Statistics & Probability

Chapter 9: TEST OF HYPOTHESES FOR A SINGLE SAMPLE

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Department of Mathematics

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Table of Contents

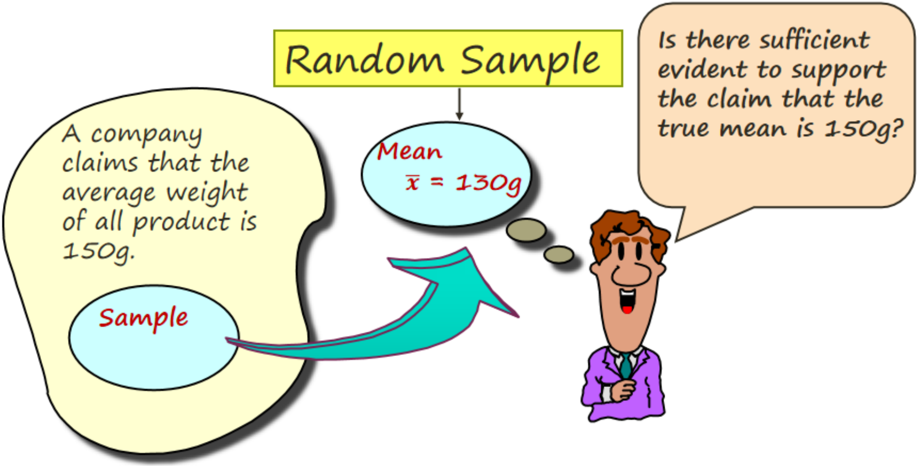
Random Sample
A company claims that the average weight of all product is 150g.
Sample
Mean
$\overline{x}$ = 130g
Is there sufficient evident to support the claim that the true mean is 150g?

Table of Contents

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Statistical Hypothesis

A **statistical hypothesis** is a statement about the parameters of one or more populations.

Hypothesis Testing

Hypothesis testing analyzes differences between a sample statistic and the results you would expect if a null hypothesis was true.

Example

1. A company claims that the mean weight all product is 150g.
   $\rightarrow$ This is a claim about the population mean $\mu = 150$g.
2. A university claims that the employment rate of its students after graduation is more than 94%.
   $\rightarrow$ This is claim about the population proportion $p > 0.94$.

Fundamental Hypothesis Testing Concepts

1. The **null hypothesis**, H_0, states a status quo claim (the hypothesis will be tested).
2. The **alternative hypothesis**, H_1, states a claim that is contrary to the null hypothesis and often represents a research claim or specific inference that an analyst seeks to prove.
3. A null and alternative pair of hypotheses are always collectively exhaustive.
   $\rightarrow$ The null and alternative hypotheses are complementary, i.e., the two alternatives together exhaust all probabilities of the values that the hypothesized parameter can assume.
4. If you reject the null hypothesis, you have strong statistical evidence that the alternative hypothesis is correct.
5. If you do not reject the null hypothesis, you have not proven the null hypothesis.
   $\rightarrow$ Rather, you have only failed to prove the alternative hypothesis.
6. The null hypothesis always refers to a population parameter such as μ and not a sample statistic such as $\overline{X}$.
7. The null hypothesis always includes an equals sign (either $=$, $\leq$ or $\geq$) when stating a claim about the population parameter.
8. The alternative hypothesis never includes an equals sign when stating a claim about the population parameter.

Stating Statistical Hypotheses

Type of equation	H_0	H_1	Type of Distribution
$=$	$\mu = 368$	$\mu \neq 368$	Two tail, two rejection regions
$\leq$	$\mu \leq 368$	$\mu > 368$	One tail, rejection region on right
$\geq$	$\mu \geq 368$	$\mu < 368$	One tail, rejection region on left

Note

Neither hypothesis testing nor statistical inference proves the hypothesis. It only indicates whether the hypothesis is supported by the data or not.

The Null and Alternative Hypothesis

You are the manager of a fast-food restaurant. You want to determine whether the waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes. State the null and alternative hypotheses.

Solution. The null hypothesis is that the population mean has not changed from its previous value of 4.5 minutes. This is stated as

$$H_0 : \mu = 4.5.$$

The alternative hypothesis is the opposite of the null hypothesis. Because the null hypothesis is that the population mean is 4.5 minutes, the alternative hypothesis is that the population mean is not 4.5 minutes. This is stated as

$$H_1 : \mu \neq 4.5.$$

Quizzes (The Null and Alternative Hypothesis)

State the null and alternative hypotheses for each the following situation:

1. A generic brand of the anti-histamine Diphenhydramine markets a capsule with a 50 milligram dose. The manufacturer is worried that the machine that fills the capsules has come out of calibration and is no longer creating capsules with the appropriate dosage.
2. A researcher is studying the effects of radical exercise program on knee surgery patients. There is a good chance the therapy will improve recovery time, but there's also the possibility it will make it worse. Average recovery times for knee surgery patients is 8.2 weeks.
3. A researcher thinks that if knee surgery patients go to physical therapy twice a week (instead of 3 times), their recovery period will be longer. Average recovery times for knee surgery patients is 8.2 weeks.

