Mathematics 227
Determinants

1. Find the determinant of the matrix:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 4 & 0 \\ -2 & 4 & 1 \end{bmatrix}.$$

2. Find the determinant of the upper triangular matrix:

$$U = \begin{bmatrix} 3 & 1 & -3 \\ 0 & -2 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

How can you quickly find the determinant of an upper triangular matrix?

What is the determinant of the identity matrix?

3. Sage can easily find the determinant of a square matrix `A` with either `A.det()` or `A.determinant()`. Find the determinant of the matrix

$$B = \begin{bmatrix} -1 & 4 & 0 \\ 6 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}.$$

4. We are interested in understanding a connection between the determinant of a matrix and the invertibility of that matrix. To understand this connection, we will study the effect of the three row operations (interchange, scaling, and row replacement) on determinants.

   Interchange: Take the matrix B above and interchange any two rows to obtain a matrix B_1. Compute $\det(B_1)$. How does it compare to $\det(B)$? This is what generally happens.

   If a matrix A has a nonzero determinant and we interchange two rows, explain why the determinant of the new matrix is nonzero.

   Scaling: Now scale the first row of B by 3 to obtain the matrix B_2. Compute $\det(B_2)$ and compare it to $\det(B)$. This is also what happens generally.

   If a matrix A has a nonzero determinant and we scale a row by a nonzero number, explain why the determinant of the new matrix is nonzero.

Row replacement: Finally, perform a row replacement operation on B to obtain B_3. Compute $\det(B_3)$ and comapre it to $\det(B)$. This is also what generally happens.

If a matrix A has a nonzero determinant and we perform a row replacement operation, explain why the determinant of the new matrix is nonzero.

5. If you have a matrix A whose determinant is nonzero, what can you guarantee about the determinant of its row echelon form? Explain your thinking.

   Consider the following two 3×3 matrices, both of which are in reduced row echelon form:

$$R_1 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   Find $\det(R_1)$ and $\det(R_2)$ (you shouldn't need Sage to do this).

   If A is a 3×3 matrix with a nonzero determinant, which of these two matrices is possible as the reduced row echelon form of A? Explain your thinking.

   What can you guarantee about the reduced row echelon form of A?

6. If A a square matrix with a nonzero determinant, explain why A is invertible.