# 1 Banach algebras

## 1.1 Basic properties

**Definice 1.1** (Algebra)

$(A, +, -, 0, \cdot_S, \cdot)$ is algebra over $\mathbb{K}$, if

- $(A, +, -, 0, \cdot_S)$ is vector space over $\mathbb{K}$;
- $(A, +, -, 0, \cdot)$ is ring (that is we have $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$);
- $\forall \lambda \in \mathbb{K}\ \forall x, y \in A : \lambda(x \cdot y) = (\lambda x)y = x(\lambda y)$.

*Důsledek*
1) $e \in A$ is left unit $\equiv e \cdot a = a$, right unit $\equiv a \cdot e = a$, unit $\equiv a \cdot e = e \cdot a = a$ $(\forall a \in A)$.

If $e_1$ is left unit and $e_2$ is right unit, then $e_1 = e_2$ is unit. $(e_1 = e_1 \cdot e_2 = e_2)$

2) (Algebra) homomorphism $\varphi : A \to B \equiv \varphi$ preserves $+, \cdot, \cdot_S$, that is $\varphi(x + y) = \varphi(x) + \varphi(y)$, $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ and $\varphi(\alpha \cdot x) = \alpha \cdot \varphi(x)$.

**Tvrzení 1.1**

*Let $A$ be algebra over $\mathbb{K}$. Put $A_e = A \times \mathbb{K}$ with operations $A_e$ defined coordinate-wise and multiplication defined by*

$$(a, \alpha) \cdot (b, \beta) := (a \cdot b + \alpha \cdot b + \beta \cdot a, \alpha \cdot \beta), \qquad a, b \in A \wedge \alpha, \beta \in \mathbb{K}.$$

*Then $A_e$ is algebra with a unit $(\mathbf{o}, 1)$ and $A \equiv A \times \{0\} \subset A_e$. Moreover, if $A$ is commutative, then $A_e$ is commutative.*

*Důkaz*
We have $A_e$ is vector space (from linear algebra). We easy proof from definition, that $A_e$ is algebra, $(\mathbf{o}, 1)$ is a unit in $A_e$ and on $A \times \{0\}$ we have $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, so $a \mapsto (a, 0)$ is homomorphism. Commutativity is easy too. □

**Definice 1.2** (Normed algebra)

$(A, \|\cdot\|)$ is normed algebra $\equiv A$ is algebra and $(A, \|\cdot\|)$ is NLS and $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ $(\forall a, b \in A)$.

**Definice 1.3** (Banach algebra)

$(A, \|\cdot\|)$ is Banach algebra $\equiv$ $(A, \|\cdot\|)$ is normed algebra nad Banach space.

*Například*

$l_\infty(I)$ is commutative Banach algebra with a unit (all ones).

If $T$ is Hausdorff topological space, then

$$\mathcal{C}_b(T) = \{f : T \to \mathbb{K} | f \text{ is continuous and bounded}\} \subseteq l_\infty(T)$$

is closed subalgebra.

If $T$ is locally compact, Hausdorff, not compact. Then

$$\mathcal{C}_0(T) = \{f : T \to \mathbb{K} \text{ continuous } | \forall \varepsilon > 0 : \{t \in T | |f(t)| \geqslant \varepsilon\} \, is \text{ compact}\} \subseteq \mathcal{C}_b(T)$$

is closed subalgebra, which doesn't have unit.

If $X$ is Banach, $\dim X > 1$, then $\mathcal{L}(X)$, with $S \cdot T := S \circ T$, $S, T \in \mathcal{L}(X)$, is Banach algebra with unit (identity), which isn't commutative.

If $X$ is Banach, $\dim X = +\infty$, then $\mathcal{K}(X) \subset \mathcal{L}(X)$ is closed subalgebra which is not commutative and doesn't have unit.

$(L_1(\mathbb{R}^d), *)$, where $*$ is convolution, is (commutative) Banach algebra (without unit).

$(l_1(\mathbb{Z}), *)$, where $x * y(n) = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}$ is (commutative) Banach algebra (with unit).

**Tvrzení 1.2**

*If $(A, \|\cdot\|)$ is normed algebra, then $\cdot : A \oplus_\infty A \to A$ is Lipschitz on bounded sets.*

*Důkaz*

$$\forall r > 0 : \forall (a, b) \in B_{A \oplus_\infty A}(\mathbf{o}, r) \ \forall (c, d) \in B_{A \oplus_\infty A}(\mathbf{o}, r) : \|ab - cd\| \leqslant$$

$$\leqslant \|a(b-d)\| + \|(a-c) \cdot d\| \leqslant \|a\| \cdot \|b-d\| + \|a-c\| \cdot \|d\| \leqslant R \cdot (\|b-d\| + \|a-c\|) \leqslant 2R \|(a, b) - (c, d)\|.$$

□

**Tvrzení 1.3**

*Let $(A, \|\cdot\|)$ be a Banach algebra. On $A_e$ we consider the norm*

$$\|(a, \alpha)\| := \|a\| + |\alpha|, \qquad (a, \alpha) \in A \times \mathbb{K} = A_e.$$

*Then $(A_e, \|\cdot\|)$ is Banach algebra.*

*Důkaz*
It is a Banach space, because $A_e = A \oplus_1 \mathbb{K}$. Now we need only check, that

$$\|(a,\alpha)\cdot(b,\beta)\| \leqslant \|(a,\alpha)\|\cdot\|(b,\beta)\|,$$

which is easy. □

*Poznámka*
There is more (natural) ways to define norm on $A_e$ (unlike $\cdot$ on $A_e$, which is natural).

$A$ has a unit … we may still consider $A_e$.

If $e \in A \setminus \{\mathbf{o}\}$ is a unit, then $\|e\| \geqslant 1$, because $\|e\| = \|e^2\| \leqslant \|e\|^2$.

## Věta 1.4

*Let $A$ be a Banach algebra, for $a \in A$ consider $L_a \in \mathcal{L}(A)$ defined as $L_a(x) := a\cdot x$, $x \in A$. Then $I : A \to \mathbb{L}(A)$, $a \mapsto L_a$ is continuous algebra homomorphism, $\|I\| \leqslant 1$.*

*Moreover, if $A$ has a unit $e$, then $I$ is isomorphism into and $I(e) = \mathrm{id}$.*

*If $\|x^2\| = \|x\|^2$, $x \in A$, then $I$ is isometry into.*

*Důkaz*
„$L_a \in \mathcal{L}(A)$ and $I \in \mathcal{L}(A, \mathcal{L}(A))$, $\|I\| \leqslant 1$“: Linearity is obvious, $\|L_a(x)\| = \|a\cdot x\| \leqslant \|a\|\cdot\|x\|$, so $\|L_a\| \leqslant \|a\|$ and so $\|I\| \leqslant 1$. Since it is easily $I$ preserves multiplication, so we are left to prove the „Moreover“ part.

„$A$ has a unit $e$“: WLOG $A \neq \{\mathbf{o}\}$.

$$\forall a \in A : \|Ia\| = \|L_a\| \geqslant \|L_a\left(\frac{e}{\|e\|}\right) = \frac{a}{\|e\|} = \frac{1}{\|e\|}\cdot a.$$

So $I$ is bounded from below, so $I$ is isomorphism.

$$I(e)(x) = L_e(x) = x, \text{ so } I(e) = \mathrm{id}\,.$$

Finally, if $\|x^2\| = \|x\|^2$, $x \in A$, then $\forall a \in A$ :

$$\|a\| \geqslant \|I(a)\| = \|L_a\| \geqslant \|L_a\left(\frac{a}{\|a\|}\right)\| = \frac{\|a^2\|}{\|a\|} = \|a\|.$$

So $I$ is isometry. □

*Poznámka*
$A \neq \{\mathbf{o}\}$ Banach algebra with a unit $\implies \exists$ equivalent norm $\|\cdot\|$ on $A$ such that $(A, \|\cdot\|)$ is Banach algebra and $\|e\| = 1$.

*Důkaz*
Let $I : A \to \mathcal{L}(A)$ be as before. Put $|\|x\|| := \|I(x)\|$, $x \in A$. Since $I$ is isomorphism, $|\|\cdot\||$ is equivalent norm. Moreover, $|\|x \cdot y\|| = \|I(x \cdot y)\| \leqslant \|I(x)\| \cdot \|I(y)\| = |\|x\|| \cdot |\|y\||$, $x, y \in A$. So $(A, |\|\cdot\||)$ is a Banach algebra. Finally

$$|\|e\|| = \|I(e)\| = \|\operatorname{id}\| = 1.$$

□

## 1.2 Inverse elements

### Definice 1.4

$(M, \cdot, e)$ is monoid ($\cdot$ is associative, $e$ is unit). Then invertible elements form a group ($e^{-1} = e, \exists x^{-1}, y^{-1} \implies (x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$); if $x \in M$, and $y \in M$ is its left inverse and $z \in M$ is its right inverse, then $y = z$ is inverse:

$$y = y \cdot e = y \cdot x \cdot z = e \cdot z = z.$$

We denote $M^{\times} := \{x \in M \mid \exists x^{-1}\}$

### Tvrzení 1.5

*If $(A, \cdot, e)$ is monoid and $x_1, \ldots, x_n \in A$ commute, then $x_1 \cdot \ldots \cdot x_n \in A^x \Leftrightarrow \{x_1, \ldots, x_n\} \subset A^x$.*

*Důkaz*
It suffices to prove it for $n = 2$ (and use induction). „If $x^{-1}$ and $y^{-1}$ exists, then $(xy)^{-1}$“ is easy from asociativity.

If we have $(xy)^{-1}$. Put $z := (xy)^{-1}x$. Then $zy = (xy)^{-1}(xy) = e$, so $z$ is left inverse to $y$. Next we show that there is also right inverse: Put $\tilde{z} := x(xy)^{-1}$: $y\tilde{z} = (xy)(xy)^{-1} = e$, so $\tilde{z}$ is right inverse. And we already know that if there is left and right inverse, then they are same and they are inverse. □

### Lemma 1.6

*Let $A$ be a Banach algebra with a unit.*

- $\|x\| < 1 \implies \exists (e-x)^{-1} \wedge (e-x)^{-1} = \sum_{n=0}^{\infty} x^n$;
- $\exists x^{-1} \wedge \|h\| < \frac{1}{\|x^{-1}\|} \implies \exists (x+e)^{-1} \wedge \|(x+h)^{-1} - x^{-1}\| \leqslant \frac{\|x^{-1}\|^2 \cdot \|h\|}{1 - \|x^{-1}\| \cdot \|h\|}$.

*Důkaz*
„First“: We have $\|x^n\| \leqslant \|x\|^n$, so $\sum_{n=0}^{\infty} x^n$ is absolute convergent series, so $\sum_{n=0}^{\infty} x^n \in A$. Moreover,

$$(e-x)\cdot\left(\sum_{n=0}^{\infty} x^n\right) = \lim_{N\to\infty}(e-x)\cdot(e+x+\ldots+x^N) = \lim_{N\to\infty} e - x^{N+1} = e,$$

because $\lim_{N\to\infty}\|x^{n+1}\| \leqslant \lim_{N\to\infty}\|x\|^N = 0$. And similarly $\left(\sum x^n\right)\cdot(e-x) = e$.

„Second item“: $x+h = x\cdot(e+x^{-1}h)$ we have $x^{-1}$ exists and $(e+x^{-1}h)^{-1}$ exists (from first item), so from previous fact $(x+h)^{-1}$ exists. Moreover

$$(x+h)^{-1} = (e+x^{-1}h)^{-1}\cdot x^{-1} \overset{1)}{=} \sum_{n=0}^{\infty}\left(-x^{-1}h\right)^n x^{-1},$$

so

$$\|(x+h)^{-1}-x^{-1}\| = \|\sum_{n=1}^{\infty}\left(-x^{-1}h\right)^n x^{-1}\| \leqslant \|x^{-1}\|\cdot\sum_{n=1}^{\infty}\|x^{-1}h\|^n \leqslant$$

$$\leqslant \|x^{-1}\|\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\|x^{-1}\|\cdot\|h\|\right)^n = \|x^{-1}\|\cdot\frac{\|x^{-1}\|\|h\|}{1-\|x^{-1}\|\cdot\|h\|}.$$

□

*Důsledek*
$A$ Banach algebra with a unit $\implies$ $A^x \subset A$ is open and $A^x$ is topological group.

*Důkaz*
$A^x \subset A$ is open by previous lemma (second item). So it remains to prove $x \mapsto x^{-1}$ is continuous:

$$A^x \ni x_n \to x \in A^x \overset{?}{\implies} x_n^{-1} \to x^{-1}.$$

$$\|x_n^{-1}-x^{-1}\| \overset{h:=x_n-x}{\leqslant} \frac{\|x^{-1}\|^2\cdot\|x_n-x\|}{1-\|x^{-1}\|\cdot\|x_n-x\|} \to 0.$$

□

## 1.3 Spectral theory

**Definice 1.5** (Resolvent set, spectrum and resolvent)

Let $A$ be a Banach algebra with a unit, $x \in A$. We define resolvent set of $x$ as $\varrho_A(x) := \{\lambda \in \mathbb{K} | \exists(\lambda\cdot e - x)^{-1}\}$. Next we define spectrum of $x$ as $\sigma_A(x) := \mathbb{K}\backslash\varrho_A(x)$. Finally we define resolvent of $x$ as $R_x : \varrho(x) \to A$, $R_x(\lambda) := (\lambda\cdot e - x)^{-1}$.

If $A$ doesn't have a unit, then notions above are defined with respect to $A_e$.

**Tvrzení 1.7**

*A Banach algebra*

a) $\forall x \in A : 0 \in \sigma_{A_e}(x)$ *(in particular, if A has no unit, then* $0 \in \sigma_A(x)$*);*

b) *A has unit* $\implies \sigma_{A_e}(x) = \sigma_A(x) \cup \{0\}$.

*Důkaz* (a))

$$\forall (b, \beta) \in A_e : (x, 0) \cdot (b, \beta) = (\ldots, 0) \neq (\mathbf{o}, 1) \implies \nexists (x, 0)^{-1} \implies 0 \in \sigma_{A_e}(x).$$

□

*Důkaz* (b))

By a) we have $0 \in \sigma_{A_e}(x)$. So it suffices: $\forall \lambda \neq 0 : \lambda \in \varrho_A(x) \Leftrightarrow \lambda \in \varrho_{A_e}(x)$. First means $(\lambda \cdot e - x)^{-1}$ exists in $A$ and second means that $((0, \lambda) - (x, 0))^{-1} = (-x, \lambda)^{-1}$ exists in $A$. We take „$x \to -x$“.

„ $\implies$ “: find $(b, \beta) \in A_e$ such that $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$. So $(x \cdot b + \lambda \cdot b + \beta \cdot x, \lambda \cdot \beta) = (\mathbf{o}, 1)$. So $\beta = \frac{1}{\lambda}$ and $b = -\frac{1}{\lambda}(\lambda e + x)^{-1} \cdot x$. Similarly we find left inverse $\left(-\frac{1}{\lambda} x (x + \lambda e)^{-1}, \frac{1}{\lambda}\right)(x, \lambda)$. And next we prove that they are really inverses.

„ $\impliedby$ “: Put $(b, \beta) := (x, \lambda)^{-1}$. Then $(\lambda e + x)^{-1} = b + \beta \cdot e$. We have $(x, \lambda) \cdot (b, \beta) = (\mathbf{o}, 1)$, so $\lambda \cdot \beta = 1$ and $x \cdot b + \lambda \cdot b + \beta \cdot x = \mathbf{o}$. Then

$$(\lambda e + x) \cdot (b + \beta \cdot e) = \lambda \cdot b + \lambda \cdot \beta \cdot e + x \cdot b + \beta \cdot x = e.$$

Similarly second inverse. □

**Věta 1.8**

$\{\mathbf{o}\} \neq A$ *complex Banach algebra,* $x \in A$*. Then* $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ *is compact, nonempty.*

*Důkaz*

After theory. □

**Definice 1.6** (Derivative)

$Y$ Banach space, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is the derivative of $f$ at $a$.

**Tvrzení 1.9** (Fact)

*$Y$ Banach, $\Omega \subset \mathbb{K}$, $f : \Omega \to Y$, $a \in \Omega$. Then $f'(a)$ exists $\implies$ $f$ is continuous at $a$ $\wedge$ $\forall x^* \in Y^*$: $(x^* \circ f)'(a) = x^*(f'(a))$.*

*Důkaz*

Continuity: $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a) = f'(a) \cdot 0 = 0$.

$x^* \in Y^*$ given, then

$$\lim_{x\to a} \frac{x^*(f(x)) - x^*(f(a))}{x-a} = \lim_{x\to a} x^* \left( \frac{f(x)-f(a)}{x-a} \right) = x^*(f'(a)).$$

□

**Tvrzení 1.10**

*$A$ Banach algebra with a unit, $x \in A$. Then*

- *$\varrho(x)$ is open set;*
- *$\forall |\lambda| > \|x\| : \lambda \in \varrho(x) \wedge R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$;*
- *(important!) $\varrho(x) \ni \lambda \mapsto R_x(\lambda)$ has derivative at each $\lambda \in \varrho(x)$;*
- *$\forall \mu, \nu \in \varrho(x) : R_x(\mu) \cdot R_x(\nu) = R_x(\nu) \cdot R_x(\mu)$;*
- *$\forall \mu, \nu \in \varrho(x) : R_x(\mu) - R_x(\nu) = (\nu - \mu) \cdot R_x(\mu) \cdot R_x(\nu)$.*

*Důkaz*

First is proved by lemma. Second by lemma we have

$$(\lambda e - x)^{-1} = \lambda^{-1} \left( e - \frac{x}{\lambda} \right)^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left( \frac{x}{\lambda} \right)^n.$$

Fourth: In general $uv = vu \implies u^{-1}v^{-1} = v^{-1}u^{-1}$ (proof: $u^{-1}v^{-1} = (vu)^{-1}$). And we apply it for $u = (\mu e - x)$ and $v = (\nu e - x)$.

Fifth: In general $u \cdot v = v \cdot u \implies u^{-1} \cdot v = v \cdot u^{-1}$ (proof: $u^{-1}v = v \cdot v^{-1}u^{-1}v = v \cdot u^{-1}v^{-1}v = vu^{-1}$) so:

$$R_x(\mu) - R_x(\nu) = R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))^{-1}R_x(\nu) =$$

$$= R_x(\mu)R_x(\nu)(R_x(\nu)^{-1}) - R_x(\mu)(R_x(\mu))R_x(\nu)^{-1} =$$
$$= R_x(\mu)R_x(\nu) \left( R_x(\nu)^{-1} - R_x(\mu)^{-1} \right) = R_x(\mu)R_x(\nu)(\nu - \mu).$$

□

*Důkaz*
For third we fix $\lambda \in \varrho(x)$ and $t \in (0,\delta)$ for $\delta$ small enough ($\lambda + t \in \varrho(x)$ and *). We shall prove that „$R_x'(\lambda) = -R_x(\lambda)^2$":

$$0 \stackrel{?}{=} \left\| \frac{R_x(\lambda+t) - R_x(\lambda)}{t} + R_x(\lambda)^2 \right\| =$$

$$= \frac{1}{|t|} \left\| (\lambda e - x + te)^{-1} - (\lambda e - x)^{-1} + (\lambda e - x)^{-1} \cdot t \cdot (\lambda e - x)^{-1} \right\| \leqslant$$

$$\stackrel{*\text{ for existence of the inverse}}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| (e + t(\lambda e - x)^{-1})^{-1} - e + (\lambda e - x)^{-1} \cdot t \right\| =$$

$$= \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \left\| \sum_{n=0}^{\infty} (-t)^n (\lambda e - x)^{-n} - e + (\lambda e - x)^{-1} \cdot t \right\| \leqslant$$

$$\stackrel{\|x^n\| \leqslant \|x\|^n}{\leqslant} \frac{1}{|t|} \|(\lambda e - x)^{-1}\| \cdot \sum_{n=2}^{\infty} \|t(\lambda e - x)^{-1}\|^n =$$

$$= \frac{1}{|t|} \|t(\lambda e - x)^{-1}\| \cdot \frac{\|t(\lambda e - x)^{-1}\|^2}{1 - \|t(\lambda e - x)^{-1}\|} \stackrel{*\text{ for denominator } \leqslant 1/2}{\leqslant} \frac{2|t|^2}{|t|} \|t(\lambda e - x)^{-1}\| \to 0.$$

□

**Věta 1.11** (Liouville for Banach space valued functions)

*$Y$ Banach space over $\mathbb{C}$, $f : \mathbb{C} \to Y$ has derivative at each point, $f$ is bounded ($\equiv \|f\|$ is bounded). Then $f \equiv$ const.*

*Důkaz*
Assume $f \not\equiv$ const, so there are $a \neq b \in \mathbb{C} : f(a) \neq f(b) \implies$ (by Hahn–Banach theorem) $\exists x^* \in Y^* : x^*(f(x)) \neq x^*(f(x))$. From fact $x^* \in f : \mathbb{C} \to \mathbb{C}$ has derivative at each point is bounded, not constant which is in contradiction with Liouville theorem for complex valued functions. □

*Důkaz* (Theorem before theory)
First case: „$A$ has a unit": Then $\sigma(x) \subseteq B_{\mathbb{C}}(0, \|x\|)$ is closed, so $\sigma(x)$ is compact. Assume that $\varrho(x) = \mathbb{C}$. By previous tvrzeni we have $R_x : \mathbb{C} \to A$ has derivative everywhere, and it is bounded because $\lim_{|} \lambda| \to \infty R_x(\lambda) = \lim_{|\lambda| \to \infty} \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}} = 0$. From previous theorem $R_x \equiv$ const so $\lim_{|\lambda| \to \infty} R_x(\lambda) = 0 \implies R_x \equiv 0$. In particular $0 = R_x(0) = (-x)^{-1}$. ↯(If $A \neq \{0\}$ then $x^{-1} \neq 0$ for $x \in A$.)

