Poznámka
There will be homework. We will discus it on practicals (particular solutions are good).

Poznámka (What it is about)
Functional analysis generalizes Linear Algebra. This lecture generalizes (real) Analysis in $\mathbb{R}^n$ ($Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is linear) by replacing $\mathbb{R}^n$ with Banach spaces.

Příklad (Calculus of variations)
Know things: $f : \mathbb{R} \to \mathbb{R}$, differentiable has minimizer at $x_0 \in \mathbb{R} \implies f'(x_0) = 0$ (in $\mathbb{R}^n$: $Df(x_0) = 0$). Generalize it:

Řešení
Trick: For example $F : u \mapsto \int_\Omega \frac{1}{2}|\nabla u|^2 - fudx$, $W_g^{1,2}(\Omega) \to \mathbb{R}$ (g means bounded values).
For any $\varphi \in W_0^{1,2}(\Omega)$ consider $\varepsilon \mapsto F(u + \varepsilon\varphi)$, $\mathbb{R} \to \mathbb{R}$.

$$0 = \frac{d}{d\varepsilon}|_{\varepsilon=0} F(u+\varepsilon\varphi) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \int_\Omega \frac{1}{2}|\nabla u + \varepsilon\nabla\varphi|^2 - f\cdot(u+\varepsilon\varphi)dx =$$

$$= \frac{d}{d\varepsilon}|_{\varepsilon=0}\left[\int_\Omega \frac{1}{2}|\nabla u|^2 - fudx + \varepsilon\int_\Omega \nabla u\nabla\varphi - f\varphi dx + \varepsilon^2\int_\Omega \frac{1}{2}|\nabla\varphi|^2 dx\right] =$$

$$= \int_\Omega \nabla u \nabla\varphi - f\varphi.$$

Assume $u \in W^{2,2}(\Omega)$:

$$\overset{\text{P.I.}}{\int}_{\partial\Omega} \frac{\partial u}{\partial n}\varphi dx - \int_\Omega (\triangle u + f)\varphi dx \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

$$\overset{\text{Fundamental lemma}}{\triangle} \qquad u + f = 0.$$

Příklad (Mapping degree)
Consider $f \in \mathcal{C}([-1,1];\mathbb{R})$. How many zeroes does f have? Let assume $f(-1) < 0 < f(1)$. Let assume $f \in \mathbb{C}^1$. And 0 is a regular value ($f(x_0) = 0 \implies f'(x_0) \neq 0$).

Řešení
From 0 to ∞. After assumption: by intermediate value theorem at least 1. After second assumption: odd and finitely many. Moreover, the number of zeros with positive derivative minus the number of zeros with the negative one is 1, which is called degree of f.

Observation: In one dimension $\deg(f) \in \{-1, 0, 1\}$. $\deg(f)$ is invariant under perturbations. $\deg f$ depends on boundary values. Can be extended from $\mathcal{C}^1$ to $\mathcal{C}$ (we take smooth perturbation).

Ad second observation: homotopy: $h : [0,1] \times [-1,1] \to \mathbb{R}$, $(s,x) \mapsto h_s(x)$ continuous $h_0 = f$, $h_1 = g$.. And it is admissible if $h_s(-1) \neq 0$ and $h_s(1) \neq 0$ for all s.

There is generalization to $\mathbb{R}^n$, to Manifolds, and to Banach spaces. And we get „corollaries“: Fix point theorems, topological statements,inability to comb a hedgehog,

1 Derivatives in Banach spaces

1.1 The notion of a derivative

Poznámka (In $\mathbb{R}^n$)
Partial derivative, directional derivative, total derivative.

Definice 1.1 (Directional and Gateaux derivative)

Let X, Y be Banach spaces, $A \subset X$ open, $f : A \to Y$. For any $x_0 \in A$, $v \in X$ if

$$\frac{\partial f}{\partial v}(x_0) := \lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

exists, we call it directional derivative (at x_0, in direction v).

