Reference for Homework 1

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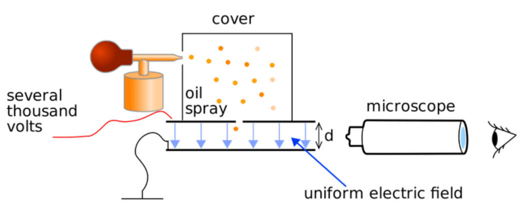
Please read the preface before reading this document!!!

1

In Milliken's experimental apparatus, oil droplets with different charge q are subject to gravitational force, buoyancy force, and drag force $F_d = 6\pi r\eta v_1$, where r is the droplet radius, η is the viscosity of air and v_1 is the terminal velocity of the droplet. When a uniform electric field E is turned on, the droplet is moving up due to the additional electric force $F_q = qE$ with a terminal speed v_2.

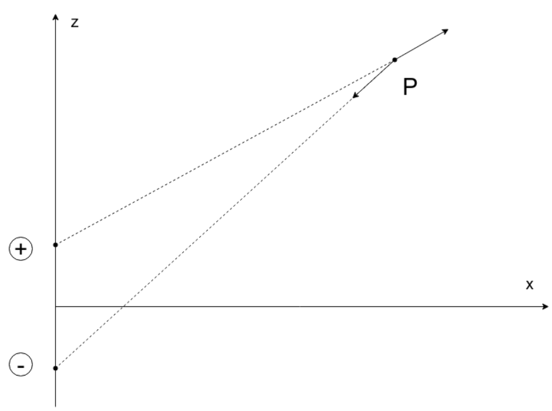


(a) Assume the droplet is the spherical. Show that the radius of the droplet is $r = \sqrt{\frac{9\eta v_1}{2g(\rho-\rho_{air})}}$, where the density of the oil droplet is ρ and the density of air is ρ_{air}. g is the gravitational acceleration.
(b) Calculate the charge of the droplet. Milliken repeated this measurement for a large number of observed droplets and found the charge to be multiples of a single number, the fundamental electric charge. Therefore, the experiment confirmed that charge is quantized.
Reference:
(a) the gravitational force on the droplet is $F_g = \rho V = \frac{4}{3}\rho\pi r^3$
the buoyancy on the droplet is $F_b = \rho_{air} V = \frac{4}{3}\rho_{air}\pi r^3$
the fraction on the droplet is $f = 6\pi r\eta v_1$
and from the equilibrium of force we have $F_g = F_b + f$
$\Rightarrow r = \sqrt{\frac{9\eta v_1}{2g(\rho-\rho_{air})}}$
(b) the fration on the droplet switches to $f^{'} = 6\pi r\eta v_2$

the Cloulomb's force on the droplet is $F_E = qE$ and from the equilibrium of force we have $F_g + f^{'} = F_b + F_E$
$\Rightarrow F_E = qE = f + f^{'} = 6\pi r\eta(v_1 + v_2)$
$\Rightarrow q = \frac{6\pi\eta}{E}(v_1 + v_2)\sqrt{\frac{9\eta v_1}{2g(\rho - \rho_{air})}}$

2

Show that the components of $\vec{E}$ due to a dipole are given, at a distant point P in the xz plane, by $E_x = \frac{1}{4\pi\epsilon_0}\frac{3pxz}{(x^2+z^2)^{5\backslash 2}}$ and $E_z = \frac{1}{4\pi\epsilon_0}\frac{\rho(2z^2-x^2)}{(x^2+z^2)^{5\backslash 2}}$ where x and z are coordinates of point P.
Reference:
with the center of the dipole as the origin, create a coordinate as the picture below



$\Rightarrow \vec{r_+} = x\vec{i} + (z - \frac{d}{2})\vec{k}$ and $\vec{r_-} = x\vec{i} + (z + \frac{d}{2})\vec{k}$

apply Cloulomb's Law and we can get that
$\vec{E_+} = \frac{q}{4\pi\epsilon_0} \cdot \frac{x}{(x^2+(z-d\backslash 2)^2)^{3\backslash 2}}\vec{i} + \frac{q}{4\pi\epsilon_0} \cdot \frac{z-d\backslash 2}{(x^2_(z-d\backslash 2)^2)^{3\backslash 2}}\vec{k}$ and
$\vec{E_-} = \frac{q}{4\pi\epsilon_0} \cdot \frac{x}{(x^2+(z+d\backslash 2)^2)^{3\backslash 2}}\vec{i} + \frac{q}{4\pi\epsilon_0} \cdot \frac{z+d\backslash 2}{(x^2_(z+d\backslash 2)^2)^{3\backslash 2}}\vec{k}$

