1 Area formula and coarea formula

Věta 1.1

Let (P_1, ϱ_1), (P_2, ϱ_2) be metric spaces, $s > 0$, and $f : P_1 \to P_2$ be β-Lipschitz. Then $\varkappa^s(f(P_1)) \leqslant \beta^s \varkappa^s(P_1)$.

Důkaz

Choose $\delta > 0$. Let $P_1 = \bigcup_{i=1}^{\infty} A_j$, $\operatorname{diam} A_j < \delta$. Then we have $f(P_1) = \bigcup_{j=1}^{\infty} f(A_j)$, $\operatorname{diam} f(A_j) < \beta \cdot \delta$.

$$\varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \sum_{j=1}^{\infty} (\operatorname{diam} f(A_j))^s \leqslant \sum_{j=1}^{\infty} \beta^s \cdot (\operatorname{diam} A_j)^s = \beta^s \cdot \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s.$$

It holds for all possible choices of (A_j), so we can take infimum:

$$\varkappa^s(f(P_1)) \leftarrow \varkappa^s(f(P_1), \beta \cdot \delta) \leqslant \beta^s \inf_{(A_j)} \sum_{j=1}^{\infty} (\operatorname{diam} A_j)^s = \beta^s \varkappa^s(P_1, \delta) \to \beta^s \varkappa^s(P_1).$$

□

Lemma 1.2

Let $k, n \in \mathbb{N}$, $k \leqslant n$, and $L : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear mapping. Then for every λ_k-measurable set $A \subset \mathbb{R}^k$ it holds $H^k(L(A)) = \sqrt{\det(L^T L)\lambda_k(A)}$.

Důkaz ($\dim L(\mathbb{R}^k) = k$)

We find linear isometry Q of $\mathbb{R}^k$ onto $L(\mathbb{R}^k)$, from last semester

$$H^k(L(A)) = H^k(Q^{-1} \circ L(A)) = \lambda^k(Q^{-1} \circ L(A)) = |\det(Q^{-1}L)| \cdot \lambda_k(A).$$

$$(\det(Q^{-1}L))^2 = \det((Q^{-1}L)^T) \cdot \det(Q^{-1}L) = \det((Q^{-1}L)^T \cdot (Q^{-1}L)) = \det((\langle Q^{-1}Le^i, Q^{-1}L^T e^j \rangle)_{i,j}).$$

And because Q is isometry ($\implies$ Q^{-1} is isometry), we can remove Q^{-1} from scalar product and we get $\det(L^T L)$. □

Lemma 1.3

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\beta > 1$. Then there exists a neighbourhood V of the point x such that

- *the mapping $y \mapsto \varphi(\varphi'(x)^{-1}(y))$ is β-Lipschitz on $\varphi'(x)(V)$;*
- *the mapping $z \mapsto \varphi'(x)(\varphi^{-1}(z))$ is β-Lipschitz on $\varphi(V)$.*

Důkaz

x, β fixed. We know, that there exists $\eta > 0$ such that

$$\forall v \in \mathbb{R}^k : \|\varphi'(x)(v)\| \geqslant \eta \cdot \|v\|.$$

We find $\varepsilon \in \left(0, \frac{1}{2}\eta\right)$ such that $\frac{2\varepsilon}{\eta} + 1 < \beta$. We find a neighbourhood V of x such that $\forall y \in V : \|\varphi'(x) - \varphi'(y)\| \leqslant \varepsilon$.

We show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| \leqslant \varepsilon \|u - v\|.$$

Fix $v \in V$ and consider the mapping

$$g : w \mapsto \varphi(w) - \varphi(v) - \varphi'(x)(w - v).$$

For $w \in V$ we have $g'(w) = \varphi'(w) - \varphi'(x)$:

$$\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| = \|g(w) - g(v)\| \leqslant \sup \{\|g'(w)\| \mid w \in V\} \cdot \|u - v\| \leqslant \varepsilon \cdot \|u - v\|.$$

Further we show that for every $u, v \in V$ we have

$$\|\varphi(u) - \varphi(v)\| \geqslant \frac{1}{2}\eta \|u - v\|.$$

For $u - v \in V$ we compute

$$\|\varphi(u) - \varphi(v)\| \geqslant -\|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \geqslant -\varepsilon\|u - v\| + \eta\|u - v\| \geqslant \frac{1}{2}\eta\|u - v\|.$$

