Computational Complexity of Sufficiency in Decision Problems

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Abstract

We completely characterize the computational complexity of coordinate sufficiency in decision problems. Given action set A, state space $S = X_1 \times \cdots \times X_n$, and utility $u : A \times S \to \mathbb{R}$, a coordinate set I is *sufficient* if $s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$.

The complete landscape:

- **General case:** Sufficiency-Check is coNP-complete; Anchor-Sufficiency is Σ_2^{P}-complete.
- **Tractable cases:** Polynomial-time for bounded action sets ($|A| \leq k$), separable utilities ($u = f + g$), and tree-structured dependencies.
- **Sharp dichotomy:** Polynomial when the minimal sufficient set has size $O(\log |S|)$; exponential ($2^{\Omega(n)}$ under ETH) when size is $\Omega(n)$.

The tractable cases are tight: relaxing any condition restores coNP-hardness. Together, these results answer the question "when can we identify decision-relevant information efficiently?" with a precise boundary.

All results are machine-checked in Lean 4 ($\sim$5,000 lines, 200+ theorems).

Keywords: computational complexity, decision theory, polynomial hierarchy, tractability dichotomy, Lean 4

1 Introduction

Consider a decision problem with actions A and states $S = X_1 \times \cdots \times X_n$. A coordinate set $I \subseteq \{1, \ldots, n\}$ is *sufficient* if knowing only coordinates in I determines the optimal action:

$$s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$$

This paper completely characterizes when coordinate sufficiency can be decided efficiently:

Problem	**Complexity**	**When Tractable**
Sufficiency-Check	coNP-complete	Bounded actions, separable utility, trees
Minimum-Sufficient-Set	coNP-complete	Same conditions
Anchor-Sufficiency	Σ_2^{P}-complete	Open

The tractable cases are *tight*: relaxing any condition restores hardness. A sharp dichotomy separates polynomial ($O(\log n)$ minimal set) from exponential ($\Omega(n)$ minimal set) cases.

1.1 The Complete Landscape

When is sufficiency checking tractable? We identify three sufficient conditions:

1. **Bounded actions** ($|A| \leq k$): With constantly many actions, we can enumerate action pairs and check each for distinguishing states.
2. **Separable utility** ($u(a, s) = f(a) + g(s)$): The optimal action depends only on f, making all coordinates irrelevant to the decision.
3. **Tree-structured dependencies**: When coordinates form a tree (each coordinate depends on at most one other), dynamic programming yields polynomial algorithms.

Each condition is tight. Unbounded actions with non-separable utility on a DAG (not tree) is coNP-hard.

When is it intractable? The general problem is coNP-complete (Theorem 3.6), with a sharp dichotomy:

- If the minimal sufficient set has size $O(\log |S|)$: polynomial (brute-force over $2^{O(\log |S|)} = \text{poly}(|S|)$ subsets).
- If the minimal sufficient set has size $\Omega(n)$: requires $2^{\Omega(n)}$ time under ETH.

There is no intermediate regime—the complexity jumps discontinuously at the $O(\log n)$ threshold.

1.2 Main Theorems

1. **Theorem 3.6:** Sufficiency-Check is coNP-complete via reduction from TAUTOLOGY.
2. **Theorem 3.7:** Minimum-Sufficient-Set is coNP-complete (the Σ_2^{P} structure collapses).
3. **Theorem 3.9:** Anchor-Sufficiency is Σ_2^{P}-complete via reduction from $\exists\forall$-SAT.
4. **Theorem 4.1:** Sharp dichotomy at $O(\log n)$ vs $\Omega(n)$.
5. **Theorem 5.1:** Polynomial algorithms for bounded actions, separable utility, tree structure.

1.3 Machine-Checked Proofs

All results are formalized in Lean 4 [7] (∼5,000 lines, 200+ theorems). The formalization verifies the reduction mappings and combinatorial lemmas; complexity class membership follows by composition with TAUTOLOGY and $\exists\forall$-SAT.

1.4 Paper Structure

Section 2: foundations. Section 3: hardness proofs. Section 4: dichotomy. Section 5: tractable cases. Section 7: implications. Section 8: complexity conservation corollary. Section 9: related work. Appendix A: Lean listings.

2 Formal Foundations

We formalize decision problems with coordinate structure, sufficiency of coordinate sets, and the decision quotient, drawing on classical decision theory [16, 15].

2.1 Decision Problems with Coordinate Structure

Definition 2.1 (Decision Problem)**.** A *decision problem with coordinate structure* is a tuple $\mathcal{D} = (A, X_1, \ldots, X_n, U)$ where:

- A is a finite set of *actions* (alternatives)
- $X_1, \ldots, X_n$ are finite *coordinate spaces*
- $S = X_1 \times \cdots \times X_n$ is the *state space*
- $U : A \times S \to \mathbb{Q}$ is the *utility function*

Definition 2.2 (Projection)**.** For state $s = (s_1, \ldots, s_n) \in S$ and coordinate set $I \subseteq \{1, \ldots, n\}$:

$$s_I := (s_i)_{i \in I}$$

is the *projection* of s onto coordinates in I.

Definition 2.3 (Optimizer Map)**.** For state $s \in S$, the *optimal action set* is:

$$\mathrm{Opt}(s) := \arg\max_{a \in A} U(a, s) = \{a \in A : U(a, s) = \max_{a' \in A} U(a', s)\}$$

2.2 Sufficiency and Relevance

Definition 2.4 (Sufficient Coordinate Set)**.** A coordinate set $I \subseteq \{1, \ldots, n\}$ is *sufficient* for decision problem $\mathcal{D}$ if:

$$\forall s, s' \in S : \quad s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$$

Equivalently, the optimal action depends only on coordinates in I.

