Poznámka
Topology…

1 Locally convex spaces

Definice 1.1 (Topological vector space (TVS))

A Topological vector space over $\mathbb{F}$ is a pair (X, τ), where X is a vector space over $\mathbb{F}$ and τ is a topology on X with the following two properties:

1. The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X;
2. The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X;

We also denote Hausdorff topological vector space by HTVS. And the symbol $\tau(\mathbf{o})$ will denote the family of all the neighbourhoods of $\mathbf{o}$ in (X, τ).

Definice 1.2 (Locally convex (LCS, HLCS))

Let (X, τ) be a TVS. The space X is said to be locally convex, if there exists a base of neighbourhoods of zero consisting of convex sets.

Poznámka (Credit)
Two homework (in Moodle) and one presentation.

Tvrzení 1.1 (Observation)

If (X, τ) is a LCS, then τ is translation invariant ($U \subset X, x \in X \implies (U \in \tau \Leftrightarrow x+U \in \tau)$). Hence τ is determined by $\tau(\mathbf{o})$.

Definice 1.3 (convex, symmetric, balanced, absolutely convex, and absorbing set)

X is a vector space, $A \subset X$. Then A is

- convex if $tx + (1-t)y \in A$ for $x, y \in A$, $t \in [0, 1]$;
- symmetric if $A = -A$;
- balanced if $\alpha A \subset A$ for $\alpha \in \mathbb{F}$, $|\alpha| \leqslant 1$;
- absolutely convex if it is convex and balanced;
- absorbing if $\forall x \in X\ \exists t > 0 : \{s \cdot x | s \in [0, t]\} \subset A$.

Definice 1.4

co(A) = convex hull, b(A) = balanced hull, aco(A) = absolutely convex hull.

Tvrzení 1.2

X is a metric space over $\mathbb{F}$, $A \subset X$. Then:

(a) If $\mathbb{F} = \mathbb{R}$, it holds A is absolutely convex $\Leftrightarrow$ A is convex and symmetric.

(b) $\operatorname{co} A = \{t_1x_1 + \ldots + t_kx_k | x_1 \ldots x_k \in A, t_1 \ldots t_k \geqslant 0, t_1 + \ldots + t_k = 1, k \in \mathbb{N}\}$.

(c) $\mathrm{b}(A) = \{\alpha x | x \in A, \alpha \in \mathbb{F}, |\alpha| \leqslant 1\}$.

(d) $\operatorname{aco}(A) = \operatorname{co}(\mathrm{b}(A))$.

(e) A is convex $\Leftrightarrow$ $(s+t)A = sA + tA$ for all $s, t > 0$.

Důkaz (a)
„ $\Longrightarrow$ “: trivial (and it also holds for $\mathbb{F} = \mathbb{C}$). „ $\Longleftarrow$ “: Assume A is convex and symmetric. We show that A is balanced:

$$x \in A, \alpha \in \mathbb{R}, |\alpha| \leqslant 1 \implies \alpha \in [-1, 1].$$

And $x \in A, -x \in A$, so the segment from x to $-x$ is contained in A ($\alpha x = \frac{1-\alpha}{2}(-x) + \frac{(1+\alpha)}{2}x \in A$). □

Důkaz (b)
„$\subseteq$“: by induction on k:

$$t_1x_1 + \ldots + t_{k+1}x_{k+1} = (t_1 + \ldots + t_k)\frac{t_1x_1 + \ldots + t_kx_k}{t_1 + \ldots + t_k} + t_{k+1}x_{k+1}.$$

„$\supseteq$“: the set on the RHS is convex and contain A. □

Důkaz (c)
„$\supseteq$“: clear. „$\subseteq$“: RHS is a balanced set. □

Důkaz (d)
„$\supseteq$“: clear. „$\subseteq$“ the set on the RHS is absolutely continuous (Clearly RHS is convex. „balanced“: using (b) and (c):

$$\operatorname{co}(\mathrm{b}(A)) = \{t_1\alpha_1x_1 + \ldots + t_k\alpha_kx_k | x_1, \ldots, x_k \in A, |\alpha_j| \leqslant 1, t_j \geqslant 0, t_1 + \ldots + t_k = 1\}$$

is clearly balanced.) □

Důkaz (e)
„ $\Longrightarrow$ “: „$\subseteq$“: always, „$\supseteq$“: $sa_1 + ta_2 = (s+t) \cdot \left(\frac{s}{s+t}a_1 + \frac{t}{s+t}a_2\right)$.

„ $\Longleftarrow$ “: in particular $\forall t \in (0,1)$: $tA + (1-t)A \subset A$, it is the definition of convexity. □

Tvrzení 1.3

Let (X,τ) be a LCS, $U \in \tau(\mathbf{o})$. Then

(i) U is absorbing.

(ii) $\exists V \in \tau(\mathbf{o}) : V + V \subset U$.

(iii) $\exists V \in \tau(\mathbf{o})$ absolutely convex, open: $V \subset U$.

Důkaz (i)

$$x \in X \implies 0 \cdot x = \mathbf{o} \in U \implies \exists V \text{ a neighbourhood of } 0 \text{ in } \mathbb{F} : V \cdot x \subset U \implies$$
$$\implies \exists t > 0 : [0,t] \subset V.$$

□

Důkaz (ii)

$$\mathbf{o} + \mathbf{o} = \mathbf{o} \in U \implies \exists W_1, W_2 \text{ neighbourhoods of } \mathbf{o} : W_1 + W \subset U.$$

Take $V = W_1 \cap W_2$. □

Důkaz

$$\exists U_0 \in \tau(\mathbf{o}) \text{ convex}, U_0 \subset U : 0 \cdot \mathbf{o} = \mathbf{o} \in U_0 \implies \exists c > 0 \; \exists W \in \tau(\mathbf{o}) \text{ open} :$$
$$\forall \lambda, |\lambda| < c : \lambda W \subset U_0.$$

$V_1 := \bigcup_{0<|\lambda|<1} \lambda W$. Then $V_1 \in \tau(0)$ open, balanced, $V_1 \subset U_0$. Let $V := \operatorname{co} V_1$. Then V is absolutely convex (the previous proposition (d)), $V \subset U_0 \subset U$ (as V_0 is convex). $V \in \tau(\mathbf{o})$ as $V \supset V_1$. „V is open“:

$$V = \bigcup \{t_1 x_1 + \ldots + t_n x_n + t_{n+1} V_1 | t_1, \ldots, t_{n+1} \geqslant 0, t_1 + \ldots + t_{n+1} = 1, x_1, \ldots, x_n \in V_1\}$$

□

Věta 1.4

1. Let (X,τ) be a LCS. Then there is $\mathcal{U}$, a base of neighbourhoods of $\mathbf{o}$ with properties:

- *the elements of $\mathcal{U}$ are absorbing, open, absolutely convex;*
- *$\forall U \in \mathcal{U} \; \exists V \in \mathcal{U} : 2V \subset U$.*

If X is Hausdorff, then $\bigcap \mathcal{U} = \{\mathbf{o}\}$.

2. Let X be a vector space, $\mathcal{U}$ a nonempty family of subsets of X satisfying:

- *the elements of $\mathcal{U}$ are absorbing and absolutely convex;*
- $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : 2V \subset U;$
- $\forall U, V \in \mathcal{U}\ \exists W \in \mathcal{U} : W \subset U \cap V.$

Then there is a unique topology τ on X such that (X, τ) is LCS and $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$. *Further, if $\bigcap \mathcal{U} = \{\mathbf{o}\}$, the τ is Hausdorff.*

Důkaz (1.)
Let $\mathcal{U}$ be the family of all open absolutely convex neighbourhoods of $\mathbf{o}$. The previous proposition (iii) gives us $\mathcal{U}$ is a base of neighbourhoods of $\mathbf{o}$, (1) gives us elements of $\mathcal{U}$ are absorbing, so the first item holds. (ii) gives us $U \in \mathcal{U} \implies \frac{1}{2}U \in \mathcal{U}$.

Assume X is Hausdorff: $x \in X \setminus \{\mathbf{o}\} \overset{\text{Hausdorff}}{\implies} \exists U \in \tau(\mathbf{o}) : x \notin U \implies \exists V \in \mathcal{U} : V \subset U : x \notin V$. □

Důkaz (2.)
Set $\tau = \{G \subset X | \forall x \in G\ \exists U \in \mathcal{U} : x + U \subset G\}$. This is a unique possibility so uniqueness is clear.

„τ is topology“: $\varnothing, X \in \tau$ and τ is closed to arbitrary union (clear). τ is closed to finite intersections by third item ($G_1, g_2 \in \tau$, $x \in G_1 \cap G_2 \dots U_1, U_2 \in \tau$, $x + U_1 \subset G_1$, $x + U_2 \subset G_2$; $\exists V \in \mathcal{U} : V \subset U_1 \cap U_2$, then $x + V \subset (x + U_1) \cap (x + U_2) \subset G_1 \cap G_2 \implies G_1 \cap G_2 \in \tau$).

„Elements of $\mathcal{U}$ are neighbourhoods of $\mathbf{o}$“: $U \in \mathcal{U}$. $V := \{x \in U | \exists W \in \mathcal{U} : x + W \subset U\}$. Then $V \subset U$, $0 \in V$ (take $W = U$). $V \in \tau$ ($x \in V \implies \exists W \in \mathcal{U} : x + W \subset U$; let $\tilde{W} \in \mathcal{U}$ such that $2\tilde{W} \subset W$, then $x + \tilde{W} \subset V$, because $y \in \tilde{W} \implies X + y + \tilde{W} \subset x + \tilde{W} + \tilde{W} \subset x + W \subset U$).

„$\mathcal{U}$ is a base of neighbourhood of $\mathbf{o}$“: now clear.

„(X, τ) is a TVS“: $x + y \in G \in \tau \implies \exists U \in \mathcal{U} : x + y + U \subset G \implies \exists V \in \mathcal{U} : 2V \subset U$. Then $(x+V)+(y+V) \subset x+y+2V \subset x+y+U \subset G$. $\lambda x \in G \in \tau \implies \exists U \in \mathcal{U} : \lambda x + U \subset G$; $\exists V \in \mathcal{U} : 2V \subset U$; V is absorbing $\implies \exists c > 0\ \forall t \in [0, c] : tx \in V$; V balanced $\implies \forall \mu \in \mathbb{F}, |\mu| \leqslant c : \mu x \in V$; assume $\lambda \in \mathbb{F}, |\mu - \lambda| < c, y \in x + \frac{1}{|\lambda|+1}V$,

$$\implies \mu y - \lambda x = \underbrace{(\mu - \lambda) y}_{(\mu-1)\cdot\left(\mu + \frac{1}{|\lambda|+1}\right)V} + \underbrace{\lambda(y - x)}_{\in \frac{\lambda}{|\lambda|+1} V \subset V}.$$

„Local convexity“: by first item: $\forall U \in \mathcal{U} : U$ is convex.

Assume $\bigcap \mathcal{U} = \{\mathbf{o}\}$. Take $x, y \in X, x \neq y \implies x - y \neq \mathbf{o} \implies \exists U \in \mathcal{U} : x - y \notin U$. Take $V \in \mathcal{U} : 2V \subset U$. Then if $(x + V) \cap (y + V) = \varnothing$, $x + v_1 = y + v_2$, $x - y = v_2 - v_1 \in V + V = 2V \subset U$ ↯. □

Věta 1.5 (On the topology generated by a family of seminorms)

Let X be a vector space and let $\mathcal{P}$ be a family of seminorms on X. The there is a unique topology τ on X such that (X,τ) is a LCS and

$$\mathcal{U} = \{\{x \in X | p_1(x) < c_1, \ldots, p_k(x) < c_k\} | p_1, \ldots, p_k \in \mathcal{P}, c_1, \ldots, c_k > 0\}$$

is a base of neighbourhood of $\mathbf{o}$.

$$(X,\tau) \text{ is Hausdorff} \Leftrightarrow \forall x \in X \setminus \{\mathbf{o}\} \ \exists p \in \mathcal{P}, p(x) > 0.$$

Důkaz

Use the previous theorem 2. on $\mathcal{U}$: The sets are absolutely convex (by properties of seminorms). „Absorbing“: $U = \{x \in X | p_1(x) < c_1, \ldots, p_k(x) < c_k\}$. Take $x \in X$?, $j \in [k]$. Then $p_j(x) \in (0,\infty)$ as for $t > 0 : p_j(t \cdot x) = t \cdot p_j(x)$ and $\exists c > 0$ such that $c \cdot p_j(x) < c_j$ for $j \in [k]$. Now for $t \in [0,c] : tx \in U$.

$$U = \{x \in X | p_1(x) < c_1, \ldots, p_k(x) < c_k\}. \text{ Take } V = \left\{x \in X | p_1(x) \subset \tfrac{c_1}{2}, \ldots, p_k(x) < \tfrac{c_k}{2}\right\}.$$

$$U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U} \text{ trivially.}$$

„Hausdorffness“:

$$\bigcap U = \{x \in X | \forall p \in \mathcal{P} : p(x) = 0\}.$$

„$\supseteq$“ clear. „$\subseteq$“: Assume $y \in X$, $p \in \mathcal{P} : p(y) > 0$: $U = \{x \in X | p(x) < p(y)\}\, in\mathcal{U} \implies y \notin U$. □

Například

$(X, \|\cdot\|)$ is a normed space, then its topology is generated by $\mathcal{P} = \{\|\cdot\|\}$.

The topology on $\mathbb{F}^\Gamma$ is generated by seminorms $p_\gamma(f) = |f(\gamma)|$, $f \in \mathbb{F}^\Gamma$ $(\gamma \in \Gamma)$.

$C(\mathbb{R},\mathbb{F})$ + topology is generated by this sequence of seminorms: $p_N(f) = \max_{x \in [-N,N]} |f(x)|$.

Definice 1.5 (Minkowski functional)

X vector space, $A \subset X$ convex absorbing. Then Minkowski functional of set A is defined by the formula $p_A(x) := \inf\{\lambda > 0 | x \in \lambda \cdot A\}$.

Lemma 1.6

Let X be LCS, $A \subset X$ convex set.

$$x \in \overline{A}, y \in \operatorname{int} A \implies \{tx + (1-t)y | t \in [0,1)\} \subset \operatorname{int} A.$$

Důkaz

WLOG $y = 0$. $t = 0$ clear, $0 \in \operatorname{int} A$. $t \in (0,1)$:

Fix U, an open absolutely convex neighbourhood of $\mathbf{o}$ such that $U \subset A$. Then $x + \frac{1-t}{t}U$ is a neighbourhood of $x \implies \exists z \in \left(x + \frac{1-t}{t}U\right) \cap A$, i.e. $\exists m \in U \; z = x + \frac{1-t}{t}m \implies -m \in U \subset m + A$. Find V an absolutely convex open neighbourhood of $\mathbf{o}$ such that $-m + V \subset U \subset A \implies tz + (1-t)(-m+V) \subset A$ (an open set containing tx). (Because $tx = tz - (1-t)a$.) And that's it. □

Tvrzení 1.7 (on the Minkowski functional of a convex neighborhood of zero)

Let X be LCS, $A \subset X$ a convex neighbourhood of $\mathbf{o}$. Then:

Clearly: $[p_A < 1] \subset A \subset [p_A \leqslant 1]$.

$[p_a < 1] = \operatorname{int} A$*:*

Důkaz

„$\subseteq$": $p_A(x) < 1 \implies \exists c > 1$ such that $cx \in A \implies x = \frac{1}{c}cx \in \operatorname{int} A$. „$\supseteq$": $x \in \operatorname{int} A \implies \exists U \in \tau(\mathbf{o}) : x + U \subset A$. U absorbing $\implies \exists \alpha > 0 : \alpha x \in U$. Then $(1+\alpha)x \in A \implies p(x) \leqslant \frac{1}{1+\alpha} < 1$.

Důkaz

$[p_A \leqslant 1] = \overline{A}$:

□

„$\subseteq$": $p_A(x) \leqslant 1 \implies \forall n \in \mathbb{N} : p_x\left(\left(1 - \frac{1}{n}\right)x\right) = \left(1 - \frac{1}{n}\right)p_A(x) \leqslant 1$. $\left(1 - \frac{1}{n}\right)x \in \operatorname{int} A \implies x \in \overline{\operatorname{int} A} \subset \overline{A}$. „$\supseteq$": $x \in \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in \operatorname{int} A$, so, $p_A\left(\left(1 - \frac{1}{n}\right)x\right) < 1 \overset{n\to\infty}{\implies} p_A(x) \leqslant 1$. □

p_A *is continuous on* X.

Důkaz

$[p_A < c] = \emptyset$ if $c \leqslant 0$ and $c \cdot \operatorname{int} A$ if $c > 0$. $[p_A > c] = X$ if $c < 0$, $X \setminus (c \cdot \overline{A})$ if $c > 0$, and $\bigcup_{t>0} X \setminus t\overline{A}$ if $c = 0$. All these sets are open. □

$p_A = p_{\overline{A}} = p_{\operatorname{int} A}$.

Důkaz

$\operatorname{int} A \subset A \subset \overline{A} \implies p_{\overline{A}} \leqslant p_A \leqslant p_{\operatorname{int} A}$. „Conversely": Assume that $p_{\overline{A}}(x) < c \implies \exists d < c : x \in d \cdot \overline{A} \implies \forall n \in \mathbb{N} : \left(1 - \frac{1}{n}\right)x \in d \operatorname{int} A \implies \left(1 - \frac{1}{n}\right)p_{\operatorname{int} A}(x) \leqslant d \implies p_{\operatorname{int} A}(x) \leqslant d < c$. □

Důsledek
Any LCS (X) is completely regular.

Důkaz
$x \in X$, U an open neighbourhood of x. Take V a convex neighbourhood of **o** such that $x + V \in U$. $f(y) := \min\{1, p_V(y-x)\}$. Then f is continuous by the previous proposition, $f(x) = 0$.

$$y \in X \backslash U \implies y - x \notin V \implies p_V(y-x) \geqslant 1 \implies f(y) = 1.$$

□

Věta 1.8 (On generating of locally convex topologies)

Let (X, τ) be a LCS. Let $\mathcal{P}_\tau$ be the family of all continuous seminorms on (X, τ). Then topology generated by $\mathcal{P}_\tau$ coincides with τ.

Důkaz
Let τ_1 be topology induced by $\mathcal{P}_\tau$. $\tau_1 \subset \tau$ (seminorms from $\mathcal{P}_\tau$ are τ-continuous, hence the sets from theorem 5? are τ-open). „$\tau \subset \tau_1$“: Let $U \in \tau(\mathbf{o}) \implies \exists V$ a neighbourhood of **o** such that $V \subset U$. The $p_V \in \mathcal{P}_\tau$ (from the previous proposition is continuous) $\implies$ $[p_V < 1] = V \subset U \implies U \in \tau_1(\mathbf{o})$. □

Tvrzení 1.9

X a vector space.

1. *p is seminorm $\implies$ $[p < 1]$ is absolutely convex, absorbing, and $p_{[p<1]} = p$.*
2. *p, q are seminorms, then $p \leqslant q \Leftrightarrow [p < 1] \supset [q < 1]$.*
3. *$\mathcal{P}$ a set of seminorms generated by a topology τ. p a seminorm on X. Then p is τ-continuous $\Leftrightarrow \exists p_1, \ldots, p_k \in \mathcal{P}\ \exists c > 0 : p \leqslant c \cdot \max\{p_1, \ldots, p_k\}$.*

Důkaz (1.)
Absolutely convex and absorbing is clear.

$$p_{[p<1]}(x) = \inf\{\lambda > 0 | x \in \lambda[p < 1]\} = \inf\{\lambda > 0 | x \in [p < \lambda]\} = p(x).$$

□

Důkaz (2.)
„$\implies$“ trivial. „$\impliedby$“: $[p < 1] \supset [q < 1] \implies p = p_{[p<1]} \leqslant p_{[q<1]} = q$. □

Důkaz (3.)
„ $\Longleftarrow$ “: $A := [p < 1] \implies A \supset [c \cdot \max\{p_1, \dots, p_k\} < 1] = \left[p_1 < \frac{1}{c}, \dots, p_k < \frac{1}{c}\right]$, which is a τ-open set $\implies A$ is a neighbourhood of $\mathbf{o} \implies p = p_A$ is continuous (by 1. and the previous proposition).

„ $\implies$ “: p is continuous $\implies [p < 1]$ is neighbourhood of $\mathbf{o}$ ($p(\mathbf{o}) = 0$) $\implies \exists p_1, \dots, p_k \in \mathcal{P}\ \exists c_1, \dots, c_k > 0$ such that $[p < 1] \supset [p_1 < c_1, \dots, p_k < c_k] \supset [p_1 < c, \dots, p_k < c] = [\frac{1}{c}\max\{p_1, \dots, p_k\} < 1]$ ($c = \min\{c_1, \dots, c_k\}$). Use 2. for seminorms $p, \frac{1}{2\max\{p_1,\dots,p_k\}}$ and get $p \leqslant \frac{1}{c}\max\{p_1, \dots, p_k\}$. □

1.1 Continuous and bounded linear mapping

Tvrzení 1.10

$(X, \tau), (Y, \mathcal{U})$ LCS, $L : X \to Y$ linear. Then the following assertions are equivalent:

1. *L is continuous;*
2. *L is continuous at $\mathbf{o}$;*
3. *L is uniformly continuous.*

Důkaz
„1. $\implies$ 2.“ trivial, „2. $\implies$ 3.“ assume L continuous at $\mathbf{o}$. Then, given $U \in \mathcal{U}(\mathbf{o})$, there is $V \in \tau(\mathbf{o})$ such that $L(V) \subset U$. Take $x, y \in X$ such that $x - y \in V$. Then $L(x) - L(y) = L(x - y) \in U$ and that's continuous. „3. $\implies$ 1.“ trivial. □

Tvrzení 1.11

$L : X \to Y$ linear. L is continuous $\Leftrightarrow \forall q$ a continuous seminorm on Y $\exists p$ a continuous seminorm on X: $\forall x \in X : q(L(x)) \leqslant p(x)$.

Důkaz
„ $\implies$ “: L continuous, q a continuous seminorm on Y, the $p(x) = q(L(x))$ is a continuous seminorm on X. „ $\Longleftarrow$ “: By the previous proposition it is enough „L is continuous at $\mathbf{o}$“: U neighbourhood of $\mathbf{o}$ in Y, $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. $q := p_V$ is a continuous seminorm. Let p be a continuous seminorm on X such that $q \circ L \leqslant p$. $W := [p < 1]$ a neighbourhood of $\mathbf{o}$ in X and $L(W) \subset V \subset U$. $x \in W \implies p(x) < 1 \implies q(L(x)) < 1 \implies L(x) \in V \subset U$. □

Tvrzení 1.12

Let (X, τ) be a LCS over $\mathbb{F}$ and let $L : X \to \mathbb{F}$ be a linear mapping. Then the following assertions are equivalent:

- *L is continuous;*
- $\operatorname{Ker} L$ *is a closed subspace of X;*
- *there exists $U \in \tau(\mathbf{o})$ such that $L(U)$ is a bounded subset of $\mathbb{F}$.*

If $\mathcal{P}$ is a family of seminorms generating the topology of X, the continuity of L is also equivalent to

$$\exists p_1, \ldots, p_k \in \mathcal{P}\ \exists c > 0\ \forall x \in X : |L(x)| \leqslant c \cdot \max\{p_1(x), \ldots, p_k(x)\}.$$

If L is discontinuous, then $\operatorname{Ker} L$ *is a dense subspace of X*

Důkaz

TODO!!! □

Definice 1.6

Let (X, τ) be a LCS and let $A \subset X$. The set A is said to be bounded in (X, τ), if for any $U \in \tau(\mathbf{o})$, there exists $\lambda > 0$ such that $A \subset \lambda U$.

Lemma 1.13

Let (X, τ) be a LCS and let $A \subset X$. Then the set A is bounded in X if and only if each continuous seminorm p on X is bounded on A. (It is enough to test it for a family of seminorms generating the topology of X.)

Důkaz

TODO? (The proof is not directly needed for the exam.) □

Tvrzení 1.14

Let (X, τ) and (Y, σ) be LCS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. Consider the following two assertions: 1. L is continuous. 2. for any bounded subset $A \subset X$ its image $L(A)$ is bounded in Y (i.e., L is a bounded mapping).

Then 1. $\implies$ 2.. *In case τ is generated by a translation invariant metric on X, then* 1. $\Leftrightarrow$ 2..

Důkaz

TODO!!! □

Definice 1.7 (Isomorphism, isomorphic LCS)

Let (X, τ) and (Y, σ) be LCS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. Then mapping L is said to be an isomorphism of X into Y, if L is continuous, one-to-one and L^{-1} is continuous on $L(X)$; and isomorphism of X onto Y if L is continuous, one-to-one, onto and L^{-1} is continuous on Y.

The spaces X and Y are said to be isomorphic if there is an isomorphism of X onto Y.

1.2 Spaces of finite and infinite dimension

Tvrzení 1.15

Let X be a HLCS of finite dimension. If Y is any LCS and $L : X \to Y$ is any linear mapping, then L is continuous. The space X is isomorphic to $\mathbb{F}^n$, where $n = \dim X$.

Důkaz

If $\dim X = 0$, i.e., $X = \{\mathbf{o}\}$, it is trivial. Assume $n := \dim X \in \mathbb{N}$. Fix a basis $x_1, \ldots, x_n$ of X. Define $T : (F^n, \|\cdot\|_2) \to X$ by $T(\lambda_1, \ldots, \lambda_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n$.

