Poznámka
Credit for giving 'small lecture'. Oral exam.

1 Meromorphic functions

Definice 1.1

We say that a function f is holomorphic in a set $F \subset \mathbb{C}$ if there is an open $G \supseteq F$ such that f is holomorphic on G.

In particular, f is holomorphic at $z_0 \in \mathbb{C}$ if f is holomorphic in some neighbour ($= U(z_0) = U(z_0, \varepsilon)$) of z_0.

Definice 1.2

Function f has at ∞ a removable singularity, if $f\left(\frac{1}{z}\right)$ has a removable singularity at 0. Similarly pole and essential singularity.

Function f is holomorphic at ∞ if $f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Let $G \subset \mathbb{S}$ be open. Then f is holomorphic on G if f is holomorphic at any z_0. Denote $\mathcal{H}(G) := \{f : G \to \mathbb{C} | f \text{ holomorphic}\}$.

Například
From Liouville theorem $\mathbb{H}(\mathbb{S})$ = constant functions. So $\mathbb{H}(G)$ is interesting only for $G \subsetneq \mathbb{S}$, so WLOG $G \subset \mathbb{C}$.

Definice 1.3 (Meromorphic function)

Let $G \subset \mathbb{S}$ be open. Then a function f on G is called meromorphic if at any $z_0 \in G$ the function f is either holomorphic at z_0 or has a pole at z_0.

Denote $\mathcal{M}(G)$ the set of meromorphic functions on G.

Důsledek

- $\mathcal{H}(G) \subset \mathcal{M}(G)$.
- Denote $P_f := \{z_0 \in G | f \text{ has a pole at } z_0\}$. Then P_f has no limit points in G.
- If $f = \infty$ on P_f, then $f : G \to \mathbb{S}$ is continuous. (We always assume, that $f \in \mathcal{H}(G)$ has this property.)

Například

$$\frac{\pi}{\sin(\pi z)} \in \mathcal{M}(\mathbb{C}), \qquad e^{\frac{1}{z}} \notin \mathcal{M}(\mathbb{C}), \qquad \Gamma \in \mathcal{M}(\mathbb{C}), \qquad \zeta \in \mathcal{M}(\mathbb{C}).$$

$\mathcal{M}(\mathbb{S})$ = rational functions. (One inclusion is clear, second: Let $f \in \mathcal{M}(\mathbb{S})$, then because $\mathbb{S}$ is compact it holds that P_f is finite (has no limit point), $P_f \cap \mathbb{C} = \{z_1, \ldots, z_n\}$, so from theorem from last semester there exists $h \in \mathcal{H}(\mathbb{C})$ such that $f(z) = h(z) + \sum_{j=1}^{n} p_j \left(\frac{1}{z-z_j}\right)$ for some polynomials p_j. f has removable singularity or pole at infinity and p_j and $\frac{1}{z-z_j}$ have removable singularity there, so $h(z)$ is polynomial, otherwise $h(z)$ has infinity Taylor polynom and $h\left(\frac{1}{z}\right)$ has essential singularity at 0.)

So $\mathcal{M}(G)$ is interesting for $G \subsetneq \mathbb{S}$, WLOG $G \subset \mathbb{C}$.

If $G \subset \mathbb{C}$ is domain, $f, g \in \mathbb{H}(G)$ and $g \equiv 0$, then $f/g \in \mathcal{M}(G)$. The inverse is also true (we will prove it) (but not for $G = \mathbb{S}$).

Lemma 1.1

Let $\mathbb{G} \subset \mathbb{C}$ be open. Then there are compacts K_n, $n \in \mathbb{N}$, in G such that $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset \operatorname{int}(K_{n+1})$ and for any compact K in G, $\exists n \in \mathbb{N} : K \in K_n$.

Důkaz

Set $K_n := \left\{z \in G \middle| \operatorname{dist}(z, \mathbb{C}\backslash G) \geqslant \frac{1}{n}\right\} \cap U(0, n)$. □

Tvrzení 1.2

Let $G \subset \mathbb{S}$ be open and $M \subset G$ has no limit point in G. Then

- *$G\backslash M$ is open;*
- *if K is a compact in G, then $K \cap M$ is finite. In particular for $G = \mathbb{S}$ we have M is finite;*
- *M is at most countable. If M is infinite, then $\emptyset \neq M' \subset \partial G$;*
- *if $G \subset \mathbb{C}$ is domain (connected), then $G\backslash M$ is domain.*

Věta 1.3 (Uniqueness of meromorphic functions)

Let $G \subset \mathbb{C}$ be a domain, $f \in \mathcal{M}(G)$ and $f \not\equiv 0$. Then $N_f := \{z \in G | f(z) = 0\}$ has no limit points in G.

Důkaz
We know this holds for holomorphic functions. Set $G_0 := G\backslash P_f$. Then $G_0 \subset \mathbb{C}$ is also domain and $f \in \mathcal{H}(G)$ and $f \not\equiv 0$ on G_0. Then $N_f \subset G_0$ has no limit points in G_0, nor in P_f. □

Věta 1.4 (Residue theorem)

Let $G \subset \mathbb{C}$ be open, φ be a closed curve (or cycle) in G and $\operatorname{int}\varphi := \{z_0 \in \mathbb{C}\backslash\langle\varphi\rangle \mid \operatorname{ind}_\varphi z_0 \neq 0\} \subset G$. Let $M \subset G\backslash\langle\varphi\rangle$ be finite and $f \in \mathcal{H}(G\backslash M)$. Then $\int_\varphi f = 2\pi i \cdot \sum_{s\in M} \operatorname{ind}_\varphi s \cdot \operatorname{res}_s f$.

Poznámka
This holds true even if instead of finiteness of M, we assume only that $M \subset G\backslash\langle\varphi\rangle$ has no limit points in G. Indeed, we have $M_0 = M \cap \operatorname{int}\varphi$ is finite, because $\langle\varphi\rangle \cup \operatorname{int}\varphi$ is compact and $G_0 := G\backslash(M\backslash M_0)$ is open and f is holomorphic on $G_0\backslash M_0$ and by R. theorem for G_0 and M_0 we get $\int_\varphi f = 2\pi i \sum_{s\in M_0} \operatorname{res}_s f \cdot \operatorname{ind}_\varphi s$.

1.1 Logarithmic integrals

Definice 1.4 (Logarithmic integral)

Let $\varphi : [a, b] \to \mathbb{C}$ be a (regular) curve and let f be a non-zero holomorphic function on $\langle\varphi\rangle$. Then we define logarithmic integrals integral as

$$I := \frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \frac{1}{2\pi i}\int_a^b \frac{f'(\varphi(t))\varphi'(t)}{f(\varphi(t))}dt = \frac{1}{2\pi i}\int_a^b \frac{(f(\varphi(t)))'}{f(\varphi(t))}dt = \frac{1}{2\pi i}\int_{f\circ\varphi} \frac{dz}{z} = \frac{1}{2\pi i}(\Phi(b)-\Phi(a)),$$

where Φ is a branch (jednoznačná větev) of logarithm of $f \circ \varphi$. If φ is, in addition, closed, then $I = \operatorname{ind}_{f\circ\varphi} 0 = \frac{1}{2\pi}(\Theta(b) - \Theta(a)) \in \mathbb{Z}$, where Θ is a branch of argument of $f \circ \varphi$.

($\frac{f'}{f}$ is called logarithmic derivative of f, because '$(\log f)' = \frac{f'}{f}$'.)

Věta 1.5 (Argument principle)

Let $G \subseteq \mathbb{C}$ be a domain, φ be a closed curve in G and $f \in \mathcal{M}(G)$. Let $\operatorname{int}\varphi \subset G$ and $\langle\varphi\rangle \cap N_f = \emptyset$, $\langle\varphi\rangle \cap P_f = \emptyset$. Then

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{s\in\operatorname{int}\varphi, f(s)=0} n_f(s) \cdot \operatorname{ind}_\varphi s - \sum_{s\in\operatorname{int}\varphi, f(s)=\infty} p_f(s) \cdot \operatorname{ind}_\varphi s,$$

where $n_f(s)$ is multiplicity of the zero point s of f and $p_f(s)$ is multiplicity of the pole s of f.

Důkaz
By Residua theorem, we have

$$\frac{1}{2\pi i}\int_\varphi \frac{f'}{f} = \sum_{s\in\operatorname{int}\varphi, s\in N_f\cup P_f} \operatorname{res}_s\left(\frac{f'}{f}\right)\cdot \operatorname{ind}_\varphi s.$$

If $s\in N_f$ then on $P(s)$:

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = n_f(s).$$

If $s\in P_f$ then on $P(s)$

$$\frac{f'(z)}{f(z)} = \frac{p\cdot c_p(z-s)^{p-1}+\dots}{c_p(z-s)^p+\dots} = \frac{p}{z-s}\cdot\frac{1+\dots}{1+\dots} \implies \operatorname{res}_s\left(\frac{f'}{f}\right) = p = -p_f(s).$$

□

Definice 1.5

$$\Sigma(f,\varphi) := \sum_{s\in\operatorname{int}\varphi, f(s)=0} n_f(s)\cdot\operatorname{ind}_\varphi s - \sum_{s\in\operatorname{int}\varphi, f(s)=\infty} p_f(s)\cdot\operatorname{ind}_\varphi s.$$

Lemma 1.6

Let $\varphi_1,\varphi_2 : [a,b]\to\mathbb{C}$ be closed curve and $s\in\mathbb{C}\setminus(\langle\varphi_1\rangle\cup\langle\varphi_2\rangle)$. Assume, for $t\in[a,b]$, $|\varphi_1(t)-\varphi_2(t)|<|\varphi_1(t)-s|$. Then $\operatorname{ind}_{\varphi_1} s = \operatorname{ind}_{\varphi_2} s$.

