Poznámka
Topology...

Definice 0.1 (Topological vector space (TVS))

A Topological vector space over $\mathbb{F}$ is a pair (X, τ), where X is a vector space over $\mathbb{F}$ and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of $\mathbf{o}$ in (X, τ).

Definice 0.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka
Two homework (in Moodle) and one presentation.

Například
Let $(X, \|\cdot\|)$ be a normed linear space. Let τ be the topology induced by $\|\cdot\|$. The (X, τ) is HLCS.

Důkaz
$\varrho(x, y) = \|x - y\|$ metric induced by $\|\cdot\|$. τ induced by ϱ. This τ is Hausdorff. Continuity of the operations: (from Funkcionalka)

$$x_n \to x, y_n \to y, t_n \to t \implies x_n + y_n \to x + y \wedge t_n x_n \to tx.$$

So, it is a HTVS. Base of neighbourhood of $\mathbf{o}$ is e. g. $U(0, r), r > 0$, which is convex. □

Let Γ be any nonempty set, $X = \mathbb{F}^\Gamma$ (= all functions $\Gamma \to \mathbb{F}$) with point-wise operations, so it is a vector space over $\mathbb{F}$. It is a HLCS.

Důkaz
„Continuity of addition:“ $x, y \in \mathbb{F}^\Gamma$, U a neighbourhood of $x+y \implies \exists F \subset \Gamma$ finite $\exists \varepsilon > 0$ such that

$$U_{\mathbf{o}} = \left\{ z \in \mathbb{F}^\Gamma \middle| \ \forall \gamma \in F : |z(\gamma) - (x(\gamma) + y(\gamma))| < \varepsilon \right\} \subset U$$

$$U_x = \left\{ z \in \mathbb{F}^\Gamma \middle| \ \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$$U_y = \left\{ z \in \mathbb{F}^\Gamma \middle| \ \forall \gamma \in F : |z(\gamma) - y(\gamma)| < \frac{\varepsilon}{2} \right\}$$

$\implies$ V_x is neighbourhood of x, and V_y is neighbourhood of y, and $U_x + U_y \subset U_0 \subset U$. Thus $z_1 \in V_x$, $z_2 \in V_y \implies z_1 + z_2 \in U_0 \subset U$.

„Continuity of multiplication“: $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^\Gamma$, U a neighbourhood of $\lambda x \implies \exists F \subset \Gamma$ finite $\exists \mu > 0$ such that

$$U_0 = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - \lambda x(\gamma)| < \varepsilon \right\} \subset U$$

$$|\mu z(\gamma) - \lambda x(\gamma)| \leqslant |\mu| \cdot |z(\gamma) - x(\gamma)| + |\mu - \lambda| \cdot |x(f)|.$$

$$M := \max_{\gamma \in F} |x(\gamma)|.$$

$$V = \left\{ \mu \in \mathbb{F} \middle| |\mu - \lambda| < \frac{\varepsilon}{2(M+1)} \right\}, \qquad W = \left\{ z \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |z(\gamma) - x(\gamma)| < \frac{\varepsilon}{2(|\lambda| + \frac{\varepsilon}{2(M+1)})} \right\}.$$

$$\mu \in V, z \in W \implies \mu z \in U_0 \subset U.$$

„Local convexity“: Base of neighbourhoods of $\mathbf{o}$: $\left\{ x \in \mathbb{F}^\Gamma \middle| \forall \gamma \in F : |x(\gamma)| < \varepsilon \right\}$, $F \subset \Gamma$ finite, $\varepsilon > 0$, consists of convex sets.

„Hausdorff“: $x \neq y \implies \exists \gamma \in \Gamma : x(\gamma) \neq y(\gamma)$. Take $\varepsilon = \frac{|x(\gamma) - y(\gamma)|}{2}$.

$$U = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - x(\gamma)| < \varepsilon \right\}, V = \left\{ z \in \mathbb{F}^\Gamma \middle| |z(\gamma) - y(\gamma)| < \varepsilon \right\} \implies U \cap V = \emptyset.$$

□

$$X = C(\mathbb{R}, \mathbb{F}) = \{ f : \mathbb{R} \to \mathbb{F} \text{ continuous} \},$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \max_{t \in [-n,n]} \{ |f(t) - g(t)| \} \right\} =: \sum_{N=1}^{\infty} \frac{1}{2^N} \min \{ 1, p_N(f - g) \}$$

is translation invariant (that implies addition is continuous, see lecture) metric.

Důkaz
$f_n \to f$ in $\varrho \Leftrightarrow \forall N : f_n \rightrightarrows f$ on $[-N, N]$.

„$f_n \to f$, $\lambda_n \to \lambda \implies \lambda_n f_n \to \lambda f$“: Let $N \in \mathbb{N}$. We will show $\lambda_n f_n \rightrightarrows \lambda f$ in $[-N, N]$. $x \in [-N, N]$:

$$|\lambda_n f_n(x) - \lambda f(x)| \leqslant |\lambda_n| \cdot |f_n(x) - f(x)| + |\lambda_n - \lambda| \cdot |f(x)| \leqslant c \cdot p_N(f_n - f) + |\lambda_n - \lambda| \cdot p_N(f) \to 0.$$

Hence, X is HTVS. „Local convexity“: $U_{N,\varepsilon} = \{f \in X | p_N(t) < \varepsilon\}$, clearly $U_{N,\varepsilon}$ is a convex set and $U_{N,\varepsilon}$ is neighbourhood of $\mathbf{o}$. If $\varepsilon < \lambda$, then $\left\{f | \varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}\right\} \subset U_{N,\varepsilon}$, because for $\varrho(f, \mathbf{o}) < \frac{\varepsilon}{2^N}$ it is $\frac{1}{2^N} p_N(f) < \frac{\varepsilon}{2^N}$. „they form a base“: $f \in U_{N,\varepsilon} \implies \varrho(f, \mathbf{o}) < \varepsilon + \frac{1}{2^N}$. Hence fix $r > 0$ and take $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \frac{r}{2}$. Then $U_{N,\frac{r}{2}} \subset \{f | \varrho(f, \mathbf{o}) < r\}$ □

(Ω, Σ, μ) a measure space, $p \in (0, 1)$. $L^p(\Omega, \Sigma, \mu) = \left\{f : \Omega \to \mathbb{F} \text{ measurable} | \int |f|^p d\mu < \infty\right\}$ (we identify functions equal almost everywhere). $\varrho(f, g) = \int |f - g|^p d\mu$ is a metric making $X = L^p(\Omega, \Sigma, \mu)$ a HTVS (but not locally convex).

Důkaz
„ϱ is a metric“: „$\triangle$-inequality“: $a, b \in [0, \infty) : (a + b)^p \leqslant a^p + b^p$. (Fix $a \geqslant 0$, take $\varphi_a(b) = (a + b)^p - a^p - b^p \implies \varphi_a$ is continuous on $[0, \infty)$, $\varphi_a(0) = 0$. For $b > 0$: $\varphi_a(b) = p(a + b)^{p-1} - pb^{p-1} = p \cdot ((a + b)^{p-1} - b^{p-1}) < 0$ as $p - 1 < 0 \implies \varphi_a$ decreasing on $[0, \infty)$ and $\varphi_a \leqslant 0$.)

φ is translation invariant $\implies$ addition is continuous. „Multiplication“: We can see that $\varrho(\lambda f, \mathbf{o}) = |\lambda|^p \varrho(f, \mathbf{o})$. $f_n \to f$, $\lambda_n \to \lambda$:

$$\varrho(\lambda_n f_n, \lambda f) \leqslant \varrho(\lambda_n f_n, \lambda_n f) + \varrho(\lambda_n f, \lambda f) = |\lambda_n|^p \varrho(f_n, f) + |\lambda_n - \lambda|^p \varrho(f, \mathbf{o}) \to 0.$$

Hence, we have a HTVS. □

Tvrzení 0.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x + U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 0.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1 - t)y \in A$ for $x, y \in A$, $t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}$, $|\alpha| \leqslant 1$;
- absolutely convex if it is convex and balanced;

- absorbing if $\forall x \in X\ \exists t > 0 : \{sX | s \in [0,t]\} \subset A$.

Definice 0.4

$\mathrm{co}(A)$ = convex hull, $\mathrm{b}(A)$ = balanced hull, $\mathrm{aco}(A)$ = absolutely convex hull.

Tvrzení 0.2

X is a metric space over $\mathbb{F}$, $A \subset X$. Then:

(a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow$ A is convex and symmetric.

(b) $\mathrm{co}\, A = \{t_1x_1 + \ldots + t_kx_k | x_1 \ldots x_k \in A, t_1 \ldots t_k \geqslant 0, t_1 + \ldots + t_k = 1, k \in \mathbb{N}\}$.

(c) $\mathrm{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leqslant 1\}$.

(d) $\mathrm{aco}(A) = \mathrm{co}(\mathrm{b}(A))$.

(e) A is convex $\Leftrightarrow$ $(s+t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)
„ $\implies$ “: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „ $\impliedby$ “: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \implies \alpha \in [-1,1].$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{(1+\alpha)}{2}x \in A$). □

Důkaz (b)
„$\subseteq$“: by induction on k:

$$t_1x_1 + \ldots + t_{k+1}x_{k+1} = (t_1 + \ldots + t_k)\frac{t_1x_1 + \ldots + t_kx_k}{t_1 + \ldots + t_k} + t_{k+1}x_{k+1}.$$

„$\supseteq$“: the set on the RHS is convex and contain A. □

Důkaz (c)
„$\supseteq$“: clear. „$\subseteq$“: RHS is a balanced set. □

Důkaz (d)
„$\supseteq$“: clear. „$\subseteq$“ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c): $\mathrm{co}(\mathrm{b}(A)) = \{t_1\alpha_1x_1 + \ldots + t_k\alpha_kx_k | x_1, \ldots, x_k \in A, |\alpha_j| \leqslant 1, t_j \geqslant 0, t_1 + \ldots + t$
is clearly balanced.) □

Důkaz (e)
„ $\Longrightarrow$ “: „$\subseteq$“: always, „$\supseteq$“: $sa_1 + ta_2 = (s+t)\cdot\left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$.

„ $\Longleftarrow$ “: in particular $\forall t \in (0,1)$: $tA+(1-t)A \subset A$, it is the definition of convexity. □

Tvrzení 0.3

Let (X,τ) be a LCS, $U \in \tau(\mathbf{o})$. Then

(i) U is absorbing.

(ii) $\exists V \in T(0) : V + V \subset U$.

(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.

Důkaz (i)

$x \in X \implies 0\cdot x = \mathbf{o} \in U \implies \exists V$ a neighbourhood of 0 in $\mathbb{F} : V\cdot x \subset U \implies \exists t > 0 : [0,t] \subset V$

□

Důkaz (ii)

$$\mathbf{o}+\mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W \subset U.$$

Take $V = W_1 \cap W_2$. □

Důkaz

$$\exists U_0 \in \tau(\mathbf{o}) \text{ convex}, U_0 \subset U : \mathbf{o}\cdot\mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0\ \exists W \in \tau(\mathbf{o}) \text{ open}:$$
$$\forall \lambda, |\lambda| < c : \lambda W \subset U_0.$$

$V_1 := \bigcup_{0<|\lambda|<1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \operatorname{co} V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „V is open“:

$$V = \bigcup \{t_1x_1 + \ldots + t_nx_n + t_{n+1}V_1 | t_1, \ldots, t_{n+1} \geqslant 0, t_1 + \ldots + t_{n+1} = 1, x_1, \ldots, x_n \in V_1\}$$

□

Věta 0.4

1. Let (X,τ) be a LCS. Then there is $\mathcal{U}$, a base of neighbourhoods of $\mathbf{o}$ with properties:

- *the elements of $\mathcal{U}$ are absorbing, open, absolutely convex;*
- *$\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U$.*

If X is Hausdorff, then $\bigcap \mathcal{U} = \{\mathbf{o}\}$.

2. Let X be a vector space, $\mathcal{U}$ a nonempty family of subsets of X satisfying:

- *the elements of $\mathcal{U}$ are absorbing and absolutely convex;*
- $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U$*;*
- $\forall U, V \in \mathcal{U}\ \exists W \in \mathcal{U} : W \subset U \cap V$*.*

Then there is a unique topology τ on X such that (X, τ) is LCS and $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$. Further, if $\bigcap \mathcal{U} = \{\mathbf{o}\}$, the τ is Hausdorff.

Důkaz (1.)
Let $\mathcal{U}$ be the family of all open absolutely convex neighbourhoods of $\mathbf{o}$. The previous proposition (iii) gives us $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$, (1) gives us elements of $\mathcal{U}$ are absorbing, so the first item holds. (ii) gives us $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$.

Assume X i Hausdorff: $x \in X \setminus \{\mathbf{o}\} \overset{\text{Hausdorff}}{\implies} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$. □

Důkaz (2.)
Set $\tau = \{G \subset X | \forall x \in G \ \exists U \in \mathcal{U} : x + U \subset G\}$. This is a unique possibility so uniqueness is clear.

„τ is topology“: $\emptyset, X \in \tau$ and τ is closed to arbitrary union (clear). τ is closed to finite intersections by third item ($G_1, g_2 \in \tau$, $x \in G_1 \cap G_2 \ldots U_1, U_2 \in \tau$, $x + U_1 \subset G_1$, $x + U_2 \subset G_2$; $\exists V \in \mathcal{U} : V \subset U_1 \cap U_2$, then $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$).

„Elements of $\mathcal{U}$ are neighbourhoods of $\mathbf{o}$“: $U \in \mathcal{U}$. $V := \{x \in U | \exists W \in \mathcal{U} : x + W \subset U\}$. Then $V \subset U$, $0 \in V$ (take $W = U$). $V \in \tau$ ($x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$; let $\tilde{W} \in \mathcal{U}$ such that $2\tilde{W} \subset W$, then $x + \tilde{W} \subset V$, because $y \in \tilde{W} \implies X + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$).

„$\mathcal{U}$ is a base of neighbourhood of $\mathbf{o}$“: now clear.

„(X, τ) is a TVS“: $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$. Then $(x+V)+(y+V) \subset x+y+2V \subset x+y+U \subset G$. $\lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G$; $\exists V \in \mathcal{U} : 2V \subset U$; V is absorbing $\implies \exists c > 0 \ \forall t \in [0, c] : tx \in V$; V balanced $\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$; assume $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1} V$,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda) y}_{(\mu-1)\cdot\left(\mu + \frac{1}{|\lambda|+1}\right) V} + \underbrace{\lambda (y - x)}_{\in \frac{\lambda}{|\lambda|+1} V \subset V}.$$

„Local convexity“: by first item: $\forall U \in \mathcal{U} : U$ is convex.

Assume $\bigcap \mathcal{U} = \{\mathbf{o}\}$. Take $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$. Take $V \in \mathcal{U} : 2V \subset U$. Then if $(x + V) \cap (y + V) = \emptyset$, $x + v_1 = y + v_2$, $x - y = v_2 - v_1 \in V + V = 2V \subset U$ ↯. □

Věta 0.5

Let X be a vector space and let $\mathcal{P}$ be a family of seminorms on X. The there is a unique topology τ on X such that (X, τ) is a LCS and $\mathcal{U} = \{\{x \in X | p_1(x) < c_1, \ldots, p_k(x) < c_k\} | p_1, \ldots, p_k \in \mathcal{P}, c_1, \ldots, c$ is a base of neighbourhood of $\mathbf{o}$.

(X, τ) is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\} \ \exists p \in \mathcal{P}, p(x) > 0$.

Důkaz
Use the previous theorem (2.) on $\mathcal{U}$: The sets are absolutely convex (by properties of seminorms). „Absorbing“: $U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $x \in X$?, $j \in [k]$. Then $p_j(x) \in (0, \infty)$ as for $t > 0 : p_j(t \cdot x) = t \cdot p_j)_x$ and $\exists c > 0$ such that $c \cdot p_j(x) < c_j$ for $j \in [k]$. Now for $t \in [0, c] : tx \in U$.

$U = \{x \in X | p_1(x) < c_1, \dots, p_k(x) < c_k\}$. Take $V = \left\{x \in X | p_1(x) \subset \frac{c_1}{2}, \dots, p_k(x) < \frac{c_k}{2}\right\}$.

$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$ trivially.

„Hausdorffness“:

$$\bigcap U = \{x \in X | \forall p \in \mathcal{P} : p(x) = 0\}.$$

„$\supseteq$“ clear. „$\subseteq$“: Assume $y \in X$, $p \in \mathcal{P} : p(y) > 0$: $U = \{x \in X | p(x) < p(y)\} in \mathcal{U} \implies y \notin U$. □

Například
$(X, \|\cdot\|)$ is a normed space, then its topology is generated by $\mathcal{P} = \{\|\cdot\|\}$.

The topology on $\mathbb{F}^\Gamma$ is generated by seminorms $p_\gamma(f) = |f(\gamma)|$, $f \in \mathbb{F}^\Gamma$ $(\gamma \in \Gamma)$.

$C(\mathbb{R}, \mathbb{F})$ the topology is generated by this sequence of seminorms: $p_N(f) = \max_{x \in [-N, N]} |f(x)|$.

Definice 0.5 (Minkowski functional)

X vector space, $A \subset X$ convex absorbing. Then

$$p_A(x) := \inf \{\lambda > 0 | x \in \lambda \cdot A\}.$$

Lemma 0.6

Let X be LCS, $A \subset X$ convex set.

$$x \in \overline{A}, y \in \operatorname{int} A \implies \{tx + (1-t)y | t \in [0, 1)\} \subset \operatorname{int} A.$$

Důkaz
WLOG $y = 0$. $t = 0$ clear, $0 \in \operatorname{int} A$. $t \in (0, 1)$:

Fix U, an open absolutely convex neighbourhood of $\mathbf{o}$ such that $U \subset A$. Then $x + \frac{1-t}{t} U$ is a neighbourhood of $x \implies \exists$

TODO!!! □

TODO!!!

