Statistics & Probability

Chapter 2. PROBABILITY

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Sample Spaces and Events

Random Experiment

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment**.

Sample Spaces

The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.

Discrete and Continuous Sample Spaces

- A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes.
- A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.

Event

An **event** is a subset of the sample space of a random experiment.

Examples

Rolling an ordinary six-sided die is a familiar example of a random experiment.

- The **sample space** is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

- The **event** A that *"an even number is obtained"* (*"an even number is rolled"*) is

$$A = \{2, 4, 6\}.$$

- The **event** B that *"a number greater than two is rolled"* is

$$B = \{3, 4, 5, 6\}.$$

Quiz

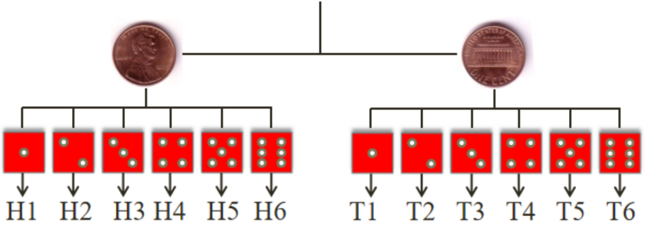
A random experiment consists of tossing two coins.

1. Construct a sample space for the situation that the coins are indistinguishable, such as two brand new pennies.
2. Construct a sample space for the situation that the coins are distinguishable, such as one a penny and the other a nickel.

Example

A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

Solution. Tree diagram



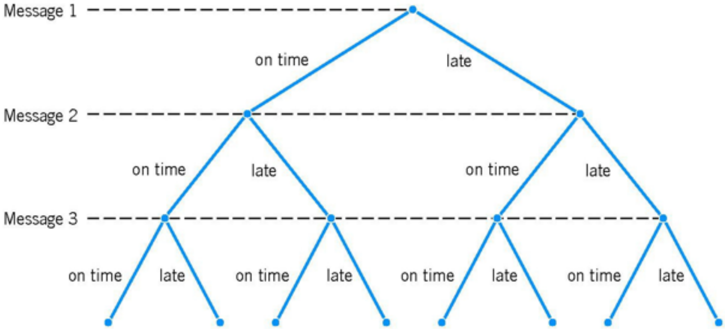
The sample space has 12 outcomes:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

Example

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

Solution.



Quiz

Construct a sample space that describes all three-child families according to the genders of the children with respect to birth order.

Basic Set Operations

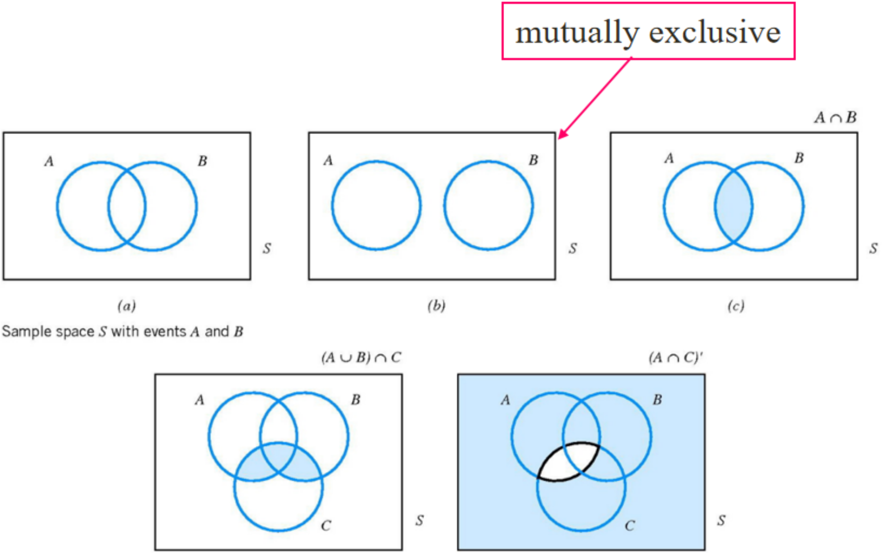
1. The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
2. The **intersection** of two events is the event that consists of all out comes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
3. The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the complement of the event E as E' or E^c.

Mutually Exclusive Events

Two events, denoted as E_1 and E_2, such that

$$E_1 \cap E_2 = \varnothing$$

are said to be **mutually exclusive**.



