

*Poznámka*

There will be homework. We will discuss it on practicals (particular solutions are good).

*Poznámka* (What it is about)

Functional analysis generalizes Linear Algebra. This lecture generalizes (real) Analysis in  $\mathbb{R}^n$  ( $Df(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear) by replacing  $\mathbb{R}^n$  with Banach spaces.

*Příklad* (Calculus of variations)

Know things:  $f : \mathbb{R} \rightarrow \mathbb{R}$ , differentiable has minimizer at  $x_0 \in \mathbb{R} \implies f'(x_0) = 0$  (in  $\mathbb{R}^n$ :  $Df(x_0) = 0$ ). Generalize it:

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*Řešení*

Trick: For example  $F : u \mapsto \int_{\Omega} \frac{1}{2} |\nabla u|^2 - f u dx$ ,  $W_g^{1,2}(\Omega) \rightarrow \mathbb{R}$  ( $g$  means bounded values). For any  $\varphi \in W_0^{1,2}(\Omega)$  consider  $\varepsilon \mapsto F(u + \varepsilon\varphi)$ ,  $\mathbb{R} \rightarrow \mathbb{R}$ .

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} F(u + \varepsilon\varphi) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_{\Omega} \frac{1}{2} |\nabla u + \varepsilon \nabla \varphi|^2 - f \cdot (u + \varepsilon\varphi) dx = \\ &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \left[ \int_{\Omega} \frac{1}{2} |\nabla u|^2 - f u dx + \varepsilon \int_{\Omega} \nabla u \nabla \varphi - f \varphi dx + \varepsilon^2 \int_{\Omega} \frac{1}{2} |\nabla \varphi|^2 dx \right] = \\ &= \int_{\Omega} \nabla u \nabla \varphi - f \varphi. \end{aligned}$$

Assume  $u \in W^{2,2}(\Omega)$ :

$$\int_{\partial\Omega} \overset{\text{P.I.}}{\frac{\partial u}{\partial n}} \varphi dx - \int_{\Omega} (\Delta u + f) \varphi dx \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

$$\underset{\text{Fundamental lemma}}{\Delta} u + f = 0.$$

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*Příklad* (Mapping degree)

Consider  $f \in \mathcal{C}([-1, 1]; \mathbb{R})$ . How many zeroes does  $f$  have? Let assume  $f(-1) < 0 < f(1)$ . Let assume  $f \in \mathcal{C}^1$ . And 0 is a regular value ( $f(x_0) = 0 \implies f'(x_0) \neq 0$ ).

Řešení

From 0 to  $\infty$ . After assumption: by intermediate value theorem at least 1. After second assumption: odd and finitely many. Moreover, the number of zeros with positive derivative minus the number of zeros with the negative one is 1, which is called degree of  $f$ .

Observation: In one dimension  $\deg(f) \in \{-1, 0, 1\}$ .  $\deg(f)$  is invariant under perturbations.  $\deg f$  depends on boundary values. Can be extended from  $\mathcal{C}^1$  to  $\mathcal{C}$  (we take smooth perturbation).

Ad second observation: homotopy:  $h : [0, 1] \times [-1, 1] \rightarrow \mathbb{R}$ ,  $(s, x) \mapsto h_s(x)$  continuous  $h_0 = f$ ,  $h_1 = g$ . And it is admissible if  $h_s(-1) \neq 0$  and  $h_s(1) \neq 0$  for all  $s$ .

There is generalization to  $\mathbb{R}^n$ , to Manifolds, and to Banach spaces. And we get „corollaries“: Fix point theorems, topological statements, inability to comb a hedgehog,

# 1 Derivatives in Banach spaces

## 1.1 The notion of a derivative

*Poznámka* (In  $\mathbb{R}^n$ )

Partial derivative, directional derivative, total derivative.

### Definition 1.1 (Directional and Gateaux derivative)

Let  $X, Y$  be Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$ . For any  $x_0 \in A$ ,  $v \in X$  if

$$\frac{\partial f}{\partial v}(x_0) := \lim_{h \rightarrow 0} \frac{f(x_0 + hv) - f(x_0)}{h}$$

exists, we call it directional derivative (at  $x_0$ , in direction  $v$ ).

If  $v \mapsto \frac{\partial f}{\partial v}(x_0)$  is a continuous linear operator from  $X$  to  $Y$ , we denote it by  $\partial f(x_0)$  and call it the Gateaux derivative (at  $x_0$ ).

*Poznámka* (Notation)

Some authors omit continuous and linear, i.e. for them directional  $\Leftrightarrow$  Gateaux.

Some use  $df$  or  $Df$  instead of  $\partial f$ .

