TODO(you should know).

TODO(motivation)

1 Sobolev spaces

Definice 1.1 (Multiindex)

α je multi-index $\equiv \alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{N}$. Length of multi-index α is $|\alpha| := \alpha_1 + \ldots + \alpha_d$. If $u \in C^k(\Omega)$ then $D^\alpha := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$, $\alpha \leqslant k$.

Definice 1.2 (Weak derivative)

Let $u, v_\alpha \in L^1_{loc}(\Omega)$ and α be a multi-index. We say that v_α is the α-th weak derivative of u in Ω iff $\forall \varphi \in C_0^\infty(\Omega) : \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int v_\alpha \varphi$.

Lemma 1.1

Weak derivative is unique. If the classical derivative exists then it is also the weak derivative.

Důkaz
Let v_α^1 and v_α^2 be two weak derivatives. Then

$$\int_\Omega (v_\alpha^1 - v_\alpha^2)\varphi = 0 \qquad \forall \varphi \in C_0^\infty(\Omega)$$

$\implies v_\alpha^1 = v_\alpha^2$ almost everywhere in Ω.

If classical $D^\alpha u$ exists, then

$$\int_\Omega \underbrace{D^\alpha u}_{v_\alpha} \varphi \overset{\text{BP}}{=} (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi.$$

□

Poznámka (Notation for this course)
D^α always means the weak derivative.

Definice 1.3 (Sobolev space)

Let $\Omega \subseteq \mathbb{R}^d$ be open, $k \in \mathbb{N}$, $p \in [1, \infty]$. We define $W^{k,p}(\Omega) = \{u \in L^p(\Omega) | \forall \alpha, |\alpha| \leqslant k : D^\alpha u \in L^p(\Omega)\}$.

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

$$\|u\|_{W^{k,\infty}(\Omega)} = \sup_{\alpha, |\alpha| \leqslant k} \|D^\alpha u\|_{L^\infty(\Omega)}$$

Lemma 1.2 (Base properties of Sobolev spaces)

Let $u, v \in W^{k,p}(\Omega)$, $k \in \mathbb{N}$ and α is multi-index. Then

- $D^\alpha u \in W^{k-|\alpha|,p}(\Omega)$, *if* $|\alpha| \leqslant k$;
- $\lambda u + \mu v \in W^{k,p}(\Omega)$ $\forall \lambda, \mu \in \mathbb{R}$ $(D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v)$;
- $\tilde{\Omega} \subset \Omega$ *open*, $u \in W^{k,p}(\tilde{\Omega})$;
- $\forall \eta \in C^\infty(\Omega) : \eta \cdot u \in W^{k,p}(\Omega)$.

TODO?

Věta 1.3 (Properties of Sobolev spaces)

$\Omega \subseteq \mathbb{R}^d$, $p \in [1, \infty]$, $k \in \mathbb{N}$:

1. $W^{k,p}(\Omega)$ *is a Banach space;*
2. *if* $p < \infty$, *then* $W^{k,p}(\Omega)$ *is separable;*
3. *if* $p \in (1, \infty)$, *then* $W^{k,p}(\Omega)$ *is reflexive.*

Důkaz (1.)
„Linear space“ is from Minkowski inequality. „Completeness“: u^n is Cauchy in $W^{k,p}(\Omega)$ $\implies$ $\exists u \in W^{k,p}(\Omega)$ $\implies$ $u^n \to u$ in $L^p(\Omega)$, $D^\alpha u^n \to v_\alpha$ in $L^p(\Omega)$ $\forall |\alpha| \leqslant k$. We must check that „$v_\alpha = D^\alpha u$“:

$$\int_\Omega v_\alpha \eta dx = \int_\Omega (v_\alpha - D^\alpha u^n)\eta + \int_\Omega D^\alpha u^n \eta =$$

$$\overset{IBP}{=} \int_\Omega (v_\alpha - D^\alpha u^n)\eta + (-1)^{|\alpha|} \int_\Omega u^n D^\alpha \eta = TODO?$$

$$|\int_\Omega (v_\alpha - D^\alpha u^n)\eta| \leqslant \|\eta\|_\infty \cdot \|v_\alpha - D^\alpha u^n\|_{L^p} \to 0.$$