If $v \mapsto \frac{\partial f}{\partial v}(x_0)$ is a continuous linear operator from X to Y, we denote it by $\partial f(x_0)$ and call it the Gateaux derivative (at x_0).

Poznámka (Notation)
Some authors omit continuous and linear, i.e. for them directional $\Leftrightarrow$ Gateaux.

Some use df or Df instead of ∂f.

We will write $\frac{\partial f}{\partial v}(x_0) = \partial f(x_0)\langle v \rangle$. ($\langle \cdot \rangle$ for linear arguments.)

Například
Consider $F : L^2([0,1]) \to L^2([0,1])$, $u \mapsto F(u)$, $F(u)(x) := \sin(u(x))$. It is continuous ($\|F(u) - F(v)\|_{L^2}^2 = \int |\sin(u(x)) - \sin(v(x))|^2 \leqslant \int |u(x) - v(x)|^2$). Fix $\varphi \in L^2([0,1])$ and calculate:

$$\frac{\partial F}{\partial \varphi}(u) = \lim_{h \to 0} \frac{\sin(u(\cdot) + h\varphi(\cdot)) - \sin(u(\cdot))}{h} = \cos(u(\cdot)) \cdot \varphi(\cdot)$$

point-wise almost everywhere and by domain convergence everywhere.

$\frac{\partial F}{\partial \varphi}$ is linear in φ and bounded $\implies$ F is Gateaux differentiable. Consider $u \mapsto \frac{\partial F}{\partial \varphi}(u)$ for fixed φ. It is continuous.

Is ∂F a good linear approximation? I.e. $\|F(u + \varphi) - F(u) - \partial F(u) \langle \varphi \rangle\|_{L^2} \overset{?}{=} o(\|\varphi\|_{L^2})$. No: Pick $u = 0$ $\varphi_k = \pi \chi_[0, \frac{1}{k}]$, then $\|\varphi_k\|_2 = \sqrt{\frac{1}{k}\pi^2} \to 0$.

$$F(u + \varphi_k)(x) = \begin{cases} \sin(0), & x > \frac{1}{k}, \\ \sin(\pi), & x \leqslant \frac{1}{k}. \end{cases} = 0.$$

$$\|\ldots\| = \|0 - 0 - \partial F(0) \langle \varphi_k \rangle\|_{L^2} = \|\varphi_k\|_{L^2} \notin o(\|\varphi_k\|_{L^2}).$$

Definice 1.2 (Fréchet derivative)

Let X, Y be Banach, $A \subset X$ open $f : A \to Y$. For any $x_0 \in A$ if there exists $Df(x_0) \in \mathcal{L}(X, Y)$ such that

$$\lim_{v \to \mathbf{o}} \frac{\|f(x_0 + v) - f(x_0)\|_Y}{\|v\|_X} = 0$$

then $Df(x_0)$ is called Fréchet derivative.

Lemma 1.1 (Fréchet $\implies$ Gateaux)

X, Y Banach spaces, $A \subset X$ open, $f : A \to Y$. If F is Fréchet differentiable at x_0, it is also Gateaux differentiable with $\partial f(x_0) = Df(x_0)$.

Důkaz
Trivial. □

Definice 1.3 (Gradient)

Let H be a Hilbert space, $A \subset H$ open $f : A \to \mathbb{R}$. If f is Gateaux differentiable at $x_0 \in A$, then the unique $\nabla f(x_0) \in H$ such that $\langle \nabla f(x_0), v \rangle_H = \partial f(x_0) \langle v \rangle \quad \forall v \in H$ is called the gradient of f at x_0.

Poznámka (Gradients in different spaces)
Derivatives are „independent“ of the space used: $X_1 \hookrightarrow X_2$, $Y_1 \hookrightarrow Y_2$ Banach, $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ such that $f_2|_{X_1} = f_1$. Then $Df_2(x_0)|_{X_1} = Df_1(x_0)$, if both exist.

