**Mathematics 327**
**Final review, part III**

1. Suppose that $W$ is a 2-dimensional subspace of $\mathbb{R}^4$ spanned by

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}, \qquad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -3 \end{bmatrix}.$$

Find a basis for $W^\perp$.

Write the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 4 \end{bmatrix}$ as the sum $\mathbf{b} = \mathbf{w} + \mathbf{v}$ where $\mathbf{w}$ is in $W$ and $\mathbf{v}$ is in $W^\perp$.

Write $\mathbf{w}$ as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Write $\mathbf{v}$ as a linear combination of the basis vectors you found for $W^\perp$.

2. Suppose that $A$ is an $m \times n$ matrix, $\mathbf{b}$ is an $m$-dimensional vector, and $\widehat{\mathbf{b}}$ is its orthogonal projection onto $\mathrm{Col}(A)$.

   Explain why $A^T(\mathbf{b} - \widehat{\mathbf{b}}) = 0$.

   If $\widehat{\mathbf{x}}$ is the least squares approximate solution to the equation $A\mathbf{x} = \mathbf{b}$, explain why the normal equations hold:

$$(A^T A)\widehat{\mathbf{x}} = A^T\mathbf{b}.$$

   Use the normal equations to find the least squares approximate solution to the equation

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -2 & -1 \\ 0 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \\ 4 \end{bmatrix}.$$

   Explain how this is related to problem 1 in this activity.

We have seen three different ways to find the least squares approximate solution to an equation $A\mathbf{x} = \mathbf{b}$. Give a brief description of each of them.

3. Suppose that we have data points $(1, 1)$, $(2, 0)$, $(3, 1.1)$, $(4, 3.9)$ that we wish to model with a quadratic function $q(x) = \beta_0 + \beta_1 x + \beta_2 x^2$. Find the best coefficients $\beta_i$ and estimate $q(2.5)$.