Mathematics 327
Symmetric matrices

1. Consider the symmetric matrix $A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$. Find the eigenvalues and corresponding eigenvectors.

   Is there an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A?

   Find an orthogonal diagonalization by finding matrices Q and D such that $A = QDQ^T$.

2. Consider the symmetric matrix $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Find the eigenvalues of A.

   For each eigenvalue, describe a basis for the corresponding eigenspace.

Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

Find an orthogonal diagonalization by writing $A = QDQ^T$.

3. **Note:** We have seen examples of square matrices that are not diagonalizable. It turns out that *symmetric* matrices, however, are always diagonalizable. In fact, a symmetric matrix A is always orthogonally diagonalizable, which means it can be written as $A = QDQ^T$ where Q has orthonormal columns and D is diagonal. This is called the *Spectral Theorem*.

4. Suppose that B is a square matrix. Explain why $A = B + B^T$ is a symmetric matrix.

   If A and B are two matrices, remember that $(AB)^T = B^T A^T$. Explain why the matrices AA^T and $A^T A$ are symmetric.

Consider the 2×3 matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$. Find an orthogonal diagonalization of $A^T A$.

Find an orthogonal diagonalization of AA^T.

5. We can see that the eigenvalues of $A^T A$ are nonnegative. For instance, suppose that $\mathbf{v}$ is an eigenvector of $A^T A$ with corresponding eigenvalue λ. Explain the following steps and why they show that $\lambda \geq 0$.

$$|A\mathbf{v}|^2 = (A\mathbf{v}) \cdot (A\mathbf{v}) = (A\mathbf{v})^T (A\mathbf{v}) = (\mathbf{v}^T A^T)(A\mathbf{v}) = \mathbf{v}^T (A^T A\mathbf{v}) = \mathbf{v} \cdot (\lambda \mathbf{v}) = \lambda \, |\mathbf{v}|^2 .$$

6. Part of the Spectral Theorem says that two eigenvectors of a symmetric matrix corresponding to different eigenvalues are orthogonal. To see this, suppose that $\mathbf{v}_1$ is an eigenvector with corresponding eigenvalue λ_1 and $\mathbf{v}_2$ is an eigenvector with corresponding eigenvalue λ_2 and that $\lambda_1 \neq \lambda_2$.

   Explain the following steps and why they show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.

   $$\lambda_1\mathbf{v}_1 \cdot \mathbf{v}_2 = (\lambda_1\mathbf{v}_1) \cdot \mathbf{v}_2 = (A\mathbf{v}_1)^T\mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 = \mathbf{v}_1^T(A\mathbf{v}_2) = \mathbf{v}_1^T(\lambda_2\mathbf{v}_2) = \lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2.$$

   Therefore,
   $$\lambda_1\mathbf{v}_1 \cdot \mathbf{v}_2 = \lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2.$$