The Slippery Slide

A differentially-recursive transition function

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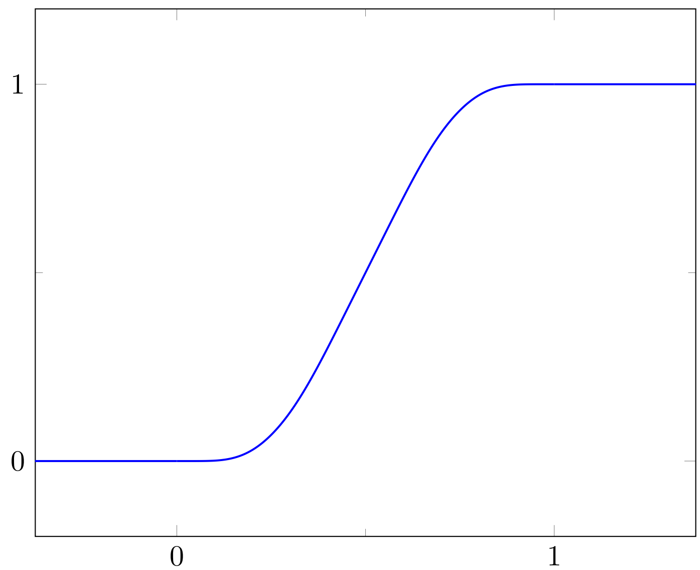

Abstract

We describe a transition function which is in some sense optimal. This function is smooth, symmetric, monotonic, and self-similar with its own derivative, in the sense that $s'(x) \propto s(2x)$ on $\left[0, \frac{1}{2}\right]$. We show it is the single unique transition function satisfying these criteria, prove it to be non-analytic on the entire unit interval, and develop an effective algorithm for calculating its values.

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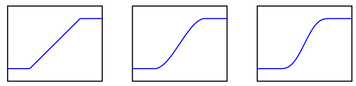

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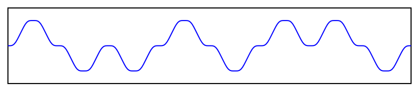

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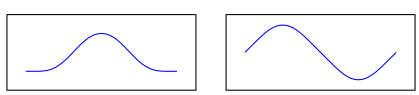

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1 Introduction

In this paper we ask "What is the smoothest possible ramp?" We treat this as a purely mathematical question, free from any real-world engineering concerns. We describe several properties that a maximally-smooth ramp should have, and investigate one possible answer in detail.

Section 1 provides an overview of the paper and its mathematical preliminaries. Section 2 describes ramps as transition functions, looks at their basic properties, and offers some examples. Section 3 introduces the main subject of the paper: a function $s(x)$ that we posit as the smoothest possible ramp. We prove this function exists and is unique, then study it in greater depth.

Section 4 shows that $s(x)$ is not analytic at any point of the ramp. Section 5 finds relationships between values of $s(x)$ across powers of two, section 6 solves them recursively, and section 7 turns those results into algorithms. Section 8 looks at the asymptotic rate that $s(x)$ approaches zero, which leads to a faster computation method for very small x. Section 9 creates modified versions of $s(x)$ with various properties. And section 10 summarizes our results.

Mathematical background

This paper makes extensive use of single-variable calculus. The reader should be familiar with the standard properties of integrals and derivatives, such as linearity and change of variables. In particular, we repeatedly use the change of variables $c - t \to t$, and $ct \to t$, where c is a constant:

$$\int_a^b f(c-t)\,dt = \int_{c-b}^{c-a} f(t)\,dt \qquad\qquad \int_a^b c\cdot f(ct)\,dt = \int_{ac}^{bc} f(t)\,dt$$

We also use factorials, $n! = 1\cdot 2\cdot 3\cdots n$, and the choose function $\binom{n}{k}$. The case of $\binom{n}{2}$ comes up often enough that we mention it specifically as well:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad\qquad \binom{n}{2} = \frac{n(n-1)}{2}$$

Our indices of summation will always be natural numbers, hence non-negative. Sometimes we sum over odd numbers only. Thus we may write:

$$\sum_{k<6}^{k \text{ odd}} f(k) = f(1) + f(3) + f(5)$$

At various points we make use of standard results, which we state without proof. For example, we use the contraction mapping theorem in section 3, and the Cauchy-Hadamard theorem in section 4. Interested readers can find a wealth of information on these results in the literature, including rigorous proofs.

2 Transition functions

When a quantity changes from one fixed value to another over the course of some interval, we call this a transition and model it with a **transition function**. Without loss of generality, we may choose a coordinate system where the initial value is 0, the final value is 1, and the duration of the transition is $[0, 1]$. Thus we define a transition function $f : \mathbb{R} \to \mathbb{R}$ to satisfy $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x \geq 1$.

The function's behavior on the unit interval $(0, 1)$ describes the progress of the transition itself. In this paper we will focus on those transition functions which are continuous, monotonic, and symmetric. By **symmetric** we mean that moving away from the midpoint $x = \frac{1}{2}$ yields equal changes in opposite directions, hence $f\left(\frac{1}{2} + x\right) - f\left(\frac{1}{2}\right)$ is an odd function.

It follows readily that $f\left(\frac{1}{2}\right) = \frac{1}{2}$, by considering the known values of a transition function at 0 and 1. As a result, the symmetry condition is equivalent to,

$$f(x) = 1 - f(1 - x) \tag{2.1}$$

as can be seen by substituting $x + \frac{1}{2}$ into equation 2.1.

Furthermore, given that the function is both monotonic and continuous, the symmetry condition ensures that its graph must partition the unit square into two identical halves. Thus,

$$\int_0^1 f(x)\, dx = \frac{1}{2} \tag{2.2}$$

Having established these properties, next we will look at some illustrative examples of transition functions. As a matter of notation, we will often denote a transition function by a formula. It is to be understood that such a formula applies on the unit interval $(0, 1)$, with the function taking the initial and final values 0 and 1 respectively outside that interval.

Polynomial transitions

The simplest transition function is given by the identity function $f(x) = x$ on the unit interval. This models a linear transition, as might apply to an object traveling at constant velocity from start to finish, or a bucket being filled from a spigot with constant flow rate.

The derivative of this linear transition function is discontinuous at 0 and 1, which makes it unsuitable for modeling many real-world physical processes. After all, a moving object must accelerate and decelerate when it starts and stops, and a faucet cannot instantaneously turn on and off.

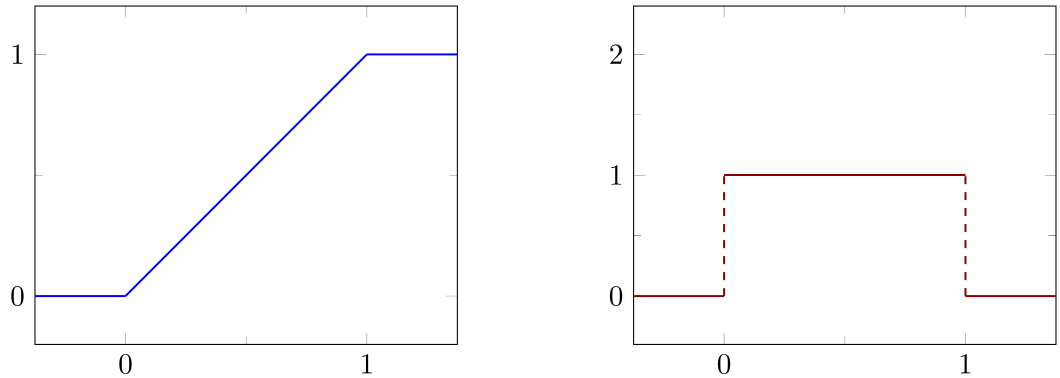


Figure 2.1: The transition function $f(x) = x$ (left) and its derivative (right).

In order to have a continuous derivative throughout the entire transition, the cubic function $f(x) = 3x^2 - 2x^2$, known as `smoothstep`, is sometimes used. This produces a characteristic "S-curve" shape, wherein the transition begins slowly, ramps up faster in the middle, then slows down toward the end.

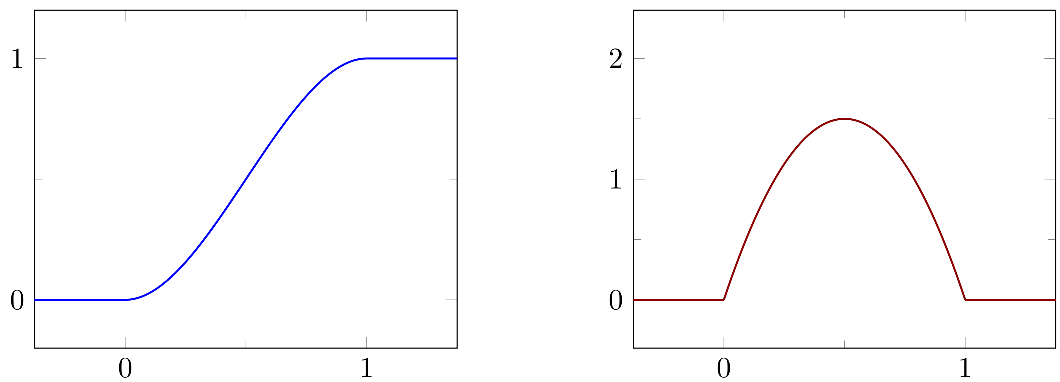


Figure 2.2: The transition function $f(x) = 3x^2 - 2x^3$ and its derivative.

Although the derivative of `smoothstep` is continuous, its second derivative is not. Thus, when used to model the position of a moving object, the acceleration would instantaneously jump between distinct values, which is not ideal. To alleviate this issue, a higher-order polynomial can produce a transition function whose second derivative is continuous. For example, the fifth-degree polynomial $f(x) = 6x^5 - 15x^4 + 10x^3$, which is sometimes called `smootherstep`.

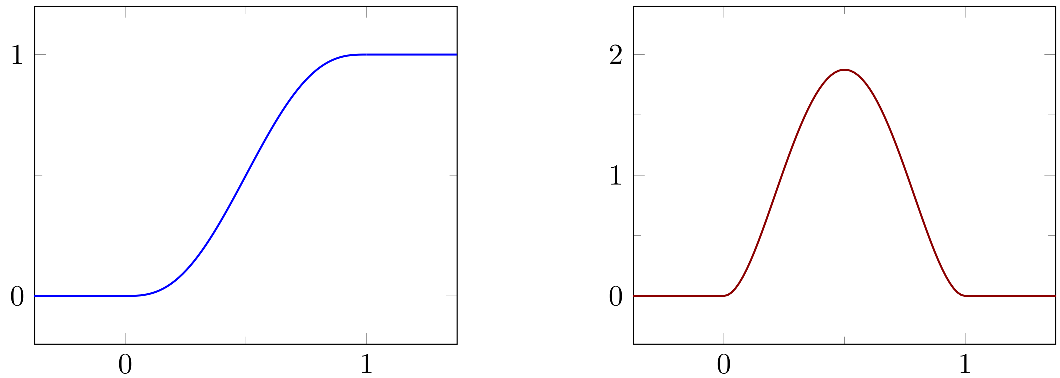


Figure 2.3: The transition function $f(x) = 6x^5 - 15x^4 + 10x^3$ and its derivative.

Trigonometric transitions

It is easy to see that every polynomial transition function must have some nth derivative which is discontinuous, namely the degree of its lowest-order term. In contrast, many real-world processes are best modeled as infinitely differentiable. One might hope that a sinusoidal transition function could perhaps achieve this goal, however that is not the case. They too exhibit discontinuous derivatives of some order at the endpoints.

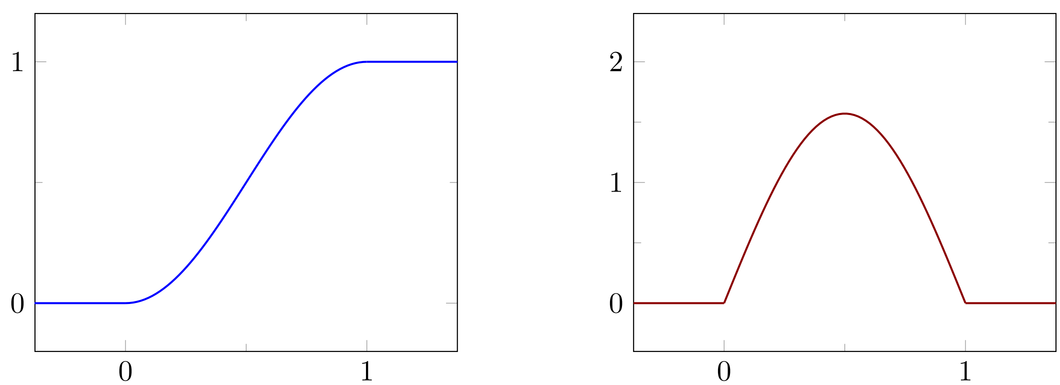


Figure 2.4: The transition function $f(x) = \frac{1}{2}(1 - \cos(\pi x))$ and its derivative.

The appearance of $f(x) = \frac{1}{2}(1 - \cos(\pi x))$ is quite similar to that of the cubic `smoothstep` function, and likewise $f(x) = x - \frac{1}{2\pi}\sin(2\pi x)$ is reminiscent of the quintic `smootherstep`. The former two have discontinuous second derivatives, while the latter two have discontinuous third derivatives.

