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Continuous Optimization

A slacks-based measure of super-efficiency in data envelopment analysis

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Abstract

In most models of Data Envelopment Analysis (DEA), the best performers have the full efficient status denoted by unity (or 100%), and, from experience, we know that usually plural Decision Making Units (DMUs) have this "efficient status". To discriminate between these efficient DMUs is an interesting subject. This paper addresses this "super-efficiency" issue by using the slacks-based measure (SBM) of efficiency, which the author proposed in his previous paper [European Journal of Operational Research 130 (2001) 498]. The method differs from the traditional one based on the radial measure, e.g. Andersen and Petersen model, in that the former deals directly with slacks in inputs/outputs, while the latter does not take account of the existence of slacks. We will demonstrate the rationality of our approach by comparing it with the radial measure of super-efficiency. The proposed method will be particularly useful when the number of DMUs are small compared with the number of criteria employed for evaluation.

Keywords: DEA; Efficiency; Super-efficiency; Slacks; Units invariant; Multiple criteria decision-making

1. Introduction

In most models of Data Envelopment Analysis (DEA) (Charnes et al., 1978; Cooper et al., 2000), the best performers have efficiency score unity, and, from experience, we know that usually there are plural Decision Making Units (DMUs) which have this "efficient status". To discriminate between these efficient DMUs is an interesting research subject. Several authors have proposed methods for ranking the best performers. See Andersen and Petersen (1993), Doyle and Green (1993, 1994), Stewart (1994), Tofallis (1996), Seiford and Zhu (1999), and Zhu (2001) among others. We will call this problem the "super-efficiency problem".

Meanwhile, the author of this paper has proposed a slacks-based measure of efficiency (SBM) (Tone, 2001), which is non-radial and deals with input/output slacks directly. The SBM returns an efficiency measure between 0 and 1, and gives unity if and only if the DMU concerned is on the frontiers of the production possibility set with no input/output slacks. In that respect, SBM differs from traditional radial measures of efficiency that do not take account of the existence of slacks.

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In this paper, we discuss the "super-efficiency" issues based on the SBM. We can rank the efficient DMUs by applying this procedure. This paper is organized as follows. Section 2 briefly introduces the SBM and then we propose a super-efficiency measure by using SBM in Section 3. We specialize our super-efficiency model to input (output) orientation in Section 4. This enables us to compare our method with the super-efficiency evaluated by the CCR (Charnes–Cooper–Rhodes) type radial super-efficiency models. In Section 5 we compare our method with the Andersen and Petersen model and demonstrate the rationality of our method. Then we extend our method to a more expanded data set and to the variable returns-to-scale environment. Some remarks will follow in Section 7.

2. Slacks-based measure of efficiency

We will deal with n DMUs with the input and output matrices $X = (x_{ij}) \in R^{m\times n}$ and $Y = (y_{ij}) \in R^{s\times n}$, respectively. We assume that the data set is positive, i.e. $X > 0$ and $Y > 0$. (This assumption will be relaxed in Section 6.)

The production possibility set P is defined as

$$P = \{(\mathbf{x}, \mathbf{y}) \,|\, \mathbf{x} \geqslant X\boldsymbol{\lambda},\ \mathbf{y} \leqslant Y\boldsymbol{\lambda},\ \boldsymbol{\lambda} \geqslant \mathbf{0}\}, \tag{1}$$

where $\boldsymbol{\lambda}$ is a non-negative vector in R^n.

We consider an expression for describing a certain DMU $(\mathbf{x}_0, \mathbf{y}_0)$ as

$$\mathbf{x}_0 = X\boldsymbol{\lambda} + \mathbf{s}^-, \tag{2}$$

$$\mathbf{y}_0 = Y\boldsymbol{\lambda} - \mathbf{s}^+ \tag{3}$$

with $\boldsymbol{\lambda} \geqslant \mathbf{0}$, $\mathbf{s}^- \geqslant \mathbf{0}$ and $\mathbf{s}^+ \geqslant \mathbf{0}$. The vectors $\mathbf{s}^- \in R^m$ and $\mathbf{s}^+ \in R^s$ indicate the *input excess* and *output shortfall* of this expression, respectively, and are called *slacks*. From the conditions $X > 0$ and $\boldsymbol{\lambda} \geqslant \mathbf{0}$, it holds

$$\mathbf{x}_0 \geqslant \mathbf{s}^-. \tag{4}$$

Using $\mathbf{s}^-$ and $\mathbf{s}^+$, we define an index ρ as follows:

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{i0}}. \tag{5}$$

It can be verified that ρ satisfies the properties (i) units invariant and (ii) monotone decreasing in input/output slacks. Furthermore, from (4), it holds

$$0 < \rho \leqslant 1. \tag{6}$$

In an effort to estimate the efficiency of $(\mathbf{x}_0, \mathbf{y}_0)$, we formulate the following fractional program [SBM] in $\boldsymbol{\lambda}$, $\mathbf{s}^-$ and $\mathbf{s}^+$.

