Mathematics 227
Linear combinations

As always, Sage cells are available at `http://gvsu.edu/s/0Ng`.

In general, given a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and weights $c_1, c_2, \ldots, c_n$, we form the linear combination

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n.$$

1. Given the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 3 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \end{bmatrix},$$

we can ask if $\mathbf{b}$ can be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Rephrase this question by writing a linear system for the weights c_1, c_2, and c_3. Then solve this linear system to determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

2. Consider the following linear system.

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ x_1 + 2x_3 &= 0 \\ -x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

Identify vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{b}$ and rephrase the question "Is this linear system consistent?" by asking "Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$?"

3. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, can $\mathbf{b}$ be written as a linear combination of these vectors in more than one way?

4. Considering the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the previous part, can we write every three-dimensional vector $\mathbf{b}$ as a linear combination of these vectors? Explain how the pivot positions of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ help answer this question.

5. Now consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix}.$$

Can $\mathbf{b}$ be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$? If so, can $\mathbf{b}$ be written as a linear combination of these vectors in more than one way?

6. Considering the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ from the previous part, can we write every three-dimensional vector $\mathbf{b}$ as a linear combination of these vectors? Explain how the pivot positions of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ help answer this question.