Mathematics 227
Finding eigenvectors and eigenvalues

If A is an $n \times n$ matrix, we can rewrite the condition $A\mathbf{v} = \lambda\mathbf{v}$ as

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

We will now learn how to find eigenvalues λ and eigenvectors $\mathbf{v}$.

1. If $\mathbf{v}$ is an eigenvector of A with associated eigenvalue λ, then $\mathbf{v}$ is a nonzero solution to the homogeneous equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. What does this imply about the pivot positions of the matrix $A - \lambda I$?

   What does this say about the invertibility of $A - \lambda I$?

   What does this say about the determinant $\det(A - \lambda I)$?

2. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and construct the matrix

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}.$$

   Find the determinant $\det(A - \lambda I)$ and then find the values λ such that $\det(A - \lambda I) = 0$. These are the eigenvalues of A.

The solution to the equation $\det(A - \lambda I) = 0$ are $\lambda = 3$ and $\lambda = -1$. These are the eigenvalues of A. Now let's find the eigenvectors, which are the solutions to the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Start with $\lambda = 3$, which gives us the matrix $A - 3I$. Find the solutions to the homogeneous equation $(A - 3I)\mathbf{x} = \mathbf{0}$. These will be the eigenvectors corresponding to $\lambda = 3$.

Now use $\lambda = -1$, which gives us the matrix $A + I$. Find the solutions to the homogeneous equation $(A + I)\mathbf{x} = \mathbf{0}$. These will be the eigenvectors corresponding to $\lambda = -1$.

Go to `http://gvsu.edu/s/0Ja` and verify that you have found the eigenvectors and eigenvalues for A.

We will call the set of all eigenvectors corresponding to an eigenvalue λ the *eigenspace* of A corresponding to λ and denote it by E_λ. Notice that $E_\lambda = \text{Nul}(A - \lambda I)$, the null space of $A - \lambda I$. For the matrix A, what are $\dim E_3$ and $\dim E_{-1}$?

3. Let's now find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

   Find the eigenvalues by solving the equation $\det(A - \lambda I) = 0$, then find a basis for the eigenspaces E_λ for each eigenvalue λ.

   You can check your results again using the interactive figure.

4. Consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and find its eigenvalues and eigenvectors. Verify your results again with the interactive figure.

5. Remember that the determinant of a triangular matrix, such as $A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$, is the product of the diagonal entries. What does this say about the eigenvalues of a triangular matrix?

6. Sage can find the eigenvalues of a matrix `A` using `A.eigenvalues()`. Use Sage to find the eigenvalues of

$$A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Then find a basis for the eigenspace E_4. What is the dimension of E_4?

Find a basis for eigenspaces E_2 and E_1.

Can you find a basis for $\mathbb{R}^4$ consisting of eigenvectors of A?