Mathematics 327
Orthonormal bases

1. Consider the orthogonal basis of $\mathbb{R}^2$ consisting of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ that are parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$.

We saw last time that a vector $\mathbf{b}$ can be written in terms of an orthogonal basis as

$$\mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{b} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2.$$

Use this fact to write $\mathbf{b} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Since $\mathbf{u}_1$ and $\mathbf{u}_2$ still form an orthogonal basis of $\mathbb{R}^2$, we can write

$$\mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{b} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2. \tag{1}$$

Write $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

How does equation 1 simplify when we have an orthonormal basis?

Let Q be the matrix $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Write Q and compute QQ^T.

Also, compute Q^TQ.

What is Q^{-1}?

2. Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis of $\mathbb{R}^2$; that is, the vectors are orthogonal and have unit length. For any vector $\mathbf{x}$, we have

$$\mathbf{x} = (\mathbf{x} \cdot \mathbf{u}_1)\, \mathbf{u}_1 + (\mathbf{x} \cdot \mathbf{u}_2)\, \mathbf{u}_2. \tag{2}$$

If $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$, explain why $Q^T\mathbf{x} = \begin{bmatrix} \mathbf{x} \cdot \mathbf{u}_1 \\ \mathbf{x} \cdot \mathbf{u}_2 \end{bmatrix}$.

Explain why equation 2 implies that $QQ^T\mathbf{x} = \mathbf{x}$.

Explain why $QQ^T = I$.

3. Suppose that

$$\mathbf{u}_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

defines a plane P in $\mathbb{R}^3$. Explain why $\mathbf{u}_1$ and $\mathbf{u}_2$ form an orthonormal basis for P.

Remember that the orthogonal projection of a vector $\mathbf{b}$ onto this plane is

$$\frac{\mathbf{b}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\,\mathbf{u}_1 + \frac{\mathbf{b}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\,\mathbf{u}_2.$$

How does this expression simplify when $\mathbf{u}_1$ and $\mathbf{u}_2$ is an orthonormal basis?

Find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ onto P.

Suppose that $Q = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$. Explain why $Q^T\mathbf{x} = \begin{bmatrix} \mathbf{x}\cdot\mathbf{u}_1 \\ \mathbf{x}\cdot\mathbf{u}_2 \end{bmatrix}$.

Explain why the orthogonal projection of $\mathbf{x}$ onto P is $QQ^T\mathbf{x}$.

Find the matrix QQ^T and then find the orthogonal projection of $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ onto P.

Find the matrix Q^TQ and explain the result that you find.