

Poznámka (Literature)
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Definice 0.1 (Polish space)

We say TS (X, τ) is polish (PTS) if X is separable and completely metrizable.

Poznámka

Complete compatible metric is not unique: $\tilde{\varrho} = \min \{1, \varrho\}$.

Například

$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, 2 := \{0, 1\}, \omega := \{0, 1, 2, \dots\}$ with discrete topology, Separable Banach space (SBS), metrizable compacts, $2^\omega, \omega^\omega$ (both with product topology).

Věta 0.1 (Baire)

X TS metrizable with complete metric. Then countable intersection of open dense subsets of X is dense in X .

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Důkaz

Without proof. (We should know it already.)

□

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Věta 0.2

X complete metric space, $\{F_n\}$ is decreasing sequence of closed subsets of X , such that $\text{diam}(F_n) \rightarrow 0$. Then $|\bigcap F_n| = 1$.

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Důkaz

Without proof. (We should know it already.)

□

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Věta 0.3

(i) If X_n are PTS, $n \in \omega$. Then $\prod_{n \in \omega} X_n$ is PTS.

(ii) X PTS, $H \subset X$. Then H is PTS $\Leftrightarrow H \in \mathcal{G}_\delta(X)$

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Důkaz ((i))

Let d_n be CCM (complete compatible metric) on X_n , $n \in \omega$. Then

$$d(x, y) := \sum_{n=0}^{\infty} \min \{2^{-n}, d_n(x_n, y_n)\}$$

is CCM on $X = \prod_{n \in \omega} X_n$, where $x = (x_n)$, $y = (y_n)$. („Definition is correct“ is trivial, „ d is metric“ straightforward, „ d is complete“ also easy, compatibility too). □

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┌ *Důkaz* ((ii))

$H = \emptyset$, $H = X$ trivial. Assume $H \neq \emptyset, X$.

„ \implies “: Fix CCM ϱ on H . $V_n := \bigcup \{V \subset X \mid V \text{ open in } X \wedge V \cap H \neq \emptyset \wedge \text{diam}_\varrho(V \cap H) < 2^{-n}\}$, $n \in \omega$. We want to show $H \stackrel{?}{=} \bigcap_{n \in \omega} (V_n \cap \overline{H}) \in \mathcal{G}_\delta$:

„ \subseteq “: $x \in H, n \in \omega, x \in B_\varrho(x, 2^{-n-2}) \subset V_n$.

„ \supseteq “: $x \in V_n \cap \overline{H}$ for every $n \in \omega \implies \exists$ open sets $G_n: x \in G_n, G_n \cap H \neq \emptyset, \text{diam}(G_n \cap H) < 2^{-n}$. We can assume: $G_{n+1} \supset G_n$ (we can use intersection: $G_{n+1} \cap G_n \cap H \stackrel{?}{\neq} \emptyset \iff x \in G_n \cap G_{n+1} \cap \overline{H} \neq \emptyset$).

$\{y\} := \bigcap_{n \in \omega} \overline{G_n \cap H}^H \in H$. For contradiction: $x \neq y \implies \exists O \subset X$ open: $x \notin \overline{O}$, $y \in O$, $G_n \cap H \subset B(y, 2^{-n})$, $n \in \omega \implies \exists n \in \omega G_n \cap H \subset O$, $x \in G_n \cap (X \setminus \overline{O}) \cap \overline{H} \implies G_n \cap (X \setminus \overline{O}) \cap H \neq \emptyset$.

„ \Leftarrow “: fix CCM d on X , $H = \bigcap_{n \in \omega} U_n$, $\emptyset = U_n \neq X$. $F_n := X \setminus U_n$, $\tilde{d}(x, y) = d(x, y) + \sum_{n=0}^{\infty} \min \left\{ 2^{-n}, \left| \frac{1}{\text{dist}(x, F_n)} - \frac{1}{\text{dist}(y, F_n)} \right| \right\}$, $x, y \in H$. Next we verified that \tilde{d} is metric, that \tilde{d} is equivalent with d on H (by convergence), and that (H, \tilde{d}) is complete metric space and separable. TODO? \square

Definition 0.2 (Notation)

$A \neq \emptyset$:

- $A^{<\omega} :=$ finite sequence of elements of $A = \bigcup_{n \in \omega} A^n$;
- $s \in A^k, t \in A^{<\omega} \cup A^\omega$: $s \wedge t := (s_0, s_1, \dots, s_{k-1}, t_0, t_1, \dots)$, where $s = (s_0, \dots, s_{k-1})$, $t = (t_0, t_1, \dots)$;
- $s \in A^{<\omega} \cup A^\omega$: $|s|$ is the number of elements of sequence s ($|s| \in \omega \cup \{\infty\}$);
- $s \in A^{<\omega} \cup A^\omega, k \in \omega, |s| \geq k$, then we denote restriction of s on first k elements as s/k ;
- $s < t$ iff $|t| \geq |s|$ and $s = t/|s|$ ($s \in A^{<\omega}, t \in A^{<\omega} \cup A^\omega$).

1 Baire space ω^ω

Definition 1.1

For $s \in \omega^{<\omega}$ we define Baire interval of s as $\mathcal{N}(s) := \{\nu \in \omega^\omega \mid s < \nu\}$.

$\mathcal{N}(s)$ are clopen ($\mathcal{N}(s) = \omega^\omega \setminus \bigcup \{\mathcal{N}(t) \mid |t| = |s|, t \neq s, t \in \omega^{<\omega}\}$).

$\{\mathcal{N}|s \in \omega^{<\omega}\}$ is base of topology of ω^ω .

Věta 1.1 (Alexandrov–Urysohn)

ω^ω is up to homeomorphism unique nonempty multi-dimension PTS such that every compact has empty interior.

Důkaz

Bez důkazu. □

Důsledek

ω^ω is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

Věta 1.2

Let $X \neq \emptyset$, PTS. Then X is continuous image of ω^ω .

Poznámka

$X \neq \emptyset$ PTS. Then there $\exists F \subset \omega^\omega$, F closed, and continuous injection $\varphi : F \rightarrow X$.

Důkaz

Find CCM on X such that $\text{diam } X \leq 1$. We inductively construct closed $\emptyset \neq A_s \subset X$ for every $s \in \omega^{<\omega}$ such that 1. $A_\emptyset = X$; 2. $\text{diam}(A_s) \leq 2^{-|s|}$; 3. $A_s = \bigcup_{i \in \omega} A_{s^\frown i}$.

Empty set is trivial. Assume we already have A_s . Find $\{x_i | i \in \omega\} \subset A_s$ dense in A_s . $A_{s^\frown i} := A_s \cap \overline{B(x_i, 2^{-|s|-2})} \neq \emptyset$ closed.

Fix $\forall \nu \in \omega^\omega : f(\nu) := x$, where $\{x\} = \bigcap_{k \in \omega} A_{\nu/k} \neq \emptyset$ (intersection of closed nonempty non-increasing sequence of sets). „ f is surjection“: $x \in A_s \xrightarrow{3.} \exists n \in \omega : x \in A_{s^\frown n} \xrightarrow{1.} \forall x \in X \exists \alpha \in \omega^\omega \forall k \in \omega : x \in A_{\alpha/k} \implies x = f(\alpha)$.

„ f continuous“: $f(\mathcal{N}_{\nu/k}) \subset A_{\nu/k}$ for every $\nu \in \omega^\omega$, $k \in \omega$, $\text{diam } A_{\nu/k} \leq 2^{-k}$. □

1.1 Cantor set 2^ω

Tvrzení 1.3

2^ω is up to homeomorphism unique nonempty nuldimensional compact metrizable space without isolated points (without isolated points is called perfect space).

Tvrzení 1.4

Let $X \neq \emptyset$ metrizable, compact. Then X is continuous image of 2^ω .

┌ *Důkaz*

Without proof, but it is similar to the previous one. □

1.2 Hilbert cube $[0, 1]^\omega$

Tvrzení 1.5

Let X be PTS. Then X is homeomorphic to G_δ subset of $[0, 1]^\omega$.

┌ *Důkaz*

X PTS, case \emptyset is trivial, so assume $X \neq \emptyset$, ϱ is CCM on X , $\varrho \leq 1$. Let $\{x_n, n \in \omega\}$ be dense in X . Define $f : [0, 1]^\omega : f(x) = (\varrho(x, x_n))_{n \in \omega}$. $\varrho \leq 1 \implies f(x) \in [0, 1]^\omega$.

„Continuity of f “: $f^{-1}(U) = \bigcap_{i=1}^n B(x_i, b_i) \setminus \overline{B(x_i, a_i)}$ open.

„Injective“: $x \neq y \implies \exists n \in \omega : \varrho(x, x_n) < \varrho(y, x_n) \implies f(x) \neq f(y)$.

„Continuity of f^{-1} “: $f(y^n) \rightarrow f(y) \stackrel{?}{\implies} y^n \rightarrow y$.

$$f(y^n) \rightarrow f(y) \stackrel{?}{\iff} \forall k \in \omega : \varrho(y^n, x_k) \rightarrow \varrho(y, x_k).$$

Let $\varepsilon > 0$ be arbitrary:

$$\exists k \in \omega : \varrho(y, x_k) < \frac{\varepsilon}{3}. \exists n_0 \forall n \geq n_0 : \varrho(y^n, x_k) < \frac{2\varepsilon}{3}.$$

Then

$$\forall n \geq n_0 : \varrho(y^n, y) \leq \varrho(y^n, x_k) + \varrho(x_k, y) < \varepsilon.$$

So $f(X)$ is homeomorphism to $X \implies f(X)$ is PTS $\implies f(X) \in \mathcal{G}_\delta([0, 1]^\omega)$. □

Důsledek

Let X be compact metrizable space. Then X is homeomorphic to some closed subset of $[0, 1]^\omega$.

┌ *Důkaz*

Compact metrizable space is Polish. And compact subset must be closed. □

1.3 $\mathcal{K}(X)$: Hyperspace of compact subsets of X

Definice 1.2

Let X be PTS, denote $\mathcal{K}(X) := \{K \subset X \mid K \text{ is compact}\}$. Vietoris topology on $\mathcal{K}(X)$ is generated by $\{K \in \mathcal{K}(X) \mid K \subset V\}$ for V open and $\{K \in \mathcal{K}(X) \mid K \cap V \neq \emptyset\} = \mathcal{K}(X) \setminus \{K \in \mathcal{K}(X) \mid K \subset X \setminus V\}$

for V open.

