HULL-WHITE N-FACTOR MODEL

ABSTRACT. Specification of Hull-White n-factor model.

CONTENTS

1. GENERAL GAUSSIAN MULTI-FACTOR INTEREST RATE MODEL

1.1. **Development from HJM Analysis.** We follow [2]. They develop the model starting from the general class of n dimensional HJM models, given by (formula 4.31 in [2])

$$(1.1) \qquad dP(t,T)/P(t,T) = r(t)dt - \sigma_P(t,T)^T dW(t)$$

under the bank account measure, where $P(t,T)$ is the zero bond price with maturity T as seen from time t, $r(t)$ is the short rate at time t and $\sigma_p(t,T,\omega)$ is a m dimensional process adapted to the filtration generated by a m-dimensional Brownian motion $W(t)$.

In terms of instantaneous forward rates 1.1 reads

$$(1.2) \qquad df(t,T) = \sigma_f(t,T)^T \int_t^T \sigma_f(t,u)dudt + \sigma_f(t,T)^T dW(t)$$

where

$$(1.3) \qquad \sigma_f(t,T) = \frac{\partial}{\partial T}\sigma_P(t,T)$$

see [2], formula 4.38. The class of Gaussian n-factor models $(n \geq m)$ is retrieved by the *separability* restriction

$$(1.4) \qquad \sigma_f(t,T) = g(t)h(T)$$

for a deterministic $m \times n$ matrix valued function g and a deterministic n-dimensional vector valued function h. In this class we can write a specific model as (see [2] proposition 12.1.2):

$$(1.5) \qquad dx(t) = (y(t)\mathbf{1} - \kappa x(t))dt + \sigma_x(t)^T dW(t)$$

$$(1.6) \qquad f(t,T) = f(0,T) + m(t,T)^T \left(x(t) + y(t)\int_t^T m(t,u)du \right)$$

$$(1.7) \qquad r(t) = f(0,t) + \sum_{i=1}^n x_i(t)$$

where $\mathbf{1}$ is a column vector containing 1 at each component and with a $n \times n$ diagonal mean reversion matrix

$$(1.8) \qquad \kappa = \mathrm{diag}(\kappa_1, \ldots, \kappa_n)$$

which we use to parametrize $H(t)$ via

$$(1.9) \qquad H(t) = \mathrm{diag}(h_1(t), \ldots, h_n(t))$$
$$(1.10) \qquad h_i(t) = e^{-\int_0^t \kappa_i(s)ds}$$

i.e. we have set

$$(1.11) \qquad \kappa_i(t) = -\frac{h_i'(t)}{h_i(t)}$$

Furthermore, the volatility σ_x is given by the $m \times n$ matrix valued function

$$(1.12) \qquad \sigma_x(t) = g(t)H(t)$$

and

$$(1.13) \qquad m(t,T) \quad := \quad H(T)H(t)^{-1}\mathbf{1}$$

$$(1.14) \qquad x(t) \quad := \quad H(t)\int_0^t g(s)^T g(s)\int_s^t h(u)du\,ds + H(t)z(t)$$

$$(1.15) \qquad y(t) \quad := \quad H(t)\left(\int_0^t g(s)^T g(s)ds\right)H(t)$$

$$(1.16) \qquad z(t) \quad := \quad \int_0^t g(s)^T dW(s)$$

The model is fully specified by the mean reversion $\kappa(t)$ and volatility $\sigma_x(t)$, since from that we can compute $y(t)$ and $M(t,T)$ as

$$(1.17) \qquad m(t,T) \quad = \quad \left(e^{-\int_t^T \kappa_1(s)ds}, e^{-\int_t^T \kappa_2(s)ds}, \ldots, e^{-\int_t^T \kappa_n(s)ds}\right)^T$$

$$(1.18) \qquad M(t,T) \quad = \quad \operatorname{diag}(m_1, \ldots, m_n)$$

and

$$(1.19) \qquad y(t) = \int_0^t M(s,t)\sigma_x(s)^T \sigma_x(s)M(s,t)ds$$

since $g(s) = \sigma_x(s)H^{-1}(s)$. This allows to evolve $x(t)$. We have a bond reconstruction formula (see [2] Corrolary 12.1.3)