Sample space S with events A and B

Quiz

Measurements of the thickness of a plastic connector might be modeled with the sample space $S = \mathbb{R}^+$, the set of positive real numbers. Let

$$E_1 = \{x \mid 10 \le x \le 12\} \quad \text{and} \quad E_2 = \{x \mid 11 < x < 15\}.$$

Find $E_1 \cap E_2,\ E_1 \cup E_2,\ E_1^c$ and $E_1^c \cap E_2$.

Properties

1. $A \cap B = B \cap A$
2. $A \cup B = B \cup A$
3. $A \cap (B \cap C) = (A \cap B) \cap C$
4. $A \cup (B \cup C) = (A \cup B) \cup C$
5. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
7. $(A \cap B)^c = A^c \cup B^c$
8. $(A \cup B)^c = A^c \cap B^c$
9. $(A^c)^c = A.$

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There are different approaches to assessing the probability of an uncertain event:

1. **A priori classical probability**: the probability of an event is based on prior knowledge of the process involved.
2. **Empirical classical probability**: the probability of an event is based on observed data.

Equally Likely Outcomes

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $\dfrac{1}{N}$.

1. **A priori classical probability**

$$\text{Probability of Occurrence} = \frac{X}{T} = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}.$$

2. **Empirical classical probability**

$$\text{Probability } y \text{ of Occurrence} = \frac{\text{number of favorable outcomes observed}}{\text{total number of outcomes observed}}.$$

Example

Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

Solution. We have

$$\text{Probability of a face card} = \frac{X}{T} = \frac{\text{number of face cards}}{\text{total number of cards}} = \frac{12 \text{ face cards}}{52 \text{ total cards}} = \frac{3}{13}.$$

Example

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	**Not Taking Stats**	**Total**
Male	84	145	229
Female	76	134	210
Total	160	279	439

Solution. We have

$$\text{Probability of Male Taking Stats} = \frac{\text{number of males taking stats}}{\text{total number of people}} = \frac{84}{439} \approx 0.191.$$

Probability of an Event

For a discrete sample space, the **probability of an event** E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

1. $P(S) = 1$
2. $0 \leq P(E) \leq 1$
3. For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

Example

A random experiment can result in one of the outcomes $S = \{a, b, c, d\}$ with probabilities $0.1,\ 0.3,\ 0.5$ and 0.1, respectively. Let

$$A = \{a, b\},\ B = \{b, c, d\},\ A^c = S \setminus A,\ B^c = S \setminus B.$$

Find $P(A),\ P(B),\ P(A^c),\ P(B^c),\ P(A \cap B),\ P(A \cup B)$.

Example

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table.

Number of contamination particles	Proportion of wafers
0	0.40
1	0.15
2	0.20
3	0.10
4 or more	0.15

What is the probability that a wafer contains three or more particles in the inspected location?

Complement Rule

If the complement of A, denoted by A^c, consists of all the outcomes in which the event A does not occur, then

$$P(A) + P(A^c) = 1.$$

Note

$$P(\varnothing) = 0.$$

Ramark

Depending on the problem, it may be easier to find $P(A^c)$ and then use the above equation to find $P(A)$.

Example

A number is chosen at random from a set of whole numbers from 1 to 50. Calculate the probability that the chosen number is not a perfect square.

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Addition Rules

1. (**Probability of a Union**) If A and B are any events, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

2. If A and B are mutually exclusive events, i.e. $A \cap B = \varnothing$,then
$$P(A \cup B) = P(A) + P(B).$$

3. If A, B, C are any events, then
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

4. A collection of events $E_1, E_2, \ldots, E_k$ is said to be mutually exclusive if for all pairs,
$$E_i \cap E_j = \varnothing.$$
For a collection of mutually exclusive events,
$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \cdots + P(E_k).$$

Example

Find the probability of selecting a male or a statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

Solution.

$$\begin{aligned} P(\text{Male or Stats}) &= P(\text{Male}) + P(\text{Stats}) - P(\text{Male and Stats}) \\ &= \frac{229}{439} + \frac{160}{439} - \frac{84}{439} = \frac{305}{439}. \end{aligned}$$

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The **conditional probability** of an event B, given that an event A already occurred, is denoted by $P(B|A)$. For instance,

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	38
	No	30	342	362
	Total	40	360	400

We have,

$$P(D|F) = \frac{10}{40} = 0.25$$
$$P(D|F^c) = \frac{18}{360} = 0.05.$$

Conditional Probability

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{for } P(A) > 0.$$

Special Case

All outcomes are equally likely $\dfrac{P(A \cap B)}{P(A)} = \dfrac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$.

