Poznámka
Credit for giving 'small lecture'. Oral exam.

1 Meromorphic functions

Definice 1.1

We say that a function f is holomorphic in a set $F \subset \mathbb{C}$ if there is an open $G \supseteq F$ such that f is holomorphic on G.

In particular, f is holomorphic at $z_0 \in \mathbb{C}$ if f is holomorphic in some neighbour ($= U(z_0) = U(z_0, \varepsilon)$) of z_0.

Definice 1.2

Function f has at ∞ a removable singularity, if $f\left(\frac{1}{z}\right)$ has a removable singularity at 0. Similarly pole and essential singularity.

Function f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Let $G \subset \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any z_0. Denote $\mathcal{H}(G) := \{f : G \to \mathbb{C} | f \text{ holomorphic}\}$.

Například
From Liouville theorem $\mathbb{H}(\mathbb{S})$ = constant functions. So $\mathbb{H}(G)$ is interesting only for $G \subsetneq \mathbb{S}$, so WLOG $G \subset \mathbb{C}$.

Definice 1.3 (Meromorphic function)

Let $G \subset \mathbb{S}$ be open. Then a function f on G is called meromorphic if at any $z_0 \in G$ the function f is either holomorphic at z_0 or has a pole at z_0.

Denote $\mathcal{M}(G)$ the set of meromorphic functions on G.

Důsledek

- $\mathcal{H}(G) \subset \mathcal{M}(G)$.
- Denote $P_f := \{z_0 \in G | f \text{ has a pole at } z_0\}$. Then P_f has no limit points in G.
- If $f = \infty$ on P_f, then $f : G \to \mathbb{S}$ is continuous. (We always assume, that $f \in \mathcal{H}(G)$ has this property.)

Například

$$\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), \qquad e^{\frac{1}{z}} \notin \mathcal{M}(\mathbb{C}), \qquad \Gamma \in \mathcal{M}(\mathbb{C}), \qquad \zeta \in \mathcal{M}(\mathbb{C}).$$

$\mathcal{M}(\mathbb{S})$ = rational functions. (One inclusion is clear, second: Let $f \in \mathcal{M}(\mathbb{S})$, then because $\mathbb{S}$ is compact it holds that P_f is finite (has no limit point), $P_f \cap \mathbb{C} = \{z_1, \dots, z_n\}$, so from theorem from last semester there exists $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^n p_j \left(\frac{1}{z-z_j}\right)$ for some polynomials p_j. f has removable singularity or pole at infinity and p_j and $\frac{1}{z-z_j}$ have removable singularity there, so $h(z)$ is polynomial, otherwise $h(z)$ has infinity Taylor polynom and $h\left(\frac{1}{z}\right)$ has essential singularity at 0.)

So $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subset \mathbb{C}$.

If $G \subset \mathbb{C}$ is domain, $f, g \in \mathbb{H}(G)$ and $g \equiv 0$, then $f/g \in \mathcal{M}(G)$. The inverse is also true (we will prove it) (but not for $G = \mathbb{S}$).

Lemma 1.1

Let $\mathbb{G} \subset \mathbb{C}$ be open. Then there are compacts K_n, $n \in \mathbb{N}$, in G such that $G = \bigcup_{n=1}^\infty K_n$, $K_n \subset \operatorname{int}(K_{n+1})$ and for any compact K in G, $\exists n \in \mathbb{N} : K \in K_n$.

Důkaz
Set $K_n := \left\{z \in G | \operatorname{dist}(z, \mathbb{C} \backslash G) \geqslant \frac{1}{n}\right\} \cap U(0, n)$. □

Tvrzení 1.2

Let $G \subset \mathbb{S}$ be open and $M \subset G$ has no limit point in G. Then

- *$G \backslash M$ is open;*
- *if K is a compact in G, then $K \cap M$ is finite. In particular for $G = \mathbb{S}$ we have M is finite;*
- *M is at most countable. If M is infinite, then $\varnothing \neq M' \subset \partial G$;*
- *if $G \subset \mathbb{C}$ is domain (connected), then $G \backslash M$ is domain.*

Věta 1.3 (Uniqueness of meromorphic functions)

Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ and $f \not\equiv 0$. Then $N_f := \{z \in G | f(z) = 0\}$ has no limit points in G.