$$(1.20) \qquad P(t,T) = \frac{P(0,T)}{P(0,t)}e^{-g(t,T)^T x(t) - \frac{1}{2}g(t,T)^T y(t)g(t,T)}$$

where we define

$$(1.21) \qquad g(t,T) \quad := \quad \int_t^T m(t,u)du$$

$$(1.22) \qquad G(t,T) \quad := \quad \int_t^T M(t,u)du$$

$$(1.23)$$

Using the results in section 1.6 of [2] for the solution of linear SDEs applied to 1.5 on the time interval $[t_i, u]$ we get

$$(1.24) \qquad x(u) = M(t_i, u)x(t_i) + \int_{t_i}^u M(s,u)y(s)ds + \int_{t_i}^u M(s,u)\sigma_x(s)^T dW(s)$$

Furthermore we can write 1.19 as

$$(1.25) \qquad y(s) = y(t_i) + \int_{t_i}^s M(v,s)\sigma_x(s)^T \sigma_x(v)M(v,s)dv$$

This allows us to write the integral of $x(u)$ between t_i and t_{i+1} as

$$\int_{t_i}^{t_{i+1}} x(u)du = \left(\int_{t_i}^{t_{i+1}} M(t_i, u)du \right) x(t_i) +$$

(1.26)
$$\int_{t_i}^{t_{i+1}} \int_{t_i}^{u} M(s, u)y(s)dsdu +$$

$$\int_{t_i}^{t_{i+1}} \int_{t_i}^{u} M(s, u)\sigma_x(s)^T dW(s)du$$

The first summand can be directly written as $G(t_i, t_{i+1})x(t_i)$ using the definition of G. Applying Fubini's theorem to the second and third summand gives

$$\int_{t_i}^{t_{i+1}} x(u)du = G(t_i, t_{i+1})x(t_i) +$$

(1.27)
$$\int_{t_i}^{t_{i+1}} G(s, t_{i+1})y(s)ds +$$

$$\int_{t_i}^{t_{i+1}} G(s, t_{i+1})\sigma_x(s)dW(s)$$

TODO derive an exact scheme to evolve x, B from this, similar to Piterbarg Lemma 10.1.1, which allows for a more precise (and unbiased) evolution of the state process and bank account numeraire

Notice that this together with 1.7 also yields a representation of the bank account since

(1.28)
$$B(t) = \frac{1}{P(0, t)} \exp \left(\mathbf{1}^T \int_0^t x(s)ds \right)$$

1.2. **Cross Currency Extension.** We consider a domestic interest rate process driven by m_d Brownians with n_d states and likewise a foreign interest rate process driven by m_f Brownians and n_f states. The interest rate processes are coupled by an FX Spot process S. The SDE system for this setup under the domestic bank account measure can be written as follows.

(1.29) $\quad dx_d(t) \quad = \quad (y_d(t)\mathbf{1} - \kappa_d x_d(t))dt + \sigma_{x,d}(t)^T dW_d(t)$

(1.30) $\quad dx_f(t) \quad = \quad (y_f(t)\mathbf{1} - \kappa_f x_f(t) - \sigma_S(t)\sigma_{x,f}^T(t)\rho_{fS})dt + \sigma_{x,f}(t)^T dW_f(t)$

(1.31) $\quad\quad dS \quad = \quad (r_d(t) - r_f(t))Sdt + \sigma_S(t)SdW_S(t)$

Here $-\sigma_S(t)\sigma_{x,f}^T(t)\rho_{fS}$ is a drift reflecting the measure change from the foreign to the domestic bank account measure, which we will derive momentarily. Notice that the drift of the FX Spot process under the domestic bank account measure is known from general principles, see [2], section 4.3.2.