T is clearly a linear bijection of $\mathbb{F}^n$ onto X (since $x_1, \ldots, x_n$ is a basis). „T is continuous“:

1. The mapping $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_j$ is continuous $\mathbb{F}^n \to \mathbb{F}$ for each j. 2. $\forall x \in X$ the mapping $x \mapsto \lambda \cdot x$ is continuous $\mathbb{F} \to X$. 3. By composition $(\lambda_1, \ldots, \lambda_n) \mapsto \lambda_j \cdot x_j$ is continuous $\mathbb{F}^n \to X$.

Hence T is the sum of n continuous mappings $\mathbb{F}^n \to X$. It is enough to show that the sum of two continuous mapping is continuous and use mathematical induction: „Ω topological space, X LCS, $f_1, f_2 : \Omega \to X$ continuous $\implies$ $f_1 + f_2$ is continuous“: $t \in \Omega$ arbitrary, let $G \subset X$ be open such that $f_1(t) + f_2(t) \in G$ $\implies$ $\exists U$ neighbourhood of $\mathbf{o}$: $V + V \subset U$. f_i continuous at t $\implies$ $\exists W_i$ open in Ω, $t \in W_i$: $f_i(W_i) \subset f_i(t) + V$. $W := W_1 \cap W_2$ open in Ω, $t \in W$. $s \in W$ $\implies$ $f_1(s) + f_2(s) \in f_1(t) + V + f_2(t) + V \subset f_1(t) + f_2(t) + U \subset G$.

So T is continuous. „T^{-1} is continuous as well“: S_{F^n} the sphere of $\mathbb{F}^n$ is compact in $\mathbb{F}^n$ $\implies$ $T(S_{F^n})$ is compact in X. X Hausdorff $\implies$ $T(S_{F^n})$ is closed. Clearly $\mathbf{o} \notin T(S_{\mathbb{F}^n})$ (T is a linear bijection and $\mathbf{o} \notin S_{F^n}$) $\implies$ $\exists U$ an absolutely convex neighbourhood of $\mathbf{o}$ in X such that $U \cap T(S_{\mathbb{F}^n}) = \emptyset$. We clam that $U \subset T(U_{\mathbb{F}^n})$ (open unit ball). Assume $x \in U \setminus T(U_{\mathbb{F}^n})$ $\implies$ $z := T^{-1}(X)$ satisfies $\|z\|_2 \geqslant 1$. Then $\frac{z}{\|z\|_2} \in S_{\mathbb{F}^n}$, $T\left(\frac{z}{\|z\|_2}\right) = \frac{1}{\|z\|_2} \cdot T(z) = \frac{1}{\|z\|_2} \cdot x \in U$ (U is balanced) $\implies$ $\frac{1}{\|z\|_2} \cdot x \in U \cap T(S_{\mathbb{F}^n})$ a contradiction. So T^{-1} is continuous at $\mathbf{o}$ ($(T^{-1})^{-1}(U_{\mathbb{F}})$ is a neighbourhood of $\mathbf{o}$ and the same for all multiplies).

$\implies$ T^{-1} is continuous. So, T is an isomorphism and second part is proven.

By the second part WLOG $X = \mathbb{F}^n$. Let $L : \mathbb{F}^n \to Y$ be linear, Y LCS. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{F}^n$. Then $L(\lambda_1, \ldots, \lambda_n) = \lambda_1 L(e_1) + \ldots + \lambda_n L(e_n)$. This is continuous (by the same argument as in the second part for T). □

Důsledek

Let X be a HLCS. Then any its finite-dimensional subspace is closed.

Definice 1.8

Let (X, τ) be a LCS and let $A \subset X$. Then set A is said to be totally bounded (or precompact), if for any $U \in \tau(\mathbf{o})$ there exists a finite set $F \subset X$ such that $A \subset F + U$.

Poznámka
Any compact set in any LCS is totally bounded. Any totally bounded set is bounded.

Lemma 1.16

Let (X, τ) be a LCS and let $A \subset X$. Then following assertions are equivalent:

1. *A is totally bounded in X.*
2. *A is totally bounded in (X, p), for any continuous seminorm p on X.*
3. *For any continuous seminorm p on X and any sequence (x_n) in A there is a subsequence x_{n_k} which is Cauchy with respect to p, i.e. $\forall \varepsilon > 0\ \exists k_0\ \forall k, l \geqslant k_0 : p(x_{n_k} - x_{n_l}) < \varepsilon$.*

Důkaz
The proof is not needed for the exam. □

Věta 1.17

Let X be a HLCS. Then following assertions are equivalent:

1. $\dim X < \infty$.
2. *There exists a compact neighbourhood of zero in X.*
3. *There exists a totally bounded neighbourhood of zero in X.*

Důkaz

„1. $\implies$ 2.“ by the proposition above. „2. $\implies$ 3.“ trivial. „3. $\implies$ 1.“: Let U be an absolutely convex totally bounded open neighbourhood of $\mathbf{o}$. Then $\frac{1}{2}U$ is also a neighbourhood of $\mathbf{o}$, so, there is $F \subset X$ finite with $U \subset F + \frac{1}{2}U$. Set $Y := \operatorname{LO} F$. We claim that $Y = X$.

„$\forall n \in \mathbb{N} : U \subset Y + 2^{-n} \cdot U$“: By induction: $n = 1$ follows by the previous. $n \mapsto n+1$: suppose $U \subset Y + 2^{-n} \cdot U$. Then

$$U \subset Y + 2^{-n} \cdot U = Y + 2^{-n+1} \cdot \left(\frac{1}{2}U\right) \subset Y + 2^{-n+1}\left(\frac{1}{2}\left(Y + \frac{1}{2}U\right)\right) =$$

$$= Y + 2^{-n+1}\left(Y + \frac{1}{4} \cdot U\right) = Y + 2^{-n+1} \cdot Y + 2^{-n-1} \cdot U = Y + 2^{-n-1} \cdot U.$$

(Y is linear subspace.)

If $Y \neq X$, then $\exists x \in X \backslash Y$. Since U is absorbing $\exists t > 0$ such that $tx \in U$. So $U \backslash Y \neq \emptyset$. Fix $x \in U \backslash Y$. X Hausdorff, $\dim Y < \infty \implies Y$ is closed. Hence there is V, an absolutely convex neighbourhood of $\mathbf{o}$ such that $x + V \subset U \backslash Y$.

Since U is totally bounded, it is also bounded, so $\exists n \in \mathbb{N} : U \subset 2^n \cdot V$, i.e. $\frac{1}{2^n} \cdot U \subset V$. It follows that $x + \frac{1}{2^n} \cdot U \cap Y = \emptyset \implies x \notin Y + \frac{1}{2^n} \cdot U$. So, $x \in U \backslash \left(Y + \frac{1}{2^n} \cdot U\right)$, a contradiction. □

1.3 Metrizability of locally convex spaces

Tvrzení 1.18

1. Let (X, τ) be a metrizable LCS. Then the topology τ is generated by a sequence of seminorms (p_n) satisfying $p_1 \leqslant p_2 \leqslant p_3 \leqslant \dots$

2. Let X be a vector space and let (p_n) be a sequence of seminorms on X satisfying conditions: $p_1 \leqslant p_2 \leqslant p_3 \leqslant \dots$ and $\forall x \in X \backslash \{\mathbf{o}\}\ \exists n : p_n(x) > 0$. Then

$$\varrho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(x - y)\}, \qquad x, y \in X$$

is a translation invariant metric on X which generates the locally convex topology on X generated by the sequence of seminorms (p_n). Moreover, given a sequence (x_k) in X we have $\varrho(x_k, x) \to 0 \Leftrightarrow \forall n \in \mathbb{N} : p_n(x_n - x) \to 0$; and the sequence (x_k) is Cauchy in ϱ if and only if it is Cauchy in each of the seminorms p_n.

Let (X, τ) be a HLCS whose topology is generated by a sequence $(p_n)_{n=1}^{\infty}$ of seminorms.

Důkaz (1. WLOG $p_1 \leqslant p_2 \leqslant p_3 \leqslant \dots$)
$q_n(x) := \max\{p_1(x), \dots, p_n(x)\}$ is also a seminorm and the family (p_n) generates the same topology as (q_n). □

Důkaz (2. ϱ is translation invariant metric)
$\varrho(x,x) = 0$ clear. $x \neq y \implies \exists n : p_n(x-y) > 0$ as X is Hausdorff. Hence $\varrho(x,y) > 0$.

$\varrho(x,y) = \varrho(y,x)$ clear, as $p_n(x-y) = p_n(y-x)$. „$\varrho(x,z) \leqslant \varrho(x,y) + \varrho(y,z)$": for each $n \in \mathbb{N}$: $p_n(x-z) \leqslant p_n(x-y) + p_n(y-z)$ and hence also

$$\min(1, p_n(x-z)) \leqslant \min(1, p_n(x-y)) + \min(1, p_n(y-z))$$

ϱ translation invariant is clear. □

Důkaz (3.)
„For each $n \in \mathbb{N}$ and $\varepsilon > 0$ we have: $\{x | p_n(x) < \varepsilon\} \subset \{x | \varrho(x, \mathbf{o}) < \varepsilon + 2^{-n}\}$":

$$p_n(x) < \varepsilon \implies \forall k \leqslant n : p_k(x) \leqslant p_n(x) < \varepsilon, \text{ so}$$

$$\varrho(x, \mathbf{o}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(1, p_k(x)) =$$

$$= \sum_{k=1}^{n} \frac{1}{2^k} \max(1, p_k(x)) + \sum_{k=n+1}^{\infty} \frac{1}{2^k} \min(1, p_k(x)) < \sum_{k=1}^{n} \frac{1}{2^k} \cdot \varepsilon + \sum_{k=n+1}^{\infty} \frac{1}{2^k} < \varepsilon + \frac{1}{2^n}.$$

□

Důkaz (4.)
„$\forall \varepsilon \in (0,1)\ \forall n \in \mathbb{N} : \left\{x | \varrho(x, \mathbf{o}) < \frac{\varepsilon}{2^n}\right\} \subset \{x | p_n(x) < \varepsilon\}$":

$$\varrho(x, \mathbf{o}) < \frac{\varepsilon}{2^n} \implies \sum_{k=1}^{\infty} \frac{1}{2^k} \min\{1, p_k(x)\} < \frac{\varepsilon}{2^n} \implies$$

$$\implies \frac{1}{2^n} \cdot \min(1, p_n(x)) < \frac{\varepsilon}{2^n} \implies \min(1, p_n(x)) < \varepsilon.$$

Since $\varepsilon < 1$, it follows $p_n(x) < \varepsilon$ □

Důkaz (ϱ generates the topology τ)
3. $\implies \forall r > 0\ \{x | \varrho(x, \mathbf{o}) < r\}$ is a neighbourhood of $\mathbf{o}$ ($r > 0$, fix $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $\varepsilon + \frac{1}{2^n} < r$. Then $\{x | \varrho(x, \mathbf{o}) < r\} \supset \left\{x | \varrho(x, \mathbf{o}) < \varepsilon + \frac{1}{2^n}\right\} \supset \{x | p_n(x) < \varepsilon\}$).

4. $\implies \{x | \varrho(x, \mathbf{o}) < r\}$, $r > 0$, is a base of neighbourhoods of $\mathbf{o}$.

Hence, the topology generated by ϱ has the same neighbourhoods of $\mathbf{o}$ as τ, so it coincides with τ. □

Důkaz (6. $\varrho(x_k, x) \to 0 \Leftrightarrow \forall n \in \mathbb{N} : p_k(x_k - x) \to 0$)
„ $\implies$ “: Assume $\varrho(x_k, x) \to 0$. Fix $n \in \mathbb{N}$. $\varepsilon \in (0,1)$, $\exists k_0\ \forall k \geqslant k_0 : \varrho(x_k, x) < \frac{\varepsilon}{2^n}$ $\overset{4.}{\implies} \forall k \geqslant k_0 : p_n(x_n - x) < \varepsilon$.

„ $\impliedby$ “: Fix $\varepsilon > 0$. Find $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$. $\exists k_0\ \forall k \geqslant k_0 : p_n(x_k - x) < \frac{\varepsilon}{2}$. By 3. we have for $k \geqslant k_0$: $\varrho(x_k, x) < \frac{\varepsilon}{2} + \frac{1}{2^n} < \varepsilon$. □

Důkaz (7. (x_n) is ϱ-Cauchy $\Leftrightarrow \forall n$: (x_n) is p_n-Cauchy)
The proof is the same as in 6. Only we work with $\varrho(x_k, x_l)$ for $k, l \geqslant k_0$ or $p_n(x_k - x_l)$ for $k, l \geqslant k_0$. □

Věta 1.19 (On metrizability of LCS)

Let (X, τ) be a HLCS. Then the following assertions are equivalent:

1. *X is metrizable (i.e., the topology τ is generated by a metric on X).*
2. *There exists a translation invariant metric on X generating the topology τ.*
3. *There exists a countable base of neighbourhoods of $\mathbf{o}$ in (X, τ).*
4. *The topology τ is generated by a countable family of seminorms.*

Důkaz
„2. $\implies$ 1.“ trivial. „1. $\implies$ 3.“ if ϱ a metric generating τ, then $U_n = \left\{x \in X | \varrho(x, 0) < \frac{1}{n}\right\}$ $\implies$ $(U_n)_n$ is a base of neighbourhoods of $\mathbf{o}$. „3. $\implies$ 4.“: (see the proof of the previous proposition, 1.) (U_n) base of neighbourhood of $\mathbf{o}$, take $V_n \subset U_n$ absolutely convex neighbourhood of $\mathbf{o}$, $p_n = p_{V_n}$ $\implies$ (p_n) generate τ. „4. $\implies$ 2.“: the previous proposition 2. □

Věta 1.20 (A characterization of normable LCS)

(X, τ) is HLCS. X is normable $\Leftrightarrow \exists U$, a bounded neighbourhood of $\mathbf{o}$.

Důkaz
„ $\implies$ “: τ generated by $\|\cdot\|$, $U := \{x \in X | \|x\| < 1\}$ is a bounded neighbourhood of $\mathbf{o}$.

„ $\impliedby$ “: U bounded neighbourhood of $\mathbf{o}$. WLOG U is absolutely convex. Then $\frac{1}{n}U$, $n \in \mathbb{N}$ is a base of neighbourhoods of $\mathbf{o}$ (V neighbourhood of $\mathbf{o}$, $W \subset V$ an absolutely convex neighbourhood of $\mathbf{o}$ $\implies \exists \lambda > 0 : U \subset \lambda W$ Take $n \in \mathbb{N}$ such that $n > \lambda$. Then $U \subset n \cdot W$ so $\frac{1}{n}U \subset W \subset V$). Finally, p_U is a norm generating the topology (U absolutely convex neighbourhood of $\mathbf{o}$ $\implies$ p_U is a continuous seminorm. $\frac{1}{n}U = [p_U < \frac{1}{n}], n \in \mathbb{N}$ is a base of neighbourhood of $\mathbf{o}$ $\implies$ p_U generated topology of X. From X Hausdorff, p_U is a norm.) □

1.4 Fréchet spaces

Definice 1.9 (Fréchet space)

A LCS whose topology is generated by a complete translation invariant metric is called Fréchet space.

Tvrzení 1.21

(X, τ) is a Fréchet space, ϱ any translation invariant metric on X generating $\tau \implies \varrho$ is complete.

Důkaz

ϱ, d two translation invariant metrics generating by τ. Idea: convergent sequences with respect to ϱ and d coincide, Cauchy sequences with respect to ϱ and d coincide. (x_n) ϱ-Cauchy: $\varepsilon > 0 \implies \{x | d(x, \mathbf{o}) < \varepsilon\}$ is a neighbourhood of $\mathbf{o} \implies \exists \delta > 0 : \{x | \varrho(x, \mathbf{o}) < \delta\} \subset \{x | d(x, \mathbf{o}) < \varepsilon\}$. $\exists n_0 \ \forall m, n > n_0$:

$$\varrho(x_m - x_n, \mathbf{o}) = \varrho(x_m, x_n) < \delta \implies d(x_m - x_n, 0) = d(x_m, x_n) < \varepsilon \implies (x_n) \text{ is } d\text{-Cauchy.}$$

□

Tvrzení 1.22

X Fréchet, $A \subset X$. A is compact $\Leftrightarrow$ A is closed and totally bounded.

Důkaz

Let ϱ be a complete translation invariant metric generating the topology. A is compact $\Leftrightarrow$ A is closed and ϱ-totally bounded. But ϱ-totally boundedness = total boundedness in X. A is totally bounded in $X \Leftrightarrow \forall U$ neighbourhood of $\mathbf{o}$ $\exists F \subset X$ finite $A \subset F + U$. A is totally bounded in $\varrho \Leftrightarrow \forall \varepsilon > 0 \ \exists F \subset X$ finite such that $A \subset \bigcup_{x \in F} U_\varrho(x, \varepsilon) = F + U_\varrho(0, \varepsilon)$. □

Tvrzení 1.23

X LCS, $A \subset X$ totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded.

Důkaz
Let U be a neighbourhood of $\mathbf{o}$. Let V be an absolutely convex neighbourhood of $\mathbf{o}$ such that $2V \subset U \implies \exists F \subset X$ finite such that $A \subset F + V$. Then clearly $\operatorname{aco} A \subset (\operatorname{aco} F) + V$. $\operatorname{aco} F$ is compact, $F = \{x_1, \dots, x_k\} \implies \operatorname{aco}(F) = \operatorname{co}(\mathrm{b}(F)) =$

$$= \operatorname{co}\{\lambda x_j | j \in [k], |\lambda| \leqslant 1\} = \left\{ t_1\lambda_1 x_1 + \dots t_n \lambda_n x_n \middle| |\lambda_j| \leqslant 1, t_j \geqslant 0, \sum t_j = 1 \right\}.$$

$$B = \left\{ (\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \middle| |\lambda_j| \leqslant 1, t_j \geqslant 0, \sum t_j = 1 \right\}$$

a compact set in $\mathbb{F}^n \times \mathbb{R}^n$. $(\lambda_1, \dots, \lambda_n, t_1, \dots, t_n) \mapsto t_1\lambda_1 x_1 + \dots + t_n\lambda_n x_n$ is a continuous map and maps B onto $\operatorname{aco} F$.

$\operatorname{aco} F$ compact $\implies$ totally bounded $\implies \exists H \subset X$ finite, $\operatorname{aco} F \subset H + V$ So $\operatorname{aco} A \subset \operatorname{aco} F + V \subset H + V + V = H + 2V < H + U$. □

Důsledek
X Fréchet space, $A \subset X$ compact $\implies \overline{\operatorname{aco} A}$ is compact.

Důkaz
A compact $\implies$ A is totally bounded $\implies$ $\operatorname{aco} A$ is totally bounded $\implies$ (because $M \subset X$ any set $\implies \overline{M} \subset M + U$) $\overline{\operatorname{aco} A}$ is totally bounded $\implies \overline{\operatorname{aco} A}$ is compact.

(M totally bounded, for any $U \in \tau(\mathbf{o})$: U is neighbourhood of $\mathbf{o}$, $x \in \overline{M}$, U absolutely convex convex neighbourhood of $\mathbf{o}$, then $V \subseteq U$ absolutely convex such that $2V \subset U$ $\implies (x + U) \cap M \neq 0 \implies x \in M + U$.)

Find F finite such that $M \subset F + V \implies \overline{M} \subset M + V \subset F + V + V \subset F + U$. □

Věta 1.24 (Banach–Steinhaus)

Let X be a Fréchet space and let Y be LCS. Let (T_n) be a sequence of continuous linear mappings $T_n : X \to Y$ such that $\forall x \in X : \lim_{n\to\infty} T_n x$ exists in Y. Then $Tx := \lim_{n\to\infty} T_n x$ define a continuous linear map $X \to Y$.

Důkaz
Clear: T is a linear map $X \to Y$. „Continuous“: Fix q any continuous sequence on Y.

$$A_m = \{x \in X | \forall n \in \mathbb{N} : q(T_n x) \leqslant m\}.$$

Then A_m is closed, absolutely convex and $\bigcup_{m=1}^{\infty} A_m = X$.

TODO?

Baire category theorem $\implies \exists m \in \mathbb{N} : \operatorname{int} A_m \neq \emptyset \implies \exists x \in A_m\ \exists U$ an absolutely convex neighbourhood of $\mathbf{o}$ such that $x+U \subset A_m \implies -(x+U) \subset A_m \implies$ (A_m convex) $U \subset A_m$ ($y \in U \implies y = \frac{1}{2}(x + y + (-x + y))$) „$\implies q(Ty) \leqslant m p_U(y)$“:

$$p_U(y) < c \implies \frac{y}{c} \in U \subset A_m \implies \forall n \in \mathbb{N} : q\left(T_n \frac{y}{c}\right) \leqslant m \implies$$
$$\implies q\left(T \frac{y}{c}\right) \leqslant m \implies q\left(Ty\right) \leqslant c \cdot m.$$

□

Věta 1.25 (Open mapping theorem)

X, Y Fréchet, $T : X \to Y$ linear continuous surjective mapping. Then T is an open mapping

Důkaz
1. It is enough to show that $\forall U$ neighbourhood of $\mathbf{o}$ in X: $T(U)$ is a neighbourhood of $\mathbf{o}$ in Y.

2. „$\forall U$ a neighbourhood of $\mathbf{o}$ in X, $\overline{TU}$ is neighbourhood of $\mathbf{o}$ in Y“: U an neighbourhood of $\mathbf{o}$ in X. $\exists V \subset U$ an absolutely convex neighbourhood of $\mathbf{o}$. V absorbing $\implies$

$$\implies X = \bigcup_{n=1}^{\infty} nV \implies Y = T(X) = T\left(\bigcup_{n=1}^{\infty} n \cdot V\right) = \bigcup_{n=1}^{\infty} n \cdot T(V).$$

Y Fréchet $\implies$ by Baire category theorem

$$\exists n \in \mathbb{N} : \emptyset \neq \operatorname{int} \overline{n \cdot T(V)} = \operatorname{int} n \cdot \overline{T(V)} = n \cdot \operatorname{int} \overline{T(V)} \implies \operatorname{int} \overline{T(V)} \neq \emptyset \implies$$

$\implies \exists y \in Y\ \exists W$ an absolutely convex neighbourhood of $\mathbf{o}$ in Y such that $y + W \subset \overline{T(V)}$. $\overline{T(V)}$ is absolutely convex $\implies -(y+w) \subset \overline{T(V)} \implies W \subset \overline{T(V)} \subset \overline{T(U)}$.

3. „$\forall U$ neighbourhood of $\mathbf{o}$ in X, TU is a neighbourhood of $\mathbf{o}$ in Y“: ϱ a translation invariant metric on X, complete, generating topology. $U_n = \left\{x \in X | \varrho(0,x) < \frac{1}{2^n}\right\}$. The U_n is a base of neighbourhoods of $\mathbf{o}$. It is enough to prove that $T(U_n)$ is a neighbourhood of $\mathbf{o}$ for each $n \in \mathbb{N}$. We know from 2. that $\forall n : \overline{TU_n}$ is a neighbourhood of $\mathbf{o}$ in Y. We will be done if we show that $TU_{n-1} \supset \overline{TU_n}$ for each $n \in \mathbb{N}$.

We will prove it for $n = 1$: So we will ? $TU_1 \subset TU_0$. Fix $y \in \overline{TU_1}$. Since $\overline{TU_2}$ is a neighbourhood of $\mathbf{o}$ $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$. So there is $x_1 \in U_1$ such that $y - Tx_1 \in \overline{TU_2}$. $\overline{TU_3}$ is a neighbourhood of $\mathbf{o}$ in Y $\implies$ $y - Tx_1 - \overline{TU_3} \subset apTU_2 = \emptyset$ so, there is $x_2 \in U_2$ such that $y - Tx_1 - Tx_2 \in \overline{TU_3}$.

By induction we find $x_n \in U_n$ such that $y - Tx_1 - Tx_2 - \ldots - Tx_n \in \overline{TU_{n+1}} \quad (n \in \mathbb{N})$.

$$x := \sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n :$$

$$M > N \implies \varrho\left(\sum_{n=1}^{M} x_n, \sum_{n=1}^{N} x_n\right) = \varrho\left(\sum_{n=N+1}^{M} x_n, \mathbf{o}\right) \leqslant$$

$$\leqslant \underbrace{\varrho\left(\sum_{n=N+1}^{M} x_n, \sum_{n=N+2}^{M}\right)}_{\varrho(x_{N+1},\mathbf{o})} + \underbrace{\varrho\left(\sum_{n=N+2}^{M} x_n, \sum_{n=N+3}^{M}\right)}_{\varrho(x_{N+2},\mathbf{o})} + \ldots + \varrho(x_M, \mathbf{o}).$$

$$Tx = y : y - Tx = \lim_{n\to\infty} (y - Tx_1 - \ldots - Tx_n)$$

$$y - Tx_1 - \ldots - Tx_n \in \overline{TU_{N+1}} \subset \overline{TU_k} \qquad \text{for } n+1 > k$$

so, $y - Tx \in \overline{TU_k}$ for each $k \in \mathbb{N}$, so $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k} = \{\mathbf{o}\}$. „Last equality“: $y \in Y \setminus \{\mathbf{o}\}$ $\implies$ $\exists V$ neighbourhood of $\mathbf{o}$ in Y such that $y \notin \overline{B}$. T continuous $\implies$ $\exists k \in \mathbb{N}$ such that $T(U_k) \subset V \implies \overline{T(U_1)} \subset \overline{V} \implies y \notin \overline{T(U_k)}$. □

1.5 Extension and separation theorems

Definice 1.10 (Dual space (the dual))

X LCS, X^* is the vector space of continuous linear functions on X.

