Poznámka
The previous semester we work with linear equation (L-M, Fredholm, Minimizing quadratic function). This semester we will have non-linear equations like $((\partial_t u)) - \Delta u + \operatorname{arctg} u = f$ or $f = -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

We don't work with $\partial_{tt} u - \Delta_p u = f$, because nobody know how to proof it has solution (for $d \geqslant 2, p > 2$).

Poznámka (Credit)
Two homework. -10 to 10 points to exam from each. (If we hand anything we get credit.)

What we must know

Poznámka
Lebesgue spaces.

Fixed point theorem: 1) Let F be continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}^d$. Assume that $\exists$ convex compact set in $\mathbb{R}^d$ such that $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega$ such that $F(x) = x$. 2) Let $F : X \to X$, where X is Banach space and F is continuous and compact and let $\exists\ \Omega \subseteq X$ convex and closed such that $F(\Omega) \subseteq \Omega$. Then $F(\Omega) \subseteq \Omega$. Then $\exists x \in X : F(x) = x$.

Luzin: Let Ω be a measurable set and $f \in L^1_{loc}(\Omega)$. Then $\forall \varepsilon > 0\ \exists U \in \Omega$, $\mu(U) \leqslant \varepsilon$, $f \in C(\Omega \backslash U)$.

Egorov: Let Ω be a measurable set and $f^n \to f$ in $L^1_{loc}(\Omega)$. Then $\forall \varepsilon > 0\ \exists U, \mu(U) \leqslant \varepsilon$ $f^n \to f$ in $C(\Omega \backslash U)$.

Lebesgue dominated convergence theorem.

Vitali convergence theorem: Let $\Omega \subseteq \mathbb{R}^d$ be bounded measurable, f^n a sequence of measurable functions, $f^n \to f$ almost everywhere in Ω. Then $\lim_{n\to\infty} \int_\Omega f^n = \int_\Omega f$, provided f^n is uniformly equi-integrable ($\forall \varepsilon > 0\ \exists \delta\ \forall U, \mu(U) \leqslant \varepsilon$).

Fatou lemma: $f^n \to f$ almost everywhere in Ω and $f^n \geqslant 0$, then $\liminf_{n\to\infty} \int_\Omega f^n \geqslant \int_\Omega f$.

Regularization: $\eta \in C_0^\infty(B_1(\mathbf{o}))$ non-negative, radially symmetric and $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Then $\forall f \in L^1_{loc}(\Omega)$ we extend f by „0“ to $\mathbb{R}^d$ and $f_\varepsilon := \eta_\varepsilon * f$, where $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d}\eta(x/\varepsilon)$. Then $f_\varepsilon \in C^\infty(\mathbb{R}^d)$ and $\forall p \in [1,\infty)$ $f \in L^p(\Omega) \implies f_\varepsilon \to f$ in $L^p(\Omega)$. (And for $p = \infty$: $f \in L^\infty(\Omega) \implies f_\varepsilon \to f$ in $L^q(\Omega)$ $\forall q \in\in [1,\infty)$).

Reflexive and separable spaces. ($L^p(\Omega)$ is a Banach space, separable for $p \in [1,\infty)$, reflexive for $p \in (1,\infty)$.)

Nemytsky operator: (Assume that for almost all $x \in \Omega$ and , $|f(x,y)| \leqslant g(x) +$

$C\sum_{i=1}^N |y_1|^{p_i/p}$ for some $p_i \in [1,\infty), p \in (1,\infty), g \in L^p(\Omega)$. Then $\forall u_i \in L^{p_i}$, the function $f(\cdot, u_1, \ldots, u_n)$ is measurable, $(u_1, \ldots, u_n) \mapsto f(\cdot, u_1, \ldots, u_n)$ is continuous $L^{p_1}(\Omega) \times \ldots \times L^{p_N}(\Omega) \to L^p(\Omega)$. This mapping is called N.O.)

1 Sobolev spaces (and Bochner spaces)

Poznámka
Ω is open bounded subset of $\mathbb{R}^d$.

Věta 1.1 (Local approximation by smooth functions)

Let $f \in W^{k,p}(\Omega)$ and extend it by „0“ outside. Define $f_\varepsilon := \eta_\varepsilon * f$ and set $\Omega_\varepsilon := \{x \in \Omega | B(x,\varepsilon) \subseteq \Omega\}$. Then $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$ almost everywhere in Ω_ε $\forall \alpha, |\alpha| \leqslant k$ and $\forall \Omega' \subseteq \overline{\Omega'} \subseteq \Omega$ and $p \in [1,\infty)$ $f_\varepsilon \to f$ in $W^{k,p}(\Omega')$. (If $p = \infty$, then $f_\varepsilon \to^* f$ in $W^{1,\infty}(\Omega')$.)

