

### *Poznámka*

The previous semester we work with linear equation (L-M, Fredholm, Minimizing quadratic function). This semester we will have non-linear equations like  $((\partial_t u)) - \Delta u + \arctg u = f$  or  $f = -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ .

We don't work with  $\partial_{tt} u - \Delta_p u = f$ , because nobody know how to proof it has solution (for  $d \geq 2, p > 2$ ).

### *Poznámka (Credit)*

Two homework. -10 to 10 points to exam from each. (If we hand anything we get credit.)

## What we must know

### *Poznámka*

Lebesgue spaces.

Fixed point theorem: 1) Let  $F$  be continuous mapping from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ . Assume that  $\exists$  convex compact set in  $\mathbb{R}^d$  such that  $F(\Omega) \subseteq \Omega$ . Then  $\exists x \in \Omega$  such that  $F(x) = x$ . 2) Let  $F : X \rightarrow X$ , where  $X$  is Banach space and  $F$  is continuous and compact and let  $\exists \Omega \subseteq X$  convex and closed such that  $F(\Omega) \subseteq \Omega$ . Then  $F(\Omega) \subseteq \Omega$ . Then  $\exists x \in X : F(x) = x$ .

Luzin: Let  $\Omega$  be a measurable set and  $f \in L^1_{loc}(\Omega)$ . Then  $\forall \varepsilon > 0 \exists U \in \Omega, \mu(U) \leq \varepsilon, f \in C(\Omega \setminus U)$ .

Egorov: Let  $\Omega$  be a measurable set and  $f^n \rightarrow f$  in  $L^1_{loc}(\Omega)$ . Then  $\forall \varepsilon > 0 \exists U, \mu(U) \leq \varepsilon f^n \rightarrow f$  in  $C(\Omega \setminus U)$ .

Lebesgue dominated convergence theorem.

Vitali convergence theorem: Let  $\Omega \subseteq \mathbb{R}^d$  be bounded measurable,  $f^n$  a sequence of measurable functions,  $f^n \rightarrow f$  almost everywhere in  $\Omega$ . Then  $\lim_{n \rightarrow \infty} \int_{\Omega} f^n = \int_{\Omega} f$ , provided  $f^n$  is uniformly equi-integrable ( $\forall \varepsilon > 0 \exists \delta \forall U, \mu(U) \leq \varepsilon$ ).

Fatou lemma:  $f^n \rightarrow f$  almost everywhere in  $\Omega$  and  $f^n \geq 0$ , then  $\liminf_{n \rightarrow \infty} \int_{\Omega} f^n \geq \int_{\Omega} f$ .

Regularization:  $\eta \in C_0^\infty(B_1(\mathbf{o}))$  non-negative, radially symmetric and  $\int_{\mathbb{R}^d} \eta(x) dx = 1$ . Then  $\forall f \in L^1_{loc}(\Omega)$  we extend  $f$  by „0“ to  $\mathbb{R}^d$  and  $f_\varepsilon := \eta_\varepsilon * f$ , where  $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta(x/\varepsilon)$ . Then  $f_\varepsilon \in C^\infty(\mathbb{R}^d)$  and  $\forall p \in [1, \infty) f \in L^p(\Omega) \implies f_\varepsilon \rightarrow f$  in  $L^p(\Omega)$ . (And for  $p = \infty$ :  $f \in L^\infty(\Omega) \implies f_\varepsilon \rightarrow f$  in  $L^q(\Omega) \forall q \in [1, \infty)$ ).

Reflexive and separable spaces. ( $L^p(\Omega)$  is a Banach space, separable for  $p \in [1, \infty)$ , reflexive for  $p \in (1, \infty)$ .)

Nemytsky operator: (Assume that for almost all  $x \in \Omega$  and ,  $|f(x, y)| \leq g(x) +$

$C \sum_{i=1}^N |y_i|^{p_i/p}$  for some  $p_i \in [1, \infty)$ ,  $p \in (1, \infty)$ ,  $g \in L^p(\Omega)$ . Then  $\forall u_i \in L^{p_i}$ , the function  $f(\cdot, u_1, \dots, u_n)$  is measurable,  $(u_1, \dots, u_n) \mapsto f(\cdot, u_1, \dots, u_n)$  is continuous  $L^{p_1}(\Omega) \times \dots \times L^{p_n}(\Omega) \rightarrow L^p(\Omega)$ . This mapping is called N.O.)

# Sobolev spaces (and Bochner spaces)

*Poznámka*

$\Omega$  is open bounded subset of  $\mathbb{R}^d$ .

## Věta 1.1 (Local approximation by smooth functions)

Let  $f \in W^{k,p}(\Omega)$  and extend it by „0“ outside. Define  $f_\varepsilon := \eta_\varepsilon * f$  and set  $\Omega_\varepsilon := \{x \in \Omega \mid B(x, \varepsilon) \subseteq \Omega\}$ . Then  $D^\alpha(f_\varepsilon) = (D^\alpha f)_\varepsilon$  almost everywhere in  $\Omega_\varepsilon$   $\forall \alpha, |\alpha| \leq k$  and  $\forall \Omega' \subseteq \overline{\Omega'} \subseteq \Omega$  and  $p \in [1, \infty)$   $f_\varepsilon \rightarrow f$  in  $W^{k,p}(\Omega')$ . (If  $p = \infty$ , then  $f_\varepsilon \rightarrow^* f$  in  $W^{1,\infty}(\Omega')$ .)

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*Důkaz*

$$\begin{aligned} \frac{\partial}{\partial x_i}(f_\varepsilon(x)) &= \frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) f(y) dy = \\ &= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i}(\eta_\varepsilon(x-y)) f(y) dy = - \int_{\mathbb{R}^d} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y) dy = \\ &= - \int_{B(x, \varepsilon)} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y) dy = - \int_{\Omega} \frac{\partial}{\partial y_i}(\eta_\varepsilon(x-y)) f(y) dy = \\ &= \int_{\Omega} \eta_\varepsilon(x-y) \frac{\partial f(y)}{\partial y_i} dy = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y) \frac{\partial f(y)}{\partial y_i} = \left( \frac{\partial f(y)}{\partial y_i} \right)_\varepsilon(x). \end{aligned}$$

Now we take sufficiently small  $\varepsilon$ , such that  $\Omega_\varepsilon \subseteq \Omega'$ . Then  $D^\alpha f_\varepsilon = (D^\alpha f)_\varepsilon \rightarrow D^\alpha f$  in  $L^p(\Omega')$ . □

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## Věta 1.2 (Composition of Lipschitz and Sobolev functions)

Let  $\Omega \subseteq \mathbb{R}^d$  be open and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be Lipschitz. Assume that  $u \in W^{1,p}(\Omega)$ . Then  $(f(u) - f(0)) \in W^{1,p}(\Omega)$  and  $\nabla f(u) = f'(u) \nabla u \chi_{x, u(x) \notin S_f}$ , where  $S_f$  are points where  $f'(s)$  doesn't exist.

Moreover define  $\Omega_a := \{x \in \Omega \mid u(x) = a\}$ , then  $\nabla u = 0$  almost everywhere in  $\Omega_a$ .

Důkaz

We know, that  $f \in C^1(\mathbb{R})$ ,  $f_{lip} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty$ .

So  $|f(u(x)) - f(0)|^p \leq f_{lip}^p \cdot |u(x)|^p$ , if  $u \in L^p(\Omega) \implies f(u) - f(0) \in L^p(\Omega)$ .

