Mathematics 327
SVD and linear systems

Today we'll investigate how to solve equations using singular value decompositions.

1. The matrix $A = \begin{bmatrix} 2 & 2 \\ 4 & -4 \end{bmatrix}$ has the singular value decomposition $A = U\Sigma V^T$ where

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}, \qquad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Suppose we want to solve the equation $A\mathbf{x} = U\Sigma V^T\mathbf{x} = \mathbf{b} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$. Since the columns of U are orthonormal, what is U^{-1}? (You shouldn't need to compute this.)

Use this to solve the equation $U\mathbf{c} = \mathbf{b}$ where $\Sigma V^T\mathbf{x} = \mathbf{c}$.

What is Σ^{-1}? Use this to solve $\Sigma\mathbf{d} = \mathbf{c}$ where $V^T\mathbf{x} = \mathbf{d}$.

Since the columns of V are orthonormal, what is $(V^T)^{-1}$? Use this to find $\mathbf{x}$.

Just as we did when we studied LU factorizations, we have traded one equation $A\mathbf{x} = \mathbf{b}$ in for three equations. The benefit is that each of these equations can be easily solved without Gaussian elimination.

2. We can apply the same thinking to understand the consistency of equations. Suppose that A is a 3×2 matrix such that $A = U\Sigma V^T$ where

$$A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}, \qquad U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}.$$

We would like to solve the equation $A = U\Sigma V^T\mathbf{x} = \mathbf{b}$.

Solve the first equation $U\mathbf{c} = \mathbf{b}$ by writing $\mathbf{c}$ as a three-dimensional vector of dot products.

Now consider the equation $\Sigma\mathbf{d} = \mathbf{c}$. What condition on $\mathbf{c}$ determines when this equation is consistent? What condition on $\mathbf{b}$ determines when this equation is consistent.

Remember that the column space Col(A) is the set of vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is consistent. Identify a basis for Col(A).

Remember that Nul$(A^T) =$ Col$(A)^\perp$. Identify a basis for Nul(A^T).

3. Suppose that $\Sigma = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. What are the dimensions of the matrix A?

   If $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}$, what is the number m?

   Identify a basis for $\mathrm{Col}(A)$. What is $\mathrm{rank}(A)$?

   Identify a basis for $\mathrm{Nul}(A^T)$.

4. Suppose that A is a 12×15 matrix with singular values

$$\sigma_1 = 100, \sigma_2 = 37, \sigma_3 = 18, \sigma_4 = 12, \sigma_5 = 0, \dots.$$

   What is $\mathrm{rank}(A)$?

5. In general, how does a singular value decomposition $A = U\Sigma V^T$ tell us

   - $\mathrm{rank}(A)$
   - A basis for $\mathrm{Col}(A)$.
   - A basis for $\mathrm{Nul}(A^T)$.