Poznámka (Literature)
Kechris.

Definice 0.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka
Complete compatible metric is not unique: $\tilde{\varrho} = \min\{1, \varrho\}$.

Například
$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^n$, $\mathbb{C}^n$, $2 := \{0, 1\}$, $\omega := \{0, 1, 2, \ldots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, 2^ω, ω^ω (both with product topology).

Věta 0.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X.

Důkaz
Without proof. (We should know it already.) □

Věta 0.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X, such that $\operatorname{diam}(F_n) \to 0$. *Then* $|\bigcap F_n| = 1$.

Důkaz
Without proof. (We should know it already.) □

Věta 0.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n\in\omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow$ $H \in \mathcal{G}_\delta(X)$

Důkaz ((i))
Let d_n be CCM (complete compatible metric) on X_n, $n \in \omega$. Then

$$d(x, y) := \sum_{n=0}^{\infty} \min\left\{2^{-n}, d_n(x_n, y_n)\right\}$$

is CCM on $X = \prod_{n\in\omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „d is metric“ straightforward, „d is complete“ also easy, compatibility too). □

Důkaz ((ii))
$H = \emptyset$, $H = X$ trivial. Assume $H \neq \emptyset, X$.

„$\implies$“: Fix CCM ϱ on H. $V_n := \bigcup\{V \subset X | V \text{ open in } X \wedge V \cap H \neq \emptyset \wedge \operatorname{diam}_\varrho(V \cap H) < 2^{-n}\}$, $n \in \omega$. We want to show $H \stackrel{?}{=} \bigcap_{n\in\omega}(V_n \cap \overline{H}) \in \mathcal{G}_\delta$:

$$\text{„}\subseteq\text{“}: x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n.$$

„$\supseteq$“: $x \in V_n \cap \overline{H}$ for every $n \in \omega \implies \exists$ open sets G_n: $x \in G_n$, $G \cap H \neq \emptyset$, $\operatorname{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \stackrel{?}{\neq} \emptyset \impliedby x \in G_n \cap G_{n+1} \cap \overline{H} \neq \emptyset$).

$\{y\} := \bigcap_{n\in\omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O$, $G_n \cap H \subset B(y, 2^{-n})$, $n \in \omega$. $\implies \exists n \in \omega G_n \cap H \subset O$, $x \in G_n \cap (X \backslash \overline{O}) \cap \overline{H} \implies G_n \cap (X \backslash \overline{O}) \cap H \neq \emptyset$.

„$\impliedby$“: fix CCM d on X, $H = \bigcap_{n\in\omega} U_n$, $\emptyset = U_n \neq X$. $F_n := X \backslash U_n$, $\tilde{d}(x,y) = d(x,y) + \sum_{n=0}^{\infty} \min\left\{2^{-n}, \left|\frac{1}{\operatorname{dist}(x,F_n)} - \frac{1}{\operatorname{dist}(y,F_n)}\right|\right\}$, $x, y \in H$. Next we verified that $\tilde{d}$ is metric, that $\tilde{d}$ is equivalent with d on H (by convergence), and that $(H, \tilde{d})$ is complete metric space and separable. TODO? □

Definice 0.2 (Notation)

$A \neq 0$:

- $A^{<\omega}$:= finite sequence of elements of $A = \bigcup_{n\in\omega} A^n$;
- $s \in A^k$, $t \in A^{<\omega} \cup A^\omega$: $s^\wedge t := (s_0, s_1, \ldots, s_{k-1}, t_0, t_1, \ldots)$, where $s = (s_0, \ldots, s_{k-1})$, $t = (t_0, t_1, \ldots)$;
- $s \in A^{<\omega} \cup A^\omega$: $|s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega$, $k \in \omega$, $|s| \geqslant k$, then we denote restriction of s on first k elements as s/k;
- $s \prec t$ iff $|t| \geqslant |s|$ and $s = t/|s|$ ($s \in A^{<\omega}$, $t \in A^{<\omega} \cup A^\omega$).

1 Baire space ω^ω

Definice 1.1

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega | s \prec \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \backslash \bigcup\{\mathcal{N}(t) | |t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N}|s \in \omega^{<\omega}\}$ is base of topology of ω^ω.

Věta 1.1 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz
Bez důkazu. □

Důsledek
ω^ω is homeomorphic to $\mathbb{R}\backslash\mathbb{Q}$.