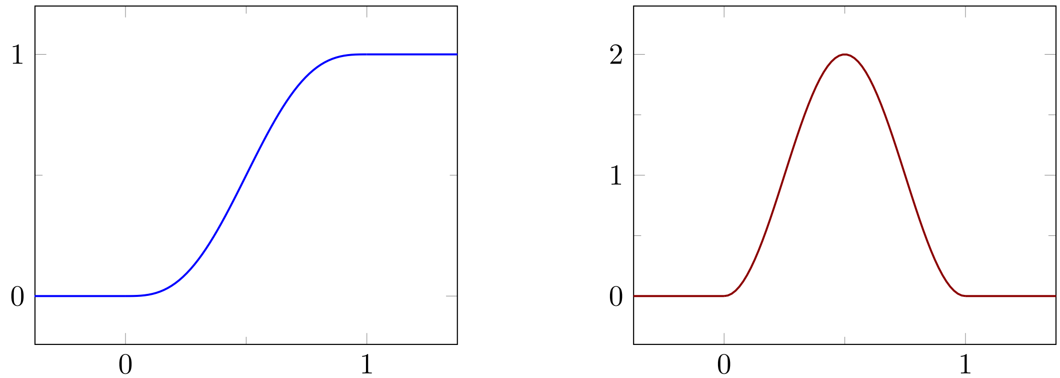


Figure 2.5: The transition function $f(x) = x - \frac{1}{2\pi}\sin(2\pi x)$ and its derivative.

At this point it might seem plausible that discontinuities in higher derivatives are unavoidable. However, perhaps surprisingly, they are not.

Smooth transitions

With care it is possible to construct a transition function which is truly **smooth**, meaning it has continuous derivatives of all orders including at the endpoints of the transition. One well-known example is:

$$f(x) = \left(1 + e^{\frac{1}{x} - \frac{1}{1-x}}\right)^{-1}$$

To see why this is smooth, let $g(x) = e^{-1/x}$ for $x > 0$, and 0 otherwise. Repeated differentiation shows that $g^{(n)}(x) = g(x) \cdot p_n\big(x^{-1}\big)$ for positive x, where p_n is a polynomial. The exponential term dominates as x approaches zero, and it follows that $g^{(n)}(0) = 0$ for all n.

Observe that $f(x) = g(x) \,/\, \big(g(x) + g(1-x)\big)$ on the unit interval, so all the derivatives of f go to zero at $x = 0$ as well. And since $f(x) + f(1-x) = 1$, by symmetry the derivatives at $x = 1$ also vanish. So $f(x)$ is indeed a smooth transition function.

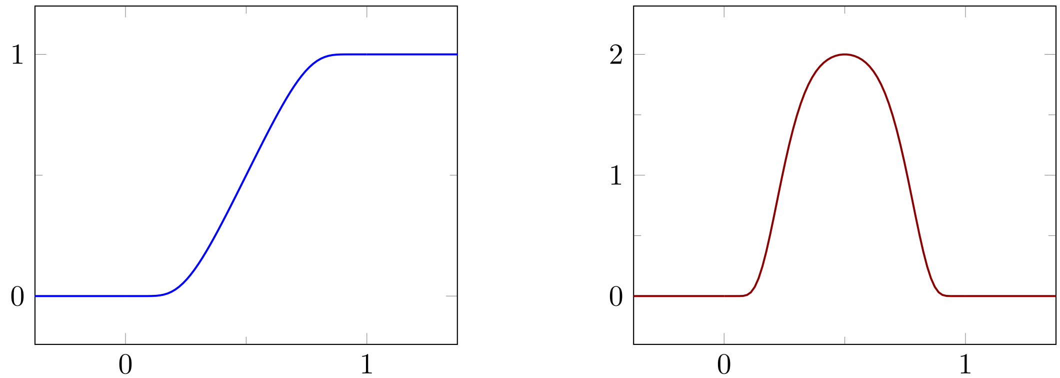


Figure 2.6: The transition function $f(x) = \left(1 + e^{\frac{1}{x} - \frac{1}{1-x}}\right)^{-1}$ and its derivative.

This function, then, begins at 0 with all its derivatives 0, yet still manages to peel off and rise up away from 0. Looking at the graph of the derivative, we see that the function's slope rises up from 0 to its maximum value at $x = \frac{1}{2}$. In other words, the derivative itself undergoes a transition on some interval, as does the second derivative, the third, and all higher derivatives as well.

By the same reasoning, every smooth transition function embodies an infinite family of transitions, one for each of its derivatives. And those transitions can each be modeled by a transition function. In the next section, we describe a function for which all those infinitely many transitions are one and the same.

3 The slippery slide

Throughout mathematics, it is often fruitful to study objects of maximal symmetry. Among rectangles, that is the square. Among ellipses, the circle. Such objects are sometimes called "natural" or "canonical" examples of their kind. Here we apply the same principle to transition functions, and set out to find one which is maximally symmetric.

We have already defined a symmetric transition function as one satisfying the relation $f(x) = 1 - f(1 - x)$, but that alone is not sufficient. We further require that the function must be continuous, monotonic, and infinitely differentiable. Yet still, one more criterion is needed.

For a transition function to be truly natural, it should stand on its own and not rely upon any other transition function. In particular, as its derivative rises up from zero to some maximum, the transition which describes the derivative should itself be a rescaled version of the original function.

This bears an additional moment of consideration. We say that a transition function is maximally symmetric if its derivative rises up with the same shape as the function itself, then drops back down symmetrically as well. That way there is only one transition function involved, even when we take higher and higher derivatives.

It is not readily apparent that such a function actually exists, nor if one does exist that it must be unique. We will demonstrate both existence and uniqueness by constructing the function, and produce a practical algorithm to calculate its value for any input.

Basic properties

Let us accept for the moment that a maximally symmetric transition function does exist, and learn what we can about it. We will denote this function by $s(x)$, and as we require its derivative to be a rescaled copy of the function itself, it follows that when $x \leq \frac{1}{2}$ there is a constant c such that,

$$s'(x) = c \cdot s(2x)$$

The value of c can be found using the fundamental theorem of calculus:

$$s\left(\tfrac{1}{2}\right) = \int_0^{1/2} s'(t)\,dt = \int_0^{1/2} c \cdot s(2t)\,dt = \tfrac{1}{2}c \int_0^1 s(t)\,dt$$

where the last equality holds by substituting $2t \to t$. Now by equation 2.2 and the symmetry of $s(x)$, both $s\left(\frac{1}{2}\right)$ and the last integral equal $\frac{1}{2}$. Thus we have $\frac{1}{2} = \frac{1}{4}c$, which means $c = 2$, and so:

$$s'(x) = 2\,s(2x) \tag{3.1}$$

Applying the fundamental theorem of calculus again, for any $x \in \left[0, \frac{1}{2}\right]$ we find:

$$s(x) = \int_0^x s'(t)\,dt = \int_0^x 2\,s(2t)\,dt = \int_0^{2x} s(t)\,dt$$

Substituting $\frac{1}{2}x$ into the above gives an identity, valid on the whole unit interval, that will be among our most powerful tools for analyzing s going forward:

$$s\left(\tfrac{1}{2}x\right) = \int_0^x s(t)\,dt \tag{3.2}$$

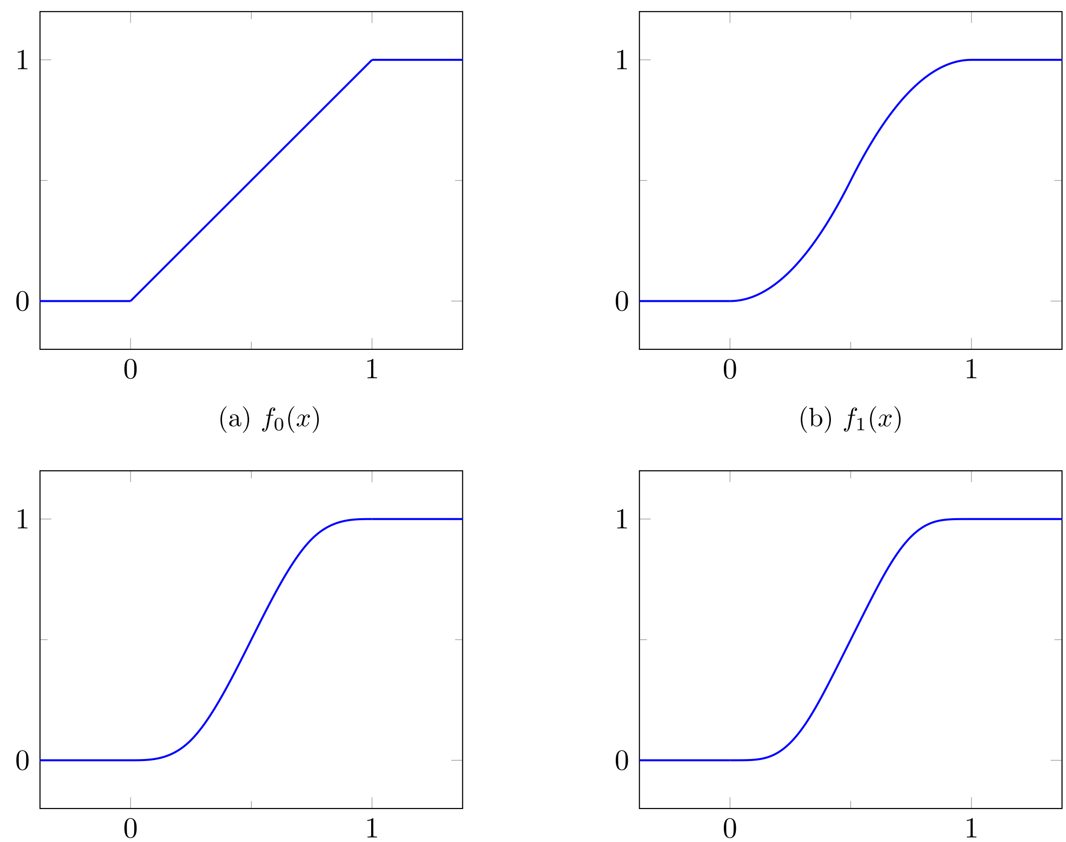
Iterated integrals

Using equation 3.2 and symmetry, we can define an iterated sequence of functions which has $s(x)$ as a fixed point:

$$f_n(x) = \begin{cases} \displaystyle\int_0^{2x} f_{n-1}(t)\,dt & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - f_n(1-x) & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \tag{3.3}$$

By induction, if f_0 is a continuous, monotonic, symmetric transition function, then so is f_n for all n. We will soon show that the iteration converges to $s(x)$ for any such f_0. (The true basin of convergence is even larger.)

Now, starting with a convenient initial function such as $f_0(x) = x$, we may construct $f_n(x)$ either symbolically or numerically. The first few iterations are shown in figure 3.1.



(a) $f_0(x)$ (b) $f_1(x)$ (c) $f_2(x)$ (d) $f_3(x)$

Figure 3.1: The first four iterations from $f_0(x) = x$.

Continuing the iteration, we observe that the sequence appears to converge to the function f_∞ shown in figure 3.2. Indeed, using numerical integration on this sequence provides our first method of calculating $s(x)$.

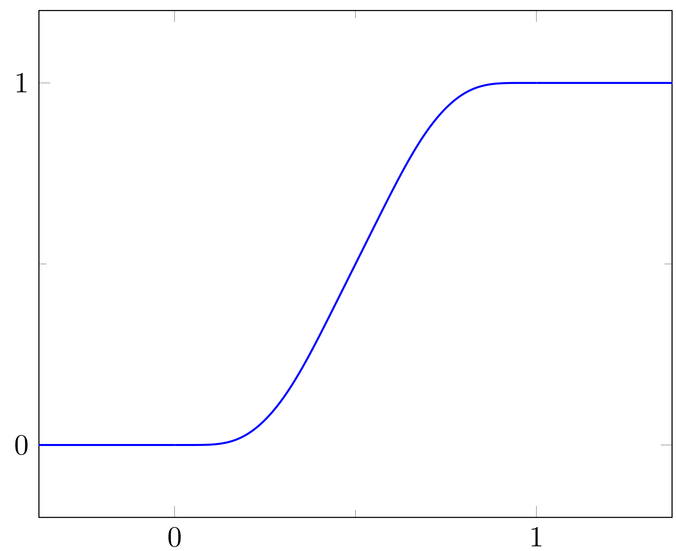


Figure 3.2: $s(x) = f_\infty(x)$

These numerical calculations shed some light on the behavior of $s(x)$. Notably, the values at multiples of $\frac{1}{8}$ appear to have repeating decimals, and pairs of values at $\frac{1}{4} \pm d$ appear to differ by $2d$ for any $0 \leq d \leq \frac{1}{4}$. Table 3.1 shows the numerical value of $s(x)$ at multiples of $\frac{1}{16}$ from 0 to $\frac{1}{2}$, and the equivalent fraction for those with repeating digits. Values above $\frac{1}{2}$ follow by symmetry.

x	**s(x)**	**fraction**
0/16	0.0	0
1/16	0.000068962...	
2/16	0.003472222...	1/288
3/16	0.022500482...	
4/16	0.069444444...	5/72
5/16	0.147500482...	
6/16	0.253472222...	73/288
7/16	0.375068962...	
8/16	0.5	1/2

Table 3.1: Numerically-computed values of $s(x)$.

