Statistics & Probability

Chapter 4: CONTINUOUS RANDOM VARIABLE AND PROBABILITY DISTRIBUTIONS

FPT University

Department of Mathematics

Quy Nhon, 2023

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Continuous Random Variable

A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples

1. The height of a student at FPT University can be any number between 150cm and 190cm.
2. The weight of a newborn can be any number between 0.5kg and 4.5kg.

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Probability Density Function

For a continuous random variable X, a **probability density function** (PDF) is a function such that

1. $f(x) \geq 0$.
2. $\displaystyle\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$.
3. $P(a \leq X \leq b) = \displaystyle\int_a^b f(x)\mathrm{d}x =$ area under $f(x)$ from a to b, for any $a, b \in \mathbb{R}$.

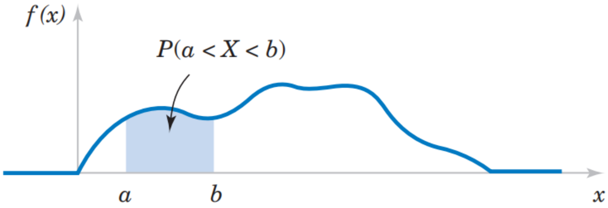

Figure. Probability determined from the area under $f(x)$, $P(a < X < b) = \displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Note

Let X be a continuous random variable. Then

$$P(x_1 \leq X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X \leq x_2) = P(x_1 < X < x_2).$$

Example

Suppose that the probability density function of a continuous random variable X is

$$f(x) = e^{-(x-3)}$$

where $x \geq 3$. Determine the following probabilities:

1. $P(1 \leq X < 5)$.
   Solution. We have
   $$P(1 \leq X < 5) = \int_3^5 f(x)\,\mathrm{d}x = \int_3^5 e^{-(x-3)}\,\mathrm{d}x = e^3(-e^{-x})\Big|_3^5 = 1 - e^{-2}.$$
2. $P(X < 8)$.
   Solution. We have
   $$P(X < 8) = \int_{-\infty}^8 f(x)\,\mathrm{d}x = \int_3^8 f(x)\,\mathrm{d}x = \int_3^8 e^{-(x-3)}\,\mathrm{d}x = e^3(-e^{-x})\Big|_3^8 = 1 - e^{-5}.$$

Quiz 1

The probability density function of the length of a metal rod is

$$f(x) = cx^2 \text{ for } 2 \leq x < 3.$$

What is the value of c? Find $P(X < 2.5 \text{ or } X \geq 2.8)$.

Quiz 2 (Hole Diameter)

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function

$$f(x) = 20e^{-20(x-12.5)} \text{ for } x \geq 12.5.$$

1. Assume a part with a diameter larger than 12.6 millimeters is scrapped. What proportion of parts is scrapped?
2. What proportion of parts is between 12.5 and 12.6 millimeters?

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Cumulative Distribution Function

The **cumulative distribution function** (CDF) of a continuous random variable is

$$F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

for $-\infty < x < +\infty$.

Remarks

1. Let X is continuous random variable with CDF $F(x)$. Then, we can use

$$P(a < X < b) = F(b) - F(a).$$

2. The probability density function (PDF) of a continuous random variable can be determined from the cumulative distribution function (CDF) by differentiating,

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x} f(t)\,\mathrm{d}t = f(x).$$

Example (Reaction Time)

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function (CDF)

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-0.01x} & \text{if } 0 \le x. \end{cases}$$

1. Determine the probability density function (PDF) of X.
   Solution. The PDF is obtained by differentiating $F(x)$, i.e.,
   $$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.01e^{-0.01x} & \text{if } 0 \le x. \end{cases}$$
2. What proportion of reactions is complete within 200 milliseconds?
   Solution. The probability that a reaction completes within 200 milliseconds is
   $$P(X < 200) = F(200) = 1 - e^{-2} = 0.8647.$$

Quiz

Supose that the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x. \end{cases}$$

Find $P(X < 2.8)$ and $P(0 < X < 1.5)$.

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Mean and Variance of a Continuous Random Variable

Assume X is a continuous random variable with probability density function (PDF) $f(x)$.

Mean

The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\,\mathrm{d}x.$$

Variance

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} x^2 f(x)\,\mathrm{d}x - \mu^2.$$

Standard Deviation

The **standard deviation** of X is $\sigma = \sqrt{V(X)}$.

