Mathematics 227
Bases

1. Explain why

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ is a basis for $\mathbb{R}^6$. (You will see why in a moment.) Construct the matrix $C_{\mathcal{B}}$ that converts $\{\mathbf{x}\}_{\mathcal{B}}$ into $\mathbf{x}$; that is, $C_{\mathcal{B}} \{\mathbf{x}\}_{\mathcal{B}} = \mathbf{x}$.

Construct the matrix $C_{\mathcal{B}}^{-1}$ that converts $\mathbf{x}$ into $\{\mathbf{x}\}_{\mathcal{B}}$; that is, $C_{\mathcal{B}}^{-1}\mathbf{x} = \{\mathbf{x}\}_{\mathcal{B}}$.

Explain how you know now that $\mathcal{B}$ is a basis for $\mathbb{R}^6$.

Suppose that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}, \{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}.$$

Express the coefficient c_2 in terms of $x_1, x_2, \ldots, x_6$. Also, express the coefficient c_4 in terms of $x_1, x_2, \ldots, x_6$.

State in words what the coefficients c_1, c_2, c_3, c_4, and c_5 measure.

Suppose that

$$\mathbf{x} = \begin{bmatrix} 25 \\ 34 \\ 30 \\ 45 \\ 190 \\ 200 \end{bmatrix}$$

represents the grayscale values in a row of pixels in an image. Find $\{\mathbf{x}\}_{\mathcal{B}}$.

Explain how $\{\mathbf{x}\}_{\mathcal{B}}$ can be used to determine the location in a jump in brightness in the pixels as we move across the row.

2. Computers represent colors using something called a *color model*. The simplest is the RGB color model, which represents colors as a 3-dimensional vector $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ describing how much red R, green G, and blue B to mix together to create a color. These quantities are represented internally with 8 bytes so, in practice, the range of values is between 0 and 255.

   A second color model is YC_bC_r, which is known as the *luminance-chrominance* color model. We introduce the basis $\mathcal{B}$ consisting of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -0.34413 \\ 1.77200 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1.40200 \\ -0.71414 \\ 0 \end{bmatrix},$$

and define

$$\left\{ \begin{bmatrix} R \\ G \\ B \end{bmatrix} \right\}_{\mathcal{B}} = \begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix}$$

The quantity Y is called the *luminance* and measures the brightness of the color, C_b is the blue chrominance and measures how much blue is mixed in, and C_r is the red chrominance and measures how much red is mixed in. The range of values is

$$\begin{aligned} 0 \leq Y &\leq 255 \\ -127.5 \leq C_b &\leq 127.5 \\ -127.5 \leq C_r &\leq 127.5 \end{aligned}$$

Go to `http://gvsu.edu/s/0Jc` where you will find some figures to experiment with color models. Remember that R, G, B, and Y run between 0 and 255 while C_b and C_r run between -127.5 and 127.5.

(a) What happens when $G = 0$ and $B = 0$ (pushed all the way to the left) and R is allowed to vary?

(b) What happens when $R = 0$ and $G = 0$ (pushed all the way to the left) and B is allowed to vary?

(c) How do you create black in the RGB color model? How do you create white?

(d) What happens when $C_b = 0$ and $C_r = 0$ (kept in the center) and Y is allowed to vary?

(e) What happens when $Y = 0$ (pushed left) and $C_r = 0$ (kept in the center) and C_b is allowed to increase from 0 to 127.5?

(f) How can you create black in the YC_bC_r color model? How do you create white?

(g) Find the matrix $C_{\mathcal{B}}$ that converts $\begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix}$ into $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$. Then find the matrix that converts $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ into YC_bC_r.

(h) Find the YC_bC_r coordinates for the following colors and use the diagrams to check that the two representations agree.

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 255 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 255 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 255 \\ 255 \\ 255 \end{bmatrix}$$

(i) Find the RGB coordinates for the following colors and use the diagrams to check that the two representations agree.

$$\begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 128 \\ 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 128 \\ 60 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 128 \\ 0 \\ 60 \end{bmatrix}.$$

(j) Write an expression for the luminance Y as it depends on R, G, and B. Explain how the luminance represents the brightness of the color.

(k) Write an expression for the blue chrominance C_b in terms of R, G, and B. Explain how the blue chrominance measures the amount of blue in the color.