Example

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD), and 20% of the cars have both. What is the probability that a car has a CD player, given that it has AC?

	CD	**No CD**	**Total**
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

Solution. We have

$$P(\text{CD}|\text{AC}) = \frac{P(\text{CD and AC})}{P(\text{AC})} = \frac{0.2}{0.7} = 0.2857.$$

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Multiplication Rule

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A).$$

Example

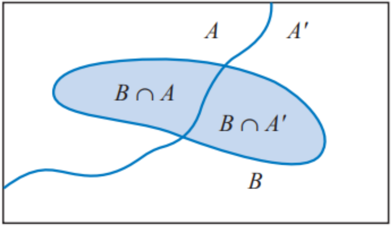
The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperature is 0.05. The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T)P(T) = 0.7 \cdot 0.05 = 0.035$$

where C: "a battery suffers low charging current" and T: "a battery is subject to high engine compartment temperature".

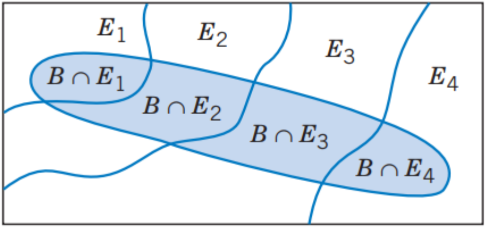
Quiz

Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$. Determine $P(A \cap B)$ and $P(A^c \cap B)$.



$$B = (B \cap A) \cup (B \cap A')$$

Figure 1. Partitioning an event into two mutually exclusive subsets.



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Figure 2. Partitioning an event into several mutually exclusive subsets.

Note

1. $A' = A^c$.
2. A collection of sets $E_1, E_2, \ldots, E_k$ such that $E_1 \cup E_2 \cup \ldots \cup E_k = S$ is said to be **exhaustive** where S is the sample space.

Total Probability Rule (Two Events)

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

Example (Total Probability Rule (Two Events))

The information of the contamination discussion is summarized in the following table:

Probability of Failure	**Level of Contamination**	**Probability of Level**
0.100	High	0.2
0.005	No high	0.8

Let F denote the event that the product fail. Find $P(F)$.

Solution. Let H denote the event that the chip is exposed to high level of contamination. So

$$P(H) = 0.2 \text{ and } P(H^c) = 0.8.$$

In addition, we have

$$P(F|H) = 0.1 \text{ and } P(F|H^c) = 0.005.$$

Therefore,

$$P(F) = P(F|H)P(H) + P(F|H^c)P(H^c) = 0.1 \cdot 0.2 + 0.005 \cdot 0.8 = 0.024.$$

Total Probability Rule (Multiple Events)

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k). \end{aligned}$$

Example (Total Probability Rule (Multiple Events)

Assume the following probabilities for product failure subject to levels of contamination in manufacturing:

Probability of Failure	**Level of Contamination**
0.10	High
0.01	Medium
0.001	Low

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails?

Example (Total Probability Rule (Multiple Events))

Let

- H denote the event that the a chip is exposed to high levels of contamination,
- M denote the event that the a chip is exposed to medium levels of contamination,
- L denote the event that the a chip is exposed to low levels of contamination,
- F denote the event that the product fails.

$$P(F) = P(F|H)P(H) + P(F|M)P(M) + P(F|L)P(L)$$
$$= 0.1 \cdot 0.2 + 0.01 \cdot 0.3 + 0.001 \cdot 0.5 = 0.0235.$$

We conclude that $P(F) = 0.0235$.

Quiz

Computer keyboard failures are due to faulty electrical connects 12% or mechanical defects 88%. Mechanical defects are related to loose keys 27% or improper assembly 73%. Electrical connect defects are caused by defective wires 35%, improper connections 13%, or poorly welded wires 52%.

1. Find the probability that a failure is due to loose keys.
2. Find the probability that a failure is due to improperly connected or poorly welded wires.