Tvrzení 1.6

Let X be PTS, ϱ CCM on X , $\varrho \leq 1$. Then mapping $h : \mathcal{K}(X) \times \mathcal{K}(X) \mapsto [0, +\infty)$ defined as:

$$h(K, L) = \begin{cases} 0, & K = L = \emptyset, \\ \max \left\{ \sup_{x \in K} \varrho(x, L), \sup_{y \in L} \varrho(y, K) \right\}, & K, L \neq \emptyset, \\ 1, & \text{other cases,} \end{cases}$$

is CCM on $\mathcal{K}(X)$ with Vietoris topology. h is known as Hausdorff metric.

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Poznámka

$\mathcal{K}(X)$ is separable if X is PTS. X is compact metrizable $\implies \mathcal{K}(X)$ is compact (totally bounded).

$$X \text{ is separable} \implies \exists D \subset X : \overline{D} = X, |D| = \omega.$$

$$M = \{K \subset D \mid |K| < \omega\} \implies |M| = \omega.$$

$\overline{M} = \mathcal{K}(X)$. $K \in \mathcal{K}(X)$ arbitrary, $\varepsilon > 0$ arbitrary. Then $\exists \frac{\varepsilon}{2}$ net $P \subset K$, $|P| < \omega$. We find $\{\tilde{x}_0, \dots, \tilde{x}_n\} \subset D : \varrho(x_i, \tilde{x}_i) < \frac{\varepsilon}{2} \wedge h(K, \{\tilde{x}_0, \dots, \tilde{x}_n\}) < \varepsilon$.

$$X \text{ is compact, } P \text{ is } \varepsilon\text{-net in } X, |P| < \omega \implies 2^P \text{ is finite } \varepsilon\text{-net in } \mathcal{K}(X).$$

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Důkaz

($\emptyset \neq K, L, P \in \mathcal{K}(X)$.) h is metric, definition is correct, $h \geq 0$ trivial, $h(K, L) = h(L, K)$ trivial, $h(K, L) = 0 \implies K = L$ ($x \notin L \implies \varrho(x, L) > 0 \implies K \subset L \wedge L \subset K$).

„ Δ “ aka „ $h(K, L) \leq h(K, P) + h(P, L)$ “: Let $x \in K, y \in L, p \in P$. Then

$$\varrho(x, L) \leq \varrho(x, y) \leq \varrho(x, p) + \varrho(p, y) \quad \inf y \in L$$

$$\varrho(x, L) \leq \varrho(x, p) + \varrho(p, L) \quad \sup p \in P$$

$$\varrho(x, L) \leq \varrho(x, p) + h(P, L) \quad \inf p \in P$$

$$\varrho(x, L) \leq \varrho(x, P) + h(P, L) \quad \inf p \in P$$

$$\sup_{x \in K} \varrho(x, L) \leq h(K, P) + h(P, L).$$

Similarly $\sup_{y \in L} \varrho(y, K) \leq h(K, P) + h(P, L)$. □

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TODO!!!

Definice 1.3

X is metrizable space, $1 \leq \alpha < \omega_1$. We define $\Sigma_\alpha^0(X)$, $\Pi_\alpha^0(X)$, and $\Delta_\alpha^0(X)$ by induction:

$$\Sigma_1^0(X) := \{U \subset X \mid U \text{ open}\},$$

$$\Pi_\alpha^0(X) := \{A \subset X \mid X \setminus A \in \Sigma_\alpha^0(X)\},$$

$$\Sigma_\alpha^0(X) := \left\{ \bigcup_{n \in \omega} A_n \mid A_n \in \Pi_{\alpha_n}^0(X), \alpha_n < \alpha, n \in \omega \right\},$$

$$\Delta_\alpha^0(X) := \Sigma_\alpha^0 \cap \Pi_\alpha^0(X).$$

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Poznámka (By induction it can be proven)

$$\Sigma_\alpha^0(X) \subset \Sigma_\beta^0(X), \Pi_\alpha^0(X) \subseteq \Pi_\beta^0(X), \quad 1 \leq \alpha < \beta < \omega_1.$$

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Poznámka

$$\forall \alpha, \beta : 1 \leq \alpha < \beta < \omega_1 : \Sigma_\alpha^0(X) \subset \Pi_\beta^0(X).$$

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Poznámka

└ If X contains homeomorphic copy of 2^ω then all inclusions are strict.

We denote $Borel(X)$ as σ -algebra of Borel sets (σ -algebra generated by $\Sigma_1^0(X)$).

Poznámka (Also non-trivial theorem)

$$Borel(X) = \bigcup_{1 \leq \alpha < \omega_1} \Sigma_\alpha^0(X) = \bigcup_{1 \leq \alpha < \omega_1} (X) = \bigcup_{1 \leq \alpha < \omega_1} \Delta_\alpha^0(X).$$

$$A_n \in \bigcup_{1 \leq \alpha < \omega_1} \Sigma_\alpha^0(X) \implies \exists 1 \leq \alpha_n < \omega_1 : A_n \in \Sigma_{\alpha_n}^0(X) \implies A_n \in \Sigma_{\sup\{\alpha_n \mid n \in \omega\}}^0 \implies \bigcup_{n \in \omega} A_n \in \Sigma_{\sup\{\alpha_n, n \in \omega\}}^0$$

Poznámka

$$F_\sigma = \Sigma_2^0, G_\delta = \Pi_2^0, F_{\sigma\delta} = \Pi_3^0, G_{\delta\sigma} = \Sigma_3^0.$$

$\Sigma_\alpha^0(X)$ is closed under countable union and $\Pi_\alpha^0(X)$ under countable intersection.

Věta 1.7

X be metrizable, $1 \leq \alpha < \omega_1$. Then

1. $\Sigma_\alpha^0(X)$ is closed under finite intersection;
2. $\Pi_\alpha^0(X)$ is closed under finite union.

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Důkaz

„1.“ Firstly for $\alpha = 1$, it is trivial. Then let $A, B \in \Sigma_\alpha^0(X)$, $\alpha > 1$. Then $A = \bigcup_{n \in \omega} A_n$, $A_n \in \Pi_{\alpha_n}^0(X)$, $\alpha_n < \alpha$, $B = \bigcup_{m \in \omega} B_m$, $B_m \in \Pi_{\beta_m}^0(X)$, $\beta_m < \alpha$. $A \cap B = \bigcup_{(m,n) \in \omega^2} A_n \cap B_m$, $A_n \cap B_m \in \Pi_{\max\{\alpha_n, \beta_n\}}^0(X) \implies A \cap B \in \Sigma_\alpha^0(X)$. „2.“ \iff de Morgan and 1. \square

Věta 1.8

X be metrizable, $A \subset Z \subset X$, $1 \leq \alpha < \omega_1$. Then $A \in \Sigma_\alpha^0(Z) \iff$ there exists $\tilde{A} \in \Sigma_\alpha^0(X) : A = \tilde{A} \cap Z$. Similarly for $\Pi_\alpha^0, \Delta_\alpha^0$.

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Důkaz

Firstly $\alpha = 1$ from definition of subspace. Then assume that it is all true for all $\beta < \alpha$. We want to prove it for α . „ \implies “:

$$A \in \Sigma_\alpha^0(Z) \implies A = \bigcup A_n, A_n \in \Pi_{\beta_n}^0(Z), \beta_n < \alpha \implies \exists \tilde{A}_n \in \Pi_{\beta_n}^0(X) : \tilde{A}_n \cap Z = A_n.$$

$$\tilde{A} = \bigcup \tilde{A}_n \in \Sigma_\alpha^0(X), \tilde{A} \cap Z = Z \cap \bigcup \tilde{A}_n = \bigcup (Z \cap \tilde{A}_n) = \bigcup A_n = A.$$

„ \impliedby “:

$$\tilde{A} \in \Sigma_\alpha^0(X), A = \tilde{A} \cap Z \implies \exists \tilde{A}_n \in \Pi_{\beta_n}^0(X), \beta_n < \alpha, \bigcup \tilde{A}_n = \tilde{A}.$$

$$\tilde{A} \cap Z \in \Pi_{\beta_n}^0(Z) \implies A = \tilde{A} \cap Z = \left(\bigcup \tilde{A}_n \right) \cap Z = \bigcup \left(\tilde{A}_n \cap Z \right) = \bigcup A_n \in \Sigma_\alpha^0(Z).$$

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Věta 1.9

X, Y be metric spaces, $f : X \rightarrow Y$ is continuous. If $A \in \Sigma_\alpha^0(Y)$ ($\Pi_\alpha^0(Y)$, $\Delta_\alpha^0(Y)$) then $f^{-1}(A) \in \Sigma_\alpha^0(X)$ ($\Pi_\alpha^0(X)$, $\Delta_\alpha^0(X)$).

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Důkaz

$\alpha = 1$ trivial. Assume it holds true for $\Sigma_\beta^0(Y)$, $\Pi_\beta^0(Y)$, $\beta < \alpha$, and we want to show for $\Sigma_\alpha^0(Y)$ ($\Pi_\alpha^0(Y)$). Let $A \in \Sigma_\alpha^0(Y)$, $\alpha > 1 \implies A = \bigcup_{n \in \omega} A_n$, $A_n \in \Pi_{\beta_n}^0(Y)$, $\beta_n < \alpha$.

$$f^{-1}(A) = f^{-1}\left(\bigcup A_n\right) = \bigcup \underbrace{f^{-1}(A_n)}_{\Pi_{\beta_n}^0(X)} \in \Sigma_\alpha^0(X),$$

$$f^{-1}(Y \setminus A) = f^{-1}(Y) \setminus f^{-1}(A) = X \setminus f^{-1}(A).$$

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Věta 1.10 (Borel classes in PTS)

X, Y be PTS, $A \in \Sigma_\alpha^0(X)$, $\alpha \geq 3$ (resp. $A \in \Pi_\alpha^0(X)$, $\alpha \geq 2$), $B \subset Y$. If B and A are homeomorphic then $B \in \Sigma_\alpha^0(Y)$ (resp. $\Pi_\alpha^0(Y)$).

