1 Introduction

Poznámka (Literature)
„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

1.1 Differentiability

Definice 1.1 (Differentiable)

A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable (also holomorphic) at a point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{|h|\to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call $f'(z_0)$ the derivative of f at z_0. A function f is differentiable on a domain (open connected subset of $\mathbb{C}$) if its differentiable for all points of this domain.

Poznámka (Writing complex numbers in cartesian cooridnates)
$z = x + iy$, for $x, y \in \mathbb{R}$, we can write a function $f : \mathbb{C} \to \mathbb{C}$ in terms of two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

Věta 1.1 (Cauchy–Riemann equations)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}$. Considering $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definice 1.2 (Orientability, orientation-preserving function)

Define and equivalence relation on the set of all bases of $\mathbb{R}^2$ by saying that $B_1 \sim B_2$ iff the determinant of the change of basis matrix is positive.

A function $f : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ is said to be orientation-preserving if on an open dense subset of U, the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Důsledek
Let f be a non-constant holomorphic function, then f is orientation-preserving.

Důsledek
Since f is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x}\frac{\partial v}{y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \overset{\text{C–R}}{=} \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geqslant 0.$$

Since f is non-constant, the inequality is strict on a dense open subset of the domain of definition.

Věta 1.2 (Open mapping theorem)

A non-constant holomorphic function f is open (that is if U is an open subset of $\mathbb{C}$, then $f(U)$ is also open).

1.2 Integration

Definice 1.3

For a path γ (smooth function, $\gamma : \mathbb{R} \supset [a,b] \to \mathbb{C}$) we define

$$\int_\gamma f(x)dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

Definice 1.4 (Continuous deformation)

For $\gamma, \mu : [a,b] \to U$ (U simply connected), paths with the same endpoints ($\gamma(a) = \mu(a)$ and $\gamma(b) = \mu(b)$). Then a continuous deformation γ into μ is a continuous function $H : [a,b] \times [0,1] \to U \subseteq \mathbb{C}$ such that $H(s,0) = \gamma(s)$, $H(s,1) = \mu(s)$, $H(a,t) = z_a := \gamma(a) = \mu(a)$ and $H(b,t) = z_b := \gamma(b) = \mu(b)$.

Věta 1.3

Suppose that $\gamma, \mu : [a,b] \to U$ (U simply connected) are related by a continuous deformation of paths H. Then for any holomorphic function f on U we have

$$\int_\gamma f(z)dz = \int_\mu f(z)dz.$$

Důkaz (Partial proof assuming H admits partial derivatives)
For any $t \in [0,1]$ we integrate the function $INT(t) = \int_{H(\cdot,t)} f(z)dz$. Consider the derivative of $INT(t)$ with respect to t:

$$\frac{d}{dt}(INT(t)) = \frac{d}{dt}\left(\int_a^b f(H(s,t))\frac{\partial H}{\partial s}(s,t)ds\right) \overset{\text{Leibniz + chain rule}}{=}$$

$$= \int_a^b f'(H(s,t))\frac{\partial H}{\partial t}(s,t) \cdot \frac{\partial H}{\partial s}(s,t) + f(H(s,t))\frac{\partial^2 H}{\partial s \partial t}(s,t)ds =$$

$$= \int_a^b \frac{d}{ds}\left[f(H(s,t))\frac{\partial H}{\partial t}\right] ds =$$

$$= f(H(s,t))\frac{\partial H}{\partial t}|_{s=a}^{s=b} \overset{\text{constant endpoints}}{=} 0.$$

Having derivative identically equal to 0, means that $INT(t)$ is a constant function and $\int_\gamma f(z)dz = INT(0) = INT(1) = \int_\mu f(z)dz$. □

Důsledek
Let U be a simply connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. For any closed path whose image is inside U, $\int_\gamma f(z)dz = 0$.

Důkaz (Sketch)
The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path c.

$$\int_\gamma f(z)dz = \int_c f(z)dz = \int_a^b f(c(z)) \cdot c'(z)dz = \int_a^b f(c(z)) \cdot 0dz = 0$$

□

Příklad
Let U be a simple connected domain and $f : U \to \mathbb{C}$ a holomorphic function on $U \setminus \{z_0\}$. For $j = 1, 2$, let γ_j be a path parametrizing a circle centered at z_0 of radius r_j, oriented counterclockwise and completely contained in U. Show that $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$.