Expected Value of a Function of a Continuous Random variable

Let X is a continuous random variable with probability density function $f(x)$. Then

$$E[h(X)] = \int_{-\infty}^{+\infty} h(x)f(x)\,\mathrm{d}x.$$

Example (Electric Current)

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20]$ and PDF is $f(x) = 0.05$ for $0 \leq x \leq 20$.

1. Determine the mean and the variance of X.
   Solution. We have

$$E(X) = \int_0^{20} xf(x)\,\mathrm{d}x = \int_0^{20} 0.05x\,\mathrm{d}x = \frac{0.05}{2}x^2\Big|_0^{20} = 10$$

$$V(X) = \int_0^{20} (x-\mu)^2 f(x)\,\mathrm{d}x = \int_0^{20} 0.05(x-10)^2\,\mathrm{d}x = \frac{0.05}{3}(x-10)^3\Big|_0^{20} = 33.33.$$

2. What is the expected value and the variance of the squared current?
   Hint. The squared current is a continuous random variable of the form $h(X) = X^2$.

Quiz 1

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function

$$f(x) = 20e^{-20(x-12.5)} \text{ for } x \geq 12.5.$$

Determine the mean and the variance of X.

Quiz 2

The cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } 3 \leq x. \end{cases}$$

Find the standard deviation of X.

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Continuous Uniform Distribution

A continuous random variable X with probability density function (PDF)

$$f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b$$

is a **continuous uniform random variable**. We can write $X \sim U(a, b)$.

Mean and Variance

Let X is a continuous uniform random variable over $a \leq x \leq b$. Then

$$\mu = E(X) = \frac{a+b}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12}.$$

Cumulative Distribution Function

The cumulative distribution function of a continuous uniform random variable X is

$$F(x) = \int_a^x f(t)\,\mathrm{d}t = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } b \leqslant x. \end{cases}$$

Example

Suppose X has a continuous uniform distribution over the interval $[1, 11]$.

1. Determine the mean, variance and standard deviation of X.
   Solution. The probability density function of X is $f(x) = \dfrac{1}{11-1} = 0.1$ and

$$\mu = \frac{1+11}{2} = 6, \quad \sigma^2 = \frac{(11-1)^2}{12} = \frac{25}{3} \quad \text{and} \quad \sigma = \sqrt{\frac{25}{3}} = \frac{5\sqrt{3}}{3}.$$

2. Find $P(X < 6.5)$.
   Solution. we have

$$P(X < 6.5) = \int_1^{6.5} f(x)\,\mathrm{d}x = \int_1^{6.5} 0.1\,\mathrm{d}x = 0.55.$$

3. Determine the cumulative distribution function of X.
   Solution. We have $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ which implies

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ \dfrac{x-1}{10} & \text{if } 1 \le x < 11 \\ 1 & \text{if } 11 \le x. \end{cases}$$

Quiz 1

Suppose X has a continuous uniform distribution over $[5, 15]$. What is the mean and variance of $Y = 8X$?

Quiz 2

Assume X has a continuous uniform distribution over the interval $[-1, 1]$.

1. Determine the mean, variance and standard deviation of X.
2. Determine the value for x such that $P(-x < X < x) = 0.9$.
3. Determine the cumulative distribution function.

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Normal Distribution

A random variable X with probability density function (PDF)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

is a **normal random variable** with parameters μ where $-\infty < \mu < +\infty$ and $\sigma > 0$. In addition,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2.$$

We can write $X \sim N(\mu, \sigma^2)$.

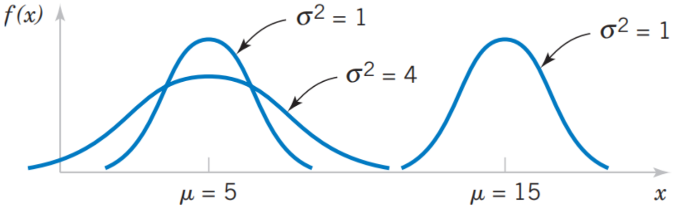


Figure. Normal probability density functions for selected values of the parameters μ, σ^2.