Věta 1.26 (Hahn–Banach extension theorem)

X LCS, $Y \Subset X$, $f \in Y^*$. Then $\exists g \in X^*$ such that $g|_Y = f$.

Poznámka

If topology of X is generated by $\mathcal{P}$ a topology of seminorms TODO?

Důkaz

1. Topology of Y: $U \subset Y$ is open $\Leftrightarrow \exists \tilde{U} \subset X$ open such that $U = \tilde{U} \cap Y$. U is a neighbourhood of $\mathbf{o}$ in $Y \Leftrightarrow \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $U = \tilde{U} \cap Y$. Lz.pat. Y is also a LSC.

2. $f \in Y^* \implies \exists p$ a continuous seminorm on Y such that $|f(y)| \subseteq p(y), y \in Y$. $U = [p < 1]$ a neighbourhood of $\mathbf{o}$ in $Y \implies \exists \tilde{U}$ a neighbourhood of $\mathbf{o}$ in X such that $\tilde{U} \cap Y = U \implies \exists \tilde{V} \subset \tilde{U}$ an absolutely convex neighbourhood of $\mathbf{o}$ in X. The $p_{\tilde{V}}$ is a continuous seminorm on X. Moreover, $p_{\tilde{V}}|_Y \geqslant p$. ($[p_{\tilde{V}}|_Y < 1] \subset \tilde{V} \cap Y \subset U = [p < 1]$). So, for $y \in Y : |f(y)| \leqslant p(y) \leqslant p_{\tilde{V}}(y) \implies$ (algebraic H–B for seminorms) $\exists g : X \to \mathbb{F}$ linear, $g|_Y = f$, $|g(x)| \leqslant p_{\tilde{V}}(x)$ for $x \in X \implies g$ is continuous by the proposition above. □

Důsledek (Separation from a subspace)

X LCS, $Y \Subset X$ closed, $x \in X \setminus Y$. Then $\exists f \in X^* : f|_Y = 0, f(x) = 1$.

Důkaz

Set $\tilde{Y} = \mathrm{LO}(Y \cup \{x\})$. Define $g(y + \lambda x) = \lambda$, $y \in Y$, $\lambda \in \mathbb{F} \implies g$ is linear functional on $\tilde{Y}$, $g|_Y = 0$, $g(x) = 1$. $\operatorname{Ker} g = Y$ is closed $\implies g$ is continuous $\implies g$ can be extended to $f \in X^*$. □

Důsledek (A proof of density using Hahn–Banach theorem)

X LCS, $Z \Subset Y \Subset X$.

$$\overline{Z} \supset Y \Leftrightarrow \forall f \in X^* : f|_Z = 0 \implies f|_Y = 0.$$

Důkaz

„$\implies$“: clear. „$\impliedby$“: $y \in Y \setminus \overline{Z} \implies \exists f \in X^* : f(y) = 1, f|_Z = 0$. □

Důsledek (The dual separates the points)
X HLCS, $x \in X \setminus \{\mathbf{o}\} \implies \exists f \in X^* : f(x) \neq 0.$

Důkaz
$Y = \{\mathbf{o}\}$ is closed linear subspace and use the first corollary. □

Věta 1.27 (Hahn–Banach separation theorem)

X LCS, $A, B \subset X$ nonempty convex, $A \cap B = \emptyset$.

- $\operatorname{int} A \neq \emptyset \implies \exists f \in X^* \setminus \{0\}\ \exists c \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < \Re f(s).$
- *A compact, B closed* $\implies \exists f \in X^*\ \exists c, d \in \mathbb{R}\ \forall a \in A\ \forall b \in B : \Re f(a) \leqslant c < d \leqslant \Re f(b).$

Důkaz
Analogous to the theorem above. Assume X is a real space ($\mathbb{F} = \mathbb{R}$). „First item“: $\operatorname{int} A \neq \emptyset \implies \operatorname{int}(B - A) \neq \emptyset$ and $\mathbf{o} \notin B - A$. Fix $z \in \operatorname{int}(B - A)$, set $U := z - (B - A)$. The U is a convex neighbourhood of $\mathbf{o}$, $z \notin U \implies p_U(z) \geqslant 1$. Define $g_0 : \mathrm{LO}\{z\} \to \mathbb{R}$ by $g_0(t \cdot z) = t \cdot p_U(z) \implies g_0$ is a linear functional, $g_0 \leqslant p_U$ on $\mathrm{LO}\{z\}$ ($t \geqslant 0 \implies g_0(t \cdot z) = t \cdot p_U(z) = p_U(t \cdot z)$, $t < 0 \implies g_0(t \cdot z) = t \cdot p_U(z) < 0 \leqslant p_U(t \cdot z)$).

From algebraic Hahn–Banach $\exists g : X \to \mathbb{R}$ linear, $g|_{\mathrm{LO}\{z\}} = g_0$, $g \leqslant p_U$ on X. g is continuous ($g \leqslant 1$ on $U \implies g \geqslant -1$ on $-U$, so $|g| \leqslant 1$ on $U \cap (-U)$, a neighbourhood of $\mathbf{o}$). $a \in A$, $b \in B \implies$

$$\implies g(z) - g(b) + g(a) = g(z - (b - a)) \leqslant p_U(z - (b - a)) \leqslant 1, \quad g(a) \leqslant g(b) + \underbrace{1 - \overbrace{g(z)}^{=p_U(z) \geqslant 1}}_{\leqslant 0}.$$

So, $\forall a \in A\ \forall b \in B : g(a) \leqslant g(b)$, $c := \sup g(A)$.

„Second item“: A compact, B closed. For $x \in A$ $\exists U_x$ an absolutely convex open neighbourhood of $\mathbf{o}$ such that $(x + U_x) \cap B = \emptyset$. The $(x + \frac{1}{2}U_x)_{x \in A}$, is an open cover of A. A is compact $\implies \exists x_1, \dots, x_n \in A : A \subset \left(x_1 + \frac{1}{2}U_{x_1}\right) \cup \dots \cup \left(x_n + \frac{1}{2}U_{x_n}\right)$. Set $V := \frac{1}{2}U_{x_1} \cap \dots \cap \frac{1}{2}U_{x_n}$ an absolutely convex open neighbourhood of $\mathbf{o}$. Then $(A + V) \cap B = \emptyset$

$$\left(a \in A \implies \exists j : a \in x_j + \frac{1}{2}U_{x_j} \implies a + V \subset x_j + \frac{1}{2}U_{x_j} + V \subset x_j + U_{x_j}\right).$$

Apply first item to $A + V$ (open convex), B (convex) $\implies \exists f \in X^* \setminus \{0\}$ such that

$$\sup f(A) + \sup f(V) = \sup(f(A) + f(V)) = \sup f(A + V) \leqslant \inf f(B),$$

observe that $\sup f(V) > 0$ ($f \neq 0$, V is neighbourhood of $\mathbf{o}$, hence absorbing).

$$c := \sup f(A), \qquad d := \sup f(A) + \sup f(V).$$

„X complex“: look at X as a real space, $f : X \to \mathbb{R}$ real-linear such that. Define $f_c(x) = f(x) - i f(ix), x \in X$. □

Důsledek
X LCS, $\emptyset \neq A \subset X$, $x \in X$.

- $x \in X \setminus \overline{\mathrm{co}} A \Leftrightarrow \exists f \in X^* : \Re f(x) > \sup\{\Re f(a) | a \in A\}$. („ $\impliedby$ “: is clear because $\{y \in X, \Re f(y) \leqslant \sup\{\Re f(x) | a \in A\}\}$ is closed convex set containing A. „ $\implies$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{co}} A$, get f and take $-f$.)

- $x \in X \setminus \overline{\mathrm{aco}} A \Leftrightarrow \exists f \in X^* : |f(x)| > \sup\{|f(a)| | a \in A\}$ („ $\impliedby$ “: Clear. „ $\implies$ “: Apply the previous theorem to $\{x\}$ and $\overline{\mathrm{aco}} A$ (and multiply by -1), $f \in X^*$: $|f(x)| \geqslant \Re f(x) > \sup\{\Re f(y) | y \in \overline{\mathrm{aco}} A\} = \sup\{|f(y)| | y \in \overline{\mathrm{aco}} A\}$. „$\leqslant$“ clear, „$\geqslant$“: $y \in \overline{\mathrm{aco}\, A} \implies \exists \alpha \in \mathbb{F}, |\alpha| = 1 : |f(y)| = \alpha f(y)$, then $|f(y)| = \lambda f(y) = \Re \alpha f(y) = \Re f(\alpha y)$, $\alpha y \in \overline{\mathrm{aco}\, A}$).

2 Weak topologies

2.1 General weak topologies and duality

Definice 2.1 (Algebraic dual, general weak topology)

X vector space. $X^\#$ is the algebraic dual of X (it means set of all linear functionals on X). $\varnothing \neq M \subset X^\#$, then $\sigma(X, M)$ is the topology on X generated by seminorms $X \mapsto |f(x)|$, $f \in M$.

Tvrzení 2.1

Properties:

1. $(X, \sigma(X, M))$ *is LCS (by the theorem above).*
2. $(X, \sigma(X, M))$ *is Hausdorff* $\Leftrightarrow \forall x \in X \setminus \{0\}\ \exists f \in M : f(x) \neq 0$ *(i.e.* M *separates points) (by the theorem above).*
3. $f \in M \implies f$ *is continuous on* $(X, \sigma(X, M))$ *(fix* $f \in M$, $p(x) = |f(x)|$, $x \in X$ *is a continuous seminorm and* $|f(x)| = p(x) \leqslant p(x)$*).*
4. $\sigma(X, M)$ *is the weakest topology on* X *making all* $f \in M$ *continuous.*
5. $\sigma(X, M) = \sigma(X, \mathrm{LO}(M))$.
6. T *a topological space,* $F : T \to X$ *mapping. Then* F *is continuous* $T \to \sigma(X, M) \Leftrightarrow \forall f \in M : f \circ F$ *is continuous* $(T \to \mathbb{F})$*.*

Důkaz (4.)
Assume τ is any topology on X such that all $f \in M$ are τ-continuous $\implies$

$$\forall x \in X\ \forall f_1, \ldots, f_n \in M\ \forall c_1, \ldots, c_n > 0 : \{y \in X \| f_j(y) - f_j(x)| < c_j\ \forall j \in [n]\} \text{ is } \tau\text{-open}$$

but these sets form a base of $\sigma(X, M) \implies \sigma(X, M) \subset \tau$. □

Důkaz (5.)
„$\subseteq$“: Clear. „$\supseteq$“: $f \in \mathrm{LO}\, M \implies f$ is $\sigma(X, M)$-continuous (the linear combination of continuous linear functionals is continuous) $f = \alpha_1 f_1 + \ldots + \alpha_n f_n$, $f_1, \ldots, f_n \in M$, $x_1, \ldots, x_n \in \mathbb{F}$.

$$|f(x)| \leqslant |\alpha_1| \cdot |f_1(x)| + \ldots + |\alpha_n| \cdot |f_n(x)| \leqslant (|\alpha_1| + \ldots + |\alpha_n|) \cdot \max\{|f_1(x)|, \ldots, |f_n(x)|\}.$$

So by the previous point we get $\sigma(X, \mathrm{LO}\, M) \subset \sigma(X, M)$. □

Důkaz (6.)
„ $\Longrightarrow$ “: $f \in M \implies f$ is $\sigma(X, M)$-continuous, so $f \circ F$ is continuous. „ $\Longleftarrow$ “: $t \in T$, U neighbourhood of $F(t)$ in $\sigma(X, M) \implies \exists f_1, \dots, f_n \in M\ \exists c_1, \dots, c_n > 0$ such that

$$F(t) \in \{y \in X | \forall j \in [n] | f_j(y) - f_j(F(t)) < c_j\} \subset U.$$

Let $G = \{u \in T | \forall j \in [n] : |(f_j \circ F)(u) - (f_j \circ F)(t)| < c_j\}$. Then G is an open neighbourhood of t (by continuity of $f_j \circ F$ and $F(G) \subset U$). □

Příklad
X LCS. Then $X^* \Subset X^\#$. So, we may consider $\sigma(X, X^*)$ „the weak topology of X“. $\sigma(X, X^*)$ is Hausdorff when X is HLCS.

TODO?

Lemma 2.2

Let X be a vector space and $f, f_1, \dots, f_k \in X^\#$. The following assertions are equivalent:

1. $f \in \mathrm{LO}\{f_1, \dots, f_k\}$;
2. $\exists C > 0\ \forall x \in X : |f(x)| \leqslant C \cdot \max\{|f_1(x)|, \dots, |f_k(x)|\}$;
3. $\bigcap_{j=1}^k \operatorname{Ker} f_j \subset \operatorname{Ker} f$.

Důkaz
TODO!!! □

Věta 2.3

Let X be a vector space and let $M \subset X^\#$ be a nonempty set. Then $(X, \sigma(X, M))^* = \mathrm{LO}\, M$.

Důkaz
TODO!!! □

Důsledek

1. Let X be a LCS and let $f \in X^\#$. Then f is continuous on X (i.e., $f \in X^*$), if and only if it is weakly continuous (i.e., $\sigma(X, X^*)$-continuous) on X.
2. Let X be a LCS. Then $(X^*, \sigma(X^*, X))^* = \varkappa(X)$.
3. Let X be a normed linear space and let $f \in X^{**}$. Then $f \in \varkappa(X)$, if and only if f is weak* continuous (i.e., $\sigma(X^*, X)$ continuous) on X^*.

Důkaz
The proof is not needed for the exam. □

2.2 Weak topologies on locally convex spaces

Věta 2.4 (Mazur theorem)

Let X be a LCS and let $A \subset X$ be a convex set. Then a) $\overline{A}^w = \overline{A}$ b) A is closed if and only if it is weakly closed.

Důkaz
TODO!!! □

Důsledek
Let X be a metrizable LCS and let (x_n) be a sequence in X weakly converging to a point $x \in X$. Then there is a sequence (y_n) in X such that $y_n \in \operatorname{co}\{x_k, k \geqslant n\}$ for each $n \in \mathbb{N}$ and $y_n \to x$ in (the original topology of) X.

Důkaz
TODO!!! □

Věta 2.5 (Boundedness and weak boundedness)

Let X be a LCS and let $A \subset X$. Then A is bounded in X if and only if it is bounded in $\sigma(X, X^*)$.

Důkaz
„ $\Longrightarrow$ “: trivial. „ $\Longleftarrow$ “: 1. A is $\sigma(X, X^*)$-bounded $\Longrightarrow$ $\forall f \in X^*$: f is bounded on A. Therefore, in case X is a normed space, this theorem is a corollary to uniform boundedness principle.

2. Let $A \subset X$ be $\sigma(X, X^*)$-bounded. To prove A is bounded, it is enough to show that any continuous seminorm on X is bounded on A (lemma above). So, let p be any continuous seminorm on X.

Set $Y := \{x \in X | p(x) = 0\}$. Then Y is a closed linear subspace of Y (Y is closed as p is continuous, $\mathbf{o} \in Y$ as $p(\mathbf{o}) = 0$, $x \in Y, \lambda \in \mathbb{F} \implies p(\lambda x) = |x| \cdot p(x) = 0 \implies \lambda x \in Y$, $x, y \in Y \implies 0 \leqslant p(x+y) \leqslant p(x) + p(y) = 0 \implies x + y \in Y$).

Let X/Y be the quotient in the linearly-algebraic sense and $q : X \to X/Y$ be the canonical quotient map.

Define a norm $\|\cdot\|$ on X/Y by $\|q(x)\| = p(x)$, $x \in X$. „Well defined“: $q(x) = q(y) \implies q(x-y) = 0 \implies x - y \in Y$, hence $p(x-y) = 0$, so,

$$p(x) \leqslant p(x-y) + p(y) = p(y) \wedge p(y) \leqslant p(y-x) + p(x) = p(x) \implies p(x) = p(y).$$

„It is a norm“:

$$\|\mathbf{o}\| = \|q(\mathbf{o})\| = p(\mathbf{o}) = 0;$$
$$\|q(x)\| = 0 \implies p(x) = 0 \implies x \in Y \implies q(x) = \mathbf{o};$$
$$\|\lambda \cdot q(x)\| = \|q(\lambda x)\| = p(\lambda x) = |\lambda| p(x) = |\lambda| \cdot \|q(x)\|;$$
$$\|q(x) + q(y)\| = \|q(x+y)\| = p(x+y) \leqslant p(x) + p(y) = \|q(x)\| + \|q(y)\|.$$

q is continuous $X \to (X/Y, \|\cdot\|)$ as $\|q(x)\| \leqslant p(x)$ (in fact equal) and p is a continuous seminorm on X. Further „the set $q(A)$ is weakly bounded in $(X/Y, \|\cdot\|)$“:

$$f \in (X/Y)^* \implies f \circ q \in X^* \implies (f \circ q)(A) \text{ is bounded},$$

and, finally, $f(q(A)) = (f \circ q)(A)$.

So, by q. (the case of normed spaces), $q(A)$ is norm-bounded on X/Y, i.e., $\exists c > 0$ $\forall x \in A : \|q(x)\| \leqslant c$. But $\|q(x)\| = p(x)$. Hence, $p \leqslant C$ on A and the proof is finished. □

Tvrzení 2.6 (Weak topology on a subspace)

Let X be a LCS and let $Y \Subset X$. Then the weak topology $\sigma(Y, Y^*)$ coincides with the restriction of the weak topology $\sigma(X, X^*)$ to Y.

Důkaz
TODO? (The proof is not directly needed for the exam.) □

2.3 Polars and their applications

Definice 2.2 (Polars, absolute polars and anihilators)

Let X be a LCS. Let $A \subset X$ and $B \subset X^*$ be nonempty sets. We define

$$\begin{aligned} A^{\rhd} &:= \{f \in X^* | \forall x \in A : \Re f(x) \leqslant 1\}, & B_{\rhd} &:= \{x \in X | \forall f \in B : \Re f(x) \leqslant 1\}, \\ A^{\circ} &:= \{f \in X^* | \forall x \in A : |f(x)| \leqslant 1\}, & B_{\circ} &:= \{x \in X | \forall f \in B : |f(x)| \leqslant 1\}, \\ A^{\perp} &:= \{f \in X^* | \forall x \in A : f(x) = 0\}, & B_{\perp} &:= \{x \in X | \forall f \in B : f(x) = 0\}. \end{aligned}$$

Příklad
Let X be a normed linear space. Then $(B_X)^{\rhd} = (B_X)^{\circ} = B_{X^*}$, $(B_{X^*})_{\rhd} = (B_{X^*})_{\circ} = B_X$.

Tvrzení 2.7 (Polar calculus)

Let X be a LCS and let $A \subset X$ be a nonempty set.

1. *The set $A^{\rhd}$ is convex and contains the zero functional, A° is absolutely convex and $A^{\perp}$ is a subspace of X^*. All the three sets are moreover weak* closed.*
2. $A^{\perp} \subset A^{\circ} \subset A^{\rhd}$.
3. *If A is balanced, then $A^{\rhd} = A^{\circ}$. If $A \Subset X$, then $A^{\rhd} = A^{\circ} = A^{\perp}$.*
4. $\{\mathbf{o}\}^{\rhd} = \{\mathbf{o}\}^{\circ} = \{\mathbf{o}\}^{\perp} = X^*$, $X^{\rhd} = X^{\circ} = X^{\perp} = \{\mathbf{o}\}$.
5. $(c \cdot A)^{\rhd} = \frac{1}{c} \cdot A^{\rhd}$ *and* $(c \cdot A)^{\circ} = \frac{1}{c} \cdot A^{\circ}$ *whenever* $c > 0$.
6. *Let $(A_i)_{i \in I}$ be a nonempty family of nonempty subsets of X. Then $\left(\bigcup_{i \in I} A_i\right)^{\circ} = \bigcap_{i \in I} A_i^{\circ}$. The analogous formulas hold for polars and anihilators too.*

Poznámka
Analogous statements hold for $B \subset X^*$. There are just two differences: The sets $B_{\rhd}$, $B_{\circ}$ and $B_{\perp}$ are weakly closed and for the validity of the second statement in 4. one needs to assume that X is Hausdorff.

Důkaz
TODO? (The proof is not directly needed for the exam.) □

Věta 2.8 (Bipolar theorem)

Let X be a LCS and let $A \subset X$ and $B \subset X^*$ be nonempty sets. Then

$$\begin{aligned} (A^{\rhd})_{\rhd} &= \overline{\mathrm{co}}(A \cup \{\mathbf{o}\}) \left(= \overline{\mathrm{co}}^w(A \cup \{\mathbf{o}\})\right), & (B_{\rhd})^{\rhd} &= \overline{\mathrm{co}}^w(B \cup \{\mathbf{o}\}), \\ (A^{\circ})_{\circ} &= \overline{\mathrm{aco}} A \left(= \overline{\mathrm{aco}}^w A\right), & (B_{\circ})^{\circ} &= \overline{\mathrm{aco}}^w B, \\ (A^{\perp})_{\perp} &= \overline{\mathrm{LO}} A \left(= \overline{\mathrm{LO}}^w A\right), & (B_{\perp})^{\perp} &= \overline{\mathrm{LO}}^w B. \end{aligned}$$

Důkaz
TODO!!! □

Důsledek
Let X and Y be normed linear spaces and let $T \in L(X, Y)$. Then $(\operatorname{Ker} T)^\perp = \overline{T^*(Y^*)}^{w^*}$.

Důkaz
The proof is not needed for the exam. □

Věta 2.9 (Goldstine)

Let X be a normed linear space and let $\varkappa : X \to X^{**}$ be the canonical embedding. Then $B_{X^{**}} = \overline{\varkappa(B_X)}^{\sigma(X^{**}, X^*)}$.

Důkaz
TODO!!! □

Věta 2.10 (Banach–Alaoglu)

Let X be a LCS and let $U \subset X$ be a neighbourhood of $\mathbf{o}$. Then

1. *U° is a weak* compact subset of X^*.*
2. *If X is moreover separable, U° is metrizable in the topology $\sigma(X^*, X)$.*

Důkaz (1.)
Consider $T : X^* \to \mathbb{F}^U$ defined by $T(f)(x) = f(x)$, $f \in X^*$, $x \in U$, i.e. $T(f) = f|_U$. Then T is a homeomorphism of (X^*, w^*) into $\mathbb{F}^U$: „T is one-to-one“: $T(x) = T(y) \implies f|_U = g|_U$. Since f, g are linear and U is absorbing, necessarily $f = g$.

„T is continuous (on $\mathbb{F}^U$ we consider the topology of point-wise convergence)“: $x \in U$ fixed $\implies f \mapsto T(f)(x) = f(x)$ is w^*-continuous by definition of the w^*-topology. „T^{-1} is continuous on $T(X^*)$“: Fix $x \in X$. Since U is absorbing, there is $t > 0$ with $t \cdot x \in U$. If $g = T(f) \in T(X^*)$, then

$$T^{-1}(g)(x) = f(x) = \frac{1}{t} f(t \cdot x) = \frac{1}{t} g(t \cdot x),$$

so $g \mapsto T^{-1}(g)(x)$ is continuous.

„Moreover $T(U^\circ) =$

$$\left\{F \in \mathbb{F}^U \middle| \forall x \in U\colon |F(x)| \leqslant 1 \wedge \forall \alpha, \beta \in \mathbb{F}\ \forall x, y \in U\colon \alpha x + \beta y \in U \implies F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)\right\}$$

“: „$\subset$“ is clear, „$\supset$“: Let F be in the set on the RHS. We will define $f : X \to \mathbb{F}$ as follows: Let $x \in X$. Find $\alpha > 0$ such that $\alpha \cdot x \in U$ and set $f(x) = \frac{1}{\alpha} F(\alpha \cdot x)$. $F(\mathbf{o}) = 0$ $(F(\mathbf{o} + \mathbf{o}) = F(\mathbf{o}) + F(\mathbf{o}))$.

„f is well defined“: $x \in X$, $\alpha, \beta > 0$, $\alpha \cdot x, \beta \cdot x \in U$. Then $\frac{1}{\alpha}(\alpha \cdot x) - \frac{1}{\beta}(\beta \cdot x) = \mathbf{o} \in U$. So,

$$0 = F(0) = F\left(\frac{1}{\alpha}(\alpha \cdot x) - \frac{1}{\beta}(\beta \cdot x)\right) = \frac{1}{\alpha} F(\alpha \cdot x) - \frac{1}{\beta} F(\beta \cdot x),$$

hence $\frac{1}{\alpha} F(\alpha \cdot x) = \frac{1}{\beta} F(\beta \cdot x)$.

„f is linear“: $x, y \in X$, $\alpha, \beta \in \mathbb{F}$. U absorbing $\implies \exists t > 0 : t \cdot x, t \cdot y, t \cdot (\alpha \cdot x + \beta \cdot y) \in U$. Then

$$f(\alpha \cdot x + \beta \cdot y) = \frac{1}{t} F(t \cdot (\alpha \cdot x + \beta \cdot y)) = \frac{1}{t} F(\alpha \cdot (t \cdot x) + \beta \cdot (t \cdot y)) =$$

$$= \frac{1}{t} \left(\alpha \cdot F(t \cdot x) + \beta \cdot F(t \cdot y)\right) = \alpha \cdot \frac{1}{t} \cdot F(t \cdot x) + \beta \cdot \frac{1}{t} \cdot F(t \cdot y) = \alpha \cdot f(x) + \beta \cdot f(y).$$

„f is continuous“: as $\forall x \in U : |f(x)| \leqslant 1$ (and U is a neighbourhood of $\mathbf{o}$), hence also $f \in U^\circ$, so $F = T(f) \in T(U^\circ)$.