The driving Brownians are coupled via correlations given by

(1.32)
$$\begin{pmatrix} dW_d \\ dW_f \\ dW_S \end{pmatrix} \begin{pmatrix} dW_d & dW_f & dW_S \end{pmatrix} = \begin{pmatrix} I_{m_d} & \rho_{df} & \rho_{dS} \\ \rho_{df}^T & I_{m_f} & \rho_{fS} \\ \rho_{dS}^T & \rho_{fS}^T & 1 \end{pmatrix}$$

where I_n denotes the n dimensional unity matrix (remember that the Brownians within the multi-factor interest rate model are independent), ρ_{df} is an $m_d \times m_f$ matrix, ρ_{dS} is an $m_d \times 1$ matrix, ρ_{fS} is an $m_f \times 1$ matrix representing the correlations between the domestic, the foreign and the FX Spot processes.

To determine the unknown drift $\mu(t)$ we consider the measure change from the foreign bank to the domestic bank account measure which has a Radon-Nikodyn derivative

$$(1.33) \qquad \zeta(t) = \frac{B_d(t)S(0)}{B_f(t)S(t)}$$

see [2], Lemma 4.3.1. From Ito's Lemma we get

$$(1.34) \qquad \ln \zeta(t) = -\frac{1}{2} \int_0^t \sigma_S(s)^2 ds - \int_0^t \sigma_S(s) dW_S(s)$$

Notice that the contributions from r_d and r_f cancel out, for concreteness one might have 1.28 in the back of our heads, although the result holds quite generally. We write the original Brownian motions in terms of uncorrelated Brownians dW_1, dW_2, dW_3 of dimension m_d, m_f, 1:

$$(1.35) \qquad \begin{pmatrix} dW_d \\ dW_f \\ dW_S \end{pmatrix} = \sqrt{\begin{pmatrix} I_{m_d} & \rho_{df} & \rho_{dS} \\ \rho_{df}^T & I_{m_f} & \rho_{fS} \\ \rho_{dS}^T & \rho_{fS} & 1 \end{pmatrix}} \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix} =: \sqrt{\rho} dW$$

and the density as

$$(1.36) \qquad \ln \zeta(t) = -\frac{1}{2} \int_0^t \sigma_S(s)^2 ds - \int_0^t \begin{pmatrix} 0 & 0 & \sigma_S(s) \end{pmatrix} \left(\sqrt{\rho}\, dW \right)$$

where $\begin{pmatrix} 0 & 0 & \sigma_S(s) \end{pmatrix}$ is a $m_d + m_f + 1$ dimensional row vector with zeros at the components belonging to W_d and W_f. We compare this last representation with the density arising from a drift $\theta(s)$ using Girsanov's theorem

$$(1.37) \qquad \ln \zeta(t) = -\frac{1}{2} \int_0^t \theta(s)^T \theta(s) ds - \int_0^t \theta(s)^T dW(s)$$

from which we conclude

$$(1.38) \qquad \begin{pmatrix} 0 & 0 & \sigma_S(s) \end{pmatrix} \sqrt{\rho} = \theta(s)^T$$

or

$$(1.39) \qquad \theta(s) = \sqrt{\rho}^T \begin{pmatrix} 0 \\ 0 \\ \sigma_S(s) \end{pmatrix}$$

We now consider the drift change associated to the measure change

$$(1.40) \qquad \begin{pmatrix} dW_d \\ dW_f \\ dW_s \end{pmatrix} \rightarrow \begin{pmatrix} dW_d \\ dW_f \\ dW_s \end{pmatrix} + \sqrt{\rho}\theta dt$$

which implies

$$(1.41) \qquad dW_f \rightarrow dW_f + \sigma_S(t)\rho_{fS}dt$$

and therefore we have, under the domestic measure (denoting the new Brownian after the measure change again as $dW_f(t)$)

$$(1.42) \qquad dx_f(t) = (y_f(t)\mathbf{1} - \kappa_f x_f(t) - \sigma_S(t)\sigma_{x,f}^T(t)\rho_{fS})dt + \sigma_{x,f}(t)^T dW_f(t)$$