Next,  $\frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \implies f(u) - f(0) \in W^{1,p}(\Omega)$ .

We take  $\eta \in C_0^\infty(\Omega)$  and  $u \in W^{1,1}(\Omega)$ .

$$\begin{aligned} \int_{\Omega} \frac{\partial \eta}{\partial x_i} f(u) &= \lim_{\varepsilon \rightarrow 0_+} \int_{\Omega} \frac{\partial \eta}{\partial x_i} f(u_\varepsilon) \stackrel{\text{IBP, both are smooth}}{=} \lim_{\varepsilon \rightarrow 0_+} \int_{\Omega} \eta \frac{\partial f(u_\varepsilon)}{\partial x_i} = \\ &= - \lim_{\varepsilon \rightarrow 0_+} \int_{\Omega} \underbrace{\eta f'(u_\varepsilon)}_{\rightarrow \eta f(u) \text{ in } L^1, \text{ so weakly in } L^\infty} \cdot \underbrace{\frac{\partial u_\varepsilon}{\partial x_i}}_{\rightarrow \frac{\partial u}{\partial x_i} \text{ in } L^1}. \end{aligned}$$

TODO?

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### Věta 1.3 (Characterization of sobolev functions)

Let  $\Omega \subseteq \mathbb{R}^d$  open, bounded. Define  $\Omega_\delta := \{x \in \Omega \mid B(x, \delta) \subseteq \Omega\}$  and  $u_i^h(x) := \frac{u(x+h \cdot e_i) - u(x)}{h}$ ,  $h > 0, i \in [d]$ .

- If  $u \in W^{1,p}(\Omega)$  then  $\forall \delta \forall h < \frac{\delta}{2} : \|u_i^h\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p}(\Omega)$ .
- If  $p \in (1, \infty]$  and  $\sup_{\delta > 0} \sup_{h < \frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leq k$ , then  $\frac{\partial u}{\partial x_i}$  exists and  $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq k$ .
- If  $p \in [1, \infty)$  and if  $u \in W^{1,p}(\Omega)$  then  $u_i^h \rightarrow \frac{\partial u}{\partial x_i}$  in  $L_{loc}^p(\Omega)$ .

(\* If  $p = 1$  and  $\sup_{\delta > 0} \sup_{h < \frac{\delta}{2}} \|u_i^h\|_{L^p(\Omega_\delta)} \leq k$ , then  $u \in BV(\Omega)$ . Moreover if  $\leq k$  and  $u_i^h$  is equiintegrable, then  $u \in W^{1,1}(\Omega)$ .)

Důkaz

„Second item“ Fix  $\Omega_1 \subset \subset \Omega$ . Fix  $\delta_0, \Omega_1 \subseteq \Omega_{\delta_0} \implies \|u_i^h\|_{L^p(\Omega_1)} \leq k$ .  $u_i^h \rightharpoonup \bar{u}$  in  $L^p(\Omega_1)$  and  $u_i^h \rightharpoonup^* \bar{u}$  in  $L^\infty(\Omega_1)$ . We want  $\|\bar{u}\|_{L^p(\Omega_1)} \leq \liminf_{h \rightarrow 0+} \|u_i^h\|_{L^p(\Omega_1)} \leq k$ .

$$\begin{aligned} \int_{\Omega_1} \bar{u} \varphi dx &= \lim_{h \rightarrow 0+} \int_{\Omega} u_i^h \varphi = \lim_{h \rightarrow 0+} \int_{\Omega_1} \frac{u(x + h \cdot e_i) - u(x)}{h} \varphi(x) dx = \\ &= \lim_{h \rightarrow 0+} \int_{\Omega} \frac{u(x + h \cdot e_i)}{h} \varphi(x) - \frac{u(x)}{h} \varphi(x) dx = \\ &= - \lim_{h \rightarrow 0+} \int_{\Omega} u(x) \frac{\varphi(x) - \varphi(x - h \cdot e_i)}{h} dx = - \int_{\Omega_1} \frac{\partial \varphi}{\partial x_i} u. \end{aligned}$$

„First item“:  $u_\varepsilon := u * \eta_\varepsilon$  (where we extend  $u$  by zero).

$$\frac{u_\varepsilon(x + h \cdot e_i) - u_\varepsilon(x)}{h} = \frac{1}{h} \int_0^1 \frac{d}{dt} u_\varepsilon(x + h e_i t) dt = \int_0^1 \frac{\partial u_\varepsilon(x + h \cdot e_i \cdot t)}{\partial x_i} dt.$$

$$\left| \frac{u_\varepsilon(x + h \cdot e_i) - u_\varepsilon(x)}{h} \right|^p \leq \left| \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i} dt \right|^p \leq \int_0^1 \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p dt.$$

$$\begin{aligned} \int_{\Omega_\delta} \left| \frac{u_\varepsilon(x + h \cdot e_i) - u_\varepsilon(x)}{h} \right|^p &\leq \int_{\Omega_\delta} \int_0^1 \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot e_i \cdot t) \right|^p dt dx = \\ \int_0^1 \int_{\Omega_\delta} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + h \cdot e_i \cdot t) \right|^p dx dt &\leq \int_0^1 \int_{\Omega_{\delta/2}} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x) \right|^p dx dt = \int_{\Omega_{\delta/2}} \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p dx \\ \varepsilon \rightarrow 0+ : \int_{\Omega_\delta} \left| \frac{u(x + h \cdot e_i) - u(x)}{h} \right|^p &\leq \int_{\Omega_{\delta/2}} \left| \frac{\partial u}{\partial x_i}(x_i) \right|^p dx. \end{aligned}$$

„Third item“: It is enough to show „ $u_i^h$  is Cauchy“:  $\varepsilon > 0$ ,  $u_\varepsilon := u * \eta_\varepsilon$ :

$$u_{\varepsilon,i}^{h_1} - u_{\varepsilon,i}^{h_2} = \frac{u_\varepsilon(x + h_1 e_i) - u_\varepsilon(x)}{h} - \frac{u_\varepsilon(x + h_2 \cdot e_i) - u_\varepsilon(x)}{h} = \int_0^1 \frac{\partial u_\varepsilon}{\partial x_i}(x + h_1 \cdot e_i t) - \frac{\partial u_\varepsilon}{\partial x_i}(x + h_2 \cdot e_i t) dt.$$

$$\begin{aligned} \int_{\Omega_\delta} |\dots|^p &\leq \int_0^1 \int_{\Omega_\delta} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x + h_1 \cdot e_i \cdot t) - \frac{\partial u_\varepsilon}{\partial x_i}(x + h_2 \cdot e_i \cdot t) \right|^p dx dt, \\ \int_{\Omega_\delta} |u_i^{h_1} - u_i^{h_2}|^p &\leq \int_0^1 \int_{\Omega_\delta} |noepsilon|^p dx dt \end{aligned}$$

□

### Věta 1.4 (Approximation by smmooth function)

Let  $\Omega \subseteq \mathbb{R}^d$  be bounded and open and  $p \in [1, \infty)$ . Then  $\forall u \in W^{k,p}(\Omega)$

- $\exists \{u^n\}_{n=1}^\infty \subset \mathcal{C}^\infty(\Omega)$  such that  $\|u^n - u\|_{W^{k,p}(\Omega)} \rightarrow 0$ ;

- if  $\Omega \in C^0$ , then  $\exists \{u^n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega})$  such that  $\|u^n - u\|_{W^{k,p}(\Omega)} \rightarrow 0$ .