1.3 Cauchy's integral formula

Věta 1.4 (Cauchy's integral formula)

Let γ be a loop around $z \in \mathbb{C}$, and $f : U \to \mathbb{C}$ a holomorphic function. For U a neighbourhood of γ,

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw.$$

Důkaz
Conway 1978, Chapter IV. □

Důsledek

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0 + z_0 - z} dw = \frac{1}{2\pi i}\oint \frac{f(w)}{w - z_0} \cdot \left(\frac{1}{1 - \frac{z-z_0}{w-z_0}} \right) dw =$$
$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0} \left(\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^n} \right) dw =$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z_0)^n} \right) (z-z_0)^n.$$

For sufficiently "small" (shrunken) γ. So f is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around z_0 converges to f in a neighbourhood of z_0).

Definice 1.5 (Pole)

Given a positive integer n, a complex function f has pole of order n at the point $z_0 \in \mathbb{C}$ if $(z - z_0)^n f(z)$ is holomorphic at z_0 but $(z - z_0)^{n-1} f(z)$ is not.

Příklad
Show that if f has a pole of order n at $z_0 \in \mathbb{C}$. Then it admits a Laurient expansion $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$ with $a_{-n} \neq 0$.

Definice 1.6 (Residue)

Let f have a pole of order n at the point $z_0 \in \mathbb{C}$. Then the residue of f at z_0 is the $k = -1$ coefficient of the Laurent expansion of f at z_0.

Příklad
Show that if f has a pole of order 1 at z_0, then the residue of f at z_0 can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)
Let $\gamma : [a, b] \to U \subset \mathbb{C}$ be a simple closed path, bounding a domain W containing the points $z_1, \ldots, z_m$. Assume that f is holomorphic on $U \setminus \{z_1, \ldots, z_m\}$ and has poles at $\{z, \ldots, z_m\}$.

Show that

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname{res}_{z=z_j} f(z).$$

TODO!!!

1.4 (Real) Projective space

Poznámka (Building structures)
Set → Topology → Differential structure (atlas) → Riemann metric → Connection…

Definice 1.7 (Real projective space)

The set $\mathbb{P}^n(\mathbb{R})$ is defined to be either of the following bijective sets: Lines through the origin in $\mathbb{R}^{n+1}$; Equivalence classes of $(n+1)$-tuples of real numbers $(x_0, \ldots, x_n) \neq (0, \ldots, 0)$, such that for any real number $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$: $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$.

Příklad
Confirm that the sets above are in bijection with each other.

Poznámka (Notation)
We will often denote a point in $\mathbb{P}^n(\mathbb{R})$ as the equivalence class $[x_0, \ldots, x_n]$.

Definice 1.8 (Topology of $\mathbb{P}^n(\mathbb{R})$)

We give a topology to $\mathbb{P}^n(\mathbb{R})$ by endowing it with following quotient topology: consider the surjection $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} \to \mathbb{P}^n(\mathbb{R})$, $(x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$. A set $U \subset \mathbb{P}^n(\mathbb{R})$ is defined to be open if $\pi^{-1}(U) := \{x \in \mathbb{R}^{n+1} \setminus \{\mathbf{o}\} | \pi(x) \in U\}$ is open in $\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}$.

That is we give $\mathbb{P}^n(\mathbb{R})$ the finest topology that makes π continuous.

Příklad
Check that for $\mathbb{C}$ we can define $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{CP}^n$ the same way.

Příklad (Projective space)
$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is an abelian group. Let $\mathbb{R}^*$ act on $\mathbb{R}^{n+1}$ by component wise multiplication. When a general group G acts on a set X we have equivalence relation $x \sim y$ if $y = g \circ x$. We call the equivalence classes the orbits of G. So $\mathbb{P}^n(\mathbb{R}) = (\mathbb{R}^{n+1} \setminus \{\mathbf{o}\}) / \mathbb{R}^*$.

Sphere quotient: Let $S^n \subseteq \mathbb{R}^{n+1}$. Denote the unit sphere. Then the group $\mathbb{Z}_2 = \{+1, -1\}$ act on the sphere by $\pm 1(x_0, \ldots, x_n) = (\pm x_0, \ldots, \pm x_n)$. Then $S^n / \mathbb{Z}_2 = \mathbb{P}^n(\mathbb{R})$.

Disk model: Consider the n-dimensional closed unit disk $\overline{\mathbb{D}^n} \subseteq \mathbb{R}^n$, and the equivalence relation on the points of the boundary: $x \sim -x$ if $\|x\| = 1$. Then $\mathbb{P}^n(\mathbb{R})$ is the quotient (collection of equivalence classes), i.e. $\overline{D^n} \backslash \sim \simeq \mathbb{P}^n(\mathbb{R})$.