1.3. **T-Forward and Rolling Spot Measure.** <mark>TODO</mark>

1.4. **Constaant parameters.** If σ_x and κ are constant it is straightforward to calculate

$$(1.43) \qquad m_i(t,T) = \exp(-\kappa_i(T-t))$$

and

$$(1.44) \qquad g_i(t,T) = \begin{cases} \frac{1-\exp(-\kappa_i(T-t))}{\kappa_i} & \kappa_i \neq 0 \\ T-t & \kappa_i = 0 \end{cases}$$

and

$$(1.45) \qquad y_{i,j}(t) = \begin{cases} \sum_{k=1}^m \sigma_{k,i}\sigma_{k,j}\frac{1-\exp(-(\kappa_i+\kappa_j)t)}{\kappa_i+\kappa_j} & \kappa_i+\kappa_j \neq 0 \\ \sum_{k=1}^m \sigma_{k,i}\sigma_{k,j}t & \kappa_i+\kappa_j = 0 \end{cases}$$

for $i,j = 1,\ldots,n$.

2. Statistical Interest Rate Model

2.1. **Model development.** We follow [2], 12.1.5 "Multi-Factor Statistical Gaussian Model". We assume a tenor structure $\tau_1,\ldots,\tau_{N_\tau}$ to discretize a single rate curve, i.e. we choose one currency and one representative curve in that currency. We have continuously compounded forward yields

$$(2.1) \qquad f_j = \frac{-\log\left(\frac{P(0,\tau_j)}{P(0,\tau_{j-1})}\right)}{\tau_j - \tau_{j-1}}$$

for $j = 1,\ldots,N_\tau$, where we set $\tau_0 := 0$. Let Σ_r denote an estimate of the covariance matrix of absolute daily returns $r_j(t) := f_j(t) - f_j(t-1)$ sourced from time series data. We assume that the mean of these returns is approximately zero. We furthermore assume that Σ_r is normalized to *annual volatilities*. See section 2.2 for more details.

Spectral decomposition (PCA) yields a orthogonal coordinate transform Γ

$$(2.2) \qquad \Sigma_r = \Gamma\Lambda\Gamma^T$$

with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{N_\tau})$ a diagonal matrix containing the eigenvalues of Σ_r and the columns of Γ being the eigenvectors of Σ_r with norm 1. We can assume that the columns of Γ are sorted in a way that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_\tau}$.

Γ defines a linear coordinate transform

$$(2.3) \qquad r^* = \Gamma^T r$$
$$(2.4) \qquad r = \Gamma r^*$$

translating the original returns r to (independent) returns r^* in terms of the principal component coordinates and vice versa. The jth eigenvector γ_j of Γ is also called the jth vector of *loadings* and translates a return $(0, \ldots, 0, 1, 0, \ldots, 0)^T$ in new coordinates with 1 at the jth component to the corresponding return in original coordinates

$$(2.5) \qquad r = \gamma_j = \Gamma \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

We now pick a number $m \leq N_\tau$ and only keep the components $1, \ldots, m$ of Γ. ==Do we want to rescale the λ_i so that we still match to total variance given by all components, i.e. avoid leaking the variance of the components that we throw away?==

In the interest rate model we aim to develop in this section the jth principal component is driven by z_j with

$$(2.6) \qquad dz_j = \sigma_j(t)dW_j(t)$$

with $j = 1, \ldots, m$. If we wish to derive $\sigma_j(t)$ from historical time series data, we set

$$(2.7) \qquad \sigma_j(t) \equiv \sqrt{\lambda_j}$$

independent of t. Alternatively, we can leave $\sigma_j(t)$ unspecified at this point and calibrate to market option quotes after having specified the reversion parameters of the model. To connect the loadings γ_j with the mean reversion parameters of the model we consider [2] formula (12.35)

$$(2.8) \qquad l_j(\tau) = \sum_{i=1}^{n_j} v_{j,i} e^{-\kappa_{j,i}\tau}$$

First we have to integrate 2.8 over τ to translate the loadings $l_j(\tau)$ for the instantaneous forward rate at $t+\tau$ as seen from t to loadings $l'_j(k)$ of the continuously compounded forward yield f_j in 2.1. This means we have to compute

$$(2.9) \qquad l'_j(k) = \frac{1}{\tau_k - \tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} l_j(\tau)d\tau = \sum_{i=1}^{n_j} \frac{v_{j,i}}{\tau_k - \tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} e^{-\kappa_{j,i}\tau}d\tau$$

for $k = 1, \ldots, N_\tau$. This is

$$(2.10) \qquad l'_j(k) = \sum_{i=1}^{n_j} v_{j,i} \frac{e^{-\kappa_{j,i}\tau_{k-1}} - e^{-\kappa_{j,i}\tau_k}}{\kappa_{j,i}(\tau_k - \tau_{k-1})}$$

