

Úvod

Poznámka (Organizační úvod)

Dnes česky, ale pravděpodobně časem přepneme do angličtiny.

Na webu přednášejícího jsou zápisky, česko-anglická skripta.

Taková bible pro lidi studující PDR je Evans (... PDE ...).

Zápočet bude za 2 velké domácí úkoly. Zkouška je písemná (požadavky jsou na stránkách): 3 části: A – nutné, B – teorie, C – praxe?

Poznámka (Konvence pro PDR)

$\Omega \subseteq \mathbb{R}^d$ je otevřená. Měřitelná = lebesgueovsky měřitelná.

$$\partial_t u := \frac{\partial u}{\partial t}$$

Poznámka

Dále se ukazovali konkrétní parciální rovnice.

Poznámka (Je potřeba znát)

- Prostory funkcí a Lebesgueův integrál: $L^p(\Omega)$, $L^p_{loc}(\Omega)$, $\|u\|_p$, $C^k(\Omega)$, $C^k(\overline{\Omega})$,

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\Omega) \mid \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}, \|u\|_{C^{0,\alpha}} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

- $\int_{\Omega} \frac{\partial u}{\partial x_i} dx = \int_{\partial\Omega} u n_i dS$, $\vec{n} = (n_1, \dots, n_d)$.
- Funkcionální analýza 1: Banachův prostor, $u^n \rightarrow u$ silná konvergence, $u^n \rightharpoonup u$ slabá konvergence, Hilbertův prostor, Věta o reprezentaci (duálů), spektrální analýza operátorů, reflexivita (+ existence slabě konvergentní podposloupnosti v omezené podmnožině reflexivního prostoru).
- Separabilita (L^p jsou separabilní až na $p = \infty$, $C^k(\overline{\Omega})$ je separabilní, $C^{0,\alpha}$ není separabilní pro $\alpha \in (0, 1]$).

Poznámka (Motivace k pojmu slabé řešení (weak solution))

$$-\Delta u = f, f \notin C(\overline{\Omega})$$

A další ukázané na přednášce.

TODO?

1 Sobolevovy prostory

Definice 1.1 (Multiindex)

α je multiindex $\equiv d = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{N}_0$. Délka α je $|\alpha| := \alpha_1 + \dots + \alpha_d$. Pro $u \in C^k(\Omega)$ definujeme $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$.

Definice 1.2 (Slabá derivace)

Buď $u, v_\alpha \in L^1_{loc}(\Omega)$. Řekneme, že v_α je α -tá slabá derivace $u \equiv$

$$\equiv \int_{\Omega} u D^\alpha \varphi = (-1)^{|\alpha|} \int_{\Omega} v_\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

Příklad

$u = \text{sign} x$ nemá slabou derivaci.

Lemma 1.1 (O smysluplnosti)

Slabá derivace je nejvýše 1. Pokud existuje klasická derivace, tak obě splývají.

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Důkaz

v_α^1, v_α^2 dvě α -té derivace u .

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^1 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$(-1)^{|\alpha|} \int_{\Omega} v_\alpha^2 \varphi = \int_{\Omega} u D^\alpha \varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

$$\int_{\Omega} (v_\alpha^1 - v_\alpha^2) \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ skoro všude v Ω .

Klasická derivace je zřejmě zároveň slabá, tedy z první části splývají.

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□

Definice 1.3 (Sobolevův prostor)

$\omega \subseteq \mathbb{R}^d$ otevřená, $k \in \mathbb{N}_0$, $p \in [1, \infty]$.

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid \forall \alpha, |\alpha| \leq k : D^\alpha u \in L^p(\Omega)\}.$$

$$\|u\|_{W^{k,p}(\Omega)} \|u\|_{k,p} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, & p < \infty, \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_\infty, & p = \infty. \end{cases}$$

Poznámka

Od teď D^α nebo $\frac{\partial}{\partial x_1}$ nebo ∂_i značí slabou derivaci.

Lemma 1.2 (Základní vlastnosti slabých derivací a Sobolevových prostorů)

Nechť $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$, a α multiindex s délkou $\leq k$.

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$ a $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta}u$, pro $|\alpha| + |\beta| \leq k$.
- $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ a $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
- $\forall \tilde{\Omega} \subseteq \Omega$ otevřená

$$u \in W^{k,p}(\Omega) \implies u \in W^{k,p}(\tilde{\Omega})$$

- $\forall \eta \in C^\infty(\Omega): \eta u \in W^{k,p}(\Omega)$ a $D^\alpha(\eta u) = \sum_{\beta_i \leq \alpha_i} D^\beta \eta D^{\alpha-\beta} u \binom{\alpha}{\beta}$, kde $\binom{\alpha}{\beta} = \prod_{i=1}^d \binom{\alpha_i}{\beta_i}$.

Důkaz

Cvičení na doma. □

Věta 1.3 (Basic properties of Sobolev spaces)

Let $\Omega \subseteq \mathbb{R}^d$ be open set, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Then

- $W^{k,p}(\Omega)$ is a Banach space;
- if $p < \infty$ it is separable space;
- if $p \in (1, \infty)$ it is reflexive space.

┌ *Důkaz*

BS means linear normed space, which is complete. Linearity and norm? are easy. Completeness: u^n is Cauchy in $L^p(\Omega)$ so $\exists u \in L^p : u^n \rightarrow u$ in L^p . $D^\alpha u^n$ is Cauchy in $L^p(\Omega)$ $\forall |\alpha| < k$ so $\exists v_\alpha \in L^p : D^\alpha u^n \rightarrow v_\alpha \in L^p$. It remains prove that $D^\alpha u = v_\alpha$.

TODO

$$\left| \int_{\Omega} (v_\alpha - D^\alpha u^n) \varphi \right| \leq \|v_\alpha - D^\alpha u^n\|_p \|\varphi\|_{p'} \leq C \|v_\alpha - D^\alpha u^n\| \rightarrow 0.$$

$$\left| \int_{\Omega} (u^n - u) D^\alpha \varphi \right| \leq \|u^n - u\|_p \|D^\alpha \varphi\|_{p'} \leq C \|u^n - u\|_p \rightarrow 0.$$

„2+3“: $W^{1,p}(\Omega) \simeq X \subseteq L^p(\Omega) \times \dots \times L^p(\Omega)$ ($d+1$ times), X closed subspace from first property. Lemma: if $X \subseteq Y$ is closed subspace then Y separable $\implies X$ separable and Y reflexive $\implies X$ reflexive. (From functional analysis and topology.) \square

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2 Approximation of Sobolev function

Věta 2.1

Let $\Omega \subseteq \mathbb{R}^d$ open, $p \in [1, \infty)$.

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

Pozor

$$\overline{\{u \in C^\infty(\Omega)\}}^{\|\cdot\|_{k,p}} \subsetneq W^{k,p}(\Omega).$$

┌ *Důkaz*

└ Summer semester. \square

Věta 2.2 (Local density)

$$\begin{aligned} \forall u \in W^{k,p}(\Omega) \exists \{u^n\}_{n=1}^\infty \\ u^n \in C_0^\infty(\mathbb{R}^d) \forall \tilde{\Omega} \text{ open}, \bar{\tilde{\Omega}} \subseteq \Omega \\ u^n \rightarrow u \text{ in } W^{k,p}(\tilde{\Omega}) \end{aligned}$$

┌

Důkaz u is extended by 0 to $\mathbb{R}^d \setminus \Omega$.

$$u^\varepsilon = u * \eta^\varepsilon \quad \eta^\varepsilon(x) = \frac{\eta(\frac{x}{\varepsilon})}{\varepsilon^d} \quad \eta \in C_0^\infty(B_1), \eta \geq 0, \eta(x) = \eta(|x|), \int_{\mathbb{R}^d} \eta(x) dx = 1.$$

$$u \in L^p(SET) \quad u^\varepsilon \rightarrow u \text{ in } L^p(SET).$$

We need: $D^\alpha u^\varepsilon \rightarrow D^\alpha u$ in $L^p(\tilde{\Omega}) \forall \alpha, |\alpha| \leq k$. Essential step: $D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon$ in $\tilde{\Omega}$ for $\varepsilon \leq \varepsilon_0$ (so that ball of radius ε_0 and center in $\tilde{\Omega}$ is in Ω):

$$\begin{aligned} (D^\alpha u)^\varepsilon(x) &= \int_{\mathbb{R}^d} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \int_{B_\varepsilon(x)} D^\alpha u(y) \eta_\varepsilon(x-y) dy = \\ &= (-1)^{|\alpha|} \int_{B_\varepsilon(x)} u(y) D_y^\alpha \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta(x-y) dy. \\ D^\alpha u^\varepsilon &= D_x^\alpha \int_{\mathbb{R}^d} u(y) \eta_\varepsilon(x-y) dy = \int_{\mathbb{R}^d} u(y) D_x^\alpha \eta_\varepsilon(x-y) dy. \end{aligned}$$

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□

Tvrzení 2.3

Ω is open connected set, $u \in W^{1,1}(\Omega)$, then $u = \text{const.} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \forall i \in [d]$.

$W^{1,1}(I) \hookrightarrow C(I)$ for I interval.

$W^{d,1}(B_1) \hookrightarrow C(B_1)$.

┌ *Důkaz*

„1. \implies “ trivial. „1. \Leftarrow “: $\tilde{\Omega} \subseteq \Omega$ connected ε_0 as before and $\varepsilon \in (0, \varepsilon_0)$. u^ε -modification of u is smooth, so

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial x_i} &= \left(\frac{\partial u}{\partial x_i} \right)^\varepsilon = 0 \quad \text{in } \tilde{\Omega} \\ \implies u^\varepsilon &= \text{const}(\varepsilon) \quad \text{in } \tilde{\Omega}. \end{aligned}$$

$$\begin{aligned} c(\varepsilon) &= \int_{\mathbb{R}} c(\varepsilon) \eta_\delta(x-y) dy = \int_{\mathbb{R}} u^\varepsilon(y) \eta_\delta(x-y) dy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u(z) \eta_\varepsilon(y-z) \eta_\delta(x-y) dz dy = \\ &= \int \int u(z+y) \eta_\varepsilon(z) \eta_\delta(y-x) dz dy = \int \int u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dz dw = \\ &= \int \int u(z+x+y) \eta_\varepsilon(z) \eta_\delta(u) dw dz = \int_{\mathbb{R}^d} u^\delta(z+x) \eta_\varepsilon(z) dz = \int c(\delta) \eta_\varepsilon(z) dz = c(\delta). \end{aligned}$$

„2.“: WLOG $I = (0, 1)$. Define $v(x) = \int_0^x \frac{\partial u}{\partial y}(y) dy$. We show: $v \in W^{1,1}(I)$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}$.

$$\begin{aligned} |v(x)| &\leq \int_0^1 \left| \frac{\partial u}{\partial x} \right| \leq \|u\|_{1,1}. \\ \varphi &\in C_0^1(0, 1) \quad \int_0^1 v(x) \frac{\partial \varphi}{\partial x}(x) dx \\ &= \int_0^1 \left(\int_0^x \frac{\partial u}{\partial y}(y) dy \right) \frac{\partial \varphi}{\partial x}(x) dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dy dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \varphi(x)}{\partial x} x_{0 < y < x} dx dy = \\ &= \int_0^1 \left(\int_y^1 \frac{\partial \varphi(x)}{\partial x} dx \right) \frac{\partial u}{\partial y}(y) dy = - \int_0^1 \varphi(y) \frac{\partial u}{\partial y}(y) dy \Leftrightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x}. \end{aligned}$$

TODO.

$$x \rightarrow y \implies \int_y^x \left| \frac{\partial u}{\partial z} \right|^\alpha \rightarrow 0 \implies |u(x) - u(y)| \rightarrow 0$$

$$\|u\|_{C(I)} \leq \|v + c\|_{C(I)} \leq \|u\|_{1,1} + |c| = \|u\|_{1,1} + |u(x) - v(x)| \forall x \in I$$

$$\|u\|_{C(I)} \leq \|u\|_{1,1} + \int_0^1 |u(x) - v(x)| dx \leq -|| - + \int_0^1 |u| + \int_0^1 |v| \leq \|u\|_{1,1}.$$

„3.“ was shown without proof. □

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3 Characterization of Sobolev function

Věta 3.1

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $\delta > 0$, $\Omega_\delta := \{x \in \Omega \mid \text{dist}(x, \delta\Omega) > \delta\}$. Then

$$\forall u \in W^{1,p}(\Omega) : \|\Delta_i^h u\|_{L^p(\Omega_{\delta})} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}, \quad \forall h, i, \delta$$

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

$$u \in L^p \implies \forall \delta, h : \|\Delta_i^h u\|_{L^p(\Omega_\delta)} \leq c.$$

$$p > 1 \implies \frac{\partial u}{\partial x_i} \text{ exists and } \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq c.$$

Definition 3.1 (Class $C^{k,\mu}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ ($\partial\Omega \in C^{k,\mu}$) iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \dots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \rightarrow \mathbb{R}$ where $\Delta_r = \{x'_r \in \mathbb{R}^{d-1} \mid |x_{r_i}| \leq \alpha\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting tr the orthogonal transformation from (x'_r, x_{r_d}) to (x', x_d) , then $\forall x \in \partial\Omega \exists r \in \{1, \dots, M\}$ such that $x = \text{tr}(x'_r, a(x_{r_d}))$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\}$$

$$V_r^- := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\}$$

$$\Lambda_r := \{(x'_r, x_{r_d}) \in \mathbb{R}^d \mid x'_r \in \Delta_r, a(x'_r) = x_{r_d}\}$$

Then $\text{tr}(V_r^+) \subset \Omega$, $\text{tr}(V_r^-) \subset \mathbb{R}^d \setminus \overline{\Omega}$, $\text{tr}(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M \text{tr}(\Lambda_r) = \partial\Omega$.

