Mathematics 327
Lab 5: Due Monday, April 8
SVD and the Supreme Court

As we've seen, the important information contained in a matrix is concentrated in the data related to the large singular values. Today we'll look at some Supreme Court decisions and see how singular value decompositions help us extract meaning from large matrices.

The makeup of the Supreme Court was unusually stable during a period from 1994-2005 when it was led by Chief Justice William Rehnquist. This is sometimes called the *second Rehnquist court*. The justices during this period were:

- William Rehnquist
- Antonin Scalia
- Clarence Thomas
- Anthony Kennedy
- Sandra Day O'Connor
- John Paul Stevens
- David Souter
- Ruth Bader Ginsburg
- Stephen Breyer

During this time, the court ruled on 911 cases in which all nine judges voted. We'd like to understand patterns in these votes.

There is a Jupyter notebook named `SVD_Supreme_Court` posted on `cocalc` that you will need to access. The first cell provides some useful functions:

- `display_column(A, k)` gives a visual display of the k^{th} column of matrix A. White corresponds to relatively large positive values, black corresponds to 0, and red corresponds to relatively small negative values.
- `display_matrix(A)` gives a visual display of matrix A.
- `plotSV(A)` creates a plot of the singular values of A.

1. Read in the file `cases.csv` to define the matrix A, which is a 9×911 matrix. Each justice corresponds to a row, in the order given above, and each of the 911 cases corresponds to a column. An entry in the matrix is 1 if the justice votes with the majority and -1 if the justice votes with the minority.

   You might want to view the first 50 columns of this matrix to get a sense of what the data looks like. What does a column in which every entry is 1 mean?

Plot the singular values of A. What is $\mathrm{rank}(A)$?

If we want to approximate A using the rank k sum of outer products

$$A_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \ldots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T,$$

what seems like a reasonable value of k? Explain your thinking.

Use Sage to form the singular value decomposition.

Describe the results of the second case (which has index `1`). You may want to do this visually: `display_column(A, 1)`.

Describe the second case as a linear combination of left singular vectors $\mathbf{u}_i$. In particular, which of the vectors is most important?

View the first left singular vector $\mathbf{u}_1$. It may help to remember that this vector will be scalar multiplied to form the columns of A. Therefore, the fact that the components of this vector are negative isn't important; instead, we should focus on the relative signs of the components. What type of case is this left singular vector describing? Explain your thinking.

Now view the 48^{th} case, whose column in A has index `47`. Describe the vote in this case.

Describe this case as a linear combination of left singular vectors $\mathbf{u}_i$. Which of the left singular vectors is most important?

View this left singular vector and explain the type of vote represented by this vector.

The following table gives the number of cases, out of 911, decided by each of the five possible votes.

Vote	Number of cases
9-0	405
8-1	89
7-2	111
6-3	118
5-4	188

Explain how this table is consistent with your findings for $\mathbf{u}_1$ and $\mathbf{u}_2$.

2. We will now focus on the 5-4 votes, which are probably the most controversial since they reflect a sharp divide among the justices. Define the matrix B by reading in the file `fivefour.csv`. This matrix has dimensions 9×188 and is similar to the matrix A we considered above except that it only has cases decided by a 5-4 vote.

Look at the first left singular vector $\mathbf{u}_1$. The justices are roughly listed in order from conservative to progressive. Would you characterize the second Rehnquist court as conservative or progressive? Explain your thinking.

We would like to know if there is a "swing vote" on the court; that is, is there a single justice whose vote determines most of the 5-4 decisions? This would be the case if there are two blocks of four justices that usually vote together with a ninth justice whose sometimes sides with one block and sometimes with the other.

Look at the seventh case in the matrix B (indexed by `6` in the matrix) and describe the vote.

Determine how this vote is expressed as a linear combination of left singular vectors. What do you notice here?

Now look at the sixth case in the matrix B (indexed by `5` in the matrix) and describe the vote.

Determine how this vote is expressed as a linear combination of left singular vectors. What do you notice?

Describe the votes represented by the first two left singular vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. What does this say about the presence of a swing vote on the second Rehnquist court?

Suppose that $\mathbf{w}$ is the 188-dimensional vector whose entries are all 1. You can construct $\mathbf{w}$ in Sage as `vector([1]*188)`. Explain the meaning of the product $B\mathbf{w}$.

Use the vector $B\mathbf{w}$ to determine which of the nine judges most frequently vote with the majority. This would be another indication of a swing vote.

Who was the swing vote on the second Rehnquist court?

How does the vector $B\mathbf{w}$ also corroborate your response to the question of whether this court was a conservative or progressive court?

Look at the vectors $\mathbf{u}_3$ and $B\mathbf{w}$ and explain why there was a second, though less prominent, swing vote.