□

Důkaz (2. + 3. for $W^{1,p}(\Omega)$)
$W^{1,p}(\Omega) = X \subseteq L^p(\Omega) \times \dots L^p(\Omega)$ ($d + 1$ times) and it is closed. □

1.1 Approximation of Sobolev functions

Věta 1.4

$\Omega \subseteq \mathbb{R}^d$ open, bounded. $k \in \mathbb{N}$, $p \in [1, \infty)$. Then

$$\overline{\mathcal{C}^\infty(\Omega)}^{\|\cdot\|_{k,p}} = W^{k,p}(\Omega).$$

Důkaz
TODO!!! □

Pozor

$$\overline{\mathcal{C}^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}} \neq W^{k,p}(\Omega).$$

Poznámka
If $\Omega \subset \mathbb{R}^d$ open, connected, then $u = \text{const} \Leftrightarrow \frac{\partial u}{\partial x_i} = 0 \; \forall_i = 1, \ldots, d$.

$W^{1,1}(I)$, I interval. Then $W^{1,1}(I) \hookrightarrow C(\overline{I})$.

Důkaz
„ $\Longrightarrow$ “: easy. „ $\Longleftarrow$ “: $u_\varepsilon = u * \eta_\varepsilon$, $\Omega_\varepsilon := \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

$$x \in \Omega_\varepsilon : \frac{\partial u_\varepsilon}{\partial x_1}(x) = \left(\frac{\partial u}{\partial x_i}\right)_\varepsilon (x) = 0 \implies u_\varepsilon \equiv \text{const in } \Omega_\varepsilon.$$

Fix $\varepsilon_0 > 0$: $\varepsilon \leqslant \varepsilon_0$: $u_\varepsilon \to u$ in $W^{1,1}(\Omega_{\varepsilon_0}) \implies u \equiv \text{const in } \Omega_{\varepsilon_0}$. $u \in W^{1,1}(I)$.

$$\tilde{u}(x) := \int_0^x \frac{\partial u(y)}{\partial y} dy, \qquad \|\tilde{u}(x)\|_\infty \leqslant \int_0^1 |\nabla u| dx.$$

Aim $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}$. $\eta \in C_0^\infty(0,1)$:

$$\int_0^1 \tilde{u}(x) \frac{\partial \eta}{\partial x}(x) dx = \int_0^1 \int_0^1 \frac{\partial u(y)}{\partial y} \frac{\partial \eta(x)}{\partial x} \chi_{\{0 \leqslant y \leqslant x\}} dx dy =$$

$$= \int_0^1 \int_y^1 \frac{\partial u}{\partial y}(y) \frac{\partial \eta(x)}{\partial x_i} dx dy =$$

$$= -\int_0^1 \frac{\partial u(y)}{\partial y} \eta(y) dy.$$

$\implies \tilde{u} - u = \text{const} =: c$.

$$|u(x_1) - u(x_2)| = |\tilde{(}x_1) - c - \tilde{u}(x_2) + c| = |\tilde{u}(x_1) - \tilde{u}(x_2)| \leqslant \int_{x_1}^{x_2} \left|\frac{\partial u}{\partial y}\right| dy \to 0.$$

$\implies C(I)$.

„$\|u\|_\infty \leqslant K \cdot \|u\|_1$“:

$$|c| = \int_0^1 |c| = \int_0^1 |\tilde{u}(x) - u(x)| \leqslant \|\tilde{u}\|_\infty + \|u\|_1 \leqslant \|u\|_{1,1}.$$

□

$W^{d,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ (for Lipschitz domain Ω).

1.2 Characterization of Sobolev functions

Věta 1.5

Let $\Omega \subset \mathbb{R}^d$, $p \in [1, \infty]$, $\Omega_d := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \delta\}$. 1. Then

$$u \in W^{1,p}(\Omega) : \|\triangle_i^n u\|_{L^p(\Omega_\delta)} \leqslant \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)},$$

where $\triangle_i^n u(x) = \frac{u(x+he_i)-u(x)}{h}$.

2. *If* $\forall h, i, \delta : \|\triangle_i^h u\|_{L^p(\Omega_\delta)} \leqslant c_i$ $(p > 1)$*. Then*

$$\exists \frac{\partial u}{\partial x_i}, \qquad \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leqslant c_i.$$

Pozor
This works only for $(p > 1)$.

Definice 1.4 (Domains of class $C^{k,\alpha}$)

Let $\Omega \subseteq \mathbb{R}^d$ open bounded set. We say that $\Omega \in C^{k,\mu}$ $(\partial\Omega \in C^{k,\mu})$ iff:

- there exist M coordinate systems $\mathbf{x} = (x_{r_1}, \ldots, x_{r_d}) = (x'_r, x_{r_d})$ and functions $a_r : \Delta_r \to \mathbb{R}$ where $\Delta_r = \left\{x'_r \in \mathbb{R}^{d-1} \middle| \, |x_{r_i}| \leqslant \alpha\right\}$ such that $a_r \in C^{k,\mu}(\Delta_r)$,
- denoting Tr the orthogonal transformation from (x'_r, x_{r_d}) to (x', x_d), then $\forall x \in \partial\Omega$ $\exists r \in \{1, \ldots, M\}$ such that $x = Tr\left(x'_{r_1}, a(x_{r_d})\right)$,
- $\exists \beta > 0$, if we define

$$V_r^+ := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) < x_{r_d} < a(x'_r) + \beta\right\}$$

$$V_r^- := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) - \beta < x_{r_d} < a(x'_r)\right\}$$

$$\Lambda_r := \left\{(x'_r, x_{r_d}) \in \mathbb{R}^d | x'_r \in \Delta_r, a(x'_r) = x_{r_d}\right\}$$

Then $Tr(V_r^+) \subset \Omega, Tr(V_r^-) \subset \mathbb{R}^d \backslash \overline{\Omega}, Tr(\Lambda_r) \subseteq \partial\Omega$ and $\bigcup_{r=1}^M Tr(\Lambda_r) = \partial\Omega$.

Věta 1.6 (Density)

Let $\Omega \in C^{0,1}$ *and* $p \neq \infty$*, then* $W^{k,p}(\Omega) = \overline{C^\infty(\overline{\Omega})}^{\|\cdot\|_{k,p}}$.

Věta 1.7 (Extension)

Let $\Omega \in C^{0,1}$*,* $k \in \mathbb{N}$*,* $p \in [1, \infty]$*. Then* $\exists$ *continuous bounded operator* $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ *such that*

1. $\|Eu\|_{W^{k,p}}(\mathbb{R}^d) \leqslant c \cdot \|u\|_{W^{k,p}(\Omega)}$ *(Eu has compact support);*
2. $Eu = u$ *almost everywhere in* Ω*.*

Věta 1.8 (Trace)

Let $\Omega \in C^{0,1}$*,* $p \in [1, \infty]$*. Then* $\exists$ *continuous bounded operator* $\mathrm{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ *such that:*

1. $\|\operatorname{tr} u\|_{L^p(\partial\Omega)} \leqslant c \cdot \|u\|_{W^{1,p}(\Omega)}$*;*

2. $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \implies \operatorname{tr} u = u|_{\partial\Omega}$.

Definice 1.5

$W_0^{k,p}(\Omega) = \overline{u \in C_0^\infty(\Omega)}^{\|\cdot\|_{k,p}}$.

Poznámka
$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) |\operatorname{tr} u = 0\}$.

TODO!!!

1.3 Existence theory via Lax–Milgram

Definice 1.6 (Eliptic forms)

Let $B : V \times V \to \mathbb{R}$ a linear form and V be a Hilbert space. We say that B is elliptic iff

1. B is V-bounded, it is $\exists c_2\ \forall u, \varphi \in V : |B(u, \varphi)| \leqslant c_2 \|u\|_V \cdot \|\varphi\|_V$;
2. B is V-coercive, it is $\exists c_1 > 0\ \forall u \in V : B(u, u) \geqslant c_1 \|u\|_V^2$.

Věta 1.9 (Lax–Milgram)

Let B be linear and satisfying two previous properties. Then $\forall F \in V^*\ \exists! u \in V : \forall \varphi \in V : B(u, \varphi) = \langle F, \varphi\rangle$.

Definice 1.7 (Lipschitz, unifomly monototne)

Let $B : V \to V^*$. We say that B is

1. Lipschitz iff $\forall u, v \in V : \|B(u) - B(v)\|_{V^*} \leqslant \overline{c_2} \|u - v\|_V$;
2. uniformly monotone iff $\forall u, v \in V : \langle B(u) - B(v), u - v\rangle \geqslant \overline{c_1} \|u - v\|_V^2$.

Věta 1.10 (Non-linear Lax–Milgram)

Let V be a Hilbert space, $B : V \to V^*$ be Lipschitz and uniformly monotone. Then $\forall F \in V^*\ \exists! u \in V : \forall \varphi \in V : \langle B(u), \varphi\rangle = \langle F, \varphi\rangle$. ($B(u) = F$.)

Důkaz (Lax–Milgram by using non-linear version)
Define $B : V \to V^* : \langle B(u), \varphi\rangle =: B(u, \varphi)$. We show that B is Lipschitz and uniformly monotone:

$$\|B(u) - B(v)\|_{V^*} = \sup_{\varphi \in V, \|\varphi\| \leqslant 1} \langle B(u) - B(v), \varphi\rangle = \sup_\varphi (B(u, \varphi) - B(v, \varphi)) \overset{\text{linear}}{=} \sup_\varphi B(u - v, \varphi) \overset{\text{bounded}}{\leqslant} \sup_\varphi c_2 \|$$

$$\langle B(u) - B(v), u - v\rangle = B(u, u - v) - B(v, u - v) = B(u - v, u - v) \geqslant c_1 \|u - v\|_V^2.$$

So $\forall F^* \ \exists! u \in V : B(u,\varphi) = \langle B(u), \varphi \rangle = \langle F, \varphi \rangle \ (\forall \varphi \in V)$. □

Důkaz (Non-linear Lax–Milgram)
„Uniqueness“: $u_1 \neq u_2$:

$$\forall \varphi : \langle B(u_1), \varphi \rangle = \langle B(u_2), \varphi \rangle = \langle F \rangle \implies \forall \varphi \in V : \langle B(u_1) - B(u_2), \varphi \rangle = 0.$$

$$\varphi = u_1 - u_2 \implies 0 = \langle B(u_1) - B(u_2), u_1 - u_2 \rangle \geqslant \overline{c_1} \|u_1 - u_2\|_V^2 \implies u_1 = u_2.$$

„Existence“: $\forall \langle B(u), \varphi \rangle = \langle F, \varphi \rangle \Leftrightarrow \exists \varepsilon > 0 : (u,\varphi)_V = (u,\varphi)_V - \varepsilon \left(\langle B(u), \varphi \rangle - \langle F, \varphi \rangle \right)$. Desire $M : V \to V, v \mapsto u$:

$$(u,v)_V = (v,\varphi)_V - \varepsilon \left(\langle B(v), \varphi \rangle - \langle F, \varphi \rangle \right).$$

If M is well-defined and if it has a fixed point then we find solution.