So, $T(U^\circ)$ is a closed subset of $\{\lambda \in \mathbb{F} | |\lambda| \leqslant 1\}^U$, which is compact by Tychonoff theorem. So, U° is ω^*-compact. □

Důkaz (2.)
Let X be moreover separable. Let $D \subset X$ be a countable dense set. Then $\sigma(X^*, D)$ is Hausdorff (D separates points of X^*: $f \in X^*$, $f|_D = 0 \implies f = 0$ as D is dense) and $\sigma(X^*)$ is metrizable (D countable $\implies \sigma(X^*, D)$ generated by a countable family of seminorms and use the theorem above) on $U^\circ : \sigma(X^*, D)$ is a weaker Hausdorff topology than $\sigma(X^*, X)$. U° $\sigma(X^*, X)$ compact $\implies \sigma(X^*, X) = \sigma(X^*, D)$ on U°. □

Důsledek (Banach–Alaoglu for normed spaces)
Let X be a normed linear space. Then (B_{X^*}, w^*) is compact. If X is separable (B_{X^*}, w^*) is moreover metrizable.

Důsledek (Reflexivity and weak compactness)
Let X be a Banach space. Then X is reflexive if and only if B_X is weakly compact. If X is reflexive and separable, (B_X, w) is moreover metrizable.

Důsledek
Let X be a reflexive Banach space and let $f : X \to \mathbb{R}$ be a function with the following properties:

- f is weakly sequentially lower semi-continuous, i.e.
$$\forall x \in X \ \forall (x_n) \subset X : x_n \xrightarrow{w} x \implies f(x) \leqslant \liminf f(x_n);$$
- $\lim_{\|x\| \to \infty} f(x) = +\infty$.

Then f attains its minimum at some point of X.

Důkaz
TODO!!! □

3 Elements of the theory of distributions

3.1 Space of test functions and weak derivatives

Definice 3.1 (Support, test functions, the space of test functions, locally integrable, approximate unit (smoothing kernel))

Let $d \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be an open set.

- If $f : \Omega \to \mathbb{F}$ is continuous, its support is the set

$$\operatorname{supp} f = \overline{\{x \in \Omega | f(x) \neq 0\}},$$

where the closure is taken in Ω.

- Let $\mathcal{D}(\Omega, \mathbb{F}) := \{f \in C^\infty(\Omega, \mathbb{F}) | \operatorname{supp} f$ is compact subset of $\Omega\}$. Elements of $\mathcal{D}(\Omega, \mathbb{F})$ are called test functions, the space $\mathcal{D}(\Omega, \mathbb{F})$ is called the space of test functions.

- A measurable function $f : \Omega \to \mathbb{F}$ is called locally integrable in Ω, if for any $x \in \Omega$ there exists $r > 0$ such that f is Lebesgue integrable on $U(\mathbf{x}, r)$ (i.e. $\int_{U(\mathbf{x},r)} |f| < \infty$). The space of all locally integrable functions in Ω is denoted by $L^1_{loc}(\Omega, \mathbb{F})$. (More precisely, the space of equivalence classes.)

- Choose a non-negative $h \in \mathcal{D}(\mathbb{R}^d)$ such that $\operatorname{supp} h \subset U(0,1)$ and $\int_{\mathbb{R}^d} h = 1$. For $j \in \mathbb{N}$ we define a function h_j by $h_j(\mathbf{x}) := j^d \cdot h(j \cdot \mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^d$. The sequence (h_j) obtained in this way is called an approximate unit in $\mathcal{D}(\mathbb{R}^d)$ or a smoothing kernel.

Lemma 3.1

Let $\Omega \subseteq \mathbb{R}^d$ be open. Then $\mathcal{D}(\Omega)$ is a dense subspace of $L^p(\Omega)$ for any $p \in [1, \infty)$.

Důkaz

TODO? (notes) □

Lemma 3.2

Let $\Omega \subset \mathbb{R}^d$ be an open set.

1. *Let μ be a (finite) signed or complex regular Borel measure on Ω. If $\int_\Omega \varphi d\mu = 0$ for any $\varphi \in \mathcal{D}(\Omega)$, then $\mu = 0$.*
2. *Let $f \in L^1_{loc}(\Omega)$ and $\int_\Omega f\varphi = 0$ for any $\varphi \in \mathcal{D}(\Omega)$. Then $f = 0$ almost everywhere on Ω.*

Důkaz

TODO? (notes) □

Definice 3.2 (Weak derivative)

Let $(a, b) \subset \mathbb{R}$ be an open interval an let $f \in L^1_{loc}((a, b))$.

- A function $g \in L^1_{loc}((a, b))$ is called a weak derivative of a function f, if for any $\varphi \in \mathcal{D}((a, b))$ we have $\int_a^b f\varphi' = -\int_a^b g\varphi$.

- Let μ be a finite regular Borel measure on (a, b) (signed or complex). The measure

μ is said to be a weak derivative of a function f if for any $\varphi \in \mathcal{D}((a,b))$ we have $\int_a^b f\varphi' = -\int_{(a,b)} \varphi d\mu$.

Tvrzení 3.3

Let $(a,b) \subset \mathbb{R}$ be an open interval and let $f \in L^1_{loc}((a,b))$. If $\int_a^b f\varphi' = 0$ for any $\varphi \in \mathcal{D}((a,b))$, the function f is constant.

Důkaz
TODO? (The proof is not directly needed for the exam.) □

Věta 3.4

Let $f \in L^1_{loc}((a,b))$.

1. *The weak derivative of f is uniquely determined.*
2. *If f is absolutely continuous on $[a,b]$, it has a finite derivative almost everywhere, $f' \in L^1((a,b))$ and f' is the weak derivative of f. Conversely, if a function f has a weak derivative $g \in L^1((a,b))$, there exists a function f_0 absolutely continuous on $[a,b]$, equal to f almost everywhere on (a,b). In this case $g = f_0'$ almost everywhere.*

   More generally, a function f has a weak derivative in $L^1_{loc}((a,b))$ if and only if there exists function f_0 locally absolutely continuous on (a,b) such that $f_0 = f$ almost everywhere.
3. *There exists a finite measure μ, which is a weak derivative of function f if and only if there exists a function f_0 of bounded variation on $[a,b]$ such that $f_0 = f$ almost everywhere on (a,b). In this case for each subinterval $(c,d) \subset (a,b)$ we have $\mu((c,d)) = \lim_{x \to d_-} f_0(x) - \lim_{x \to c_+} f_0(x)$.*

   Moreover, μ is real-valued if and only if f_0 may be real-valued and μ is non-negative if and only if f_0 may be non-increasing.

Důkaz (1. and 2.)
TODO? (notes) □

3.2 Distributions – basic properties and operations

Definice 3.3 (Convergence in $\mathcal{D}$)

Let $\Omega \subset \mathbb{R}^d$ be an open set, (φ_n) a sequence in $\mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. We say that the sequence (φ_n) converges to φ in $\mathcal{D}(\Omega)$, if the following two conditions are fulfilled:

- There exists $K \subset \Omega$ compact such that $\operatorname{supp} \varphi_n \subset K$ for each $n \in \mathbb{N}$.

- $D^\alpha \varphi_n \rightrightarrows D^\alpha \varphi$ on K for each multiindex $\alpha \in \mathbb{N}_0^d$.

This is expressed by writing $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$.

Lemma 3.5

Let $\Omega \subset \mathbb{R}^d$ be an open set.

a) $\|\cdot\|_N$ is a norm on $\mathcal{D}(\Omega)$;

b) $\mathcal{D}_K(\Omega)$ is a Fréchet space when equipped with $(\|\cdot\|_N)_{N\in\mathbb{N}_0}$.

Důkaz (a))
TODO!!! $\square$

Důkaz (b))
$\|\cdot\|_0 \leqslant \|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \dots \implies \mathcal{D}_K(\Omega)$ is a metrizable LCS (by translation invarinat metric ϱ from the proposition above).

$(\varphi_n) \subset \mathcal{D}_k(\Omega)$ ϱ-cauchy, then $\forall N \in \mathbb{N}_0$: (φ_n) is $\|\cdot\|_N$-cauchy $\implies \forall \alpha$: $(D^\alpha \varphi_n)$ is $\|\cdot\|_\infty$-cauchy $\implies \forall \alpha\ \exists \psi_n$ such that $D^\alpha \varphi_n \rightrightarrows \psi_\alpha$ on Ω. The ψ_α is continuous, $\varphi_\alpha = 0$ on $\Omega \backslash K$. Fix $\alpha \in \mathbb{N}_0^d$ and $j \in [d]$. Then

$$D^\alpha \varphi_n \rightrightarrows \psi_\alpha \wedge \frac{\partial}{\partial x_j} D^\alpha \varphi_n = D^{\alpha+e_j} \varphi_n \rightrightarrows \psi_{\alpha+e_j} \implies \psi_{\alpha+e_j} = \frac{\partial}{\partial x_j} \psi_\alpha.$$

$\implies \psi_\alpha = D^\alpha \psi_0.$

$\implies \psi_0 \in \mathcal{D}_K(\Omega)$, $\forall \alpha : D^\alpha \varphi_n \rightrightarrows D^\alpha \psi_0$, i.e. $\forall N : \varphi_n \to \psi_0$ in $\|\cdot\|_N$. $\implies \varphi_n \to \psi_0$ in ϱ. $\square$

Tvrzení 3.6

$\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ linear then following assertions are equivalent:

1. *$\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to \Lambda(\varphi)$;*
2. *$\varphi_n \to 0$ in $\mathcal{D}(\Omega) \implies \Lambda(\varphi_n) \to 0$;*
3. *$\forall K \subset \Omega$ compact and $\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous;*
4. *$\forall K \subset \Omega$ compact $\exists N \in \mathbb{N}_0\ \exists C > 0$ such that*

$$|\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \qquad \varphi \in \mathcal{D}_K(\Omega).$$

Důkaz
„1. $\implies$ 2.“ is trivial. „2. $\implies$ 3.“: Fix $K \subset \Omega$ compact. $\varphi_n \to 0$ on $\mathcal{D}_K(\Omega) \implies \varphi_n \to 0$ in $\mathcal{D}(\Omega) \overset{2.}{\implies} \Lambda(\varphi_n) \to 0$. Thus $\Lambda|_{\mathcal{D}_K(\Omega)}$ is continuous at $\mathbf{o}$, so it is continuous.

„3. $\implies$ 1.“ $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \implies \exists K \subset \Omega$ compact such that $\operatorname{supp} \varphi_n \subset K$ for each n. Then $(\varphi_n) \subset \mathcal{D}_K(\Omega) \implies \varphi_n \to \varphi$ in $\mathcal{D}_K(\Omega) \overset{3.}{\implies} \Lambda(\varphi_n) \to \varphi(\varphi)$.

„3. $\Leftrightarrow$ 4.“. By the proposition above. □

Definice 3.4 (Distribution, finite order)

A distribution on Ω is a linear functional $\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ satisfying assertions from the previous proposition. We will denote distributions on Ω by $\mathcal{D}'(\Omega)$.

$\Lambda \in \mathcal{D}'(\Omega)$ is of finite order, if $N \in \mathbb{N}_0$ in 4. of the previous proposition can be chosen independently on K.

Například
$f \in L^1_{loc}(\Omega)$. $\Lambda_f(\varphi) = \int_\Omega f \cdot \varphi$ $(\varphi \in \mathcal{D}(\Omega)) \implies \Lambda_f$ is a distribution of order 0. Because $K \subset \Omega$ compact $\implies \int_K |f| < \infty$, $\varphi \in D_K(\Omega)$:

$$|\Lambda_f(\varphi)| = \left|\int_\Omega f \cdot \varphi\right| = \left|\int_K f \cdot \varphi\right| \leqslant \int_K |f\varphi| \leqslant \|\varphi\|_\infty \cdot \int_K |f| = \|\varphi\|_0 \cdot \int_K |f|.$$

$\mu \geqslant 0$ regular Borel measure, finite on compacts. $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution on Ω of order 0. Because if $K \subset \Omega$, $\varphi \in \mathcal{D}_K(\Omega)$, then

$$|\Lambda_\mu(\varphi)| = \left|\int_\Omega \varphi d\mu\right| = \left|\int_K \varphi d\mu\right| \leqslant \|\varphi\|_\infty \mu(K).$$

μ is a signed (or complex) finite measure $\Lambda_\mu(\varphi) = \int_\Omega \varphi d\mu$ is a distribution of order 0:

$$\left|\int_K \varphi d\mu\right| \leqslant \int_K |\varphi| d|\mu| \leqslant |\mu|(K) \cdot \|\varphi\|_\infty \leqslant \|\mu\| \cdot \|\varphi\|_\infty.$$

$\Lambda(\varphi) = \varphi'(0)$, $\varphi \in \mathcal{D}(\mathbb{R})$ is a distribution of order 1. (Clearly $|\Lambda(\varphi)| \leqslant \|\varphi'\|_\infty \leqslant \|\varphi\|_1$.) Λ not of order 0: Find $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi'(0) = 1$, $\operatorname{supp} \varphi \subset [-c, c]$ for some $c > 0$. $\varphi_n(x) = \varphi(nx)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\implies \varphi_n \in \mathcal{D}(\mathbb{R})$. $\operatorname{supp} \varphi_n \subset [-c/n, c/n] \subset [-c, c]$. $\|\varphi_n\|_0 = \|\varphi\|_0$. $\Lambda(\varphi_n) = \varphi_n'(0) = \varphi'(0) \cdot n = n$.

$\Lambda(\varphi) = \sum_{n=0}^\infty \varphi^{(n)}(n)$, $\varphi \in \mathcal{D}(\mathbb{R}) \implies \Lambda$ is a distribution on $\mathbb{R}$, not of finite order ($\operatorname{supp} \varphi \subset [-k, k], k \in \mathbb{N}, \implies |\Lambda(\varphi)| \leqslant (k+1)\|\varphi\|_K$.)

Poznámka
If $f, g \in L^1_{loc}(\Omega)$, $\Lambda_f = \Lambda_g$, then $f = g$ almost everywhere. If μ, ν measures, $\Lambda_\mu = \Lambda_\nu$, then $\mu = \nu$.

If $f \in L^1(\Omega)$, μ finite measure, $\Lambda_f = \Lambda_\mu$, then $\mu(A) = \int_A f$, for each $A \subset \Omega$ Borel.

Definice 3.5

$\Lambda \in \mathcal{D}'(\Omega)$.

- For $\alpha \in \mathbb{N}_0^d$ define $D^\alpha\Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)
- For $f \in C^\infty(\Omega)$ define $(f\Lambda)(\varphi) = \Lambda(f\varphi)$. (For any $\varphi \in \mathcal{D}(\Omega)$.)

Tvrzení 3.7

a) $\Lambda \in \mathcal{D}'(\Omega)$, $\alpha \in \mathbb{N}_0^d \implies D^\alpha\Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
Clear: $D^\alpha\Lambda : \mathcal{D}(\Omega) \to \mathbb{F}$ linear, $K \subset \Omega$ compact $\implies \exists N \in \mathbb{N}_0, C > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega)$. Then $\forall \varphi \in \mathcal{D}_k(\Omega)$:

$$|D^\alpha\Lambda(\varphi)| = |\Lambda(D^\alpha\varphi)| \leqslant C \cdot \|D^\alpha\varphi\|_N \leqslant C \cdot \|\varphi\|_{|\alpha|+N}$$

. □

b) $f \in C^\infty(\Omega) \implies D^\alpha\Lambda_f = \Lambda_{D^\alpha f}$

Důkaz (For $\partial/\partial x_1$)

$$\frac{\partial}{\partial x_1}\Lambda_f(\varphi) = -\Lambda_f\left(\frac{\partial\varphi}{\partial x_1}\right) =?= -\int_\Omega f \cdot \frac{\partial\varphi}{\partial x_1}$$

TODO!?

$$= \int_{\bigcup_j (a_j,b_j)} f\frac{\partial\varphi}{\partial x_1}dx_1 =$$

$$= \sum_{j=1}^n \left([f\cdot\varphi]_{a_j}^{b_j} - \int_{a_j}^{b_j}\frac{\partial f}{\partial x_1}\varphi dx_1\right) = \int_\Omega \frac{\partial f}{\partial x_1}\varphi = \Lambda_{\frac{\partial f}{\partial x_1}}(\varphi).$$

□

c) $d = 1$, $\Omega = (a,b)$, $f \in L^1_{loc}(\Omega)$. *Then* $(\Lambda_f)' = \Lambda_g \Leftrightarrow g$ *is the weak derivative of* f. *And* $(\Lambda_f)' = \Lambda\mu \Leftrightarrow \mu$ *is the weak derivative of* f.

Důkaz
By definitions. □

d) $\Lambda \in \mathcal{D}'(\Omega),\ f \in C^\infty(\Omega) \implies f\Lambda \in \mathcal{D}'(\Omega)$.

Důkaz
clear: $f\Lambda : \mathcal{D}(\Omega) \implies$ IF linear □

Tvrzení 3.8

a) $\Lambda \in \mathcal{D}'((a,b)), \Lambda' = 0 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
We will prove $\operatorname{Ker}\Lambda_1 \subset \operatorname{Ker}\Lambda$. Then $\exists c : \Lambda = c \cdot \Lambda_1 = \Lambda c$.

$$\varphi \in \operatorname{Ker}\Lambda_1 \implies \Lambda_1(\varphi) = 0, i.e. \int_a^b \varphi = 0.$$

Define $\varphi(t) = \int_a^t \varphi,\ t \in (a,b)$. Then $\psi \in \mathcal{D}((a,b)),\ \psi' = \varphi$ ($\psi' = \varphi$ ... differentiation of indefinite integral $\implies \psi \in C^\infty((a,b)),\ \psi = 0$ on $(a, \min \operatorname{supp}\varphi)$ and $(\max \operatorname{supp}\varphi, b)$ $\implies \psi \in \mathcal{D}((a,b))$). Hence $\Lambda(\varphi) = \Lambda(\psi') = -\Lambda'(\psi) = 0$, so $\varphi \in \operatorname{Ker}\Lambda$. □

b) $\Omega \subset \mathbb{R}^d$ *open connected,* $\Lambda \in \mathcal{D}'(\Omega),\ D^\alpha\Lambda = 0$ *for* $|\alpha| = 1 \implies \exists c \in \mathbb{F} : \Lambda = \Lambda_c$.

Důkaz
„Step 1: $\Omega = \prod_{j=1}^{d}(a_j, b_j)$“: Induction on d. For $d = 1$ use a). Assume it holds for $d-1$, denote $\Omega' = \prod_{j=1}^{d-1}(a_j, b_j)$, $x \in \Omega \implies x = (x', x_d)$ ($x' \in \mathbb{R}^{d-1}$, $x_d \in \mathbb{R}$), $\alpha \in N_0^d \implies \alpha = (\alpha', \alpha_d)$.

$\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$ for $|\alpha| = 1$. It means: $\forall \varphi \in \mathcal{D}(\Omega)\ \forall j \in [d] : \Lambda\left(\frac{\partial \varphi}{\partial x_j}\right) = 0$.

Claim: $\psi \in \mathcal{D}(\Omega)$. Then $\exists \varphi \in \mathcal{D}(\Omega) : \frac{\partial \varphi}{\partial x_d} = \psi \Leftrightarrow \forall x' \in \Omega' : \int_{a_d}^{b_d} \psi(x', x_d) dx_d = 0$. („$\implies$“ clear, „$\impliedby$“: define $\varphi(x', x_d) = \int_{a_d}^{x_d} \psi(x', t) dt$). Define

$$T : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega'), \qquad T\varphi(x') = \int_{a_d}^{b_d} \varphi(x', x_d) dx_d, \quad \varphi \in \mathcal{D}(\Omega).$$

T is linear, $\operatorname{Ker} T \subset \operatorname{Ker} \Lambda$ ($T\varphi = 0 \implies \exists \psi \in \mathcal{D}(\Omega) : \varphi = \frac{\partial \psi}{\partial x_d}$, thus $\Lambda(\varphi) = 0$). Fix $\eta \in \mathcal{D}((a_d, b_d))$, $\int_{a_d}^{b_d} \eta = 1$. For $\varphi \in \mathcal{D}(\Omega')$ define $(\varphi\eta)(x) = \varphi(x')\eta(x_d)$. Then $\varphi\eta \in \mathcal{D}(\Omega)$. $\tilde{\Lambda}(\varphi) = \Lambda(\varphi\eta)$, $\varphi \in \mathcal{D}(\Omega')$. Then $\tilde{\Lambda} \in \mathcal{D}'(\Omega')$.

Moreover, $\forall \alpha'$ with $|\alpha'| = 1 : D^{\alpha'}\tilde{\Lambda} = 0$.

$$\left(\forall j \in [d-1] : \frac{\partial}{\partial x_j}\tilde{\Lambda}(\varphi) = -\tilde{\Lambda}\left(\frac{\partial \varphi}{\partial x_j}\right) = -\Lambda\left(\frac{\partial \varphi}{\partial x_j}\eta\right) = -\Lambda\left(\frac{\partial}{\partial x_j}(\varphi\eta)\right) = 0.\right)$$

$\implies \exists c \in \mathbb{F} : \tilde{\Lambda} = \Lambda_c$ in $\mathcal{D}'(\Omega')$. Then $\Lambda = \Lambda_c$ (in $\mathcal{D}(\Omega)$) cause

$$\varphi \in \mathcal{D}(\Omega) \implies \varphi - (T\varphi)\eta \in \mathcal{D}(\Omega), \varphi - (T\varphi)\eta \in \operatorname{Ker} T \subset \operatorname{Ker} \Lambda, \text{ so, } \Lambda(\varphi) = \Lambda((T\varphi)\eta) =$$

$$= \tilde{\Lambda}(T\varphi) = \Lambda_c(T\varphi) = \int_{\Omega'} c \cdot T\varphi = \int_{\Omega'} c \cdot \int_{a_d}^{b_d} \varphi(x', x_d) dx_d dx' \overset{\text{FUBINI}}{=} \int_{\Omega} c \cdot \varphi = \Lambda_c(\varphi).$$

„Step 2: Ω is open connected, $\Lambda \in \mathcal{D}'(\Omega)$, $D^\alpha \Lambda = 0$, $|\alpha| = 1$.“: Step 1 $\implies \forall Q \subset \Omega$ cuboid $\exists c : \Lambda|_{\mathcal{D}(Q)} = \Lambda_c$. Fix one cuboid $Q_0 \subset \Omega$ and the respective c.

$$A := \left\{x \in \Omega | \exists Q \subset \Omega \text{ cuboid}, x \in Q, \Lambda|_{\mathcal{D}(Q)} = \Lambda_c\right\}.$$

Fix $A \neq \emptyset$ ($Q_0 \subset A$), A is open, A is closed in Ω ($x \in \overline{A} \cap \Omega$, $Q \cap A \neq \emptyset$, $\Lambda|_{\mathcal{D}(Q)} = \Lambda_d$, $y \in Q \cap A \implies \Lambda|_{\mathcal{D}(Q_y)} = \Lambda_c \implies$ on $\mathcal{D}(Q \cap Q_y) : \Lambda = \Lambda_c = \Lambda_d \implies c = d \implies x \in A$.). So $A = \Omega$ as Ω is connected. The $\Lambda = \Lambda_c$ in $\mathcal{D}'(\Omega)$. (Proof of this was skipped, it remains that for every $\varphi \in \mathcal{D}(\Omega)$, not only for every $\varphi \in \mathcal{D}(Q)$, it holds $\Lambda(\varphi) = \Lambda_c(\varphi)$.) □

3.3 A bit more on distributions

Definice 3.6 (Convergence in distributions (in $\mathcal{D}'$))

$\Lambda_n \to \Lambda$ in $\mathcal{D}(\Omega) \equiv \forall \varphi \in \mathcal{D}(\Omega) : \lim_{n\to\infty} \Lambda_n(\varphi) = \Lambda(\varphi)$.

Tvrzení 3.9 (On the convergence of distributions)

a) $\Lambda_n \to \Lambda$ *in* $\mathcal{D}(\Omega)$*, then:*

- $\forall \alpha : D^\alpha \Lambda_n \to D^\alpha \Lambda$*;*

  Důkaz

  $$D^\alpha \Lambda_n(\varphi) = (-1)^{|\alpha|} \Lambda_n(D^\alpha \varphi) \to (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = D^\alpha \Lambda(\varphi).$$

  □

- $f \in C^\infty(\Omega) : f\Lambda_n \to f\Lambda$*.*

  Důkaz

  $$f\Lambda_n(\varphi) = \Lambda_n(f\varphi) \to \Lambda(f\varphi) = f\Lambda(\varphi).$$

  □

b) $f_n \to f$ *in* $L^1_{loc}(\Omega)$ *(*$\forall K \subset \Omega$ *compact:* $\int_K |f_n - f| \to 0$*). Then* $\Lambda_{f_n} \to \Lambda_f$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi \in \mathcal{D}(\Omega) : |\Lambda_{f_n}(\varphi) - \Lambda_f(\varphi)| = \left| \int_\Omega f_n \varphi - \int_\Omega f\varphi \right| \leqslant \int_\Omega |f_n - f| \cdot |\varphi| =$$
$$= \int_{\operatorname{supp}\varphi} |f_n - f| \cdot |\varphi| \leqslant \|\varphi\|_\infty \int_{\operatorname{supp}\varphi} |f_n - f| \to 0.$$

□

c) $f_n \to f$ *in* $L^p(\Omega)$ *for some* $p \in [1, \infty]$*. Then* $\Lambda_{f_n} \to \Lambda_f$*.*

Důkaz

Let $K \subset \Omega$ be compact, q the dual exponent. Then use b) with

$$\int_K |f_n - f| \leqslant \|f_n - f\|_{L^p(K)} \cdot \|1\|_{L^q(K)} \to 0.$$

□

d) $\varphi_n \to \varphi$ *in* $\mathcal{D}(\Omega)$*. Then* $\Lambda_{\varphi_n} \to \Lambda_\varphi$ *in* $\mathcal{D}'(\Omega)$*.*

Důkaz

$$\varphi_n \to \varphi \text{ in } \mathcal{D}(\Omega) \implies \varphi_n \to \varphi \text{ in } C^\infty(\Omega), \text{ and use c).}$$

□

Věta 3.10 (Banach—Steinhaus for distributions)

$(\Lambda_n) \subset \mathcal{D}'(\Omega)$ and $\forall \varphi \in \mathcal{D}(\Omega) : (\Lambda_n(\varphi))$ converges in $\mathbb{F}$. Then $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$ is a distribution on Ω.