The next step is to determine the parameters $v_{j,i}$ and $\kappa_{j,i}$ such that the model implied loadings $l'_j(k)$ match the loadings γ_j from the PCA as closely as possible:

$$(2.11) \qquad l'_j \approx \gamma_j$$

Here, the parameter n_j is the number of "basis functions" available to approximate γ_j. We specify n_j as a fixed external parameter, while $v_{j,i}$ and $\kappa_{j,i}$ have to be determined in an numerical optimization procedure. Notice that the total number of state variables in the final model will be $n = \sum_{j=1}^{m} n_j$ and the numerical optimization will be run over $2n$ free parameters, i.e. there is a trade off between the accuracy of the approximation of the PCA loadings and the number of state variables in the model. Also notice that the number of driving Brownian motions in the model will be m.

[2], proposition 12.15 summarizes the model dynamics we arrive at. The short rate is given by

$$(2.12) \qquad r(t) = f(0, t) + (1, 1, \ldots, 1)x(t)$$

where the driving vector $x(t)$ of size n follows the dynamics

$$(2.13) \qquad dx(t) = (y(t)(1, 1, \ldots, 1)^T - \kappa x(t))dt + \sigma_x(t)^T dW(t)$$

with an m-dimensional vector $W(t)$ of independent Brownian motions, a diagonal $n \times n$ mean reversion matrix κ

$$(2.14) \qquad \kappa = \text{diag}(\kappa_{1,1}, \ldots, \kappa_{1,n_1}, \ldots, \kappa_{m,1}, \ldots, \kappa_{m,n_m})$$

containing the calibrated mean reversion parameters from from 2.8, a diagonal $n \times n$ matrix σ

$$(2.15) \qquad \sigma = \text{diag}(\sigma_1, \ldots, \sigma_1, \ldots, \sigma_m, \ldots, \sigma_m)$$

containing the historical volatilities of the m factors from 2.7 or unspecified values to be calibrated to market option quotes, a column vector h of size n

$$(2.16) \qquad h(t) = (e^{-\kappa_{1,1}t}, \ldots, e^{-\kappa_{1,n_1}}, \ldots, e^{-\kappa_{m,1}}, \ldots, e^{-\kappa_{m,n_m}})^T$$

The $m \times n$ matrix σ_x is given by

$$(2.17) \qquad \sigma_x(t) = v\sigma(t)$$

where v is a $m \times n$ matrix containing the calibrated parameters from 2.8 in the following form

$$(2.18) \qquad v = \begin{pmatrix} v_{1,1} & \cdots & v_{1,n_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & v_{2,1} & \cdots & v_{2,n_2} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & 0 & \vdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & v_{m,1} & \cdots & v_{m,n_m} \end{pmatrix}$$

The $n \times n$ matrix y is given by

$$(2.19) \qquad y(t) = H(t) \left(\int_0^t H(s)^{-1} \sigma(s) v^T v \sigma(s) H(s)^{-1} ds \right) H(t)$$

where $H(t)$ is the diagonal $n \times n$ matrix $\mathrm{diag}(h_1(t), \ldots, h_n(t))$. This model is a special form of the model described in section 1 with the parameter g set to

$$(2.20) \qquad g(t) = v\sigma(t)H(t)^{-1}$$

2.2. **Covariance matrix estimation.** To compute the covariance matrix, we will use the exponential-weighted approach from RiskMetrics with the normal covariance matrix as a special case.

