Mathematics 327
Singular value decomposition

Go to `http://gvsu.edu/s/0YZ` for a page of Sage cells that will be helpful. In particular, you will have commands:

- `orthonormal(A)`, which returns the matrix that results from applying Gram-Schmidt to `A`.
- `shufflecolumns(A, list)`, which returns a matrix whose columns have been reordered. For instance, `shufflecolumns(A, [1,0])` will return a matrix whose columns are the second, then first, of `A`.

1. Begin with the matrix $A = \begin{bmatrix} 6 & 21 & 30 \\ 22 & 17 & -10 \end{bmatrix}$ and form the Gram matrix $G = A^T A$.

   Find the eigenvalues of G and the singular values of A. Remember that the singular values of A are ordered from largest to smallest.

   Use this to construct the matrix Σ that appears in the singular value decomposition of A.

   Find an orthonormal basis of eigenvectors of G. Use this to find the matrix V, which is an orthonormal basis of right singular vectors for A. Make sure that the columns of A are in the proper order so that the associated singular values are decreasing.

Find the matrix U consisting of left singular vectors.

Verify that the singular value decomposition $A = U\Sigma V^T$ holds.

2. If A has the singular value decomposition $A = U\Sigma V^T$, what is the singular value decomposition of A^T?

3. Let's now consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & -2 \\ 0 & 1 \end{bmatrix}$. What are the dimensions of U, Σ, and V?

Find the Gram matrix, and use it to find the singular values and right singular vectors of A.

How many left singular vectors does our usual process give?

As you can see, we are missing some left singular vectors. For this reason, we will use the observation from the previous problem and note that, if $A = U\Sigma V^T$, then $A^T = V\Sigma^T U^T$. That is, the left singular vectors of A are the right singular vectors of A^T and vice-versa.

Form A^T and its Gram matrix.

Find the singular values Σ^T and right singular vectors U of A^T.

Use this to find the left singular vectors V of A^T.

Now verify that $A = U\Sigma V^T$.