1 Introduction

Poznámka (Literature)
„Riemann surfaces and algebraic curves“, Renzo Cavalieri and Eric Miles

1.1 Differentiability

Definice 1.1 (Differentiable)

A function $f : \mathbb{C} \to \mathbb{C}$ is differentiable (also holomorphic) at a point $z_0 \in \mathbb{C}$ if the following limit exists

$$\lim_{|h| \to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0) \in \mathbb{C}.$$

We call $f'(z_0)$ the derivative of f at z_0. A function f is differentiable on a domain (open connected subset of $\mathbb{C}$) if its differentiable for all points of this domain.

Poznámka (Writing complex numbers in cartesian cooridnates)
$z = x + iy$, for $x, y \in \mathbb{R}$, we can write a function $f : \mathbb{C} \to \mathbb{C}$ in terms of two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x, y) = u(x, y) + i \cdot v(x, y).$$

Věta 1.1 (Cauchy–Riemann equations)

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on an open subset of $\mathbb{C}$. Considering $f = u + iv$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definice 1.2 (Orientability, orientation-preserving function)

Define and equivalence relation on the set of all bases of $\mathbb{R}^2$ by saying that $B_1 \sim B_2$ iff the determinant of the change of basis matrix is positive.

A function $f : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ is said to be orientation-preserving if on an open dense subset of U, the determinant of the Jacobi matrix is positive. Jacobi matrix:

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Důsledek
Let f be a non-constant holomorphic function, then f is orientation-preserving.

Důsledek
Since f is holomorphic, the Cauchy-Riemann equations implies that

$$\det(J(f)) = \frac{\partial u}{\partial x}\frac{\partial v}{y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} \overset{\text{C–R}}{=} \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geqslant 0.$$

Since f is non-constant, the inequality is strict on a dense open subset of the domain of definition.

Věta 1.2 (Open mapping theorem)

A non-constant holomorphic function f is open (that is if U is an open subset of $\mathbb{C}$, then $f(U)$ is also open).

1.2 Integration

Definice 1.3

For a path γ (smooth function, $\gamma : \mathbb{R} \supset [a, b] \to \mathbb{C}$) we define

$$\int_\gamma f(x)dx := \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt$$

Definice 1.4 (Continuous deformation)

For $\gamma, \mu : [a, b] \to U$ (U simply connected), paths with the same endpoints ($\gamma(a) = \mu(a)$ and $\gamma(b) = \mu(b)$). Then a continuous deformation γ into μ is a continuous function $H : [a, b] \times [0, 1] \to U \subseteq \mathbb{C}$ such that $H(s, 0) = \gamma(s)$, $H(s, 1) = \mu(s)$, $H(a, t) = z_a := \gamma(a) = \mu(a)$ and $H(b, t) = z_b := \gamma(b) = \mu(b)$.

Věta 1.3

Suppose that $\gamma, \mu : [a, b] \to U$ (U simply connected) are related by a continuous deformation of paths H. Then for any holomorphic function f on U we have

$$\int_\gamma f(z)dz = \int_\mu f(z)dz.$$

Důkaz (Partial proof assuming H admits partial derivatives)
For any $t \in [0,1]$ we integrate the function $INT(t) = \int_{H(\cdot,t)} f(z)dz$. Consider the derivative of $INT(t)$ with respect to t:

$$\frac{d}{dt}(INT(t)) = \frac{d}{dt}\left(\int_a^b f(H(s,t))\frac{\partial H}{\partial s}(s,t)ds\right) \overset{\text{Leibniz + chain rule}}{=}$$

$$= \int_a^b f'(H(s,t))\frac{\partial H}{\partial t}(s,t) \cdot \frac{\partial H}{\partial s}(s,t) + f(H(s,t))\frac{\partial^2 H}{\partial s \partial t}(s,t)ds =$$

$$= \int_a^b \frac{d}{ds}\left[f(H(s,t))\frac{\partial H}{\partial t}\right] ds =$$

$$= f(H(s,t))\frac{\partial H}{\partial t}|_{s=a}^{s=b} \overset{\text{constant endpoints}}{=} 0.$$

Having derivative identically equal to 0, means that $INT(t)$ is a constant function and $\int_\gamma f(z)dz = INT(0) = INT(1) = \int_\mu f(z)dz$. □

Důsledek
Let U be a simply connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. For any closed path whose image is inside U, $\int_\gamma f(z)dz = 0$.

Důkaz (Sketch)
The definition of simply connected is (essentially) the same as saying that any closed path can be continuously deformed to a constant path c.

$$\int_\gamma f(z)dz = \int_c f(z)dz = \int_a^b f(c(z)) \cdot c'(z)dz = \int_a^b f(c(z)) \cdot 0dz = 0$$

□

Příklad
Let U be a simple connected domain and $f : U \to \mathbb{C}$ a holomorphic function on $U \setminus \{z_0\}$. For $j = 1, 2$, let γ_j be a path parametrizing a circle centered at z_0 of radius r_j, oriented counterclockwise and completely contained in U. Show that $\oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$.