Existence and uniqueness

We are now ready to prove that $s(x)$ exists and is unique. To do so, we will proceed as with the iteration above, only restricted to the interval $0 \leq x \leq \frac{1}{2}$ where it is readily shown to be a contraction map. Then we will invoke the contraction mapping theorem to establish that a unique fixed point exists.

To begin, let $f : \left[0, \frac{1}{2}\right] \to \mathbb{R}$ be a continuous function, and define T as the transformation of functions which is equivalent to the iteration in equation 3.3 restricted to the half-interval:

$$T(f) = \begin{cases} \displaystyle\int_0^{2x} f(t)\,dt & \text{if } 0 \le x \le \frac{1}{4} \\ \displaystyle\int_0^{1/2} f(t)\,dt + \int_{1/2}^{2x} \Big(1 - f(1-t)\Big)\,dt & \text{if } \frac{1}{4} < x \le \frac{1}{2} \end{cases} \tag{3.4}$$

Note that this definition ensures $T(f)$ is continuous.

In the last integral of equation 3.4, the substitution $(1-t) \to t$ yields,

$$\int_{1/2}^{2x} \Big(1 - f(1-t)\Big)\,dt = 2x - \tfrac{1}{2} + \int_{1/2}^{1-2x} f(t)\,dt$$

which lets us combine the two integrals in the second case since they share the integrand $f(x)$ and their bounds meet at $\frac{1}{2}$. This gives:

$$T(f) = \begin{cases} \displaystyle\int_0^{2x} f(t)\,dt & \text{if } 0 \le x \le \frac{1}{4} \\ 2x - \dfrac{1}{2} + \displaystyle\int_0^{1-2x} f(t)\,dt & \text{if } \frac{1}{4} < x \le \frac{1}{2} \end{cases}$$

A further simplification follows by observing that in both integrals, the upper bound may be written as $\frac{1}{2} - \left|2x - \frac{1}{2}\right|$. When $x \le \frac{1}{4}$ this gives $2x$, and when $x > \frac{1}{4}$ it gives $1 - 2x$. Similarly, the terms outside the integrals may in both cases be written as $x - \frac{1}{4} + \left|x - \frac{1}{4}\right|$. This lets us combine the two cases and obtain:

$$T(f) = x - \tfrac{1}{4} + \left|x - \tfrac{1}{4}\right| + \int_0^{\frac{1}{2} - \left|2x - \frac{1}{2}\right|} f(t)\,dt \tag{3.5}$$

Note in particular that this implies $T(f)\left(\frac{1}{2}\right) = \frac{1}{2}$, regardless of f.

Next we will show that T is a contraction map, which means it reduces the distance between functions by at least a multiplicative factor $0 \le k < 1$. For this purpose, we use the uniform norm (also called the sup-norm) to measure the distance between functions:

$$\left|f - g\right| = \sup\left|f(x) - g(x)\right|$$

It is a standard result that the continuous functions on a closed interval form a complete metric space under the uniform norm. The contraction mapping theorem states that a contraction map on a complete metric space admits a unique fixed point, and the iteration of the map from any initial value converges to it. Thus, by showing that T is a contraction map, it will follow that exactly one continuous function is a fixed point of T, and we can find it by iteration.

Consider two continuous functions f and g from $\left[0, \frac{1}{2}\right]$ to $\mathbb{R}$, and observe that for any value of x, the formulas for $T(f)$ and $T(g)$ given by equation 3.5 have identical bounds on their integrals, and identical terms outside the integrals. Thus when we subtract them we find,

$$T(f)(x) - T(g)(x) = \int_0^{\frac{1}{2}-\left|2x-\frac{1}{2}\right|} f(t) - g(t)\,dt$$

Now let us calculate the distance between transformed functions:

$$\begin{aligned}
\left|T(f) - T(g)\right| &= \sup\left|T(f)(x) - T(g)(x)\right| && \text{(uniform norm)} \\
&= \sup\left|\int_0^{\frac{1}{2}-\left|2x-\frac{1}{2}\right|} f(t) - g(t)\,dt\right| && \text{(apply transform } T\text{)} \\
&\leq \sup\int_0^{\frac{1}{2}-\left|2x-\frac{1}{2}\right|} \left|f(t) - g(t)\right| dt && \text{(triangle inequality)} \\
&= \int_0^{1/2} \left|f(t) - g(t)\right| dt && \text{(maximize bounds)} \\
&\leq \int_0^{1/2} \sup\left|f(x) - g(x)\right| dt && \text{(overestimate integrand)} \\
&= \tfrac{1}{2}\sup\left|f(x) - g(x)\right| && \text{(integral of constant)} \\
&= \tfrac{1}{2}\left|f - g\right| && \text{(uniform norm)}
\end{aligned}$$

The end result is,

$$\left|T(f) - T(g)\right| \le \tfrac{1}{2}\left|f - g\right|$$

which shows that T is a contraction map with $k = \frac{1}{2}$. By the contraction mapping theorem, T has a unique fixed point f_T, and the iteration of T from any continuous function converges to it. In particular, this is convergence under the uniform norm, hence it is uniform convergence.

We define $s(x)$ as the symmetric transition function which equals the fixed point of T on the interval $\left[0, \frac{1}{2}\right]$:

$$s(x) = \begin{cases} 0 & \text{if } x < 0 \\ f_T(x) & \text{if } 0 \le x \le \frac{1}{2} \\ 1 - s(1-x) & \text{if } x > \frac{1}{2} \end{cases} \tag{3.6}$$

Note that s is continuous, as $s(0) = 0$ and $s\left(\frac{1}{2}\right) = \frac{1}{2}$, so the piecewise components match where they meet. Furthermore, by the definition of T in equation 3.4 and the symmetry condition $s(1-t) = 1 - s(t)$, we find that on $\left[0, \frac{1}{2}\right]$,

$$s(x) = \int_0^{2x} s(t)\, dt = \int_0^{x} 2\, s(2t)\, dt$$

The lower bound of those integrals may be adjusted from 0 to any negative value, because $s(x) = 0$ for all $x < 0$. The fundamental theorem of calculus then implies that s, being the integral of a continuous function, is differentiable on $\left(-\infty, \frac{1}{2}\right)$, and its derivative is $s'(x) = 2\, s(2x)$ in that region. This is equation 3.1, though we have not yet shown it to hold at $x = \frac{1}{2}$.

By symmetry, s is also differentiable on $\left(\frac{1}{2}, \infty\right)$, and the slopes match when approaching $x = \frac{1}{2}$ from either side. Thus, taking $g(x) = s'(x)$ on $\mathbb{R}\backslash\left\{\frac{1}{2}\right\}$ and filling the removable discontinuity at $x = \frac{1}{2}$ with its limiting value $g\left(\frac{1}{2}\right) = 2\, s(1) = 2$, we obtain a continuous function on all of $\mathbb{R}$ whose integral is $s(x)$. The fundamental theorem of calculus then implies that s is differentiable everywhere, and so equation 3.1 holds on all of $\left[0, \frac{1}{2}\right]$.

By the definition of T in equation 3.4, any symmetric transition function that satisfies $f'(x) = 2f(2x)$ (equation 3.1) on $\left[0, \frac{1}{2}\right]$ must, when restricted to that interval, be a fixed point of T. But we have shown that T has only one fixed point among the continuous functions, and every function satisfying equation 3.1 is continuous (because it is differentiable), so $s(x)$ is the only symmetric transition function to satisfy equation 3.1 on $\left[0, \frac{1}{2}\right]$.

Furthermore, s is non-negative and non-decreasing. To see this, notice that if $f\colon \left[0, \frac{1}{2}\right] \to \mathbb{R}$ has all its values in the range $\left[0, \frac{1}{2}\right]$, then $T(f)$ is non-decreasing and has all its values in that same range. As this condition is true for $f(x) = x$, the result holds for all iterations T^n thereof.

The limit of a sequence of functions with values in $\left[0, \frac{1}{2}\right]$ also has values in $\left[0, \frac{1}{2}\right]$, and the limit of a sequence of non-decreasing functions is also non-decreasing. So the fixed point of T, which is the limit of this iteration, is non-decreasing with values in $\left[0, \frac{1}{2}\right]$. Therefore $s(x)$ is non-negative and non-decreasing, with values in $[0, 1]$.

4 Non-analyticity

Now that we have proven there exists a unique symmetric transition function $s(x)$ which is self-similar to its own derivative, meaning it satisfies $s'(x) = 2\,s(2x)$ (equation 3.1), let us further investigate the properties of this function. We begin by looking at its higher derivatives, and conclude that $s(x)$ is infinitely differentiable everywhere, even at 0 and 1.

Then we consider its Taylor series, and find that despite being a smooth function, $s(x)$ is not analytic at any point in the unit interval. Moreover, there is a dense set of points where the Taylor series of s has zero radius of convergence, and another dense set where it has infinite radius of convergence yet does not converge to $s(x)$ on any neighborhood.

Even so, we consider $s(x)$ to be quite well-behaved, appearing just as smooth and curving just as gently as the analytic functions one regularly encounters. It may thus be seen as something of a counter-counterexample, demonstrating that a non-analytic function need not exhibit any overtly pathological behavior.

Higher derivatives

Applying equation 3.1 recursively yields:

$$
\begin{aligned}
s'(x) &= 2\,s(2x) && \text{if } 0 \le x \le \tfrac{1}{2} \\
s''(x) &= 8\,s(4x) && \text{if } 0 \le x \le \tfrac{1}{4} \\
s^{(3)}(x) &= 64\,s(8x) && \text{if } 0 \le x \le \tfrac{1}{8} \\
s^{(4)}(x) &= 1024\,s(16x) && \text{if } 0 \le x \le \tfrac{1}{16} \\
&\vdots \\
s^{(n)}(x) &= 2^{\binom{n+1}{2}}\,s(2^n x) && \text{if } 0 \le x \le 2^{-n}
\end{aligned}
\tag{4.1}
$$

The symmetry condition $s(x) = 1 - s(1 - x)$ implies that the nth derivative of s ramps up and down on intervals of length 2^{-n}, following the pattern of the Thue-Morse sequence[1]. The Thue-Morse sequence begins with 0, then continues

[1] The Thue-Morse sequence is entry A010060 in the Online Encyclopedia of Integer Sequences (OEIS), `https://oeis.org/A010060`

by repeatedly taking all elements found so far and appending a copy with each 0 replaced by a 1 and each 1 by a 0, thereby doubling in length at each step.

This is analogous to the process of repeatedly taking higher derivatives of $s(x)$, as each derivative begins with a copy of the previous one (rescaled in both height and width) followed by a vertically-flipped copy of the same. So the rise and fall of the derivatives of $s(x)$ match the 0s and 1s at the start of the Thue-Morse sequence.

The first several entries of the Thue-Morse sequence are:

$$0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, \ldots$$

With zero-based indexing, the values of this sequence correspond to the parity of the number of 1-bits in the binary representation of the index. Thus, on each interval $\left[k\,2^{-n}, (k+1)\,2^{-n}\right] \subseteq [0, 1]$, the function $s^{(n)}(x)$ ramps up if the number of 1s in the binary representation of k is even, and down if it is odd.

From this and equation 4.1, it follows that s is infinitely differentiable everywhere (including at 0 and 1 where all its derivatives are zero), so it is indeed a smooth transition function. Figure 4.1 shows $s(x)$ and its derivative, while figure 4.2 shows the next several higher derivatives of s, where the pattern of the Thue-Morse sequence can be seen taking shape.

