Mathematics 227
Review

1. (a) What does it mean for a set of vectors to be linearly dependent?

   (b) Is the following set of vectors linearly dependent?

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 5 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

   (c) If so, find one vector that is a linear combination of the others.

   (d) Explain why the columns of a matrix are linearly dependent if the matrix has a column without a pivot.

2. Suppose that people either live in urban or rural areas and that every year,

   - 97% of people living in urban areas remain in urban areas, and
   - 95% of people living in rural areas remain in rural areas.

   Let $\mathbf{x} = \begin{bmatrix} U \\ R \end{bmatrix}$ be the vector describing the fraction of people U living in urban areas and the fraction of people R living in rural areas in one year. The matrix transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ describes the fractions the next year.

Find a matrix A that represents the matrix transformation.

Suppose that in 2010, 60% of the population lived in urban areas and 40% in rural areas. How are the populations distributed in 2011.

How were the populations distributed in 2009?

How are the populations distributed in 2018?

3. Find the matrix that reflects vectors in the plane across the line $y = -x$.

Find the matrix that rotates vectors by 90° in the clockwise direction.

What geometric operation is performed if we first reflect in the line $y = -x$ and then rotate by 90° in the clockwise direction?

4. Suppose that A is an invertible $n \times n$ matrix. Explain why the columns of A form a basis for $\mathbb{R}^n$.

Suppose that $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$. Explain how you know that A is invertible?

Explain an algorithm for finding A^{-1}.

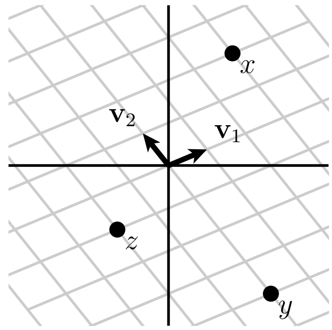
Find the solution to the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ -5 \\ 10 \end{bmatrix}$ using the inverse of A.

Suppose that A and B are both 2×2 invertible matrices and that

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}.$$

Without finding A and B, find the solution to the equation $BA\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

5. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$, the vectors shown below, form a basis $\mathcal{B}$.



Indicate the vector $\mathbf{w}$ where $\{\mathbf{w}\}_{\mathcal{B}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Find

- $\{\mathbf{v}_1\}_{\mathcal{B}}$
- $\{\mathbf{x}\}_{\mathcal{B}}$
- $\{\mathbf{y}\}_{\mathcal{B}}$
- $\{\mathbf{y}\}_{\mathcal{B}}$

6. Suppose that

$$\mathbf{v}_1 = \begin{bmatrix} -4 \\ 7 \\ -6 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -8 \\ 7 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 5 \\ -4 \end{bmatrix}.$$

Explain why $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis $\mathcal{B}$ for $\mathbb{R}^3$.

Find the coordinate representations $\{\mathbf{e}_1\}_{\mathcal{B}}$ and $\{\mathbf{e}_2\}_{\mathcal{B}}$.

If $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, find $\mathbf{x}$.