Mathematics 227
Subspaces

Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & -4 & -6 & 0 \\ -4 & -1 & 7 & 11 & 2 \\ 0 & -1 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Give a parametric description of the null space $\text{Nul}(A)$, the solution space to the equation $A\mathbf{x} = \mathbf{0}$.

   Using your parametric description, state a set of vectors that span $\text{Nul}(A)$.

   Check that your set of vectors is also linearly independent.

   A *basis* for a subspace is a set of vectors that span the subspace and that are linearly independent. State a basis for $\text{Nul}(A)$.

The null space $\mathrm{Nul}(A)$ is a subpsace of $\mathbb{R}^p$ for which p? How is this number related to the dimensions of the matrix A?

The *dimension* of a subspace equals the number of vectors in a basis for that subspace. What is the dimension of $\mathrm{Nul}(A)$? How is this number related to the dimensions of A and the number of pivot positions?

Suppose you have a 10×15 matrix B that has 7 pivot positions. Complete the sentence: $\mathrm{Nul}(B)$ is a ____________-dimensional subspace of $\mathbb{R}^p$ for $p =$____________.

2. Suppose that we denote the columns of A by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$. The column space $\mathrm{Col}(A)$ is the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$.

   Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5$ linearly independent? Do they form a basis for $\mathrm{Col}(A)$?

   Explain how to write $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{v}_5$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   Then explain why $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_5\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ are linearly independent?

State a basis for $\mathrm{Col}(A)$.

The column space $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^p$ for what p? How is this number related to the dimensions of the matrix A?

What is the dimension of $\mathrm{Col}(A)$? How is this number related to the dimensions of the A and the number of pivot positions.

Suppose you have a 10×15 matrix B that has 7 pivot positions. Complete the sentence: $\mathrm{Col}(B)$ is a ____________-dimensional subspace of $\mathbb{R}^p$ for $p =$____________.

3. Suppose that

$$A = \begin{bmatrix} 2 & 0 & -2 & -4 \\ -2 & -1 & 1 & 2 \\ 0 & -1 & -1 & -2 \end{bmatrix}.$$

(a) Is the vector $\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ in $\mathrm{Col}(A)$?

(b) Is the vector $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ in Col(A)?

(c) Is the vector $\begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$ in Nul(A)?

(d) Is the vector $\begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$ in Nul(A)?

(e) Is the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ in Nul(A)?