Příklad
Conclude from either of these models of $\mathbb{P}^n(\mathbb{R})$ that as a topological space, $\mathbb{R}^n(\mathbb{P})$ is compact and Hausdorff.

Poznámka
Now we come to the smooth manifold structures. Let's start with $\mathbb{P}^1(\mathbb{R})$. Define

$$U_x := \mathbb{P}^1(\mathbb{R}) \backslash \left\{ [x, y] \in \mathbb{P}^1(\mathbb{R}) | x = 0 \right\}, \qquad \varphi_x : U_x \to \mathbb{R}, \quad \varphi_x([x, y]) = \frac{x}{y}.$$

Similarly, we define a second chart:

$$U_y := \mathbb{P}^1(\mathbb{R}) \backslash \left\{ [x, y] \in \mathbb{P}^1(\mathbb{R}) | y = 0 \right\}, \qquad \varphi_y : U_y \to \mathbb{R}, \quad \varphi_y([x, y]) = \frac{y}{x}.$$

Příklad
Check that U_x, U_y are open and that φ_x, φ_y are homeomorphisms.

Důkaz
Consider the translation functions:

$$U = U_x \cap U_y = \left\{ [x, y] \in \mathbb{P}^1(\mathbb{R}) | x, y \neq 0 \right\}, \qquad \varphi_x(U) = \varphi_y(U) = \mathbb{R} \backslash \left\{ 0 \right\}.$$

The translation function $T_{x,y} := \varphi_y \circ (\varphi_x)^{-1}$ sends, for $y \neq 0$:

$$T_{x,y} : y \overset{(\varphi_x)^{-1}}{\mapsto} [1, y] = \left[\frac{1}{y}, 1 \right] \overset{\varphi_y}{\mapsto} \frac{1}{y}.$$

Which is smooth on the domain $\mathbb{R} \backslash \{0\}$.

TODO smooth. Thus $\mathbb{P}^1(\mathbb{R})$ is a smooth manifold. □

Příklad
Show that $\mathbb{P}^1(\mathbb{R})$ is homomorphic to the circle S^1. We call $\mathbb{P}^1(\mathbb{R})$ the real projective line.

Příklad
Try to show that $\mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$ is a smooth manifold.

Příklad
For $\mathbb{P}^2(\mathbb{R})$ the followings charts form atlas:

$$U_x := \{[x,y,z] | x \neq 0\}, \qquad \varphi_x : U_x \to \mathbb{R}, \quad \varphi_x([x,y,z]) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

$$U_y := \{[x,y,z] | y \neq 0\}, \qquad \varphi_y : U_y \to \mathbb{R}, \quad \varphi_y([x,y,z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$U_z := \{[x,y,z] | z \neq 0\}, \qquad \varphi_z : U_z \to \mathbb{R}, \quad \varphi_z([x,y,z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Check these are open subsets and homeomorphisms, with smooth transformation functions. And extend this to $\mathbb{P}^n(\mathbb{R})$.

1.5 Compact surfaces

Definice 1.9 (Surface)

A surface is a manifold of real dimension 2.

Například
$\mathbb{R}^2$, $\mathbb{C}$, and any of their open subsets are surfaces. S^2 is a compact surface, as is $\mathbb{P}^2(\mathbb{R})$.

Definice 1.10 (Connected surface)

Given two connected surfaces S_1 and S_2, the connected surface $S_1 \# S_2$ is the surface obtained by removing an open disc from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

Příklad
At the level of topological spaces, show that the operation # is well defined up to homeomorphism, that is, show that the choice of disks in S_1 and S_2 does not change the definition of $S_1 \# S_2$ / homeomorphism.

Příklad
Show that # gives the set of homeomorphism classes of connected compact surfaces the structure of a monoid. (Which surface is the identity of the monoid?)

Věta 1.5 (Classification of compact surfaces)

Any connected, compact surfaces is homeomorphic to exactly one surface in the following list:

- S^2;
- $T^{\# g} := T \# \ldots \# T,\ g \in \mathbb{N}_0$;

- $\mathbb{P}^2(\mathbb{R})^{\#n} := \mathbb{P}^2(\mathbb{R})\# \dots \#\mathbb{P}^2(\mathbb{R}),\ n \in \mathbb{N}_0$.