Second case: „$A$ hasn't a unit", then $\sigma(x) := \sigma_{A_e}((x, 0))$ so we apply the already proven case. □

*Poznámka* (Convention)
If not said otherwise, in chapter about Banach algebras, all Banach spaces are complex.

**Věta 1.12** (Gelfand–Mazur)

*$\{\mathbf{o}\} \neq A$ Banach algebra with a unit. Assume $\forall x \in A \setminus \{\mathbf{o}\} : \exists x^{-1}$. Then $A$ is isomorphic to $\mathbb{C}$. If moreover $e$ is a unit in $A$ and $\|e\| = 1$, then $A$ is isometrically isomorphic to $\mathbb{C}$.*

*Důkaz*
Consider $\psi : \mathbb{C} \to A$ defined as $\psi(\lambda) := \lambda \cdot e$. This is algebraic homomorphism and $\|\psi(\lambda)\| = |\lambda| \cdot \|e\|$, so it is isomorphism (and isometry, if $\|e\| = 1$).

It remains „$\varphi$ is surjective“: Pick $a \in A$. From previously proved theorem $\exists \lambda \in \sigma(a)$, then $(\lambda e - a) \notin A^x$. So, $\lambda \cdot e - a = 0$, then $\psi(\lambda) = a$. □

**Definice 1.7** (Spectral radius)

A Banach algebra, $x \in A$. Then $r(x) := \sup \{|\lambda|, \lambda \in \sigma(x)\}$ is called spectral radius of $x$.

**Věta 1.13** (Beurling–Gelfand)

*$A$ Banach algebra, $x \in A \implies r(x) = \inf_{n\in\mathbb{N}} \sqrt[n]{\|x^n\|} = \lim_n \sqrt[n]{\|x^n\|}$.*

**Lemma 1.14**

*$A$ Banach algebra with a unit, $x \in A$. For $p(z) = \sum_{j=1}^n \alpha_j z^j \in \mathbb{C}$ a polynom (with complex coefficients) we put $p(x) = \sum_{j=1}^n \alpha_j x^j \in A$. Then $\sigma(p(x)) = p(a(x))$.*

*Důkaz*
Fix $\lambda \in \mathbb{C}$ and write $(\lambda - p)(z) = c \cdot \prod_{i=1}^m (z - z_i)$, where $z_1, \dots, z_m$ are roots of $\lambda - p$. Then $\lambda \in \sigma(p(x)) \Leftrightarrow (\lambda e - p(x))^{-1}$ does not exists. $(\lambda e - p(x))^{-1} = c \cdot \prod_{i=1}^m (x - z_i \cdot e)$, so it does'nt exists if and only if $\exists i \in [m]$, such that $(x - z_i \cdot e)^{-1}$ doesn't exists $\Leftrightarrow z_i \in \sigma(x) \Leftrightarrow$ $\exists$ root $\nu$ of $\lambda - p$ such that $\nu \in \sigma(x) \Leftrightarrow \exists \nu \in \sigma(x) : p(\nu) = \lambda \Leftrightarrow \lambda \in p(\sigma(x))$. □

*Důkaz* (Beurling–Gelfand)
WLOG $A$ has a unit. Step 1, „$r(x) \leqslant \inf_n \sqrt[n]{\|x^n\|}$“: fix $\lambda \in \sigma(x)$. By previous lemma $\forall n : \lambda^n \in \sigma(x^n)$. By theorem 'Before theory' we have $\forall n : |\lambda|^n \leqslant \|x^n\|$.

Step 2, „$r(x) \geqslant \limsup_n \sqrt[n]{\|x^n\|}$“: Pick $r > r(x)$. Claim: „$\frac{x^n}{r^n} \to^w 0$“: Fix $x^* \in A^*$ and put $f(\lambda) := \lambda \cdot x^*(R_x(\lambda))$. By fact and tvrzeni after it, $f$ has derivative at each $\lambda \in \varrho(x)$. Moreover for $|\lambda| \geqslant \|x\|$ we have $f(\lambda) = \lambda \cdot x^* \left(\sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}}\right) = \sum_{n=0}^\infty \frac{x^*(x^n)}{\lambda^n}$. Thus $f(\lambda) = \sum_{n=0}^\infty \frac{x^*(x^n)}{\lambda^n}$, $\lambda \in P(0, r(x), \infty)$. From Complex analysis $f \in H(P(0, r, \infty))$ is uniquely given by Laurent series. In particular $f(r) = \sum_{n=0}^\infty \frac{x^*(x^n)}{r^n}$, so $x^* \left(\frac{x^n}{r^n}\right) \to 0$.

From princip of unique boundedness (last semester): $\frac{x^n}{r^n}$ if $\|\cdot\|$-bounded, so $\exists c > 0$ : $\|x^n\| \leqslant cr^n$, $\sqrt[n]{\|x^n\|} \leqslant \sqrt[n]{c} \cdot r \to r$. So $\limsup \sqrt[n]{\|x^n\|} \leqslant r$. □

*Důsledek*
$A$ Banach algebra, $x \in A$ and $|\lambda| > r(x)$. Then $\sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}$ is absolutely convergent and $R_x(\lambda) = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$.

*Důkaz*
Fix $q$, such that $\frac{r(x)}{|\lambda|} < q < 1$. By previous theorem, $\exists n_0 \ \forall n \geqslant n_0 : \frac{\sqrt[n]{\|x^n\|}}{\lambda} < q$, so $\frac{\|x^n\|}{|\lambda|^n} < q^n$, $n \geqslant n_0$. Thus $\sum \left\| \frac{x^n}{\lambda^n} \right\| \leqslant \infty$, so the sum is absolutely convergent.

Now we easily check that $(\lambda e - x)^{-1} = \sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$. □

## 1.4 Subalgebra

**Věta 1.15**

*$A$ Banach algebra with a unit $e$, $B \subset A$ is closed subalgebra such that $e \in B$. Fix $x \in B$. Then*

- *$C \subset \varrho_A(x)$ is component (maximum connected subset) $\implies$ $C \subseteq \sigma_B(x)$ or $C \cap \sigma_B(x) = \varnothing$;*
- *$\partial \sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$;*
- *$\varrho_A(x)$ is connected $\implies$ $\sigma_A(x) = \sigma_B(x)$;*
- *$\operatorname{int} \sigma_B(x) = \varnothing \implies \sigma_A(x) = \sigma_B(x)$.*

*Důkaz*
„$\sigma_A(x) \subseteq \sigma_B(x)$“: $(\lambda e - x)^{-1}$ exists in $B$ implies it exists (it's same) in $A$.

„First item“: Let $C \subset \varrho_A(x)$ be component. Pick $\lambda_0 \in C \cap \sigma_B(x)$. Wanted: „$C \backslash \sigma_B(x) = \varnothing$“. Pick $x^* \in A^* : x^*|_B = 0 \wedge x^*(R_x(\lambda)) = 1$ (separate $B$ and $R_x(\lambda) \notin B$). Then $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is holomorphic function on open (because maximum) connected set $C$. Which is zero[a] on $C \backslash \sigma_B(x)$.

Since $C \backslash \sigma_B(x)$ is open, if it is nonempty it contains a ball, so it has cluster point. Thus $C \ni \lambda \mapsto x^*(R_x(\lambda))$ is such that $\{\lambda \in C | x^*(R_x(\lambda))\} = 0$ has a cluster point, so from complex analysis (uniqueness theorem) it is constant zero. ↯with $x^*(R_x(\lambda_0)) = 1$.

„Second item“: Pick $\lambda \in \sigma_B(x) \backslash \sigma_A(x)$ and let $C \subset \varrho_A(x)$ be a component containing $\lambda$. By first item, $C \subseteq \sigma_B(x)$, $C$ is open, so $\lambda \in C \subseteq \operatorname{int}(\sigma_B(x))$. □

[a] For $\lambda \in C \backslash \sigma_B(x)$, $(\lambda e - x)^{-1}$ exists in $B$ so $R_x(\lambda) \in B$ and therefore, $x^*(R_x(\lambda)) = 0$

*Důkaz*
„Third item“: If $\varrho_A(x)$ is connected, we can apply first item to $C = \varrho_A(x)$, we have either $\varrho_A(x) \subseteq \sigma_B(x)$ or $\varrho_A(x) \cap \sigma_B(x) = \emptyset$. But first is not possible, because $\varrho_A(x)$ is unbounded and $\sigma_B(x)$ is bounded. Therefore $\sigma_B(x) \subseteq \sigma_A(x)$.

„Fourth item“: If $\operatorname{int}(\sigma_B(x)) = \emptyset$, then (by second item) $\sigma_B(x) \subseteq \partial\sigma_B(x) \subseteq \sigma_A(x) \subseteq \sigma_B(x)$. □

*Důsledek*
$A$ Banach algebra, $B \subseteq A$ closed subalgebra, $x \in B$. Then all items from previous theorem hold as well if we replace $\sigma_A(x)$ and $\sigma_B(x)$ by $\sigma_A(x) \cup \{0\}$ and $\sigma_B(x) \cup \{0\}$.

*Důkaz*
Without proof. (Basically same that previous; we add unit to $A$ and $B$, so this unit is same $((\mathbf{o}, 1))$, etc.) □

## 1.5 Holomorphic calculus

### Definice 1.8

$X$ Banach, $\gamma : [a, b] \to \mathbb{C}$ path (continuous, piecewise smooth ($C^1$)), $f : \langle\gamma\rangle \to X$ continuous. Then

$$\int_\gamma f := \int_{[a,b]} \gamma'(t) f(\gamma(t)) dt. \qquad \text{(As Bochner integral.)}$$

If $\Gamma = \gamma_1 + \ldots + \gamma_n$ is chain in $\mathbb{C}$, $f : \langle\Gamma\rangle \to X$ continuous, then

$$\int_\Gamma f := \sum_{i=1}^{n} \int_{\gamma_i} f.$$

### Lemma 1.16

*$\Gamma$ chain in $\mathbb{C}$, $X$ Banach, $f : \langle\Gamma\rangle \to X$, $x \in X$. Then*

$$\int_\Gamma f = x \Leftrightarrow \forall x^* \in X^* : x^*(x) = \int_\Gamma x^* \circ f.$$

*Důkaz*
„ $\Longleftarrow$ “ by Hahn–Banach theorem. „ $\Longrightarrow$ “: (by previous semester $x^*$ and $\int$ ”commutes”)

$$x^* \left( \int_\Gamma f \right) = \sum_{i=1}^{n} x^* \left( \int_{\gamma_i} f \right) = \sum_{i=1}^{n} \int_{[a_i,b_i]} \gamma_i'(t) x^*(f(\gamma_i(t))) dt = \int_\Gamma x^* \circ f.$$

□

*Poznámka* (Recall)
If $\Omega \subset \mathbb{C}$ open, $K \subset \Omega$ compact. Then there is a cycle $\Gamma$ such that $\langle\Gamma\rangle \subset \Omega \backslash K$ and $\operatorname{ind}_\Gamma z = 1$ if $z \in K$ and 0 if $z \notin \Omega$.

Then we say that $\Gamma$ circulates $K$ in $\Omega$.

## Definice 1.9

Let $A$ be a Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open and $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $f(x) := \frac{1}{2\pi i}\int_\Gamma f \cdot R_x$, where is any cycle which circulates $\sigma(x)$ in $\Omega$.

*Poznámka*
$f(x)$ exists ($f \cdot R_x$ is continuous on $\langle\Gamma\rangle$), $f(x)$ does not depend on the choice of $\Gamma$ (Pick $x^* \in X^*$, then $(x^* \circ f \cdot R_x)(\lambda) = f(\lambda) \cdot x^*(R_x(\lambda))$ is holomorphic. Pick $\Gamma_1, \Gamma_2$ cycles circulating $\sigma(x)$ in $\Omega$, then $\int_{\Gamma_1 - \Gamma_2} x^* \circ f \cdot R_x = 0$ from Cauchy).

## Věta 1.17 (Holomorphic calculus)

*A Banach algebra with unit, $x \in A$, $\Omega \subset \mathbb{C}$ open such that $\sigma(x) \subset \Omega$, $f \in \mathcal{H}(\Omega)$. Then $\Phi : \mathcal{H}(\Omega) \to A$ defined as $\Phi(f) = f(x)$ (from definition above) has the following properties:*

- *$\Phi$ is algebra homomorphism, $\Phi(1) = e$, $\varphi(\mathrm{id}) = x$;*
- *$f_n \overset{loc.}{\rightrightarrows} f$ in $H(\Omega)$, then $f_n(x) \to f(x)$;*
- *$f(x)^{-1}$ exists $\Leftrightarrow f \neq 0$ on $\sigma(x)$, in this case $f(x)^{-1} = \frac{1}{f}(x)$;*
- *$\sigma(f(x)) = f(\sigma(x))$;*
- *if $\Omega_1$ is open and $f(\sigma(x)) \in \Omega_1$, $g \in \mathcal{H}(\Omega_1)$, then $(g \circ f)(x) = g(f(x))$;*
- *if $y \in A$ commutes with $x$, then $y$ commutes with $f(x)$.*

*Moreover, if $\psi : \mathcal{H}(\Omega) \to A$ satisfy first two item, then $\psi = \Phi$.*

## Lemma 1.18

*$(\Omega, \mu)$ complete measurable space, $A$ Banach algebra, $f \in L_1(\mu, A)$. Let $x \in A$ and $E \subset \Omega$ is measurable. Then*

$$x \cdot \left(\int_E f(t) d\mu(t)\right) = \int_E x \cdot f(t) d\mu(t), \qquad \left(\int_E f(t) d\mu(t)\right) \cdot x = \int_E f(t) \cdot x d\mu(t).$$

*Důkaz*
Easy (by commutation of integral and linear operator from last semester), skipped. □

*Důkaz* (Holomorphic calculus)
„1st item“: „$\Phi$ is linear“ is easy, „$\Phi$ is multipicative“: Pick $f, g \in \mathcal{H}(\Omega)$, open set $U$ such that $\sigma(x) \subset U \subset \overline{U} \subset \Omega$. Let $\Gamma$ cycle circulating $\sigma(x)$ in $U$, $\Lambda$ cycle circulating $\overline{U}$ in $\Omega$. Then

$$f(x) \cdot g(x) = \left( \frac{1}{2\pi i} \int_\Gamma f \cdot R_x \right) \cdot g(x) \overset{\text{lemma}}{=}$$

$$= \frac{1}{2\pi i} \int_\Gamma f(t) R_x(t) g(x) dt = \frac{1}{2\pi i} \int_\Gamma f(t) \cdot R_x(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(s) ds dt \overset{\text{lemma}}{=}$$

$$= \frac{1}{2\pi i} \int_\Gamma f(t) \cdot \frac{1}{2\pi i} \int_\Lambda g(s) \cdot R_x(t) \cdot R_x(s) ds dt =$$

because $\langle \Lambda \rangle \cap \langle \Gamma \rangle = \varnothing$, we can use theorem after definition of $R_x$:

$$= \frac{1}{(2\pi i)^2} \int_\Gamma \int_\Lambda f(t) \cdot g(s) \cdot \frac{R_x(t) - R_x(s)}{s - t} ds dt \overset{\text{Fubini to } x^*(\dots) \text{ and lemma}}{=}$$

$$= \frac{1}{(2\pi i)^2} \int_\Gamma f(t) \left( \int_\Lambda \frac{g(s)}{s-t} ds \right) R_x(t) dt - \frac{1}{(2\pi i)^2} \int_\Lambda g(s) \left( \int_\Gamma \frac{f(t)}{s-t} \right) R_x(s) ds =$$

(Now we use Cauchy theorem ($f(z) \operatorname{ind}_\Gamma z = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw$). $\forall s \in \langle \Lambda \rangle : (t \mapsto \frac{f(t)}{s-t}) \in \mathcal{H}(U) \wedge \operatorname{ind}_\Gamma z = 0, z \notin U$, so $\int_\Gamma \frac{f(t)}{s-t} dt = 0$. $\forall t \in \langle \Gamma \rangle : \operatorname{ind}_\Lambda t = 1 \wedge (s \mapsto g(s)) \in \mathcal{H}(\Omega) \implies g(t) = \frac{1}{2\pi i} \int_\Lambda \frac{g(s)}{s-t} ds$.)

$$= \frac{1}{2\pi i} \int_\Gamma f(t) g(t) R_x(t) dt - 0.$$

It remains that „if $f(z) = z^k$, $k \in \mathbb{N} \cup \{0\}$ then $f(x) = x^k$“ (we want it for $k = 0$ and $k = 1$). Put $\Gamma(t) = r \cdot e^{it}$, $t \in [0, 2\pi]$, where $r > \|x\|$ arbitrary. By some theorem:

$$R_x(\lambda) = \sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}}, \qquad |\lambda| > \|x\|.$$

Thus (we switch integral and sum, because later we realize that sum of integral of absolute value is finite)

$$\forall x^* \in A^* : x^*(f(x)) = \frac{1}{2\pi i} \int_\Gamma \lambda^k x^* (\sum_{n=0}^\infty \frac{x^n}{\lambda^{n+1}}) d\lambda = \frac{1}{2\pi i} \int_\Gamma \sum_{n=0}^\infty \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{n=0}^\infty \int_\Gamma \frac{x^*(x^n)}{\lambda^{n-k+1}} d\lambda = \frac{1}{2\pi i} \sum_{n=0}^\infty x^*(x^n) \int_\Gamma \frac{1}{\lambda^{n-k+1}} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{n=0}^\infty x^*(x^n) \int_0^{2\pi} i \frac{1}{\Gamma(t)^{n-k}} dt = x^*(x^k) + \sum 0,$$

because $\Gamma$ (is $2\pi$ periodic).

„2nd item“: For $\Gamma = \gamma_1 + \ldots + \gamma_N$:

$$\|f_n(x) - f(x)\| = \frac{1}{2\pi i}\left\|\int_\Gamma (f_n(\lambda) - f(\lambda))R_x(\lambda)d\lambda\right\| \leqslant \frac{1}{2\pi}\int_\Gamma |f_n(\lambda) - f(\lambda)| \cdot \|R_x(\lambda)\|d\lambda \leqslant$$

$$\leqslant \frac{1}{2\pi}\sum_{i=1}^{N}\int_{a_i}^{b_i} |\gamma_i'(t)| \sup_{z\in\langle\Gamma\rangle} |f_n(z) - f(z)| \cdot \|R_x(\gamma_i(t))\|dt =$$

$$= \sup_{z\in\langle\Gamma\rangle} |f_n(z) - f(z)| \cdot \frac{1}{2\pi}\sum_{i=1}^{N}\int_{a_i}^{b_i} \|R_x(\gamma_i(t))\| \cdot |\gamma_i'(t)|dt \to 0.$$

„Moreover part“: By Runge theorem (and second item) it is enough prove it for rational functions. If $R$ was polynom, then $\Phi(R) = \Psi(R)$ by second item. So it suffices „$\forall p$ polynom: $\frac{1}{p} \in \mathcal{H}(\Omega) \implies \Phi(\frac{1}{p}) = \psi(\frac{1}{p})$“. Pick $p$ polynom. Then $e = \psi(1) = \psi(p \cdot \frac{1}{p}) = \psi(p) \cdot \psi(\frac{1}{p}) = \Phi(p) \cdot \psi(\frac{1}{p})$ (similarly for $\frac{1}{p} \cdot p$). So $\psi(\frac{1}{p}) = \Phi(p)^{-1} = \Phi(\frac{1}{p})$.

„3rd item“: „ $\implies$ “ Let $f(z) = 0$ for some $z \in \sigma(x)$. Then exists $g \in H(\Omega) : f(u) = (z-u)g(z)$. By item one, we have $(ze-x)g(x) = f(x) = g(x)(ze-x)$. But $(ze-x)^{-1}$ does not exist, so $f(x)^{-1}$ does not exists.

„ $\impliedby$ “ Suppose $f \neq 0$ on $\sigma(x)$ by compactness. $\exists\Omega_1 \subset \Omega$ open: $\sigma(x) \subset \Omega_1$ and $f \neq 0$ on $\Omega_1$. Then $\frac{1}{f} \in H(\Omega_1)$ and by first item we have $e = (f \cdot \frac{1}{f})(x) = f(x)\frac{1}{f}(x) = \ldots = \frac{1}{f}(x) \cdot f(x) \implies f(x)^{-1} = \frac{1}{f}(x)$. □

*Poznámka*
$f = g$ on a neighbourhood of $\sigma(x) \implies f(x) = g(x)$ (from definition), other implication doesn't hold!

## 1.6 Multiplicative functionals

**Definice 1.10** (Multiplicative functional)

Let $A$ be a Banach algebra. We say $\varphi : A \to \mathbb{C}$ is multiplicative linear functional $\equiv \varphi$ preserves $+, \cdot, \cdot_S$.

$$\Delta(A) := \{\varphi : A \to \mathbb{C} | \varphi \text{ multiplicative linear functional }, \varphi \not\equiv 0\}.$$

**Tvrzení 1.19**

*$A$ Banach algebra, $\varphi \in \Delta(A) \cup \{0\}$. Then*

- $\exists!\tilde{\varphi} \in \Delta(A_e) : \tilde{\varphi}((x, 0)) = \varphi(x), \forall x \in A$. *It is given by* $\tilde{\varphi}((x, \lambda)) = \varphi(x) + \lambda$. *Moreover,* $\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$.
- $\forall x \in A : \varphi(x) \in \sigma(x)$ *whenever* $\varphi \not\equiv 0$.
- $\Delta(A) \subseteq B_{A^*}$.
- *$A$ has a unit,* $\varphi \not\equiv 0 \implies \|\varphi\| \geqslant \frac{1}{\|e\|}$. *In particular if* $\|e\| = 1$, *then* $\|\varphi\| = 1$.