Věta 3.2 (Density of smooth functions)

Let $\Omega \in C^0$. Then $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}$, $p \in [1, \infty)$.

Věta 3.3 (Extension of Sobolev functions)

Let $\Omega \in C^{0,1}$ (Ω is Lipschitz) and $k \in \mathbb{N}$, $p \in [1, \infty]$. Then there exists a continuous linear operator $E : W^{k,p}(\Omega) \rightarrow W^{k,p}(\mathbb{R}^d)$ such that:

- $\|Eu\|_{W^{k,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{k,p}(\Omega)}$ (C is independent of u)
- $Eu = u$ almost everywhere in Ω .

Věta 3.4 (Trace theorem)

Let $\Omega \in C^{0,1}$, $p \in [1, \infty]$. Then there exists a continuous linear operator $\text{tr} : W^{1,p}(\Omega) \rightarrow L^p(\partial\Omega)$ such that:

- $\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leq c \|u\|_{1,p},$
- $\forall u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) : \operatorname{tr} u|_{\partial\Omega} = u|_{\partial\Omega}.$

Definice 3.2

$$W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}.$$

Věta 3.5

Let $\Omega \in C^{0,1}$ and let $p \in [1, \infty]$. Then

- if $p < d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq \frac{dp}{d-p}$,
- if $p = d$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q < \infty$,
- if $p > d$, then $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\overline{\Omega})$.

Moreover

- if $p < d$, then $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$ for all $1 \leq \frac{dp}{d-p}$,
- if $p = d$, then $W^{1,p}(\Omega) \hookleftrightarrow L^q(\Omega)$ for all $q < \infty$,
- if $p > d$, then $W^{1,p}(\Omega) \hookleftrightarrow C^{0,\alpha}(\overline{\Omega})$ for all $\alpha < 1 - \frac{d}{p}$.

$$X \hookleftrightarrow Y \Leftrightarrow X \leq Y \wedge (A \subseteq X \text{ is bounded in } X \implies A \text{ is precompact in } Y).$$

$$X \hookleftrightarrow Y \implies X \subseteq Y \wedge (\{u^n\}_{n=1}^\infty, \exists c : \|u^n\|_{1,p} \leq c \implies \exists u^{n_j} : u^{n_j} \rightarrow u \text{ in } Y).$$

Důsledek (Trace theorem)

Let $\Omega \in C^{0,1}$. Then $\forall u \in W^{1,p}(\Omega)$ and $v \in W^{1,p'}(\Omega)$ we have integration by parts:

$$\int_{\Omega} u \frac{\partial v}{\partial x_i} dx = - \int_{\Omega} v \frac{\partial u}{\partial x_i} dx + \int_{\partial\Omega} uv|_{u=\operatorname{tr} u, v=\operatorname{tr} v} n_i ds.$$

Věta 3.6 (Poincaré)

Let $\Omega \in C^{0,1}$ and $p \in [1, \infty]$. Let $\Omega_1, \Omega_2 \subseteq \Omega$, $|\Omega_i| > 0$ and $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$, $|\Gamma_i|_{d-1} > 0$. Let $\alpha_1, \alpha_2 \geq 0$ and $\beta_1, \beta_2 \geq 0$ and at least one of $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

Then there exist $c_1, c_2 > 0$ such that $\forall u \in W^{1,p}(\Omega)$

$$c_1 \|u\|_{1,p}^p \leq \|\nabla u\|_p^p + \alpha_1 \int_{\Omega_1} |u|^p + \alpha_2 \int_{\Omega_2} |u|^p + \beta_1 \int_{\Gamma_1} |u|^p + \beta_2 \int_{\Gamma_2} |u|^p \leq c_2 \|u\|_{1,p}^p.$$

$$(\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p.)$$

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Důkaz (Of the first (the only difficult) inequality)
└ TODO!!!

□

4 Linear elliptic PDEs

Definice 4.1 (Elliptic)

Let $a_{ij}, b, c_i, d_i \in L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is bounded. We say that L is elliptic if $\exists c_1 > 0$ such that $\forall \zeta \in \mathbb{R}^d$ and almost all $x \in \Omega$

$$A\zeta \cdot \zeta \geq c_1 |\zeta|^2.$$

Lemma 4.1

If u is classical solution, then $\forall \varphi \in C^1(\overline{\Omega}), \varphi = 0$ on Γ_1 : $B_{L,\delta}(u, \varphi) = \int_{\Omega} f\varphi + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$.

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Důkaz
└ TODO!!!

□

Lemma 4.2

If $u \in C^2(\overline{\Omega})$ and $A, b, \mathbf{c}, \mathbf{d}$ are smooth and previous lemma holds $\forall \varphi \in C^1, \varphi|_{\Gamma_1} = 0$ and $u = u_0$ on Γ_1 , then u is a classical solution.

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Důkaz
└ TODO!!!

□

Definice 4.2 (Weak solution)

Let $\Omega \subseteq \mathbb{R}^d$ Lipschitz, L be an elliptic operator, $u_0 \in W^{1,2}(\Omega)$, $f \in (W^{1,2}(\Omega))^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$. We say that $u \in W^{1,2}(\Omega)$ is a weak solution iff

- $\text{tr } u = \text{tr } u_0$ on Γ_1 and
- $B_{L\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \forall \varphi \in V$, where $V := \{\varphi \in W^{1,2}(\Omega) | \text{tr } \varphi = 0 \text{ on } \Gamma_1\}$.

4.1 Existence of solution for coercive operators

Definice 4.3 (Elliptic form)

Let $B : V \times V \rightarrow \mathbb{R}$ bilinear nad V be a Hilbert space, $c_1, c_2 > 0$. We say that B is elliptic if it is

- V -bounded $\Leftrightarrow |B(u, \varphi)| \leq c_2 \|u\|_V \|\varphi\|_V$ and
- V -coercive $\Leftrightarrow B(u, u) \geq c_1 \|u\|_V^2$.

Věta 4.3 (Lax-Milgram)

Let B be a bilinear elliptic form. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : B(u, \varphi) = \langle F, \varphi \rangle.$$

Definice 4.4

Let $B : V \rightarrow V^*$. We say that B is

- Lipschitz $\equiv \forall u, v \in V : \|B(u) - B(v)\|_{V^*} \leq c_2 \|u - v\|_V, c_2 > 0$;
- Uniformly monotone $\equiv \forall u, v \in V : \langle B(u) - B(v), u - v \rangle_V \geq c_1 \|u - v\|_V^2, c_1 > 0$.

Věta 4.4 (Non-linear Lax-Milgram)

Let B be Lipschitz continuous and uniformly monotone. Then

$$\forall F \in V^* \exists! u \in V \forall \varphi \in V : \langle B(u), \varphi \rangle = \langle F, \varphi \rangle.$$

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Důkaz
└ TODO!!!

□

Důkaz (Lax-Milgram)
TODO!!!

□

Věta 4.5

If $B_{L,\sigma}$ is bilinear, V -bounded and V -elliptic. Then there exists a unique weak solution u .

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Důkaz
└ TODO!!!

□

4.2 Existence via Fredholm alternative

TODO!!!

Věta 4.6

Let $\Omega \in C^{0,1}$, L be an elliptic operator and $\Gamma_1 = \partial\Omega$. Then

1. Σ is at most countable and if infinite $\{\lambda_k\}_{k=1}^{\infty} \implies \lambda_k \rightarrow \infty$;
2. $(\lambda \notin \Sigma) \Leftrightarrow \forall f \in L^1 \exists! u : Lu = f + \lambda u$;
3. $\forall \lambda \notin \Sigma \exists C > 0 \forall f \in L^2 \exists! u \in W_0^{1,2}(\Omega) : Lu = f + \lambda u$ and $\|u\|_{1,2} \leq C\|f\|_2$;

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Důkaz

3) TODO improve convergence of u^{n_k} and show

$$u^{n_k} \rightarrow u \text{ in } W_0^{1,2}(\Omega) \text{ Strongly!};$$

show $\{u^{n_k}\}$ is Cauchy in $W_0^{1,2}(\Omega)$

$$v^{n,m} = u^n - u^m$$

$$C_1 \|\nabla(u^n - u^m)\|_2^2 \leq \int_{\Omega} A \nabla v^{n,m} \nabla v^{n,m} = V_l(v^{n,m}, v^{n,m}) -$$

$$\int_{\Omega} \mathbf{c} \nabla v^{n,m} v^{n,m} - b(v^{n,m})^2 + \mathbf{d} \nabla v^{n,m} v^{n,m} =$$

$$= \int_{\Omega} (f^n - f^m) v^{n,m} + \lambda (v^{n,m})^2 \pm - \leq$$

$$\leq \|v^{n,m}\|_2 (\|f^n - f^m\|_2 + \lambda \|v^{n,m}\|_2 + \|\mathbf{c}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla v^{n,m}\|_2 + \|b\|_{\infty} \|v^{n,m}\|_2) \leq$$

$$\leq \|v^{n,m}\| C(\lambda) \stackrel{u^n \text{ is Cauchy}}{\leq} C(\lambda) \varepsilon$$

$$\implies \nabla u^n \text{ is Cauchy sequence} \implies u^n \rightarrow u \text{ in } W_0^{1,2}(\Omega) \implies \|?\|_{n_k} = 1$$

$$\int_{\Omega} A \nabla a u^n \nabla a \varphi + b u^n \varphi + \mathbf{c} \nabla u^n \varphi - \mathbf{d} \nabla u^n \varphi = \int_{\Omega} f^n \varphi + \lambda u^n \varphi.$$

$$n \rightarrow \infty$$

$$\int_{\Omega} A \nabla u \nabla \varphi + b u \varphi + \mathbf{c} \nabla u \varphi - \mathbf{d} \nabla u \varphi = \lambda \int_{\Omega} u \varphi \Leftrightarrow Lu = \lambda u$$

└ But $\lambda \notin \Sigma$.

□

Poznámka

Next we discussed homework.

