Příklad (1.)
As we have discussed in class, a mapping cylinder M_f for a map $f : X \to Y$ deformation retracts to the subspace Y by sliding each point (x, t) along the segment $\{x\} \times [0, 1] \subset M_f$ to the endpoint $f(x) \in Y$. Show continuity of this deformation retraction. Hint: Check the discussion on this subject on p. 2 of Hatcher's book.

Důkaz
We have deformation retraction $F(x, t) = (x, t)$. Firstly we show continuity of F from $X \times [0, 1]$ to space $X \times [0, 1]$ (without Y and quotient). When we compose F with projection to X or $[0, 1]$, we get identity, so F is continuous to $X \times [0, 1]$.

Now we do the quotient. In domain of F (now $X \times [0, 1] \coprod Y / \sim$), there are even more open sets. Open sets in $X \times [0, 1)$ is unchanged. In $Y = X \times \{1\}$ we have in addition open sets „created" by Y. But $f(x)$ is continuous, so if $U \subset Y$ is open, $f^{-1}(U)$ is open and so $U \subset X \times \{1\}$ was already open. Thus continuity to $X \times [0, 1] \coprod Y / \sim$ is the same as the continuity to $X \times [0, 1]$. □

Příklad (2.)
Construct an explicit deformation retraction of $\mathbb{R}^n \setminus \{\mathbf{o}\}$ onto S^{n-1}.

Řešení

$$F(t, x) = f_t(x) = t \cdot \frac{x}{\|x\|} + (1 - t) \cdot x, \qquad \|x\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

Obviously for $t = 0$: $f_0(x) = x$, so $f_0 = \mathrm{id}_X$, and $f_1(x) = \frac{x}{\|x\|} \in S^{n-1}$, so $f_1(X) = S^2$ and $f_1|_A = \mathrm{id}_A$ ($x \in A \implies \|x\| = 1$).

It remains to show continuity: multiplication, addition, and square root are continuous. $\|x\| \neq 0$ ($x \neq \mathbf{o}$), thus $\frac{1}{\|x\|}$ is continuous.

Příklad (3.)
Given positive integers v (for vertices), e (for edges), and f (for faces) satisfying $v-e+f=2$, construct a cell structure on S^2 having v 0-cells, e 1-cells, and f 2-cells.

Řešení
We take „chain“ of vertex–edge–vertex–edge–...–vertex so we use all vertices ($e \geqslant v-1$ from equality $v-e+f=2$ so we have enough edges): Take $e_1^0, \ldots, e_v^0$, $e_1^1, \ldots, e_{v-1}^1$, such that we have $\varphi_i^1 : \partial e_i^1 \to \bigcup_j e_j^0$, $\varphi_i^1(0) := e_i^0$, $\varphi_i^1(1) := e_{i+1}^0$.

Now we add „balloons“ aka faces wrapped with one edge starting and ending in first vertex. So we add e_v^1 ($e_{v+1}^1, \ldots, e_{v-1+f-1}^1$) and e_1^2 ($e_2^2, \ldots, e_{f-1}^2$) with $\varphi_i^1 : \partial e_i^1 \to \bigcup_j e_j^0$, $\varphi_i^1(0) := \varphi_i^1(1) := e_1^0$ a $\varphi_i^2 : \partial e_i^2 \to \bigcup_j e_j^0 \cup \bigcup_j e_j^1$ homeomorphism on e_{v-1+i}^1.

Finally we add e_f^2 and $\varphi_f^2 : \partial e_f^2 \to \bigcup_j e_j^0 \cup \bigcup_j e_j^1$ which "goes through"

$$e_1^1, e_2^1, \ldots, e_{v-1}^1, e_{v-1}^1, e_{v-2}^1, \ldots, e_1^1, e_v^1, e_{v+1}^1, \ldots, e_{v-1+f-1}^1.$$

Příklad (4.)
Show that the change-of-basepoint homomorphism β_h (for fundamental groups) depends only on the homotopy class of h.

Důkaz
We define β_h as $[f] \mapsto [h \cdot f \cdot h^{-1}]$. When we have $\tilde{h}$ and h and homotopy $F(\cdot, 0) = h$ and $F(\cdot, 1) = \tilde{h}$, than we can define

$$G(x,t) = \begin{cases} F\left(3x, t\right), & \text{for } x \leqslant \frac{1}{3} \\ f(3x-1), & \text{for } \frac{1}{3} \leqslant x \leqslant \frac{2}{3} \\ F\left(3-3x, t\right), \text{for } \frac{2}{3} \leqslant x, \end{cases}$$

where $x \in [0,1]$ and $f, h : [0,1] \to X$, $h \cdot f \cdot h^{-1} : [0,3] \to X$. G is homotopy $h \cdot f \cdot h^{-1}$ to $\tilde{h} \cdot f \cdot \tilde{h}^{-1}$ $\square$