Důkaz
For $t\in[a,b]$, we have $|(\varphi_1(t)-s)-(\varphi_2(t)-s)|<|\varphi_1(t)-s|$. Divide by $|\varphi_1(t)-s|$:

$$|1-\psi(t)|<1, \qquad \psi(t) := \frac{\varphi_2(t)-s}{\varphi_1(t)-s}.$$

Then ψ is a closed curve, $<\psi>\subset U(1,1)$, and so

$$0 = \operatorname{ind}_\psi 0 = \frac{1}{2\pi i}\int_a^b \frac{\psi'}{\psi} = \frac{1}{2\pi i}\int_a^b \frac{\frac{\varphi_2'(\varphi_1-s)-\varphi_1'(\varphi_2-s)}{(\varphi_1-s)^2}}{\frac{\varphi_2-s}{\varphi_1-s}} = \frac{1}{2\pi i}\int_a^b\frac{\varphi_2'}{\varphi_2-s} - \frac{1}{2\pi i}\int_a^b\frac{\varphi_1'}{\varphi_1-s} = \operatorname{ind}_{\varphi_2} s - \operatorname{ind}_{\varphi_1} s$$

□

Věta 1.7 (Rouché)

Let $G\subset\mathbb{C}$ be a domain, $f_1,f_2\in\mathcal{M}(G)$ and φ be closed curve in G such that $\operatorname{int}\varphi\subset G$. Assume $\forall z\in\langle\varphi\rangle$:

$$|f_1(z)-f_2(z)|<|f_1(z)|<+\infty$$

Then $\Sigma(f_1,\varphi)=\Sigma(f_2,\varphi)$.

Důkaz
Set $\varphi_j = f_j \circ \varphi$. Then

$$\operatorname{ind}_{\varphi_j} 0 = \frac{1}{2\pi i} \int_{\varphi} \frac{f_j'}{f_j} = \Sigma(f_j, \varphi).$$

By previous lemma we have for $s = 0$: $\operatorname{ind}_{\varphi_1} 0 = \operatorname{ind}_{\varphi_2} 0$. □

Důsledek
Let f_1, f_2 be holomorphic functions on $\overline{U(z_0, r)}$ and $\forall z \in \partial U(z_0, r) : |f_1(z) - f_2(z)| < |f_1(z)|$. Then $\Sigma_1 = \Sigma_2$, where $\Sigma_j := \sum_{s \in U(z_0,r), f(s)=0} n_{f_j}(s)$.

Důkaz
Apply Rouché's theorem to $\varphi(t) := z_0 + r \cdot e^{it}$, $t \in [0, 2\pi]$. □

Příklad
$f_2 = p$, $f_1(z) = a_0 z^n$ and big enough $U(0, r)$.

Definice 1.6 (Notation)

Let f be a function holomorphic at $z_0 \in \mathbb{C}$. We say that $f(z_0) = w_0 \in \mathbb{C}$ p times for $p \in \mathbb{N}$ if z_0 is a zero point of $f - w_0$ of order p.

Poznámka
Following statements are equivalent to each other:

- $f(z_0) = w_0$ p times;
- $f(z_0) = w_0$, $f'(z_0) = 0 = \ldots = f^{(p-1)}(z_0)$, $f^{(p)}(z_0) \neq 0$;
- $f(z) = w_0 + \sum_{k=p}^{+\infty} c_k (z - z_0)^k$ on some neighbourhood of z_0 and $c_p \neq 0$.

We say that $f(z_0) = \infty$ p times if z_0 is a zero point of $\frac{1}{f}$ of order p. (It's the same as z_0 is pole of f of order p.) And we say that $f(\infty) = w_0 \in \mathbb{S}$ p times if $f(1/z)$ attains w_0 p times at 0.

Věta 1.8 (On a multiple value)

Let $z_0, w_0 \in \mathbb{S}$, f be a holomorphic function on a $P(z_0)$ and $f(z_0) = w_0$ p times for some $p \in \mathbb{N}$. Let $\delta_0 > 0$. Then there are $\varepsilon > 0$ and $\delta \in (0, \delta_0)$ such that, for any $w \in P(w_0, \varepsilon)$ there are just p different points $z_1, \ldots, z_p$ in $P(z_0, \delta)$ with $f(z_j) = w$. In addition, $f(z_j) = 0$ once.

Důkaz
WLOG, assume $z_0 = 0 = w_0$.Then $z_0 = 0$ is a zero point of f of order p. Choose $\delta \in (0, \delta_0)$ such that $f \neq 0$ and $f' \neq 0$ on $P(0, 2\delta)$. Set $\varepsilon := \min_{|z|=\delta} |f(z)| > 0$.

Let $w \in P(0, \varepsilon)$. Use Rouché's theorem for $f_1 := f$, $f_2 := f - w$ and $\varphi := \delta e^{it}$, $t \in [0, 2\pi]$. Of course, $|f_1 - f_2| = |w| < \varepsilon < |f_1|$ on $\langle \varphi \rangle$.

Since in $U(0, \delta)$ the function $f = f_1$ has the only zero point of order p at origin, $f - w = f_2$ has just p simple zero points in $P(0, \delta)$. □

Důsledek
Let $G \subset \mathbb{S}$ be a domain, $f \in \mathcal{M}(G)$ and f be not constant on G. Then $f : G \to \mathbb{S}$ is an open map (for any open $\Omega \subset G$, $f(\Omega)$ is open).

Důkaz
Let $\Omega \subset G$ be open and $w_0 \in f(\Omega)$. Then there is a $z_0 \in \Omega$ and $p \in \mathbb{N}$ such that $f(z_0) = w_0$ p times. Choose $\delta_0 > 0$ such that $U(z_0, \delta_0) \subset \Omega$. By the previous theorem, there is $\varepsilon > 0$, $\delta \in (0, \delta_0)$ such that $P(w_0, \varepsilon) \subset f(P(z_0, \delta))$, so $U(w_0, \varepsilon) \subset f(U(z_0, \delta)) \subset f(\Omega)$. □

Poznámka
This is true for $\mathcal{H}(G)$ too.

Důsledek
Let f be a function holomorphic at $z_0 \in \mathbb{C}$. Then $f'(z_0) \neq 0$ if and only if there is $U(z_0)$ such that $f|_{U(z_0)}$ is one-to-one.

Důkaz
„ $\implies$ “: Let $f'(z_0) \neq 0$. Then $f(z_0) = w_0$ once, so we choose $\delta_0 > 0$ such that $f \neq w_0$ on a $P(z_0, \delta_0)$. By the previous theorem choose $\varepsilon > 0$, $\delta \in (0, \delta_0)$. Moreover, due to the continuity of f at z_0 choose $\delta_1 \in (0, \delta)$ such that $f(U(z_0, \delta_1)) \subset U(w_0, \varepsilon)$. Then $f|_{U(z_0, \delta_1)}$ is one-to-one.

„ $\impliedby$ “: Let $f'(z_0) = 0$ and let f be not constant on any neighbourhood of z_0. Then $f(z_0) = w_0$ p times ($p \in \mathbb{N} \setminus \{1\}$). By the previous theorem f is not one-to-one on any neighbourhood of z_0. □

Věta 1.9 (On holomorphic inverse)

Let $G \subset \mathbb{C}$ be open and $f : G \to \mathbb{C}$ be a one-to-one holomorphic[a] function, then $f' \neq 0$ on G, $\Omega := f(G)$ is open and $f_{-1} : \Omega \overset{onto}{\to} G$ is holomorphic.

In addition, $(f_{-1})' = \frac{1}{f' \circ f_{-1}}$ on Ω.

Důkaz

WLOG, $G \subset \mathbb{C}$ is a domain. By first „dusledek“ of previous theorem f is an open map, so $\Omega := f(G)$ is open and $f_{-1} : \Omega \to G$ is continuous. Let $z_0 \in G$ and $w_0 = f(z_0)$. By second „dusledek“ we have $f'(z_0) \neq 0$, and

$$\frac{1}{f'(z_0)} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} \overset{*}{=} \lim_{w \to w_0} \frac{f_{-1}(w) - f_{-1}(w_0)}{w - w_0} = f'_{-1}(w_0).$$

The equality $*$ follows from theorem on limits of composite functions because f_{-1} is continuous and $f_{-1}(w) \neq f_{-1}(w_0)$ for $w \neq w_0$. □

[a] One-to-one holomorphic function is sometimes called conformal.

Věta 1.10 (Hurwitz)

Let $G \subset \mathbb{C}$ be a domain, $f_n \in \mathcal{H}(G)$, $f_n \overset{loc.}{\rightrightarrows} f$ on G and $f \not\equiv 0$. Let $z_0 \in G$ be a zero point of f. Then $\exists\ \{z_n\}_{n=1}^{\infty} \subset G$ and a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ such that $z_n \to 0$ and $f_{k_n}(z_n) = 0$.

Poznámka

Not true in $\mathbb{R}$! The assumption $f \not\equiv 0$ is important! ($f_n(z) := z/n$)

Důsledek

Let $G \subset \mathbb{C}$ be a domain, f_n be one-to-one holomorphic functions on G and $f_n \overset{\text{loc}}{\rightrightarrows} f$ on G. Then f is either one-to-one and holomorphic, or constant.

Důkaz (Hurwitz theorem)

Choose $\delta > 0$ such that $U(z_0, \delta) \subset G$ and $f \neq 0$ on $P(z_0, \delta)$. For $n \in \mathbb{N}$ put $\varrho_n := \frac{\delta}{n+1}$ and $\varphi_n(t) := z_0 + \varrho_n e^{it}$, $t \in [0, 2\pi]$. Of course, $\tau_n := \min_{\langle \varphi_n \rangle} |f| > 0$. For a given n, there is (from uniformly convergence) $k_n \in \mathbb{N}$ such that $\forall z \in \langle \varphi_n \rangle : |f_{k_n}(z) - f(z)| < \tau_n \leqslant |f|$.