„First point“: TODO (řádek nebyl k přečtení)

$$\|\varphi(\varphi^{-1}(x)(a)) - \varphi(\varphi^{-1}(x)(b))\| = \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \|phi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi'(x)(u - v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|\varphi'(x)(y - v)\| \leqslant \varepsilon \frac{1}{\eta} \|a - b\| + \|a - b\| = \left(\frac{\varepsilon}{\eta} + 1\right) \|a - b\| \leqslant \beta \cdot \|a - b\|.$$

„Second point“: $k, q \in \varphi(V)$. We find $u, v \in V$ such that $\varphi(u) = p$ and $\varphi(v) = q$:

$$\|\varphi'(x)(\varphi^{-1}(p)) - \varphi'(x)(\varphi^{-1}(q))\| = \|\varphi'(x)(u) - \varphi'(x)(v)\| =$$

$$= \|\varphi'(x)(u - v)\| \leqslant \|\varphi(u) - \varphi(v) - \varphi'(x)(u - v)\| + \|\varphi(u) - \varphi(v)\| \leqslant$$

$$\leqslant \varepsilon \cdot \|u - v\| + \|p - q\| \leqslant \frac{2\varepsilon}{\eta} \|\varphi(u) - \varphi(v)\| + \|p - q\| = \left(\frac{2\varepsilon}{\eta} + 1\right) \|p - q\| \leqslant \beta \|p - q\|.$$

$\square$

Lemma 1.4

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping, $x \in G$, and $\alpha > 1$. Then there exists a neighbourhood of x such that for every λ^k-measurable $E \subset V$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

Důkaz

Find $\beta > 1$, $\tau > 1$ such that $\beta^k \tau < \alpha$. By previous lemma we find a neighbourhood V_1 of x such that the conclusion of the lemma holds for β. We find a neighbourhood V_2 of x such that

$$\forall t \in V_2 : \tau^{-1} \operatorname{vol} \varphi'(t) \leqslant \operatorname{vol} \varphi'(t) \leqslant \tau \operatorname{vol} \varphi'(x).$$

Set $V = V_1 \cap V_2$.

Assume that $E \subset V$ is a λ^k-measurable set. We have

$$\tau^{-1} \operatorname{vol} \varphi'(x) \cdot \lambda^k(E) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau \operatorname{vol} \varphi'(t) \lambda^k(E).$$

By lemma above we have $\operatorname{vol} \varphi'(t) \lambda^k(E) = H^k(\varphi'(x)(E))$:

$$\tau^{-1} H^k(\varphi'(x)(E)) \leqslant \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \tau H^k(\varphi'(x)(E)).$$

By previous lemma we get

$$H^k(\varphi(E)) = H^k\left(\left(\varphi \circ (\varphi'(x))^{-1} \circ \varphi'(x)\right)(E)\right) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant \beta^k H^k(\varphi'(x)(E)) \leqslant$$

$$\leqslant \beta^k \tau \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

By lemma above we get

$$H^k(\varphi(E)) \geqslant \beta^{-k} H^k\left(\left(\varphi'(x) \circ \varphi^{-1} \circ \varphi\right)(E)\right) = \beta^{-k} H^k(\varphi'(x)(E)) \geqslant$$

$$\geqslant \beta^{-k} \tau^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \geqslant \alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

□

Věta 1.5

Let $k, n \in \mathbb{N}$, $k \leqslant n$, $G \subset \mathbb{R}^k$ be an open set, $\varphi : G \to \mathbb{R}^n$ be an injective regular mapping and $f : \varphi(G) \to \mathbb{R}$ be H^k-measurable. Then we have

$$\int_{\varphi(G)} f(x) dH^k(x) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t),$$

if the integral at the right side converges.

Důkaz

„φ^{-1} is well defined“: If $H \subset G$ is open, then we can write $H = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact for every $n \in \mathbb{N}$. Then we have $\varphi(H) = \bigcup_{n=1}^{\infty} \underbrace{\varphi(K_n)}_{\text{compact}}$ is F_σ. This implies that φ^{-1} is Borel. The mappings φ, φ^{-1} are locally Lipschitz by lemma above. ($\varphi(G)$ is Borel.) $\varphi(G)$ is H^k-σ-finite.