[SBM]

$$\begin{aligned} \min \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0}}{1 + \frac{1}{s}\sum_{i=1}^{s} s_i^+/y_{i0}} \\ \text{subject to} \quad & \mathbf{x}_0 = X\boldsymbol{\lambda} + \mathbf{s}^-, \\ & \mathbf{y}_0 = Y\boldsymbol{\lambda} - \mathbf{s}^+, \\ & \boldsymbol{\lambda} \geqslant \mathbf{0},\ \ \mathbf{s}^- \geqslant \mathbf{0},\ \ \mathbf{s}^+ \geqslant \mathbf{0}. \end{aligned} \tag{7}$$

[SBM] can be transformed into a linear program using the Charnes–Cooper transformation in a similar way to the CCR model (see Charnes and Cooper, 1962; Charnes et al., 1978). Refer to Tone (2001) and Cooper et al. (2000) for details.

Let an optimal solution for [SBM] be $(\rho^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$. Based on this optimal solution, we define a DMU as being *SBM-efficient* as follows:

Definition 1 (*SBM-efficient*). A DMU $(\mathbf{x}_0, \mathbf{y}_0)$ is SBM-efficient, if $\rho^* = 1$.

This condition is equivalent to $\mathbf{s}^{-*} = \mathbf{0}$ and $\mathbf{s}^{+*} = \mathbf{0}$, i.e., no input excesses and no output shortfalls in any optimal solution.

3. Super-efficiency evaluated by SBM

In this section, we discuss the super-efficiency issues under the assumption that the DMU $(\mathbf{x}_0, \mathbf{y}_0)$ is SBM-efficient, i.e. $\rho^* = 1$.

3.1. Super-efficiency score

Let us define a production possibility set $P\backslash(\mathbf{x}_0, \mathbf{y}_0)$ spanned by (X, Y) excluding $(\mathbf{x}_0, \mathbf{y}_0)$, i.e.

$$P \setminus (\mathbf{x}_0, \mathbf{y}_0) = \left\{ (\bar{\mathbf{x}}, \bar{\mathbf{y}}) \middle| \bar{\mathbf{x}} \geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j,\ \bar{\mathbf{y}} \leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j,\ \bar{\mathbf{y}}. \geqslant \mathbf{0},\ \boldsymbol{\lambda} \geqslant \mathbf{0} \right\}. \tag{8}$$

Further, we define a subset $\bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$ of $P \setminus (\mathbf{x}_0, \mathbf{y}_0)$ as

$$\bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0) = P \setminus (\mathbf{x}_0, \mathbf{y}_0) \bigcap \left\{ \bar{\mathbf{x}} \geqslant \mathbf{x}_0 \text{ and } \bar{\mathbf{y}} \leqslant \mathbf{y}_0 \right\}. \tag{9}$$

By the assumption $X > 0$ and $Y > 0$, $\bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$ is not empty.

As a weighted l_1 distance from $(\mathbf{x}_0, \mathbf{y}_0)$ to $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$, we employ the index δ as defined by

$$\delta = \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{i0}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{r0}}. \tag{10}$$

From (9), this distance is not less than 1 and attains 1 if and only if $(\mathbf{x}_0, \mathbf{y}_0) \in \bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$, i.e. exclusion of the DMU $(\mathbf{x}_0, \mathbf{y}_0)$ has no effect on the original production possibility set P.

We can interpret this index as follows. The numerator is a weighted l_1 distance from $\mathbf{x}_0$ to $\bar{\mathbf{x}}$ ($\geqslant \mathbf{x}_0$), and hence it expresses an average expansion rate of $\mathbf{x}_0$ to $\bar{\mathbf{x}}$ of the point $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$. The denominator is a weighted l_1 distance from $\mathbf{y}_0$ to $\bar{\mathbf{y}}$ ($\leqslant \mathbf{y}_0$), and hence it is an average reduction rate of $\mathbf{y}_0$ to $\bar{\mathbf{y}}$ of $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$. The smaller the denominator is, the farther $\mathbf{y}_0$ is positioned to $\bar{\mathbf{y}}$. Its inverse can be interpreted as an index of the distance from $\mathbf{y}_0$ to $\bar{\mathbf{y}}$. Therefore, δ is a product of two indices: one, the distance in the input space, and the other, that in the output space. Both indices are dimensionless.