1.3 Cauchy's integral formula

Věta 1.4 (Cauchy's integral formula)

Let γ be a loop around $z \in \mathbb{C}$, and $f : U \to \mathbb{C}$ a holomorphic function. For U a neighbourhood of γ,

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}dw.$$

Důkaz
Conway 1978, Chapter IV. □

Důsledek

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0 + z_0 - z}dw = \frac{1}{2\pi i}\oint \frac{f(w)}{w - z_0} \cdot \left(\frac{1}{1 - \frac{z - z_0}{w - z_0}}\right) dw =$$
$$= \frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w - z_0}\left(\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^n}\right) dw =$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w - z_0)^n}\right)(z - z_0)^n.$$

For sufficiently "small" (shrunken) γ. So f is smooth (infinitely differentiable). Moreover, it is analytic (that is, its Taylor expansion around z_0 converges to f in a neighbourhood of z_0).

Definice 1.5 (Pole)

Given a positive integer n, a complex function f has pole of order n at the point $z_0 \in \mathbb{C}$ if $(z - z_0)^n f(z)$ is holomorphic at z_0 but $(z - z_0)^{n-1} f(z)$ is not.

Příklad
Show that if f has a pole of order n at $z_0 \in \mathbb{C}$. Then it admits a Laurent expansion $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$ with $a_{-n} \neq 0$.

Definice 1.6 (Residue)

Let f have a pole of order n at the point $z_0 \in \mathbb{C}$. Then the residue of f at z_0 is the $k = -1$ coefficient of the Laurent expansion of f at z_0.

Příklad
Show that if f has a pole of order 1 at z_0, then the residue of f at z_0 can be computed as the following limit:

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z).$$

Příklad (Residue theorem)
Let $\gamma : [a, b] \to U \subset \mathbb{C}$ be a simple closed path, bounding a domain W containing the points $z_1, \ldots, z_m$. Assume that f is holomorphic on $U \setminus \{z_1, \ldots, z_m\}$ and has poles at $\{z, \ldots, z_m\}$.

Show that

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{m} \operatorname{res}_{z=z_j} f(z).$$

TODO!!!

1.4 (Real) Projective space

Poznámka (Building structures)
Set → Topology → Differential structure (atlas) → Riemann metric → Connection...

Definice 1.7 (Real projective space)

The set $\mathbb{P}^n(\mathbb{R})$ is defined to be either of the following bijective sets: Lines through the origin in $\mathbb{R}^{n+1}$; Equivalence classes of $(n+1)$-tuples of real numbers $(x_0, \ldots, x_n) \neq (0, \ldots, 0)$, such that for any real number $\lambda \in \mathbb{R}^* = \mathbb{R}\setminus\{0\}$: $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$.

Příklad
Confirm that the sets above are in bijection with each other.

Poznámka (Notation)
We will often denote a point in $\mathbb{P}^n(\mathbb{R})$ as the equivalence class $[x_0, \ldots, x_n]$.

Definice 1.8 (Topology of $\mathbb{P}^n(\mathbb{R})$)

We give a topology to $\mathbb{P}^n(\mathbb{R})$ by endowing it with following quotient topology: consider the surjection $\pi : \mathbb{R}^{n+1}\setminus\{\mathbf{o}\} \to \mathbb{P}^n(\mathbb{R})$, $(x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$. A set $U \subset \mathbb{P}^n(\mathbb{R})$ is defined to be open if $\pi^{-1}(U) := \{x \in \mathbb{R}^{n+1}\setminus\{\mathbf{o}\} | \pi(x) \in U\}$ is open in $\mathbb{R}^{n+1}\setminus\{\mathbf{o}\}$.

That is we give $\mathbb{P}^n(\mathbb{R})$ the finest topology that makes π continuous.

Příklad
Check that for $\mathbb{C}$ we can define $\mathbb{P}^n(\mathbb{C})$ or $\mathbb{CP}^n$ the same way.

Příklad (Projective space)
$\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ is an abelian group. Let $\mathbb{R}^*$ act on $\mathbb{R}^{n+1}$ by component wise multiplication. When a general group G acts on a set X we have equivalence relation $x \sim y$ if $y = g \circ x$. We call the equivalence classes the orbits of G. So $\mathbb{P}^n(\mathbb{R}) = \left(\mathbb{R}^{n+1}\setminus\{\mathbf{o}\}\right)/\mathbb{R}^*$.