For Hilbert spaces $H_1 \hookrightarrow H_2$:

$$\langle a, v \rangle_{H_1} = \langle b, v \rangle_{H_2} \; \forall v \in H_1 \not\Rightarrow a = b.$$

$\implies$ ∇f depends on the space! Notation $\nabla_H f(x_0)$.

One can define „formal gradients“: Let X Banach, H Hilbert, $X \hookrightarrow H$. $f : A \subset X \to \mathbb{R}$ Gateaux differentiable. Then there might be $\nabla f(x_0) \in H$ such that

$$\langle v, \nabla f(x_0) \rangle_H = Df(x_0)(v) \qquad \forall v \in X.$$

If X is dense in H, then $\nabla f(x_0)$ is unique.

Classically gradients are associate inner product, but principle works with dual pairings, $(\langle \cdot, \cdot \rangle_{L^p \times L^q}, \frac{1}{p} + \frac{1}{q} = 1)$.

1.2 Calculation rules

Tvrzení 1.2 (Chain rule)

Let X, Y, Z be Banach, $A \subset X$, $B \subset Y$ open, $f : B \to Z$, $g : A \to B$, $x_0 \in A$, $y_0 := g(x_0)$.

1. *If f is Fréchet differentiable at y_0 and g is Gateaux differentiable at x_0, then $f \circ g$ is Gateaux differentiable at x_0 with*

$$\partial(f \circ g)(x_0)\langle v \rangle = Df(x_0)\langle \partial g(x_0)\langle v \rangle\rangle \qquad \forall v \in X.$$

2. *If g is additionally Fréchet differentiable, then so is $f \circ g$.*

Důkaz (1.)

$$\lim_{h \to 0} \left\| \frac{f(g(x_0 + hv)) - f(g(x_0))}{h} - Df(y_0)\langle \partial g(x_0)\langle v \rangle\rangle \right\|_Z \leqslant$$

$$\leqslant \lim_{h \to 0} \left\| \frac{f(g(x_0 + hv) + y_0 - g(x_0)) - f(y_0) - Df(y_0)\langle g(x_0 + hv) - g(x_0)\rangle}{h} \right\|_Z +$$

$$+ \lim_{h \to 0} \underbrace{\left\| Df(y_0) \left\langle \partial g(x_0)\langle v \rangle - \frac{g(x_0 + hv) - g(x_0)}{h} \right\rangle \right\|_Z}_{\to 0} =$$

$$= \lim_{h \to 0} \frac{\| f(x_0 + g(x_0 + hv) - g(x_0)) - f(y_0) - Df(x_0)\langle g(x_0 + hv) - g(x_0)\rangle \|_Z}{\|g(x_0 + hv) - g(x_0)\|_Y} \cdot \frac{\|g(x_0 + hv) - g(x_0)\|_Y}{h} =$$

□

Důkaz (2.)

Last convergence in 1. is independent of v. □

Lemma 1.3 (Mean value)

Let $I \subset \mathbb{R}$ be an interval, Y Banach, $f : I \to Y$ differentiable, $a \in Y$. Then $\forall x, y \in I$, $x > y$, $\exists \xi \in [y, x]$ such that

$$\left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y \leqslant \|f'(\xi) - a\|_Y.$$

Důkaz
By Hahn–Banach $\exists \varphi \in Y^*$ such that

$$* := \left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y = \varphi \left\langle \frac{f(x) - f(y)}{x - y} - a \right\rangle \wedge \|\varphi\|_{Y^*} = 1.$$

Define $\Psi : [y, x] \to \mathbb{R}$, $s \mapsto \varphi\langle f(s) - s \cdot a \rangle$. Then

$$* = \frac{\varphi\langle f(x)\rangle - \varphi\langle f(y)\rangle}{x - y} - \frac{x - y}{x - y}\varphi\langle a\rangle = \frac{\psi(x) - \psi(y)}{x - y} \overset{\text{Mean value theorem}}{\psi'}(\xi) \overset{\text{Chain rule}}{=} \varphi\langle f'(\xi) - a\rangle \leqslant \|f'(\xi) -$$