Based on the above observations, we define the super-efficiency of $(\mathbf{x}_0, \mathbf{y}_0)$ as the optimal objective function value δ^* of the following program:

[SuperSBM]

$$\begin{aligned} \delta^* = \min \delta = {} & \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i / x_{i0}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r / y_{r0}} \\ \text{subject to} \quad & \bar{\mathbf{x}} \geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\ & \bar{\mathbf{y}} \leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\ & \bar{\mathbf{x}} \geqslant \mathbf{x}_0 \text{ and } \bar{\mathbf{y}} \leqslant \mathbf{y}_0, \\ & \bar{\mathbf{y}} \geqslant \mathbf{0},\ \boldsymbol{\lambda} \geqslant \mathbf{0}. \end{aligned} \tag{11}$$

We have the following two propositions:

Proposition 1. *The super-efficiency score δ^* is units invariant, i.e. it is independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.*

Proof. This proposition holds, since both the objective function and constraints are units invariant. □

Proposition 2. *Let $(\alpha\mathbf{x}_0, \beta\mathbf{y}_0)$ with $\alpha \leqslant 1$ and $\beta \geqslant 1$ be a DMU with reduced inputs and enlarged outputs than $(\mathbf{x}_0, \mathbf{y}_0)$. Then, the super-efficiency score of $(\alpha\mathbf{x}_0, \beta\mathbf{y}_0)$ is not less than that of $(\mathbf{x}_0, \mathbf{y}_0)$.*

Proof. The super-efficiency score $(\hat{\delta}^*)$ of $(\alpha\mathbf{x}_0, \beta\mathbf{y}_0)$ is evaluated by solving the following program:

[SuperSBM′]

$$\begin{aligned} \hat{\delta}^* = \min \hat{\delta} = {} & \frac{\frac{1}{m}\sum_{i=1}^{m} \hat{x}_i / (\alpha x_{i0})}{\frac{1}{s}\sum_{r=1}^{s} \hat{y}_r / (\beta y_{r0})} \\ = {} & \min \frac{\beta}{\alpha} \frac{\frac{1}{m}\sum_{i=1}^{m} \hat{x}_i / x_{i0}}{\frac{1}{s}\sum_{r=1}^{s} \hat{y}_r / y_{r0}} \\ \text{subject to} \quad & \hat{\mathbf{x}} \geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\ & \hat{\mathbf{y}} \leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\ & \hat{\mathbf{x}} \geqslant \alpha\mathbf{x}_0 \text{ and } \mathbf{0} \leqslant \hat{\mathbf{y}} \leqslant \beta\mathbf{y}_0, \\ & \boldsymbol{\lambda} \geqslant \mathbf{0}. \end{aligned} \tag{12}$$

It can be observed that, for any feasible solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ for [SuperSBM′], $(\hat{\mathbf{x}}/\alpha, \hat{\mathbf{y}}/\beta)$ is feasible for [SuperSBM]. Hence it holds

$$\delta^* \leqslant \frac{\frac{1}{m}\sum_{i=1}^{m}(\hat{x}_i/\alpha)/x_{i0}}{\frac{1}{s}\sum_{r=1}^{s}(\hat{y}_r/\beta)/y_{r0}} = \frac{\beta}{\alpha}\frac{\frac{1}{m}\sum_{i=1}^{m}\hat{x}_i/x_{i0}}{\frac{1}{s}\sum_{r=1}^{s}\hat{y}_r/y_{r0}}. \tag{13}$$

Comparing (12) with (13) we see that

$$\delta^* \leqslant \hat{\delta}^*.$$

Thus, the super-efficiency score of $(\alpha\mathbf{x}_0, \beta\mathbf{y}_0)$ ($\alpha \leqslant 1$ and $\beta \geqslant 1$) is not less than that of $(\mathbf{x}_0, \mathbf{y}_0)$. □

3.2. Solving super-efficiency

The fractional program [SuperSBM] can be transformed into a linear programming problem using the Charnes–Cooper transformation as

[LP]

$$\tau^* = \min \tau = \frac{1}{m}\sum_{i=1}^{m}\frac{\widetilde{x}_i}{x_{i0}}$$

$$\begin{aligned} \text{subject to} \quad & 1 = \frac{1}{s}\sum_{r=1}^{s}\frac{\widetilde{y}_r}{y_{r0}}, \\ & \widetilde{\mathbf{x}} \geqslant \sum_{j=1,\neq 0}^{n}\Lambda_j\mathbf{x}_j, \\ & \widetilde{\mathbf{y}} \leqslant \sum_{j=1,\neq 0}^{n}\Lambda_j\mathbf{y}_j, \\ & \widetilde{\mathbf{x}} \geqslant t\mathbf{x}_0 \text{ and } \widetilde{\mathbf{y}} \leqslant t\mathbf{y}_0, \\ & \mathbf{\Lambda} \geqslant \mathbf{0}, \ \widetilde{\mathbf{y}} \geqslant \mathbf{0}, \quad t > 0. \end{aligned} \tag{14}$$