from $\vec{E} = \vec{E_+} + \vec{E_-}$ we can get that
$E_x = \frac{qx}{4\pi\epsilon_0}[(x^2 + (z - \frac{d}{2})^2)^{-\frac{3}{2}} - (x^2 + (z + \frac{d}{2})^2)^{-\frac{3}{2}}]$
apply Taylor's Law we can get that
$(x^2 + (z - t)^2)^{-\frac{3}{2}} = (x^2 + z^2)^{-\frac{3}{2}} + 3zt(x^2 + z^2)^{-\frac{5}{2}}$
$\Rightarrow E_x = \frac{3qxzd}{4\pi\epsilon_0(x^2+z^2)^{-\frac{5}{2}}} = \frac{3pxz}{4\pi\epsilon_0}(x^2 + z^2)^{-\frac{5}{2}}$

also we can get that
$E_z = \frac{q}{4\pi\epsilon_0}[(z - \frac{d}{2})(x^2 + (z - \frac{d}{2}))^{-\frac{3}{2}} - (z + \frac{d}{2})(x^2 + (z + \frac{d}{2}))^{-\frac{3}{2}}]$

3

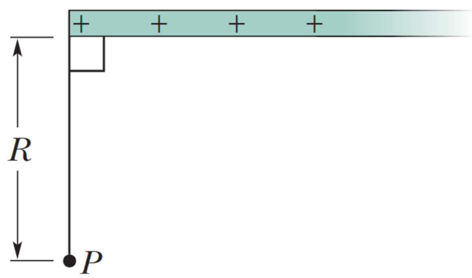
Suppose N electrons can be placed in either of two configurations. In configuration 1, they are all placed on the circumference of a narrow ring of radius R and are uniformly distributed so that the distance between adjacent electrons is the same everywhere. In configuration 2, $N-1$ electrons are uniformly distributed on the ring and one electron is placed in the center of the ring.
(a) What is the smallest value of N for which the second configuration is less energetic than the first?
(b) For that value of N, consider any one circumference electron - call it e_0. How many other circumference electrons are closer to e_0 than the central electron is?
Reference:
(a) in the first configuration,
if N is odd: $\theta = \frac{\pi}{N}$
for every two electrons with $\pi = \theta$: $E_\theta^{'} = \frac{N}{4\pi\epsilon_0} \cdot \frac{e^2}{2R\sin(k\theta)}$
there're N pairs of such electrons: $E_\theta = \frac{N}{4\pi\epsilon_0} \cdot \frac{e^2}{2R\sin(k\theta)}$
so in configuration 1, $E_1(N) = \frac{Ne^2}{8\pi\epsilon_0 R}\Sigma_{k=1}^{\frac{N-1}{2}} \frac{1}{\sin(k\theta)}$
similarly, if N is even, $E_1(N) = \frac{Ne^2}{8\pi\epsilon_0 R}\Sigma_{k=1}^{\frac{N-2}{2}} \frac{1}{\sin(k\theta)} + \frac{Ne^2}{16\pi\epsilon_0 R}$
configuration 2 can be seen as configuration 1 with one more central electron
$E_2(N) = E_1(N-1) + \frac{(N-1)e^2}{4\pi\epsilon_0 R}$
$\Rightarrow N = 12$
(b) when $N = 12$, $\theta = \frac{\pi}{11}$
solve the inquality $R < 2R\sin(k\theta) \Rightarrow k = 2$
so there're $(2-1) \times 2 = 2$ electrons

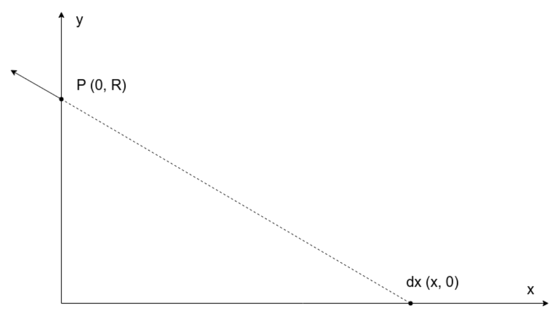
4

A semi-infinite nonconducting rod has uniform linear charge density λ. Show that the electric field $\vec{E_p}$ at point P makes an angle of $45°$ with the rod and that this result is independent of the distance R.