Sphere quotient: Let $S^n \subseteq \mathbb{R}^{n+1}$. Denote the unit sphere. Then the group $\mathbb{Z}_2 = \{+1, -1\}$ act on the sphere by $\pm 1(x_0, \ldots, x_n) = (\pm x_0, \ldots, \pm x_n)$. Then $S^n/\mathbb{Z}_2 = \mathbb{P}^n(\mathbb{R})$.

Disk model: Consider the n-dimensional closed unit disk $\overline{\mathbb{D}^n} \subseteq \mathbb{R}^n$, and the equivalence relation on the points of the boundary: $x \sim -x$ if $\|x\| = 1$. Then $\mathbb{P}^n(\mathbb{R})$ is the quotient (collection of equivalence classes), i.e. $\overline{D^n} \setminus \sim \simeq \mathbb{P}^n(\mathbb{R})$.

Příklad
Conclude from either of these models of $\mathbb{P}^n(\mathbb{R})$ that as a topological space, $\mathbb{R}^n(\mathbb{P})$ is compact and Hausdorff.

Poznámka
Now we come to the smooth manifold structures. Let's start with $\mathbb{P}^1(\mathbb{R})$. Define

$$U_x := \mathbb{P}^1(\mathbb{R}) \setminus \left\{ [x,y] \in \mathbb{P}^1(\mathbb{R}) | x = 0 \right\}, \qquad \varphi_x : U_x \to \mathbb{R}, \quad \varphi_x([x,y]) = \frac{x}{y}.$$

Similarly, we define a second chart:

$$U_y := \mathbb{P}^1(\mathbb{R}) \setminus \left\{ [x,y] \in \mathbb{P}^1(\mathbb{R}) | y = 0 \right\}, \qquad \varphi_y : U_y \to \mathbb{R}, \quad \varphi_y([x,y]) = \frac{y}{x}.$$

Příklad
Check that U_x, U_y are open and that φ_x, φ_y are homeomorphisms.

Důkaz
Consider the transition functions:

$$U = U_x \cap U_y = \left\{ [x,y] \in \mathbb{P}^1(\mathbb{R}) | x, y \neq 0 \right\}, \qquad \varphi_x(U) = \varphi_y(U) = \mathbb{R} \setminus \{0\}.$$

The translation function $T_{x,y} := \varphi_y \circ (\varphi_x)^{-1}$ sends, for $y \neq 0$:

$$T_{x,y} : y \overset{(\varphi_x)^{-1}}{\mapsto} [1, y] = \left[\frac{1}{y}, 1 \right] \overset{\varphi_y}{\mapsto} \frac{1}{y}.$$

Which is smooth on the domain $\mathbb{R} \setminus \{0\}$.

TODO smooth. Thus $\mathbb{P}^1(\mathbb{R})$ is a smooth manifold. □

Příklad
Show that $\mathbb{P}^1(\mathbb{R})$ is homomorphic to the circle S^1. We call $\mathbb{P}^1(\mathbb{R})$ the real projective line.

Příklad
Try to show that $\mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C})$ is a smooth manifold.

Příklad
For $\mathbb{P}^2(\mathbb{R})$ the followings charts form atlas:

$$U_x := \{[x,y,z] | x \neq 0\}, \qquad \varphi_x : U_x \to \mathbb{R}, \quad \varphi_x([x,y,z]) = \left(\frac{y}{x}, \frac{z}{x}\right),$$

$$U_y := \{[x,y,z] | y \neq 0\}, \qquad \varphi_y : U_y \to \mathbb{R}, \quad \varphi_y([x,y,z]) = \left(\frac{x}{y}, \frac{z}{y}\right),$$

$$U_z := \{[x,y,z] | z \neq 0\}, \qquad \varphi_z : U_z \to \mathbb{R}, \quad \varphi_z([x,y,z]) = \left(\frac{x}{z}, \frac{y}{z}\right).$$

Check these are open subsets and homeomorphisms, with smooth transformation functions. And extend this to $\mathbb{P}^n(\mathbb{R})$.

1.5 Compact surfaces

Definice 1.9 (Surface)

A surface is a manifold of real dimension 2.

Například
$\mathbb{R}^2$, $\mathbb{C}$, and any of their open subsets are surfaces. S^2 is a compact surface, as is $\mathbb{P}^2(\mathbb{R})$.

Definice 1.10 (Connected surface)

Given two connected surfaces S_1 and S_2, the connected surface $S_1 \# S_2$ is the surface obtained by removing an open disc from each of the surfaces and identifying the resulting boundaries via a homeomorphism.

Příklad
At the level of topological spaces, show that the operation # is well defined up to homeomorphism, that is, show that the choice of disks in S_1 and S_2 does not change the definition of $S_1 \# S_2$ / homeomorphism.

Příklad
Show that # gives the set of homeomorphism classes of connected compact surfaces the structure of a monoid. (Which surface is the identity of the monoid?)

