1 Question I

1.1 Numerical Experiment

By modify the code from last homework, we get the following result:

```
clc,clear
n=1e7;
GAM=rand(n,1)+1;
rho=rand(n,1);
p=rand(n,1);
v=(2*rand(n,1)-1);
t=(2*rand(n,1)-1);
ga=1./sqrt(1-v.^2);
h=1+(GAM./(GAM-1)).*(p./rho);
D=rho.*ga.*(1+t.*v);
m=rho.*h.*ga.^2.*v.*(1+t.*v)+t.*p;
E=rho.*h.*ga.^2-p+t.*m;
Yes = D.^2+m.^2<E.^2;

plot( D./E, m./E,'.')
title(sprintf('out of bound: %d',sum(Yes)-n))
axis equal
```

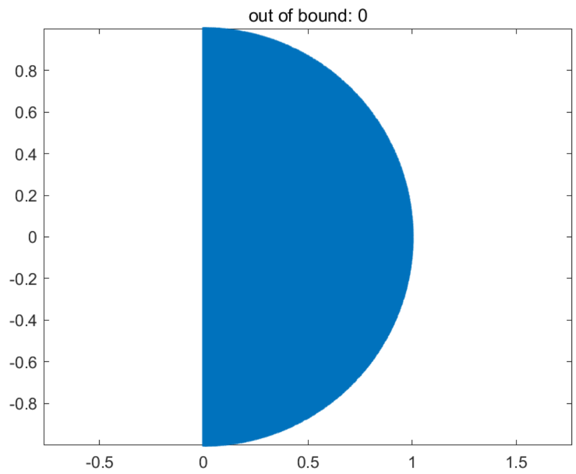

Figure 1.

which shows that in every experiment out of 10^7 times, the point lands in the expected region.

1.2 Rigorous Proof

By definition, we have:

$$Ev = m - pv$$

$$\begin{aligned}\tilde{D}^2 + \tilde{m}^2 - \tilde{E}^2 &= D^2(1+tv)^2 + m^2(1+tv)^2 + 2mtp(1+tv) + t^2p^2 - E^2 - 2tmE - t^2m^2 \\ &= D^2(1+tv)^2 + m^2(1+tv)^2 - (1+tv)^2E^2 + t^2v^2E^2 + 2tvE^2 \\ &\quad +2mtp(1+tv) - 2Evtp(1+tv) + 2Evtp(1+tv) - 2Etm\end{aligned}$$

$$
\begin{aligned}
&+t^2p^2-t^2p^2v^2+t^2p^2v^2-t^2m^2\\
&=D^2(1+tv)^2+m^2(1+tv)^2-(1+tv)^2E^2+t^2(m-pv)^2+2tE(m-pv)\\
&\quad+2mtp(1+tv)-2Evtp(1+tv)+2tp(1+tv)(m-pv)-2Etm\\
&\quad+t^2p^2-t^2p^2v^2+t^2p^2v^2-t^2m^2\\
&=(1+tv)^2(D^2+m^2-E^2)+t^2m^2-2pvmt^2+t^2p^2v^2+2tE(m-pv)\\
&\quad+2tp(1+tv)(m-Ev)+2(pvmt^2-t^2p^2v^2)+2tpEv-2Etm\\
&\quad+t^2p^2(1-v^2)+t^2p^2v^2-t^2m^2\\
&=(1+tv)^2(D^2+m^2-E^2)+2tE(m-pv)\\
&\quad+2tp(1+tv)(m-Ev)+2tpEv-2Etm\\
&\quad+t^2p^2(1-v^2)\\
&=(1+tv)^2(D^2+m^2-E^2)+2tp(1+tv)(m-Ev)+t^2p^2(1-v^2)\\
&=(1+tv)^2(D^2+m^2-E^2)+p^2(1+tv)^2-p^2+t^2p^2\\
&=(1+tv)^2(D^2+m^2-E^2)+p^2(t^2-1)^2\\
&\leqslant(1+|tv|)^2(D^2+m^2-E^2)\\
&\leqslant 0
\end{aligned}
$$

2 Question II

2.1 Numerical Experiment

Define function $\Phi(x):=\frac{m^2}{E+x}+D\sqrt{1-\frac{m^2}{(E+x)^2}}+Cx-E-x$.

We apply different values as the initial values of the Newton method to solve $\Phi(x)=0$, the result is as followed, where the blue dots stand for the solutions to the equations, and the yellow dots stand for the maximum difference between the solutions to the same equation with differenct initial values.