Let an optimal solution of [LP] be $(\tau^*, \widetilde{\mathbf{x}}^*, \widetilde{\mathbf{y}}^*, \mathbf{\Lambda}^*, t^*)$. Then we have an optimal solution of [SuperSBM] as expressed by

$$\begin{aligned} &\delta^* = \tau^*, \quad \boldsymbol{\lambda}^* = \mathbf{\Lambda}^*/t^*, \\ &\bar{\mathbf{x}}^* = \widetilde{\mathbf{x}}^*/t^*, \quad \bar{\mathbf{y}}^* = \widetilde{\mathbf{y}}^*/t^*. \end{aligned} \tag{15}$$

3.3. An example

We will illustrate the slacks-based super-efficiency using an example. Table 1 exhibits data for seven DMUs using two inputs (x_1, x_2) to produce a single output $(y = 1)$. The input-oriented CCR (Charnes–Cooper–Rhodes) model identifies F and G as weakly efficient $(\theta^* = 1)$, although they have slacks in x_1 against C. The SBM model reveals these slacks and identifies only C, D and E as efficient $(\rho^* = 1)$. We evaluated the super-efficiency of these three DMUs and obtained the scores listed in column δ^*. It is observed that E has the largest distance as measured by the weighted l_1-norm, from the remaining production possibility set. Actually, E has the optimal solution $\bar{x}_1^* = 4$, $\bar{x}_2^* = 4$, $\bar{y}^* = 1$ and hence

$$\rho_E^* = \frac{1}{2}\left(\frac{4}{2} + \frac{4}{4}\right)\Big/ 1 = 1.5.$$

4. Input/output oriented super-efficiency

In order to adapt our super-efficiency model to input (output) orientation, we can modify the preceding program as follows.

For input orientation, we deal with the weighted l_1-distance only in the input space, keeping the outputs status quo. Thus, the program turns out to be

Table 1
Data and results of CCR (θ^*), SBM (ρ^*) and super-efficiency (δ^*)

DMU	Data			CCR	SBM				
	x_1	x_2	y	θ^*	ρ^*	δ^*	$\bar{x}_1^*$	$\bar{x}_2^*$	$\bar{y}^*$
A	4	3	1	0.8571	0.8333				
B	7	3	1	0.6316	0.6191				
C	8	1	1	1	1	1.125	10	1	1
D	4	2	1	1	1	1.25	4	3	1
E	2	4	1	1	1	1.5	4	4	1
F	10	1	1	1	0.9				
G	12	1	1	1	0.8333				

[SuperSBM(I)]

$$\begin{aligned}
\delta_{\mathrm{I}}^* &= \min \delta = \frac{1}{m} \sum_{i=1}^{m} \bar{x}_i / x_{i0} \\
\text{subject to} \quad \bar{\mathbf{x}} &\geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\
\bar{\mathbf{y}} &\leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\
\bar{\mathbf{x}} &\geqslant \mathbf{x}_0 \text{ and } \bar{\mathbf{y}} = \mathbf{y}_0, \\
\boldsymbol{\lambda} &\geqslant \mathbf{0}.
\end{aligned} \tag{16}$$

The following proposition holds for this program:

Proposition 3. *If inputs* $\mathbf{x}_0$ *decrease to* $\mathbf{x}_0 - \Delta\mathbf{x}$ $(\geqslant \mathbf{0},\ \Delta\mathbf{x} \geqslant \mathbf{0})$, *then the optimal objective function value* $\delta_{\mathrm{I}}^*(\Delta\mathbf{x})$ *corresponding to this change satisfies*

$$\delta_{\mathrm{I}}^*(\Delta\mathbf{x}) \geqslant \delta_{\mathrm{I}}^*. \tag{17}$$

Furthermore, the equality holds if and only if $\Delta x_i = 0$ or $\bar{x}_i^* = x_{i0} - \Delta x_i$ *holds for every* i $(= 1, \ldots, m)$, *where* $\bar{\mathbf{x}}_i^*$ *is an optimal solution of the above program* (16).

Proof. The linear program for this perturbed problem is expressed as

$$\begin{aligned}
\delta_{\mathrm{I}}^*(\Delta\mathbf{x}) &= \min \delta = \frac{1}{m} \sum_{i=1}^{m} \bar{x}_i / (x_{i0} - \Delta x_i) \\
\text{subject to} \quad \bar{\mathbf{x}} &\geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\
\bar{\mathbf{y}} &\leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\
\bar{\mathbf{x}} &\geqslant \mathbf{x}_0 - \Delta\mathbf{x} \text{ and } \bar{\mathbf{y}} = \mathbf{y}_0, \\
\boldsymbol{\lambda} &\geqslant \mathbf{0}.
\end{aligned} \tag{18}$$

