Riemann surface is centered at  $x$  if  $\varphi_x(x) = 0$ .

## Věta 5.1

Let  $f : X \rightarrow Y$  be a non-constant holomorphic map of Riemann surfaces. For any  $x \in X$   $\exists$  charts centered at  $x$  such that the local expression of  $f$  in terms of these charts is  $z \mapsto z^h$ , for some  $h \geq 1$ .

┌

*Důkaz*

Let  $\varphi$  and  $\psi$  be two charts centered at  $x$ , and  $f(x)$ , respectively. This gives the local expression  $F = \psi \circ f \circ \varphi^{-1}$ . Consider the Taylor expansion of  $F$  at 0 and let  $k$  be the smallest positive integer such that the coefficient of  $z^k$  does not vanish. Since  $F(0) = 0$ ,  $k > 1$ , and  $F(z) = z^k (\sum_{n=0}^{\infty} a_{k+n} z^n)$ . Denote by  $G(z) = a_k + a_{k+1}z + \dots$  the second factor in that equation. The function  $G(z)$  is holomorphic at 0 and  $G(0) = a_k \neq 0$ . Thus we can make a choice of root  $\sqrt[k]{G(z)}$  such that the associated map is well-defined and holomorphic around  $0 = \varphi(z)$ .

Defining  $h(z) = z \cdot \sqrt[k]{G(z)}$  we have that  $F = h^k$ . The function  $h$  is holomorphic in a neighbourhood  $U$  of  $0 = \varphi(x)$ ,  $h(0) = 0$ , and  $h'(0) \neq 0$ . The inverse function theorem now implies that  $h$  is bi-holomorphic on a neighbourhood  $U' \subseteq U$  of  $\varphi(x)$ , and therefore the composition  $\tilde{\varphi} = h \circ \varphi$  gives local chart for  $x$  centered at  $x$ . The local coordinate  $\tilde{z}$  coming from  $\tilde{\varphi}$  is related to  $z$  with  $\tilde{z} = h(z)$ . The local expression for  $f$  allows  $x$  is now obtained by changing coordinates from  $z$  to  $\tilde{z}$  in  $F$ :

$$\tilde{F}(\tilde{z}) = F(z(\tilde{z})) = h(z)^k = (\tilde{z})^k.$$

└

□

## Definice 5.4

Let  $f : X \rightarrow Y$  be a non-constant holomorphic map of Riemann surfaces

- Given a point  $x \in X$ , the integer  $k_x$  such that  $\exists$  a local expression centered at  $x$  of the form  $F(z) = z^{k_x}$  is called the Ramification index of  $f$  at  $x$ .
- The quantity  $V_x = k_x - 1$  the differential length of  $f$  at  $x$ .
- A point  $x$  with Ramification index  $k_x = 1$  is called unramified.
- A point for which  $k_x \geq 2$  is ramified (ramification point).
- If  $x$  is a ramification point, then  $f(x) \in Y$  is called a branch point.
- The ramification locus is the subset of  $X$  consisting of all ramification points of  $f$ .
- The branch locus is  $f(R)$ .

*Příklad*

Let  $f : X \rightarrow Y$  be a non-constant holomorphic map of two Riemann surfaces, and let  $x \in X$ . Suppose that  $f$  has two local expressions around  $x$  of the form  $F(z) = z^k$ ,  $\tilde{F}(\tilde{z}) = \tilde{z}^{\tilde{k}}$ , then  $k = \tilde{k}$ .



Řešení (Hint)

Let  $\varphi, \tilde{\varphi}, \psi, \tilde{\psi}$  be charts. Observe that the change of coordinates near  $x$  must be of the form  $\tilde{z} = z\alpha(z)$ , where  $\alpha(z)$  is holomorphic in a neighbourhood of 0, and  $\alpha(0) \neq 0$ . (Try to draw this in picture.)

Similarly (but going in the opposite direction)  $\omega = \tilde{\omega}\beta(\tilde{\omega})$  for some holomorphic function  $\beta$  with  $\beta(0) \neq 0$ . Write down the Taylor expansions of  $\alpha$  and  $\beta$  and use the fact that  $F$  is obtained from  $\tilde{F}$  via the above changes of coordinates.

### Lemma 5.2

*The ramification locus  $R$  is a discrete subset of open subsets (i.e.  $\exists U_i \subset X$ , such that  $U_i$  contains exactly one element of  $R$ ).*

Poznámka

A discrete subset of a compact set is finite (an infinite set has a limit point, thus a discrete subset, which has no limit points, must be finite).

