Orthogonality_and_Sage

May 26, 2019

1 Lab 2: Orthogonality with Sage

This lab will be due on **Wednesday, February 13**. You may save it as a pdf and submit it in class. Please double click here and enter the names of everyone in your group.

Names:

We've been looking at some concepts like orthogonal and orthonormal bases and orthogonal projections. I'd like for us to go over some of these concepts again using Sage to facilitate the computations.

First, remember the projection formula: if $\mathbf{b}$ is a vector in $\mathbf{R}^p$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for an n-dimensional subspace V of $\mathbf{R}^p$, then the orthogonal projection of $\mathbf{b}$ onto V is

$$\frac{\mathbf{b}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{b}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \ldots + \frac{\mathbf{b}\cdot\mathbf{v}_n}{\mathbf{v}_n\cdot\mathbf{v}_n}\mathbf{v}_n.$$

The following cell will define some important functions for you. There are a couple of things for you to know. First of all, we can bundle things together in a list by enclosing them in square brackets [thing1, thing2, ..., lastthing]. In particular, we can put vectors into a list [v1, v2, v3] to represent a basis.

The first function defined below is projection. If basis is a list of *orthogonal* vectors, then projection(b, basis) will project b onto the subspace defined by the vectors in basis.

Second, the function unit(v) will return the unit vector defined by the vector v.

Finally, the function vectors2matrix(vectors) forms the matrix whose columns are in the list defined by vectors.

Evaluate the cell below.

```
In [1]: def projection(b, basis):
            return sum([b.dot_product(v)/v.dot_product(v)*v for v in basis])

        def unit(v):
            return v/v.norm()

        def vectors2matrix(vectors):
            return matrix(vectors).transpose()
```

Problem 1 The vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ form a basis for the plane V in $\mathbf{R}^3$. Define them below and verify they are orthogonal.

```
In [0]:
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Find the vector $\mathbf{z}$ that is the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$ onto V.

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In [0]:
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Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ that form an orthonormal basis for V.

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In [0]:
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Form the matrix Q whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$.

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In [0]:
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Now find the matrix $P = QQ^T$ that orthogonally projects vectors in $\mathbf{R}^3$ onto V.

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In [0]:
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Multiply P by $\mathbf{b}$ and verify that you obtain $\mathbf{z}$, the orthogonal projection of $\mathbf{b}$ onto V that you found above.

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In [0]:
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Find the product Q^TQ and explain the result.

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In [0]:
```

Can you explain this result? Double click here and enter your thoughts:

Problem 2 Define the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -5 \\ -3 \\ 5 \end{bmatrix}$. Apply Gram-Schmidt orthogonalization to find an orthogonal basis $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$.

```
In [22]:
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Beginning with the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, form an orthonormal basis $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

```
In [23]:
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Form the orthogonal matrix Q whose columns are the orthonormal basis vectors, and verify that $QQ^T = Q^TQ = I$.

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In [0]:
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Problem 3 The following cell defines a function gs that implements the Gram-Schmidt orthogonalization algorithm so that gs(basis) forms an orthonormal basis from a given basis. Be sure to evaluate this cell.

```
In [30]: def gs(basis):
             onbasis = []
             for b in basis:
                 if len(onbasis) == 0: onbasis.append(b)
                 else: onbasis.append(b-projection(b, onbasis))
             return map(unit, onbasis)
```

Suppose that V is a 2-dimensional subspace of $\mathbf{R}^4$ with basis

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 2 \end{bmatrix}.$$

Find an orthonormal basis for V.

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In [0]:
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Find the matrix P that projects vectors orthogonally onto V.

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In [0]:
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Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} 4 \\ 10 \\ -6 \\ 2 \end{bmatrix}$ onto V using P.

```
In [0]:
```

Explain why $\mathbf{b}$ should be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and then express $\mathbf{b}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

```
In [0]:
```

Suppose that A is the matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$; that is, $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. Find the matrix $R = Q^T A$. What is special about R?

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In [0]:
```

Verify that $A = QR$. This will be another important matrix factorization, called the QR-factorization of A, that we will work with.

```
In [0]:
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Problem 4 If A is an $m \times n$ matrix with linearly independent columns, then we can write $A = QR$ where the columns of Q are orthonormal and R is upper triangular. We will see why this is true in our next class period. For now, define vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 5 \end{bmatrix}.$$

These vectors are linearly independent so they form a basis for a 3-dimensional subspace V of $\mathbf{R}^4$. Let A the matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Use Gram-Schmidt orthogonalization to find an orthonormal basis $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ of V.

In [0]:

Define Q to be the matrix whose columns are $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. Then define $R = Q^T A$.

In [0]:

Verify once again that $A = QR$.

In [0]: