# 1 Area formula and coarea formula

**Věta 1.1**

*Let $(P_1, \varrho_1)$, $(P_2, \varrho_2)$ be metric spaces, $s > 0$, and $f : P_1 \to P_2$ be $\beta$-Lipschitz. Then $\varkappa^s(f(P_1)) \leqslant \beta^s \varkappa^s(P_1)$.*

*Důkaz*

Choose $\delta > 0$. Let $P_1 = \bigcup_{i=1}^{\infty} A_j$, $\operatorname{diam} A_j < \delta$. Then we have $f(P_1) = \bigcup_{j=1}^{\infty} f(A_j)$, $\operatorname{diam} f(A_j) < \beta \cdot \delta$.

$$\varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \sum_{j=1}^{\infty} (\operatorname{diam} f(A_j))^s \leqslant \sum_{j=1}^{\infty} \beta^s \cdot (\operatorname{diam} A_j)^s = \beta^s \cdot \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s.$$

It holds for all possible choices of $(A_j)$, so we can take infimum:

$$\varkappa^s(f(P_1)) \leftarrow \varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \beta^s \inf_{(A_j)} \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s = \beta^s \varkappa^s(P_1, \delta) \to \beta^s \varkappa^s(P_1).$$

□

**Lemma 1.2**

*Let $k, n \in \mathbb{N}$, $k \leqslant n$, and $L : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear mapping. Then for every $\lambda_k$-measurable set $A \subset \mathbb{R}^k$ it holds $H^k(L(A)) = \sqrt{\det(L^T L)} \lambda_k(A)$.*

*Důkaz* ($\dim L(\mathbb{R}^k) = k$)

We find linear isometry $Q$ of $\mathbb{R}^k$ onto $L(\mathbb{R}^k)$, from last semester

$$H^k(L(A)) = H^k(Q^{-1} \circ L(A)) = \lambda^k(Q^{-1} \circ L(A)) = |\det(Q^{-1}L)| \cdot \lambda_k(A).$$

$$(\det(Q^{-1}L))^2 = \det((Q^{-1}L)^T) \cdot \det(Q^{-1}L) = \det((Q^{-1}L)^T \cdot (Q^{-1}L)) = \det((\langle Q^{-1}Le^i, Q^{-1}L^T e^j \rangle)_{i,j}).$$

And because $Q$ is isometry ( $\implies$ $Q^{-1}$ is isometry), we can remove $Q^{-1}$ from scalar product and we get $\det(L^T L)$. □

**Lemma 1.3**

*Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood $V$ of the point $x$ such that*

- *the mapping $y \mapsto \varphi(\varphi'(x)^{-1}(y))$ is $\beta$-Lipschitz on $\varphi'(x)(V)$;*
- *the mapping $z \mapsto \varphi'(x)(\varphi^{-1}(z))$ is $\beta$-Lipschitz on $\varphi(V)$.*

*Důkaz*
$x$, $\beta$ fixed. We know, that there exists $\eta > 0$ such that

$$\forall v \in \mathbb{R}^k : \|\varphi'(x)(v)\| \geqslant \eta \cdot \|v\|.$$

We find $\varepsilon \in \left(0, \frac{1}{2}\eta\right)$ such that $\frac{2\varepsilon}{\eta} + 1 < \beta$. We find a neighbourhood $V$ of $x$ such that $\forall y \in V : \|\varphi'(x) - \varphi'(y)\| \leqslant \varepsilon$.

We show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leqslant \varepsilon\|u - v\|.$$

Fix $v \in V$ and consider the mapping

$$g : w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v).$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$:

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| = \|g(w) - g(v)\| \leqslant \sup\{\|g'(w)\| \mid w \in V\} \cdot \|u - v\| \leqslant \varepsilon \cdot \|u - v\|.$$

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

For $u - v \in V$ we compute

$$\|\varphi(u) - \varphi(v)\| \geqslant -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \geqslant -\varepsilon\|u - v\| + \eta\|u - v\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

„First point“: TODO (řádek nebyl k přečtení)

$$\|\varphi(\varphi^{-1}(x)(a)) - \varphi(\varphi^{-1}(x)(b))\| = \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \|phi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|\varphi'(x)(y - v)\| \leqslant \varepsilon\frac{1}{\eta}\|a - b\| + \|a - b\| = \left(\frac{\varepsilon}{\eta} + 1\right)\|a - b\| \leqslant \beta \cdot \|a - b\|.$$

„Second point“: $k, q \in \varphi(V)$. We find $u, v \in V$ such that $\varphi(u) = p$ and $\varphi(v) = q$:

$$\|\varphi'(x)(\varphi^{-1}(p)) - \varphi'(x)(\varphi^{-1}(q))\| = \|\varphi'(x)(u) - \varphi'(x)(v)\| =$$

$$= \|\varphi'(x)(u - v)\| \leqslant \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|p - q\| \leqslant \frac{2\varepsilon}{\eta}\|\varphi(u) - \varphi(v)\| + \|p - q\| = \left(\frac{2\varepsilon}{\eta} + 1\right)\|p - q\| \leqslant \beta\|p - q\|.$$

□