By Rouché's theorem there is $z_n \in U(z_0, \varrho_n)$ such that $f_{k_n}(z_n) = 0$. Of course, we can choose $\{k_n\}$ to be increasing. □

Důkaz (Corollary)

Assume that there is $w_0 \in \mathbb{C}$ such that $f \neq w_0$ but, for different $z', z'' \in G$ we have $f(z') = w_0 = f(z'')$. WLOG $w_0 = 0$. Choose $\delta > 0$ such that $U(z', \delta) \cap U(z'', \delta) = \varnothing$. By Hurwitz, there are $\{z'_n\} \subset U(z', \delta)$ and $\left\{f_{k'_n}\right\}$ of $\{f_n\}$ such that $z'_n \to z'$ and $f_{k'_n}(z'_n) = 0$. By Hurwitz, there are also $\{z''_n\} \subset U(z'', \delta)$ and $\left\{f_{k''_n}\right\} \subset \left\{f_{k'_n}\right\}$ such that $z''_n \to z''$ and $f_{k''_n}(z''_n) = 0$.

Every $f_{k''_n}$ has at least two different zero points which is contradiction. □

Věta 1.11 (Mittag-Leffler)

Let $\{s_j\} \subset \mathbb{C}$ be one-to-one, $s_j \to \infty$ and

$$s_0 := 0 < |s_1| \leqslant |s_2| \leqslant |s_3| \leqslant \ldots \leqslant |s_j| \leqslant \ldots$$

Let $P_0, P_1, \ldots, P_j, \ldots$ be polynomials such that $P_j(0) = 0$. Then the function

$$f(z) := P_0\left(\frac{1}{z}\right) + \sum_{j=1}^{\infty}\left(P_j\left(\frac{1}{z - s_j}\right) - Q_j(z)\right)$$

for some polynomials Q_j satisfies:

1. *series in definition converges locally uniformly on $\mathbb{C}$, i. e., on any compact $K \subset \mathbb{C}$, the series converges uniformly if we omit finitely many terms which have poles.*
2. *$f \in \mathcal{M}(\mathbb{C})$ and f has poles just at $s_0, s_1, \ldots, s_j, \ldots$, while at s_j the function f has its principal part equal to $P_j\left(\frac{1}{z-s_j}\right)$.*
3. *If $g \in \mathcal{M}(\mathbb{C})$ satisfies previous property, then there is $h \in \mathcal{H}(\mathbb{C})$ such that $g = f + h$ on G.*

Důkaz

Let $k \in \mathbb{N}$. Then $H_k(z) := P_k\left(\frac{1}{z-s_k}\right) \in \mathcal{H}(U(0, |s_k|))$, $H_k(z) = \sum_{n=0}^{\infty} c_n^k z^n$ for $|z| < |s_k|$. There is $n_k \in \mathbb{N}$ such that $Q_k(z) = \sum_{n=1}^{n_k} c_n^k z^n$ satisfies $|H_k(z) - Q_k(z)| < \frac{1}{2^k}$, $|z| \leqslant \frac{|s_k|}{2}$ (*).

Let $K \subset \mathbb{C}$ be a compact. Choose $k_0 \in \mathbb{N}$ such that $K \subset \overline{U(0, |s_{k_0}|/2)}$. If $k > k_0$, (*) holds on K which implies 1. obviously, 2. is valid.

3. follow from the fact that $g - f \in \mathcal{M}(\mathbb{C})$ has all isolated singularities removable. $\square$

2 Zero points of holomorphic functions

Tvrzení 2.1

Let f be non-zero holomorphic function on a simply connected domain (G is domain, and $\mathbb{S}\backslash G$ is connected) $G \subset \mathbb{C}$. Then there is $L \in \mathcal{H}(G)$ such that $f = e^L$ on G.

Důkaz
1) Let $L \in \mathcal{H}(G)$ and $f = e^L$ on G. Then $f' = L' \cdot e^L$ and $f'/f = L'$.

2) Since G is a simply connected domain and $f'/f \in \mathcal{H}(G)$, by Cauchy theorem, there is $L_0 \in \mathcal{H}(G)$ such that $L_0' = f'/f$.

3) On G we have $\left(f \cdot e^{-L_0}\right)' = e^{-L_0} \cdot (f' - L_0' \cdot f) = 0$ on G, hence $f \cdot e^{-L_0} = e^c$ is constant, i. e. $c \in \mathbb{C}$. Put $L := L_0 + c$. □

Poznámka
Polynomial $f(z) = \prod_{j=1}^n (z - z_j)$ has zero points just at $z_1, \dots, z_n$ and their multiplicity corresponds to their occurrence.

Let $g \in \mathcal{H}(\mathbb{C})$ have the same zero points including multiplicity as f. Then there is $L \in \mathcal{H}(\mathbb{C})$ such that $g = f \cdot e^L$ on $\mathbb{C}$. (Proof: use previous tvrzeni for g/f.)

Poznámka (Notation)
Let $\{a_j\} \subset \mathbb{C}$. Then we define

$$\prod_{j=1}^{\infty} a_j := \lim_{n \to \infty} \prod_{j=1}^{n} a_j,$$

if the limit on the right-hand side exists.

Tvrzení 2.2

Let $0 \neq z_j \to \infty$ and $k \in \mathbb{N}_0$ (multiplicity of 0 as zero point). Then consider

$$f(z) := z^k \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right).$$

It sometimes converges and then f has zero points in z_j with right multiplicities.

Věta 2.3 (On infinite product)

Let M be a set (in $\mathbb{C}$), $u_j : M \to \mathbb{C}$ be bounded and $\sum_{j=1}^{\infty} |u_j|$ converges uniformly on M. Then $p_n := \prod_{j=1}^{n}(1 + u_j)$ converge uniformly to a function $f : M \to \mathbb{C}$, and it holds that $f = \prod_{j=1}^{\infty}(1 + u_{n(j)})$ on M, where n is bijection onto $\mathbb{N}$.

If $z_0 \in M$, then $f(z_0) = 0$ if and only if $u_{j_0}(z_0) = -1$ for some $j_0 \in \mathbb{N}$.

Důkaz

Denote $p_n^* := \prod_{j=1}^n (1+|u_j|)$. Then $p_n^* \leqslant \exp\left(\sum_{j=1}^n |u_j|\right)$ and $|p_n - 1| \leqslant p_n^* - 1$ (from $1 + x \leqslant e^x$ and the second inequality by induction on n: $n = 1$ yes, $p_{n+1} - 1 = p_n(1 + u_{n+1}) - 1 = (p_n - 1)\cdot(1 + u_{n+1}) + u_{n+1}$ so $|p_{n+1} - 1| \leqslant (p_n^* - 1)\cdot(1 + |u_{n+1}|) + |u_{n+1}| = p_{n+1}^* - 1$).

$\sum_{j=1}^\infty |u_j|$ is bounded on M, because there is $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^\infty |u_j| < 1$. By inequalities there is $C \in (0, +\infty)$ such that $|p_n| \leqslant C\ \forall n \in \mathbb{N}$.

Let $0 < \varepsilon < \frac{1}{2}$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty |u_n| < \varepsilon$ on M. Let $\{n_1, n_2, \dots\}$ be a permutation of $\mathbb{N}$ and $q_m := \prod_{j=1}^m (1 + u_{n_j})$, $m \in \mathbb{N}$. Let $n \geqslant n_0$ and $m \in \mathbb{N}$ be such that $\{n_1, \dots, n_m\} \supseteq [n]$. Then

$$|q_m - p_n| = |p_n \cdot \left(\prod_{n_j > n, j \in [m]} (1 + u_{n_j}) - 1\right) \leqslant |p_n| \left(\prod_{\dots} (1 + |u_{n_j}|) - 1\right) \leqslant |p_n| \cdot \left(e^{\sum_{\dots} |u_{n_j}|} - 1\right) \leqslant |p_n| \cdot (e^\varepsilon -$$

If $n_j = j\ \forall j \in \mathbb{N}$, then $q_m = p_m$ and we get $\forall m > n : |q_m - p_n| < 2C\varepsilon$, so $p_n \rightrightarrows f$ on M. Moreover we have, for $n \geqslant n_0$, $|p_n - p_{n_0}| \leqslant 2\varepsilon |p_{n_0}|$, so $|p_n| \geqslant |p_{n_0}| - |p_n - p_{n_0}| \geqslant (1 - 2\varepsilon)|p_{n_0}|$. For $n \to \infty$: $|f| \geqslant (1 - 2\varepsilon)|p_{n_0}|$, hence $f(z_0) = 0 \Leftrightarrow p_{n_0}(z_0) = 0$.

If n_j is any, then $q_m \rightrightarrows f$ on M. □

Důsledek

Let $G \subset \mathbb{C}$ be open, $f_n \in \mathcal{H}(G)$ and $f_n \not\equiv 0$ on any component of G. We assume $\sum_{n=1}^\infty |1 - f_n|$ converges locally uniformly on G. Then $f = \prod_{n=1}^\infty f_n$ converges locally uniformly on G, $f \in \mathcal{H}(G)$ and the resulting infinite product f does not depend on the order of functions f_n. Moreover, we have

$$n_f(s) = \sum_{k=1}^\infty n_{f_k}(s), \qquad s \in G$$

where $n_f(s)$ is multiplicity of a zero point s of f. Here we put $n_f(s) = 0$ if $f(s) \neq 0$.

Poznámka

Moreover the ? in previous sum contains only finitely many non-zero terms for any $s \in G$.

Důkaz

Sufficient to prove previous equality. Let $s \in G$. There is a neighbourhood V of s such that $f_n \rightrightarrows 1$ on V. Choose $n_0 \in \mathbb{N}$ such that $f_n \neq 0$ on V for $n > n_0$. By previous theorem, we get $\prod_{n=n_0+1}^\infty f_n \neq 0$ on V. Since $f = (\prod_{n=1}^{n_0} f_n) \cdot \left(\prod_{n=n_0+1}^\infty f_n\right)$ we get $n_f(s) = \sum_{k=1}^{n_0} n_{f_k}(s) = \sum_{k=1}^\infty n_{f_k}(s)$. □

Příklad (Homework)

Under the assumption of previous corollary prove that

$$\frac{f'}{f} = \sum_{n=1}^{\infty} \frac{f'_n}{f_n} \text{ on } G\backslash N_f.$$

Například (Euler formula)

$$\sin(\pi z) = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

Lemma 2.4 (Weierstrass's factor)

Let $E_0(z) := (1-z)$ *and* $E_m(z) := (1-z)\cdot e^{z+\ldots+\frac{z^m}{m}}$, $z \in \mathbb{C}$, $m \in \mathbb{N}$. *Then* $|1-E_m(z)| \leqslant |z|^{m+1}$, $|z| \leqslant 1$.