We will write  $\frac{\partial f}{\partial v}(x_0) = \partial f(x_0) \langle v \rangle$ . ( $\langle \cdot \rangle$  for linear arguments.)

### *Například*

Consider  $F : L^2([0, 1]) \rightarrow L^2([0, 1])$ ,  $u \mapsto F(u)$ ,  $F(u)(x) := \sin(u(x))$ . It is continuous ( $\|F(u) - F(v)\|_{L^2}^2 = \int |\sin(u(x)) - \sin(v(x))|^2 \leq \int |u(x) - v(x)|^2$ ). Fix  $\varphi \in L^2([0, 1])$  and calculate:

$$\frac{\partial F}{\partial \varphi}(u) = \lim_{h \rightarrow 0} \frac{\sin(u(\cdot) + h\varphi(\cdot)) - \sin(u(\cdot))}{h} = \cos(u(\cdot)) \cdot \varphi(\cdot)$$

point-wise almost everywhere and by domain convergence everywhere.

$\frac{\partial F}{\partial \varphi}$  is linear in  $\varphi$  and bounded  $\implies F$  is Gateaux differentiable. Consider  $u \mapsto \frac{\partial F}{\partial \varphi}(u)$  for fixed  $\varphi$ . It is continuous.

Is  $\partial F$  a good linear approximation? I.e.  $\|F(u + \varphi) - F(u) - \partial F(u) \langle \varphi \rangle\|_{L^2} \stackrel{?}{=} o(\|\varphi\|_{L^2})$ .  
No: Pick  $u = 0$   $\varphi_k = \pi \chi_{[0, \frac{1}{k}]}$ , then  $\|\varphi_k\|_2 = \sqrt{\frac{1}{k} \pi^2} \rightarrow 0$ .

$$F(u + \varphi_k)(x) = \begin{cases} \sin(0), & x > \frac{1}{k}, \\ \sin(\pi), & x \leq \frac{1}{k}. \end{cases} = 0.$$

$$\|\dots\| = \|0 - 0 - \partial F(0) \langle \varphi_k \rangle\|_{L^2} = \|\varphi_k\|_{L^2} \notin o(\|\varphi_k\|_{L^2}).$$

### **Definice 1.2** (Fréchet derivative)

Let  $X, Y$  be Banach,  $A \subset X$  open  $f : A \rightarrow Y$ . For any  $x_0 \in A$  if there exists  $Df(x_0) \in \mathcal{L}(X, Y)$  such that

$$\lim_{v \rightarrow 0} \frac{\|f(x_0 + v) - f(x_0)\|_Y}{\|v\|_X} = 0$$

then  $Df(x_0)$  is called Fréchet derivative.

### **Lemma 1.1** (Fréchet $\implies$ Gateaux)

$X, Y$  Banach spaces,  $A \subset X$  open,  $f : A \rightarrow Y$ . If  $F$  is Fréchet differentiable at  $x_0$ , it is also Gateaux differentiable with  $\partial f(x_0) = Df(x_0)$ .

┌ *Důkaz*  
└ Trivial. □

### **Definice 1.3** (Gradient)

Let  $H$  be a Hilbert space,  $A \subset H$  open  $f : A \rightarrow \mathbb{R}$ . If  $f$  is Gateaux differentiable at  $x_0 \in A$ , then the unique  $\nabla f(x_0) \in H$  such that  $\langle \nabla f(x_0), v \rangle_H = \partial f(x_0) \langle v \rangle \quad \forall v \in H$  is called the gradient of  $f$  at  $x_0$ .

### *Poznámka* (Gradients in different spaces)

Derivatives are „independent“ of the space used:  $X_1 \hookrightarrow X_2$ ,  $Y_1 \hookrightarrow Y_2$  Banach,  $f_1 : X_1 \rightarrow Y_1$ ,  $f_2 : X_2 \rightarrow Y_2$  such that  $f_2|_{X_1} = f_1$ . Then  $Df_2(x_0)|_{X_1} = Df_1(x_0)$ , if both exist.

For Hilbert spaces  $H_1 \hookrightarrow H_2$ :

$$\langle a, v \rangle_{H_1} = \langle b, v \rangle_{H_2} \quad \forall v \in H_1 \Rightarrow a = b.$$

$\Rightarrow \nabla f$  depends on the space! Notation  $\nabla_H f(x_0)$ .

One can define „formal gradients“: Let  $X$  Banach,  $H$  Hilbert,  $X \hookrightarrow H$ .  $f : A \subset X \rightarrow \mathbb{R}$  Gateaux differentiable. Then there might be  $\nabla f(x_0) \in H$  such that

$$\langle v, \nabla f(x_0) \rangle_H = Df(x_0)(v) \quad \forall v \in X.$$

If  $X$  is dense in  $H$ , then  $\nabla f(x_0)$  is unique.