„M well defined“:

$$\forall v \in V \ \exists \tilde{F} \in V^* \left\langle \tilde{F}, \varphi \right\rangle_V = (v,\varphi)_V - \varepsilon \left(\langle B(v), \varphi \rangle - \langle F, \varphi \rangle \right).$$

Riesz $\implies \exists! u \in V (u,\varphi) = \left\langle \tilde{F}, \varphi \right\rangle$.

„M contraction“: We want bound $\|M(u) - M(v)\|_V^2$.

$$M(u) = \overline{u}, \qquad (\overline{u}, \varphi) = (u, \varphi) - \varepsilon \left(\langle B(u), \varphi \rangle - \langle F, \varphi \rangle \right).$$

$$M(v) = \overline{v}, \qquad (\overline{v}, \varphi) = (v, \varphi) - \varepsilon \left(\langle B(v), \varphi \rangle - \langle F, \varphi \rangle \right).$$

$$(\overline{u} - \overline{v}, \varphi) = (u - v, \varphi) - \varepsilon \langle B(u) - B(v), \varphi \rangle.$$

Riesz: $\exists! w_1 : \langle B(u), \varphi \rangle = (w_1, \varphi)$ and $\exists! w_2 : \langle B(v), \varphi \rangle = (w_2, \varphi)$.

$$(*) \|\overline{u} - \overline{v}\|_V^2 = \|u - v - \varepsilon(w_1 - w_2)\|_V^2 = \|u - v\|_V^2 + \varepsilon^2 \|w_1 - w_2\|_V^2 - 2\varepsilon (w_1 - w_2, u - v).$$

$(w_1 - w_2, w_1 - w_2) = \|w_1 - w_2\|_V^2 = \langle B(u) - B(v), w_1 - w_2 \rangle \leqslant \|B(u) - B(v)\|_{V^*} \cdot \|w_1 - w_2\|_V \implies \|w_1 - w_2\|_V$

$$(w_1 - w_2, u - v) = \langle B(u) - B(v), u - v \rangle \overset{\text{Uniformly monotone}}{\geqslant} c_1 \|u - v\|_V^2$$

By (*):

$$\|M(u) - M(v)\|_V^2 \leqslant \|u - v\|_V^2 + \varepsilon^2 \overline{c}_2^2 \|u - v\|_V^2 - 2\varepsilon \overline{c}_1 \|u - v\|_V^2 =$$
$$= (1 + \varepsilon^2 \overline{c}_2 - 2\varepsilon \overline{c}_1) \|u - v\|_V^2.$$

$\implies$ M is contraction and has a fixed point (for ε such that this constant is less than 1, so $\varepsilon(\varepsilon c_2^2 - 2c_1) < 0$, so $0 < \varepsilon < \frac{2c_1}{c_2^2}$). □

Věta 1.11

Let $B_{L,\sigma}$ be bilinear, V-bounded and V-elliptic. Then $\exists! u$ weak solution:

$$u_0 \in W^{1,2}, \qquad u - 0 \in V, \qquad B_{L,\sigma}(u, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi.$$

Důkaz

„Uniqueness“: $u_1, u_2 \implies u_1 - u_0 \in V, u_2 - u_0 \in V$ TODO!!!

„Existence“: $w \in V$ $(u - u_0 = w)$:

$$B_{L,\sigma}(w, \varphi) = \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi - B_{L,\sigma}(u_0, \varphi) := \left\langle \overline{F}, \varphi \right\rangle.$$

Find $w \in V$.

$$\left\langle \overline{F}, \varphi \right\rangle := \langle f, \varphi \rangle + \int_{\Gamma_2 \cup \Gamma_3} g\varphi + \int_\Omega -A\nabla u \cdot \nabla\varphi - bu_0\varphi - \mathbf{c} \cdot \nabla u_0 \varphi + \mathbf{d} \cdot \nabla\varphi u_0 - \int_{\Gamma_2} \sigma u_0 \varphi.$$

