Mathematics 327
2^{nd} review, part I

1. Suppose that U is an $n \times n$ matrix whose columns form an orthonormal basis for $\mathbb{R}^n$. Such matrices are often called *orthogonal*. Explain why $U^T U = I$.

   Remembering that $|\mathbf{y}|^2 = \mathbf{y}^T\mathbf{y}$, explain why $|U\mathbf{x}| = |\mathbf{x}|$.

2. Consider the matrix $A = \begin{bmatrix} -6 & 5 \\ -10 & 9 \end{bmatrix}$. Find the eigenvalues of A.

   Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors of A?

   Can A be diagonalized; that is, can we write $A = PDP^{-1}$ where D is a diagonal matrix? If so, give an example of an appropriate D and P.

   Can A be orthogonally diagonalized; that is, can we write $A = QDQ^T$ where Q is an orthogonal matrix? Explain your thinking.

3. Consider the matrix $B = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$. Find the eigenvalues of B.

Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors of B?

Can B be diagonalized?

Can B be orthogonally diagonalized?

4. Consider the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ where

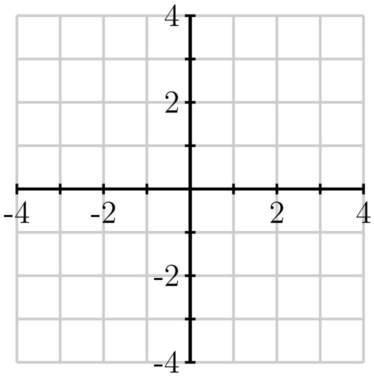
$$Q\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = x_1^2 + 4x_1x_2 - 2x_2^2.$$

Find a matrix A such that $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

If we restrcit x to lie on the unit circle—that is, $|\mathbf{x}| = 1$—what is the maximum value of $Q(\mathbf{x})$? In what direction does it occur?

If we restrcit $\mathbf{x}$ to lie on the unit circle—that is, $|\mathbf{x}| = 1$—what is the minimum value of $Q(\mathbf{x})$? In what direction does it occur?

5. Consider the points $\mathbf{x}_1 = (3, 1)$, $\mathbf{x}_2 = (1, 4)$, $\mathbf{x}_3 = (-1, 2)$, $\mathbf{x}_4 = (1, 5)$. Find de-meaned data points $\widetilde{\mathbf{x}}_i$ and plot them below.



Write the quadratic form $Q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ that expresses the variance in the direction defined by $\mathbf{x}$.

Find the direction in which the variance is greatest? What is the variance in this direction?

If $\mathbf{u}$ is the direction in which the variance is greatest, explain how to write the projection of a de-meaned data point $\widetilde{\mathbf{x}}_i$ as $s_i\mathbf{u}$.

How do the scalars s_i relate to the variance in the direction $\mathbf{u}$?

What is the residual in this direction?

6. Suppose that $A = \begin{bmatrix} \widetilde{\mathbf{x}}_1 & \widetilde{\mathbf{x}}_2 & \dots & \widetilde{\mathbf{x}}_{10} \end{bmatrix}$ is a matrix whose columns are de-meaned data points in $\mathbb{R}^8$. Explain why to find the covariance matrix C. What are the dimensions of C?

   Suppose that the eigenvalues of C are, in decreasing order, $\lambda_1 = 103.2, \lambda_2 = 92.1, \lambda_3 = 2.1, \dots, \lambda_8 = 0.2$ with associated eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_8$. Explain a reasonable strategy for performing principal component analysis.