For any optimal solution $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^* = \mathbf{y}_0)$ of the above perturbed LP, $(\bar{\mathbf{x}}^* + \Delta\mathbf{x}, \bar{\mathbf{y}}^* = \mathbf{y}_0)$ is feasible for [SuperSBM(I)]. Hence it holds

$$\delta_{\mathrm{I}}^* \leqslant \frac{1}{m} \sum_{i=1}^{m} \frac{\bar{x}_i^* + \Delta x_i}{x_{i0}} \leqslant \frac{1}{m} \sum_{i=1}^{m} \frac{\bar{x}_i^*}{x_{i0} - \Delta x_i} = \delta_{\mathrm{I}}^*(\Delta\mathbf{x}).$$

The last inequality holds since

$$\frac{\bar{x}_i^*}{x_{i0} - \Delta x_i} - \frac{\bar{x}_i^* + \Delta x_i}{x_{i0}} = \frac{\Delta x_i(\bar{x}_i^* - x_{i0} + \Delta x_i)}{x_{i0}(x_{i0} - \Delta x_i)} \geqslant 0.$$

The equality holds if and only if $\Delta x_i = 0$ or $\bar{x}_i^* = x_{i0} - \Delta x_i$ holds for every i $(= 1, \ldots, m)$. □

In a similar way we can develop the output-oriented super-efficiency model as follows:

[SuperSBM(O)]

$$\begin{aligned}
\delta_{\mathrm{O}}^* &= \min \delta = \frac{1}{\frac{1}{s} \sum_{r=1}^{s} \bar{y}_r / y_{r0}} \\
\text{subject to} \quad \bar{\mathbf{x}} &\geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\
\bar{\mathbf{y}} &\leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\
\bar{\mathbf{x}} &= \mathbf{x}_0 \text{ and } \mathbf{0} \leqslant \bar{\mathbf{y}} \leqslant \mathbf{y}_0, \\
\boldsymbol{\lambda} &\geqslant \mathbf{0}.
\end{aligned} \tag{19}$$

Since the above two models behave in the restricted feasible region of [SuperSBM], we have:

Proposition 4. $\delta_{\mathrm{I}}^* \geqslant \delta^*$ *and* $\delta_{\mathrm{O}}^* \geqslant \delta^*$.

5. Comparisons with the Andersen and Petersen model

In this section, we compare our method with the super-efficiency model proposed by Andersen and Petersen (1993), and point out remarkable differences between them.

5.1. Andersen and Petersen model

This model can be described, in the input-oriented CCR case, as follows:

[SuperCCR]

$$\begin{aligned}
\theta^* &= \min \theta \\
\text{subject to} \quad \theta \mathbf{x}_0 &= \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j + \mathbf{s}^-, \\
\mathbf{y}_0 &= \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j - \mathbf{s}^+, \\
\boldsymbol{\lambda} &\geqslant \mathbf{0},\ \mathbf{s}^- \geqslant \mathbf{0},\ \mathbf{s}^+ \geqslant \mathbf{0},
\end{aligned} \tag{20}$$

where $\mathbf{s}^-$ and $\mathbf{s}^+$ represent input and output slacks, respectively. Let an optimal solution of [SuperCCR] be $(\theta^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$. For an efficient DMU $(\mathbf{x}_0, \mathbf{y}_0)$, θ^* is not less than unity, and this value indicates "super-efficiency". Regarding this measure we have the following proposition:

Proposition 5. *The [SuperCCR] model returns the same super-efficiency score θ^* for any DMUs represented by $(\mathbf{x}_0 - \alpha\mathbf{s}^{-*}/\theta^*,\ \mathbf{y}_0)$ for the range $0 \leqslant \alpha \leqslant 1$.*

This contradicts our common understanding that a reduction of input values usually increases super-efficiency. This irrationality is caused by the fact that this model deals only with the radial measure and neglects the existence of input slacks as represented by $\mathbf{s}^{-*}$.

Furthermore, we have the following relationships between [SuperCCR] and [SuperSBM(I)].

Lemma 1. *Let us define*

$$\alpha^* = \min_i \left\{ \frac{(\theta^* - 1)x_{i0}}{s_i^{-*}} \mid s_i^{-*} > 0 \right\} = 0$$

$$\text{if } \mathbf{s}^{-*} = \mathbf{0}. \tag{21}$$

Then, $(\widetilde{\mathbf{x}} = \theta^*\mathbf{x}_0 - \alpha^*\mathbf{s}^{-*},\ \widetilde{\mathbf{y}} = \mathbf{y}_0,\ \widetilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*)$ is a feasible solution for [SuperSBM(I)].