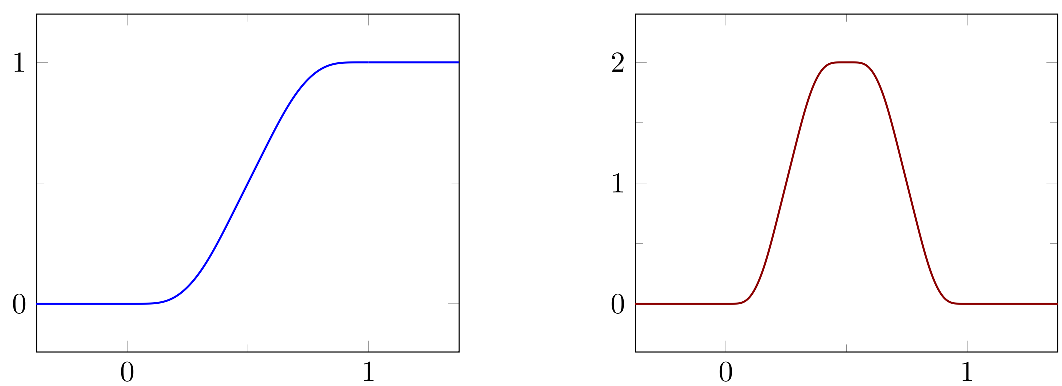
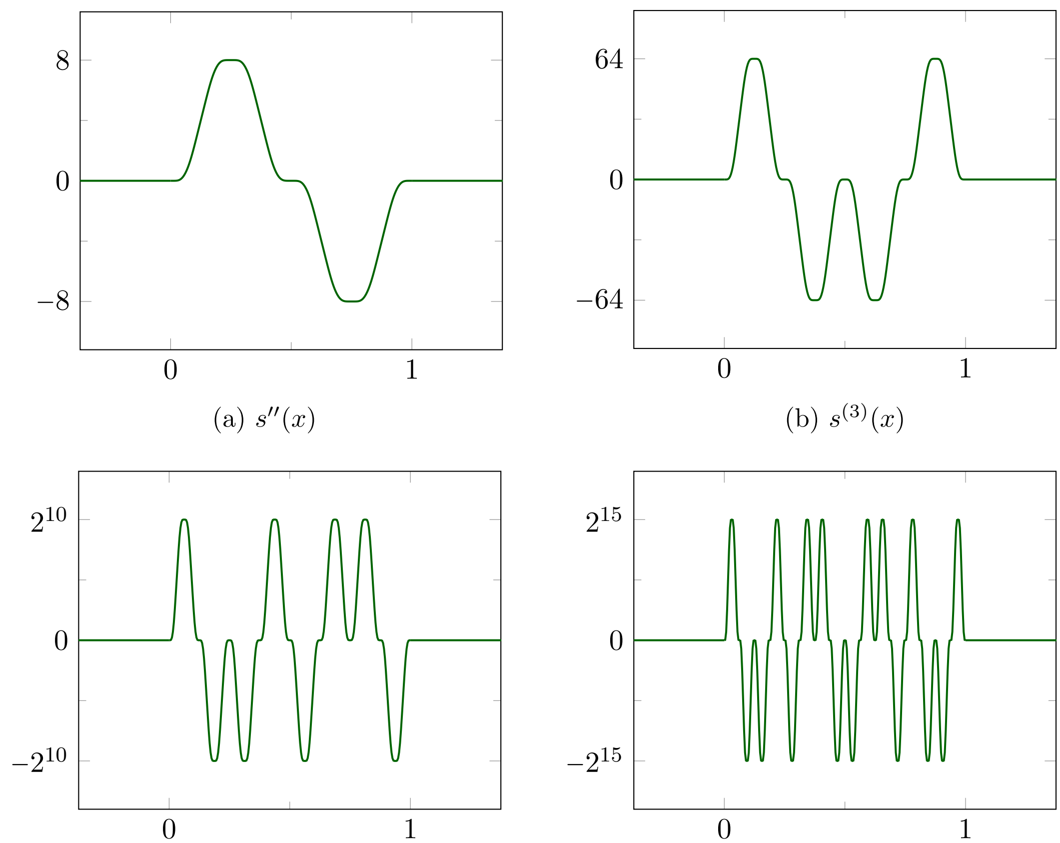


Figure 4.1: The transition function $s(x)$ and its derivative.



(a) $s''(x)$ (b) $s^{(3)}(x)$

(c) $s^{(4)}(x)$ (d) $s^{(5)}(x)$

Figure 4.2: Higher derivatives of s.

First proof of non-analyticity

Let us now consider whether s is analytic on the unit interval. By definition, a function is analytic at a point if its Taylor series about that point converges to the function on some neighborhood around it. The points where a function is analytic form an open set, as the Taylor series may be shifted to a new center anywhere within the neighborhood. We will show that every open subset of $[0, 1]$ contains a point where $s(x)$ is not analytic, and therefore $s(x)$ must be non-analytic on the whole unit interval.

First we observe that at every dyadic rational $x_0 = k \cdot 2^{-n}$ (with k odd) in the unit interval, the Taylor series of s about x_0 is a polynomial of degree n, as its higher derivatives are all 0 in accordance with equation 4.1 and the Thue-Morse pattern.

Within any neighborhood around x_0, we can find another dyadic rational x_1 with a larger exponent $m > n$, where the Taylor series is a polynomial of higher degree. Shifting the original Taylor series to be centered on x_1 would result in a polynomial of the same degree, so the fact that the degrees are different implies the Taylor series about x_0 does not converge to s around x_1.

Since we may choose x_1 as close to x_0 as we like, it follows that there is no neighborhood of x_0 where the Taylor series converges to s. In other words, s is not analytic at any dyadic rational in $[0, 1]$. That is a dense subset, hence s is non-analytic on the entire unit interval.

Second proof of non-analyticity

We have just seen that the Taylor series of s around any dyadic rational in $[0, 1]$ is a polynomial, so it has an infinite radius of convergence. Yet what it converges to—namely the polynomial itself—does not match the function s on any neighborhood of that point.

Interestingly, there is also a dense set where the Taylor series of s has zero radius of convergence, which constitutes a second and entirely separate proof that s is non-analytic on the unit interval. We prove this using the Cauchy-Hadamard theorem, which states that a power series,

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

has radius of convergence R given by:

$$\frac{1}{R} = \limsup_{n \to \infty} \left| c_n \right|^{\frac{1}{n}} \tag{4.2}$$

We will show that within every neighborhood of any point in the unit interval, there exists a point where the lim sup in equation 4.2 is unbounded. Our strategy is to find a nested sequence of closed intervals, each of which places a sufficiently large lower bound on the magnitude of one of the derivatives of s. Because these intervals are closed and nested, their intersection is non-empty, so we can find a point which is in every interval. The Cauchy-Hadamard theorem will then ensure the radius of convergence of s is zero at that point.

To begin, observe that per equation 4.1, the maximum magnitude of $s^{(n)}$ is,

$$\max \left| s^{(n)}(x) \right| = 2^{\binom{n+1}{2}} = 2^{\frac{n(n+1)}{2}}$$

which is achieved at odd multiples of 2^{-n} in $[0, 1]$. By continuity of $s^{(n)}$, there is a neighborhood of every such point where for all x in the neighborhood,

$$\left| s^{(n)}(x) \right| \geq 2^{\frac{1}{2} n^2} \tag{4.3}$$

Without loss of generality, we may take that neighborhood to be a closed interval of positive length.

Given a neighborhood of a point in the unit interval $[0,1]$, choose a dyadic rational d_0 interior to both that neighborhood and $(0,1)$. Then d_0 is an odd multiple of some power of 2, say 2^{-n_0}. Take a positive-length closed-interval neighborhood of d_0 where $s^{(n_0)}$ satisfies inequality 4.3 with $n = n_0$, and which is contained in both $(0,1)$ and the original neighborhood.

From the interior of that interval, choose a new dyadic rational d_1 with exponent $n_1 > n_0$, and repeat the process. That is, take a positive-length closed-interval neighborhood of d_1 where $s^{(n_1)}$ satisfies inequality 4.3 with $n = n_1$, and which is contained in the previous interval, then choose a dyadic rational d_2 from its interior with larger exponent $n_2 > n_1$, and so forth.

Continuing in this manner yields a nested sequence of closed intervals, where inequality 4.3 applies on each interval (and thus on all remaining intervals in the tail of the sequence) with increasing values of n_k. The intersection of these intervals is non-empty, so it contains a point d. Because d is in each interval, it satisfies inequality 4.3 for all the n_k:

$$\left|s^{(n_k)}(d)\right| \geq 2^{\frac{1}{2}n_k^2}$$

The Taylor series of s centered at d therefore has coefficients of magnitude,

$$\left|c_{n_k}\right| = \frac{1}{n_k!}\left|s^{(n_k)}(d)\right| \geq \frac{1}{n_k!}2^{\frac{1}{2}n_k^2}$$

Taking nth roots, and observing that $n! \leq n^n$, we find,

$$\left|c_{n_k}\right|^{\frac{1}{n_k}} \geq \frac{1}{n_k}2^{\frac{1}{2}n_k}$$

This grows without bound, as the n_k are an increasing sequence of integers. So the $\limsup$ in equation 4.2 is infinite, and by the Cauchy-Hadamard theorem the Taylor series of s centered at d therefore has zero radius of convergence. We can find such a d in any neighborhood of a point in the unit interval, so they are dense.

We have thus shown in two different ways that s is non-analytic on the whole of $[0,1]$. There is a dense subset where its Taylor series has an infinite radius of convergence but does not converge to s on any neighborhood, and another dense subset where its Taylor series has zero radius of convergence.

5 Reflection formulas

We have seen one way to evaluate $s(x)$, by iterating the numerical integration of equation 3.3. However, that approach is slow and computationally intensive, as it involves approximating and refining estimates along the entire unit interval repeatedly. We would instead like to find a direct method of calculating $s(x)$ for any given value of x.

The symmetry condition $s(x) = 1 - s(1-x)$ means that we need only consider the interval $\left[0, \frac{1}{2}\right]$, as the values of s on $\left[\frac{1}{2}, 1\right]$ follow directly. Our strategy will be to find similar relations that allow us to "fold over" or "reflect" the interval across reciprocal powers of two, so that we need only understand s on a tiny interval $\left[0, 2^{-n}\right]$ where the values can be made arbitrarily small by choosing a sufficiently large n.

Reflecting across 1/4

Let us attempt to find $s(x)$ for $x \in \left[\frac{1}{4}, \frac{1}{2}\right]$. First we write $x = \frac{1}{4} + h$, where $h \in \left[0, \frac{1}{4}\right]$. Now by equation 3.2,

$$s\left(\tfrac{1}{4} + h\right) = \int_0^{\frac{1}{2}+2h} s(t)\,dt$$

Separating that integral into two pieces, with one symmetric about $\frac{1}{2}$, we have:

$$s\left(\tfrac{1}{4} + h\right) = \int_0^{\frac{1}{2}-2h} s(t)\,dt + \int_{\frac{1}{2}-2h}^{\frac{1}{2}+2h} s(t)\,dt = s\left(\tfrac{1}{4} - h\right) + \int_{\frac{1}{2}-2h}^{\frac{1}{2}+2h} s(t)\,dt$$

By symmetry, the values of s at t and $1-t$ sum to one. So between $\frac{1}{2} - 2h$ and $\frac{1}{2} + 2h$ we can pair up the values to the left of $\frac{1}{2}$ with those to the right, and see that the area under the curve is just half the area of a rectangle with the same width and unit height. This is illustrated in figure 5.1, where we note the symmetry of the areas above and below the curve.

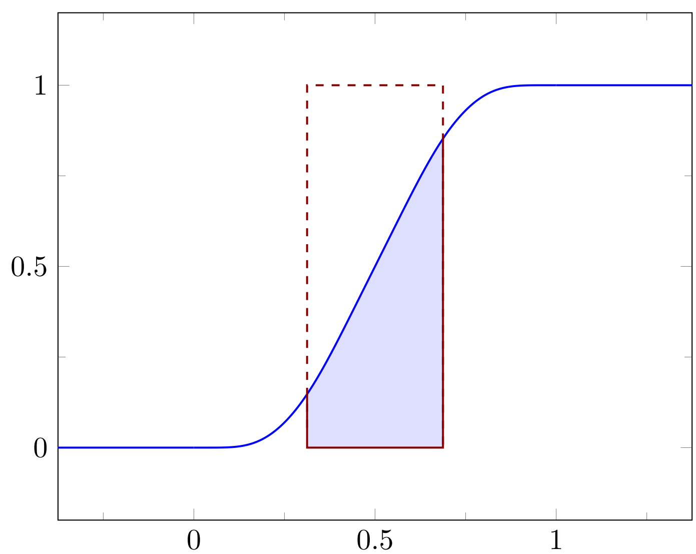


Figure 5.1: The integral of s between symmetric bounds.

The width of the rectangle is $4h$, so half its area is $2h$. Thus we find,

$$s\left(\tfrac{1}{4}+h\right) = 2h + s\left(\tfrac{1}{4}-h\right) \tag{5.1}$$

This is one of the reflection formulas we were after. With it, we now need only consider values of x up to $\frac{1}{4}$. The value of $s(x)$ for larger x can be found from there by reflecting across $\frac{1}{4}$ with equation 5.1, and across $\frac{1}{2}$ with equation 2.1.