Důkaz

$$E'_m(z) = e^{z+\ldots+\frac{z^m}{m}} \cdot (-1 + (1-z)\cdot(1+\ldots+z^m)) = -z^m \cdot e^{z+\ldots+\frac{z^m}{m}} = -z^m \cdot \sum_{k=0}^{\infty} b_k z^k,$$

where $b_0 = 1$, $b_k \geqslant 0$, $k \in \mathbb{N}$. Hence

$$E_m(0) - E_m(z) = 1 - E_m(z) = -\int_{[0,z]} E'_m(w)dw = +\sum_{k=0}^{\infty} c_k z^{k+m+1}$$

with $c_k = \frac{b_k}{m+k+1} \geqslant 0$.

By this, if $|z| \leqslant 1$, $z \neq 0$, then $\left|\frac{1-E_m(z)}{z^m}\right| \leqslant \sum_{k=0}^{\infty} c_k = 1 - E_m(1) = 1$. □

Věta 2.5 (Weierstrass factorization in $\mathbb{C}$)

Let $k \in \mathbb{N}_0$ *and* $0 \neq z_i \to \infty$. *Then there is* $\{m_j\} \subset \mathbb{N}_0$ *such that*

$$f(z) = z^k \cdot \prod_{j=1}^{\infty} E_{m_j}\left(\frac{z}{z_j}\right)$$

converges locally uniformly on $\mathbb{C}$, $f \in \mathcal{H}(\mathbb{C})$ *and* f *has at* 0 *zero point of multiplicity* K *and 'non-zero' zero points just at* $z_1, z_2, \ldots, z_j, \ldots$, *and their multiplicity corresponds to their occurrence in* $\{z_j\}$. *We can always take* $m_j := j-1$, $j \in \mathbb{N}$.

If $g \in \mathcal{H}(\mathbb{C})$ *has the same zero points as* f *including multiplicities, then there is* $L \in \mathcal{H}(\mathbb{C})$ *such that* $g = f \cdot e^L$ *on* $\mathbb{C}$.

Důkaz
By the previous corollary, we know the product converges locally uniformly in $\mathbb{C}$ if $\sum_{j=1}^{\infty} |1-E_{m_j}\left(\frac{z}{z_j}\right)|$ converges locally uniformly on $\mathbb{C}$. By lemma, this is true if $\sum_{j=1}^{\infty} \left|\frac{z}{z_j}\right|^{m_j+1}$ converges locally uniformly on $\mathbb{C}$.

Let $r > 0$ and $|z| \leqslant r$. Choose $j_0 \in \mathbb{N}$ such that $\frac{r}{|z_j|} < \frac{1}{2}$ for $j \geqslant j_0$. If $m_j := j-1$, then $\left|\frac{z}{z_j}\right|^j \leqslant \frac{1}{2^j}$, $j \geqslant j_0$ and $|z| \leqslant r$. So, for $m_j := j-1$, sum converges uniformly on $|z| \leqslant r$. ▫

Poznámka
If $\sum_{j=1}^{\infty} \frac{1}{|z_j|} < +\infty$, take $m_j = 0$. If $\sum_{j=1}^{\infty} \frac{1}{|z_j|^2} < +\infty$, take $m_j = 1$. Etc.

Věta 2.6 (Weierstrass factorization in a general open set)

Let $G \subsetneq \mathbb{S}$ be open, $N \subset G$ have no limit points in G and $n : N \to \mathbb{N}$. Then there is $f \in \mathcal{H}(G)$ such that $N_f = N$ and $n_f(s) = n(s)$, $s \in N_f$.

Důkaz
WLOG $\infty \in G \backslash N$. Then $K := \mathbb{S}\backslash G = \mathbb{C}\backslash G$ is compact in $\mathbb{C}$. For a finite N it is obvious. Assume that N is (infinite) countable. We put points of N into the sequence $s_1, s_2, \ldots, s_n$ such that any $s \in N$ occurs in $\{s_n\}$ just $n(s)$ times. For any n, take $t_n \in K$ such that $|s_n - t_n| = \operatorname{dist}(s_n, K)$, $n \in \mathbb{N}$.

Then „$|s_n - t_n| \to 0$“: Let $\varepsilon > 0$ and $\{n_k\} \subset \mathbb{N}$ such that $|s_{n_k} - t_{n_k}| \geqslant \varepsilon$, i. e., $\operatorname{dist}(s_{n_k}, K) \geqslant \varepsilon$. If s_∞ is a limit point of s_{n_k}, then $\operatorname{dist}(s_\infty, K) \geqslant \varepsilon$. Hence $s_\infty \in G$, a contradiction.

Put $f(z) := \prod_{n=1}^{\infty} E_n\left(\frac{s_n - t_n}{z - t_n}\right)$, $z \in G$. The infinite product converges locally uniformly on G. In fact, let L be a compact in G. Put $r_n := 2 \cdot |s_n - t_n|$. Since $\operatorname{dist}(L, K) > 0$, there is $n_0 \in \mathbb{N}$ such that $|z - t_n| > r_n$, $\forall z \in L$, $\forall n \geqslant n_0$. So

$$\left|\frac{s_n - t_n}{z - t_n}\right| < \frac{1}{2} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

By lemma on Weierstrass factors, we get

$$\left|1 - E_n\left(\frac{s_n - t_n}{z - t_n}\right)\right| < \frac{1}{2^n} \qquad \forall z \in L \ \forall n \geqslant n_0.$$

Now use theorem on infinite product. ▫

Lemma 2.7

If $G \subseteq \mathbb{C}$ is open and $f \in \mathcal{M}(G)$, then there are $g, h \in \mathcal{H}(G)$ such that $f = \frac{g}{h}$ on G.

Důkaz
Let P_f be the set of poles of f. By Weierstrass factorization, we construct $h \in \mathcal{H}(G)$ such that $N_h = P_f$ and $n_h = p_f$ on P_f. Put $g := f \cdot h$. Then $g \in \mathcal{H}(G)$ because at the points of P_f g has a removable singularities. □

3 The space H(G)

Poznámka (Arzela–Ascoli theorem)
Let $\mathcal{F} \subset \mathcal{C}(K)$ and let the functions of $\mathcal{F}$ be equibounded (i.e. $\exists M \in (0, +\infty)\ \forall f \in \mathcal{F} : |f| \leqslant M$ on K) and equicontinuous (i.e. $\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in \mathcal{F}\ \forall x, y \in K : \varrho(x,y) < \delta \implies |f(x) - f(y)| < \varepsilon$, where ϱ is metric on K). Then every $\{f_n\} \subset \mathcal{F}$ has $\{f_{n_k}\}$ which is uniformly convergent on K.

3.1 The space C(G)

Definice 3.1

Let $G \subseteq \mathbb{C}$, then $\mathcal{C}(G) := \{f : G \to \mathbb{C} | f \text{ continuous}\}$.

Tvrzení 3.1

For $f_n, f \in \mathcal{C}(G)$ and K_m compact in G such that $\bigcup_{m=1}^{\infty} K_m = G$ and $\forall m \in \mathbb{N} : K_m \subseteq \operatorname{int} K_{m+1}$, TSAE:

- $f_n \overset{loc.}{\rightrightarrows}$ *on G;*
- *for any compact K in G, $\|f_n - f\| \to 0$, where $\|f\|_K := \sup_K |f|$ is a seminorm on $\mathcal{C}(G)$;*
- $\forall m \in \mathbb{N} : \|f_n - f\|_{K_m} \to 0$ *for $u \to \infty$;*
- $\varrho(f_n, f) \to 0$, *where* $\varrho(f_n, f) := \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|f_n - f\|_{K_m}}{1 + \|f_n - f\|_{K_m}}$.

Důkaz
„1 ⇔ 2 $\implies$ 3“ is obvious. „2 $\impliedby$ 3“: Let K be a compact in G. Then $K \subset K_{m_0}$ for come $m_0 \in \mathbb{N}$. Then $\|f_n - f\|_K \leqslant \|f_n - f\|_{K_{m_0}}$. „3 ⇔ 4“ homework. □

Poznámka
$(\mathcal{C}(G), \varrho)$, where ϱ is defined in previous tvrzeni, is complete metric space and $\mathcal{H}(G)$ is closed subspace.

ϱ is not canonical, it depends on the choice of $\{K_m\}$.

The convergence / the topology on $\mathcal{C}(G)$ is given by the system of seminorms $\|\cdot\|_K$ for any compact K in G.

Věta 3.2 (Moore–Osgood, Montöl)

Let $G \subset \mathbb{C}$ be open and let $\{f_n\} \subset \mathcal{H}(G)$ be locally equibounded (i.e. on every compact K in G $\{f_n\}$ is equibounded). Then there is $\{f_{n_k}\}$ which converges locally uniformly on G.

Důkaz

First step: Let $\overline{U(z_0, 2r)} \subset G$ and $\varphi(t) := z_0 + 2re^{it}$, $t \in [0, 2\pi]$. Let $z_1, z_2 \in \overline{U(z_0, r)}$. Then by the Cauchy formula we get $f_n(z_j) = \frac{1}{2\pi i} \int_\varphi \frac{f_n(z)}{z - z_j} dz$. There is $M \in (0, +\infty)$ such that $\forall n \in \mathbb{N}$ $|f_n| \leqslant M$ on $\langle\varphi\rangle$. Then we have

$$|f_n(z_1) - f_n(z_2)| = \frac{1}{2\pi} \left| \int_\varphi f_n(z) \cdot \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz \right| \leqslant$$

$$\leqslant \frac{2\pi \cdot 2r}{2\pi} \cdot M \cdot \frac{|z_1 - z_2|}{r^2}$$

$\left(\left|\frac{1}{z-z_1} - \frac{1}{z-z_2}\right| = \left|\frac{z_1 - z_2}{(z-z_1)\cdot(z-z_2)}\right| \leqslant \frac{|z_1-z_2|}{r^2}\right)$.

