Mathematics 327
Singular value decomposition

1. Begin with the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$. Can A be diagonalized?

   Form the Gram matrix $A^T A$.

   Find an orthogonal diagonalization of the Gram matrix to find eigenvalues λ_1 and λ_2 and eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

   Find the singular values and the right singular vectors of A. Remember that $\sigma_1 \geq \sigma_2$.

   Now define *left singular vectors* $\mathbf{u}_1$ and $\mathbf{u}_2$ to be unit vectors so that $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$.

   What is the angle between the left singular vectors?

Form the matrices

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \qquad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}.$$

Verify that $U\Sigma V^T = A$. This is called the *singular value decomposition* of A.

2. Suppose that σ_i and $\mathbf{v}_i$ are the singular values and right singular vectors of A. Suppose that the vectors $\mathbf{u}_i$ are unit vectors such that $A\mathbf{v}_i = \sigma_i\mathbf{u}_i$. Notice that $A\mathbf{v}_i \cdot A\mathbf{v}_j = (A\mathbf{v}_i)^T(A\mathbf{v}_j)$. Explain why $\mathbf{u}_i$ and $\mathbf{u}_j$ are orthogonal if $i \neq j$ and the singular values are nonzero.

3. Define the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Find the singular values and right singular vectors of A.

Find the left singular vectors of A.

Construct the matrix Σ to have the same dimensions (2×3) as A and verify that A has the singular value decomposition $A = U\Sigma V^T$.

4. Given the singular value decomposition $A = U\Sigma V^T$, what is the singular value decomoposition of A^T?

5. Find the singular value decomposition of the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

6. Suppose that A is symmetric so that $A^T = A$. We know that A is orthogonally diagonalizable; that is, A has eigenvalues λ_i and unit eigenvectors $\mathbf{v}_i$ so that $A = QDQ^T$ where

$$Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

Explain why the Gram matrix of A is A^2.

Explain why the singular values of A are $|\lambda_i|$ and the right singular vectors of A are $\mathbf{v}_i$.

How are the left singular vectors of A related to $\mathbf{v}_i$?