Classically gradients are associate inner product, but principle works with dual pairings,  $(\langle \cdot, \cdot \rangle_{L^p \times L^q}, \frac{1}{p} + \frac{1}{q} = 1)$ .

## 1.2 Calculation rules

### **Tvrzení 1.2** (Chain rule)

Let  $X, Y, Z$  be Banach,  $A \subset X$ ,  $B \subset Y$  open,  $f : B \rightarrow Z$ ,  $g : A \rightarrow B$ ,  $x_0 \in A$ ,  $y_0 := g(x_0)$ .

1. If  $f$  is Fréchet differentiable at  $y_0$  and  $g$  is Gateaux differentiable at  $x_0$ , then  $f \circ g$  is Gateaux differentiable at  $x_0$  with

$$\partial(f \circ g)(x_0) \langle v \rangle = Df(y_0) \langle \partial g(x_0) \langle v \rangle \rangle \quad \forall v \in X.$$

2. If  $g$  is additionally Fréchet differentiable, then so is  $f \circ g$ .

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Důkaz (1.)

$$\begin{aligned} & \lim_{h \rightarrow 0} \left\| \frac{f(g(x_0 + hv)) - f(g(x_0))}{h} - Df(y_0) \langle \partial g(x_0) \langle v \rangle \rangle \right\|_Z \leq \\ & \leq \lim_{h \rightarrow 0} \left\| \frac{f(g(x_0 + hv) + y_0 - g(x_0)) - f(y_0) - Df(y_0) \langle g(x_0 + hv) - g(x_0) \rangle}{h} \right\|_Z + \\ & \quad + \lim_{h \rightarrow 0} \underbrace{\left\| Df(y_0) \left\langle \partial g(x_0) \langle v \rangle - \frac{g(x_0 + hv) - g(x_0)}{h} \right\rangle \right\|_Z}_{\rightarrow 0} = \\ & = \lim_{h \rightarrow 0} \frac{\|f(x_0 + g(x_0 + hv) - g(x_0)) - f(y_0) - Df(x_0) \langle g(x_0 + hv) - g(x_0) \rangle\|_Z \cdot \|g(x_0 + hv) - g(x_0)\|_Y}{\|g(x_0 + hv) - g(x_0)\|_Y} \cdot \frac{1}{h} = \end{aligned}$$

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Důkaz (2.)

└ Last convergence in 1. is independent of  $v$ . □

**Lemma 1.3** (Mean value)

Let  $I \subset \mathbb{R}$  be an interval,  $Y$  Banach,  $f : I \rightarrow Y$  differentiable,  $a \in Y$ . Then  $\forall x, y \in I$ ,  $x > y$ ,  $\exists \xi \in [y, x]$  such that

$$\left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y \leq \|f'(\xi) - a\|_Y.$$

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By Hahn–Banach  $\exists \varphi \in Y^*$  such that

$$* := \left\| \frac{f(x) - f(y)}{x - y} - a \right\|_Y = \varphi \left\langle \frac{f(x) - f(y)}{x - y} - a \right\rangle \wedge \|\varphi\|_{Y^*} = 1.$$

Define  $\Psi : [y, x] \rightarrow \mathbb{R}$ ,  $s \mapsto \varphi \langle f(s) - s \cdot a \rangle$ . Then

$$* = \frac{\varphi \langle f(x) \rangle - \varphi \langle f(y) \rangle}{x - y} - \frac{x - y}{x - y} \varphi \langle a \rangle = \frac{\psi(x) - \psi(y)}{x - y} \stackrel{\text{Mean value theorem}'}{\underset{\psi}{=}} (\xi) \stackrel{\text{Chain rule}}{=} \varphi \langle f'(\xi) - a \rangle \leq \|f'(\xi) - a\|_Y.$$

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□

**Tvrzení 1.4** (Product spaces)

Let  $X_1, X_2, Y$  be Banach,  $f : X_1 \times X_2 \rightarrow Y$ . Let  $x_1 \in X_1$ ,  $x_2 \in X_2$ , and denote by  $\partial_1 f(x_1, x_2)$  the Gateaux derivative of  $x \mapsto f(x, x_2)$  at  $x_1$ , by  $\partial_2 f(x_1, x_2)$  the Gateaux derivative of  $x \mapsto f(x_1, x)$  and similarly  $D_1 f(x_1, x_2)$  and  $D_2 f(x_1, x_2)$ .