The Hypothesis Testing Process

We wish to test

$$H_0 : \mu = 50$$
$$H_1 : \mu \neq 50.$$

Assume that $H_0 : \mu = 50$ is true. A random sample of $n = 10$ objects is selected and the sample mean $\overline{x}$ is observed.

1. If $\overline{x}$ falls close to the hypothesized value of $\mu = 50$, we fail to reject H_0, it is evidence in support of the null hypothesis.
2. If $\overline{x}$ is considerably different from 50, we reject H_0, it is evidence in support of the alternative hypothesis.

$\rightarrow$ Since the alternative hypothesis specifies values of μ that could be either greater or less than 50, it is called **two-sided alternative hypothesis**.

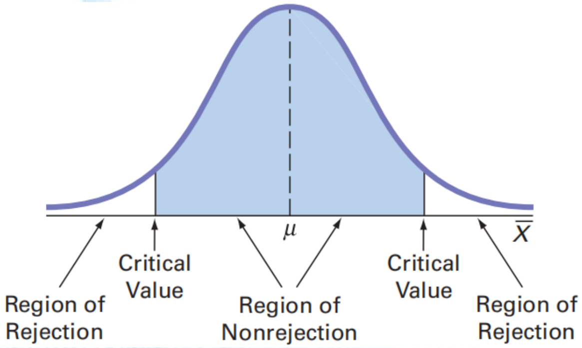
In some situations, we may wish to formulate a **one-sided alternative hypothesis**,

$$H_0 : \mu = 50 \qquad\qquad H_0 : \mu = 50$$
$$H_1 : \mu < 50 \qquad \text{or} \qquad H_1 : \mu > 50.$$

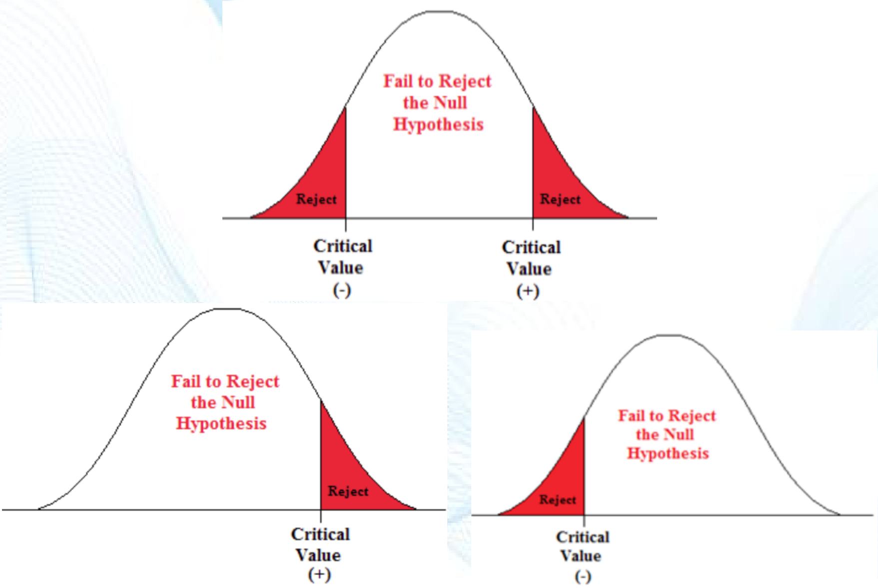
Question: How far is "far enough" to reject H_0?

$\rightarrow$ The critical value of a test statistic creates a "line in the sand" for decision making – it answers the question of how far is far enough.

Regions of Rejection and Nonrejection



- The sampling distribution of the test statistic is divided into two regions, a **region of rejection** (or critical region) and a **region of nonrejection** (or acceptance region).
- If the test statistic falls into the region of nonrejection, you do not reject the null hypothesis.
- To make a decision concerning the null hypothesis, you first determine the critical value of the test statistic.