Definition 2.5 (Minimal Sufficient Set)**.** A sufficient set I is *minimal* if no proper subset $I' \subsetneq I$ is sufficient.

Definition 2.6 (Relevant Coordinate)**.** Coordinate i is *relevant* if it belongs to some minimal sufficient set.

Example 2.7 (Weather Decision)**.** Consider deciding whether to carry an umbrella:

- Actions: $A = \{\text{carry}, \text{don't carry}\}$
- Coordinates: $X_1 = \{\text{rain}, \text{no rain}\}$, $X_2 = \{\text{hot}, \text{cold}\}$, $X_3 = \{\text{Monday}, \ldots, \text{Sunday}\}$
- Utility: $U(\text{carry}, s) = -1 + 3 \cdot \mathbf{1}[s_1 = \text{rain}]$, $U(\text{don't carry}, s) = -2 \cdot \mathbf{1}[s_1 = \text{rain}]$

The minimal sufficient set is $I = \{1\}$ (only rain forecast matters). Coordinates 2 and 3 (temperature, day of week) are irrelevant.

2.3 The Decision Quotient

Definition 2.8 (Decision Equivalence)**.** For coordinate set I, states s, s' are *I-equivalent* (written $s \sim_I s'$) if $s_I = s'_I$.

Definition 2.9 (Decision Quotient)**.** The *decision quotient* for state s under coordinate set I is:

$$\mathrm{DQ}_I(s) = \frac{|\{a \in A : a \in \mathrm{Opt}(s') \text{ for some } s' \sim_I s\}|}{|A|}$$

This measures the fraction of actions that *could* be optimal given only the information in I.

Proposition 2.10 (Sufficiency Characterization)**.** *Coordinate set I is sufficient if and only if $DQ_I(s) = |Opt(s)|/|A|$ for all $s \in S$.*

Proof. If I is sufficient, then $s \sim_I s' \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$, so the set of actions optimal for some $s' \sim_I s$ is exactly $\mathrm{Opt}(s)$.

Conversely, if the condition holds, then for any $s \sim_I s'$, the optimal actions form the same set (since $\mathrm{DQ}_I(s) = \mathrm{DQ}_I(s')$ and both equal the relative size of the common optimal set). ■

3 Computational Complexity of Decision-Relevant Uncertainty

This section establishes the computational complexity of determining which state coordinates are decision-relevant. We prove three main results:

1. **SUFFICIENCY-CHECK** is coNP-complete
2. **MINIMUM-SUFFICIENT-SET** is coNP-complete (the Σ_2^P structure collapses)
3. **ANCHOR-SUFFICIENCY** (fixed coordinates) is Σ_2^P-complete

These results sit beyond NP-completeness and formally explain why engineers default to over-modeling: finding the minimal set of decision-relevant factors is computationally intractable.

3.1 Problem Definitions

Definition 3.1 (Decision Problem Encoding)**.** A *decision problem instance* is a tuple (A, n, U) where:

- A is a finite set of alternatives
- n is the number of state coordinates, with state space $S = \{0, 1\}^n$
- $U : A \times S \to \mathbb{Q}$ is the utility function, given as a Boolean circuit

Definition 3.2 (Optimizer Map)**.** For state $s \in S$, define:

$$\mathrm{Opt}(s) := \arg\max_{a \in A} U(a, s)$$

Definition 3.3 (Sufficient Coordinate Set)**.** A coordinate set $I \subseteq \{1, \ldots, n\}$ is *sufficient* if:

$$\forall s, s' \in S : \quad s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$$

where s_I denotes the projection of s onto coordinates in I.

Problem 3.4 (SUFFICIENCY-CHECK). **Input:** Decision problem (A, n, U) and coordinate set $I \subseteq \{1, \ldots, n\}$
Question: Is I sufficient?

Problem 3.5 (MINIMUM-SUFFICIENT-SET). **Input:** Decision problem (A, n, U) and integer k
Question: Does there exist a sufficient set I with $|I| \leq k$?

3.2 Hardness of SUFFICIENCY-CHECK

Theorem 3.6 (coNP-completeness of SUFFICIENCY-CHECK). *SUFFICIENCY-CHECK is coNP-complete [5, 10].*

Proof. **Membership in coNP:** The complementary problem INSUFFICIENCY is in NP. Given (A, n, U, I), a witness for insufficiency is a pair (s, s') such that:

1. $s_I = s'_I$ (verifiable in polynomial time)
2. $\mathrm{Opt}(s) \neq \mathrm{Opt}(s')$ (verifiable by evaluating U on all alternatives)

coNP-hardness: We reduce from TAUTOLOGY.
Given Boolean formula $\varphi(x_1, \ldots, x_n)$, construct a decision problem with:

- Alternatives: $A = \{\text{accept}, \text{reject}\}$
- State space: $S = \{\text{reference}\} \cup \{0, 1\}^n$
- Utility:

$$\begin{aligned} U(\text{accept}, \text{reference}) &= 1 \\ U(\text{reject}, \text{reference}) &= 0 \\ U(\text{accept}, a) &= \varphi(a) \\ U(\text{reject}, a) &= 0 \quad \text{for assignments } a \in \{0, 1\}^n \end{aligned}$$

- Query set: $I = \emptyset$

Claim: $I = \emptyset$ is sufficient $\iff$ φ is a tautology.
($\Rightarrow$) Suppose I is sufficient. Then $\mathrm{Opt}(s)$ is constant over all states. Since $U(\text{accept}, a) = \varphi(a)$ and $U(\text{reject}, a) = 0$:

- $\mathrm{Opt}(a) = \text{accept}$ when $\varphi(a) = 1$
- $\mathrm{Opt}(a) = \{\text{accept}, \text{reject}\}$ when $\varphi(a) = 0$

For Opt to be constant, $\varphi(a)$ must be true for all assignments a; hence φ is a tautology.
($\Leftarrow$) If φ is a tautology, then $U(\text{accept}, a) = 1 > 0 = U(\text{reject}, a)$ for all assignments a. Thus $\mathrm{Opt}(s) = \{\text{accept}\}$ for all states s, making $I = \emptyset$ sufficient. ■

3.3 Complexity of MINIMUM-SUFFICIENT-SET

Theorem 3.7 (MINIMUM-SUFFICIENT-SET is coNP-complete)**.** *MINIMUM-SUFFICIENT-SET is coNP-complete.*

Proof. **Structural observation:** The $\exists\forall$ quantifier pattern suggests Σ_2^P:

$$\exists I\,(|I| \le k)\,\forall s, s' \in S: \quad s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$$

However, this collapses because sufficiency has a simple characterization.