Důkaz

$f : A \rightarrow B$ is homeomorphism A onto B . The theorem above (name ?) there is extension \tilde{f} of f , \tilde{f} is homeomorphism \tilde{A} onto \tilde{B} , $A \subset \tilde{A}$, $B \subset \tilde{B}$, $\tilde{A} \in \Pi_2^0(X)$, $\tilde{B} \in \Pi_2^0(Y)$. Then $B \in \Sigma_\alpha^0(\tilde{B})$ (because $B = (f^{-1})^{-1}(A)$). From the theorem above, $\exists \hat{B} \in \Sigma_\alpha^0(Y) : B = \hat{B} \cap \tilde{B} \in \Sigma_\alpha^0(Y) \iff \alpha \geq 3$. \square

1.4 Analytic sets

Definice 1.4

X PTS, $A \subset X$. We say that A is analytic set in X if there exists PTS Y and continuous mapping $\varphi : Y \rightarrow X$ such that $\varphi(Y) = A$.

We denote collection of analytic subsets of X as $\Sigma_1^1(X)$. We say that A is coanalytic in X if $X \setminus A \in \Sigma_1^1(X)$ and we denote this collection as $\Pi_1^1(X)$. $\Delta_1^1(X) = \Sigma_1^1(X) \cap \Pi_1^1(X)$.

Například

$$Q = \{\alpha \in 2^\omega \mid \exists n \in \omega \forall j \geq n : \alpha_j = 0\} = 2^{<\omega} \in \Sigma_2^0(2^\omega) \setminus \Pi_2^0(2^\omega)$$

TODO?

Poznámka

X PTS, $F : X \rightarrow \mathcal{K}(X)$ by $F(x) = \{x\}$. Then F is continuous, $F^{-1}(\mathcal{K}(A)) = A \implies$ if $\mathcal{K}(A) \in \Sigma_\alpha^0(\mathcal{K}(X))$ ($\Pi_\alpha^0, \Delta_\alpha^0$) then $A \in \Sigma_\alpha^0(X)$ ($\Pi_\alpha^0, \Delta_\alpha^0$). A open $\implies \mathcal{K}(A)$ is open, A is closed $\implies \mathcal{K}(A)$ is closed. $\mathcal{K}(\bigcap A_n) = \bigcap \mathcal{K}(A_n)$. Thus for $A \in \Pi_2^0(X) : \mathcal{K}(A) \in \Pi_2^0(\mathcal{K}(X))$. $A \in \Sigma_1^0(X)$ ($\Pi_1^0(X), \Pi_2^0(X)$) $\iff \mathcal{K}(A) \in \Sigma_1^0(\mathcal{K}(X))$ ($\Pi_1^0(\mathcal{K}(X)), \Pi_2^0(\mathcal{K}(X))$).

Věta 1.11

X PTS, $|X| > \omega$. Assume $I \subset \mathcal{K}(X)$, I is σ -ideal ($K \in I, L \subset K \implies L \in I$; $K_n \in I, \bigcup K_n \in \mathcal{K}(X) \implies \bigcup K_n \in I$). If $I \in \Pi_2(\mathcal{K}(X))$, then $I \in \Sigma_1^1(\mathcal{K}(X))$.

Důsledek

$A \notin \Pi_2^0(X) \implies \mathcal{K}(A) \notin \Sigma_1^1(\mathcal{K}(X))$.

Poznámka

$A \in \Pi_1^1(X)$, $\mathcal{K}(A) = \mathcal{K}(X) \setminus \{K \in \mathcal{K}(X) \mid \exists x \in (X \setminus A) \cap K\}$ $\{(K, x) \in \mathcal{K}(X) \times X \mid x \in K\}$ is closed.

Definice 1.5

$$\Sigma_1^1(X) := \{A \subset X \mid \exists Y \text{ PTS}, f : Y \rightarrow X \text{ continuous} : f(Y) = A\}.$$

Poznámka • $\emptyset \in \Sigma_1^1$;

- $\Pi_2^0(X) \subset \Sigma_1^1(X)$, $f = \text{id}$;
- $X, Z \text{ PTS}$, $\psi : X \rightarrow Z \text{ continuous}$, $A \in \Sigma_1^1(X) \implies \psi(A) \in \Sigma_1^1(Z)$;
- $\Sigma_{n+1}^1(X) = \{A \subset X \mid \exists Y \text{ PTS}, \psi : Y \rightarrow X \text{ continuous}, B \in \Pi_n^1(X), A = \psi(B)\}$, $n \in \omega \setminus \{0\}$;
- $\Pi_n^1(X) = \{A \subset X \mid X \setminus A \in \Sigma_n^1(X)\}$, $\Delta_n^1(X) = \Sigma_n^1(X) \cap \Pi_n^1(X)$;
- $\bigcup_{n \in \mathbb{N}} \Sigma_n^1(X) = \bigcup_{n \in \mathbb{N}} \Pi_n^1 = \bigcup_{n \in \mathbb{N}} \Delta_n^1(x) = \mathbb{P}(X)$;
- $\#\mathbb{P}(X) \leq 2^\omega$, $\mathbb{P}(X)$ is closed under continuous images and inverse images;
- $\Sigma_1^1(X) = \{A \subset X \mid \exists \psi : \omega^\omega \rightarrow X \text{ continuous} : \psi(\omega^\omega) = A\}$; $Y \text{ PTS}$, $f : Y \rightarrow X$: $f(Y) = A$, $g : \omega^\omega \rightarrow Y : g(\omega^\omega) = Y$, g, f are constant. So $\psi = f \circ g$.

Věta 1.12

$X \text{ PTS}$, $A_n \in \Sigma_1^1(X)$, $n \in \omega$. Then $\bigcup_{n \in \omega} A_n, \bigcap_{n \in \omega} A_n \in \Sigma_1^1(X)$.

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Důsledek

Similar for $\Pi_1^1(X)$.

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Důkaz

„Union“: Assume $A_n \neq \emptyset$, $n \in \omega \implies \varphi_n : \omega^\omega \rightarrow X : \varphi_n(\omega^\omega) = A_n$ continuous. Define $\varphi : \omega^\omega \rightarrow X$ by $\varphi(\nu_0, \nu_1, \dots) = \varphi_{\nu_0}(\nu_1, \nu_2, \dots)$. „ φ is continuous“: $\nu^j \rightarrow \nu \implies \exists n_0 \in \omega \forall j \geq n_0 : \nu_0^j = \nu_0$.

$$\lim_{j \rightarrow \infty} \varphi(\nu^j) = \lim_{j \rightarrow \infty} \varphi_{\nu_0^j}(\nu_1^j, \nu_2^j, \dots) = \lim_{j \rightarrow \infty} \varphi_{\nu_0}(\nu_1^j, \dots) = \varphi_{\nu_0}(\nu_1, \dots) = \varphi(\nu).$$

„ $\varphi(\omega^\omega) = \bigcup_{n \in \omega} A_n$ “:

$$x \in \bigcup A_n \implies \exists n \in \omega : x \in A_n \implies \exists \nu \in \omega^\omega : \varphi_n(\nu) = x \implies \varphi(n^\wedge \nu) = x.$$

$$x \in \varphi(\omega^\omega) \implies \exists \tilde{\nu} \in \omega^\omega : \varphi(\tilde{\nu}) = x \implies x = \varphi_{\tilde{\nu}_0}(\tilde{\nu}_1, \dots) \implies z \in A_{\tilde{\nu}_0} \implies x \in \bigcup A_n.$$

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□

Poznámka (Intersection)

WLOG: $A_n \neq \emptyset$, $n \in \omega$. $Y := (\omega^\omega)^\omega$, $Y \text{ PTS}$ by the theorem above (first item). $\varphi_n : \omega^\omega \rightarrow$

X , meh that $\varphi_n(\omega^\omega) = A_n$.

$$F := \{y = (y_0, y_1, \dots) \in Y \mid \forall n, m \in \omega : \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap_{n, m \in \omega} \{y \in Y \mid \varphi_n(y_n) = \varphi_m(y_m)\} = \bigcap_{n, m \in \omega} (\bigcap_{k \in \omega} \{y \in Y \mid \varphi_k(y_k) = \varphi_m(y_m)\})$$

intersection of closed, so F is closed and is PTS.

$$„\varphi_0 \circ \pi_0(F) = \bigcap_{n \in \omega} A_n“:$$

$$x \in \varphi_0 \circ \pi_0(F) \implies \exists y \in F : x = \varphi_0(y_0) = \varphi_1(y_1) = \varphi_2(y_2) = \dots \implies x \in \bigcap_{n \in \omega} A_n.$$

$$x \in \bigcap A_n \implies \exists y_0, y_1, \dots \in \omega^\omega : \varphi_0(y_0) = x, \varphi_1(y_1) = x, \dots \implies y = (y_0, y_1, \dots) \in F, \varphi_0 \circ \pi_0(y) = x \implies$$

Poznámka

$\Sigma_1^1(X)$ is not closed under complement: $\sigma(\Sigma_1^1(X)) \supset \Sigma_1^1(X) \cup \Pi_1^1(x)$.

$$\text{Borel}(X) \subset \Sigma_1^1(X) \cap \Pi_1^1(X) = \Delta_1^1(X).$$

Věta 1.13

X, Y PTS, $A \in \Sigma_1^1(X)$ (respective $\Pi_1^1(X)$), $B \subset Y$, A and B are homeomorphism. Then $B \in \Sigma_1^1(Y)$ (resp. $\Pi_1^1(Y)$).

┌

Důkaz

For Σ_1^1 trivial. $A \in \Pi_1^1(X)$, $\varphi : A \rightarrow B$ homeomorphism. Then from the theorem above, $\exists \tilde{A} \in \Pi_2^0(X)$, $\tilde{B} \in \Pi_2^0(Y)$ and $\tilde{\varphi} : \tilde{A} \rightarrow \tilde{B}$ homeomorphism extending φ , $A \subset \tilde{A}$, $B \subset \tilde{B}$.

Then $\tilde{A} \setminus A = (X \setminus A) \cap \tilde{A} \in \Sigma_1^1(X) \implies \tilde{B} \setminus B \in \Sigma_1^1(Y)$. $B = Y \setminus (\tilde{B} \setminus B \cup Y \setminus \tilde{B}) \in \Pi_1^1(Y)$. \square

└

Věta 1.14

X PTS. Then $\text{Borel}(X) \subset \Delta_1^1(X)$.

┌

Důkaz

Trivial. \square

└

1.5 Luzin theorem

Věta 1.15 (Luzin)

X PTS, $A_1, A_2 \in \Sigma_1^1(X)$, $A_1 \cap A_2 = \emptyset$. Then there exists $B \in \text{Borel}(X)$, such that $A_1 \subset B \subset X \setminus A_2$.