Fail to Reject
the Null
Hypothesis
Reject
Reject
Critical
Value
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Critical
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Fail to Reject
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Value
(+)
Fail to Reject
the Null
Hypothesis
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Critical
Value
(-)

Risks in Decision Making Using Hypothesis Testing

Type I and Type II Errors

1. A **Type I error** occurs if you reject the null hypothesis, H_0, when it is true and should not be rejected.
   $\rightarrow$ A Type I error is a "false alarm". The probability of a Type I error occurring is α which is the **significance level** (α-error) preset by the researcher. It means that
   $$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}).$$
2. A **Type II error** occurs if you do not reject the null hypothesis, H_0, when it is false and should be rejected.
   $\rightarrow$ A Type II error represents a "missed opportunity" to take some corrective action. The probability of a Type II error occurring is β (β **risk** or β-error). It means that
   $$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}).$$

Example of Type I and Type II Errors

You decide to get tested for COVID-19 based on mild symptoms. There are two errors that could potentially occur:

1. **Type I error**: the test result says you have coronavirus, but you actually do not.
2. **Type II error**: the test result says you do not have coronavirus, but you actually do.

Complements of Type I and Type II Errors

1. The **confidence coefficient**, $(1-\alpha)$, is the probability that you will not reject the null hypothesis, H_0, when it is true and should not be rejected.
2. The **power of a statistical test**, $(1-\beta)$, is the probability that you will reject the null hypothesis when it is false and should be rejected.

Statistical Decision	**Actual Situation**: H_0 True	**Actual Situation**: H_0 False
Fail to reject H_0	Correct decision Confidence coefficient: $1-\alpha$	Type II error $P(\text{Type II error}) = \beta$
Reject H_0	Type I error $P(\text{Type I error}) = \alpha$	Correct decision Power $= 1-\beta$

To (indirectly) reduce the risk of a Type II error, you can increase the sample size or the significance level.

Central Limit Theorem (CLT)

$$z = \frac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

where

- $\overline{x}$: sample mean
- μ_0: hypothesized parameter - population mean
- $\sigma/\sqrt{n}$: standard deviation of $\overline{x}$ which is the relevant statistic.

Example (Computing the Type I Error Probability)

We wish to test

$$H_0 : \mu = 50 \quad \text{and} \quad H_1 : \mu \neq 50.$$

Suppose that if $48.5 \leq \overline{x} \leq 51.5$, we will not reject the null hypothesis $H_0 : \mu = 50$; and if either $\overline{x} < 48.5$ or $\overline{x} > 51.5$, we will reject the null hypothesis in favor of the alternative hypothesis $H_1 : \mu \neq 50$. Thus, $\alpha = P(\overline{x} < 48.5$ or $\overline{x} > 51.5 \mid \mu = 50)$, i.e.,

$$\alpha = P(\overline{x} < 48.5 \mid \mu = 50) + P(\overline{x} > 51.5 \mid \mu = 50).$$

With $\sigma = 2.5$ and $n = 10$, by using CLT we have

$$\alpha = P\left(z < \frac{48.5 - 50}{2.5/\sqrt{10}}\right) + P\left(z > \frac{51.5 - 50}{2.5/\sqrt{10}}\right) \approx 0.0574.$$

$\rightarrow$ The probability of rejecting the true null hypothesis $H_0 : \mu = 50$ is 5.74%.

Note

1. The values $\overline{x} < 48.5$ and $\overline{x} > 51.5$ constitute the critical region for the test.
2. The values $48.5 \le \overline{x} \le 51.5$ form a region for which we will fail to reject H_0, is called acceptance region.
3. The boundaries between the critical region and the acceptance region are called the critical values.
   $\rightarrow$ In the above example, the critical values are $48.5,\ 51.5$.
4. We can reduce α by widening the acceptance region or increasing the sample size n.
   $\rightarrow$ In the above example, we make the critical values 48 and 52, or choose $n = 16$.
5. We select a small value of α such as $0.1,\ 0.05,\ 0.01$ to make the probability of rejecting a true null hypothesis small.

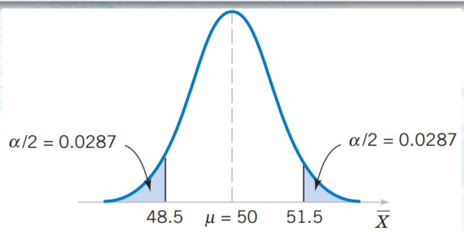


Figure. The critical region for $H_0 : \mu = 50$ versus $H_1 : \mu \neq 50$ and $n = 10$.

