1 Q1

Gaussian quadrature is given by

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{i=1}^{n+1} w_i f(x_i)$$

Given that $P_i = \frac{w_i \sum_{j=0}^{n} \phi_j^2(x_i)}{n+1}$, we know that

$$P_i = \frac{w_i \sum_{j=0}^{n} \phi_j^2(x_i)}{n+1}$$

where

$$w_i = \frac{2n+3}{(1-x_i^2)\phi'_{n+1}(x_i)}$$

thus

$$\sum_{i=1}^{n+1} P_i = \frac{1}{n+1} \sum_{i=1}^{n+1} \sum_{j=0}^{n} w_i \phi_j^2(x_i)$$

$$\begin{aligned}
\sum_{i=1}^{n+1} P_i =& \frac{1}{n+1} \sum_{i=1}^{n+1} \sum_{j=0}^{n} w_i \phi_j^2(x_i) \\
=& \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=1}^{n+1} w_i \phi_j^2(x_i) \\
\approx& \frac{1}{n+1} \sum_{j=0}^{n} \int_{-1}^{1} \phi_j^2(x)\mathrm{d}x \\
=& \frac{1}{n+1} \sum_{j=0}^{n} 1 \\
=& 1
\end{aligned}$$

2 Q2

The problem can be rewrite as:

$$\arg\min_{c} \|\Phi \cdot \hat{c} - b\|$$

where $\Phi = \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \phi_j(x_i) & \vdots \\ \phi_0(x_{n+1}) & \phi_1(x_{n+1}) & \cdots & \phi_n(x_{n+1}) \end{bmatrix}$, and $b = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n+1}) \end{bmatrix}$

Apply Least Square Method to the problem, we get:

$$\hat{c} = (\Phi^T \Phi)^{-1} \Phi^T b = \Phi^{-1} b$$

Notice that $\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{i=1}^{n+1} w_i f(x_i)$, and

$$\int_{-1}^{1} \phi_i(x)\phi_j(x)\mathrm{d}x = 1 \text{ if } i = j$$
$$0 \text{ if } i \neq j$$

$$\begin{bmatrix} w_1\phi_0(x_1) & w_2\phi_0(x_2) & \cdots & w_{n+1}\phi_0(x_{n+1}) \\ w_1\phi_1(x_1) & w_2\phi_1(x_2) & \cdots & w_{n+1}\phi_1(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ w_1\phi_n(x_1) & w_2\phi_n(x_2) & \cdots & w_{n+1}\phi_n(x_{n+1}) \end{bmatrix} \begin{bmatrix} \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \phi_j(x_i) & \vdots \\ \phi_0(x_{n+1}) & \phi_1(x_{n+1}) & \cdots & \phi_n(x_{n+1}) \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^{n+1} w_i\phi_0^2(x_i) & \sum_{i=1}^{n+1} w_i\phi_0(x_i)\phi_1(x_i) & \cdots & \sum_{i=1}^{n+1} w_i\phi_0(x_i)\phi_n(x_i) \\ \sum_{i=1}^{n+1} w_i\phi_0(x_i)\phi_1(x_i) & \sum_{i=1}^{n+1} w_i\phi_1^2(x_i) & \cdots & \sum_{i=1}^{n+1} w_i\phi_1(x_i)\phi_n(x_i) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n+1} w_i\phi_0(x_i)\phi_n(x_i) & \sum_{i=1}^{n+1} w_i\phi_1(x_i)\phi_n(x_i) & \cdots & \sum_{i=1}^{n+1} w_i\phi_n^2(x_i) \end{bmatrix}$$
$$= I$$

Therefore, we can solve out Φ^{-1}b is

$$\Phi^{-1}b = \begin{bmatrix} w_1\phi_0(x_1) & w_2\phi_0(x_2) & \cdots & w_{n+1}\phi_0(x_{n+1}) \\ w_1\phi_1(x_1) & w_2\phi_1(x_2) & \cdots & w_{n+1}\phi_1(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ w_1\phi_n(x_1) & w_2\phi_n(x_2) & \cdots & w_{n+1}\phi_n(x_{n+1}) \end{bmatrix} \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n+1}) \end{bmatrix}$$

Therefore, we have proven that

$$\hat{c_k} = \sum_{i=1}^{n+1} \phi_k(x_i) w_i f(x_i)$$

3 Q3

3.1 Numerical Result

The numerical result is as followed:

Notice that the margin only drops significantly at the final step, suggesting that the difference is kept until the final step.

