**Mathematics 227**
**Introduction to eigenvectors and eigenvalues**

If $A$ is an $n \times n$ matrix, we say that a nonzero vector $\mathbf{v}$ is an eigenvector of $A$ with associated eigenvalue $\lambda$ if

$$A\mathbf{v} = \lambda\mathbf{v}.$$

1. Consider the matrix $A = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$. Find the product $A\mathbf{v}$ to verify that $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of $A$. What is the associated eigenvalue $\lambda$?

2. If $\lambda$ is a scalar, what is the geometric relationship between $\mathbf{v}$ and $\lambda\mathbf{v}$?

   If $A\mathbf{v} = \lambda\mathbf{v}$, what is the geometric relationship between $\mathbf{v}$ and $A\mathbf{v}$?

3. Go to `http://gvsu.edu/s/0Ja` where you will find an interactive diagram that allows you to choose a matrix $A$ using the sliders along the top. The red vector $\mathbf{v}$ may be moved by clicking in the head of the vector and dragging it to a new location. The gray vector is $A\mathbf{v}$.

   Choose the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Move the red vector $\mathbf{v}$ so that the eigenvector condition holds. What is the eigenvector $\mathbf{v}$ and what is the associated eigenvalue?

By algebraically computing $A\mathbf{v}$, verify that your vector $\mathbf{v}$ is an eigenvector.

If you multiply your eigenvector $\mathbf{v}$ by the scalar 2, do you still have an eigenvector? If so, what is the associated eigenvector?

Are you able to find another linearly independent eigenvector $\mathbf{v}$? If so, what is the eigenvector and what is the associated eigenvalue?

Now consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. Use the diagram to describe any eigenvectors and their associated eigenvalues.

Now consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Use the diagram to describe any eigenvectors and their associated eigenvalues. What geometric transformation does this matrix perform on vectors? How does this explain the presence of any eigenvectors?

4. So why do we care about any of this? Good question. We'll now look at an example, but it helps to remember that matrix multiplication satisfies a linearity condition:

$$A\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\right) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2.$$

Suppose we work for a car rental company that has two locations, $P$ and $Q$. You find that

- 80% of cars rented at location $P$ are returned to $P$ at the end of the day with the rest returned to $Q$.
- 40% of cars rented at location $Q$ are returned to $Q$ at the end of the day with the rest returned to $P$.

If we write the state vector $\mathbf{x} = \begin{bmatrix} P \\ Q \end{bmatrix}$ to record the number of cars at location $P$ and $Q$ on one day, find a matrix $A$ such that $A\mathbf{x}$ describes the distribution of cars the next day.

Verify that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are eigenvectors of the matrix $A$. What are their associated eigenvalues?

Suppose that there are initially 1000 cars at location $P$ and none at location $Q$ so that the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$. Write $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Using the linearity of matrix multiplication, write $\mathbf{x}_1 = A\mathbf{x}_0$, the distribution of cars the next day, as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Continue into the future by writing the distributions the next few days, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ as linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.

What happens to the distribution of cars after a very long time?