Důsledek

X PTS. $\Delta_1^1(X) = \text{Borel}(X)$.

┌

Důkaz

$\Delta_1^1(X) \subseteq \text{Borel}(X)$ we already have.

$A \in \Delta_1^1(X) \implies A \in \Sigma_1^1(X), X \setminus A \in \Sigma_1^1 \implies \exists B \in \text{Borel}(X) : A \subset B \subset X \setminus (X \setminus A) = A \implies A = B$

└

□

Lemma 1.16

$C_n, D_n \subset X$, $n, m \in \omega$ and $\forall n, m \in \omega$ we can separate C_n, D_m by some Borel set. Then we can separate $\bigcup_{n \in \omega} C_n$ and $\bigcup_{m \in \omega} D_m$ by Borel set.

┌

Důkaz

Let $B_{n,m} \in \text{Borel}(X)$ separating C_n from D_m ($C_n \subset B_{n,m} \subset X \setminus D_m$). Put $B := \bigcup_{n \in \omega} \bigcap_{m \in \omega} B_{n,m}$.

└

□

Důkaz (Luzin theorem)

Assume $A_1, A_2 \neq \emptyset$. Then exists $\varphi_1, \varphi_2 : \omega^\omega \rightarrow X$ $\varphi_i(\omega^\omega) = A_i$. We assume A_1 can't be separated from A_2 by any Borel set.

$A_i = \varphi_i(\omega^\omega) \implies A_i = \bigcup_{n \in \omega} \varphi_i(\mathcal{N}(n)) \implies \exists \nu_0, \mu_0 \in \omega : \varphi_i(\mathcal{N}(\mu_0))$ can't be separated from $\varphi_2(\mathcal{N}(\nu_0))$.

We use lemma again and obtain $\mu, \nu \in \omega^\omega$ such that $\forall k \in \omega : \varphi_1(\mathcal{N}(\mu/k))$ can't be separated from $\varphi_2(\mathcal{N}(\nu/k))$

$\varphi_1(\mu) \in A_1, \varphi_2(\nu) \in A_2 \implies \varphi_1(\mu) \neq \varphi_2(\nu) \implies \exists G_1, G_2$ open, $G_1 \cap G_2 = \emptyset$

such that $\varphi_1(\mu) \in G_1, \varphi_2(\nu) \in G_2, \varphi_1, \varphi_2$ are continuous $\implies \exists k \in \omega : \varphi_1(\mathcal{N}(\mu/k)) \subset G_1, \varphi_2(\mathcal{N}(\nu/k)) \subset G_2$ which is continuous. □

Například

$\{f \in C([0, 1]) \mid \forall x \in [0, 1] : f'(x) \in \mathbb{R}\} \in \Pi_1^1 \setminus \Delta_1^1$.

$\{f \in C([0, 2\pi]) \mid \text{Fourier series converges to } f \text{ for every } x \in [0, 2\pi]\} \in \Pi_1^1 \setminus \Delta_1^1$.

$\{K \in \mathcal{K}([0, 1]) \mid |K| \leq \omega\}, \{K \in \mathcal{K}(\mathbb{R}) \mid K \subset \mathbb{Q}\} \in \Pi_1^1 \setminus \Delta_1^1$.

Například

$\{x \in X \mid \exists y \in Y : (x, y) \in B\} \in \Sigma_1^1(X)$.

TODO!!!

Lemma 1.17

(X, τ) PTS, $F \in \Pi_1^0(X)$. Let τ_F be topology generated by $\tau \cup \{F\}$. Then τ_F is Polish, $F \in \Delta_1^0(X, \tau_F)$, $\Delta_1^1(X, \tau_F) = \Delta_1^1(X, \tau)$.

┌

Důkaz

(X, τ_F) is homeomorphic with $((X \setminus F) \times \{0\}) \cup (F \times \{1\}) \subset X \times \{0, 1\}$ which is PTS and those two subsets are G_δ in $X \times \{0, 1\}$, so, (X, τ_F) is Polish.

$$\Delta_1^1(((X \setminus F) \times \{0\}) \cup (F \times \{1\})) \leftrightarrow \Delta_1^1(\tau_F) = \{A \cup B \mid A \in \Delta_1^1(X \setminus F, \tau), B \in \Delta_1^1(F, \tau)\} \subset \Delta_1^1(\tau) \subset \Delta_1^1(\tau_F).$$

└

□

Lemma 1.18

(X, τ) PTS, $(\tau_n)_{n \in \omega}$ Polish topology, $\tau \subset \tau_n$, $n \in \omega$. Then topology τ_∞ generated by $\bigcup_{n \in \omega} \tau_n$ is polish. If $\forall n \in \omega : \tau_n \subset \Delta_1^1(\tau)$, then $\Delta_1^1(\tau) = \Delta_1^1(\tau_\infty)$.

┌

Důkaz

Set $X_n := (X, \tau_n)$, $\varphi : X \rightarrow \prod_{n \in \omega} X_n$, $\varphi(x) = (x, x, x, x, \dots)$. φ is homomorphism (X, τ_∞) on $\varphi(X)$. ($U \in \text{base of } \tau_\infty \implies \exists n \in \omega : U \in \tau_n, \varphi(U) = x_1 \times x_2 \times \dots \times x_{n-1} \times U \times x_{n+1} \times \dots \cap \varphi(X)$ is open. $\varphi(X) \in \Pi_1^0(\prod X_n) \implies \varphi(X)$ PTS $\implies (X, \tau_\infty)$ PTS.)

$$\Delta_1^1(\tau) = \Delta_1^1(\tau_\infty) \iff \sigma(\sigma(M)) = \sigma(M). (\tau_\infty \subset \Delta_1^1(\tau) = \Delta_1^1(\tau_n).) \tau_\infty \subset \bigcup \Delta_1^1(\tau_n). \quad \square$$

└

Věta 1.19

(X, τ) PTS, $A \in \Delta_1^1(X, \tau)$. There exists polish topology τ_A such that $\tau \subset \tau_A$, $\Delta_1^1(\tau_A) = \Delta_1^1(\tau)$ and $A \in \Delta_1^0(X, \tau_A)$.

┌

Důkaz

$\mathcal{S} := \{D \in \Delta_1^1(X) \mid \text{exists polish topology } \tau_D \supset \tau \text{ and } \Delta_1^1(\tau_D) = \Delta_1^1(\tau), D \in \Delta_1^0(X, \tau_D)\}$. We know that $\tau \subset \mathcal{S}$ and that \mathcal{S} is closed under complements. Moreover, \mathcal{S} is closed under countable union ($A_n \in \mathcal{S} \rightarrow \tau_{A_n} \rightarrow \tau_\infty = \tau_{\bigcup A_n}$). So $\mathcal{S} = \Delta_1^1(X, \tau)$. □

└

Lemma 1.20

X, Y PTS. $f : X \rightarrow Y$ Borel. Then $\text{graph}(f) \in \Delta_1^1(X \times Y)$.

┌
Důkaz

Fix compatible complete metric ϱ on Y . U_n , $n \in \omega$, countable collection of open balls with $\text{diam} < 2^{-n}$ covering Y .

$$\text{graph } f \stackrel{?}{=} \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U \in \Delta_1^1(X \times Y).$$

„ \subseteq “: $(x, y) \in \text{graph}(f) \Leftrightarrow f(x) = y \implies \forall n \in \omega \exists U \in U_n : y \in U \wedge x \in f^{-1}(U) \implies (x, y) \in \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U.$

„ \supseteq “: $(x, y) \notin \text{graph}(f) \Leftrightarrow f(x) \neq y \implies \exists n \in \omega : \varrho(f(x), y) > \frac{1}{n} \implies \exists n \in \omega \forall U \in U_n \neg (x \in f^{-1}(U) \cap y \in U) \implies (x, y) \notin \bigcap_{n \in \omega} \bigcup_{U \in U_n} f^{-1}(U) \times U. \quad \square$

└

Poznámka (Notation)

If f is Borel, we write $f \in \Delta_1^1$.

Věta 1.21

X, Y PTS, $f \in \Delta_1^1(X \times Y)$. If $A \in \Delta_1^1(X)$ and $f|_A$ is injective, then $f(A) \in \Delta_1^1(Y)$.

┌ *Důkaz*

If $f : X \rightarrow Y$ is injective, then $f(A) = \prod_Y(\text{graph}(f) \cap A \times Y) \in \Sigma_1^1(Y)$.

$$Y \setminus F(A) = \prod_Y(\text{graph}(f) \cap (X \setminus A) \times Y) \in \Sigma_1^1(Y) \implies f(A) \in \Delta_1^1(Y).$$

Assume f is continuous, $A \in \Pi_1^0(X)$. From the theorem above $A \subset \omega^\omega$, $B_s := f(\mathcal{N}(s) \cap A)$. $\forall s \in \omega^{<\omega} \forall i, j, i \neq j : B_{s \wedge i} \cap B_{s \wedge j} = \emptyset \iff f$ is injection. $\forall s \in \omega^{<\omega} : B_s = \bigcup_{i \in \omega} B_{s \wedge i}$.

From Luzin separation theorem, there exists (by induction) $(B'_s)_{s \in \omega^{<\omega}}$ of Borel sets:

$$\forall s \in \omega^{<\omega} \forall i, j \in \omega, i \neq j : B'_{s \wedge i} \cap B'_{s \wedge j} = \emptyset.$$

(separation $B_{s \wedge i}, \bigcup_{j < i} B_{s \wedge j} \cup \bigcup_{l > i} B_{s \wedge l}$) $\forall s \in \omega^{<\omega} : B_s \subset B'_s$.

Put: $B_\emptyset^* = Y$, $B_{s \wedge j}^* = B_{s \wedge j} \cap \overline{B_{s \wedge j}'} \cap B_s^*$. $\forall s \in \omega^{<\omega} : B_s^* \in \Delta_1^1(Y)$, $B_s \subset B_s^* \subset \overline{B_s}$, $B_{s \wedge j}^* \subset B_s^*$, $B_{s \wedge j}^* \cap B_{s \wedge i}^* = \emptyset$, $s \in \omega^{<\omega}$, $i, j \in \omega, i \neq j$. We proof: $f(A) \stackrel{?}{=} \bigcup_{s \in \omega^{<\omega}} \bigcap_{k \in \omega} B_{s/k}^* = \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B_s^* \in \Delta_1^1(Y)$.

$$B_s^*, s \in \omega^{<\omega}, B_s^* \in \Delta_1^1(Y). f(A) = \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B_s^*:$$

„ \subseteq “: $x \in f(A) \implies \exists \nu \in A : f(\nu) = x$. Then $x \in f(\mathcal{N}_{\nu/k} \cap A) = B_{\nu/k} \subset B_{\nu/k}^*$, $k \in \omega \implies x \in \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B_s^*$.