By this $\{f_n\}$ are equicontinuous on $\overline{U(z_0, r)}$, and by Arzela–Ascoli, there is $\{f_{n_k}\}$ which is uniformly convergent on $\overline{U(z_0, r)}$.

Second step: Let us cover the set G by $U_j = U(z_j, r_j)$, $j \in \mathbb{N}$, such that $\overline{U(z_j, 2r_j)} \subset G$. Then use a diagonal choice: 1. By first step choose $\left\{f_{n_k^1}\right\}$ of $\{f_n\}$ such that $\left\{f_{n_k^1}\right\}$ converges uniformly on $\overline{U_1}$. 2. By first step choose $\left\{f_{n_k^2}\right\}$ subsequence of $\left\{f_{n_k^1}\right\}$ such that $\left\{f_{n_k^2}\right\}$ converges uniformly on $\overline{U_2}$ and so on.

Then $\left\{f_{n_k^k}\right\}_{k=1}^{\infty}$ converges uniformly on any $\overline{U_j}$, i.e., locally uniformly on G. □

Definice 3.2

Let E be a (complex) linear space and let $\mathcal{P}$ be a system of seminorms on E. Then $(E, \mathcal{P})$ is called locally convex space (LCS). In $(E, \mathcal{P})$ we define:

- convergence: $f_n \to f \Leftrightarrow \forall p \in \mathcal{P} : p(f_n - f) \to 0$;
- topology τ is the weakest topology on E for which all $p \in \mathcal{P}$ are continuous;
- $\mathcal{F} \subset E$ is bounded if $\mathcal{F}$ is bounded with respect to any $p \in \mathcal{P}$, i.e.,

$$\forall p \in \mathcal{P}\ \exists C \in (0, +\infty) : p(f) \leqslant C\ \forall f \in \mathcal{F};$$

- the dual space to $(E, \mathbb{P})$ is defined as

$$E^* := \{L : E \to \mathbb{C} | L \text{ linear and continuous}\}.$$

Poznámka
$\mathcal{C}(G)$ is the so-called Fréchet space, i.e., completely metrizable LCS. So is $\mathcal{H}(G)$ because $\mathcal{H}(G)$ is closed subspace of $\mathcal{C}(G)$.

Topology τ on $\mathcal{C}(G)$ is generated by the system of seminorms

$$\mathcal{P} := \{\| \cdot \|_K | K \text{ is compact in } G\}.$$

$U \subset \mathcal{C}(G)$ is neighbourhood of $f \in \mathcal{C}(G)$ iff there are a compact $K \in G$ and $\varepsilon > 0$ such that

$$U \supset U_{K,\varepsilon}(f) := \{g \in \mathcal{C}(G) | \|g - f\|_K < \varepsilon\}.$$

Důkaz
„ $\Longleftarrow$ “: obvious. „ $\Longrightarrow$ “: There are $m \in \mathbb{N}$, compact, $K_1, \dots, K_m$ in G and $\varepsilon_1, \dots, \varepsilon_m > 0$ such that

$$U \supset \bigcap_{j=1}^{m} U_{K_j,\varepsilon_j}(f) \supset U_{K,\varepsilon}(f),$$

where $K := K_1 \cup \dots \cup K_m$ and $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_m\} > 0$. $\square$

Poznámka
Let $X = \mathcal{H}(G)$. Then in the sense of (LCS) $\mathcal{F} \subset \mathcal{H}(G)$ is bounded iff in the functions of $\mathcal{F}$ are locally equibounded on G. By the Montal theorem, we get $\overline{\mathcal{F}}$ is a compact in $\mathcal{H}(G)$. Easily we get that $\mathcal{F} \subset X$ is compact iff $\mathcal{F}$ is closed and bounded in X.

4 The dual space $\mathcal{H}^*(G)$

Poznámka
1. Let $G = \mathbb{D} := \{z \in \mathbb{C} | |z| < 1\}$. Let $L \in \mathcal{H}^*(\mathbb{D})$. Let $f \in \mathcal{H}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, and $R := \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}} \geqslant 1$. Then

$$L(f) = L(\sum_{n=0}^{\infty} a_n z^n) = L\left(\lim_{n\to\infty} \sum_{k=0}^{n} a_k z^k\right) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k L(z^k) = \sum_{n=0}^{\infty} a_n \cdot b_n,$$

where $b_n := L(z^n) \in \mathbb{C}$. We show $r := \limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$:

If $r > 1$, then for $a_n := 1$, $n \in \mathbb{N}_0$, we get $\sum_{n=0}^{\infty} a_n \cdot b_n$ is divergent. If $r = 1$, then there is $\{n_k\}$ such that such that $0 \neq \sqrt[n_k]{|b_{n_k}|} \to 1$. Putting $a_n = \frac{1}{b_{n_k}}$, $n = n_k$, we get $\sum_{n=0}^{\infty} a_n b_n$ is divergent.

Conclusion: $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\{b_n\} \subset \mathbb{C}$ such that $\limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$ and $L(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}(\mathbb{D})$. In addition, $b_n = L(z^n)$, $n \in \mathbb{N}_0$. ($\Longleftarrow$ obvious, HW.)

Poznámka (Integral form of L)
Let $\{b_n\} \subset \mathbb{C}$ and $r := \limsup_{n\to\infty} \sqrt[n]{|b_n|} < 1$. Define

$$\lambda(z) := \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}, \qquad |z| > r.$$

Of course, $\lambda \in \mathcal{H}(\mathbb{S} \setminus \overline{U(0,r)})$, $\lambda(\infty) = 0$ and $b_n = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$. Here $\lambda^{(k)}(\infty) := \left(\lambda\left(\frac{1}{z}\right)\right)^{(k)}(0)$.

Let $R \in (r,1)$ and $\varphi(t) := Re^{it}$, $t \in [0,2\pi]$. Let $f \in \mathcal{H}(\mathbb{D})$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Then

$$\frac{1}{2\pi i}\int_\varphi f(z)\cdot\lambda(z)dz = \frac{1}{2\pi i}\int_\varphi \left(\sum_{n=0}^{\infty} a_n \cdot z^n\right)\cdot\left(\sum_{m=0}^{\infty}\frac{b_m}{z^{m+1}}\right)dz =$$

$$= \frac{1}{2\pi i}\int_\varphi \sum_{n,m=0}^{\infty} a_n b_m z^{n-m-1} dz = \sum_{n,m=0}^{\infty} a_n \cdot b_m \cdot \frac{1}{2\pi i}\int_\varphi z^{n-m-1}dz = \sum_{n=0\infty} a_n \cdot b_n = L(f).$$

Definice 4.1 (Notation)

Let $A \subset \mathbb{S}$. Then a function f is holomorphic on A if f is holomorphic on some open superset $U \supset A$. Let f_1, f_2 be holomorphic function on A. We say that $f_1 \sim f_2$ if there are open $U_1, U_2 \subset \mathbb{S}$ such that $A \subset U_1 \cap U_2$, $f_1 \in \mathcal{H}(U_1)$, $f_2 \in \mathcal{H}(U_2)$ and $f_1 = f_2$ on $U_1 \cap U_2$. Denote $\mathcal{H}(A) := \{[f] | f \text{ is holomorphic on } A\}$, where $[f]$ is an equivalence class for $\sim$. As usual, we do not often distinguish between $[f]$ and f.

We have that $\lambda \in \mathcal{H}_0(\mathbb{S}\setminus\mathbb{D}) := \{\mu \in \mathcal{H}(\mathbb{S}\setminus\mathbb{D}) | \mu(\infty) = 0\}$. Moreover, we have

$$(*) L(f) = \frac{1}{2\pi i}\int_\varphi f(z)\cdot\lambda(z)dz, \qquad f \in \mathcal{H}(\mathbb{D});$$

$$L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, \qquad n \in \mathbb{N}_0;$$

$$\lambda(w) = L\left(\frac{1}{w-z}\right), \qquad |w| \geqslant 1.$$

Důkaz
In fact, we have

$$L\left(\frac{1}{w-z}\right) = L\left(\sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}\right) = \sum_{n=0}^{\infty} \frac{b_n}{w^{n+1}} = \lambda(w),$$

because $\frac{1}{w-z} = \frac{1}{w} \cdot \frac{1}{1-\frac{1}{w}} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$, $z \in \mathbb{D}$. □

Poznámka (Conclusion)

$$\mathcal{H}^*(\mathbb{D}) = \mathcal{H}_0(\mathbb{S}\backslash\mathbb{D}).$$

In particular, $L \in \mathcal{H}^*(\mathbb{D})$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash\mathbb{D})$ such that $(*)$ hold true.

Příklad (Birkhoff)
There is a universal entire function, i.e., $f \in \mathcal{H}(\mathbb{C})$ such that $\overline{\{\tau_\gamma(f) | \gamma \in \mathbb{C}\}} = \mathcal{H}(\mathbb{C})$, where $\tau_\gamma(f) := f(z-\gamma)$, $z, \gamma \in \mathbb{C}$.

Řešení
HW.

Poznámka
2. Let $G = \bigcup_{j=1}^n D_j$ with $D_j = U(z_j, r_j)$ and $D_j \cap D_k = \varnothing$ for $j = k$.

Let $L \in \mathcal{H}^*(G)$. For $j \in [n]$, put $L_j(d) := L(\tilde{f})$ for $f \in \mathcal{H}(D_j)$ and $\tilde{f} := f$ on D_j and $\tilde{f} := 0$ on D_k, $k \neq j$. Then

$$L(f) = \sum_{j=1}^{n} L_j(f|_{D_j}), \qquad f \in \mathcal{H}(G).$$

By 1., for each $j \in [n]$, there are $\tilde{r}_j \in (0, r_j)$ and $\lambda_j \in \mathcal{H}_0(\mathbb{S}\backslash\overline{U(z_j, \tilde{r}_j)})$ such that

$$L_j(f) = \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz, \qquad f \in \mathcal{H}(D_j),$$

where $\varphi_j(t) := z_j + R_j e^{it}$, $t \in [0, 2\pi]$ for some $R_j \in (\tilde{r}_j, r_j)$.