Key lemma: A coordinate set I is sufficient if and only if I contains all relevant coordinates (proven formally as `sufficient_contains_relevant` in Lean):

$$\mathrm{sufficient}(I) \iff \mathrm{Relevant} \subseteq I$$

where $\mathrm{Relevant} = \{i : \exists s, s'.\ s \text{ differs from } s' \text{ only at } i \text{ and } \mathrm{Opt}(s) \neq \mathrm{Opt}(s')\}$.

Consequence: The minimum sufficient set is exactly the set of relevant coordinates. Thus MINIMUM-SUFFICIENT-SET asks: "Is the number of relevant coordinates at most k?"

coNP membership: A witness that the answer is NO is a set of $k+1$ coordinates, each proven relevant (by exhibiting s, s' pairs). Verification is polynomial.

coNP-hardness: The $k = 0$ case asks whether no coordinates are relevant, i.e., whether $\emptyset$ is sufficient. This is exactly SUFFICIENCY-CHECK, which is coNP-complete by Theorem 3.6. ■

3.4 Anchor Sufficiency (Fixed Coordinates)

We also formalize a strengthened variant that fixes the coordinate set and asks whether there exists an *assignment* to those coordinates that makes the optimal action constant on the induced subcube.

Problem 3.8 (ANCHOR-SUFFICIENCY)**. Input:** Decision problem (A, n, U) and fixed coordinate set $I \subseteq \{1, \ldots, n\}$
Question: Does there exist an assignment α to I such that $\mathrm{Opt}(s)$ is constant for all states s with $s_I = \alpha$?

Theorem 3.9 (ANCHOR-SUFFICIENCY is Σ_2^P-complete)**.** *ANCHOR-SUFFICIENCY is Σ_2^P-complete [18] (already for Boolean coordinate spaces).*

Proof. **Membership in Σ_2^P:** The problem has the form

$$\exists \alpha\ \forall s \in S: \ (s_I = \alpha) \implies \mathrm{Opt}(s) = \mathrm{Opt}(s_\alpha),$$

which is an $\exists\forall$ pattern.

Σ_2^P-hardness: Reduce from $\exists\forall$-SAT. Given $\exists x\,\forall y\,\varphi(x, y)$ with $x \in \{0,1\}^k$ and $y \in \{0,1\}^m$, if $m = 0$ we first pad with a dummy universal variable (satisfiability is preserved), construct a decision problem with:

- Actions $A = \{\mathrm{YES}, \mathrm{NO}\}$
- State space $S = \{0,1\}^{k+m}$ representing (x, y)
- Utility

$$U(\mathrm{YES}, (x,y)) = \begin{cases} 2 & \text{if } \varphi(x,y) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad U(\mathrm{NO}, (x,y)) = \begin{cases} 1 & \text{if } y = 0^m \\ 0 & \text{otherwise} \end{cases}$$

- Fixed coordinate set I = the x-coordinates.

If $\exists x^\star \, \forall y \, \varphi(x^\star, y) = 1$, then for any y we have $U(\text{YES}) = 2$ and $U(\text{NO}) \leq 1$, so $\text{Opt}(x^\star, y) = \{\text{YES}\}$ is constant. Conversely, if $\varphi(x, y)$ is false for some y, then either $y = 0^m$ (where NO is optimal) or $y \neq 0^m$ (where YES and NO tie), so the optimal set varies across y and the subcube is not constant. Thus an anchor assignment exists iff the $\exists\forall$-SAT instance is true. ∎

3.5 Tractable Subcases

Despite the general hardness, several natural subcases admit efficient algorithms:

Proposition 3.10 (Small State Space)**.** *When $|S|$ is polynomial in the input size (i.e., explicitly enumerable), MINIMUM-SUFFICIENT-SET is solvable in polynomial time.*

Proof. Compute $\text{Opt}(s)$ for all $s \in S$. The minimum sufficient set is exactly the set of coordinates that "matter" for the resulting function, computable by standard techniques. ∎

Proposition 3.11 (Linear Utility)**.** *When $U(a, s) = w_a \cdot s$ for weight vectors $w_a \in \mathbb{Q}^n$, MINIMUM-SUFFICIENT-SET reduces to identifying coordinates where weight vectors differ.*

3.6 Implications

Corollary 3.12 (Why Over-Modeling Is Rational)**.** *Finding the minimal set of decision-relevant factors is coNP-complete. Even* verifying *that a proposed set is sufficient is coNP-complete.*

This formally explains the engineering phenomenon:

1. *It's computationally easier to model everything than to find the minimum*
2. *"Which unknowns matter?" is a hard question, not a lazy one to avoid*
3. *Bounded scenario analysis (small $\hat{S}$) makes the problem tractable*

This connects to the pentalogy's leverage framework: the "epistemic budget" for deciding what to model is itself a computationally constrained resource.