„ \supseteq “: $x \in \bigcap_{k \in \omega} \bigcup_{s \in \omega^k} B_s^* \implies \forall k \in \omega \exists \nu^k \in \omega^\omega : x \in B_{\nu^k/k}^*$. $\exists \nu \in \omega^\omega : \nu^k = \nu$, $k \in \omega$. $\implies \forall k \in \omega \exists \nu \in \omega^\omega : f(\mathcal{N}(\nu/k) \cap A) \neq \emptyset \implies \exists \nu^k \in \mathcal{N}(\nu/k) \cap A, \nu^k \rightarrow \nu \implies \nu \in A$ (A is closed). $f(\nu) = x$? Assume $f(\nu) \neq x \implies \exists U$ neighbourhood of $f(\nu)$, such that $x \notin \overline{U} \implies (f \text{ is continuous}) \exists k_0 \in \omega : x \in B_{\nu/k_0}^* \subset f(\mathcal{N}_{\nu/k_0}) \cap A = \overline{B_{\nu/k_0}} \subset \overline{U}$ which is contradiction.

a) Let f is continuous and $A \in \Delta_1^1(X)$. On X we find Polish topology τ_A such that $A \in \Delta_1^0(\tau_A)$, $\tau \subset \tau_A$ (so f is continuous with respect to τ_A), $\Delta_1^1(\tau) = \Delta_1^1(\tau_A)$.

b) Let $f \in \Delta_1^1$. Then $f(A) = \pi_Y(\text{graph}(f) \cap A \times Y)$. Observe that π_Y is injective on $(\text{graph}(f) \cap A \times Y)$ if f is injective on A . □

Věta 1.22

X, Y PTS, $f \in \Delta_1^1(X \times Y)$.

1. $A \in \Sigma_1^1(X) \implies f(A) \in \Sigma_1^1(Y)$;
2. $B \in \Sigma_1^1(Y) \implies f^{-1}(B) \in \Sigma_1^1(X)$;
3. $B \in \Pi_1^1(Y) \implies f^{-1}(B) \in \Pi_1^1(X)$.

┌ *Důkaz*

„1.“: $f(A) = \pi_Y((\text{graph}(f) \cap A \times Y))$ is continuous image of Σ_1^1 set.

„2.“: $f^{-1}(B) = \pi_X((\text{graph}(f) \cap X \times B))$ is continuous image of Σ_1^1 set.

„3.“: $f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(Y \setminus B)$.

└ □

1.6 Standard Borel spaces (SBS)

Definition 1.6 (Standard Borel space (SBS))

Measurable space (X, \mathcal{S}) is called standard Borel space (SBS) if there exists Polish topology τ on X such that $\Delta_1^1(X, \tau) = \mathcal{S}$.

Definition 1.7 (Effros Borel space)

Let X be PTS and $\mathcal{F}(X) := \Pi_1^0(X)$. Let \mathcal{S} be σ -algebra generated by sets of form $\{F \in \mathcal{F}(X) | F \cap U \neq \emptyset\} =: M_U$, where $U \in \Sigma_1^0(X)$. $(\mathcal{F}(X), \mathcal{S})$ is called Effros Borel space.

Věta 1.23

X PTS. Then $(\mathcal{F}(X), \mathcal{S})$ is SBS.

┌ *Důkaz*

Without proof.

└ □

Poznámka

X be measurable compact. Then $\mathcal{F}(X)$ can be equipped by Vietoris topology.

Příklad

$SB := \{Y \in \mathcal{F}(C([0, 1])) | Y \text{ is Banach subspace of } C([0, 1])\}$. If we restrict Effros σ -algebra on SB then SB is SBS.

$$SD = \{Y \in SB | Y \text{ has separable dual}\},$$

$$NU = \{Y \in SB | Y \text{ is not universal}\},$$

$$REFL = \{Y \in SB | Y \text{ is reflexive}\},$$

$$NL_1 = \{Y \in SB | Y \text{ does not contain } l_1\}.$$

2 Regularity of Σ_1^1 sets

2.1 Sets with Baire property (BP)

Definice 2.1 (Baire property (BP))

X TS, $A \subset X$ has Baire property (BP) in X if there exists open $U \subset X$ and set of 1. category $M \subset X$ such that $A = U \Delta M := (U \setminus M) \cup (M \setminus U)$. Collection of all sets with BP we denote as $Baire(X)$.

Věta 2.1

X TS. Then $Baire(X)$ is σ -algebra and $Baire(X) \supset \Delta_1^1(X)$.

┌

Důkaz

1. „ $Baire(X) \supset \Sigma_1^0(X)$ “ trivial. 2. „ $Baire(X)$ is σ -algebra“: a) „ $A \in Baire(X) \xrightarrow{?} X \setminus A \in Baire(X)$ “: $A \in Baire(X) \implies \exists G \in \Sigma_1^0(X)$ and M meagre such that $A = G \Delta M$.

$$\begin{aligned} X \setminus A &= X \setminus (G \Delta M) = (X \setminus G) \Delta M = (\text{int}(X \setminus G) \cup (X \setminus G) \setminus \text{int}(X \setminus G)) \Delta M = \\ &= (V \cup M_1) \Delta M_2 = V \Delta M \quad (M = M_1 \Delta M_2). \end{aligned}$$

b) „ $A_n \in Baire(X) \xrightarrow{?} \bigcup A_n \in Baire(X)$ “: $A_n = G_n \Delta M_n$, $G_n \in \Sigma_1^0(X)$, M_n meager. $M'_n = G_n \cap M_n$ (meager), $M''_n = M_n \setminus G_n$ (meager).

$$\bigcup A_n = \bigcup ((G_n \setminus M'_n) \cup M''_n) = ((\bigcup G_n) \setminus M''') \cup \bigcup M''_n,$$

where $M''' \subset \bigcup_{n \in \omega} M'_n$.

└

□

Lemma 2.2

X TS, $A \subset X$. Then A is meager iff $\forall x \in A \exists V \in \Sigma_1^0(X)$ such that $x \in V$ and $A \cap V$ is meager.

┌ *Důkaz*

„ \implies “ trivial. „ \impliedby “ \mathcal{U} denote as maximal collection of disjoint Σ_1^0 sets such that $U \cap A$ is meager for $U \in \mathcal{U}$. We show that $A \cap \bigcup \mathcal{U}$ is meager, $X \setminus \bigcup \mathcal{U}$ is nowhere dense, so meager.

„ $X \setminus \bigcup \mathcal{U}$ is nowhere dense“: By contradiction we assume that there exists $\emptyset \neq V \in \Sigma_1^0(X)$, $V \subset X \setminus \bigcup \mathcal{U}$. Now we have 2 cases: $A \cap V = \emptyset \implies V \in \mathcal{U}$ contradiction, or $A \cap V \neq \emptyset \implies \exists x \in A \cap V \implies \exists W \in \Sigma_1^0(X) : x \in W, W \cap A$ is meager $\implies x \in W \cap V \neq \emptyset, W \cap V \cap A$ is meager $\implies W \cap V \in \mathcal{U}$ contradiction.

„ $\bigcup \mathcal{U} \cap A$ is meager“: $\mathcal{U} := \{U_\alpha | \alpha \in I\}$, $U_\alpha \cap A$ meager \implies exist? $F_n^\alpha \in \Pi_1^0(X)$ nowhere dense: $U_\alpha \cap A \subset \bigcup F_n^\alpha \subset \overline{U_\alpha}$. We show that $\bigcup_{\alpha \in I} F_n^\alpha$ is nowhere dense:

$$a) \bigcup_{\alpha \in I} U_\alpha \setminus F_n^\alpha \in \Sigma_1^0(X), \quad \left(\bigcup_{\alpha \in I} U_\alpha \setminus F_n^\alpha \right) \cap \left(\bigcup_{\alpha \in I} F_n^\alpha \right) = \emptyset \iff F_n^\alpha \subset \overline{U_\alpha}, \quad \overline{U_\alpha} \cap U_\beta = \emptyset, \alpha \neq \beta$$

So \mathcal{U} is disjoint collection, so $\bigcup_{\alpha \in I} U_\alpha F_n^\alpha \cap \overline{\bigcup_{\alpha \in I} F_n^\alpha} = \emptyset$.

$$\implies \overline{\bigcup_{\alpha \in I} F_n^\alpha} \subset \left(\bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \right) \cup (X \setminus \bigcup \mathcal{U}).$$

b) We assume $\exists V \in \Sigma_1^0(X)$, $V \neq \emptyset$, $V \subset \overline{\bigcup_{\alpha \in I} F_n^\alpha}$.

$$? \implies V \not\subset X \setminus \bigcup \mathcal{U} \xrightarrow{a)} V \cap \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \neq \emptyset \implies \exists \alpha \in I : V \cap U_\alpha \neq \emptyset.$$

$$a) \implies V \cap U_\alpha \subset \bigcup_{\alpha \in I} (U_\alpha \cap F_n^\alpha) \xrightarrow{\mathcal{U} \text{ disjoint}} V \cap U_\alpha \subset F_n^\alpha \nexists.$$

└

□

TODO!!!