*Důkaz*

„1. uniqueness“: For $\tilde{\varphi} \in \Delta(A_e)$ such that $\tilde{\varphi}((x,0)) = \varphi(x)$, $x \in A$:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda\tilde{\varphi}((\mathbf{o},1)) = \varphi(x) + \lambda,$$

second equality by $\varphi \in \Delta(A) \implies \varphi(e) = \varphi(e^2) = \varphi^2(e)$. „1. existence“ is proven by check that defined $\tilde{\varphi}$ is multiplicative linear functional (and it is nonzero, but $\tilde{\varphi}((0,1)) = 1 \neq 0$). This is easy (omitted).

„$\Delta(A_e) = \{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}$“: „$\subseteq$“: $\varphi \in LHS$, put $\varphi(x) := \psi((x,0))$. Then $\varphi \in \Delta(A) \cup \{0\}$ and $\tilde{\varphi} = \psi$ became:

$$\tilde{\varphi}((x,\lambda)) = \varphi(x) + \lambda = \psi((x,0)) + \lambda = \psi((x,\lambda)).$$

„$\supseteq$“: We know already that $\tilde{\varphi} \in \Delta(A_e)$.

„2. with $A$ has unit $e$“: $\varphi \neq 0$, $\varphi \in \Delta(A)$: If $\lambda \in \varrho(x)$, then $\varphi(\lambda e - x) \neq 0$ ($\varphi(x) \neq 0$ if $x^{-1}$ exists). $0 \neq \varphi(\lambda e - x) = \lambda - \varphi(x) \implies \lambda \neq \varphi(x)$. Thus $\varphi(x) \notin \varrho(x)$, so $\varphi(x) \in \sigma(x)$. „2. with $A$ hasn't unit“, then $\varphi(x) = \tilde{\varphi}((x,0)) \in \sigma_{A_e}((x,0)) = \sigma_A(x)$.

„3.“: $\varphi \in \Delta(A)$. Then $\forall x \in A : \varphi(x) \in \sigma(x) \subseteq B(\mathbf{o}, \|x\|)$, so $|\varphi(x)| \leqslant \|x\|$.

„4.“: $A$ has a unit $e$, then $\|\varphi\| \geqslant \left|\varphi\left(\frac{e}{\|e\|}\right)\right| = \frac{1}{\|e\|}$. □

## Věta 1.20

*$A$ Banach algebra, $M := \Delta(A) \cup \{0\}$. Then $M \subset (B_{A^*}, w^*)$ is compact, $\Delta(A)$ is locally compact and if $A$ has u unit, then $\Delta(A)$ is compact. The mapping $\Phi : M \to \Delta(A_e)$, $\Phi(\varphi) = \tilde{\varphi}$ is $w^*$–$w^*$ homeomorphism.*

*Důkaz*

By previous proposition, $M \subset (B_{A^*}, w^*)$ ($(B_{A^*}, w^*)$ is compact by previous semester). So, it suffices to check that $M$ is $w^*$-closed.

$$M = \bigcap_{x,y \in A} \{\varphi \in A^* | \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\}.$$

Sets from RHS is closed by previous semester, so, $M$ is closed. Thus $M$ is compact.

$\Delta \subset M$ is open, so $\Delta(A)$ is locally compact (and $M$ is 1-point compactification of $\Delta(A)$). If $\Delta$ has a unit, then $\Delta(A) = \{\varphi \in M | \varphi(e) = 1\}$ is $w^*$-closed, so $\Delta(A)$ is compact (and 0 is isolated in $M$).

Finally, by previous proposition, $\Phi$ is bijection. $\Phi$ is $w^*$-continuous:

$$\varphi_i \xrightarrow{w^*} \varphi \implies \forall (x,\lambda) : \tilde{\varphi}_i((x,\lambda)) = \varphi_i(x) + \lambda \to \varphi(x) + \lambda = \tilde{\varphi}((x,\lambda)) \implies \tilde{\varphi}_i \xrightarrow{w^*} \tilde{\varphi}$$

So, $\Phi$ is homeomorphism (continuous bijection on compact, last semester?). □

*Například*
$\Delta(\mathcal{C}(K)) = \{\delta_x | x \in K\}$. ($f \mapsto f(x)$ is multiplicative. Suppose $\varphi \in \Delta(\mathcal{C}(K))$, $\varphi \notin \{\delta_x | x \in K\}$. So for $x \in K$ there is $g_x \in C(x) : \varphi(g_x) \neq g_x(x)$. Consider $f_x = g_x - \varphi(g_x)$. Then $\varphi(f_x) = 0$, $f_x(x) \neq 0$. So there is $U_x$ open neighbourhood of $x$ such that $f_x \neq 0$ on $U_x$. Compactness implies $\exists x_1, \ldots, x_n \in K : K \subset \bigcup_{i=1}^n U_{x_i}$. Consider $h := \sum_{i=1}^n |f_{x_i}|^2$. Then $h > 0$ on $K$, so $h^{-1}$ exists and therefore $\varphi(h) \neq 0$. But $\varphi(h) = \sum_{i=1}^n \varphi(f_{x_i})\overline{\varphi_{x_i}} = 0$.)

$\Delta\{M_n\} = \emptyset$, $n \geqslant 2$, where $M_n$ is (non-commutative) algebra of $n \times n$ matrices. ($M_n = \mathrm{LO}\{E^{i,j}\}$. $E^{ij} \cdot E^{kl} = E^{il}$ if $j = k$, else 0. So $\varphi(E^{ij}) \cdot \varphi(E^{ij}) = \varphi(E^{ij} \cdot E^{ij}) = 0$ if $i \neq j$. $\varphi(E^{ii}) = \varphi(E^{in}E^{ni}) = \varphi(E^{in})\varphi(E^{in}) = 0$. $\varphi(E^{nn}) = \varphi(E^{n1}E^{1n}) = 0$.)

## **Definice 1.11** (Ideal, maximal ideal)

$A$ Banach algebra. Ideal in $A$ is a subspace $I \subset A$ if $\forall x \in I\ \forall y \in A : x \cdot y \in I \wedge y \cdot x \in I$.

Maximal ideal $\equiv$ proper ($I \neq A$) ideal and it is maximal proper ideal with respect to inclusion.

*Například* (2021, Johnson-Schetman, Acta mathematica)
$\mathcal{L}(L_p)$ has $2^{2^\omega}$ non-isomorphic closed ideals.

## **Tvrzení 1.21**

*A Banach algebra with a unit. Then:*

- *Any proper ideal is contained in a maximum ideal. (From Zorn's lemma. And $I \subset A$ ideal is proper $\Leftrightarrow e \notin I$.)*
- *$I \subset A$ proper ideal $\implies \overline{I} \in A$ is proper ideal. In particular, maximal ideals are closed. (Easy: $\overline{I}$ is ideal. Moreover, $I \cap A^* = \emptyset$ (if $x \in I$ was invertible thus $e = x \cdot x^{-1} \in I$, but $e \notin I$). So ($A^*$ is open) $\overline{I} \cap A^* = \emptyset$ and therefore $e \notin \overline{I}$.)*

## **Tvrzení 1.22**

*A Banach algebra, $I \subseteq A$ closed ideal $\implies A/I$ is Banach algebra ($[x] \cdot [y] := [x \cdot y]$).*

*Důkaz*
Straightforward from definition. (Omitted.) □

*Poznámka*
From now on, $A$ will be commutative.

Step 1: „Hahn-Banach“: $I \subset A$ closed ideal $\implies \exists \varphi \in \Delta(A) : \varphi / I \equiv \ldots$.

## Věta 1.23

*A commutative Banach algebra with a unit. Then* $\Phi : \Delta(A) \to \{$*maximal ideals in* $A\}$, $\Phi(\varphi) := \operatorname{Ker}\varphi$, *is bijection.*

*Důkaz*
Pick $\varphi \in \Delta(A)$. Then „$\operatorname{Ker}\varphi$ is maximal ideal“: ideal: $y \in \operatorname{Ker}\varphi, x \in A : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y) = \ldots \cdot 0 = 0$, proper: $\varphi \not\equiv 0$, maximal: $\operatorname{codim}\operatorname{Ker}\varphi = 1$: pick $x_0 : \varphi(x_0) \neq 0$, $a = a - \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} + \varphi(a) \cdot \frac{x_0}{\varphi(x_0)} \in \operatorname{Ker}\varphi \oplus \mathbb{R}$.

„$\Phi$ is one-to-one“: Pick $\varphi, \psi \in \Delta(A)$: $\operatorname{Ker}\varphi = \operatorname{Ker}\psi$. Then (by lemma from previous semester) $\varphi = c \cdot \psi$ for some $c \in \mathbb{K}$. But $\varphi(e) = 1 = \psi(1)$ so $\varphi = \psi$.

„$\Phi$ is surjective“: Let $I \subset A$ be maximal ideal ( $\implies$ closed). Step 1 „Any nonzero element in $A/I$ is invertible“: For contradiction assume $\exists q(x) \in A/I$ ($q(x) = [x]$), $q(x) \neq 0 \wedge q(x)^{-1}$ does not exist. By next lemma $q(x)(A/I)$ is proper ideal. Then $q^{-1}(q(x)(A/I))$ is an ideal in $A$ which is proper and $I \subsetneq q^{-1}(q(x)(A/I))$, which contradicts maximality of $I$. It follows from: ideal: follows from the fact that $q$ is algebra homomorphism; proper: $q(e) = [e] \notin q(x)A/I$; $I \subseteq q^{-1}(\ldots)$: $0 \in q(x)A/I$; $I \neq q^{-1}(\ldots)$: $q(x) \neq 0 \implies x \notin I$, but $q(x) = q(x)q(e) \in q(x)(A/I)$, so $x \in q^{-1}(\ldots)$.

From Gelfand–Mazur theorem $\exists$ surjective isomorphism $j : A/I \to \mathbb{C}$. Then $\varphi := j \circ q \in \Delta(A)$. It remains „$I = \operatorname{Ker}\varphi$“: $x \in \operatorname{Ker}\varphi \Leftrightarrow j(q(x)) = 0 \Leftrightarrow q(x) = 0 \Leftrightarrow x \in I$. □

## Lemma 1.24

*A* commutative *Banach algebra with a unit,* $x \in A$ *does not have inverse* $\implies$ $xA$ *is proper ideal.*

*Důkaz*
$xA$ is ideal, because $A$ is commutative. Then $xA$ is proper ($e \notin xA$). □

*Důsledek* (Hahn–Banach like theorem)
$A$ is commutative Banach algebra with a unit, $I \subset A$ proper ideal. Then $\exists \varphi \in \Delta(A) : \varphi/I \equiv 0$.

*Důkaz*
Let $\tilde{I} \supseteq I$ be maximal ideal. By previous theorem there is $\varphi \in \Delta(A) : \tilde{I} = \operatorname{Ker}\varphi$. □

## Tvrzení 1.25

*A, B Banach algebras,* $\Phi : A \to B$ *algebraic isomorphism. Then* $\Phi^{\#} : \Delta(B) \to \Delta(A)$ *defined as* $\Phi^{\#}(\varphi) := \varphi \circ \Phi$ *is homeomorphism.*

*Důkaz*
„$\Phi^\#(\varphi) \in \Delta(A)$“: $\Phi^\#(\varphi) = \varphi \circ \Phi \in \Delta(A) \cup \{0\}$ and $\varphi \not\equiv 0 \wedge \Phi$ is onto $\implies \varphi \circ \Phi \neq 0$.

„$\Phi^\#$ is $w^*$-$W^*$ continuous“: $\varphi_i \xrightarrow{w^*} \varphi \implies \varphi_i \circ \Phi \xrightarrow{w^*} \varphi \circ \Phi$.

Apply the proven part to $\Phi^{-1}$, obtain that $(\Phi^{-1})^\# : \Delta(A) \to \Delta(B)$ is $w^*$-$W^*$ continuous. Moreover we have $\Phi^\# \circ (\Phi^{-1})^\# = \mathrm{id} \wedge (\Phi^{-1})^\# \circ \Phi^\#$. □

## Tvrzení 1.26

*$L$ locally compact $T_2$. Then $\delta : L \to \Delta(C_0(L)), x \mapsto \delta_x$ is homeomorphism onto.*

*Důkaz*
„Case 1: $L$ is compact“: By example $\delta$ is onto. Of course, $\delta$ is one-to-one, continuous. So $\delta$ is homeomorphism.

„Case 2: $L$ is not compact“: Then there is $K = L \cup \{\infty\}$, one-point compactification, and $\{f \in \mathcal{C}(K) | f(\infty) = 0\} \ni f \mapsto f|_L \in C_0(L)$ is isometric isomorphism. Moreover $\Phi : \mathcal{C}_0(L)_e \to \mathcal{C}(K)$, $\Phi(f, \lambda) := f + \lambda$, is algebraic isomorphism.

So, we have $K \xrightarrow{\eta} \Delta(C(K)) \xrightarrow{\Phi^\#} \Delta(C_0(L)_e) \xrightarrow{\psi} \Delta(C_0(L)) \cup \{0\}$, where $\eta$ is homeomorphism from case 1 and $\psi(\varphi) := \varphi|_{C_0(L)}$.

Thus $\delta := \psi \circ \Phi^\# \circ \eta$ is homeomorphism between $L \cup \{\infty\}$ and $\Delta(C_0(L)) \cup \{0\}$. Finally, for $x \in K$ and $f \in \mathcal{C}_0(L)$:

$$\Phi^\# \circ \eta(x)(f) = (\eta(x) \circ \Phi)(f) = f(x),$$

so $\psi \circ \Phi^\# \circ \eta(x) = \Phi^\# \circ \eta(x)|_{C_0(L)} = \delta_x|_{C_0(L)}$. □

## Věta 1.27

*$K$, $L$ locally compact $T_2$. Then following is ekvivalent*

- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as Banach algebra;*
- *$\mathcal{C}_0(K) \equiv \mathcal{C}_0(L)$ as algebras;*
- *$K \approx L$ as topological spaces.*

*Důkaz*
„1 $\implies$ 2“ trivial. „2 $\implies$ 3“: $K \approx \Delta(\mathcal{C}_0(K)) \approx \Delta(\mathcal{C}_0(L)) \approx L$ from previous two tvrzeni. „3 $\implies$ 1“: Given $h : K \to L$ homeomorphism, $f \mapsto f \circ h$ is isometry between Banach algebras. □

**Definice 1.12** (Semi-simple Banach algebra)

$A$ commutative Banach algebra. It is semi-simple $\equiv$ $\Delta(A)$ separates points of $A$. ($\Leftrightarrow$ $\bigcap\{\operatorname{Ker}\varphi|\varphi \in \Delta(A)\} = \{\mathbf{o}\}$.)

*Poznámka*
Semi-simple $\implies$ commutative. (Semi-simple and $x \cdot y \neq y \cdot x \implies \exists\varphi \in \Delta(A)$ : $\varphi(x) \cdot \varphi(y) = \varphi(x \cdot y) \neq \varphi(y \cdot x) = \varphi(y) \cdot \varphi(x)$ ↯.)

**Věta 1.28**

*$A, B$ Banach algebras, $B$ is semi-simple, then every (algebra) homomorphism $\Phi : A \to B$ is continuous.*

*Důkaz*
Use Closed graph theorem. Pick $x_n \to x$, $\varphi(x_n) \to y$. Wanted „$\Phi(x) = y$“ ($\Leftrightarrow \forall\varphi \in \Delta(B) : \varphi(\Phi(x)) = \varphi(y)$). For $\varphi \in \Delta(B)$ we have $\varphi(y) = \lim_n \varphi(\Phi(x_n)) \overset{\varphi\circ\Phi\in\Delta(A)\subseteq A^*}{=} \varphi \circ \Phi(\lim_n x_n) = \varphi(\Phi(x))$. □

*Důsledek*
$(A, \|\cdot\|)$ semi-simple Banach algebra and $(A, \|\|\cdot\|\|)$ is Banach algebra (with other norm), then $\|\cdot\|$ and $\|\|\cdot\|\|$ are equivalent.

*Důkaz*
We have that $\mathrm{id} : (A, \|\|\cdot\|\|) \to (A, \|\cdot\|)$ is algebra homomorphism, so continuous by previous theorem. Similarly inverse is continuous (semi-simplicity doesn't depend on norm). So, id is isomorphism. □

# 2 Gelfand transformation

**Definice 2.1** (Gelfand transformation)

$A$ Banach algebra. For $x \in A$ we define $\hat{x} : \Delta(A) \to \mathbb{C}$, $\hat{x}(\varphi) := \varphi(x)$. We say that $\hat{x}$ is Gelfand transformation of $x$.

*Poznámka*
$\hat{x} \in \mathcal{C}_0(\Delta(A))$.

$A = \mathcal{C}_0(L) \implies \Delta(A) = \{\delta_x|x \in L\} \implies \forall f \in A : \hat{f}(\delta_x) = f(x), x \in L$. So, $\hat{f} = f$.

$A = L_1(\mathbb{R}^d) \implies \Delta(A) = \{e^{it\cdot x}|x \in \mathbb{R}\} \subseteq L_\infty(\mathbb{R}^d) = A^*$ and $\hat{f}$ is Fourier transformation.

## Věta 2.1

*A commutative Banach algebra, $x \in A$. Then*

- *A has a unit* $\implies \sigma(x) = \operatorname{Rng} \hat{x}$;
- *A doesn't have a unit* $\implies \sigma(x) = \operatorname{Rng} \hat{x} \cup \{0\}$;
- $\|\hat{x}\|_\infty = r(x) = \sup \{|\lambda| | \lambda \in \sigma(x)\}$.

*Důkaz*

„a) $\subseteq$“: $\lambda \in \sigma(x) \Leftrightarrow (\lambda \cdot e - x)^{-1}$ does not exists $\implies$ (Lemma above) $(\lambda e - x)A$ is proper ideal $\implies \exists \varphi \in \Delta(A) : \varphi|_{(\lambda e - x)A} \equiv 0 \implies \exists \varphi \in \Delta(A) : 0 = \varphi(\lambda e - x) = \lambda - \varphi(x) = \lambda - \hat{x}(\varphi) \implies \lambda \in \operatorname{Rng} \hat{x}$.

„$\supseteq$“ follows from Tvrzeni above, $\varphi(x) \in \sigma(x)$ for $\varphi \in \Delta(A)$.

„b)“ For $x \in A$ :

$$\sigma(x) = \sigma_{A_e}((x,0)) \overset{\text{a)}}{=} \operatorname{Rng} \widehat{(x,0)} = (\{\tilde{\varphi} | \varphi \in \Delta(A) \cup \{0\}\}) =$$

$$= \{\varphi(x) | \varphi \in \Delta(A) \cup \{0\}\} = \operatorname{Rng} \hat{x} \cup \{0\}\,.$$

„c)“ $\|\hat{x}\|_\infty = \sup \{|\lambda| | \lambda \in \operatorname{Rng} \hat{x}\} = \sup \{|\lambda| | \lambda \in \operatorname{Rng} \hat{x} \cup \{0\}\} = \sup \{|\lambda| | \lambda \in \sigma(x)\} = r(x)$. □

## Definice 2.2 (Gelfand transformation of algebra)

$A$ Banach algebra, then $\Gamma : A \to \mathcal{C}_0(\Delta(A))$, $\Gamma(x) := \hat{x}$ is the Gelfand transformation of $A$.

## Věta 2.2

*A commutative Banach algebra, $\Gamma$ Gelfand transformation. Then*

- $\Gamma$ *is algebra transformation, continuous,* $\|\Gamma\| \leqslant 1$;
- $\Gamma(A)$ *separates the points of* $\Delta(A)$;
- $\Gamma$ *is one-to-one* $\Leftrightarrow$ *A is semi-simple;*
- $\Gamma$ *is an isomorphism into* $\Leftrightarrow \exists K > 0 : \|x^2\| \geqslant K \cdot \|x\|^2$, $x \in A$; ($\Leftrightarrow \Gamma$ *is one-to-one and* $\Gamma(A)$ *is closed;*)
- $\Gamma$ *is an isometry into* $\Leftrightarrow \|x^2\| = \|x\|^2$, $x \in A$.

*Důkaz*

„a)“: $\Gamma$ is linear (obvious), $\Gamma$ preserves multiplication (obvious). Finally, $\|\Gamma(x)\|_\infty = \|\hat{x}\|_\infty = r(x) \leqslant \|x\|$. So $\|\Gamma\| \leqslant 1$.

„b)“: Let $\varphi \neq \psi \in \Delta(A)$ and $x \in A : \hat{x}(\varphi) = \varphi(x) \neq \psi(x) = \hat{x}(\psi)$.

„c)“: $\Gamma(x) = 0 \Leftrightarrow \hat{x}(\varphi) = 0, \varphi \in \Delta(A) \Leftrightarrow \varphi(x) = 0, \varphi \in \Delta(A)$. So, $\Gamma$ is one-to-one $\Leftrightarrow \forall x \neq 0\ \exists \varphi \in \Delta(A) : \varphi(x) \neq 0 \Leftrightarrow A$ is semi-simple.

„d) second“: $\Gamma$ is isomorphism into $\Leftrightarrow$ $\Gamma$ is bijection between $A$ and $\Gamma(A) \wedge \Gamma(A)$ is closed. ($\Gamma(A)$ is closed, then we use Open mapping theorem; if $\Gamma$ is isomorphism, $\Gamma(A)$ is a Banach space.).

