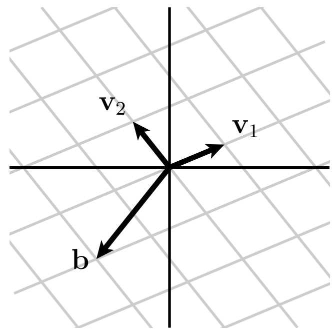
Mathematics 327
MTH 227 Review

1. **Matrix multiplication and linear combinations:** Shown below are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Sketch the linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ with weights $c_1 = 1$ and $c_2 = 3$.



Suppose that A is the 2×2 matrix $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$. Sketch the product $A \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Find the solution to the equation $A\mathbf{x} = \mathbf{b}$.

If $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & -2 & 3 \end{bmatrix}$, find the product $A \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ without using technology.

2. **Reduced row echelon form:** The following augmented matrices are in reduced row echelon form. For each matrix, indicate the pivot positions and identify basic and free variables. Then describe the solution space to the associated linear system, using a parametric form when appropriate.

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right].$$

What condition on the pivots of A guarantee that every equation of the form $A\mathbf{x} = \mathbf{b}$ is consistent?

What condition on the pivots of A guarantee that a consistent equation $A\mathbf{x} = \mathbf{b}$ has a unique solution?

Suppose that a matrix A has more rows than columns, like $A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$. Sometimes a matrix like this is called "tall" or the system $A\mathbf{x} = \mathbf{b}$ is called *over-determined*. What is most likely true about the solution space to the equation $A\mathbf{x} = \mathbf{b}$?

3. **Software:** Go to the website `gvsu.edu/s/0Ng` ("zero-capital N-lower case g") where you will find some "Sage cells" you can use to perform computations. Sup-

pose we want to row reduce the matrix $A = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 2 & 4 & -6 & 4 \\ 1 & 1 & -2 & 1 \end{bmatrix}$. We may define the matrix and then ask for its reduced row echelon form:

```
A = matrix(3, 4, [-1, -1, 2, 0, 2, 4, -6, 4, 1, 1, -2, 1])
A.rref()
```

When defining the matrix, the first entry in parentheses is the number of rows, the second is the number of columns, and the third is a list of the entries found by reading across the rows.

After entering these two lines in a cell, you may either click the **Evaluate** button or press **Shift-Enter**.

What is the reduced row echelon form of A?

You may define a vector by saying

```
v = vector([0, -1, 2, 4])
```

Matrix multiplication is defined by `A*v`. What is $A\mathbf{v}$?

4. **Gaussian elimination:** Let's recall the Gaussian elimination algorithm, which is used to find the reduced row echelon form of a matrix using three operations: row replacement, scaling, and interchange.

   Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 1 & 0 & -2 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$. The first step in the Gaussian elimination operation is to use a row operation to produce a zero in the first column and second row. Perform this operation to produce the matrix A_1.

   Define the matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and verify that $E_1 A = A_1$.

The second step is to begin with A_1 and perform a row replacement operation to create a zero in the first column and third row. Defining the matrix $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$, verify that $A_2 = E_2A_1$ is the next step in the Gaussian elimination algorithm. What is A_2?

The next step in the Gaussian elimination algorithm is to perform a row replacement operation that puts a zero in the second column and third row. Find a matrix E_3 that performs this operation by creating $U = E_3A_2$. What are E_3 and U?

We denote the resulting matrix by U because it is upper triangular, which means that the entries below the diagonal are zero.

We have now created matrices E_1, E_2, and E_3 so that

$$E_3E_2E_1A = U.$$

Find the matrix $L = E_3E_2E_1$ and verify that $LA = U$. We denote this matrix by L because it is lower triangular.

Suppose that we want to scale the second row of A_3 by $-\frac{1}{2}$. Find a matrix S that performs this scaling operation as SA_3.

Suppose we would like to interchange the first and second rows of A. Find a matrix P that performs this interchange as PA.