Důkaz

$$\frac{\partial}{\partial x_i}(f_\varepsilon(x)) = \frac{\partial}{\partial x_i}\int_{\mathbb{R}^d} \eta_\varepsilon(x-y)f(y)dy =$$
$$= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i}(\eta_\varepsilon(x-y))f(y)dy = -\int_{\mathbb{R}^d} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy =$$
$$= -\int_{B(x,\varepsilon)} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy = -\int_{\Omega} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y))f(y)dy =$$
$$= \int_{\Omega} \eta_\varepsilon(x-y)\frac{\partial f(y)}{\partial y_i}dy = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)\frac{\partial f(y)}{\partial y_i} = \left(\frac{\partial f(y)}{\partial y_i}\right)_\varepsilon (x).$$

Now we take sufficiently small ε, such that $\Omega_\varepsilon \subseteq \Omega'$. Then $D^\alpha f_\varepsilon = (D^\alpha f)_\varepsilon \to D^\alpha f$ in $L^p(\Omega')$. □

Věta 1.2 (Composition of Lipschitz and Sobolev functions)

Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Assume that $u \in W^{1,p}(\Omega)$. Then $(f(u) - f(0)) \in W^{1,p}(\Omega)$ and $\nabla f(u) = f'(u)\nabla u \chi_{x, u(x) \notin S_f}$, where S_f are points where $f'(s)$ doesn't exists.

Moreover define $\Omega_a := \{x \in \Omega | u(x) = a\}$, then $\nabla u = 0$ almost everywhere in Ω_a.

Důkaz
We know, that $f \in C^1(\mathbb{R})$, $f_{lip} := \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$.

So $|f(u(x)) - f(0)|^p \leqslant f_{lip}^p \cdot |u(x)|^p$, if $u \in L^p(\Omega) \implies f(u) - f(0) \in L^p(\Omega)$.

Next, $\frac{\partial f(u)}{\partial x_i} = f'(u)\frac{\partial u}{\partial x_i} \implies f(u) - f(0) \in W^{1,p}(\Omega)$.

We take $\eta \in C_0^\infty(\Omega)$ and $u \in W^{1,1}(u)$.

$$\int_\Omega \frac{\partial \eta}{\partial x_i} f(u) = \lim_{\varepsilon \to 0_+} \int_\Omega \frac{\partial \eta}{\partial x_i} f(u_\varepsilon) \overset{\text{IBP, both are smooth}}{-} \lim_{\varepsilon \to 0_+} \int_\Omega \eta \frac{\partial f(u_\varepsilon)}{\partial x_i} =$$

$$= -\lim_{\varepsilon \to 0_+} \int_\Omega \underbrace{\eta f'(u_\varepsilon)}_{\to \eta f(u_\varepsilon) \text{ in } L^1\text{, so weakly in } L^\infty} \cdot \underbrace{\frac{\partial u_\varepsilon}{\partial x_i}}_{\to \frac{\partial u}{\partial x_i} \text{ in } L^1}.$$

TODO? □

Věta 1.3 (Characterization of sobolev functions)

Let $\Omega \subseteq \mathbb{R}^d$ open, bounded. Define $\Omega_\delta := \{x \in \Omega | B(x,\delta) \subseteq \Omega\}$ and $u_i^h(x) := \frac{u(x+h\cdot e_i)-u(x)}{h}$, $h > 0, i \in [d]$.

- *If $u \in W^{1,p}(\Omega)$ then $\forall \delta\ \forall h < \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p}(\Omega)$.*
- *If $p \in (1,\infty]$ and $\sup_{\delta>0} \sup_{h<\frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant k$, then $\frac{\partial u}{\partial x_i}$ exists and $\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)} \leqslant k$.*
- *If $p \in [1,\infty)$ and if $u \in W^{1,p}(\Omega)$ then $u_i^h \to \frac{\partial u}{\partial x_i}$ in $L^p_{loc}(\Omega)$.*

(If $p = 1$ and $\sup_{\delta>0} \sup_{h<\frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leqslant k$, then $u \in BV(\Omega)$. Moreover if $\leqslant k$ and u_i^h is equiintegrable, then $u \in W^{1,1}(\Omega)$.)*

Důkaz
„Second point“ Fix $\Omega_1 \subset\subset \Omega$. Fix δ_0, $\Omega_1 \subseteq \Omega_{\delta_0} \implies \|u_i^h\|_{L^p(\Omega_1)} \leqslant k$. $u_i^h \rightharpoonup \overline{u}$ in $L^p(\Omega_1)$ and $u_i^h \rightharpoonup^* \overline{u}$ in $L^\infty(\Omega_1)$. We want $\|\overline{u}\|_{L^p(\Omega_1)} \leqslant \liminf_{h\to 0_+} \|u_i^h\|_{L^p(\Omega_1)} \leqslant k$.

$$\int_{\Omega_1} \overline{u}\varphi dx = \lim_{h\to 0_+} \int_{\Omega} u_i^h \varphi = \lim_{h\to 0_+} \int_{\Omega_1} \frac{u(x+h\cdot e_i)-u(x)}{h}\varphi(x)dx =$$

$$= \lim_{h\to 0_+} \int_{\Omega} \frac{u(x+h\cdot e_i)}{h}\varphi(x) - \frac{u(x)}{h}\varphi(x)dx =$$

$$= -\lim_{h\to 0_+} \int_{\Omega} u(x)\frac{\varphi(x)-\varphi(x-h\cdot e_i)}{h}dx = -\int_{\Omega_1} \frac{\partial\varphi}{\partial x_i}u.$$

□