Proof. From (21), we have $\widetilde{\mathbf{x}} \geqslant \mathbf{x}_0$, and hence, the above $(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$ satisfies the constraints of [SuperSBM(I)]. □

Let an optimal solution of [SuperCCR] be $(\theta^*, \boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$, the optimal objective value of [SuperSBM(I)] be δ_{I}^*, and α^* as defined by (21). Then we have:

Theorem 1

$$\delta_{\mathrm{I}}^* \leqslant \theta^* - \frac{\alpha^*}{m} \sum_{i=1}^{m} \frac{s_i^{-*}}{x_{i0}}. \tag{22}$$

Proof. The objective function value corresponding to $\widetilde{\mathbf{x}}$ in Lemma 1 for [SuperSBM(I)] is given by

$$\frac{1}{m} \sum_{i=1}^{m} \frac{\widetilde{x}_i}{x_{i0}} = \theta^* - \frac{\alpha^*}{m} \sum_{i=1}^{m} \frac{s_i^{-*}}{x_{i0}}.$$

δ_{I}^* is not greater than this value, since δ_{I}^* is optimal for [SuperSBM(I)]. □

5.2. Comparisons using a numerical example

In this section, we compare the SuperCCR and SuperSBM(I) models using an example which is often referred to in papers dealing with discernment in DEA, e.g. Doyle and Green (1993, 1994), Stewart (1994), Tofallis (1996). We restrict our comparisons only within the above two models and do not discuss the practical aspects of the problem, i.e. we deal with the structure of the problem as represented by the production possibility set assumptions. This example consists of six "efficient" DMUs (power plant locations) with four inputs and two outputs as listed below:

Input

x_1 = manpower required

x_2 = construction costs in millions of dollars

x_3 = annual maintenance costs in millions of dollars

x_4 = number of villages to be evacuated

Output

y_1 = power generated in megawatts

y_2 = safety level

Table 2 exhibits the data and Table 3 displays the CCR super-efficiency scores, along with the projected point $(\theta^*\mathbf{x}_0)$, input slacks $(\mathbf{s}^{-*})$ and the reference set $\{j \mid \lambda^* > 0\}$.

Table 2
Data

DMU	Data					
	x_1	x_2	x_3	x_4	y_1	y_2
D1	80	600	54	8	90	5
D2	65	200	97	1	58	1
D3	83	400	72	4	60	7
D4	40	1000	75	7	80	10
D5	52	600	20	3	72	8
D6	94	700	36	5	96	6

It can be observed that D2, D4 and D5 have projected points comparatively larger than the inputs, i.e.

$$\text{D2:}\quad x_1 = 65 \to 157, \quad x_2 = 200 \to 483,$$
$$x_3 = 97 \to 234, \quad x_4 = 1 \to 2.4;$$

$$\text{D4:}\quad x_1 = 40 \to 65, \quad x_2 = 1000 \to 1626,$$
$$x_3 = 75 \to 122, \quad x_4 = 7 \to 11.4;$$

$$\text{D5:}\quad x_1 = 52 \to 125, \quad x_2 = 600 \to 1442,$$
$$x_3 = 20 \to 48, \quad x_4 = 3 \to 7.$$

These large projection values tally with their large super-efficiency scores: $\theta_2^* = 2.4167$ (rank 1), $\theta_4^* = 1.625$ (rank 3) and $\theta_5^* = 2.4026$ (rank 2).

At the same time, they have relatively large input slacks against their respective referent composed of a positive combination of their reference DMUs, as designated by the number in parentheses. This means that their super-efficiency scores are evaluated by referring to points far apart from the efficient portions of the production possibility set $\bar{P} \setminus (\mathbf{x}_0, \mathbf{y}_0)$. Of course, this is caused by the "radial" characteristics of the CCR efficiency measure. However, is it legitimate for D2 to have a super-efficiency score which is more than double of that of D1, considering the existence of such large slacks?

Table 4 presents the results obtained by applying the SuperSBM(I) model to this problem. As expected, the super-efficiency score δ_I^* dropped from that of the super CCR score θ^*. The most notable examples are that D2, D4 and D5 lost about 30% efficiency and the ranking of D2 and D5 is reversed.

We will now examine this change in the case of D2 in more detail. The [SuperCCR] model for D2 gives the solution

$$\theta_2^* \mathbf{x}_2 = \lambda_5^* \mathbf{x}_5 + \mathbf{s}^{-*}.$$

Thus, the projected point $\theta_2^* \mathbf{x}_2$ has slacks $\mathbf{s}^{-*} = (115.2, 0, 218.3, 0)^{\mathrm{T}}$ against the referent $\lambda_5^* \mathbf{x}_5 = (41.91, 483.33, 16.12, 2.42)^{\mathrm{T}}$ which is on the efficient frontiers of $\bar{P} \setminus (\mathbf{x}_2, \mathbf{y}_2)$. Referring to Lemma 1, from (21), we have, for $(\mathbf{x}_2, \mathbf{y}_2)$, $\alpha^* = 0.629$ and $\tilde{\mathbf{x}} = (84.61, 483.33, 97.00, 2.42)^{\mathrm{T}}$. As demonstrated in Lemma 2, this $\tilde{\mathbf{x}}$ is feasible for [SuperSBM(I)], and has slacks $\tilde{\mathbf{s}}^{-*} = (42.7, 0, 80.9, 0)^{\mathrm{T}}$ against $\theta_2^* \mathbf{x}_2$. This reduction of slacks from $\mathbf{s}^{-*}$ to $\tilde{\mathbf{s}}^{-*}$ indicates that the resulting $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}} = \mathbf{y}_2)$ is closer to the efficient portion of $\bar{P} \setminus (\mathbf{x}_2, \mathbf{y}_2)$ than the super CCR projection $(\theta_2^* \mathbf{x}_2, \mathbf{y}_2)$.