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Důkaz

„First item“ at home. „Second item“: Lemma (Partition of unite): „Let  $\{\Omega_r\}_{r=1}^{M+1}$  be open covering of  $\overline{\Omega}$ . Then  $\exists \varphi_r \in C_0^\infty(\Omega_r)$ , such that  $\forall x \in \overline{\Omega} : \sum_{r=1}^{M+1} \varphi_r(x) = 1$ .“ Proof at home.

Define  $u_r(x) := u(x)\varphi_r(x)$ . TODO!!!

1)  $u_{M+1}$  is supported in  $\Omega_{M+1} \subseteq \Omega$ , so it can be extended by 0. So  $u_{M+1}^n = u_{M+1} * \eta_{\frac{1}{n}}$ . 2)  $u_r$  for  $r \in [M]$ . Set  $u_i^h(x) := u_i(x_1, \dots, x_{d-1}, x_d - h)$ ,  $u_i^n := u_i * \eta_{\frac{1}{n}}$ . We need  $h, \varepsilon$  edependending on  $n$ :  $\|u_i^n - u_i\|_{W^{k,p}(V_1^+)} \rightarrow 0$  (because  $a_i$  is continuous).  $\varphi_i \in C_0^\infty$   $\exists \delta' > 0$ .  $\varphi_1$  is positive on the set  $x_d < a_1(x') + \beta - \delta' \wedge x_d > a_1(x') - \beta + \delta'$ ,  $h < h_0 < \delta'$ . Take  $(x, \dots, x_{d-1}, x_d - h)$ , where  $(x_1, \dots, x_d) \in \partial\Omega$ ,  $\text{dist}((x_1, \dots, x_{d-1}, x_d - h), \partial\Omega) < \delta$ . Denote this  $h$  as  $h_{max}$ , so for  $h < h_{max}$   $\text{dist}(\dots) < \delta$ .

Give me  $\delta > 0$ , I find  $h_0, h_{max}$  and define  $u_i^h = u_i^h * \eta^\delta$ , where  $h < \min(h_0, h_{max})$ . Then  $\|u_i - u_i^h\|_{W^{k,p}(V_i^+)} \rightarrow 0$ ,  $\|u_i^h - (u_i^h)_\delta\|_{W^{k,p}(V_i^+)} \rightarrow 0$  □

## Věta 1.5

Let  $\Omega \in C^{0,1}$  and  $p \in [1, \infty]$ . Then there exists a linear operator  $E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$  such that

- $Eu = u$  in  $\Omega$ ;
- $\exists B_R \subset \mathbb{R}^d$  such that  $Eu \equiv 0$  in  $\mathbb{R}^d \setminus B_R$ ;
- $\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq c(p, \Omega) \cdot \|u\|_{W^{1,p}(\Omega)}$ .

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Důkaz

Focus only on  $V_1$  (previous proof),  $u = \sum u_r$  ( $u_{M+1}$  is done) and only for  $u_1$ .

TODO images!!!

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□

## 2 Proof of $W^{1,p} \hookrightarrow C^{0,d}$

### Lemma 2.1 (Morrey)

Let  $u \in W^{1,1}(B_R(0))$  and 0 be the Lebesgue point of  $u$ .

$$\left| \int_{B_R} u(x) dx - u(0) \right| \leq R^A c(A, d) \sup_{\varrho \leq R} \int_{B_\varrho} \frac{|\nabla u(x)|}{\varrho^{d-1+A}} dx \quad A > 0.$$

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Důkaz

$$\begin{aligned}
|\oint_{B_R} u - u(0)| &= \lim_{r \rightarrow 0_+} |\oint_{B_R} u - \oint_{B_r} u| = \lim_{r \rightarrow 0_+} \left| \int_r^R \frac{d}{d\varrho} \oint_{B_\varrho} u(x) dx d\varrho \right| = \lim_{r \rightarrow 0_+} \left| \int_r^R \frac{d}{d\varrho} \oint_{B_1(0)} u(\varrho x) dx d\varrho \right| = \\
&\leq \lim_{r \rightarrow 0_+} \int_r^R \oint_{B_1(0)} |\nabla u(\varrho x)| dx d\varrho = \lim_{r \rightarrow 0_+} \int_r^R \oint_{B_\varrho} |\nabla u(x)| dx d\varrho = \lim_{r \rightarrow 0_+} \int_r^R \kappa_d \int_{B_\varrho} \frac{|\nabla u(x)| dx}{\varrho^{d-1+A}} \varrho^{A-1} d\varrho \leq \\
&\leq c(d) \sup_{\varrho < R} \int_{B_\varrho} \frac{|\nabla u|}{\varrho^{d-1+A}} \cdot \lim_{r \rightarrow 0_+} \int_r^R \varrho^{A-1} d\varrho = c(d, A) R^A \sup \dots
\end{aligned}$$

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Poznámka

Replace  $u$  by  $Eu$ .

Důsledek

$x, y$  Lebesgue points of  $u$ .

$$|u(x) - u(y)| \leq |x - y|^\alpha c(\alpha, \Omega, p) \max \{I_x, I_y\},$$

where

$$I_x := \sup_{r \leq 2|x-y|} \int_{B_r(x)} \frac{|\nabla u|}{\varrho^{d-1+A}}$$

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*Důkaz*

$$R := |x - y|$$

$$|u(x) - u(y)| \leq \left| \int_{B_R(x)} u - u(x) \right| + \left| \int_{B_R(y)} u - u(y) \right| + \left| \int_{B_R(x)} u - \int_{B_R(y)} u \right| \leq c(\varrho, \alpha) r^\alpha (I_x + I_y).$$

$$\begin{aligned} ?|| &= \left| \int_0^1 \frac{d}{dt} \int_{B_R(tx + (1-t)y)} u(z) dz \right| = \left| \int_0^1 \frac{d}{dt} \int_{B_R(0)} u(tx + (1-t)y + z) dz \right| \leq \\ &\leq \int_0^1 \int_{B_R(0)} |\nabla u(\dots) \cdot (x - y)| dz dt \leq c(d) \int_0^1 \int_{B_R(0)} \frac{|\nabla u(\dots)|}{R^{d-1}} dz dt \leq \\ &\leq c(d) \int_0^1 \int_{B_{2R}(x)} \frac{|\nabla u(z)| dz}{R^{d-1}} dt = c(d) \int_{B_{2R}(x)} \frac{|\nabla u(z)| dz}{(2R)^{d-1+\alpha}} (2R)^\alpha \leq c(d, \alpha) I_x (2R)^\alpha. \end{aligned}$$

$$\sup_{R>0, x \in \mathbb{R}^d} I_R(x) = \sup_{R, x} \int_{B_\varrho(x)} \frac{|\nabla E u(z)|}{R^{d-1+\alpha}} dz \leq \sup \left( \int_{B_R} |\nabla E u|^p \right)^{1/p} \left( \int_{B_R} R^{(1-d-\alpha)p'} \right)^{1/p'} \leq c \|u\|_{W^{1,p}(\Omega)} \left( R^{(1-d-\alpha)p'} \right)^{1/p'}$$

It remains  $\|u\|_{L^\infty(\Omega)}$ :

$$|u(x)| \leq |u(x) - u(y)| + |u(y)|$$

$$|u(x)| = \int_\Omega |u(x)| dy \leq \int_\Omega |u(y)| dy + K \|u\|_{W^{1,p}(\Omega)} \leq C(\Omega) \|u\|_{1,p},$$

$$\left\| \frac{u(x) - u(y)}{|x - y|^\alpha} \right\| \leq c \|u\|_{1,p}.$$

$x, y$  Lebesgue points, so estimates TODO?