2.2 Solecky theorem

Poznámka (Notation)

X PTS, $\mathcal{I} \subset \Pi_1^0(X)$.

$$\mathcal{I}^{ext} := \left\{ A \subset X | \exists \mathcal{F} \subset \mathcal{I}, |\mathcal{F}| = \omega, A \subset \bigcup \mathcal{F} \right\}.$$

Například

$$\mathcal{I} = \{A \subset X | |A| < \omega\}, \quad \mathcal{I} = \{A \subset X | A \text{ nowhere dense}\}.$$

$$\mathcal{I}^{perf} = \{A \subset X \mid A \neq \emptyset, \forall U \in \Sigma_1^0(X) : U \cap A \neq \emptyset \implies U \cap A \notin \mathcal{I}^{ext}\}.$$

$$\begin{aligned} \text{Ker } A &:= A \setminus \bigcup \{U \subset X \mid U \in \Sigma_1^0(X), U \cap A \in \mathcal{I}^{ext}\} = \\ &= \text{max perfect subset of } A \iff X \text{ has countable base.} \end{aligned}$$

$$MGR(A) = \{Z \subset A \mid Z \text{ be meager in } A\}, \quad A \subset X.$$

Věta 2.3 (Solecki)

X PTS, $A \in \Sigma_1^1(A)$, $\mathcal{I} \subset \Pi_1^0(X)$. $A \notin \mathcal{I}^{ext} \implies \exists H \in \Pi_2^0(X), H \subset A, H \notin \mathcal{I}^{ext}$

Lemma 2.4 (For proof of Solecki)

$A \in \Sigma_1^1(X) \setminus \mathcal{I}^{ext}$. Then there exists Suslin scheme $(A_s)_{s \in \omega^{<\omega}}$ of closed subsets of X such that:

$$A_\emptyset = \emptyset, \quad a_s A_s \subset A, \quad A_s \neq \emptyset \implies A \cap A_s \in \mathcal{I}^{perf}, \overline{A \cap A_s} = A_s, \quad \overline{\bigcup_{n \in \omega} A_{s \wedge n}} = A_s.$$

┌

Důkaz

$(H_s)_{s \in \omega^{<\omega}}$ closed subsets of X , decreasing $(H_s \supset H_{s \wedge n}, n \in \omega)$, $A = a_s H_s \iff A \in \Sigma_1^1(X)$.
For $s \in \omega^{<\omega} : L_s := a_t H_{s \wedge t}, A_s := \overline{\text{Ker}(L_s)}$.

1. $A_\emptyset = \overline{\text{Ker}(L_\emptyset)} = \overline{\text{Ker}(A)} \neq \emptyset \iff A \notin \mathcal{I}^{ext}$ (X has countable base).
2. $H_s \searrow \implies L_s \subset H_s \implies \text{Ker}(L_s) \subset H_s \xrightarrow{H_s \in \Pi_1^0(X)} A_s \subset H_s \implies a_s A_s \subset a_s H_s = A$.
3. $\text{Ker}(L_s) \subset A_s, L_s \subset A : (A = \bigcup_{|s|=k} L_s, k \in \omega \iff H_s \searrow) \implies \text{Ker}(L_s) \subset A_s \cap A, \overline{\text{Ker}(L_s)} = A_s$.

$$A_s = \overline{\text{Ker}(L_s)} \subset \overline{A_s \cap A} \subset \overline{A_s} = A_s.$$

Assume $A_s \neq \emptyset \implies A \cap A_s \neq \emptyset$. $U \in \Sigma_1^0(X), U \cap A \cap A_s \neq \emptyset \implies U \cap \text{Ker}(L_s) \neq \emptyset \implies U \cap \text{Ker}(L_s) \notin \mathcal{I}^{ext} \implies U \cap A \cap A_s \notin \mathcal{I}^{ext}$.

4. $\bigcup_{n \in \omega} A_{s \wedge n} \subset A_s \iff (H_s \searrow \implies L_s \searrow \implies A_s \searrow)$. Let $U \in \Sigma_1^0(X), U \cap A_s \neq \emptyset \implies U \cap \text{Ker}(L_s) \neq \emptyset \implies U \cap L_s \notin \mathcal{I}^{ext}$.

$$L_s = \bigcup_{n \in \omega} L_{s \wedge n} \implies \exists n_0 \in \omega : U \cap L_{s \wedge n_0} \notin \mathcal{I}^{ext} \implies U \cap \text{Ker}(L_{s \wedge n_0}) \notin \mathcal{I}^{ext} \implies U \cap A_{s \wedge n_0} \neq \emptyset.$$

└

□

Důkaz (Solecki theorem, not in exam)

$A \in \Sigma_1^1(X) \setminus \mathcal{I}^{ext}$, $(A_s)_{s \in \omega^{<\omega}}$ from the previous lemma. There are 2 cases:

„1st case $\exists s \in \omega^{<\omega} \exists U \in \Sigma_1^0(X) : A_s \cap U \neq \emptyset \wedge MGR(A_s \cap U) \subset \mathcal{I}^{ext}$ “: Put $\tilde{A} := A \cap A_s \cap U$. Then from the third item of the previous lemma $\tilde{A} \in \mathcal{I}^{perf}, \tilde{A} \in \Sigma_1^1(X)$. $A_s \neq \emptyset$,

$$A \cap A_s \in \mathcal{I}^{perf}, U \cap A_s \neq \emptyset \implies U \cap A \cap A_s \neq \emptyset \iff \overline{A \cap A_s} = A_s.$$

$$\implies \tilde{A} \in \text{Baire}(A_s \cap U) \iff (A_s \cap U \in \Pi_2^0(X)), A_s \cap U \text{ PTS}.$$

$$\tilde{A} = H \cup M, H \in \Pi_2^0(A_s \cup U), M \in \text{MGR}(A_s \cap U) \subset \mathcal{I}^{ext} \implies H \notin \mathcal{I}^{ext}, H \subset A.$$

„2nd case $\forall s \in \omega^{<\omega} \forall U \in \Sigma_1^0(X), U \cap A_s \neq \emptyset : \text{MGR}(A_s \cap U) \setminus \mathcal{I}^{ext} \neq \emptyset$ “: Notation: $\mathcal{F} \subset 2^X : \mathcal{F}^d := \overline{\bigcup \mathcal{F}} \setminus \bigcup \{\overline{F} \mid F \in \mathcal{F}\}$. Choose CCM ≤ 1 on X . We will inductively construct $\varphi : \omega^{<\omega} \rightarrow \omega^{<\omega}, U_s \subset X, s \in \omega^{<\omega}$ such that:

1. $|\varphi(s)| = |s|$ TODO
2. $U_s \in \Sigma_1^0(X)$;
3. $\text{diam } U_s \leq 2^{-|s|}$;
4. $\lim_{n \rightarrow \infty} \text{diam}(U_{s \wedge n}) = 0$;
5. $\forall t, s \in \omega^{<\omega}, t < s, t \neq s : \overline{U_s} \subset U_t$;
6. $\forall s \in \omega^{<\omega} \forall m, n \in \omega, m \neq n : U_{s \wedge m} \cap U_{s \wedge n} = \emptyset$;
7. $U_a \cap A_{\varphi(s)} \neq \emptyset$;
8. $\{U_{s \wedge n} \mid n \in \omega\}^d \notin \mathcal{I}^{ext}$;
9. $\{U_{s \wedge n} \mid n \in \omega\}^d \subset U_s$;
10. (9. + 5.) $\overline{\bigcup_{n \in \omega} U_{s \wedge n}} \subset U_s$.

Construction: $\varphi(\emptyset) = \emptyset, U_\emptyset$ be arbitrary open subset of X : $U_\emptyset \cap A_\emptyset \neq \emptyset$. Then all items are satisfied. We assume that U_s, φ_s are constructed for all $s \in \omega^{<\omega}, |s| \leq N \in \omega$. Let $s \in \omega^{<\omega}, |s| \leq N$ be arbitrary. From 7th item $U_s \cap A_{\varphi(s)} \neq \emptyset, \text{MGR}(A_{\varphi(s)} \cap U_s) \notin \mathcal{I}^{ext} \implies \exists K \subset A_{\varphi(s)} \cap U_s, K \in \Pi_1^0(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s, K \notin \mathcal{I}^{ext}$. Because

$$\exists L \in \text{MGR}(A_{\varphi(s)} \cap U_s) \setminus \mathcal{I}^{ext} \implies \exists H \in \Sigma_2^0(X), H \supset L, H \in \Sigma_2^0(A_{\varphi(s)} \cap U_s), H \notin \mathcal{I}^{ext},$$

so $H = \bigcup F_n, F_n \in \Pi_1^0(X)$, nowhere dense in $A_{\varphi(s)} \cap U_s \implies \exists n_0 \in \omega : F_{n_0} = K \notin \mathcal{I}^{ext}$.

Find $D \subset A_{\varphi(s)} \cap U_s$: D is discrete in $X \setminus K$. $D \cap K = \emptyset$. $\overline{D} = K \cup D$. Let $\{y_n\} \subset K, \overline{\{y_n\}} = K$, and every element of $\{y_n\}$ repeats infinitely many times. Find $x_n \in (A_{\varphi(s)} \cap U_s) \setminus K$ such that $\varrho(x_n, y_n) < \frac{1}{n}$ (it exists $\iff K$ is nowhere dense in $A_{\varphi(s)} \cap U_s$). Then $D = \{x_n \mid n \in \omega\}, D \cap K = \emptyset, \overline{D} \supset \overline{D \cup \{y_n \mid n \in \omega\}} \supset D \cup K, x \notin K \cup D \implies \exists n \in \omega \setminus \{0\} : \varrho(x, K) > \frac{1}{n} \implies \#(B(x, 1/2n) \cap D) \leq 2n \implies x \notin \overline{D} \implies \overline{D} = D \cup K, D$ is discrete in $X \setminus K$. Assume $x_n \neq x_m, n \neq m$.

Define $U_{s \wedge n}$ as open ball with center x_n : $\overline{U_{s \wedge n}} \subset U_s. U_{s \wedge n} \cap U_{s \wedge m} = \emptyset$ (D is discrete), $\text{diam } U_{s \wedge n} \leq 2^{-|s| - 1}, \lim_{n \rightarrow \infty} \text{diam } U_{s \wedge n} = 0, \bigcup_{n \in \omega} U_{s \wedge n} \setminus \bigcup_{n \in \omega} \overline{U_{s \wedge n}} = \{U_{s \wedge n} \mid n \in \omega\} = K \iff \overline{U_{s \wedge n}} \cap K = \emptyset, \overline{D} = K \cup D. x_n \in A_{\varphi(s)} \implies U_{s \wedge n} \cap A_{\varphi(s)} \neq \emptyset, \bigcup_{k \in \omega} A_{\varphi(s) \wedge k} = A_{\varphi(s)} \implies \exists k \in \omega : U_{s \wedge n} \cap A_{\varphi(s) \wedge k} \neq \emptyset$.