Reference:
build a coordinate system as the figure shows

consider the electric field generated by the line element at $(x, 0)$ of length dx at point P
$\vec{r} = -x\vec{i} + y\vec{j}$, $dq = \lambda dx \Rightarrow d\vec{E} = -\frac{\lambda dx}{4\pi\epsilon_0} \cdot \frac{x}{(x^2+R^2)^{3\backslash 2}}\vec{i} + \frac{\lambda dx}{4\pi\epsilon_0} \cdot \frac{R}{(x^2+R^2)^{3\backslash 2}}\vec{j}$
$\Rightarrow dE_x = -\frac{\lambda}{4\pi\epsilon_0} \cdot \frac{xdx}{(x^2+R^2)^{3\backslash 2}}$ and $dE_y = \frac{\lambda}{4\pi\epsilon_0} \cdot \frac{Rdx}{(x^2+R^2)^{3\backslash 2}}$
do integral and we'll get that $E_x = \int_0^{+\infty} dE_x = -\frac{\lambda}{4\pi\epsilon_0 R}$ and $E_y = \int_0^{+\infty} dE_y =$

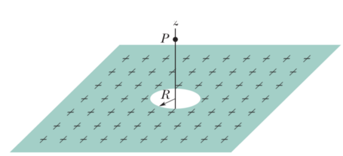




$\frac{\lambda}{4\pi\epsilon_0 R}$
so that $|E_x| = |E_y|$, the angle is $45°$

5

A small circular hole of radius $R = 1.80$ cm has been cut in the middle of an infinite, flat, nonconducting surface that has uniform charge density $\sigma = 4.50$ pC/m^2 A z axios, with its origin at the hole's center, is perpendicular to the surface. In unit vector notation, what is the electric field at point P at $z = 2.56$ cm?



Reference:
first consider the electric field generated by a loop of radius r, width dr with its center the same as the center of the hole at point P
$\Rightarrow dq = 2\pi\sigma r dr$

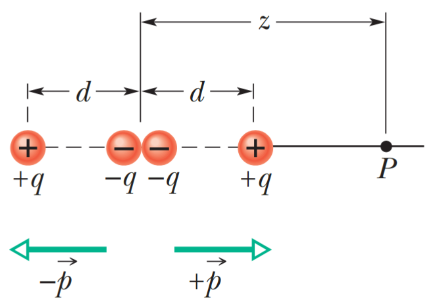
according to the symmetry of the loop, the electric field should be in the direction of the z axis

$\Rightarrow d\vec{E} = \frac{1}{4\pi\epsilon_0} \cdot \frac{h \cdot \vec{k}}{(r^2+h^2)^{3\backslash 2}} \cdot 2\pi\sigma r dr = -\vec{k} \cdot \frac{\sigma h}{2\epsilon_0} \cdot d(r^2+h^2)^{-1\backslash 2}$

$\Rightarrow \vec{E} = \int_R^{+\infty} d\vec{E} = \vec{k} \cdot \frac{\sigma h}{2\epsilon_0 \sqrt{R^2+h^2}} = 0.208 \text{ N/C} \cdot \vec{k}$

6

A type of electric quadrupole consists of two dipoles with dipole moments that are equal in magnitude but opposite in direction. Show that the value of E on the axis of the quadrupole for a point P at a distance z from its center (assume $z \gg d$) is given by $E = \frac{3Q}{4\pi\epsilon_0 z^4}$ in which $Q = 2qd^2$ is known as the quadrupole moment of the charge distribution.