Example (Computing the Probability of Type II Error)

Suppose that $48.5 \leq \overline{x} \leq 51.5$, the null hypothesis $H_0 : \mu = 50$ and the alternative hypothesis $H_1 : \mu \neq 50$. We now consider the case the alternative hypothesis is true and the value of mean is $\mu = 52$ (or $H_1 : \mu = 52$). In addition, $\sigma = 2.5$ and $n = 10$. Then,

$$\beta = P(48.5 \leq \overline{x} \leq 51.5 \mid \mu = 52)$$

which implies

$$\beta = P\left(\frac{48.5 - 52}{2.5/\sqrt{10}} \leq z \leq \frac{51.5 - 52}{2.5/\sqrt{10}}\right) = P(z \leq -0.63) - P(z \leq -4.43) = 0.2643.$$

$\rightarrow$ If we are testing $H_0 : \mu = 50$ against $H_1 : \mu \neq 50$ with $n = 10$ and the true value of the mean is $\mu = 52$, the probability that we will fail to reject the false null hypothesis is 26.43%.

Acceptance region	**Sample size**	α	β **at** $\mu = 52$	β **at** $\mu = 50.5$
$48.5 < \overline{x} < 51.5$	10	0.0576	0.2643	0.8923
$48 < \overline{x} < 52$	10	0.0114	$\boxed{0.5000}$	$\boxed{0.9705}$
$48.81 < \overline{x} < 51.19$	16	$\boxed{0.0576}$	$\boxed{0.0966}$	$\boxed{0.8606}$
$48.42 < \overline{x} < 51.58$	16	$\boxed{0.0114}$	$\boxed{0.2515}$	$\boxed{0.9578}$

Relationship between Type I and Type II Errors

1. The size of the critical region, and consequently the probability of a type I error, can always be reduced by appropriate selection of the critical values.
2. Type I and type II errors are related. A decrease in the probability of one type of error always results in an increase in the probability of the other, provided that the sample size n does not change.
3. An increase in sample size reduces β, provided that α is held constant.
4. When the null hypothesis is false,

   + β increases as the true value of the parameter approaches the value hypothesized in the null hypothesis.

   + β decreases as the difference between the true mean and the hypothesized value increases.

A widely used procedure in hypothesis testing is to use a type I error or significance level of $\alpha = 0.05$. This value has evolved through experience, and may not be appropriate for all situations.

One-Sided and Two-Sided Hypotheses

Example. Consider the propellant burning rate problem. Suppose that if the burning rate is less than 50 cm/s, we wish to show this with a strong conclusion. The hypotheses should be stated as

$$H_0 : \mu = 50 \qquad H_1 : \mu < 50.$$

- The critical region lies in the lower tail of the distribution of $\overline{X}$.
- Since the rejection of H_0 is always a strong conclusion, this statement of the hypotheses will produce the desired outcome if H_0 is rejected.
- **Notice** that, although the null hypothesis is stated with an equals sign, it is understood to include any value of not specified by the alternative hypothesis.
- Therefore, failing to reject H_0 does not mean that cm/s exactly, but only that we do not have strong evidence in support of H_1.

In formulating one-sided alternative hypotheses, we should remember that rejecting H_0 is always a strong conclusion. Consequently, we should put the statement about which it is important to make a strong conclusion in the alternative hypothesis. In real-world problems, this will often depend on our point of view and experience with the situation.

P-Value

The P-**value** (or **observed significance level**) is the smallest level of significance that would lead to rejection of the null hypothesis H_0 which the given data.

Example. Consider the two-sided hypothesis test for burning rate

$$H_0 : \mu = 50 \qquad H_1 : \mu \neq 50$$

with $n = 16$ and $\sigma = 2.5$. Suppose that the observed sample mean is $\overline{x} = 51.3$ cm/s.

The following **Figure** is a critical region for this test with the value of $\overline{x} = 51.3$ and the symmetric value 48.7. The P-value of the test is the probability above 51.3 plus the probability below 48.7. The P-value is easy to compute after the test statistic is observed.

$$\begin{aligned} P\text{-value} &= 1 - P(48.7 < \overline{X} < 51.3) \\ &= 1 - P\left(\frac{48.7 - 50}{2.5/\sqrt{16}} < Z < \frac{51.3 - 50}{2.5/\sqrt{16}}\right) \\ &= 1 - P(-2.08 < Z < 2.08) = 0.038. \end{aligned}$$

The P-value tells us that if $H_0 = 50$ is true, the probability of obtaining a random sample whose mean is at least as far from 50 as 51.3 (or 48.7) is 0.038.