Věta 1.5 (Classification of compact surfaces)

Any connected, compact surfaces is homeomorphic to exactly one surface in the following list:

- S^2;
- $T^{\#g} := T \# \dots \# T$, $g \in \mathbb{N}_0$;

- $\mathbb{P}^2(\mathbb{R})^{\# n} := \mathbb{P}^2(\mathbb{R})\# \dots \#\mathbb{P}^2(\mathbb{R})$, $n \in \mathbb{N}_0$.

Poznámka (Deep fact)
For $d \leqslant 3$, if two d-dimensional manifolds are homeomorphic, then they are diffeomorphic.

2 Riemann surfaces

Definice 2.1 (Riemann surface)

A Riemann surface is a complex analytic manifold of dimension 1:

- X is a Hausdorff, connected topological space;
- for all $x \in X$, there is a homeomorphism $\varphi_x : U_x \to V_x$, where U_x is an open neighbourhood of $x \in X$, V_x is an open set in $\mathbb{C}$;
- for any U_x, U_y such that $U_x \cap U_y \neq \varnothing$, the transition function $T_{x,y} := \varphi_y \circ \varphi_x^{-1} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y)$ is holomorphic.

Poznámka
We saw in the first lecture that a holomorphic preserves orientation when thought of as a function from the real plane to itself. Since our transition functions are holomorphic, any Riemann surface is orientable.

Příklad (The complex projective line)
Just as for $\mathbb{P}^1(\mathbb{R})$, we define $\mathbb{P}^1(\mathbb{C})$ to be the set whose elements are complex 1-dimensional subspaces of $\mathbb{C}^2$.

Let $U_1 = U_2 := \mathbb{C}$ and define $g : U_1 \setminus \{0\} \to U_2 \setminus \{0\}$, $z \mapsto \frac{1}{z}$. We define $\mathbb{P}^1(\mathbb{C})$ to be the quotient $\mathbb{P}^1(\mathbb{C}) := U_1 \coprod U_2/(z \sim g(z))$.

Příklad (Show that)
As a set $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{C}$ plus a point.

As a topological space $\mathbb{P}^1(\mathbb{C})$ is the one point compactification of $\mathbb{C}$.

Conclude from the previous sentence that $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the two sphere.

Poznámka
In complex analysis $\mathbb{P}^1(\mathbb{C})$ is known as the Riemann sphere.

Poznámka
For $i = 1, 2$, we denote the image of U_i in the quotient $U_1 \coprod U_2/(z \sim g(z))$ by $[U_i]$. Note that U_i define the local coordinate functions: $\varphi_i : [U_i] \to U_i$, $p \mapsto z_i$, where z_i is the complex numbers in U_i such that $[z_i] = p_i$. Both φ_1, φ_2 are homeomorphisms.

We now consider the transition functions: the intersection

$$[U_1] \cap [U_2] = [U_i \backslash \{0\}] = [U_2 \backslash \{0\}].$$

The image of the intersection under φ_1 is $\varphi_1([U_i] \cap [U_2]) = \mathbb{C} \backslash \{0\}$ (*). Thus (*) is the domain of our single transition function $T = \varphi_2 \circ \varphi_1^{-1}$. For $z_1 \neq 0$, we have $T : z_1 \overset{\varphi_1^{-1}}{\mapsto} [z_1] = [z_2 := g(z_1) = 1/z_1] \overset{\varphi_1}{\mapsto} z_2 = 1/z_1$. Thus

$$T : \mathbb{C} \backslash \{0\} \to \mathbb{C} \qquad z \mapsto \frac{1}{z}.$$

Since T has a pole only at $z_1 = 0$, we see that it is holomorphic on $\mathbb{C} \backslash \{0\}$. A symmetric (exchange 1 for 2) calculation shows that T^{-1} is also holomorphic. So $\mathbb{P}^1(\mathbb{C})$ is a Riemann surface.

Příklad (Hopf fibration)
Consider the 3-dimensional real sphere $S^3 \subseteq \mathbb{C}^2 = \mathbb{R}^4$. Given a point $p \in S^3$, $\exists$ a unique line l_p through the origin and p. Thus we got a function $H : S^3 \to \mathbb{P}^1(\mathbb{C})$, $p \mapsto l_p$.

Check that H is continuous and surjective. Since S^3 is closed and bounded, it is compact. Moreover, since the image of a compact set under a continuous function is compact, $\mathbb{P}^1(\mathbb{C})$ is compact.

What is the fiber of the surjective map $H : S^3 \to \mathbb{P}^1(\mathbb{C})$, i.e. what is $H^{-1}(p)$, for any point $p \in \mathbb{P}^1(\mathbb{C})$. (Hint: It is S^1.)

This gives us $S^2 \times S^1 = S^3$ as set. (Not as topological space!)