4.3 Variational approach – minimization

Poznámka

$B_{L,\sigma}(u, v)$ must be symmetric! ($B_{L,\sigma}(u, v) = B_{L,\sigma}(v, u)$)

$$L = -\operatorname{div}(A\nabla u) + bu + \mathbf{c}\nabla u + \operatorname{div}(\mathbf{d}u)$$

$$B_{L,\sigma}(u, v) := \int_{\Omega} A\nabla u \cdot \nabla v + Buv + \mathbf{c} \cdot \nabla uv - \mathbf{d}\nabla vu + \int_{\Gamma} \sigma uv$$

$$B_{L,\sigma}(v, u) := \int_{\Omega} A\nabla v \cdot \nabla u + Bvu + \mathbf{c} \cdot \nabla vu - \mathbf{d}\nabla uv + \int_{\Gamma} \sigma vu$$

$$\implies A = A^T, \quad \mathbf{c} = -\mathbf{d}$$

Věta 4.7

Let $B_{L,\sigma}$ be linear symmetric V -elliptic and V -bounded. $f \in V^*$, $g \in L^2(\Gamma_2 \cup \Gamma_3)$, $u \in V$. Then the following is equivalent:

- $u - u_0 \in V$ and $B_{L,\sigma}(u, v) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi$;
- $u - u_0 \in V \quad \forall v \in W^{1,2}(\Omega), \quad v, u_0 \in V$

$$\frac{1}{2}B_{L,\sigma}(u, u) - \langle f, u \rangle - \int_{\Gamma_2 \cup \Gamma_3} gu \leq \frac{1}{2}B_{L,\sigma}(v, v) - \langle f, v \rangle - \int_{\Gamma_2 \cup \Gamma_3} gv.$$

┌
Důkaz („1 \implies 2“)

$$\begin{aligned} 0 &\stackrel{V\text{-elliptic}}{\leq} \frac{1}{2}B_{L,\sigma}(v-u, v-u) \stackrel{\text{linearity}}{=} \frac{1}{2}B_{L,\sigma}(v, v) + \frac{1}{2}B_{L,\sigma}(u, u) - \frac{1}{2}B_{L,\sigma}(u, v) - \frac{1}{2}B_{L,\sigma}(v, u) = \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u) - B_{L,\sigma}(u, v) = \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + B_{L,\sigma}(u, u - v) \stackrel{\text{weak formulation}}{=} \\ &= \frac{1}{2}(B_{L,\sigma}(v, v) - B_{L,\sigma}(u, u)) + \langle f, u - v \rangle + \int_{\Gamma_2 \cup \Gamma_3} g(u - v) \end{aligned}$$

└

□

┌ *Důkaz („2 \implies 1“)*

u is minimizer, so set $v = u + \varepsilon\varphi$, $\varphi \in V$

$$\begin{aligned} \frac{1}{2}B_{L,\sigma}(u, u) - \langle j, u \rangle - \int gu &\leq \frac{1}{2}B_{L,\sigma}(u + \varepsilon\varphi, u + \varepsilon\varphi) - \langle j, u + \varepsilon\varphi \rangle - \int g(u + \varepsilon\varphi) = \\ &= \frac{1}{2}B_{L,\sigma}(u, u) + \frac{1}{2}\varepsilon\frac{1}{2}B_{L,\sigma}(\varphi, \varphi) + \varepsilon B_{L,\sigma}(u, \varphi) - \langle f, u \rangle - \varepsilon \langle f, \varphi \rangle - \int ga - \varepsilon \int g\varphi \end{aligned}$$

divide by ε and $\varepsilon \rightarrow 0_+$

$$0 \leq B_{L,\sigma}(u, \varphi) - \langle j, \varphi \rangle - \int_{\Gamma_2 \cup \Gamma_3} g\varphi, \quad \forall \varphi \in V$$

(Euler-Lagrange inequality?), which is true also for $-\varphi \implies 0 = -||- \implies u$ is weak solution. \square

Věta 4.8 (Duel formulation)

Let $Lu = -\operatorname{div}(A\nabla u)$ with A elliptic, bounded and symmetric, $\Gamma_1 \neq \emptyset$, $\Gamma = \emptyset$, $f \in V^*$, $g \in L^2(\Gamma_2)$, $u_0 \in W^{1,2}(\Omega)$. Then the following are equivalent:

- u is a weak solution;
- $\nabla u = A^{-1}\mathbf{T}$, where \mathbf{T} minimizes $\int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} = \nabla u_0 \mathbf{T}$ over the set $\tilde{V} := \{\mathbf{T} \in L^2(\Omega, \mathbb{R}^d)\}$, $\forall \varphi \in V$.

$$\int_{\Omega} \mathbf{T} \cdot \nabla \varphi = \langle f, \varphi \rangle + \int_{\Gamma_2} g\varphi \Leftrightarrow -\operatorname{div} \mathbf{T} = f \text{ in } \Omega, T\mathbf{u} = g \text{ on } \Gamma_2$$

┌ *Důkaz („1 \implies 2“)*

Let $\mathbf{V} \in \tilde{V}$ and $\mathbf{T} := A\nabla u \in \tilde{V}$.

$$\begin{aligned} 0 &\leq \frac{1}{2} \int_{\Omega} A^{-1}(\mathbf{V} - \mathbf{T}) \cdot (\mathbf{V} - \mathbf{T}) = \int \frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} \int_{\Omega} A^{-1}\mathbf{T} \cdot \mathbf{T} - A^{-1}\mathbf{T}\mathbf{V} = \\ &= \int \left(\frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \left(\frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \right) + \int_{\Omega} (\nabla u_0(\mathbf{V} - \mathbf{T}) + A^{-1}\mathbf{T}(\mathbf{T} - \mathbf{V})) = \\ &= \int \left(\frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} (A^{-1}\mathbf{T} - \nabla u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left(\frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} - \int_{\Omega} \nabla(u - u_0) \cdot (\mathbf{V} - \mathbf{T}) = \\ &\quad \int \left(\frac{A^{-1}\mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V} \right) - \int \frac{A^{-1}\mathbf{T} \cdot \mathbf{T}}{2} + 0. \end{aligned}$$

┌ So \mathbf{T} is minimizer of the formula above. \square

┌

Důkaz („2 \implies 1“)
 $\mathbf{T} \in \tilde{V} \quad \forall V \in \tilde{V}: \int_{\Omega} \frac{1}{2} A^{-1} \mathbf{T} \cdot \mathbf{T} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{V} \cdot \mathbf{V}}{2} - \nabla u_0 \mathbf{V}, \quad \mathbf{V} = \mathbf{T} + \varepsilon \mathbf{W}, \quad \mathbf{W} \in L^2(\Omega, \mathbb{R}^d)$
 $\forall \varphi \in V: \int_{\Omega} \mathbf{W} \cdot \nabla \varphi = 0.$

$$\int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T}}{2} - \nabla u_0 \mathbf{T} \leq \int_{\Omega} \frac{A^{-1} \mathbf{T} \cdot \mathbf{T} + \varepsilon^2 A^{-1} \mathbf{W} \cdot \mathbf{W} + 2\varepsilon A^{-1} \mathbf{T} \cdot \mathbf{W}}{2} - \nabla u_0 \mathbf{T} - \varepsilon \nabla u_0 \mathbf{W}$$

divide by ε and $\varepsilon \rightarrow 0_+$:

$$0 \leq \int_{\Omega} A^{-1} \mathbf{T} \cdot \mathbf{W} - \nabla u_0 \cdot \mathbf{W}.$$

This also holds for $-\mathbf{W}$, so $0 = -||-$.

Now we find unique $u \in W^{1,2}$ $u - u_0 \in V: \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} A^{-1} \mathbf{T} \cdot \nabla \varphi$ ($\langle F, \varphi \rangle_V$).

$$\begin{aligned} \int_{\Omega} |A^{-1} \mathbf{T} - \nabla u|^2 &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u)(A^{-1} \mathbf{T} - \nabla u) = \\ &= \int_{\Omega} (A^{-1} \mathbf{T} - \nabla u_0) \cdot (A^{-1} \mathbf{T} - \nabla u) + \int_{\Omega} \nabla(u_0 - u)(A^{-1} \mathbf{T} - \nabla u) = 0 + 0 = 0 \end{aligned}$$

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□

Lemma 4.9

Let X be a reflexive space and $\{u^n\}_{n=1}^{\infty}$ be a bounded sequence, $\|u^n\|_X \leq c < \infty$. Then $\exists u^{n_k}, \exists u \in X: u^{n_k} \rightharpoonup u$ ($\forall F \in X^*: \langle F, u^{n_k} \rangle \rightarrow \langle F, u \rangle$).

Věta 4.10 (Spectrum of symmetric operator)

V Hilbert infinity-dimensional space. Let B be linear, symmetric, V -elliptic and V -bounded operator. Then there exist $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m$ and corresponding $\{u_i\}_{i=1}^{\infty}$ such that

- $B(u^k, \varphi) = \lambda_k \int_{\Omega} u^k \varphi;$
- $\lambda_k \rightarrow \infty;$
- $\{u^k\}_{k=1}^{\infty}$ is basis in V and fulfils

$$\int_{\Omega} u^i u^j = \delta_{ij}, \quad B(u^i, u^j) = 0 \forall i \neq j;$$

- $P^n u := \sum_{i=1}^n u^i (\int_{\Omega} u u^i),$ then $\forall n: \|P^n u\|_2 \leq \|u\|_2$ and $B(P^n u, P^n u) \leq B(u, u).$

|

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Dikaz

Step 1: Construct λ_k, u^k : $\lambda_1 := \inf_{u \in V, \|u\|_2=1} B(u, u)$ and denote u^1 function, where infimum is obtained. Then for $V^N = \{u \in V | \forall k \in [N] : B(u, u^k) = 0\}$ we do the same.

Step 2: The construction is OK:

$$\begin{aligned} 0 < \lambda_1 &= \lim_{n \rightarrow \infty} B(u^n, u^n), \|u^n\|_2 = 1 \implies \\ &\implies \|u^n\|_V \leq C \implies u^{n_k} \rightharpoonup u \text{ in } V \\ V \hookrightarrow L^2 &\implies u^{n_k} \rightarrow u \text{ in } L^2(\Omega) \implies \|u\|_2 = 1 \\ \lambda_1 &= \lim_{n_k \rightarrow \infty} B(u^{n_k}, u^{n_k}) \geq B(u, u) \geq \lambda_1. \end{aligned}$$

Step 3: λ_k, u^k eigenvalues, eigen functions: $\forall v \in V, \|v\|_2 = 1, \lambda_1 = B(u^1, u^1) \leq B(v, v), \|u^1\|_2 = 1$

$$v = \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \quad \varphi \in V, 0 < \varepsilon \ll 1.$$

$$\lambda_1 \leq B\left(\frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}, \frac{u^1 + \varepsilon \psi}{\|u^1 + \varepsilon \psi\|_2}\right)$$

$$\lambda_1 \|u^1 + \varepsilon \psi\|_2 \leq B(u^1 + \varepsilon \psi, u^1 + \varepsilon \psi) = B(u_1, u_1) + \varepsilon^2 B(\psi, \psi) + 2\varepsilon B(u, \psi) \leq \lambda_1 \|u^1\|_2^2 + \lambda_1 \varepsilon^2 \|\psi\|_2^2 + 2\varepsilon \lambda_1 \int_{\Omega} u^1 \psi$$

$$\varepsilon \rightarrow 0_+ \implies 2\lambda_1 \int_{\Omega} u^1 \psi \leq 2B(u, \psi).$$

So $\lambda_1 \int_{\Omega} u^1 \psi = B(u, \psi)$.

The same way we obtain $\lambda_k \int_{\Omega} u^k \psi \leq B(u, \psi)$ for $\psi \in V^N$.

$$u^1 : \lambda_1 \int_{\Omega} u^1 \psi = B(u^1, \psi) \implies \psi = u^k \int_{\Omega} u^1 u^k = V(u_1, u^k).$$

But $u^k \in V^k \implies B(u^k, u^i) = 0 \forall i \in [k-1]$, so $\int u^1 u^k = B(u^1, u^k) = 0$.

$$\implies \forall i \in [k-1] : \int_{\Omega} u^k u^1 = B(u^k, u^i) = 0.$$

Step 4: $\lambda_k \nearrow \infty$. We already know $\lambda_1 \leq \lambda_2 \leq \dots$. Assume a contradiction $\lambda_k \leq C < \infty$. $c_1 \|u^k\|_V^2 \leq B(u^k, u^k) = \lambda_k \|u^k\|_2^2 = \lambda_k < C$.

$$\implies u^k \rightharpoonup u \text{ in } V,$$

$$u^k \rightarrow u \text{ in } L^2 \implies u^k \text{ is Cauchy in } L^2$$

$$\|u^n - y^m\|_2^2 = \|u^n\|_2^2 + \|u^m\|_2^2 - 2 \int u^n u^m =$$

$$= 2 - \frac{2}{\lambda_1 n} B(u^n, u^m) = 2 \implies \text{not Cauchy.}$$

Step 5: λ_k are all eigenvalues (u^k is basis of V and of L^2). Assume that $\lambda \neq \lambda_j$ is also eigenvalue, so $\exists u : B(u, \varphi) = \lambda \int_{\Omega} u \varphi \forall \varphi$. We can find $i \in \mathbb{N}$, so $\lambda_i < \lambda < \lambda_{i+1}$.

$$B(u, u^j) = \lambda \int u u^j \wedge B(u^j, u) = \lambda_j \int u^j u \implies B(u, u_j) = 0$$

4.4 Regularity of weak solution

Poznámka

We assume that we have $u \in W^{1,2}(\Omega)$ a weak solution

$$-\operatorname{div} A \nabla u + Vu + \mathbf{c} \cdot \nabla u + \operatorname{div}(\mathbf{d}u) = Lu = f.$$

When $u \in W_{loc}^{2,2}(\Omega)$, when $u \in W^{2,2}(\Omega)$, when $u \in W_{loc}^{k,2}(\Omega)$, $u \in W^{k,2}(\Omega)$.

Simplify $-\operatorname{div} A \nabla u = f - bu - \mathbf{c} \nabla u - u \operatorname{div} \mathbf{d} - \nabla u \cdot \mathbf{d} = \tilde{f}$. If $u \in W^{1,2}$, $f \in L^2$, $b \in L^\infty$, $\mathbf{d} \in W^{1,\infty} \implies \tilde{f} \in L^2(\Omega)$.