### Lemma 5.3

*If  $X$  is a compact Riemann surface and  $f : X \rightarrow Y$  is a non-constant holomorphic map of two Riemann surfaces, the ramification locus  $R$  is a finite set. Moreover, the branch locus is also finite.*

## 6 Holomorphic maps between compact Riemann surfaces

### Věta 6.1

*Let  $f : X \rightarrow Y$  be a holomorphic of Riemann surfaces. If  $X$  is compact and  $f$  is non-constant, then  $f$  is onto / surjective.*

Důkaz

Consider the image  $f(X) \subset Y$ . By Liouville's theorem (from complex analysis)  $f(x)$  is open. On the other hand, since  $X$  is compact and  $f$  is continuous,  $f(X)$  is compact. Since  $Y$  is Hausdorff, we must have that  $f(X)$  is closed.

Finally, since  $f(X)$  is an open, closed, and non-empty subset of a connected topological space, we must have that  $f(X) = Y$ .  $\square$

Důsledek

Hence  $Y$  is also compact.

### *Důsledek*

If  $X$  is a compact Riemann surface, the only holomorphic functions  $f : X \rightarrow \mathbb{C}$  are the constant functions.

### *Poznámka (Discussion)*

Let  $f : X \rightarrow Y$  be a non-constant map of compact Riemann surfaces. Consider a point  $x \in f^{-1}(y)$ . We know that  $\exists$  a neighbourhood  $U_x$  and coordinate charts centered at  $x$  such that we have local expression  $z \rightarrow z^{k_x}$ .

Since  $F^{-1}(0) = 0$ , there are no other pre-images of  $y$  in  $U_x$ . Thus the pre-image set  $f^{-1}(y)$  is discrete, and since  $X$  is compact, the pre-image is a finite set.

## **Věta 6.2**

Let  $f : X \rightarrow Y$  be a non-constant holomorphic map of compact Riemann surfaces. If  $y_0, y_1$  are not in the branch locus of  $f$  then  $|f^{-1}(y_0)| = |f^{-1}(y_1)|$ .

┌

*Důkaz*

└

Next week.

□

## **Definice 6.1**

For  $f : X \rightarrow Y$  a non-constant map of compact Riemann surfaces, the degree of  $f$  is the cardinality of the fiber  $f^{-1}(y)$ , for any  $y \notin$  branch locus of  $f$ . If  $f$  is constant, then we say that its degree is 0.

### *Důkaz (next week proof)*

Let  $B$  be the branch locus of  $f$ . Since  $B$  is finite,  $Y \setminus B$  is a connected topological space ( $Y$  is a surface). Hence it cannot have a proper subset which is both closed and open. Let  $y_0 \in Y \setminus B$  and set  $d := |f^{-1}(y_0)|$ .

The set  $A := \{y \in Y \setminus B \mid |f^{-1}(y)| = d\}$  is non-empty since it contains  $y_0$ . We claim that „ $A$  is open in  $Y \setminus B$ “: For any  $y \in A$  we show that  $y$  is an interior point of  $A$ . Denote by  $\{x_1, \dots, x_d\}$  the inverse images of  $y$  and by  $U_{x_1}, \dots, U_{x_d}$  pairwise disjoint charts around each inverse such that  $f$  admits the local expression  $F(z) = z$  on each chart. We can choose (see exercise below)  $V_y \subseteq F(U_{x_1}) \cap \dots \cap f(U_{x_d})$  a connected neighbourhood of  $y$  homeomorphic to a disk, such that the inverse image  $f^{-1}(V_y)$  consists of  $d$  connected components  $\tilde{U}_{x_i}$  (each contains one of the  $x_i$  and is contained in  $U_{x_i}$ ). Then  $f$  restricted to each  $\tilde{U}_{x_i}$  is bijective onto  $V_y$ . This implies that every point of  $V_y$  has  $d$  inverse images, and hence that  $y$  is an interior point for  $A$ .

Now consider  $A^c = \{y \in Y \setminus B \mid |f^{-1}(y)| \neq d\}$ . With essentially the same argument, one shows that  $A^c$  is also open in  $Y \setminus B$ . Thus we can conclude that  $A$  is closed. Since  $A$  is a non-empty, open and closed subset of a connected topological space,  $A = Y \setminus B$ . □

*Příklad*

Show that a  $V_y$  (as in the previous proof) exists by showing that otherwise one may construct a sequence  $\{\xi_n\}$  of points of  $X$  such that  $\xi_n \rightarrow \xi$ ,  $f(\xi_n) \rightarrow y$  but  $f(\xi) \neq y$  violating continuity.

