1 Ramdom points in circles

1.1 Numerical Experiment

The points are generated with vectorized codes:

```
n = 1e4;

X1 = 2*rand(n,1)-1;
Y1 = (2*rand(n,1)-1).*sqrt(1-X1.^2);

R=rand(n,1);
Theta=2*pi*rand(n,1);
X2 = R.*cos(Theta);
Y2 = R.*sin(Theta);
```

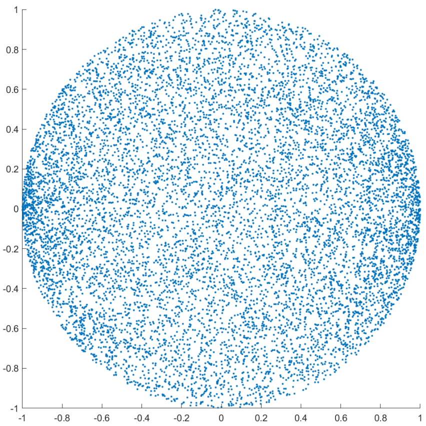

Fig. 1. First Method

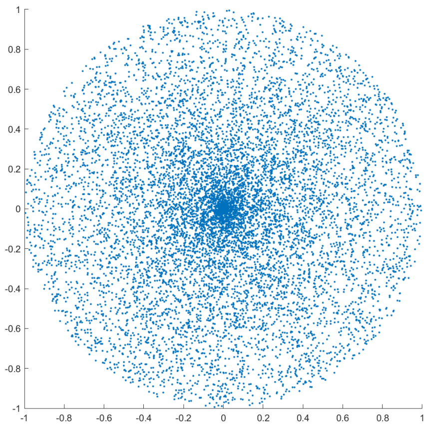

Fig. 2. Second Method

Through numerical experiments, it is easy to notice that neither methods seem to be able to produce uniform distribution on a circle: the first method is denser on the edge of the left and right sides, while the second one is denser at the centre. The proof of the observation will be covered in the following section.

1.2 Distribution Density

1.2.1 First method

In the first method, let the probability density of rand() devoted by

$$p(u)=p(v)=1$$

$$p(u,v)=1=p(u)p(v)$$

For the first method:

$$\begin{aligned} x &= 2u-1=g_1(u,v) \\ y &= (2v-1)\cdot(1-x^2)=(2v-1)(4u-4u^2)=g_2(u,v) \end{aligned}$$

thus:

$$\begin{aligned} u= & \frac{x-1}{2} = h_1(x,y) \\ v= & \frac{y}{2-2x^2}+\frac{1}{2} = h_2(x,y) \end{aligned}$$

$$J=\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} = \frac{\partial g_1}{\partial u}\frac{\partial g_2}{\partial v}-\frac{\partial g_2}{\partial u}\frac{\partial g_1}{\partial v}=2\cdot 2(1-x^2)-0\cdot(4-8u)=4(1-x^2)$$

$$\rightarrow |J^{-1}|=\frac{1}{4(1-x^2)}$$

$$\begin{aligned} p(x,y) &= p(h_1(u,v),h_2(u,v))|J^{-1}| \\ &= \frac{1}{4(1-x^2)}p\left(\frac{x-1}{2},\frac{y}{2-2x^2}+\frac{1}{2}\right) \\ &= \frac{1}{4(1-x^2)} \end{aligned}$$

which is not a constant, thus the distribution is not uniform.

1.3 *Second Method*

In the second method, let the probability density of rand() devoted by

$$p(r) = 1, p(\theta) = \frac{1}{2\pi}$$

$$p(r,\theta) = \frac{1}{2\pi} = p(r)p(\theta)$$

For the second method:

$$\begin{aligned} x &= r\cos\theta = g_1(r,\theta) \\ y &= r\sin\theta = g_2(r,\theta) \end{aligned}$$

thus:

$$\begin{aligned} r &= \sqrt{x^2+y^2} = h_1(x,y) \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) = h_2(x,y) \end{aligned}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} = r\cos\theta \cdot \cos\theta + r\sin\theta \cdot \sin\theta = r \\ &\rightarrow |J^{-1}| = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}} \end{aligned}$$

$$\begin{aligned} p(x,y) &= p\left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right)|J^{-1}| \\ &= |J^{-1}| \\ &= \frac{1}{2\pi\sqrt{x^2+y^2}} \end{aligned}$$

which is not a constant aswell.

1.4 Alternative Method

A few methods to generate uniform distribution in a circle are eleborated in math - Generate a random point within a circle (uniformly) - Stack Overflow, one of which will be demostrated below.

```
R=sqrt(rand(n,1));
Theta=2*pi*rand(n,1);
X2 = R.*cos(Theta);
Y2 = R.*sin(Theta);
```

The result is as followed, this time, no obvious cluster feature is observed:

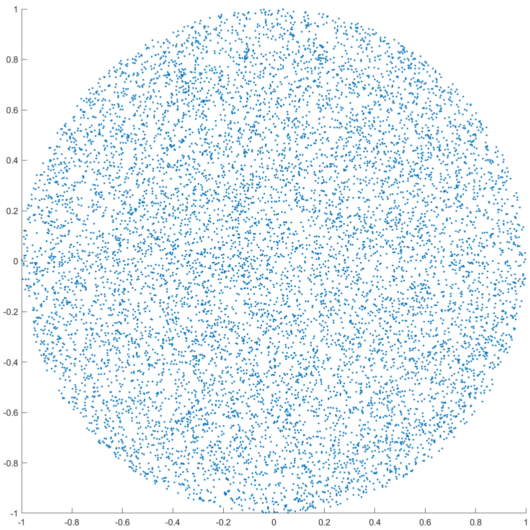

Fig. 3. Alternative

For the alternative method:

$$x = \sqrt{r}\cos\theta = g_1(r,\theta)$$
$$y = \sqrt{r}\sin\theta = g_2(r,\theta)$$

thus:

$$r = x^2 + y^2 = h_1(x,y)$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = h_2(x, y)$$

$$J = \begin{vmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \theta} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \theta} \end{vmatrix} = \frac{1}{2}$$

$$\rightarrow |J^{-1}| = 2$$

$$\begin{aligned} p(x, y) &= p\left(x^2 + y^2, \tan^{-1}\left(\frac{y}{x}\right)\right)|J^{-1}| \\ &= 2 \end{aligned}$$

2 Fluid Dynamics

2.1 Numerical Experiment

Many methods are eleaborated in math - Generate a random point within a circle (uniformly) - Stack Overflow, here one simple method will be covered.

Though program Main_Exp48.m, we could get the following plot

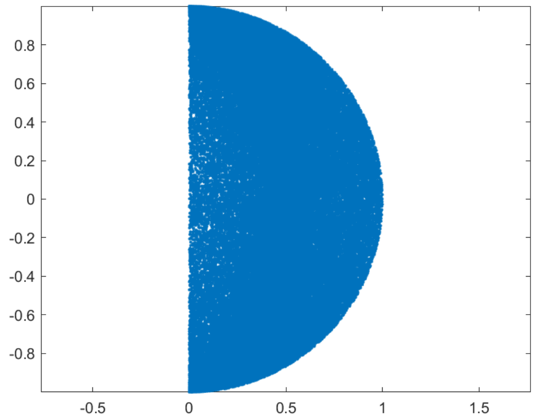

Fig. 4. Numerical Result

Our Initial conjecture would be that $D^2+m^2<E^2$, however, this equation can not show the whole picture.

2.2 Dimensional Analysis

If one apply dimensional analysis to D, m, E , we get the following result:

$$D \equiv \rho\gamma \equiv \mathrm{kg}/m^3 \cdot 1 \equiv \mathrm{kg} \cdot m^{-3}$$

$$m \equiv \rho h \gamma^2 v \equiv \mathrm{kg}/m^3 \cdot 1 \cdot 1^2 \cdot m/s \equiv \mathrm{kg} \cdot m^{-2} \cdot s^{-1}$$

$$E \equiv p \equiv N \cdot m^{-2} \equiv \mathrm{kg} \cdot m^{-1} \cdot s^{-2}$$

Since c is assumed as 1 in the program, and the unit of c is $m \cdot s^{-1}$. It is reasonable to propose the following claim:

$$D^2c^4 + m^2c^2 < E^2$$

2.3 Rigorous Proof

Proposition 1 $D^2c^4 + m^2c^2 < E^2$

Proof For $\forall S > 0$, since $|v| < 1$, it is easy to notice that

$$(1 - v^2)S^2 + 2S + 1 > 1$$

Plug in $c = 1$:

$$\left(\frac{4c^4}{c^4} - \frac{4c^2}{c^2} + \frac{c^2}{c^2} - \frac{v^2}{c^2}\right)S^2 + (4c^2 - 2c^2)S + c^4 > c^4$$

Rearrange the equation:

$$\left(1 + \frac{4S^2}{c^4} + \frac{4S}{c^2}\right)c^4 - \left(2S + \frac{4S^2}{c^2}\right)c^2 + S^2\frac{(c^2 - v^2)}{c^2} > c^4$$

Multiply γ^2 on both side:

$$\left(1 + \frac{4S^2}{c^4} + \frac{4S}{c^2}\right)\gamma^2c^4 - \left(2S + \frac{4S^2}{c^2}\right)\gamma^2c^2 + S^2 > \gamma^2c^4$$

$$\left(1 + \frac{2S}{c^2}\right)^2\gamma^4c^2(c^2 - v^2) - 2S\left(1 + \frac{2S}{c^2}\right)\gamma^2c^2 + S^2 > \gamma^2c^4$$

Since $h = 1 + \frac{\Gamma}{\Gamma - 1}\frac{p}{\rho c^2}$ where $1 < \Gamma \leqslant 2$, we can derive that $h \geqslant 1 + \frac{2S}{c^2}$, thus

$$h^2\gamma^4c^2(c^2 - v^2) - 2hS\gamma^2c^4 + S^2 > \gamma^2c^4$$

$$h^2\gamma^4c^4 - 2h\gamma^2c^2S + S^2 > \gamma^2c^4 + h^2\gamma^4v^2c^2$$

Multily ρ^2 on both side of the inequality:

$$\rho^2h^2\gamma^4c^4 - 2\rho^2h\gamma^2c^2S + \rho^2S^2 > \rho^2\gamma^2c^4 + \rho^2h^2\gamma^4v^2c^2$$

$$\rho^2h^2\gamma^4c^4 - 2\rho p h\gamma^2c^2 + p^2 > \rho^2\gamma^2c^4 + \rho^2h^2\gamma^4v^2c^2$$

$$(\rho h\gamma^2c^2 - p)^2 > (\rho\gamma)^2c^4 + (\rho h\gamma^2v)^2c^2$$

$$E^2 > D^2c^4 + m^2c^2 \qquad \square$$

then the claim is proved.