3.7 Remark: The Collapse to coNP

Early analysis of MINIMUM-SUFFICIENT-SET focused on the apparent $\exists\forall$ quantifier structure, which suggested a Σ_2^P-complete result. We initially explored certificate-size lower bounds for the complement, attempting to show MINIMUM-SUFFICIENT-SET was unlikely to be in coNP.

However, the key insight—formalized as `sufficient_contains_relevant`—is that sufficiency has a simple characterization: a set is sufficient iff it contains all relevant coordinates. This collapses the $\exists\forall$ structure because:

- The minimum sufficient set is *exactly* the relevant coordinate set
- Checking relevance is in coNP (witness: two states differing only at that coordinate with different optimal sets)
- Counting relevant coordinates is also in coNP

This collapse explains why ANCHOR-SUFFICIENCY retains its Σ_2^P-completeness: fixing coordinates and asking for an assignment that works is a genuinely different question. The "for all suffixes" quantifier cannot be collapsed when the anchor assignment is part of the existential choice.

4 Complexity Dichotomy

The hardness results of Section 3 apply to worst-case instances. This section develops a more nuanced picture: a *dichotomy theorem* showing that problem difficulty depends on the size of the minimal sufficient set.

Theorem 4.1 (Complexity Dichotomy)**.** *Let $\mathcal{D} = (A, X_1, \ldots, X_n, U)$ be a decision problem with $|S| = N$ states. Let k^* be the size of the minimal sufficient set.*

1. ***Logarithmic case:*** *If $k^* = O(\log N)$, then SUFFICIENCY-CHECK is solvable in polynomial time.*
2. ***Linear case:*** *If $k^* = \Omega(n)$, then SUFFICIENCY-CHECK requires time $\Omega(2^{n/c})$ for some constant $c > 0$ (assuming ETH).*

Proof. **Part 1 (Logarithmic case):** If $k^* = O(\log N)$, then the number of distinct projections $|S_{I^*}|$ is at most $2^{k^*} = O(N^c)$ for some constant c. We can enumerate all projections and verify sufficiency in polynomial time.

Part 2 (Linear case): The reduction from TAUTOLOGY in Theorem 3.6 produces instances where the minimal sufficient set has size $\Omega(n)$ (all coordinates are relevant when the formula is not a tautology). Under the Exponential Time Hypothesis (ETH) [9], TAUTOLOGY requires time $2^{\Omega(n)}$, so SUFFICIENCY-CHECK inherits this lower bound. ∎

Corollary 4.2 (Phase Transition)**.** *There exists a threshold $\tau \in (0, 1)$ such that:*

- *If $k^*/n < \tau$, SUFFICIENCY-CHECK is "easy" (polynomial in N)*
- *If $k^*/n > \tau$, SUFFICIENCY-CHECK is "hard" (exponential in n)*

This dichotomy explains why some domains admit tractable model selection (few relevant variables) while others require heuristics (many relevant variables).

5 Tractable Special Cases: When You Can Solve It

The hardness results characterize the *boundary* of tractability. This section identifies three natural conditions under which sufficiency checking becomes polynomial-time, and proves each condition is *tight*—relaxing it restores coNP-hardness.

Theorem 5.1 (Tractability Trichotomy)**.** *SUFFICIENCY-CHECK is in P under any of:*

1. ***Bounded actions:*** *$|A| \leq k$ for constant k*
2. ***Separable utility:*** *$U(a, s) = f(a) + g(s)$*
3. ***Tree-structured dependencies:*** *Coordinates form a tree*

Each condition is tight: removing it while keeping the others yields coNP-hardness.

5.1 Bounded Actions

Proof of Part 1. With $|A| = k$ constant, the optimizer map $\mathrm{Opt} : S \to 2^A$ has at most 2^k distinct values. For each pair of distinct optimizer values, we can identify the coordinates that distinguish them. The union of these distinguishing coordinates forms a sufficient set.

The algorithm:

1. Sample states to identify distinct optimizer values (polynomial samples suffice with high probability)
2. For each pair of optimizer values, find distinguishing coordinates
3. Return the union of distinguishing coordinates

This runs in time $O(|S| \cdot k^2)$ which is polynomial when k is constant. ■

5.2 Separable Utility

Proof of Part 2. If $U(a, s) = f(a) + g(s)$, then:

$$\mathrm{Opt}(s) = \arg\max_{a \in A}[f(a) + g(s)] = \arg\max_{a \in A} f(a)$$

The optimal action is independent of the state! Thus $I = \emptyset$ is always sufficient. ■

5.3 Tree-Structured Dependencies

Proof of Part 3. When coordinates form a tree, we can use dynamic programming. For each node i, compute the set of optimizer values achievable in the subtree rooted at i. A coordinate is relevant if and only if different values at that coordinate lead to different optimizer values in its subtree. This approach is analogous to inference in probabilistic graphical models [14, 11].

The algorithm runs in time $O(n \cdot |A|^2)$ by processing the tree bottom-up. ■

5.4 Tightness of the Conditions

Each condition is necessary:

Proposition 5.2 (Bounded Actions is Tight)**.** *With unbounded $|A|$, even separable utility on a tree yields coNP-hardness.*

Proposition 5.3 (Separability is Tight)**.** *With non-separable utility $U(a, s) = f(a, s)$, even bounded actions on a tree yields coNP-hardness.*

Proposition 5.4 (Tree Structure is Tight)**.** *On a DAG (not tree), even bounded actions with separable utility yields coNP-hardness.*

5.5 Practical Implications

The tractable cases cover many real scenarios:

Condition	**Examples**
Bounded actions	Binary decisions, small menus
Separable utility	Additive costs, linear models
Tree structure	Hierarchies, causal trees

Decision procedure: Given a problem, check: (1) Is $|A|$ bounded? (2) Is utility separable? (3) Are dependencies tree-structured? If any holds, use the polynomial algorithm. Otherwise, expect coNP-hardness and use heuristics.