Příklad (Complex tori)
Definition: Let τ_1 and τ_2 be two complex numbers, that are linearly independent. The set of all integral linear combinations of τ_1 and τ_2:

$$\Lambda := \{n\tau_1, m\tau_2 | n, m \in \mathbb{Z}\} \subseteq \mathbb{C}$$

is called the lattice of τ_1 and τ_2.

Observe that we can assume that $\tau_1 = 1$, and $\Im(\tau_2) > 0$, allowing to make simplifying assumption: that our lattice has the form $\Lambda = \{n + m\tau | n, m \in \mathbb{Z}, \tau \in \mathbb{H}\}$, where $\mathbb{H}$ is the upper half plane.

Consider the quotient space $T = \mathbb{C}/\Lambda$. THat is the quotient space with respect to the equivalence relation $z_2 \sim z_1 \Leftrightarrow z_2 = z_1 + w$ for $w \in \Lambda$.

The canonical projection map $\pi : \mathbb{C} \to T$ (i.e. $\pi(z) = [z]$) induces a quotient topology on T (i.e. $V \subseteq T$ is open iff $p^{-1}(V)$ is open in $\mathbb{C}$).

Příklad
For P the closed parallelogram with vertices $0, 1, \tau, 1+\tau$, show that for any $z \in \mathbb{C}$ $\exists z' \in P$ that $\pi|_p \to T$ is surjective. Hence we can restrict our attention to p.

Poznámka
By considering the identification points in p, we see that T is topologically a torus.

Příklad
Prove that π (from previous exercise) is an open map, i.e. that V an open subset of $\mathbb{C}$ implies that $\pi(V)$ is open in T.

Poznámka
Now to the complex structure: from the previous exercise, we see that if π restricted to a subset $V \subseteq \mathbb{C}$ is bijective, then it is a homeomorphism onto its image in T. In this case, $(\pi|_V)^{-1}$ is also a homeomorphism from the image of $\pi|_V$ to V. Hence we can use $(\pi|_V)^{-1}$ as a chart for T.

Příklad
Find a real number r (depends on t) such that for any $z \in \mathbb{C}$: π restricted to $B_r(z)$ is a bijective map.

Given this r, define $U_z := \pi(B_r(z)) \subseteq T$ and $\varphi_z := (\pi|_{B_r(z)})^{-1}$. We claim that the collection $\mathcal{A} = \{U_z, \varphi_z | z \in \mathbb{C}\}$ forms an atlas for T. It is clear that $\mathcal{A}$ gives a cover for T. Moreover, by definition the maps φ_z are homeomorphic to their images. Assume that $U_{z_1} \cap U_{z_2} \neq \varnothing$. For $j \in [2]$ denote by (α_j, β_j) the unique pair of real numbers such that $z_j = \alpha_j + t\beta_j$. We have that $T_{21}(z) = \left(\varphi_{z_1} \circ \varphi_{z_2}^{-1}\right)(z) = z + k$, where $k = ([\alpha_2] - [\alpha_1]) + ([\beta_1] - [\beta_2])t$ is just a constant depending on z_1 and z_2. Therefore the transition function T_{21} is holomorphic $\implies$ T is a Riemann surface.

3 Graph of complex functions

Definice 3.1 (Graph)

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function. The graph of f is the set

$$\Gamma_f := \{(z, f(z)) | z \in \mathbb{C}\} \subseteq \mathbb{C} \times \mathbb{C},$$

given the subset topology.

Poznámka

Note that Γ_f is Hausdorff since $\mathbb{C} \times \mathbb{C}$ is Hausdorff. The graph of f is naturally given the structure of a Riemann surface by an atlas, with one chart, namely Γ_f: the local coordinate function is the projection map $\varphi := \pi_1|_{\Gamma_f}$, i.e. $(z, f(z)) \mapsto z$. TODO!!!(Jedna celá tabule)

Definice 3.2 (Affine plane curve)

For any polynomial $p(x,y) \in \mathbb{C}[x,y]$, the set $V(p) := \{(x,y)|p(x,y) = 0\} \subseteq \mathbb{C}^2$, is called an affine plane curve. We say that $V(p)$ is smooth if $\nexists (x_0, y_0) \in V(p)$ such that $\frac{\partial p}{\partial x}(x_0, y_0) = 0 = \frac{\partial p}{\partial y}(x_0, y_0)$.

Věta 3.1

A smooth affine plane curve is a Riemann surface.

Důkaz

Let $(x_0, y_0) \in V(p)$. Since $V(p)$ is smooth, then for at least one of $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$ is non-zero at (x_0, y_0). Assume (WLOG) that $\frac{\partial p}{\partial y}(x_0, y_0) \neq 0$. Then by the implicit function theorem, exists a neighbourhood $U_{(x_0,y_0)} \subseteq \mathbb{C}^2$, and a neighbourhood $V_{x_0} \subseteq \mathbb{C}$ and a holomorphic function $f : V_{x_0} \to \mathbb{C}$ such that $V(p) \cap U_{(x_0,y_0)} = \{(x, f(x))|x \in V_{x_0}\}$. We call this the graph of f.