Důkaz (Continuity of multiplication? Theorem 4. TODO?)
„U is a neighbourhood of $\mathbf{o}$ in τ, $\lambda > 0 \implies \lambda U$ is neighbourhood of $\mathbf{o}$“: $\lambda \geqslant 1$: $\exists V \in \mathcal{U} : V \subset U \implies V \subset \lambda V \subset \lambda U$ (V is absolutely convex) $\implies \lambda U$ is neighbourhood of $\mathbf{o}$. $\lambda = \frac{1}{2}$: $\exists V \in \mathcal{U} : V \subset U$, then $\exists W \in \mathcal{U} : 2W \subset V$, then $W \subset \frac{1}{2}V \subset \frac{1}{2}U \implies \frac{1}{2}U$ is a neighbourhood of $\mathbf{o}$. Now by induction for $\lambda = \frac{1}{2^n}$. For $\lambda > 0$ find $n \in \mathbb{N}$ such that $\lambda > \frac{1}{2^n}$.

$\lambda x \in G$ ($\lambda \in \mathbb{F}, x \in X, G \in \tau$) $\implies \exists U \in \mathcal{U} : \lambda x + U \in G$. Find $V \in \mathcal{U} : 2V \subset U$ such that V is absorbing ($\implies \exists c > 0\ \forall t \in [0, c] : tx \in V$) and V is balanced ($\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$). Let $\mu \in F, y \in X$ such that

$$|\mu - \lambda| < c \wedge y \in x + \frac{1}{|\lambda| + c} V \text{ (a neighbourhood of } \mathbf{o})$$

$$\implies \mu y - \lambda x = \mu(y - x) + (\mu - \lambda)x \in V + V = 2V \subset U \implies \mu y \in \lambda x + U \subset G.$$

□

Tvrzení 0.7 (8. see notes of lecturer)

Let X be LCS, $A \subset X$ a convex neighbourhood of $\mathbf{o}$.

Clearly: $[p_A \subset 1] \subset A \subset [p_A \leqslant 1]$.

Důkaz
„$[p_a < 1] = \operatorname{int} A$“: „$\subseteq$“: $p_A(x) < 1 \implies \exists c > 1$ such that $cx \in A \implies x = \frac{1}{c} cx \in \operatorname{int} A$. „$\supseteq$“: $x \in \operatorname{int} A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$. U absorbing $\implies \exists \alpha > 0 : \alpha x \in U$. Then $(1 + \alpha)x \in A \implies p(x) \leqslant \frac{1}{1+\alpha} < 1$.

„$[p_A \leqslant 1] = \overline{A}$“: „$\subseteq$“: $p_A(x) \leqslant 1 \implies \forall n \in \mathbb{N} : p_x\left(\left(1 - \frac{1}{n}\right)x\right) = \left(1 - \frac{1}{n}\right) p_A(x) \leqslant 1$. $\left(1 - \frac{1}{n}\right) x \in \operatorname{int} A \implies x \in \overline{\operatorname{int} A} \subset \overline{A}$. „$\supseteq$“: $x \in \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right) x \in \operatorname{int} A$, so, $p_A\left(\left(1 - \frac{1}{n}\right)x\right) < 1 \overset{n \to \infty}{\implies} p_A(x) \leqslant 1$. □

p_A is continuous on X.

Důkaz
$[p_A < c] = \emptyset$ if $c \leqslant 0$ and $c \cdot \operatorname{int} A$ if $c > 0$. $[p_A > c] = X$ if $c < 0$, $X \backslash (c \cdot \overline{A})$ if $c > 0$, and $\bigcup_{t>0} X \backslash t\overline{A}$ if $c = 0$. All these sets are open. □

$p_A = p_{\overline{A}} = p_{\operatorname{int} A}$.

Důkaz
$\operatorname{int} A \subset A \subset \overline{A} \implies p_{\overline{A}} \leqslant p_A \leqslant p_{\operatorname{int} A}$. „Conversely“: Assume that $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right) x \in d \operatorname{int} A \implies \left(1 - \frac{1}{n}\right) p_{\operatorname{int} A}(x) \leqslant d \implies p_{\operatorname{int} A}(x) \leqslant d < c$. □

Důsledek
Any LCS (X) is completely regular.

Důkaz
$x \in X$, U an open neighbourhood of x. Take V a convex neighbourhood of $\mathbf{o}$ such that $x + V \in U$. $f(y) := \min\{1, p_V(y-x)\}$. The f is continuous by the previous proposition, $f(x) = 0$.

$$y \in X \backslash U \implies y - x \notin V \implies p_V(y-x) \geqslant 1 \implies f(y) = 1.$$

□

Věta 0.8

TODO!!! The topology generated by $\mathcal{P}_\tau$ coincides with τ.

Důkaz
Let τ_1 be topology induced by $\mathcal{P}_\tau$. $\tau_1 \subset \tau$ (seminorms from $\mathcal{P}_\tau$ are τ-continuous, hence the sets from theorem 5? are τ-open). „$\tau \subset \tau_1$“: Let $U \in \tau(\mathbf{o}) \implies \exists V$ a neighbourhood of $\mathbf{o}$ such that $V \subset U$. The $p_V \in \mathcal{P}_\tau$ (from the previous proposition is continuous) $\implies$ $[p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$. □

Tvrzení 0.9

X a vector space.

1. *p is seminorm $\implies$ $[p < 1]$ is absolutely convex, absorbing, and $p_{[p<1]} = p$.*
2. *p, q are seminorms, then $p \leqslant q \Leftrightarrow [p < 1] \supset [q < 1]$.*
3. *$\mathcal{P}$ a set of seminorms generated by a topology τ. p a seminorm on X. Then p is τ-continuous $\Leftrightarrow \exists p_1, \ldots, p_k \in \mathcal{P}\ \exists c > 0 : p \leqslant c \cdot \max\{p_1, \ldots, p_k\}$.*

Důkaz (1.)
Absolutely convex and absorbing is clear.

$$p_{[p<1]}(x) = \inf\{\lambda > 0 | x \in \lambda[p < 1]\} = \inf\{\lambda > 0 | x \in [p < \lambda]\} = p(x).$$

□

Důkaz (2.)
„ $\implies$ “ trivial. „ $\impliedby$ “: $[p < 1] \supset [q < 1] \implies p = p_{[p<1]} \leqslant p_{[q<1]} = q$. □

Důkaz (3.)
„ $\Longleftarrow$ “: $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = \left[p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}\right]$, which is a τ-open set $\implies A$ is a neighbourhood of $\mathbf{o} \implies p = p_A$ is continuous (by 1. and the previous proposition).

„ $\implies$ “: p is continuous $\implies [p < 1]$ is neighbourhood of $\mathbf{o}$ ($p(\mathbf{o}) = 0$) $\implies \exists p_1, \dots, p_k \in \mathcal{P}\ \exists c_1, \dots, c_k > 0$ such that $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c} \max\{p_1, \dots, p_k\} < 1]$ ($c = \min\{c_1, \dots, c_k\}$). Use 2. for seminorms p, $\frac{1}{2\max\{p_1,\dots,p_k\}}$ and get $p \leqslant \frac{1}{c} \max\{p_1, \dots, p_k\}$. □

1 Continuous and bounded linear mapping

Tvrzení 1.1

$(X, \tau), (Y, \mathcal{U})$ LCS, $L : X \to Y$ linear. Then the following assertions are equivalent:

1. *L is continuous;*
2. *L is continuous at $\mathbf{o}$;*
3. *L is uniformly continuous.*

Důkaz
„1. $\implies$ 2.“ trivial, „2. $\implies$ 3.“ assume L continuous at $\mathbf{o}$. Then, given $U \in \mathcal{U}(\mathbf{o})$, there is $V \in \tau(\mathbf{o})$ such that $L(V) \subset U$. Take $x, y \in X$ such that $x - y \in V$. Then $L(x) - L(y) = L(x - y) \in U$ and that's continuous. „3. $\implies$ 1.“ trivial. □

Tvrzení 1.2

$L : X \to Y$ linear. L is continuous $\Leftrightarrow \forall q$ a continuous seminorm on Y $\exists p$ a continuous seminorm on X: $\forall x \in X : q(L(x)) \leqslant p(x)$.

Důkaz
„ $\implies$ “: L continuous, q a continuous seminorm on Y, the $p(x) = q(L(x))$ is a continuous seminorm on X. „ $\Longleftarrow$ “: By the previous proposition it is enough „L is continuous at $\mathbf{o}$“: U neighbourhood of $\mathbf{o}$ in Y, $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. $q := p_V$ is a continuous seminorm. Let p be a continuous seminorm on X such that $q \circ L \leqslant p$. $W := [p < 1]$ a neighbourhood of $\mathbf{o}$ in X and $L(W) \subset V \subset U$. $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$. □

TODO!!!

TODO!!!

Věta 1.3

TODO[Theorem 22]!!!

Důkaz

„2. $\implies$ 1." trivial. „1. $\implies$ 3." if ϱ a metric generating τ, then $U_n = \left\{x \in X | \varrho(x, 0) < \frac{1}{n}\right\}$ $\implies$ $(U_n)_n$ is a base of neighbourhoods of $\mathbf{o}$. „3. $\implies$ 4.": (see the proof of the previous proposition, 1.) (U_n) base of neighbourhood of $\mathbf{o}$, take $V_n \subset U_n$ absolutely convex neighbourhood of $\mathbf{o}$, $p_n = p_{V_n}$ $\implies$ (p_n) generate τ. „4. $\implies$ 2.": the previous proposition 2. □

Věta 1.4

(X, τ) is HLCS. X is normable $\Leftrightarrow$ $\exists U$, a bounded neighbourhood of $\mathbf{o}$.

Důkaz

„ $\implies$ ": τ generated by $\|\cdot\|$, $U := \{x \in X | \|x\| < 1\}$ is a bounded neighbourhood of $\mathbf{o}$.

„ $\impliedby$ ": U bounded neighbourhood of $\mathbf{o}$. WLOG U is absolutely convex. Then $\frac{1}{n}U$, $n \in \mathbb{N}$ is a base of neighbourhoods of $\mathbf{o}$ (V neighbourhood of $\mathbf{o}$, $W \subset V$ an absolutely convex neighbourhood of $\mathbf{o}$ $\implies$ $\exists \lambda > 0 : U \subset \lambda W$ Take $n \in \mathbb{N}$ such that $n > \lambda$. Then $U \subset n \cdot W$ so $\frac{1}{n}U \subset W \subset V$). Finally, p_U is a norm generating the topology (U absolutely convex neighbourhood of $\mathbf{o}$ $\implies$ p_U is a continuous seminorm. $\frac{1}{n}U = [p_U < \frac{1}{n}], n \in \mathbb{N}$ is a base of neighbourhood of $\mathbf{o}$ $\implies$ p_U generated topology of X. From X Hausdorff, p_U is a norm.) □

2 Fréchet spaces

Definice 2.1 (Fréchet space)

A LCS whose topology is generated by a complete translation invariant metric is called Fréchet space.

Například

X Banach space $\implies$ X Fréchet space. $\mathbb{F}^{\mathbb{N}}, C(\mathbb{R}, \mathbb{F}), H(\Omega)$ are Fréchet spaces.

Důkaz ($\mathbb{F}^{\mathbb{N}}$)

$$p_n((x_k)) = \max\{|x_k| | k \in [n]\}$$

seminorms generating the topology, $p_1 \leqslant p_2 \leqslant \dots$

$$\varrho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x-y)\}$$

is translation invariant metric generating the topology. It is complete: $((x_k^m)_k)_{m=1}^{\infty}$ a ϱ-Cauchy sequence $\implies \forall n \in \mathbb{N} : ((x_k^m))_m$ is p_n-Cauchy $\implies$ it is $\|\cdot\|_\infty$-Cauchy in $\mathbb{F}^{\mathbb{N}} \implies$ (because $\mathbb{F}^{\mathbb{N}}$ is complete) $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \xrightarrow{m\to\infty} (y_1^n, \dots, y_n^n) \in \mathbb{F}^n$.

Moreover, if $i \leqslant n_1 \leqslant n_2$, then $y_i^{n_1} = y_i^{n_2}$. So, we have $y = (y_k)_{k=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$, such that $\forall n \in \mathbb{N} : (x_k^m)_{k=1}^n \xrightarrow{m} (y_k)_{k=1}^n$

$$\implies \forall n \in \mathbb{N} : p_n(x^n - y) \xrightarrow{m} 0 \implies \varrho(x^n, y) \to 0, \text{ i.e. } x^n \to y \text{ in } X.$$

□

Důkaz ($\mathbb{C}(\mathbb{R}, \mathbb{F})$)

$$p_n(f) = \max_{x\in[-n,n]} |f(x)|.$$

(f_k) ϱ-Cauchy $\implies \forall n : (f_k)$ is p_n-Cauchy $\implies \forall n : (f_k|_{[-n,n]})$ is $\|\cdot\|_\infty$-Cauchy in $C([-n,n) \implies \forall n\, \exists g_n \in C([-n,n])$ such that $f_k|_{[-n,n]} \xrightarrow{k} g_n$ in $C([-n,n])$.

$\forall n_1 \leqslant n_2 : g_{n_2}|_{[-n_1,n_1]} = g_{n_1}$ so, we have one function $g : \mathbb{R} \to \mathbb{F}$ such that $\forall n \in \mathbb{N} : g|_{[-n,n]=g_n}$. Then g is continuous, i.e. $g \in C(\mathbb{R}, \mathbb{F})$ and $\forall n \in \mathbb{N} : p_n(f_k - g) \xrightarrow{k} 0$. So $p_n(f_k, g) \to 0 \implies f_n \to g$. □

Tvrzení 2.1

(X,τ) is a Fréchet space, ϱ any translation invariant metric on X generating $\tau \implies \varrho$ is complete.

Důkaz

ϱ, d two translation invariant metrics generating by τ. Idea: convergent sequences with respect to ϱ and d coincide, Cauchy sequences with respect to ϱ and d coincide. (x_n) ϱ-Cauchy: $\varepsilon > 0 \implies \{x | d(x, \mathbf{o}) < \varepsilon\}$ is a neighbourhood of $\mathbf{o} \implies \exists \delta > 0 : \{x | \varrho(x, \mathbf{o}) < \delta\} \subset \{x | d(x, \mathbf{o}) < \varepsilon\}$.

$\exists n_0\, \forall m, n > n_0 : \varrho(x_m - x_n, \mathbf{o}) = \varrho(x_m, x_n) < \delta \implies d(x_m - x_n, 0) = d(x_m, x_n) < \varepsilon \implies (x_n)$ is d-Cauch

□

Tvrzení 2.2

X Fréchet, $A \subset X$. A is compact $\Leftrightarrow$ A is closed and totally bounded.

Důkaz

Let ϱ be a complete translation invariant metric generating the topology. A is compact $\Leftrightarrow$ A is closed and ϱ-totally bounded. But ϱ-totally boundedness = total boundedness in X. A is totally bounded in $X \Leftrightarrow \forall U$ neighbourhood of $\mathbf{o}$ $\exists F \subset X$ finite $A \subset F + U$. A is totally bounded in $\varrho \Leftrightarrow \forall \varepsilon > 0\ \exists F \subset X$ finite such that $A \subset \bigcup_{x \in F} U_\varrho(x, \varepsilon) = F + U_\varrho(0, \varepsilon)$. □

Tvrzení 2.3

X LCS, $A \subset X$ totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded.

Důkaz

Let U be a neighbourhood of $\mathbf{o}$. Let V be an absolutely convex neighbourhood of $\mathbf{o}$ such that $2V \subset U \implies \exists F \subset X$ finite such that $A \subset F + V$. Then clearly $\operatorname{aco} A \subset (\operatorname{aco} F) + V$. $\operatorname{aco} F$ is compact,

$$F = \{x_1, \dots, x_k\} \implies \operatorname{aco}(F) = \operatorname{co}(\operatorname{b}(F)) = \operatorname{co}\{\lambda x_j | j \in [k], |\lambda| \leqslant 1\} = \left\{ t_1\lambda_1 x_1 + \dots t_n \lambda_n x_n \middle| |\lambda_j| \leqslant 1, t_j \right.$$

$$B = \left\{ (\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \middle| |\lambda_j| \leqslant 1, t_j \geqslant 0, \sum t_j = 1 \right\}$$

a compact set in $\mathbb{F}^n \times \mathbb{R}^n$. $(\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mapsto t_1\lambda_1 x_1 + \dots + t_n \lambda_n x_n$ is a continuous map and maps B onto $\operatorname{aco} F$.

$\operatorname{aco} F$ compact $\implies$ totally bounded $\implies$ $\exists H \subset X$ finite, $\operatorname{aco} F \subset H + V$ So $\operatorname{aco} A \subset \operatorname{aco} F + V \subset H + V + V = H + 2V < H + U$. □

Důsledek

X Fréchet space, $A \subset X$ compact $\implies$ $\overline{\operatorname{aco} A}$ is compact.

Důkaz

A compact $\implies$ A is totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded $\implies$ (because $M \subset X$ any set $\implies \overline{M} \subset M + U$) $\overline{\operatorname{aco} A}$ is totally bounded $\implies$ $\overline{\operatorname{aco} A}$ is compact.

(M totally bounded, for any $U \in \tau(\mathbf{o})$: U is neighbourhood of $\mathbf{o}$, $x \in \overline{M}$, U absolutely convex convex neighbourhood of $\mathbf{o}$, then $V \subseteq U$ absolutely convex such that $2V \subset U$ $\implies$ $(x + U) \cap M \neq 0$ $\implies$ $x \in M + U$.)

Find F finite such that $M \subset F + V \implies \overline{M} \subset M + V \subset F + V + V \subset F + U$. □

Věta 2.4 (Banach–Steinhaus)

Let X be a Fréchet space and let Y be LCS. Let (T_n) be a sequence of continuous linear mappings $T_n : X \to Y$ such that $\forall x \in X : \lim_{n\to\infty} T_n x$ exists in Y. Then $Tx := \lim_{n\to\infty} T_n x$

define a continuous linear map $X \to Y$.

Důkaz

Clear: T is a linear map $X \to Y$. „Continuous“: Fix q any continuous sequence on Y.

$$A_m = \{x \in X | \forall n \in \mathbb{N} : q(T_n x) \leqslant m\} .$$

Then A_m is closed, absolutely convex and $\bigcup_{m=1}^{\infty} A_m = X$.