Důkaz
We know this holds for holomorphic functions. Set $G_0 := G \backslash P_f$. Then $G_0 \subset \mathbb{C}$ is also domain and $f \in \mathcal{H}(G)$ and $f \not\equiv 0$ on G_0. Then $N_f \subset G_0$ has no limit points in G_0, nor in P_f. □

Věta 1.4 (Residue theorem)

Let $G \subset \mathbb{C}$ be open, φ be a closed curve (or cycle) in G and $\operatorname{int}\varphi := \{z_0 \in \mathbb{C}\backslash\langle\varphi\rangle \mid \operatorname{ind}_\varphi z_0 \neq 0\} \subset G$. Let $M \subset G\backslash\langle\varphi\rangle$ be finite and $f \in \mathcal{H}(G\backslash M)$. Then $\int_\varphi f = 2\pi i \cdot \sum_{s\in M} \operatorname{ind}_\varphi s \cdot \operatorname{res}_s f$.

Poznámka
This holds true even if instead of finiteness of M, we assume only that $M \subset G\backslash\langle\varphi\rangle$ has no limit points in G. Indeed, we have $M_0 = M \cap \operatorname{int}\varphi$ is finite, because $\langle\varphi\rangle \cup \operatorname{int}\varphi$ is compact and $G_0 := G\backslash(M\backslash M_0)$ is open and f is holomorphic on $G_0\backslash M_0$ and by R. theorem for G_0 and M_0 we get $\int_\varphi f = 2\pi i \sum_{s\in M_0} \operatorname{res}_s f \cdot \operatorname{ind}_\varphi s$.

1.1 Logarithmic integrals

Definice 1.4 (Logarithmic integral)

Let $\varphi : [a,b] \to \mathbb{C}$ be a (regular) curve and let f be a non-zero holomorphic function on $\langle\varphi\rangle$. Then we define logarithmic integrals integral as

$$I := \frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \frac{1}{2\pi i}\int_a^b \frac{f'(\varphi(t))\varphi'(t)}{f(\varphi(t))}dt = \frac{1}{2\pi i}\int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))}dt = \frac{1}{2\pi i}\int_{f\circ\varphi}\frac{dz}{z} = \frac{1}{2\pi i}(\Phi(b)-\Phi(a)),$$

where Φ is a branch (jednoznačná větev) of logarithm of $f \circ \varphi$. If φ is, in addition, closed, then $I = \operatorname{ind}_{f\circ\varphi} 0 = \frac{1}{2\pi}(\Theta(b) - \Theta(a)) \in \mathbb{Z}$, where Θ is a branch of argument of $f \circ \varphi$.

($\frac{f'}{f}$ is called logarithmic derivative of f, because '$(\log f)' = \frac{f'}{f}$'.)

Věta 1.5 (Argument principle)

Let $G \subseteq \mathbb{C}$ be a domain, φ be a closed curve in G and $f \in \mathcal{M}(G)$. Let $\operatorname{int}\varphi \subset G$ and $\langle\varphi\rangle \cap N_f = \emptyset$, $\langle\varphi\rangle \cap P_f = \emptyset$. Then

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{s\in\operatorname{int}\varphi, f(s)=0} n_f(s)\cdot\operatorname{ind}_\varphi s - \sum_{s\in\operatorname{int}\varphi, f(s)=\infty} p_f(s)\cdot\operatorname{ind}_\varphi s,$$

where $n_f(s)$ is multiplicity of the zero point s of f and $p_f(s)$ is multiplicity of the pole s of f.

Důkaz
By Residua theorem, we have

$$\frac{1}{2\pi i}\int_{\varphi}\frac{f'}{f} = \sum_{s\in\operatorname{int}\varphi, s\in N_f\cup P_f} \operatorname{res}_s\left(\frac{f'}{f}\right)\cdot \operatorname{ind}_{\varphi} s.$$

If $s \in N_f$ then on $P(s)$:

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = n_f(s).$$

If $s \in P_f$ then on $P(s)$

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = -p_f(s).$$

□