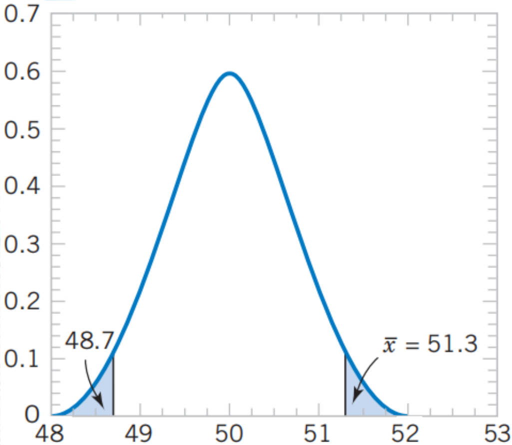


Figure. P-value is the area of the shaded region when $\overline{x} = 51.3$.

Connection Between Hypothesis Tests and Confidence Intervals

Read more in Textbook page 206.

General Procedure For Hypothesis Tests

The following sequence of steps in applying hypothesis-testing methodology:

1. **Parameter of interest**: From the problem context, identify the parameter of interest.
2. **Null hypothesis**, H_0: State the null hypothesis, H_0.
3. **Alternative hypothesis**, H_1: Specify an appropriate alternative hypothesis, H_1.
4. **Test statistic**: Determine an appropriate test statistic.
5. **Reject** H_0 **if**: State the rejection criteria for the null hypothesis.
6. **Computations**: Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
7. **Draw conclusions**: Decide whether or not H_0 should be rejected and report that in the problem context.

In practice, only three steps are really required:

1. Specify the test statistic to be used (such as Z_0).
2. Specify the location of the critical region (two-tailed, upper-tailed, or lower-tailed).
3. Specify the criteria for rejection (typically, the value of α, or the P-value at which rejection should occur).

Statistical versus Practical Significance

Read more in Textbook page 207.

Note

Be careful when interpreting the results from hypothesis testing when the sample size is large, because any small departure from the hypothesized value μ_0 will probably be detected, even when the difference is of little or no practical significance.

Table of Contents

Tradition Method (Two-tailed Test)

S1. Form the two hypotheses $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$.

S2. Find the **test statistic** $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$.

S3. Identify acceptance region.

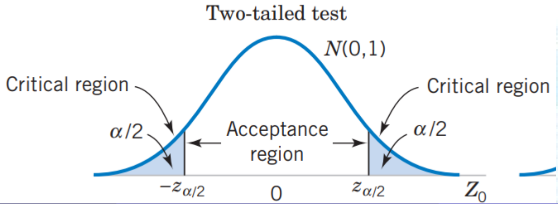
S4. Make a decision:

+ If the test statistic z_0 is in critical region , then reject H_0. Namely,

$$z_0 < -z_{\alpha/2} \quad \text{or} \quad z_0 > z_{\alpha/2}.$$

+ If the test statistic z_0 is in acceptance region, then we fail to reject H_0. Namely,

$$-z_{\alpha/2} \leq z_0 \leq z_{\alpha/2}.$$



Example (Tradition Method (Two-tailed Test))

You are the manager of a fast-food restaurant. The business problem is to determine whether the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes. Assume the population is normally distributed, with a standard deviation of 1.2 minutes. You select a sample of 36 orders during a one-hour period. The sample mean is 5.1 minutes. Determine whether there is evidence at the 0.05 level of significance that the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes.

1. State the null and the alternative hypotheses: $H_0 : \mu = 4.5$ and $H_1 : \mu \neq 4.5$.
2. Test statistic: $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{5.1 - 4.5}{1.2/\sqrt{36}} = +3.$
3. For $\alpha = 0.05$, the acceptance region is $-1.96 \leq z_0 \leq +1.96$.
4. Since $z_0 = +3 > +1.96$, then we reject H_0. We conclude that there is evidence that the population mean waiting time to place an order has changed from its previous value of 4.5 minutes. The mean waiting time for customers is longer now than it was last month.

Quiz

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and their average height is 1.64m. Test the hypothesis that the true average height of all adults in the community is 1.7m, at $\alpha = 0.05$.

The P-**value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the data given.

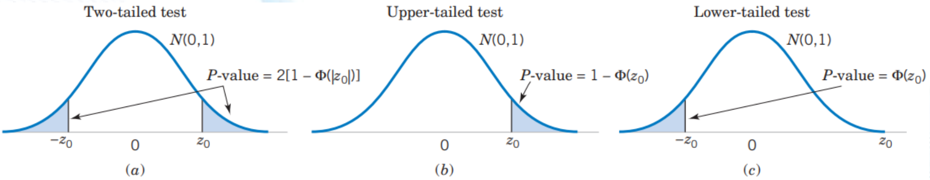


Figure. The P-value for z-test. (a) The two-sided alternative $H_1 : \mu \neq \mu_0$. (b) The one-sided alternative $H_1 : \mu > \mu_0$. (c) The one-sided alternative $H_1 : \mu < \mu_0$.

