Mathematics 327
SVD and approximating matrices

1. Suppose that $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, and $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ where
$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$
Write the outer products $\mathbf{u}_1\mathbf{v}_1^T$ and $\mathbf{u}_2\mathbf{v}_2^T$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$. What is the rank of these matrices?

   Write the product $UV^T = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} =$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$.

   Explain why $UV^T = \mathbf{u}_1\mathbf{v}_1^T + \mathbf{u}_2\mathbf{v}_2^T$.

   Notice that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Use this observation to write $UV^T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

   Explain why UV^T has rank 2.

2. Suppose that $A = U\Sigma V^T$ where

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{u}_2 & \mathbf{v}_3 \end{bmatrix}.$$

Find the product $U\Sigma$.

Explain why $A = U\Sigma V^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$.

3. For a general matrix A, we have

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \ldots \sigma_r \mathbf{u}_r \mathbf{v}_r^T.$$

Remember also that the singular values are such that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r.$$

We therefore have a sequence of improving approximations:

$$\begin{array}{rcl} A & \approx & A_1 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T \\ A & \approx & A_2 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T \\ A & \approx & A_3 = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \sigma_3 \mathbf{u}_3 \mathbf{v}_3^T \\ A & \approx & \vdots \\ A & = & A_r = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \ldots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T \end{array}$$

Consider the matrix: $A = \begin{bmatrix} 12 & 25 & -29 & 16 & 14 \\ -17 & 0 & -37 & 41 & 58 \\ -8 & -11 & -27 & 23 & 28 \\ 29 & 36 & -15 & -2 & -2 \end{bmatrix}$. Find the singular value decomposition of A using `A.SVD()`. What are the singular values?

We are now going to form the approximations shown above. A convenient way to do this is to modify Σ so that the last singular values are set to zero. For instance, for the first approximation, you want to set the singular values $\sigma_2 = \sigma_3 = \sigma_4 = 0$. The catch is that Sage will not let you modify Σ directly. Instead, you can create the approximation `A1` as:

```
S = copy(Sigma)
S[2,2] = 0
S[3,3] = 0
S[4,4] = 0
A1 = U*S*V.T
```

How do the matrices A and A_1 compare to one another?

You can find the error in the approximation by finding `A - A1`. To find the entry with the largest absolute value, you can say

```
import numpy as np
np.max(np.abs( A - A1 ))
```

What is the entry with the largest absolute value in the matrix $A - A_1$?

Now form the matrix A_2. What is the entry with the largest absolute value in $A - A_2$?

Do the same with A_3.

And again with A_4. Why does this result make sense?