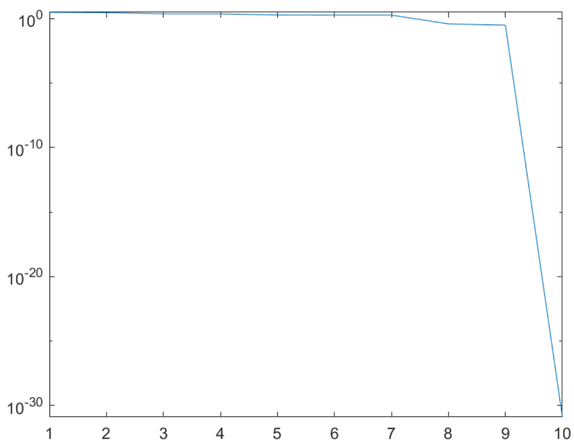

Figure 1. Numerical Result for n=9

If you can rewrite

$$\begin{aligned}
(\Phi\Phi^T)^{-1} =&(\Phi^T)^{-1}\Phi^{-1}\\
=&\begin{bmatrix} w_1\phi_0(x_1) & w_1\phi_1(x_1) & \cdots & w_1\phi_n(x_1) \\ w_1\phi_1(x_1) & w_2\phi_1(x_2) & \cdots & w_2\phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ w_{n+1}\phi_0(x_{n+1}) & w_{n+1}\phi_1(x_{n+1}) & \cdots & w_{n+1}\phi_n(x_{n+1}) \end{bmatrix}\\
&\times\begin{bmatrix} w_1\phi_0(x_1) & w_2\phi_0(x_2) & \cdots & w_{n+1}\phi_0(x_{n+1}) \\ w_1\phi_1(x_1) & w_2\phi_1(x_2) & \cdots & w_{n+1}\phi_1(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ w_1\phi_n(x_1) & w_2\phi_n(x_2) & \cdots & w_{n+1}\phi_n(x_{n+1}) \end{bmatrix}\\
=&\begin{bmatrix} \sum_{j=0}^n w_i^2\phi_j^2(x_1) & \sum_{j=0}^n w_i^2\phi_j(x_1)\phi_j(x_2) & \cdots & \sum_{j=0}^n w_i^2\phi_j(x_1)\phi_j(x_{n+1}) \\ \sum_{j=0}^n w_i^2\phi_j(x_1)\phi_j(x_2) & \sum_{j=0}^n w_i^2\phi_j^2(x_2) & \cdots & \sum_{j=0}^n w_i^2\phi_j(x_2)\phi_j(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^n w_i^2\phi_j(x_1)\phi_j(x_{n+1}) & \sum_{j=0}^n w_i^2\phi_j(x_2)\phi_j(x_{n+1}) & \cdots & \sum_{j=0}^n w_i^2\phi_j^2(x_{n+1}) \end{bmatrix}\\
=&\begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_{n+1} \end{bmatrix}
\end{aligned}$$

Thus

$$\begin{aligned}
&\begin{bmatrix} \sum_{j=0}^n \phi_j^2(x_1) & \sum_{j=0}^n \phi_j(x_1)\phi_j(x_2) & \cdots & \sum_{j=0}^n \phi_j(x_1)\phi_j(x_{n+1}) \\ \sum_{j=0}^n \phi_j(x_1)\phi_j(x_2) & \sum_{j=0}^n \phi_j^2(x_2) & \cdots & \sum_{j=0}^n \phi_j(x_2)\phi_j(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=0}^n \phi_j(x_1)\phi_j(x_{n+1}) & \sum_{j=0}^n \phi_j(x_2)\phi_j(x_{n+1}) & \cdots & \sum_{j=0}^n \phi_j^2(x_{n+1}) \end{bmatrix}\\
=&\Phi^T\Phi\\
=&\begin{bmatrix} 1/w_1 & & & \\ & 1/w_2 & & \\ & & \ddots & \\ & & & 1/w_{n+1} \end{bmatrix}
\end{aligned}$$

Therefore, Φ is orthogonal, thus Φ_i are perpendicular to each others.

