Mathematics 327
Singular values

Given a matrix A, we will study the function $f(\mathbf{x}) = |A\mathbf{x}|$. Notice that this is not a quadratic form; if $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, then we have

$$f\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) = \sqrt{4x_1^2 + x_2^2}.$$

You will find a useful figure at `http://gvsu.edu/s/0YE` that allows you to select a matrix A and vary the red vector $\mathbf{x}$ on the left. You will see $A\mathbf{x}$ on the right. The height of the blue rectangle shows you the value of $f(\mathbf{x}) = |A\mathbf{x}|$.

1. Let's start with our favorite matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, which is special because it is symmetric. You should remember the eigenvalues and eigenvectors of this matrix, but if you don't, go ahead and find them or ask someone.

   Move the vector $\mathbf{x}$ around the unit circle. What is the shape of the curve traced out by $A\mathbf{x}$?

   What is the maximum value of $f(\mathbf{x}) = |A\mathbf{x}|$? State a vector $\mathbf{v}_1$ at which the maximum value occurs.

   What is the minimum value of $f(\mathbf{x}) = |A\mathbf{x}|$? State a vector $\mathbf{v}_2$ at which the minimum value occurs.

The *singular values* σ_i of A are the largest and smallest values of $f(\mathbf{x})$. For instance, σ_1 is the largest value of $f(\mathbf{x})$ and σ_2 is the smallest value. What is σ_1 and σ_2 and how do they relate to the eigenvalues of A?

We call the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *right singular vectors*. How are they related to the eigenvectors of A?

What is the angle between these vectors $\mathbf{v}_1$ and $\mathbf{v}_2$?

What is the angle between $A\mathbf{v}_1$ and $A\mathbf{v}_2$?

2. Let's now look at an example of a non-symmetric matrix: $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$. What are the eigenvalues of A?

Is A diagonalizable; that is, can we write $A = PDP^{-1}$?

Once again, what is the shape of the curve traced out by $A\mathbf{x}$?

What is the first singular value σ_1, the maximum value of the function $f(\mathbf{x}) = |A\mathbf{x}|$? State a vector $\mathbf{v}_1$ at which this maximum occurs.

What is the second singular value σ_2, the minimum value of the function $f(\mathbf{x}) = |A\mathbf{x}|$? State a vector $\mathbf{v}_2$ at which this minimum occurs.

What is the angle between the right singular vectors $\mathbf{v}_1$ and $\mathbf{v}_2$?

What is the angle between $A\mathbf{v}_1$ and $A\mathbf{v}_2$?

3. Still working with the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$, form the symmetric matrix $A^T A$.

Find an orthogonal diagonalization of $A^T A$.

What is the relationship between the singular values of A and the eigenvalues of $A^T A$?

What is the relationship between the singular vectors of A and the eigenvectors of $A^T A$?

Consider the quadratic form $q(\mathbf{x}) = |A\mathbf{x}|^2 = (f(\mathbf{x}))^2$. We have

$$(f(\mathbf{x}))^2 = q(\mathbf{x}) = |A\mathbf{x}|^2 = (A\mathbf{x}) \cdot (A\mathbf{x}) = (A\mathbf{x})^T (A\mathbf{x}) = x^T A^T A\mathbf{x}.$$

Use this fact to explain why the singular values of A are the square roots of the eigenvalues of $A^T A$.

Also, use this fact to explain why the right singular vectors of A are the eigenvectors of $A^T A$.