Mathematics 227
Some more examples of subspaces

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix A is 3×3 so both $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$ will be subspaces of $\mathbb{R}^3$.

Solving the homogeneous equation, we have

$$\begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3. \end{aligned}$$

This means that the null space $\mathrm{Nul}(A)$, which is the solution space to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the vector $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ forms a basis for $\mathrm{Nul}(A)$, which is 1-dimensional. This agrees with our work in class where we saw that the dimension of the null space equals the number of columns minus the number of pivots. The null space $\mathrm{Nul}(A)$ is just a line in $\mathbb{R}^3$, which is a 1-dimensional subspace of $\mathbb{R}^3$.

For $\mathrm{Col}(A)$, denote the columns of A by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. The reduced row echelon form of A shows that

$$\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2.$$

Therefore, any linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In addition, the reduced row echelon form shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and therefore form a basis for $\mathrm{Col}(A)$. Therefore, $\mathrm{Col}(A)$ is a 2-dimensional subspace of $\mathbb{R}^3$, which you can think of as a plane in $\mathbb{R}^3$.

2. For another example, consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 2 & 3 \\ 1 & 0 & 0 & 2 \\ -2 & 2 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that there are two pivots so that the rank of the matrix A is two: $\mathrm{rank}(A) = 2$. Since A has four columns, the null space $\mathrm{Nul}(A)$ will be a subspace of $\mathbb{R}^4$ having dimension $4 - 2 = 2$.

To find a basis, write the solution space to the homogeneous equation as

$$\begin{aligned} x_1 &= -2x_4 \\ x_2 &= 2x_3 - x_4, \end{aligned}$$

which means that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that

$$\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for Nul(A), which agrees with our finding that the null space should be two-dimensional.

For the column space, denote the columns of A by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and notice that

$$\begin{aligned} \mathbf{v}_3 &= -2\mathbf{v}_1 \\ \mathbf{v}_4 &= 2\mathbf{v} + \mathbf{v}_2. \end{aligned}$$

This shows that $\mathbf{v}_3$ and $\mathbf{v}_4$ can be written as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ and therefore $\mathbf{v}_1$ and $\mathbf{v}_2$ span Col(A). They are also linearly independent so they form a basis for Col(A), which is a two-dimensional subspace of $\mathbb{R}^3$ (which would be a plane in $\mathbb{R}^3$).

3. In general, the rank of a matrix is defined to be the number of pivot positions. The dimension of the column space Col(A) equals the rank and a basis is found from the columns that A that have pivots. The dimension of Nul(A) equals the number of columns minus the rank; a basis is found by solving the homogeneous equation $A\mathbf{x} = \mathbf{0}$.