P-value Method (Two-tailed Test)

S1. Form the two hypotheses $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$.

S2. Find the **test statistic** $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$.

S3. Find P-value where P-value $= 2 * \text{NORM.S.DIST}(-|z_0|)$

S4. Make a decision:

+ If P-value $< \alpha$, then reject H_0.

+ If P-value $\geq \alpha$, then fail to reject H_0.

Example (The P-Value Approach to Hypothesis Testing)

You are the manager of a fast-food restaurant. The business problem is to determine whether the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes. Assume the population is normally distributed, with a standard deviation of 1.2 minutes. You select a sample of 36 orders during a one-hour period. The sample mean is 5.1 minutes. Determine whether there is evidence at the 0.05 level of significance that the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes.

1. State the null and the alternative hypotheses: $H_0 : \mu = 4.5$ and $H_1 : \mu \neq 4.5$.
2. Test statistic: $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{5.1 - 4.5}{1.2/\sqrt{36}} = +3.$
3. P-value for two tails test is

$$P\text{-value} = P(z_0 > +3) + P(z_0 < -3) = 0.00135 + 0.00135 = 0.0027.$$

4. Since P-value $= 0.0027 < 0.05 = \alpha$, then we reject H_0. We conclude that there is evidence that the population mean waiting time to place an order has changed from its previous value of 4.5 minutes. The mean waiting time for customers is longer now than it was last month.

Quiz

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and their average height is 1.64m. Test the hypothesis that the true average height of all adults in the community is 1.7m, at $\alpha = 0.05$.

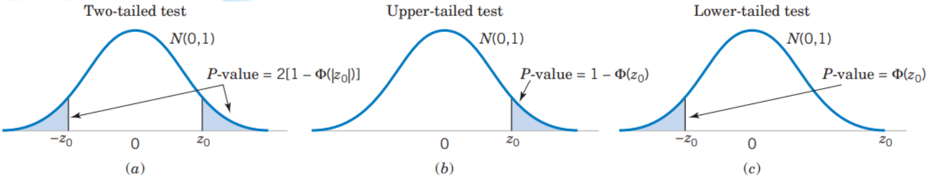


Figure. The P-value for z-test. (a) The two-sided alternative $H_1 : \mu \neq \mu_0$. (b) The one-sided alternative $H_1 : \mu > \mu_0$. (c) The one-sided alternative $H_1 : \mu < \mu_0$.

P-value Method (One-tailed Test)

S1. Form the two hypotheses

+ Right-tailed test: $H_0 : \mu \leq \mu_0$ and $H_1 : \mu > \mu_0$.

+ Left-tailed test: $H_0 : \mu \geq \mu_0$ and $H_1 : \mu < \mu_0$.

S2. Find the **test statistic** $z_0 = \dfrac{\overline{x} - \mu_0}{\sigma/\sqrt{n}}$.

S3. Find P-value.

S4. Make a decision:

+ If P-value $< \alpha$, then reject H_0.

+ If P-value $\geq \alpha$, then fail to reject H_0.

Connection Between Confidence Interval Estimation & Hypothesis Testing

Problem. You are the manager of a fast-food restaurant. The business problem is to determine whether the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes. Assume the population is normally distributed, with a standard deviation of 1.2 minutes. You select a sample of 36 orders during a one-hour period. The sample mean is 5.1 minutes. Determine whether there is evidence at the 0.05 level of significance that the population mean waiting time to place an order has changed in the past month from its previous population mean value of 4.5 minutes.

Solution. $\mu = 4.5,\ \sigma = 1.2,\ n = 36,\ \overline{x} = 5.1$ and $\alpha = 0.05$.

- Instead of testing the null hypothesis that $\mu = 4.5$, you can reach the same conclusion by constructing a confidence interval estimate of μ.
- If the hypothesized value of $\mu = 4.5$ is contained within the interval, you do not reject the null hypothesis because 4.5 would not be considered an unusual value.
- $\overline{x} \pm z_{\alpha/2}\dfrac{\sigma}{\sqrt{n}} = 5.1 \pm 1.96\dfrac{1.2}{\sqrt{36}} = (4.708, 5.492)$, this interval does not include $\mu = 4.5$, thus, we reject H_0. The mean waiting time for customers is longer now than it was last month.

Quiz

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and their average height is 1.64m. Test the hypothesis that the true average height of all adults in the community is 1.7m, at $\alpha = 0.05$.

Quiz 1

1. If you use a 0.05 level of significance in a two-tail hypothesis test, what decision will you make if $z_0 = +2.21$?
2. If you use a 0.01 level of significance in a two-tail hypothesis test, what is your decision rule for rejecting $H_0 : \mu = 12.5$ if you use the Z-test?
3. What is the P-value if, in a two-tail hypothesis test, $z_0 = +2.00$?