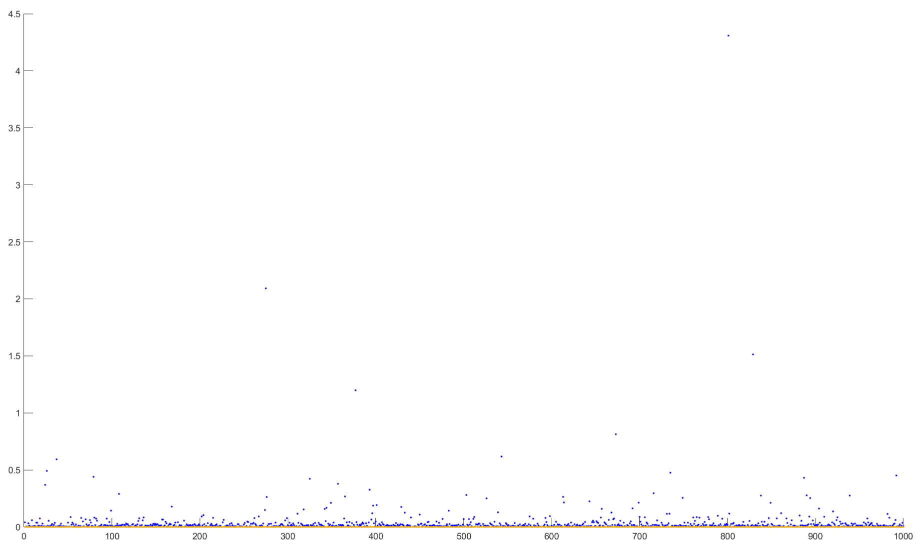

Figure 2.

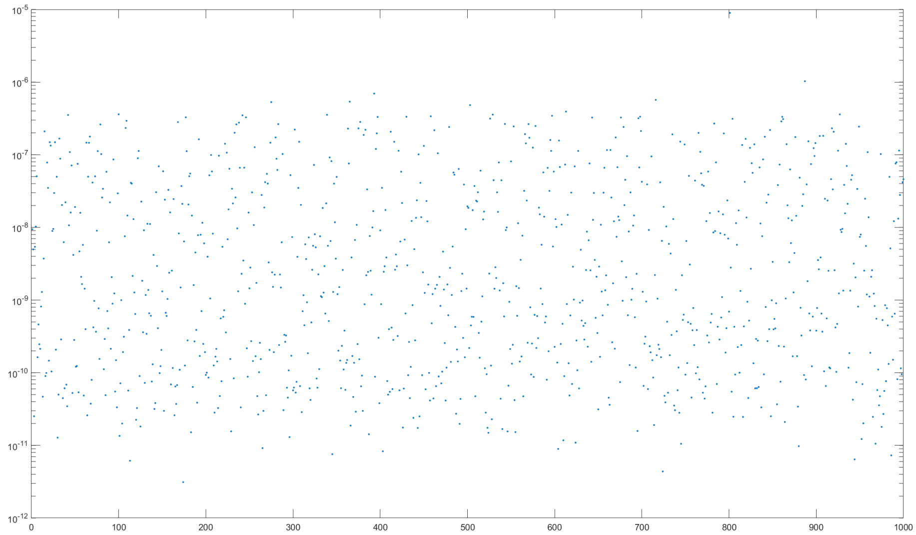

Figure 3. Max Differences

If we take a closer look at the max differences, we can see that all the solutions of the same equation is very close to each other, suggesting that the equation probably has only one positive solution.

2.2 Proof

From the assignment last time we know that $C = \frac{\Gamma}{\Gamma - 1}$ for $\Gamma \in (1, 2]$, therefore $C - 1 \geqslant 1$.

Define function $\Phi(x) := \frac{m^2}{E + x} + D\sqrt{1 - \frac{m^2}{(E + x)^2}} + Cx - E - x$, then take the derivative of $\Phi(x)$ over x , we get:

$$\Phi'(x) = (C - 1) - \frac{m^2}{(E + x)^2}\left(1 - \frac{D}{\sqrt{(E + x)^2 - m^2}}\right)$$

When $x \geqslant 0$, we get the following inequality:

$$\Phi'(x) = (C - 1) - \frac{m^2}{(E + x)^2}\left(1 - \frac{D}{\sqrt{(E + x)^2 - m^2}}\right) \geqslant 1 - \frac{m^2}{(E + x)^2} > 0, \forall x \in [0, +\infty)$$

Since we know

$$\Phi(0) = \frac{m^2}{E} + D\sqrt{1 - \frac{m^2}{E^2}} - E = \frac{D^2 + m^2 - E^2}{E} < 0$$

Combine with the fact that

$$\lim_{x \to +\infty} \frac{\Phi(x)}{x} = C - 1 \geqslant 1$$

The solution of $\Phi(x) = 0$ must be unique and between $(0, -\Phi(0))$.