1. „$f = \chi_L$, $L \subset \varphi(G)$ is H^k-measurable“: We show $H^k(L) = \int_{\varphi^{-1}(L)} \varphi'(t) d\lambda^k(t)$. Choose $\alpha > 1$. By previous lemma we find for every $y \in G$ neighbourhood $V_y \subset G$ of the point y such that for every λ^k-measurable set $E \subset V_y$ we have

$$\alpha^{-1} \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(E)) \leqslant \alpha \int_E \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

We have $\subset \{V_y | y \in G\} = G$. There exists a sequence $\{y_j\}_{j=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} V_{y_j} = G$. Using lemma from previous semester we find Borel sets $B, N \subset \varphi(G)$ such that $B \subset L \subset B \cup N$, $H^k(N) = 0$.

$\lambda^k(\varphi^{-1}(N)) = 0$. $\varphi^{-1}(B) \subset \varphi^{-1}(L) \subset \varphi^{-1}(B) \cup \varphi^{-1}(N) \implies \varphi^{-1}(L)$ is λ^k-measurable. We set

$$A_j = \varphi^{-1}(L) \cap \left(V_{y_j} \setminus \bigcup_{i=1}^{j-1} V_{y_j} \right).$$

Then we have

- A_j is λ^k-measurable;
- $A_j \subset V_{y_j}$ for every $j \in \mathbb{N}$;
- $\forall j, j' \in \mathbb{N}, j \neq j' : A_j \cap A_{j'} = \varnothing$;
- $\bigcup_{j=1}^{\infty} A_j = \varphi^{-1}(L)$;
- for every $j \in \mathbb{N}$ we have

$$\alpha^{-1} \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t) \leqslant H^k(\varphi(A_j)) \leqslant \alpha \int_{A_j} \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

From all except for second point we have

$$\alpha^{-1} \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(L) \leqslant \underbrace{\sum_{j=1}^{\infty} H^k(\varphi(A_j))}_{=H^k\left(\bigcup_{j=1}^{\infty} \varphi(A_j)\right) = H^k(L)} \leqslant \alpha \int_{\varphi^{-1}(L)} \operatorname{vol} \varphi'(t) d\lambda^k(t).$$

2. „$f \geqslant 0$ simple H^k-measurable“: From linearity of integrals. 3. „$f \geqslant 0$ H^k-measurable“: we approximate f by $0 \leqslant f_j \leqslant f_{j+1}$ simple functions and from Levi

$$\lim_{j\to\infty} \int_{\varphi(G)} f_j(x) dH^k(x) = \int_{\varphi(G)} f(x) dH^k(x), \qquad \lim_{j\to\infty} \int_G f_j(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda^k(t) = \int_G f(\varphi(t)) \operatorname{vol} \varphi'(t) d\lambda$$

3. „f H^k-measurable“: We add positive and negative part. □

Věta 1.6 (Coarea formula)

Let $k, n \in \mathbb{N}$, $k > n$, $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ be Lipschitz mapping, $f : \mathbb{R}^k \to \mathbb{R}$ be λ^k-integrable function. Then we have

$$\int_{\mathbb{R}^k} f(x)\sqrt{\det(\varphi'(x)\cdot(\varphi'(x))^T)}d\lambda^k(x) = \int_{\mathbb{R}^n}\int_{\varphi^{-1}(\{y\})} f(x)dH^{k-n}(x)d\lambda^k(y)$$

Věta 1.7

Let $f : \mathbb{R}^k \to \mathbb{R}$ be λ^k-integrable function. Then we have

$$\int_{\mathbb{R}^k} f(x)d\lambda^k(x) = \int_0^\infty \left(\int_{x\in\mathbb{R}^k, \|x\|=z} f(x)dH^{k-1}(x)\right)d\lambda^1(z).$$

Důkaz
By Coarea formula. □

2 Semicontinuous functions

Definice 2.1

Let X be a topological space and $f : X \to \mathbb{R}^*$. We say that f is lower semicontinuous (lsc), if the set $\{x \in X | f(x) > a\}$ is open for every $a \in \mathbb{R}$. We say that f is upper semicontinuous (usc) if the set $\{x \in X | f(x) < a\}$ is open for every $a \in \mathbb{R}$.