Quiz 2

A bottled water distributor wants to determine whether the mean amount of water contained in 1-gallon bottles purchased from a nationally known water bottling company is actually 1 gallon. You know from the water bottling company specifications that the standard deviation of the amount of water per bottle is 0.02 gallon. You select a random sample of 50 bottles, and the mean amount of water per 1-gallon bottle is 0.995 gallon.

1. Is there evidence that the mean amount is different from 1.0 gallon? (Use $\alpha = 0.01$.)
2. Compute the P-value and interpret its meaning.
3. Construct a 99% confidence interval estimate of the population mean amount of water per bottle.
4. Compare the results of (1) and (3). What conclusions do you reach?

Quiz 3

The heights of all adults in a community is known to have standard deviation of 0.03m. A random sample of 43 adults are collected, and the average height of this sample is 1.64m. Use both tradition and P-value methods, at the significance level of 5%, test the hypothesis that the average height of all adults in the community is greater than or equal to 1.6m.

Summary of Tests on the Mean, Variance Known

Testing Hypotheses on the Mean, Variance Known (*Z*-Tests)

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon \mu \neq \mu_0$	Probability above $\lvert z_0\rvert$ and probability below $-\lvert z_0\rvert$, $P = 2[1 - \Phi(\lvert z_0\rvert)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon \mu > \mu_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1\colon \mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

The *P*-values and critical regions for these situations are shown in Figs. 9-7 and 9-8.

Can You Ever Know the Population Standard Deviation?

1. Probably not!
2. In virtually all real world business situations, σ is not known.
3. If there is a situation where σ is known, then μ is also known since to calculate σ you need to know μ.
4. If you truly know μ there would be no need to gather a sample to estimate it.

Table of Contents

Tradition Method

S1. Form the two hypotheses $H_0 : \mu = \mu_0$ and $H_1 : \mu \neq \mu_0$.

S2. Compute the **test statistic** $t_0 = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$.

S3. Identify acceptance region, use t-distribution with d$f = n - 1$.

S4. Make a decision:

+ If the test statistic t_0 is in critical region, the reject H_0.

+ If the test statistic t_0 is in acceptance region, then fail to reject H_0.

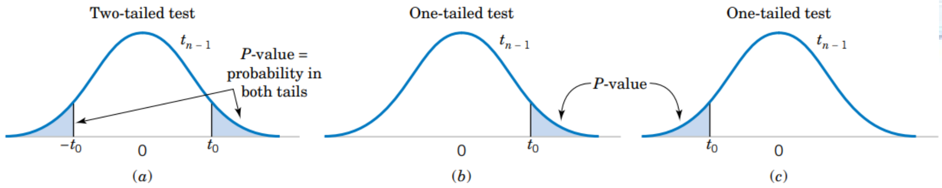


Figure. Calculating the P-value for a t-test.
(a) $H_1 : \mu \neq \mu_0$, (b) $H_1 : \mu > \mu_0$, (c): $H_1 : \mu < \mu_0$.

Note.

- $t_{1-\alpha,n} = -t_{\alpha,n}$.
- To use the t-test, we must assume the population is normal. As long as the sample size is not very small and the population is not very skewed, the t-test can be used.

Quiz 1

1. If, in a sample of $n = 16$ selected from a normal population, $\overline{X} = 56$ and $S = 12$, what is the value of t_0 if you are testing the null hypothesis $H_0 : \mu = 50$?
2. If, in a sample of $n = 16$ selected from a left-skewed population, $\overline{X} = 65$, and $S = 21$, would you use the t-test to test the null hypothesis $H_0 : \mu = 60$? Discuss.

Quiz 2

The heights of all adults in a community is known to have a normal distribution. A random sample are collected and the heights (in meters) are recorded as follows:

$$1.55 \quad 1.60 \quad 1.58 \quad 1.62 \quad 1.65 \quad 1.70 \quad 1.68.$$

Test the hypothesis that the average height of all adults in the community is at most 1.60m, at the significance level of 5%.

Quiz 3

You are the manager of a restaurant for a fast-food franchise. Last month, the mean waiting time at the drive-through window for branches in your geographic region, as measured from the time a customer places an order until the time the customer receives the order, was 3.7 minutes. You select a random sample of 64 orders. The sample mean waiting time is 3.57 minutes, with a sample standard deviation of 0.8 minute.

(a) At the 0.05 level of significance, is there evidence that the population mean waiting time is different from 3.7 minutes?