Note

For any normal random variable, i.e., $X \sim N(\mu, \sigma^2)$

$$P(\mu < X) = P(X < \mu) = 0.5$$

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973.$$

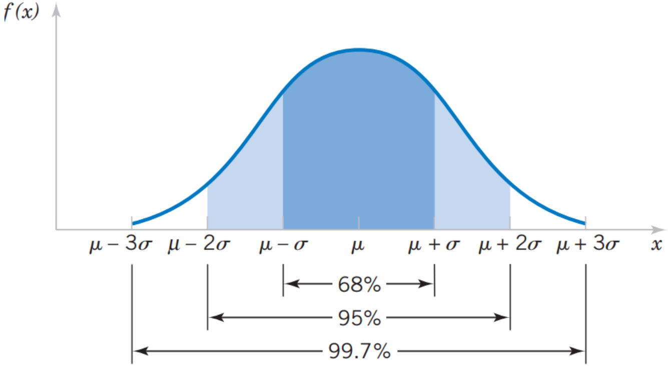


Figure. Probabilities associated with a normal distribution.

Standard Normal Random Variable

Standard Normal Random Variable

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$, i.e.,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < +\infty$$

is called a **standard normal random variable** and is denoted as Z or $Z \sim N(0, 1)$.

Cumulative Distribution Function

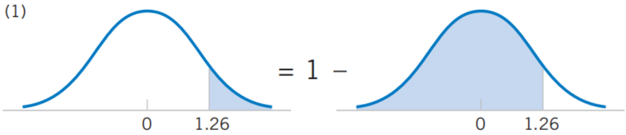
The cumulative distribution function of a standard normal random variable is denoted as

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, \mathrm{d}x.$$

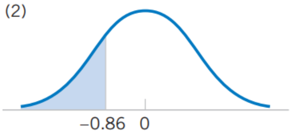
Note that. Use Appendix Table III in page 708 and 709 of the textbook *Applied Statistics and Probability for Engineers* to find $\Phi(z)$.

Examples.

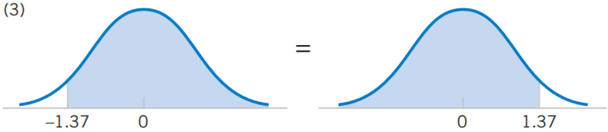
1. $P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - 0.89616 = 0.10384.$



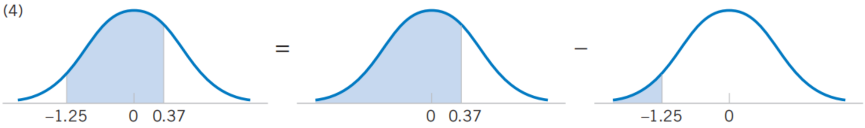
2. $P(Z < -0.86) = 0.19490.$



3. $P(-1.37 < Z) = P(Z < 1.37) = 0.91465.$



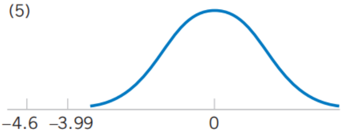
4. $P(-1.25 < Z < 0.37) = P(Z < 0.37) - P(Z < -1.25) = 0.64431 - 0.10565.$



5. $P(Z \leq -4.6)$ cannot be found exactly from Appendix Table III. However, we can use

$$P(Z \leq -4.6) < P(Z \leq -3.99) = 0.00003,$$

meaning that $P(Z \leq -4.6)$ is nearly zero.

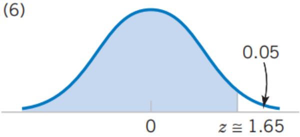


6. $\boxed{\text{Find the value } z \text{ such that } P(Z > z) = 0.05.}$

We have

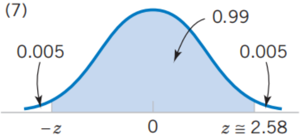
$$P(Z \le z) = 1 - P(Z > z) = 0.95.$$

Appendix Table III is used in reverse, we search the value corresponds to 0.95. We do not find 0.95 exactly, but the nearest value is 0.95053, corresponding to $z = 1.65$.



7. $\boxed{\text{Find the value of } z \text{ such that } P(-z < Z < z) = 0.99.}$

Using the symmetry of the normal distribution, the area in each tail of the distribution must equal 0.005. Thus, we need to find the value for z corresponds to 0.995. The nearest probability in Appendix Table III is 0.99506, when $z = 2.58$.



Standardizing a Normal Random Variable

Let X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. Then, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a **standard normal random variable**.

Standardizing to Calculate a Probability

Suppose that X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \le x) = P\left(\frac{X - \mu}{\sigma} \le \frac{x - \mu}{\sigma}\right) = P(Z \le z)$$

where Z is a **standard normal random variable**, and $z = \dfrac{x - \mu}{\sigma}$ is the z-**value** (z-score) obtained by **standardizing** X.