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*Poznámka*

$W^{1,d}(\Omega) \not\hookrightarrow C^0(\overline{\Omega})$ , but  $W^{1,d}(\Omega) \hookrightarrow BMO(\Omega)(VMO)$ .

$W^{d,1}(\Omega) \hookrightarrow W^{1,d} \not\hookrightarrow C^0(\overline{\Omega})$ , but  $W^{d,1}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ .

## Věta 2.2

Let  $\Omega \in C^{0,1}$ ,  $p \in [1, \infty)$ . Then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  if

- for  $p \in [1, d)$  and  $q \leq \frac{dp}{d-p}$ ;
- for  $p = d$  and  $q \in [1, \infty)$ ;
- for  $p > d$  and  $q \in [1, \infty]$ .

And  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  if previous holds, except for  $p \in [1, d)$  and  $q = \frac{dp}{d-p}$ .

┌ *Dikaz* (Scheme of the proof)

We use extension  $W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$  compactly supported. Mollification  $W^{1,p}(\mathbb{R}^d) \rightarrow C_0^\infty(\mathbb{R}^d)$ . Show all estimates only for smooth functions. □

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**Lemma 2.3** (Gagliardo–Nirenberg inequality)

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$\exists C(d), C(d, p)$  such that  $\forall u \in C_0^\infty(\mathbb{R}^d)$ :

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq C(d) \|\nabla u\|_{L^1(\mathbb{R}^d)},$$

$$\|u\|_{L^{\frac{dp}{d-p}}(\mathbb{R}^d)} \leq C(d, p) \|\nabla u\|_{L^p(\mathbb{R}^d)}.$$



Důkaz (Proof of lemma)

Firstly we show that first inequality implies second. Define  $v := |u|^q$  for some  $q > 1$ . Then from first inequality  $\|v\|_{\frac{d}{d-1}} \leq c \cdot \|\nabla v\|_1$ :

$$\begin{aligned} \left( \int_{\mathbb{R}^d} |u|^{\frac{qd}{d-1}} \right)^{\frac{d-1}{d}} &\leq C(d) \int_{\mathbb{R}^d} |\nabla |u|^q| \leq c(d, q) \int_{\mathbb{R}^d} |u|^{q-1} |\nabla u| \stackrel{\text{Hölder}}{\leq} \\ &\leq c(d, q) \|\nabla u\|_p \cdot \| |u|^{q-1} \|_{p'} = c(d, q) \|\nabla u\|_p \left( \int_{\mathbb{R}^d} |u|^{\frac{p(q-1)}{p-1}} \right)^{\frac{p-1}{p}}. \end{aligned}$$

Choose  $q$  such that  $\frac{qd}{d-1} = \frac{p(q-1)}{p-1}$ , i. e.  $q = \frac{p(d-1)}{d-p}$ .

$$\left( \int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}} \right)^{\frac{d-1}{d}} \leq C(d, p) \|\nabla u\|_p \left( \int_{\mathbb{R}^d} |u|^{\frac{dp}{d-p}} \right)^{\frac{p-1}{p}} \implies \|u\|_{\frac{dp}{d-p}} \leq C(d, p) \|\nabla u\|_p.$$

Then we proof first inequality by next lemma: ( $u$  is smooth, compactly supported)

$$\begin{aligned} u(x) &= \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d) ds, \\ |u(x)| &\leq \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds, \\ |u(x)|^d dx &\leq \prod_{i=1}^d \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds, \\ \int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx &\leq \int_{\mathbb{R}^d} \prod_{i=1}^d \left( \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, x_{i-1}, s, x_{i+1}, \dots, x_d)| ds \right) dx =: \int_{\mathbb{R}^d} \prod_{i=1}^d v_i \stackrel{\text{Gagliardo}}{\leq} \\ &\leq \prod_{i=1}^d \left( \int_{\mathbb{R}^{d-1}} \left[ \left( \int_{-\infty}^{\infty} |\nabla u(x_1, \dots, s, \dots, x_d)| ds \right)^{\frac{1}{d-1}} \right]^{d-1} dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_d \right)^{\frac{1}{d-1}} \leq \\ &\leq \prod_{i=1}^d \|\nabla u\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}} = \|\nabla u\|_1^{\frac{d}{d-1}}. \end{aligned}$$

### Lemma 2.4 (Gagliardo)

Let  $u_i \in C_0^\infty(\mathbb{R}^{d-1})$ ,  $i \in [d]$ . Define  $v_i(x_1, \dots, x_d) := u_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$ . Then

$$\int_{\mathbb{R}^d} \prod_{i=1}^d |v_i(x)| dx \leq \prod_{i=1}^d \|u_i\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

┌  
*Důkaz* (Proof of lemma)  
 By induction: 1) „ $d = 2$ “:

$$\int_{\mathbb{R}^d} \prod_{i=1}^2 |v_i(x)| dx = \int_{\mathbb{R}^2} |u_1(x_2)| \cdot |u_2(x_1)| dx_1 dx_2 = \|u_1\|_{L^1(\mathbb{R})} \cdot \|u_2\|_{L^1(\mathbb{R})}.$$

2) „ $d \implies d + 1$ “:

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} \prod_{i=1}^{d+1} |v_i(x)| dx &= \int_{\mathbb{R}^d} |v_{d+1}(x)| \cdot \left( \int_{\mathbb{R}} \prod_{i=1}^d |v_i(x)| dx_{d+1} \right) dx_1 \dots dx_d \leq \\ &\leq \|v_{d+1}\|_{L^d(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} \prod_{i=1}^d |v_i(x)| dx_{d+1} \right)^{d'} dx_1 \dots dx_d \right)^{\frac{1}{d}} = RHS \\ (\dots)^{d'} &= \left( \int_{\mathbb{R}} |v_1| \cdot \dots \cdot |v_d| dx_{d+1} \right)^{d'} \stackrel{\text{Hölder}}{\leq} \left( \prod_{i=1}^d \left( \int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{\frac{1}{d}} \right)^{d'} . \\ RHS &\leq \|v_{d+1}\|_{L^d} \left( \int_{\mathbb{R}^d} \left( \prod_{i=1}^d \left( \int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{\frac{1}{d}} \right)^{\frac{d}{d-1}} dx_1 \dots dx_d \right)^{\frac{d-1}{d}} \leq \\ &\leq \|u_{d+1}\|_d \left( \int_{\mathbb{R}^d} \prod_{i=1}^d \left( \int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{\frac{1}{d-1}} dx_1 \dots dx_d \right)^{\frac{d-1}{d}} \stackrel{\text{Induction step}}{\leq} \\ &\leq \|u_{d+1}\|_d \cdot \prod_{i=1}^d \left\| \left( \int_{\mathbb{R}} |v_i|^d dx_{d+1} \right)^{\frac{1}{d-1}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R}^{d-1})} = \prod_{i=1}^d \|u_i\|_{L^d} \end{aligned}$$

└

□

„Důkaz

If  $p < d$  Gagliardo–Nirenberg finishes  $W^{1,p} \hookrightarrow L^{\frac{dp}{d-p}}$ . If  $p = d$ ,  $W^{1,d} \hookrightarrow W^{1,d-\varepsilon} \hookrightarrow L^{\frac{d(d-\varepsilon)}{d-(d-\varepsilon)}} = L^{\frac{d(d-\varepsilon)}{\varepsilon}}$ . If  $p > d$  forget G–N and use  $W^{1,p} \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega) \hookrightarrow L^q(\Omega)$ .