We get a local chart on $V(p)$ around (x_0, y_0) (as in the previous example) by setting $\varphi_{(x_0,y_0)} : V(p) \cap U_{(x_0,y_0)} \to V_{x_0}$, $(x, f(x)) \mapsto x$. Finally, we show that the transition functions are holomorphic: for $U_{(x_0,y_0)} \cap U_{(x,y)} \cap V(p) \neq 0$, if $\varphi_{(x_0,y_0)}$ and $\varphi_{(x,y)}$ are both projections to the same axis, the transition function $\varphi_{(x_0,y_0)} \circ \varphi^{-1}_{(x_0,y_0)}$ is the identity function restricted to the appropriate domain in $\mathbb{C}$. Assume now that $\varphi_{(x_0,y_0)}$ is projection onto the x-axis and that $\varphi_{(x,y)}$ is projection to the axis y. Then set $U_{(x_0,y_0)} \cap U_{(x,y)} \cap V(p)$ is simultaneously on the graph of a holomorphic function f_0 and of a holomorphic function f_1. Then functions all $\varphi_{(x,y)} \circ \varphi^{-1}_{(x_0,y_0)} = f_0(x)$ and $\varphi_{(x_0,y_0)} \circ \varphi^{-1}_{(x,y)} = f_1(x)$ restricted to the appropriate domains, which are holomorphic. □

4 Projective curves

Příklad

Consider the polynomial $p(x,y,z) = x^2 + y + z + 1$. Note that $p(1,1,1) = 4 \neq 7 = p(2,2,2)$ since $[1,1,1] = [2,2,2]$ in $\mathbb{P}^2(\mathbb{C})$ p does not restrict to $\mathbb{P}^2(\mathbb{C})$.

Definice 4.1 (Homogeneous polynomial)

A polynomial $p \in \mathbb{C}[x,y,z]$ is said to be homogeneous of degree l, if the following equivalent conditions hold

- every monomial of p has degree l;
- for each $t \in \mathbb{C}$: $p(tx, ty, tz) = t^l p(x, y, z)$;
- $x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} + z\frac{\partial p}{\partial z} = lp$.

Důsledek
If p is homogeneous, $V(p) \subset \mathbb{P}^2(\mathbb{C})$ is well-defined.

Příklad
Confirm that these three conditions are equivalent.

Příklad
Show that if $p \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial, then the set of points $[x, y, z] \in \mathbb{P}^2(\mathbb{C})$ satisfying $p(x, y, z) = 0$ is well-defined.

Definice 4.2

We call

$$V(p) := \left\{[x, y, z] \in \mathbb{P}^2(\mathbb{C}) | p(x, y, z) = 0\right\}$$

the vanishing locus of p. Moreover, we call $V(p)$ a (plane) projective curve of degree l.

If

$$\left\{(x, y, z) \in \mathbb{C}^3 | \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0\right\}$$

the $V(p)$ is said to be smooth.

Tvrzení 4.1

A smooth projective plane curve $V(p)$ is a compact Riemann surface.

Důkaz
We first show that $V(p)$ is compact by showing that $V(p)$ is closed set in $\mathbb{P}^2(\mathbb{C})$ which is a compact space.

Consider the diagram $\mathbb{P}^2(\mathbb{C}) \xleftarrow{\pi} \mathbb{C}^3 \setminus \{(0,0,0)\} \xrightarrow{p} \mathbb{C}$, where π is the natural projection function and p is the continuous function defined by the homogeneous polynomial $p : \mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbb{C}$, $(x,y,z) \mapsto p(x,y,z)$ by definition $V(p)$ is a closed subset of $\mathbb{P}^2(\mathbb{C})$ if $\pi^{-1}(V(p))$ is closed in $\mathbb{C}^3 \setminus \{(0,0,0)\}$. But $\pi^{-1}(V(p)) = p^{-1}(0)$ is the inverse image of the closed set $\{0\} \subseteq \mathbb{C}$. Thus since p is continuous, $p^{-1}(0)$ is closed, in other words, $\pi^{-1}(V(p))$ is closed. Thus $V(p)$ is compact.

So, to show that $V(p)$ is Riemann surface, we need to show that its intersection with any of the coordinate open sets of $\mathbb{P}^2(\mathbb{C})$ is a Riemann surface. So let us consider (WLOG) the chart $U_z = \{[x,y,z] | z \neq 0\} \subseteq \mathbb{P}^2(\mathbb{C})$ with affine coordinates $\varphi_z(x,y,z) = \left(\frac{x}{z}, \frac{y}{z}\right)$. The set $\varphi_z(V(p) \cap U_z)$ is equal to $V(\tilde{p})$ where $\tilde{p}(x,y) = p(x,y,1)$.