„d) + e), $\implies$ “: Suppose $\exists c > 0$: $\|\Gamma(x)\| \geqslant c \cdot \|x\|$, $x \in A$. Then $\forall x \in A : \|x^2\| \overset{a)}{\geqslant} \|\Gamma(x^2)\| = \|\Gamma(x)\|^2 \geqslant c^2 \cdot \|x\|^2$.

„d) + e), $\impliedby$ “: Let d) hold with $K$ (this holds in every algebra). Then (proven by induction)

$$\forall x \in A : \|x^{2^n}\| \geqslant K^{2^{n-1}} \|x\|^{2^n}, \qquad n \in \mathbb{N}.$$

$$\implies \sqrt[2^n]{\|x^{2^n}\|} \geqslant K^{1-2^{-n}} \|x\|,$$

where left side converges (by Beurling) to $r(x)$ and right side converges to $\|x\|$. So $r(x) \geqslant K \cdot \|x\|$ and from previous theorem $r(x) \geqslant \|\hat{x}\|_\infty = \|\Gamma(x)\|$. □

## 2.1 $C^*$-algebras

**Definice 2.3** (Involution)

$A$ is a Banach algebra. Involution is a mapping $* : A \to A$ such that

$$\forall x, y \in A\ \forall \lambda \in \mathbb{C} :$$

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x^*, \qquad (xy)^* = y^* \cdot x^*, \qquad (x^*)^* = x.$$

**Definice 2.4** ($C^*$-algebra)

Banach algebra with involution $*$ is a $C^*$-algebra, if

$$\forall x \in A : \|x \cdot x^*\| = \|x\|^2, x \in A.$$

**Definice 2.5** (Self-adjoint element, normal element)

For $A$ with involution $*$ and $x \in A$ we say that $x$ is self-adjoint $\equiv x = x^*$, and $x$ is normal $\equiv x \cdot x^* = x^* \cdot x$.

**Tvrzení 2.3** (Properties)

*A Banach algebra with involution, $x \in A$. Then*

- *$e$ is left/right unit $\implies$ $e$ is unit and $e = e^*$. ($e$ is left unit $\Leftrightarrow$ $e^*$ is right unit. So there is unit.)*
- *$A$ is $C^*$-algebra $\Leftrightarrow$ $\|x \cdot x^*\| \geqslant \|x\|^2$, $x \in A$. Then $\|x^*\| = \|x\|$, $x \in A$. („ $\implies$ “: clear, „ $\impliedby$ “: Then $\forall x \in A : \|x\|^2 \leqslant \|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\|$, so $\|x\| \leqslant \|x^*\|$, and applying to $x^*$ we get $\|x^*\| \leqslant \|x\|$. But then we have $\|x \cdot x^*\| \leqslant \|x\| \cdot \|x^*\| = \|x\|^2$.)*
- *Let $A$ has a unit. then $x^{-1}$ exists $\Leftrightarrow$ $(x^*)^{-1}$ exists. Then $(x^*)^{-1} = (x^{-1})^*$. („ $\implies$ “: $x^* \cdot (x^{-1})^* = (x^{-1}x)^* = e^* = e$, analogically $(x^{-1})^* x^* = e$. „ $\impliedby$ “: Apply the proven part to $x^*$.)*
- *$\lambda \in \sigma(x) \Leftrightarrow \overline{\lambda} \in \sigma(x^*)$. ($A$ has a unit: $\lambda \notin \sigma(x) \Leftrightarrow \exists(\lambda e - x)^{-1} \Leftrightarrow \exists\left((\lambda e - x)^*\right)^{-1} \Leftrightarrow \overline{\lambda} \notin \sigma(x^*)$. If $A$ has not a unit, then we use previous sentence and next theorem?)*
- *$x + x^*$, $x^* \cdot x$, $x \cdot x^*$, $i \cdot (x - x^*)$ are self-adjoint. (Easy, omitted.)*
- *$\exists! u, v \in A$ self-adjoint: $x = u + i \cdot v$. Then $x^* = u - i \cdot v$, and $x$ is normal $\Leftrightarrow$ $uv = vu$. („Existence“: $u := \frac{1}{2}(x + x^*), v := \frac{1}{2i}(x - x^*)$. Then $x = u + iv$. „Formulas“: $(u + i \cdot v)^* = u^* + \overline{i} v^*$. „Uniqueness“: Pick $a, b \in A_{sa} : x = a + i \cdot b$. Then $a + i \cdot b = x = u + i \cdot v$, $a - i \cdot b = x^* = u - i \cdot v$. By subtracting or summing equation we get $a = u$ and $b = v$. „Normality“: $x$ normal $\Leftrightarrow (u + i \cdot v)(u - i \cdot v) = (u - i \cdot v)(u + i \cdot v)$ $\Leftrightarrow -i \cdot u \cdot v + i \cdot v \cdot u = i \cdot u \cdot v - i \cdot v \cdot u \Leftrightarrow u \cdot v = v \cdot u$.)*

**Věta 2.4**

*$A$ is $C^*$-algebra, $x \in A$ is normal. Then $r(x) = \|x\|$.*

*Důkaz*

„Step 1: $\|x^2\| = \|x\|^2$“:

$$\|x\|^4 = \|x^* x\|^2 = \|(x^* x)^* (x^* x)\| = \|x^* x x^* x\| = \|x^* x^* x x\| = \|(xx)^* xx\| = \|xx\|^2 = \|x^2\|^2.$$

Thus inductively, we obtain $\|x^{2^k}\| = \|x\|^{2^k}$, $k \in \mathbb{N}$. Thus, Beurling gives

$$r(x) = \lim_k \sqrt[2^k]{\|x^{2^k}\|} = \|x\|.$$

□

*Důsledek*

$A$ (Banach) algebra with involution. Then there is at most one norm $\|\cdot\|$ on $A$, such that $(A, \|\cdot\|)$ is $C^*$-algebra.

*Důkaz*
If $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $A$ such that $(A, \|\cdot\|)$ is $C^*$-algebra, then by previous theorem

$$\forall x \in A : \|x\|_1^2 = \|x^*x\|_1 = r(x^*x) = \|x^*x\|_2 = \|x\|_2^2.$$

□

### Věta 2.5

*$(A, \|\cdot\|)$ Banach algebra.*

- *$(a, \lambda)^* = (a^*, \overline{\lambda})$, $(a, \lambda) \in A_e$ defines an involution on $A_e$. (Trivial.)*
- *If $A$ is $C^*$-algebra, then on $A_e$ there exists a norm $|||\cdot|||$ (equivalent to the norm from $A \oplus_1 \mathbb{K}$) such that $(A_e, |||\cdot|||)$ is $C^*$-algebra and $|||(a, 0)||| = \|a\|$, $a \in A$.*

### Věta 2.6

*$A$ is $C^*$-algebra, $x \in A$. Then*

- *$x = x^* \implies \sigma(x) \subseteq \mathbb{R}$;*
- *$A$ has a unit and $x^* = x^{-1}$ (that is, $x$ is unitary) $\implies \sigma(x) \subseteq \{\lambda | |\lambda| = 1\}$.*

*Důkaz*
By previous theorem, WLOG $A$ has a unit.

„a)“: Let $\alpha + i\beta \in \sigma(x)$, $\alpha, \beta \in \mathbb{R}$. We want $\beta = 0$. Trick: $x_t := x + i \cdot t \cdot e$, $t \in \mathbb{R}$. Then

$$\alpha + i \cdot (\beta + t) \in \sigma(x_t) ( \impliedby (\alpha + i(\beta + t))e - x_t = (\alpha + i \cdot \beta)e - x),$$

$$\alpha^2 + (\beta+t)^2 = |\alpha + i(\beta+t)|^2 \leqslant \|x_t\|^2 = \|x_t^* x_t\| = \|(x - i \cdot t \cdot e) \cdot (x + i \cdot t \cdot e)\| = \|x^2 + (t \cdot e)^2\| \leqslant$$
$$\leqslant \|x^2\| + t^2.$$

So, $\alpha^2 + (\beta + t)^2 - t^2 \leqslant \|x^2\|$, $t \in \mathbb{R} \implies \beta = 0$ (Otherwise $LHS \to +\infty$ for $t \to \pm\infty$.)

„b)“: ($\|e\| = \|e^2\| = \|e\|^2$.) $1 = \|e\| = \|x^*x\| = \|x\|^2$, so $\|x\| = 1$. Then, for $\lambda \in \sigma(x)$, we have $|\lambda| \leqslant \|x\| = 1$. On the other hand $\frac{1}{\lambda} \in \sigma(x^{-1})$ (because if not, then $\frac{1}{\lambda}e - x^{-1}$ ha inverse $\implies \lambda e - x = (\lambda e - x)x^{-1}x = (\lambda x^{-1} - e)x = -\lambda(\frac{1}{\lambda}e - x^{-1})x \implies \lambda e - x$ has inverse.) So

$$\left|\frac{1}{\lambda}\right| \leqslant \|x^{-1}\| = \|x^*\| = \|x\| = 1.$$

□

**Definice 2.6**

$A, B$ are $C^*$-algebras, then $\Phi : A \to B$ is $*$-homomorphism if $\Phi$ is homomorphism preserving $*$ (that is, $\Phi(x^*) = (\Phi(x))^*$).

*Důsledek*

Let $A$ be a $C^*$-algebra and $\Phi \in \Delta_A$. Then $\Phi$ is $*$-homomorphism.

*Důkaz*

„If $x = x^*$“, then $\Phi(x) \in \sigma(x) \subseteq \mathbb{R}$, so $\Phi(x^*) = \Phi(x) = \overline{\Phi(x)}$.

„In general“, if $x = u + i \cdot v$ ($u = u^*$, $v = v^*$), then $\Phi(x^*) = \Phi(u - i \cdot v) = \Phi(u) - i \cdot \Phi(v) = \overline{\Phi(u) + i \cdot \Phi(v)} = \overline{\Phi(x)}$. □

**Tvrzení 2.7** (Automatical continuous)

*Let $A, B$ be $C^*$-algebras, $\Phi : A \to B$ is $*$-homomorphism. Then $\Phi$ is continuous and $\|\Phi\| \leqslant 1$.*

*Důkaz*

$\forall x \in A : \|\Phi(x)\|^2 = \|\Phi(x)^* \cdot \Phi(x)\| = r(\Phi(x^*) \cdot \Phi(x)) = r(\Phi(x^* x)) \stackrel{*}{=} r(x^* x) = \|x^* x\| = \|x\|^2$.

Thus it suffices to show that (by following lemma)

$$\sigma(\Phi(x^* x)) \subseteq \sigma(x^* x) \cup \{0\} .$$

□

**Lemma 2.8**

*Let $A, B$ be Banach algebras, $\Phi : A \to B$ algebra homomorphism. Then $\forall x \in A : \sigma_B(\varphi(x)) \subseteq \sigma_A(x) \cup \{0\}$.*

*Důkaz*

Consider $\tilde{\Phi} : A_e \to B_e$ defined as $\tilde{\Phi}(a, \lambda) := (\Phi(a), \lambda)$. Then $\tilde{\Phi}$ is algebra homomorphism preserving unit. Moreover $\sigma_B(\Phi(x)) \subseteq \sigma_{B_e}((\Phi(x), 0)) \cup \{0\}$ and $\sigma_{A_e}((x, 0)) \subseteq \sigma_A(x) \cup \{0\}$. Thus, WLOG $A, B$ have units and $\Phi(e_A) = e_B$.

But then for $\lambda \neq 0$ and $x \in A : \lambda e - x$ has inverse in $A$, then $\Phi(\lambda e - x) = \lambda \Phi(e) - \Phi(x)$ has inverse in $B$. So, $\lambda \notin \sigma_A(x) \cup \{0\} \implies \lambda \notin \sigma_B(\Phi(x))$. □

**Věta 2.9** (Gelfand–Naimark)

*$A$ commutative $C^*$-algebra. Then the Gelfand transformation $\Gamma : A \to \mathcal{C}_0(\Delta(A))$ is isometric $*$-isomorphism onto.*

*Důkaz*
By proposition above, $\Gamma$ is algebra homomorphism, $\|\Gamma\| \leqslant 1$ and from theorem above $\|\Gamma(x)\|_\infty = r(x)$, $x \in A$. „$\Gamma$ is $*$-homomorphism“:

$$\forall a \in A\ \forall \varphi \in \Delta(A) : \Gamma(a^*)(\varphi) = \varphi(a^*) = \overline{(\varphi(a))} = \overline{\Gamma(a)}(\varphi).$$

„$\Gamma$ is isometry“:

$$\forall x \in A : \|\Gamma(x)\|^2 = \|\overline{\Gamma(x)} \cdot \Gamma(x)\| = \|\Gamma(x^*x)\| = r(x^*x) = \|x^x\| = \|x\|^2.$$

„$\Gamma$ is onto“: $\Gamma(A)$ is Banach space so $\Gamma(A) \subseteq \mathcal{C}_0(\Delta(A))$ is closed and $*$-subalgebra. And $\Gamma(A)$ separates points of $\Delta(A)$. So from Stone–Weierstrass theorem ($A \subset \mathcal{C}_0(K)$ is $*$-subalgebra separating the points, then $\overline{A}^{\|\cdot\|} = \mathcal{C}_0(K)$) $\Gamma(A) = \mathcal{C}_0(\Delta(A))$. □

*Důsledek*
$A, B$ commutative $C^*$-algebras. Then the following items are equivalent:

- $A$ and $B$ are isometrically $*$-isomorphic;
- $A$ and $B$ are algebraically isomorphic;
- $\Delta(A)$ and $\Delta(B)$ are homeomorphic.

*Důkaz*
„2. $\Leftrightarrow$ 3.“ follows from theorem above (where it is proved for $\mathcal{C}_0(K)$-spaces). „1. $\implies$ 2.“: trivial.

„3. $\implies$ 1.“: easy for $\mathcal{C}_0(K)$-spaces, because if $h : K \to L$ is homeomorphism, then $f \mapsto f \circ h$ is isometrical $*$-isomorphism. □

## Definice 2.7

$A$ Banach algebra, $M \subset A$. Then $alg(M) = \bigcap \{B \supseteq M | B \text{ is subalgebra of } A\}$.

*Poznámka* (Easy)

$$= \left\{ \sum_{i=1}^{n} \alpha_i \prod_{j=1}^{m} x_{ij} | n, m \in \mathbb{N}, \alpha_i \in \mathbb{C}, x_{ij} \in M \right\}.$$

Moreover $\overline{alg} M = \bigcap \{B \supseteq M | B \text{ is closed subalgebra of } A\}$.

*Poznámka* (Easy)

$$= \overline{algM}^{\|\cdot\|}.$$

**Tvrzení 2.10** (Fact)

*$A$ is $C^*$-algebra, $M \subset A$ is commutative and closed under $*$, then $\overline{alg} M$ is commutative $C^*$-subalgebra of $A$.*

**Věta 2.11**

*$A, B$ are $C^*$-algebras, $h : A \to B$ is $*$-homomorphism, one-to-one. Then $h$ is isometry.*

*Důkaz*
WLOG $A$, $B$ have units and $h(e)$ is a unit ($(a, \lambda) \mapsto (h(a), \lambda)$ is 1-to-1 $*$-homomorphism). Suffices: $\forall x \in A$ self-adjoint: $\|x\| = \|h(x)\|$ ($\forall y \in A : \|h(y)\|^2 = \|h(y^*y)\| = \|y^*y\| = \|y\|^2$). Let $x \in A$ be self-adjoint. Put $A_0 := \overline{alg}\{e, x\} = \overline{\mathrm{LO}}\{e, x, x^2, x^3, \dots\}$ is commutative and $C^*$-subalgebra.

$$B_y := \overline{alg}\{e, h(x)\} = \overline{\mathrm{LO}}\left\{e, h(x), h(x^2), \dots\right\}$$

is commutative and $C^*$-subalgebra. So, we have $A_0 \xrightarrow{h} B_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(B_0))$, $A_0 \xrightarrow{\Gamma} \mathcal{C}(\Delta(A_0))$. So there is $\tilde{h} : \mathcal{C}(\Delta(A_0)) \to \mathcal{C}(\Delta(B_0))$ one-to-one $*$-homeomorphism, $\tilde{h}(1) = 1$. So, it suffices to prove the following lemma. □

**Lemma 2.12**

*Let $K, L$ be $T_2$ compact spaces, $\varphi : \mathcal{C}(K) \to \mathcal{C}(L)$ $*$-homomorphism, $\varphi(1) = 1$. Then $\exists \alpha : L \to K$ continuous mapping such that $\varphi(f) := f \circ \alpha$, $f \in \mathcal{C}(K)$.*

*Moreover, if $\varphi$ is one-to-one, then $\alpha$ is onto and so $\varphi$ is isometry.*

*Důkaz*
By proposition above $\|\varphi\| \leqslant 1$ and $\varphi$ is continuous. Consider $\varphi^* : \mathcal{M}(L) \to \mathcal{M}(K)$. Then „$\varphi^*(\Delta(\mathcal{C}(L)) \subseteq \Delta(\mathcal{C}(K))$“:

$$\forall h \in \Delta(\mathcal{C}(L))\ \forall f, g : \varphi^* h(fg) = h(\varphi(fg)) = h(\varphi(f))h(\varphi(g)) = \varphi^* h(f)\varphi^* h(g).$$

So, we have: $L \xrightarrow{\delta} \Delta(\mathcal{C}(L)) \xrightarrow{\varphi^*} \Delta(\mathcal{C}(K)) \xrightarrow{\delta^{-1}} K$. So, $\alpha(x) := \delta^{-1}(\varphi^*(\delta(x)))$, $x \in L$ is continuous from $L$ to $K$.

For this $\alpha$ we have:

$$\forall x \in L\ \forall f \in \mathcal{C}(K) : \varphi(f)(x) = \delta_x(\varphi(f)) = (\varphi^* \circ \delta_x)(f) = f(\delta^{-1}\varphi^*\delta_x) = f(\alpha(x)).$$

Moreover, „if $\varphi$ is one-to-one, then $\alpha$ is onto“: Suppose $\alpha(L) \subsetneq K \implies \exists f \in C(K) \backslash \{0\} : f|_{\alpha(L)} \equiv 0$. But then $\varphi(f) \equiv 0$, but $f \neq 0$. ↯($\varphi$ should be one-to-one.) Thus $\varphi$ is isometry. □

*Poznámka* (GNS construction)
$A$ is $C^*$-algebra $\implies \exists H$ Hilbert $\exists\varphi : A \to B(H)$ $*$-isomorphism into.

*Důkaz* (Sketch)
$f \geqslant 0$ $(\sigma(f) \geqslant 0)$ on $A|_{\{a|f(a*a)=0\}}$ constructs inner product $\langle [x],[y]\rangle := f(y^*x)$. Put $H := \overline{A|_{\{a|f(a*a)=0\}}}$. Then $\varphi(a)([x]) = [ax]$. □

# 3 Continuous calculus for formal elements of $C^*$-algebras

*Poznámka*
Idea: $\varphi(\sigma(x)) \ni f \mapsto f(x) \in A$.

For $A = C(K)$:

$$g \in \mathcal{C}(K), \varphi(\sigma(x)) \ni f \implies g \circ f \in C(K).$$

Let $A$ be $C^*$-algebra with a unit, $x \in A$ normal. Consider

$$B = \overline{alg}\,\{e, x, x^*\} \in A \implies \Gamma_B : B \to \mathcal{C}(\Delta(B)) \wedge f(x) := \Gamma_B^{-1}(f \circ \Gamma_B(x)), f \in \mathcal{C}(\sigma_A(x)).$$

Problem is when $\Gamma_B(x) \subseteq \sigma_A(x)$.

**Lemma 3.1**

*$A$ is $C^*$-algebra, $B \subset A$ is $C^*$-algebra. Then*

- *If $A$ and $B$ have the same unit $\implies \forall x \in B : \exists x^{-1} \in B \Leftrightarrow \exists x^{-1} \in A$;*
- *$\forall x \in B : B$ has a unit, which is not a unit in $A \implies \sigma_A(x) = \sigma_B(x) \cup \{0\}$, otherwise $\sigma_A(x) = \sigma_B(x)$;*
- *(In any case $\sigma_B(x) = \sigma_A(x)$).*

*Důkaz*
„1.“: Pick $x \in B$. „ $\implies$ “: easy. „ $\impliedby$ “: If $x^{-1}$ exists in $A$, then $(x^*x)^{-1}$ exists in $A$. So $0 \notin \sigma_A(x^*x) = \sigma_B(x^*x) \implies (x^*x)^{-1}$ exists in $B$. $x^{-1} = x^{-1}(x^*)^{-1}x^* = x^{-1}(x^*x)^{-1}x^*$.

„2.“: If $A$ and $B$ have the same unit, we have $\sigma_A(x) = \sigma_B(x)$. WLOG $A$ has a unit $e \notin B$ (Because $B \in A_e$ and $\sigma_{A_e}(x) = \sigma_A(x)$ if $A$ has not unit). Then $\sigma_A(x) = \sigma_{B+\mathrm{LO}(e)}(x) \overset{*}{=} \sigma_{B_e}((x,0)) = \sigma_B(x)$ if $B$ has no unit and $\sigma_B(x) \cup \{0\}$ if $B$ has a unit.