Důkaz

Clearly Λ is a linear functional on $\mathcal{D}(\Omega)$. Further: $K \subset \Omega$ compact $\implies \forall n : \Lambda_n|_{\mathcal{D}_K(\Omega)}$ is continuous. $\mathcal{D}_K(\Omega)$ is a Fréchet space $\overset{\text{the lemma above, b)}}{\implies} \Lambda|_{\mathcal{D}_K(\Omega)}$ continuous $\implies \Lambda \in \mathcal{D}'(\Omega)$. □

Definice 3.7 (Vanishing of distributions, support of distribution, distribution with compact support)

$\Lambda \in \mathcal{D}'(\Omega)$.

- $G \subset \Omega$ open. Λ vanishes on G if $\Lambda(\varphi) = 0$ whenever $\varphi \in \mathcal{D}(\Omega), \operatorname{supp}\varphi \subset G$.
- $\operatorname{supp}\Lambda = \Omega \setminus \{G \subset \Omega \text{ open } | \Lambda \text{ vanishes on } G\} =$

$$= \{x \in \Omega | \forall \varepsilon > 0 \exists \varphi \in \mathcal{D}(\Omega) : \operatorname{supp}\varphi \subset U(x,\varepsilon) \wedge \Lambda(\varphi) \neq \emptyset\}.$$

- Λ has compact support if $\operatorname{supp}\Lambda$ is a compact subset of Ω.

Tvrzení 3.11 (On the support of a distribution)

a) $\Lambda = \Lambda_f$ for some $f \in L^1_{loc}(\Omega)$. Then

$$\operatorname{supp}\Lambda_f = \operatorname{supp} f = \left\{x \in \Omega | \forall \varepsilon > 0 : \lambda^d\left(\{y \in U(x,\varepsilon) \cap \Omega | f(y) \neq 0\}\right) > 0\right\}$$

Důkaz

„$\subseteq$“: $X \notin \operatorname{supp} f \implies \exists \varepsilon > 0 : f = 0$ almost everywhere on $U(x,\varepsilon) \cap \Omega \implies \Lambda_f$ vanishes on $U(x,\varepsilon) \cap \Omega \implies x \notin \operatorname{supp}\Lambda_f$.

„$\supseteq$“: $x \in$ supp. Let $\varepsilon > 0$. Then f is not 0 almost everywhere on $U(x,\varepsilon) \cap \Omega \implies \exists \varphi \in \mathcal{D}(U(x,\varepsilon) \cap \Omega)$ □

b) $\Lambda = \Lambda_\mu$. Then $\operatorname{supp}\Lambda = \operatorname{supp}\mu = \Omega \setminus \bigcup\{G \subset \Omega$ *open* $|\forall B \subset G$ *Borel* $\mu(B) = 0\}$.

Důkaz

$G \subset \Omega$ open the $\forall B \subset G$ Borel $\mu(B) = 0 \Leftrightarrow \forall \varphi \in \mathcal{D}(G) : \int \varphi d\mu = 0 \Leftrightarrow \Lambda_\mu$ vanishes on G. □

Poznámka

f is continuous $\implies \operatorname{supp} f = \overline{\{x | f(x) \neq 0\}} \cap \Omega$.

c) $\varphi \in \mathcal{D}(\Omega)$, $\operatorname{supp}\varphi \cap \operatorname{supp}\Lambda = \emptyset \implies \Lambda(\varphi) = 0$.

Důkaz

Máme $\operatorname{supp}\varphi \cap \operatorname{supp}\Lambda = \varnothing \implies \operatorname{supp}\varphi \subset \bigcup\{G \subset \Omega \text{ open } | \Lambda \text{ vanishes on } G\} \implies \exists G_1, G_2, \ldots, G_n \subset \Omega$ open such that Λ vanishes on each G_j and $\operatorname{supp}\varphi \subset G_1 \cup \ldots \cup G_n$. We will be done if we show that Λ vanishes on $G_1 \cup \ldots \cup G_n$. □

Důkaz (Λ vanishes on $G_1, G_2 \implies$ vanishes on $G_1 \cup G_2$)

$\psi \in \mathcal{D}(\Omega)$, $\operatorname{supp}\psi \subset G_1 \cup G_2$. If $\operatorname{supp}\psi \subset G_1$ or $\operatorname{supp}\psi \subset G_2$, then $\Lambda(\psi) = 0$. Assume $\operatorname{supp}\psi \not\subset G_1$ and $\operatorname{supp}\psi \not\subset G_2$. Then $L := \operatorname{supp}\varphi \setminus G_2 \implies L$ is compact, nonempty, $L \subset G_1$. Fix $\delta > 0$ such that $3\delta < \operatorname{dist}(L, \mathbb{R}^d \setminus G_1)$, h_k smooth kernel.

Fix $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$, $\xi := h_k * \chi_{L+B(0,2\delta)} \implies \xi \in C^\infty(\mathbb{R}^d)$. $\operatorname{supp}\xi \subset L + B(0,2\delta) + U(0,1/k) \subset L + U(0,3\delta) \subset G_1$, $\xi = 1$ on $L + B(0,\delta)$. Set $\psi_1 = \xi \cdot \psi$, $\psi_2 = (1-\xi)\psi \implies \psi_1, \psi_2 \in \mathcal{D}(\Omega)$, $\operatorname{supp}\psi_1 \subset \xi \subset G_1$, $\operatorname{supp}\psi_2 \subset \operatorname{supp}\psi \setminus (L + B(0,\delta)) \subset \operatorname{supp}\psi \setminus (L + U(0,\delta)) \subset \operatorname{supp}\psi \setminus L \subset G_2 \implies \Lambda(\psi_1) = \Lambda(\psi_2) = 0$. $\psi = \psi_1 + \psi_2 \implies \Lambda(\psi) = \Lambda(\psi_1) + \Lambda(\psi_2) = 0$. □

d) Λ has compact support $\implies \exists N \in \mathbb{N}_0\ \exists C > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N$ for $\varphi \in \mathcal{D}(\Omega)$. In particular, Λ has finite order.

Důkaz

$\operatorname{supp}\Lambda$ is a compact subset of $\Omega \implies \exists\delta > 0 : K := \operatorname{supp}\Lambda + B(0,3\delta) \subset \Omega \implies K \subset \Omega$ is compact $\implies$

$$\exists N \in \mathbb{N}_0\ \exists C > 0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$\xi := h_k * \chi_{\operatorname{supp}\Lambda + B(0,2\delta)}$. $(1/k < \delta.)$ $\xi \in C^\infty(\mathbb{R}^d), \operatorname{supp}\xi \subset \operatorname{supp}\Lambda + B(0,2\delta) + U(0,1/k) \subset K$. $\xi = 1$ on $\operatorname{supp}\Lambda + B(0,\delta)$.

$\forall\varphi \in \mathcal{D}(\Omega) : \Lambda(\varphi) = \Lambda(\varphi\xi)$. $(1-\xi)\varphi \in \mathcal{D}(\Omega) = 0$ on $\operatorname{supp}\Lambda + B(0,\delta) \implies \operatorname{supp}(1-\xi)\varphi \cap \operatorname{supp}\Lambda = \varnothing. \implies \Lambda((1-\xi)\varphi) = 0 \implies \Lambda(\varphi) = \Lambda(\xi\varphi)$.

Then

$$|\Lambda(\varphi)| = |\Lambda(\varphi\xi)| \leqslant C \cdot \|\xi \cdot \varphi\|_N \leqslant C \cdot 2^N \cdot \|\xi\|_N \cdot \|\varphi\|_N.$$

□

e) $\operatorname{supp}\Lambda = \{p\} \Leftrightarrow \exists N \in \mathbb{N}_0, C_\alpha \in \mathbb{F}, |\alpha| \leqslant N, \Lambda = \sum_{|\alpha| \leqslant N} C_\alpha D^\alpha \Lambda_{\delta_p}$.

Důkaz

„ $\Longleftarrow$ “: trivial. „ $\implies$ “: $\{p\}$ is compact $\implies \exists N, C : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_M$, $\varphi \in D(\Omega)$. The Λ is a linear combination of $D^\alpha\Lambda_{\delta_p}$, $|\alpha| \leqslant N$. To prove this, we use lemma above and show

$$\bigcap_{|\alpha| \leqslant N} \operatorname{Ker} D^\alpha \Lambda_{\delta_p} \subset \operatorname{Ker}\Lambda,$$

i.e. $\forall\varphi \in \mathcal{D}(\Omega) : D^\alpha\varphi(p) = 0$ for each $|\alpha| \leqslant N \implies \Lambda(\varphi) = 0$. □

3.4 Convolution of distribution

Definice 3.8 (Notation: translate, reflexion and derivative in direction)

$M \subset \mathbb{R}^d$, $f : M \to \mathbb{F}$

- $y \in \mathbb{R}^d$, $\tau_y f(x) = f(x-y)$, $x \in y + M$;
- $\hat{f}(x) = f(-x)$, $x \in -M$;
- $a, e \in \mathbb{R}^d$: $\partial_e f(a) = \lim_{r\to 0} : \frac{f(a+re)-f(a)}{r}$.

Lemma 3.12

$\varphi \in \mathcal{D}(\mathbb{R}^d)$.

a) $x_n \to x$ *in* $\mathbb{R}^d \implies \tau_{x_n}\varphi \to \tau_x \varphi$ *in* $\mathcal{D}(\mathbb{R}^d)$.

Důkaz

$\operatorname{supp}\varphi \subset U(0,r_1)$ for some $r_1 > 0$, $\{x_n, n \in \mathbb{N}\} \subset U(0,r_2)$ for some $r_2 > 0$. $K := \overline{U(0, r_1+r_2)} \implies K$ is compact and $\operatorname{supp}\tau_{x_n}\varphi \subset K$ for each n. $\alpha \in \mathbb{N}_0^d$:

$$\|D^\alpha \tau_{x_n}\varphi - D^\alpha \tau_x \varphi\|_\infty = \sup_{y\in\mathbb{R}^d} |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)| = \sup_{y\in K} |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)|.$$

Thus $D^\alpha\varphi$ is continuous, so it is uniformly continuous on $\overline{U(2r_2+r_1)}$.

$$\varepsilon > 0 \implies \exists\delta > 0\ \forall y_1, y_2 \in \overline{U(2r_2+r_1)} : (\|y_1 - y_2\| < \delta \implies |D^\alpha\varphi(y_1) - D^\alpha\varphi(y_2)| < \varepsilon)\,.$$

$$x_n \to x \implies \exists n_0\ \forall n \geqslant n_0 : \|x_n - x\| < \delta.$$

$$n \geqslant n_0, y \in K \implies y - x_n, y - x \in \overline{U(2r_2+r_1)}, \|(y-x_n)-(y-x)\| = \|x_n - x\| < \delta \implies$$
$$\implies |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)| < \varepsilon \implies D^\alpha\tau_{x_n}\varphi \rightrightarrows D^\alpha\tau_x\varphi.$$

□

b) $e \in \mathbb{R}^d \implies \partial_e\varphi \in \mathcal{D}(\mathbb{R}^d)$. *Moreover, set*

$$\varphi_r(x) := \frac{1}{r}(\varphi(x+re) - \varphi(x)), \qquad x \in \mathbb{R}^d,$$

then $\varphi_r \xrightarrow{r\to 0} \partial_e\varphi$ *in* $\mathcal{D}(\mathbb{R}^d)$.

Důkaz ($e \in \mathbb{R}^d \implies \partial_e\varphi \in \mathcal{D}(\mathbb{R}^d)$)
$x \in \mathbb{R}^d$. $g_x(t) := \varphi(x+te)$, $t \in \mathbb{R}$. Then $g_x \in C^\infty(\mathbb{R})$.

$$\partial_e\varphi(x) = g_x'(0) = \sum_{j=1}^d \frac{\partial\varphi}{\partial x_j}(x+te)\cdot e_j|_{t=0} =$$

$$= \sum_{j=1}^d \frac{\partial\varphi}{\partial x_j}(x)e_j \implies \partial_e\varphi = \sum_{j=1}^d e_j\frac{\partial\varphi}{\partial x_j} \in \mathcal{D}(\mathbb{R}^d).$$

□

Důkaz (Moreover part)
Fix $c > 0$, such that $\operatorname{supp}\varphi \subset U(0,c)$, and $0 < |r| < 1$. Then $\operatorname{supp}\varphi_r \subset \overline{U(0, c+\|e\|)}$.

$$|\varphi_r(x) - \partial_e\varphi(x)| = \left|\frac{1}{r}(g_x(r) - g_x(0)) - g_x'(0)\right| = \left|\frac{1}{r}\int_0^r g_x' - g_x'(0)\right| =$$

$$= \left|\frac{1}{r}\int_0^r (g_x'(t) - g_x'(0))dt\right| = \left|\frac{1}{r}\int_0^r \sum_{j=1}^d e_j\left(\frac{\partial\varphi}{\partial x_j}(x+te) - \frac{\partial\varphi}{\partial x_j}(x)\right)dt\right| \leqslant$$

$$\leqslant \left|\frac{1}{r}\int_0^r \|e\|\left(\sum_{j=1}^d \left\|\frac{\partial\varphi}{\partial x_j}(x+te) - \frac{\partial\varphi}{\partial x_j}(x)\right\|^2\right)^{1/2} dt\right| \leqslant$$

$$\leqslant \left|\frac{1}{r}\int_0^r \|e\|\left(\sum_{j=1}^d \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty^2\right)^{1/2} dt\right|.$$

$$\varepsilon > 0 \implies \exists\delta\ \forall y, \|y\| < \delta : \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty < \varepsilon.$$

If $0 < |t|\cdot\|e\|\cdot c$, then

$$\|e\|\left(\sum_{j=1}^d \left\|\tau_{-te}\frac{\partial\varphi}{\partial x_j} - \frac{\partial\varphi}{\partial x}\right\|_\infty^2\right)^{1/2} \leqslant \|e\|\cdot\sqrt{d}\cdot\varepsilon.$$

So $\varphi_r \rightrightarrows \partial_e\varphi$, $D^\alpha\varphi_r = (D^\alpha\varphi)_r \rightrightarrows \partial_e(D^\alpha\varphi) = D^\alpha(\partial_e\varphi)$. □

Tvrzení 3.13

$\varphi \in \mathcal{D}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$.

a) $\Lambda \in \mathcal{D}'(\mathbb{R}^{d_1})$. *Define* $\psi(y) = \Lambda(x \mapsto \varphi(x,y))$ $(y \in \mathbb{R}^{d_2})$. *Then* $\psi \in \mathcal{D}(\mathbb{R}^{d_2})$.

Důkaz
Fix $c > 0$ such that $\operatorname{supp}\varphi \subset \overline{U(\mathbf{o}, c)}$. 1. „$\psi$ is well defined“: given $y \in \mathbb{R}^{d_2}$, $x \mapsto \varphi(x, y)$ belongs to $\mathcal{D}(\mathbb{R}^{d_1})$, i.e. it is C^∞ and $\operatorname{supp} \subset \overline{U(0, c)}$. 2. $\operatorname{supp}\psi \subset \overline{U(\mathbf{o}, c)}$, so it is compact.

3. $y \in \mathbb{R}^{d_2}$, $\varphi_y(x) = \varphi(x, y)$ $(x \in \mathbb{R}^{d_1})$. Then „$y_n \to y$ in $\mathbb{R}^{d_2} \implies \varphi_{y_n} \to \varphi_y$ in $\mathcal{D}(\mathbb{R}^{d_2})$“: Assume $y_n \to y$ in $\mathbb{R}^{d_2}$. WLOG $\|y_n\| \leqslant c$ for each n. $\forall n : \operatorname{supp}\varphi_{y_n} \subset \overline{U(\mathbf{o}, c)}$. Fix $\alpha \in \mathbb{N}_0^{d_1}$. Then „$D^\alpha\varphi_{y_n} \rightrightarrows D^\alpha\varphi_y$“:

$D^\alpha\varphi_{y_n}(x) = D^{(\alpha,0)}\varphi(x, y_n)$. $D^{(\alpha,0)}\varphi$ is continuous, hence uniformly continuous on $\overline{U(\mathbf{o}, c)}$. So, give $\varepsilon. > 0$ $\exists\delta > 0$ $\forall(u_1, u_2), (v_1, v_2) \in \overline{U(\mathbf{o}, c)}$:

$$\|(u_1, v_1) - (u_2, v_2)\| < \delta \implies |D^{(\alpha,0)}\varphi(u_1, v_1) - D^{(\alpha,0)}\varphi(u_2, v_2) < \varepsilon.$$

Fix $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0 : \|y - y_n\| < d$. If $n \geqslant n_0$ and $x \in \overline{U_{\mathbb{R}^{d_1}}(\mathbf{o}, c)}$, then

$$|D^{(\alpha,0)}\varphi(x, y_n) - D^{(\alpha,0)}\varphi(x, y)| < \varepsilon \qquad \Longleftarrow \|(x, y_n) - (x, y)\| < \delta.$$

Hence $\|D^\alpha\varphi_{y_n} - D^\alpha\varphi_y\| \leqslant \varepsilon$ for $n \geqslant n_0$.

4. „ψ is continuous“:

$$y_n \to y \overset{3.}{\implies} \varphi_{y_n} \to \varphi_y \text{ in } \mathcal{D}(\mathbb{R}^{d_1}) \implies \psi(y_n) = \Lambda(\varphi_{y_n}) \to \Lambda(\varphi_y) = \psi(y).$$

5. „$\frac{\partial\psi}{\partial y_j}(y) = \Lambda(x \mapsto \frac{\partial\varphi}{\partial y_j}(x, y))$“:

$$\frac{\partial\psi}{\partial y_j}(y) = \lim_{t\to0} \frac{\psi(y + te_j) - \psi(y)}{\tau} \overset{\Lambda \text{ linear}}{=} \lim_{t\to0} \Lambda\left(x \mapsto \frac{\varphi(x, y + te_j) - \varphi(x, y)}{t}\right) =$$

$$= \lim_{t\to0} \Lambda(x \mapsto \varphi_t(x, y)).$$

We know $\varphi_t \to \partial_{(0,y_j)}\varphi$ in $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. So we have $\varphi_t \to \frac{\partial\varphi}{\partial y_j}$ in $\mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Hence, for each $y \in \mathbb{R}^{d_2}$: $(\varphi_t)_y \to \left(\frac{\partial\varphi}{\partial y_j}\right)_y$ in $\mathcal{D}(\mathbb{R}^{d_1}) \implies \Lambda((\varphi_t)_y) \to \Lambda\left(\left(\frac{\partial\varphi}{\partial y_j}\right)_y\right)$.

$$(*) = \Lambda\left(\left(\frac{\partial\varphi}{\partial y_j}\right)_y\right) = \Lambda(x \mapsto \frac{\partial\varphi}{\partial y_j}(x, y)).$$

6. „$\psi \in C^\infty(\mathbb{R}^{d_2})$ and $\forall\alpha : D^\alpha\psi(y) = \Lambda(x \mapsto D^{(0,\alpha)}\varphi((x, y)))$“: 5. $\implies$ for $|\alpha| = 1$. 4. applied to $\frac{\partial\varphi}{\partial y_j}$ implies $\psi \in C^1(\mathbb{R}^{d_2})$. Induction: Assume it holds for $|\alpha| \leqslant k$, take $|\alpha| = k+1$. Then $\alpha = \beta + e_j$, $|\beta| = k$, $j \in [d]$.

$$D^\alpha\psi(y) = \frac{\partial}{y_j}(D^\beta\psi)(y) = \frac{\partial}{\partial y_j}\left(y \mapsto \Lambda\left(x \mapsto D^{(0,\beta)}\varphi(x, y)\right)\right) \overset{5.}{=}$$

$$= \Lambda(x \mapsto \frac{\partial}{\partial y_j} D^{(0,\beta)}\varphi(x, y)) = \Lambda(x \mapsto D^{(0,\alpha)}\varphi(x, y)).$$

□

Lemma 3.14

$\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}(\Omega)$, $K \subset \Omega$ compact. Then $\exists N \in \mathbb{N}_0$, $\exists \mu_\alpha$, $|\alpha| \leqslant N$, finite (signed or complex) Borel measure on K such that

$$\Lambda(\varphi) = \sum_{|\alpha| \leqslant N} \int_K D^\alpha \varphi d\mu_\alpha, \qquad \varphi \in \mathcal{D}_K(\Omega).$$

Důkaz (of lemma, sketch)
From the proposition above $\exists N, C$ such that

$$|\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N, \varphi \in \mathcal{D}_K(\Omega).$$

$X := (C(K))^{\{\alpha||\alpha|\leqslant N\}}$. $T : \mathcal{D}_K(\Omega) \to X$ by $T\varphi = (D^\alpha \varphi)_{|\alpha| \leqslant N} \implies \Lambda \circ T^{-1}$ is continuous on $T(\mathcal{D}_K(\Omega)) \implies$ extend to $X \implies$ (by Riesz) find $\mu_\alpha, |\alpha| \leqslant N$. □

b) $\Lambda_1 \in \mathcal{D}'(\mathbb{R}^{d_1})$, $\Lambda_2 \in \mathcal{D}'(\mathbb{R}^{d_2})$. Then

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))) = \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x,y))).$$

Důkaz
By a) both sides are well defined. $\operatorname{supp} \varphi \subset \overline{U(\mathbf{o}, c)}$. From the previous lemma: Λ_1 (resp. Λ_2) on $\overline{U(\mathbf{o}, c)}$ is equal to μ_α (resp. ν_α) for some $|\alpha| \leqslant N_1$ (resp. $|\alpha| \leqslant N_2$).

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))) = \sum_{|\beta| \leqslant N_2} \int D^\beta \lambda_1(x \mapsto \varphi(x,y)) d\nu_\beta(y) =$$

$$= \sum_{|\beta| \leqslant N_2} \int \Lambda_1(x \mapsto D^{(0,\beta)} \varphi(x,y)) d\nu_\beta(y) =$$

$$= \sum_{|\beta| \leqslant N_2} \sum_{|\alpha| \leqslant N_1} \int \int D^{(\alpha,\beta)} \varphi(x,y) d\mu_\alpha(x) d\nu_\beta(y) \overset{\text{FUBINI}}{=}$$

$$= \sum_{|\beta| \leqslant N_2} \sum_{|\alpha| \leqslant N_1} \int \int D^{(\alpha,\beta)} \varphi(x,y) d\nu_\beta(y) d\mu_\alpha(x) \dots$$

□

Definice 3.9 (Convolution in distributions)

$U \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x-y))$ $(x \in \mathbb{R}^d)$.

Věta 3.15 (On the convolution of a distribution and a test function)

a) $f \in L^1_{loc} \implies \Lambda_f * \varphi = f * \varphi$.

Důkaz

$$\Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x-y)) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)dy = f * \varphi(x).$$

□

b) $U * \varphi \in C^\infty(\mathbb{R}^d),\ D^\alpha(U * \varphi) = D^\alpha U * \varphi = U * D^\alpha \varphi.$

Důkaz
„$U * \varphi$ is continuous“:

$$x_n \to x \text{ in } \mathbb{R}^d \implies \tau_{x_n}\check{\varphi} \to \tau_x\check{\varphi} \text{ in } \mathcal{D}(\mathbb{R}^d) \implies U*\varphi(x_n) = U(\tau_{x_n}\check{\varphi}) \to U(\tau_x\check{\varphi}) = U*\varphi(x).$$

$$\frac{\partial}{\partial x_j}(U * \varphi)(x) = \lim_{t\to 0} \frac{U * \varphi(x+te_j) - U * \varphi(x)}{t} =$$

$$= \lim_{t\to 0} U\left(\frac{\tau_{x+te_j}\check{\varphi} - \tau_x\check{\varphi}}{t}\right) \overset{\psi:=\tau_x\check{\varphi}}{=} \lim_{t\to 0} U\left(\frac{\tau_{te_j}\psi - \psi}{t}\right) = U(\partial_{-e_j}\psi) =$$

$$= U\left(\tau_x\left(\frac{\partial\varphi}{\partial x_j}\right)\right) = U * \frac{\partial\varphi}{\partial x_j}(x).$$

$$\partial_{-e_j}\psi = -\partial_{e_j}\psi = -\frac{\partial\psi}{\partial y_j} = -\frac{\partial}{\partial y_j}(\tau_x\check{\varphi}) = \tau_x\left(\frac{\partial\varphi}{\partial y_j}\right)^v.$$

$$\frac{\partial}{\partial x_j}(U * \varphi) = U * \frac{\partial\varphi}{\partial x_j}.$$

$$\frac{\partial U}{\partial x_j} * \varphi(x) = \frac{\partial U}{\partial x_j}\tau_x\check{\varphi} = -U\left(\frac{\partial\tau_x\check{\varphi}}{\partial x}\right) = U * \frac{\partial\varphi}{\partial x_j}(x).$$

So, we have it for $|\alpha| = 1$. The general case by induction. □

c) $\operatorname{supp}(U * \varphi) \subset \operatorname{supp} U + \operatorname{supp}\varphi.$

Důkaz

$$U * \varphi(x) \neq 0 \implies U(\tau_x\check{\varphi}) \neq 0 \implies \operatorname{supp}(\tau_x\check{\varphi}) \cap \operatorname{supp} U \neq \varnothing \implies x \in \operatorname{supp}\varphi + \operatorname{supp} U.$$

□

Důsledek
So U has compact support $\implies U * \varphi$ has compact support.

d) h_j *smoothing kernel. Then* $\Lambda_{U*h_j} \to U$ *in* $\mathcal{D}'(\mathbb{R}^d)$.