1. If  $f$  is Gateaux differentiable at  $(x_1, x_2)$  then  $\partial_1 f(x_1, x_2)$ ,  $\partial_2 f(x_1, x_2)$  exists and we have

$$\forall v_1 \in X_1, v_2 \in X_2 : \partial f(x_1, x_2) \langle (v_1, v_2) \rangle = \partial_1 f(x_1, x_2) \langle v_1 \rangle + \partial_2 f(x_1, x_2) \langle v_2 \rangle.$$

2. If  $\partial_1 f$  and  $\partial_2 f$  exists at  $(x_1, x_2)$  and one of them is continuous (as a function  $X_1 \times X_2 \mapsto \mathcal{L}(X_i; Y)$ ) then  $f$  is Gateaux differentiable.
3. The previous points hold also for Fréchet derivation.

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*Důkaz* (1.)

From definition:

$$\partial_1 f(x_1, x_2) \langle (v_1, 0) \rangle = \lim_{h \rightarrow 0} \frac{f(x_1 + hv_1, x_2) - f(x_1, x_2)}{h}.$$

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□

Důkaz (2.)

WLOG  $\partial_2 f$  is continuous.

$$\begin{aligned}
& \lim_{h \rightarrow 0} \left\| \frac{f(x_1 + hv_1, x_2 + hv_2) - f(x_1, x_2)}{h} - \partial_1 f(x_1, x_2) \langle v_1 \rangle - \partial_2 f(x_1, x_2) \langle v_2 \rangle \right\|_Y \leq \\
& \leq \lim_{h \rightarrow 0} \underbrace{\left\| \frac{f(x_1 + hv_1, x_2) - f(x_1, x_2)}{h} - \partial_1 f(x_1, x_2) \langle v_1 \rangle \right\|_Y}_{\rightarrow 0} + \\
& + \lim_{h \rightarrow 0} \underbrace{\left\| \frac{f(x_1 + hv_1, x_2 + hv_2) - f(x_1 + hv_1, x_2)}{h} - \partial_2 f(x_1 + hv_1, x_2) \langle v_2 \rangle \right\|_Y}_{*} + \\
& + \lim_{h \rightarrow 0} \underbrace{\| \partial_2 f(x_1 + hv_1, x_2) \langle v_2 \rangle - \partial_2 f(x_1, x_2) \langle v_2 \rangle \|_Y}_{\rightarrow 0} = 0
\end{aligned}$$

Consider  $\psi : s \mapsto f(x_1 + hv_1, x_2 + sv_2)$ .

$$* \leq \sup_{\xi \in [0, h]} \| \partial_2 f(x_1 + hv_1, x_2 + \xi v_2) \langle v_2 \rangle - \partial_2 f(x_1 + hv_1, x_2) \langle v_2 \rangle \| \rightarrow 0$$

by continuous of  $\partial_2 f$ . □

Důkaz (3.)

Similarly. □

## 1.3 Inverse and implicit function theorem

### Věta 1.5 (Inverse function theorem)

Let  $X, Y, A \subset X$  open,  $f : A \rightarrow Y$  continuously Fréchet differentiable. If  $x_0 \in A$  such that  $Df(x_0) : X \rightarrow Y$  is an isomorphism then there exists  $U \subset A, V \subset Y$  such that  $f|_U : U \rightarrow V$  is bijection and  $(f|_U)^{-1}$  is Fréchet differentiable with

$$D(f^{-1})(y_0) = (Df(x_0))^{-1}, \quad y_0 := f(x_0).$$

Důkaz

Given  $\hat{y}$  close to  $f(x_0)$  find  $\hat{x}$  such that  $f(\hat{x}) = \hat{y}$ . Idea: fix  $\hat{y}$  try  $x$ : error in  $y$  is  $f(x) - \hat{y}$  and error in  $x$  is  $(Df(x_0))^{-1} \langle f(x) - \hat{y} \rangle$ . Therefore try iteration:

$$F_{\hat{y}}(x) := x - (Df(x_0))^{-1} \langle f(x) - \hat{y} \rangle.$$

If  $F_{\hat{y}}$  has fix point  $\hat{x}$  then  $\hat{x} = F_{\hat{y}}(\hat{x}) = \hat{x} - (Df(x_0))^{-1} \langle f(\hat{x}) - \hat{y} \rangle \implies f(\hat{x}) = \hat{y}$ . So we use Banach fixed point theorem: „ $F_{\hat{y}}$  is contraction“:  $(x_1, x_2 \in B_\delta(x_0))$

$$\begin{aligned} \|F_{\hat{y}}(x_1) - F_{\hat{y}}(x_2)\|_X &= \|x_1 - x_2 - (Df(x_0))^{-1} \langle f(x_1) - f(x_2) \rangle\|_X = \\ &= \|(Df(x_0))^{-1} \langle Df(x_0) \langle x_1 - x_2 \rangle + f(x_1) - f(x_2) \rangle\|_X \leq \\ &\leq \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y, X)} \cdot \|Df(x_0) \langle x_1 - x_2 \rangle + f(x_1) - f(x_2)\|_Y = * \end{aligned}$$

