Mathematics 327
Orthogonal complements and projections

1. Suppose that L is the line in $\mathbb{R}^3$ defined by scalar multiples of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

   What kind of geometric object do you expect $L^\perp$ to be?

   Find a basis for $L^\perp$. You may wish to first form the 3×1 matrix A whose only column is $\mathbf{v}$. We then have $L = \text{Col}(A)$.

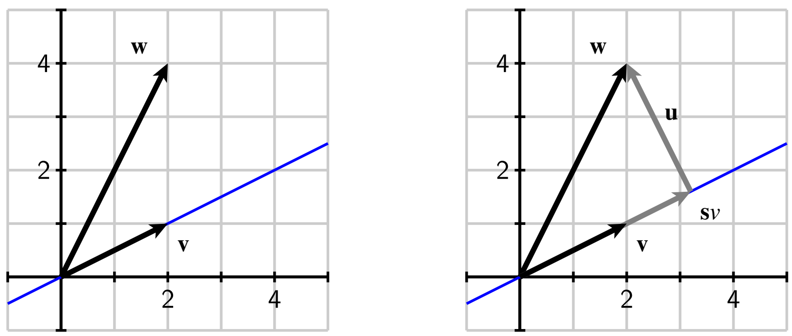
2. Suppose that W is the subspace of $\mathbb{R}^5$ having basis

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 0 \\ -6 \end{bmatrix}, \qquad \mathbf{w}_3 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -3 \\ 7 \end{bmatrix}.$$

   If we form the matrix A whose columns are $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, we then have $W = \text{Col}(A)$. Use this to find a basis for $W^\perp$.

   What are $\dim W$ and $\dim W^\perp$?

3. **Orthogonal projections:** Shown below on the left are two vectors $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ together with the line L defined by $\mathbf{v}$. We would like to find the vector on L that is closest to $\mathbf{w}$. Because this vector will be a scalar multiple of $\mathbf{v}$, we call it $s\mathbf{v}$ as seen on the right below.



The closest point $s\mathbf{v}$ to $\mathbf{w}$ on L is found by noting that $\mathbf{u} = \mathbf{w} - s\mathbf{v}$ is perpendicular to $\mathbf{v}$. What does this mean about $\mathbf{u} \cdot \mathbf{v}$?

Use this fact and the fact that $\mathbf{u} = \mathbf{w} - s\mathbf{v}$ to find the scalar s.

What is the vector $s\mathbf{v}$? We call this vector the *projection of* $\mathbf{w}$ *onto the line* L.

Suppose you have general vectors $\mathbf{v}$ and $\mathbf{w}$. Find the expression for the orthogonal projection of $\mathbf{w}$ onto L.

Under what conditions is the projection $\mathbf{0}$?