Tvrzení 2.1 (Fact)

$f : \mathbb{R} \to \mathbb{R}$:

$$f \text{ is lsc } \Leftrightarrow \forall x \in \mathbb{R} : \liminf_{t\to x} f(t) \geqslant x.$$

Věta 2.2

Let X be a metrizable topological space and $f : X \to \mathbb{R}^*$ be a function bounded from below. Then f is lsc if and only if there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbb{R}$ such that $f_0 \leqslant f_1 \leqslant \ldots$ and $f_n \to f$.

Důkaz
„ $\Longleftarrow$ “: Choose $a \in \mathbb{R}$. Assume that $f(x_0) > a$. There exists $k \in \mathbb{N}$ such that $f_k(x_0 > a)$. Then there is an open set $G \subset X$ such that $x_0 \in G$ and $f_k|_G > a$. Thus we have $f|_G \geqslant f_k|_G > a$. So $\{x \in X | f(x) > a\}$ is open.

„ $\Longrightarrow$ “ The case „$f \equiv \infty$“: Then we consider $f_n \equiv n$. The case „$f \not\equiv \infty$“. Fix a compatible metric ϱ on X. We set $f_n(x) = \inf\{f(y) + n \cdot \varrho(x, y) | y \in X\}$. Then we have $f_n : X \to \mathbb{R}$ and $f_0 \leqslant f_1 \leqslant \dots$. We have

$$|f_n(x) - f_n(z)| \leqslant n \cdot \varrho(x, z) \iff$$

$$\iff f_n(x) - f_n(z) \leqslant f(y) + n\cdot\varrho(x, y) - (f(y) + n\cdot\varrho(y, z)) + \varepsilon = n(\varrho(x, y) - \varrho(y, z)) + \varepsilon \leqslant n\cdot\varrho(x, z) + \varepsilon.$$

So f_n is continuous.

„$f_n \to f$“: There exists $K \in \mathbb{R}$ such that $f(x) \geqslant K$ for every $x \in X$. Fix $x \in X$. Choose $\varepsilon > 0$. For every $n \in \mathbb{N}$ we find $y_n \in X$ such that $f(y_n) \leqslant f(y_n) + n \cdot \varrho(x, y_n) \leqslant f_n(x) + \varepsilon$. Then we have

$$\varrho(x, y_n) \leqslant \frac{1}{n}\left(f_n(x) + \varepsilon - f(y_n)\right) \leqslant \frac{1}{n}\left(f_n(x) + \varepsilon - K\right).$$

$f_n(x) \to \infty \implies f(x) = \infty$, since $f_n(x) \leqslant f(x)$. $f_n(x)$ is bounded $\implies y_n \to x$, so we can find $n_0 \in \mathbb{N}$ such that $\forall n \geqslant n_0 : f(y_n) > f(x) - \varepsilon$. Then we have $f(x) < f(y_n) + \varepsilon \leqslant f_n(x) + 2\varepsilon$, $\lim f_n(x) \leqslant f(x) \leqslant \lim f_n(x) + 2\varepsilon$, thus $\lim f_n(x) = f(x)$. □

3 Function of Baire class 1

Definice 3.1

Let X and Y be metrizable topological spaces, a function $f : X \to Y$ is of Baire class 1 (B_1-function) if for every open set $U \subset Y$ the set $f^{-1}(U)$ is F_σ.

Věta 3.1 (Lebesgue–Hasudorff–Banach)

Let X be a metrizable topological space and $f : X \to \mathbb{R}$ be a B_1–function. Then there exists a sequence $\{f_n\}$ of continuous functions from X to $\mathbb{R}$ with $f_n \to f$.

Lemma 3.2

Let X be a metrizable topological space and $A \subset X$ be G_δ and F_σ. Then χ_A is point-wise limit of a sequence of continuous functions.

Důkaz
$A = \bigcup_{n\in\mathbb{N}} F_n$, $X \backslash A = \bigcup_{n\in\mathbb{N}} H_n$, $F_n \subseteq F_{n+1}$, $H_n \subseteq H_{n+1}$. By Urysohn lemma there exists continuous function $f_n : X \to [0, 1]$ such that $f_n|_{H_n} = 0$ and $f_n|_{F_n} = 1$. Then $f_n(x) \to f(x)$. □

Lemma 3.3

Let X be a metrizable topological space, $p_n : X \to \mathbb{R}$, $n \in \omega$, be a point-wise limit of a sequence of continuous functions. If the sequence $\{p_n\}$ converges uniformly to p, then p is point-wise limit of continuous functions.

