Mathematics 327
Gram-Schmidt orthogonalization

1. Suppose that V is a three-dimensional subspace of $\mathbb{R}^4$ with basis:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -3 \\ -3 \end{bmatrix}.$$

We would like to create an orthogonal basis $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ for V.

Notice that the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ is not an orthogonal basis by computing $\mathbf{v}_1 \cdot \mathbf{v}_2$.

To begin forming a new basis, simply define $\mathbf{w}_1 = \mathbf{v}_1$. Now find the vector $\mathbf{z}_2$ that is the orthogonal projection of $\mathbf{v}_2$ onto the line defined by $\mathbf{w}_1$.

Form the vector $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{z}_2$ and verify that it is orthogonal to $\mathbf{w}_1$.

Explain why any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is also a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$. Use this fact to explain why $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are an orthogonal basis for a two-dimensional subspace W_2 of $\mathbb{R}^4$. Find the vector $\mathbf{z}_3$ that is the orthogonal projection of $\mathbf{v}_3$ onto W_2.

Verify that $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{z}_3$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$.

Explain why $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ form an orthogonal basis for V.

Use this orthogonal basis to find the orthogonal projection of $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ onto V.

Find an orthonormal basis for V.

2. The vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$$

form a basis for a plane P in $\mathbb{R}^3$.

Find an orthonormal basis for P.

Find the matrix $A = QQ^T$ that projects vectors orthogonally onto P.

Find the orthogonal projection of $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ onto P.

Give a description of the orthogonal complement $P^{\perp}$.