□

Tvrzení 1.4 (Product spaces)

Let X_1, X_2, Y be Banach, $f : X_1 \times X_2 \to Y$. Let $x_1 \in X_1$, $x_2 \in X_2$, and denote by $\partial_1 f(x_1, x_2)$ the Gateaux derivative of $x \mapsto f(x, x_2)$ at x_1, by $\partial_2 f(x_1, x_2)$ the Gateaux derivative of $x \mapsto f(x_1, x)$ and similarly $D_1 f(x_1, x_2)$ and $D_2 f(x_1, x_2)$.

1. *If f is Gateaux differentiable at (x_1, x_2) then $\partial_1 f(x_1, x_2)$, $\partial_2 f(x_1, x_2)$ exists and we have*

$$\forall v_1 \in X_1, v_2 \in X_2 : \partial f(x_1, x_2)\langle (v_1, v_2)\rangle = \partial_1 f(x_1, x_2)\langle v_1\rangle + \partial_2 f(x_1, x_2)\langle v_2\rangle.$$

2. *If $\partial_1 f$ and $\partial_2 f$ exists at (x_1, x_2) and one of them is continuous (as a function $X_1 \times X_2 \mapsto \mathcal{L}(X_i; Y)$) then f is Gateaux differentiable.*

3. *The previous points hold also for Fréchet derivation.*

Důkaz (1.)
From definition:

$$\partial_1 f(x_1, x_2) = \partial f(x_1, x_2)\langle (v_1, 0)\rangle = \lim_{h \to 0} \frac{f(x_1 + hv_1, x_2) - f(x_1, x_2)}{h}.$$

□

Důkaz (2.)
WLOG $\partial_2 f$ is continuous.

$$\lim_{h\to 0}\left\|\frac{f(x_1+hv_1,x_2+hv_2)-f(x_1,x_2)}{h}-\partial_1 f(x_1,x_2)\langle v_1\rangle-\partial_2 f(x_1,x_2)\langle v_2\rangle\right\|_Y \leqslant$$

$$\leqslant \lim_{h\to 0}\underbrace{\left\|\frac{f(x_1+hv_1,x_2)-f(x_1,x_2)}{h}-\partial_1 f(x_1,x_2)\langle v_1\rangle\right\|_Y}_{\to 0} +$$

$$+\lim_{h\to 0}\underbrace{\left\|\frac{f(x_1+hv_1,x_2+hv_2)-f(x_1+hv_1,x_2)}{h}-\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle\right\|_Y}_{*} +$$

$$+\underbrace{\lim_{h\to 0}\left\|\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle-\partial_2 f(x_1,x_2)\langle v_2\rangle\right\|_Y}_{\to 0} = 0$$

Consider $\psi : s \mapsto f(x_1+hv_1, x_2+sv_2)$.

$$* \leqslant \sup_{\xi\in[0,h]}\left\|\partial_2 f(x_1+hv_1,x_2+\xi v_2)\langle v_2\rangle-\partial_2 f(x_1+hv_1,x_2)\langle v_2\rangle\right\| \to 0$$

by continuous of $\partial_2 f$. □

Důkaz (3.)
Similarly. □

1.3 Inverse and implicit function theorem

Věta 1.5 (Inverse function theorem)