Reflecting across 1/8

To reflect across $\frac{1}{8}$, we proceed in a similar, but not quite identical, manner. Rather than separating the bounds to produce a symmetric integral, instead we separate at $\frac{1}{4}$ and apply equation 5.1:

$$\begin{aligned}
s\left(\tfrac{1}{8}+h\right) &= \int_0^{\frac{1}{4}+2h} s(t)\,dt && \text{(by equation 3.2)} \\
&= \int_0^{1/4} s(t)\,dt + \int_{1/4}^{\frac{1}{4}+2h} s(t)\,dt \\
&= s\left(\tfrac{1}{8}\right) + \int_0^{2h} s\left(\tfrac{1}{4}+t\right)dt \\
&= s\left(\tfrac{1}{8}\right) + \int_0^{2h} \left(2t + s\left(\tfrac{1}{4}-t\right)\right) dt && \text{(by equation 5.1)} \\
&= s\left(\tfrac{1}{8}\right) + \int_0^{2h} 2t\,dt + \int_{\frac{1}{4}-2h}^{1/4} s(t)\,dt \\
&= 2s\left(\tfrac{1}{8}\right) + 4h^2 - s\left(\tfrac{1}{8}-h\right) && (5.2)
\end{aligned}$$

This is promising, but it requires the value of $s\left(\frac{1}{8}\right)$. Let us find it:

$$\begin{aligned}
s\left(\tfrac{1}{8}\right) &= \int_0^{1/4} s(t)\,dt && \text{(by equation 3.2)} \\
&= \int_0^{1/8} s(t)\,dt + \int_{1/8}^{1/4} s(t)\,dt \\
&= s\left(\tfrac{1}{16}\right) + \int_0^{1/8} s\left(\tfrac{1}{8} + t\right)dt \\
&= s\left(\tfrac{1}{16}\right) + \int_0^{1/8} \Big(2s\left(\tfrac{1}{8}\right) + 4t^2 - s\left(\tfrac{1}{8} - t\right)\Big)\,dt && \text{(by equation 5.2)} \\
&= s\left(\tfrac{1}{16}\right) + \tfrac{1}{4}s\left(\tfrac{1}{8}\right) + \tfrac{4}{3}\left(\tfrac{1}{8}\right)^3 - \int_0^{1/8} s(t)\,dt \\
&= \tfrac{1}{4}s\left(\tfrac{1}{8}\right) + \tfrac{1}{384}
\end{aligned}$$

which implies,

$$s\left(\tfrac{1}{8}\right) = \tfrac{1}{288} \tag{5.3}$$

This is our first non-trivial value of $s(x)$, and it matches what we calculated in table 3.1. Substituting it into equation 5.2 gives the reflection formula,

$$s\left(\tfrac{1}{8} + h\right) = \tfrac{1}{144} + 4h^2 - s\left(\tfrac{1}{8} - h\right) \tag{5.4}$$

This enables us to find $s\left(\frac{1}{4}\right)$ as well, by plugging in $h = \frac{1}{8}$:

$$s\left(\tfrac{1}{4}\right) = \tfrac{1}{144} + 4\left(\tfrac{1}{8}\right)^2 - s(0) = \tfrac{5}{72} \tag{5.5}$$

which also matches our numerical result.

Reflecting across 2^{-n}

Let us review the reflection formulas we have found so far:

$$s\left(\tfrac{1}{2} + h\right) = 1 - s\left(\tfrac{1}{2} - h\right)$$

$$s\left(\tfrac{1}{4} + h\right) = 2h + s\left(\tfrac{1}{4} - h\right)$$

$$s\left(\tfrac{1}{8} + h\right) = \tfrac{1}{144} + 4h^2 - s\left(\tfrac{1}{8} - h\right)$$

These suggest the shape of a general reflection formula to be a polynomial plus or minus the reflected value:

$$s\left(2^{-n}+h\right) = P_n(h) + (-1)^n s\left(2^{-n}-h\right) \tag{5.6}$$

We already have base cases up to $n = 3$, and we will use induction to prove this is valid in general. The strategy is similar to that for $\frac{1}{8}$. Namely, we first split the integral at a power of two:

$$\begin{aligned}
s\left(2^{-n}+h\right) &= \int_0^{2(2^{-n}+h)} s(t)\,dt && \text{(by equation 3.2)}\\
&= \int_0^{2\cdot 2^{-n}} s(t)\,dt + \int_{2\cdot 2^{-n}}^{2(2^{-n}+h)} s(t)\,dt\\
&= s\left(2^{-n}\right) + \int_0^{2h} s\left(2^{-(n-1)}+t\right) dt
\end{aligned}$$

and then apply the reflection formula 5.6 for $n-1$:

$$\begin{aligned}
s\left(2^{-n}+h\right) &= s\left(2^{-n}\right) + \int_0^{2h} \left(P_{n-1}(t) + (-1)^{n-1} s\left(2^{-(n-1)}-t\right)\right) dt\\
&= s\left(2^{-n}\right) + \int_0^{2h} P_{n-1}(t)\,dt + (-1)^{n-1} \int_{2(2^{-n}-h)}^{2\cdot 2^{-n}} s(t)\,dt\\
&= s\left(2^{-n}\right) + \int_0^{2h} P_{n-1}(t)\,dt + (-1)^{n-1}\left(s\left(2^{-n}\right) - s\left(2^{-n}-h\right)\right)\\
&= \left[\left(1-(-1)^n\right) s\left(2^{-n}\right) + \int_0^{2h} P_{n-1}(t)\,dt\right] + (-1)^n s\left(2^{-n}-h\right)
\end{aligned}$$

Comparing this to equation 5.6, we see that the inductive step will be valid if $P_n(h)$ equals the part in square brackets. Thus we have:

$$P_n(h) = \left(1-(-1)^n\right) s\left(2^{-n}\right) + \int_0^{2h} P_{n-1}(t)\,dt \tag{5.7}$$

This lets us recursively obtain the polynomials P_n to use in the reflection formula of equation 5.6. Doing so requires the values of $s(2^{-n})$ for odd n, as the factor $1-(-1)^n$ vanishes when n is even. For brevity we will write $s_n = s(2^{-n})$. Now:

$$\begin{aligned}
s_n &= \int_0^{2 \cdot 2^{-n}} s(t)\,dt && \text{(by equation 3.2)} \\
&= \int_0^{2^{-n}} s(t)\,dt + \int_{2^{-n}}^{2 \cdot 2^{-n}} s(t)\,dt \\
&= s_{n+1} + \int_0^{2^{-n}} s\big(2^{-n}+t\big)\,dt \\
&= s_{n+1} + \int_0^{2^{-n}} \Big(P_n(t) + (-1)^n s\big(2^{-n}-t\big)\Big)\,dt && \text{(by equation 5.6)} \\
&= s_{n+1} + \int_0^{2^{-n}} P_n(t)\,dt + (-1)^n \int_0^{2^{-n}} s(t)\,dt \\
&= \big(1+(-1)^n\big)\, s_{n+1} + \int_0^{2^{-n}} P_n(t)\,dt && (5.8)
\end{aligned}$$

This at first appears as though it may be problematic, since s_n depends on s_{n+1}. However, the factor $1+(-1)^n$ vanishes for odd n, which is all we need to find P_n.

6 Solving recursively

We could proceed directly to obtain formulas for s_n and P_n, but they would turn out to be rather unwieldy. It will be more convenient to introduce new sequences Q_n and z_n defined by:

$$Q_n\big(2^n x\big) = 2^{\binom{n}{2}} P_n(x) \tag{6.1}$$

$$z_n = 2^{\binom{n}{2}+1} s_n \tag{6.2}$$

Converting between s_n and z_n, or P_n and Q_n, is computationally straightforward, because the scaling factors are powers of two. The values of z_n and Q_n are easier to work with, both because they do not shrink as fast as s_n and P_n, and because the formulas relating them to each other end up being simpler.

Evaluating $\mathbf{Q_n}$

From the symmetry condition of equation 2.1, we have $P_1(x) = 1 = 2s_1$, which implies $Q_1(x) = 1 = z_1$. Now to find a general expression for Q_n, we multiply both sides of equation 5.7 by $2^{\binom{n}{2}}$:

$$2^{\binom{n}{2}} P_n(h) = \left(1 - (-1)^n\right) 2^{\binom{n}{2}} s_n + \int_0^{2h} 2^{\binom{n}{2}} P_{n-1}(t)\, dt$$

Substituting in Q_n and z_n, and noting that $\binom{n}{2} - \binom{n-1}{2} = n - 1$, gives:

$$\begin{aligned} Q_n\left(2^n h\right) &= \frac{1 - (-1)^n}{2} z_n + \int_0^{2h} 2^{n-1} Q_{n-1}\left(2^{n-1} t\right) dt \\ &= \frac{1 - (-1)^n}{2} z_n + \int_0^{2^n h} Q_{n-1}(t)\, dt \end{aligned}$$

Replacing $2^n h$ with x, we then have:

$$Q_n(x) = \frac{1 - (-1)^n}{2} z_n + \int_0^{x} Q_{n-1}(t)\, dt \tag{6.3}$$

Repeatedly applying equation 6.3 lets us express each Q_n in terms of the z_n:

$$\begin{aligned} Q_1(x) &= z_1 \\ Q_2(x) &= z_1 \frac{x}{1!} \\ Q_3(x) &= z_1 \frac{x^2}{2!} + z_3 \\ Q_4(x) &= z_1 \frac{x^3}{3!} + z_3 \frac{x}{1!} \\ Q_5(x) &= z_1 \frac{x^4}{4!} + z_3 \frac{x^2}{2!} + z_5 \\ Q_6(x) &= z_1 \frac{x^5}{5!} + z_3 \frac{x^3}{3!} + z_5 \frac{x}{1!} \\ Q_7(x) &= z_1 \frac{x^6}{6!} + z_3 \frac{x^4}{4!} + z_5 \frac{x^2}{2!} + z_7 \\ &\vdots \end{aligned}$$

So by induction,

$$Q_n(x) = \sum_{k \le n}^{k \text{ odd}} \frac{x^{n-k}}{(n-k)!} z_k \tag{6.4}$$

Evaluating $\mathbf{z_n}$

Equation 6.4 will be our key to calculating $s(x)$. To use it, we need the values of z_n for odd n. To find z_n, we multiply equation 5.8 by $2^{\binom{n}{2}+1}$:

$$2^{\binom{n}{2}+1} s_n = \Big(1 + (-1)^n\Big)\, 2^{\binom{n}{2}+1} s_{n+1} + \int_0^{2^{-n}} 2^{\binom{n}{2}+1} P_n(t)\, dt$$

and substitute in z_n and Q_n:

$$\begin{aligned} z_n &= \Big(1 + (-1)^n\Big)\, 2^{\binom{n}{2} - \binom{n+1}{2}} z_{n+1} + \int_0^{2^{-n}} 2\, Q_n\big(2^n t\big)\, dt \\ &= \Big(1 + (-1)^n\Big)\, 2^{-n} z_{n+1} + \int_0^1 2^{-(n-1)} Q_n(t)\, dt \\ &= 2^{-(n-1)} \left[\frac{1 - (-1)^{n+1}}{2} z_{n+1} + \int_0^1 Q_n(t)\, dt \right] \end{aligned}$$

By equation 6.3 we recognize the part in square brackets as $Q_{n+1}(1)$, which means:

$$z_n = 2^{-(n-1)} Q_{n+1}(1)$$

and so by equation 6.4:

$$z_n = 2^{-(n-1)} \sum_{k \le n+1}^{k \text{ odd}} \frac{z_k}{(n+1-k)!} \tag{6.5}$$

We see that the z_n are given by weighted sums of z_k for odd k. When n is odd, z_n appears on both sides of equation 6.5 so we must solve for it. This can be done by separating the final term from the sum, where $k = n$, and rearranging:

$$z_n = 2^{-(n-1)} \left(z_n + \sum_{k < n}^{k \text{ odd}} \frac{z_k}{(n+1-k)!} \right) \qquad (n \text{ odd})$$

$$= \frac{1}{2^{n-1} - 1} \sum_{k < n}^{k \text{ odd}} \frac{z_k}{(n+1-k)!} \qquad (n \text{ odd}) \tag{6.6}$$

Now using equation 6.6 we can calculate z_n for odd n:

$$
\begin{aligned}
z_1 &= 1 \\
z_3 &= \frac{1}{3}\left(\frac{z_1}{3!}\right) = \frac{1}{18} \\
z_5 &= \frac{1}{15}\left(\frac{z_1}{5!} + \frac{z_3}{3!}\right) = \frac{19}{16{,}200} \\
z_7 &= \frac{1}{63}\left(\frac{z_1}{7!} + \frac{z_3}{5!} + \frac{z_5}{3!}\right) = \frac{583}{42{,}865{,}200} \\
&\vdots
\end{aligned}
$$

And thus too, using equation 6.5, for even n:

$$
\begin{aligned}
z_2 &= \frac{1}{2}\left(\frac{z_1}{2!} + \frac{z_3}{0!}\right) = \frac{5}{18} \\
z_4 &= \frac{1}{8}\left(\frac{z_1}{4!} + \frac{z_3}{2!} + \frac{z_5}{0!}\right) = \frac{143}{16{,}200} \\
z_6 &= \frac{1}{32}\left(\frac{z_1}{6!} + \frac{z_3}{4!} + \frac{z_5}{2!} + \frac{z_7}{0!}\right) = \frac{1{,}153}{8{,}573{,}040} \\
&\vdots
\end{aligned}
$$

These of course immediately yield values for $s_n = z_n \cdot 2^{-\binom{n}{2}-1}$, and both are included in table A.3 of appendix A.

7 Algorithms

We now have all the tools required to calculate $s(x)$ for any x. We can find z_n for odd n using equation 6.6, and then for even n using equation 6.5. We can use the values of z_n to calculate $Q_n(x)$ via equation 6.4, and then $P_n(x)$ via equation 6.1. And we can use $P_n(x)$ to reflect $s(x)$ across powers of two using equation 5.6.

Let us first consider the task of computing values of z_n. We know that $z_1 = 1$, and if we have already found the previous values of z up to some odd index k, then we can compute two more: first z_{k+2} at the next odd index, then z_{k+1} at the even index between. And we can repeat this process until we have found as many values of z as we like.

The zList function in algorithm 7.1 carries out this process starting at z_1 and continuing through z_n. However, it is a simple modification to write a function that instead begins at the last odd index for which z_k has previously been calculated, and extends the list of cached values as needed. We may assume such a modification has been made.