$*$: $\varphi : B + \mathrm{LO}(e) \to B_e, b + \lambda e \mapsto (b, \lambda)$ is algebra homomorphism. □

## Věta 3.2

*Let $A$ be a $C^*$-algebra with a unit, $x \in A$ normal, $f \in \mathcal{C}(\sigma(x))$. Then the mapping*

$$\Phi : \mathcal{C}(\sigma(x)) \to A, \qquad \Phi(g) := g(x) := \Gamma_B^{-1}(g \circ \Gamma_B(x))$$

*has the following properties:*

1. *$\Phi$ is isometric $*$-isomorphism onto $B = \overline{alg}\{e, x, x^*\}$, $\Phi(1) = e$ and $\Phi(\mathrm{id}) = x$.*
2. *If $\psi : \mathcal{C}(\sigma(x)) \to A$ is $*$-homomorphism, $\psi(1) = e$, $\psi(\mathrm{id}) = x$, then $\psi = \Phi$.*
3. *If $g \in \mathcal{H}(\Omega)$, where $\Omega \subset \mathbb{C}$ open, $\sigma(x) \subset \Omega$, then $\Phi(g|_{\sigma(x)}) = \psi(g)$, where $\psi$ is from holomorphic calculus.*
4. *$f(x)^{-1}$ exists in $A \Leftrightarrow f \neq 0$ on $\sigma(x)$. In this case $f(x)^{-1} = \left(\frac{1}{f}\right)(x)$.*
5. *$\sigma(f(x)) = f(\sigma(x))$.*
6. *$\forall g \in \mathcal{C}(f(\sigma(x))) : (g \circ f)(x) = g(f(x))$.*
7. *$\forall y \in A : yx = xy : yf(x) = f(x)y$.*

*Důkaz*
„1.“: Recall theorem above $\Gamma_B(x) : \Delta(B) \to \mathcal{C}$ continuous and onto $\sigma_B(x)$. And it is „one-to-one“:

$$\forall \varphi_1, \varphi_2 \in \Delta(B) : \varphi_1(x) = \varphi_2(x) \implies \varphi_1 = \varphi_2 \text{ on } B.$$

So $\Gamma_B(x) : \Delta(B) \to \sigma(x)$ is homeomorphism, then $\mathcal{C}(\sigma(x)) \ni g \mapsto g \circ \Gamma_B(x) \in \mathcal{C}(\Delta(A))$ is isometric $*$-isomorphism onto. Thus $\mathcal{C}(\sigma(x)) \ni g \mapsto \Gamma_B^{-1}(g \circ \Gamma_B(x)) \in B$ is isometric $*$-isomorphism onto.

Moreover, $\Phi(1) = \Gamma_B^{-1}(1) = e$ $(\forall \varphi \in \Delta(B) : \varphi(e) = 1)$. $\Phi(\mathrm{id}) = \Gamma_B^{-1}(\Gamma_B(x)) = x$.

„2.“: By theorem above, $\psi$ is continuous (because it is $*$-isomorphism), moreover $\psi = \Phi$ on complex polynomials. Since complex polynomials are dense in $\mathcal{C}(\sigma(x))$ by (S-W), by continuity $\Phi = \psi$ everywhere.

„3.“: Omitted (on polynomials, on inverse, on rationals, rationals are dense in $\mathcal{H}$).

„4.“: Since $f(x) \in B$, we have $f(x)^{-}1$ exists in $B \Leftrightarrow f(x)^{-1}$ exists in $A \overset{\Phi \text{ is ?}}{\Leftrightarrow} f^{-1}$ exists in $\mathcal{C}(\sigma(x)) \Leftrightarrow f \neq 0$ on $\sigma(x)$. And if $f \neq 0$ on $\sigma(x)$, then $f(x)^{-1} = \Phi(f^{-1}) = \Phi\left(\frac{1}{f}\right) = \left(\frac{1}{f}\right)(x)$.

„5.“: $f(x) \in B$, so $\sigma_A(f(x)) \overset{\text{Lemma}}{=} \sigma_B(f(x)) = \sigma_B(\Phi(f)) \overset{\Phi \text{is isomorphism}}{=} \sigma_{\mathcal{C}(\sigma(x))} = \operatorname{Rng} f = f(\sigma(x))$.

„6.“: Omitted.

„7.“: TODO!!! □

**Věta 3.3** (Bent Fuglede (1950), Calvin R. Putnam (1951))

*Let $A$ be complex $C^*$-algebra, $x \in A$ and $a, b \in A$ be complex such that $ax = xb$. Then $a^* x = x b^*$.*

*Důkaz*
TODO!!! (Maybe omitted?) □

# 4 Operators on Hilbert spaces

**Definice 4.1** (Sesquilinear map, sesquilinear form)

Let $X, Y$ be vector spaces over $\mathbb{C}$. Map $S : X \times X \to Y$ is called sesquilinear, if it is linear in the first variable and conjugate-linear in the second one. If $Y = \mathbb{C}$, $S$ is a sesquilinear form.

**Tvrzení 4.1** (Polarization identity)

*$X, Y$ vector spaces over $\mathbb{C}$ and $S : X \times X \to Y$ is a sesquilinear map. Then for all $x, y \in X$, it holds that*

$$S(x,y) = \frac{1}{4}(S(x+y,x+y) - S(x-y,x-y) + iS(x+iy,x+iy) - iS(x-iy,x-iy)).$$

*Důkaz*
TODO!!! □

*Důsledek*
$\{\mathbf{o}\} \neq H$ Hilbert space, $T, S \in \mathcal{L}(H)$. Then $T = S$ iff $\forall x \in H : \langle Tx, x\rangle = \langle Sx, x\rangle$.

*Důkaz*
TODO!!! □

**Věta 4.2**

*$\{\mathbf{o}\} \neq H$ Hilbert space and $T \in \mathcal{L}(H)$. Then*

- *$T$ is self-adjoint iff $\forall x \in H : \langle Tx, x\rangle \in \mathbb{R}$;*
- *$T$ is normal iff $\forall x \in H : \|Tx\| = \|T^*x\|$;*
- *$\forall x \in H : \langle Tx, x\rangle \geqslant 0$ iff $T$ is self-adjoint and $\sigma(T) \subseteq [0, \infty)$.*

*Důkaz*
TODO!!! □

**Definice 4.2** (Non-negative)

$A$ $C^*$-algebra and $x \in A$. We say that $x$ is non-negative ($x \geqslant 0$), if $x$ is self-adjoint and $\sigma(x) \subseteq [0, +\infty)$.

**Věta 4.3**

*$H$ Hilbert space and $T \in \mathcal{L}(H)$ normal. Then*

- *$\operatorname{Ker} T = \operatorname{Ker} T^*$ and $\operatorname{Ker} T = (\operatorname{Rng} T)^{\perp}$;*
- *$\operatorname{Rng} T$ is dense in $H$ iff $T$ is one-to-one;*
- *$\lambda \in \sigma_P(T)$ iff $\overline{\lambda} \in \sigma(T^*)$, eigenspace of $T$ associated with $\lambda$ is equal to eigenspace of $T^*$ associated with $\overline{\lambda}$;*
- *if $\lambda_1 \neq \lambda_2$ are eigenvalues of $T$, then $\operatorname{Ker}(\lambda_1 I - T) \perp \operatorname{Ker}(\lambda_2 I - T)$.*

*Důkaz*
TODO!!! □

**Věta 4.4** (Hilbert–Schmidt)

*$H$ Hilbert space and $T \in \mathcal{K}(H)$ nonzero normal. Then exists orthonormal basis $B$ of space $H$ consisting of eigenvectors of $T$. The set of vectors from $B$ associated with nonzero eigenvalues of $T$ is at most countable and we can arrange them to sequence $\{e_n\}_{n=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, then $\{e_n\}$ is orthonormal basis of $\overline{\operatorname{Rng} T}$ and for every $x \in H$:*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, e_n \rangle e_n,$$

*where $\lambda_n$ is eigenvalue associated with the eigenvector $e_n$.*

*Důkaz*
Omitted. „`OM4/Funkcionalka.pdf`“ □

**Věta 4.5** (Schmidt)

*$H$ Hilbert space and $T \in \mathcal{L}(H)$ nonzero compact. Then exists $N \in \mathbb{N}_0 \cup \{\infty\}$, sequence of positive numbers $\{\lambda_n\}_{n=1}^N$ and orthonormal systems $\{u_n\}_{n=1}^N$ and $\{v_n\}_{n=1}^N$ such that for every $x \in H$:*

$$Tx = \sum_{n=1}^{N} \lambda_n \langle x, u_n \rangle v_n.$$

*Důkaz*
TODO!!! □

**Věta 4.6**

*$H$ Hilbert space and $P \in \mathcal{L}(H)$ projection. Then following are equivalent: $P$ is orthogonal ($\operatorname{Rng} P \perp \operatorname{Ker} P$); $P \geqslant 0$; $P$ is self-adjoint; $P$ is normal.*

*Moreover, if $P, Q \in \mathcal{L}(H)$ are orthogonal projections, then $\operatorname{Rng}(P) \perp \operatorname{Rng}(Q)$ iff $PQ = 0$.*

*Důkaz*
TODO!!! □

**Definice 4.3** (Unitary operator)

$H, K$ Hilbert spaces. Operator $T \in \mathcal{L}(H, K)$ is called unitary, if $T^{-1} = T^*$, i.e., $T^* \circ T = I_H$ and $T \circ T^* = I_K$.

**Tvrzení 4.7**

*$H, K$ Hilbert spaces and $T \in \mathcal{L}(H, K)$. Then $T$ is unitary $\Leftrightarrow$ $T$ is isometry onto $\implies$ $T$ is isometry $\Leftrightarrow$ $\langle Tx, Ty \rangle = \langle x, y \rangle$ for every $x, y \in H$. Moreover if $T$ is onto, then all propositions are equivalent.*

*Důkaz* (TODO!!!) □

**Definice 4.4** (Partial isometry, initial subspace)

$H$ Hilbert space. Operator $U \in \mathcal{L}(H)$ is called partial isometry, if there is closed subspace $K \subset H$ (initial subspace of $U$) such that $U|_K$ is isometry and $U|_{K^\perp} \equiv 0$.

**Věta 4.8** (Polararization decomposition)

*$H$ Hilbert space, $T \in \mathcal{L}(H)$.*

*Exists unique operators $P, U \in \mathcal{L}(H)$ such that $P \geqslant 0$, $U$ is partial isometry with initial subspace $\overline{\operatorname{Rng} P}$ and $T = UP$. Moreover $P = \sqrt{T^*T} = U^*T$.*

*If $T$ is invertible, then exists unique $P, U \in \mathcal{L}(H)$ such that $P \geqslant 0$ is invertible, $U$ is unitary and $T = UP$.*

# 5 Borel measurable calculus

**Lemma 5.1** (Lax–Milgram)

*$H$ Hilbert, $S : H \times H \to C$ sesquilinear, $\|S\| := \sup_{x,y \in S_H} |S(x, y)| < \infty$. Then $\exists! T \in \mathcal{L}(H) : \|T\| = \|S\| \wedge \langle Tx, y \rangle = S(x, y)$.*

*Důkaz*

Fix $y \in H$. Then $H \ni x \mapsto S(x, y)$ is a point in $H^* \implies \exists! U(y) \in H : S(x, y) = \langle x, U(y) \rangle$, $x \in H$. Then $U \in \mathcal{L}(H)$, $\|U\| = \|S\|$.

„Linearity“: Easy: $\forall y, z \in H, \alpha \in \mathbb{K} \implies$

$$\forall x \in H : \langle x, U(\alpha y + z) \rangle = S(x, \alpha y + z) = \overline{\alpha} S(x, y) + S(x, z) = \overline{\alpha} \langle x, Uy \rangle + \langle x, Uz \rangle .$$

„$\|U\| \leqslant \|S\|$“:

$$\forall y \in H : \|Uy\|^2 = \langle Uy, Uy \rangle = S(Uy, y) \leqslant \|S\| \cdot \|Uy\| \cdot \|y\| \implies \|Uy\| \leqslant \|S\| \cdot \|y\|.$$

„$\|U\| \geqslant \|S\|$“:

$$\forall x, y \in S_H : \|S(x, y)| = |\langle x, Uy \rangle| \leqslant \|x\| \cdot \|U\| \cdot \|y\| = \|U\|.$$

„Uniqueness“: Bounded operator is given by values of $\langle Tx, y \rangle$. □

## Definice 5.1

$H$ Hilbert, $T \in \mathcal{L}(H)$ normal, $\Phi : \mathcal{C}(\sigma(T)) \to \mathcal{L}(H)$ continuous ?.

- $\forall x, y \in H$: $\mu_{x,y} \in M(\sigma(T))$ is the unique measure satisfying
$$\int_{\sigma(T)} f d\mu_{x,y} = \langle \Phi(f)x, y \rangle, \qquad f \in \mathcal{C}(\sigma(T)).$$
- $\forall f \in Bor_0(\sigma(T))$ (bounded, Borel) we get $\Phi(f) \in \mathcal{L}(H)$ be the unique operator such that
$$\langle \Phi(f)x, y \rangle = \int_{\sigma(T)} f d\mu_{x,y}, \qquad x, y \in H$$

*Důkaz*

„1.“: $f \mapsto \langle \Phi(f)x, y \rangle$ is linear and $|\langle \Phi(f)x, y \rangle| \leqslant \|\Phi(f)\| \cdot \|x\| \cdot \|y\|$. So $f \mapsto \langle \Phi(f)x, y \rangle \in \mathcal{C}(\sigma(T))^* = M(\sigma(T)) \implies \mu$ exists by Riesz representation theorem.

„2.“: $\forall x, x_2, y \in H\ \forall \alpha \in \mathbb{K}\ \forall f \in \mathcal{C}(\sigma(T))$ :
$$\langle \Phi(f)(\alpha x_1 + x_2), y \rangle = \alpha \langle \Phi(f)x_1, y \rangle + \langle \Phi(f)x_2, y \rangle = \alpha\mu_{x,y}(f) + \mu_{x_2,y}(f).$$

Thus $\cdot \mapsto \mu_{\cdot,y}$ is linear (for each $y$). Analogously $\cdot \mapsto \mu_{x,\cdot}$ is conjugate-linear.

Thus, $(x, y) \mapsto \mu_{x,y}(f) \in \mathbb{C}$ is sesquilinear form.
$$\forall x, y \in S_H : |\mu_{x,y}(f)| \leqslant \int |f| d|\mu_{x,y}| \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\| = \|f\|_\infty.$$

And from Lax–Milgram:
$$\exists! \Phi(f) \in \mathcal{L}(H) : \langle \Phi(f)x, y \rangle = \mu_{x,y}.$$

Moreover $\|\Phi(f)\| \leqslant \|f\|_\infty$. □

*Poznámka*

$H$ Hilbert, $T \in \mathcal{L}(H)$ normal:

- Mapping $H \times H \ni (x, y) \mapsto \mu_{x,y}$ is sesquilinear, so
$$\mu_{x,y} = \frac{1}{4}\left(\mu_{x+y,x+y} - \mu_{x-y,x-y} + i\mu_{x+iy,x+iy} - i\mu_{x-iy,x-iy}\right).$$
- $\forall x \in H : \mu_{x,x} \geqslant 0$. (Proof: „$f \geqslant 0 \implies \mu_{x,x}(f) \geqslant 0, f \in \mathcal{C}(\sigma(T))$“: $f \geqslant 0 \implies \Phi(f) \geqslant 0$ $(\sigma(\Phi(f)) = f(\sigma(T)) \subseteq [0, \infty) \implies \Phi(f) \geqslant 0.)$ So $\int_{\sigma(T)} f d\mu_{x,x} = \Phi(f)x, x \geqslant 0.)$

- $Bor_b(\sigma(T)) \subseteq l_\infty(\sigma(x)) \mapsto \mathcal{L}(H)$ is $C^*$-subalgebra.
- The mapping $\Phi : Bor_b(\sigma(x)) \to \mathcal{L}(H)$ from previous definition, is extension of continuous calculus from theorem above.

## Věta 5.2

*Let $P$ be a metric space, $\Phi$ be the smallest system of functions such that $\mathcal{C}_b(P) \subset \Phi$ and $\Phi$ is closed under point-wise bounded convergent sequences. Then $\Phi = Bor_b(P)$.*

*Důkaz* (Sketch)
Suffices: „$\forall A \subset P$ Borel: $\chi_A \in \Phi$."

$$\mathcal{F} := \{A \subset P \text{ Borel } | \chi_A \in \Phi\}$$

is $\sigma$-algebra containing closed sets $\implies \mathcal{F} = Bor(P)$. □

## Definice 5.2

Let $X, Y$ be normed linear spaces. On $\mathcal{L}(X, Y)$ we define the following two Hausdorff locally convex topologies:

- $\tau_{SOT}$ generated by pseudonorms $\{P_x(T) = \|T_x\| | x \in X\}$ (so, $T_i \overset{\text{SOT}}{\to} T \Leftrightarrow \forall x \in X : T_i x \overset{\|\cdot\|}{\to} Tx$);
- $\tau_{WOT}$ generated by pseudonorms $\{P_{x,y^*}(T) = y^*(Tx) | x \in X \wedge y^* \in Y^*\}$ (so, $T_i \overset{\text{WOT}}{\to} T \Leftrightarrow \forall x \in X : T_i x \overset{w}{T} x$) (in $X = Y = H$ Hilbert: $\Leftrightarrow \forall x, y \in H : \langle T_i x, y\rangle \to \langle Tx, y\rangle$).

*Poznámka*

$$T_i \overset{\|\cdot\|}{\to} T \implies T_i \overset{\text{SOT}}{\to} T \implies T_i \overset{\text{WOT}}{\to} T.$$

*Například*
$R_n x := (0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)$, $x \in l_2$. Then $R_n \in \mathcal{L}(l_2)$, $n \in \mathbb{N}$, and $R_n \overset{\|\cdot\|}{\nrightarrow} 0$, because $\|R_n(e_{n+1})\| = 1$, $n \in \mathbb{N}$. But $R_n \overset{\text{SOT}}{\to} 0$, because $\forall x \in l_2 : \|R_n x\|_2^2 = \sum_{i=n+1}^\infty |x_i|^2 \to 0$.

$S_n x := (0, 0, \ldots, 0, x_1, x_2, \ldots)$, $x \in l_2$. Then $S_n \in \mathcal{L}(l_2)$ is isometry $\forall n \in \mathbb{N}$. But $S_n \overset{\text{SOT}}{\nrightarrow} 0$, because $\|S_n(e_1)\| = 1 \nrightarrow 0$. But $S_n \overset{\text{WOT}}{\to} 0$, because $\forall x, y \in l_2$:

$$|\langle S_n x, y\rangle| = |\sum_{i=1}^\infty x_i y_{n+i}| \leqslant \|x\|_2 \sqrt{\sum_{i=n+1}^\infty |y_i|^2} \to 0.$$

## Věta 5.3

*H Hilbert, $T \in \mathcal{L}(H)$ normal, $f \in Bor_b(\sigma(T))$, $\Phi : Bor_b(\sigma(T)) \to \mathcal{L}(H)$ as in definition above. Then*

$\Phi$ *is continuous $*$-homomorphism and $\|\Phi\| = 1$;*

*Důkaz*

$\Phi$ is linear (easy from definition). $\|\Phi\| \leqslant 1$ follows from the second point of the previous theorem, and $\|\Phi(1)\| = \|\operatorname{id}\| = 1$, so $\|\Phi\| = 1$.

„$\Phi$ is multiplicative“: Step 1: „$\mathcal{F} \coloneqq \{g \in Bor_b(\sigma(T)) | \forall f \in \mathcal{C}(\sigma(t)) \colon \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$, then $\mathcal{F} = Bor_b(\sigma(T))$“: $\mathcal{F} \subseteq \mathcal{C}(\sigma(T))$ follows from continuous calculus, „$\mathcal{F}$ closed under point-wise limits of bounded sequences“: Let $\mathcal{F} \ni g_n \to g$ and $f \in \mathcal{C}(\sigma(T))$. Then $g_n f \to gf$ point-wise. So, for $x, y \in H$:

$$\langle \Phi(g, f)x, y \rangle = \int_{\sigma(T)} gf d\mu_{x,y} = \lim \int g_n f d\mu_{x,y} = \lim \langle \Phi(g_n)x, y \rangle =$$

$$= \lim \langle \Phi(g_n)(\Phi(f)x), y \rangle = \lim \int g_n d\mu_{\Phi(f)x,y} = \langle \Phi(g)(\Phi(f)x), y \rangle .$$

$\implies \mathcal{F} = Bor_b(\sigma(T))$.

Step 2: „$\mathcal{H} := \{g \in Bor_b(\sigma(T)) | \forall f \in Bor_b(\sigma(t)) : \Phi(gf) = \Phi(g) \cdot \Phi(f)\}$, then $\mathcal{H} = Bor_b(\sigma(T))$“: „$\mathcal{H}$ is closed under point-wise limits of bounded sequences“: $\mathcal{H} \ni f_n \overset{\tau_p}{f}$, $f_n$ bounded, then $\forall x, y \in H \ \forall g \in Bor_b(\sigma(T))$:

$$\langle \Phi(gf)x, y \rangle \overset{\text{Lebesgue}}{=} \lim_n \langle \varphi(gf_n)x, y \rangle = \lim_n \langle \Phi(g)\Phi(f_n)x, y \rangle = \lim_n \langle \Phi(f_n)x, \Phi(g)^* y \rangle =$$

$$= \lim_n \int f_n d\mu_{x,\Phi(g)^*y} \overset{\text{Lebesgue}}{=} \int f d\mu_{x,\Phi(g)^*y} = \langle \Phi(f)x, \Phi(g)^* y \rangle = \langle \Phi(g)\Phi(f)x, y \rangle .$$

Thus $\Phi(gf) = \Phi(g)\Phi(f)$.

„$\Phi$ preserves $*$“: $\mathcal{F} := \{g \in Bor_b(\sigma(T)) | \Phi(g)^* = \Phi(\overline{g})\}$. Then $\mathcal{F} \subseteq \mathcal{C}(\sigma)$ by continuous calculus and $\mathcal{F}$ is ”closed under taking limits” analogously as above. $\implies \mathcal{F} = Bor_b(\sigma(T))$.

