Mathematics 327
Review

There is a page of Sage cells available at:

1. Use Gaussian elimination to find the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -2 \\ -4 & -2 & 1 \end{bmatrix}.$$

Use your LU factorization to solve the equation

$$A\mathbf{x} = \begin{bmatrix} 6 \\ -6 \\ -9 \end{bmatrix}$$

without using technology.

2. Suppose that A is an 8×2 matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$ and that $\mathbf{x}$ is an 8-dimensional vector. Suppose also that $A^T\mathbf{x} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. What are $\mathbf{v}_1 \cdot \mathbf{x}$ and $\mathbf{v}_2 \cdot \mathbf{x}$?

   What can you conclude if $A^T\mathbf{x} = \mathbf{0}$?

3. Suppose that Q is a 3×3 matrix whose columns, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, form an orthonormal basis for $\mathbb{R}^3$. What are the dimensions of Q^TQ? What is Q^TQ? Explain your thinking.

   What are the dimensions of QQ^T? What is QQ^T? Explain your thinking.

   Suppose that Q is a 3×2 whose columns $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for a plane V in $\mathbb{R}^3$. What are the dimensions of Q^TQ? What is Q^TQ? Explain your thinking.

   What are the dimensions of QQ^T? What is QQ^T? Explain your thinking.

4. Suppose that $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$ and that $\mathbf{b} = \begin{bmatrix} 10 \\ 8 \\ 2 \end{bmatrix}$. Write

$$\mathbf{b} = \mathbf{b}^{\perp} + \widehat{\mathbf{b}},$$

where $\mathbf{b}^{\perp}$ is orthogonal to $\mathrm{Col}(A)$ and $\widehat{\mathbf{b}}$ is in $\mathrm{Col}(A)$.

5. Consider the linearly independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ -4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -3 \\ -4 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 6 \\ -8 \\ 2 \\ 4 \end{bmatrix},$$

which span a three-dimensional subspace of $\mathbb{R}^4$. Find an orthonormal basis for V.

Find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$ onto V.

6. Suppose that we have data points

x_1	x_2	y
0	1	2
1	-1	3
2	1	4
3	3	7

Suppose we would like to fit the function

$$f(x_1, x_2) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 = y$$

to this data.

Set up a linear system $A\mathbf{x} = \mathbf{b}$ for the parameters $\mathbf{x} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$.

Find the least squares approximate solution $\widehat{\mathbf{x}}$ for this system.

If $x_1 = 4$ and $x_2 = 3$, what is your prediction for y?

7. Explain how to use a QR factorization to find the least squares approximate solution $\widehat{\mathbf{x}}$ to the over-determined linear system $A\mathbf{x} = \mathbf{b}$.

8. Find the best quadratic function $q(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ fitting the data

x	1	2	3	4
y	-1	0	-1	2

What is your prediction for $q(2.5)$?