Mathematics 327
2^{nd} review, part II

1. Suppose that A is an $m \times n$ matrix with singular value decomposition $A = U\Sigma V^T$ where

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}.$$

   What are the values of m and n?

   What is rank(A)?

   How can you find a basis for

   - Nul(A)?
   - Nul(A^T)?
   - Col(A)?

   What is a singular value decomposition of A^T? How does this show that rank$(A) =$ rank(A^T)?

   How are the singular values and right singular vectors $\mathbf{v}_i$ found by an eigenvalue-eigenvector computation?

How are the left singular vectors related to the eigenvectors of the covariance matrix defined by A?

2. Suppose that we have the demeaned data set $\mathbf{x}_i$ and that we construct the matrix

$$A = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 & \mathbf{x}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 2 & 3 & 1 & -6 \\ 0 & -2 & -1 & 3 \end{bmatrix}.$$

Construct the covariance matrix C and find its eigenvalues.

Applying a principal component analysis, use the eigenvectors corresponding to the two largest eigenvalues to project the demeaned data onto a plane.

For what fraction of the variance does this plane account?

Suppose that we find a singular value decomposition $A = U\Sigma V^T$. How do the eigenvectors that you used in the principal component analysis arise in this singular value decomposition?

Explain why the points that you projected onto the plane can also be found by computing

$$\begin{bmatrix} \sigma_1 \mathbf{v}_1^T \\ \sigma_2 \mathbf{v}_2^T \end{bmatrix}.$$

3. Suppose that $A = U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Explain why A must be invertible.

Explain why the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ has a unique solution.

Express this unique solution in terms of U, V, and Σ.

4. Suppose that $A = U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}.$$

Explain why the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ most likely has no solution.

Explain how to use the singular value decomposition to find the approximation least squares solution $\widehat{\mathbf{x}}$ to the equation $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$.