Mathematics 327
LU factorizations

Remember that there is a page of Sage cells at `gvsu.edu/s/0Ng`.

1. Consider the matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 0 & 2 \\ -4 & 0 & -2 \end{bmatrix}$. Find an elementary 3×3 matrix E_1 that performs the first row replacement operation in the Gaussian elimination algorithm. This operation should create matrix A_1 with a zero in the second row and first column.

   Use Sage to verify that $A_1 = E_1 A$.

   Now find an elementary matrix E_2 that performs the second row replacement operation on A_1. This will create a matrix A_2 with zeroes in the second and third rows of the first column. You should now have $A_2 = E_2 A_1 = E_2 E_1 A$.

   Finally, find an elementary matrix E_3 that performs the third row operation creating the upper-triangular matrix U. You should now have $U = E_3 A_2 = E_3 E_2 E_1 A$.

   Use Sage to find the product $K = E_3 E_2 E_1$. What type of matrix is K?

   Verify that $U = E_3 E_2 E_1 A = KA$.

Use Sage to find the inverse $L = K^{-1}$. What kind of matrix is L?

Explain why $A = LU$ and use Sage to verify that it is true.

2. This is known as the LU factorization of A, and it is helpful because we can solve equations having the form $A\mathbf{x} = \mathbf{b}$ more efficiently than with the inverse A^{-1}.

   Suppose we want to solve the equation $A\mathbf{x} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$. We can rewrite this as

$$A\mathbf{x} = L(U\mathbf{x}) = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}.$$

   We will define the unknown vector $\mathbf{c} = U\mathbf{x}$ so that this equation becomes

$$L\mathbf{c} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}.$$

   Find the vector $\mathbf{c}$.

   Now solve the equation $U\mathbf{x} = \mathbf{c}$ to find the vector $\mathbf{x}$, which is the solution to the original equation $A\mathbf{x} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$ that we sought.

The LU factorization of A is helpful because solving two equations with triangular matrices is much faster than solving one equation with a general matrix. Of course, you can't get something for nothing: finding the LU factorization of A is about the same amount of work as row reducing A. The benefit is that we can now solve lots of equations of the form $A\mathbf{x} = \mathbf{b}$ using the LU factorization. In essence, we can reuse the work we have done in row reducing A.

3. Remember that we sometimes need to interchange two rows in the Gaussian elimination algorithm, which will interfere with the factorization. There is a way around this, but we won't discuss it here. Instead, just know that Sage can compute LU factorizations; if we have defined a matrix A, we can say

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P, L, U = A.LU()
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which produces the lower-triangular matrix L and the upper-triangular matrix U and a permutation matrix P, which swaps out rows as needed. We then have

$$A = PLU.$$

Still using the matrix A from the first part of this activity, use Sage to find the LU factorization. What are P, L, and U?

Verify that $A = PLU$.

What is the product LU and then what is the effect of multiplying LU by P?

Suppose that $\mathbf{b} = \begin{bmatrix} 8 \\ -2 \\ -2 \end{bmatrix}$ as before and suppose we would like to solve the equation $A\mathbf{x} = PLU\mathbf{x} = \mathbf{b}$. As before, we will view the equation as

$$A\mathbf{x} = P(L(U\mathbf{x})) = \mathbf{b}$$

and peel off one matrix at a time from the left-hand side.

Define the unknown vector $\mathbf{d}$ so that $P\mathbf{d} = \mathbf{b}$, and find $\mathbf{d}$.

Define the unknown vector $\mathbf{c}$ so that $L\mathbf{c} = \mathbf{d}$ and find $\mathbf{c}$.

Finally, solve the equation $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$. Then verify that it is the same solution that you found before.