Given that $\Phi_i = [\ \phi_0(x_1) \quad \phi_1(x_1) \quad \cdots \quad \phi_n(x_1)\ ]$, we can use the conclusion in **Q2**

$$\hat{c_k} = \sum_{i=1}^{n+1} \phi_k(x_i) w_i f(x_i) = \sum_{i=1}^{n+1} \frac{\phi_k(x_i) f(x_i)}{\sum_{j=0}^n \phi_j^2(x_i)}$$

$$\hat{c} = \sum_{i=1}^{n+1} \Phi_i w_i f(x_i) = \sum_{i=1}^{n+1} \frac{\Phi_i f(x_i)}{\|\Phi_i\|_2^2}$$

Notice that plug $\hat{c}$ into the iteration formula will produce itself.

Rewrite the iteration formula as:

$$\begin{aligned}
c^{(i+1)} =&c^{(i)} + \frac{f(x_i) - \Phi_i (c^{(i)})^T}{\|\Phi_i\|_2^2}\Phi_i\\
=&\left(1 - \frac{\Phi_i^T\Phi_i}{\Phi_i\Phi_i^T}\right)c^{(i)} + \frac{f(x_i)\Phi_i}{\Phi_i\Phi_i^T}
\end{aligned}$$

With this rewriten formula in mind, combining the fact that Φ_i are perpendicular to each others, the whole idea of the iteration becomes quite obivous each step corrects c on the direction component with respect to Φ_i:

$$
\begin{aligned}
c^{(i+1)} =& \left(1 - \frac{\Phi_i^T \Phi_i}{\Phi_i \Phi_i^T}\right) c^{(i)} + \frac{f(x_i)\Phi_i}{\Phi_i \Phi_i^T} \\
=& \left(1 - \frac{\Phi_{s-1}^T \Phi_i}{\Phi_i \Phi_i^T}\right)\left(\left(1 - \frac{\Phi_{i-1}^T \Phi_{i-1}}{\Phi_{i-1} \Phi_{i-1}^T}\right) c^{(i-1)} + \frac{f(x_{i-1})\Phi_{i-1}}{\Phi_{i-1} \Phi_{i-1}^T}\right) + \frac{f(x_i)\Phi_i}{\Phi_i \Phi_i^T} \\
=& \left(1 - \frac{\Phi_{s-1}^T \Phi_i}{\Phi_i \Phi_i^T} - \frac{\Phi_{i-1}^T \Phi_{i-1}}{\Phi_{i-1} \Phi_{i-1}^T}\right) c^{(i-1)} + \frac{f(x_{i-1})\Phi_{i-1}}{\Phi_{i-1} \Phi_{i-1}^T} + \frac{f(x_i)\Phi_i}{\Phi_i \Phi_i^T} \\
=& \cdots = \\
=& c^{(1)} - \sum_{m=1}^{i} \frac{\Phi_m^T \Phi_m}{\Phi_m \Phi_m^T} c^{(1)} + \sum_{m=1}^{i} \frac{f(x_m)\Phi_m}{\Phi_m \Phi_m^T}
\end{aligned}
$$

Plug in $i = n$, we get

$$
c^{(n+1)} = c^{(1)} - \sum_{m=1}^{n} \tfrac{\Phi_m^T \Phi_m}{\Phi_m \Phi_m^T} c^{(1)} + \sum_{m=1}^{n} \frac{f(x_m)\Phi_m}{\Phi_m \Phi_m^T}
$$

since Φ_i are perpendicular to each others, we know

$$
c^{(1)} = \sum_{m=1}^{n} \frac{\Phi_m^T \Phi_m}{\Phi_m \Phi_m^T} c^{(1)}
$$

Therefore,

$$
c^{(n+1)} = \sum_{m=1}^{n} \frac{f(x_m)\Phi_m}{\Phi_m \Phi_m^T} = \hat{c}
$$

as desired.