TODO?

Baire category theorem $\implies \exists m \in \mathbb{N} : \operatorname{int} A_m \neq \varnothing \implies \exists x \in A_m \; \exists U$ an absolutely convex neighbourhood of $\mathbf{o}$ such that $x+U \subset A_m \implies -(x+U) \subset A_m \implies$ (A_m convex) $U \subset A_m$ $(y \in U \implies y = \frac{1}{2}(x + y + (-x + y)))$ „$\implies q(Ty) \leqslant m p_U(y)$“:

$$p_U(y) < c \implies \frac{y}{c} \in U \subset A_m \implies \forall n \in \mathbb{N} q(T_n \frac{y}{c}) \leqslant m \implies q(T\frac{y}{c}) \leqslant m \implies q(Ty) \leqslant cm.$$

□

Věta 2.5 (Open mapping theorem)

X, Y Fréchet, $T : X \to Y$ linear continuous surjective mapping. Then T is an open mapping

Důkaz
1. It is enough to show that $\forall U$ neighbourhood of $\mathbf{o}$ in X: $T(U)$ is a neighbourhood of $\mathbf{o}$ in Y.

2. „$\forall U$ a neighbourhood of $\mathbf{o}$ in X, $\overline{TU}$ is neighbourhood of $\mathbf{o}$ in Y“: U an neighbourhood of $\mathbf{o}$ in X. $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. V absorbing $\implies$

$$\implies X = \bigcup_{n=1}^{\infty} nV \implies Y = T(X) = T\left(\bigcup_{n=1}^{\infty} n \cdot V\right) = \bigcup_{n=1}^{\infty} n \cdot T(V).$$

Y Fréchet $\implies$ by Baire category theorem

$$\exists n \in \mathbb{N} : \varnothing \neq \operatorname{int} \overline{n \cdot T(V)} = \operatorname{int} n \cdot \overline{T(V)} = n \cdot \operatorname{int} \overline{T(V)} \implies \operatorname{int} \overline{T(V)} \neq \varnothing \implies$$

$\implies \exists y \in Y\ \exists W$ an absolutely convex neighbourhood of $\mathbf{o}$ in Y such that $y + W \subset \overline{T(V)}$. $\overline{T(V)}$ is absolutely convex $\implies -(y+w) \subset \overline{T(V)} \implies W \subset \overline{T(V)} \subset \overline{T(U)}$.

3. „$\forall U$ neighbourhood of $\mathbf{o}$ in X, TU is a neighbourhood of $\mathbf{o}$ in Y“: ϱ a translation invariant metric on X, complete, generating topology. $U_n = \left\{x \in X | \varrho(0,x) < \frac{1}{2^n}\right\}$. The U_n is a base of neighbourhoods of $\mathbf{o}$. It is enough to prove that $T(U_n)$ is a neighbourhood of $\mathbf{o}$ for each $n \in \mathbb{N}$. We know from 2. that $\forall n : \overline{TU_n}$ is a neighbourhood of $\mathbf{o}$ in Y. We will be done if we show that $TU_{n-1} \supset \overline{TU_n}$ for each $n \in \mathbb{N}$.

We will prove it for $n = 1$: So we will ? $TU_1 \subset TU_0$. Fix $y \in \overline{TU_1}$. Since $\overline{TU_2}$ is a neighbourhood of $\mathbf{o}$ $(y - \overline{TU_2}) \cap TU_1 \neq \varnothing$. So there is $x_1 \in U_1$ such that $y - Tx_1 \in \overline{TU_2}$. $\overline{TU_3}$ is a neighbourhood of $\mathbf{o}$ in $Y \implies y - Tx_1 - \overline{TU_3} \subset apTU_2 = \varnothing$ so, there is $x_2 \in U_2$ such that $y - Tx_1 - Tx_2 \in \overline{TU_3}$.

By induction we find $x_n \in U_n$ such that

$$y - Tx_1 - Tx_2 - \ldots - Tx_n \in \overline{TU_{n+1}} \quad (n \in \mathbb{N}).$$

$$x := \sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n :$$

$$M > N \implies \varrho\left(\sum_{n=1}^{M} x_n, \sum_{n=1}^{N} x_n\right) = \varrho\left(\sum_{n=N+1}^{M} x_n, \mathbf{o}\right) \leqslant \underbrace{\varrho\left(\sum_{n=N+1}^{M} x_n, \sum_{n=N+2}^{M}\right)}_{\varrho(x_{N+1},\mathbf{o})} + \underbrace{\varrho\left(\sum_{n=N+2}^{M} x_n, \sum_{n=N+3}^{M}\right)}_{\varrho(x_{N+2},\mathbf{o})} + \ldots$$

$$Tx = y : y - Tx = \lim_{n\to\infty} (y - Tx_1 - \ldots - Tx_n)$$

$$y - Tx_1 - \ldots - Tx_n \in \overline{TU_{N+1}} \subset \overline{TU_k} \qquad \text{for } n+1 > k$$

so, $y - Tx \in \overline{TU_k}$ for each $k \in \mathbb{N}$, so $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k} = \{\mathbf{o}\}$. „Last equality“: $y \in Y \setminus \{\mathbf{o}\}$ $\implies \exists V$ neighbourhood of $\mathbf{o}$ in Y such that $y \notin \overline{B}$. T continuous $\implies \exists k \in \mathbb{N}$ such that $T(U_k) \subset V \implies \overline{T(U_1)} \subset \overline{V} \implies y \notin \overline{T(U_k)}$. □

3 Extension and separation theorems

Definice 3.1

X LCS, X^* is the vector space of continuous linear functions on X.

Věta 3.1

X LCS, $Y \Subset X$, $f \in Y^*$. Then $\exists g \in X^*$ such that $g|_Y = f$.

Poznámka

If topology of X is generated by $\mathcal{P}$ a topology of seminorms TODO!!!

Důkaz

1. Topology of Y: $U \subset Y$ is open $\Leftrightarrow \exists \tilde{U} \subset X$ open such that $U = \tilde{U} \cap Y$. U is a neighbourhood of $\mathbf{o}$ in $Y \Leftrightarrow \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $U = \tilde{U} \cap Y$. Lz.pat. Y is also a LSC.

2. $f \in Y^* \implies \exists p$ a continuous seminorm on Y such that $|f(y)| \subseteq p(y), y \in Y$. $U = [p < 1]$ a neighbourhood of $\mathbf{o}$ in $Y \implies \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $\tilde{U} \cap Y = U \implies \exists \tilde{V} \subset \tilde{U}$ an absolutely convex neighbourhood of $\mathbf{o}$ in X. The $p_{\tilde{V}}$ is a continuous seminorm on X. Moreover, $p_{\tilde{V}}|_Y \geqslant p$. ($[p_{\tilde{V}}|_Y < 1] \subset \tilde{V} \cap Y \subset U = [p < 1]$). So, for $y \in Y : |f(y)| \leqslant p(y) \leqslant p_{\tilde{V}}(y) \implies$ (algebraic H–B for seminorms) $\exists g : X \to \mathbb{F}$ linear, $g|_Y = f$, $|g(x)| \leqslant p_{\tilde{V}}(x)$ for $x \in X \implies g$ is continuous by the proposition above. □

Důsledek

X LCS, $Y \Subset X$ closed, $x \in X \setminus Y$. Then $\exists f \in X^* : f|_Y = 0, f(x) = 1$.

Důkaz

Set $\tilde{Y} = \mathrm{LO}(Y \cup \{x\})$. Define $g(y + \lambda x) = \lambda$, $y \in Y$, $\lambda \in \mathbb{F} \implies g$ is linear functional on $\tilde{Y}$, $g|_Y = 0$, $g(x) = 1$. $\operatorname{Ker} g = Y$ is closed $\implies g$ is continuous $\implies g$ can be extended to $f \in X^*$. □

Důsledek

X LCS, $Z \Subset Y \Subset X$.

$$\overline{Z} \supset Y \Leftrightarrow \forall f \in X^* : f|_Z = 0 \implies f|_Y = 0.$$

Důkaz

„$\implies$“: clear. „$\impliedby$“: $y \in Y \setminus \overline{Z} \implies \exists f \in X^* : f(y) = 1, f|_Z = 0$. □

Důsledek

X HLCS, $x \in X \setminus \{\mathbf{o}\} \implies \exists f \in X^* : f(x) \neq 0.$

Důkaz

$Y = \{\mathbf{o}\}$ is closed linear subspace and use the first corollary. □

Věta 3.2 (Hahn–Banach separation theorem)

X LCS, $A, B \subset X$ nonempty convex, $A \cap B = \emptyset$.

- $\operatorname{int} A \neq \emptyset \implies \exists f \in X^* \setminus \{0\}\ \exists c \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < \Re f(s)$.
- *A compact, B closed* $\implies \exists f \in X^*\ \exists c, d \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < d \leqslant \Re f(b)$.

Důkaz
Analogous to the theorem above. Assume X is a real space ($\mathbb{F} = \mathbb{R}$). „First item“: $\operatorname{int} A \neq \varnothing \implies \operatorname{int}(B - A) \neq \varnothing$ and $- \notin B - A$. Fix $z \in \operatorname{int}(B - A)$, set $U := z - (B - A)$. The U is a convex neighbourhood of $\mathbf{o}$, $z \notin U \implies p_U(z) \geqslant 1$. Define $g_0 : \mathrm{LO}\{z\} \to \mathbb{R}$ by $g_0(t \cdot z) = t \cdot p_U(z) \implies g_0$ is a linear functional, $g_0 \leqslant p_U$ on $\mathrm{LO}\{z\}$ ($t \geqslant 0 \implies g_0(t \cdot z) = t \cdot p_U(z) = p_U(t \cdot z)$, $t < 0 \implies g_0(t \cdot z) = t \cdot p_U(z) < 0 \leqslant p_U(t \cdot z)$).

From algebraic Hahn–Banach $\exists g : X \to \mathbb{R}$ linear, $g|_{\mathrm{LO}\{z\}} = g_0$, $g \leqslant p_U$ on X. g is continuous ($g \leqslant 1$ on $U \implies g \geqslant -1$ on $-U$, so $|g| \leqslant 1$ on $U \cap (-U)$, a neighbourhood of $\mathbf{o}$). $a \in A$, $b \in B \implies$

$$\implies g(z) - g(b) + g(a) = g(z - (b - a)) \leqslant p_U(z - (b - a)) \leqslant 1,$$

$$g(a) \leqslant g(b) + \underbrace{1 - \overbrace{g(z)}^{=p_U(z) \geqslant 1}}_{\leqslant 0}.$$

So, $\forall a \in A\ \forall b \in B : g(a) \leqslant g(b)$, $c := \sup g(A)$.

„Second item“: A compact, B closed. For $x \in A$ $\exists U_x$ an absolutely convex open neighbourhood of $\mathbf{o}$ such that $(x + U_x) \cap B = \varnothing$. The $(x + \frac{1}{2}U_x)_{x \in A}$, is an open cover of A. A is compact $\implies \exists x_1, \dots, x_n \in A : A \subset \left(x_1 + \frac{1}{2}U_{x_1}\right) \cup \ldots \cup \left(x_n + \frac{1}{2}U_{x_n}\right)$. Set $V := \frac{1}{2}U_{x_1} \cap \ldots \cap \frac{1}{2}U_{x_n}$ an absolutely convex open neighbourhood of $\mathbf{o}$. Then $(A + V) \cap B = \varnothing$

$$\left(a \in A \implies \exists j : a \in x_j + \frac{1}{2}U_{x_j} \implies a + V \subset x_j + \frac{1}{2}U_{x_j} + V \subset x_j + U_{x_j}\right).$$

Apply first item to $A + V$ (open convex), B (convex) $\implies \exists f \in X^* \setminus \{0\}$ such that

$$\sup f(A) + \sup f(V) = \sup(f(A) + f(V)) = \sup f(A + V) \leqslant \inf f(B),$$

observe that $\sup f(V) > 0$ ($f \neq 0$, V is neighbourhood of $\mathbf{o}$, hence absorbing).

$$c := \sup f(A), \qquad d := \sup f(A) + \sup f(V).$$

„X complex“: look at X as a real space, $f : X \to \mathbb{R}$ real-linear such that. Define $f_c(x) = f(x) - i f(ix), x \in X$. □

Důsledek
X LCS, $\varnothing \neq A \subset X$, $x \in X$.

- $x \in X \setminus \overline{\mathrm{co}} A \Leftrightarrow \exists f \in X^* : \Re f(x) > \sup\{\Re f(a) | a \in A\}$. („ $\Longleftarrow$ “: Clear as $\{y \in X, \Re f(y) \leqslant \sup\{\Re f(x$ is closed convex set containing A. „ $\Longrightarrow$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{co}} A$, get f and take $-f$.)
- $x \in X \setminus \overline{\mathrm{aco}} A \Leftrightarrow \exists f \in X^* : |f(x)| > \sup\{|f(a)| | a \in A\}$ („ $\Longleftarrow$ “: Clear. „ $\Longrightarrow$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{aco}} A$ (and multiply by -1), $f \in X^*$:

$|f(x)| \geqslant \Re f(x) > \sup\{\Re f(y) | y \in \overline{\text{aco}} A\} = \sup\{|f(y)| | y \in \overline{\text{aco}} A\}$. „$\leqslant$“ clear, „$\geqslant$“: $y \in \overline{\text{aco}\, A} \implies \exists \alpha \in \mathbb{F}, |\alpha| = 1 : |f(y)| = \alpha f(y)$, then $|f(y)| = \lambda f(y) = \Re \alpha f(y) = \Re f(\alpha y)$, $\alpha y \in \overline{\text{aco}\, A}$).

4 Weak topologies

4.1 General weak topologies and duality

Definice 4.1 (Algebraic dual, general weak topology)

X vector space. $X^{\#}$ is the algebraic dual of X (it means set of all linear functionals on X). $\emptyset \neq M \subset X^{\#}$, then $\sigma(X, M)$ is the topology on X generated by seminorms $X \mapsto |f(x)|$, $f \in M$.

Tvrzení 4.1

Properties:

1. *$(X, \sigma(X, M))$ is LCS (by the theorem above).*
2. *$(X, \sigma(X, M))$ is Hausdorff $\Leftrightarrow \forall x \in X \setminus \{0\}\ \exists f \in M : f(x) \neq 0$ (i.e. M separates points) (by the theorem above).*
3. *$f \in M \implies f$ is continuous on $(X, \sigma(X, M))$ (fix $f \in M$, $p(x) = |f(x)|$, $x \in X$ is a continuous seminorm and $|f(x)| = p(x) \leqslant p(x)$).*
4. *$\sigma(X, M)$ is the weakest topology on X making all $f \in M$ continuous.*
5. *$\sigma(X, M) = \sigma(X, \mathrm{LO}(M))$.*
6. *T a topological space, $F : T \to X$ mapping. Than F is continuous $T \to \sigma(X, M) \Leftrightarrow \forall f \in M : f \circ F$ is continuous ($T \to \mathbb{F}$).*

Důkaz (4.)
Assume τ is any topology on X such that all $f \in M$ are τ-continuous $\implies$

$$\implies \forall x \in X\ \forall f_1, \dots, f_n \in M\ \forall c_1, \dots, c_n > 0 : \{y \in X | |f_j(y) - f_j(x)| < c_j\ \forall j \in [n]\} \text{ is } \tau\text{-open}$$

but these sets form a base of $\sigma(X, M) \implies \sigma(X, M) \subset \tau$. □

Důkaz (5.)
„$\subseteq$“: Clear. „$\supseteq$“: $f \in \mathrm{LO}\, M \implies f$ is $\sigma(X, M)$-continuous (the linear combination of continuous linear functionals is continuous) $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$, $f_1, \ldots, f_n \in M$, $x_1, \ldots, x_n \in \mathbb{F}$.

$$|f(x)| \leqslant |\alpha_1| \cdot |f_1(x)| + \ldots + |\alpha_n| \cdot |f_n(x)| \leqslant (|\alpha_1| + \ldots + |\alpha_n|) \cdot \max\{|f_1(x)|, \ldots, |f_n(x)|\}.$$

So by the previous point we get $\sigma(X, \mathrm{LO}\, M) \subset \sigma(X, M)$. □

Důkaz (6.)
„ $\implies$ “: $f \in M \implies f$ is $\sigma(X, M)$-continuous, so $f \circ F$ is continuous. „ $\impliedby$ “: $t \in T$, U neighbourhood of $F(t)$ in $\sigma(X, M) \implies \exists f_1, \ldots, f_n \in M\ \exists c_1, \ldots, c_n > 0$ such that

$$F(t) \in \{y \in X | \forall j \in [n] | f_j(y) - f_j(F(t)) < c_j\} \subset U.$$

Let $G = \{u \in T | \forall j \in [n] : |(f_j \circ F)(u) - (f_j \circ F)(t)| < c_j\}$. Then G is an open neighbourhood of t (by continuity of $f_j \circ F$ and $F(G) \subset U$). □

Příklad
X LCS. Then $X^* \Subset X^\#$. So, we may consider $\sigma(X, X^*)$ „the weak topology of X“. $\sigma(X, X^*)$ is Hausdorff when X is HLCS.

TODO!!!

5 Distributions

TODO!!!

TODO!!!

Lemma 5.1

TODO

a) $\|\cdot\|_N$ *is a norm on* $\mathcal{D}(\Omega)$*;*

b) $\mathcal{D}_K(\Omega)$ *is a Fréchet space when equipped with* $(\|\cdot\|_N)_{N \in \mathbb{N}_0}$*.*

Důkaz (a))
TODO!!! □

Důkaz (b))
$\|\cdot\|_0 \leqslant \|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \dots \implies \mathcal{D}_K(\Omega)$ is a metrizable LCS (by translation invarinat metric ϱ from the proposition above).

