*Poznámka* (Exam)
Oral, similar as in FA1.

*Poznámka* (Credit)
Similar as in FA1.

# 1 Banach algebras

## 1.1 Basic properties

**Definice 1.1** (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over $\mathbb{K}$, if

- $(A, +, -, 0, \cdot_S)$ is vector space over $\mathbb{K}$;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K}\ \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

*Důsledek*
1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ $(\forall a \in A)$.

If $e_1$ is left unit and $e_2$ is right unit, then $e_1 = e_2$ is unit. $(e_1 = e_1 \cdot e_2 = e_2)$

2) (Algebra) homomorphism $\varphi : A \to B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

**Tvrzení 1.1**

*Let $A$ be algebra over $\mathbb{K}$. Put $A_e = A \times \mathbb{K}$ with operations $A_e$ defined coordinate-wise and multiplication defined by*

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \qquad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

*Then $A_e$ is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if $A$ is commutative, then $A_e$ is commutative.*

*Důkaz*
We have $A_e$ is vector space (from linear algebra). We easy proof from definition, that $A_e$ is algebra, $(\mathbf{o}, 1)$ is a unit in $A_e$ and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. □

**Definice 1.2** (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv$ $A$ is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ $(\forall a, b \in A)$.

**Definice 1.3** (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv$ $(A, \|\cdot\|)$ is normed algebra nad Banach space.

*Například*
$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If $T$ is Hausdorff topological space, then $\mathcal{C}_b(T) = \{f : T \to \mathbb{K} | f \text{ is continuous and bounded}\} \subseteq l_\infty(T)$ is closed subalgebra.

If $T$ is locally compact, Hausdorff, not compact. Then $\mathcal{C}_0(T) = \{f : T \to \mathbb{K} \text{ continuous } | \forall \varepsilon > 0 : \{t \in T$ $\mathcal{C}_b(T)$ is closed subalgebra, which doesn't have unit.

If $X$ is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If $X$ is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

**Tvrzení 1.2**

*If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \to A$ is Lipschitz on bounded sets.*

*Důkaz*

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r)\ \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) :$$
$$\|ab - cd\| \leqslant \|a(b-d)\| + \|(a-c) \cdot d\| \leqslant \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leqslant R \cdot (\|b-d\| + \|a-c\|) \leqslant 2R\|(a, b) - (c, d)\|.$$
□

## Tvrzení 1.3

*Let $(A, \|\cdot\|)$ be a Banach algebra. On $A_e$ we consider the norm*

$$\|(a, \alpha)\| := \|a\| + |\alpha|, \qquad (a, \alpha) \in A \times \mathbb{K} = A_e.$$

*Then $(A_e, \|\cdot\|)$ is Banach algebra.*

*Důkaz*
It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a, \alpha) \cdot (b, \beta)\| \leqslant \|(a, \alpha)\| \cdot \|(b, \beta)\|,$$

which is easy. □

*Poznámka*
There is more (natural) ways to define norm on $A_e$ (unlike $\cdot$ on $A_e$, which is natural).

$A$ has a unit … we may still consider $A_e$.

If $e \in A \setminus \{\mathbf{o}\}$ is a unit, then $\|e\| \geqslant 1$, because $\|e\| = \|e^2\| \leqslant \|e\|^2$.

## Věta 1.4

*Let $A$ be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a \cdot x$, $x \in A$. Then $I : A \to \mathbb{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leqslant 1$.*

*Moreover, if $A$ has a unit $e$, then $I$ is isomorphism into and $I(e) = \mathrm{id}$.*

*If $\|x^2\| = \|x\|^2$, $x \in A$, then $I$ is isometry into.*

*Důkaz*
„$L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leqslant 1$“: Linearity is obvious, $\|L_a(x)\| = \|a \cdot x\| \leqslant \|a\| \cdot \|x\|$, so $\|L_a\| \leqslant \|a\|$ and so $\|I\| \leqslant 1$. Since it is easily $I$ preserves multiplication, so we are left to prove the „Moreover“ part.

„$A$ has a unit $e$“: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geqslant \|L_a \left(\frac{e}{\|e\|}\right) = \frac{a}{\|e\|} = \frac{1}{\|e\|} \cdot a.$$

So $I$ is bounded from below, so $I$ is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \operatorname{id}.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$ :

$$\|a\| \geqslant \|I(a)\| = \|L_a\| \geqslant \|L_a \left(\frac{a}{\|a\|}\right) \| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So $I$ is isometry. □

*Poznámka*
$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies \exists$ equivalent norm $\|\cdot\|$ on $A$ such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

*Důkaz*
Let $I : A \to \mathcal{L}(A)$ be as before. Put $|\|x\|| := \|I(x)\|$, $x \in A$. Since $I$ is isomorphism, $|\| \cdot \||$ is equivalent norm. Moreover, $|\|x \cdot y\|| = \|I(x \cdot y)\| \leqslant \|I(x)\| \cdot \|I(y)\| = |\|x\|| \cdot |\|y\||$, $x, y \in A$. So $(A, |\| \cdot \||)$ is a Banach algebra. Finally

$$|\|e\|| = \|I(e)\| = \|\operatorname{id}\| = 1.$$

□

## 1.2 Inverse elements

### Definice 1.4

$(M, \cdot, e)$ is monoid ($\cdot$ is associative, $e$ is unit). Then invertible elements form a group ($e^{-1} = e, \exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^\times := \{x \in M \mid \exists x^{-1}\}$