Note

The probability is obtained by using Appendix Table III with $z = \dfrac{x - \mu}{\sigma}$.

Example (Normally Distributed Current)

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement will exceed 13 milliamperes?

Solution. Let X denotes the current in milliamperes. We have $\mu = 10,\ \sigma^2 = 4$ and

$$P(X > 13) = P\left(\frac{X-\mu}{\sigma} > \frac{13-\mu}{\sigma}\right) = P\left(Z > \frac{13-10}{2}\right) = P(Z > 1.5) = 0.06681.$$

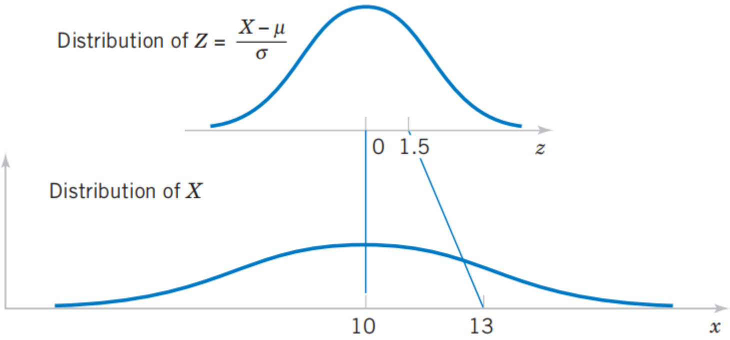


Figure. Standardizing a normal random variable.

Quiz 1

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2.

1. What is the probability that a current measurement is between 9 and 11 milliampares?
2. Determine the value for which the probability that a current measurement is below this value is 0.98.

Quiz 2

A machine pours beer into 16 oz. bottles. Experience has shown that the number of ounces poured is normally distributed with a standard deviation of 1.3 ounces. Find the probabilities that the amount of beer the machine will pour into the next bottle will be more than 16.25 ounces.

Quiz 3

The tread life of a particular brand of tire is a random variable best described by a normal distribution with a mean of $65,000$ miles and a standard deviation of $1,500$ miles. What warranty should the company use if they want 95% of the tires to outlast the warranty?

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Normal Approximation to the Binomial Distribution

Let X be a binomial random variable with parameters n and p. Then,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable.

Continuity Correction

To approximate a binomial distribution with a normal distribution, a **continuity correction** is applied as follows

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z\right).$$

Note. The approximation is good for $np > 5$ and $n(1-p) > 5$.

Example

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit received in error is 1×10^{-5}. If 16 million bits are transmitted, what is the probability that 150 or fewer errors occur?

Solution. Let X denote the numbers of errors. Then X is a binomial random variables and

$$P(X \le 150) = \sum_{x=0}^{150} \binom{16,000,000}{x} (10^{-5})^x (1 - 10^{-5})^{16,000,000-x}.$$

The computational difficulty is clear. Thus, the probability can be approximated as

$$\begin{aligned} P(X \le 150) &= P(X \le 150.5) \\ &= P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} \le \frac{150.5 - 160}{\sqrt{160(1 - 10^{-5})}} \right) \\ &\cong P(Z \le -0.75) = 0.227. \end{aligned}$$

Note. Since $np = (16 \times 10^6)(1 \times 10^{-5}) = 160 > 5$ and $n(1-p) \gg 5$ (much larger), the approximation is expected to work well in this case.

Example (Normal Approximation to Binomial)

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit received in error is $p = 0.1$. If only $n = 50$ bits are transmitted, what is the probability that 2 or less errors occur?

Solution. The exact probability that 2 or less errors occur is

$$P(X \leq 2) = \binom{50}{0} 0.1^0 (0.9^{50}) + \binom{50}{1} 0.1 (0.9^{49}) + \binom{50}{2} 0.1^2 (0.9^{48}) = 0.112.$$

Based on the normal approximation,

$$P(X \leq 2) = P\left(\frac{X-5}{\sqrt{50(0.1)(0.9)}} \leq \frac{2.5-5}{\sqrt{50(0.1)(0.9)}}\right) = P(Z < -1.118) = 0.119.$$

Note. We can even approximate $P(X = 5) = P(5 \leq X \leq 5) = P(4.5 \leq X \leq 5.5)$.