**Definition 6.2** (Degree)

For  $f : X \rightarrow Y$  a non-constant holomorphic map of compact Riemann surfaces, the degree of  $f$  is the cardinality of the fiber  $|f^{-1}(y)|$ , for any  $y \in Y$  such that  $y \notin \text{branch locus}$ . If  $f$  is constant then we say that it has degree 0.

*Příklad*

Let  $f : X \rightarrow Y$  be a holomorphic map of compact Riemann surfaces, of degree  $d > 0$ ,  $y \in Y$  and  $f^{-1}(y) = \{x_1, \dots, x_n\}$ . Prove that  $\sum_{i=1}^n k_{x_i} = d$ .

*Poznámka* (Recall the classification of compact Riemann surfaces)

Any compact, connected surface is homeomorphic to one surface in the following list:  $S^2$  (two sphere, also  $\mathbb{C}$ -projective plane  $\mathbb{P}\mathbb{C}$ ),  $T^{\#g}$ ,  $\mathbb{P}^2(\mathbb{R})^{\#n}$ .

The first two are Riemann surfaces (Ex: the connected sum of Riemann surfaces is again a Riemann surface). The third are not orientable, so not Riemann surfaces.

**Definition 6.3** (Genus)

Define the genus of a compact connected Riemann surface as 0 for 2 sphere and  $g$  for connected sum of  $g$  tori.

┌

*Poznámka* (Alternative definition of genus)

A vector field  $X$  on a differential manifold is a derivation on  $C^\infty(m)$  i.e. a  $\mathbb{R}$  linear map  $X : C^\infty(m) \rightarrow C^\infty(m)$  such that  $X(f \cdot g) = X(f) \cdot g + f \cdot X(g) \forall f, g \in C^\infty(m)$ . The space of vector fields is denoted by  $\mathcal{X}(m)$ . Note that  $\mathcal{X}(m)$  is a  $C^\infty(m)$ -module.  $(fX)(g) = f(X(g))$ .  $((fX)(g \cdot h) = f(X(g \cdot h)) = f(X(g) \cdot h + g \cdot X(h)) = f(X(g)) \cdot h + g \cdot f(X(h))$ .)

Denote  $\Omega^1(m) = \mathcal{X}(m)^* = \text{Mor}_{C^\infty(m)}(\mathcal{X}(m), C^\infty(m))$ . This is again  $C^\infty(m)$ -module (just as for any dual module)

$$(f\omega)(X) := f(\omega(X)) \quad ((f\omega)(gX) = f(\omega(gX)) = g(f\omega(X)) = g((f\omega)(X))).$$

Consider the map  $d : C^\infty(m) \rightarrow \Omega^1(m)$ ,  $f \mapsto df$ , where  $df : \mathcal{X}(m) \rightarrow C^\infty(m)$ ,  $X \mapsto X(f)$ . (Note that  $d(fg)(X) = X(fg) = X(f) \cdot g + f \cdot X(g) = df(X) \cdot g + f dg(X) = (g \cdot df)(X) + (f \cdot dg)(X) \implies d(f \cdot g) = g \cdot df + f \cdot dg$ .)

Now let's restrict to surfaces:

$$\Omega^2(m) := \frac{\Omega^1(m) \otimes_{C^\infty(m)} \Omega^1(m)}{\{\omega \otimes \nu + \nu \otimes \omega\}}.$$

We denote the product in  $\Omega^2(m)$  by  $\wedge$ .  $\implies \omega \wedge \nu = -\nu \wedge \omega$ .

TODO example?

### Věta 6.3 (Riemann–Hurwitz)

For  $f : X \rightarrow Y$  be non-constant, degree  $d$ , holomorphic map of compact Riemann surfaces. Denote by  $g_x$ , resp.  $g_y$ , the genus of  $X$ , and  $Y$ . Then  $2g_X - 2 = d(2g_Y - 2) + \sum_{x \in X} v_x$ , where  $v_x = h_x - 1$  is the differential length of  $f$  at  $x$ .

### Příklad

Observe that  $\sum_{x \in X} v_x = \sum_{x \in \text{Ram. locus}} v_x$  and hence conclude that this sum is well-defined.