In addition, we have

$$L_j(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}, \qquad n \in \mathbb{N}_0.$$

If $f \in \mathcal{H}(G)$, then $L(f) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\varphi_j} f(z) \cdot \lambda_j(z) dz$.

$\overset{?}{\Longrightarrow}$ $L(f) = \frac{1}{2\pi i} \int_\Gamma f(z) \cdot \lambda(z) dz$, where $\Gamma := \{\varphi_1, \dots, \varphi_n\}$ and $\lambda := \sum_{j=1}^n \lambda_j$.

? holds true because $\int_{\varphi_j} f(z) \cdot \lambda_k(z) dz = 0$ for $k \neq j$ by Cauchy ($f(z) \cdot \lambda_k(z) \in \mathcal{H}(D_j)$).

We have $L(z^n) = \frac{\lambda^{(n+1)}(\infty)}{(n+1)!}$, $n \in \mathbb{N}_0$.

Poznámka (Conclusion)
($G = \bigcup_{j=0}^n D_j$.) $\mathcal{H}^*(G) = \mathcal{H}_0(\mathbb{S} \backslash G)$. Indeed, $L \in \mathcal{H}^*(G)$ iff there is a unique $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash G)$ such that last 2 equation hold true.

5 Hahn–Banach theorem

Lemma 5.1

Let $L : E \to \mathbb{C}$ be linear. Then $L \in E^*$ iff there is a compact K in G and $M \in [0, +\infty)$ such that $|L(f)| \leqslant M \cdot \|f\|_K$, $f \in E$.

Důkaz
„ $\Longleftarrow$ “ from continuity of $\|\cdot\|_K$. „ $\Longrightarrow$ “: Since $U := L^{-1}(\mathbb{D})$ is a neighbourhood of $\mathbf{o}$ in E, there are a compact K in G and $\varepsilon > 0$ such that $U \supseteq U_{K,\varepsilon}(0) := \{f \in E | \|f\|_K < \varepsilon\}$. Let $f \in E$.

1. Let $\|f\|_K \neq 0$. Then

$$\left| L \left(\frac{f}{\|f\|_K} \cdot \frac{\varepsilon}{2} \right) \right| < 1,$$

hence $|L(f)| < \frac{2}{\varepsilon} \|f\|_K$. Put $M := \frac{2}{\varepsilon}$.

2. Let $\|f\|_K = 0$. Then for any $n \in \mathbb{N}$, we have $\|nf\|_K = 0$, so $|L(n \cdot f)| < 1$, $|L(f)| < \frac{1}{n} \to 0$, $L(f) = 0$. □

Věta 5.2 (Hahn–Banach)

Let A be a linear subspace of E. Then

- *if $L \in A^*$, then there is $\tilde{L} \in E^*$ such that $\tilde{L}|_A = L$;*
- *if A is closed and $0 \neq b \in E \backslash A$, then there is $L \in E^*$ such that $L(b) = 1$ and $L = 0$ on A;*
- *$\overline{A} = E$ iff ($L \in E^*$, $L = 0$ on $A \implies L = 0$ on E).*

Důkaz
„1.“ Use lemma and algebraic version of HB theorem.

„2. + 3.“ can be proved as for Banach space. □

Věta 5.3 (Runge (special))

Let $G \subset \mathbb{C}$ be a finite union of pairwise open discs as in above "poznamka"s. Then for each $f \in \mathcal{H}(G)$ there are polynomials P_n, $n \in \mathbb{N}$, such that $P_n \overset{loc.}{\rightrightarrows} f$ on G.

Důkaz
Let $\mathcal{P} := \mathrm{LO}\{1, z, \ldots\}$ be the space of polynomials. Then $\mathcal{P} \subset \mathcal{H}(G)$. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathcal{P}$. We know that there is $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash G)$ such that ? is valid. So, $\lambda^{(n)}(\infty) = 0$, $n \in \mathbb{N}_0$. By the uniqueness theorem, we get $\lambda \equiv 0$, so $L = 0$ on $\mathcal{H}(G)$ (because $L = 0$ fits and is uniquely determined by λ). By HB theorem, $\overline{\mathcal{P}} = \mathcal{H}(G)$. □

Věta 5.4 (Cauchy formula for compact)

Let $G \subset \mathbb{C}$ be open, $K \subset G$ compact. Then there is a cycle $\Gamma \subset G$, $K \subseteq \operatorname{int}\Gamma \subseteq G$ and $\forall a \in \operatorname{int}\Gamma : \operatorname{ind}_\Gamma a = 1$.

In addition

$$\forall f \in \mathcal{H}(G) : \int_\Gamma f = 0 \wedge \forall a \in \operatorname{int}\Gamma : f(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz.$$

Poznámka
„In addition“ follows from the properties of Γ and residue's theorem for cycles, but we prove it directly.

Důkaz
Choose $0 < \delta < \frac{1}{2}\operatorname{dist}(K, \mathbb{C}\backslash G)$, if $G \subsetneq \mathbb{C}$, otherwise, if $G = \mathbb{C}$, take $\delta := 1$. For $m, n \in \mathbb{Z}$ let $Q_{m,n}$ be the closed square with edges (parallel to the axes) with length δ, and such that $m\delta + in\delta$ is the lower left vertex of $Q_{m,n}$.

Denote $Q^* := \{Q_{n,m} | Q_{n,m} \cap K \neq \emptyset\}$, $U := \int(\bigcup Q^*)$. Q^* is finite because of compactness of K. Of course, $K \subseteq U \subseteq \bigcup Q^* \subseteq G$ (by choice of δ).

We understand $\partial Q_{m,n}$ as a positively oriented curve (piece-wise linear curve). Let Γ be the system of all edges $\Gamma_1, \ldots, \Gamma_k$ of squares of Q^* when we omit those edges which occur twice ($\pm$). Of course, $U = \bigcup Q^* \backslash \operatorname{Im} \Gamma$.

a) Let $f \in \mathcal{H}(G)$. Then $\int_\Gamma f := \sum_{j=1}^k \int_{\Gamma_j} f = \sum_{Q_{m,n} \subset Q^*} \int_{\partial Q_{m,n} f} = 0$.

b) Γ can be viewed as a cycle. In fact the edges $\Gamma_1, \ldots, \Gamma_k$ form finitely many closed simple piece-wise linear curves.

For $j \in [k]$ put $\Gamma_j =: [a_j, b_j]$.

(*) „Every point $c \in \mathbb{C}$ is the starting point of some edge of Γ as many times as it is the ending point of some edge in Γ“:

Take a polynomial P such that $p(c) = 1$ and $p(a) = 0$, if $a \neq c$ and $[a, b] \in \Gamma$ for some b. $p(b) = 0$, if $b \neq c$ and $[a, b] \in \Gamma$ for some a. By a):

$$0 = \int_\Gamma p' = \sum_{j=1}^k \int_{\Gamma_j} p' = \sum_{j=1}^k (p(b_j) - p(a_j)) = \sum_{j=1}^k p(b_1) - \sum_{j=1}^k p(a_j) = \# \text{ } c \text{ is the ending point} - \# \text{ } c \text{ is the sta}$$

„Γ can be viewed as a cycle“: Let L be longest (one of the longest) simple piecewise linear curve consisting of edges of Γ which begins with $\Gamma 1$, i. e.,

- $L = [c_1, c_2, \ldots, c_l] := [c_1, c_2] + [c_2, c_3] + \ldots + [c_{l-1}, c_l]$;
- $\Gamma 1 = [c_1, c_2]$;
- $c_i \neq c_j$ for $i \neq j$ (simple curve);
- l is the biggest.

Since we have (*) there is an index $j \in [l-2]$ such that $[c_l, c_j] \in \Gamma$ (otherwise we would have a longer curve).

$$L' := [c_j, c_{j+1}] + \ldots + [c_{l-2}, c_l] + [c_l, c_j] \subseteq L$$

$\implies L'$ is simple closed piece-wise linear curve. The proper subset Γ', which we get from Γ_k by omitting the edges of L' has again (*). We can process in this fashion for Γ', by finitely many steps we get what we want.

c) Let $f \in \mathcal{H}(G)$ and $a \in U = \operatorname{int}(\bigcup Q^*)$. c1) $a \in \operatorname{int}\left(\tilde{Q}\right)$ for some $\tilde{Q} \in Q^*$. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} dz = \sum_{Q_{m,n} \in Q^*} \frac{1}{2\pi i} \int_{\partial Q_{m,n}} \frac{f(z)}{z-a} dz = f(a)$$

Věta 5.5 (Description of $\mathcal{H}^*(G)$)

Let $G \subset \mathbb{C}$ be open subset. Then $\mathcal{H}^*(G) \simeq \mathcal{H}_0(\mathbb{S}\backslash G)$.

In more detail, let $L \in \mathcal{H}^*(G)$. Then there are a compact $K \subset G$ and $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash K)$ such that

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z)\lambda(z)dz, \qquad f \in \mathcal{H}(G),$$

where Γ is a cycle in $G\backslash K$ with $K \subset \operatorname{int}\Gamma \subset G$ and $\forall z_0 \in \operatorname{int}\Gamma : \operatorname{ind}_\Gamma z_0 = 1$.

In addition, as an element of $\mathcal{H}_0(\mathbb{S}\backslash G)$, λ is uniquely determined by

$$\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k), k \in \mathbb{N}_0, \frac{\lambda^{(k)}(z_0)}{k!} = -L\left(\frac{1}{(z-z_0)^{k+1}}\right), z_0 \in \mathbb{C}\backslash G, k \in \mathbb{N}_0.$$

Důkaz (Step 1)
Let $L \in \mathcal{H}^*(G)$.

Step 1: There are a compact $K \subset G$ and $L_1 \in (\mathcal{C}(K))^* =: \mathcal{C}^*(K)$ such that $L(f) = L_1(f|_K), f \in \mathcal{H}(G)$.

