Mathematics 227
Stochastic matrices and Markov chains

A *probability vector* is a vector whose entries are non-negative and add to 1. For instance, $\mathbf{v} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ is a probability vector. A *stochastic* matrix is a matrix whose columns are all probability vectors. For instance $A = \begin{bmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$ is a stochastic matrix.

1. Find the product $A\mathbf{v}$ and verify that it is a probability vector.

   It is always the case that we obtain a probability vector when we multiply a stochastic matrix by a probability vector.

   Find the eigenvalues of A and a basis for each eigenspace.

   Find a probability vector $\mathbf{q}$ in E_1. To do this, remember that every vector in E_1 is a scalar multiple of the basis vector. What scalar multiple of your basis vector gives a probability vector?

   Write $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors of A.

Now consider the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$. What happens to $\mathbf{x}_k$ as k grows increasingly large?

How is the result related to the probability vector $\mathbf{q}$ that you found in E_1?

Rather than $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, suppose you began with another initial state vector $\mathbf{x}_0$. What happens to $\mathbf{x}_k$ as k grows increasingly large?

2. Now consider the stochastic matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace.

Write the initial state vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of eigenvectors of A.

What happens to $\mathbf{x}_k$ as k grows increasingly large?