$(\varphi_n) \subset \mathcal{D}_k(\Omega)$ ϱ-cauchy, then $\forall N \in \mathbb{N}_0$: (φ_n) is $\|\cdot\|_N$-cauchy $\implies \forall \alpha$: $(D^\alpha \varphi_n)$ is $\|\cdot\|_\infty$-cauchy $\implies \forall \alpha\ \exists \psi_n$ such that $D^\alpha \varphi_n \rightrightarrows \psi_\alpha$ on Ω. The ψ_α is continuous, $\varphi_\alpha = 0$ on $\Omega \backslash K$. Fix $\alpha \in \mathbb{N}_0^d$ and $j \in [d]$. Then

$$D^\alpha \varphi_n \rightrightarrows \psi_\alpha \wedge \frac{\partial}{\partial x_j} D^\alpha \varphi_n = D^{\alpha + e_j} \varphi_n \rightrightarrows \psi_{\alpha + e_j} \implies \psi_{\alpha + e_j} = \frac{\partial}{\partial x_j} \psi_\alpha.$$

$\implies \psi_\alpha = D^\alpha \psi_0.$

TODO!!! □

Tvrzení 5.2

$\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ *linear then following assertions are equivalent:*

1. $\varphi_n \to \varphi$ *in* $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to \Lambda(\varphi)$*;*
2. $\varphi_n \to 0$ *in* $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to 0$*;*
3. $\forall K \subset \Omega$ *compact and* $\Lambda|_{\mathcal{D}_K(\Omega)}$ *is continuous;*
4. $\forall K \subset \Omega$ *compact* $\exists N \in \mathbb{N}_0\ \exists C > 0$ *such that*

$$|\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \qquad \varphi \in \mathcal{D}_K(\Omega).$$

Důkaz
„1. $\implies$ 2." is trivial. „2. $\implies$ 3.": Fix $K \subset \Omega$ compact. $\varphi_n \to 0$ on $\mathcal{D}_K(\Omega) \implies \varphi_n \to 0$ in $\mathcal{D}(\Omega) \overset{2.}{\implies} \Lambda(\varphi_n) \to 0$. Thus $\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous at $\mathbf{o}$, so it is continuous.

„3. $\implies$ 1." $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \implies \exists K \subset \Omega$ compact such that $\operatorname{supp} \varphi_n \subset K$ for each n. Then $(\varphi_n) \subset \mathcal{D}_K(\Omega) \implies \varphi_n \to \varphi$ in $\mathcal{D}_K(\Omega) \overset{3.}{\implies} \Lambda(\varphi_n) \to \varphi(\varphi)$.

„3. $\Leftrightarrow$ 4.". By the proposition above. □

Definice 5.1 (Distribution, finite order)

A distribution on Ω is a linear functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ satisfying assertions from the previous proposition. We will denote distributions on Ω by $\mathcal{D}'(\Omega)$.

$\Lambda \in \mathcal{D}'(\Omega)$ is of finite order, if $N \in \mathbb{N}_0$ in 4. of the previous proposition can be chosen independently on K.

Například
$f \in L^1_{loc}(\Omega)$. $\Lambda_f(\varphi) = \int_\Omega f \cdot \varphi$ $(\varphi \in \mathcal{D}(\Omega)) \implies \Lambda_f$ is a distribution of order 0. Because $K \subset \Omega$ compact $\implies \int_K |f| < \infty$, $\varphi \in D_K(\Omega)$:

$$|\Lambda_f(\varphi)| = |\int_\Omega f \cdot \varphi| = |\int_K f \cdot \varphi| \leqslant \int_K |f\varphi| \leqslant \|\varphi\|_\infty \cdot \int_K |f| = \|\varphi\|_0 \cdot \int_K |f|.$$

$\mu \geqslant 0$ regular Borel measure, finite on compacts. $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution on Ω of order 0. Because if $K \subset \Omega$, $\varphi \in \mathcal{D}_K(\Omega)$, then

$$|\Lambda_\mu(\varphi)| = |\int_\Omega \varphi d\mu| = |\int_K \varphi d\mu| \leqslant \|\varphi\|_\infty \mu(K).$$

μ is a signed (or complex) finite measure $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution of order 0:

$$|\int_K \varphi d\mu| \leqslant \int_K |\varphi| d|\mu| \leqslant |\mu|(K) \cdot \|\varphi\|_\infty \leqslant \|\mu\| \cdot \|\varphi\|_\infty.$$

$\Lambda(\varphi) = \varphi'(0)$, $\varphi \in \mathcal{D}(\mathbb{R})$ is a distribution of order 1. (Clearly $|\Lambda(\varphi)| \leqslant \|\varphi'\|_\infty \leqslant \|\varphi\|_1$.) Λ not of order 0: Find $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi'(0) = 1$, $\operatorname{supp} \varphi \subset [-c, c]$ for some $c > 0$. $\varphi_n(x) = \varphi(nx)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\implies \varphi_n \in \mathcal{D}(\mathbb{R})$. $\operatorname{supp} \varphi_n \subset [-c/n, c/n] \subset [-c, c]$. $\|\varphi_n\|_0 = \|\varphi\|_0$. $\Lambda(\varphi_n) = \varphi_n'(0) = \varphi'(0) \cdot n = n$.

$\Lambda(\varphi) = \sum_{n=0}^\infty \varphi^{(n)}(n)$, $\varphi \in \mathcal{D}(\mathbb{R}) \implies \Lambda$ is a distribution on $\mathbb{R}$, not of finite order ($\operatorname{supp} \varphi \subset [-k, k], k \in \mathbb{N}, \implies |\Lambda(\varphi)| \leqslant (k+1)\|\varphi\|_K$.)

Poznámka
If $f, g \in L^1_{loc}(\Omega)$, $\Lambda_f = \Lambda_g$, then $f = g$ almost everywhere. If μ, ν measures, $\Lambda_\mu = \Lambda_\nu$, then $\mu = \nu$.

If $f \in L^1(\Omega)$, μ finite measure, $\Lambda_f = \Lambda_\mu$, then $\mu(A) = \int_A f$, for each $A \subset \Omega$ Borel.

Definice 5.2

$\Lambda \in \mathcal{D}'(\Omega)$.

- For $\alpha \in \mathbb{N}_0^d$ define $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)
- For $f \in C^\infty(\Omega)$ define $(f\Lambda)(\varphi) = \Lambda(f\varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)

Tvrzení 5.3

a) $\Lambda \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}_0^d \implies D^\alpha \Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
Clear: $D^\alpha\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ linear, $K \subset \Omega$ compact $\implies \exists N \in \mathbb{N}_0, C > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega)$. Then $\forall \varphi \in \mathcal{D}_k(\Omega)$:

$$|D^\alpha\Lambda(\varphi)| = |\Lambda(D^\alpha\varphi)| \leqslant C \cdot \|D^\alpha\varphi\|_N \leqslant C \cdot \|\varphi\|_{|\alpha|+N}$$

. □

b) $f \in C^\infty(\Omega) \implies D^\alpha\Lambda_f = \Lambda_{D^\alpha f}$

Důkaz (For $\partial/\partial x_1$)

$$\frac{\partial}{\partial x_1}\Lambda_f(\varphi) = -\Lambda_f\left(\frac{\partial\varphi}{\partial x_1}\right) =?= -\int_\Omega f \cdot \frac{\partial\varphi}{\partial x_1}$$

TODO □

c) $d = 1$, $\Omega = (a,b)$, $f \in L^1_{loc}(\Omega)$. Then $(\Lambda_f)' = \Lambda_g \Leftrightarrow g$ is the weak derivative of f. And $(\Lambda_f)' = \Lambda\mu \Leftrightarrow \mu$ is the weak derivative of f.

Důkaz
By definitions. □

d) $\Lambda \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega) \implies f\Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
clear: $f\Lambda : \mathcal{D}(\Omega) \implies$ IF linear □

Tvrzení 5.4

a) $\Lambda \in \mathcal{D}'((a,b)), \Lambda' = 0 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
We will prove $\operatorname{Ker}\Lambda_1 \subset \operatorname{Ker}\Lambda$. Then $\exists c : \Lambda = c \cdot \Lambda_1 = \Lambda c$.

$$\varphi \in \operatorname{Ker}\Lambda_1 \implies \Lambda_1(\varphi) = 0, i.e. \int_a^b \varphi = 0.$$

Define $\varphi(t) = \int_a^t \varphi$, $t \in (a,b)$. Then $\psi \in \mathcal{D}((a,b))$, $\psi' = \varphi$ ($\psi' = \varphi$ ... differentiation of indefinite integral $\implies \psi \in C^\infty((a,b))$, $\psi = 0$ on $(a, \min\operatorname{supp}\varphi)$ and $(\max\operatorname{supp}\varphi, b)$ $\implies \psi \in \mathcal{D}((a,b))$). Hence $\Lambda(\varphi) = \Lambda(\psi') = -\Lambda'(\psi) = 0$, so $\varphi \in \operatorname{Ker}\Lambda$. □

b) $\Omega \subset \mathbb{R}^d$ open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha\Lambda = 0$ for $|\alpha| = 1 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
„Step 1: $\Omega = \prod_{j=1}^{d}(a_j, b_j)$“: Induction on d. For $d = 1$ use a). Assume it holds for $d-1$, denote $\Omega' = \prod_{j=1}^{d-1}(a_j, b_j)$, $x \in \Omega \implies x = (x', x_d)$ ($x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$), $\alpha \in N_0^d \implies \alpha = (\alpha', \alpha_d)$.

$\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$ for $|\alpha| = 1$. It means: $\forall \varphi \in \mathcal{D}(\Omega)\ \forall j \in [d] : \Lambda\left(\frac{\partial \varphi}{\partial x_j}\right) = 0$.

Claim: $\psi \in \mathcal{D}(\Omega)$. Then $\exists \varphi \in \mathcal{D}(\Omega) : \frac{\partial \varphi}{\partial x_d} = \psi \Leftrightarrow \forall x' \in \Omega' : \int_{a_d}^{b_d} \psi(x', x_d) dx_d = 0$. („$\implies$“ clear, „$\impliedby$“: define $\varphi(x', x_d) = \int_{a_d}^{x_d} \psi(x', t) dt$). Define

$$T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega'), \qquad T\varphi(x') = \int_{a_d}^{b_d} \varphi(x', x_d) dx_d, \quad \varphi \in \mathcal{D}(\Omega).$$

T is linear, $\operatorname{Ker} T \subset \operatorname{Ker} \Lambda$ ($T\varphi = 0 \implies \exists \psi \in \mathcal{D}(\Omega) : \varphi = \frac{\partial \psi}{\partial x_d}$, thus $\Lambda(\varphi) = 0$). Fix $\eta \in \mathcal{D}((a_d, b_d))$, $\int_{a_d}^{b_d} \eta = 1$. For $\varphi \in \mathcal{D}(\Omega')$ define $(\varphi\eta)(x) = \varphi(x')\eta(x_d)$. Then $\varphi\eta \in \mathcal{D}(\Omega)$. $\tilde{\Lambda}(\varphi) = \Lambda(\varphi\eta)$, $\varphi \in \mathcal{D}(\Omega')$. Then $\tilde{\Lambda} \in \mathcal{D}'(\Omega')$.

Moreover, $\forall \alpha'$ with $|\alpha'| = 1 : D^{\alpha'} \tilde{\Lambda} = 0$.

$$\left(\forall j \in [d-1] : \frac{\partial}{\partial x_j} \tilde{\Lambda}(\varphi) = -\tilde{\Lambda}\left(\frac{\partial \varphi}{\partial x_j}\right) = -\Lambda\left(\frac{\partial \varphi}{\partial x_j}\eta\right) = -\Lambda\left(\frac{\partial}{\partial x_j}(\varphi\eta)\right) = 0.\right)$$

$\implies \exists c \in \mathbb{F} : \tilde{\Lambda} = \Lambda_c$ in $\mathcal{D}'(\Omega')$. Then $\Lambda = \Lambda_c$ (in $\mathcal{D}(\Omega)$) cause

$$\varphi \in \mathcal{D}(\Omega) \implies \varphi - (T\varphi)\eta \in \mathcal{D}(\Omega), \varphi - (T\varphi)\eta \in \operatorname{Ker} T \subset \operatorname{Ker} \Lambda, \text{ so,}$$

$$\Lambda(\varphi) = \Lambda((T\varphi)\eta) = \tilde{\Lambda}(T\varphi) = \Lambda_c(T\varphi) = \int_{\Omega'} c \cdot T\varphi = \int_{\Omega'} c \cdot \int_{a_d}^{b_d} \varphi(x', x_d) dx_d dx' \overset{\text{FUBINI}}{=} \int_\Omega c \cdot \varphi = \Lambda_c(\varphi).$$

„Step 2: Ω is open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$, $|\alpha| = 1$.“: Step 1 $\implies \forall Q \subset \Omega$ cuboid $\exists c : \Lambda|_{\mathcal{D}(Q)} = \Lambda_c$. Fix one cuboid $Q_0 \subset \Omega$ and the respective c.

$$A := \left\{x \in \Omega | \exists Q \subset \Omega \text{ cuboid}, x \in Q, \Lambda|_{\mathcal{D}(Q)} = \Lambda_c\right\}.$$

Fix $A \neq \emptyset$ ($Q_0 \subset A$), A is open, A is closed in Ω ($x \in \overline{A} \cap \Omega$, $Q \cap A \neq \emptyset$, $\Lambda|_{\mathcal{D}(Q)} = \Lambda_d$, $y \in Q \cap A \implies \Lambda|_{\mathcal{D}(Q_y)} = \Lambda_c \implies$ on $\mathcal{D}(Q \cap Q_y) : \Lambda = \Lambda_c = \Lambda_d \implies c = d \implies x \in A$.). So $A = \Omega$ as Ω is connected. The $\Lambda = \Lambda_c$ in $\mathcal{D}'(\Omega)$. (Proof of this was skipped, it remains that for every $\varphi \in \mathcal{D}(\Omega)$, not only for every $\varphi \in \mathcal{D}(Q)$, it holds $\Lambda(\varphi) = \Lambda_c(\varphi)$.) □

5.1 A bit more on distributions

Definice 5.3

$\Lambda_n \to \Lambda$ in $\mathcal{D}(\Omega) \equiv \forall \varphi \in \mathcal{D}(\Omega) : \Lambda_n(\varphi) = \Lambda(\varphi)$.

Tvrzení 5.5

a) $\Lambda_n \to \Lambda$ *in* $\mathcal{D}(\Omega)$*, then:*

- $\forall \alpha : D^\alpha \Lambda_n \to D^\alpha \Lambda$*;*

  Důkaz

  $$D^\alpha \Lambda_n(\varphi) = (-1)^{|\alpha|} \Lambda_n(D^\alpha \varphi) \to (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = D^\alpha \Lambda(\varphi).$$

  □

- $f \in C^\infty(\Omega) : f\Lambda_n \to f\Lambda$.

  Důkaz

  $$f\Lambda_n(\varphi) = \Lambda_n(f\varphi) \to \Lambda(f\varphi) = f\Lambda(\varphi).$$

  □

b) $f_n \to f$ *in* $L^1_{loc}(\Omega)$ *(*$\forall K \subset \Omega$ *compact:* $\int_K |f_n - f| \to 0$*). Then* $\Lambda_{f_n} \to \Lambda_f$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi \in \mathcal{D}(\Omega) : |\Lambda_{f_n}(\varphi) - \Lambda_f(\varphi)| = \left| \int_\Omega f_n \varphi - \int_\Omega f\varphi \right| \leqslant \int_\Omega |f_n - f| \cdot |\varphi| = \int_{\operatorname{supp} \varphi} |f_n - f| \cdot |\varphi| \leqslant \|\varphi\|_\infty \int_{\operatorname{supp} \varphi} |f_n -$$

□

c) $f_n \to f$ *in* $L^p(\Omega)$ *for some* $p \in [1, \infty]$*. Then* $\Lambda_{f_n} \to \Lambda_f$*.*

Důkaz

Let $K \subset \Omega$ be compact, q the dual exponent. Then use b) with

$$\int_K |f_n - f| \leqslant \|f_n - f\|_{L^p(K)} \cdot \|1\|_{L^q(K)} \to 0.$$

□

d) $\varphi_n \to \varphi$ *in* $\mathcal{D}(\Omega)$*. Then* $\Lambda_{\varphi_n} \to \Lambda_\varphi$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) \implies \varphi_n \to \varphi \text{ in } C^\infty(\Omega), \text{ and use c).}$$

□

Věta 5.6

$(\Lambda_n) \subset \mathcal{D}'(\Omega)$ and $\forall \varphi \in \mathcal{D}(\Omega) : (\Lambda_n(\varphi))$ converges in $\mathbb{F}$. Then $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$ is a distribution on Ω.

Důkaz

Clearly Λ is a linear functional on $\mathcal{D}(\Omega)$. Further: $K \subset \Omega$ compact $\implies \forall n : \Lambda_n|_{\mathcal{D}_K(\Omega)}$ is continuous. $\mathcal{D}_K(\Omega)$ is a Fréchet space $\overset{\text{the lemma above, b)}}{\implies}$ $\Lambda|_{\mathcal{D}_K(\Omega)}$ continuous $\implies \Lambda \in \mathcal{D}'(\Omega)$. □

Definice 5.4

$\Lambda \in \mathcal{D}'(\Omega)$.

- $G \subset \Omega$ open. Λ vanishes on G if $\Lambda(\varphi) = 0$ whenever $\varphi \in \mathcal{D}(\Omega), \operatorname{supp} \varphi \subset G$.
- $\operatorname{supp} \Lambda = \Omega \backslash \{G \subset \Omega$ open $|\Lambda$ vanishes on $G\} = \{x \in \Omega | \forall \varepsilon > 0 \exists \varphi \in \mathcal{D}(\Omega) : \operatorname{supp} \varphi \subset U(x, \varepsilon) \wedge \Lambda(\varphi$
- Λ has compact support if $\operatorname{supp} \Lambda$ is a compact subset of Ω.

