Mathematics 327
Transpose matrices and orthogonality

1. **Why do we care about the matrix transpose?**

   Suppose we have vectors

   $$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}.$$

   Find the dot products $\mathbf{v}_1 \cdot \mathbf{w}$ and $\mathbf{v}_2 \cdot \mathbf{w}$.

   We may view $\mathbf{v}_1$ as a 3×1 matrix. Find $\mathbf{v}_1^T$ and then compute $\mathbf{v}_1^T\mathbf{w}$.

   Notice that $\mathbf{v}_1^T\mathbf{w}$ is a 1×1 matrix whose sole entry is the dot product $\mathbf{v}_1 \cdot \mathbf{w}$. For this reason, we will often write $\mathbf{v}_1^T\mathbf{w} = \mathbf{v}_1 \cdot \mathbf{w}$.

   Now write the matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ and its transpose A^T.

   Find the product $A^T\mathbf{w}$ and describe how this product computes both dot products $\mathbf{v}_1 \cdot \mathbf{w}$ and $\mathbf{v}_2 \cdot \mathbf{w}$.

   Suppose that $\mathbf{x}$ is a vector that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. What does this say about $\mathbf{v}_1 \cdot \mathbf{v}_1$ and $\mathbf{v}_2 \cdot \mathbf{x}$?

Remember that dot products satisfy a distributive property:

$$\mathbf{x} \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\mathbf{x} \cdot \mathbf{v}_1 + c_2\mathbf{x} \cdot \mathbf{v}_2.$$

If $\mathbf{x}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, explain why $\mathbf{x}$ is orthogonal to any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

If $\mathbf{x}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, what does this say about $A^T\mathbf{x}$?

Give a parametric description of all vectors $\mathbf{x}$ that are orthogonal to all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$. How would you describe this set of vectors geometrically?

The set of all linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ is called the column space of A. It is, in this example, a two-dimensional subspace of $\mathbb{R}^3$. How would you describe this set of vectors geometrically?

If $\mathbf{w}$ is a vector in $\mathrm{Nul}(A^T)$ and $\mathbf{v}$ is a vector in $\mathrm{Col}(A)$, what is $\mathbf{v} \cdot \mathbf{w}$?

2. **Properties of the transpose:** In Sage, we can compute the transpose of a matrix `A` with `A.transpose()`. Suppose that

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & -2 & 1 \\ 3 & 2 & 0 \end{bmatrix}.$$

Sums: Find $(A + B)^T$ and $A^T + B^T$. What do you notice about the relationship between these two?

Products: Find the product AC and its transpose $(AC)^T$.

Is it possible to compute the product $A^T C^T$? Explain why or why not.

Find the product $C^T A^T$ and compare to $(AC)^T$. What do you notice?

Determinants: Find $\det(C)$ and $\det(C^T)$. What do you notice?

Inverses: We know that $C^{-1}C = I$. Explain why $(C^T)^{-1} = (C^{-1})^T$.

3. Suppose that A is a matrix that is not necessarily invertible or even square and suppose that $A = BC$ where C is invertible.

   Suppose that $\mathbf{b}$ is some vector. If $A\mathbf{x} = \mathbf{b}$ is consistent and hence $BC\mathbf{x} = \mathbf{b}$ is consistent, explain why $B\mathbf{y} = \mathbf{b}$ is consistent. To do this, you may want to express $\mathbf{y}$ in terms of $\mathbf{x}$.

   Notice that if $A = BC$, then $B = AC^{-1}$. If $B\mathbf{y} = \mathbf{b}$ is consistent, explain why $A\mathbf{x} = \mathbf{b}$ is consistent. To do this, you may want to express $\mathbf{x}$ in terms of $\mathbf{y}$.

   Explain how these two facts tell us that $\text{Col}(A) = \text{Col}(B)$.

   Explain how we know that $\text{rank}(A) = \text{rank}(B)$.