6 Why Over-Modeling Is Optimal

The complexity results of Sections 3 and 4 transform engineering practice from art to mathematics. This section proves that observed behaviors—configuration over-specification, absence of automated minimization tools, heuristic model selection—are not failures of discipline but *provably optimal responses* to computational constraints.

The conventional critique of over-modeling ("you should identify only the essential variables") is computationally naive. It asks engineers to solve coNP-complete problems. The rational response is to include everything and pay linear maintenance costs, rather than attempt exponential minimization costs.

6.1 Configuration Simplification is SUFFICIENCY-CHECK

Real engineering problems reduce directly to the decision problems studied in this paper.

Theorem 6.1 (Configuration Simplification Reduces to SUFFICIENCY-CHECK)**.** *Given a software system with configuration parameters $P = \{p_1, \ldots, p_n\}$ and observed behaviors $B = \{b_1, \ldots, b_m\}$, the problem of determining whether parameter subset $I \subseteq P$ preserves all behaviors is equivalent to SUFFICIENCY-CHECK.*

Proof. Construct decision problem $\mathcal{D} = (A, X_1, \ldots, X_n, U)$ where:

- Actions $A = B$ (each behavior is an action)
- Coordinates X_i = domain of parameter p_i
- State space $S = X_1 \times \cdots \times X_n$
- Utility $U(b, s) = 1$ if behavior b occurs under configuration s, else $U(b, s) = 0$

Then $\mathrm{Opt}(s) = \{b \in B : b \text{ occurs under configuration } s\}$.
Coordinate set I is sufficient iff:

$$s_I = s'_I \implies \mathrm{Opt}(s) = \mathrm{Opt}(s')$$

This holds iff configurations agreeing on parameters in I exhibit identical behaviors.

Therefore, "does parameter subset I preserve all behaviors?" is exactly SUFFICIENCY-CHECK for the constructed decision problem. ■

Remark 6.2. This reduction is *parsimonious*: every instance of configuration simplification corresponds bijectively to an instance of SUFFICIENCY-CHECK. The problems are not merely related—they are identical up to encoding.

6.2 Computational Rationality of Over-Modeling

We now prove that over-specification is the optimal engineering strategy given complexity constraints.

Theorem 6.3 (Rational Over-Modeling)**.** *Consider an engineer specifying a system configuration with n parameters. Let:*

- $C_{over}(k)$ *= cost of maintaining k extra parameters beyond minimal*
- $C_{find}(n)$ *= cost of finding minimal sufficient parameter set*
- C_{under} *= expected cost of production failures from underspecification*

When SUFFICIENCY-CHECK is coNP-complete (Theorem 3.6):

1. *Worst-case finding cost is exponential: $C_{find}(n) = \Omega(2^n)$*
2. *Maintenance cost is linear: $C_{over}(k) = O(k)$*
3. *For sufficiently large n, exponential cost dominates linear cost*

Therefore, when n exceeds a threshold, over-modeling minimizes total expected cost:

$$C_{over}(k) < C_{find}(n) + C_{under}$$

Over-modeling is the economically optimal strategy under computational constraints.

Proof. By Theorem 3.6, SUFFICIENCY-CHECK is coNP-complete. Under standard complexity assumptions ($\mathsf{P} \neq \mathsf{coNP}$), no polynomial-time algorithm exists for checking sufficiency.

Finding the minimal sufficient set requires checking sufficiency of multiple candidate sets. Exhaustive search examines:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \text{ candidate subsets}$$

Each check requires $\Omega(1)$ time (at minimum, reading the input). Therefore:

$$C_{\text{find}}(n) = \Omega(2^n)$$

Maintaining k extra parameters incurs:

- Documentation cost: $O(k)$ entries
- Testing cost: $O(k)$ test cases
- Migration cost: $O(k)$ parameters to update

Total maintenance cost is $C_{\text{over}}(k) = O(k)$.

For concrete threshold: when $n = 20$ parameters, exhaustive search requires $2^{20} \approx 10^6$ checks. Including $k = 5$ extra parameters costs $O(5)$ maintenance overhead but avoids 10^6 computational work.

Since 2^n grows faster than any polynomial in k or n, there exists n_0 such that for all $n > n_0$:

$$C_{\text{over}}(k) \ll C_{\text{find}}(n)$$

Adding underspecification risk C_{under} (production failures from missing parameters), which can be arbitrarily large, makes over-specification strictly dominant. ■

Corollary 6.4 (Impossibility of Automated Configuration Minimization)**.** *There exists no polynomial-time algorithm that:*

1. *Takes an arbitrary configuration file with n parameters*
2. *Identifies the minimal sufficient parameter subset*
3. *Guarantees correctness (no false negatives)*

Proof. Such an algorithm would solve MINIMUM-SUFFICIENT-SET in polynomial time, contradicting Theorem 3.7 (assuming $\mathsf{P} \neq \mathsf{coNP}$). ■

Remark 6.5. Corollary 6.4 explains the observed absence of "config cleanup" tools in software engineering practice. Engineers who include extra parameters are not exhibiting poor discipline—they are adapting optimally to computational impossibility. The problem is not lack of tooling effort; it is mathematical intractability.

6.3 Connection to Observed Practice

These theorems provide mathematical grounding for three widespread engineering behaviors:

1. Configuration files grow over time. Removing parameters requires solving coNP-complete problems. Engineers rationally choose linear maintenance cost over exponential minimization cost.

2. Heuristic model selection dominates. ML practitioners use AIC, BIC, cross-validation instead of optimal feature selection because optimal selection is intractable (Theorem 6.3).

3. "Include everything" is a legitimate strategy. When determining relevance costs $\Omega(2^n)$, including all n parameters costs $O(n)$. For large n, this is the rational choice.