For an interest rate curve, we have a tenor structure $\tau_1, \ldots, \tau_{N_\tau}$. The continuously compounded forward yields can be computed by

$$(2.21) \qquad f_j(t) = \frac{-\log\left(\frac{P_t(0,\tau_j)}{P_t(0,\tau_{j-1})}\right)}{\tau_j - \tau_{j-1}}$$

for $j = 1, 2, \ldots, N_\tau$, $t = 1, 2, \ldots, T$. T, here, is the days in the historical time series data. The absolute difference of the daily returns will be computed by

$$(2.22) \qquad r_{t,j} := f_j(t) - f_j(t-1)$$

The covariance matrix Σ_r can be computed as

$$(2.23) \qquad \Sigma_r = AC^T C$$

A here is the annualizing factor that normalize the daily covariance matrix to annual covariance. Normally, $A = 252$. For matrix C here, we use RiskMetrics (3.23).

$$(2.24) \qquad C = \left(\frac{1}{\Lambda}\right)^{\frac{1}{2}} \begin{pmatrix} r_{1,1}\lambda^{\frac{T-1}{2}} & r_{1,2}\lambda^{\frac{T-1}{2}} & \cdots & r_{1,N_\tau}\lambda^{\frac{T-1}{2}} \\ r_{2,1}\lambda^{\frac{T-2}{2}} & r_{2,2}\lambda^{\frac{T-2}{2}} & \cdots & r_{2,N_\tau}\lambda^{\frac{T-2}{2}} \\ \vdots & \vdots & \ddots & \vdots \\ r_{T,1} & r_{T,2} & \cdots & r_{T,N_\tau} \end{pmatrix}$$

Here, λ is the factor for exponential weight, and $\Lambda = \sum_{m=1}^T \lambda^{m-1}$. When $\lambda = 1$, the covariance matrix will become equal-weighted.

When using C to compute the covariance matrix Σ_r, we provide an option to choose whether to subtract the actual mean or to use the zero-mean assumption.

If user choosed to subtract the actual mean, the exponentially-weight will be apply to the dataset after the mean is subtracted.

3. Statistical Interest Rate - FX Model

3.1. **Model development.** For multiple currencies $c_1, \ldots, c_C$ the calibration of the statistical interest rate / fx model proceeds as follows:

(1) Check if the all data are available for all IR curves and FX spot currency pairs on dates that have data. If the data is missing, we will use the data from the previous available date. If there is no previous date (i.e. the missing date is the start date of the period), we will use the data from the next available date.

(2) Calibrate the parameters of the interest rate model in each currency with m_i Brownians and n_i factors, $i = 1, \ldots, C$. See section 2 for more details.

(3) Calibrate the parameters of the fx model for each currency pair $c_{i+1} - c_1$, $i = 1, \ldots, C - 1$. See section 3.2 for more details.

(4) Calibrate the interest rate correlation matrices $\rho_{i,j}^{IR-IR}$ of size $m_i \times m_j$ for $i, j = 1, \ldots, C$, $i \neq j$:
   (a) transform the rate returns for currency c_i and c_j to PC-coordinates using 2.3
   (b) compute a historical correlation estimate between each PC of currency c_i with each PC of currency c_j

(5) Calibrate the interest rate - fx correlation matrices $\rho_{i,j}^{IR-FX}$ of size $m_i \times 1$ for $i = 1, \ldots, C$, $j = 1, \ldots, C - 1$
   (a) transform the rate returns for currency c_i to PC-coordinates using 2.3
   (b) compute a historical correlation estimate between each PC of c_i with the FX Spot log-returns for the currency pair $c_{j+1} - c_1$

(6) Calibrate the fx - fx correlations $\rho_{i,j}^{FX-FX}$ of size 1×1 for $i = 1, \ldots, C - 1$, $j = 1, \ldots, C - 1$
   (a) compute a historical correlation estimate between the FX Spot log-returns for currency pairs $c_{i+1} - c_1$, $c_{j+1} - c_1$

3.2. **FX spot sigma.** For each currency pairs $c_{i+1} - c_1$, $i = 1, \ldots, C - 1$, we have a set of time-series spot rates $s_{t,i+1}$, for $t = 1, \ldots, T$. If the currency pair is in the format of $c_1 - c_{i+1}$, an inverse will be done on the spot rates (i.e. $1/s_{t,i+1}$). The daily log-return is then computed by

$$(3.1) \qquad r_{t,i+1} := log(\frac{s_{t+1,i+1}}{s_{t,i+1}})$$

Before computing the variance for the log-returns, user can choose whether to subtract the actual mean or to use the zero-mean assumption. If user choosed to sutract the actual mean, $r_{t,i+1} - \bar{r}_{i+1}$ will be used in the next step to compute the variance.