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Independence (Two Events)

Two events are called **independent** if the occurrence of one event does not change the probability of the other event. Equivalently, any one of the following equivalent statements is true:

1. $P(A|B) = P(A)$.
2. $P(B|A) = P(B)$.
3. $P(A \cap B) = P(A)P(B)$.

Remark

If A and B are independent events, then so are events A and B^c, events A^c and B, and events A^c and B^c.

Example

A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch. Let $A =$ "the first part is defective", and $B =$ "the second part is defective". We suspect that these two events are not independent because knowledge that the first part is defective suggests that it is less likely that the second part selected is defective.

Solution. We have

$$P(B|A) = \frac{49}{849}.$$

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= \frac{49}{849} \cdot \frac{50}{850} + \frac{50}{849} \cdot \frac{800}{850} = \frac{50}{850} \end{aligned}$$

which implies

$$P(B|A) \neq P(B).$$

We conclude that the two events are not independent, as we suspected.

Independent (Multiple Events)

The events $E_1, E_2, \ldots, E_k$ are **independent** if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = P(E_{i_1})P(E_{i_2}) \cdots P(E_{i_k}).$$

Example

Assume that the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle is not dependent on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Solution. Let E_i denote the event that the i^{th} wafer contains no large particle, $i = 1, 2, \ldots, 15$. Then $i = 1, 2, \ldots, 15$, $P(E_i) = 0.99$ and we have

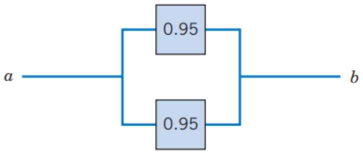
$$P(E_1 \cap E_2 \cap \ldots \cap E_{15}) = P(E_1)P(E_2) \cdots P(E_{15}) = (0.99)^{15} = 0.86.$$

Quiz

Two coins are tossed. Let $A =$ "at most one head on the two tosses" and $B =$ "one head and one tail in both tosses". Are A and B independent events?

Example

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Solution. Let T, B denote the events that the top and bottom devices operate, respectively. The probability that the circuit operates is

$$\begin{aligned} P(T \cup B) &= 1 - P(T \cup B)^c \\ &= 1 - P(T^c \cap B^c) \\ &= 1 - P(T^c)P(B^c) \\ &= 1 - 0.05^2 \\ &= 0.9975. \end{aligned}$$

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From the definition of conditional probability,

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B) = P(B \cap A).$$

Thus, for $P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \qquad (*)$$

By using Total Probability Rule (multiple events), we obtain the following general result:

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{\sum\limits_{i=1}^{k} P(B|E_i)P(E_i)}$$

for $P(B) > 0$.

Note. From (*), we have

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{implies} \quad P(A|B)P(B) = P(B|A)P(A).$$

Quiz

Suppose that $P(A|B) = 0.7,\ P(A) = 0.5$ and $P(B) = 0.2.$ Determine $P(B|A)$.

Quiz

Assume that two events A and B are such that $P(A \cap B) = 0.15,\ P(A \cup B) = 0.65$ and $P(A|B) = 0.5.$ Find $P(B|A)$.

Quiz

In a state where cars have to be tested for the emission of pollutants, 25% of all cars emit excessive amount of pollutants. When tested, 99% of all cars that emit excessive amount of pollutants will fail, but 17% of all cars that do not emit excessive amount of pollutants will also fail. What is the probability that a car that fails the test actually emits excessive amounts of pollutants?

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Random Variable

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

Note

A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milli-amperes.

Discrete Random Variable

A **discrete random variable** is a random variable with a finite (or countably infinite) range.

Continuous Random Variable

A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range.

Discrete Random Variable

- Number of scratches on a surface.
- Proportion of defective parts among 1000 tested.
- Number of transmitted bits received in error.

Continuous Random Variable

- Electrical current
- Length.
- Pressure.
- Temperature.
- Time.
- Voltage.
- Weight.

Quiz

Decide whether a discrete or continuous random variable is the best model for each of the following variables:

1. The time until a projectile returns to earth.
2. The number of times a transistor in a computer memory changes state in one operation.
3. The volume of gasoline that is lost to evaporation during the filling of a gas tank.
4. The outside diameter of a machined shaft.
5. The number of cracks exceeding one-half inch in 10 miles of an interstate highway.
6. The weight of an injection-molded plastic part.
7. The number of molecules in a sample of gas.
8. The concentration of output from a reactor.
9. The current in an electronic circuit.

Thank you!

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