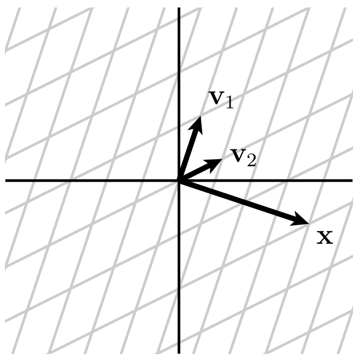
Mathematics 227
Eigenvalues and Eigenvectors

1. Suppose that A is a 2×2 matrix and that $\mathbf{v}_1$ is an eigenvector with associated eigenvalue $\lambda_1 = -1$ and $\mathbf{v}_2$ is an eigenvector with associated eigenvalue $\lambda_2 = \frac{1}{2}$. Sketch the vectors $A\mathbf{x}$ and $A^2\mathbf{x}$.



2. We have seen that the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and associated eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$.

   If $\mathbf{x} = 2\mathbf{v}_1 - 3\mathbf{v}_2$, write $A\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

   The eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis $\mathcal{B}$ for $\mathbb{R}^2$. If $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, find $\{A\mathbf{x}\}_{\mathcal{B}}$.

   If $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, find $\{A\mathbf{x}\}_{\mathcal{B}}$.

Find a matrix D such that $\{A\mathbf{x}\}_{\mathcal{B}} = D\,\{\mathbf{x}\}_{\mathcal{B}}$.

3. Consider the matrix $A = \begin{bmatrix} 5 & -4 & 6 \\ -1 & 3 & -1 \\ -2 & 3 & -3 \end{bmatrix}$. Use Sage to find the eigenvectors using the command `A.eigenvalues()`.

For each eigenvector λ, find a basis for the eigenspace E_λ. State the dimension of each eigenspace.

Can you form a basis for $\mathbb{R}^3$ consisting of eigenvectors of A? If so, give such a basis.

Write the vector $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$ as a linear combination of eigenvectors.

Suppose that $x_1 = A\mathbf{x}_0$ and $x_2 = A\mathbf{x}_1$ and so forth. Express $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ as a linear combination of eigenvectors of A.

Suppose that $\mathcal{B}$ is the basis of $\mathbb{R}^3$ consisting of eigenvectors that you found above. Find $\{\mathbf{x}\}_{\mathcal{B}}$, the $\mathcal{B}$-coordinate representation of $\mathbf{x} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$.

Find $\{A\mathbf{x}\}_{\mathcal{B}}$.

Find $\{A^2\mathbf{x}\}_{\mathcal{B}}$.

Find $\left\{A^k\mathbf{x}\right\}_{\mathcal{B}}$.

Suppose that $\mathbf{x}$ is a vector such that $\{\mathbf{x}\}_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. What is $\{A\mathbf{x}\}_{\mathcal{B}}$?

Find a matrix D such that $D\,\{\mathbf{x}\}_{\mathcal{B}} = \{A\mathbf{x}\}_{\mathcal{B}}$.