The σ_{i+1} for currency pair $c_{i+1} - c_1$ is computed by

$$(3.2) \qquad \sigma_{i+1}^2 = A \sum_{t=1}^{T} r_{t,i+1}^2 \lambda^{T-t} / \Lambda$$

Here, A is the annualizing factor that normalize the variance of daily log-return into annualized variance. Normally, $A = 252$. λ is the factor for exponential weight, and $\Lambda = \sum_{m=1}^{T} \lambda^{m-1}$. When $\lambda = 1$, it will become equal-weighted.

4. COMMODITY MODEL

4.1. **Simple Model: Black-Scholes and Black76.** The simplest approach is a Black-Scholes model

$$(4.1) \qquad dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t)S(t)dW(t)$$

with r_d denoting the risk free interest rate, r_f denoting the convenience yield. We usually set r_d to match the overnight curve in the commodity currency and use r_f to match the quoted future curve of the commodity. See [3], 2.2.1.

A variation is the Black76 model under which the future with maturity T evolves as

$$(4.2) \qquad dF(t, T) = \sigma(t)F(t, T)dW(t)$$

See [3], 2.2.4.

4.2. **Model variants: Shift and normal volatility.** The shifted Black-Scholes model reads

$$(4.3) \qquad dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t)(S(t) + d)dW(t)$$

with a shift $d \geq 0$. This allows to model negative commodity prices up to $-d$, as they might be occasionally observed for some commodities.

The normal Black-Scholes model is

$$(4.4) \qquad dS(t) = (r_d(t) - r_f(t))S(t)dt + \sigma(t)dW(t)$$

This allows for arbitrary negative prices.

The Black76 model and often also more sophisticated models allow for both a shift-extension and a variant with normal volatility in a similar way.

4.3. **Modeling of spreads.** Spreads occur as spreads between different commodity names and / or different future expiries of one of two involved commodity names.

4.3.1. *Separate component modeling.* The standard procedure to model spreads is to model the two spread components separately and compute the spread from the two components.

4.3.2. *Direct modeling of the spread.* ... TODO: for which concrete spreads is that necessary or convenient? Which market quotes are available in this case? To be further discussed...

4.4. **Advanced Model: 2 Factor model.** We follow [1]. We model future prices $F(t, T)$ at t with maturity T as

$$(4.5) \qquad dF(t, T) = \sigma_1(t, T)dW_1(t) + \sigma_2(t, T)dW_2(t)$$

with independent W_i, $i = 1, 2$. See [1] section 7.1.

4.5. **PCA-based model calibration.** We follow [1]. Section 6.2.2 presents a PCA for USD Gas Futures, in particular note Figure 6. The PCA is performed separately per calendar month to account for seasonality. The paper is focussed on a calibration for to liquid option prices pricing purposes. We aim to get a full model calibration from historical data on the other hand. We therefore use a more general form of the parametrization proposed at the beginning of [1], section 7.1:

$$(4.6) \qquad \sigma_i(t, T) = e^{b(T)} h_i e^{-\kappa(T-t)} + e^{a(T)} h_{i,\infty}$$

for $i = 1, 2$. This idea of this form is to mimic the exponentially decreasing shape of the PCA with increasing future maturity and a certain long term level $h_{i,\infty}$. Here, κ is a decay rate and $a(T)$ and $b(T)$ are functions oscicalling around zero to account for the seasonality. The parameters h_i and $h_{\infty,i}$ are additionally scaled to match the variance of the pricincipal component, i.e. the ith eigenvalue.