Quiz

The manufacturing of semiconductor chips produces 3% defective chips. Assume the chips are independent and that a lot contains 800 chips. Approximate the probability that more than 30 chips are defective.

Normal Approximation to the Poisson Distribution

Let X be a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$. Then,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard normal random variable.

Continuity Correction

To approximate a Poisson probability with a normal distribution, a **continuity correction** is applied as follows

$$P(X \leq x) = P(X \leq x + 0.5) \cong P\left(Z \leq \frac{x + 0.5 - \lambda}{\sqrt{\lambda}}\right)$$

and

$$P(x \leq X) = P(x - 0.5 \leq X) \cong P\left(\frac{x - 0.5 - \lambda}{\sqrt{\lambda}} \leq Z\right).$$

Note that. The approximation is good for $\lambda > 5$.

Example (Normal Approximation to Poisson)

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that 950 or fewer particles are found?

Solution. Since $\lambda = 1000$, this probability can be expressed exactly as

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} 1000^x}{x!}.$$

It is really difficult to compute. So, the probability can be approximated as

$$\begin{aligned} P(X \leq 950) &= P(X \leq 950.5) \\ &\cong P\left(Z \leq \frac{950.5 - 1000}{\sqrt{1000}}\right) \\ &= P(Z \leq -1.57) = 0.058. \end{aligned}$$

Quiz

The number of customers that arrive at a fast-food business during a one-hour period is known to be Poisson distributed with a mean equal to 9.6. What is the probability that more than 10 customers will arrive in a one-hour period?

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Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval is an **exponential random variable** with parameter λ. The probability density function (PDF) of X is

$$f(x) = \lambda e^{-\lambda x}$$

for $0 \leq x < +\infty$.

Mean and Variance

Let X be a random variable which has an exponential distribution with parameter λ. Then

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2}.$$

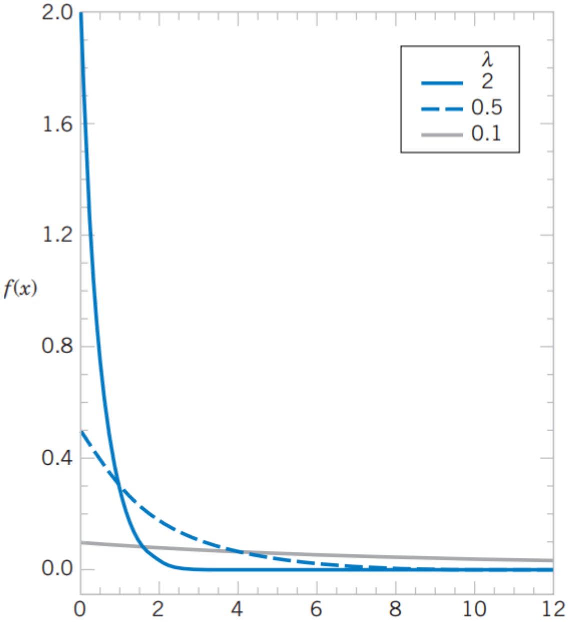

Figure. PDF of exponential random variables for selected values of λ.

Example (Computer Usage)

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour.

1. What is the probability that there are no log-ons in an interval of 6 minutes?

   Solution. Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes (0.1 h).

$$P(X > 0.1) = \int_{0.1}^{+\infty} 25e^{-25x}\,\mathrm{d}x = e^{-25\cdot 0.1} = 0.082.$$

2. What is the probability that the time until the next log-on is between 2 and 3 minutes?

   Solution. Converting all units to hours, we have

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,\mathrm{d}x = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152.$$

   An alternative solution is given as

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152.$$

Example (Computer Usage)

3. Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90.

Solution. We need to find x satisfies

$$P(X > x) = 0.90$$

which implies

$$\int_x^{+\infty} 25e^{-25t}\, \mathrm{d}t = e^{-25x} = 0.90.$$

Therefore,

$$x = 0.00421 \text{ hours} = 0.25 \text{ minutes}.$$

Quiz 1

The time between customer arrivals at a furniture store has an approximate exponential distribution with mean of 9 minutes. If a customer just arrived, find the probability that the next customer will not arrive for at least 15 minutes.

Quiz 2

The time between patients arriving at an outpatient clinic follows an exponential distribution at a rate of 15 patients per hour. What is the probability that a randomly chosen arriving interval will not exceed 6 minutes?

Thank you!