1 Introduction

Define function $\Phi(x) := \frac{m^2}{E+x} + D\sqrt{1 - \frac{m^2}{(E+x)^2}} + Cx - E - x$

By Homework 6, we already know that

$$\Phi'(x) = (C-1) - \frac{m^2}{(E+x)^2}\left(1 - \frac{D}{\sqrt{(E+x)^2 - m^2}}\right) \geqslant 1 - \frac{m^2}{(E+x)^2} > 0, \forall x \in [0, +\infty)$$

Note that:

$$\Phi'(x) = (C-1) - \frac{m^2}{(E+x)^2}\left(1 - \frac{D}{\sqrt{(E+x)^2 - m^2}}\right) < C - 1$$

2 Bisection method

By Homework 6, we already know that

$$\Phi'(x) = (C-1) - \frac{m^2}{(E+x)^2}\left(1 - \frac{D}{\sqrt{(E+x)^2 - m^2}}\right) \geqslant (C-1) - \frac{m^2}{(E+x)^2}$$

Solve equation

$$-\Phi(0) = \int_0^t (C-1) - \frac{m^2}{(E+x)^2} \mathrm{d}x$$

we get

$$t(C-1) + \frac{m^2}{E+t} - \frac{m^2}{E} = -\Phi(0)$$

which is the upper bound of p , since $p(C-1) > C-1$, we can derive our bisection method in

$$\left(-\frac{\Phi(0)}{C-1}, t\right)$$

But the equation might be hard to solve, so instead, because $t(C-1) < \frac{m^2}{E} - \Phi(0)$, the upper bound could be $\frac{1}{C-1}\left(\frac{m^2}{E} - \Phi(0)\right)$.

2.1 Numerical Result

The numerical results of a thousand times experiments give a convincing output, where all the error is below 0.01.

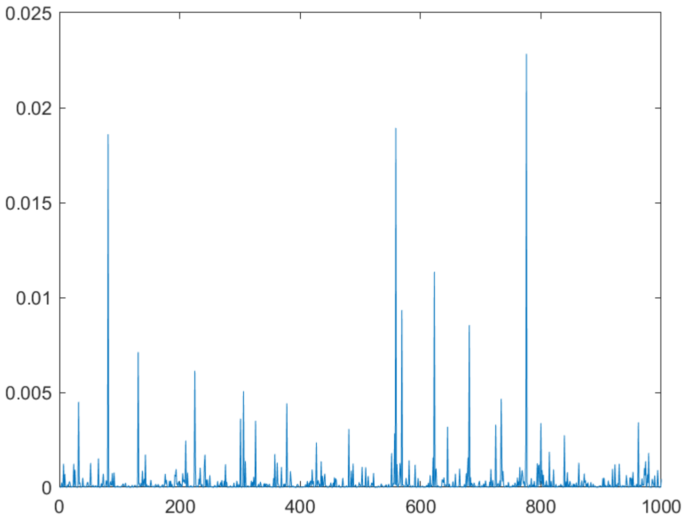

Figure 1.

3 Fixed Point Iteration

Construct the function by:

$$f(x) = x - \frac{\Phi(x)}{C}$$

then we have $f'(x) = 1 - \frac{\Phi'(x)}{C}$, which has the property that:

$$0 < 1 - \frac{\Phi'(x)}{C} < 1$$

Since we had proved that $\Phi(x)$ has a unique positive solution, point p must be the only fixed point of equation

$$x = f(x)$$

Note that we have $f'(x) = 1 - \frac{\Phi'(x)}{C}$, which implies that:

$$0 < f'(x) = 1 - \frac{\Phi'(x)}{C} < 1$$

Therefore point p is attractive, garantee the convergence of the method.

3.1 Numerical Result

3.1.1 Graph Analysis

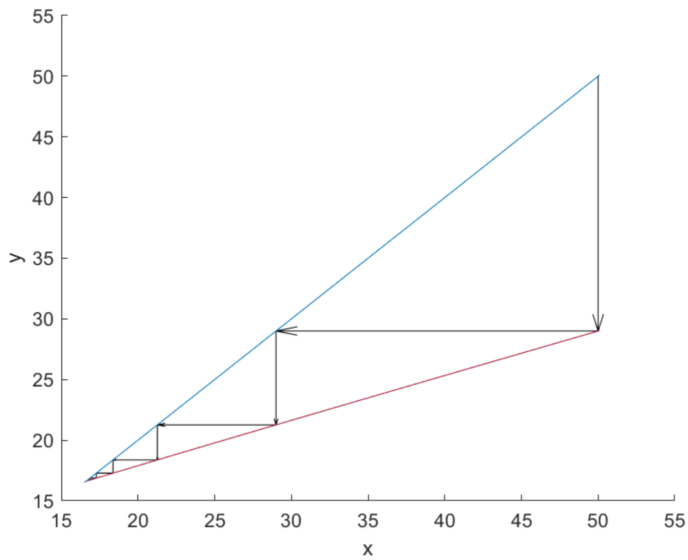

Figure 2. Graph Analysis of a single run

3.1.2 Ramdom Experiments

```
n = 1e3;
GAM = rand(n,1)+1;
C = (GAM./(GAM-1));
clear GAM
rho = rand(n,1);
p = rand(n,1)*100;
```

```
v = (2*rand(n,1)-1);
t = (2*rand(n,1)-1);
ga = 1./sqrt(1-v.^2);
h = 1+C.*(p./rho);
D = rho.*ga;
m = rho.*h.*ga.^2.*v;
E = rho.*h.*ga.^2-p;
Phi0 = m.^2./(E) + D.*sqrt(1-m.^2./(E).^2) - E;
X = -Phi0.*(C-1);
for k = 1:200
 X = -(m.^2./(E+X) + D.*sqrt(1-m.^2./(E+X).^2) - E-X)./C;
end
semilogy(abs(X-p))
```

Run the experiment above, we get the following result:

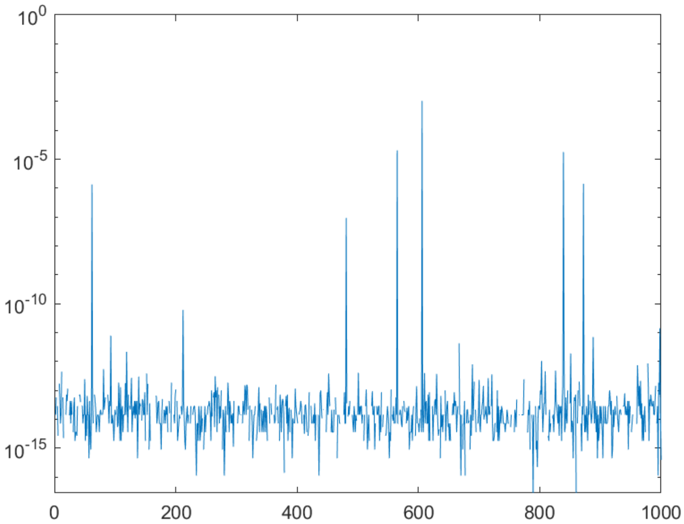

Figure 3. 1000 times experiments

We can see that the result converges quite well.