Problem is reduced to

$$\begin{aligned} -\operatorname{div}(A \nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_1, \\ (A \nabla u) \cdot \mathbf{v} &= g \text{ on } \Gamma_2, \\ (A \nabla u) \cdot \mathbf{v} + \sigma u &= g \text{ on } \Gamma_3. \end{aligned}$$

Definition 4.5 (Interior regularity)

$u \in W_{loc}^{2,2}(\Omega)$; assumptions: $A \in W^{k+1,\infty}$, $f \in W^{k,2}(\Omega) \implies u \in W_{loc}^{k+1,2}(\Omega)$.

Definition 4.6 (Boundary regularity)

$u \in W^{2,2}(\Omega)$; assumptions: on $\Omega \in C^{k+1,\infty}$, $g \in W^{\frac{1}{2},2}(\partial\Omega)$ and $\overline{\Gamma_2} \cap \overline{\Gamma_1} = \{\emptyset\} \implies u \in W^{2,2}(\Omega)$.

Věta 4.11 (Interior regularity)

Let A be an elliptic operator and $u \in W^{1,2}$ solves

$$\int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega) \quad \forall f \in L^2(\Omega).$$

Then if $A \in W^{k+1,\infty}(\Omega, \mathbb{R}^{d,d})$, $f \in W^{k,2}(\Omega)$ then $u \in W_{loc}^{k+2,2}(\Omega)$.

Moreover $\forall \tilde{\Omega} \subset \overline{\tilde{\Omega}} \subseteq \Omega \exists c(\tilde{\Omega}, A)$:

$$\|u\|_{W^{k+2,2}(\tilde{\Omega})} \leq c(\|f\|_2 + \|u\|_{W^{1,2}(\Omega)}).$$

Důkaz

$k = 0$: Recall $v \in W^{1,2}(\Omega) \Leftrightarrow \{v \in L^2(\Omega) \wedge \Delta_k^n v \in L^2(\Omega_h) \forall h\}$

$$\int_{\Omega_h} \frac{|v(x + he_k) - v(x)|^2}{h^2} \leq c.$$

$$u \in W^{2,2}(\tilde{\Omega}) \Leftrightarrow \left\{ u \in W^{1,2}(\Omega) \wedge \Delta_k^n \frac{\partial u}{\partial x_i} \in L^2 \right\}.$$

We want:

$$\int_{\tilde{\Omega}_h} \frac{\left| \frac{\partial u(x+he_i)}{\partial x_j} - \frac{\partial u(x)}{\partial x_j} \right|^2}{h^2} \leq c,$$

$$\int_{\Omega_h} \left| \frac{\nabla u(x + he_i) - \nabla u(x)}{h} \right|^2 \leq c.$$

$$\int_{\Omega} A \nabla u \nabla \varphi = \int_{\Omega} f \varphi$$

$$h > 0, \varphi \in W_0^{1,2}(\Omega), \varphi(x) = 0 \text{ if } \text{dist}(x, \partial\Omega) \subset h.$$

Set $\varphi(x) := \psi(x - he_k)$.

$$\begin{aligned} \implies \int_{\Omega} A(x) \nabla u(x) \nabla \psi(x - he_k) &= \int_{\Omega} f(x) \psi(x - he_k) = \\ &= \int_{\Omega} A(x + he_k) \nabla u(x + he_k) \cdot \nabla \psi(x) dx. \end{aligned}$$

Set $\varphi(x) := \psi(x)$:

$$\int_{\Omega} A(x) \cdot \nabla u(x) \cdot \nabla \psi(x) = \int_{\Omega} f(x) \psi(x) dx.$$

$$\begin{aligned} &\int_{\Omega} A(x + he_k) (\nabla u(x + he_k) - \nabla u(x)) \cdot \nabla \psi(x) = \\ &= - \int_{\Omega} (A(x + he_k) - A(x)) \nabla u(x) \cdot \nabla \psi(x) + \int_{\Omega} f(x) (\psi(x - he_k) - \psi(x)). \end{aligned}$$

Set $\psi := (u(x + he_k) - u(x)) \tau^2(x)$, $\tau(x) = 0$, if $\text{dist} \in (x, \partial\Omega)$, $\tau \in C^1(\tilde{\Omega})$.

Evaluate all terms ($w^{h,i} = u(x + he^i) - u(x)$):

$$\begin{aligned} &\int_{\Omega} A(x + he_i) \nabla w^{h,i} \cdot (\nabla w^{h,i} \tau^2 + 2w^{h,i} \tau \nabla \tau) \geq \\ &\stackrel{ellip.}{\geq} c_1 \int_{\Omega} |\nabla w^{hi}|^2 \tau^2 - \int_{\Omega} \frac{2 \|A\|_{\infty} |w^{h,i}| - |\nabla \tau| (|\nabla w^{hi}| \sqrt{c_1} \tau)}{\sqrt{c_1}} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2}{c_1} \|A\|_{\infty}^2 \|\nabla \tau\|_{\infty}^2 h^2 \int_{\Omega_h} \frac{|u(x + he_i) - u(x)|^2}{h^2} \geq \\ &\geq \frac{c_1}{2} \int_{\Omega} |\nabla w^{h,i}|^2 \tau^2 - \frac{2 \|A\|_{\infty}^2 \|\nabla \tau\|_{\infty}^2 h^2 c_1 \|\nabla u\|_2^2}{c_1} \end{aligned}$$

TODO?

Věta 4.12 (Regularity up to the boundary)

Let u be a weak solution $-\operatorname{div}(A\nabla u) = f$ in Ω , $A\nabla u \cdot \mathbf{v} = g$ on Γ_2 , $A\nabla u \cdot \mathbf{v} + \sigma u = g$ on Γ_3 , $u = u_0$ on Γ_1 .

Assume that $\Omega \in C^{k+1,\infty}$, $A \in W^{k,\infty}$, $f \in W^{k-1,2}$, $g \in W^{-\frac{1}{2}+k,2}(\partial\Omega)$, $\sigma \in W^{k,\infty}(\partial\Omega)$ and $\Gamma_1, \Gamma_2, \Gamma_3$ are smooth open (in partial Ω) and $\overline{\Gamma_i} \cap \overline{\Gamma_j} = \emptyset \ \forall i \neq j$.

Then $u \in W^{k+1,2}(\Omega)$.

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Důkaz (Step 1: Flat boundary)

$\Omega = (-1, 1)^{d-1} \times (0, 1)$. Assume that $u \in W^{1,2}(\Omega)$ and $u = 0$ on $(x, 0)$. We want that $u \in W^{2,3}((-1 + \delta, 1 - \delta)^{d-1} \times (0, 1 - \delta))$.

1a tangential derivatives $\frac{\partial u}{\partial x_1} \in W^{1,2}(-||-)$. 1b normal derivative $\frac{\partial^2 u}{\partial x_d^2} \in L^2(-||-)$.

1a: $\operatorname{WF} - \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \ \forall \varphi \in W_0^{1,2}(\Omega)$. Take continuous $\tau = 1$ in $-||-$ and $\tau = 0$ in $\Omega \setminus \text{"inflated"} -||-$.

$$\varphi(x) = \psi(x - he_i)\tau, \quad i \in [d-1], \psi \in W_0^{1,2}(\Omega \setminus \text{"inflated"} -||-)$$

Redefine interior regularity

$$\int_{\Omega} (A(x + he_i) \nabla u(x + he_i) - A(x) \nabla u(x)) \nabla \varphi(x) = \int_{\Omega} f(\psi(x - he_i) - \psi(x)).$$

Set $\psi = (u(x + he_i) - u(x))\tau^2 \in W_0^{1,2}$ and apply local regularity.

1b: $\varphi \in C_0^{\infty}(-||-)$

$$\begin{aligned} - \int_{\Omega} \sum_{i,j}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) \varphi &= - \int_{\Omega} \operatorname{div}(A \nabla u) \varphi = \int_{\Omega} A \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \\ - \int_{\Omega} a_{dd} \frac{\partial^2 u}{\partial x_d^2} \varphi &= \underbrace{\int_{\Omega} f \varphi}_{\in L^2(\Omega)} + \int_{\Omega} \varphi \left(\sum_{i,j=1}^d \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i,j=1, \neg(i=j=d)}^d \right) a_{ij} \frac{\partial u}{\partial x_i x_j}. \\ \|a_{dd} \frac{\partial^2 u}{\partial x_d^2}\|_2^2 &\leq \|f + \sum \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{\neg(i=j=d)} a_{ij} \frac{\partial^2 u}{\partial x_i x_j}\|_2^2 \leq C. \end{aligned}$$

A is elliptic

$$c_1 |\zeta|^2 \leq a_{ij} \zeta_i \zeta_j$$

Special choice $\zeta = (0, \dots, 0, 1)$, $0 < c_1 \leq a_{dd}(x) \implies \|\frac{\partial^2 u}{\partial x_d^2}\|_{L^2}^2 \leq C(\text{DATA?})$ □

└

┌ *Důkaz* (Step 2: Transfer from flat to small parts of $\partial\Omega$)
 TODO!!!

└ TODO!!! □

┌ *Důkaz* (Step 3: Introduce a proper covering of $\partial\Omega$ and use step 2)

$$-\operatorname{div}(A\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

? $\Omega \ u \in W_{loc}^{2,2}(\Omega)$. ? of $\partial\Omega$, apply step 2.

Define $w := u - u_0 \in W_0^{1,2}(\Omega)$.

$$-\operatorname{div}(A\nabla w) = f + \operatorname{div}(A\nabla u_0)$$

└ if $f \in L^2$ and $\operatorname{div}(A\nabla u_0) \in L^2$, e.g. $A \in W^{1,\infty} \wedge u_0 \in W^{2,2}(\Omega)$. □

5 Bochner integral

Definition 5.1 (Measurability)

We say that $f : I \rightarrow X$ is measurable (strongly, Bochner) if $\exists \{s_j\}_{j=1}^\infty$ simple functions, $\|f(t) - s_n(t)\|_X \rightarrow 0$ as $n \rightarrow \infty$ for almost every $t \in I$.

Věta 5.1 (Measurability)

$f : I \rightarrow X$ is measurable iff

1. f is almost separably valued;

$$\exists E \subset I : |E| = 0, f(I \setminus E) \text{ is separable.}$$

2. f is weakly measurable;

$$\forall F \in X^* : \langle F^*, u(t) \rangle_X \text{ is Lebesgue measure w.r.t } t \in I.$$

Definition 5.2 (Bochner integral for simple function)

Let $s : I \rightarrow X$ be a simple function on I . We define

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

Definice 5.3 (Bochner integral for measurable functions)

Let $s : I \rightarrow X$ be a Bochner measurable function. We say that f is Bochner integrable if $\exists \{s^n\}_{n=1}^\infty$ such that $s^n(t) \rightarrow f(t)$ a. a. t and $\int_I \|s^n(t) - f(t)\|_X dt \rightarrow 0$ as $n \rightarrow \infty$ and we set

$$X \ni \int_I f(t) dt = \lim_{n \rightarrow \infty} \int_I s^n(t) dt.$$

$$\int_I s(t) dt := \sum_{j=1}^n X_j |I_j|$$

Definice 5.4 ($L^p(O, T, X)$ space)

Let X be a Banach space

$$L^p(O, T, X) = \left\{ f : (O, T) \rightarrow X \text{ bochner integrable} \mid \int_I \|f(t)\|_X^p < \infty \right\}$$

$$\|f\|_{L^p(O, T, X)} = \left(\int_I \|f(t)\|_X^p dt \right)^{\frac{1}{p}}.$$

Věta 5.2 (Dual space)

Let X be a Banach space, separable and $p \in [1, \infty)$, then

$$(L^p(O, T, X))^* = L^{p'}(O, T, X^*)$$

5.1 Sobolev-Bochner spaces

Definice 5.5

Let $f : I \rightarrow X$ be Bochner integrable. We say that $g : I \rightarrow X$ is a weak derivative of f w. r. t. iff g is Bochner integrable and $\forall \tau \in C_0^\infty(I) : \int_I f(t) \tau'(t) dt = - \int_I g(t) \tau(t) dt$.

Poznámka

If $f \in L^1(I, x)$ and $\frac{\partial f}{\partial t} \in L^1(I, x)$, then $f \in C(I, x)$.

