Mathematics 327
Final review, part II

1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Form the matrix $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and evalute $V^T V$.

Explain how you know that these vectors form an orthogonal basis $\mathcal{B}$ for $\mathbb{R}^3$.

Express the vector $\mathbf{b} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_1$, and $\mathbf{v}_3$.

Is $\mathcal{B}$ an orthonormal basis? If no, form an orthonormal basis from it.

Suppose that $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are the orthonormal basis and that Q is the matrix having these vectors as columns. What is $Q^T Q$?

What is QQ^T?

2. Suppose that $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ -3 & -1 \end{bmatrix}$.

   Find an orthonormal basis for $\mathrm{Col}(A)$.

   Let Q be the matrix whose columns are the orthonormal basis you found. What is Q^TQ?

   Find the matrix QQ^T. What does this matrix represent and what is its rank?

   Use the work you have done here to write $A = QR$.

Use your QR factorization to find the least squares approximation solution $\widehat{\mathbf{x}}$ to the equation $A\mathbf{x} = \begin{bmatrix} 42 \\ 70 \\ -28 \end{bmatrix}$.