Mathematics 227
Invertible matrices

1. Consider the matrices

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & -4 \\ -1 & -1 & 3 \\ 0 & -1 & 2 \end{bmatrix}.$$

   (a) Define these matrices in Sage and verify that $BA = I$ so that $B = A^{-1}$. What is A^{-1}?

   (b) Find the solution to the equation $A\mathrm{x} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ using A^{-1}.

   (c) Remember that the product of matrices usually depends on the order in which you multiply; that is, if C and D are matrices, it usually happens that $CD \neq DC$. In this example, we have seen that $BA = I$. What is the product AB?

2. Suppose that A is an invertible $n \times n$ matrix. We know that every equation $A\mathbf{x} = \mathbf{b}$ has a solution $x = A^{-1}\mathbf{b}$. What does this say about the span of the columns of A?

   What does this say about the pivot positions of A?

If A is an invertible 4×4 matrix, what is its reduced row echelon form?

3. Let's begin with the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. We would like to construct the inverse of A, which we'll denote by $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$. This means that we need to solve

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = I,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the columns of the identity matrix. We therefore have the two equations

$$A\mathbf{b}_1 = \mathbf{e}_1, \qquad A\mathbf{b}_2 = \mathbf{e}_2$$

that we can solve for $\mathbf{b}_1$ and $\mathbf{b}_2$. Find these vectors and then write the matrix B.

Verify that $AB = I$ and that $BA = I$.

Instead of solving the two equations $A\mathbf{b}_1 = \mathbf{e}_1$ and $A\mathbf{b}_2 = \mathbf{e}_2$ separately, we may as well solve them at the same time. To do this, build the augmented matrix

$$\left[\begin{array}{c|c} A & I \end{array}\right]$$

and find the reduced row echelon form. How does this produce $B = A^{-1}$?