Věta 5.3

$$W^{1,p}(I, X) := \{f \in L^p(I, x), \partial_t f \in L^p(I, X)\}, \quad \|f\|_{W^{1,p}(I, X)} = \begin{cases} \left(\int_I \|f\|_X^p + \|\partial_t f\|_X^p \right)^{\frac{1}{p}}, & p \in [1, \infty) \\ \text{esssup}_{t \in I} (\|f(t)\|_X + \|\partial_t f\|_X), & p = \infty \end{cases}$$

Then $W^{1,p}(I, X)$ is a Banach space, is separable for $p < \infty$ and X separable and

is reflexive if $p \in (1, \infty)$ and X is reflexive and separable.

5.2 Time derivative in heat/wave equations – Gelfand triple

Poznámka (Motivation)

$$\partial_t u - \Delta u = 0 \text{ in } (0, T) \times \Omega, u = 0 \text{ on } (0, T) \times \partial\Omega, x(0, x) = u_0(x) \text{ for } x \in \Omega, \quad \Omega \subseteq \mathbb{R}^d$$

Definition 5.6 (Gelfand triple)

We say that X, H, X^* is Gelfand triple iff $X \xrightarrow{\text{dense}} H \cong H^* \xrightarrow{\text{dense}} X^*$.

┌ *Například*

$$X = W_0^{1,2}(\Omega), H = L^2(\Omega), X^* = (W_0^{1,2}(\Omega))^*,$$

Neboť $W_0^{1,2}$ is dense in $C_0 \xrightarrow{\text{dense}} L^2(\Omega)$ and $f \in (W_0^{1,2}(\Omega))^* \implies \exists! u \in W_0^{1,2}(\Omega) : -\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$.

$$\forall \varphi \in W_0^{1,2}(\Omega) \langle f, \varphi \rangle = \int_{\Omega} \nabla u \cdot \nabla \varphi = \lim_{n \rightarrow \infty} \int_{\Omega} \nabla u^n \cdot \nabla \varphi = \lim_{n \rightarrow \infty} - \int_{\Omega} \Delta u^n \varphi = \lim_{n \rightarrow \infty} (f^n, \varphi)_{L^2(\Omega)},$$

where $\{u^n\}_{n=1}^{\infty} \subset C_0^{\infty}(\Omega)$, $u^n \rightarrow u$ in $W_0^{1,2}(\Omega)$,

$$(X = W_0^{1,p}(\Omega \cap L^2(\Omega)), H = L^2(\Omega))$$

Definition 5.7

Let X, H, X^* be Gelfand triple, $\varphi : H \rightarrow H^*$ is Riesz representation and define $i : X \rightarrow X^*$, such that $\forall x_0, x \in X$:

$$\langle i(x_0), x \rangle_X := (\text{id}(x_0), \text{id}(x))_H = \langle \varphi \text{id}(x_0), \text{id}(x) \rangle_{H^*},$$

i maps X densely onto X^* .

Lemma 5.4

Let $u \in L^1(0, T, H)$, $\partial_t u \in L^1(0, T, X^*)$ and X, H, X^* be a Gelfand triple. Then $\forall w \in X \forall \tau \in C_0^1(0, T)$ we have

$$\begin{aligned} \int_0^T \langle \partial_t u, w \rangle \tau dt &= \langle \int_0^T \partial_t u \tau dt, w \rangle_X = \\ &= - \langle \int_0^T u \tau' dt, w \rangle_X = - \int_0^T \langle u \tau', w \rangle_X dt = \end{aligned}$$

$$= - \int_0^T (u\tau', w)_H dt \stackrel{\text{if } \partial_t u \in L^1(0,T)}{=} \int_0^T (\partial_t u \tau, w)_H.$$

Věta 5.5 (Integration by parts for Sobolev-Bochner function)

Let $p \in (1, \infty)$, X, H, X^* a Gelfond triple, $u, v \in L^p(0, T, X)$, $\partial_t u, \partial_t v \in L^{p'}(0, T, X^*)$. Then $u, v \in C([0, T], H)$ and $\forall 0 \leq t_1 < t_2 \leq T$.

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

┌
Důkaz

Step 1) Modify u, v in terms of the Steklov ar? $u_h = \int_t^{t+h} u(\tau) d\tau$.

Step 2) Prove for u_h, v_h from step 1).

Step 3) $h \rightarrow 0_+$.

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Důkaz („Step 1“)

Define $u_h(t) := \frac{1}{h} \int_t^{t+h} u(\tau) d\tau$, $\forall t \in (0, T-h)$. $u_h \rightarrow u$ in $L^p(0, T-h, X)$, $\forall h_0 \in (0, T)$. We want „ $(\partial_t u)_h = \partial_t u_h = \frac{u(t+h)-u(t)}{h}$ “.

$$(\partial_t u)_h \rightarrow \partial_t u \text{ in } L^{p'}(0, T-h_0, X^*), \quad \forall h_0 \in (0, T).$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^{T-h} u_h(t) \varphi'(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \int_t^{t+h} u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi'(t) \left(\int_0^{t+h} u(\tau) d\tau - \int_0^t u(\tau) d\tau \right) = \\ &= -\frac{1}{h} \int_0^{T-h} \varphi(t) (u(t+h) - u(t)) dt \Leftrightarrow \partial_t u_h = \frac{u(t+h) - u(t)}{h}. \end{aligned}$$

$$\begin{aligned} \varphi \in C_0^\infty(0, T-h) : \int_0^T \varphi(t) (\partial_t u)_h(t) dt &= \frac{1}{h} \int_0^{T-h} \varphi(t) \int_t^{t+h} \partial_t u(\tau) d\tau dt = \\ &= \frac{1}{h} \int_0^{T-h} \varphi(t) \left(\int_0^{t+h} \partial_t u(\tau) d\tau - \int_0^t \partial_t u(\tau) d\tau \right) dt = (*) \end{aligned}$$

$$\frac{1}{h} \int_0^{T-h} \varphi(t) \left(\int_0^t \partial_t u(\tau) d\tau \right) dt = \int_0^{T-h} \int_0^{T-h} \varphi(t) \partial_t u(\tau) \chi_{\tau \leq t} d\tau dt = \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \left(\int_t^{T-h} \varphi(t) dt \right) d\tau.$$

$$(*) = \frac{1}{h} \int_0^{T-h} \partial_t u(\tau) \underbrace{\left(\int_{\tau-h}^\tau \varphi(t) dt \right)}_{C_0^\infty(0, T)} d\tau = -\frac{1}{h} \int_0^{T-h} u(\tau) (\varphi(\tau) - \varphi(\tau-h)) d\tau dt.$$

└

□

┌ *Dikaz* („Step 2“)

We want

$$\begin{aligned} \int_{t_1}^{t_2} < \partial_t u_{h_1}, v_{h_2} >_X + < \partial_t v_{h_2}, u_{h_1} >_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \Leftrightarrow \\ \Leftrightarrow \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H + (\partial_t v_{h_2}, u_{h_1})_H dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H \end{aligned}$$

$$\begin{aligned} \int_{t_1}^{t_2} (\partial_t u_{h_1}, v_{h_2})_H &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_t^{t+h_2} v(\tau) d\tau)_H dt = \frac{1}{h_1 h_2} \int_{t_1}^t (u(t+h_1) - u(t), \int_{t_1}^{t+h_2} v(\tau) d\tau - \int_{t_1}^t v(\tau) d\tau)_H dt \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^t v(\tau+h_2) d\tau - \int_{t_1}^t v(\tau) d\tau)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau)_H dt + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_1}^{t_2} u(t+h_1) - u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt = \\ &= \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h_1} u(t) - \int_{t_2}^{t_2+h_1} u(t) dt)_H d\tau + \int_{t_1}^{t_2} \left(u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau+h_2) d\tau \right)_H dt \\ &\quad + \int_{t_1}^{t_2} \left(\frac{v(\tau+h_2) - v(\tau)}{h_2}, \int_{\tau}^{\tau+h_1} u(t) dt \right)_H d\tau + \\ &\quad + \frac{1}{h_1 h_2} \int_{t_1}^{t_2} (v(\tau+h_2) - v(\tau), \int_{t_2}^{t_2+h_1} u(t) dt)_H d\tau + \int_{t_1}^{t_2} (u(t+h_1) - u(t), \int_{t_1-h_2}^{t_1} v(\tau) d\tau)_H dt = \\ &\quad - \int_{t_2}^{t_1} (\partial_t v_{h_2}(\tau), u_{h_1}(\tau))_H d\tau + REST \\ REST &= \frac{1}{h_1 h_2} \left(\int_{t_2}^{t_2+h_2} v(t) dt - \int_{t_1}^{t_1+h_2} v(t) dt, \int_{t_2}^{t_2+h_2} u(t) dt \right) + SIMILAR = \\ &= (v_{h_2}(t_2) - v_{h_2}(t_1), u_{h_1}(t_2))_H - SIMILAR = (v_{h_2}(t_2), u_{h_2}(t_2))_H - \dots \end{aligned}$$

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□

┌ *Důkaz („Step 3“)*

We have

$$\int_{t_1}^{t_2} \langle \partial_t u_{h_1}, v_{h_2} \rangle_X + \langle \partial_t v_{h_2}, u_{h_1} \rangle_X dt = (u_{h_1}(t_2), v_{h_2}(t_2))_H - (u_{h_1}(t_1), v_{h_2}(t_1))_H$$

Let $h_1 \rightarrow 0_+$ and $h_2 \rightarrow 0_+$. We have $\partial_t u_{h_1} \rightarrow \partial_t u$ in $L^{p'}(0, T, X^*)$, $\partial_t v_{h_2} \rightarrow \partial_t v$ in $L^{p'}(0, T, X^*)$, $u_{h_1} \rightarrow u$ in $L^p(0, T, X)$, $v_{h_2} \rightarrow v$ in $L^p(0, T, X)$. So for almost all t in $(0, T)$: $v_{h_2}(t) \rightarrow v(t)$ in $X \hookrightarrow H$ and $u_{h_1}(t) \rightarrow u(t)$ in $X \hookrightarrow H$.

$$\int_{t_1}^{t_2} \langle \partial_t u, v \rangle_X + \langle \partial_t v, u \rangle_X = (u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H.$$

Now, it is enough to show $u, v \in C([0, T], H)$. We show that u_h is Cauchy in $C([0, T], H)$.

Use IBP $u_{h_1} = u_{h^n} - u_{h^m}$, $v_{h_2} = u_{h^n} - u_{h^m}$:

$$\|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H = \|u_{h^m}(t_1) - u_{h^n}(t_1) + 2 \int_{t_1}^{t_2} \langle \partial_t(u_{h^n} - u_{h^m}), u_{h^n} - u_{h^m} \rangle_X \|$$

$$\begin{aligned} \|u_{h^n} - u_{h^m}\|_{C([\frac{T}{4}, T], L^2(\Omega))}^2 &= \sup_{t_2 \in (\frac{T}{2}, T)} \|u_{h^n}(t_2) - u_{h^m}(t_2)\|_H^2 \leq \\ &\leq \|u_{h^m}(t_1) - u_{h^n}(t_1)\|_H^2 + \int_0^T \|\partial_t(u_{h^n}) - \partial_t(u_{h^m})\|_{X^*} \|u_{h^n} - u_{h^m}\|_X dt. \end{aligned}$$

Choose t_1 such that $u_h(t_1) \rightarrow u(t_1)$ in H :

$$\leq \|u_h(t_1) - u_{h^m}(t_1)\|_H^2 + \|\partial_t u_{h^m} - \partial_t u_{h^n}\|_{L^p(X^*)} \cdot \dots$$

$$u \in C([\frac{T}{4}, T], L^2(\Omega)) \wedge u \in C([0, \frac{3T}{4}], L^2(\Omega)) \rightarrow u \in C([0, T], L^2(\Omega)) (u(t_1), v(t_1))_H$$

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□

6 Parabolic equations

Poznámka

Ω open set in \mathbb{R}^d , $T > 0$, L elliptic operator,

$$\partial_t u + Lu = f \text{ in } Q = (0, T) \times \Omega, \quad u = 0 \text{ on } (0, T) \times \partial\Omega, \quad u(0, x) = u_0(x) \text{ } x \in \Omega.$$

$$Lu = -\operatorname{div}_x(A(t, x)\nabla_x u(t, x)) + b(t, x)u(t, x) + \mathbf{c}(t, x)\nabla u(t, x) + \operatorname{div}(\mathbf{d}(t, x)u(t, x)),$$

$$A, b, \mathbf{c}, \mathbf{d} \in L^\infty(\Omega).$$

$$A(t, x) \cdot \xi \cdot \xi \geq c_1 |\xi|^2, \forall \xi \in \mathbb{R}^d \text{ and almost all } (t, x) \in Q.$$

6.1 Formal a priory estimates

Poznámka

Multiply by u and $\int_{\Omega} dx$ and use IBP.

$$\int_{\Omega} \partial_t u u + \int_{\Omega} A \nabla u \nabla u = \int_{\Omega} f u - b u^2 - \mathbf{c} \nabla u u + \mathbf{d} u \nabla u.$$