Consider  $a := Df(x_0) \langle x_1 - x_2 \rangle$ .  $\psi : [0, 1] \rightarrow Y$ ,  $f(1 - \xi)x_1 + \xi x_2$  and apply Mennroltz? lemma.

$$\begin{aligned} * &\leq \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y, X)} \cdot \|Df(x_0) \langle x_1 - x_2 \rangle - Df((1 - \xi)x_1 + \xi x_2) \langle x_2 - x_1 \rangle\|_Y \leq \\ &\leq \|(Df(x_0))^{-1}\|_{\mathcal{L}(Y, X)} \cdot \sup_{x \in B_\delta(x_0)} \|Df(x_0) - Df(x)\|_{\mathcal{L}(X, Y)} \cdot \|x_1 - x_2\|_X \ll 1 \end{aligned}$$

$$\begin{aligned} \|F_{\hat{y}}(x) - x_0\|_X &= \|F_{\hat{y}}(x) - F_{\hat{y}}(x_j)\|_X + \|F_{\hat{y}}(x_0) - x_0\|_X \leq \\ &\leq \frac{1}{2} \|x - x_0\|_X + \|(Df(x_0))^{-1}\| \cdot \|\hat{y} - x_0\| \end{aligned}$$

$\|\hat{y} - x_0\|$  can chosen to be small  $\implies F_{\hat{y}}$  maps  $\overline{B_\delta(x_0)}$  to  $\overline{B_\delta(x_0)}$   $\implies F_{\hat{y}}$  has unique fix point.

Next „regularity“:  $(y_1 := f(x_1), y_2 := f(x_2))$

$$\begin{aligned} \|f^{-1}(y_1) - f^{-1}(y_2)\|_X &= \|F_{y_1}(x_1) - F_{y_2}(x_2)\|_X \leq \\ &\leq \|F_{y_1}(x_1) - F_{y_1}(x_2)\|_X + \|F_{y_1}(x_2) - F_{y_2}(x_2)\|_X \leq \\ &\leq \frac{1}{2} \|x_1 - x_2\|_X + \|(Df(x_0))^{-1} \langle y_1 - y_2 \rangle\|_X \leq \frac{1}{2} \underbrace{\|x_1 + x_2\|_X}_{=\|f^{-1}(y_1) - f^{-1}(y_2)\|} + c \cdot \text{TODO!!!} \\ &\implies \frac{1}{2} \|f^{-1}(x_1) - f^{-1}(x_2)\|_X \leq c \cdot \|y_1 - y_2\|_Y \implies f^{-1} \text{ is Lipschitz.} \end{aligned}$$

Pick  $\delta$  so small that

$$\|Df(x) - Df(x_0)\| \leq \frac{1}{2} \cdot \frac{1}{\|(Df(x_0))^{-1}\|} \quad \forall x \in B_\delta(x_0).$$

$\implies (Df(x))^{-1}$  exists and is uniformly bounded (from functional analysis).

$$\underbrace{\|f^{-1}(y + w) - f^{-1}(y) - (Df(x))^{-1} \langle w \rangle\|}_{=: v}$$

$$(f(x + v) + f(x) = f(f^{-1}(y + w)) - y = w)$$

$$\|v - (Df(x))^{-1} \langle f(x + v) - f(x) \rangle\| = \|(Df(x))^{-1} \langle Df(x) \langle v \rangle - f(x + v) + f(x) \rangle\| \leq \|(Df(x))^{-1}\| \cdot \sigma(\|v\|) \leq$$

because  $f^{-1}$  is Lipschitz

### Věta 1.6 (Global inverse function theorem)

Let  $X, Y$  Banach,  $f : X \rightarrow Y$  continuously Fréchet differentiable and  $(Df(x))^{-1}$  exists, depends continuously on  $x$  and  $c > 0$  such that  $\|(Df(x))^{-1}\| < c \forall x \in X$ . Then  $f : X \rightarrow Y$  is a diffeomorphism.

