Mathematics 227
Linear independence

1. Consider the matrix

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & -2 \\ 2 & 1 & 1 \end{bmatrix}$$

whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(a) Give a description of the solution space to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

(b) Find one set of nonzero weights c_1, c_2, c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}.$$

(c) Use these weights to show that one of the vectors can be written as a linear combination of the others.

(d) Explain why the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent.

(e) Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span $\mathbb{R}^3$?

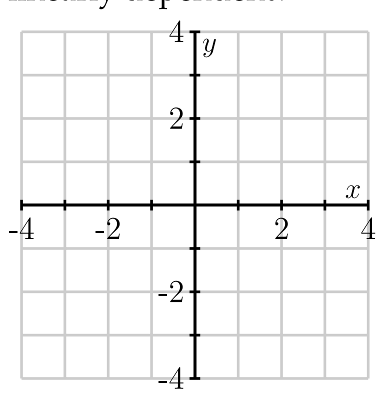
2. (a) If A is any matrix, explain why the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is consistent.

(b) What condition on the pivots of A guarantees that there is solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ besides the solution $\mathbf{x} = \mathbf{0}$?

(c) In the example above, we saw that having a nonzero solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ enabled us to conclude that the columns of A are linearly dependent. What condition on the pivots of A will guarantee that the columns of A are linearly dependent?

3. Consider the matrix $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

(a) Sketch the vectors below. Do you think the vectors are linearly independent or linearly dependent?



(b) Explain how we know that there is a nonzero solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

(c) Find a nonzero solution to the homogeneous solution $A\mathbf{x} = \mathbf{0}$.

(d) Use this solution to express one of the vectors as a linear combination of the others.

4. (a) What condition on a matrix A will guarantee that its columns are linearly independent?

(b) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent vectors in $\mathbb{R}^9$. What can you say about the number of vectors in this set?

(c) What is the smallest number of vectors that can span $\mathbb{R}^{13}$?

(d) What is the largest number of vectors in $\mathbb{R}^{13}$ that are linearly independent?

(e) Suppose that a set of vectors in $\mathbb{R}^{13}$ is linearly independent and spans $\mathbb{R}^{13}$. What can you say about the number of vectors?