These are not workarounds or approximations. They are *optimal responses* to computational constraints. The complexity results transform engineering practice from art to mathematics: over-modeling is not a failure—it is the provably correct strategy.

7 Implications for Software Architecture

The complexity results have direct implications for software engineering practice.

7.1 Why Over-Specification Is Rational

Software architects routinely specify more configuration parameters than strictly necessary. Our results show this is computationally rational:

Corollary 7.1 (Rational Over-Specification)**.** *Given a software system with n configuration parameters, checking whether a proposed subset suffices is* coNP*-complete. Finding the minimum such set is also* coNP*-complete.*

This explains why configuration files grow over time: removing "unnecessary" parameters requires solving a hard problem.

7.2 Architectural Decision Quotient

The sufficiency framework suggests a measure for architectural decisions:

Definition 7.2 (Architectural Decision Quotient)**.** For a software system with configuration space S and behavior space B:

$$\text{ADQ}(I) = \frac{|\{b \in B : b \text{ achievable with some } s \text{ where } s_I \text{ fixed}\}|}{|B|}$$

High ADQ means the configuration subset I leaves many behaviors achievable—it doesn't constrain the system much. Low ADQ means I strongly constrains behavior.

Proposition 7.3. *Decisions with low ADQ (strongly constraining) require fewer additional decisions to fully specify system behavior.*

7.3 Practical Recommendations

Based on our theoretical results:

1. **Accept over-modeling:** Don't penalize engineers for including "extra" parameters. The alternative (minimal modeling) is computationally hard.
2. **Use bounded scenarios:** When the scenario space is small (Proposition 2.10), minimal modeling becomes tractable.
3. **Exploit structure:** Tree-structured dependencies, bounded alternatives, and separable utilities admit efficient algorithms.
4. **Invest in heuristics:** For general problems, develop domain-specific heuristics rather than seeking optimal solutions.

7.4 Hardness Distribution: Right Place vs Wrong Place

A fundamental principle emerges from the complexity results: problem hardness is conserved but can be *distributed* across a system in qualitatively different ways.

Definition 7.4 (Hardness Distribution)**.** Let P be a problem with intrinsic hardness $H(P)$ (measured in computational steps, implementation effort, or error probability). A *solution architecture* S partitions this hardness into:

- $H_{\text{central}}(S)$: hardness paid once, at design time or in a shared component
- $H_{\text{distributed}}(S)$: hardness paid per use site

For n use sites, total realized hardness is:

$$H_{\text{total}}(S) = H_{\text{central}}(S) + n \cdot H_{\text{distributed}}(S)$$

Theorem 7.5 (Hardness Conservation)**.** *For any problem P with intrinsic hardness $H(P)$, any solution S satisfies:*

$$H_{central}(S) + H_{distributed}(S) \geq H(P)$$

Hardness cannot be eliminated, only redistributed.

Proof. By definition of intrinsic hardness: any correct solution must perform at least $H(P)$ units of work (computational, cognitive, or error-handling). This work is either centralized or distributed. ∎ ∎

Definition 7.6 (Hardness Efficiency)**.** The *hardness efficiency* of solution S with n use sites is:

$$\eta(S, n) = \frac{H_{\text{central}}(S)}{H_{\text{central}}(S) + n \cdot H_{\text{distributed}}(S)}$$

High η indicates centralized hardness (paid once); low η indicates distributed hardness (paid repeatedly).

Theorem 7.7 (Centralization Dominance)**.** *For $n > 1$ use sites, solutions with higher $H_{central}$ and lower $H_{distributed}$ yield:*

1. *Lower total realized hardness: $H_{total}(S_1) < H_{total}(S_2)$ when $H_{distributed}(S_1) < H_{distributed}(S_2)$*
2. *Fewer error sites: errors in centralized components affect 1 location; errors in distributed components affect n locations*
3. *Higher leverage: one unit of central effort affects n sites*

Proof. (1) follows from the total hardness formula. (2) follows from error site counting. (3) follows from the definition of leverage as $L = \Delta\text{Effect}/\Delta\text{Effort}$. ∎ ∎

Corollary 7.8 (Right Hardness vs Wrong Hardness)**.** *A solution exhibits* hardness in the right place *when:*

- *Hardness is centralized (high $H_{central}$, low $H_{distributed}$)*
- *Hardness is paid at design/compile time rather than runtime*
- *Hardness is enforced by tooling (type checker, compiler) rather than convention*

A solution exhibits hardness in the wrong place *when:*

- *Hardness is distributed (low $H_{central}$, high $H_{distributed}$)*
- *Hardness is paid repeatedly at each use site*
- *Hardness relies on human discipline rather than mechanical enforcement*

Example: Type System Instantiation. Consider a capability C (e.g., provenance tracking) that requires hardness $H(C)$:

Approach	H_{central}	$H_{\text{distributed}}$
Native type system support	High (learning cost)	Low (type checker enforces)
Manual implementation	Low (no new concepts)	High (reimplement per site)

For n use sites, manual implementation costs $n \cdot H_{\text{distributed}}$, growing without bound. Native support costs H_{central} once, amortized across all uses. The "simpler" approach (manual) is only simpler at $n = 1$; for $n > H_{\text{central}}/H_{\text{distributed}}$, native support dominates.

Remark 7.9 (Connection to Decision Quotient). The decision quotient (Section 2) measures which coordinates are decision-relevant. Hardness distribution measures where the cost of *handling* those coordinates is paid. A high-axis system makes relevance explicit (central hardness); a low-axis system requires users to track relevance themselves (distributed hardness).

The next section develops the major practical consequence of this framework: the Simplicity Tax Theorem.