Reference:

$\Rightarrow E_1 = \frac{qd}{2\pi\epsilon_0}(z-\frac{d}{2})^{-3}$ and $E_2 = -\frac{qd}{2\pi\epsilon_0}(z+\frac{d}{2})^{-3}$

from Taylor's Law we can get that $(z-t)^{-3} = z^{-3} + 3z^{-4}t$

$\Rightarrow E = E_1 - E_2 = \frac{qd}{2\pi\epsilon_0} \cdot 3dz^{-4} = \frac{3Q}{4\pi\epsilon_0 z^4}$

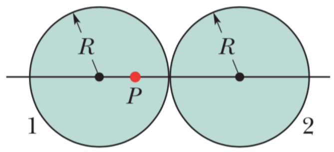
7

In cross section, two solid spheres with uniformly distributed charge throughout their volumes. Each has radius R. Point P lies on a line connecting the centers of the sphere 1. If the net electric field at point P is zero, what is the ratio $q_2 \backslash q_1$ fof the total charges?

Reference:

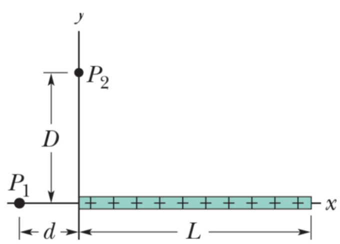
$\Rightarrow E_1 = \frac{q_1}{4\pi\epsilon_0}(\frac{R}{2})^{-2}$ and $E_2 = -\frac{q_2}{4\pi\epsilon_0}(\frac{3R}{2})^{-2}$

$E_1 + E_2 = 0 \Rightarrow q_2 \backslash q_1 = 9$



8

A thin plastic rod of length $L = 10.0$ cm has a nonuniform linear charge density $\lambda = cx$, where $c = 49.9$ pC/m^2.
(a) With $V = 0$ at infinity, find the electric potential at point P_2 on the y axis at $y = D = 3.56$ cm.
(b) Find the electric field component E_y at P_2.
(c) Why cannot the field component E_x at P_2 be found using the result of (a)?

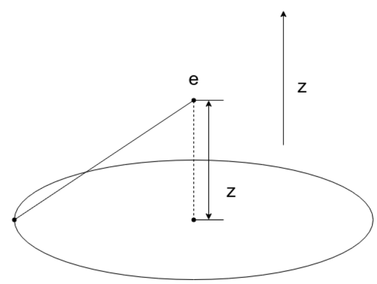


Reference:
(a) the electric charge of a line element of length dx is $dq = cxdx$
$\Rightarrow dV = \frac{1}{4\pi\epsilon_0} \cdot \frac{cxdx}{(x^2+D^2)^{1\setminus 2}} = \frac{c}{4\pi\epsilon_0} d(x^2 + D^2)^{\frac{1}{2}}$
$\Rightarrow$ the electric potential of at point P_2 is $\phi = \int_0^L dV = \frac{c}{4\pi\epsilon_0}(\sqrt{L^2 + D^2} - D) = 3.16 \times 10^{-2}$ V
(b) $\Rightarrow$ the electric field the line element generated at point P_2 is
$dE_y = \frac{1}{4\pi\epsilon_0} \cdot \frac{cxdx}{x^2+D^2} D(x^2 + D^2)^{-\frac{1}{2}} = -\frac{cD}{4\pi\epsilon_0} d(x^2 + D^2)^{-\frac{1}{2}}$
$\Rightarrow$ the electric field component $E_y = \int_0^L dE_y = \frac{cD}{4\pi\epsilon_0}(\frac{1}{D} - \frac{1}{\sqrt{L^2+D^2}}) = \frac{1}{\sqrt{L^2+D^2}} \cdot \frac{c}{4\pi\epsilon_0}(\sqrt{L^2 + D^2} - D) = \frac{\phi}{\sqrt{L^2+D^2}} = 0.298$ N/C
(c) $\Rightarrow dE_x = -\frac{1}{4\pi\epsilon_0} \cdot \frac{cxdx}{x^2+D^2} \cdot \frac{x}{\sqrt{x^2+D^2}}$,
this cannot be resolved to a function of $\sqrt{x^2 + D^2}$, so the field component cannot be found using the result of (a)

9

An electron is constrained to the central axis of the ring of charge of radius R, with $z \ll R$. Show that the electrostatic force on the electro can cause it