Hölder's inequality:

$$\frac{d}{dt} \frac{\|u\|_2^2}{2} + C_1 \|\nabla u\|_2^2 \leq \|f\|_2 \|u\|_2 + \|b\|_{\infty} \|u\|_2^2 + \|\mathbf{c}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 + \|\mathbf{d}\|_{\infty} \|\nabla u\|_2 \|\nabla u\|_2 \leq C_1 \frac{\|\nabla u\|_2^2}{2} + C(\mathbf{c})$$

Poincaré's inequality:

$$\frac{d}{dt} \|u\|_2^2 + \mathbf{c} \|u\|_{1,2}^2 \leq C(\mathbf{c}, \mathbf{d}, b) \|f\|_2^2 + K \|u\|_2^2.$$

Grönwall's inequality:

$$\sup_{t \in (0, T)} \|u(t)\|_2^2 \leq + \int_0^T \|f\|_2^2$$

$$\int_0^T \|u\|_{1,2}^2 dt \leq C$$

$$\|\partial_t u\|_{(W_0^{1,2})^*} = \sup_{\|\varphi\| \leq 1} \langle \partial_t u, \varphi \rangle = \sup \langle f - Lu, \varphi \rangle =$$

$$= \sup_{\|\varphi\| \leq 1} \int_{\Omega} (f - ? - bu - \mathbf{c} \nabla u) \varphi - \int_{\Omega} (A \nabla u - \mathbf{d} u) \nabla \varphi \leq \int_0^T \|f\|_2^2 + c(\|u\|_2^2 + \|\nabla u\|_2^2).$$

Definice 6.1

Let $\Omega \subseteq \mathbb{R}^d$ open and bounded, L be an elliptic operator, $u_0 \in L^2(\Omega)$, $f \in L^2(0, T, V^*)$ ($V = W_0^{1,2}(\Omega)$). We say that u is a weak solution to

$$\partial_t u + Lu = f \text{ in } (0, T) \times \Omega,$$

$$u = 0 \text{ on } (0, T) \times \partial\Omega,$$

$$u(0) = u_0 \text{ in } \Omega$$

iff $u \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$, $u(0) = u_0$ and for almost all $t \in (0, T)$ and $\forall \varphi \in V$:

$$\langle \partial_t u, \varphi \rangle_V + \int_{\Omega} A \nabla u \cdot \nabla \varphi + bu \varphi + \mathbf{c} \cdot \nabla u \varphi - \mathbf{d} \nabla \varphi u = \langle f, \varphi \rangle_V.$$

6.2 Existence and uniqueness

Věta 6.1

Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded, $f \in L^2(0, T, V^*)$, $u_0 \in L^2(\Omega)$ and L be elliptic operator. Then $\exists! u$ – weak solution.

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Důkaz (Uniqueness)

u_1, u_2 are weak solutions. Define $w := u_1 - u_2 \in L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$. WF for u_1 – WF for u_2 :

$$\langle \partial_t w, \varphi \rangle + \int_{\Omega} A \nabla w \cdot \nabla \varphi = \int_{\Omega} -b w \varphi - \mathbf{c} \nabla w \varphi + \mathbf{d} \cdot \nabla \varphi w.$$

Follow almost everywhere, replace u by w . Set $\varphi = w \implies$

$$\langle \partial_t w, w \rangle + \mathbf{c} \|w\|_{1,2}^2 \leq c \|w\|_2^2.$$

Integrate in respect of time, use IBP-formula for $\langle \cdot, \cdot \rangle$:

$$\int_0^t \langle \partial_t w, w \rangle = \frac{1}{2} \|w(t)\|_2^2 - \frac{1}{2} \|w(0)\|_2^2 = \frac{1}{2} \|w(t)\|_2^2.$$

$$\implies \|w(t)\|_2^2 \leq c \int_0^t \|w(\tau)\|_2^2 d\tau$$

$$\underbrace{\frac{d}{dt} \int_0^t \|w(\tau)\|_2^2 d\tau}_{=: g(t) \geq 0} \leq c \int_0^t \|w(\tau)\|_2^2 d\tau$$

$$g' \leq c \cdot g$$

From Grönwall's inequality

$$g(t) \leq e^{c \cdot t} g(0) \implies \int_0^t \|w(\tau)\|_2^2 d\tau \leq e^{ct} \int_0^0 \|w(\tau)\|_2^2 d\tau = 0 \implies w(t) = 0.$$

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Důkaz (Existence (via Galerkin approximation))

We know $\exists \{w_j\}_{j=1}^{\infty}$ basis of \mathbf{V} , which is ortonormal in L^2 and $\|P^N u\|_V \leq c \|u\|_V$, where P^N is orthogonal projection in $L^2(\Omega)$ onto $\{w_j\}_{j=1}^N$.

We solve for $u^n(t, x) = \sum_{i=1}^n a_i^n(t) w_i(x)$. We want

$$\langle \partial_t u^n, w_j \rangle = - \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n + \langle f, w_j \rangle$$

for $j \in [n]$ for almost all $t \in (0, T)$ (weak formulation of the problem for n, call it WF_n). □

└

┌ *Důkaz* („Existence of u^n “)

LHS of WFn:

$$\sum_{i=1}^n \langle \partial_t a_i^n w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n(t) \langle w_i, w_j \rangle_V = \sum_{i=1}^n \partial_t a_i^n \delta_{ij} = \partial_t a_j^n(t).$$

RHS of WFn:

$$\sum_{i=1}^n a_i^n(t) \left(\underbrace{- \int_{\Omega} A \nabla w_i \nabla w_j + b w_i w_j + \mathbf{c} \nabla w_i w_j - \mathbf{d} \nabla w_j w_i}_{G_{ij}(k) - \text{bounded and measurable?}} \right) + \underbrace{\langle f(t), w_j \rangle}_{g_j^{(t)} - \text{measurable? on } g \in L^2(0,T)}.$$

So

$$\frac{d}{dt} a_j^n(t) = \sum_{i=1}^n a_i^n(t) G_{ij}(t) + g_j(t), \quad j \in [n].$$

Initial data: $u^n(0) := P^n u_0$ ($a_j^n(0) := \int_{\Omega} u_0 w_j$).

ODE $\implies \exists \tilde{T} \leq T$ and $a_i^n(t) \in AC$ on $[0, \tilde{T})$ and solve for almost all $t \in (0, \tilde{T})$.
Moreover either we can set $\tilde{T} = T$ or $|a^n(t)| \xrightarrow{t \rightarrow \tilde{T}} \infty$.

Now we prove $\tilde{T} = T$. We show $|a^n(t)| \leq c$.

Multiply WFn for j by $a_j^n(t)$ and sum it:

$$LHS = \sum_{j=1}^n a_j^n \langle \partial_t u_j^n, w_j \rangle = \langle \partial_t u^n, \sum_{j=1}^n a_j^n w_j \rangle = \langle \partial_t u^n, u^n \rangle.$$

$$RHS = \underbrace{\langle \partial_t u^n, u^n \rangle}_{= \frac{d}{dt} \|u^n\|_2^2} + c_1 \|u^n\|_V^2 \leq c(\|f\|_{V^*}^2 + \|u^n\|_2^2).$$

Grönwall: $\|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c(\|u^n(0)\|_2^2) + \int_0^T \|f\|_{V^*}^2$.

$$\forall t < \tilde{T} : \|u^n(t)\|_2^2 + \int_0^{\tilde{T}} \|u^n\|_{1,2}^2 \leq c \left(\int_0^{\tilde{T}} \|f\|_{V^*}^2 + \|u_0\|_2^2 \right) \leq \tilde{c}.$$

$$\lim_{t \rightarrow \tilde{T}_-} |a^n(t)|^2 = \lim_{t \rightarrow \tilde{T}_-} \|u^n(t)\|_2^2 \leq \tilde{c}.$$

└

□

┌ *Důkaz*

$$\|u^n\|_{L^2(0,T,V)} + \|u^n\|_{L^\infty(0,T,L^2(\Omega))} \leq c(f, u_0).$$

Time derivative

$$\|\partial_t u^n\|_{V^*} = \sup_{w \in V, \|w\| \leq 1} \langle \partial_t u^n, w \rangle = \sup_{w \in V, \|w\| \leq 1} \int_{\Omega} \partial_t u^n w = \sup_{\dots} \int_{\Omega} \partial_t u^n P^n w.$$

WFn:

$$\begin{aligned} &\leq \sup_{\dots} c(\|f\|_{V^*} + \|u^n\|_V) \|P^n w\|_V \leq \sup_{\dots} \tilde{c}(\|f\|_{V^*} + \|u^n\|_V) \|w\|_V \leq \\ &\leq (\tilde{c}(\|f\|_{V^*} + \|u^n\|_V)). \end{aligned}$$

$$\int_0^T \|\partial_t u^n(t)\|_{V^*}^2 \leq \tilde{c} \int_0^T (\|f(t)\|_{V^*}^2 + \|u^n(t)\|_V^2) \leq c(f, u_0).$$

u^n is a bounded sequence in $L^2(0, T, V) \cap W^{1,2}(0, T, V^*)$ so \exists a subsequence u^{m_n} :

$$u^{m_n} \rightharpoonup u \text{ in } L^2(0, T, V), \quad \partial_t u^{m_n} \rightharpoonup \partial_t u \text{ in } L^2(0, T, V^*).$$

To show u is a weak solution.

└ TODO?

□

TODO!!!

┌ *Důkaz* (Initial condition)

$\tau \in C_0^\infty(-\infty, T)$:

$$\begin{aligned} & - \int_0^T \int_{\Omega} u^n w_j \partial_t \tau - \int_{\Omega} u^n(0) w_j \tau(0) + \int_0^T (\dots) \tau \dots = 0. \\ & \rightarrow - \int_0^T \int_{\Omega} u w_j \partial_t \tau - \int_{\Omega} u_0 w_j \tau(0) + \int_0^T (\dots) \tau = 0. \end{aligned}$$

Integration by parts in time:

$$\begin{aligned} & u \in L^2(W_0^{1,2}(\Omega)) \ni \tau w_j, \\ & \partial_t u \in L^2((W_0^{1,2}(\Omega))^*) \ni \partial_t(\tau w_j) = \partial_t \tau w_j \in L^2, \\ & - \int_0^T \int_{\Omega} u w_j \partial_t \tau - \int_{\Omega} w_j \tau(0) = - \int_0^T \langle u, \partial_t(\tau w_j) \rangle - \int_{\Omega} u_0 w_j \tau(0) = \\ & = \int_0^T \langle \partial_t u, \tau w_j \rangle + \int_{\Omega} u(0) \tau(0) w_j - \int_{\Omega} u_0 w_j \tau(0). \end{aligned}$$

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□

6.3 Regularity of parabolic equations

TODO Example?

Věta 6.2

Let $\mathbf{b}, \mathbf{c}, \mathbf{d} \in L^\infty$, $\operatorname{div} \mathbf{d} \in L^\infty$, $A, \nabla A, \partial_t A \in L^\infty$, $f \in L^2(0, T, L^2(\Omega))$, then $\forall \delta > 0$:

$$\int_\delta^T \|\partial_t u\|_2^2 + \sup_{t \geq \delta} \|\nabla u(t)\|_2^2 \leq \frac{c}{\delta}$$

.

Moreover if $u_0 \in W_0^{1,2}(\Omega)$, then $\partial_t u \in L^2(0, T, L^2(\Omega))$, $u \in L^\infty(0, T, W_0^{1,2}(\Omega))$.

Moreover $u \in L^2(0, T, W_{loc}^{1,2}(\Omega))$ and if $\Omega \in C^{1,1}$, then $u \in L^2(0, T, W^{2,2}(\Omega))$.