Now for any $(x,y) \in \mathbb{C}^2$:

$$(*) : \frac{\partial \tilde{p}}{\partial x}(x,y) = \frac{\partial p}{\partial x}(x,y,1), \qquad (**) : \frac{\partial \tilde{p}}{\partial y}(x,y) = \frac{\partial p}{\partial y}(x,y,1).$$

We claim $\nexists$ an $(\hat{x}, \hat{y}) \in \mathbb{C}^2$ such that

$$\tilde{p}(\hat{x},\hat{y}) = \frac{\partial \tilde{p}}{\partial x}(\hat{x},\hat{y}) = \frac{\partial \tilde{p}}{\partial y}(\hat{x},\hat{y}) = 0.$$

This claim implies that $V(\tilde{p})$ is a smooth affine plane curve and hence a Riemann surface. Since the restriction of $V(p)$ with any affine chart is a Riemann surface, then so is $V(p)$.

So it remains to prove the claim: Assume $\exists(\hat{x}, \hat{y}) \in \mathbb{C}$ satisfying condition above. By $(*)$ and $(**)$, together smoothness of $V(p)$, which would imply that $\frac{\partial p}{\partial z}(\hat{x},\hat{y},1) \neq 0$. But now Euler's identity implies $0 \neq \frac{\partial p}{\partial x}(\hat{x},\hat{y},1) + \frac{\partial p}{\partial y}(\hat{x},\hat{y},0) + \frac{\partial p}{\partial z}(\hat{x},\hat{y},1) = lp(\hat{x},\hat{y},1) = 0 \implies$ contradiction $\implies$ we are done. $\square$

Příklad
Confirm that $V(p)$ is Hausdorff.

Například (Elliptic curves)
Consider a polynomial p of the form $p(x,y,z) = y^2 z - (x - \alpha_1 z)(x - \alpha_2 z)(x - \alpha_3 z)$ where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ are distinct complex numbers. Note that the partial derivative with respect to y satisfies $\frac{\partial p}{\partial y} = 2yz$, which is zero only if $y = 0$ or $z = 0$. We show that $V(p)$ is a smooth projective curve by considering the case $z = 0$, $y = 0$, and finding in each chase a non--vanishing partial derivative.

„Case $z = 0$“: Then the only part in $\mathbb{P}^2(\mathbb{C})$ belonging to $V(p)$ is $[0,1,0]$. But we have $\frac{\partial p}{\partial z} = y^2 + Q(x,z) = 1 + 0 \neq 0$.

„Case $y = 0$": Then the parts belonging to $V(p)$ are $[\alpha_1, 0, 1]$, $[\alpha_2, 0, 1]$, $[\alpha_3, 0, 1]$. For $j \in [3]$: $\frac{\partial p}{\partial x}(\alpha_j, 0, 1) \neq 0$, follows from the fact that the α_1, α_2, α_3 are distinct.

So $V(p)$ is a smooth projective curve of degree 3.

Například
Consider the function $\varphi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C})$ defined in homogeneous coordinates by $\varphi[s,t] = [s^3, s^2t, st^2, t^3]$. We call the image of φ the twisted cubic in $\mathbb{P}^3(\mathbb{C})$.

5 Holomorphic maps of Riemann surfaces

Definice 5.1

Let X, Y be two Riemann surfaces, and $f : X \to Y$ a function (of sets).

- We say that f is holomorphic at $x \in X$ if for every choice of charts φ_x, $\varphi_{f(x)}$, the function $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic.
- If $U \subseteq X$ is open, we say that f is holomorphic on U if f is holomorphic at each $x \in U$.
- If f is holomorphic on $U = X$, then we say that f is a holomorphic map.

The function $F := \varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is called a local expression for f.

Příklad
Show that a map of Riemann surfaces $f : X \to Y$ is holomorphic at $x \in X \Leftrightarrow \exists$ a choice of charts φ_x, $\varphi_{f(x)}$ such that $\varphi_{f(x)} \circ f \circ \varphi_x^{-1}$ is holomorphic at x.

Příklad (Eg)
Let X, Y be two Riemann surfaces and choose a point $y \in Y$. Denote associated constraint map $c : X \to Y$, $x \mapsto y$. We see that c is a holomorphic map.

Příklad (Eg)
Let X be a Riemann surface. The identity map on X is the function $I_X : X \to X$, $x \mapsto x$. We see that I_x is a holomorphic map.

Příklad (Eg)
Consider an arbitrary rational function $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z) \in \mathbb{C}[z]$ are two polynomials with distinct roots. We can extend f to a function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$ as follows:

- $f(\alpha) := \infty$ (point at infinity of the Riemann sphere) where α is a root of q;
- $f(\infty) := \lim_{z\to\infty} \frac{p(z)}{q(z)}$.

