Mathematics 227
Final Review, Part I

1. What does it mean to say that a vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

   Consider the vectors

   $$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}.$$

   Can $\mathbf{b}$ be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, describe all the ways in which it can be so written.

   Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$? If not, find a vector that is not in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

   Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

2. We earlier defined the *rank* of a matrix A to be the number of pivot positions in A. Suppose that A is a 3×4 matrix.

   If $\text{rank}(A) = 3$, what can you say about $\text{Col}(A)$?

   If $\text{rank}(A) = 2$, give a geometric description of $\text{Col}(A)$.

   If $\text{rank}(A) = 1$, give a geometric description of $\text{Col}(A)$.

   Give an example of a 3×4 matrix A whose rank is 1.

3. What does it mean for a set of vectors to be linearly independent?

   Are the set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the last part of this activity linearly independent?

If $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$, give a basis of $\mathrm{Nul}(A)$.

Write one of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ as a linear combination of the others.

4. Suppose that A is a 3×4 matrix with $\mathrm{rank}(A) = 3$. What can you say about the span of the columns of A? What can you say about the linear independence of the columns of A?

5. Suppose that A is a 7×5 matrix and that the equation $A\mathbf{x} = \mathbf{0}$ has only one solution. What is the solution?

   What can you guarantee about the solution space of the equation $A\mathbf{x} = \mathbf{b}$ for some other vector $\mathbf{b}$?

   What can you guarantee about the solution space of the equation $A\mathbf{x} = \mathbf{b}$ for some other vector $\mathbf{b}$ that is in $\mathrm{Col}(A)$?