„Compact embeddings:“ 1. step:  $W^{1,1}(\Omega) \hookrightarrow L^1(\Omega)$ . 2. step  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ .

„1  $\implies$  2“:  $\|u\|_q \leq \|u\|_{\frac{dp}{d-p}}^\alpha \|u\|_1^{1-\alpha} \leq c \|u\|_{1,p}^\alpha \|u\|_1^{1-\alpha}$ . Let  $B$  be a bounded set in  $W^{1,p}(\Omega)$ .

Use 1. step: (for  $\frac{1}{q} = 1 - \alpha + \frac{\alpha(d-p)}{dp}$ , i.e.  $1 \leq q < \frac{dp}{d-p}$ )

$$\exists \{u_i\}_{i=1}^N \subseteq W^{1,p}(\Omega) \quad \forall u \in B : \min_{i \in [N]} \|u - u_i\|_{L^1} \leq \tilde{\varepsilon}.$$

$$\|u - u_i\|_q \leq c \cdot \|u - u_i\|_{1,p}^\alpha \cdot \|u - u_i\|_1^{1-\alpha} \leq c(\alpha, B)(\tilde{\varepsilon})^{1-\alpha}.$$

„1. step“: Let  $B$  be a bounded set in  $W^{1,1}(\Omega)$ ,  $EB$  be bounded set in  $W^{1,1}(\mathbb{R}^d)$  and compactly supported in  $B_\Omega$ ?

$$\forall u \in EB : u_\delta := u * \eta_\delta :$$

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x) - u_\delta(x)| dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} u(x) \eta_\delta(y) - u(x+y) \eta_\delta(y) dy \right| dx \leq \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(x+y)|}{|y|} \eta_\delta(y) |y| dx dy \stackrel{\text{we already had it}}{\leq} \\ &\leq \int_{\mathbb{R}^d} \|\nabla u\|_1 \eta_\delta(y) |y| dy \leq \|\nabla u\|_1 \delta \int_{\mathbb{R}^d} \eta_\delta(y) dy = \|\nabla u\|_1 \delta \leq C(B) \delta. \end{aligned}$$

Given  $\varepsilon > 0$  choose  $\delta > 0$ ,  $C(B)\delta < \frac{\varepsilon}{2}$  and use Arzela–Ascoli. □

„Poznámka (Lipschitz domain is necessary)

$$u = \frac{1}{(1 + |x|)^{3/2}}.$$

## 2.1 Compact embedding in Bochner spaces

**Lemma 2.5** (Aubin–Lions)

$V_1 \hookrightarrow V_2 \hookrightarrow V_3$  Banach spaces.  $p \in [1, \infty)$ . Then  $U := \{u \in L^p(0, T, V_1) \mid \partial_t u \in L^1(0, T, V_3)\} \hookrightarrow L^p(0, T, V_2)$ .

**Lemma 2.6** (Ehring (start of proof of Aubin–Lions))

Let  $V_1 \hookrightarrow V_2 \hookrightarrow V_3$  be Banach spaces. Then

$$\forall \varepsilon > 0 \exists c > 0 \forall u \in V_1 : \|u\|_{V_2} \leq \varepsilon \|u\|_{V_1} + c \cdot \|u\|_{V_3}.$$

┌ *Dũkaz* (By contradiction)

$$\exists u^n \in V_1 : \|u^n\|_{V_2} > \varepsilon \|u^n\|_{V_1} + n \|u^n\|_{V_3}.$$

$$u^n \neq 0 \implies v^n := \frac{u^n}{\|u^n\|_{V_2}} \implies 1 = \|v^n\|_{V_2} > \varepsilon \cdot \|v^n\|_{V_1} + n \cdot \|v^n\|_{V_3}.$$

$$v^n \rightarrow \text{ in } V_3 \implies v^n \text{ bounded in } V_1 \hookrightarrow V_2 \implies v^n \rightarrow v \text{ in } V_2 \implies \\ \implies \|v\|_{V_2} = 1 \implies v \neq 0. \zeta$$

└

□

*Dũkaz* (Aubin–Lions)

$M \subseteq U$  bounded set:  $\exists c^* \forall u \in M : \int_0^T \|u\|_{V_1}^p + \|\partial_t u\|_{V_3} \leq c^*.$

We want  $M$  is precompact in  $L^p(0, T; V_2) \Leftrightarrow$

$$\forall \varepsilon > 0 \exists \{w_i\}_{i=k}^N \forall u \in M \exists i \in [N] : \int_0^T \|u - w_i\|_{V_2}^p \leq \varepsilon.$$

1. Mollify with respect to time and use Arselà–Ascoli for  $C^1(0, T; V_1) \hookrightarrow C^0(0, T < V_2).$

2. Mollification is „not far“ from  $u$  in  $L^1(0, T; V_3).$

3, Ehrling interpolation to  $L^p(0, T; V_2).$

┌ *Dũkaz* (1.)

$u \in M$  extend to  $(0, 2T)$  as  $\tilde{u}(t) = u(t)$  if  $t < T$  and  $\tilde{u}(t) = u(2T - t)$  if  $t > T$ .

$$\int_0^{2T} \|\tilde{u}(t)\|_{V_1}^p + \|\partial_t \tilde{u}(t)\|_{V_3} = 2 \int_0^T \|u\|_{V_1} + \|\partial_t u\|_{V_3} \leq 2C^*.$$

$\forall 0 < \delta < T$  and  $t \in (0, T)$ ,  $u_\delta(t) = \int_0^\delta \tilde{u}(t+s) \varphi_\delta(s) ds = \int_{\mathbb{R}} \tilde{u}(s) \varphi_\delta(s-t)$ , where  $\varphi \in C_0^\infty(0, 1)$ ,  $\varphi \geq 0$ .

$$\|u_\delta(t)\|_{V_1} \leq \frac{c}{\delta} \int_0^{2T} \|\tilde{u}\|_{V_1} \leq \frac{c \cdot c^*}{\delta}.$$

$$\|\partial_t u_\delta(t)\|_{V_1} \leq \int_{\mathbb{R}} \|\tilde{u}\|_{V_1} |\varphi'_\delta| \leq c(\delta) \cdot c^*.$$

$M_\delta := \{u_\delta, u \in M\} \implies M_\delta$  is bounded in  $C^1(0, T; V_1).$   $\forall \tilde{\varepsilon} > 0 \exists \{w_k\}_{k=1}^N \subseteq L^p(0, T; V_1)$  such that  $\forall u_\delta \in M_\delta \exists k : \int_0^T \|w_\delta - w_k\|_{V_2}^p < \tilde{\varepsilon}.$  □

└

Důkaz (2.)