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Last theorem  $\implies$   $f$  is a local diffeomorphism. Left to show:  $f$  is bijective. „Surjectivity“: Fix  $x_0 \in X, y_0 \in Y$ . Let  $y \in Y, \varphi(t) = y_0 + t(y - y_0), t \in [0, 1]$ . Goal: find  $\psi(t)$  continuous, such that  $\varphi(t) = f(\psi(t))$  (then  $y = f(\varphi(t))$ ) (so called lifting). Local diffeomorphism implies  $\psi$  exists on  $[0, \delta]$ , in fact if  $Y$  is defined on  $[0, t_0]$ , it can be extended to  $[0, t_0 + \delta]$ . Similarly, if  $\psi$  is defined on  $[0, t_0]$ , per chain rule:

$$\|\psi'(t)\| = \|Df^{-1}(\varphi(t))\langle\varphi'(t)\rangle\| < c.$$

$\psi$  is Lipschitz,  $\lim_{t \nearrow t_0} \psi(t)$  is well defined and  $\psi$  can be extended to  $[0, t_0]$ . From Zorn lemma  $\Psi$  is defined on  $[0, 1]$ .

„Injectivity“: Assume  $f(x_1) = f(x_2) = y$ . Pick  $\psi_1(t) := x_1 + t(x_2 - x_1)$ .  $\varphi_1(t) = f(\psi_1(t))$ . Define  $\varphi_s(t) = s\varphi_1(t) + (1 - s)y$  ( $t, s \in [0, 1]$ ). Similar to before (homework)  $\exists \psi_s(t)$  continuous in  $s$  and  $t$ , such that  $f(\psi_s(t)) = \varphi_s(t)$ . But then

$$x_1 = \psi_1(0) = \psi_s(0) = \psi_0(0) = \psi_0(t) = \psi_0(1) = \psi_s(1) = \psi_1(1) = x_2.$$

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### Věta 1.7 (Implicit function theorem)

Let  $X_1, X_2, Y$  Banach,  $A_1 \subset X_1, A_2 \subset X_2$  open,  $f : A_1 \times A_2 \rightarrow Y$  continuously Fréchet differentiable and exists  $\hat{x}_1 \in A_1$  and  $\hat{x}_2 \in A_2$  with  $f(\hat{x}_1, \hat{x}_2) = 0$ . If  $D_2f(\hat{x}_1, \hat{x}_2)$  is an isomorphism (between  $X_2$  and  $Y$ ), then are neighbourhoods  $U_1, U_2$  of  $\hat{x}_1, \hat{x}_2$  such that  $\forall \hat{x}_1 \in U_1 \exists! \hat{x}_2 \in U_2$  with  $f(\hat{x}_1, \hat{x}_2) = 0$ .

If we call  $\hat{x}_2 = g(\hat{x}_1)$ , then  $g$  is continuously Fréchet differentiable with  $Dg(\hat{x}_1) = -(D_2f(\hat{x}_1, g(\hat{x}_1)))^{-1} \circ D_1f(\hat{x}_1, g(\hat{x}_1))$ .

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Apply the inverse function theorem to

$$F(x_1, x_2) := (x_1, (D(f(\hat{x}_1, \hat{x}_2)))^{-1}\langle f(x_1, x_2) \rangle)$$

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□

TODO!!!



**Tvrzení 1.8** (Noether–type theorem)

Let  $\Omega \subset \mathbb{R}^n$ ,  $F(u) := \int_{\Omega} f(x, u, Du)$  with  $f \in C^2(\Omega \times \mathbb{R}^n \times \mathbb{R}^{m \times n})$  and  $(\psi_s)_{s \in \mathbb{R}} \subset C^2(\mathbb{R}^n, \mathbb{R}^n)$  is a smooth family with  $\psi_0 = \text{id}$ , such that

$$f(x, \psi_s \circ u, D(\psi_s \circ u)) = f(x, u, Du).$$

Then there exists a conservation  $0 \neq Q : \Omega \times \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^n$  such that  $\text{div}(Q(x, u, Du)) = 0 \forall$  critical points of  $u$ .

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Důkaz

$$\begin{aligned} 0 &= \frac{d}{ds} \Big|_{s=0} f(x, \psi_s \circ u, D(\psi_s \circ u)) = \\ &= \sum_i \frac{\partial \psi^i}{\partial s} \Big|_{s=0} \frac{\partial f}{\partial z^i}(x, u, Du) + \sum_{ij} \frac{\partial^2 \psi_j}{\partial s \partial y^j} \frac{\partial u^i}{\partial x_j} \frac{\partial f}{\partial p^{ij}}(x, u, Du) = \\ &= \sum_i \frac{\partial \psi^i}{\partial s} \Big|_{s=0} \sum_j \frac{\partial}{\partial x^j} \left( \frac{\partial f}{\partial p^{ij}}(x, u, Du) \right) + \sum_{ij} \frac{\partial^2 \psi_s}{\partial s \partial y^j} \frac{\partial u^j}{\partial x_i} \frac{\partial f}{\partial p^{ij}}(x, y, Du) = \\ &= \sum_j \frac{\partial}{\partial x^j} \left( \sum_i \frac{\partial(\psi^i \circ u)}{\partial s} \Big|_{s=0} \frac{\partial f}{\partial p^{ij}}(x, u, Du) \right). \end{aligned}$$