Tvrzení 5.7

a) $\Lambda = \Lambda_f$ for some $f \in L^1_{loc}(\Omega)$. Then

$$\operatorname{supp} \Lambda_f = \operatorname{supp} f = \left\{x \in \Omega | \forall \varepsilon > 0 : \lambda^d \left(\{y \in U(x, \varepsilon) \cap \Omega | f(y) \neq 0\}\right) > 0\right\}$$

Důkaz

„$\subseteq$“: $X \notin \operatorname{supp} f \implies \exists \varepsilon > 0 : f = 0$ almost everywhere on $U(x, \varepsilon) \cap \Omega \implies \Lambda_f$ vanishes on $U(x, \varepsilon) \cap \Omega \implies x \notin \operatorname{supp} \Lambda_f$.

„$\supseteq$“: $x \in$ supp. Let $\varepsilon > 0$. Then f is not 0 almost everywhere on $U(x, \varepsilon) \cap \Omega \implies \exists \varphi \in \mathcal{D}(U(x, \varepsilon) \cap \Omega)$ □

b) $\Lambda = \Lambda_\mu$. Then $\operatorname{supp} \Lambda = \operatorname{supp} \mu = \Omega \backslash \bigcup \{G \subset \Omega$ *open* $|\forall B \subset G$ *Borel* $\mu(B) = 0\}$.

Důkaz

$G \subset \Omega$ open the $\forall B \subset G$ Borel $\mu(B) = 0 \Leftrightarrow \forall \varphi \in \mathcal{D}(G) : \int \varphi d\mu = 0 \Leftrightarrow \Lambda_\mu$ vanishes on G. □

Poznámka

f is continuous $\implies \operatorname{supp} f = \overline{\{x | f(x) \neq 0\}} \cap \Omega$.

c) $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp} \varphi \cap \operatorname{supp} \Lambda = \varnothing \implies \Lambda(\varphi) = 0$.

Důkaz
$\operatorname{supp}\varphi \cap \operatorname{supp}\Lambda = \varnothing \implies \operatorname{supp}\varphi \subset \bigcup\{G \subset \Omega \text{ open } | \Lambda \text{ vanishes on } G\} \implies \exists G_1, G_2, \ldots, G_n \subset \Omega$ open such that Λ vanishes on each G_j and $\operatorname{supp}\varphi \subset G_1 \cup \ldots \cup G_n$. We will be done if we show that Λ vanishes on $G_1 \cup \ldots \cup G_n$. ▫

Důkaz (Λ vanishes on $G_1, G_2 \implies$ vanishes on $G_1 \cup G_2$)
$\psi \in \mathcal{D}(\Omega)$, $\operatorname{supp}\psi \subset G_1 \cup G_2$. If $\operatorname{supp}\psi \subset G_1$ or $\operatorname{supp}\psi \subset G_2$, then $\Lambda(\psi) = 0$. Assume $\operatorname{supp}\psi \not\subset G_1$ and $\operatorname{supp}\psi \not\subset G_2$. Then $L := \operatorname{supp}\varphi \backslash G_2 \implies L$ is compact, nonempty, $L \subset G_1$. Fix $\delta > 0$ such that $3\delta < \operatorname{dist}(L, \mathbb{R}^d \backslash G_1)$, h_k smooth kernel.

Fix $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$, $\xi := h_k * \chi_{L+B(0,2\delta)} \implies \xi \in C^\infty(\mathbb{R}^d)$. $\operatorname{supp}\xi \subset L + B(0,2\delta) + U(0,1/k) \subset L + U(0,3\delta) \subset G_1$, $\xi = 1$ on $L + B(0,\delta)$. Set $\psi_1 = \xi \cdot \psi$, $\psi_2 = (1-\xi)\psi \implies \psi_1, \psi_2 \in \mathcal{D}(\Omega)$, $\operatorname{supp}\psi_1 \subset \xi \subset G_1$, $\operatorname{supp}\psi_2 \subset \operatorname{supp}\psi \backslash (L + B(0,\delta)) \subset \operatorname{supp}\psi \backslash (L + U(0,\delta)) \subset \operatorname{supp}\psi \backslash L \subset G_2 \implies \Lambda(\psi_1) = \Lambda(\psi_2) = 0$. $\psi = \psi_1 + \psi_2 \implies \Lambda(\psi) = \Lambda(\psi_1) + \Lambda(\psi_2) = 0$. ▫

d) Λ has compact support $\implies \exists N \in \mathbb{N}_0\ \exists c > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N$ for $\varphi \in \mathcal{D}(\Omega)$. In particular, Λ has finite order.

Důkaz
$\operatorname{supp}\Lambda$ is a compact subset of $\Omega \implies \exists\delta > 0 : K := \operatorname{supp}\Lambda + B(0,3\delta) \subset \Omega \implies K \subset \Omega$ is compact $\implies$

$$\exists N \in \mathbb{N}_0\ \exists c > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$\xi := h_k * \chi_{\operatorname{supp}\Lambda + B(0,2\delta)}$. ($1/k < \delta$.) $\xi \in C^\infty(\mathbb{R}^d), \operatorname{supp}\xi \subset \operatorname{supp}\Lambda + B(0,2\delta) + U(0,1/k) \subset K$. $\xi = 1$ on $\operatorname{supp}\Lambda + B(0,\delta)$.

$\forall\varphi \in \mathcal{D}(\Omega) : \Lambda(\varphi) = \Lambda(\varphi\xi)$. $(1-\xi)\varphi \in \mathcal{D}(\Omega) = 0$ on $\operatorname{supp}\Lambda + B(0,\delta) \implies \operatorname{supp}(1-\xi)\varphi \cap \operatorname{supp}\Lambda = \varnothing$. $\implies \Lambda((1-\xi)\varphi) = 0 \implies \Lambda(\varphi) = \Lambda(\xi\varphi)$.

Then

$$|\Lambda(\varphi)| = |\Lambda(\varphi\xi)| \leqslant C \cdot \|\xi \cdot \varphi\|_N \leqslant C \cdot 2^N \cdot \|\xi\|_N \cdot \|\varphi\|_N.$$

▫

e) $\operatorname{supp}\Lambda = \{p\} \Leftrightarrow \exists N \in \mathbb{N}_0, C_\alpha \in \mathbb{F}, |\alpha| \leqslant N, \Lambda = \sum_{|\alpha| \leqslant N} C_\alpha D^\alpha \Lambda_{\delta_p}$.

Důkaz
„$\Longleftarrow$“: trivial. „$\Longrightarrow$“: $\{p\}$ is compact $\implies \exists N, C : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_M$, $\varphi \in D(\Omega)$. The Λ is a linear combination of $D^\alpha \Lambda_{\delta_p}$, $|\alpha| \leqslant N$. To prove this, we use lemma above and show

$$\bigcap_{|\alpha| \leqslant N} \operatorname{Ker} D^\alpha \Lambda_{\delta_p} \subset \operatorname{Ker}\Lambda,$$

i.e. $\forall\varphi \in \mathcal{D}(\Omega) : D^\alpha\varphi(p) = 0$ for each $|\alpha| \leqslant N \implies \Lambda(\varphi) = 0$. ▫

6 Convolution of distribution

Definice 6.1 (Notation)

$M \subset \mathbb{R}^d$, $f : M \to \mathbb{F}$

- $y \in \mathbb{R}^d$, $\tau_y f(x) = f(x-y)$, $x \in y + M$;
- $\hat{f}(x) = f(-x)$, $x \in -M$;
- $a, e \in \mathbb{R}^d$: $\partial_e f(a) = \lim_{r\to 0} : \frac{f(a+re)-f(a)}{r}$.

Lemma 6.1

$\varphi \in \mathcal{D}(\mathbb{R}^d)$.

a) $x_n \to x$ in $\mathbb{R}^d \implies \tau_{x_n}\varphi \to \tau_x\varphi$ in $\mathcal{D}(\mathbb{R}^d)$.

Důkaz

$\operatorname{supp}\varphi \subset U(0,r_1)$ for some $r_1 > 0$, $\{x_n, n \in \mathbb{N}\} \subset U(0,r_2)$ for some $r_2 > 0$. $K := \overline{U(0,r_1+r_2)} \implies K$ is compact and $\operatorname{supp}\tau_{x_n}\varphi \subset K$ for each n.

$$\alpha \in \mathbb{N}_0^d : \|D^\alpha\tau_{x_n}\varphi - D^\alpha\tau_x\varphi\|_\infty = \sup_{y\in\mathbb{R}^d}|D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)| = \sup_{y\in K}|D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)|.$$

Thus $D^\alpha\varphi$ is continuous, so it is uniformly continuous on $\overline{U(2r_2+r_1)}$.

$$\varepsilon > 0 \implies \exists\delta > 0\ \forall y_1, y_2 \in \overline{U(2r_2+r_1)} : (\|y_1 - y_2\| < \delta \implies |D^\alpha\varphi(y_1) - D^\alpha\varphi(y_2)| < \varepsilon).$$

$$x_n \to x \implies \exists n_0\ \forall n \geqslant n_0 : \|x_n - x\| < \delta.$$

$$n \geqslant n_0, y \in K \implies y - x_n, y - x \in \overline{U(2r_2+r_1)}, \|(y-x_n)-(y-x)\| = \|x_n - x\| < \delta \implies$$
$$\implies |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)| < \varepsilon \implies D^\alpha\tau_{x_n}\varphi \rightrightarrows D^\alpha\tau_x\varphi.$$

□

b) $e \in \mathbb{R}^d \implies \partial_e\varphi \in \mathcal{D}(\mathbb{R}^d)$. Moreover, set

$$\varphi_r(x) := \frac{1}{r}(\varphi(x+re) - \varphi(x)), \qquad x \in \mathbb{R}^d,$$

then $\varphi_r \xrightarrow{r\to 0} \partial_e\varphi$ in $\mathcal{D}(\mathbb{R}^d)$.

Důkaz ($e \in \mathbb{R}^d \implies \partial_e\varphi \in \mathcal{D}(\mathbb{R}^d)$)
$x \in \mathbb{R}^d$. $g_x(t) := \varphi(x+te)$, $t \in \mathbb{R}$. Then $g_x \in C^\infty(\mathbb{R})$.

$$\partial_e\varphi(x) = g_x'(0) = \sum_{j=1}^d \frac{\partial\varphi}{\partial x_j}(x+te)\cdot e_j|_{t=0} =$$

$$= \sum_{j=1}^d \frac{\partial\varphi}{\partial x_j}(x)e_j \implies \partial_e\varphi = \sum_{j=1}^d e_j\frac{\partial\varphi}{\partial x_j} \in \mathcal{D}(\mathbb{R}^d).$$

□

Důkaz (Moreover part)
Fix $c > 0$, such that $\operatorname{supp}\varphi \subset U(0,c)$, and $0 < |r| < 1$. Then $\operatorname{supp}\varphi_r \subset \overline{U(0, c+\|e\|)}$.

$$|\varphi_r(x) - \partial_e\varphi(x)| = \left|\frac{1}{r}(g_x(r)-g_x(0)) - g_x'(0)\right| = \left|\frac{1}{r}\int_0^r g_x' - g_x'(0)\right| = \left|\frac{1}{r}\int_0^r (g_x'(t)-g_x'(0))dt\right| =$$

$$= \left|\frac{1}{r}\int_0^r \sum_{j=1}^d e_j\left(\frac{\partial\varphi}{\partial x_j}(x+te) - \frac{\partial\varphi}{\partial x_j}(x)\right)dt\right| \leqslant$$

$$\leqslant \left|\frac{1}{r}\int_0^r \|e\|\left(\sum_{j=1}^d \left\|\frac{\partial\varphi}{\partial x_j}(x+te) - \frac{\partial\varphi}{\partial x_j}(x)\right\|^2\right)^{1/2} dt\right| \leqslant$$

$$\leqslant \left|\frac{1}{r}\int_0^r \|e\|\left(\sum_{j=1}^d \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty^2\right)^{1/2} dt\right|.$$

$$\varepsilon > 0 \implies \exists\delta\ \forall y, \|y\| < \delta : \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty < \varepsilon.$$

If $0 < |t|\cdot\|e\|\cdot c$, then

$$\|e\|\left(\sum_{j=1}^d \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty^2\right)^{1/2} \leqslant \|e\|\cdot\sqrt{d}\cdot\varepsilon.$$

So $\varphi_r \rightrightarrows \partial_e\varphi$, $D^\alpha\varphi_r = (D^\alpha\varphi)_r \rightrightarrows \partial_e(D^\alpha\varphi) = D^\alpha(\partial_e\varphi)$. □

Tvrzení 6.2

$\varphi \in \mathcal{D}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$.

a) $\Lambda \in \mathcal{D}'(\mathbb{R}^{d_1})$. *Define* $\psi(y) = \Lambda(x \mapsto \varphi(x,y))$ $(y \in \mathbb{R}^{d_2})$. *Then* $\psi \in \mathcal{D}(\mathbb{R}^{d_2})$.

Důkaz
Fix $c > 0$ such that $\operatorname{supp} \varphi \subset \overline{U(\mathbf{o}, c)}$. 1. „$\psi$ is well defined“: given $y \in \mathbb{R}^{d_2}$, $x \mapsto \varphi(x, y)$ belongs to $\mathcal{D}(\mathbb{R}^{d_1})$, i.e. it is C^∞ and $\operatorname{supp} \subset \overline{U(0, c)}$. 2. $\operatorname{supp} \psi \subset \overline{U(\mathbf{o}, c)}$, so it is compact.

3. $y \in \mathbb{R}^{d_2}$, $\varphi_y(x) = \varphi(x, y)$ $(x \in \mathbb{R}^{d_1})$. Then „$y_n \to y$ in $\mathbb{R}^{d_2} \implies \varphi_{y_n} \to \varphi_y$ in $\mathcal{D}(\mathbb{R}^{d_2})$“: Assume $y_n \to y$ in $\mathbb{R}^{d_2}$. WLOG $\|y_n\| \leqslant c$ for each n. $\forall n : \operatorname{supp} \varphi_{y_n} \subset \overline{U(\mathbf{o}, c)}$. Fix $\alpha \in \mathbb{N}_0^{d_1}$. Then „$D^\alpha \varphi_{y_n} \rightrightarrows D^\alpha \varphi_y$“:

$D^\alpha \varphi_{y_n}(x) = D^{(\alpha,0)} \varphi(x, y_n)$. $D^{(\alpha,0)} \varphi$ is continuous, hence uniformly continuous on $\overline{U(\mathbf{o}, c)}$. So, give $\varepsilon. > 0$ $\exists \delta > 0$ $\forall (u_1, u_2), (v_1, v_2) \in \overline{U(\mathbf{o}, c)}$:

$$\|(u_1, v_1) - (u_2, v_2)\| < \delta \implies |D^{(\alpha,0)} \varphi(u_1, v_1) - D^{(\alpha,0)} \varphi(u_2, v_2) < \varepsilon.$$

Fix $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0 : \|y - y_n\| < d$. If $n \geqslant n_0$ and $x \in \overline{U_{\mathbb{R}^{d_1}}(\mathbf{o}, c)}$, then

$$|D^{(\alpha,0)} \varphi(x, y_n) - D^{(\alpha,0)} \varphi(x, y)| < \varepsilon \qquad \Longleftarrow \|(x, y_n) - (x, y)\| < \delta.$$

Hence $\|D^\alpha \varphi_{y_n} - D^\alpha \varphi_y\| \leqslant \varepsilon$ for $n \geqslant n_0$.

4. „ψ is continuous“:

$$y_n \to y \overset{3.}{\implies} \varphi_{y_n} \to \varphi_y \text{ in } \mathcal{D}(\mathbb{R}^{d_1}) \implies \psi(y_n) = \Lambda(\varphi_{y_n}) \to \Lambda(\varphi_y) = \psi(y).$$

5. „$\frac{\partial \psi}{\partial y_j}(y) = \Lambda(x \mapsto \frac{\partial \varphi}{\partial y_j}(x, y))$“:

$$\frac{\partial \psi}{\partial y_j}(y) = \lim_{t \to 0} \frac{\psi(y + te_j) - \psi(y)}{\tau} \overset{\Lambda \text{ linear}}{=} \lim_{t \to 0} \Lambda \left(x \mapsto \frac{\varphi(x, y + te_j) - \varphi(x, y)}{t} \right) =$$

$$= \lim_{t \to 0} \Lambda(x \mapsto \varphi_t(x, y)).$$

We know $\varphi_t \to \partial_{(0, y_j)} \varphi$ in $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. So we have $\varphi_t \to \frac{\partial \varphi}{\partial y_j}$ in $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Hence, for each $y \in \mathbb{R}^{d_2}$: $(\varphi_t)_y \to \left(\frac{\partial \varphi}{\partial y_j} \right)_y$ in $\mathcal{D}(\mathbb{R}^{d_1}) \implies \Lambda((\varphi_t)_y) \to \Lambda\left(\left(\frac{\partial \varphi}{\partial y_j} \right)_y \right)$.

$$(*) = \Lambda \left(\left(\frac{\partial \varphi}{\partial y_j} \right)_y \right) = \Lambda(x \mapsto \frac{\partial \varphi}{\partial y_j}(x, y)).$$

6. „$\psi \in C^\infty(\mathbb{R}^{d_2})$ and $\forall \alpha : D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0,\alpha)} \varphi((x, y)))$“: 5. $\implies$ for $|\alpha| = 1$. 4. applied to $\frac{\partial \varphi}{\partial y_j}$ implies $\psi \in C^1(\mathbb{R}^{d_2})$. Induction: Assume it holds for $|\alpha| \leqslant k$, take $|\alpha| = k+1$. Then $\alpha = \beta + e_j$, $|\beta| = k$, $j \in [d]$.

$$D^\alpha \psi(y) = \frac{\partial}{y_j}(D^\beta \psi)(y) = \frac{\partial}{\partial y_j} \left(y \mapsto \Lambda \left(x \mapsto D^{(0,\beta)} \varphi(x, y) \right) \right) \overset{5.}{=}$$

$$= \Lambda(x \mapsto \frac{\partial}{\partial y_j} D^{(0,\beta)} \varphi(x, y)) = \Lambda(x \mapsto D^{(0,\alpha)} \varphi(x, y)).$$

□

Lemma 6.3

$\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}(\Omega)$, $K \subset \Omega$ compact. Then $\exists N \in \mathbb{N}_0$, $\exists \mu_\alpha$, $|\alpha| \leqslant N$, finite (signed or complex) Borel measure on K such that

$$\Lambda(\varphi) = \sum_{|\alpha| \leqslant N} \int_K D^\alpha \varphi d\mu_\alpha, \qquad \varphi \in \mathcal{D}_K(\Omega).$$

Důkaz (of lemma, sketch)
From the proposition above $\exists N, C$ such that

$$|\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$X := (C(K))^{\{\alpha | |\alpha| \leqslant N\}}$. $T : \mathcal{D}_K(\Omega) \to X$ by $T\varphi = (D^\alpha \varphi)_{|\alpha| \leqslant N} \implies \Lambda \circ T^{-1}$ is continuous on $T(\mathcal{D}_K(\Omega)) \implies$ extend to $X \implies$ (by Riesz) find $\mu_\alpha, |\alpha| \leqslant N$. □

b) $\Lambda_1 \in \mathcal{D}'(\mathbb{R}^{d_1})$, $\Lambda_2 \in \mathcal{D}'(\mathbb{R}^{d_2})$. Then

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x, y))) = \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x, y))).$$

Důkaz
By a) both sides are well defined. $\operatorname{supp} \varphi \subset \overline{U(\mathbf{o}, c)}$. From the previous lemma: Λ_1 (resp. Λ_2) on $\overline{U(\mathbf{o}, c)}$ is equal to μ_α (resp. ν_α) for some $|\alpha| \leqslant N_1$ (resp. $|\alpha| \leqslant N_2$).