Algorithm 7.1 Computing z_n

```
function zList(n) {                          ▷ Computes z_1 through z_n
    z_1 ← 1                                    (and z_{n+1} if n is even)
    k ← 1
    while k < n {
        z_{k+2} ← zOdd(k + 2)
        z_{k+1} ← zEven(k + 1)
        k ← k + 2
    }
}

function zOdd(n) {                           ▷ Computes z_n for odd n
    return zSum(n) / (2^{n-1} − 1)
}

function zEven(n) {                          ▷ Computes z_n for even n
    return (zSum(n) + z_{n+1}) / 2^{n-1}
}

function zSum(n) {                           ▷ Computes Σ_{k<n, k odd} z_k / (n + 1 − k)!
    result ← 0
    k ← 1
    while k < n {
        result ← (result + z_k) / ((n − k)(n − k + 1))
        k ← k + 2
    }
    return result
}
```

When working at a fixed level of precision, the number of required z values can be determined ahead of time. For example, in double-precision arithmetic the values of s_n round to zero when $n \geq 43$, thus it is sufficient to precompute z_1 through z_{42}. Otherwise, an extensible cache may be used. In either case, we can use the values of z_n to calculate $Q_n(x)$. The implementation in algorithm 7.2 is a straightforward translation of equation 6.4 using Horner's method.

Algorithm 7.2 Computing $Q_n(x)$

```
function Q(n, x) {                                   ▷ Computes  Σ_{k ≤ n, k odd} x^{n−k}/(n−k)! z_k
    result ← 0
    k ← 1
    while k ≤ n {
        result ← (result · x^2) / ((n − k + 1)(n − k + 2)) + z_k
        k ← k + 2
    }
    if n is even { result ← result · x }
    return result
}
```

▷ Computes $\displaystyle\sum_{\substack{k \le n \\ k \text{ odd}}} \frac{x^{n-k}}{(n-k)!} z_k$

$$result \leftarrow \frac{result \cdot x^2}{(n-k+1)(n-k+2)} + z_k$$

Of course, when computing $s(x)$, the values we need for the reflection formulas are s_n and $P_n(x)$, rather than z_n and $Q_n(x)$. However we defined the latter pair so that the conversion is a simple matter of rescaling by powers of 2, as shown in algorithm 7.3. Note that to avoid ambiguity, we have given the name sBinade to the function which computes s_n.

Algorithm 7.3 Computing s_n and $P_n(x)$

```
function sBinade(n) {                                ▷ Computes s_n
    c ← n · (n − 1) / 2
    return z_n / 2^{c+1}
}

function P(n, x) {                                   ▷ Computes P_n(x)
    c ← n · (n − 1) / 2
    return Q(n, 2^n x) / 2^c
}
```

And now, finally, we are ready to compute $s(x)$. Although equation 5.6 naturally gives rise to a recursive implementation, we have chosen to recast it as an iterative loop in algorithm 7.4, which may be easier for compilers to optimize. We have also introduced a second parameter, ε, for the acceptable tolerance, so that arbitrary-precision computations will have a finite stopping point. With minor changes, a relative tolerance may be used instead of an absolute one.

Algorithm 7.4 Computing $s(x)$

```
function sIterative(x, ε) {                      ▷ Computes s(x) to within ε
    if x ≤ 0 { return 0 }
    if x ≥ 1 { return 1 }

    result ← 0
    sign ← 1

    repeat
        n ← −⌊log₂(x)⌋
        sₙ ← sBinade(n)
        h ← x − 2⁻ⁿ
        if h = 0 { return result + sₙ · sign }

        x ← 2⁻ⁿ − h
        result ← result + P(n, h) · sign
        if n is odd { sign ← −sign }
    until sₙ ≤ ε

    return result
}
```

Correctness

To show that algorithm 7.4 is correct when $\varepsilon > 0$, recall that $s(x)$ is continuous, non-negative, monotonically increasing, and $s(0) = 0$. Thus if $x \le 2^{-n}$, we have $0 \le s(x) \le s_n$. For n large enough, $s_n \le \varepsilon$, and we can calculate s_n via sBinade to check. Therefore, if we can reduce the calculation to a sum of known terms plus a value of $s(x)$ for sufficiently small x, then the error will be bounded by ε. In order to reduce the calculation in that manner, we use the reflection formula of equation 5.6, which we reproduce here:

$$s\big(2^{-n}+h\big) = P_n(h) + (-1)^n s\big(2^{-n}-h\big)$$

If x is outside $(0, 1)$ the result is trivial, and if $x = 2^{-n}$ then $s(x) = s_n$. Otherwise, we may write $x = 2^{-n} + h$ with $0 < h < 2^{-n}$. Then we can calculate $P_n(h)$ via algorithm 7.3, and we must either add or subtract $s\big(2^{-n} - h\big)$ depending on the parity of n. Now $2^{-n} - h < 2^{-n}$, so we have moved into a smaller binade and are making progress. We continue in the same manner, checking first if we have landed on a power of two, and if not then calculating $P_n(h)$ for the new n and h.

Let us denote the sequence of x values encountered in this process as $\{x_0, x_1, x_2, \ldots\}$, the corresponding powers of two as $\{n_0, n_1, n_2, \ldots\}$, and the offsets as $\{h_0, h_1, h_2, \ldots\}$. So we have $x = x_0$, and each $x_i = 2^{-n_i} + h_i$ with $0 \le h_i < 2^{-n_i}$.

If at any time we find $h_i = 0$, then we have reached a power of 2 and $s(x_i) = s_{n_i}$, which we know how to calculate. This occurs when x has a finite binary expansion, although it is possible we might achieve the desired level of accuracy first.

If the sequence n_i never lands on a power of 2, then every $h_i > 0$ and the reflection formula tells us that $x_{i+1} = 2^{-n_i} - h_i$. In particular, $x_{i+1} < 2^{-n_i}$, so $n_{i+1} > n_i$. The sequence of integers n_i is then strictly increasing, so it is unbounded and 2^{-n_i} approaches zero. By continuity of s, the corresponding values s_{n_i} approach $s(0) = 0$, so eventually they will be bounded by ε.

Now, to see that the loop in algorithm 7.4 carries out this process, let us write $p_i = P_{n_i}(h_i)$ and expand the reflection formula recursively:

$$s(x) = p_0 + (-1)^{n_0}\Big(p_1 + (-1)^{n_1}\Big(p_2 + (-1)^{n_2}\Big(p_3 + \cdots\Big)\Big)\Big)$$

Distributing the factors of (-1), we obtain:

$$\begin{aligned} s(x) = p_0 &+ p_1 \cdot (-1)^{n_0} \\ &+ p_2 \cdot (-1)^{n_0+n_1} \\ &+ p_3 \cdot (-1)^{n_0+n_1+n_2} \\ &+ \cdots \end{aligned}$$

It follows that when the sum of all the previous n_i is even we must add the next p value, and when that sum is odd we must subtract. Thus we begin by adding p_0, and we continue adding until some n_k is odd. Then we subtract p_{k+1}, and continue subtracting until another odd n_i is reached. In other words, we toggle between adding and subtracting after each odd n_i.

The loop in algorithm 7.4 is a direct translation of this process. We keep the running total in the *result* variable, and use *sign* to track whether to add or subtract. When $h = 0$, we have landed on a power of two and the process stops. Otherwise it continues until $s_n \leq \varepsilon$, at which point the remaining amount to add or subtract is less than ε. The only remaining technicality is to show that $n = -\lfloor \log_2(x) \rfloor$. This is easily done, for if $2^{-n} \leq x < 2^{-(n-1)}$ then $-n \leq \log_2(x) < -(n-1)$. So the function sIterative from algorithm 7.4 does indeed calculate $s(x)$ to within ε.

As an implementation detail, note that n may be extracted directly from a binary floating-point representation of x, as the exponent field. Similarly, $2^n h$, which is used as input to Q_n, is one less than the binary significand of x.

When working in finite-precision arithmetic, such as `double` or `float`, ε may be set to zero and the calculation will proceed to the working precision. It is not guaranteed that the result will be correctly rounded, nor even that the error will be less than one ulp, due to the accumulated rounding of intermediate results. However, the rapid convergence of the iteration means the error should be quite small even with a naive implementation.

8 Asymptotic behavior

Now that we have an algorithm for calculating $s(x)$, let us turn our attention to the behavior of s at small positive values of x. We know that s goes to zero at $x = 0$, as do all its derivatives, however we would like to be more precise about its asymptotic behavior. Just how quickly *does* it diminish?

First, observe that a factor of $n!$ may be removed from the denominators of z_n and Q_n, by defining $w_n = n! \cdot z_n$ and $R_n(x) = n! \cdot Q_n(x)$. Then multiplying equations 6.4 and 6.6 by $n!$ implies:

$$R_n(x) = \sum_{k \le n}^{k \text{ odd}} \binom{n}{k} x^{n-k} w_k \tag{8.1}$$

$$w_n = \frac{1}{(n+1)\left(2^{n-1}-1\right)} \sum_{k<n}^{k \text{ odd}} \binom{n+1}{k} w_k \qquad (n \text{ odd}) \tag{8.2}$$

In the previous sections we used Q_n and z_n because the scaling factors to convert between them and P_n and s_n are powers of two. However when n is large, R_n and w_n will be more convenient because their denominators remain smaller. For the purpose of asymptotic analysis it suffices to consider only values of x which are powers of two, thus we need not involve R_n at all and may focus on w_n.

Writing w_n in terms of $s(x)$:

$$w_n = 2^{\binom{n}{2}+1} \, n! \, s\left(2^{-n}\right)$$

suggests a continuous version of w,

$$w(x) = 2^{\binom{n}{2}+1} \, \Gamma(n+1) \, s(x)$$

where we let $n = -\log_2(x)$ be continuous as well. Henceforth we will not be too particular about whether w is discrete or continuous.

A first approximation

We begin by plotting $v = -\log_2(w)$ against $m = \log_2(n)$ in figure 8.1, and observe that the graph closely matches the parabola $y = 0.5x^2 - 0.5x + 0.6 = \binom{x}{2} + \frac{3}{5}$. This is our first approximation for v, which we call A_1:

$$A_1 = \binom{m}{2} + \frac{3}{5}$$

It immediately gives rise to an approximation for $w \approx 2^{-A_1}$ and hence s as well:

$$s\left(2^{-n}\right) \approx \frac{1}{n!} \, 2^{-\binom{n}{2}-1-A_1}$$

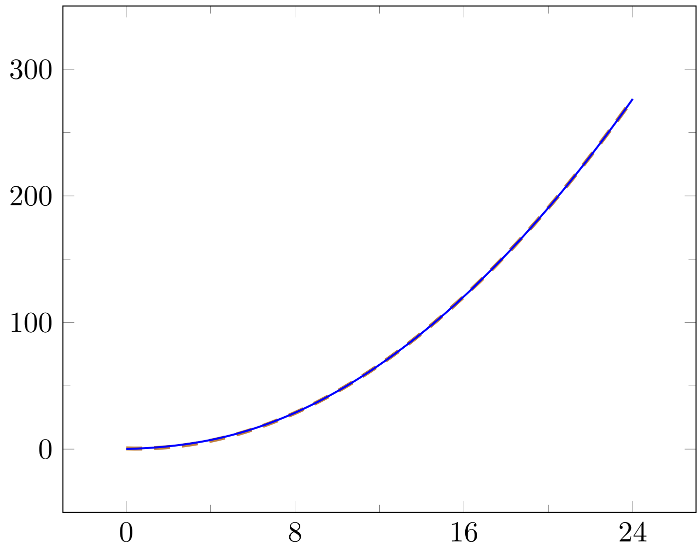


Figure 8.1: v vs m (solid), and the parabola $y = \binom{x}{2} + \frac{3}{5}$ (dashed)

The difference between v and A_1 is shown in figure 8.2. It appears to approach zero as m grows, thus in the limit of large n (and so also large m), we conjecture that the asymptotic equivalence holds:

$$v \sim \binom{m}{2} + \frac{3}{5} \qquad \text{(conjecture)} \tag{8.3}$$

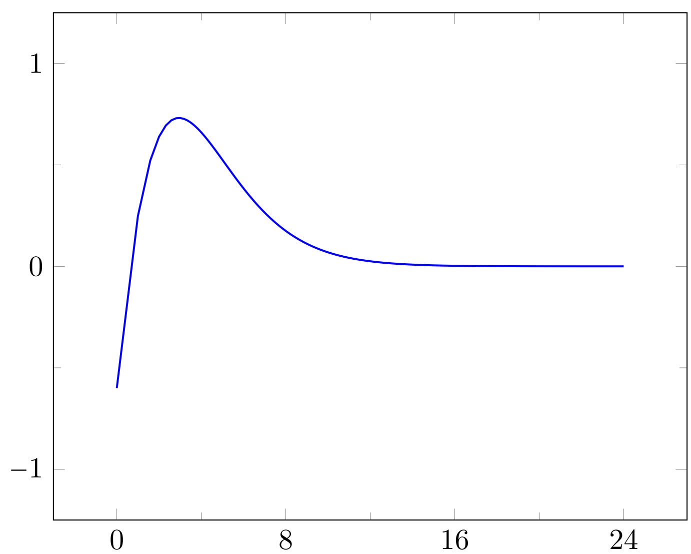


Figure 8.2: $v - A_1$ vs m

Motivation for the conjecture

We found the approximation $v \approx A_1 = \binom{m}{2} + \frac{3}{5}$ empirically, and we conjecture that it is valid asymptotically. We do not attempt to prove this, but some reasonable estimates show why it is plausible. Binomial coefficients obey the addition formula,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Using that, the formula for w_n from equation 8.2 can be rewritten as a sum across row n of Pascal's triangle, rather than the odd entries of row $(n+1)$. Then, after distributing a factor of 2^{-n} into the sum, we obtain:

$$w_n = \frac{1}{(n+1)\left(2^{-1} - 2^{-n}\right)} \sum_{k<n} \tbinom{n}{k} 2^{-n} w_{2\lfloor \frac{k}{2} \rfloor + 1} \qquad (n \text{ odd})$$

Now let us make some simplifying approximations. First, the fraction outside the summation becomes $2/n$. Next, for the subscript of w inside the sum, replace $2\lfloor k/2 \rfloor + 1$ with k. Finally, we know by the De Moivre-Laplace theorem that as n grows, $\binom{n}{k}2^{-n}$ approaches a normal distribution with mean $n/2$ and standard deviation $\sqrt{n}/2$. Thus the sum can be approximated by an integral of w times that normal distribution. This is a "gaussian blur" of w, so the integral approximates the value of w at its midpoint, namely $w_{n/2}$.