Furthermore, for this activity $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}} = \mathbf{y}_2)$, the corresponding objective value for [SuperSBM(I)] is calculated as

$$\theta_2^* - \frac{\alpha}{m} \sum_{i=1}^{m} \frac{s_i^{-*}}{x_{i2}} = 1.7839,$$

Table 3
Results of SuperCCR model

DMU	CCR	Rank	Projected point (input slacks)				Reference		
	θ^*		$\theta^* x_1^*$ (s_1^{-*})	$\theta^* x_2^*$ (s_2^{-*})	$\theta^* x_3^*$ (s_3^{-*})	$\theta^* x_4^*$ (s_4^{-*})	Referent (λ^*)		
D1	1.0283	6	82.26 (0)	616.95 (0)	55.53 (0)	8.23 (4.56)	D2 (.33)	D5 (.46)	D6 (.39)
D2	2.4167	1	157.08 (115.19)	483.33 (0)	234.41 (218.30)	2.42 (0)	D5 (.81)		
D3	1.3125	4	108.94 (63.43)	525.00 (0)	94.50 (77.00)	5.25 (2.63)	D5 (.88)		
D4	1.6250	3	65.00 (0)	1626.00 (875.00)	121.88 (96.88)	11.38 (7.63)	D5 (1.25)		
D5	2.4026	2	124.93 (0)	1441.54 (508.78)	48.05 (0)	7.21 (0.55)	D4 (.003)	D6 (1.33)	
D6	1.0628	5	99.90 (25.12)	743.95 (0)	38.26 (0)	5.31 (0)	D1 (.32)	D2 (.03)	D5 (.91)

Table 4
Results of SuperSBM(I) model

DMU	SBM	Rank	Projected point (input slacks)				Reference		
	δ_{I}^{*}		$\bar{x}_1^{-*}$ (s_1^{-*})	$\bar{x}_2^{-*}$ (s_2^{-*})	$\bar{x}_3^{-*}$ (s_3^{-*})	$\bar{x}_4^{-*}$ (s_4^{-*})	Referent (λ^*)		
D1	1.0116	6	80.00	627.88	54.00	8.00	D2	D5	D6
			(0)	(0)	(0)	(4.40)	(.32)	(.58)	(.31)
D2	1.7083	2	65.00	483.33	97.00	2.42	D5		
			(23.11)	(0)	(80.89)	(0)	(.81)		
D3	1.0781	4	83.00	525.00	72.00	4.00	D5		
			(37.5)	(0)	(54.50)	(1.38)	(.88)		
D4	1.1563	3	65.00	1000.00	75.00	7.00	D5		
			(0)	(250.00)	(50.00)	(3.25)	(1.25)		
D5	1.7988	1	52.00	828.57	57.43	5.83	D4	D6	
			(0)	(0)	(0)	(0)	(.63)	(.29)	
D6	1.0198	5	94.00	755.47	36.00	5.00	D1	D2	D5
			(20.36)	(0)	(0)	(0)	(.25)	(.03)	(1.01)

which is still larger than the $\delta_{\mathrm{I}}^{*} = 1.7083$ of D2, as Theorem 1 asserts.

As for the rank reversal of D2 and D5, D2 is closer to the respective production possibility frontiers than D5, as measured by the weighted l_1 distance, and this ranking is rational as long as we employ the proposed measure of distance function.

6. Extensions

So far, we have discussed the super-efficiency issues under the positive data set assumption within the constant returns-to-scale environment. In this section, we will relax this assumption and extend our results to the variable returns-to-scale case.

First, we relax the restriction $X > 0$ and $Y > 0$ to $X \geqslant 0$ and $Y \geqslant 0$. However, we assume, without becoming too specific, that:

(A1) For every input $i\ (= 1, \ldots, m)$, at least two DMUs have positive values.

(A2) For every output $r\ (= 1, \ldots, s)$, at least two DMUs have positive values.

(A3) For every DMU, the maximum value of inputs as well as outputs is positive.