3 Question III

3.1 Numerical Result

If we plot the figure for $|p - \Phi(p)|$, we can see that all values are very close to zero, suggesting that the equation probably holds.

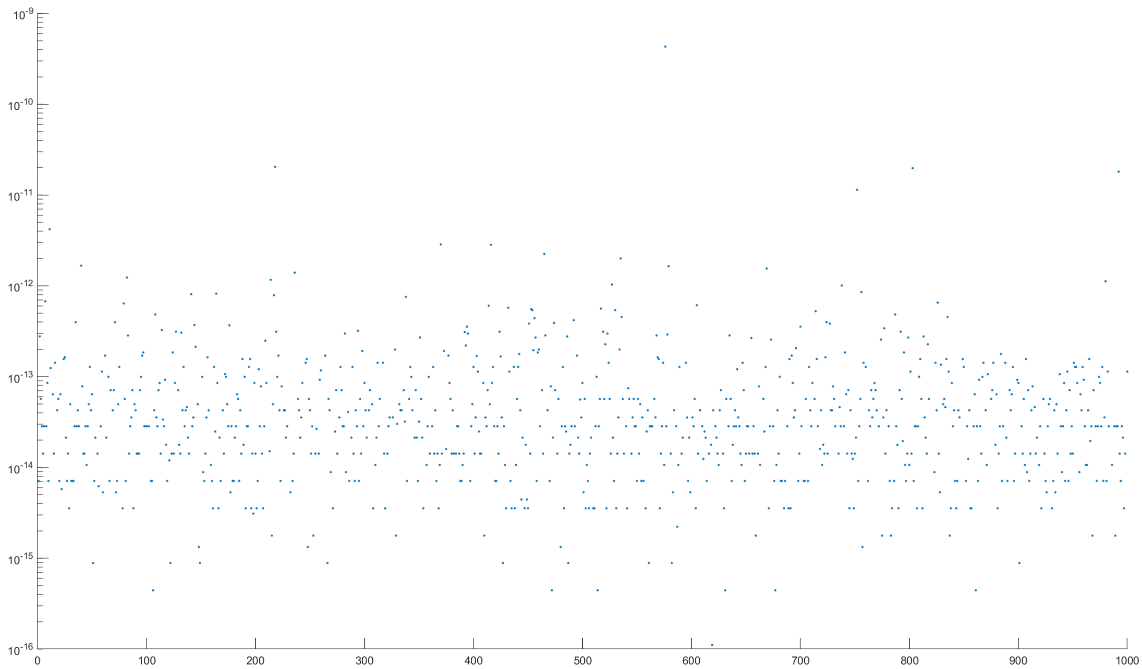

Figure 4. $|p - \Phi(p)|$

By definition we know that $E + p = \frac{m}{v}$

$$Cp + \rho = \frac{(\rho + Cp)}{1 - v^2}(1 - v^2)$$

$$Cp + \frac{(\rho + Cp)v^2}{1 - v^2} + \frac{\rho}{\sqrt{1 - v^2}} \cdot \sqrt{1 - v^2} = \frac{(\rho + Cp)}{1 - v^2}$$

$$Cp + \frac{m^2}{E + p} + D \cdot \sqrt{1 - v^2} = E + p$$

$$Cp + \frac{m^2}{E + p} + D \cdot \sqrt{1 - \left(\frac{m}{E + p}\right)^2} = E + p$$

as desired.

4 Question IV

By question III we know that for a given tuple $\{D, m, E\}$, equation

$$Cp + \frac{m^2}{E + p} + D \cdot \sqrt{1 - \left(\frac{m}{E + p}\right)^2} = E + p$$

has only one solution, p must be unique;

$\Rightarrow E + p$ is unique and non-zero, therefore $\frac{m}{v} = E + p$ is unique;

$\Rightarrow v$ is unique;

$\Rightarrow \rho = D(1 - v^2)$ is unique.

We can know conclude that $\{\rho, p, v\}$ is unique for each tuple $\{D, m, E\}$

Conversely, by the definition of $\{D, m, E\}$, we know it is unique for each $\{\rho, p, v\}$.

5 References

K. Wu and H. Tang

Physical-constraint-preserving central discontinuous Galerkin methods for special relativistic hydrodynamics with a general equation of state

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High-order accurate physical-constraints-preserving finite difference WENO schemes for special relativistic hydrodynamics

Journal of Computational Physics, 298:539–564, 2015.