└

□

*Příklad* (Particle in potential well)

$y : I \rightarrow \mathbb{R}^n$  position of a particle,  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  a physical potential.  $F(u) := \int_I \frac{m}{2} |\dot{y}|^2 - V(y) dt$  (Physics: critical points are behaviour of a ion particle). El eg:  $\frac{\partial V}{\partial x_i} + \frac{d}{dt}(m\dot{y}^i) = 0 \implies m\ddot{y} = -\nabla V(y)$ .

Assume that  $V$  is invariant under rotations, i.e.  $V(R(\theta)y) = V(y)$ , where  $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & I \end{pmatrix}$ . And always  $|\frac{d}{dt} R(\theta)y|^2 = y^T R(\theta)^T R(\theta)y$ .  $\implies$  (Noether)

$$\begin{aligned} 0 &= \frac{d}{dt} \left( \frac{dR(\theta)}{d\theta} \Big|_{\theta=0} \frac{\partial f}{\partial p}(y, \dot{y}) \right) = \\ &= \frac{d}{dt} \cdot \left( \begin{pmatrix} 0 & -1 & \dots \\ 1 & 0 & \dots \\ \dots & \dots & 0 \end{pmatrix} y \right) \cdot m\dot{y} = m(y_1 \dot{y}_2 - y_2 \dot{y}_1). \end{aligned}$$

(Which is angular momentum.)

*Poznámka* (Conservation law in  $n + 1$  dimensions)

If we single out one direction as time, e.g.

$$(t, x) = (t, x_1, \dots, x_n),$$

then the conservation law reads as

$$\frac{\partial}{\partial t} Q_0 + \operatorname{div}_x(\overline{Q}) = 0.$$

( $Q_0$  – conserved quantity,  $\overline{Q}$  – conservation current.) And

$$\frac{d}{dt} \int_{\Omega} Q_0 = \int_{\Omega} \operatorname{div}_x \overline{Q}.$$

### **Tvrzení 1.9** (2nd Variation)

Let  $X$  be Banach space  $A \subset X$  open,  $F : A \rightarrow \mathbb{R}$ .

1. If  $x_0 \in A$  is local minimizer of  $F$  and  $F$  is twice Gateaux differentiable in  $x_0$ , then  $\partial^2 F(x) \langle v, v \rangle \geq 0 \ \forall v \in X$ ;
2. If  $x_0$  is critical point of  $F$  and  $F$  is twice Fréchet differentiable and  $D^2 F(x_0) \langle v, v \rangle \geq c \cdot \|v\|^2 \ \forall v \in X$  with  $c$  independent of  $v$ , then  $x_0$  is a local minimum.

┌

*Důkaz*

„1.“: Consider  $\varphi : \varepsilon \mapsto F(x_0 + \varepsilon \cdot v)$ , if  $x_0$  is local minimum of  $F$ , then 0 is local minimum of  $\varphi \implies$

$$\implies 0 \leq \varphi''(0) = \frac{d^2}{d\varepsilon^2} \big|_{\varepsilon} F(x_0 + \varepsilon v) = \partial^2 F(x_0) \langle v, v \rangle.$$

„2.“: By continuity  $\exists \delta > 0$  such that  $D^2 F(x) \langle v, v \rangle \geq \frac{c}{2} \|v\|^2 \ \forall v \in X \ \forall x \in B_{\delta}(x_0)$ . Pick  $x \in B_{\delta}(x_0)$ , define  $\psi(t) := x_0 + t(x - x_0)$ ,  $H(t) := J(\psi(t))$ .

$$H(j) - H(0) = \int_0^1 1 \cdot H'(t) dt \stackrel{BP'}{=} H'(0) + \int_0^1 (1-t) H''(t) dt = (*).$$

$$H'(t) = DF(\psi(t)) \langle x - x_0 \rangle \implies H'(0) = 0.$$

$$H''(t) = D^2 F(\psi(t)) \langle x - x_0, x - x_0 \rangle \geq 0.$$

$$\implies (*) \geq 0 \implies F(x) \geq F(x_0) \ \forall x \in B_{\delta}(x_0).$$