TODO: Perform our own PCAs on the relevant commodity names to see how well the approach works. We possibly need to make the parametrization more flexible or even dependent on the specific commodity name?

TODO: Do we want more than 2 factors? Which portion of the variance is explained by two principal components in our data?

4.6. **Future Covariance estimation.** TODO: details on covariance matrix estimate, e.g. rolling future maturities / interpolation ...

5. Appendix

Here, we provide a proof that the results of subtracting mean and applying ewma on the pc-adjusted non ewma-adjusted absolute return is the same as multiply the ewma-adjusted return with the eigenvectors.

We denote the absolute return matrix as

$$(5.1) \qquad R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1\tau} \\ r_{21} & r_{22} & \dots & r_{2\tau} \\ \vdots & \vdots & \ddots & \vdots \\ r_{T1} & r_{T2} & \dots & r_{T\tau} \end{pmatrix}$$

The mean of each column as $\bar{r}_1, \bar{r}_2, \dots, \bar{r}_\tau$.

Eigenvector of the first n principal components as

$$(5.2) \qquad V = \begin{pmatrix} e_{11} & e_{12} & \dots & e_{1n} \\ e_{21} & e_{22} & \dots & e_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\tau 1} & e_{\tau 2} & \dots & e_{\tau n} \end{pmatrix}$$

Ewma-adjusted vector as

$$(5.3) \qquad \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_T \end{pmatrix}$$

First, we compute the results for subtracting mean and applying ewma on pc-adjusted non ewma-adjusted absolute reture. For element at ith row and jth column in the $R \cdot V$ matrix, we have

$$(5.4) \qquad RV_{i,j} = \sum_{k=1}^{\tau} r_{ik} e_{kj}$$

The mean of column j column of $R \cdot V$ can be written as

$$\begin{aligned}
\bar{RV}_j &= \frac{1}{T} \sum_{i=1}^{T} \sum_{k=1}^{\tau} r_{ik} e_{kj} \\
(5.5) \qquad &= \frac{1}{T} \sum_{k=1}^{\tau} e_{kj} \sum_{i=1}^{T} r_{ik} \\
&= \sum_{k=1}^{\tau} e_{kj} \bar{r}_k
\end{aligned}$$

After subtracting the mean and applying the ewma factor, we have

$$(5.6) \qquad (RV_{i,j} - \bar{RV}_j)\omega_i = \sum_{k=1}^{\tau} e_{kj}(r_{ik} - \bar{r}_k)\omega_i$$

Next, we compute the results for directly multiply ewma-adjusted return and the eigenvectors.

The ewma-adjusted absolute return can be written as

$$(5.7) \qquad R_{ewma} = \begin{pmatrix}
(r_{11} - \bar{r}_1)\omega_1 & (r_{12} - \bar{r}_2)\omega_1 & \dots & (r_{1\tau} - \bar{r}_\tau)\omega_1 \\
(r_{21} - \bar{r}_1)\omega_2 & (r_{22} - \bar{r}_2)\omega_2 & \dots & (r_{2\tau} - \bar{r}_\tau)\omega_2 \\
\vdots & \vdots & \ddots & \vdots \\
(r_{T1} - \bar{r}_1)\omega_T & (r_{T2} - \bar{r}_2)\omega_T & \dots & (r_{T\tau} - \bar{r}_\tau)\omega_T
\end{pmatrix}$$

For element at ith row and jth column in the $R_{ewma} \cdot V$ matrix, we have

$$(5.8) \qquad (R_{ewma}V)_{i,j} = \sum_{k=1}^{\tau} (r_{ik} - \bar{r}_k)\omega_i e_{kj}$$

Thus, we proved that the results from the 2 approaches are the same.

References

[1] Andersen, Leif B.G., Markov Models for Commodity Futures: Theory and Practice (September 30, 2008). Available at SSRN: https://ssrn.com/abstract=1138782 or http://dx.doi.org/10.2139/ssrn.1138782
[2] Andersen, L., and Piterbarg, V. (2010): Interest Rate Modeling, Volume I-III
[3] Clark, Iain J.: Commodity option pricing, A Practioner's Guide, Wiley 2014