Mathematics 327
Quadratic forms

If A is a symmetric matrix, we call $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ the *quadratic form* associated to A.

1. Suppose that $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Evaluate the quadratic form $q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ associated to D.

   Also evaluate $q\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ and $q\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

   Suppose that we denote $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Write the value of the quadratic form $q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

   Go to the webpage I've indicated and you will find a figure that allows you to choose a matrix A and move a vector $\mathbf{x}$ around the unit circle. The height of the rectangle will show the value of the quadratic form $q(\mathbf{x})$.

   If $q(\mathbf{x})$ is the quadratic form associated to the matrix D, what is the maximum value of $q(\mathbf{x})$ on the unit circle?

   What is the minimum value of $q(\mathbf{x})$?

Since we have $-1 < 2$, we have $-x_2^2 \leq 2x_2^2$ and hence

$$2x_1^2 - x_2^2 \leq 2x_1^2 + 2x_2^2.$$

Use this, along with the fact that $x_1^2 + x_2^2 = 1$ on the unit circle, to explain why $q(\mathbf{x}) \leq 2$ on the unit circle.

In the same way, we have $-x_1^2 \leq 2x_1^2$. Use this to explain why $q(\mathbf{x}) \geq -1$ on the unit circle.

2. Now consider the symmetric matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and its associated quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

   Evaluate $q\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $q\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and $q\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$.

   Write the value of the quadratic form $q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

   Use the figure on the webpage to study the values of this quadratic form. What is the maximum value of q on the unit circle? In what directions does it occur?

What is the minimum value of q on the unit circle? In what directions does it occur?

Find an orthogonal diagonalization of A by finding Q and D such that $A = QDQ^T$.

We will perform a change of coordinates by introducing a new coordinate $\mathbf{y} = Q^T\mathbf{x}$ or $\mathbf{x} = Q\mathbf{y}$. Write the quadratic form in the new variable $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ by noting that

$$q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = \mathbf{x}^T QDQ^T\mathbf{x} = (Q^T\mathbf{x})^T D(Q^T\mathbf{x}) = \mathbf{y}^T D\mathbf{y}.$$

What is the maximum value of q on the unit circle? In what direction does the maximum value occur?

What is the minimum value of q on the unit circle? in what direction does the minimum value occur?