Mathematics 227
Bases

A set of vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ and is linearly independent is called a *basis* of $\mathbb{R}^n$.

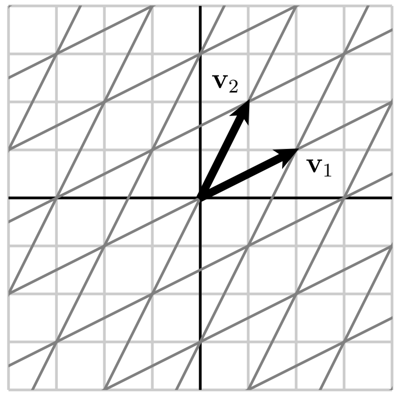
1. Explain why the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ form a basis for $\mathbb{R}^2$.

   Explain why the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for $\mathbb{R}^3$.

   How many vectors will be in a basis for $\mathbb{R}^{12}$? Explain your thinking.

   Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis for $\mathbb{R}^n$. Explain why every vector $\mathbf{b}$ in $\mathbb{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in exactly one way.

2. Suppose that $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The basis of $\mathbb{R}^2$ formed by $\mathbf{v}_1$ and $\mathbf{v}_2$ will be denoted by $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. We now know that every vector in $\mathbb{R}^2$ can be expressed in two different ways: in its usual form as a column vector and as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. We are going to use this to define a new coordinate system for $\mathbb{R}^2$.



Express the vector $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ as linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Express the vector $2\mathbf{v}_1 - \mathbf{v}_2$ in standard form.

If we have a vector $\mathbf{x}$ in $\mathbb{R}^2$, we can write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. We will use c_1 and c_2 as new coordinates for $\mathbf{x}$. For instance, $\begin{bmatrix} -3 \\ 0 \end{bmatrix} = -2\mathbf{v}_1 + \mathbf{v}_2$. In the coordinate system defined by $\mathbf{v}_1$ and $\mathbf{v}_2$, this vector has coordinates -2 and 1. We will write this as

$$\left\{\begin{bmatrix} -3 \\ 0 \end{bmatrix}\right\}_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

In general, if $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$$\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Express $\begin{bmatrix} -4 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ to find the coordinates $\left\{\begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}_{\mathcal{B}}$.

Find the vector $\mathbf{x}$ so that $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

3. Explain why the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

form a basis $\mathcal{B}$ for $\mathbb{R}^3$.

Find the vector $\mathbf{x}$ such that $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

Find the coordinates $\left\{ \begin{bmatrix} -2 \\ 1 \\ -8 \end{bmatrix} \right\}_{\mathcal{B}}$.

4. Suppose that $\mathcal{B}$ is the basis for $\mathbb{R}^2$ consisting of the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Let's form the matrix

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Explain why $A\{\mathbf{x}\}_{\mathcal{B}} = \mathbf{x}$.

Find a matrix B such that $B\mathbf{x} = \{\mathbf{x}\}_{\mathcal{B}}$.