Put $\varphi(s^\wedge n) = \varphi(s)^\wedge k$. And then all items are satisfied. $H = \bigcap_{n \in \omega} \bigcup_{|s|=n, s \in \omega^{<\omega}} U_s \in \Pi_2^0(X)$, $H \subset A$, $H \notin \mathcal{I}^{ext}$.

$$H := \bigcap_{n \in \omega} \bigcup \{U_s | s \in \omega^{<\omega}, |s| = n\} \in \Pi_2^0(\Leftarrow 2.).$$

$H \subset A?$, $H \notin \mathcal{I}^{ext}$. „ $H \subset A$ “: 5. and 6. $\implies H = \bigcup_{\nu \in \omega^\omega} \bigcap_{n \in \omega} U_{\nu/n} = a_s U_s?$

$$x \in H \Leftrightarrow \forall n \in \omega \exists \omega^{<\omega}, |s| = n : x \in U_s \stackrel{5. \wedge 6.}{\Leftrightarrow} \exists s \in \omega^\omega \forall n \in \omega x \in U_{s/n} \Leftrightarrow a_s U_s.$$

(3. $\implies \text{diam}(U_{\nu/n}) \leq 2^{-n}$, (7. $\implies U_{\nu/m} \cap A_{\varphi(\nu/n)} \neq \emptyset$)) $\implies \bigcap_{n \in \omega} \overline{U_{\nu/n}} \subset \bigcap_{n \in \omega} A_{\varphi(\nu/n)} \subset A \implies H \subset A$

„ $H \notin \mathcal{I}^{ext}$ “: $\forall \nu \in \omega^\omega : \bigcap_{n \in \omega} U_{\nu/n} \neq \emptyset \Leftarrow 3. \wedge 5.$, so $\forall s \in \omega^{<\omega} : U_s \cap H \neq \emptyset$. Assume $H \subset \bigcup_{m \in \omega} F_m$, $F_m \in \mathcal{I}$. $H \in G_\delta \implies \exists n_0 \in \omega : F_{n_0}$ is not meager in $H \implies \exists U \neq \emptyset$ open in X : $\emptyset \neq U \cap H \subset F_{n_0}$. Let $x \in U \cap H$. Then there exist $\nu \in \omega^\omega \forall n \in \omega : x \in U_{\nu/n} \implies \exists m_0 \in \omega : U_{\nu/m_0} \subset U \implies \emptyset \neq U_{\nu/m_0} \cap H \subset F_{n_0}$. Denote $\nu/m_0 =: s$. For contradiction assume 9. \wedge 3. $\stackrel{?}{\implies} \{U_{s^\wedge n}, n \in \omega\}^d \subset F_{n_0}$ which contradicts 8. So 9. $\implies \{U_{s^\wedge n}, n \in \omega\}^d \subset U_s$,

$$x \in \{U_{s^\wedge n}, n \in \omega\}^d \implies \exists n_k \nearrow, x_k \in U_{s^\wedge n_k} : x_k \rightarrow x, y_k \in U_{s^\wedge n_k} \cap H \subset F_{n_0} \implies$$

$$\implies 4. \implies y_k \rightarrow x \implies x \in F_{n_0}. \quad \square$$

TODO!!!

3 Infinite games

3.1 Baire definitions

Definition 3.1

Assume $A \neq \emptyset$, $X \subset A^\omega$. In game $G(X)$, there are 2 players I, II and those players choose $a_i, i \in \omega, a_i \in A$:

$$I : a_0, a_2, a_4, \dots$$

$$II : a_1, a_3, a_5, \dots$$

Player I wins $\equiv (a_i) \in X$, otherwise player II wins.

Strategy for I is mapping $\mathcal{S} : A^{<\omega} \rightarrow A^{<\omega}$ such that $\forall s \in A^{<\omega} : |\mathcal{S}(s)| = |s| + 1$ and $\forall s, t \in A^{<\omega}, s < t : \mathcal{S}(s) < \mathcal{S}(t)$.

Definition 3.2 (Notation)

$\sigma \subset A^{<\omega}$ be tree iff $\forall s, t \in A^{<\omega}, s < t : t \in \sigma \implies s \in \sigma$

Let σ be tree, $s \in \sigma$. Then s is leaf iff $\forall n \in A : s \wedge n \notin \sigma$.

Let σ be tree. Then σ is pruned $\equiv \sigma$ does not have leaves ($\forall s \in \sigma : \exists a \in A : s \wedge a \in \sigma$).

$$[\sigma] = \{\nu \in A^\omega \mid \forall n \in \omega : \nu/n \in \sigma\}, \quad [\sigma] \in \Pi_1^0(A^\omega).$$

$\forall F \in \Pi_1^0(A^\omega) \exists!$ pruned tree $\sigma \subset A^{<\omega} : [\sigma] = F$.

Strategy for I is pruned tree $\sigma \subset A^{<\omega}$ such that $\sigma \neq \emptyset$, $(a_0, a_1, \dots, a_{2j}) \in \sigma \implies \forall a \in A : (a_0, \dots, a_{2j}, a) \in \sigma$, and $(a_0, \dots, a_{2j+1}) \in \sigma \implies \exists! a \in A : (a_0, \dots, a_{2j+1}, a) \in \sigma$.

Definition 3.3

Strategy σ for I is winning \Leftrightarrow I wins if he follows this strategy $[\sigma] \subset X$.

Poznámka

In Game $G(A, X)$ at most one player has winning strategy. It can happen (ZFC) that nobody has winning strategy.

Definition 3.4 (Game with rulse)

$T \subset A^{<\omega}$ be pruned tree, $X \subset [T]$.

$$I : a_0, a_2, a_4, \dots$$

$$II : a_1, a_3, a_5, \dots$$

such that $\forall n \in \omega : (a_0, \dots, a_n) \in T$ (T is tree of rules). Other notions are similar.

Poznámka

Assume $X = \{x \in A^\omega \mid x \in X \cap [T] \cup (\exists n \in \omega : x/n \notin T \text{ and the least } n \in \omega : x/n \notin T \text{ is odd})\}$ then I (resp. II) has winning strategy in $G(X') \Leftrightarrow$ I (resp. II) has winning strategy in $G(T, X)$.

Například

$SG(A, B_0, B_1)$. Let S, T be nonempty pruned trees on ω , $A \subset [S]$, $B_0, B_1 \subset [T]$.

$$I : x(0), x(1), x(2), \dots$$

$$II : y(0), y(1), y(2), \dots$$

$x(i), y(i) \in \omega, x/n \in S, y/n \in T$. Player II wins $\Leftrightarrow (x \in A \implies y \in B_0) \wedge (x \notin A \implies y \in B_1)$.

I has winning strategy $\Leftrightarrow \exists f : [T] \rightarrow [S]$ continuous: $f(B_0) \cap A = \emptyset$, $f(B_1) \subset A \Leftrightarrow f^{-1}(A)$ separates B_1 from B_2 .

II has winning strategy $\Leftrightarrow \exists g : [S] \rightarrow [T]$ continuous: $g(A) \subset B_0, g(A^c) \subset B_1$.

Věta 3.1 (H?)

X Polish topological space, $A \in \Sigma_1^1(X) \setminus \Sigma_2^0(X)$. Then there exists C , the cantor set, $C \subset X$, such that $\overline{C \setminus A} = C$, $|C \setminus A| = \omega$ ($C \cap A$ is homeomorphic to \mathcal{N}).

Poznámka (Similarly)

$A \in \Pi_1^1(X)$. Then A is polish space or it contains rel. closed set homeomorphic to \mathbb{Q} .

3.2 Banach–Mazur game

Definice 3.5 (Banach–Mazur game $G^{**}(M, Y)$)

Let Y be PTS, $M \subset Y$, $Y \neq \emptyset$.

$$I : U_0, U_1, U_2, \dots$$

$$II : V_0, V_1, V_2, \dots$$

$U_i, V_i \neq \emptyset$ base open sets (some \mathcal{V} , the countable base of Y , $V_i, U_i \in \mathcal{V}$).

Poznámka

There is some ccm on Y and $\text{diam}(U_i) \rightarrow 0$.

$$U_i \supset V_i \supset U_{i+1}, i \in \omega.$$

$$\text{II player wins} \equiv \bigcap V_n = \bigcap u_n \subset M.$$

Věta 3.2

$X \neq \emptyset$ topological space: 1. II has wining strategy $\Leftrightarrow M$ is co-meager. 2. X Polish topological space: I has wining strategy $\Leftrightarrow (\exists U \subset X \text{ open, } U \neq \emptyset : M \cap U \text{ is meager})$.

⌈

Důkaz

„1.“: „ \Leftarrow “: Let W_n be open, dense: $M \supseteq \bigcap W_n$. II plays $V_n \subseteq U_n \cap W_n \wedge \bigcap V_n \subset \bigcap W_n \subset M$.

„ \Rightarrow “:

$\mathcal{M}_0 := \{V_0 | V_0 \text{ is answer of II on some } U_n \text{ from I}\},$

$\tilde{\mathcal{M}}_0$ maximal disjoint subsystem of \mathcal{M}_0 . $W_0 := \bigcup \tilde{\mathcal{M}}_0$ is open and dense.

$\mathcal{M}_n := \{V_n | V_n \text{ is answer of II on some } U_n \text{ from base which is continued in some element of } \tilde{\mathcal{M}}_{n-1}\},$

$\tilde{\mathcal{M}}_n$ is maximal disjoint subsystem of \mathcal{M}_n . $W_n := \bigcup \tilde{\mathcal{M}}_n$ is open and dense.

$\bigcap W_n \subset M \Rightarrow M$ is co-meager.

„2.“: „ \Rightarrow “: I has winning strategy. Put $U := V_0$. Then rest of the game is played inside of U_0 and I wins $\Leftrightarrow \bigcap U_n \subset U_0 \setminus A \Rightarrow U_0 \setminus A$ is co-meager in $U_0 \Rightarrow A$ is meager in $U_0 = U$.

„ \Leftarrow “: $M \subset \bigcup F_n$, F_n are closed, nowhere dense in $U_0 = U$. $U_i \subseteq V_{i-1} \setminus F_{i-1}$, $i \geq 1$.
 $\bigcap U_n \not\subset \bigcup F_n \Rightarrow \bigcap U_n \not\subset M$. $\overline{U_i} \subset \overline{V_i}$, $|V_i| \rightarrow 0$. □

⌋

3.3 Game of J. Malý (point-line game)

Definice 3.6 (Game of J. Malý)

Let B be unit ball in \mathbb{R}^2 .