Putting this all together, and writing $\mathcal{N}\left(\mu, \sigma^2, x\right)$ for the normal distribution with mean μ and variance σ^2 evaluated at x, we find for large n:

$$\begin{aligned} w_n &\approx \frac{2}{n} \sum_{k<n} \tbinom{n}{k} 2^{-n} w_k \\ &\approx \frac{2}{n} \int_0^n \mathcal{N}\left(\tfrac{n}{2}, \tfrac{n}{4}, k\right) w_k \, dk \\ &\approx \frac{2}{n} w_{n/2} \end{aligned}$$

And substituting $n \to 2n$, we get:

$$w_{2n} \approx \frac{w_n}{n} \tag{8.4}$$

Although we have derived this approximation under the limit of large n, let us consider what would happen if it were true for all n:

$$
\begin{aligned}
w_1 &= 1 & w_{32} &\approx \tfrac{1}{16} w_{16} \approx 2^{-10} \\
w_2 &\approx \tfrac{1}{1} w_1 \approx 2^{-0} & w_{64} &\approx \tfrac{1}{32} w_{32} \approx 2^{-15} \\
w_4 &\approx \tfrac{1}{2} w_2 \approx 2^{-1} & w_{128} &\approx \tfrac{1}{64} w_{64} \approx 2^{-21} \\
w_8 &\approx \tfrac{1}{4} w_4 \approx 2^{-3} & &\vdots \\
w_{16} &\approx \tfrac{1}{8} w_8 \approx 2^{-6} & w_{2^n} &\approx 2^{-\binom{n}{2}}
\end{aligned}
$$

Thus, taking logarithms of both n and w, we have:

$$\log_2(w_n) \approx -\binom{\log_2(n)}{2} \tag{8.5}$$

which matches conjecture 8.3 up to an added constant. This is not a proof, but it does provide some heuristic plausibility in support of the numerical evidence.

A second approximation

We do not propose a theoretical basis for the value of the constant $\frac{3}{5}$ in conjecture 8.3. However, the error in the approximation $v \approx A_1 = \binom{m}{2} + \frac{3}{5}$ seems well-behaved, as we saw in figure 8.2. Indeed, we see in figure 8.3 that the negative log (base 2) of the approximation error appears to approach the line $\frac{5}{6}m - 5$.

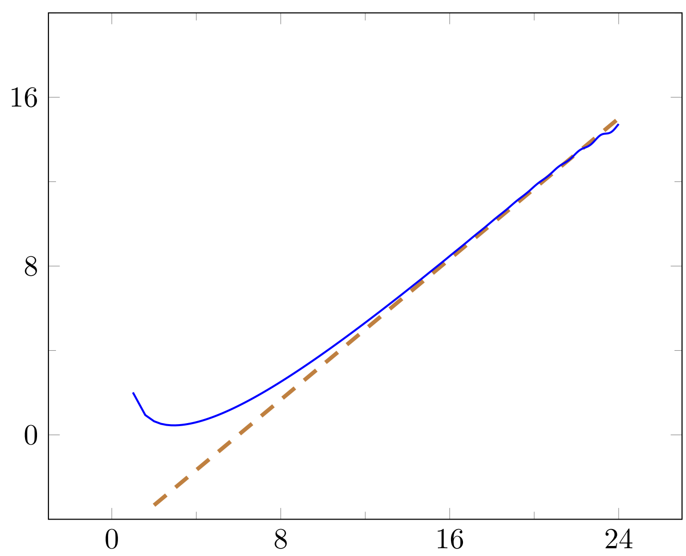


Figure 8.3: $-\log_2\big(v - A_1\big)$ vs m (solid), and the line $y = \frac{5}{6}x - 5$ (dashed)

Toward the right-hand end of this graph, we observe small oscillations. We are unsure if these represent genuine behavior, or if they are numerical artifacts that result from exhausting the precision of our calculated values for w. We do not propose any heuristic reason to expect the line $\frac{5}{6}x - 5$ here, and we do not place high confidence in the precise values of the coefficients. Nonetheless, we take it as the basis for our second approximation:

$$v \approx A_2 = \binom{m}{2} + \frac{3}{5} + 2^{5-\frac{5}{6}m} \tag{8.6}$$

We note in passing that $\frac{5}{6}\ln(2) = 0.5776\ldots \approx \gamma = 0.5772\ldots$, which implies that $2^{5-\frac{5}{6}m} \approx 32e^{-\gamma m}$. We do not have a reason to expect the Euler-Mascheroni constant γ here, yet neither do we have enough confidence in the value $\frac{5}{6}$ to rule it out.

Computing with large n

In the course of our investigations of $s(x)$, we developed a variety of strategies for calculating its values. We found exact formulas for s_n, and simple algorithms to compute $s(x)$ accurately. We factored out powers of two, which lets us calculate z_n far beyond where s_n would underflow to zero in floating-point. Similarly, factoring out $n!$ lets us calculate w_n beyond where z_n would underflow.

These techniques enable us to compute values of s_n for quite large n, and at a certain point the speed of computation becomes a limiting factor. Notably, the method of calculating z_n from algorithm 7.1 takes linear time for each z_n, thus it is quadratic overall. A similar algorithm for w_n naturally arises from equation 8.2, and it too is quadratic. This is not a problem when n is a few hundred or even a few thousand. But if we want n in the millions, then we may have difficulty.

It is not immediately clear how to improve the situation. We can maintain an array of each term $t_n(k) = \binom{n+1}{k}w_k$, but the work to update the array and sum the terms to find w_n remains linear at each step. However, the observations we made while deriving the heuristic estimate for the asymptotic behavior of w hold a clue. When we approximated the summation with a normal distribution, its standard deviation was $\sqrt{n}/2$. That implies the terms $t_n(k)$ drop off rapidly away from the midpoint, and the number of terms with significant magnitude is of order $\sqrt{n}$.

This suggests that we may omit the tiny terms in the tails, and only update the larger ones near the center. If the number of significant terms is in fact of order $\sqrt{n}$, then the time complexity to calculate w_1 through w_n would be $O(n^{3/2})$ instead of $O(n^2)$. We used this strategy when calculating w_n, and our timing measurements agree with $O(n^{3/2})$. However we do not have a rigorous proof at this time.

9 Related functions

Now that we have developed some familiarity with $s(x)$, let us consider ways in which it may be extended or modified. First, we observe that $s(x) \cdot s(1-x)$ forms a smooth bell curve on the unit interval, as depicted in figure 9.1.

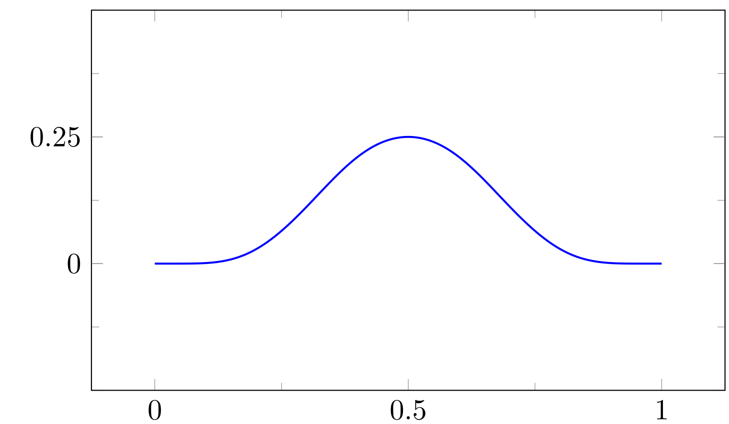


Figure 9.1: $s(x) \cdot s(1-x)$ forms a bell curve with finite support.

This bell curve is non-analytic on the entire unit interval, as can be seen by adapting the argument from our first proof that s is non-analytic. That is, at every dyadic rational in $[0, 1]$ its Taylor series is a polynomial, and we can find arbitrarily-close points where it is a polynomial of a different degree.

Next, we notice that $x - s(x)$ makes a nearly-sinusoidal wave, which is also non-analytic on all of $[0, 1]$. It achieves a maximum of $13/72$ at $x = 1/4$, and its slope ranges between 1 and -1, as seen in figure 9.2.

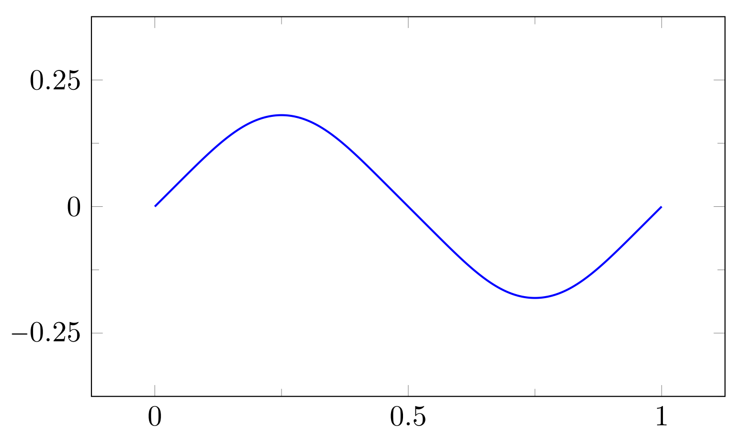


Figure 9.2: $x - s(x)$

Numerical calculations put the total harmonic distortion of this wave at 2.80%. By translating copies of it to every integer, we can make a function which is nearly sinusoidal, yet non-analytic on the entire real line, as seen in figure 9.3.

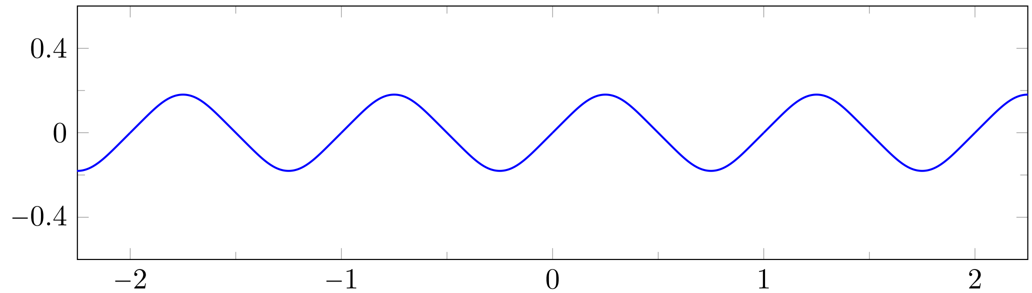


Figure 9.3: Repeating $x - s(x)$ for a periodic wave

Another way to extend $s(x)$ is by continuing the defining formula $f'(x) = 2f(2x)$ beyond $x = \frac{1}{2}$. This results in repeated copies of the ramp from $s(x)$ on $[0, 1]$ rising and falling as per the Thue-Morse sequence along the entire positive x-axis. The pattern does not continue to the left of zero though, because for negative x the height and slope would have opposite signs. However, if we make this an odd function, then it will satisfy $f'(x) = 2f\big(|2x|\big)$. The resulting curve is graphed in figure 9.4.

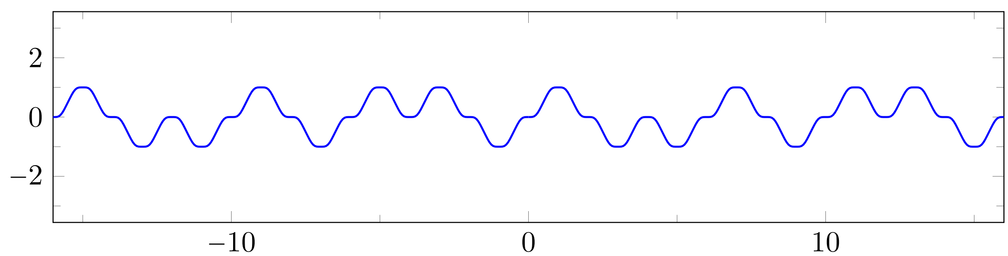


Figure 9.4: $f'(x) = 2f\big(|2x|\big)$ extended from $s(x)$ on $[0, 1]$ to $\mathbb{R}$

A family of functions

The defining feature of $s(x)$ is the relation $s'(x) = 2s(2x)$. This corresponds to a transition whose speed ramps up to a maximum in the middle, then immediately ramps back down. However, in real-world applications it is often the case that a process will maintain its maximum speed for some duration before ramping down again. For example, a vehicle may accelerate from rest up to some velocity, continue at that speed for most of its journey, then decelerate to a stop at the destination.