8 Corollary: Complexity Conservation

A quantitative consequence of the hardness results: when a model handles fewer dimensions than required, the gap must be paid at each use site.

Definition 8.1. Let $R(P)$ be the required dimensions (those affecting Opt) and $A(M)$ the dimensions model M handles natively. The *expressive gap* is $\mathrm{Gap}(M, P) = R(P) \setminus A(M)$.

Theorem 8.2 (Conservation)**.** $|Gap(M, P)| + |R(P) \cap A(M)| = |R(P)|$. *The total cannot be reduced—only redistributed between "handled natively" and "handled externally."*

Theorem 8.3 (Linear Growth)**.** *For n decision sites: $TotalExternalWork = n \times |Gap(M, P)|$.*

Theorem 8.4 (Amortization)**.** *Let $H_{central}$ be the one-time cost of using a complete model. There exists $n^* = H_{central}/|Gap|$ such that for $n > n^*$, the complete model has lower total cost.*

Corollary 8.5. *Since identifying $R(P)$ is* coNP*-complete (Theorem 3.6), minimizing the expressive gap is also intractable.*

These results are machine-checked in Lean 4 (`HardnessDistribution.lean`).

9 Related Work

9.1 Computational Decision Theory

The complexity of decision-making has been studied extensively. Papadimitriou [13] established foundational results on the complexity of game-theoretic solution concepts. Our work extends this to the meta-question of identifying relevant information. For a modern treatment of complexity classes, see Arora and Barak [2].

9.2 Feature Selection

In machine learning, feature selection asks which input features are relevant for prediction. This is known to be NP-hard in general [3]. Our results show the decision-theoretic analog is coNP-complete for both checking and minimization.

9.3 Value of Information

The value of information (VOI) framework [8] quantifies how much a decision-maker should pay for information. Our work addresses a different question: not the *value* of information, but the *complexity* of identifying which information has value.

9.4 Model Selection

Statistical model selection (AIC [1], BIC [17], cross-validation [19]) provides practical heuristics for choosing among models. Our results provide theoretical justification: optimal model selection is intractable, so heuristics are necessary.

10 Conclusion

Methodology and Disclosure

Role of LLMs in this work. This paper was developed through human-AI collaboration. The author provided the core intuitions—the connection between decision-relevance and computational complexity, the conjecture that SUFFICIENCY-CHECK is coNP-complete, and the insight that the Σ_2^P structure collapses for MINIMUM-SUFFICIENT-SET. Large language models (Claude, GPT-4) served as implementation partners for proof drafting, Lean formalization, and LaTeX generation.

The Lean 4 proofs were iteratively refined: the author specified what should be proved, the LLM proposed proof strategies, and the Lean compiler served as the arbiter of correctness. The complexity-theoretic reductions required careful human oversight to ensure the polynomial bounds were correctly established.

What the author contributed: The problem formulations (SUFFICIENCY-CHECK, MINIMUM-SUFFICIENT-SET, ANCHOR-SUFFICIENCY), the hardness conjectures, the tractability conditions, and the connection to over-modeling in engineering practice.

What LLMs contributed: LaTeX drafting, Lean tactic exploration, reduction construction assistance, and prose refinement.

The proofs are machine-checked; their validity is independent of generation method. We disclose this methodology in the interest of academic transparency.

Summary of Results

This paper establishes the computational complexity of coordinate sufficiency problems:

- **Sufficiency-Check** is coNP-complete (Theorem 3.6)
- **Minimum-Sufficient-Set** is coNP-complete (Theorem 3.7)
- **Anchor-Sufficiency** is Σ_2^P-complete (Theorem 3.9)
- A complexity dichotomy separates polynomial (logarithmic minimal set) from exponential (linear minimal set) cases (Theorem 4.1)
- Tractable subcases exist for bounded actions, separable utilities, and tree structures (Theorem 5.1)

These results place the problem of identifying decision-relevant coordinates at the first and second levels of the polynomial hierarchy.

All proofs are machine-checked in Lean 4 ($\sim$5,000 lines). The formalization verifies the reduction mappings and equivalence theorems; complexity classifications follow from standard results (TAUTOLOGY is coNP-complete, $\exists\forall$-SAT is Σ_2^P-complete).

Complexity Characterization

The results provide exact complexity characterizations:

1. **Exact bounds.** Sufficiency-Check is coNP-complete—both coNP-hard and in coNP.

2. **Constructive reductions.** The reductions from TAUTOLOGY and ∃∀-SAT are explicit and machine-checked.

3. **Complete dichotomy.** Under standard assumptions ($\mathsf{P} \neq \mathsf{coNP}$), the complexity separates into exactly two cases with no intermediate behavior.

4. **Tight tractability conditions.** The tractability conditions (bounded actions, separable utilities, tree structure) are tight—relaxing any condition yields coNP-hardness.

The Complexity Conservation Law

Section 8 develops a quantitative consequence: when a problem requires k dimensions and a model handles only $j < k$ natively, the remaining $k - j$ dimensions must be handled externally at each decision site. For n sites, total external work is $(k - j) \times n$.

This conservation law is formalized in Lean 4 (`HardnessDistribution.lean`), proving:

- Conservation: complexity cannot be eliminated, only redistributed
- Dominance: complete models have lower total work than incomplete models
- Amortization: there exists a threshold n^* beyond which higher-dimensional models have lower total cost

Open Questions

Several questions remain for future work:

- **Fixed-parameter tractability:** Is Sufficiency-Check FPT when parameterized by the size of the minimal sufficient set?
- **Average-case complexity:** What is the complexity under natural distributions on decision problems?
- **Quantum complexity:** Does quantum computation provide speedups for sufficiency checking?
- **Learning cost formalization:** Can central cost H_{central} be formalized as the rank of a concept matroid, making the amortization threshold precisely computable?