□

$(f_n) \in Bor_b(\sigma(T))^{\mathbb{N}}$ *bounded and* $f_n \overset{\tau_p}{f}$*, then* $\Phi(f_n) \overset{SOT}{\to} \Phi(f)$.

*Důkaz*

Step 1: „$\Phi(f_n) \overset{\mathrm{WOT}}{\Phi} (f)$“:

$$\forall x, y \in H : \langle \Phi(f_n)x, y \rangle \overset{\text{Lebesgue}}{\to} \langle \Phi(f)x, uy \rangle .$$

Step 2: „$\|\Phi(f_n)x\| \to \|\Phi(f)x\|, x \in H$“:

$$\|\Phi(f_n)x\|^2 = \left\langle \Phi(\overline{f_n})\Phi(f_n)x, x \right\rangle = \left\langle \Phi(\overline{f_n}f_n)x, x \right\rangle \overset{\text{Lebesgue}}{\to} \left\langle \Phi(\overline{f}f)x, x \right\rangle = \|\Phi(f)x\|^2.$$

Step 3: From steps 1 and 2:

$$\|\Phi(f_n)x - \Phi(f)x\|^2 \overset{\text{Cos. věta}}{=} \|\Phi(f_n)x\|^2 + \|\Phi(f)x\|^2 - 2\Re \langle \Phi(f_n)x, \Phi(f)x \rangle \to 0.$$

□

*If* $K \subset \mathbb{C}$ *is compact,* $K \supseteq \sigma(T)$*,* $\psi : Bor_b(K) \to \mathcal{L}(H)$ *is continuous* $*$*-homomorphism,* $\psi(1) = \mathrm{id}$*,* $\psi(\mathrm{id}) = T$ *and* $f_n \overset{\tau_p}{\to} f \implies \psi(f_n) \overset{WOT}{\to} \psi(f)$*. Then* $\psi(g) = \Phi(g|_{\sigma(T)})$*,* $g \in Bor_b(K)$*.*

*Důkaz*

Skipped. Using characterization of $Bor_b$. □

$\Phi(f)$ *is normal,* $\Phi(f)$ *is self-adjoint* $\Leftrightarrow$ $f$ *is real.*

*Důkaz*

Skipped. Easy from first part of theorem. □

$\sigma(\Phi(f)) \subseteq \overline{f(\sigma(T))}$.

$g \in Bor_b(\overline{\operatorname{Rng} f}) \implies (g \circ f)(T) = g(f(T))$.

$\forall S \in \mathcal{L}(H), ST = TS : Sf(T) = f(T)S$.

# 6 Spectral decomposition of normal operator

**Definice 6.1** (Spectral measure)

$H$ Hilbert space, $(X, \mathcal{A})$ measurable space. Then $E : \mathcal{A} \mapsto \mathcal{L}(H)$ is spectral measure for $(X, \mathcal{A}, H)$ if

- $\forall A \in \mathcal{A} : E(A)$ is orthogonal projection;
- $E(X) = \text{id}$, $E(\emptyset) = 0$;
- if $\{A_n, n \in \mathbb{N}\} \subset \mathcal{A}$ is point-wise disjoint, then

$$E(\bigcup A_n)x = \sum_{n=1}^{\infty} E(A_n)x, x \in H.$$

**Tvrzení 6.1** (Properties of spectral measure)

*$H$ Hilbert, $(X, \mathcal{A})$ measurable space, $E$ is spectral measure for $(X, \mathcal{A}, H)$. Then*

1. $\forall A, B \in \mathcal{A}, A \subset B : E(A) \leqslant E(B)$ *(that's $E(B) - E(A) \geqslant 0$);*
2. $\forall A, B \in \mathcal{A} : E(A \cap B) = E(A) \cdot E(B)$, *in particular, if $A \cap B = \emptyset$, then $E(A) \cdot E(B) = \emptyset$.*
3. $\forall x, y \in H : \mathcal{A} \ni A \mapsto \langle E(A)x, y\rangle$ *is a complex measure (denoted by $E_{x,y}$), with total variation $\|E_{x,y}\| \leqslant \|x\| \cdot \|y\|$.*
4. $(x, y) \mapsto E_{x,y}$ *is sesquilinear mapping.*
5. $\forall x, y \in H$ $\forall A \in \mathcal{A}$:
$$|E_{x,y}(A)| \leqslant \frac{1}{2} \left(E_{x,x}(A) + E_{y,y}(A)\right).$$
6. $\forall x, y \in H$:
$$E_{x+y,x+y} \leqslant 2 \left(E_{x,x} + E_{y,y}\right).$$

*Důkaz*
„1.“: $E(A)+E(B\backslash A)=E(B)$, so $E(B)-E(A)=E(B\backslash A)\geqslant 0$.

„2.“: „Step 1: $A\cap B=\varnothing$“:

$$\mathrm{id}=E(X)=E(A)+E(A^c)\geqslant E(A)+E(B),$$

so $E(B)\leqslant \mathrm{id}-E(A)$, which is orthogonal projection onto $(\operatorname{Rng}E(A))^\perp$. Thus („$P,Q\in\mathcal{L}(A)$ orthogonal projections, $Q-P\geqslant 0$, then $\operatorname{Rng}P\subset(\operatorname{Rng}Q)^{\perp}$“: $\|Px\|^2=$

$$=\|QPx\|^2+\|(\mathrm{id}-Q)Px\|^2=\langle QPx,Px\rangle+\|(\mathrm{id}-Q)Px\|^2\geqslant\underbrace{\langle PPx,Px\rangle}_{\|Px\|^2}+\|(\mathrm{id}-Q)Px\|^2,$$

thus, $(\mathrm{id}-Q)Px=0$, so $\operatorname{Rng}P\subseteq\operatorname{Ker}(\mathrm{id}-Q)=\operatorname{Rng}Q$.) $\operatorname{Rng}E(B)\subseteq(\operatorname{Rng}E(A))^\perp$. Thus $\operatorname{Rng}E(A)\perp\operatorname{Rng}E(B)$, so $E(A)\cdot E(B)=0$.

„Step 2: In general“:

$$E(A)=E(A\cap B)+E(A\backslash B),\qquad E(B)=E(A\cap B)+E(B\backslash)\implies$$

$$E(A)\cdot E(B)=(E(A\backslash B)+E(A\backslash B))\cdot(E(A\cap B)+E(B\backslash A))=E^2(A\cap B)+3\cdot 0=E(A\cap B).$$

„3.“: „$E_{x,y}$ is countably additive“ is easy. By this it is a complex measure. „Calculation of $\|E_{x,y}\|$“: Fix $A_1,\dots,A_n\in\mathcal{A}$ disjoint such that $\bigcup_{i=1}^n A_i=X$. For $i\in[n]$ pick $\alpha_i\in S_{\mathbb{C}}:\alpha_i\langle E(A_i)x,y\rangle=|\langle E(A_i)x,y\rangle|$. Then

$$\sum_{i=1}^n|E_{x,y}(A_i)|=\sum_{i=1}^n\alpha_i\langle E(A_i)x,y\rangle\overset{\text{Cauchy–Schwartz}}{\leqslant}\|\sum_{i=1}^n\alpha_iE(A_i)x\|\cdot\|y\|.$$

$$\left\|\sum_{i=1}^nE(A_i)(\alpha_ix)\right\|^2\overset{\text{Pythagoras}}{=}\sum_{i=1}^n\|E(A_i)(\alpha_ix)\|=\sum_{i=1}^n\|E(A_i)(x)\|=\sum_{i=1}^n\langle E(A_i)x,x\rangle=$$
$$=\left\langle E\left(\bigcup A_i\right)x,x\right\rangle=\langle x,x\rangle=\|x\|^2.$$

„4.“: Easy, using definition. „5.“:

$$|E_{x,y}(A)|=|\langle E(A)x,y\rangle|=|\langle E(A)x,E(A)y\rangle|\overset{\text{Cauchy-Schwartz}}{\leqslant}\|E(A)x\|\cdot\|E(A)y\|=$$
$$=\sqrt{E_{x,x}(A)}\cdot\sqrt{E_{y,y}(A)}\overset{\text{A–G}}{\leqslant}\frac{1}{2}\left(E_{x,x}(A)+E_{y,y}(A)\right).$$

„6.“:

$$E_{x+y,x+y}(A)=E_{x,x}(A)+E_{y,x}(A)+E_{x,y}(A)+E_{y,y}(A)\leqslant E_{x,x}(A)+2\Re E_{y,x}(A)+E_{y,y}(A)\leqslant$$
$$\leqslant E_{x,x}(A)+2\cdot\frac{1}{2}\left(E_{x,x}(A)+E_{y,y}(A)\right)+E_{y,y}(A)=2\left(E_{x,x}(A)+E_{y,y}(A)\right).$$

$\square$

*Poznámka*
From 4. we get $E_{x,y}(A) = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle E(A)(x+i^k y), x+iky \right\rangle$. Thus 3. is equivalent to $\forall x \in H : E_{x,x} \geqslant 0$ is measure.

**Definice 6.2** (Integral)

$H$ Hilbert space, $(X,\mathcal{A})$ measurable space, $E$ spectral measure for $(X,\mathcal{A},H)$. $f : X \to \mathbb{C}$ bounded $\mathcal{A}$-measurable function. Then integral of $f$ with respect to $E$ is the operator $T \in \mathcal{L}(H)$ such that

$$\langle Tx, y\rangle = \int_X f dE_{x,y}, \qquad x,y \in H.$$

Notation: Then $\int f dE := T$.

*Poznámka*
It always exists due to Lax–Milgram: $(x,y) \mapsto \int f dE_{x,y}$ is sesquilinear and $\left|\int f dE_{x,y}\right| \leqslant \|f\|_\infty \cdot \|E_{x,y}\| \leqslant \|f\|_\infty \cdot \|x\| \cdot \|y\|$. So $T$ exists and $\|T\| \leqslant \|f\|_\infty$.

**Tvrzení 6.2**

*$H$ Hilbert, $(X,\mathcal{A})$ measurable space, $E$ spectral measure for $(X,\mathcal{A},H)$, $f : X \to \mathbb{C}$ bounded $\mathcal{A}$-measurable. Then for $\varepsilon > 0$ pick $A_1,\dots,A_m \in \mathcal{A}$ disjoint partition of $X$ such that $\operatorname{diam} f(A_i) < \varepsilon$ and for $x_i \in A_i$, $i \in [n]$*

$$\left\|\int f dE - \sum_{i=1}^{n} f(x_i)E(A_i)\right\| < \varepsilon.$$

*Důkaz*
Denote $T = \int f dE$. For $x,y \in H : |\langle Tx,y\rangle - \langle \sum f(x_i)E(A_i)x,y\rangle| =$

$$= \left|\sum_{i=1}^{n}\int_{A_i} (f(t)-f(x_i))\, dE_{x,y}\right| \leqslant \sum_{i=1}^{n}\int_{A_i} |f(t)-f(x_i)| d|E_{x,y}| \leqslant \varepsilon \int_X d|E_{x,y}| \leqslant \varepsilon \cdot \|x\| \cdot \|y\|.$$

This finishes the proof. ($|\langle Sx,y\rangle| \leqslant \varepsilon \cdot \|x\| \cdot \|y\| \implies \|S\| < \varepsilon$.) □

**Definice 6.3** (Notation)

$(X,\mathcal{A})$ measurable space, $B(X,\mathcal{A}) \subset l_\infty(X)$ $c^*$ algebra consisting of bounded $f : X \to \mathcal{C}$ $\mathcal{A}$-measurable functions.

**Tvrzení 6.3**

*$H$ Hilbert, $(X,\mathcal{A})$ measurable space, $E$ spectral measure for $(X,\mathcal{A},H)$. Consider $\varrho : B(X,\mathcal{A}) \to \mathcal{L}(H)$, $\varrho(f) = \int f dE$. Then*

1. *$\varrho$ is continuous $*$-homomorphism, $\|\varrho\| = 1$, $\varrho(1) = \mathrm{id}$.*

2. *$\forall f \in B(X, \mathcal{A}) : \varrho(f)$ is normal. $f$ is real $\implies$ $\varrho(f)$ is self-adjoint, $f \geqslant 0 \implies \varrho(f) \geqslant 0$.*

3. *$f_n \in B(X, \mathcal{A})^n$ bounded, $f_n \to f$ point-wise $\implies$ $\varrho(f_n) \overset{WOT}{\to} \varrho(f)$.*

4. *$\forall f \in B(X, \mathcal{A})$ $\forall x \in H$: $\|\varrho(f)x\| = \sqrt{\int |f|^2 dE_{x,x}}.i$*

5. *$\int f dE$ is the unique $T \in \mathcal{L}(H)$: $\langle Tx, y\rangle = \int f dE_{x,y}$, $x, y \in H$.*

*Důkaz*

1.) „$\varrho$ is linear“: easy. „$\|\varrho\| \leqslant 1$“: easy as well. „$\varrho$ preserves $*$“:

$$\forall x \in H : \langle \varrho(f)^* x, x\rangle = \langle x, \varrho(f)x\rangle = \overline{\langle \varrho(f)x, x\rangle} = \overline{\int f dE_{x,x}} = \int \overline{f} dE_{x,x} = \left\langle \varrho(\overline{f})x, x\right\rangle.$$

„$\varrho$ is multiplicative“: For $f, g \in B(X, \mathcal{A})$, $\varepsilon > 0$. Find disjoint partition $A_1, \dots, A_n \in \mathcal{A}$ of $X$ such that for $\omega \in \{f, g, f \cdot g\}$ we have $\operatorname{diam} \omega(A_i) < \varepsilon$ for $i \in [n]$. Pick $x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$. Thus using previous proposition we have

$$\left\| \int fgdE - \left(\int fdE\right)\left(\int gdE\right)\right\| \leqslant \varepsilon +$$

$$+\left\|\sum_{i=1}^{n}(f \cdot g)(x_i)E(A_i) - \left(\sum f(x_i)E(A_i)\right)\left(\sum g(x_i)E(A_i)\right)\right\| +$$

$$+\left\|\left(\sum f(x_i)E(A_i)\right)\left(\sum g(x_i)E(A_i)\right) - \left(\int fdE\right)\left(\int gdE\right)\right\| \leqslant \varepsilon + 0 +$$

$$+\left\|\left(\sum f(x_i)E(A_i)\right)\left(\sum g(x_i)E(A_i) - \int gdE\right)\right\| + \|(\sum f(x_i)E(A_i))TODO <$$

$$< \|f\|_\infty \cdot \varepsilon + \varepsilon \cdot \|g\|_\infty.$$

„$\|\varrho\| = 1$“: TODO!!!

„$\varrho(1) = \mathrm{id}$“: $\forall x \in H : \langle \varrho(1)x, x\rangle = \int_X 1 dE_{x,x} = \langle E(X)x, x\rangle = \langle x, x\rangle = \langle \mathrm{id}\, x, x\rangle$.

2.)
$$\varrho(f)^*\varrho(f) = \varrho(\overline{f}f) = \varrho(f\overline{f}) = \varrho(f)\varrho(f)^* \implies \varrho(f) \text{ is normal.}$$
$$f \text{ is real} \implies f = \overline{f} \implies \varrho(f) = \varrho(f)^*.$$
$$f \geqslant 0 \implies \forall x \in H : \langle \varrho(f)x, x\rangle = \int fdE_{x,x} \geqslant 0 \implies \varrho(f) \geqslant 0.$$

3.)
$$\forall x, y \in H : \langle \varrho(f_n)x, y\rangle = \int f_n dE_{x,y} \overset{\text{Lebesgue}}{\to} \int fdE_{x,y} = \langle \varrho(f)x, y\rangle.$$

4.)
$$\|\varrho(f)x\|^2 = \langle \varrho(f)x, \varrho(f)x\rangle = \left\langle \varrho(\overline{f}f)x, x\right\rangle = \int \overline{f}fdE_{x,x} = \int |f|^2 dE_{x,x}.$$

□

*Důsledek* (Spectral decomposition of normal operator)
$H$ Hilbert, $T \in \mathcal{L}(H)$ normal $\implies \exists!$ spectral measure $E$ for $(\sigma(T), Bor(\sigma(T)), H)$: $T = \int \mathrm{id}\, dE$. Moreover $E(A) = \Phi(\chi_A)$ for any $A \in Bor(\sigma(T))$, where $\Phi : Bor_b(\sigma(T)) \to \mathcal{L}(H)$ is borel calculus from definition above.

*Důkaz*
Whenever $E$ is spectral measure for $(\sigma(T), Bor(\sigma(T)), H)$ satisfying $T = \int \mathrm{id}\, dE$, then $\int f dE = \Phi(f)$, $f \in \mathcal{B}(\sigma(T), Bor(\sigma(T)))$. This proves uniqueness.

„Existence“: Put $E(A) := \Phi(A)$, $A \subset \sigma(T)$ borel. Then $E$ is spectral measure: $E(A)$ is orthogonal projection ($\chi_A^2 = \chi_A$, $\chi_A$ is real), $E(\sigma(T)) = \mathrm{id}$, $E(\emptyset) = 0$ ($\chi_{\sigma(T)} = 1$ and $\Phi(1) = \mathrm{id}$, $\chi_\emptyset = 0$), $A_i \in Borel(\sigma(T))$ disjoint, $x \in H$, then

$$\left\| E\left(\bigcup A_n\right) x - \sum E(A_i) x \right\| = \left\langle E\left(\bigcup A_i\right) x, E\left(\bigcup A_i\right) x \right\rangle = \left\langle E\left(\bigcup A_i\right) x, x \right\rangle =$$

$$= \int \chi_{\bigcup A_i} d\mu_{x,x} = \sum_{N+1}^{\infty} \mu_{x,x}(A_i) \to 0.$$

„$T = \int \mathrm{id}\, dE$“: $E_{x,y} = \mu_{x,y}$ ($E_{x,y}(A) = \langle E(A)x, y \rangle = \int \chi_A d\mu_{x,y} = \mu_{x,y}(A)$). Thus

$$\left\langle \int \mathrm{id}\, dEx, y \right\rangle = \int \mathrm{id}\, dE_{x,y} = \int \mathrm{id}\, d\mu_{x,y} = \langle \Phi(\mathrm{id})x, y \rangle = \langle Tx, y \rangle .$$

□

# 7 Unbounded operators

## Definice 7.1

$X, Y$ Banach spaces. Operator from $X$ to $Y$ is a linear mapping defined on a linear space $D(T) \subset X$ with values in $R(T) \subset Y$. If $X = Y$, we say $T$ is operator on $X$. Then graph of $T$ is $G(T) = \{(x, Tx) | x \in D(T)\} \subseteq X \times Y$.

We say that $T$ is densely defined $\equiv \overline{D(T)} = X$. We say that $T$ is closed $\equiv G(T) \subset X \times Y$ is closed.

## Definice 7.2 (Notations)

$X, Y$ Banach spaces. If $T, S$ is operator from $X$ to $Y$, then $S + T$ is operator from $X$ to $Y$ defined as $(S+T)(x) = Sx + T(x)$ for $x \in D(S+T) = D(S) \cap D(T)$.

If $T$ is operator from $X$ to $Y$ and $S$ is operator from $Y$ to a Banach space $Z$, then $ST$ is operator with $D(ST) = \{x \in D(T) | Tx \in D(S)\}$ defined as $(ST)x = S(Tx)$ for $x \in D(ST)$.

Operator $S$ from $X$ to $Y$ is extension of $T$, if $G(S) \supset G(T)$ (and we write $T \subset S$).

*Například*
$D(T) = c_{00} \subset l_2 = X$, $Tx = (\sum_{n=1}^{\infty} x_n, 0, 0, 0, 0, \dots)$. Then $T$ is densely defined, butt it doesn't have closed extension.

*Důkaz*
Consider $x^n = \left(\frac{1}{2^n}, \dots, \frac{1}{2^n}, 0, \dots\right)$ then $(x_n, Tx_n) \to (\mathbf{o}, e_1)$, so if there is extension, then $(\mathbf{o}, e_1) \in G(S)$, but $S\mathbf{o} = \mathbf{o}$, because of linearity. □

*Poznámka*
It is easy to check:

$$(S+T)+V = S+(T+V),$$
$$(ST)V = S(TV),$$
$$(S+T)V = SV+TV.$$

*Pozor*

$$V(S+T) \supseteq VS+VT.$$

## Lemma 7.1

*$X$, $Y$ Banach and $L \subseteq X \times Y$. Then $\exists$ operator $T$ from $X$ to $Y$ such that $L = G(T) \Leftrightarrow L$ is a subspace and $\{(x,y) \in L | x = 0\} = \{(0,0)\}$.*

*Důkaz*
„ $\implies$ “: Easy.

„ $\impliedby$ “: Put $D(T) = \{x \in X | \exists y \in Y : (x,y) \in L\}$. Then $\forall x \in D(T)\ \exists! y \in Y$: $(x,y) \in L$. $((x,y_1),(x,y_2) \in L \implies (0, y_1 - y_2) \in L.)$ So, we put $Tx := y$, where $y \in L$ is such that $(x,y) \in L$. Then $T$ is linear and $G(T) = L$ □

## Tvrzení 7.2

*$X$, $Y$ Banach spaces, $T$ operator from $X$ to $Y$.*

- *$D(T) = X \wedge T$ is closed $\implies T \in \mathcal{L}(X,Y)$.*
- *Equivalence:*
  1. *$T$ has closed extension;*
  2. *$(x_n, Tx_n) \to (0,y)$ in $D(T) \times Y \implies y = 0$;*
  3. *$\overline{G(T)} \subset X \times Y$ is graph of an operator from $X$ to $Y$.*
- *$T$ is one-to-one and closed $\implies T^{-1}$ is closed.*

*Důkaz*
First point follows immediately from closed graph theorem.

