*Příklad* (1.)
We have argued that the right concept of time derivative for a second order tensor that maps normal-like vectors to tangent-like vectors is introduced by the formula

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t)|_{\mathbf{x}=\chi(\mathbf{X},t)} := \mathbb{F}(\mathbf{X},t)\left[\frac{d}{dt}\left(\mathbb{F}^{-1}(\mathbf{X},t)\mathbb{A}(\chi(\mathbf{X},t),t)\mathbb{F}^{-T}(\mathbf{X},t)\right)\right]\mathbb{F}(\mathbf{X},t)^{T}.$$

Use the formula for the time derivative of the deformation gradient $\frac{d\mathbb{F}}{dt} = \mathbb{L}\mathbb{F}$ and the concept of material time derivative, and show that this reduces to

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t) = \frac{d\mathbb{A}}{dt}(\mathbf{x},t) - \mathbb{L}(\mathbf{x},t)\mathbb{A}(\mathbf{x},t) - \mathbb{A}(\mathbf{x},t)\mathbb{L}(\mathbf{x},t)^{T}.$$

*Důkaz*
První spočítáme $\mathbb{F}\frac{d}{dt}\mathbb{F}^{-1}$ a to tak, že zderivujeme jednotku jako součin:

$$\mathbb{O} = \frac{d}{dt}\mathbb{I} = \left(\frac{d}{dt}\mathbb{F}\right)\mathbb{F}^{-1} + \mathbb{F}\frac{d}{dt}\mathbb{F}^{-1},$$

$$\mathbb{F}\frac{d}{dt}\mathbb{F}^{-1} = -\left(\frac{d}{dt}\mathbb{F}\right)\mathbb{F}^{-1} = -\mathbb{L}\mathbb{F}\mathbb{F}^{-1} = -\mathbb{L}.$$

Transponováním $\left(\frac{d}{dt}\mathbb{F}^{-T}\right)\mathbb{F}^{T} = \left(\mathbb{F}\frac{d}{dt}\mathbb{F}^{-1}\right)^{T} = -\mathbb{L}^{T}$.

Nyní z derivace součinu

$$\overset{\triangledown}{\mathbb{A}} := \mathbb{F}\left[\frac{d}{dt}\left(\mathbb{F}^{-1}\mathbb{A}\mathbb{F}^{-T}\right)\right]\mathbb{F}^{-T} =$$

$$= \mathbb{F}\left(\frac{d}{dt}\mathbb{F}^{-1}\right)\mathbb{A}\mathbb{F}^{-T}\mathbb{F}^{T} + \mathbb{F}\mathbb{F}^{-1}\left(\frac{d}{dt}\mathbb{A}\right)\mathbb{F}^{-T}\mathbb{F}^{T} + \mathbb{F}\mathbb{F}^{-1}\mathbb{A}\left(\frac{d}{dt}\mathbb{F}^{-T}\right)\mathbb{F}^{T} =$$

$$= -\mathbb{L}\mathbb{A}\mathbb{I} + \mathbb{I}\left(\frac{d}{dt}\mathbb{A}\right)\mathbb{I} - \mathbb{I}\mathbb{A}\mathbb{L}^{T} = \frac{d}{dt}\mathbb{A} - \mathbb{L}\mathbb{A} - \mathbb{A}\mathbb{L}^{T}.$$

Teď už by stačilo jen vzít tyto veličiny ne vůči $\mathbf{X}$, ale vůči $\mathbf{x}$. Jen $\mathbb{A}$ musíme správně zderivovat. Podle řetízkového pravidla:

$$\frac{d}{dt}\mathbb{A}(\chi(\mathbf{X},t),t) = \frac{\partial\mathbb{A}}{\partial t}(\chi(\mathbf{X},t),t) + \frac{\partial\mathbb{A}}{\partial x_i}(\chi(\mathbf{X},t),t)\cdot\frac{\partial\chi_i}{\partial t}(\chi(\mathbf{X},t),t) =$$

$$= \frac{\partial\mathbb{A}}{\partial t}(\chi(\mathbf{X},t),t) + (\mathbf{v}(\chi(\mathbf{X},t),t)\cdot\nabla_{\mathbf{x}})\mathbb{A}(\chi(\mathbf{X},t),t).$$

To je ale přesně to, co jsme dostali v časové derivaci materiálu, tedy $\frac{d\mathbb{A}}{dt}(\mathbf{x},t)$, čímž opravdu dostáváme

$$\overset{\triangledown}{\mathbb{A}}(\mathbf{x},t) = \frac{d\mathbb{A}}{dt}(\mathbf{x},t) - \mathbb{L}(\mathbf{x},t)\mathbb{A}(\mathbf{x},t) - \mathbb{A}(\mathbf{x},t)\mathbb{L}(\mathbf{x},t)^{T}.$$

□