$I : a_0, a_1, a_2, \dots$

$II : p_0, p_1, \dots$

$a_i \in B$, p_i line in \mathbb{R}^2 , $a_i, a_{i+1} \in p_i$. II player wins $\equiv (a_i)$ is convergent.

Věta 3.3 (From Real functions)

II player has winning strategy in this game.

3.4 Determinacy of games

Definice 3.7

We say that game $G(A, X)$ is determined, if I or II player has winning strategy.

Poznámka (Remark)

I has wining strategy if $\exists a_0 \forall a_1 \exists a_2 \forall a_3 \dots (a_0, a_1, \dots) \in X$.

II has wining strategy if $\forall a_0 \exists a_1 \forall a_2 \exists a_3 \dots (a_0, a_1, \dots) \notin X$.

Věta 3.4 (Gale–Steward, 1953)

$A \neq \emptyset$, $T \subset A^{<\omega}$, $T \neq \emptyset$ pruned tree, $X \subset [T] \subset A^\omega$ be closed in $[T]$. Then $G(T, X)$ is determined.

┌

Důkaz

Assume II does not have winning strategy in $G(T, X)$. We say that $p = (a_0, a_1, \dots, a_{2n-1}) \in T$ is non-losing for I, if II player does not have winning strategy in game with this beginning. II does not have winning strategy $\implies \emptyset$ is non-losing. If p is non-losing than there exist $a_{2n} \in A$ such that $(a_0, a_1, \dots, a_{2n}) \in T$ and $\forall a = a_{2n+1} \in A$ such that $(a_0, a_1, \dots, a_{2n+1}) \in T$ this position is non-losing for I. The strategy for I player is choosing non-losing positions. Assume that this strategy is not winning strategy. So, there exists $(a_0, a_1, \dots) \in [T] \setminus X$ and for every $n \in \omega$: (a_0, \dots, a_{2n+1}) are non-losing position. But from fact that $[T] \setminus X$ is open, $\exists n \in \omega : \mathcal{N}(a_0, a_1, \dots, a_{2n+1}) \cap [T] \subset [T] \setminus X \implies$ this is losing position for I. \square

└

Příklad („Closed“ is necessary)

$A = \{0, 1\}$, $T = 2^{<\omega}$, $X = 2^\omega \setminus \{0, 0, \dots\}$. This game is determined, but the previous proof cannot be used:

$I : 0, 0, 0, 0, \dots$

$II : 0, 0, 0, 0, \dots$

is always non-losing, but $(0, 0, 0, \dots) \notin X$.

Důsledek

$T \neq \emptyset$ pruned tree on A , $X \in [T]$ is open in $[T]$. Then $G(T, X)$ is determined.

┌

Důkaz

I player plays a_0 and we assume game where II player starts and we play with set $(\mathcal{N}(a_0) \cap A^{<\omega})$. Using the previous theorem we have that game is determined. \square

└

Poznámka (Historical remark)

$G(T, X)$ be determined for:

- $X \in \Pi_1^0$: 1953 Gale-Stewnar.
- $X \in \Pi_2^0$: 1955 Woolfe.
- $X \in \Pi_3^0$: 1964 Davis.

- $X \in \Pi_4^0$: 1972 Paris.
- Martin: $T \neq \emptyset$ pruned tree on A , $X \subset [T]$, $X \in \Delta_1^1([T])$. Then $G(T, X)$ is determined.

Analytic determinacy is independent of ZFC.

3.5 Game $G^*(A)$

Definice 3.8 (Game $G^*(A)$)

$X \neq \emptyset$ Polish topological space, d is compatible compact norm on X , $V = (V_n)$ countable bare of X , $V_n \neq \emptyset$, $A \subset X$.

$$I : (U_0^{(0)}, U_1^{(0)}), (U_0^{(1)}, U_1^{(1)}), \dots$$

$$II : i_0, i_1, \dots$$

$$i_n \in \{0, 1\}. U_i^{(n)} \in V, \text{diam } U_i^{(n)} < 2^{-n}. \overline{U_0^{(n)}} \cap \overline{U_1^{(n)}} = \emptyset, \overline{U_0^{(n+1)}} \cup \overline{U_1^{(n+1)}} \subset U_{i_n}^{(n)}.$$

We define $x \in X$ by $\{x\} = \bigcap_{n \in \omega} \bigcup_{i_n} U_{i_n}^{(n)}$, then I wins $\Leftrightarrow x \in A$.

Poznámka

X Polish topological space, $|x| > \omega$. Then there exist $\tilde{X} \in \Pi_1^0(X)$ such that $|X \setminus \tilde{X}| = \omega$, \tilde{X} is perfect. ($\text{Ker}(X), I = \{M \subset X, |M| < \omega\}$).

Věta 3.5

$X \neq \emptyset$, perfect Polish topological space, $A \subset X$. Then 1. I player has winning strategy $\Leftrightarrow A$ contains homomorphic copy of 2^ω . 2. II player has winning strategy $\Leftrightarrow |A| \leq \omega$.

Důkaz

„1. \implies “: Winning strategy for I gives scheme $(U_s)_{s \in 2^\omega \setminus \{\emptyset\}}$ such that $U_s \neq \emptyset$ open $\overline{U_s \wedge 0} \cup \overline{U_s \wedge 1} \subset U_s$, $\overline{U_s \wedge 0} \cap \overline{U_s \wedge 1} = \emptyset$, $\text{diam } U_s < \sum 2^{|s|+1} \forall y \in 2^\omega: \bigcap_{n \in \omega} \bigcup y|_n = \{x\} \subset A$. ($\Leftarrow f(y) = x$ clearly homeomorphism.).

$$C = \bigcap_{n=1}^{\infty} \bigcup_{|s|=n} U_s = \bigcup_{s \in 2^\omega} \bigcap_{k \in \omega \setminus \{0\}} U_{s/k} \subset A.$$

„1. \Leftarrow “: A continuous copy C of 2^ω (homeomorphic). I player choose $U_i^{(n+1)}$: $\text{diam}(U_i^{(n+1)}) < 2^{-n-1}$, $U_i^{(n+1)} \cap C \neq \emptyset$, $\overline{U_0^{(n+1)}} \cap \overline{U_1^{(n+1)}} = \emptyset$, $\overline{U_0^{(n+1)}} \cup \overline{U_1^{(n+1)}} \subset U_{i_n}^{(n)}$.

„2. \Leftarrow “: $|A| \leq \omega$: $A = \{x_0, x_1, \dots\}$. II in n -th move choose $i_n \in \{0, 1\}$ to have $x_n \notin U_{i_n}^{(n)}$. $\implies \bigcap_{n \in \omega} \bigcup_{i_n} U_{i_n}^{(n)} \not\subset A$.

„2. \implies “ Let σ be winning strategy for II player. Let $x \in A$. We say that position $p = ((U_0^{(0)}, U_1^{(0)}), i_0, \dots, (U_0^{(n-1)}, U_1^{(n-1)}), i_{n-1})$ is good for x if we get to this position following one winning strategy and $x \in U_{i_{n-1}}^{(n-1)}$. We assume empty sequence be good for x . Assume that every good position p for x has nontrivial extension, which is still good for x . Then there exists some play following σ , such that the result is $x \in A$, which is const. So $\forall x \in A \exists$ maximal good position p for x . $A_p = \{y \in U_{i_{n-1}}^{(n-1)} \mid \text{for every extension following rules } (U_0^{(n)}, U_1^{(n)}) \text{ if } i \text{ is the move following } \sigma, \text{ then } y \notin U_i^{(n)}\} = \{y \mid p \text{ maximal good for } y\}$.

$A \subset \bigcup_p A_p$, $|A_p| = 1$? By constr. $y_0 \neq y_1$, $y_0, y_1 \in A_p$. We choose $U_0^{(n)}, U_1^{(n)} : y_i \in U_{i_n}^{(n)} \implies y_0$ or $y_1 \in U_{i_n}^{(n)}$ and p is not maximal for this element. $\implies |A| = \omega$. \square

3.6 Game $G_u^*(F)$

Definice 3.9 (Game $G_u^*(F)$)

$X \neq \emptyset$, perfect, Polish topological space, $F \subset X \times \omega^\omega$.

$$I : y(0), (U_0^{(0)}, U_1^{(0)}), y(1), (U_0^{(1)}, U_1^{(1)}), \dots$$

$$II : i_0, i_1, \dots$$

$U_i^{(n)}, i_n$ are same like in the previous game, $U_i^{(n)} \in V$ (fix constant bare), $y(u) \in \omega$, $x \in X$ like before, $y = (y(0), y(1), \dots)$. I wins $\Leftrightarrow (x, y) \in F$.

Věta 3.6

$X \neq \emptyset$, perfect, Polish topological space, $F \subset X \times \omega^\omega$, $A = \prod_X(F)$. Then 1. I has winning strategy in $G_u^* \implies A$ contains 2^ω . 2. II has winning strategy in $G_u^*(F) \implies |A| \leq \omega$.

┌ *Důkaz*

„1.“ proof is same like proof of 1. in the previous theorem.

„2.“ Let II has winning strategy in $G_n^*(F)$. Assume $(x, y) \in F$. Similarly like before we find maximal good position for (x, y) ($p = (y(0), (U_0^{(0)}, U_1^{(0)}), i_0, y(1), (U_0^{(1)}, U_1^{(1)}), i_1, \dots, y(n-1), (U_0^{(n-1)}, U_1^{(n-1)}), i_{n-1})$). Put $A_{p,\sigma} = \{z \in U_{i_{n-1}}^{(n-1)} \mid \text{for any possible extension } (a, U_0^{(n)}, U_1^{(n)}), \text{ if } i \text{ is played winning } \sigma, \text{ then } z \notin U_i^{(n)}\}$. If $a = y_0(n)$ and p as above, there $x \in A_{p,a}$.

Similarly like before $|A'_{p,n}| \leq 1, A \subset \bigcup_{p \in \sigma, a \in \omega} A_{p,a} \implies |A| \leq \omega$. □

Definice 3.10

Let Γ be class of subsets of Polish topological spaces, $A \subset X$ (X Polish topological space) be Γ -hard $\equiv \forall B \in \Gamma(\omega^\omega) \exists f : \omega^\omega \rightarrow X : f^{-1}(A) = B$.

A is Γ -complete $\equiv A \in \Gamma \wedge A$ be Γ -hard.