Mathematics 327
Orthogonal bases

In this activity, it will be really helpful to remember that distributive property of dot products:

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\cdot\mathbf{w} = c_1\mathbf{v}_1\cdot\mathbf{w} + c_2\mathbf{v}_2\cdot\mathbf{w}.$$

1. Consider the vectors

$$\mathbf{v}_1 = \left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right], \qquad \mathbf{v}_2 = \left[\begin{array}{r} 1 \\ 1 \\ 0 \end{array}\right], \qquad \mathbf{v}_3 = \left[\begin{array}{r} 1 \\ -1 \\ -2 \end{array}\right].$$

Check that they form an orthogonal basis of $\mathbb{R}^3$.

Suppose that $\mathbf{b} = \left[\begin{array}{r} 2 \\ 4 \\ -4 \end{array}\right]$ and that we would like to find weights c_1, c_2, and c_3 so that

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

Of course, we could do that by forming the augmented matrix $\left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b} \end{array}\right]$ and row reducing. Instead, take the dot product of both sides of the equation above with $\mathbf{v}_1$:

$$\mathbf{b}\cdot\mathbf{v}_1 = c_1\mathbf{v}_1\cdot\mathbf{v}_1 + c_2\mathbf{v}_2\cdot\mathbf{v}_1 + c_3\mathbf{v}_3\cdot\mathbf{v}_1.$$

What does this give for the coefficient c_1?

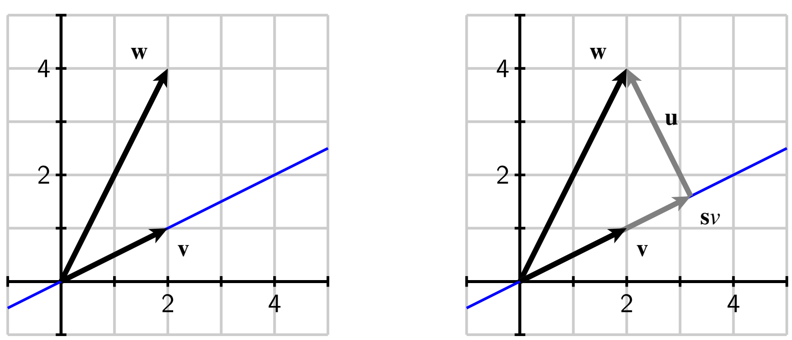
Find the coefficients c_2 and c_3 in the same way.

Now check that $\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

2. Suppose that $\mathbf{b}$ is some other vector. Explain why

$$\mathbf{b} = \frac{\mathbf{b}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{b}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{b}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3.$$

3. Shown below on the left are two vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ together with the line L defined by $\mathbf{v}$. We would like to find the vector on L that is closest to $\mathbf{w}$. Because this vector will be a scalar multiple of $\mathbf{v}$, we call it $s\mathbf{v}$ as seen on the right below.



The closest point $s\mathbf{v}$ to $\mathbf{w}$ on L is found by noting that $\mathbf{u} = \mathbf{w} - s\mathbf{v}$ is perpendicular to $\mathbf{v}$. What does this mean about $\mathbf{u}\cdot\mathbf{v}$?

Use this fact and the fact that $\mathbf{u} = \mathbf{w} - s\mathbf{v}$ to find the scalar s.

What is the vector $s\mathbf{v}$? We call this vector the *projection of* $\mathbf{w}$ *onto the line* L.

Suppose you have general vectors $\mathbf{v}$ and $\mathbf{w}$. Find the expression for the orthogonal projection of $\mathbf{w}$ onto L.

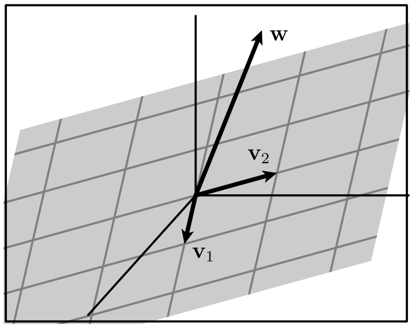
Under what conditions is the projection $\mathbf{0}$?

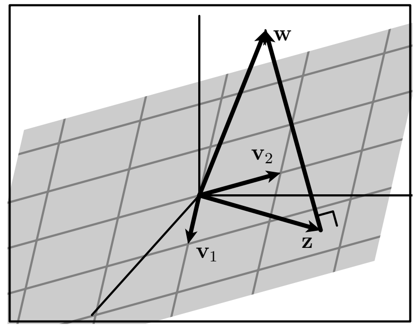
4. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

which define a plane P in $\mathbb{R}^3$. Check that $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthogonal basis for P.

Suppose that $\mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ is another vector as shown on the left below and suppose that we want to find $\mathbf{z}$, the vector in the plane closest to $\mathbf{w}$, as shown on the right. We call $\mathbf{z}$ the orthogonal projection of $\mathbf{w}$ onto P.





Because $\mathbf{z}$ is in the plane, we can write it as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{z} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2.$$

Because $\mathbf{z}$ is closest to $\mathbf{w}$, we know that

$$\mathbf{w} - \mathbf{z} = \mathbf{w} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2$$

is orthogonal to the plane P and hence $\mathbf{v}_1$ and $\mathbf{v}_2$. Use this fact to find both c_1 and c_2.

If $\mathbf{w}$ is another vector in $\mathbb{R}^3$, find the expression for the orthogonal projection of $\mathbf{w}$ onto $\mathbf{z}$.

5. If $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, find a vector $\mathbf{u}$ that is parallel to $\mathbf{v}$ and whose length is one. To get started, you can write $\mathbf{u} = s\mathbf{v}$ for some scalar. What should the scalar be?