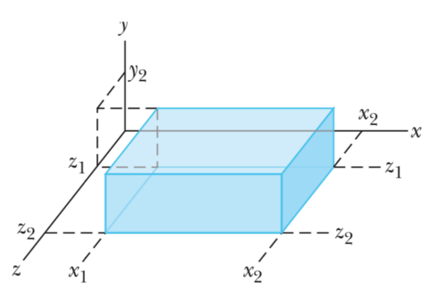
to oscilate through the ring center with an angular frequency $\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 m R^3}}$ where q is the ring's charge and m is the electron's mass.
Reference:
draw a figure to demonstrate the situation as below



when the electron is z away from the center, the Cloulomb force on the electron is $F_e = \frac{1}{4\pi\epsilon_0} \cdot eq \cdot \frac{-z}{(R^2+Z^2)^{3\backslash 2}}$
when $z \ll R$, $R^2 + z^2 \approx R^2 \Rightarrow F_e = -\frac{eq}{4\pi\epsilon_0 R^3} z$
this is a linear restoring force, so the electron will oscillate with an angular frequency $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{eq}{4\pi\epsilon_0 R^3 m}}$

10

A box-like Gaussian surface encloses a net charge of $+24.0\epsilon_0$ C and lies in an electric field given by $\vec{E} = [(10.0 + 2.00x)\hat{i} - 3.00\hat{j} + bz\hat{k}]$ N/C, with x and z in meters and b a constant. THe bottom face is in the xz plane; the top face is the horizontal plane passing through $y_2 = 1.00m$. For $x_1 = 1.00$ m, $x_2 = 4.00$ m, $z_1 = 1.00$ m, $z_2 = 3.00$ m, what is b?



Reference:
let Ω_1 be the area $z = z_1, x_1 \le x \le x_2, 0 \le y \le y_2$,

$\Rightarrow \iint_{\Omega_1} EdS = -\int_0^{y_2} dy \int_{x_1}^{x_2} bz_1 dx = -bz_1 y_2(x_2 - x_1)$

let Ω_2 be the area $z = z_2, x_1 \leq x \leq x_2, 0 \leq y \leq y_2$,

$\Rightarrow \iint_{\Omega_2} EdS \int_0^{y_2} dy \int_{x_1}^{x_2} bz_2 dx = bz_2 y_2(x_2 x_1)$

let Ω_3 be the area $y = y_2, x_1 \leq x \leq x_2, z_1 \leq z \leq z_2$,

$\Rightarrow \iint_{\Omega_3} EdS = -\int_{z_1}^{z_2} dz \int_{x_1}^{x_2} 3dx = -3(x_2 - x_1)(z_2 - z_1)$

let Ω_4 be the area $y = 0, x_1 \leq x \leq x_2, z_1 \leq z \leq z - 2$,

$\Rightarrow \iint_{\Omega_4} EdS = \int_{z_1}^{z_2} dz \int_{x_1}^{x_2} 3dx = 3(x_2 - x_1)(z_2 - z_1)$

let Ω_5 be the area $x = x_1, 0 \leq y \leq y_2, z_1 \leq z \leq z_2$,

$\Rightarrow \iint_{\Omega_5} EdS = -\int_{z_1}^{z_2} dz \int_{y_1}^{y_2} (10 + 2x_1) dy = -(10 + 2x_1) y_2(z_2 - z_1)$

let Ω_6 be the area $x = x_2, 0 \leq y \leq y_2, z_1 \leq z \leq z_2$,

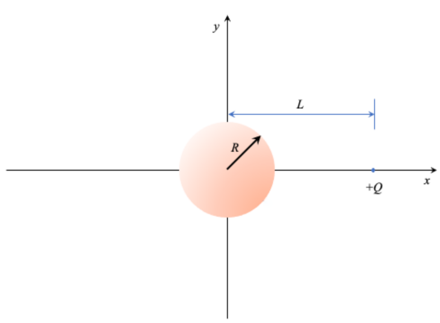
$\Rightarrow \iint_{\Omega_6} EdS = \int_{z_1}^{z_2} dz \int_{y_1}^{y_2} (10 + 2x_2) dy = (10 + 2x_2) y_2(z_2 - z_1)$

$\Rightarrow \oiint_{\partial V} EdS = \iint_{\Omega_1+\Omega_2+\Omega_3+\Omega_4+\Omega_5+\Omega_6} EdS = (b+2)(x_2 - x_1) y_2(z_2 - z_1) = \frac{q}{\epsilon_0}$