Důkaz

$$\Lambda_{U*h_j}(\varphi) = \int (U*h_j)(x)\varphi(x)dx = \int U(y \mapsto h_j(x-y))\varphi(x)dx =$$

$$= \int U(y \mapsto \varphi(x)h_j(x-y))dx = \Lambda_1(y \mapsto \varphi(x)h_j(x-y)) =$$

$$= U(y \mapsto \Lambda_1(x \mapsto \varphi(x)h_j(x-y))) = U(y \mapsto \int \varphi(x)h_j(x-y)dx) = U(\varphi * \check{h}_j) \to \Lambda(\varphi).$$

Because $\varphi * \check{h}_j \to \varphi$ in $\mathcal{D}(\mathbb{R}^d)$ and

$$\operatorname{supp}(\varphi * \check{h}_j) \subset \operatorname{supp}\varphi + U(0,1/j) \subset \varphi + \overline{U(0,1)},$$

$$D^\alpha(\varphi * \check{h}_j) = (D^\alpha\varphi) * h_j \rightrightarrows D^\alpha\varphi.$$

□

e) $\tau_x(U*\varphi) = \tau_x U * \varphi = U * \tau_x\varphi$

Důkaz

$$\tau_x(U*\varphi)(z) = (U*\varphi)(z-x) = U(\tau_{z-x}\check{\varphi}) = U(\tau_{-x}\tau_z\check{\varphi}) = \tau_x U(\tau_z\check{\varphi}) = \tau_x U * \varphi(z).$$

$$\tau_x(U*\varphi)(z) = (U*\varphi)(z-x) = U(\tau_{z-x}\check{\varphi}) = U(\tau_z(\tau_{-x}\check{\varphi})) = U(\tau_z(\widecheck{\tau_x\varphi})) = U * \tau_x\varphi(z).$$

$$(\tau_{-x}\check{\varphi}(y) = \check{\varphi}(y+x) = \varphi(-y-x) = \tau_x\varphi(-y) = (\widecheck{\tau_x\varphi})(y).)$$

□

f) $U*(\varphi*\psi) = (U*\varphi)*\psi$ *(*$U \in \mathcal{D}'(\mathbb{R}^d), \varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$*).*

Důkaz

$$U*(\varphi*\psi)(x) = U(y \mapsto (\varphi*\psi)(x-y)) = U(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y-z)\psi(z)dz) =$$

$$= U(y \mapsto \Lambda_1(z \mapsto \varphi(x-y-z)\psi(z))) = \Lambda_1(z \mapsto U(y \mapsto \varphi(x-y-z)\psi(z))) =$$

$$= \Lambda_1(z \mapsto \psi(z) \cdot U(y \mapsto \varphi(x-y-z))) = \Lambda_1(z \mapsto \psi(z) \cdot (U*\varphi)(x-z)) =$$

$$= \int \psi(z) \cdot (U*\varphi(x-z))dz = (U*f)*\psi(x).$$

□

Poznámka

$$\check{U}(\varphi) = U(\check{\varphi}), \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\tau_x U$ and $\check{U}$ are distributions, $\tau_x\Lambda_f = \Lambda_{\tau_x f}$, $\check{\Lambda}_f = \Lambda_{\check{f}}$, $f \in L^1_{loc}(\mathbb{R}^d)$ (standard one page of computations or less).

Poznámka
U, V distributions, $U * V(\varphi) = U(\check{V} * \varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$:

- It is natural formula:
$$V = \Lambda_\psi, \psi \in \mathcal{D}(\mathbb{R}^d) \implies \Lambda_{U*\psi}(\varphi) = U(\check{\psi} * \varphi).$$

Důkaz

$$\Lambda_{U*\psi}(\varphi) = \int_{\mathbb{R}^d} U * \psi(x)\varphi(x)dx = \int_{\mathbb{R}^d} U(y \mapsto \psi(x-y))\varphi(x)dx =$$
$$= \int_{\mathbb{R}^d} U(y \mapsto \psi(x-y)\varphi(x))dx = U(y \mapsto \int_{\mathbb{R}^d} \psi(x-y)\varphi(x)dx) = U(y \mapsto \check{\psi} * \varphi(y)).$$

□

- This formula does not work in general because $\check{V} * \varphi$ is a C^∞-function but it need not have compact support.

Poznámka (1.)
$\operatorname{supp} V$ is compact, then $V * \varphi \in \mathcal{D}(\mathbb{R}^n)$ for each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ ($\operatorname{supp} \check{V} * \varphi \subset \operatorname{supp} \check{V} + \operatorname{supp} \varphi$, so it is compact). Then $U * V$ is linear functional on $\mathcal{D}(\mathbb{R}^d)$. Moreover, „it is a distribution“:

Fix $K \subset \mathbb{R}^d$ compact. Set $L := \operatorname{supp} \check{V} + K \implies$
$$\implies \exists C > 0, N \in \mathbb{N}_0 : |V(\psi)| \leqslant C \cdot \|\psi\|, \qquad \forall \psi \in \mathcal{D}_L(\mathbb{R}^d).$$

$\varphi \in \mathcal{D}_K(\mathbb{R}^d) \implies \check{V} * \varphi \in \mathcal{D}_L(\mathbb{R}^d) \implies |(U*V)(\varphi)| = |U(\check{V}*\varphi)| \leqslant C \cdot \|\check{V}*\varphi\|_N \leqslant C \cdot D \cdot \|\varphi\|_{N+M}$.
($\check{V} * \varphi(x) = V(y \mapsto \varphi(x+y))$, V has compact support $\implies \exists D, M : |V(\eta)| \leqslant D \cdot \|\eta\|_M$, $\forall \eta \in \mathcal{D}(\mathbb{R}^d)$.)

Poznámka (2.)
$\operatorname{supp} U$ is compact $\implies \exists \psi \in \mathcal{D}(\mathbb{R}^d)$ such that $U(\varphi) = U(\psi \cdot \varphi), \varphi \in \mathcal{D}(\mathbb{R}^d)$. (Proof of the theorem above item d.) So, define $(U * V)(\varphi) = U(\psi \cdot (\check{V} * \varphi))$. Again $U * V \in \mathcal{D}'(\mathbb{R}^d)$. (Proof skipped.)

Poznámka (3.)
$\forall r > 0 : (\overline{U(\mathbf{o}, r)} - \operatorname{supp} V) \cap \operatorname{supp} U$ is compact. For $r > 0$ let $\psi_r \in \mathcal{D}(\mathbb{R}^d)$, $\psi_r = 1$ on a neighbourhood of this set. Then U may be extended to
$$Y = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \operatorname{supp} f \subset \overline{U(\mathbf{o}, r)} - \operatorname{supp} V \text{ for some } r > 0 \right\}.$$

$\tilde{U}(f) = U(\psi_r \cdot f)$ if $\operatorname{supp} f \subset \overline{U(\mathbf{o}, r)} - \operatorname{supp} V$.

Then define $U * V(\varphi) = \tilde{U}(\check{V} * \varphi)$ ($\operatorname{supp} \check{V} * \varphi \subset \operatorname{supp} \varphi - \operatorname{supp} V$).

Poznámka (4.)
Assume $\exists m, n \in \mathbb{N}_0,\ c, d > 0$:

$$|U(\varphi)| \leqslant c \cdot \|\varphi\|_n \wedge |V(\varphi)| \leqslant d \cdot \|\varphi\|_m, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R}^d).$$

$\implies \mu_\alpha, |\alpha| \leqslant n$ measures (finite ...):

$$U(\varphi) = \sum_{|\alpha| \leqslant n} \int_{\mathbb{R}^d} D^\alpha \varphi d\mu_\alpha, \varphi \in \mathcal{D}(\mathbb{R}^d) \implies$$

$$\implies (U * V)(\varphi) = \sum_{|\alpha| \leqslant n} \int_{\mathbb{R}^d} D^\alpha (\check{V} * \varphi) d\mu_\alpha.$$

$$|(U * V)(\varphi)| \leqslant c \cdot d \cdot \|\varphi\|_{n+m}.$$

3.5 Tempered distributions

Definice 3.10 (Schwartz space)

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in C^\infty(\mathbb{R}^d) \middle| \forall \alpha \in \mathbb{N}_0^d \ \forall N \in \mathbb{N} : x \mapsto (1 + \|x\|^2)^N D^\alpha f(x) \text{ is bounded on } \mathbb{R}^d \right\}.$$

$$f \in \mathcal{S}(\mathbb{R}^d), \quad N \in \mathbb{N}_0, \quad p_N(f) := \max_{|\alpha| \leqslant N} \|x \mapsto (1 + \|x\|^2)^N D^\alpha f(x)\|_\infty.$$

Then $(p_N)_{N=0}^\infty$ is sequence of norms on $\mathcal{S}(\mathbb{R}^d)$, $p_0 \leqslant p_1 \leqslant p_2 \leqslant \ldots$. ($p_0(f) = \|f\|_\infty$)

Tvrzení 3.16

a) $\mathcal{S}(\mathbb{R}^d)$ *is a Fréchet space when equipped with* $(p_N)_{N=0}^\infty$.

Důkaz
$\mathcal{S}(\mathbb{R}^d)$ is a metrizable LCS. Let ϱ be the respective translation invariant metric. „Completeness“: Assume (f_n) is ϱ-Cauchy $\implies$ $\forall N$: (f_n) is p_N-Cauchy $\implies$ $\forall N\ \forall \alpha, |\alpha| \leqslant N : (x \mapsto (1 + \|x\|^2) D^\alpha f_k(x))_{k=1}^\infty$ is $\|\cdot\|_\infty$-Cauchy $\implies$ $\forall N, \alpha, |\alpha| \leqslant N\ \exists g_{N,\alpha}$ such that $(1 + \|x\|^2)^N D^\alpha f_n(x) \rightrightarrows g_{N,\alpha}(x)$ on $\mathbb{R}^d$. $D^\alpha f_k(x) \rightrightarrows \frac{g_{N,\alpha}(x)}{(1+\|x\|^2)^N}$. $\implies$ $\forall \alpha\ \exists h_\alpha$ continuous such that $g_{N,\alpha}(x) = (1 + \|x\|^2)^N h_\alpha(x)$ if $N \geqslant |\alpha|$. $D^\alpha f_k \rightrightarrows h_\alpha$ $\implies$ $h_\alpha = D^\alpha h_\alpha$ $\implies$ $h_\alpha \in C^\infty(\mathbb{R}^d)$.

„$h_0 \in \mathcal{S}(\mathbb{R}^d)$“:

$$(1 + \|x\|^2)^N D^\alpha h_0(x) = g_{N,\alpha}(x),$$

which is bounded (uniform limit of bounded functions). Moreover $f_k \to h_0$ in p_N, hence by the theorem above $f_n \to h_0$ in $\mathcal{S}(\mathbb{R}^d)$ (in ϱ). □

b) $\mathcal{D}(\mathbb{R}^d)$ is a dense subset of $\mathcal{S}(\mathbb{R}^d)$.

Důkaz
Clearly $\mathcal{D}(\mathbb{R}^d) \Subset \mathcal{S}(\mathbb{R}^d)$. „Density“: Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ na $U(\mathbf{o}, 1)$. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $f_n(x) = f(x) \cdot \varphi(x/n)$, $x \in \mathbb{R}^d$. Then $f_n \in \mathcal{D}(\mathbb{R}^d)$. Moreover, „$f_n \to f$ in $\mathcal{S}(\mathbb{R}^d)$“: Let $N \in \mathbb{N}_0$, $d \in \mathbb{N}_0^d$, $|\alpha| \leqslant N$:

$$\left|(1+\|x\|^2)^N(D^\alpha f(x) - D^\alpha f_n(x))\right| = (1+\|x\|^2)^N \left|D^\alpha((1-\varphi(x/n)))f(x)\right| =$$

$$= (1+\|x\|^2)^N \left|(1-\varphi(x/n))D^\alpha f(x) + \sum_{0 \neq \beta \leqslant \alpha} \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d} (-1) \frac{1}{n^{|\beta|}} D^\beta \varphi(x/n) D^{\alpha-\beta} f(x)\right|$$

$$\begin{cases} = 0, & \|x\| \leqslant n \\ \leqslant \sup_{\|x\| \geqslant n, |\gamma| \leqslant N} \frac{(1+\|x\|^2)^{N+1}|D^\gamma f(x)|}{1+\|x\|^2}, & \|x\| > n \end{cases}$$

$$\left(\sup_{\|x\| \geqslant n} \left(1 + \sum_{0 \neq \beta \leqslant \alpha} \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_d}{\beta_d} \cdot \underbrace{\frac{1}{n^{|\beta|}}}_{\leqslant 1} \underbrace{|D^\beta \varphi(x/n)|}_{\leqslant \|\varphi\|_N}\right)\right) \leqslant 1 + 2^N \|\varphi\|_N.$$

$$\leqslant (1 + 2^N \cdot \|\varphi\|_n) \cdot \frac{p_{N+1}(f)}{1+n^2} \to 0.$$

□

c) $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \implies \varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$.

Důkaz
Assume $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R}^d) \implies \exists R > 0$ such that $\operatorname{supp} \varphi_n \subset \overline{U(\mathbf{o}, R)}$. Then

$$p_n(\varphi_n - \varphi) = \max_{|\alpha| \leqslant N} \left\|x \mapsto (1+\|x\|^2)^N (D^\alpha \varphi_n(x) - D^\alpha \varphi(x))\right\|_\infty \leqslant (1+R^2)^N \cdot \|\varphi_n - \varphi\|_N \to 0.$$

□

Definice 3.11 (A tempered distribution on $\mathbb{R}^d$)

A tempered distribution on $\mathbb{R}^d$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^d)$. Notation: $\mathcal{S}'(\mathbb{R}^d)$.

Poznámka
$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \implies \Lambda|_{\mathcal{D}(\mathbb{R}^d)} \in \mathcal{D}'(\mathbb{R}^d)$. (By the previous theorem item c.)

$\mathcal{D}'(\mathbb{R}^d) \Subset \mathcal{D}'(\mathbb{R}^d)$. (By item a. and b.)

We say that distribution is tempered, if it can be extended to $\mathcal{S}(\mathbb{R}^d)$.

Tvrzení 3.17 (A characterization of tempered distributions) *a)* $\Lambda : \mathcal{S}(\mathbb{R}^d) \to \mathbb{F}$ *linear. Then*

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0 \ \exists C > 0 : |\Lambda(\varphi)| \leqslant C \cdot p_N(\varphi), \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Důkaz

By the proposition above. □

b) Assume $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$*. Then* Λ *is tempered iff*

$$\exists N \in \mathbb{N}_0 \ \exists c > 0 : |\Lambda(\varphi)| \leqslant C \cdot p_N(\varphi), \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Důkaz

„ $\Longrightarrow$ “: a). „ $\Longleftarrow$ “: For example by Hahn–Banach and a). □

Definice 3.12 (Convergence in $\mathcal{S}'$)

$\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d) \equiv \forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \Lambda_n(\varphi) \to \Lambda(\varphi)$, i.e. $\Lambda_n \overset{w*}{\to} \Lambda$.

Věta 3.18 (Banach–Steinhaus theorem for tempered distribution)

$(\Lambda_n) \subset \mathcal{S}'(\mathbb{R}^d)$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$: $(\Lambda_n(\varphi))$ *converges in* $\mathbb{F}$*. Then* $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *is tempered distribution.*

Důkaz

Use the previous proposition item a) and the theorem above. □

Tvrzení 3.19 (Examples of tempered distributions)

a) $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$, $\operatorname{supp}\Lambda$ *is compact* $\Longrightarrow$ Λ *is tempered.*

Důkaz

Λ has compact support $\Longrightarrow$ $\exists C > 0 \ \exists N \in \mathbb{N}_0 : |\Lambda(\varphi)| \leqslant C \cdot \|\varphi\|_N \leqslant C \cdot p_N(\varphi)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$. □

b) $f \in L^p(\mathbb{R}^d)$ *for some* $p \in [1, \infty]$*. Then* $\Lambda_f \in \mathcal{S}(\mathbb{R}^d)$ *and, moreover,* $L_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, \varphi \in \mathcal{S}(\mathbb{R}^d)$*.*

Důkaz
Theorem IV.11(a) $\implies$ $\mathcal{S}(\mathbb{R}^d) \subset \bigcap_{p\in[1,\infty]} L^p(\mathbb{R}^d)$. (It was stated and almost proven at chapter IV, but full proof is not easy.) So, fix $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^d)$. Let p' be the dual exponent. Then $\forall \varphi \in \mathcal{S}(\mathbb{R}^d) : \varphi \in L^{p'}(\mathbb{R}^d)$, hence $f\varphi \in L^1(\mathbb{R}^d)$.

So $\tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is a well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$: „continuity“:

$$p = 1 : |\tilde{\Lambda}(\varphi)| = |\int_{\mathbb{R}^d} f\varphi| \leqslant \|f\|_1 \cdot \|\varphi\|_\infty = \|f\|_1 \cdot p_0(\varphi);$$

$$p > 1 : \forall n \in \mathbb{N} : f \cdot \chi_{U(\mathbf{o},n)} \in L^1(\mathbb{R}^d) \implies \Lambda_{f\cdot\chi_{U(\mathbf{o},n)}} \in \mathcal{S}(\mathbb{R}^d) \text{ by the first case } \implies$$

$$\implies \tilde{\Lambda}(\varphi) = \int_{\mathbb{R}^d} f\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^d} f \cdot \chi_{U(\mathbf{o},n)}\varphi = \lim_{n\to\infty} \Lambda_{f\cdot\chi_{U(\mathbf{o},n)}}(\varphi) = \Lambda(\varphi).$$

□

c) f measurable on $\mathbb{R}^d$, $|f| \leqslant |p|$ for some polynomial p on $\mathbb{R}^d$. Then $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$ and $\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, f \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
p polynomial $\implies$ $p(x) = \sum_{|\alpha|\leqslant N} c_\alpha x^\alpha$ $(c_\alpha \in \mathbb{F}, x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_d^{\alpha_d})$.

$$\implies |p(x)| \leqslant c \cdot (\sqrt{2})^{dN}(1 + \|x\|^2)^{N\cdot\frac{d}{2}}, \qquad c = \max_\alpha |c_\alpha|.$$

So, if $|f| \leqslant |p|$, then $\frac{|f(x)|}{(1+\|x\|^2)^m} \leqslant c \cdot (\sqrt{2})^{d\cdot N} \cdot (1 + \|x\|^2)^{N\cdot\frac{d}{2}-m}$. If m is large enough (such that $N \cdot \frac{d}{2} - m < -\frac{d}{2}$), then $f(x)/(1+\|x\|^2)^m$ is integrable in $\mathbb{R}^d$. ($1/(1+\|x\|^2)^k$ is integrable for $k > \frac{d}{2}$ see the comment before theorem IV.11). Then:

$$\left|\int_{\mathbb{R}^d} f \cdot \varphi\right| = \left|\int_{\mathbb{R}^d} \frac{f(x) \cdot (1 + \|x\|^2)^m}{(1 + \|x\|^2)^m}\right| \leqslant \left(\int_{\mathbb{R}^d} \frac{|f(x)|}{(1 + \|x\|^2)^m}\right) \cdot p_m(f).$$

□

d) μ is a finite measure $\implies$ $\Lambda_\mu \in \mathcal{S}'(\mathbb{R}^d)$, $\Lambda_m(\varphi) = \int_{\mathbb{R}^d} \varphi d\mu$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$\varphi \in \mathcal{S}(\mathbb{R}^d)$ $\implies$ φ is continuous and bounded.

$$\left|\int_{\mathbb{R}^d} \varphi d\mu\right| \leqslant \int_{\mathbb{R}^d} |\varphi| d|\mu| \leqslant \|f\|_\infty \cdot \|\mu\| = p_0(\varphi) \cdot \|\mu\|.$$

□

Lemma 3.20 (Continuity of operations on the Schwartz space)

$f \mapsto D^\alpha f$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$f \in \mathcal{S}(\mathbb{R}_6 d)$, $\alpha \in \mathbb{N}_0^d \implies D^\alpha f \in L^\infty()\mathbb{R}^d$. Fix $N \in \mathbb{N}_0$ and β, $|\beta| \leqslant N$:

$$|(1+\|x\|^2)^N D^\beta(D^\alpha f)(x)| = (1+\|x\|^2)^N |D^{\beta+\alpha} f(x)| \leqslant p_{N+|\alpha|}(f) \implies$$

$\implies p_N(D^\alpha f) \leqslant p_{N+|\alpha|}(f)$. □

p is polynomial $\implies f \mapsto p \cdot f$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
Clearly $p \cdot f \in C^\infty(\mathbb{R}^d)$. Fix $N \in \mathbb{N}_0$. Then $\exists c > 0, m \in \mathbb{N}$ such that

$$\forall \alpha, |\alpha| \leqslant N,\ \forall x \in \mathbb{R}^d L |D^\alpha p(x)| \leqslant c \cdot (1+\|x\|^2)^m.$$

Fix α, $|\alpha| \leqslant N$, $x \in \mathbb{R}^d$:

$$|(1+\|x\|^2)^N D^\alpha(p \cdot f)(x)| = (1+\|x\|^2)^N |\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta p(x) D^{\alpha-\beta} f(x)| \leqslant$$

$$\leqslant c \cdot (1+\|x\|^2)^{N+M} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta} f(x)| \leqslant c \cdot \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} p_{N+M}(f) \leqslant c \cdot 2^N p_{N+M}(f) \implies$$

$$\implies p_N(p \cdot f) \leqslant c \cdot 2^N p_{N+M}(f).$$

□

$g \in \mathcal{S}(\mathbb{R}^d) \implies f \mapsto f \cdot g$ is continuous $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$g \in \mathcal{S}(\mathbb{R}^d) \implies \forall \alpha : D^\alpha g$ is bounded on $\mathbb{R}^d$. Fix $N \in \mathbb{N}_0$. Set $C := \max_{|\alpha| \leqslant N} \|D^\alpha g\|_\infty$. Fix α, $|\alpha| \leqslant N$, $x \in \mathbb{R}^d$.

$$\left|(1+\|x\|^2)^N D^\alpha(f \cdot g)(x)\right| = (1+\|x\|^2)^N \left|\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} D^\beta g(x) D^{\alpha-\beta} f(x)\right| \leqslant$$

$$C \cdot \sum_{\beta \leqslant \alpha} p_N(f) \leqslant C \cdot 2^N \cdot p_N(f) \implies p_N(g \cdot f) \leqslant C \cdot 2^N p_N(f).$$

□

Poznámka
Similarly one may probe that: $g \in C^\infty(\mathbb{R}^d)$, $\forall \alpha\ \exists P_\alpha$ polynomial: $|D^\alpha g| \leqslant |P_\alpha|$ on $\mathbb{R}^d \implies f \mapsto f \cdot g$ is a continuous mapping $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Tvrzení 3.21 (Operations with tempered distributions)

Let $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$.

a) $\forall\alpha : D^\alpha\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ *and* $D^\alpha\Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies D^\alpha\varphi \in \mathcal{S}(\mathbb{R}^d)$. So, $\tilde{\Lambda}(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, is well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$ whose restriction to $\mathcal{D}(\mathbb{R}^d)$ is $D^\alpha\Lambda$. „Continuity:“ $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d) \implies$ (by the previous lemma) $D^\alpha\varphi_n \to D^\alpha\varphi$ in $\mathcal{S}(\mathbb{R}^d)$, so $\tilde{\Lambda}(\varphi_n) = (-1)^{|\alpha|}\Lambda(D^\alpha\varphi_n) \to (-1)^{|\alpha|}\Lambda(D^\alpha\varphi) = \tilde{\Lambda}(\varphi)$. □

b) $f \in \mathcal{S}(\mathbb{R}^d)$ *and* f *is a polynomial* $\implies f\cdot\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ *and* $f\Lambda(\varphi) = \Lambda(f\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz (Skipped on lecture)
Completely analogous to a). □

c) $y \in \mathbb{R}^d \implies \tau_y\Lambda \in \mathcal{S}'(\mathbb{R}^d), \tau_y\Lambda(\varphi) = \Lambda(\tau_{-y}\varphi), \varphi \in \mathcal{S}'(\mathbb{R}^d)$.