Let X, Y, $A\subset X$ open, $f : A \to Y$ continuously Fréchet differentiable. If $x_0 \in A$ such that $Df(x_0) : X \to Y$ is an isomorphism then there exists $U \subset A$, $V \subset Y$ such that $f|_U : U \to V$ is bijection and $(f|_U)^{-1}$ is Fréchet differentiable with

$$D(f^{-1})(y_0) = (Df(x_0))^{-1}, \qquad y_0 := f(x_0).$$

Důkaz
Given $\hat{y}$ close to $f(x_0)$ find $\hat{x}$ such that $f(\hat{x}) = \hat{y}$. Idea: fix $\hat{y}$ try x: error in y is $f(x) - y$ and error in x is $(Df(x_0))^{-1} \langle f(x) - y \rangle$. Therefore try iteration:

$$F_{\hat{y}}(x) := x - (Df(x_0)) < f(x) - y > .$$

If $F_{\hat{y}}$ has fix point $\hat{x}$ then $\hat{x} = F_{\hat{y}}(\hat{x}) = \hat{x} - (Df(x_0)) \langle f(\hat{x} - y) \rangle \implies f(\hat{x}) = \hat{y}$. So we use Banach fixed point theorem: „$F_{\hat{y}}$ is contraction“: $(x_1, x_2 \in B_\delta(x_0))$

$$\|F_{\hat{y}}(x_1) - F_{\hat{y}}(x_2)\|_X = \|x_1 - x_2 - (Df(x_0))^{-1} \langle f(x_1) - f(x_2) \rangle \|_X =$$

$$= \|(Df(x_0))^{-1} \langle Df(x_0) \langle x_1, x_2 \rangle + f(x_1) - f(x_2) \rangle \|_X \leqslant$$

$$\leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \|Df(x_0) \langle x_1 - x_2 \rangle + f(x_1) - f(x_2)\|_Y = *$$

Consider $a := Df(x_0) \langle x_1 - x_2 \rangle$. $\psi : [0,1] \to Y,\ f(1-\xi)x_1 + \xi x_2)$ and apply Mennroltz? lemma.

$$* \leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \|Df(x_0) < x_1 - x_2 > - Df((1-\xi)x_1 + \xi x_2) \langle x_2 - x_1 \rangle \|_Y \leqslant$$

$$\leqslant \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y,X)} \cdot \sup_{x \in B_0(x_0)} \|Df(x_0) - Df(x)\|_{\mathcal{L}(X,Y)} \cdot \|x_1 - x_2\|_X \ll 1$$

$$\|F_{\hat{y}}(x) - x_0\|_X = \|F_{\hat{y}}(x) - F_{\hat{y}}(x_j)\|_X + \|F_{\hat{y}}(x_0) - x_0\|_X \leqslant$$

$$\leqslant \frac{1}{2}\|x - x_0\|_X + \|(Df(x_0))^{-1}\| \cdot \|\hat{y} - x_0\|$$

$\|\hat{y} - x_0\|$ can chosen to be small $\implies$ $F_{\hat{y}}$ maps $\overline{B_\delta(x_0)}$ to $\overline{B_\delta(x_0)}$ $\implies$ $F_{\hat{y}}$ has unique fix point.

Next „regularity“: $(y_1 := f(x_1),\ y_2 := f(x_2))$

$$\|f^{-1}(y_1) - f^{-1}(y_2)\|_X = \|F_{y_1}(x_1) - F_{y_2}(x_2)\|_X \leqslant$$

$$\leqslant \|F_{y_1}(x_1) - F_{y_1}(x_2)\|_X + \|F_{y_1}(x_2) - F_{y_2}(x_2)\|_X \leqslant$$

$$\leqslant \frac{1}{2}\|x_1 - x_2\|_X + \|(Df(x_0))^{-1} \langle y_1 - y_2 \rangle \|_X \leqslant \frac{1}{2} \underbrace{\|x_1 + x_2\|_X}_{=\|f^{-1}(y_1) - f^{-1}(y_2)\|} + c \cdot TODO!!!$$

$$\implies \frac{1}{2}\|f^{-1}(x_1) - f^{-1}(x_2)\|_X \leqslant c \cdot \|y_1 - y_2\|_Y \implies f^{-1} \text{ is Lipschitz.}$$

Pick δ so small that

$$\|Df(x) - Df(x_0)\| \leqslant \frac{1}{2} \cdot \frac{1}{\|(Df(x_0))^{-1}\|} \qquad \forall x \in B_\delta(x_0).$$

