Mathematics 227
Invertible matrices

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix}.$$

Explain why the matrix A is invertible.

Find the inverse A^{-1} by row reducing $\left[\begin{array}{c|c} A & I \end{array}\right]$.

Note: In Sage, you may create the 3×3 identity matrix by `identity_matrix(3)`.
You may augment A using `A.augment(identity_matrix(3))`

Use the inverse A^{-1} to solve the equation

$$A\mathbf{x} = \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}.$$

Note: In Sage, you may find the inverse of a matrix `B` using `B.inverse()`.

2. Suppose that A is an invertible 5×5 matrix. What can you guarantee about the span of the columns of A?

What can you guarantee about the linear independence of the columns of A?

3. If A is a 7×7 matrix whose columns are linearly independent, can you guarantee that A is invertible? Explain your thinking.

4. Suppose that both A and B are invertible $n \times n$ matrices. Explain why $(AB)^{-1} = B^{-1}A^{-1}$ by multiplying this matrix by AB.

   Explain why the product of two invertible $n \times n$ matrices is invertible.

5. We will now look at a special type of matrix having the form

$$L = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}, \qquad U = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}.$$

   The matrix L is called *lower triangular* because all the entries above the diagonal are zero; in the same way, U is called *upper triangular*.

   Consider the following two lower triangular matrices

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}, \qquad L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 1 \end{bmatrix}.$$

Imagine where the pivot positions in these matrices occur. Which one of them is invertible and which is not invertible?

What condition on the diagonal entries of a triangular matrix determines whether the matrix is invertible or not?

6. We will now revisit Gaussian elimination, the algorithm we learned at the beginning of the class that we have been using to row reduce matrices.

   Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & -1 \end{bmatrix}$. Perform a row operation to make the first entry in the second row zero.

   If $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, verify that $L_1 A$ is the same matrix you obtained from this row operation.

   Explain how you know that L_1 is invertible.

Find L_1^{-1} and explain what row operation it performs.

Suppose that $L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find L_2A and explain what row operation it performs.

What is L_2^{-1} and explain what row operation it performs.

Suppose that $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find PA and explain what row operation it performs.

What is P^{-1} and explain what row operation it performs.