Důkaz
„$\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\tau_{-y}\varphi(x) = \varphi(x-y)$“: Clearly $\tau_{-y}\varphi \in C^\infty(\mathbb{R}^d)$.

$$|\alpha| \leqslant N : (1+\|x\|^2)^N D^\alpha\tau_{-y}\varphi(x) = (1+\|\alpha\|^2)^N D^\alpha\varphi(x+y) =$$

$$= \left(\frac{1+\|x\|^2}{1+\|x+y\|^2}\right)^N \cdot (1+\|x+y\|^2)^N D^\alpha\varphi(x+y) \leqslant \left(\frac{1+\|x\|^2}{1+\|x+y\|^2}\right)^N \cdot p_N(\varphi) \leqslant M^N,$$

where $M = \sup_{t\in[0,\infty)} \frac{1+t^2}{1+(t-\|y\|)^2} < \infty$. $\implies \tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $p_N(\tau_{-y}\varphi) \leqslant M^N p_N(\varphi)$. So $\varphi \mapsto \tau_{-y}\varphi$ is continuous and then continue as in a). □

d) $\check{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$, $\check{\Lambda}(\varphi) = \Lambda(\check{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Důkaz
Observe that $\varphi \in \mathcal{S}(\mathbb{R}^d) \implies \check{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ ($\check{\varphi}(x) = \varphi(-x)$) and $p_N(\check{\varphi}) = p_N(\varphi)$. □

Tvrzení 3.22

$\Lambda_n \to \Lambda$ *in* $\mathcal{S}'(\mathbb{R}^d)$. *a)* $\forall\alpha : D^\alpha\Lambda_n \to D^\alpha\Lambda$, *b)* $f \in \mathcal{S}(\mathbb{R}^d)$ *and* f *is polynomial* $\implies f\Lambda_n \to f\Lambda$.

Důkaz
„a)“: $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$D^\alpha\Lambda_n(\varphi) = (-1)^{|\alpha|}\Lambda_n(D^\alpha\varphi) \to (-1)^{|\alpha|}\Lambda(D^\alpha\varphi) = D^\alpha\Lambda(\varphi).$$

„b)“ similarly. □

3.6 Convolution and the Fourier transform of tempered distributions

Poznámka (Recall)
$f \in L^1(\mathbb{R}^d) \implies \hat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-i\langle t,x\rangle} dm_d(x)$.

Fourier transform maps $L^1(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

$$\hat{\hat{f}} = \check{f}, \qquad f \in \mathcal{S}(\mathbb{R}^d), \qquad \left(\hat{\hat{\hat{\hat{f}}}} = f\right).$$

Lemma 3.23

Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

Důkaz
1. The theorem above $\implies$ Fourier transform is a linear bijection $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$.

2. $m := \left\lfloor \frac{d}{2} \right\rfloor + 1$. Then $C := \int_{\mathbb{R}^d} \frac{1}{(1+\|x\|^2)^m} dm_d(x) \leqslant \infty$. $f \in \mathcal{S}(\mathbb{R}^d) \implies$

$$\implies \|\hat{f}\|_\infty \leqslant \|f\|_{L^1} = \int_{\mathbb{R}^d} |f(x)| dm_d(x) \leqslant \int_{\mathbb{R}^d} \frac{(1+\|x\|^2)^m |f(x)|}{(1+\|x\|^2)^m} dm_d(x) \leqslant C \cdot p_m(f).$$

TODO!?

3. Fix $N \in \mathbb{N}_0$, α, $|\alpha| \leqslant N$.

$$f \in \mathcal{S}(\mathbb{R}^d) : (1+\|x\|^2)^N D^\alpha \hat{f}(x) = (1+\|x\|^2)^N (y \mapsto \widehat{(-1)^{|\alpha|} y^\alpha f(y)})(x) =$$

$$= (-i)^{|\alpha|} (y \mapsto \widehat{\check{(D)}(y^\alpha (f(y)))}\!\!p(x) = (-i)^{|\alpha|} (y \mapsto \sum_{|\beta| \leqslant 2N} \widehat{a_\beta D^\beta (y^\alpha f(y))})(x),$$

where $\check{p}(x) = p(ix)$, $p(D)f = \sum_{c_\alpha} D^\alpha f$ if $p(x) = \sum c_\alpha x^\alpha$. $\breve{\check{p}}(x) = (1 - \sum_{j=1}^d x_j^2)^N$ a polynom of degree $2N$.

So, $\|x \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \leqslant c \cdot p_m(y \mapsto \sum_{|\beta| \leqslant 2N} a_\beta D^\beta (y^\alpha f(y)))$. From the previous lemma $f \mapsto \sum_{|\beta| \leqslant N} a_\beta D^\beta (y^\alpha f(y))$ is continuous.

So, $\exists M = M_{N,\alpha} > 0$, $\exists m = m_{N,\alpha} \in \mathbb{N}_0$:

$$p_m(y \mapsto \sum_{|\beta| \leqslant 2N} a_\beta D^\beta (y^\alpha f(y))) \leqslant M \cdot p_n(f) \implies$$

$$\implies \|x\| \mapsto (1+\|x\|^2)^N D^\alpha \hat{f}(x)\|_\infty \leqslant C \cdot M \cdot p_m(f).$$

4. So, $p_N(\hat{f}) \leqslant C \cdot \tilde{M} \cdot p_{\tilde{m}}(t)$, where $\tilde{M} = \max_{|\alpha| \leqslant N} M_{N,\alpha}$, $\tilde{m} = \max_{|\alpha| \leqslant N} m_{N,\alpha}$. □

Definice 3.13 (Fourier transform of tempered distribution)

$\Lambda \in \mathcal{S}'(\mathbb{R}^d)$: $\hat{\Lambda}(\varphi) = \Lambda(\hat{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Poznámka
$\hat{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$: $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d) \implies \hat{\varphi_n} \to \hat{\varphi}$ in $\mathcal{S}(\mathbb{R}^d) \implies \hat{\Lambda}(\varphi_n) = \Lambda(\hat{\varphi_n}) \to \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi)$.

Věta 3.24 (Properties of Fourier transform)

a) Fourier transform is a linear bijection of $\mathcal{S}'(\mathbb{R}^d)$ onto $\mathcal{S}'(\mathbb{R}^d)$.

Důkaz
$\hat{\hat{\Lambda}} = \check{\Lambda}$, $\hat{\hat{\hat{\hat{\Lambda}}}} = \Lambda$ for $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$, $\check{(\Lambda)}(\varphi) = \Lambda(\check{\varphi})$, $\hat{\Lambda_1} = \hat{\Lambda_2} \implies \hat{\hat{\hat{\hat{\Lambda}}}}_1 = \hat{\hat{\hat{\hat{\Lambda}}}}_2 \implies \Lambda_1 = \Lambda_2$. □

b) $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d) \implies \hat{\Lambda_n} \to \hat{\Lambda}$ in $\mathcal{S}'(\mathbb{R}^d)$.

Důkaz

$$\hat{\Lambda_n}(\varphi) = \Lambda_n(\hat{\varphi}) \to \Lambda(\hat{\varphi}) = \hat{\Lambda}(\varphi).$$

□

c) $f \in C^1(\mathbb{R}^d) \implies \hat{\Lambda_f} = \Lambda_{\hat{f}}$.

Důkaz

$$\hat{\Lambda_f}(\varphi) = \Lambda_f(\hat{\varphi}) = \int f\hat{\varphi}dm_d = \int \hat{f}\varphi dm_d = \Lambda_{\hat{f}}(\varphi).$$

□

d) $f \in L^2(\mathbb{R}^d) \implies \hat{\Lambda_f} = \Lambda_{\mathcal{P}(f)}$, where $\mathcal{P}$ is the Plancherel transform.

Důkaz
$f_n := f \cdot \chi_{U(\mathbf{o},n)}$. Then $f_n \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $f_n \to f$ in $L^2(\mathbb{R}^d)$, and, moreover, $\hat{f_n} \to \mathcal{P}(f)$ in $L^2(\mathbb{R}^d)$. So, $\hat{\Lambda_f}(\varphi) = \Lambda_f(\hat{\varphi}) =$

$$= \int_{\mathbb{R}^d} f\hat{\varphi}dm_d = \lim_{n\to\infty}\int_{\mathbb{R}^d} f_n\hat{\varphi}dm_d = \lim_{n\to\infty}\int_{\mathbb{R}^d}\hat{f}_n\varphi = \int_{\mathbb{R}^d}\mathcal{P}(f)\varphi dm_d = L_{\mathcal{P}(f)}(\varphi).$$

□

e) p polynomial $\implies \widehat{p(D)\Lambda} = \tilde{p}\hat{\Lambda}$, $\widehat{p\cdot\Lambda} = \check{p}(D)\hat{\Lambda}$.

$$\Big(\tilde{p}(t) = p(it), \qquad \check{p}(t) = p(-t), \qquad p(t) = \sum c_\alpha t^\alpha \implies p(D)f = \sum c_\alpha D^\alpha f.\Big)$$

Důkaz

$$\widehat{p(D)\Lambda}(\varphi) = p(D)\Lambda(\hat{\varphi}) = \Lambda(\check{(}D)\hat{\varphi}) = \Lambda\left(\widehat{\check{\ }p\varphi}\right) = \hat{\Lambda}(\check{p}\varphi) = \check{p}\hat{\Lambda}(\varphi).$$

$$\widehat{p\cdot\Lambda}(\varphi) = p\cdot\Lambda(\hat{\varphi}) = \Lambda(p\hat{\varphi}) = \Lambda\left(\widehat{\check{\check{\ }}(D)p\varphi}\right) = \hat{\Lambda}(\check{\check{p}}(D)\varphi) = \check{p}(D)\hat{\Lambda}(\varphi).$$

$\square$

Poznámka

In particular

$$\widehat{D^\alpha\Lambda} = (x\mapsto c^{|\alpha|}x^\alpha)\hat{\Lambda}, \qquad (x\widehat{\mapsto x^\alpha})L = c^{|\alpha|}D^\alpha\hat{\Lambda}.$$

Poznámka

Next two lemmata are analogues of Lemmata above.

Lemma 3.25

a) $\varphi\in\mathcal{S}(\mathbb{R}^d)$, $x_n\to x$ *in* $\mathbb{R}^d \implies \tau_{x_n}\varphi\to\tau_x\varphi$ *in* $\mathcal{S}(\mathbb{R}^d)$.

Důkaz

Fix $N\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^d$, $|\alpha|\leqslant N$. $x,z\in\mathbb{R}^d$:

$$\left|(1+\|x\|^2)^N D^\alpha\tau_z\varphi(y) - (1+\|y\|^2)^N D^\alpha\tau_x\varphi(y)\right| =$$

$$= (1+\|y\|^2)^N\left|\int_0^1 \frac{d}{dt}D^\alpha\varphi(y-x-t(z-x))dt\right| \leqslant$$

$$\leqslant (1+\|y\|^2)^N\int_0^1\left|\sum_{j=1}^d D^{\alpha+e_j}\varphi(y-x-t(z-x))(x_j-z_j)\right|dt \leqslant$$

$$\leqslant (1+\|y\|^2)^N\int_0^1\left|\sum_{j=1}^d D^{\alpha+e_j}\varphi(y-x-t(z-x))^2\right|^{1/2}\|x-z\|dt \leqslant$$

$$\leqslant \|x-z\|p_{N+1}(\varphi)(1+\|y\|^2)^N\int_0^1\frac{\sqrt{d}dt}{(1+\|y-x-t(z-x)\|^2)^{N+1}} \leqslant$$

$$\leqslant \|x-z\|p_{N+1}(\varphi)\cdot\sqrt{d}\frac{(1+\|y\|^2)^N}{(1+\|y-x\|^2-\|y-x\|)^{N+1}}.$$

$$\left(1+\|y-x\|-t\|z-x\|^2 \geqslant 1+\|y-x\|^2-\|y-x\|,\ \text{if } \|z-x\|\leqslant\frac{1}{2}.\right)$$

$\square$

b) skipped and proof skipped too.

Lemma 3.26 (RT)

$$\Lambda \in \mathcal{S}'(\mathbb{R}^d) \Leftrightarrow \exists N \in \mathbb{N}_0, \mu_\alpha, |\alpha| \leqslant N_0 : \Lambda(\varphi) = \sum_{|\Lambda| \leqslant N} \int_{\mathbb{R}^d} (1+\|x\|^2)^N D^\alpha \varphi(x) d\mu_\alpha(x), \varphi \in \mathcal{S}(\mathbb{R}^d).$$

(Finite signed/complex measure on $\mathbb{R}^d$.)

Definice 3.14 (Convolution of function and tempered distribution)

$$U \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{S}(\mathbb{R}^d) \implies U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x-y)).$$

Věta 3.27 (Analogues to the theorem above)

TODO? (Fubini; It is not directly needed for the exam.)

a) $U * \varphi \in C^\infty(\mathbb{R}^d)$ and $D^\alpha(U * \varphi) = (D^\alpha U) * \varphi = U * D^\alpha \varphi$ for each multi-index α.

Důkaz
Skipped. □

b) $\Lambda_{U*\varphi}$ is a tempered distribution.

Důkaz
The proposition above $\implies$ $\exists N, C$ such that $|U(\psi)| \leqslant c \cdot p_N(\psi)$

$$\implies |(U * \varphi)(x)| = |U(\tau_x \check{\varphi})| \leqslant C \cdot p_N(\tau_x \check{\varphi}).$$

$|\alpha| \leqslant N : |(1+\|y\|^2)^N D^\alpha \varphi(x-y)| \leqslant p_N(\varphi) \cdot \left(\dfrac{1+\|y\|^2}{1+\|x-y\|^2} \right)^N \leqslant p_N(\varphi)(1+\|y\|+\|x\|^2)^N \implies$

$\implies \Lambda'_{U*\varphi}$ is tempered.

$$\frac{1+\|y\|^2}{1+\|x-y\|^2} = \frac{1+\|y-x\|^2 + 2\langle y-x, x\rangle + \|x\|^2}{1+\|x-y\|^2} =$$

$$= 1 + \frac{2 \cdot \|y-x\| \cdot \|x\|}{1+\|x-y\|^2} + \frac{\|x\|^2}{1+\|x-y\|^2} \leqslant 1 + \|x\| + \|x\|^2.$$

□

c) If $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then $\Lambda_f * \varphi = f * \varphi$.

Důkaz
Skipped. □

d) $\widehat{\Lambda_{U*\varphi}} = \hat{\varphi} \cdot \hat{U}$, $\widehat{\varphi \cdot U} = \Lambda_{\hat{\varphi} * \hat{U}}$.

Důkaz

$$\widehat{\Lambda_{U*\varphi}}(\psi) = \Lambda_{U*\varphi}(\hat{\psi}) = \int_{\mathbb{R}^d} (U * \varphi)(x)\hat{\psi}(x) f m_d(x) = \int_{\mathbb{R}^d} U(y \mapsto \varphi(x-y))\hat{\psi}(x) dm_d(x) =$$

$$= U(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y)\hat{\psi}(x)dm_d(x)) = U(\check{\varphi}*\hat{\psi}) = U(\hat{\hat{\varphi}}+\hat{\psi}) = U(\widehat{\hat{\ } \cdot \varphi\psi}) = \hat{U}(\hat{\varphi}\cdot\psi) = \hat{\varphi}\cdot\hat{U}(\psi).$$

$$\Lambda_{\hat{\varphi}*\hat{U}} = \widehat{\widehat{\widehat{\widehat{\Lambda_{\hat{\varphi}+\hat{U}}}}}} = \widehat{\widehat{\widehat{\hat{\hat{\varphi}} \cdot \hat{\hat{U}}}}} = \widehat{\widehat{\check{\varphi} \cdot \check{U}}} = \widehat{\check{\varphi} \cdot \check{U}} = \widehat{\varphi U}.$$

□

e) $U * (\varphi * \psi) = (U * \varphi) * \psi$.

Důkaz
Skipped. □

4 Elements of vector integration

Poznámka
$(M, \mathcal{A})$ is measure space, (Ω, Σ, μ) is a complete measure space ($\mu \geqslant 0$), X is a Banach space.

4.1 Measurability

Definice 4.1 (Simple, simple measurable, (strongly) $\mathcal{A}$-measurable, Borel $\mathcal{A}$-measurable, weakly $\mathcal{A}$-measurable)

$f : M \to X$.

- f is simple, if $f(M)$ is finite , i.e. $f = \sum_{j=1}^{k} x_j \chi_{A_j}$, where $x_j \in X$, $A_j \subset M$ pairwise disjoint;
- f is simple measurable, if f is a simple and, moreover, $A_j \in \mathcal{A}$;
- f is (strongly) $\mathcal{A}$-measurable if $\exists (u_n)$ simple measurable: $u_n \to f$ point-wise, i.e. $\forall x \in M : u_n(x) \to f(x)$ in $(X, \|\cdot\|)$;
- f is Borel $\mathcal{A}$-measurable, if $\forall U \subset X$ open: $f^{-1}(U) \in \mathcal{A}$;
- f is weakly $\mathcal{A}$-measurable if $\forall \varphi \in X^* : \varphi \circ f$ is $\mathcal{A}$-measurable.

Tvrzení 4.1

a) Simple functions, simple measurable functions, strongly $\mathcal{A}$*-measurable functions, and* *weakly $\mathcal{A}$-measurable functions form vector spaces.*

Důkaz
$f, g : M \to X$, $\alpha, \beta \in \mathbb{F}$.

- „f, g simple $\implies$ $\alpha f + \beta g$ is simple“: $(\alpha f + \beta g)(M) \subset \alpha f(M) + \beta g(M)$.
- „f, g simple measurable $\alpha f + \beta g$ is simple measurable“:

$$f = \sum_{j=1}^{k} x_j \chi_{A_j}, \quad g = \sum_{l=1}^{m} y_l \chi_{B_l}, \qquad \alpha f + \beta g = \sum_{j=1}^{k} \sum_{l=1}^{m} (\alpha x_j + \beta y_l) \cdot \chi_{A_j \cap B_l},$$

  $A_j, B_l \in \mathcal{A} \implies A_j \cap B_l \in \mathcal{A}$.
- „f, g strongly $\mathcal{A}$-measurable $\implies$ $\alpha f + \beta g$ is strongly $\mathcal{A}$-measurable“: $f = \lim u_n$, $g = \lim v_n$, u_n, v_n simple measurable, $\alpha f + \beta g = \lim(\alpha u_n + \beta v_n)$.
- „f, g weakly $\mathcal{A}$-measurable $\implies$ $\alpha f + \beta g$ weakly $\mathcal{A}$-measurable“: $\forall \varphi \in X^* : \varphi \circ (\alpha f + \beta g) = \alpha \varphi \circ f + \beta \varphi \circ g$ (measurable by the scakercah?).

□

b) $f_n \to f$ point-wise, f_n Borel $\mathcal{A}$-measure (resp. weakly $\mathcal{A}$-measurable) $\implies$ f is Borel $\mathcal{A}$-measurable (resp. weakly $\mathcal{A}$-measurable).

Důkaz
Assume that $\forall n$: f_n is Borel $\mathcal{A}$-measurable. $U \subset X$ open:

$$f^{-1}(U) = \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{k=m}^{\infty} f_k^{-1}(\left\{ x \in X |\, \mathrm{dist}(x, X \backslash U) > \frac{1}{n} \right\}),$$

$$f(x) \in U \Leftrightarrow \exists n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall k \geqslant m : \mathrm{dist}(f_k(x), X \backslash U) > \frac{1}{n}.$$

f_n are wakly $\mathcal{A}$-measurable, $\varphi \in X^* \implies \forall n : \varphi \circ f_n$ is Borel $\mathcal{A}$-measurable and $\varphi \circ f_n \to \varphi \circ f$, so, $\varphi \circ f$ is Borel $\mathcal{A}$-measurable. □

c) f is strongly $\mathcal{A}$-measurable $\implies$ f is Borel $\mathcal{A}$-measurable $\implies$ f is weakly $\mathcal{A}$-measurable.

Důkaz
f is simple $\implies$ (f is simple measurable $\Leftrightarrow$ f is Borel $\mathcal{A}$-measurable).

„f strongly $\mathcal{A}$-measurable $\implies$ f Boreal $\mathcal{A}$ measurable“: $f = \lim u_n$, u_n simple measurable, then u_n are Borel $\mathcal{A}$-measurable, so by b), f is Borel $\mathcal{A}$-measurable.

„f Borel $\mathcal{A}$-measurable $\implies$ f is weakly $\mathcal{A}$-measurable“: $\varphi \in X^*$, $U \subset \mathbb{F}$ open $\implies$ $(\varphi \circ f)^{-1}(U) = f^{-1}(\underbrace{\varphi^{-1}(U)}_{\text{open}}) \in \mathcal{A}$.

„f simple, weakly $\mathcal{A}$-measurable $\implies$ f is simple measurable“: $f(M) = \{x_1, \dots, x_k\}$, distinct points. „$f^{-1}(x_1) \in \mathcal{A}$“: for $j \in \{2, \dots, k\}$ find $\varphi_j \in X^*$, $\varphi_j(x_1) \neq \varphi_j(x_j)$. Then

$$f^{-1}(x_1) = \bigcap_{j=2}^{k} \{t \in M | \varphi_j(f(t) - x_j) \neq 0\} = \bigcap_{j=2}^{k} \overbrace{(\varphi_j \circ f)^{-1}(\underbrace{\mathbb{F} \setminus \{\varphi_j(x_j)\}}_{\text{open}})}^{\in \mathcal{A}}.$$

□

d) $f: M \to X$ strongly $\mathcal{A}$-measurable $\implies$ $f(M)$ is separable.

Důkaz
$f = \lim u_n$, u_n simple measurable. $f(M) \subset \overline{\bigcup_n u_n(M)}$. □

e) f Borel $\mathcal{A}$-measurable $\implies$ $t \mapsto \|f(t)\|$ measurable.

Důkaz
$h(x) = \|x\|$, $x \in X$, is continuous, hence $h \circ f$ is measurable:

$$U \text{ open} : (h \circ f)^{-1}(U) = f^{-1}(\underbrace{h^{-1}}_{\text{open}}) \in \mathcal{A}.$$

□

Lemma 4.2

(f_n) strongly $\mathcal{A}$-measurable, $f_n \to f$ point-wise $\implies$ f is strongly $\mathcal{A}$-measurable.

Důkaz
$u_{m,n}$ simple measurable, $u_{m,n} \xrightarrow{m} f_n$. $C = \bigcup\limits_{m,n} u(m,n)(M)$ is countable, so, $C = \{x_k, k \in \mathbb{N}\}$.
For $k \in \mathbb{N}$ define $g_k : M \to X$ by $g_k(x) =$ the point from $\{x_1, \dots, x_k\}$ nearest to $f(x)$ (the first such point). Then g_k is simple, $g_k \to f$ point-wise ($t \in M$, $\varepsilon > 0 \implies \exists n_0\ \forall n \geqslant n_0 :$ $\|f_n(t) - f(t)\| < \varepsilon/2$. Fix one $n \geqslant n_0 \implies \exists m_9\ \forall m \geqslant m_0 : \|u_{m,n}(t) - f_n(t)\| < \varepsilon/2$. Fix one $m \geqslant m_0 \implies \|u_{m,n}(t) - f(t)\| < \varepsilon$, and there is k_0 such that $u_{m,n}(t) = x_{k_0}$. Then for $k \geqslant k_0$: $\|f(t) - g_k(t)\| \leqslant \|f(x) - x_{k_0}\| < \varepsilon$).

„g_k are also simple measurable“: f is Borel $\mathcal{A}$–measurable $\implies \forall x \in X$: $f - x$ is Borel $\mathcal{A}$-measurable $\implies \forall x \in X$: $t \mapsto \|f(t) - x\|$ is measurable, $g_k(t) = x_j$

$$\Leftrightarrow \forall i \in [k] : \|x_j - f(x)\| \leqslant \|x_i - f(x)\| \wedge \forall i < j : \|x_j - f(x)\| < \|x_i - f(x)\| \Leftrightarrow \forall i \in [k]\ \forall g \in \mathbb{Q} :$$

$$\|x_j - f(x)\| \leqslant q \vee \|x_i - f(x)\| \geqslant q \wedge \forall i < j\ \exists q \in \mathbb{Q} : \|x_j - f(x)\| < q \wedge \|x_i - f(x)\| > q.$$

□

Věta 4.3 (Pettis)

$f : M \to X$. Then following assertions are equivalent:

1. *f is strongly $\mathcal{A}$-measurable;*
2. *f is Borel $\mathcal{A}$-measurable and $f(M)$ is separable;*
3. *f is weakly $\mathcal{A}$-measurable and $f(M)$ is separable.*

Důkaz
„1. $\implies$ 2. $\implies$ 3.“ from the previous proposition. „3. $\implies$ 1.“: Firstly WLOG X is separable (replace X by $\overline{\mathrm{LO}\, f(M)}$). Secondly let (x_n) be a dense sequence in X.

$$\forall n : \text{ fix } \varphi_n \in X^*, \|\varphi_n\| = 1, \varphi_n(x_n) = \|x_n\|.$$

Thirdly $\forall x \in X : \|x\| = \sup_n |\varphi_n(x)|$ („$\geqslant$“: clear as $\|\varphi_n\| = 1$, „$\leqslant$“: it holds for $x = x_n$, so on dense set, LHS is continuous, RHS is continuous (supremum of 1-Lipschitz functions), so it holds on X). Fourthly $\forall x \in X : t \mapsto \|f(t) - x\|$ is measurable ($\|f(t) - x\| = \sup_n |(\varphi_n \circ f)(t) - \varphi_n(x)|$, so supremum from $\mathcal{A}$-measurable functions).