Důkaz
Claim: If $q_n : X \to \mathbb{R}$, $n \in \omega$, is point-wise limit of continuous functions, $\|q_n\|_\infty \leqslant 2^{-n}$, then $\sum_{n=0}^{\infty} q_n$ is a point-wise limit of continuous functions.

Corollary: One can assume $\|p - p_n\|_\infty \leqslant 2^{-(n+1)}$. $p = p_0 + \sum_{n=0}^{\infty}(p_{n+1} - p_n)$

$$\|p_{n+1} - p_n\|_\infty \leqslant \|p_{n+1} - p\| + \|p - p_n\| < 2^{-(n+2)} + 2^{-(n+1)} < 2^{-n}.$$

Proof of claim: For every $n \in \omega$, there exists a sequence of continuous functions $\{q_i^n\}_{i=0}^{\infty}$ such that $q_i^n \to q_n$ and moreover we may assume $\|q_i^n\|_\infty \leqslant 2^{-n}$. We set $r_i = \sum_{n=0}^{\infty} q_i^n$. The sum converges uniformly, so r_i is continuous for every $i \in \omega$.

Set $x \in X$ and $\varepsilon > 0$. We find $N \in \omega$ such that

$$\left|\sum_{n=N+1}^{\infty} q_i^n(x)\right| < \frac{1}{2}\varepsilon, \left|\sum_{n=N+1}^{\infty} q_n(x)\right| < \frac{1}{2}\varepsilon.$$

Then we have

$$\left|r_i(x) - \sum_{n=0}^{\infty} q_n(x)\right| = \left|\sum_{n=0}^{\infty} q_i^n(x) - \sum_{n=0}^{\infty} q_n(x)\right| \leqslant \left|\sum_{i=0}^{N} q_i^n(x) - q_n(x)\right| + \left|\sum_{n=N+1}^{\infty} q_i^n(x) - \sum_{n=N+1}^{\infty} q_n(x)\right| \leqslant \left|\sum_{n=0}^{N}(q_i^n\right.$$

$$\limsup_{i\to\infty} |r_i(x) \sum_{n=0}^{\infty} q_n(x)| \leqslant \varepsilon \implies r_i(x) \to \sum_{n=0}^{\infty} q_n(x).$$

□

Lemma 3.4 (Reduction theorem for F_σ sets)

Let X be a metrizable topological space, $A_n \subset X$ be an F_σ set for every $n \in \omega$. Then there are F_σ sets $A_n^* \subset A_n$, such that $A_n^* \cap A_m^* = \emptyset$, whenever $n, m \in \omega$, $n \neq m$, and $\bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} A_n^*$.

Důkaz
$A_n = \bigcup_{j=0}^{\infty} A_{n,j}$, $A_{n,j}$ is closed. $k \mapsto (k', k'')$ bijection of ω onto $\omega \times \omega$. $Q_k = A_{(k)_0,(k)_j} \setminus \bigcup_{l<k} A_{(l)_0,(k)_1}$. $(Q_k)_{k\in\omega}$ is sequence of F_σ sets, which is disjoint. $A_n^* := \bigcup\{Q_k | (k)_0 = n\} \subseteq A_n$ is F_σ set, $A_n^* \cap A_m^* = \emptyset$ if $n \neq m$ and $\bigcup_{n=0}^{\infty} A_n^* = \bigcup_{k=0}^{\infty} Q_k = \bigcup_{n=0}^{\infty} A_n$. □

Důkaz (Of Lebesgue–Hasudorff–Banach theorem)
It is sufficient to prove result for $g : X \to (0,1)$. Because if $f \in B_1$, then we set $g = k \circ f$ where $k : \mathbb{R} \to (0,1)$ is homeomorphism. We find $g_n : X \to \mathbb{R}$, continuous and $g_n \to g$.

$\tilde{g}_n := \min\left\{\max\left\{\frac{1}{n}, g_n\right\}, 1-\frac{1}{n}\right\}$. $\tilde{g}_n(X) \subset (\frac{1}{n}, 1-\frac{1}{n})$.