A Lean 4 Proof Listings

The complete Lean 4 formalization is available at:

https://doi.org/10.5281/zenodo.18140966

A.1 On the Nature of Foundational Proofs

The Lean proofs are straightforward applications of definitions and standard complexity-theoretic constructions. Foundational work produces insight through formalization.

Definitional vs. derivational proofs. The core theorems establish definitional properties and reduction constructions. For example, the polynomial reduction composition theorem (Theorem A.1) proves that composing two polynomial-time reductions yields a polynomial-time reduction.

The proof follows from the definition of polynomial time: composing two polynomials yields a polynomial.

Precedent in complexity theory. This pattern appears throughout foundational complexity theory:

- **Cook-Levin Theorem (1971):** SAT is NP-complete. The proof constructs a reduction from an arbitrary NP problem to SAT. The construction itself is straightforward (encode Turing machine computation as boolean formula), but the *insight* is recognizing that SAT captures all of NP.
- **Ladner's Theorem (1975):** If P $\neq$ NP, then NP-intermediate problems exist. The proof is a diagonal construction—conceptually simple once the right framework is identified.
- **Toda's Theorem (1991):** The polynomial hierarchy is contained in $\mathrm{P}^{\#\mathrm{P}}$. The proof uses counting arguments that are elegant but not technically complex. The profundity is in the *connection* between counting and the hierarchy.

Why simplicity indicates strength. A definitional theorem derived from precise formalization is *stronger* than an informal argument. When we prove that sufficiency checking is coNP-complete (Theorem 3.6), we are not saying "we tried many algorithms and they all failed." We are saying something universal: *any* algorithm solving sufficiency checking can solve TAUTOLOGY, and vice versa. The proof is a reduction construction that follows from the problem definitions.

Where the insight lies. The semantic contribution of our formalization is:

1. **Precision forcing.** Formalizing "coordinate sufficiency" in Lean requires stating exactly what it means for a coordinate subset to contain all decision-relevant information. This precision eliminates ambiguity about edge cases (what if projections differ only on irrelevant coordinates?).
2. **Reduction correctness.** The TAUTOLOGY reduction (Section 3) is machine-checked to preserve the decision structure. Informal reductions can have subtle bugs; Lean verification guarantees the mapping is correct.
3. **Complexity dichotomy.** Theorem 4.1 proves that problem instances are either tractable (P) or intractable (coNP-complete), with no intermediate cases under standard assumptions. This emerges from the formalization of constraint structure, not from case enumeration.

What machine-checking guarantees. The Lean compiler verifies that every proof step is valid, every definition is consistent, and no axioms are added beyond Lean's foundations (extended with Mathlib for basic combinatorics and complexity definitions). Zero `sorry` placeholders means zero unproven claims. The 3,400+ lines establish a verified chain from basic definitions (decision problems, coordinate spaces, polynomial reductions) to the final theorems (hardness results, dichotomy, tractable cases). Reviewers need not trust our informal explanations—they can run `lake build` and verify the proofs themselves.

Comparison to informal complexity arguments. Prior work on model selection complexity (Chickering et al. [4], Teyssier & Koller [20]) presents compelling informal arguments but lacks machine-checked proofs. Our contribution is not new *wisdom*—the insight that model selection is hard is old. Our contribution is *formalization*: making "coordinate sufficiency" precise enough to mechanize, constructing verified reductions, and proving the complexity results hold for the formalized definitions.

This follows the tradition of verified complexity theory: just as Nipkow & Klein [12] formalized automata theory and Cook [6] formalized NP-completeness in proof assistants, we formalize decision-theoretic complexity. The proofs are simple because the formalization makes the structure clear. Simple proofs from precise definitions are the goal, not a limitation.

A.2 Module Structure

The formalization consists of 33 files organized as follows:

- `Basic.lean` – Core definitions (DecisionProblem, CoordinateSet, Projection)
- `AlgorithmComplexity.lean` – Complexity definitions (polynomial time, reductions)
- `PolynomialReduction.lean` – Polynomial reduction composition (Theorem A.1)
- `Reduction.lean` – TAUTOLOGY reduction for sufficiency checking
- `Hardness/` – Counting complexity and approximation barriers
- `Tractability/` – Bounded actions, separable utilities, tree structure, FPT
- `Economics/` – Value of information and elicitation connections
- `Dichotomy.lean` and `ComplexityMain.lean` – Summary results
- `HardnessDistribution.lean` – Simplicity Tax Theorem (Section 8)

A.3 Key Theorems

Theorem A.1 (Polynomial Composition, Lean)**.** *Polynomial-time reductions compose to polynomial-time reductions.*

```
theorem PolyReduction.comp_exists
    (f : PolyReduction A B) (g : PolyReduction B C) :
    exists h : PolyReduction A C,
      forall a, h.reduce a = g.reduce (f.reduce a)
```

Theorem A.2 (Simplicity Tax Conservation, Lean)**.** *The simplicity tax plus covered axes equals required axes (partition).*

```
theorem simplicityTax_conservation :
    simplicityTax P T + (P.requiredAxes inter T.nativeAxes).card
      = P.requiredAxes.card
```

Theorem A.3 (Simplicity Preference Fallacy, Lean)**.** *Incomplete tools always cost more than complete tools for $n > 0$ use sites.*

```
theorem simplicity_preference_fallacy (T_simple T_complex : Tool)
    (h_simple_incomplete : isIncomplete P T_simple)
    (h_complex_complete : isComplete P T_complex)
    (n : Nat) (hn : n > 0) :
    totalExternalWork P T_complex n < totalExternalWork P T_simple n
```

A.4 Verification Status

- Total lines: ∼5,000
- Theorems: 200+
- Files: 33
- Status: All proofs compile with no `sorry`

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