└

□

*Poznámka* (Lebesgue–Hadamard)

If  $F(u) = \int_{\Omega} f(x, u, Du)$ , then  $D^2 F(u) \langle \varphi, \varphi \rangle$  includes

$$\int_{\Omega} \sum_{ijkl} \frac{\partial}{\partial p_{ij}} \frac{\partial f}{\partial p_{kl}}(x, u, Du) \frac{\partial \varphi_i}{\partial x_j} \frac{\partial \varphi_k}{\partial x_l} ds.$$

This is the dominant term. Even more, its enough:

$$\sum_{ijkl} \frac{\partial}{\partial p_{ij}} \frac{\partial}{\partial p_{kl}} f(x, u, Du) \xi^i \xi^j \eta^k \eta^l \geq c \cdot ?.$$

## 1.4 Lagrange multipliers

### **Tvrzení 1.10** (Lagrange multipliers)

Let  $X$  Banach,  $A \subset X$  open  $F, G : A \rightarrow \mathbb{R}$  continuous Fréchet differentiable. Let  $x_0$  be a local minimizer of  $F|_{\{G=0\}}$  with  $DG(x) \neq 0$ . Then  $\exists \lambda \in \mathbb{R}$  such that  $DF(x_0) + \lambda DG(x_0) = 0$ .

$\lambda$  is called the Lagrange multiplier, any  $x_0$  that satisfies this equation is called critical point.

TODO!!!

*Příklad* (Principal eigenvalue of  $\Delta$ )

Consider  $\Omega \subset \mathbb{R}^n$  domain, bounded. Minimize  $F(u) := \int_{\Omega} \frac{1}{2} |Du|^2$ ,  $u \in W_0^{1,2}(\Omega)$ , under constraint  $\frac{1}{2} \int_{\Omega} |u|^2 = 1$ , i.e.  $G(u) = \frac{1}{2} \int |u|^2 dx - 1 = 0$ .

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*Řešení*

We are looking for  $u_1 \in W_0^{1,2}(\Omega)$  such that

$$\begin{aligned} \forall \varphi \in W_0^{2,2}(\Omega) : 0 &= DF(u_1) \langle \varphi \rangle + \lambda_1 DG(u_1) \langle \varphi \rangle = \\ &= \langle \nabla u_1, \nabla \varphi \rangle_{L^2} + \lambda_1 \langle u_1, \varphi \rangle. \end{aligned}$$

I.e. a weak solution to  $\Delta u_1 = \lambda_1 u_1$  in  $\Omega$  and  $u_1 = 0$  on  $\partial\Omega$ . Additionally take  $\varphi = u_1 \implies \lambda_1 = -\frac{\int_{\Omega} |\nabla u_1|^2}{\int_{\Omega} |u_1|^2} \implies \lambda_1$  is largest eigenvalue.

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*Příklad* (Stokes problem)

Minimize  $F(u) := \int_{\Omega} \frac{1}{2} |\nabla u|^2 - f u dx$  in  $W_0^{1,2}(\Omega, \mathbb{R}^3)$  under the constant  $\operatorname{div}(u) = 0$ .

TODO!!!!

## 2 The direct method on convex integrands

### 2.1 Direct method

**Tvrzení 2.1** (Direct method in the calculus of variations)

Let  $X$  be topological space,  $f : X \rightarrow \mathbb{R}$  such that

1. All sublevel sets  $(\{x \in X | F(x) \leq c\})$  are sequentially precompact;
2.  $F$  is sequentially lower-semi-continuous  $(x_k \rightarrow x_0 \implies \liminf_{k \rightarrow \infty} F(x_k) \geq F(x_0).)$

Then  $F$  has a minimizer in  $X$ .

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*Důkaz*

Let  $s := \inf_X F$ . Pick sequence  $(x_k)_k \subset X$  such that  $F(x_k) \rightarrow s$ . For  $k_0$  large enough  $(x_i)_{i \geq k_0} \subset x \in X : F(x) \leq s + 1$ .  $\xrightarrow{1.} \exists$  subsequenc (not relabeled) and  $x_0 \in X$  such that  $x_k \rightarrow x_0$ .  $s = \inf F \leq F(x_0) \leq \liminf_{k \rightarrow \infty} F(x_k) = s$ . □

*Poznámka* (The three c's of the direct method)

Equivalent conditions: Coercivity (sublevel sets are bounded with respect to metric), Compactness (bounded sets are compact with respect to some topology) and lower-semi-Continuity (As before.)

Sometimes also Convexity (if  $F$  is strictly convex, then the minimum is unique).