1

By applying the numerical scheme in the power point:

```
g(n+1) = g(n) - (f(n) + f(n).*(f(n).^2-g(n).^2))*h;
f(n+1) = f(n)*(n/(n+1))^2 - h*(n+0.5)^2/(n+1)^2*(v*g(n) ...
                                        + g(n).*(f(n).^2-g(n).^2));
```

Since each states can be directly calculated from the previous step, the method is easy to calculate and requires little memory. Also, the scheme does not require the steps to be equal length.

2

We could see the behavior of the equations by ploting them, where the blue line indicates f and the red line indicates g:

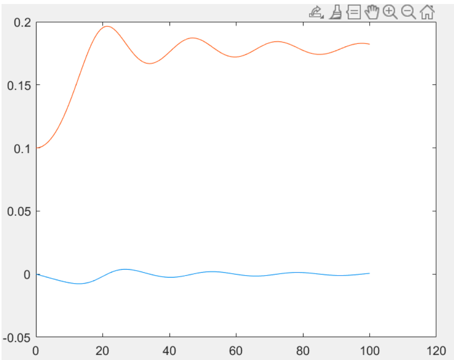

Figure 1.

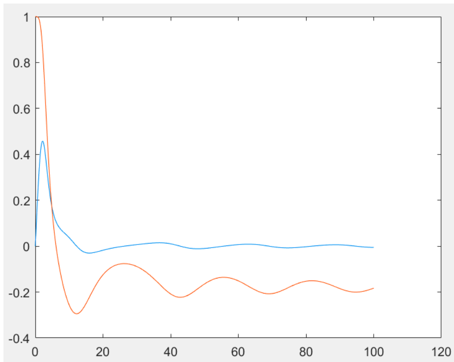

Figure 2.

Notice that if g_0 is small enough, then g will converges to around 0.17; if g_0 is larger, then g will converges to around -0.17; if g_0 is even larger, then g will goes to infinity:

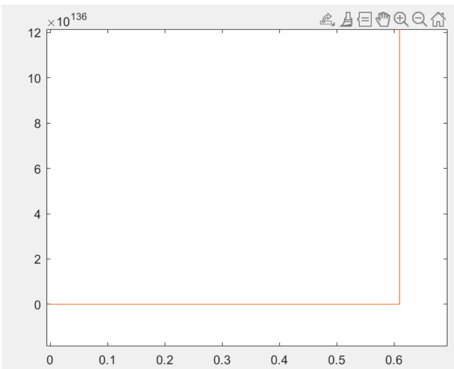

Figure 3.

After some research, will know the two transition point is around 0.65 and 1.37.

3

It is easy to verify that $f + f(f^2 - g^2)$ and $\frac{2f}{r} + vg + g(f^2 - g^2)$ satisfy Lipschitz condition on $[\eta, A]$ for $\forall \eta > 0$, conbine the fact that $\lim_{r\to 0^+} \frac{2f}{r} = 0$, we know that the equations have Lipschitz continuity on $[0, A]$, therefore by Existene and Uniqueness Theorem, f, g must exist and continuous on the inteval.

Since the functions are continuous on $[0, A]$, then they must be of Lipschitz continuous.

By the following theorem from GTM182, page307, the result is proved as desired:

Theorem. *Let* $\mathbf{f} : D \to \mathbb{R}^n$ *be continuous and locally Lipschitz continuous with respect to* $\mathbf{y}$. *Let the set* M *of all solutions of* (1) *that exist in all of the interval* $[a, b]$ *be nonempty. Then the sets* $M_a = \{\mathbf{y}(a) : \mathbf{y} \in M\}$ *and* $M_b = \{\mathbf{y}(b) : \mathbf{y} \in M\}$ *are open, and the Poincaré map* $P : M_a \to M_b$ *is a homeomorphism (i.e.,* P *is bijective,* P *and* P^{-1} *are continuous).*

Proof. Let $J = [a, b]$ and $\mathbf{z}(t) \in M$. As in 13.X, we first determine an $\alpha > 0$ with $S_\alpha = \{(t, \mathbf{y}) : t \in J, |\mathbf{y} - \mathbf{z}(t)| \le \alpha\} \subset D$ and extend $\mathbf{f}$ to the set $J \times \mathbb{R}^n$ while preserving the values in S_α, say, by setting

$$\mathbf{f}^*(t, \mathbf{y}) = \mathbf{f}(t, \mathbf{z}(t) + (\mathbf{y} - \mathbf{z}(t))h(|\mathbf{y} - \mathbf{z}(t)|))$$

with $h(s) = 1$ for $0 < s \le \alpha$ and $h(s) = \alpha/s$ for $s > \alpha$. For any $(t, \mathbf{y}) \in J \times \mathbb{R}^n$, the argument of $\mathbf{f}$ appearing in the above formula belongs to S_α, i.e., $\mathbf{f}^*$ is defined in all of $J \times \mathbb{R}^n$. Further, $\mathbf{f} = \mathbf{f}^*$ in S_α, and $\mathbf{f}^*$ is Lipschitz continuous with respect to $\mathbf{y}$ in $J \times \mathbb{R}^n$.