„1.) $\implies$ 2.)“: Let $S \supset T$ be closed. If $(x_n, Tx_n) \to (\mathbf{o}, y)$, then $(\mathbf{o}, y) \in G(S)$, so $\mathbf{o} = S\mathbf{o} = y$.

„2.) $\implies$ 3.)“ We will show, using the previous lemma, that $\overline{G(T)}$ is graph of an operator: $\overline{G(T)}$ is linear, because $G(T)$ is linear. If $(\mathbf{o}, y) \in \overline{G(T)}$, then $\exists (x_n) \in D(T)^{\mathbb{N}}$ : $(x_n, Tx_n) \to (\mathbf{o}, y)$, so $y = \mathbf{o}$ from 2)..

„3.) $\implies$ 1.)“: Clear.

Third point $\Phi : X \times Y \to Y \times X$ defined as $(x, y) \mapsto (y, x)$ is homeomorphism, so, $G(T)$ is closed $\Leftrightarrow \Phi(G(T)) = G(T^{-1})$ is closed. □

**Definice 7.3** (Closure of operator)

$X, Y$ Banach spaces, $T$ operator from $X$ to $Y$, $T$ has closed extension. Then $\overline{T}$ is operator satisfying $\overline{T} \supset T$ and $G(\overline{T}) = \overline{G(T)}$.

**Tvrzení 7.3**

*$X, Y, Z$ Banach spaces, $T$ operator from $X$ to $Y$, which is closed.*

- *If $S \in \mathcal{L}(X, Y)$, then $S + T$ is closed and $D(S + T) = D(T)$.*
- *If $S \in \mathcal{L}(Y, Z)$, then $D(ST) = D(T)$ and if $S$ is isomorphism into, then $ST$ is closed.*
- *If $S = \mathcal{L}(Z, X)$, then $TS$ is closed.*

*Důkaz*
Of course $D(S + T) = D(S) \cap D(T) = D(T)$. If $(x_n, (S + T)x_n) \to (x, y)$, then $Tx_n = (S+T)x_n - Sx_n \to y - Sx$. So $(x_n, Tx_n) \to (x, y - Sx) \in G(T)$, so $Tx = y - Sx \implies y = (T + S)x$.

$$D(ST) = \{x \in D(T) | Tx \in D(S) = Y\} = D(T).$$

Suppose $S$ is isomorphism into, $(x_n, STx_n) \to (x, z)$, then $Tx_n = S^{-1}STx_n \to S^{-1}z$. So $(x_n, Tx_n) \to (x, S^{-1}z) \in G(T)$, so $Tx = S^{-1}z$, then $STx = z$.

$(z_n, TSz_n) \to (x, y)$, then $Sz_n \to Sx$, so $(Sz_n, TSz_n) \to (Sx, y) \in G(T)$, thus $TSx = y$. □

TODO example?

**Tvrzení 7.4**

*$X, Y$ Banach, $T$ one-to-one closed operator from $X$ to $Y$. Then following statements are equivalent:*

$$\operatorname{Rng} T = Y \wedge T^{-1} \in \mathcal{L}(Y, X); \quad \operatorname{Rng} T = Y; \quad \operatorname{Rng} T \textit{ is dense and } T^{-1} \in \mathcal{L}(\operatorname{Rng} T, X).$$

*Důkaz*

„1) $\implies$ 2)“: trivial. „2) $\implies$ 3)“: $\operatorname{Rng} T$ is dense and $T^{-1}(\operatorname{Rng} T, X)$ due to previous proposition (by which $T^{-1}$ is closed).

„3) $\implies$ 1)“: Let $S \in \mathcal{L}(Y, X)$ be continuous extension of $T^{-1}$. Pick $y \in Y$. Since $\overline{\operatorname{Rng} T} = Y$, there is $(x_n) \in X^{\mathbb{N}}$ such that $Tx_n \to y$. Then $STx_n = T^{-1}Tx_n = x_n \to Sy$. So $(x_n, Tx_n) \to (Sy, y) \in G(T)$, thus $TSy = y \in \operatorname{Rng} T$. □

**Definice 7.4** (Resolvent set, resolvent function, spectrum of operator)

$X$ Banach, $T$ linear operator on $X$. Then resolvent set is

$$\varrho(T) := \{\lambda \in \mathbb{K} | \lambda I - T \text{ has inverse which belongs to } \mathcal{L}(X)\};$$

resolvent function is $R_T(\lambda) := (\lambda I - T)^{-1}$, $\lambda \in \varrho(T)$; spectrum of $T$ is $\sigma(T) := \mathbb{K} \backslash \varrho(T)$.

**Věta 7.5**

*$X$ Banach, $T$ linear operator on $X$. Then $\varrho(T)$ is open, $\varrho(T)$ is closed and $R_T$ has derivative at each point of $\varrho(T)$. (So, if $X$ is complex, then $R_t$ is holomorphic on $\varrho(T)$).*

*Důkaz*

„$\varrho(T)$ is open“: Pick $\lambda \in \varrho(T)$ and $h \in \mathbb{K}$ small ($|\cdot|$) enough: $|h| < \frac{1}{\|(\lambda I - T)^{-1}\|}$. Then $h(\lambda I - T)^{-1} =: S \in \mathcal{L}(X)$, $\|S\| < 1$. Thus, $(I + S)^{-1}$ exists, so $(\lambda + h)I - T = (I + S) \cdot (\lambda I - T)$ has inverse $(\lambda I - T)^{-1} \circ (I + S)^{-1} \in \mathcal{L}(X)$. $(\lambda I - T)^{-1} \circ (I + S)^{-1} \in \mathcal{L}(X)$. So $U(\lambda, \frac{1}{\|(\lambda I - T^{-1})\|}) \subset \varrho(T)$.

„$R_T$ has derivative at each $\lambda \in \varrho(T)$“: $R_T'(\lambda) = -R_T(\lambda)^2$:

$$\forall h \text{ small enough}: \left\| \frac{R_t(\lambda + h) - R_t(\lambda)}{h} + R_T(\lambda)^2 \right\| =$$

$$\frac{1}{h} \|R_T(\lambda + h) - R_T(\lambda) + R_T(\lambda) h R_T(\lambda)\| = \frac{\|R_T(\lambda)\|}{\|h\|} \cdot \|(I + S)^{-1} - I + hR_T(\lambda)\| =$$

$$\left( (I + S)^{-1} = \sum_{n=0}^{\infty} (-S)^n = I - S + \sum_{n=2}^{\infty} (-S)^n = I - hR_T(\lambda) + \sum_{n=2}^{\infty} (-hR_T(\lambda))^n \right)$$

$$= \frac{\|R_T(\lambda)\|}{|h|} \cdot \left\| \sum_{n=2}^{\infty} (-hR_T(\lambda))^n \right\| \leqslant \frac{\|R_t(\lambda)\|}{|h|} \sum_{n=2}^{\infty} \|hR_t(\lambda)\|^n = \frac{\|R_T(\lambda)\|}{|h|} \cdot \frac{\|hR_T(\lambda)\|^2}{1 - \|hR_T(\lambda)\|} \leqslant$$

$$\leqslant \frac{\|R_T(\lambda)\|}{|h|} \cdot \frac{|h|^2 \|R_T(\lambda)\|^2}{1/2} = 2|h| \cdot \|R_T(\lambda)\|^3 \to 0.$$

□

**Lemma 7.6**

*$X$ Banach space, $T$ operator in $X$, $0 \notin \sigma(T)$. Then $\forall \lambda \neq 0 : \lambda \in \sigma(T) \Leftrightarrow \frac{1}{\lambda} \in \sigma(T^{-1})$.*

*Důkaz*

Since $0 \in \varrho(T)$, so $T^{-1} \in \mathcal{L}(X)$. Moreover, $T = (T^{-1})^{-1}$ is closed (by proposition above). In the same time, since $T$ is closed, we have $\lambda \in \varrho(T) \Leftrightarrow \lambda I - T$ is bijection („ $\Longrightarrow$ “: trivial, „ $\Longleftarrow$ “: $\lambda I - T$ is bijection and closed operator, so by previous proposition $(\lambda I - T)^{-1} \in \mathcal{L}(X)$).

So, it suffices: „$\forall \lambda \neq 0$: $\lambda I - T$ bijection $\Leftrightarrow \frac{1}{\lambda} I - T^{-1}$ bijection“:

$$\frac{1}{\lambda} I - T^{-1} = -\frac{1}{\lambda}(\lambda I - T)T^{-1} \qquad \left(\text{so } (\lambda I - T)^{-1} \text{ exists } \Longrightarrow (\frac{1}{\lambda} I - T^{-1})^{-1} \text{ exists}\right)$$

$$\lambda I - T = -\lambda(\frac{1}{\lambda} I - T^{-1})T \qquad \left(\text{so } (\frac{1}{\lambda} I - T^{-1})^{-1} \text{ exists } \Longrightarrow (\lambda I - T)^{-1} \text{ exists}\right).$$

□

*Důsledek*

$X$ complex Banach, $T$ operator on $X$, $\sigma(T) = \emptyset$. Then $T^{-1} \in \mathcal{L}(X)$ and $\sigma(T^{-1}) = \{0\}$.

*Důkaz*

$0 \in \varrho(T) \implies T^{-1} \in \mathcal{L}(x)$. By previous lemma, $\forall \lambda \neq 0 : \frac{1}{\lambda} \notin \sigma(T^{-1})$. So $\sigma(T^{-1}) \subset \{0\}$. Since $\sigma(T^{-1}) \neq \emptyset$, we have $\sigma(T^{-1}) = \{0\}$. □

## 7.1 Unbounded operators in Hilbert spaces

**Definice 7.5** (Convention)

From now, all Banach spaces are over $\mathbb{K} = \mathbb{C}$ (if not said otherwise).

**Definice 7.6** (Hilbert adjoint of operator)

$H$ Hilbert, $T$ densely defined operator on $H$. Hilbert adjoint of $T$, denoted as $T^*$, is defined on $D(T^*) := \{y \in H | x \mapsto \langle Tx, y \rangle \text{ is continuous linear on } D(T)\}$. For $y \in D(T^*)$, $T^*y$ is the unique point from $H$ satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$, $x \in D(T)$.

*Důkaz*

„$T^*y$ exists“: any $\varphi \in D(T)^*$ can be extended to $H^* = H$. □

## Tvrzení 7.7

*$H$ Hilbert, $S$ and $T$ densely defined in $H$.*

- *$S \subset T \implies T^* \subset S^*$.*

  *Důkaz*
  $D(T^*) = \{y | x \mapsto \langle Tx, y\rangle = \langle Sx, y\rangle$ is continuous on $D(T) \supset D(S)\} \subset D(S^*)$. And for $y \in D(T^*)$:

  $$\forall x \in D(S) : \langle x, T^*y\rangle = \langle Tx, y\rangle = \langle Sx, y\rangle = \langle x, S^*y\rangle \implies T^*y = S^*y.$$

  □

- *$S + T$ is densely defined $\implies S^* + T^* \subset (S + T)^*$ and if $S \in \mathcal{L}(H)$, then there is equality.*

  *Důkaz*
  For $y \in D(S^* + T^*) = D(S^*) \cap D(T^*)$ and $x \in D(S + T)$:

  $$\langle (S + T)x, y\rangle = \langle x, S^*y\rangle + \langle x, T^*y\rangle = \langle x, (S^* + T^*)y\rangle.$$

  So, $y \in D((S + T)^*)$ and $(S + T)^*y = (S^* + T^*)(y)$. This proves the inclusion.

  „If $S \in \mathcal{L}(H)$“ For $y \in D((S + T)^*)$ and for $x \in D(S + T) = D(T)$:

  $$D(T) \ni x \mapsto \langle Tx, y\rangle = \langle (S + T)x, y\rangle - \langle Sx, y\rangle$$

  is constant on $D(T)$. So, $y \in D(T^*) = D(T^*) \cap D(S^*) = D(S^* + T^*)$. Thus, $D(S^* + T^*) = D((S + T)^*) \wedge S^* + T^* \subset (S + T)^*$, so $S^* + T^* = (S + T)^*$. □

- *$ST$ is densely defined $\implies T^*S^* \subset (ST)^*$ and if $S \in \mathcal{L}(H)$ then there is equality.*

  *Důkaz*
  Pick $y \in D(T^*S^*)$. Then for $x \in D(ST)$:

  $$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

  So, $y \in D((ST)^*)$ and $(ST)^*y = T^*S^*y$.

  „If $S \in \mathcal{L}(H)$“: Then $D(ST) = D(T)$ and for $y \in D((ST)^*)$ we want „$S^*y \in D(T^*)$“ (then $y \in D(T^*S^*)$ and we are done):

  $$D(T) \ni\mapsto \langle Tx, S^*y\rangle = \langle STx, y\rangle = \langle x, (ST)^*y\rangle.$$

  So, $x \mapsto \langle Tx, S^*y\rangle$ is continuous on $D(T)$. □

## Tvrzení 7.8

*H Hilbert, T densely defined on H.*

- *$T^*$ is closed operator on $H$;*
- *$T$ has closed extension $\Leftrightarrow$ $T^*$ is densely defined. Then $(T^*)^* = \overline{T}$.*
- *$T$ is closed $\Leftrightarrow$ $T^*$ is densely defined and $T = (T^*)^*$.*

## Lemma 7.9

*$H$ Hilbert, $T$ densely defined on $H$. Consider $V \in \mathcal{L}(H \oplus H)$ such that $V(x, y) := (-y, x)$. Then $V$ is unitary and $G(T^*) = V(G(T))^\perp$.*

*Důkaz*
„$V$ is unitary:“ obvious ($V$ is isometry onto).

„$G(T^*) \subseteq V(G(T))^\perp$“: Pick $y \in D(T^*)$ and $x \in D(T)$. Then

$$\langle (y, T^*y), V(x, Tx) \rangle = \langle (y, T^*y), (-Tx, x) \rangle = \langle y, -Tx \rangle + \langle T^*y, x \rangle = 0.$$

„$V(G(T))^\perp \subseteq G(T^*)$“: Pick $(x, y) \in V(G(T))^\perp$. Then for $z \in D(T)$:

$$0 = \langle (x, y), (-Tz, z) \rangle = -\langle x, Tz \rangle + \langle y, z \rangle,$$

so $\langle x, Tz \rangle = \langle y, z \rangle$, so $D(T) \ni z \mapsto \langle Tz, x \rangle$ $(= \langle z, y \rangle)$ is continuous. So $x \in D(T^*)$ and $T^*x = y$, co $(x, y) \in G(T^*)$. □

*Poznámka*
$U \in \mathcal{L}(H)$ unitary, $A \subset H$. Then $U(A^\perp) = U(A)^\perp$.

*Důkaz*
$x \in U(A)^\perp \Leftrightarrow \forall a \in A : 0 = \langle x, Ua \rangle = \langle U^*x, a \rangle \Leftrightarrow U^*x \in A^\perp \Leftrightarrow x \in U(A^\perp)$. □

*Důkaz* (Of the previous proposition)
First point follows from the previous lemma.

„Second point, $\implies$ “: Pick $y_0 \in D(T^*)^\perp$. Wanted: $y_0 = 0$. We have $(y_0, 0) \in G(T^*)^\perp$ $(\forall z \in D(T^*) : \langle (z, T^*z), (y_0, 0) \rangle = 0)$. $G(T^*)^\perp = V(G(T))^{\perp\perp} = \overline{V(G(T))} = V(\overline{G(T)})$. So $(0, -y_0) = V^*(y_0, 0) \in V^*V(\overline{G(T)}) = \overline{G(T)}$. Thus $y_0 = 0$ (because $T$ is closed).

„Second point, $\impliedby$ “: $T^*$ is densely defined. Then $(T^*)^*$ is defined and, by first point, it is closed. Moreover, „$T \subset (T^*)^*$“: Pick $x \in D(T)$. Then $D(T^*) \ni y \mapsto \langle T^*y, x \rangle = \langle y, Tx \rangle$, so $x \in D((T^*)^*)$ and $(T^*)^*x = Tx$.

„Second point, then part“: $T \subseteq (T^*)^*$ is done, „$(T^*)^* \subseteq \overline{T}$“: it suffices to prove

„$G((T^*)^*) = \overline{G(T)}$“: By previous lemma, $G((T^*)^*) = V(G(T^*))^\perp = V^*(G(T^*))^\perp = V^*(V(G(T))^\perp)^\perp = V^*V(G(T)^{\perp\perp}) = \overline{G(T)}$.

„Third point“: „ $\implies$ “ follows directly from second point, „ $\impliedby$ “ by second point, $T$ has closed extension and $\overline{T} = (T^*)^* = T$, so ti is closed. □

## Tvrzení 7.10

*$H$ Hilbert, $T$ densely defined on $H$. Then*

- $\mathrm{Rng}(T)^\perp = \mathrm{Ker}\, T^*$*;*

  *Důkaz*
  $y \in \mathrm{Ker}\, T^* \Leftrightarrow T^*y = 0 \Leftrightarrow \forall x \in D(T) : \langle Tx, y\rangle = 0 \Leftrightarrow y \in \mathrm{Rng}\, T^\perp$. □

- *If $T$ is moreover closed, then* $\mathrm{Ker}\, T = (\mathrm{Rng}\, T^*)^\perp$.

  *Důkaz*
  By the previous proposition $T^*$ is densely defined and $T^{**} = T$. By the previous point, $\mathrm{Ker}\, T = \mathrm{Ker}\, T^{**} = (\mathrm{Rng}\, T^*)^\perp$. □

## Tvrzení 7.11

*$H$ Hilbert, $T$ is one-to-one densely defined on $H$, $\overline{\mathrm{Rng}\, T} = H$. Then $T^*$ is one-to-one and $(T^*)^{-1} = (T^{-1})$.*

*Důkaz*
Proof omitted (using the previous proposition and lemma). □

## Definice 7.7 (Self-adjoint operator, symmetric operator, maximally symetric operator)

$H$ Hilbert, $T$ operator on $H$. $T$ is self-adjoint $\equiv T = T^*$. $T$ is symmetric $\equiv \forall x, y \in D(T) : \langle Tx, y\rangle = \langle x, Ty\rangle$. $T$ is maximally symmetric $\equiv$ $T$ is symmetric, and there is no $S \supsetneq T$ symmetric.

*Poznámka*
$T$ is self-adjoint $\implies$ $T$ is densely defined. $T$ is densely defined, then it is symmetric $\Leftrightarrow$ $T \subseteq T^*$. If $T$ is densely defined, then $T$ is self-adjoint $\implies$ symmetric. (And the other implication doesn't hold.)

## Tvrzení 7.12

*$H$ Hilbert, $T$ densely defined and symmetric.*

- *$T$ has closed extension and $\overline{T}$ is symmetric;*
- *$R(T)$ is dense $\implies$ $T$ is one-to-one;*
- *$D(T) = H \implies T = T^*$ and $T \in \mathcal{L}(H)$;*
- *$R(T) = H \implies T$ is one-to-one, self-adjoint and $T^{-1} \in \mathcal{L}(H)$;*
- *$T$ is self-adjoint $\implies$ $T$ is maximally symmetric.*

*Důkaz*
Omitted. □

## Věta 7.13

*$H$ Hilbert space, $H \neq \{0\}$, $T$ is self-adjoint operator on $H$. Then $\varnothing \neq \sigma(T) \subseteq \mathbb{R}$.*

*Důkaz*
Let $T \neq 0$ be self-adjoint. „$\sigma(T) \neq 0$“: If $\sigma(T) = 0$, then by corollary above, $T^{-1} \in \mathcal{L}(H)$ and $\sigma(T^{-1}) = \{0\}$. Moreover $T^{-1}$ is self-adjoint by the previous proposition (third point). So $0 = r(T^{-1}) = \|T^{-1}\|$, so $T^{-1} = 0$. ↯.

TODO? (Tady se něco zjednoduší: BÚNO $0 \not\equiv T = T^*$. Kdyby $\sigma(T) = \varnothing$, pak $T^{-1} \in \mathcal{L}(H)$. Pak $T^{-1}$ je samoadjungovaný $((T^{-1})^* = (T^*)^{-1} = T^{-1}.)$.)

„$\sigma(T) \subseteq \mathbb{R}$“: Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. Then

$$\overline{\operatorname{Rng}(\lambda I - T)} = \operatorname{Ker}((\lambda I - T)^*)^\perp = \operatorname{Ker}(\overline{\lambda} I - T^*)^\perp = \{\mathbf{o}\}^\perp = H.$$

By next lemma, $\lambda I - T$ is onto. (Because $T$ is closed because $T$ is self-adjoint) and $(\lambda I - T)^{-1}$ is continuous. Thus $\lambda \notin \sigma(T)$.

□

## Lemma 7.14

*$T$ is symmetric on Hilbert $H$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Then $(\lambda I - T)$ is one-to-one, $(\lambda I - T)^{-1}$ is continuous on $R(\lambda I - T)$, and moreover $T$ is closed $\Leftrightarrow R(\lambda I - T)$ is closed.*

*Důkaz*
$\lambda = \alpha + i \cdot \beta$, $\beta \neq 0$, $\alpha, \beta \in \mathbb{R}$. Then $\alpha I - T$ is symmetric, so $\forall x \in D(T)$:

$$\|(\lambda I - T)x\|^2 = \|(\alpha I - T)x + i \cdot \beta x\|^2 = \|i \cdot \beta \cdot x\|^2 + \|(\alpha I - T)x\|^2 + 2\Re\langle i \cdot \beta \cdot x, (\alpha I - T)x\rangle =$$

$$= |\beta|^2 \cdot \|x\|^2 + \|(\alpha I - T)x\|^2 + 0 \geqslant |\beta|^2 \cdot \|x\|^2,$$

cause $S$ is symmetric, then $\langle Sx, x\rangle \in \mathbb{R}$, $x \in D(S)$. So, $\|(\lambda I - T)x\| \geqslant |\beta| \cdot \|x\|$, $x \in D(T)$, thus $(\lambda I - T)$ is one-to-one. And $(\lambda I - T)^{-1}$ is bounded on its domain, so continuous on its domain.