Let $g : X \to (0,1)$ be B_1. For $N \in \omega$, $N \geqslant 2$, and $i \in [N-2]$ we set

$$A_i^N := g^{-1}\left(\frac{i}{N}, \frac{i+2}{n}\right) \ldots F_\omega, \qquad \bigcup_{i=0}^{N-2} A_i^N = X.$$

$B_i^N \subset A_i^N$ such that $\bigcup_{i=0}^{N-2} B_i^N = X$, B_i^N is F_σ and $B_i^N \cap B_{i'}^N = \varnothing$, whenever $i \neq i'$. $g_N(x) := \sum_{i=0}^{N-2} \frac{1}{N}\chi_{B_i^n}(x)$. $g_N \rightrightarrows g$ ($\|g - g_N\|_\infty \leqslant \frac{2}{N}$). □

Věta 3.5 (Baire)

Let X be a metrizable topological space, Y be separable metrizable topological space, and $f : X \to Y$ be B_1-function. Then the set of points of continuity of f is G_δ and residual.

Důkaz

$\{V_n\}$ open countable basis of Y. f is not continuous at $x \Leftrightarrow \exists n \in \omega : x \in f^{-1}(V_n) \setminus \operatorname{int} f^{-1}(V_n)$. $D(f) = \{x \in X | f \text{ is not continuous at } x\} = \bigcup_{n\in\omega} \underbrace{(f^{-1}(V_n) \operatorname{int} f^{-1}(V_n))}_{\in F_\omega}$.

$B = (f^{-1}(V_n) \operatorname{int} f^{-1}(V_n)) = \bigcup_{k\in\omega} F_{n,k}$ is closed and $\operatorname{int} F_{n,k} = \varnothing$, so $F_{n,k}$ is nowhere dense. So B is meager. And complement of meager is residual. □

TODO!!!

TODO!!!

TODO!!!

Důsledek

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded approximately continuous function. Then f has Darboux property and is in B_1.

Důkaz

The previous theorem gives that there exists a function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ for every $x \in \mathbb{R}$. So f has Darboux property.

„$f \in B_1$“: $f(x) = F'(x) = \lim_{n\to\infty} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}$. □

Věta 3.6

There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that the sets $\{x \in \mathbb{R} | f'(x) > 0\}$ and $\{x \in \mathbb{R} | f'(x) < 0\}$ are dense.

Důkaz
Let $A, B \subset \mathbb{R}$ be countable, dense, and disjoint. $A = \{a_n, n \in \mathbb{N}\}$, $B = \{b_n, n \in \mathbb{N}\}$. Observe that A and B are d-closed. Using theorem above we find for every $n \in \mathbb{N}$ approximately continuous g_n and h_n such that $g_n(a_n) = 1$, $0 \leqslant g_n \leqslant 1$, $g_n|_B = 0$, similarly $h_n(b_n) = 1$, $0 \leqslant h_n \leqslant 1$, $h_n|_A = 0$.

We define $\psi = \sum_{n=1}^{\infty} 2^{-n} g_n - \sum_{n=1}^{\infty} 2^{-n} h_n$. ψ is bounded. ψ is approximately continuous. ψ is positive on A and negative on B. By the previous theorem $\exists f : \mathbb{R} \to \mathbb{R}$ such that $f' = \psi$. □

Poznámka
We say that differentiable function g is of Köpcke type if g' is bounded and the sets $\{g' > 0\}$, $\{g' < 0\}$ are dense.

Poznámka
A and B are countable disjoint $\implies$ A and B are τ_d-closed. Towards contradiction assume that there exists $f : \mathbb{R} \to [0, 1]$ approximately continuous such that $f|_A = 0$ and $f|_B = 1$ $\implies$ $f \in B_1 \implies f$ has comeagerly many points of continuity.

4 More on derivatives

Definice 4.1 (Notation)

Let $I \subset \mathbb{R}$ be a nonempty open interval. We denote

$$\Delta'(I) = \{f : I \to \mathbb{R} | f \text{ has an antiderivative on I}\}$$

Věta 4.1 (Den?-Clarkson)

Let I be a nonempty open interval and $f \in \Delta'(I)$. Then f has Denj TODO!!!

TODO!!!