Mathematics 227
Lab 5, Due: December 7, 2018

Instructions: The exercises here should be completed in groups of 2 or 3 students with one write-up submitted from each group.

So far, we have found eigenvalues by solving the characteristic equation $\det(A-\lambda I) = 0$ and eigenvectors by solving the homogeneous equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$. In practice, the matrices we'll be working with could be $10,000 \times 10,000$ and this approach is not practical: finding the roots of a $10,000$ degree polynomial is not practical and solving the homogeneous equation is plagued by the fact that computers only perform approximate arithmetic. Forunately, there's a way to numerically find eigenvectors using what's called the *power method*.

1. Let's look at the example $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.4 & 0.7 \end{bmatrix}$ and the initial vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   Find the vector $\mathbf{x}_1 = A\mathbf{x}_0$.

   Then identify m_1, the component of $\mathbf{x}_1$ that has the largest absolute value, and form the new vector $\overline{\mathbf{x}}_1 = \frac{1}{m_1}\mathbf{x}_1$. What is $\overline{\mathbf{x}}_1$? Notice that the component of $\overline{\mathbf{x}}_1$ having the largest absolute value is 1.

   Repeat this process by finding $\mathbf{x}_2 = A\overline{\mathbf{x}}_1$. Then identify m_2, the component of $\mathbf{x}_2$ having the largest absolute value and form $\overline{\mathbf{x}}_2 = \frac{1}{m_2}\mathbf{x}_2$.

   This is clearly a process that can be automated. There is a collection of Sage cells at the page `http://gvsu.edu/s/0TD`. The first cell there contains some useful commands so you should evaluate it now.

   In the second cell, define the matrix A and vector $\mathbf{x}_0$ from above. You can repeat the process we've outlined above by saying `power(A, x0, 20)`. When you do this, you will see the multiplier m_j and the vectors $\overline{\mathbf{x}}_j$ appear.

To what vector $\mathbf{p}$ does the sequence of vectors $\overline{\mathbf{x}}_j$ converge? To what value do the multipliers m_j converge?

Find $A\mathbf{p}$ and verify that $\mathbf{p}$ is an eigenvector. What is the corresponding eigenvalue?

This technique is called the power method. Explain how the power method computes the eigenvalue of A having the largest absolute value and a corresponding eigenvector.

2. The power method finds the eigenvalue with the largest absolute value. Suppose now that we want to find the eigenvalue with the smallest absolute value. We can use the *inverse power method* to do this.

   If λ is an eigenvalue of A with eigenvector $\mathbf{v}$, then we know that

   $$A\mathbf{v} = \lambda\mathbf{v}.$$

   If we multiply both sides by $\lambda^{-1}A^{-1}$, then we have

   $$A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}.$$

   This means that λ^{-1} is an eigenvalue of A^{-1} with eigenvector $\mathbf{v}$. Therefore, if λ is the eigenvalue of A having the smallest possible absolute value, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1} having the largest possible absolute value. This means that we can find $\frac{1}{\lambda}$, and hence λ, by applying the power method to A^{-1}. This is called the inverse power method.

   Back on the page of Sage cells, the command `inverse_power(A, x0, N)` applies the power method to A^{-1}. Find the smallest eigenvalue of A and a corresponding eigenvector.

3. Now define the matrix

$$A = \begin{bmatrix} 3.6 & 1.6 & 4.0 & 7.6 \\ 1.6 & 2.2 & 4.4 & 4.1 \\ 3.9 & 4.3 & 9.0 & 0.6 \\ 7.6 & 4.1 & 0.6 & 5.0 \end{bmatrix}.$$

Use the power method and inverse power method to find the eigenvalues with the largest and smallest absolute values.

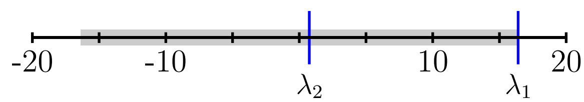
Since A is a 4×4 matrix, we expect that there should be two more eigenvalues. How can we find them?

Suppose that s is some scalar. Suppose also that $\mathbf{v}$ is an eigenvalue of A with associated eigenvalue λ. Explain why $\mathbf{v}$ is also an eigenvector of $A - sI$.

What is the eigenvalue of $A - sI$ associated to the eigenvector $\mathbf{v}$?

The eigenvalue of A closest to s is the eigenvalue of $A - sI$ closest to 0, which is the eigenvalue of $A - sI$ having the smallest absolute value. This means that we can find the eigenvalue of A closest to some number s by applying the inverse power method to $A - sI$. You can find the eigenvalues of A closest to some number s by saying `find_closest_eigenvalue(A, s, x0, 20)`.

For the matrix A above, you should have found the eigenvalues $\lambda_1 = 16.35$ and $\lambda_2 = 0.75$, which may be represented on a number line as shown. The other eigenvalues must be in the gray shaded area.



Use the command `find_closest_eigenvalue` to probe into the gray areas to find the other two eigenvalues. What do you find?

4. The first Sage cell on that page defined the matrix

$$B = \begin{bmatrix} -14.6 & 9.0 & -14.1 & 5.8 & 13.0 \\ 27.8 & -4.2 & 16.0 & 0.9 & -21.3 \\ -5.5 & 3.4 & 3.4 & 3.3 & 1.1 \\ -25.4 & 11.3 & -15.4 & 4.7 & 20.3 \\ -33.7 & 14.8 & -22.5 & 9.7 & 26.6 \end{bmatrix}.$$

Use the power method to find the eigenvalue with the largest absolute value, the eigenvalue with the smallest absolute value, and the other three eigenvalues. You can keep track of your results on the number line below. State your result by listing the eigenvalues in increasing order.