Fifthly $k, n \in \mathbb{N}$: $A_n^k := f^{-1}(U(x_n, 1/k)) = \{t \in M | \|f(t) - x_n\| < 1/k\} \in \mathcal{A}$ by fourthly.

$$\bigcup_n A_n^k = M, \qquad B_n^k = A_n^k \setminus \bigcup_{j<n} A_j^k \in \mathcal{A}, \qquad \bigcup_n B_n^k = M,$$

and $\left\{B_n^k, n \in \mathbb{N}\right\}$ is pair-wise disjoint.

Define $g_k(t) = x_n$, $t \in B_n^k$. Then $\|g_k(t) - f(t)\| < \frac{1}{k}$. So $g_k \rightrightarrows f$ on M. g_n is strongly measurable. $g_k = \lim_{n\to\infty} \sum_{j=1}^n x_j \chi_{B_j^k}$, so, by the previous lemma f is strongly $\mathcal{A}$-measurable. □

Definice 4.2 (Strongly μ-measurable)

(Ω, Σ, μ) complete measure space, $f : \Omega \to X$ is strongly μ-measurable if $\exists (u_n)$ simple measurable such that $u_n \to f$ point-wise μ-almost everywhere.

Poznámka

f is strongly μ-measurable $\Leftrightarrow \exists g$ strongly Σ-measurable: $f = g$ almost everywhere.

Důkaz

„ $\Longleftarrow$ “ obvious. „ $\Longrightarrow$ “: u_n simple measurable, $u_n \to f$ almost everywhere. $\exists N, \mu(N) = 0 : u_n \to f$ on $\Omega \backslash N$. Modify u_n, f: $v_n = 0$ on N and u_n on $\Omega \backslash N$, $g = 0$ on N and f on $\Omega \backslash N$. v_n simple measurable, $v_n \to g$. □

Definice 4.3 (Essentially separably valued)

(If $f : \Omega \to X$ is (strongly) μ-measurable, then)

$$\exists Y \leqslant X \text{ separable } \exists N \in \Sigma : \mu(N) = 0 \wedge f(\Omega \backslash N) \subset Y.$$

Lemma 4.4

Let (f_n) be a sequence of strongly μ-measurable functions $f_n : M \to X$ almost everywhere converging to a function $f : M \to X$. Then f is strongly μ-measurable as well.

Důkaz

TODO!!! (notes) □

Věta 4.5 (Pettis)

Let $f : \Omega \to X$ be a function. Then following assertions are equivalent

1. *f is strongly μ-measurable.*
2. *f is Borel μ-measurable and essentially separably valued.*
3. *f is weakly μ-measurable and essentially separably valued.*

Důkaz

TODO!!! (Notes) □

4.2 Integrability of vector-valued functions

Definice 4.4 (Integrable over set, integral over set, integrable, Bochner integrable, Bochner integral, weakly integrable)

Let $f : \Omega \to X$ be a simple measurable function of the form $f = \sum_{j=1}^{k} x_j \chi_{E_j}$. Let $E \in \Sigma$. We say that f is integrable over E, if for each $j \in [k]$ one has either $\mu(E \cap E_j) < \infty$ or $x_j = \mathbf{o}$. By the integral of f over E we mean the element of X defined by the formula

$$\int_E f d\mu = \sum_{j=1}^{k} \mu(E \cap E_j) x_j, \qquad (\infty \cdot \mathbf{o} = \mathbf{o}).$$

If f is integrable over Ω, it is called integrable.

Let $f : \Omega \to X$ be strongly μ-measurable. Then function f is said to be Bochner integrable if there exists a sequence (f_n) of simple integrable functions such that

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\| d\mu(\omega) = 0.$$

By Bochner integral of f we then mean the element of X defined by

$$(B) \int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

A function $f : \Omega \to X$ is said to be weakly integrable if $\varphi \circ f$ is integrable (i.e., $\varphi \circ f \in L^1(\mu)$) for each $\varphi \in X^*$.

Tvrzení 4.6 (Basic properties of the Bochner integral)

a) Integrable functions form a vector space, and the mapping assigning to a simple integrable function f its integral $\int_\Omega f d\mu$ is linear.

b) Let f be a simple measurable function. Then f is integrable if and only if the function $\omega \mapsto \|f(\omega)\|$ is integrable. In this case $\left\|\int_\Omega f d\mu\right\| \leqslant \int_\Omega \|f(\omega)\| d\mu(\omega)$.

c) The limit defining the Bochner integral does exist and does not depend on the choice of the sequence (f_n).

d) Bochner integrable functions form a vector space and the mapping assigning to a Bochner integrable function its Bochner integral is linear.

e) If $f : \Omega \to X$ is Bochner integrable, then the function $\omega \mapsto \|f(\omega)\|$ is integrable and $\left\|(B) \int_\Omega f d\mu\right\| \leqslant \int_\Omega \|f(\omega)\| d\mu(\omega)$.

f) If $f : \Omega \to X$ Bochner integrable, then $\chi_E \cdot f$ is Bochner integrable for each $E \in \Sigma$. (The value $(B) \int_\Omega \chi_E \cdot f d\mu =: (B) \int_E f d\mu$ is called the Bochner integral of f over E.)

Důkaz

TODO!!! □

Věta 4.7 (A characterization of Bochner integrability)

Let $f : \Omega \to X$ be a strongly μ-measurable function. Then f is Bochner integrable if and only if $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$.

Důkaz

TODO!!! (notes) □

Věta 4.8 (Lebesgue dominated convergence theorem for Bochner integral)

Let (f_n) be a sequence of Bochner integrable functions $f_n : \Omega \to X$ almost everywhere converging to a function $f : \Omega \to X$. Let $g : \Omega \to \mathbb{R}$ be an integrable function such that for each $n \in \mathbb{N}$ one has $\|f_n(\Omega)\| \leqslant g(\omega)$ for almost all $\omega \in \Omega$. Then f is Bochner integrable and $(B) \int_\Omega f d\mu = \lim_{n\to\infty} (B) \int_\Omega f_n d\mu$.

Důkaz

TODO!!! (notes) □

Tvrzení 4.9 (Absolute continuity of Bochner integral)

Let $f : \Omega \to X$ be Bochner integrable. Then

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall E \in \Sigma : \mu(E) < \delta \implies \left\| \int_E f d\mu \right\| < \varepsilon.$$

Důkaz

The proof is not needed for the exam. □

Poznámka

(Ω, Σ, μ) complete measure space, X Banach space, $f : \Omega \to X$ is weakly integrable $\equiv$ $\forall \varphi \in X^* : \varphi \circ f$ is integrable.

Tvrzení 4.10 (Weak integral)

f weakly integrable $\implies$ $F(\varphi) = \int_\Omega \varphi \circ f d\mu$, $\varphi \in X^*$ belongs to X^{**}.

Poznámka

Then

- $(D) \int_\Omega f d\mu := F$;
- $E \in \Sigma$, then $(D) \int_E f d\mu := (D) \int_\Omega \chi_E \cdot f d\mu$;
- f is Pettits-integrable if $\forall E \in \Sigma : (D) \int_E f d\mu \in \varkappa(X)$.

Důkaz
TODO!!! □

Tvrzení 4.11

Let $f : \Omega \to X$ be Bochner integrable. Let $L : X \to Y$ be bounded linear operator. Then $L \circ f$ is Bochner integrable and $(B)\int_\Omega L \circ f d\mu = L((B)\int_\Omega f d\mu)$.

Důkaz
„1. $L \circ f$ is strongly μ-measurable.“: f strongly μ-measurable $\implies \exists s_n$ simple measurable $s_n \to f$ almost everywhere. The $L \circ s_n$ are simple measurable and $L \circ s_n \to L \circ f$ almost everywhere.

2. f Bochner integrable $\implies \exists (s_n)$ simple integrable such that $\int_\Omega \|s_n - f\| d\mu \to 0$. The $L \circ s_n$ are simple integrable

$$\int \|(L \circ s_n)(\omega) - (L \circ f)(\omega)\| d\mu(\omega) = \int \|L(s_n(\omega) - f(\omega))\| d\mu(\omega) \leqslant$$

$$\leqslant \int \|L\| \cdot \|s_n(\omega) - f(\omega)\| d\mu(\omega) \to 0 \implies$$

$\implies L \circ f$ is Bochner integrable.

$$3.(B)\int_\Omega L \circ f d\mu = \lim_{n\to\infty} \overbrace{\int_\Omega L \circ s_n d\mu}^{\sum L(x_j)\mu(E_j)} = \lim_{n\to\infty} L \underbrace{\overbrace{\left(\int_\Omega s_n d\mu\right)}^{\sum x_j \mu(E_j)}}_{\to (B)\int_\Omega f d\mu} = L\left((B)\int_\Omega f d\mu\right),$$

where $s_n = \sum_{j=1}^k x_j \chi_{E_j}$, $L \circ s_n = \sum_{j=1}^n L(x_j), \chi_{E_j}$. □

Důsledek
f Bochner integrable $\implies$ f is Pettis integrable.

Příklad
$\Omega = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$, μ the contg measure. $f : \Omega \to X$

a) f is Bochner integrable $\sum f(n)$ is absolutely convergent.

Důkaz
$\sum f(n)$ absolutely convergent $\Leftrightarrow \int_\Omega \|f(n)\| d\mu(n) = \sum \|f(n)\| < \infty \Leftrightarrow f$ Bochner integrable. □

b) If $\sum f(n)$ is unconditionally convergent then f is Pettis integrable

Důkaz
$\sum_{n\in\mathbb{N}} f(n)$ is unconditionally convergent $\implies$ $\forall A \subset \mathbb{N}$: $\sum_{n\in A} f(n)$ unconditionally convergent $\implies$

$$\implies \forall A \subset \mathbb{N}\ \forall \varphi \in X^* : \sum_{n\in A} \varphi(f(n)) \text{ unconditionally convergent } \implies \sum_{n\in A} |\varphi(f(n))| < \infty.$$

$\implies$ f is weakly integrable. Moreover $\sum_{n\in A} \varphi(f(n)) = \varphi\left(\sum_{n\in A} f(n)\right)$ $\implies$ f is Pettis integrable. □

Příklad
$f : \mathbb{N} \to l_2$, $f(n) = \frac{e_n}{n}$. Then f is Pettis integrable, not Bochner integrable (f strongly measurable).

$g : \mathbb{N} \to C_0$, $g(n) = e_n$. Then g is weakly integrable not Pettis integrable. The weak integral is $(1)_{n=1}^{\infty} \in l^{\infty}$.

4.3 Lebesgue–Bochner spaces

Definice 4.5

$f : \Omega \to X$ strongly μ-measurable.

$p \in [1, \infty)$: $f \in L^p(\mu; X)$ if $\int_\Omega \|f(\omega)\|^p d\mu(\omega) < \infty$, $\|f\|_p = \left(\int_\Omega \|f(\omega)\|^p d\mu(\omega)\right)^{1/p}$.

$p = \infty$: $f \in L^\infty(\mu; X)$ if $\omega \mapsto \|f(\omega)\|^p$ is essentially bounded, $\|f\|_\infty = \operatorname{esssup}_{\omega\in\Omega} \|f(\omega)\|^p$.

Poznámka
1. Simple integrable functions belong to $L^p(\mu; X)$, $p \in [1, \infty)$.

$$\left\| \sum_{j=1}^{n} x_i \chi_{e_i} \right\|_p = \left(\sum_{j=1}^{n} \|x_j\|^p \mu(E_j) \right)^{1/p}.$$

2. Simple measurable functions belong to $L^\infty(\mu; X)$.

$$\| \sum_{j=1}^{n} x_j \chi_{E_j} \|_\infty = \max \left\{ \|x_j\|, \mu(E_j) > 0 \right\}.$$

3. $p \in [1, \infty]$, $h \in C^p(\mu)$, $x \in X$. Then $f(\omega) = h(\omega) \cdot x$, $\omega \in \Omega$ ($f = h \cdot x$), $f \in L^p(\mu; X)$, $\|f\|_p = \|h\|_p \cdot \|x\|$.

Věta 4.12

a) $(L^p(\mu; X), \|\cdot\|_p)$ *is a Banach space.*

Důkaz

It is a NLS by the Minkowski inequality. Completeness: $p = \infty$ easy, $p \in [1, \infty)$: Assume $(f_n) \subset L^p(\mu; X)$, $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$. Define $g_n(\omega) = \|f_n(\omega)\|$, $\omega \in \Omega$. Then $g_n \in L^p(\mu)$, $\|g_n\|_p = \|f_n\|_p$. So, $\sum_{n=1}^{\infty} \|g_n\|_p < \infty$, $L^p(\mu)$ complete $\implies$ $g := \sum_{n=1}^{\infty} g_n \in L^p(\mu)$ $\implies$ $\exists$ a subsequence of the sequence of partial sums which converges almost everywhere. But $g_n \geqslant 0$, so $g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$ almost everywhere. So, for almost all $\omega \in \Omega$. We have $\sum g_n(\omega) < \infty$, hence $\sum \|f_n(\omega)\| < \infty$, so $f(n) = \sum_{n=1}^{\infty} f_n(\omega)$ is defined almost everywhere.

Then f is strongly μ-measurable. $\|f(\omega)\| \leqslant \sum_{n=1}^{\infty} \|f_n(\omega)\| = g(\omega)$. Since $g \in L^p(\omega)$, we deduce $f \in L^p(\mu; X)$.

$$\|f(\omega) - \sum_{k=1}^{n} f_k(\omega)\| = \|\sum_{k>n} f_k(\omega)\| \leqslant \sum_{k>n} \|f_k(\omega)\| \text{ almost everywhere} \implies$$

$$\implies \|f - \sum_{k=1}^{n} f_n\|_p \leqslant \|\omega \mapsto \sum_{k>n} \|f_k(\omega)\|\|_p \leqslant$$
$$\leqslant \sum_{k>n} \|\omega \mapsto \|f_n(\omega)\|\|_p = \sum_{k>n} \|f_n\|_p \xrightarrow{n\to\infty} 0.$$

So $f = \sum_{k=1}^{\infty} f_k$ is $L^p(\mu; X)$. □

b) $L^1(\mu; X) =$ *Bochner-integrable functions. (see the definition and a theorem above.)*

c) If X *is a Hilbert space. Then* $L^2(\mu; X)$ *is also a Hilbert space.*

$$\langle f, g\rangle := \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega).$$

Důkaz

$\omega \mapsto \langle f(\omega), g(\omega)\rangle$ is (strongly) measurable. $\omega \mapsto \langle f(\omega), g(\omega)\rangle$ is integrable

$$\int |\langle f(\omega), g(\omega)\rangle| \leqslant \int \|f(\omega)\| \cdot \|g(\omega)\| \leqslant \left(\int \|f(\omega)\|^2\right)^{\frac{1}{2}} \cdot \left(\int \|g(\omega)\|^2\right)^{\frac{1}{2}} = \|f\|_2 \cdot \|g\|_2 < \infty.$$

□

Věta 4.13

$p \in [1, \infty)$.

a) Simple integrable function are dense in $L^p(\mu; X)$.

Důkaz
$f \in L^p(\mu; X) \implies f$ is strongly measurable. Find (s_n) simple measurable, $s_n \to f$ almost everywhere. Define $f_n : \Omega \to X$:

$$f_n(\omega) := \begin{cases} s_n(\omega) & \text{if } \|f(\omega) - s_n(\omega)\| < 2 \cdot \|f(\omega)\|, \\ 0 & \text{otherwise.} \end{cases}$$

Then f_n are simple measurable, $f_n \to f$ almost everywhere (a proof above). $\|f_n(\omega) - f(\omega)\| \leqslant 2 \cdot \|f(\omega)\|$. So,

$$f_n - f \in L^p(\mu; X) \implies f_n \in L^p(\mu; X) \implies f_n \text{ simple integrable.}$$

$\|f_n(\omega) - f(\omega)\|^p \to 0$ almost everywhere, $2 \cdot \|f(\omega)\|^p$ is a majorant $\implies$

$$\int \|f_n(\omega) - f(\omega)\|^p d\mu(\omega) \to 0 \qquad f_n \to f \text{ in } L^p(p; X).$$

$\square$

b) $L^p(\mu)$, X *separable* $\implies$ $L^p(\mu; X)$ *is separable.*

Důkaz
$\{z_n, n \in \mathbb{N}\}$ dense in X, $\{k_n, n \in \mathbb{N}\}$ dense in $L^p(\mu)$.

$$A = \left\{ \sum_{j=1}^{k} z_{n_k} \cdot k_{m_j}, n_1, \dots, n_k \in \mathbb{N}, m_1, \dots, m_k \in \mathbb{N}, k \in p \right\}$$

is a countable dense set of $L^p(\mu; X)$

1. A is countable subset of $L^p(\mu; X)$.

2. $f \in L^p(\mu; X)$, $\varepsilon > 0$ by a): $\exists g_1$ simple integrable $\|f - g_1\|_p < \frac{\varepsilon}{3}$. Then $g_1 = \sum_{j=1}^{k} x_j \chi_{E_j}$ ($E_j \in \Sigma$, disjoint, $x_j \in X$, $\mu(E_j) < \infty$).

3. Find $n_1, \dots, n_k \in \mathbb{N}$ such that $\|x_j - z_{n_j}\|$ is so small, such that

$$\sum_{j=1}^{k} \|x_j - z_{n_j}\|^p \mu(E_j) < \left(\frac{\varepsilon}{3}\right)^p.$$

Set $g_2 := \sum_{j=1}^{k} z_{n_j} \chi_{E_j}$. Then $\|g_2 - g_1\|_p < \frac{\varepsilon}{3}$.

4. Find $m_1, \dots, m_k \in \mathbb{N}$ such that $\|\chi_{E_j} - k \cdot m_j\|_p$ is so small that

$$\sum_{j=1}^{k} \|z_{n_j}\| \cdot \|\chi_{E_j} - k_{m_j}\|_p \leqslant \frac{\varepsilon}{3}.$$

Then $g_3 := \sum_{j=1}^{k} z_{n_j} \cdot k_{m_j} \in A$, $\|g_3 - g_2\|_p < \frac{\varepsilon}{3}$.

To summarize: $\|g_3 - f\| < \varepsilon$. $\square$

Příklad
1. $G \subset \mathbb{R}^n$ Lebesgue measurable. $\mu := \lambda^n|_G$ for some $\mu(G) > 0$. $L^p(G; X) := L^p(\mu; X)$. $p \in [1, \infty)$, X separable $\implies L^p(G; X)$ is separable.

2. $\mu =$ the counting measure on $\mathbb{N} \implies L^p(\mu, X) = l^p(X)$. X separable, $p \in [1, \infty)$ $\implies l^p(X)$ separable.

Poznámka (Representation of duals)
p, p^* dual exponents, $p \in [1, \infty)$ ($p^* \in (1, \infty]$).

$$(l^p(X))^* \approx l^{p^*}(X^*).$$

X reflexive, μ σ-finite:

$$(L^p(\mu; X))^* = L^{p^*}(\mu; X^*).$$

X reflexive, μ σ-finite, $p \in (1, \infty)$: $L^p(\mu; X)$ is also reflexive

5 Compact convex sets

Poznámka (Convention)
In this chapter we consider only vector spaces over $\mathbb{R}$. It causes no harm, as all the definitions and results can be used for complex spaces as well, because only the structure of real version of the space in question is used.

Definice 5.1 (Extreme point)

Let X be a vector space and let $A \subset X$ be a convex set. A point $x \in A$ is said to be an extreme point of A if it is not an interior point of any segment in A, i.e. if

$$\forall a, b \in A \ \forall t \in (0,1) : x = t \cdot a + (1-t) \cdot b \implies a = b = x.$$

The set of all extreme point of A is denoted $\operatorname{ext} A$.

TODO?

Definice 5.2 (Face)

Let X be a vector space and let $A \subset X$ be a convex set. A subset $F \subset A$ is said to be a face of A if the following two conditions are fulfilled: 1. F is a nonempty convex subset of A; 2. $\forall a, b \in A : \frac{1}{2}(a+b) \in F \implies a \in F \wedge b \in F$.

Lemma 5.1 (Properties of faces)

Let X be a vector space and let $A \subset X$ be a convex set

1. *$x \in A$ is an extreme point of A if and only if $\{x\}$ is a face of A.*
2. *If $F_1 \subset A$ is a face of A and $F_2 \subset F_1$ is a face of F_1, then F_2 is a face of A.*
3. *If, moreover, X is a HLCS and A is a compact set containing at least two points, then there is a closed face $F \subsetneq A$.*

Důkaz
„1." clear from definitions. „2.": clearly F_2 is nonempty and convex, $x, y \in A \wedge \frac{x+y}{2} \in F_2 \implies x, y \in F_1$ (as $F_2 \subset F_1$ and F_1 is a face of A.) Hence $x, y \in F_2$ (as F_2 is a face of F_1).

„3.": $x, y \in A$, $x \neq y$ by Hahn–Banach: $\exists f \in X^*$: $f(x) \neq f(y)$. Since A is compact and f continuous, if attains maximum on A. Let $F = \{a \in A | f(a) = \max f(A)\}$. Then

- $0 \neq F \subsetneq K$ (as $f(x) \neq f(y)$, f is not constant on A);
- F is closed and convex (as f is continuous and linear);
- $\frac{x+y}{2} \in F$, $x, y \in A \implies$

$$\max f(A) = f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x) + f(y)) \leqslant \frac{1}{2}(\max f(A) \max f(A)) = \max f(A)$$

$\implies$ there are equalities, i.e. $x, y \in F$.

So, F is a closed face. □

Věta 5.2 (Krein–Milman)

Let X be a HLCS and let $K \subset X$ be a convex compact set. Then $K = \overline{\operatorname{co} \operatorname{ext} K}$. (In particular $\operatorname{ext} K \neq \emptyset$ *whenever K is nonempty.)*

Důkaz
Suppose $K \neq \emptyset$. Denote by $\mathcal{F}$ the family of closed faces in K. If $\mathcal{R} \subset \mathcal{F}$ is linearly ordered by „$\subset$", then $\bigcap \mathcal{R} \in \mathcal{F}$. (The intersection if compact nonempty, convex, the second property of a face is clear.) Then by Zorn's lemma, there is a minimal $F \in \mathcal{F}$.

From the previous lemma F is a singleton, i.e. $F = \{x\}$ for some $x \in K$ and $x \in \operatorname{ext} K$.

Thus $\operatorname{ext} K \neq \emptyset$. If $K \setminus \overline{\operatorname{co} \operatorname{ext} K} \neq \emptyset$, fix $x \in K \setminus \overline{\operatorname{co} \operatorname{ext} K}$. By Hahn–Banach separation theorem $\exists f \in X^* : f(x) > \sup f(\operatorname{ext} K)$.

Let $F = \{y \in K | f(y) = \max f(K)\}$. The F is a closed face. By the first part, we know that $\operatorname{ext} F \neq \emptyset$, thus there is $y \in \operatorname{ext} F$.

Since $F \cap \operatorname{ext} K = \emptyset$, we have $y \notin \operatorname{ext} K$. But by the previous lemma, we deduce $y \in \operatorname{ext} K$. ↯. □

Tvrzení 5.3 (Minkoski–Carathéodory)

TODO?

Tvrzení 5.4 (Milman)

Let X be a HLCS and $K \subset X$ a convex compact set. If $A \subset K$ is such that $K = \overline{\operatorname{co} A}$, then $\operatorname{ext} K \subset \overline{A}$.

Důkaz

1. If $A \subset K$ is such that $K = \operatorname{co} A$, then $\operatorname{ext} K \subset A$ by the definition of extreme points.

2. Let U be an absolutely convex open neighbourhood of $\mathbf{o}$ in X. Then there is a finite set $F \subset \overline{A}$ with $F + U \supset \overline{A}$ (using compactness of $\overline{A}$).

$$\implies K = \overline{\operatorname{co} A} \subset \overline{\operatorname{co}((F+U) \cap K)} =$$

$$= \overline{\operatorname{co}\left(\bigcup_{x \in F} (x + \overline{U}) \cap K\right)} = \operatorname{co}(\bigcup_{x \in F} (x + \overline{U}) \cap K).$$

($(x + \overline{U}) \cap K$, $x \in F$, are compact convex sets, they are finitely many, so the convex hull of their union is compact.)

Then by 1. we get $\operatorname{ext} K \subset \bigcup_{x \in F} (x + \overline{U}) \cap K \subset \overline{A} + \overline{U}$. Since U is arbitrary, we get $\operatorname{ext} K \subset \overline{A}$ and we are done. ($x \notin \overline{A} \implies \exists U$ absolutely convex open neighbourhood of $\mathbf{o}$ such that $(x + U) \cap \overline{A} = \emptyset$, then $\left(x + \frac{1}{2}\overline{U}\right) \cap \overline{A} = \emptyset$, so $x \notin \overline{A} + \frac{1}{2}\overline{U}$.) □

Důkaz (Union of compact convex sets is compact)

The set $H_1 \times \ldots \times H_n \times \{(t_1, \ldots, t_n) \in [0,1]^n | t_1 + \ldots + t_n = 1\}$ is compact and the mapping $(x_1, \ldots, x_n, (t_1, \ldots, t_n)) \mapsto t_1 x_1 + \ldots + t_n x_n$ is continuous, its range is $\operatorname{co}(H_1 \cup \ldots \cup H_n)$. □

Tvrzení 5.5 (On the barycenter of a measure)

TODO?

Věta 5.6 (Krein–Milman theorem on integral representation)

TODO?

Tvrzení 5.7

TODO?