6.1. Zeros in input data

Suppose that the DMU $(\mathbf{x}_0, \mathbf{y}_0)$ has zero elements in the input $\mathbf{x}_0$, e.g. $x_{10} = 0$. Then, there are two conceivable cases:

(Case 1) DMU $(\mathbf{x}_0, \mathbf{y}_0)$ has no function as to the input 1, e.g. in the case of banks, some have no drive-through service. In this case, the variable $\bar{x}_1$ in (11) must be set to zero. Correspondingly, the term $\bar{x}_1/x_{10}$ in the objective function may be assigned the value 1 and the term regarding to $\bar{x}_1$ in the constraints of (11) should be deleted, since $\bar{x}_1$ is no longer a variable.

(Case 2) DMU $(\mathbf{x}_0, \mathbf{y}_0)$ has the function 1 but incidentally its observed value is zero. In this case, we set x_{10} to a small positive number ε and continue the ordinary process. The value of ε is set, for example, to:

$$\varepsilon = (\text{the smallest positive } x_1 \text{ value in the data set})/100.$$

6.2. Zeros in output data

This case can be dealt with in a similar way as the zeros in the input data case above.

6.3. Extension to the variable returns-to-scale case

We now extend our analysis to the variable returns-to-scale case, i.e. we impose the following constraint:

$$\sum_{j=1}^{n} \lambda_j = 1. \tag{23}$$

Thus, the super-efficiency is evaluated by solving the following program:

[SuperSBMVRS]

$$\delta^* = \min \delta = \frac{\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i/x_{i0}}{\frac{1}{s}\sum_{r=1}^{s} \bar{y}_r/y_{r0}}$$

$$\begin{aligned}
\text{subject to} \quad & \bar{\mathbf{x}} \geqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{x}_j, \\
& \bar{\mathbf{y}} \leqslant \sum_{j=1,\neq 0}^{n} \lambda_j \mathbf{y}_j, \\
& \bar{\mathbf{x}} \geqslant \mathbf{x}_0 \text{ and } \bar{\mathbf{y}} \leqslant \mathbf{y}_0, \\
& \sum_{j=1,\neq 0}^{n} \lambda_j = 1, \\
& \bar{\mathbf{y}} \geqslant \mathbf{0}, \;\; \boldsymbol{\lambda} \geqslant \mathbf{0}.
\end{aligned} \tag{24}$$

The zero data issue can be dealt with in the same way as in the constant returns-to-scale case. Therefore we can assume that $\mathbf{x}_0 > \mathbf{0}$ and $\mathbf{y}_0 > \mathbf{0}$. We will demonstrate that the [SuperSBMVRS] is feasible and has a finite optimum. By the assumption (A2), for each output r, there is at least one positive element in the set $\{y_{rj}\}$ $(j = 1, \ldots, n;$ $j \neq 0)$. By assigning a certain positive number to λ_j for the positive output, we can choose $\bar{\boldsymbol{\lambda}}$ such that

$$\sum_{j=1,\neq 0}^{n} \bar{\lambda}_j y_{rj} > 0 \;\; (\forall r), \qquad \sum_{j=1,\neq 0}^{n} \bar{\lambda}_j = 1. \tag{25}$$

For this $\bar{\boldsymbol{\lambda}}$, we define

$$\widetilde{x}_i = \max\left\{ x_{i0}, \sum_{j=1,\neq 0}^{n} \bar{\lambda}_j x_{ij} \right\} \quad (i = 1, \ldots, m), \tag{26}$$

$$\widetilde{y}_r = \min\left\{ y_{r0}, \sum_{j=1,\neq 0}^{n} \bar{\lambda}_j y_{rj} \right\} \quad (r = 1, \ldots, s). \tag{27}$$

Thus, the set $(\bar{\mathbf{x}} = \widetilde{\mathbf{x}}, \bar{\mathbf{y}} = \widetilde{\mathbf{y}}, \boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}})$ is feasible for the [SuperSBMVRS] and satisfies

$$\widetilde{x}_i > 0 \;\; (\forall i) \quad \text{and} \quad \widetilde{y}_r > 0 \;\; (\forall r).$$

Hence, [SuperSBMVRS] is always feasible with a finite optimum.

7. Concluding remarks

In this paper, we proposed a super-efficiency measure based on input/output slacks and demonstrated its characteristics theoretically and empirically by numerical examples. The rationality for this measure is to minimize a sort of weighted l_1 distance from an efficient DMU to the production possibility set excluding the DMU. The measure is thus in sharp contrast to other methods proposed so far. In particular, when specialized in input/output orientation, it can be directly compared with the super-efficiency measures using the radial expansion/reduction of input/output.

However, we need to study the possibility of other types of distance, e.g. l_2 norm and Chebychev metric. For this purpose, Appendix A of Charnes and Cooper (1961) will be informative.

Another direction for future research includes study of the dual side of the associated linear programs, which connect the virtual input/output weights with the efficiency scores. Incorporation of weight restrictions to our model will enhance the power of real super-efficiency discernment.

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