It suffices: For $S := \lambda I - T$: $S$ is closed $\Leftrightarrow R(S)$ is closed. And proof of this is omitted.

„Moreover“: Denote $S := \lambda I - T$ ($S$ closed $\Leftrightarrow$ $T$ closed). „ $\implies$ “: Let $S$ be closed, then „$\operatorname{Rng} S$“ is closed: $\operatorname{Rng} S \ni y_n \to y \implies (S^{-1}(y_n))$ is Cauchy, so there is $x \in D(S)$ : $S^{-1}y_n \to x$. Then $(S^{-1}y_n, y_n) \to (x, y)$, so $Sx = y$.

„ $\impliedby$ “: Let $\operatorname{Rng} S$ be closed. Then „$G(S)$ is closed“: $(x_n, Sx_n) \to (x, y) \implies x_n = S^{-1}Sx_n \to S^{-1}y$. So $S^{-1}y = x$. □

*Důsledek* (Of the previous theorem)
$H$ Hilbert, $T$ operator on $H$. Then next propositions are equivalent

- $T$ is self-adjoint;
- $T$ is densely defined, symmetric and $\sigma(T) \subseteq \mathbb{R}$;
- $T$ is densely defined, symmetric and there is $\lambda \in \mathbb{C}\backslash\mathbb{R} : \lambda, \overline{\lambda} \in \sigma(T)$.

*Důkaz*
„1. $\implies$ 2.“ use the previous theorem. „2. $\implies$ 3.“ easy. „3. $\implies$ 1.“: $T \subset T^*$ by third point. Wanted: „$D(T^*) \subset D(T)$“: Pick $x \in D(T^*)$. Put

$$y := (\lambda I - T)^{-1}\left((\lambda I - T^*)x\right) \in \operatorname{Rng}((\lambda I - T)^{-1}) = D(\lambda I - T).$$

Then

$$(\lambda I - T^*)x = (\lambda I - T)y = \lambda y - Ty = \lambda y - T^*y = (\lambda I - T^*)y.$$

$\lambda I - T^*$ is one-to-one ($\operatorname{Ker}(\lambda I - T^*) = \operatorname{Ker}((\overline{\lambda} I - T)^*) = \operatorname{Rng}(\overline{\lambda} I - T)^{\perp} = H^{\perp} = \{\mathbf{o}\}$). So, $x = y \in D(T)$. □

# 8 Cayley transform

*Poznámka* (Motivation)
$T$ self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$ and $M(z) = \frac{z-i}{z+i}$, $z \in \mathbb{R}$ is bijection between $\mathbb{R}$ and $\mathbb{D} \setminus \{1\}$.

**Definice 8.1** (Cayley transform of operator)

$H$ Hilbert, $T$ symmetric operator on $H$. Then Cayley transform of $T$ is the operator $\mathcal{C}(T) := (T - iI) \cdot (T + i \cdot I)^{-1}$.

*Poznámka*
$\mathcal{C}(T)$ is well defined: $T + iI$ is one-to-one, $\operatorname{Rng}(T + iI)^{-1} = D(T + iI) = D(T - iI)$.

$$Tx + ix \overset{\mathcal{C}(T)}{T} x - ix.$$

**Věta 8.1**

*$H$ Hilbert, $T$ symmetric operator on $H$, $\mathcal{C}(T)$ Cauchy transform. Then*

- *$\mathcal{C}(T)$ is linear isometry $D(\mathcal{C}(T)) = R(T+iI)$ onto $R(\mathcal{C}(T)) = R(T-iI)$;*

  *Důkaz*
  $D(\mathcal{C}(T)) = R(T + iI)$ by definition. $R(\mathcal{C}(T)) = R(T - iI)$ by definition too.

  For $y = Tx + ix \in D(\mathcal{C}(T))$ we have

  $$\|\mathcal{C}(T)y\|^2 = \|Tx + ix\|^2 \overset{\text{COS}}{=} \|Tx\|^2 + \|x\|^2 + 2\Re\langle Tx, -ix\rangle = \|Tx\|^2 + \|x\|^2$$

  $$\|y\|^2 = \|Tx + ix\|^2 = \ldots = \|Tx\|^2 + \|x\|^2.$$

  So, $\mathcal{C}(T)$ is isometry. □

- *$I - \mathcal{C}(T) = 2i(T + iI)^{-1}$, and so $I - \mathcal{C}(T)$ is one-to-one and $R(I - \mathcal{C}(T)) = D(T)$;*

  *Důkaz*
  Let $y = Tx + ix \in D(\mathcal{C}(T))$, then

  $$(I - \mathcal{C}(T))y = y - \mathcal{C}(T)y = Tx + ix - (Tx - ix) = 2ix = (T + iI)^{-1}y$$

  $\implies$ formula holds.

  Since $(T + iI)^{-1}$ is one-to-one, $I - \mathcal{C}(T)$ is one-to-one. Moreover, $R(I - \mathcal{C}(T)) = R((T + iI)^{-1}) = D(T + iI) = D(T)$. □

- $T = i\,(I + \mathcal{C}(T)) \cdot (I - \mathcal{C}(T))^{-1}$.

*Důkaz*
We know $D(T) = R(I - \mathcal{C}(T))$ and $R\left((I - \mathcal{C}(T))^{-1}\right) = D(I - \mathcal{C}(T)) = D(I + \mathcal{C}(T))$. So operator on RHS is well-defined and LHS have same domain as RHS.

Pick $y \in D(T)$ and $x \in D(\mathcal{C}(T))$ such that $(I - \mathcal{C}(T))x = y$. Then

$$y - (I - \mathcal{C}(T))x = 2i(T + iI)^{-1}x,$$

so $i(I + \mathcal{C}(T)) \cdot (I - \mathcal{C}(T))\,y = i(I + \mathcal{C}(T))x = i\left(x + (T - iI)(T + iI)^{-1}x\right) =$

$$= i\left(x + (T - iI) \cdot (y/2i)\right) = \frac{i}{2i}\left(2ix + (T - iI)y\right) = \frac{1}{2}((T + iI)y + (T - iI)y) = Ty.$$

□

- *$T$ closed $\Leftrightarrow$ $\mathcal{C}(T)$ closed $\Leftrightarrow$ $D(\mathcal{C}(T))$ closed $\Leftrightarrow$ $R(\mathcal{C}(T))$ closed.*

*Důkaz* (Omitted.) □

## Věta 8.2

*Let $H$ be a Hilbert space and $U$ isometry form $D(U)$ onto $R(U)$. Let $I - U$ be one-to-one. Then $T := i(I + U)(I - U)^{-1}$ is symmetric and $\mathcal{C}(T) = U$. Moreover $T$ is densely defined if and only if $R(I - U)$ is dense.*

*Důkaz*
$T$ is well-defined: $R((I - U)^{-1}) = D(I - U) = D(I + U)$. $D(T) = R(I - U)$, so $T$ is densely defined iff $R(I - U)$ is dense.

„$T$ is symmetric“: Let $x = (I - U)x' \in D(T)$, $y = (I - U)y' \in D(T)$.

$$\langle Tx, y\rangle = \langle i(I + U)x', y\rangle = i\,\langle x' + Ux', y' - Uy'\rangle \overset{U \text{ isometry}}{=} i\left(-\langle x'Uy'\rangle + \langle Ux', y'\rangle\right),$$

$$\langle x, Ty\rangle = \ldots = \langle x, i(I + u)y'\rangle = -i\,\langle x' - Ux'\rangle = -i\left(\langle x', Uy'\rangle - \langle Ux', y'\rangle\right).$$

„$\mathcal{C}(T) = U$“: Let $x = (I - U)x' \in D(T)$:

$$(T - iI)x = i(I + U)x' - ix = i(x' + Ux') - i(x' - Ux') = 2iUx',$$

$$(T + iI)x = \ldots + ix = \ldots + \ldots = 2ix'.$$

So, $x' \in R(T + iI) = D(\mathcal{C}(T))$ and $D(U) \subseteq D(\mathcal{C}(T))$ and $D(\mathcal{C}(T)) = R(T + iI) \subseteq D(U)$. Thus, $D(U) = D(\mathcal{C}(T))$. Finally, for $x \in D(T)$:

$$U(Tx + ix) = U(2ix') = 2iUx' = (T - iI)x = Tx - ix.$$

□

## Věta 8.3

*H Hilbert:*

*a) Let $T$ be a symmetric operator on $H$. Then $T$ is self-adjoint $\Leftrightarrow \mathcal{C}(T)$ is unitary (i.e. $D(\mathbb{C}(T)) = H = R(\mathcal{C}(T))$).*

*b) $U \in \mathcal{U}(H)$ such that $I - U$ is one-to-one, then*

$$T := i(I + U)(I - U)^{-1}$$

*is self-adjoint and $\mathcal{C}(T) = U.i$*

*Důkaz*

„a) $\implies$“: Since $\sigma(T) \subseteq \mathbb{R}$, we have $\pm i \in \varrho(T)$, so $T \pm iI$ are onto, so $D(\mathcal{C}(T)) = H = R(\mathcal{C}(T))$ by the theorem above.

„a) $\impliedby$“: We have $D(T)^\perp = R(I - \mathcal{C}(T))^\perp = \operatorname{Ker}(I - \mathcal{C}(T))^* = \operatorname{Ker}(I - \mathcal{C}(T)) = \{\mathbf{o}\}$, co $T$ is densely defined. Moreover, $T \pm iI$ is onto, so $Ii \in \varrho(T)$. Thus, from the corollary above, $T$ is self-adjoint.

„b)“: $\mathcal{C}(T) = U$ by the previous theorem. Moreover $D(T)^\perp = R(I - U)^\perp = \ldots = \{\mathbf{o}\}$, so $T$ is densely-defined. It remains „$T \pm iI$ is onto“: Fix $y \in H$, put $zi = (I - U)y$, then:

$$(T + iI)z = Tz + iz = i(I + U)y + i(I - U)y = 2iy,$$

$$(T - iI)z = Tz - iz = i(I + U)y - i(I - U)y = 2iUy.$$

So, (Since $D(U) = H = R(U)$), we have $T \pm iI$ is onto. □

## Definice 8.2 ($n_+$ and $n_i$ (deficiency indices))

Let $T$ be a symmetric closed operator in a Hilbert space H. Then

$$n_+(T) = \dim(\operatorname{Rng}(T + iI))^\perp = \dim D(\mathcal{C}(T))^\perp,$$

$$n_-(T) = \dim(\operatorname{Rng}(T - iI))^\perp = \dim \operatorname{Rng}(\mathcal{C}(T))^\perp$$

are called deficiency indices of the operator $T$.

## Věta 8.4

*$T$ symmetric, densely defined, closed operator on separable (we prove it only for separable) $H$. Then*

*a) $T$ is self-adjoint $\Leftrightarrow n_+(T) = n_-(T) = 0$;*

*b) ($T$ is maximal symmetric $\Leftrightarrow \min(n_+(T), n_-(T)) = 0$;)*

*c) $T$ has self-adjoint extension $\Leftrightarrow n_+(T) = n_-(T)$.*

*Důkaz*
„a)“: $T$ self-adjoint $\Leftrightarrow \mathcal{C}(T)$ is unitary $\Leftrightarrow D(\mathcal{C}(T)) = R(\mathcal{C}(T)) = H \overset{*}{\Leftrightarrow} n_+(T) = 0 = n_-(T)$.
*) $T$ is closed, so $D(\mathcal{C}(T)) \neq H \Leftrightarrow n_+(T) > 0$ and $R(\mathcal{C}(T)) \neq 0 \Leftrightarrow n_-(T) > 0$ (from item d) from the theorem above).

„b)“ omitted.

„c) $\implies$ “: Let $S \supseteq T$ be self-adjoint. Then $\mathcal{C}(S) \supseteq \mathcal{C}(T)$ and $\mathcal{C}(S)$ is unitary and $\mathcal{C}(S)(D(\mathcal{C}(T))) = R(\mathcal{C}(T))$, $\mathcal{C}(S)(\dots^\perp) = R(\mathcal{C}(T))^\perp$ ($U$ unitary, $U(A) = B \overset{\text{easy}}{\implies} U(A^\perp) = B^\perp$). So,

$$n_+(T) = \dim D(\mathcal{C}(T))^\perp = \dim R(\mathcal{C}(T))^\perp = n_-(T)$$

Since $H$ is separable, we have $n_+(T) = n_-(T) \Leftrightarrow \exists$ isometry between $D(\mathcal{C}(T))^\perp$ and $R(\mathcal{C}(T))^\perp$ (because Hilbert spaces are isometric to right $l_2$). Let $V \supseteq \mathcal{C}(T)$ is unitary operator such that $V(R(\mathcal{C}(T))^\perp) = R(\mathcal{C}(T))^\perp$.

Then „$R(I - V)$ is dense and $I - V$ is one-to one.“:

$$R(I - V) \supseteq R(I - \mathcal{C}(T)) = D(T),$$

so $R(I - V)$ is dense. Fix $x \in \operatorname{Ker}(I - V)$ and $y \in D(V)$. Then

$$\langle x, (I - V)y\rangle = \langle x, y\rangle - \langle x, Vy\rangle = \langle Vx, Vy\rangle - \langle x, Vy\rangle = \langle Vx - x, Vy\rangle = \langle \mathbf{o}, Vy\rangle = 0.$$

Thus, $x \in R(I - V)^\perp = \{\mathbf{o}\}$.

$\implies$ $\exists S$ symmetric and densely defined such that $\mathcal{C}(S) = V \supseteq \mathcal{C}(T)$, so $S \supseteq T$ ($S = i(I + V)(I - V)^{-1} \supseteq i(I + \mathcal{C}(T))(I - \mathcal{C}(T))^{-1} = T$). □

# 9 Integral of unbounded function with respect to a spectral measure

**Definice 9.1**

$H$ Hilbert, $(X, \mathcal{A})$ is measurable space, $E$ spectral measure for $(X, \mathcal{A}, H)$, $E$ spectral measure for $(X, \mathcal{A}, H)$, $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable. Then $\int f dE$ is the operator on $H$ such that

$$D(\int f dE) := \left\{ x \in H \middle| \int |f|^2 dE_{x,x} < \infty \right\}, \qquad \langle Tx, y\rangle := \int_X f dE_{x,y}, \quad x, y \in D(T).$$

**Věta 9.1**

*$H$ Hilbert, $(X, \mathcal{A})$ is measurable space, $E$ spectral measure for $(X, \mathcal{A}, H)$, $E$ spectral measure for $(X, \mathcal{A}, H)$, $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable. Then $D := \left\{x \in H \middle| \int_X |f|^2 dE_{x,x} < \infty\right\}$ is dense subspace of $H$, $\int f dE$ exists (and it is unique).*

*Moreover,* $\|Tx\|^2 = \int_X |f(\lambda)| dE_{x,x}$, $x \in D(\int f dE)$.

*Důkaz*
„$D$ is subspace“: From proposition (basic properties of spectral measure) sixth item (addition) and fourth point (multiplication).

„For $A_n := f^{-1}(B(\mathbf{o}, n))$ we have $\operatorname{Rng} E(A_n) \subseteq D(\int f dE)$, $n \in \mathbb{N}$“: $\forall x \in \operatorname{Rng} E(A_n)$:

$$E_{x,x}(A_n) = \langle E(A_n)x, x\rangle = \langle x, x\rangle = \langle E(X)x, x\rangle = E_{x,x}(X).$$

So, $E_{x,x}(X \backslash A_n) = 0$, so $|f| \leqslant n$ $E_{x,x}$-almost everywhere, so

$$\int_X |f|^2 dE_{x,x} \leqslant n^2 \int_X 1 \cdot E_{x,x} < \infty.$$

„$D$ is dense“: Pick $y \in H$, then $D \ni E(A_n)y \to y$ ($\|E(a_n)y - y\|^2 = \|E(X\backslash A_n)y\|^2 = E_{y,y}(X\backslash A_n) \to 0$.)

„$\forall x, y \in D : \int f dE_{x,y} \in \mathbb{C}$“: $(x, y) \mapsto E_{x,y}$ is sesquilinear, so it suffices to check it for $x = y$. But $f \in L^2(E_{x,x}) \subseteq L^1(E_{x,x})$, so $\int f dE_{x,x} \in \mathbb{C}$.

„Definition of $T$“: For $x \in D$ put $Tx := \lim_{n\to\infty} \left(\int_X f\chi_{A_n} dE\right) x$. „$T$ well defined“: limit exists, because the sequence is cauchy:

$$\forall m < n : \| \int f\chi_{A_n} dEx - \int f\chi_{A_m} dEx\|^2 = \| \int f\chi_{A_n\backslash A_m} dEx\|^2 = \int_{A_n\backslash A_m} |f|^2 dE_{x,x} \to 0.$$

„$T$ linear“: easy (VAL + Linearity of the integral). „For $T$ equation holds“: By sesquilinearity, suffices to check for $x = y \in D$:

$$\langle Tx, x\rangle = \lim \left\langle \int_X f\chi_{A_n} dEx, x \right\rangle = \lim \int f\chi_{a_n} dE_{x,x} \overset{\text{Lebesgue}}{=} \int \lim f\chi_{a_n} dE_{x,x} = \int f dE_{x,x}.$$

„$\|Tx\| = \sqrt{\ldots}$“:

$$\|Tx\|^2 = \lim \left\langle \int f\chi_{A_n} dEx, \int f\chi_{A_n} dEx \right\rangle = \lim \int |f\chi_{A_n}|^2 dE_{x,x} \overset{\text{Lebesgue}}{=} \int |f|^2 dE_{x,x}.$$

„Uniqueness“: $\langle Tx, y\rangle = \langle z, y\rangle$, $y \in D \implies Tx = z$ on $H$, because $D$ is dense. □

## Věta 9.2

*Let $H$ Hilbert space, $(X, \mathcal{A})$ measurable space, $E$ spectral measure for $(X, \mathcal{A}, H)$ and $f, g : X \to \mathbb{C}$ be $\mathcal{A}$-measurable functions. Then the following assertions hold:*

$\int f dE + \int g dE \subset \int f + g dE$;

*Důkaz* (Omitted. (From definition.)) □

$\left(\int fdE\right)\left(\int gdE\right) \subset \int fgdE$ *and* $D\left(\left(\int fdE\right)\left(\int gdE\right)\right) = D\left(\int gdE\right) \cap D\left(\int fgdE\right)$*;*

*Důkaz* (Omitted. (Technical, difficult, from definition of bounded version.)) □

$(\int fdE)^* = \int \overline{f}dE$ *and* $\int fdE\left(\int fdE\right)^* = \int |f|^2 dE = (\int fdE)^* \int fdE$*, that is,* $\int fdE$ *is normal;*

*Důkaz* (Omitted.) □

$\int fdE$ *is closed;*

*Důkaz*
From the previous item: $\int fdE = \int \overline{\overline{f}}dE = \left(\int \overline{f}dE\right)^* \implies$ (by the proposition above) $\int fdE$ is closed. □

$\int fdE \in \mathcal{L}(H) \Leftrightarrow \exists A \in \mathcal{A}$*:* $E(X\backslash A) = \mathbf{o} \wedge f$ *is bounded on* $A$*.*

*Důkaz*
„ $\Longleftarrow$ “: „$D(\int fdE) = H$“: $\forall x \in H : \int_X |f|^2 dE_{x,x} = \int_A |f|^2 dE_{x,x} < \infty$. „$\forall x \in H : \|\int fdEx\|^2 \leqslant C \cdot \|x\|^2$“: from the previous theorem:

$$\|\int fdEx\|^2 = \int_X |f|^2 dE_{x,x} = \int_A |f|^2 dE_{x,x} \leqslant \|f|_A\|_\infty \cdot E_{x,x}(X) \leqslant \|f|_A\|_\infty \cdot \|x\|^2.$$

„ $\Longrightarrow$ “: Put $K := \|\int |f|dE\| < \infty$, $A := \{t||f(t)| \leqslant K+1\}$. Then „$E(X\backslash A) = 0$“: If not, $\exists x \in S_H \cap \operatorname{Rng} E(X\backslash A)$ and then

$$K+1 = \int (K+1)dE_{x,x} \leqslant \int_{A^c} |f|dE_{x,x} = \int |f|\chi_{A^c} dE_{x,x} = \left\langle \int \chi_{A^c} dE \int |f|dEx, x \right\rangle =$$

$$= \left\langle E(A^c) \cdot \int |f|dEx, x \right\rangle = \left\langle \int |f|dE_{x,x}, E(A^c)x \right\rangle = \left\langle \int |f|dEx, x \right\rangle \leqslant$$

$$\leqslant \left\| \int |f|dEx \right\| \cdot 1 \leqslant \left\| \int |f|dE \right\| \cdot 1 \cdot 1 = K.$$

□

## Věta 9.3

*Let* $H$ *be a Hilbert space,* $(X, \mathcal{A})$ *measurable space,* $E$ *spectral measure for* $(X, \mathcal{A}, H)$ *and* $f : X \to \mathbb{C}$ *be* $\mathcal{A}$*-measurable function. Then*

$$\sigma\left(\int fdE\right) = ess \operatorname{Rng} f := \left\{\lambda \in \mathcal{C} | \forall r > 0 : E(f^{-1}(U(\lambda, r))) \neq 0\right\}.$$

*Moreover, for* $\lambda \in \mathbb{C}$ *we have* $\operatorname{Ker}(\lambda I - \int fdE) = \operatorname{Rng}(E(f^{-1}(\{\lambda\})))$*. Thus* $\lambda \in \sigma_P(\int TODO)$ *TODO.*