Applying Theorem 13.II with $\mathbf{k} = \mathbf{f}^*$, $\lambda = \boldsymbol{\eta}$, $\mathbf{g}(x, \lambda) = \boldsymbol{\eta}$, $\alpha(\lambda) = a$ leads to the conclusion that the solution $\mathbf{y}^*(t; a, \boldsymbol{\eta})$ of (1) depends continuously on $(t; a, \boldsymbol{\eta})$. In particular, there exists $\delta > 0$ such that if $|\boldsymbol{\eta} - \mathbf{z}(a)| < \delta$, then $|\mathbf{y}^*(t; a, \boldsymbol{\eta}) - \mathbf{z}(t)| < \alpha$ in J. Therefore, if $\boldsymbol{\eta}$ is in this range, then $\mathbf{y}^*(t; a, \boldsymbol{\eta}) = \mathbf{y}(t; a, \boldsymbol{\eta}) \in M$ and $P\boldsymbol{\eta} = \mathbf{y}(b; a, \boldsymbol{\eta})$ is continuous. Since $\mathbf{z}(t) \in M$ is arbitrary, it follows that M_a is open and P is continuous in M_a. The continuity of P^{-1} and the openness of M_b follow in a corresponding manner. ∎

Figure 4.

4

The most straight forward method would be bisection method:

```
a = 0;
b = 1;
v = 0.032;
gx = 1;
while abs(gx)>1e-5||abs(b-a)>1e-15
  x = (b+a)./2;
  sprintf('%0.55f',x)
  gx = fg(x,v);
  if gx<=0
   b=x;
  else
   a=x;
  end
end

function gt = fg(g0,v)
steps = 1e8;
f(1) = 0;
g(1) = g0;
```

```
r = 100;
h = r/steps;
f = zeros(steps,1);
g = zeros(steps,1);
f(1) = - h*(0.5)^2*(v*g0 + g0.*(-g0.^2));
g(1) = g0;
for n = 1:steps
 g(n+1) = g(n) - (f(n) + f(n).*(f(n).^2-g(n).^2))*h;
 f(n+1) = f(n)*(n/(n+1))^2 - h*(n+0.5)^2/(n+1)^2*(v*g(n) +
g(n).*(f(n).^2-g(n).^2));
end
gt = g(end);
end
```

Run the code, we can get the

$$g_0 = 0.652037247666157782077789306640625$$

and following result:

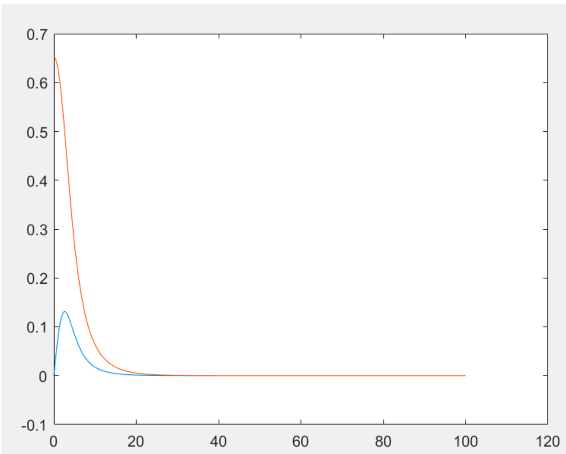

Figure 5.

5

Similar result to the previous problem, taking

$$g_0 = 1.088415735446411680698020063573494553565979003906 25$$

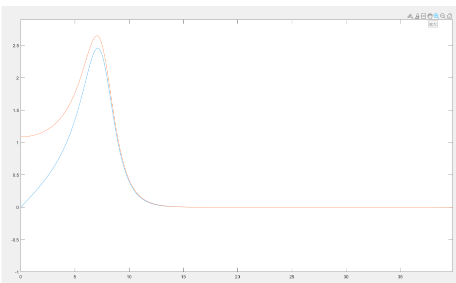

Figure 6.