$$\begin{aligned}
u(t) - u_\delta(t) &= u(t) - \int_0^\delta \tilde{u}(t+s) \varphi_\delta(s) ds = \\
&= \int_0^\delta (u(t) - \tilde{u}(t+s)) \varphi_\delta(s) ds = - \int_0^\delta (u(t) - \tilde{u}(t+s)) \frac{d}{ds} \left( \int_s^\delta \varphi_\delta(\tau) d\tau \right) ds = \\
&= - \int_0^\delta \partial_t \tilde{u}(t+s) \int_s^\delta \varphi_\delta(\tau) d\tau ds = - \int_0^\delta \int_0^\tau \partial_t \tilde{u}(t+s) \varphi_\delta(\tau) ds d\tau.
\end{aligned}$$

$$\int_0^T \|u(t) - u_0(t)\|_{V_3} dt \leq \int_0^T \int_0^\delta \int_0^\tau \|\partial_t \tilde{u}(t+s)\|_{V_3} \varphi_\delta(\tau) ds d\tau dt \leq \int_0^T \int_0^\delta \int_0^\tau \|\partial_t \tilde{u}(t+s)\|_{V_3} \varphi_\delta(\tau) ds d\tau dt \leq c$$

$$\|u(t) - u_\delta(t)\|_{V_3} \leq \int_0^\delta \int_0^\tau \|\partial_t u(t+s)\|_{V_3} \varphi_\delta(\tau) ds d\tau \leq c^*.$$

□

Důkaz (3.)

$$\begin{aligned}
\int_0^T \|u - w_k\|_{V_2}^p &\stackrel{\text{Ehrling}}{\leq} \tilde{\varepsilon} \int_0^T \|u - w_k\|_{V_1}^p + c(\tilde{\varepsilon}) \int_0^T \|u - w_k\|_{V_3}^p \leq k(c^*) \tilde{\varepsilon} + c(\tilde{\varepsilon}) \int_0^T \|u - w_k\|_{V_3}^p \leq \\
&\leq k(c^*) \tilde{\varepsilon} + c(\tilde{\varepsilon}, p) \int_0^T \|u - u_\delta\|_{V_3}^p \leq \\
&\leq k(c^*) \tilde{\varepsilon} + c(\tilde{\varepsilon}, p) \sup_{t \in (0, T)} \{ \|u(t) - u_\delta(t)\|_{V_3}^{p-1} \} \int_0^T \|u - u_\delta\|_{V_3} + -||- \leq \\
&\leq k(c^*) \tilde{\varepsilon} + c(c^*, p, \tilde{\varepsilon}) \delta + c(\tilde{\varepsilon}, p, c^*) \int_0^T \|u_\delta - w_k\|_{V_3}^p.
\end{aligned}$$

Find  $\tilde{\varepsilon}$  such that  $k(c^*) \tilde{\varepsilon} \leq \frac{\varepsilon}{3}$ . Find  $\delta > 0$  such that  $c(c^*, p, \varepsilon) < \frac{\varepsilon}{3}$ . Find  $N$ -covering  $\{U_i\}_{i=1}^N$  such that  $\min_k c(\tilde{\varepsilon}, p, c^*) \int_0^T \|u_\delta - w_k\| \leq \frac{\varepsilon}{3}$ .

□

## 2.2 Trace theorem

### Věta 2.7 (Trace theorem)

Let  $\Omega \in C^{0,1}$ . Then there exists a linear operator  $\text{tr} : W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$  ( $\hookrightarrow L^q(\partial\Omega)$ ,  $q = \frac{(d-1)p}{d-p}$ ) for all  $p \in [1, \infty]$ , such that for all  $u \in C(\overline{\Omega})$   $\text{tr} u = u|_{\partial\Omega}$ .

Důkaz (Trace theorem, part one: on cube for smooth functions)

$$\begin{aligned} \int_{-1}^1 \cdots \int_{-1}^1 |u(x_1, \dots, x_{d-1}, 0)|^q dx_1 \dots dx_{d-1} &= \int_{-1}^1 \cdots \int_{-1}^1 \int_0^1 \frac{d}{ds} |u(x_1, \dots, x_{d-1}, s)|^q ds dx_1 \dots dx_{d-1} \leq q \cdot \int_{-1}^1 \cdots \int_{-1}^1 |u(x_1, \dots, x_{d-1}, 1)|^q dx_1 \dots dx_{d-1} \\ &\leq C(\Omega) \cdot \|\nabla u\|_{L^p(\Omega)} \cdot \|u\|_{W^{1,p}(\Omega)}^{q-1} = C(q, p, \Omega) \|u\|_{W^{1,p}(\Omega)}^q. \end{aligned}$$

We use  $q = \frac{(d-1)p}{d-p}$ . Lemma: „Let  $p < d$ “, then for all  $u \in C^1(\text{some interior of cube?})$  there holds  $\|u\|_{L^q(\text{„line“})} \leq C(p) \|u\|_{W^{1,p}(\text{cube})}$ .  $\square$

*Důkaz* (Trace theorem, general boundary)

## Definice 2.1

Let  $\Omega \in C^{0,1}$  and  $f : \partial\Omega \rightarrow \mathbb{R}$ . We say that  $f \in L^p(\partial\Omega)$  ( $p \in [1, \infty]$ ) if  $\forall i \in [N]$ ,  $f \circ T \in L^p(-\alpha, \alpha)^{d-1}$ .

## Definice 2.2

Let  $f \in L^1(\partial\Omega)$ ,  $\{\varphi_i\}$  be a partition of unity (smooth, compactly supported,  $\sum_i \varphi_i(x) = 1$ ) corresponding to  $V_i$ . Then

$$\int_{\partial\Omega} f ds := \int_{\partial\Omega} \sum_{i=1}^N (f \varphi_i) = \sum_{i=1}^N \int_{V_i} f \varphi_i := \int_{(-\alpha, \alpha)^{d-1}} f(T_i(y)) \sqrt{1 + |\nabla a_i|^2} \varphi_i(T(y)).$$

$\mathbf{n} := \frac{(\nabla a, 1)}{\sqrt{1 + |\nabla a_i|^2}}$  is outer unit normal vector (to  $\partial\Omega$ ). (It exists for almost all  $x \in \partial\Omega$ .)

### Lemma 2.8 (IBP)

Let  $\Omega \in C^{0,1}$  and  $f \in C^1(\overline{\Omega})$ . Then  $\int_{\Omega} \frac{\partial f}{\partial x_i} dx = \int_{\partial\Omega} f u_i dS$ .

*Důkaz* (Trace theorem, general boundary + Sobolev functions)

Case „ $p > d$ “: nothing to prove,  $W^{1,p} \hookrightarrow C(\overline{\Omega})$ .

Case „ $p \leq d$ “:  $C^1(\bar{\Omega})$  is dense in  $W^{1,p}(\Omega)$ ,  $\forall u \in W^{1,p}(\Omega) \exists \{u^n\}_{n=1}^\infty \subseteq C^1(\bar{\Omega}) : u^n \rightarrow u$  in  $W^{1,p}(\Omega)$ . We have  $u^n$  is well defined in  $\partial\Omega$ . So we define  $\text{tr}$  as the limit. It remains that the limit exists, so we show that „ $u^n$  is Cauchy in  $L^q(\partial\Omega)$ “:

$$\begin{aligned} \int_{\partial\Omega} |u^n - u^m|^q ds &= \sum_i \int_{V_i} |u^n - u^m|^q \varphi_i dS = \sum_i \underbrace{\int \dots \int}_{d-1} |u^n \circ T_i - u^m \circ T_i|^q \varphi_i \circ T_i \sqrt{1 + |\nabla a_i|^2} \leq \\ &\leq C(q, \Omega) \sum_{i=1}^N \|u^n \circ T_i - u^m \circ T_i\|_{W^{1,p}(\text{cube})}^q \leq C(\Omega, q, p) \|u^n - u^m\|_{W^{1,p}(\Omega)}^q. \end{aligned}$$

**Věta 2.9** (IBP for  $W^{1,p}(\Omega)$ )

Let  $\Omega \in C^{0,1}$ ,  $u \in W^{1,p}(\Omega)$ ,  $v \in W^{1,q}(\Omega)$ ,  $W^{1,p}(\Omega) \hookrightarrow L^{q'}(\Omega)$  and  $W^{1,q}(\Omega) \hookrightarrow L^{p'}$ . Then

$$\int_{\Omega} \frac{\partial u}{\partial x_i} v = - \int_{\Omega} \frac{\partial v}{\partial x_i} u dx + \int_{\partial\Omega} \text{tr}(u) \text{tr}(v) n_i dS.$$

*Poznámka*

Question: If  $u \in L^q(\partial\Omega)$ , is there extension into  $\Omega$  so that  $u \in W^{1,p}(\Omega)$ . So for  $\text{tr} : W^{1,p} \rightarrow \text{Range}$  is  $\text{Range} = L^q(\partial\Omega)$ . And answer is NO.