(b) Because the sample size is 64, do you need to be concerned about the shape of the population distribution when conducting the t-test in (a)? Explain.

Summary for the One-Sample t-Test.

Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon \mu \neq \mu_0$	Probability above $\lvert t_0\rvert$ and probability below $-\lvert t_0\rvert$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1\colon \mu > \mu_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1\colon \mu < \mu_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

The calculations of the *P*-values and the locations of the critical regions for these situations are shown in Figs. 9-10 and 9-12, respectively.

Table of Contents

Tradition Method

S1. Construct the two hypotheses $H_0 : p = p_0$ and $H_1 : p \neq p_0$.

S2. Find the test statistic $z_0 = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = \dfrac{x - np_0}{\sqrt{np_0(1-p_0)}}$.

S3. Identify acceptance region, use $Z \sim N(0,1)$.

S4. Make a decision:

+ If the test statistic is in critical region, then reject H_0.
+ If the test statistic is in acceptance region, then fail to reject H_0.

Note. The z_0 test statistic approximately follows a standardized normal distribution when $np_0 \geq 5$ and $n(1-p_0) \geq 5$).

Example: A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = 0.05$ significance level.

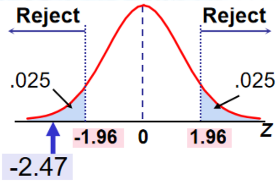
Check: $np_0 = 500 \cdot 0.08 = 40 > 5$ and $n(1-p_0) = 500\dot{0}.92 = 460 > 5$.

1. **Form hypothesis test**: $H_0 : p = 0.08$ and $H_1 : p \neq 0.08$.
2. **Find the rejection region**: Since $\alpha = 0.05$, then $\pm z_{\alpha/2} = \pm 1.96$. We reject H_0 if

$$z_0 > +1.96 \quad \text{or} \quad z_0 < -1.96.$$

3. **Test statistic**: $\hat{p} = 0.05$ and $n = 500$.

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}} = \frac{0.05 - 0.08}{\sqrt{\dfrac{0.08(1-0.08)}{500}}} = -2.47.$$



4. **Decision**: Reject H_0 at $\alpha = 0.05$. We conclude that there is sufficient evidence to reject the company's claim of 8% response rate.

Quiz 1

A company claims that the percentage of defective products is kept under control, that is less than 3%. In a random sample of 135 products it is found out that 6 of them are defective. Test the claim of the company at the significance level of 5%.

Quiz 2

A cellphone provider has the business objective of wanting to determine the proportion of subscribers who would upgrade to a new cellphone with improved features if it were made available at a substantially reduced cost. Data are collected from a random sample of 500 subscribers. The results indicate that 135 of the subscribers would upgrade to a new cellphone at a reduced cost.

(a) At the 0.05 level of significance, is there evidence that more than 20% of the customers would upgrade to a new cellphone at a reduced cost?

(b) How would the manager in charge of promotional programs concerning residential customers use the results in (a)?

Summary of Approximate Tests on a Binomial Proportion

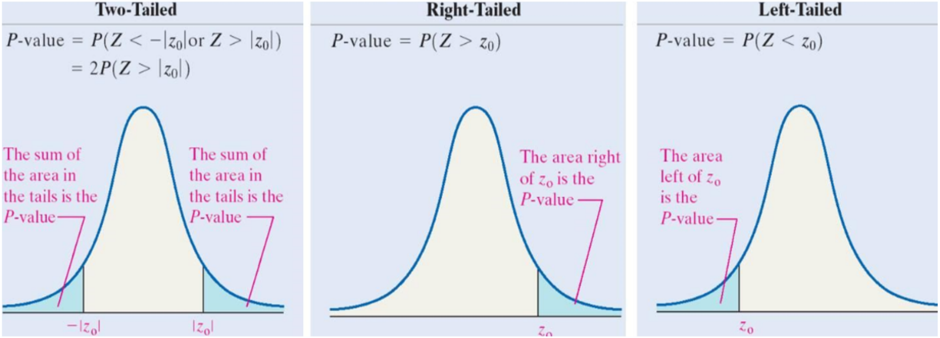
Testing Hypotheses on a Binomial Proportion

Null hypotheses: $H_0\colon p = p_0$

Test statistic: $$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon p \neq p_0$	Probability above $\lvert z_0\rvert$ and probability below $-\lvert z_0\rvert$ $P = 2[1 - \Phi(z_0)]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon p > p_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1\colon p < p_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

P-value



Quiz

A company claims that the percentage of defective products is kept under control, that is less than 3%. In a random sample of 135 products it is found out that 6 of them are defective. Test the claim of the company at the significance level of 5%. Using P-value method to solve.

Thank you!