Příklad (1.)
Show that the linearisation of

$$P_{out} - P_{in} = \mu \int_{\sqrt{R_{in}^2 + c \cdot R_{in}^2}}^{\sqrt{R_{out}^2 + c \cdot R_{in}^2}} \frac{(r^2 - c \cdot R_{in}^2)^2 - r^4}{r^3 (r^2 - c \cdot R_{in}^2)} \, dr$$

with respect to c indeed yields

$$P_{out} - P_{in} \approx -\mu \int_{R_{in}}^{R_{out}} \frac{2c \cdot R_{in}^2}{R^3} \, dR.$$

Důkaz
Úpravou a linearizací $c \cdot f(c) \approx c \cdot f(0)$:

$$P_{out} - P_{in} = \mu \int_{\sqrt{R_{in}^2 + c \cdot R_{in}^2}}^{\sqrt{R_{out}^2 + c \cdot R_{in}^2}} \frac{r^4 + c^2 \cdot R_{in}^4 - 2c \cdot r^2 \cdot R_{in}^2 - r^4}{r^5 - c \cdot r^3 \cdot R_{in}^2} \, dr =$$

$$= c \cdot \mu \int_{\sqrt{R_{in}^2 + c \cdot R_{in}^2}}^{\sqrt{R_{out}^2 + c \cdot R_{in}^2}} \frac{c \cdot R_{in}^4 - 2r^2 \cdot R_{in}^2}{r^5 - c \cdot r^2 \cdot R_{in}^2} \, dr = c \cdot \mu \int_{\sqrt{R_{in}^2}}^{\sqrt{R_{out}^2}} \frac{-2r^2 \cdot R_{in}^2}{r^5} \, dr \approx -\mu \int_{R_{in}}^{R_{out}} \frac{2cR_{in}^2}{R^3} \, dR.$$

□

Příklad (2.)
It is known that the identity $\operatorname{div} \operatorname{cof} \mathbb{F} = \mathbf{o}$ holds for every deformation χ, $\mathbb{F} := \nabla \chi$. (This identity is usually referred to as the Piola identity, and it is straightforward to show using similar manipulations that lead to the relation between the Cauchy stress tensor and the first Piola–Kirchhoff stress tensor. Direct differentiation is not a good idea.) Use the identity and show that the functional

$$(\chi) := \int_{V(t_0)} \det \mathbb{F} dV$$

is a null Lagrangian. The null Lagrangian is a functional for which the corresponding Euler–Lagrange equations are identically satisfied.

Důkaz
Máme (použijeme rozepsání determinantu podle sloupce/řádku) pro libovolné $i \in [n]$

$$L \left(\chi_1, \chi_2, \ldots, \frac{\partial}{\partial x_1} \chi_1, \frac{\partial}{\partial x_1} \chi_2, \ldots, \frac{\partial}{\partial x_2} \chi_1, \ldots \right) = \det(\nabla \chi) = \sum_{j=1}^{n} (\nabla \chi)_{ij} (\operatorname{cof}(\nabla \chi))_{ij}.$$

A příslušné Eulerovy–Lagrangeovy rovnice jsou (postupně pro každé $i \in [n]$):

$$\underbrace{\frac{\partial L}{\partial \chi_i}}_{=0} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial L}{\frac{\partial \chi_i}{\partial x_j}} \right) = - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (\operatorname{cof}(\nabla \chi))_{ij} = (\operatorname{div}(\operatorname{cof}(\nabla \chi)))_i = 0.$$

□