Mathematics 227
Linear independence

1. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 & -1 & 1 \\ 1 & -1 & 3 & 7 \\ 3 & -2 & 1 & 5 \\ -1 & 2 & 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & -1 & 4 \\ 1 & -1 & 3 & -1 \\ 3 & -2 & 1 & 3 \\ -1 & 2 & 2 & 1 \end{bmatrix}.$$

   (a) Are the columns of A linearly independent? Are the columns of B linearly independent? Explain your thinking.

   (b) For the matrix above whose columns are linearly dependent, express one of the vectors as a linear combination of the others.

2. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are vectors in $\mathbb{R}^8$ and that $\mathbf{v}_2 = \mathbf{0}$. What can you say about the linear dependence/independence of this set of four vectors in $\mathbb{R}^8$?

   If the vectors are linearly dependent, express one of them as a linear combination of the others.

3. Explain why seven vectors in $\mathbb{R}^5$ cannot be linearly independent.

   What is the largest number of vectors in $\mathbb{R}^5$ that can be linearly independent?

   If you have three vectors in $\mathbb{R}^5$, can you guarantee that they are linearly independent?

4. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are vectors in $\mathbb{R}^{24}$ that are linearly independent and that span $\mathbb{R}^{24}$. What can you say about the pivot positions of the matrix A whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$?

   How many vectors are there in this set?

   Suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^{24}$. Explain how you know that $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Suppose that $\mathbf{b}$ is a vector in $\mathbb{R}^{24}$. Explain how you know that $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in exactly one way.

What can you say about the solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$?

5. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are a set of vectors and that $\mathbf{v}_3$ is a scalar multiple of $\mathbf{v}_7$. What can you say about the linear independence/dependence of this set of vectors?

   If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a linearly independent set of vectors, is it necessarily true that one vector is a scalar multiple of another one?

6. Suppose that A is a matrix such that the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$. What can you say about the linear dependence/independence of the columns of A?

   What can you say about the span of the columns of A?