$\Rightarrow b = 2.00\ \mathrm{V/m^2}$

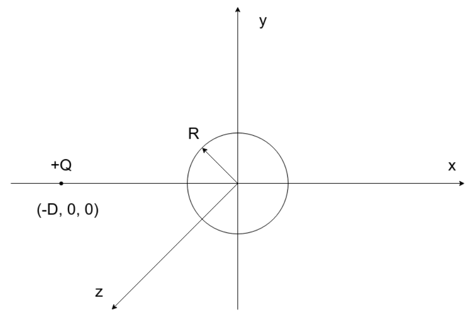
11 *

(You can try if you are interested.) Solve the electro-static potential function $V(x, y, z)$ and the electrostatic field $\mathbf{E}(x, y, z)$ in space, with a positive charge $+Q$ located at $(-D, 0, 0)$ and a conducting sphere of radius R (not grounded) at origin $(R < D)$.



Reference:
the figure is not compatible with the question, so redraw the figure as below

first consider the situation in which the sphere is grounded



to solve the question, we can use the method of image charges
we find an image charge $-q$, so that the electric field generated by $+Q$ and $-q$ are equal to that generated by the conducting sphere
the surface of the conducting sphere is a zero potential surface, suppose $-q$ is located at $(-a, 0, 0) \Rightarrow \frac{Q}{d_2} = \frac{q}{d_1}$, in which d_1, d_2 is the distance between $+Q$, $-q$ and a point on the surface of the sphere
$\Rightarrow \frac{Q}{q} = \frac{d_2}{d_1}$ is constant, according to Appollonius' Theorem, $a = \frac{R}{D}, \frac{Q}{q} = \frac{d_2}{d_1} = \frac{R}{a} = \frac{D}{R} \Rightarrow q = \frac{R}{D}Q$
then consider the situation in which the sphere is not grounded
the sphere is electircally neutral, the rest charge $+q$ is uniformly distributed on the surface of the sphere
so we introduce an image charge of $+q$, located at the origin
thus the potential is the sum of the potential generated by $+Q$, $-q$ and $+q$
when $(x, y, z) \neq (-D, 0, 0)$ and $x^2 + y^2 + z^2 \geq R^2$

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}(Q \cdot \frac{1}{((x+D)^2+y^2+z^2)^{1\backslash 2}} - \frac{R}{D}Q \cdot \frac{1}{((x+R^2\backslash D)^2+y^2+z^2)^{1\backslash 2}} + \frac{R}{D}Q \cdot \frac{1}{(x^2+y^2+z^2)^{1\backslash 2}})$$

$$E_x(x, y, z) = \frac{1}{4\pi\epsilon_0}(Q \cdot \frac{x+D}{((x+D)^2+y^2+z^2)^{3\backslash 2}} - \frac{R}{D}Q \cdot \frac{x+R^2\backslash D}{((x+R^2\backslash D)^2+y^2+z^2)^{3\backslash 2}} + \frac{R}{D}Q \cdot \frac{x}{(x^2+y^2+z^2)^{3\backslash 2}})$$

$$E_y(x, y, z) = \frac{Qy}{4\pi\epsilon_0}(\frac{1}{((x+D)^2+y^2+z^2)^{3\backslash 2}} - \frac{R}{D} \cdot \frac{1}{((x+R^2\backslash D)^2+y^2+z^2)^{3\backslash 2}} + \frac{R}{D} \cdot \frac{1}{(x^2+y^2+z^2)^{3\backslash 2}})$$

$$E_z(x, y, z) = \frac{Qz}{4\pi\epsilon_0}(\frac{1}{((x+D)^2+y^2+z^2)^{3\backslash 2}} - \frac{R}{D} \cdot \frac{1}{((x+R^2\backslash D)^2+y^2+z^2)^{3\backslash 2}} + \frac{R}{D} \cdot \frac{1}{(x^2+y^2+z^2)^{3\backslash 2}})$$

$$E(x, y, z) = E_x(x, y, z) \cdot \vec{i} + E_y(x, y, z) \cdot \vec{j} + E_z(x, y, z) \cdot \vec{k}$$

when $x^2 + y^2 + z^2 < R^2$, the sphere is equipotential
$\Rightarrow E(x, y, z) = \vec{0}, V(x, y, z) = \frac{Q}{4\pi\epsilon_0 D}$