**Věta 2.10**

Let  $\Omega \in C^{0,1}$ .  $p \in (1, \infty]$ ,  $s \in (\frac{1}{p}, 1]$ . Then the trace  $\text{tr}$  is bounded map from  $W^{s,p}(\Omega)$  to  $W^{s-\frac{1}{p},p}(\partial\Omega)$ .

Moreover,  $\exists \text{tr}^{-1} : W^{s-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{s,p}(\Omega)$ , linear, bounded, and  $\text{tr}^{-1}(\text{tr } u) = u$ .

For  $p = 1$ ,  $\exists \text{tr}^{-1} : L^1(\partial\Omega) \rightarrow W^{1,1}$ ,  $\text{tr}(\text{tr}^{-1}(u)) = u$ . But this  $\text{tr}^{-1}$  is nonlinear!

**Definice 2.3** (Sobolev-Stobodeckij spaces)

We say  $u \in W^{s,p}(\Omega)$  for  $s \in (0, 1)$  if

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} dx dy < \infty.$$

Moreover  $\|u\|_{s,p}^p = \|u\|_p^p + \int \dots$

**Definice 2.4** (Nikolski spaces)

...,  $p \in [1, \infty)$ ,  $s \in (0, 1)$ ,  $u \in N^{s,p}(\Omega)$  if

$$\forall i \forall h : \int_{2h} \frac{|u(x + h \cdot e_i) - u(x)|^p}{h^{s \cdot p}} dx < \infty.$$

*Poznámka* (You can prove)

$$W^{s,p}(\Omega) \hookrightarrow N^{s,p}(\Omega) \hookrightarrow W^{s-\varepsilon,p}(\Omega) \quad \forall \varepsilon > 0.$$

### 3 Nonlinear elliptic equation as compact perturbation

*Poznámka*

$$-\Delta u + g(u, \nabla u) = f \text{ in } \Omega \wedge u = 0 \text{ on } \partial\Omega.$$

*Poznámka*

Case  $g : \mathbb{R} \rightarrow \mathbb{R}$  continuous,  $|g(s)| \leq c(1 + |s|)^\alpha$ ,  $\alpha \in [0, 1)$ . Nem? if  $u \in L^2(\Omega) \implies g(u) \in L^2(\Omega)$ .

Apriori estimates:

$$\begin{aligned} \int_{\Omega} -\Delta u \cdot u + g(u) \cdot u &= \int_{\Omega} \cdot \\ \int_{\Omega} |\nabla u|^2 &\leq \int_{\Omega} |f| \cdot |u| + c \cdot |u| \cdot (1 + |u|)^\alpha \leq \int_{\Omega} |f| \cdot |u| + \tilde{c}|u|^{1+\alpha} \cdot 1 + \tilde{c} \leq \\ &\leq \varepsilon \int_{\Omega} |u|^2 + c(\varepsilon, \alpha) \left( \int_{\Omega} |f|^2 + 1 \right) \implies \|u\|_{1,k}^2 \leq \tilde{c}(\alpha)(1 + \|f\|_2^2). \end{aligned}$$

### **Lemma 3.1**

Let  $\Omega \in C^{0,1}$ ,  $\alpha \in [0, 1)$ . Then  $\exists C(\Omega, \alpha)$ ,  $\forall g \in L^2(\Omega) \exists u \in W_0^{1,2}(\Omega) \|u\|_{1,2} \leq c(1 + \|f\|_2)$ :

$$\int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \cdot \varphi.$$



┌ *Dikaz*

Define a new problem:

$$-\Delta u = f - g(w) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

For  $w$  find  $u$ . Now we check that  $M : L^2(\Omega) \rightarrow L^2(\Omega)$ ,  $w \mapsto u$ , satisfies assumptions from Schauder fixed-point theorem (and so there exists solution):

- $M$  is continuous (homework);
- $M$  is compact:  $w \in B_R(L^2(\Omega)) \implies u \in \tilde{B}_R(W_0^{1,2}(\Omega))$  but  $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ ;
- find a convex set  $K$ :  $M(K) \subseteq K$ . Here it will be proper ball:

$$\exists R > 0 : \|w\|_2 \leq R \implies \|u\|_{1,2} \leq R :$$

$$\int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} |f| \cdot |u| + |g(w)| \cdot |u| \leq \varepsilon \|u\|_2^2 + c(2)(\|f\|_2^2 + \|g(w)\|_2^2),$$

$$\|u\|_{1,2}^2 \leq C(\|f\|_2^2 + \|g(w)\|_2^2) \leq$$

$$\leq c \cdot \|f\|_2^2 + c \cdot \int_{\Omega} (1 + |w|)^{2\alpha} \leq c\|f\|_2^2 + \delta\|w\|_2^2 + c(\delta).$$

$$\|w\|_2^2 \leq R^2 \implies \|u\|_{1,2}^2 \leq \delta R^2 + c \cdot \|f\|_2^2 + c(\delta) \leq R^2.$$

$$\text{Set } \delta = \frac{1}{2} \leq \frac{R^2}{2} + c(\|f\|_2^2 + c(\delta)) \leq R^2.$$

Uniqueness??  $u_1, u_2$  solutions:  $(\varphi = u_1 - u_2 \in W_0^{1,2}(\Omega))$

$$\int_{\Omega} \nabla(u_1 - u_2) \cdot \nabla \varphi = \int_{\Omega} (g(u_2) - g(u_1))\varphi,$$

$$\lambda_1 \|u_1 - u_2\|_2^2 \leq \int_{\Omega} |\nabla u_1 - \nabla u_2|^2 = \int_{\Omega} (g(u_2) - g(u_1))(u_1 - u_2) \leq \varepsilon \lambda_1 \|u_1 - u_2\|_2^2.$$

└ So if  $g$  is non-decreasing, then uniqueness holds. Moreover it holds, when  $-g' < \lambda_1$ . □

TODO?

TODO!!!

TODO!!!

## 4 PDE vs. Minimizers

TODO?

### Lemma 4.1

Let  $A(u, \xi)$  and  $B(u, \xi)$  be  $C^1$ -mappings. Then the following is equivalent:

- $\exists F$  such that  $A = \frac{\partial F}{\partial \xi}$ ,  $B = \frac{\partial F}{\partial u}$ ;
- $\forall i, j : \frac{\partial A_i}{\partial \xi_j} = \frac{\partial A_j}{\partial \xi_i}$ ,  $\frac{\partial B}{\partial \xi_i} = \frac{\partial A_i}{\partial u}$ .