Is $\overline{F}$ in V^*? $(\varphi \in V \subseteq W^{1,2}(\Omega))$

$$|\left\langle \overline{F}, \varphi \right\rangle| \leqslant \|\varphi\|_V c \left(\|f\|_{V*} + \|g\|_{L^2(\partial\Omega)} + \|A\|_\infty \cdot \|u_0\|_{1,2} + \|b\|_\infty \|u_0\|_2 + \|c\|_\infty \|u_0\|_{1,2} + \dots \right).$$

□

TODO!!!

1.4 Existence theory via Fredholm alternative

Lemma 1.12 (Fredholm alternative)

Let H be a Hilbert space and $K : H \to H$ be linear compact.

- *F1* $\mathrm{Ker}(I - K)$ *has finite dimension* $(u \in \mathrm{Ker}(I-K) \Leftrightarrow (I-K)(u) = 0)$;
- *F2* $\mathrm{Rng}(I - K)$ *is closed* $(u \in \mathrm{Rng}(I-K) \Leftrightarrow \exists w \in H (I-K)w = u)$;
- *F3* $\mathrm{Rng}(I - K) = (\mathrm{Ker}(I - K^*))^\perp$ $(u \in \mathrm{Rng}(I-K), w \in \mathrm{Ker}(I-K^*) \Leftrightarrow (u, w) = 0)$;
- *F4!* $\mathrm{Ker}(I - K) = \{\mathbf{o}\} \Leftrightarrow \mathrm{Rng}(I - K) = H$;
- *F5* $\dim(\mathrm{Ker}(I - K)) = \dim(\mathrm{Ker}(I - K^*)) < \infty$;
- *F6* *spectrum of K is at most countable and if it is infinite then zero is the only attracting point.*

Věta 1.13 (Fredholm alternative for PDR)

Let $\Omega \in C^{0,1}$, $u_0 = 0$ *and* $\Gamma_1 = \partial\Omega$ *and* L *be an elliptic operator.*

1. *Either* $\forall f \in L^2(\Omega)\ \exists! u \in W_0^{1,2}(\Omega) : Lu = f$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$, *or* $\exists u \neq 0 : Lu = 0$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$.
2. $N_L := \{u \in V | Lu = \mathbf{o}\}$: $B_L(u,\varphi) = 0\ \forall \varphi \in W_0^{1,2}(\Omega)$, $N_{L*} := \{\varphi \in V | L^*\varphi = \mathbf{o}\}$. *Then* N_L *and* N_{L*} *are closed subspaces of* $W_0^{1,2}(\Omega)$, $\dim N_L = \dim N_{L*} < \infty$.
3. *For* $f \in L^2(\Omega)$: $(\exists u \in W_0^{1,2} : Lu = f) \Leftrightarrow (\forall \varphi \in N_{L*} : \int_\Omega f\varphi = 0)$.

Where

$$Lu = -\operatorname{div}(A\nabla u) + \mathbf{c} \cdot \nabla u + \operatorname{div}(\mathbf{d}u) + bu,$$

$$L^*\varphi = -\operatorname{div}(A^T\nabla\varphi) - \mathbf{d} \cdot \nabla\varphi - \operatorname{div}(\mathbf{c}\varphi) + b\varphi.$$

$$Lu = f \Leftrightarrow \forall\varphi : B_L(u,\varphi) = \int_\Omega f\varphi, \qquad L^*\varphi = g \Leftrightarrow \forall u : B_L(\varphi,u) = \int_\Omega gu.$$

Důkaz

From Lax–Milgram $\exists p > 0$:

$$\forall f \in L^2\ \exists! u \in W_0^{1,2}(\Omega) L_p u := Lu + pu = f.$$

$$B_{L,p}(u,\varphi) = B_L(u,\varphi) + p \cdot \int_\Omega u\varphi.$$

□