Dukaz

Consider u^n -Galeikin approximation

$$u^n(t, x) = \sum_{i=1}^n a_i^n w_i : \int_{\Omega} \partial_t u^n w_j + \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n = \langle f, w_j \rangle.$$

Multiply by $\partial_t a_j^n(t)$ and

$$\sum_{i=1}^n \int_{\Omega} \partial_t u^n \partial_t a_i^n w_j = \int_{\Omega} \partial_t u^n \left(\sum \partial_t a_i^n w_j \right) = \int_{\Omega} \partial_t u^n (\partial_t u^n).$$

$$\int_{\Omega} \partial_t u^n \partial_t u^n + \int_{\Omega} A \nabla u^n \cdot \nabla \partial_t u^n + b u^n \partial_t u^n + \mathbf{c} \cdot \nabla u^n \partial_t u^n - \mathbf{d} \nabla \partial_t u^n u^n = \langle f, \partial_t u^n \rangle.$$

$$\text{Good guy: } \int_{\Omega} \partial_t u^n \partial_t u^n = \|\partial_t u^n\|_2^2.$$

$$\text{First half of other guy: } \int_{\Omega} A \nabla u \nabla \partial_t u =$$

$$\begin{aligned} & \int_{\Omega} \frac{(A + A^T)}{2} \nabla u \nabla \partial_t u + \int_{\Omega} \frac{A - A^T}{2} \nabla u \nabla \partial_t u = \\ &= \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \frac{A + A^T}{2} \nabla u \nabla u - \frac{1}{2} \int_{\Omega} \frac{\partial_t (A - A^T)}{2} \nabla u \nabla u - \int_{\Omega} \sum_{i,j} \frac{\partial}{\partial x_i} \frac{(A_{ij} - A_{ji})}{2} \frac{\partial u}{\partial x_j} \partial_t u - \underbrace{\sum_{i,j} \frac{A_{ij} - A_{ji}}{2} \frac{\partial^2 u}{\partial x_i \partial x_j}}_{\text{sum with symmetric dot antis}} \end{aligned}$$

$$\text{Worst guy: } \int_{\Omega} \mathbf{d} \cdot \nabla \partial_t u^n u^n =$$

$$= - \int_{\Omega} \partial_t u^n \operatorname{div}(\mathbf{d} u^n) = - \int_{\Omega} \partial_t u^n (\operatorname{div} \mathbf{d} u^n + \mathbf{d} \cdot \nabla u^n).$$

$$\|\partial_t\|_2^2 + \frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} \frac{A + A^T}{2} \nabla u \nabla u \leq \int_{\Omega} |\partial_t u^n| (|f| + \dots) + |\nabla u|^2 |\partial_t A|.$$

From Young's inequality:

$$\begin{aligned} & \leq \frac{1}{2} \int_{\Omega} |\partial_t u^n|^2 + C \int_{\Omega} |b|^2 |u^n|^2 + |\mathbf{c}|^2 |\nabla u^n|^2 + |\operatorname{div} \mathbf{d}|^2 |u^n|^2 + |\mathbf{d}|^2 |\nabla u^n|^2 + |f|^2 + \left| \nabla \frac{A - A^T}{2} \right|^2 |\nabla u^n|^2 + \left| \partial_t \frac{A + A^T}{2} \right|^2 |\nabla u^n|^2 \\ & \leq \frac{1}{2} \|\partial_t u^n\|_2^2 \leq c(b, \mathbf{c}, \mathbf{d}) (\|f\|_2^2 + \|u^n\|_{1,2}^2). \\ & \implies \|\partial_t u^n\|_2^2 + \frac{d}{dt} \int_{\Omega} A \nabla u^n \cdot \nabla u^n \leq c(\dots) (\|f\|_2^2 + \|u^n\|_{1,2}^2). \end{aligned}$$

We want to know, if right hand side is integrable in time:

$$\int_{\tau}^t \|\partial_t u^n\|_2^2 + \int_{\Omega} A \nabla u^n(t) \nabla u^n(t) \leq \int_{\Omega} A \nabla u^n(\tau) \nabla u^n(\tau) + c \cdot \int_{\tau}^t \|f\|_2^2 + \|u^n\|_{1,2}^2.$$

With $\tau \leq \delta$ we add $\int_0^{\delta} \cdot d\tau$:

$$\int_{\delta}^t \|\partial_t u^n\|_2^2 + \int_{\Omega} A \nabla u^n(t) \nabla u^n(t) \leq \int_0^{\delta} \int_{\Omega} A \nabla u^n(\tau) \nabla u^n(\tau) d\tau + C(\text{DATA})$$

Věta 6.3

Let $\partial_t f \in L^2(0, T, L^2(\Omega))$, $\partial_t A, \partial_t b, \partial_t \mathbf{c}, \partial_t d \in L^\infty$. Then $\forall \delta > 0 : \partial_{tt} u \in L^2(\delta, T, V^*), \partial_t u \in L^2(\delta, T, W_0^{1,2}(\Omega))$. If $-Lu_0 + f(0) \in L^2(\Omega)$, then

$$\partial_{tt} u \in L^2(0, T, V^*), \quad \partial_t u \in L^2(0, T, W_0^{1,2}(\Omega)).$$

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Důkaz (Sketch)

Take u^n – Galerkin approximation. Apply ∂_t to it:

$$\int_{\Omega} \partial_t u^n w_j + \int_{\Omega} A \nabla u^n \nabla u^n w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j - \mathbf{d} \nabla w_j u^n = \int_{\Omega} f w_j, \quad \forall j \in [n] \text{ and almost every } t \in (0, T)$$

$$\int_{\Omega} \partial_{tt} u^n w_j + \int_{\Omega} A \nabla \partial_t u^n \nabla w_j = \int_{\Omega} -\partial_t A \nabla u^n \nabla w_j + (\partial_t b u^n + b \partial_t u^n) w_j + \partial_t \mathbf{c} \nabla u^n + \mathbf{c} \nabla \partial_t u^n w_j.$$

Similar as before we replace $w_j, b_j, \partial_t u^n$:

$$\frac{1}{2} \frac{d}{dt} \|\partial_t u^n\|_2^2 + c_1 \|\nabla \partial_t u^n\|_2^2 \leq \int_{\Omega} \|\nabla \partial_t u^n\| (SOMETHING).$$

$$\implies \frac{d}{dt} \|\partial_t u^n\|_2^2 + \|\nabla \partial_t u^n\|_2^2 \leq C(\|\partial_t u^n\|_2^2 + \dots).$$

$$t \geq 2\delta : \|\partial_t u(t)\|_2^2 + \int_{\tau}^t \|\partial_t u^n\|_2^2 \leq C(1 + \int_{\tau}^t \|\partial_t u^n\|_2^2) + \|\partial_t u^n(\tau)\|_2^2.$$

Add $\int_{\delta}^2 \delta d\tau$:

$$\|\partial_t u(t)\|_2^2 + \int_{2\delta}^T \|\nabla u\|_2^2 \leq X \left(\int_{\delta}^T \|\partial_t u^n\|_2^2 + 1 + \int_{\delta}^{2\delta} \|\partial_t u^n(\tau)\|_2^2 \right) \leq C(1 + \frac{c}{\delta} + \frac{c}{\delta^2}).$$

$$\rightarrow C \left(\int_0^T \|\partial_t u^n\|_2^2 + \|\partial_t u^n(0)\|_2^2 + 1 \right) \leq$$

$$\leq C + C \|\partial_t u^n(0)\|_2^2 = C + C \|-Lu_0^n + f(0)\|_2^2 \leq \text{const}.$$

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□

7 Linear hyperbolic equations

Poznámka (Prototype)

$$\begin{aligned} \frac{\partial u^2}{\partial t^2} - \Delta u &= 0 \text{ in } (0, T) \times \Omega, & u &= 0 \text{ on } (0, T) \times \partial\Omega. \\ u(0, x) &= u_0(x) \in W_0^{1,2}(\Omega), & \partial_t u(0, x) &= u_1(x) \in L^2(\Omega). \end{aligned}$$

Poznámka (Formal a priory estimate)

Test by $\partial_t u$:

$$\begin{aligned} \int_{\Omega} \partial_{tt} u \partial_t u - \Delta u \partial_t u &= 0 \\ \frac{1}{2} \frac{d}{dt} \|\partial_t u\|_2^2 + \int_{\Omega} \underbrace{\nabla u \nabla \partial_t u}_{\frac{1}{2} \partial_t \|\nabla u\|^2} &= 0 \end{aligned}$$

$$\frac{d}{dt} (\|\partial_t u\|_2^2 + \|\nabla u\|_2^2) = 0$$

$$\|\partial_t u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq \|\partial_t u(0)\|_2^2 + \|\nabla u(0)\|_2^2 = \|u_1\|_2^2 + \|\nabla u_0\|_2^2.$$

$$\|\partial_{tt}^2 u\|_{(W_0^{1,2}(\Omega))^*} = \sup_{\|\varphi\| \leq 1} \langle \partial_{tt}^2 u, \varphi \rangle \sim \sup \int_{\Omega} \partial_{tt}^2 u \varphi = \sup \int_{\Omega} \nabla u \varphi.$$

Věta 7.1

L be an elliptic operator such that $\int_0^T (\|\partial_t u\|_{\infty} + \|A\|_{1,\infty} + \|b\|_{\infty} + \|\mathbf{c}\|_{\infty} + \|\mathbf{u}\|_{1,\infty}) < \infty$ and $f \in L^2(0, T, L^2(\Omega))$. Assume that $u_0 \in W_0^{1,2}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then there $\exists! u \in L^2(0, T, W_0^{1,2}(\Omega)) \cap W^{1,2}(0, T, L^2(\Omega)) \cap W^{2,2}(0, T, V^*)$.

And $u(t) \rightarrow u_0$ in $L^2(\Omega)$, $\partial_t u(t) \rightarrow u_1$ in V^* .

┌

Důkaz (Existence)

Step one: Galleikin approximation. Step two: Uniform estimates. Step three: $n \rightarrow \infty$. \square

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Důkaz (Step one)

$\{w_j\}_{j=1}^{\infty}$ base of $W_0^{1,2}$ ($\|P^n u\|_{1,2} \leq c \|u\|_{1,2}$). ? for $u^n(t, x) = \sum_{j=1}^n a_j^n(t) w_j(x)$.

$$\int_{\Omega} \partial_{tt} u^n w_j + \int_{\Omega} A \nabla u^n \nabla w_j + b u^n w_j + \mathbf{c} \nabla u^n w_j + \mathbf{d} \nabla w_j u^n = \int_{\Omega} f w_j.$$

Weak formulation for n -th coord. (WFn). $\partial_t u^n(0) = P^n u_1$ and $u^n(0) = P^n u_0$.

$$(a_j^n)'(0) = \int u_1 w_j, \quad a_j^n(0) = \int u_0 w_j, \quad (a_j^n)''(t) = F_j(a^n, t) + b_j(t).$$

┌

Assume there exists u^n a solution to (WFn). \square

┌ *Dukaz* (Step two)

Uniform (N -independent) estimates: Multiply WFn by $(e_j^n)'(t)$ and $\sum_{j=1}^n$:

$$\begin{aligned} & \int_{\Omega} \partial_{tt} u^n \partial_t u^n + \int_{\Omega} A \nabla u^n \partial \nabla u^n = \int_{\Omega} f \partial_t u^n + \mathbf{d} \nabla \partial_t u^n u^n - \mathbf{c} \nabla u^n \partial_t u^n - b u^n \partial_t u^n. \\ & \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |\partial_t u^n|^2 + \frac{A + A^T}{2} \nabla u^n \nabla u^n \right) = - \underbrace{\int_{\Omega} \partial_t \left(\frac{A + A^T}{2} \nabla u^n \nabla u^n \right) - \int_{\Omega} \frac{A - A^T}{2} \nabla u^n \partial \nabla u^n}_{\text{by part}} \stackrel{\text{Hölder}}{\leq} \\ & \leq c(\|f\|_2^2 + \|\partial_t u^n\|_2^2 + \|\nabla u^n\|_2^2) \leq \tilde{c} \left(\|f\|_2^2 + \int_{\Omega} |\partial_t u^n|^2 + \int_{\Omega} \frac{A + A^T}{2} \nabla u^n \nabla u^n \right). \end{aligned}$$

Gronwall's lemma:

$$\begin{aligned} \int_{\Omega} |\partial_t u(t)|^2 + \frac{A + A^T}{2} \nabla u^n(t) \nabla u^n(t) & \leq C(T) \cdot \left(\int_0^T \|f\|_2^2 + \int_{\Omega} |\partial_t u^n(0)|^2 + \frac{A + A^T}{2} \nabla u^n(0) \nabla u^n(0) \right) \leq C_t \cdot \\ & \leq \tilde{C}_T (1 + \|v_0\|_2^2 + \|u_0\|_{1,2}^2). \end{aligned}$$

$$\begin{aligned} & \sup_{t \in (0, T)} \|\partial_t u^n(t)\|_2^2 + \|u^n(t)\|_{1,2}^2 \leq C(\text{DATA}). \\ & \|\partial_{tt} u^n\|_{(W_0^{1,2}(\Omega))^*} := \sup_{\varphi \in W_0^{1,2}(\Omega)} \langle \partial_{tt} u^n, \varphi \rangle \stackrel{\text{Gelfand}}{=} \\ & = \sup_{\varphi} \int_{\Omega} \partial_{tt} u^n \varphi \stackrel{\text{basis}}{=} \sup_{\varphi} \int_{\Omega} \partial_{tt} u^n P^n(\varphi) \stackrel{\text{WFn}}{=} \\ & = - \int_{\Omega} A \nabla u^n \nabla P^n(\varphi) \dots \stackrel{\text{Hölder}}{\leq} C \cdot \|P^n(\varphi)\|_{1,2} (\|f\|_2 + \|u^n\|_{1,2}) \leq \\ & \leq \tilde{c} \|\varphi\|_{1,2} (\|f\|_2 + \|u^n\|_{1,2}). \end{aligned}$$

$$\int_0^T \|\partial_{tt} u^n\|_{(W_0^{1,2}(\Omega))^*}^2 \leq \tilde{c} \cdot 2 \cdot \int_0^T (\|f\|_2^2 + \|u^n\|_{1,2}^2) \leq C(\text{DATA}).$$

└

□

┌ *Důkaz* (Step three)

$u^n \rightharpoonup^* u$ in $W^{1,\infty}(0, T, L^2(\Omega)) \cap L^\infty(0, T, W_0^{1,2}(\Omega))$. $u^n \rightharpoonup u$ in $W^{2,2}(0, T, (W_0^{1,2}(\Omega))^*)$.