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x, y))) = \sum_{|\beta| \leqslant N_2} \int D^\beta \lambda_1(x \mapsto \varphi(x, y)) d\nu_\beta(y) =$$

$$= \sum_{|\beta| \leqslant N_2} \int \Lambda_1(x \mapsto D^{(0,\beta)} \varphi(x, y)) d\nu_\beta(y) =$$

$$= \sum_{|\beta| \leqslant N_2} \sum_{|\alpha| \leqslant N_1} \int \int D^{(\alpha,\beta)} \varphi(x, y) d\mu_\alpha(x) d\nu_\beta(y) \overset{\text{FUBINI}}{=}$$

$$= \sum_{|\beta| \leqslant N_2} \sum_{|\alpha| \leqslant N_1} \int \int D^{(\alpha,\beta)} \varphi(x, y) d\nu_\beta(y) d\mu_\alpha(x) \ldots$$

□

Definice 6.2 (Konvoluce v distribucích)

$U \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x - y))$ $(x \in \mathbb{R}^d)$.

Věta 6.4

a) $f \in L^1_{loc} \implies \Lambda_f * \varphi = f * \varphi$.

Důkaz

$$\Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x-y)) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)dy = f * \varphi(x).$$

□

b) $U * \varphi \in C^\infty(\mathbb{R}^d),\ D^\alpha(U * \varphi) = D^\alpha U * \varphi = U * D^\alpha \varphi.$

Důkaz

„$U * \varphi$ is continuous“:

$$x_n \to x \text{ in } \mathbb{R}^d \implies \tau_{x_n}\check{\varphi} \to \tau_x\check{\varphi} \text{ in } \mathcal{D}(\mathbb{R}^d) \implies U*\varphi(x_n) = U(\tau_{x_n}\check{\varphi}) \to U(\tau_x\check{\varphi}) = U*\varphi(x).$$

$$\frac{\partial}{\partial x_j}(U * \varphi)(x) = \lim_{t\to 0} \frac{U * \varphi(x+te_j) - U * \varphi(x)}{t} =$$

$$= \lim_{t\to 0} U\left(\frac{\tau_{x+te_j}\check{\varphi} - \tau_x\check{\varphi}}{t}\right) \overset{\psi:=\tau_x\check{\varphi}}{=} \lim_{t\to 0} U\left(\frac{\tau_{te_j}\psi - \psi}{t}\right) = U(\partial_{-e_j}\psi) =$$

$$= U\left(\tau_x\left(\frac{\partial\varphi}{\partial x_j}\right)\right) = U * \frac{\partial\varphi}{\partial x_j}(x).$$

$$\partial_{-e_j}\psi = -\partial_{e_j}\psi = -\frac{\partial\psi}{\partial y_j} = -\frac{\partial}{\partial y_j}(\tau_x\check{\varphi}) = \tau_x\left(\frac{\partial\varphi}{\partial y_j}\right)^v.$$

$$\frac{\partial}{\partial x_j}(U * \varphi) = U * \frac{\partial\varphi}{\partial x_j}.$$

$$\frac{\partial U}{\partial x_j} * \varphi(x) = \frac{\partial U}{\partial x_j}\tau_x\check{\varphi} = -U\left(\frac{\partial\tau_x\check{\varphi}}{\partial x}\right) = U * \frac{\partial\varphi}{\partial x_j}(x).$$

So, we have it for $|\alpha| = 1$. The general case by induction. □

c) $\operatorname{supp}(U * \varphi) \subset \operatorname{supp} U + \operatorname{supp}\varphi.$

Důkaz

$$U * \varphi(x) \neq 0 \implies U(\tau_x\check{\varphi}) \neq 0 \implies \operatorname{supp}(\tau_x\check{\varphi}) \cap \operatorname{supp} U \neq \varnothing \implies x \in \operatorname{supp}\varphi + \operatorname{supp} U.$$

□

Důsledek

So U has compact support $\implies$ $U * \varphi$ has compact support.

d) h_j *smoothing kernel. Then* $\Lambda_{U*h_j} \to U$ *in* $\mathcal{D}'(\mathbb{R}^d)$.

Důkaz

$$\Lambda_{U*h_j}(\varphi) = \int (U*h_j)(x)\varphi(x)dx = \int U(y \mapsto h_j(x-y))\varphi(x)dx =$$

$$= \int U(y \mapsto \varphi(x)h_j(x-y))dx = \Lambda_1(y \mapsto \varphi(x)h_j(x-y)) = U(y \mapsto \Lambda_1(x \mapsto \varphi(x)h_j(x-y))) =$$

$$= U(y \mapsto \int \varphi(x)h_j(x-y)dx) = U(\varphi * \check{h}_j) \to \Lambda(\varphi).$$

Because $\varphi * \check{h}_j \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ and

$$\operatorname{supp}(\varphi * \check{h}_j) \subset \operatorname{supp}\varphi + U(0, 1/j) \subset \varphi + \overline{U(0,1)},$$

$$D^\alpha(\varphi * \check{h}_j) = (D^\alpha\varphi) * h_j \rightrightarrows D^\alpha\varphi.$$

□

e) $\tau_x(U*\varphi) = \tau_x U * \varphi = U * \tau_x\varphi$

Důkaz

$$\tau_x(U*\varphi)(z) = (U*\varphi)(z-x) = U(\tau_{z-x}\check{\varphi}) = U(\tau_{-x}\tau_z\check{\varphi}) = \tau_x U(\tau_z\check{\varphi}) = \tau_x U * \varphi(z).$$

$$\tau_x(U*\varphi)(z) = (U*\varphi)(z-x) = U(\tau_{z-x}\check{\varphi}) = U(\tau_z(\tau_{-x}\check{\varphi})) = U(\tau_z(\widetilde{\tau_x\varphi})) = U * \tau_x\varphi(z).$$

$$(\tau_{-x}\check{\varphi}(y) = \check{\varphi}(y+x) = \varphi(-y-x) = \tau_x\varphi(-y) = (\widetilde{\tau_x\varphi})(y).)$$

□

f) $U * (\varphi * \psi) = (U * \varphi) * \psi$ *(*$U \in \mathcal{D}'(\mathbb{R}^d), \varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$*).*

Důkaz

$$U * (\varphi * \psi)(x) = U(y \mapsto (\varphi*\psi)(x-y)) = U(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y-z)\psi(z)dz) =$$

$$= U(y \mapsto \Lambda_1(z \mapsto \varphi(x-y-z)\psi(z))) = \Lambda_1(z \mapsto U(y \mapsto \varphi(x-y-z)\psi(z))) =$$

$$= \Lambda_1(z \mapsto \psi(z) \cdot U(y \mapsto \varphi(x-y-z))) = \Lambda_1(z \mapsto \psi(z) \cdot (U*\varphi)(x-z)) =$$

$$= \int \psi(z) \cdot (U * \varphi(x-z))dz = (U * f) * \psi(x).$$

□

Poznámka

$$\check{U}(\varphi) = U(\check{\varphi}), \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\tau_x U$ and $\check{U}$ are distributions, $\tau_x\Lambda_f = \Lambda_{\tau_x f}$, $\check{\Lambda}_f = \Lambda_{\check{f}}$, $f \in L^1_{loc}(\mathbb{R}^d)$ (standard one page of computations or less).

Poznámka
U, V distributions, $U * V(\varphi) = U(\check{V} * \varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$:

- It is natural formula:

$$V = \Lambda_\psi, \psi \in \mathcal{D}(\mathbb{R}^d) \implies \Lambda_{U*\psi}(\varphi) = U(\check{\psi} * \varphi).$$

Důkaz

$$\Lambda_{U*\psi}(\varphi) = \int_{\mathbb{R}^d} U * \psi(x)\varphi(x)dx = \int_{\mathbb{R}^d} U(y \mapsto \psi(x-y))\varphi(x)dx =$$
$$= \int_{\mathbb{R}^d} U(y \mapsto \psi(x-y)\varphi(x))dx = U(y \mapsto \int_{\mathbb{R}^d} \psi(x-y)\varphi(x)dx) = U(y \mapsto \check{\psi} * \varphi(y)).$$

□

- This formula does not work in general because $\check{V} * \varphi$ is a C^∞-function but it need not have compact support.

Poznámka (1.)
$\operatorname{supp} V$ is compact, then $V * \varphi \in \mathcal{D}(\mathbb{R}^n)$ for each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ ($\operatorname{supp} \check{V} * \varphi \subset \operatorname{supp} \check{V} + \operatorname{supp} \varphi$, so it is compact). Then $U * V$ is linear functional on $\mathcal{D}(\mathbb{R}^d)$. Moreover, „it is a distribution“:

Fix $K \subset \mathbb{R}^d$ compact. Set $L := \operatorname{supp} \check{V} + K \implies$

$$\implies \exists C > 0, N \in \mathbb{N}_0 : |V(\psi)| \leqslant C \cdot \|\psi\|, \qquad \forall \psi \in \mathcal{D}_L(\mathbb{R}^d).$$

$\varphi \in \mathcal{D}_K(\mathbb{R}^d) \implies \check{V} * \varphi \in \mathcal{D}_L(\mathbb{R}^d) \implies |(U * V)(\varphi)| = |U(\check{V} * \varphi)| \leqslant C \cdot \|\check{V} * \varphi\|_N \leqslant C \cdot D \cdot \|\varphi\|_{N+M}$.
($\check{V} * \varphi(x) = V(y \mapsto \varphi(x+y))$, V has compact support $\implies \exists D, M : |V(\eta)| \leqslant D \cdot \|\eta\|_M$, $\forall \eta \in \mathcal{D}(\mathbb{R}^d)$.)

Poznámka (2.)
$\operatorname{supp} U$ is compact $\implies \exists \psi \in \mathcal{D}(\mathbb{R}^d)$ such that $U(\varphi) = U(\psi \cdot \varphi), \varphi \in \mathcal{D}(\mathbb{R}^d)$. (Proof of the theorem above item d.) So, define $(U * V)(\varphi) = U(\psi \cdot (\check{V} * \varphi))$. Again $U * V \in \mathcal{D}'(\mathbb{R}^d)$. (Proof skipped.)

Poznámka (3.)
$\forall r > 0 : (\overline{U(\mathbf{o}, r)} - \operatorname{supp} V) \cap \operatorname{supp} U$ is compact. For $r > 0$ let $\psi_r \in \mathcal{D}(\mathbb{R}^d)$, $\psi_r = 1$ on a neighbourhood of this set. Then U may be extended to $Y = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \operatorname{supp} f \subset \overline{U(\mathbf{o}, r)} - \operatorname{supp} V \right.$ for som
$\tilde{U}(f) = U(\psi_r \cdot f)$ if $\operatorname{supp} f \subset \overline{U(\mathbf{o}, r)} - \operatorname{supp} V$.

Then define $U * V(\varphi) = \tilde{U}(\check{V} * \varphi)$ $(\operatorname{supp} \check{V} * \varphi \subset \operatorname{supp} \varphi - \operatorname{supp} V)$.

Poznámka (4.)
Assume $\exists m, n \in \mathbb{N}_0$, $c, d > 0$:

$$|U(\varphi)| \leqslant c \cdot \|\varphi\|_n \wedge |V(\varphi)| \leqslant d \cdot \|\varphi\|_m, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\implies \mu_\alpha, |\alpha| \leqslant n$ measures (finite ...):

$$U(\varphi) = \sum_{|\alpha| \leqslant n} \int_{\mathbb{R}^d} D^\alpha \varphi d\mu_\alpha, \varphi \in \mathcal{D}(\mathbb{R}^d) \implies$$

$$\implies (U * V)(\varphi) = \sum_{|\alpha| \leqslant n} \int_{\mathbb{R}^d} D^\alpha (\check{V} * \varphi) d\mu_\alpha.$$

$$|(U * V)(\varphi)| \leqslant c \cdot d \cdot \|\varphi\|_{n+m}.$$

6.1 Tempered distributions

Definice 6.3 (Schwartz space)

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha \in \mathbb{N}_0^d \; \forall N \in \mathbb{N} : x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \text{ is bounded on } \mathbb{R}^d \right\}.$$

$$f \in \mathcal{S}(\mathbb{R}^d), \quad N \in \mathbb{N}_0, \quad p_N(f) := \max_{|\alpha| \leqslant N} \|x \mapsto (1 + \|x\|^2)^N D^\alpha f(x)\|_\infty.$$

Then $(p_N)_{N=0}^\infty$ is sequence of norms on $\mathcal{S}(\mathbb{R}^d)$, $p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$ $(p_0(f) = \|f\|_\infty)$

Tvrzení 6.5

a) $\mathcal{S}(\mathbb{R}^d)$ *is a Fréchet space when equipped with* $(p_N)_{N=0}^\infty$.

Důkaz
$\mathcal{S}(\mathbb{R}^d)$ is a metrizable LCS. Let ϱ be the respective translation invariant metric. „Completeness“: Assume (f_n) is ϱ-Cauchy $\implies$ $\forall N$: (f_n) is p_N-Cauchy $\implies$ $\forall N \; \forall \alpha, |\alpha| \leqslant N : (x \mapsto (1 + \|x\|^2) D^\alpha f_k(x))_{k=1}^\infty$ is $\|\cdot\|_\infty$-Cauchy $\implies$ $\forall N, \alpha, |\alpha| \leqslant N \; \exists g_{N,\alpha}$ such that $(1 + \|x\|^2)^N D^\alpha f_n(x) \rightrightarrows g_{N,\alpha}(x)$ on $\mathbb{R}^d$. $D^\alpha f_k(x) \rightrightarrows \frac{g_{N,\alpha}(x)}{(1+\|x\|^2)^N}$. $\implies$ $\forall \alpha \; \exists h_\alpha$ continuous such that $g_{N,\alpha}(x) = (1 + \|x\|^2)^N h_\alpha(x)$ if $N \geqslant |\alpha|$. $D^\alpha f_k \rightrightarrows h_\alpha$ $\implies$ $h_\alpha = D^\alpha h_\alpha$ $\implies$ $h_\alpha \in C^\infty(\mathbb{R}^d)$.

„$h_0 \in \mathcal{S}(\mathbb{R}^d)$“:

$$(1 + \|x\|^2)^N D^\alpha h_0(x) = g_{N,\alpha}(x),$$

which is bounded (uniform limit of bounded functions). Moreover $f_k \to h_0$ in p_N, hence by the theorem above $f_n \to h_0$ in $\mathcal{S}(\mathbb{R}^d)$ (in ϱ). □

b) $\mathcal{D}(\mathbb{R}^d)$ *is a dense subset of* $\mathcal{S}(\mathbb{R}^d)$.

Důkaz

Clearly $\mathcal{D}(\mathbb{R}^d) \subset\subset \mathcal{S}(\mathbb{R}^d)$. „Density“: Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ na $U(\mathbf{o}, 1)$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $f_n(x) = f(x) \cdot \varphi(x/n)$, $x \in \mathbb{R}^d$. Then $f_n \in \mathcal{D}(\mathbb{R}^d)$. Moreover, „$f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$“: Let $N \in \mathbb{N}_0$, $d \in \mathbb{N}_0^d$, $|\alpha| \leqslant N$:

$$\left|(1 + \|x\|^2)^N (D^\alpha f(x) - D^\alpha f_n(x))\right| = (1 + \|x\|^2)^N \left|D^\alpha((1 - \varphi(x/n)))f(x)\right| =$$

$$= (1+\|x\|^2)^N \left|(1 - \varphi(x/n))D^\alpha f(x) + \sum_{0 \neq \beta \leqslant \alpha} \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d} (-1) \frac{1}{n^{|\beta|}} D^\beta \varphi(x/n) D^{\alpha-\beta} f(x)\right|$$

$$\begin{cases} = 0, & \|x\| \leqslant n \\ \leqslant \sup_{\|x\| \geqslant n, |\gamma| \leqslant N} \frac{(1+\|x\|^2)^{N+1} |D^\gamma f(x)|}{1+\|x\|^2}, & \|x\| > n \end{cases}$$

$$\left(\sup_{\|x\| \geqslant n} \left(1 + \sum_{0 \neq \beta \leqslant \alpha} \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d} \cdot \underbrace{\frac{1}{n^{|\beta|}}}_{\leqslant 1} \underbrace{|D^\beta \varphi(x/n)|}_{\leqslant \|\varphi\|_N}\right)\right) \leqslant 1 + 2^N \|\varphi\|_N.$$

$$\leqslant (1 + 2^N \cdot \|\varphi\|_n) \cdot \frac{p_{N+1}(f)}{1 + n^2} \to 0.$$

□

c) $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \implies \varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$.