We know that there are a compact $K \subseteq G$ and $C \in (0, +\infty)$ such that $\forall f \in \mathcal{H}(G)$: $|L(f)| \leqslant \|f\|_K \cdot C$.

By the Hahn–Banach theorem we can extend L (from $\mathcal{H}^*(G)$ to $\mathcal{C}^*(G)$) to $\tilde{L} \Leftrightarrow \tilde{L} \in \mathcal{C}^*(G)$ such that $\tilde{L}_2|_{\mathcal{H}}(G) = L$ and $|L(f)| \leqslant \|f\|_K \cdot C$, $f \in \mathbb{C}(G)$.

For each $f \in \mathcal{C}(K)$ put $L_1(f) := \tilde{L}_1(\tilde{f})$, where $\tilde{f} \in \mathcal{C}(G)$ and $\tilde{f}|_K = f$.

Is definition of L_1 correct?

i) by Tietze theorem: $f \in \mathcal{C}(K)$ can be extended to $f \in \mathcal{C}(G)$,

$$\forall f \in \mathcal{C}(K)\ \exists \tilde{f} \in \mathcal{C}(\mathbb{C})\ (\mathcal{C}(G)) : \tilde{f}|_K = f;$$

ii) for any extension we want to get the same result. $\tilde{f}_1, \tilde{f}_2 \in \mathcal{C}(G)$, $\tilde{f}_i|_U = f$, $i = 1, 2$.

$$\Longrightarrow\ |\tilde{L}_1(\tilde{f}_1) - \tilde{L}_1(\tilde{f}_2)| = |\tilde{L}_1(\tilde{f}_1 - \tilde{f}_2)| \leqslant C \cdot \|\tilde{f}_1 - \tilde{f}_2\|_K = C\|f - f\|_K = 0.$$

□

Poznámka ($\mathcal{C}^*(K)$)
By the Riesz representation theorem, for each $L_1 \in \mathcal{C}^*(K)$ there is a unique complex Borel measure μ on K such that

$$L_1(f) = \int_K f d\mu, \qquad \forall f \in \mathcal{C}(K).$$

Důkaz
Step 2: By the Cauchy formula for compact, there is a cycle $\Gamma \subset G$ such that $K \subset \operatorname{int}\Gamma \subset G$, $\forall a \in \operatorname{int}\Gamma : \operatorname{ind}_\Gamma a = 1$ and we have, $\forall f \in \mathcal{H}(G)$:

$$f(z_1) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z_2)dz_2}{z_2 - z_1}, \qquad z_1 \in K.$$

Denote

$$L_2(f) := \frac{1}{2\pi i}\int_\Gamma f(z_2)dz_2, f \in \mathcal{C}(\langle\Gamma\rangle), \qquad F(z_1, z_2) := \frac{f(z_2)}{z_2 - z_1}.$$

Of course $L_2 \in \mathcal{C}^*(\langle\Gamma\rangle)$ and $f(z_1) = L_2(F(z_1, z_3))$, $z_1 \in K$.

Step 3: For a given $f \in \mathcal{H}(G)$,

$$L(f) = L_1(f(z_1)) = L_1(L_2(F(z_1, z_2))) \overset{\text{Fubini}}{L}{}_2(L_1(F(z_1, z_2))),$$

hence

$$L(f) = \frac{1}{2\pi i}\int_\Gamma f(z_2)\cdot\lambda(z_2)dz_2,$$

where

$$\lambda(z_2) := L_1\left(\frac{1}{z_2 - z_1}\right), \qquad z_2 \in \mathbb{C}\backslash K.$$

Step 4: $\lambda \in \mathcal{H}_0(\mathbb{S}\backslash K)$ satisfies „in addition“: Let $U(\infty, \varepsilon) \subset \mathbb{S}\backslash K$. For $u \in P(0, \varepsilon)$, we have

$$\lambda\left(\frac{1}{u}\right) = L_1(\frac{u}{1 - u\cdot z_1}) = L_1\left(\sum_{k=0}^{\infty} z_1^k u^{k+1}\right) = \sum_{k=0}^{\infty} L_1(z_1^k)u^{k+1},$$

hence $\lambda(\infty) = 0$ and

$$\forall k \in \mathbb{N}_0 : \frac{\lambda^{(k+1)}(\infty)}{(k+1)!} = L_1\left(z_1^k\right).$$

Let $U(z_0, \varepsilon) \subset \mathbb{C}\backslash K$. Then $\forall z_2 \in U(z_0, \varepsilon)$:

$$\lambda(z_2) = L_1\left(\frac{1}{z_2 - z_1}\right) = -L_1\left(\sum_{k=0}^{\infty}\frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}\right) = -\sum_{k=0}^{\infty} L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right)(z_2 - z_0)^k;$$

$$\forall z_1 \in K : \frac{1}{z_2 - z_1} = \frac{1}{(z_2 - z_0) - (z_1 - z_0)} = -\frac{1}{z_1 - z_0}\cdot\frac{1}{1 - \frac{z_2 - z_0}{z_1 - z_0}} = -\sum_{k=0}^{\infty}\frac{(z_2 - z_0)^k}{(z_1 - z_0)^{k+1}}.$$

Hence $\frac{\lambda^{(k)}(z_0)}{k!} = -L_1\left(\frac{1}{(z_1 - z_0)^{k+1}}\right)$, $k \in \mathbb{N}_0$.

Step 5: As an element of $\mathcal{H}_0(\mathbb{S}\backslash G)$, λ is uniquely determined by „in addition“. (Proof below.) □

Lemma 5.6

Let $G \subset \mathbb{C}$ be open and K be a compact in G. There is a compact K_1 such that $K \subset K_1 \subset G$ and each component of $\mathbb{S}\backslash K_1$ contains some component of $\mathbb{S}\backslash G$.

Důkaz
Take $n \in \mathbb{N}$ such that $K_1 := \left\{z \in G | \operatorname{dist}(z, \mathbb{C}\backslash G) \geqslant \frac{1}{n}\right\} \cap \overline{U(0,n)} \supset K$. In addition, we have

$$\mathbb{S}\backslash K_1 = \bigcup_{z_0 \in \mathbb{S}\backslash G} U(z_0, \frac{1}{n}).$$

Let V be a component of $\mathbb{S}\backslash K_1$. There is $z_0 \in \mathbb{S}\backslash G$ such that $U\left(z_0, \frac{1}{n}\right) \subset V$. If W is a component of $\mathbb{S}\backslash G$ containing z_0, then $W \subset V$. □

Důkaz (Step 5)
Let $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\backslash G)$ satisfying „in addition". Then there is a compact $K \subset G$ such that $\lambda_1, \lambda_2 \in \mathcal{H}_0(\mathbb{S}\backslash K)$.

By the previous lemma, WLOG we assume that each component V of $\mathbb{S}\backslash K$ intersect $\mathbb{S}\backslash G$. We show $\lambda_1 = \lambda_2$ on $\mathbb{S}\backslash K$.

Let V be any component of $\mathbb{S}\backslash K$ and $z_0 \in V \cap (\mathbb{S}\backslash G) \neq 0$. By „in addition" we have $\lambda_1^{(k)}(z_0) = \lambda_2^{(k)}(z_0)\ \forall k \in \mathbb{N}_0$. By the uniqueness theorem $\lambda_1 = \lambda_2$ on the domain B, so $\lambda_1 = \lambda_2$ on $\mathbb{S}\backslash K$. □

Lemma 5.7 (Fubini)

Let $K_1, K_2 \subset \mathbb{C}$ be compact, $L_j \in \mathcal{C}^*(K_j)$ for $j = 1, 2$ and $F \in \mathcal{C}(K_1 \times K_2)$. Then we have

$$L_1(L_2(F(z_1, z_2))) = L_2(L_1(F(z_1, z_2))).$$

Důkaz (Sketch)
Obviously it holds true for the functions of the following form: $F(z_1, z_2) = f(z_1) \cdot g(z_2)$ for $f \in \mathcal{C}(K_1)$, $G \in \mathbb{C}(K_2)$.

Now we can use the Stone–Weierstrass theorem which show that the linear span of the functions of this form is dense in $\mathcal{C}(K_1 \times K_2)$. □

6 Runge's theorem

Definice 6.1 (Notation)

Let $E \subset \mathbb{C}$ and $m : E \to \mathbb{N} \cup \{\infty\}$. We call $m(e)$ the multiplicity of $e \in E$. We say that (E, m) has a limit point $e \in \mathbb{S}$ if e is a limit point of E, or $e \in E$ with $m(e) = \infty$.

Denote by $\mathcal{F}(E,m)$ system of functions which consists of

- $\frac{1}{z-e}$ if $e \in E \cap \mathbb{C}$, $m(e) < \infty$;
- $\frac{1}{(z-e)^k}$, $k \in \mathbb{N}$ if $e \in E \cap \mathbb{C}$, $m(e) = \infty$;
- z^k, $k \in \mathbb{N}_0$ if $\infty \in E$, $m(\infty) = \infty$.

Věta 6.1 (Runge)

Let $G \subset \mathbb{C}$ be open, $E \subset \mathbb{S} \backslash G$ and $m : E \to N \cup \{\infty\}$. If (E,m) has a limit point in every component of $\mathbb{S} \backslash G$, then the linear span of $\mathbb{F}(E,m)$ is dense in $\mathcal{H}(G)$.

Důkaz

We shall use Hahn–Banach theorem. Let $L \in \mathcal{H}^*(G)$ and $L = 0$ on $\mathbb{F}(E,m)$. We need to show $L = 0$ on $\mathcal{H}(G)$. Let $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash G)$ which represents L in the sense of theorem describing $\mathcal{H}^*(G)$.

If $e \in E \cap \mathbb{C}$, $m(e) < \infty$, then $\lambda(e) = -L\left(\frac{1}{z-e}\right) = 0$. If $e \in E \cap \mathbb{C}$, $m(e) = \infty$, then $\frac{\lambda^{(k)}(e)}{k!} = -L\left(\frac{1}{(z-e)^k}\right) = 0$ $\forall k \in \mathbb{N}_0$. If $\infty \in E$, $m(\infty) = \infty$, then $\frac{\lambda^{k+1}(\infty)}{(k+1)!} = L(z^k) = 0$ $\forall k \in \mathbb{N}_0$.