Limits:

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \partial_{tt} u^n w_j \tau dx dt = \int_0^T \langle \partial_{tt} u^n, w_j \tau \rangle dt = \langle \partial_{tt} u^n, w_j \tau \rangle_{L^2(0,T,W_0^{1,2}(\Omega))}.$$

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} A \nabla u^n \nabla w_j \tau = \langle \nabla u^n, A^T \nabla w_j \tau \rangle_{L^2(0,T,L^2(\Omega))} \rightarrow \langle \nabla u, A^T \nabla w_j \tau \rangle = \int_0^T \int_{\Omega} A \nabla u \nabla w_j \tau.$$

$$WF : \int_0^T \langle \partial_{tt} u, w_j \rangle \tau + \int_{\Omega} A \nabla u \nabla w_j \tau + bu w_j \tau + \mathbf{c} \nabla u w_j \tau - \mathbf{d} \nabla w_j \tau = \int_0^T \int_{\Omega} f w_j \tau.$$

TODO?

└ TODO!!! □

8 Semigrupy

Definice 8.1 (Značení)

$$\mathcal{L}(X) := \{L : X \rightarrow X \mid L \text{ lineární omezený operátor}\}, \quad \|L\|_{\mathcal{L}} = \sup_{\|x\| < 1} \|Lx\|_X.$$

Dvojice $(A, D(A))$ je neomezený operátor, kde $D(A) \subset X$ je definiční obor A – podprostor X . $A : D(A) \rightarrow X$ je lineární.

Definice 8.2 (Semigrupa (jednparametrická lineární semigrupa))

$S(t) : [0, \infty] \rightarrow \mathcal{L}(X)$ se nazývá semigrupa \equiv

- $S(0) = \text{id}$, neboli $S(0)x = x \ \forall x \in X$;
- $\forall t, s \geq 0 : S(t)S(s) = S(t+s)$.
- Pokud navíc $S(t)x \rightarrow x$ pro $t \rightarrow 0_+$, pak $S(t)$ nazýváme C_0 -semigrupa.

Lemma 8.1

Nechť $S(t)$ je C_0 -semigrupa. Potom

1. $\exists M \geq 1 \ \exists \omega \geq 0 \ \forall t \in [0, \infty) : \|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}$;

2. $\forall x$ pevné $t \mapsto S(t)x$ je spojité zobrazení z $[0, \infty)$ do X .

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Důkaz

Krok 1): „ $\exists M \exists \delta > 0 \forall t \in [0, \delta] : \|S(t)\|_{\mathcal{L}(X)} \leq M$ “: Pro spor nechť toto neplatí. Tedy $\exists t_n \rightarrow 0_+ : \|S(t_n)\|_{\mathcal{L}(X)} \rightarrow \infty$. Víme, že $\forall x : S(t_n)x \rightarrow x$. To implikuje $\forall x \sup_{t_n} \|S(t_n)x\|_X < \infty$. Z Principu stejnoměrné omezenosti (Věta Banach-Steinhaus, z úvodu do funkcionaly) $\exists M > 0 \|S(t_n)\|_{\mathcal{L}(X)} \leq M$. \nexists .

„1.“: Definujeme $\omega = \frac{1}{\delta} \ln M$. $t \geq 0 \exists \varepsilon \in (0, \delta) : t = n\delta + \varepsilon$.

$$\|S(t)\|_{\mathcal{L}(X)} = \|S(\delta) \cdot \dots \cdot S(\delta) \cdot S(\varepsilon)\|_{\mathcal{L}(X)} \leq \|S(\delta)\|_{\mathcal{L}(X)}^n \cdot \|S(\varepsilon)\|_{\mathcal{L}(X)} \leq M e^{\omega t}.$$

„2.“: Spojitost v 0_+ plyne z třetího bodu definice semigrupy. Pro $t_0 > 0, t \rightarrow (t_0)_+$:

$$\lim_{h \rightarrow 0_+} S(t_0 + h)x = \lim_{h \rightarrow 0_+} S(t_0)S(h)x = S(t_0) \lim_{h \rightarrow 0} S(h)x = S(t_0)x.$$

Pro $t \rightarrow (t_0)_-$ (bereme h dostatečně malé, aby $t_0 - h > 0$):

$$\|S(t_0 - h)x - S(t_0)x\|_X = \|S(t_0 - h)(x - S(h)x)\|_X \leq \|S(t_0 - h)\|_{\mathcal{L}} \cdot \|x - S(h)x\|_X \rightarrow 0.$$

└

□

Definice 8.3 (Generator of semigroup)

An unbounded operator $(A, D(A))$ is called a generator of $S(t)$ iff

$$Ax := \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h}, \quad D(A) := \left\{ x \in X \mid \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} \text{ exists in } X \right\}.$$

Věta 8.2 (Properties of generator)

Let $(A, D(A))$ be a generator of a C_0 -semigroup $S(t)$. Then

1. $x \in D(A) \implies S(t)x \in D(A) \forall t \geq 0$;
2. $x \in D(A) \implies AS(t)x = S(t)Ax = \frac{d}{dt}(S(t)x) \forall t \geq 0$;
3. $x \in X, t > 0 \implies x_t := \int_0^t S(s)x ds \in D(A), A(x_t) = S(t)x - x$.

┌

Poznámka (Použití)

$u_0 \in D(A) \subseteq X, u(t) := S(t)u_0, 2. \implies \frac{d}{dt}u(t) = \frac{d}{dt}(S(t)u_0) = AS(t)u_0 = Au(t)$. E. g. if $S(t)$ corresponds to the solution operator of $\partial_t u = \Delta u$, then generator $S(t)$ is Laplace.

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Důkaz

$$A(S(t)x) \stackrel{\text{if exists}}{=} \lim_{h \rightarrow 0_+} \frac{S(h)(S(t)x) - S(t)x}{h} = \lim_{h \rightarrow 0_+}$$

TODO!!!

$$\frac{d}{dt}(S(t)x)_{t \rightarrow t_0+} = \lim_{h \rightarrow 0_+} \frac{S(t_0+h)x - S(t_0)x}{h} = S(t_0) \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} = S(t_0)Ax.$$

$$\frac{d}{dt}(S(t)x)_{t \rightarrow t_0-} = \lim_{h \rightarrow 0_+} \frac{S(t_0-h)x - S(t_0)x}{h} = \lim_{h \rightarrow 0_+} \left(S(t_0-h) \left[\frac{x - S(h)x}{h} - S(h)Ax \right] \right) + \lim_{h \rightarrow 0_+} S(t_0-h)S$$

because

$$\begin{aligned} \|S(t_0-h) \left(\frac{x - S(h)x}{h} - S(h)Ax \right)\|_X &\leq \|S(t_0-h)\|_{\mathcal{L}(X)} \cdot \left\| \frac{x - S(h)x}{h} - Ax + Ax - S(h)Ax \right\|_X \leq \\ &\leq Me^{\omega t_0} \left(\left\| \frac{S(h)x - x}{h} - Ax \right\|_X + \|Ax - S(h)Ax\|_X \right) \rightarrow 0. \end{aligned}$$

3.)

$$\begin{aligned} \frac{S(h)x_t - x_t}{h} &= \frac{S(h) \int_0^t S(s)x ds - \int_0^t S(s)x ds}{h} = \frac{\int_0^t S(h+s)x ds - \int_0^t S(s)x ds}{h} = \\ &= \frac{\int_h^{t+h} S(s)x ds - \int_0^t S(s)x ds}{h} = \frac{1}{h} \int_t^{t+h} S(s)x ds - \frac{1}{h} \int_0^h S(s)x ds. \end{aligned}$$

└

□

Definice 8.4 (Closed operator)

We say that $(A, D(A))$ is closed iff

$$\begin{aligned} (u_n \in D(A), \quad u_n \rightarrow u \text{ in } X, \quad A(u_n) \rightarrow v \text{ in } X) \\ \implies u \in D(A), \quad Au = v. \end{aligned}$$

Věta 8.3 (Density and closedness of generator)

Let $(A, D(A))$ be a generator to a C_0 -semigroup $S(t)$ in X . Then $D(A)$ is dense in X and $(A, D(A))$ is closed.

┌ *Důkaz*

$x \in X$ arbitrary $\implies x_t \in D(A)$.

$$\left[\frac{x_t}{t}\right] = \frac{1}{t} \int_0^t S(s)x ds \rightarrow x \implies D(A) \text{ is dense in } X.$$

$x_n \in D(A)$, $x_n \rightarrow x$ in X , $Ax_n \rightarrow y$ in X . We want $x \in D(A)$ and $Ax = y$.

$$S(h)x_n - S(0)x_n \int_0^t \frac{d}{dt}(S(t)x_n)dt = \int_0^h S(t)(Ax_n)dt$$

$$n \rightarrow \infty : \frac{S(h)x - x}{h} = \int_0^h S(t)y dt.$$

$$A(x) = \lim_{h \rightarrow 0_+} \frac{S(h)x - x}{h} = \lim_{n \rightarrow \infty} \int_0^h S(t)y dt = S(0)y = y.$$

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□

Poznámka

We have $A(\int_0^t S(s)x ds) = S(t)x - x = \int_0^t AS(t)x dt$.

Lemma 8.4 (Uniqueness of $S(t)$)

Let $S(t)$ and $\tilde{S}(t)$ be C_0 semigroup with the same generator. Then $S(t) = \tilde{S}(t) \forall t \geq 0$.

┌ *Důkaz*

$$y(t) = S(T-t)\tilde{S}(t)x, \quad x \in D(A), \quad T > 0 \text{ fixed.}$$

$$\frac{d}{dt}y(t) = -S(T-t)A\tilde{S}(t)x + S(T-t)A\tilde{S}(t)x = 0.$$

$$y(t) = y(0) = y(T) \implies S(T)x = \tilde{S}(T)x.$$

An $D(A)$ is dense, so $S = \tilde{S}$ on X .

└

□

Definice 8.5 (Resolvent)

Let $(A, D(A))$ be unbounded operator. We define resolvent set $\varrho(A) := \{\lambda \in \mathbb{R} | (\lambda I - A) : D(A) \rightarrow X \text{ is on}\}$
 $\forall \lambda \in \varrho(A)$ we define the resolvent operator

$$R(\lambda, A) := (\lambda I - A)^{-1} : X \rightarrow D(A).$$

┌ *Poznámka*

$(A, D(A))$ closed $\implies R(\lambda, A)$ is continuous $\implies R(\lambda, A) \in \mathcal{L}(X)$.

└

Lemma 8.5 (Properties of $R(\lambda, A)$)

Let $(A, D(A))$ be a generator of C_0 -semigroup $S(t)$ and let $\|S(t)\| \leq Me^{\omega t}$.

1. $AR(\lambda, A)x = \lambda R(\lambda, A)x - x \quad \forall x \in X.$
2. $R(\lambda, A)Ax = \lambda R(\lambda, A)x - x \quad \forall x \in D(A).$
3. $R(\lambda, A)x - R(\mu, A)x = (\mu - \lambda)R(\lambda, A)R(\mu, A)x.$
4. $\forall \lambda > \omega \quad \lambda \in \varrho(A): R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x dt \quad \text{and} \quad \|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}.$

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Důkaz

„1.“: $AR(\lambda, A)x = [(A - \lambda I) + \lambda I] R(\lambda, A)x = -x + \lambda R(\lambda, A)x.$ „2. + 3.“: obdobně.

„4.“: Rescale and define $\tilde{S}(t) = e^{-\omega t} S(t)$. We will prove the result for $\tilde{S}(t)$ and transform it to $S(t)$:

$$\|\tilde{S}(t)\| \leq e^{-\omega t} \|S(t)\| \leq Me^{0 \cdot t} = M.$$

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