Důkaz

Assume $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \implies \exists R > 0$ such that $\operatorname{supp} \varphi_n \subset \overline{U(\mathbf{o}, R)}$. Then

$$p_n(\varphi_n - \varphi) = \max_{|\alpha| \leqslant N} \left\|x \mapsto (1 + \|x\|^2)^N (D^\alpha \varphi_n(x) - D^\alpha \varphi(x))\right\|_\infty \leqslant (1 + R^2)^N \cdot \|\varphi_n - \varphi\|_N \to 0.$$

□

Definice 6.4 (A tempered distribution on $\mathbb{R}^d$)

A tempered distribution on $\mathbb{R}^d$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$. Notation: $\mathcal{S}'(\mathbb{R}^d)$.

Poznámka

$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \implies \Lambda|_{\mathcal{D}(\mathbb{R}^d)} \in \mathcal{D}'(\mathbb{R}^d)$. (By the previous theorem item c.)

$\mathcal{D}'(\mathbb{R}^d) \subset\subset \mathcal{D}'(\mathbb{R}^d)$. (By item a. and b.)

We say that distribution is tempered, if it can be extended to $\mathcal{S}(\mathbb{R}^d)$.

Tvrzení 6.6

a) $\Lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{F}$ linear. Then

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0\ \exists C > 0 : |\Lambda(\varphi)| \leqslant C \cdot p_N(\varphi), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Důkaz
By the proposition above. □

b) Assume $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$. Then Λ is tempered iff

$$\exists N \in \mathbb{N}_0 \ \exists c > 0 : |\Lambda(\varphi)| \leqslant C \cdot p_N(\varphi), \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Důkaz
„ $\Longrightarrow$ “: a). „ $\Longleftarrow$ “: For example by Hahn–Banach and a). □

Definice 6.5

$\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d) \equiv \forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \Lambda_n(\varphi) \to \Lambda(\varphi)$, i.e. $\Lambda_n \xrightarrow{w^*} \Lambda$.

Věta 6.7

$(\Lambda_n) \subset \mathcal{S}'(\mathbb{R}^d)$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$: $(\Lambda_n(\varphi))$ converges in $\mathbb{F}$. Then $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is tempered distribution.

Důkaz
Use the previous proposition item a) and the theorem above. □

Tvrzení 6.8

a) $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$, $\operatorname{supp} \Lambda$ is compact $\Longrightarrow$ Λ is tempered.

Důkaz
Λ has compact support $\Longrightarrow$ $\exists C > 0 \ \exists N \in \mathbb{N}_0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N \leqslant C \cdot p_N(\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. □

b) $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$. Then $\Lambda_f \in \mathcal{S}(\mathbb{R}^d)$ and, moreover, $L_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, \varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
Theorem IV.11(a) $\implies$ $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{p\in[1,\infty]} L^p(\mathbb{R}^d)$. (It was stated and almost proven at chapter IV, but full proof is not easy.) So, fix $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^d)$. Let p' be the dual exponent. Then $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \varphi \in L^{p'}(\mathbb{R}^d)$, hence $f\varphi \in L^1(\mathbb{R}^d)$.

So $\tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is a well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$: „continuity“:

$$p = 1 : |\tilde{\Lambda}(\varphi)| = |\int_{\mathbb{R}^d} f\varphi| \leqslant \|f\|_1 \cdot \|\varphi\|_\infty = \|f\|_1 \cdot p_0(\varphi);$$

$p > 1 : \forall n \in \mathbb{N} : f \cdot \chi_{U(\mathbf{o},n)} \in L^1(\mathbb{R}^d) \implies \Lambda_{f\cdot\chi_{U(\mathbf{o},n)}} \in \mathcal{S}(\mathbb{R}^d)$ by the first case $\implies$

$$\implies \tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^d} f \cdot \chi_{U(\mathbf{o},n)}\varphi = \lim_{n\to\infty} \Lambda_{f\cdot\chi_{U(\mathbf{o},n)}}(\varphi) = \Lambda(\varphi).$$

$\square$

c) f measurable on $\mathbb{R}^d$, $|f| \leqslant |p|$ for some polynomial p on $\mathbb{R}^d$. Then $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$ and $\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, f \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
p polynomial $\implies$ $p(x) = \sum_{|\alpha|\leqslant N} c_\alpha x^\alpha$ $(c_\alpha \in \mathbb{F}, x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d})$.

$$\implies |p(x)| \leqslant c \cdot (\sqrt{2})^{dN}(1 + \|x\|^2)^{N\cdot\frac{d}{2}}, \qquad c = \max_\alpha |c_\alpha|.$$

So, if $|f| \leqslant |p|$, then $\frac{|f(x)|}{(1+\|x\|^2)^m} \leqslant c \cdot (\sqrt{2})^{d\cdot N} \cdot (1 + \|x\|^2)^{N\cdot\frac{d}{2}-m}$. If m is large enough (such that $N \cdot \frac{d}{2} - m < -\frac{d}{2}$), then $f(x)/(1+\|x\|^2)^m$ is integrable in $\mathbb{R}^d$. ($1/(1+\|x\|^2)^k$ is integrable for $k > \frac{d}{2}$ see the comment before theorem IV.11). Then:

$$\left|\int_{\mathbb{R}^d} f \cdot \varphi\right| = \left|\int_{\mathbb{R}^d} \frac{f(x) \cdot (1 + \|x\|^2)^m}{(1 + \|x\|^2)^m}\right| \leqslant \left(\int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + \|x\|^2)^m}\right) \cdot p_m(f).$$

$\square$

d) μ is a finite measure $\implies$ $\Lambda_\mu \in \mathcal{S}'(\mathbb{R}^d)$, $\Lambda_m(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$\varphi \in \mathcal{S}(\mathbb{R}^d)$ $\implies$ φ is continuous and bounded.

$$\left|\int_{\mathbb{R}^d} \varphi d\mu\right| \leqslant \int_{\mathbb{R}^d} |\varphi| d|\mu| \leqslant \|f\|_\infty \cdot \|\mu\| = p_0(\varphi) \cdot \|\mu\|.$$

$\square$

Lemma 6.9

$f \mapsto D^\alpha f$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$f \in \mathcal{S}(\mathbb{R}_6 d)$, $\alpha \in \mathbb{N}_0^d \implies D^\alpha f \in L^\infty()\mathbb{R}^d$. Fix $N \in \mathbb{N}_0$ and β, $|\beta| \leqslant N$:

$$|(1+\|x\|^2)^N D^\beta(D^\alpha f)(x)| = (1+\|x\|^2)^N |D^{\beta+\alpha} f(x)| \leqslant p_{N+|\alpha|}(f) \implies p_N(D^\alpha f) \leqslant p_{N+|\alpha|}(f).$$

□

p is polynomial $\implies$ $f \mapsto p \cdot f$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
Clearly $p \cdot f \in C^\infty(\mathbb{R}^d)$. Fix $N \in \mathbb{N}_0$. Then $\exists c > 0, m \in \mathbb{N}$ such that

$$\forall \alpha, |\alpha| \leqslant N,\ \forall x \in \mathbb{R}^d L |D^\alpha p(x)| \leqslant c \cdot (1 + \|x\|^2)^m.$$

Fix α, $|\alpha| \leqslant N$, $x \in \mathbb{R}^d$:

$$|(1 + \|x\|^2)^N D^\alpha(p \cdot f)(x)| = (1 + \|x\|^2)^N |\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta p(x) D^{\alpha-\beta} f(x)| \leqslant$$

$$\leqslant c \cdot (1 + \|x\|^2)^{N+M} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta} f(x)| \leqslant c \cdot \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} p_{N+M}(f) \leqslant c \cdot 2^N p_{N+M}(f) \implies$$

$$\implies p_N(p \cdot f) \leqslant c \cdot 2^N p_{N+M}(f).$$

□

$g \in \mathcal{S}(\mathbb{R}^d) \implies f \mapsto f \cdot g$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$g \in \mathcal{S}(\mathbb{R}^d) \implies \forall \alpha : D^\alpha g$ is bounded on $\mathbb{R}^d$. Fix $N \in \mathbb{N}_0$. Set $C := \max_{|\alpha| \leqslant N} \|D^\alpha g\|_\infty$. Fix α, $|\alpha| \leqslant N$, $x \in \mathbb{R}^d$.

$$\left|(1 + \|x\|^2)^N D^\alpha(f \cdot g)(x)\right| = (1 + \|x\|^2)^N \left|\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta g(x) D^{\alpha-\beta} f(x)\right| \leqslant$$

$$C \cdot \sum_{\beta \leqslant \alpha} p_N(f) \leqslant C \cdot 2^N \cdot p_N(f) \implies p_N(g \cdot f) \leqslant C \cdot 2^N p_N(f).$$

□

Poznámka
Similarly one may probe that:

$$g \in C^\infty(\mathbb{R}^d), \forall \alpha\ \exists P_\alpha \text{ polynomial } : |D^\alpha g| \leqslant |P_\alpha| \text{ on } \mathbb{R}^d \implies f \mapsto f \cdot g \text{ is a continuous mapping } \mathcal{S}(\mathbb{R}^d) \to$$

Tvrzení 6.10

Let $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$.

a) $\forall \alpha : D^\alpha \Lambda \in \mathcal{S}'(\mathbb{R}^d)$ *and* $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies D^\alpha \varphi \in \mathcal{S}(\mathbb{R}^d)$. So, $\tilde{\Lambda}(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, is well--defined linear functional on $\mathcal{S}(\mathbb{R}^d)$ whose restriction to $\mathcal{D}(\mathbb{R}^d)$ is $D^\alpha\Lambda$. „Continuity:“ $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d) \implies$ (by the previous lemma) $D^\alpha\varphi_n \to D^\alpha\varphi$ in $\mathcal{S}(\mathbb{R}^d)$, so $\tilde{\Lambda}(\varphi_n) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi_n) \to (-1)^{|\alpha|}\Lambda(D^\alpha\varphi) = \tilde{\Lambda}(\varphi)$. □

b) $f \in \mathcal{S}(\mathbb{R}^d)$ *and* f *is a polynomial* $\implies f \cdot \Lambda \in \mathcal{S}'(\mathbb{R}^d)$ *and* $f\Lambda(\varphi) = \Lambda(f\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz (Skipped on lecture)
Completely analogous to a). □

c) $y \in \mathbb{R}^d \implies \tau_y\Lambda \in \mathcal{S}'(\mathbb{R}^d), \tau_y\Lambda(\varphi) = \Lambda(\tau_{-y}\varphi), \varphi \in \mathcal{S}'(\mathbb{R}^d)$.

Důkaz
„$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\tau_{-y}\varphi(x) = \varphi(x-y)$“: Clearly $\tau_{-y}\varphi \in C^\infty(\mathbb{R}^d)$.

$|\alpha| \leqslant N : (1+\|x\|^2)^N D^\alpha \tau_{-y}\varphi(x) = (1+\|\alpha\|^2)^N D^\alpha\varphi(x+y) = \left(\dfrac{1+\|x\|^2}{1+\|x+y\|^2}\right)^N \cdot (1+\|x+y\|^2)^N D^\alpha\varphi(x+y)$

where $M = \sup_{t\in[0,\infty)} \frac{1+t^2}{1+(t-\|y\|)^2} < \infty$. $\implies \tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $p_N(\tau_{-y}\varphi) \leqslant M^N p_N(\varphi)$. So $\varphi \mapsto \tau_{-y}\varphi$ is continuous and then continue as in a). □

d) $\check{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$, $\check{\Lambda}(\varphi) = \Lambda(\check{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
Observe that $\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \check{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ ($\check{\varphi}(x) = \varphi(-x)$) and $p_N(\check{\varphi}) = p_N(\varphi)$. □

Tvrzení 6.11

$\Lambda_n \to \Lambda$ *in* $\mathcal{S}'(\mathbb{R}^d)$. *a)* $\forall\alpha : D^\alpha\Lambda_n \to D^\alpha\Lambda$, *b)* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* f *is polynomial* $\implies f\Lambda_n \to f\Lambda$.

Důkaz
„a)“: $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$D^\alpha\Lambda_n(\varphi) = (-1)^{|\alpha|}\Lambda_n(D^\alpha\varphi) \to (-1)^{|\alpha|}\Lambda(D^\alpha\varphi) = D^\alpha\Lambda(\varphi).$$

„b)“ similarly. □

6.2 Convolution and the Fourier transform of tempered distributions

Poznámka (Recall)
$f \in L^1(\mathbb{R}^d) \implies \hat{f}(t) = \int_{\mathbb{R}^d} f(x)e^{-i\langle t,x\rangle} dm_d(x)$.

Fourier transform maps $L^1(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

$$\hat{\hat{f}} = \check{f}, \qquad f \in \mathcal{S}(\mathbb{R}^d), \qquad \left(\hat{\hat{\hat{\hat{f}}}} = f\right).$$

Lemma 6.12

Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

Důkaz
1. The theorem above $\implies$ Fourier transform is a linear bijection $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.

2. $m := \left\lfloor \frac{d}{2} \right\rfloor + 1$. Then

$$C := \int_{\mathbb{R}^d} \frac{1}{(1+\|x\|^2)^m} dm_d(x) \leqslant \infty$$

$$f \in \mathcal{S}(\mathbb{R}^d) \implies \|\hat{f}\|_\infty \leqslant \|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)| dm_d(x) \leqslant \int_{\mathbb{R}^d} \frac{(1+\|x\|^2)^m |f(x)|}{(1+\|x\|^2)^m} dm_d(x) \leqslant C \cdot p_m(f).$$

TODO!!!

3. Fix $N \in \mathbb{N}_0$, α, $|\alpha| \leqslant N$.

$$f \in \mathcal{S}(\mathbb{R}^d) : (1+\|x\|^2)^N D^\alpha \hat{f}(x) = (1+\|x\|^2)^N (y \mapsto \widehat{(-1)^{|\alpha|}} y^\alpha f(y))(x) =$$

$$= (-1)^{|\alpha|} (y \mapsto \widehat{\breve{(D)}(y^\alpha (f(y)))})(x) = (-1)^{|\alpha|} (y \mapsto \widehat{\sum_{|\beta| \leqslant 2N} a_\beta D^\beta (y^\alpha f(y))})(x),$$

where $\breve{p}(x) = p(ix)$, $p(D)f = \sum_{c_\alpha} D^\alpha f$ if $p(x) = \sum c_\alpha x^\alpha$. $\breve{\breve{p}}(x) = (1 - \sum_{j=1}^d x_j^2)^N$ a polynom of degree $2N$.

So, $\|x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \leqslant c \cdot p_m(y \mapsto \sum_{|\beta| \leqslant 2N} a_\beta D^\beta (y^\alpha f(y)))$. From the previous lemma $f \mapsto \sum_{|\beta| \leqslant @N} a_\beta D^\beta (y^\alpha f(y))$ is continuous.

So, $\exists M = M_{N,\alpha} > 0$, $\exists m = m_{N,\alpha} \in \mathbb{N}_0$:

$$p_m(y \mapsto \sum_{|\beta| \leqslant 2N} a_\beta D^\beta (y^\alpha f(y))) \leqslant M \cdot p_n(f) \implies$$

$$\implies \|x\| \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \leqslant C \cdot M \cdot p_m(f).$$

4. So, $p_N(\hat{f}) \leqslant C \cdot \tilde{M} \cdot p_{\tilde{m}}(t)$, where $\tilde{M} = \max_{|\alpha| \leqslant N} M_{N,\alpha}$, $\tilde{m} = \max_{|\alpha| \leqslant N} m_{N,\alpha}$. □

Definice 6.6

$\Lambda \in \mathcal{S}'(\mathbb{R}^d)$: $\hat{\Lambda}(\varphi) = \Lambda(\hat{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Poznámka
$\hat{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$: $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d) \implies \hat{\varphi_n} \to \hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^d) \implies \hat{\Lambda}(\varphi_n) = \Lambda(\hat{\varphi_n}) \to \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi)$.

Věta 6.13

a) Fourier transform is a linear bijection of $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$.

Důkaz
$\hat{\hat{\Lambda}} = \check{\Lambda}$, $\hat{\hat{\hat{\hat{\Lambda}}}} = \Lambda$ for $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$, $\check{(\Lambda)}(\varphi) = \Lambda(\check{\varphi})$, $\hat{\Lambda_1} = \hat{\Lambda_2} \implies \hat{\hat{\hat{\hat{\Lambda}}}}_1 = \hat{\hat{\hat{\hat{\Lambda}}}}_2 \implies \Lambda_1 = \Lambda_2$. □

b) $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d) \implies \hat{\Lambda_n} \to \hat{\Lambda}$ in $\mathcal{S}'(\mathbb{R}^d)$.

Důkaz

$$\hat{\Lambda_n}(\varphi) = \Lambda_n(\hat{\varphi}) \to \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi).$$

□

c) $f \in C^1(\mathbb{R}^d) \implies \hat{\Lambda_f} = \Lambda_{\hat{f}}$.

Důkaz

$$\hat{\Lambda_f}(\varphi) = \Lambda_f(\hat{\varphi}) = \int f\hat{\varphi}dm_d = \int \hat{f}\varphi dm_d = \Lambda_{\hat{f}}(\varphi).$$

□

d) $f \in L^2(\mathbb{R}^d) \implies \hat{\Lambda_f} = \Lambda_{\mathcal{P}(f)}$, where $\mathcal{P}$ is the Plancherel transform.