We can model this behavior with a small change to the definition of s. Given a constant $c \in \big[0, \frac{1}{2}\big]$, define a smooth symmetric transition function f_c which is linear on $[c, 1 - c]$ and satisfies $f'(x) \propto f(x/c)$ on $[0, c]$. A quick calculation shows the constant of proportionality to be $1/(1 - c)$, hence:

$$f_c'(x) = \frac{1}{1 - c} f_c\Big(\frac{x}{c}\Big) \tag{9.1}$$

The slope of f_c on $[c, 1-c]$ is thus $1/(1-c)$, which means $f_c(1-c) - f_c(c) = ((1-c)-c)/(1-c)$. We also have $f_c(1-c) = 1 - f_c(c)$ by the symmetry condition. Together, these equations let us find $f_c(c)$, which has the value:

$$f_c(c) = \frac{c}{2(1-c)} \tag{9.2}$$

It is clear from the definition that $s = f_{1/2}$, and many of the same techniques we used on s are applicable to all the f_c. We can prove existence and uniqueness with a similar fixed-point iteration; there are reflection formulas to reduce the function input toward zero; and except on the interiors of intervals where f_c matches a polynomial, non-analyticity follows by finding arbitrarily-close points at which its Taylor series are polynomials of different degree.

Of particular note, the function $f_{1/3}$ is non-analytic on the middle-thirds Cantor set, and it equals a polynomial of degree n on each interval removed at step n in the construction of that set. That is to say, it is linear on $\left[\frac{1}{3}, \frac{2}{3}\right]$, quadratic on $\left[\frac{1}{9}, \frac{2}{9}\right]$ and $\left[\frac{7}{9}, \frac{8}{9}\right]$, and so forth. We leave further investigation of the family of functions f_c to future work.

10 Conclusion

We have shown there exists a unique transition function $s(x)$ which is smooth, symmetric, and self-similar with its own derivative in the sense that $s'(x) \propto s(2x)$ on $\left[0, \frac{1}{2}\right]$. We proved this function is non-analytic at every point of the unit interval, and we developed an effective algorithm to calculate its values. Based on numerical evidence and heuristic estimates, we conjecture that as x approaches zero from above, the asymptotic behavior of s is described by approximation 8.3.

We leave for future work a detailed analysis of our algorithms, the development of faster algorithms to compute $s(x)$ at high precision, and a proof or disproof of the conjectured asymptotic behavior near zero. It may also be of interest to consider the class of transition functions defined similarly to $s(x)$, but with a central linear section.

Appendix A — Tables of values

Table A.1: Exact values of $w_n = 2^{\binom{n}{2}+1} n!\, s\left(2^{-n}\right)$ for small n

n	$\mathbf{w_n}$	**decimal**
1	1	1
2	$\frac{5}{9}$	0.555555555555...
3	$\frac{1}{3}$	0.333333333333...
4	$\frac{143}{675}$	0.211851851851...
5	$\frac{19}{135}$	0.140740740740...
6	$\frac{1{,}153}{11{,}907}$	0.096833795246...
7	$\frac{583}{8{,}505}$	0.068547912992...
8	$\frac{1{,}616{,}353}{32{,}531{,}625}$	0.049685590559...
9	$\frac{132{,}809}{3{,}614{,}625}$	0.036742124010...
10	$\frac{134{,}926{,}369}{4{,}881{,}045{,}015}$	0.027642926583...
11	$\frac{46{,}840{,}699}{2{,}218{,}656{,}825}$	0.021112187550...
12	$\frac{67{,}545{,}496{,}213{,}157}{4{,}133{,}856{,}862{,}760{,}625}$	0.016339582732...
13	$\frac{4{,}068{,}990{,}560{,}161}{317{,}988{,}989{,}443{,}125}$	0.012796010853...
14	$\frac{411{,}124{,}285{,}571{,}171}{40{,}594{,}391{,}797{,}766{,}625}$	0.010127612888...
15	$\frac{1{,}204{,}567{,}303{,}451{,}311}{148{,}846{,}103{,}258{,}477{,}625}$	0.008092702980...

Table A.2: Values of w_n, $v_n = -\log_2(w_n)$, and $A_1 = \binom{m}{2} + \frac{3}{5}$, for $n = 2^m$

$\mathbf{n}$	$\mathbf{w_n}$	$\mathbf{v_n}$	$\mathbf{A_1}$
2^0	1	0	0.6
2^1	0.555555555556	0.847996906555	0.6
2^2	0.211851851852	2.238872355160	1.6
2^3	$4.968559055996 \cdot 10^{-2}$	4.331028676857	3.6
2^4	$6.523107305793 \cdot 10^{-3}$	7.260224923462	6.6
2^5	$4.476759516974 \cdot 10^{-4}$	11.125257558066	10.6
2^6	$1.542408358139 \cdot 10^{-5}$	15.984455699932	15.6
2^7	$2.617870301941 \cdot 10^{-7}$	21.865103041100	21.6
2^8	$2.177259107006 \cdot 10^{-9}$	28.774839746805	28.6
2^9	$8.887036809814 \cdot 10^{-12}$	36.711434674870	36.6
2^{10}	$1.787384367929 \cdot 10^{-14}$	45.669143416441	45.6
2^{11}	$1.778655043325 \cdot 10^{-17}$	55.641990875128	55.6
2^{12}	$8.787369200956 \cdot 10^{-21}$	66.625058682650	66.6
2^{13}	$2.160754302087 \cdot 10^{-24}$	78.614739243652	78.6
2^{14}	$2.648951405282 \cdot 10^{-28}$	91.608565278975	91.6
2^{15}	$1.620873684652 \cdot 10^{-32}$	105.604927370973	105.6
2^{16}	$4.953775470413 \cdot 10^{-37}$	120.602811030836	120.6
2^{17}	$7.565244927298 \cdot 10^{-42}$	136.601593194344	136.6
2^{18}	$5.774601970784 \cdot 10^{-47}$	153.600898951198	153.6
2^{19}	$2.203435268989 \cdot 10^{-52}$	171.600506419411	171.6
2^{20}	$4.203361186155 \cdot 10^{-58}$	190.600286073842	190.6
2^{21}	$4.008979107724 \cdot 10^{-64}$	210.600163173702	210.6
2^{22}	$1.911720518756 \cdot 10^{-70}$	231.600095016117	231.6
2^{23}	$4.558015702387 \cdot 10^{-77}$	253.600057411624	253.6
2^{24}	$5.433655914500 \cdot 10^{-84}$	276.600036760548	276.6

Table A.3: Values of $s_n = s(2^{-n})$, $z_n = 2^{\binom{n}{2}+1} s_n$, and $w_n = n!\, z_n$

$\mathbf{n}$	$\mathbf{s_n}$	$\mathbf{z_n}$	$\mathbf{w_n}$
1	0.5	1	1
2	$6.944444444444 \cdot 10^{-2}$	0.277777777778	0.555555555556
3	$3.472222222222 \cdot 10^{-3}$	$5.555555555556 \cdot 10^{-2}$	0.333333333333
4	$6.896219135802 \cdot 10^{-5}$	$8.827160493827 \cdot 10^{-3}$	0.211851851852
5	$5.726755401235 \cdot 10^{-7}$	$1.172839506173 \cdot 10^{-3}$	0.140740740741
6	$2.052175633038 \cdot 10^{-9}$	$1.344913822868 \cdot 10^{-4}$	$9.683379524649 \cdot 10^{-2}$
7	$3.242677780955 \cdot 10^{-12}$	$1.360077638737 \cdot 10^{-5}$	$6.854791299236 \cdot 10^{-2}$
8	$2.295303180634 \cdot 10^{-15}$	$1.232281511904 \cdot 10^{-6}$	$4.968559055996 \cdot 10^{-2}$
9	$7.367012151470 \cdot 10^{-19}$	$1.012514440314 \cdot 10^{-7}$	$3.674212401010 \cdot 10^{-2}$
10	$1.082533106206 \cdot 10^{-22}$	$7.617649521444 \cdot 10^{-9}$	$2.764292658342 \cdot 10^{-2}$
11	$7.340028734446 \cdot 10^{-27}$	$5.289048107735 \cdot 10^{-10}$	$2.111218755068 \cdot 10^{-2}$
12	$2.311502077449 \cdot 10^{-31}$	$3.411174979884 \cdot 10^{-11}$	$1.633958273244 \cdot 10^{-2}$
13	$3.399574992880 \cdot 10^{-36}$	$2.054916992304 \cdot 10^{-12}$	$1.279601085335 \cdot 10^{-2}$
14	$2.346060133557 \cdot 10^{-41}$	$1.161712709561 \cdot 10^{-13}$	$1.012761288848 \cdot 10^{-2}$
15	$7.628066133779 \cdot 10^{-47}$	$6.188622472362 \cdot 10^{-15}$	$8.092702980336 \cdot 10^{-3}$
16	$1.172750032486 \cdot 10^{-52}$	$3.117704350477 \cdot 10^{-16}$	$6.523107305793 \cdot 10^{-3}$
17	$8.552339519043 \cdot 10^{-59}$	$1.490027691137 \cdot 10^{-17}$	$5.299841172523 \cdot 10^{-3}$
18	$2.966704142587 \cdot 10^{-65}$	$6.774754627919 \cdot 10^{-19}$	$4.337451089254 \cdot 10^{-3}$
19	$4.907580283679 \cdot 10^{-72}$	$2.937830039523 \cdot 10^{-20}$	$3.573726301419 \cdot 10^{-3}$
20	$3.880167156974 \cdot 10^{-79}$	$1.217810199731 \cdot 10^{-21}$	$2.962812880505 \cdot 10^{-3}$
21	$1.469321144508 \cdot 10^{-86}$	$4.835549278912 \cdot 10^{-23}$	$2.470527685773 \cdot 10^{-3}$
22	$2.669806684842 \cdot 10^{-94}$	$1.842632853374 \cdot 10^{-24}$	$2.071120668220 \cdot 10^{-3}$
23	$2.331759454696 \cdot 10^{-102}$	$6.749982471452 \cdot 10^{-26}$	$1.745006598392 \cdot 10^{-3}$
24	$9.804190849699 \cdot 10^{-111}$	$2.380787072144 \cdot 10^{-27}$	$1.477155533779 \cdot 10^{-3}$
25	$1.987437770819 \cdot 10^{-119}$	$8.096964479769 \cdot 10^{-29}$	$1.255937167591 \cdot 10^{-3}$
26	$1.944963158798 \cdot 10^{-128}$	$2.658826299224 \cdot 10^{-30}$	$1.072281943096 \cdot 10^{-3}$
27	$9.200371837019 \cdot 10^{-138}$	$8.440416029919 \cdot 10^{-32}$	$9.190658825700 \cdot 10^{-4}$
28	$2.106094898746 \cdot 10^{-147}$	$2.593261382831 \cdot 10^{-33}$	$7.906551701569 \cdot 10^{-4}$
29	$2.335595308380 \cdot 10^{-157}$	$7.719796096416 \cdot 10^{-35}$	$6.825659972471 \cdot 10^{-4}$
30	$1.256036887664 \cdot 10^{-167}$	$2.228848264400 \cdot 10^{-36}$	$5.912083762194 \cdot 10^{-4}$
31	$3.278708332422 \cdot 10^{-178}$	$6.247133168731 \cdot 10^{-38}$	$5.136916809764 \cdot 10^{-4}$
32	$4.157997017276 \cdot 10^{-189}$	$1.701343548002 \cdot 10^{-39}$	$4.476759516974 \cdot 10^{-4}$
33	$2.563947768513 \cdot 10^{-200}$	$4.505851556349 \cdot 10^{-41}$	$3.912574020700 \cdot 10^{-4}$
34	$7.693422511542 \cdot 10^{-212}$	$1.161387499073 \cdot 10^{-42}$	$3.428796821210 \cdot 10^{-4}$
35	$1.124185933806 \cdot 10^{-223}$	$2.915516896653 \cdot 10^{-44}$	$3.012646749161 \cdot 10^{-4}$
36	$8.005161795395 \cdot 10^{-236}$	$7.133413149599 \cdot 10^{-46}$	$2.653582088886 \cdot 10^{-4}$
37	$2.779742976483 \cdot 10^{-248}$	$1.702204559426 \cdot 10^{-47}$	$2.342872326671 \cdot 10^{-4}$
38	$4.709947243068 \cdot 10^{-261}$	$3.963994014026 \cdot 10^{-49}$	$2.073258524838 \cdot 10^{-4}$
39	$3.896412346453 \cdot 10^{-274}$	$9.014085923033 \cdot 10^{-51}$	$1.838682617278 \cdot 10^{-4}$
40	$1.574704738363 \cdot 10^{-287}$	$2.002745422922 \cdot 10^{-52}$	$1.634070599017 \cdot 10^{-4}$
41	$3.110676228472 \cdot 10^{-301}$	$4.349919806088 \cdot 10^{-54}$	$1.455158080783 \cdot 10^{-4}$
42	$3.005087170000 \cdot 10^{-315}$	$9.240880140827 \cdot 10^{-56}$	$1.298349313128 \cdot 10^{-4}$