$\implies$ $(Df(x))^{-1}$ exists and is uniformly bounded (from functional analysis).

$$\| \underbrace{f^{-1}(y+w) - f^{-1}(y)}_{=:v} - (Df(x))^{-1} \langle w \rangle \|$$

$(f(x+v) + f(x) = f(f^{-1}(y+w)) - y = w)$

$\|v - (Df(x)) \langle f(x+v) - f(x) \rangle \| = \|(Df(x))^{-1} \langle Df(x) \langle v \rangle - f(x+v) + f(x) \rangle \leqslant \|(Df(x))^{-1}\| \cdot \sigma(\|v\|) \leqslant$

because f^{-1} is Lipschitz

Věta 1.6 (Global inverse function theorem)

Let X, Y Banach, $f : X \to Y$ continuously Fréchet differentiable and $(Df(x))^{-1}$ exists, depends continuously on X and $c > 0$ such that $\|(Df(x))^{-1}\| < c \ \forall x \in X$. Then $f : X \to Y$ is a diffeomorphism.

Důkaz

Last theorem $\implies$ f is a local diffeomorphism. Left to show: f is bijective. „Surjectivity“: Fix $x_0 \in X$, $y_0 \in Y$, . Let $y \in Y$, $\varphi(t) = y_0 + t(y - y_0)$, $t \in [0,1]$. Goal: find $\psi(t)$ continuous, such that $\varphi(t) = f(\psi(t))$ (then $y = f(\varphi(t))$) (so called lifting). Local diffeomorphism implies ψ exists on $[0,\delta]$, in fact if Y is defined on $[0,t_0]$, it can be extended to $[0,t_0+\delta]$. Similarly, if ψ is defined on $[0,t_0]$, per chain rule:

$$\|\psi'(t)\| = \|Df^{-1}(\varphi(t))\langle\varphi'(t)\rangle\| < c.$$

ψ is Lipschitz, $\lim_{t \nearrow t_0} \psi(t)$ is well defined and ψ can be extended to $[0,t_0]$. From Zorn lemma Ψ is defined on $[0,1]$.

„Injectivity“: Assume $f(x_1) = f(x_2) = y$. Pick $\psi_1(t) := x_1 + t(x_2 - x_1)$. $\varphi_1(t) = f(\psi_1(t))$. Define $\varphi_s(t) = s\varphi_1(t) + (1-s)y$ $(t, s \in [0,1])$. Similar to before (homework) $\exists \psi_s(t)$ continuous in s and t, such that $f(\psi_s(t)) = \varphi_s(t)$. But then

$$x_1 = \psi_1(0) = \psi_s(0) = \psi_0(0) = \psi_0(t) = \psi_0(1) = \psi_s(1) = \psi_1(1) = x_2.$$

□

Věta 1.7 (Implicit function theorem)

Let X_1, X_2, Y Banach, $A_1 \subset X_1$, $A_2 \subset X_2$ open, $f : A_1 \times A_2 \to Y$ continuously Fréchet differentiable and exists $\hat{x}_1 \in A_1$ and $\hat{x}_2 A_2$ with $f(x_1,x_2) = 0$. If $D_2 f(\hat{x}_1,\hat{x}_2)$ is an isomorphism (between X_2 and Y), then are neighbourhoods U_1, U_2 of x_1, x_2 such that $\forall \hat{x}_1 \in U_1 \ \exists! \hat{x}_2 \in U_2$ with $f(\hat{x}_1,\hat{x}_2) = 0$.

If we call $\hat{x}_2 = g(x_1)$, then g is continuously Fréchet differentiable with $Dg(x) = -(D_2 f(x, g(x)))^{-1} \circ D_1 f(x, g(x))$.

Důkaz

Apply the inverse function theorem to

$$F(x_1,x_2) := (x_1, (D(f(\hat{x}_1,\hat{x}_2)))^{-1})\langle f(x_1,x_2)\rangle$$

□