Důkaz
$f_n := f \cdot \chi_{U(\mathbf{o},n)}$. Then $f_n \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $f_n \to f$ in $L^2(\mathbb{R}^d)$, and, moreover, $\hat{f_n} \to \mathcal{P}(f)$ in $L^2(\mathbb{R}^d)$. So,

$$\hat{\Lambda_f}(\varphi) = \Lambda_f(\hat{\varphi}) = \int_{\mathbb{R}^d} f\hat{\varphi}dm_d = \lim_{n\to\infty}\int_{\mathbb{R}^d} f_n\hat{\varphi}dm_d = \lim_{n\to\infty}\int_{\mathbb{R}^d} \hat{f_n}\varphi = \int_{\mathbb{R}^d} \mathcal{P}(f)\varphi dm_d = L_{\mathcal{P}(f)}(\varphi).$$

□

e) p polynomial $\implies \widehat{p(D)\Lambda} = \tilde{p}\hat{\Lambda}$, $\widehat{p \cdot \Lambda} = \breve{p}(D)\hat{\Lambda}$.

$$\Big(\tilde{p}(t) = p(it), \qquad \breve{p}(t) = p(-t), \qquad p(t) = \sum c_\alpha t^\alpha \implies p(D)f = \sum c_\alpha D^\alpha f.\Big)$$

Důkaz

$$\widehat{p(D)\Lambda}(\varphi) = p(D)\Lambda(\hat{\varphi}) = \Lambda(\check{(}D)\hat{\varphi}) = \Lambda\left(\widehat{\check{\ } p\varphi}\right) = \hat{\Lambda}(\check{p}\varphi) = \check{p}\hat{\Lambda}(\varphi).$$

$$\widehat{p\cdot\Lambda}(\varphi) = p\cdot\Lambda(\hat{\varphi}) = \Lambda(p\hat{\varphi}) = \Lambda\left(\widehat{\check{\check{\ }}(D)p\varphi}\right) = \hat{\Lambda}(\check{\check{p}}(D)\varphi) = \check{p}(D)\hat{\Lambda}(\varphi).$$

□

Poznámka
In particular

$$\widehat{D^\alpha\Lambda} = (x \mapsto c^{|\alpha|}x^\alpha)\hat{\Lambda}, \qquad (x \widehat{\mapsto x^\alpha})L = c^{|\alpha|}D^\alpha\hat{\Lambda}.$$

Poznámka
Next two lemmata are analogues of Lemmata above.

Lemma 6.14

a) $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $x_n \to x$ *in* $\mathbb{R}^d \implies \tau_{x_n}\varphi \to \tau_x\varphi$ *in* $\mathcal{S}(\mathbb{R}^d)$.

Důkaz
Fix $N \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leqslant N$. $x, z \in \mathbb{R}^d$:

$$\left|(1+\|x\|^2)^N D^\alpha\tau_z\varphi(y) - (1+\|y\|^2)^N D^\alpha\tau_x\varphi(y)\right| = (1+\|y\|^2)^N \left|\int_0^1 \frac{d}{dt}D^\alpha\varphi(y-x-t(z-x))dt\right| \leqslant$$

$$\leqslant (1+\|y\|^2)^N \int_0^1 \left|\sum_{j=1}^d D^{\alpha+e_j}\varphi(y-x-t(z-x))(x_j-z_j)\right| dt \leqslant$$

$$\leqslant (1+\|y\|^2)^N \int_0^1 \left|\sum_{j=1}^d D^{\alpha+e_j}\varphi(y-x-t(z-x))^2\right|^{1/2} \|x-z\|dt \leqslant$$

$$\leqslant \|x-z\|p_{N+1}(\varphi)(1+\|y\|^2)^N \int_0^1 \frac{\sqrt{d}dt}{(1+\|y-x-t(z-x)\|^2)^{N+1}} \leqslant$$

$$\leqslant \|x-z\|p_{N+1}(\varphi)\cdot\sqrt{d}\frac{(1+\|y\|^2)^N}{(1+\|y-x\|^2-\|y-x\|)^{N+1}}.$$

$$\left(1+\|y-x\|-t\|z-x\|^2 \geqslant 1+\|y-x\|^2-\|y-x\|, \text{ if } \|z-x\| \leqslant \frac{1}{2}.\right)$$

□

b) skipped and proof skipped too.

Lemma 6.15 (RT)

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0, \mu_\alpha, |\alpha| \leqslant N_0 : \Lambda(\varphi) = \sum_{|\Lambda| \leqslant N} \int_{\mathbb{R}^d} (1+\|x\|^2)^N D^\alpha \varphi(x) d\mu_\alpha(x), \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(Finite signed/complex measure on $\mathbb{R}^d$.)

Definice 6.7

$$U \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{S}(\mathbb{R}^d) \implies U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x-y)).$$

Věta 6.16 (Analogues to the theorem above)

a) skipped.

b) $\Lambda_{U*\varphi}$ is a tempered distribution.

Důkaz

The proposition above $\implies$ $\exists N, C$ such that $|U(\psi)| \leqslant c \cdot p_N(\psi)$

$$\implies |(U * \varphi)(x)| = |U(\tau_x \check{\varphi})| \leqslant C \cdot p_N(\tau_x \check{\varphi}).$$

$$|\alpha| \leqslant N : |(1+\|y\|^2)^N D^\alpha \varphi(x-y)| \leqslant p_N(\varphi) \cdot \left(\frac{1+\|y\|^2}{1+\|x-y\|^2} \right)^N \leqslant p_N(\varphi)(1+\|y\|+\|x\|^2)^N \implies \Lambda'_{U*\varphi} \text{ is tem}$$

$$\frac{1+\|y\|^2}{1+\|x-y\|^2} = \frac{1+\|y-x\|^2 + 2\langle y-x, x\rangle + \|x\|^2}{1+\|x-y\|^2} =$$

$$= 1 + \frac{2 \cdot \|y-x\| \cdot \|x\|}{1+\|x-y\|^2} + \frac{\|x\|^2}{1+\|x-y\|^2} \leqslant 1 + \|x\| + \|x\|^2.$$

□

c) skipped.

d) $\widehat{\Lambda_{U*\varphi}} = \hat{\varphi} \cdot \hat{U}$, $\widehat{\varphi \cdot U} = \Lambda_{\hat{\varphi} * \hat{U}}$.

Důkaz

$$\widehat{\Lambda_{U*\varphi}}(\psi) = \Lambda_{U*\varphi}(\hat{\psi}) = \int_{\mathbb{R}^d} (U * \varphi)(x)\hat{\psi}(x) f m_d(x) = \int_{\mathbb{R}^d} U(y \mapsto \varphi(x-y))\hat{\psi}(x) dm_d(x) =$$

$$= U(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y)\hat{\psi}(x) dm_d(x)) = U(\check{\varphi} * \hat{\psi}) = U(\hat{\hat{\varphi}} + \hat{\psi}) = U(\widehat{\hat{\ } \cdot \varphi\psi}) = \hat{U}(\hat{\varphi} \cdot \psi) = \hat{\varphi} \cdot \hat{U}(\psi).$$

$$\Lambda_{\hat{\varphi} * \hat{U}} = \widehat{\widehat{\widehat{\widehat{\Lambda_{\hat{\varphi}+\hat{U}}}}}} = \widehat{\widehat{\widehat{\hat{\hat{\varphi}} \cdot \hat{\hat{U}}}}} = \widehat{\widehat{\widehat{\check{\varphi} \cdot \check{U}}}} = \widehat{\check{\varphi} \cdot \check{U}} = \widehat{\varphi U}.$$

□

7 Elements of vector integration

Poznámka
$(M, \mathcal{A})$ is measure space, (Ω, Σ, μ) is a complete measure space ($\mu \geqslant 0$), X is a Banach space.

7.1 Measurability

Definice 7.1

$f : M \to X$.

- f is simple, if $f(M)$ is finite , i.e. $f = \sum_{j=1}^{k} x_j \chi_{A_j}$, where $x_j \in X$, $A_j \subset M$ pairwise disjoint;
- f is simple measurable, if f is a simple and, moreover, $A_j \in \mathcal{A}$;
- f is (strongly) $\mathcal{A}$-measurable if $\exists (u_n)$ simple measurable: $u_n \to f$ point-wise, i.e. $\forall x \in M : u_n(x) \to f(x)$ in $(X, \|\cdot\|)$;
- f is Borel $\mathcal{A}$-measurable, if $\forall U \subset X$ open: $f^{-1}(U) \in \mathcal{A}$;
- f is weakly $\mathcal{A}$-measurable if $\forall \varphi \in X^* : \varphi \circ f$ is $\mathcal{A}$-measurable.

Tvrzení 7.1

a) Simple functions, simple measurable functions, strongly $\mathcal{A}$-measurable functions, and weakly $\mathcal{A}$-measurable functions form vector spaces.

Důkaz

$f, g : M \to X$, $\alpha, \beta \in \mathbb{F}$.

- „f, g simple $\implies$ $\alpha f + \beta g$ is simple“: $(\alpha f + \beta g)(M) \subset \alpha f(M) + \beta g(M)$.
- „f, g simple measurable $\alpha f + \beta g$ is simple measurable“:

$$f = \sum_{j=1}^{k} x_j \chi_{A_j}, \quad g = \sum_{l=1}^{m} y_l \chi_{B_l}, \qquad \alpha f + \beta g = \sum_{j=1}^{k} \sum_{l=1}^{m} (\alpha x_j + \beta y_l) \cdot \chi_{A_j \cap B_l}, \qquad A_j, B_l \in \mathcal{A} \implies A_j \cap B$$

- „f, g strongly $\mathcal{A}$-measurable $\implies$ $\alpha f + \beta g$ is strongly $\mathcal{A}$-measurable“: $f = \lim u_n$, $g = \lim v_n$, u_n, v_n simple measurable, $\alpha f + \beta g = \lim(\alpha u_n + \beta v_n)$.
- „f, g weakly $\mathcal{A}$-measurable $\implies$ $\alpha f + \beta g$ weakly $\mathcal{A}$-measurable“: $\forall \varphi \in X^* : \varphi \circ (\alpha f + \beta g) = \alpha \varphi \circ f + \beta \varphi \circ g$ (measurable by the scakercah?).

$\square$

b) $f_n \to f$ point-wise, f_n Borel $\mathcal{A}$-measure (resp. weakly $\mathcal{A}$-measurable) $\implies$ f is Borel $\mathcal{A}$-measurable (resp. weakly $\mathcal{A}$-measurable).

Důkaz

Assume that $\forall n$: f_n is Borel $\mathcal{A}$-measurable. $U \subset X$ open:

$$f^{-1}(U) = \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k=m}^{\infty} f_k^{-1}(\left\{ x \in X |\, \operatorname{dist}(x, X \backslash U) > \frac{1}{n} \right\}),$$

$$f(x) \in U \Leftrightarrow \exists n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall k \geqslant m : \operatorname{dist}(f_k(x), X \backslash U) > \frac{1}{n}.$$

f_n are wakly $\mathcal{A}$-measurable, $\varphi \in X^*$ $\implies$ $\forall n : \varphi \circ f_n$ is Borel $\mathcal{A}$-measurable and $\varphi \circ f_n \to \varphi \circ f$, so, $\varphi \circ f$ is Borel $\mathcal{A}$-measurable. $\square$

c) f is strongly $\mathcal{A}$-measurable $\implies$ f is Borel $\mathcal{A}$-measurable $\implies$ f is weakly $\mathcal{A}$-measurable.

Důkaz
f is simple $\implies$ (f is simple measurable $\Leftrightarrow$ f is Borel $\mathcal{A}$-measurable).

„f strongly $\mathcal{A}$-measurable $\implies$ f Boreal $\mathcal{A}$ measurable“: $f = \lim u_n$, u_n simple measurable, then u_n are Borel $\mathcal{A}$-measurable, so by b), f is Borel $\mathcal{A}$-measurable.

„f Borel $\mathcal{A}$-measurable $\implies$ f is weakly $\mathcal{A}$-measurable“: $\varphi \in X^*$, $U \subset \mathbb{F}$ open $\implies$ $(\varphi \circ f)^{-1}(U) = f^{-1}(\underbrace{\varphi^{-1}(U)}_{\text{open}}) \in \mathcal{A}$.

„f simple, weakly $\mathcal{A}$-measurable $\implies$ f is simple measurable“: $f(M) = \{x_1, \dots, x_k\}$, distinct points. „$f^{-1}(x_1) \in \mathcal{A}$“: for $j \in \{2, \dots, k\}$ find $\varphi_j \in X^*$, $\varphi_j(x_1) \neq \varphi_j(x_j)$. Then

$$f^{-1}(x_1) = \bigcap_{j=2}^{k} \{t \in M | \varphi_j(f(t) - x_j) \neq 0\} = \bigcap_{j=2}^{k} \overbrace{(\varphi_j \circ f)^{-1}(\underbrace{\mathbb{F} \setminus \{\varphi_j(x_j)\}}_{\text{open}})}^{\in \mathcal{A}}.$$

□

d) $f : M \to X$ strongly $\mathcal{A}$-measurable $\implies$ $f(M)$ is separable.

Důkaz
$f = \lim u_n$, u_n simple measurable. $f(M) \subset \overline{\bigcup_n u_n(M)}$. □

e) f Borel $\mathcal{A}$-measurable $\implies$ $t \mapsto \|f(t)\|$ measurable.

Důkaz
$h(x) = \|x\|$, $x \in X$, is continuous, hence $h \circ f$ is measurable:

$$U \text{ open} : (h \circ f)^{-1}(U) = f^{-1}(\underbrace{h^{-1}}_{\text{open}}) \in \mathcal{A}.$$

□

Lemma 7.2

(f_n) strongly $\mathcal{A}$-measurable, $f_n \to f$ point-wise $\implies$ f is strongly $\mathcal{A}$-measurable.

Důkaz
$u_{m,n}$ simple measurable, $u_{m,n} \overset{m}{\to} f_n$.

$$C = \bigcup_{m,n} u(m,n)(M) \text{ is countable, so, } C = \{x_k, k \in \mathbb{N}\}.$$

For $k \in \mathbb{N}$ define $g_k : M \to X$ by $g_k(x) =$ the point from $\{x_1, \dots, x_k\}$ nearest to $f(x)$ (the first such point). Then g_k is simple, $g_k \to f$ point-wise ($t \in M$, $\varepsilon > 0 \implies \exists n_0\ \forall n \geqslant n_0 :$ $\|f_n(t) - f(t)\| < \varepsilon/2$. Fix one $n \geqslant n_0 \implies \exists m_9\ \forall m \geqslant m_0 : \|u_{m,n}(t) - f_n(t)\| < \varepsilon/2$. Fix one $m \geqslant m_0 \implies \|u_{m,n}(t) - f(t)\| < \varepsilon$, and there is k_0 such that $u_{m,n}(t) = x_{k_0}$. Then for $k \geqslant k_0$: $\|f(t) - g_k(t)\| \leqslant \|f(x) - x_{k_0}\| < \varepsilon$).

„g_k are also simple measurable“: f is Borel $\mathcal{A}$–measurable $\implies \forall x \in X$: $f - x$ is Borel $\mathcal{A}$-measurable $\implies \forall x \in X$: $t \mapsto \|f(t) - x\|$ is measurable, $g_k(t) = x_j$

$$\Leftrightarrow \forall i \in [k] : \|x_j - f(x)\| \leqslant \|x_i - f(x)\| \wedge \forall i < j : \|x_j - f(x)\| < \|x_i - f(x)\|$$

$$\Leftrightarrow \forall i \in [k]\ \forall g \in \mathbb{Q} : \|x_j - f(x)\| \leqslant q \vee \|x_i - f(x)\| \geqslant q \wedge \forall i < j\ \exists q \in \mathbb{Q} : \|x_j - f(x)\| < q \wedge \|x_i - f(x)\| > q.$$

□

Věta 7.3

$f : M \to X$. Then following assertions are equivalent:

1. *f is strongly $\mathcal{A}$-measurable;*
2. *f is Borel $\mathcal{A}$-measurable and $f(M)$ is separable;*
3. *f is weakly $\mathcal{A}$-measurable and $f(M)$ is separable.*

Důkaz
„1. $\implies$ 2. $\implies$ 3.“ from the previous proposition. „3. $\implies$ 1.“: Firstly WLOG X is separable (replace X by $\overline{\mathrm{LO}\, f(M)}$). Secondly let (x_n) be a dense sequence in X.

$$\forall n: \text{ fix } \varphi_n \in X^*, \|\varphi_n\| = 1, \varphi_n(x_n) = \|x_n\|.$$

Thirdly $\forall x \in X : \|x\| = \sup_n |\varphi_n(x)|$ („$\geqslant$“: clear as $\|\varphi_n\| = 1$, „$\leqslant$“: it holds for $x = x_n$, so on dense set, LHS is continuous, RHS is continuous (supremum of 1-Lipschitz functions), so it holds on X). Fourthly $\forall x \in X : t \mapsto \|f(t) - x\|$ is measurable ($\|f(t) - x\| = \sup_n |(\varphi_n \circ f)(t) - \varphi_n(x)|$, so supremum from $\mathcal{A}$-measurable functions).

Fifthly $k, n \in \mathbb{N}$: $A_n^k := f^{-1}(U(x_n, 1/k)) = \{t \in M | \|f(t) - x_n\| < 1/k\} \in \mathcal{A}$ by fourthly.

$$\bigcup_n A_n^k = M, \qquad B_n^k = A_n^k \setminus \bigcup_{j<n} A_j^k \in \mathcal{A}, \qquad \bigcup_n B_n^k = M,$$

and $\left\{B_n^k, n \in \mathbb{N}\right\}$ is pair-wise disjoint.

Define $g_k(t) = x_n$, $t \in B_n^k$. Then $\|g_k(t) - f(t)\| < \frac{1}{k}$. So $g_k \rightrightarrows f$ on M. g_n is strongly measurable. $g_k = \lim_{n\to\infty} \sum_{j=1}^n x_j \chi_{B_j^k}$, so, by the previous lemma f is strongly $\mathcal{A}$-measurable. □

Definice 7.2

(Ω, Σ, μ) complete measure space, $f : \Omega \to X$ is strongly μ-measurable if $\exists (u_n)$ simple measurable such that $u_n \to f$ point-wise μ-almost everywhere.

Poznámka
f is strongly μ-measurable $\Leftrightarrow$ $\exists g$ strongly Σ-measurable: $f = g$ almost everywhere.

Důkaz
„ $\impliedby$ “ obvious. „ $\implies$ “: u_n simple measurable, $u_n \to f$ almost everywhere. $\exists N, \mu(N) = 0 : u_n \to f$ on $\Omega \setminus N$. Modify u_n, f: $v_n = 0$ on N and u_n on $\Omega \setminus N$, $g = 0$ on N and f on $\Omega \setminus N$. v_n simple measurable, $v_n \to g$. □

TODO!!!