We show that $\lambda = 0$ in $\mathcal{H}_0(\mathbb{S} \backslash G)$. There is a compact $K \subset G$ such that $\lambda \in \mathcal{H}_0(\mathbb{S} \backslash K)$ and every component of $\mathbb{S} \backslash K$ contains some component of $\mathbb{S} \backslash G$.

Let V be any component of $\mathbb{S} \backslash K$. Then V is domain and V contains a limit point e of (E,m). By the uniqueness theorem, we get $\lambda = 0$ on V, so on $\mathbb{S} \backslash K$. □

Věta 6.2 (Runge, classical version)

Let $G \subset \mathbb{C}$ be open and $f \in \mathcal{H}(G)$. Then there are rational functions R_n, $n \in \mathbb{N}$ with poles outside G such that $R_n \overset{Loc.}{f}$ on G.

If, in addition, $\mathbb{S} \backslash G$ is connected, then there are polynomials P_n, $n \in \mathbb{N}$, such that $P_n \overset{Loc.}{f}$ on G.

Důkaz

„Second part“: Let $E = \{\infty\}$ and put $m(\infty) = \infty$. Then

$$\mathbb{F}(E,m) = \left\{1, z, \dots, z^k, \dots\right\}$$

and by the previous theorem, the polynomials are dense in $\mathcal{H}(G)$.

„First part“: Let $E \subset \mathbb{S} \backslash G$ containing at least one point of every component of $\mathbb{S} \backslash G$. Put $m = \infty$ on E. Then $\mathrm{LO}(\mathcal{F}(E,m))$ is dense in $\mathcal{H}(G)$ and it is a subspace of rational functions with poles outside G. □

Důsledek (Cauchy's theorem for simply connected domains)
Let $G \subset \mathbb{C}$ be open a nd $\mathbb{S} \backslash G$ be connected. If $f \in \mathcal{H}(G)$ and φ is a closed curve in G, then $\int_\varphi f = 0$.

Důkaz
By Runge, there are polynomials P_n such that $P_n \overset{\text{Loc.}}{\rightrightarrows} f$ on G. Then (P_n has a primitive function in $\mathbb{C}$) $0 = \int_\varphi P_n \to \int_\varphi f$. □

Důsledek (Cauchy's theorem for cycles)
Let $G \subset \mathbb{C}$ be open and Γ be a cycle in G (i.e., $\langle\Gamma\rangle \subset G$). Then

$$\left(\forall f \in \mathcal{H}(G) : \int_\Gamma f = 0\right) \Leftrightarrow \operatorname{int}\Gamma \subset G.$$

Důkaz
„ $\Longrightarrow$ “: If $z_0 \in \mathbb{C} \backslash G$, then $f(z) := \frac{1}{z-z_0} \in \mathcal{H}(G)$ and $\operatorname{ind}_\Gamma z_0 = \frac{1}{2\pi i} \int_\Gamma f = 0$.

„ $\Longleftarrow$ “: Let $f \in \mathcal{H}(G)$. By Runge, there are rational R_n with poles outside G such that $R_n \overset{\text{Loc.}}{\rightrightarrows} f$. Then $0 = \int_\Gamma R_n \to \int_\Gamma f$. (First equality is from: Let $\Gamma = \{\varphi_1, \ldots, \varphi_m\}$, where φ_j are closed curves in G. Then $\int_\Gamma R_n = \sum_{j=1}^m \int_{\varphi_j} R_n = \sum_{j=1}^m 2\pi i \sum_{R_n(s)=\infty} \operatorname{res}_s R_n \operatorname{ind}_{\varphi_j} s = 2\pi i \cdot \sum_{R_n(s)=\infty} \operatorname{res}_s R_n \cdot \operatorname{ind}_\Gamma s$, but s lies outside of G, so it is equal to 0.) □

Věta 6.3 (Runge, for compacts)

Let K be a compact in $\mathbb{C}$ and let $S \subset \mathbb{S} \backslash K$ contain at least one point of any component of $\mathbb{S} \backslash K$. Let f be a holomorphic function on K. Then there are rational functions R_n with poles in S such that $R_n \rightrightarrows f$ on K.

Poznámka (Technique: pushing poles)
Each rational function R can be uniquely expressed in the form (rational function has $n \in \mathbb{N}$ poles, and we will write the principal part of Laurent expansion around the pole z_k):

$$R(z) = \sum_{k=1}^{n} \sum_{j=1}^{n_k} \frac{A_j^k}{(z - z_k)^j} + C_0 + C_1 z + \ldots + C_m z^m,$$

where $n, m, n_k \in \mathbb{N}$, $z_k \in \mathbb{C}$ and $A_{n_k}^k \neq 0$, $C_m \neq 0$. Then z_k is a pole of R of multiplicity n_k and ∞ is a pole of R of multiplicity m. A rational function R is a polynomial iff R has a pole at most at ∞.

Notation: Let K be a compact in $\mathbb{C}$, $U \subset \mathbb{S}$ and $U \cap K = \emptyset$. Put $B(K,U) = \overline{\{R|_K | R \text{ is rational with poles in } U\}}^{\mathcal{C}(K)}$. (Remark: $B(K,U)$ is a closed subalgebra of $\mathcal{C}(K)$.)

Theorem (pushing poles): Let K be a compact in $\mathbb{C}$, $U \subset \mathbb{S}$ be a domain, $K \cap U = \emptyset$ and $z_0 \in U$. If R is rational function with poles in U, then $R \in R(K, \{z_0\})$.

Corollary: By theorem, we have $B(K,U) = B(K,z_0)$.

Proof: Put $V := \left\{\xi \in U | \frac{1}{z-\xi} \in B(K,x_0), \text{ for } \xi \in \mathbb{C} \text{ and } z \in B(K,z_0) \text{for } \xi = \infty\right\}$. Of course $B(K,z_0) = B(K,V)$. Indeed, if $\xi \in V$, then $\frac{1}{(z-\xi)^k} \in B(K,z_0)$, for $\xi \in \mathbb{C}$ and $k \in \mathbb{N}$, and $z^k \in B(K,z_0)$ for $\xi = \infty$, $k \in \mathbb{N}$.

Then each rational R with poles in V is contained in $B(K,z_0)$. Hence $B(K,V) \subset B(K,z_0)$. Since $z_0 \in V$, we have $B(K,z_0) \subset B(K,V)$.

„V is closed in U“: Let $\xi_n \in V$, $\xi_n \to \xi_0$ and $\xi_0 \in U$. We need to show that $\xi_0 \in V$. WLOG $\forall n \in \mathbb{N} : \xi_n \in \mathbb{C}$.

„$\xi_0 \in \mathbb{C}$“. Then put $\delta := \operatorname{dist}(\xi_0, K) > 0$. Choose $n_0 \in \mathbb{N}$ such that $\operatorname{dist}(\xi_n, K) \geqslant \frac{\delta}{2}$ for $n > n_0$. Then

$$\frac{1}{z - \xi_n} \rightrightarrows \frac{1}{z - \xi_0}, \qquad z \in K,$$

$$\Longleftarrow \left| \frac{1}{z-\xi_n} - \frac{1}{z-\xi_0} \right| = \frac{|\xi_n - \xi_0|}{|z-\xi_n| \cdot |z - \xi_0|} \leqslant \frac{2}{\delta^2} \cdot |\xi_n - \xi_0| \to 0,$$

if $n > n_0$ and $z \in K$. Hence $\frac{1}{z-\xi_n} \in B(K,z_0)$, so $\xi_0 \in V$.

„$\xi_0 = \infty$“. Then

$$\frac{\xi_n z}{\xi_n - z} = -\xi_n \left(\frac{\xi_n}{z - \xi_n} + 1 \right) \in B(K, z_0).$$

Take $C > 0$ with $\forall z \in K : |z| \leqslant C$. Take $n_0 \in \mathbb{N}$ such that $\forall n > n_0 : |\xi_n| > C$. Then $\forall z \in K : \frac{\xi_n z}{\xi_n - z} \rightrightarrows z$, because

$$\left| \frac{\xi_n z}{\xi_n - z} - z \right| = \frac{|z|^2}{|\xi_n - z|} \leqslant \frac{C^2}{|\xi_n| - C} \to 0.$$

if $n > n_0$ and $z \in K$. Hence $z \in B(K,z_0)$, so $\infty \in V$.

„V is open (so $V = U$)“: Let $\xi_0 \in V$.

„$\xi \in \mathbb{C}$“: Put $\delta := \operatorname{dist}(\xi_0, K) > 0$. Let $\xi \in U(\xi \in U(\xi_0, \delta/2))$. Then

Důkaz
Let f be a holomorphic function on an open set $G \supset K$. Using Runge's theorem for "open sets", there are rational functions $\tilde{R}_n$ with poles outside G such that $\tilde{R}_n \rightrightarrows f$ on K.

„$\tilde{R}_n \in B(K, S)$“: All poles of $\tilde{R}_n$ are contained in a finitely many components $C_1, \dots, C_k$ of $\mathbb{S} \backslash K$. Express $\tilde{R}_n = \tilde{Q}_1 + \dots + \tilde{Q}_k$, where $\tilde{Q}_j$ is a rational function with poles in the domain C_j. For $j \in [k]$ take $s_j \in S \cap C_j$. By pushing poles we have $\tilde{Q}_j \in B(K, s_j)$. For given $\varepsilon > 0$ and $j \in [k]$, there is a rational function Q_j with a pole at s_j such that $|\tilde{Q}_j - Q_j| \leqslant \frac{\varepsilon}{k}$ on K. Put $R_n := Q_1 + \dots + Q_k \in B(K, S)$. Then $|R_n - \tilde{R}_n| \leqslant \varepsilon$ on K. Hence $\tilde{R}_n \in B(K, S)$. $\square$