Mathematics 327
Eigenvectors and diagonalization

1. Suppose that we have a 2×2 matrix A having eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with associated eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$.

What are the vectors:

$A\mathbf{v}_1$?

$A\mathbf{v}_2$?

$A(3\mathbf{v}_1 + 7\mathbf{v}_2)$?

Suppose that $\mathbf{x} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Express $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Write $A\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

What is $A\mathbf{x}$?

The last exercise shows us that knowing the eigenvalues and eigenvectors of A tells us everything we need to know about A. We will form the matrices

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Express $P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Express $AP\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

What is $D\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$?

Express $PD\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

What do you notice when you compare $AP\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ to $PD\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$?

Explain why $A = PDP^{-1}$ and use this to find the matrix A.

2. Suppose that we have an $n \times n$ matrix A and that we can find a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If we construct

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}, \qquad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

then $A = PDP^{-1}$ and we say that A is *diagonalizable*.

3. Now consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Verify that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of A. What are their corresponding eigenvalues?

Form the matrices P and D and verify that $A = PDP^{-1}$.

Find the dot product $\mathbf{v}_1 \cdot \mathbf{v}_2$. What does this tell you about the basis of $\mathbb{R}^2$ formed by $\mathbf{v}_1$ and $\mathbf{v}_2$?

Find an orthonormal basis $\mathbf{u}_1$ and $\mathbf{u}_2$ consisting of eigenvectors of A.

Suppose that $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Verify that $A = QDQ^T$ and explain why this relationship must hold.