Show that f is a holomorphic function from $\mathbb{P}^1(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$.

Recall that $\mathbb{P}^1(\mathbb{C})$ is $\mathbb{C} \cup \infty$ with a topology. Identifying $\mathbb{C}$ with image of the first affine chart $\varphi_1([U_1])$, the additional chart point corresponds to the image of zero in the second affine chart. We denote this point by ∞.

This we have that $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. Using this identification, define the function $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, $z \mapsto z^2 =: \omega, \infty \mapsto \infty$. We will now show that f is a holomorphic map.

We denote by z the local coordinate on the source and by ω the local coordinate on the range. Then since $\omega = f(z) = z^2$ is a holomorphic function on all of $\mathbb{C}$, f is holomorphic on the image of U_1.

It remains to show that f is holomorphic at ∞. We consider the local expression for f using the chart U_2 whose image contains ∞. We denote by $\tilde{z} := \frac{1}{z}$ the corresponding local coordinate for the source and $\tilde{\omega} := \frac{1}{\omega}$ for the range.

The local expression $\tilde{F}$ for f in these coordinates is then obtained by composing $F(z)$ with the transition functions for the local coordinate:

$$\tilde{F}(\tilde{z}) = \tilde{\omega} = \frac{1}{\omega} = \frac{1}{z^2} = (\tilde{z})^2.$$

Since the point at ω corresponding to $\tilde{z} = \tilde{\omega} = 0$, and we have that $f(\infty) = \infty$ and $\tilde{F}(0) = 0$, the local expression extends to the whole chart, and in particular, it is a holomorphic function at 0. So f is holomorphic at every point of $\mathbb{P}^1(\mathbb{C})$.

Definice 5.2

Two Riemann surfaces X, Y are called isomorphic (or bi-holomorphic) if $\exists$ holomorphic maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$. In this case we write $X \simeq Y$. We call f and g isomorphisms (or bi-holomorphisms). An isomorphism $h : X \to X$ from a Riemann surface to itself is called an automorphism.

Příklad
Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $\Gamma_f \subseteq \mathbb{C}^2$ its graph. Show that $\Gamma_f \simeq \mathbb{C}$.

Definice 5.3

We say that a chart (U_x, φ_x) for a Riemann surface is centered at x if $\varphi_x(x) = 0$.

Věta 5.1

Let $f : X \to Y$ be a non-constant holomorphic map of Riemann surfaces. For any $x \in X$ $\exists$ charts centered at x such that the local expression of f in terms of these charts is $z \mapsto z^h$, for some $h \geqslant 1$.

Důkaz

Let φ and ψ be two charts centered at x, and $f(x)$, respectively. This gives the local expression $F = \psi \circ f \circ \varphi^{-1}$. Consider the Taylor expansion of F at 0 and let k be the smallest positive integer such that the coefficient of z^h does not vanish. Since $F(0) = 0$, $k > 1$, and $F(z) = z^k \left(\sum_{n=0}^{\infty} a_{k+n} z^n\right)$. Denote by $G(z) = a_k + a_{k+1}z + \dots$ the second factor in that equation. The function $G(z)$ is holomorphic at 0 and $G(0) = a_k \neq 0$. Thus we can make a choice of root $\sqrt[n]{G(z)}$ such that the associated map is well-defined and holomorphic around $0 = \varphi(z)$.

Defining $h(z) = z \cdot \sqrt[k]{G(z)}$ we have that $F = h^k$. The function h is holomorphic in a neighbourhood U of $0 = \varphi(x)$, $h(0) = 0$, and $h'(0) = 0$. The inverse function theorem now implies that h is bi-holomorphic on a neighbourhood $U' \subseteq U$ of $\varphi(x)$, and therefore the composition $\tilde{\varphi} = h \circ \varphi$ gives local chart for x centered at x. The local coordinate $\tilde{z}$ coming from $\tilde{\varphi}$ is related to z with $\tilde{z} = h(z)$. The local expression for f allows x is now obtained by changing coordinates from z to $\tilde{z}$ in F:

$$\tilde{F}(\tilde{z}) = F(z(\tilde{z})) = h(z)^k = (\tilde{z})^k.$$

□

Definice 5.4

Let $f : X \to Y$ be a non-konstant holomorphic map of Riemann surfaces

- Given a point $x \in X$, the integer k_x such that $\exists$ a local expression centered at x of the form $F(z) = z^{k_x}$ is called the Ramification index of f at x.
- The quantity $V_x = k_x - 1$ the differential length of f at x.
- A point x with Ramification index $k_x = 1$ is called unramified.
- A point for which $k_x \geqslant 2$ is ramified.
- If x is a ramification point, then $f(x) \in Y$ is called a branch point.