┌

*Důkaz*

„2. is necessary to have  $F$ “:

$$\frac{\partial A_i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \left( \frac{\partial F}{\partial \xi_i} \right) = \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} = \frac{\partial}{\partial \xi_i} \left( \frac{\partial F}{\partial \xi_j} \right) = \frac{\partial}{\partial \xi_i} A_j.$$

$$\frac{\partial B}{\partial \xi_i} = \frac{\partial}{\partial \xi_i} \left( \frac{\partial F}{\partial u} \right) = \frac{\partial}{\partial u} \left( \frac{\partial F}{\partial \xi_i} \right) = \frac{\partial A_i}{\partial u}$$

„2.  $\implies$  1.“: define

$$F(u, \xi) := \int_0^1 A(tu, t\xi) \cdot \xi dt + \int_0^1 B(tu, t\xi) u dt.$$

We need  $\frac{\partial F}{\partial \xi} = A$ ,  $\frac{\partial F}{\partial u} = B$ :

$$\begin{aligned} \frac{\partial F}{\partial u}(u, \xi) &= \int_0^1 \frac{\partial A(tu, t\xi)}{\partial(tu)} \xi t dt + \int_0^1 \frac{\partial B(tu, t\xi)}{\partial(tu)} t u dt + \int_0^1 B(tu, t\xi) dt = \\ &= \int_0^1 \frac{\partial A}{\partial(tu)} \cdot \xi t dt + \int_0^1 B(tu, t\xi) dt + \int_0^1 t \frac{d}{dt} \left( B(tu, t\xi) - \frac{\partial B}{\partial(t\xi)} t \xi \right) dt = [B(tu, t\xi) t]_{t=0}^{t=1} = B(u, \xi). \end{aligned}$$

└

□

### Věta 4.2

Let  $A, B$  be Caratheodory and  $\frac{\partial A}{\partial \xi_j} = \frac{\partial A}{\partial \xi_i}$ ,  $\frac{\partial A}{\partial u} = \frac{\partial B}{\partial \xi}$ . Let  $B(u, 0) = 0$ , and  $A$  be monotone with respect to  $\xi$  and

1.  $|A(u, \xi)| \leq c(1 + |\xi|)^{p-1}$ ,  $A(u, \xi)\xi \geq c \cdot |\xi|^p$
2.  $|B(u, \xi)| \leq c[(1 + |\xi|)^p + |u|^p]$ .

Then  $\forall f \in (W_0^{1,p}(\Omega))^* \exists u \in W_0^{1,p}(\Omega)$ :

$$\int_{\Omega} A(u, \nabla u) \cdot \nabla v + B(u, \nabla u) v = \langle f, v \rangle, \quad \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

┌

*Důkaz* •  $\exists F$ , such that  $\frac{\partial F}{\partial \xi} = A$ ,  $\frac{\partial F}{\partial u} = B$ ;

- $A$ -monotone with respect to  $\xi \implies F$  is convex with respect to  $\xi$ ;
- $F$  is coercive?

TODO!!!

□

### Věta 4.3

Let  $A, B, F$  be as above + assumptions and

$$(A(u_1, \xi_1) - A(u_2, \xi_2))(\xi_1 - \xi_2) + (B(u_1, \xi_1) - B(u_2, \xi_2))(u_1 - u_2) \geq 0.$$

(The whole operator is monotone.) Then any weak solution is minimizer.

┌

*Důkaz*

The whole operator is monotone  $\implies F(u, \xi)$  is convex with respect to all variables.

$$F(u_2, \xi_2) - F(u_1, \xi_1) \geq \frac{\partial F}{\partial \xi}(u_1, \xi_1) \cdot (\xi_2 - \xi_1) + \frac{\partial F}{\partial u}(u_1, \xi_1)(u_2 - u_1) = A(u_1, \xi_1) \cdot (\xi_2 - \xi_1) + B(u_1, \xi_1)(u_2 - u_1).$$

$$\begin{aligned} \int_{\Omega} F(v, \nabla v) &= F(\text{tr}(u), \nabla \text{tr}(u)) \geq \int_{\Omega} A(\text{tr}(u), \nabla \text{tr}(u)) \cdot (\nabla v - \nabla \text{tr}(u)) + B(u, \nabla \text{tr}(u)) \cdot (v, \text{tr}(u)) = \\ &= \int_{\Omega} A(\text{tr}(u), \nabla \text{tr}(u)) \cdot \nabla v + B(\text{tr}(u), \nabla \text{tr}(u))v - \int_{\Omega} A(\text{tr}(u), \nabla \text{tr}(u)) \cdot \nabla \text{tr}(u) + B(\text{tr}(u), \nabla \text{tr}(u)) \text{tr}(u) = \\ &\quad \left( \int_{\Omega} A(\text{tr}(u), \nabla \text{tr}(u)) \nabla v + B(\text{tr}(u), \nabla \text{tr}(u))v - \int_{\Omega} A(\dots) \cdot \nabla v + B(\dots) \cdot v = 0. \right) \\ &= - \int_{\Omega} A(\text{tr}(u), \nabla \text{tr}(u)) \cdot \nabla \text{tr}(u) + B(\text{tr}(u), \nabla \text{tr}(u)) \text{tr}(u) = \\ &= - \int_{\Omega} (A(\text{tr}(u), \nabla \text{tr}(u)) - A(u, \nabla u)) \cdot \nabla \text{tr}(u) - (B(\text{tr}(u), \nabla \text{tr}(u)) - B(u, \nabla u)) \text{tr}(u) - \int_{\Omega} A(u, \nabla u) \nabla \text{tr}(u) \end{aligned}$$

□

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## 4.1 Monotone operator – dual approach

*Poznámka*

1.  $u$ -weak solution to  $-\text{div}(A \nabla u) = f$  in  $\Omega$  and  $u = u_0$  on  $\partial\Omega$ ,  $A = A^T$ .
2.  $u$ -minimizer to  $\min_{\bar{S}} \int_{\Omega} \frac{A \nabla u \nabla u}{2} - f u$ , where  $\bar{S} = W^{1,2}(\Omega)$  such that  $u = u_0$  on  $\partial\Omega$ .

3. Dual formulation  $S := \{\xi \in L^2(\Omega, \mathbb{R}^d), \forall v \in W_0^{1,2}(\Omega) : \int_{\Omega} \xi \nabla v = \int_{\Omega} f v\}$ . Minimize  $\int_{\Omega} \frac{A^{-1} \xi \xi}{2} - \nabla u_0 \xi$ , where  $\xi = A \nabla u$ .

#### Věta 4.4

Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\frac{\partial F}{\partial \xi} = A(\xi)$ ,  $F$  strictly convex and  $F(0) = 0$ ,  $A(0) = 0$ . Let

1.  $|F(\xi)| \leq C(1 + |\xi|)^p, \quad |A(\xi)| \leq c(1 + |\xi|)^{p-1}, \quad p \in (1, \infty);$
2.  $F(\xi) \geq c_1 |\xi|^p - c_2, \quad A(\xi) \xi > c_1 |\xi|^p - c_2.$

Let  $u_0 \in W^{1,p}(\omega)$ ,  $g \in L^{p'}(\Gamma_i)$ ,  $\Gamma_i \subseteq \partial B$  TODO!!!

TODO!!!