## Tvrzení 1.5

*If $(A, \cdot, e)$ is monoid and $x_1, \ldots, x_n \in A$ commute, then $x_1 \cdot \ldots \cdot x_n \in A^x \Leftrightarrow \{x_1, \ldots, x_n\} \subset A^x$.*

*Důkaz*

It suffices to prove it for $n = 2$ (and use induction). „If $x^{-1}$ and $y^{-1}$ exists, then $(xy)^{-1}$" is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so $z$ is left inverse to $y$. Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so $\tilde{z}$ is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. □

## Lemma 1.6

*Let $A$ be a Banach algebra with a unit.*

- $\|x\| < 1 \implies \exists (e-x)^{-1} \wedge (e-x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x+e)^{-1} \wedge \|(x+h)^{-1} - x^{-1}\| \leqslant \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

*Důkaz*

„First point": We have $\|x^n\| \leqslant \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0} |^\infty x^n \in A$. Moreover,

$$(e-x) \cdot \left( \sum_{n=0}^{\infty} x^n \right) = \lim_{N \to \infty} (e-x) \cdot (e + x + \ldots + x^N) = \lim_{N \to \infty} e - x^{N+1} = e,$$

because $\lim_{N \to \infty} \|x^{n+1}\| \leqslant \lim_{N \to \infty} \|x\|^N = 0$. And similarly $\left(\sum x^n\right) \cdot (e - x) = e$.

„Second point": $x + h = x \cdot (e + x^{-1}h)$ we have $x^{-1}$ exists and $(e + x^{-1}h)^{-1}$ exists (from first point), so from previous fact $(x+h)^{-1}$ exists. Moreover

$$(x+h)^{-1} = (e + x^{-1}h)^{-1} \cdot x^{-1} \overset{1)}{=} \sum_{n=0}^{\infty} \left(-x^{-1}h\right)^n x^{-1},$$

so

$$\|(x+h)^{-1} - x^{-1}\| = \|\sum_{n=1}^{\infty} \left(-x^{-1}h\right)^n x^{-1}\| \leqslant \|x^{-1}\| \cdot \sum_{n=1}^{\infty} \|x^{-1}h\|^n \leqslant$$

$$\leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \left(\|x^{-1}\| \cdot \|h\|\right)^n = \|x^{-1}\| \cdot \frac{\|x^{-1}\| \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}.$$

□

*Důsledek*
$A$ Banach algebra with a unit $\implies A^x \subset A$ is open and $A^x$ is topological group.

*Důkaz*
$A^x \subset A$ is open by previous lemma (second point). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^x \ni x_n \to x \in A^x \overset{?}{\implies} x_n^{-1} \to x^{-1}.$$

$$\|x_n^{-1} - x^{-1}\| \overset{h:=x_n-x}{\leqslant} \frac{\|x^{-1}\|^2 \cdot \|x_n - x\|}{1 - \|x^{-1}\| \cdot \|x_n - x\|} \to 0.$$

□

## 1.3 Spectral theory

**Definice 1.5** (Resolvent set, spectrum and resolvent)

$A$ Banach algebra with a unit, $x \in A$. We define resolvent set of $x$ as $S_A(x) := \{\lambda \in \mathbb{K} | \exists (\lambda \cdot e - x)^{-1}\}$. Next we define spectrum of $x$ as $\sigma_A(x) := \mathbb{K} \backslash S_A(x)$. Finally we define resolvent of $x$ as $R_x : S(x) \to A$, $R_x(\lambda) := (\lambda \cdot e - x)^{-1}$.

If $A$ doesn't have a unit, then notions above are defined with respect to $A_e$.

**Tvrzení 1.7**

*A Banach algebra*

*a)* $\forall x \in A : 0 \in \sigma_{A_e}(x)$ *(in particular, if $A$ has no unit, then $0 \in \sigma_A(x)$);*

*b) $A$ has unit* $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

*Důkaz* (a))

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\dots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

□

*Důkaz* (b))
By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in S_A(x) \Leftrightarrow \lambda \in S_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in $A$ and second means that $((0,\lambda)-(x,0))^{-1} = (-x,\lambda)^{-1}$ exists in $A$. We take „$x \to -x$“.

„ $\Longrightarrow$ “: find $(b,\beta) \in A_e$ such that $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$. So $(x\cdot b+\lambda\cdot b+\beta\cdot x, \lambda\cdot\beta) = (\mathbf{o},1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e+x)^{-1}\cdot x$. Similarly we find left inverse $\left(-\frac{1}{\lambda}x(x+\lambda e)^{-1}, \frac{1}{\lambda}\right)(x,\lambda)$. And next we prove that they are really inverses.

„ $\Longleftarrow$ “: Put $(b,\beta) := (x,\lambda)^{-1}$. Then $(\lambda e+x)^{-1} = b+\beta\cdot e$. We have $(x,\lambda)\cdot(b,\beta) = (\mathbf{o},1)$, so $\lambda\cdot\beta = 1$ and $x\cdot b+\lambda\cdot b+\beta\cdot x = \mathbf{o}$. Then

$$(\lambda e + x)\cdot(b+\beta\cdot e) = \lambda\cdot b+\lambda\cdot\beta\cdot e+x\cdot b+\beta\cdot x = e.$$

Similarly second inverse. $\square$