# Dynamics of relay systems with band-pass filtered feedback

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# Motivation

Illing, Ryan & Amann; (ArXived June, 2023)

First-order linear systems with time-delayed relay feedback are well understood.

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However, many real-world systems are not adequately described by first-order systems, requiring second-order systems to capture their dynamics.

$$\alpha \ddot{y} + \beta \dot{y} + \gamma y = \delta \operatorname{sgn}\left[g\left(y\left(t-\tau\right), \dot{y}\left(t-\tau\right)\right)\right]$$

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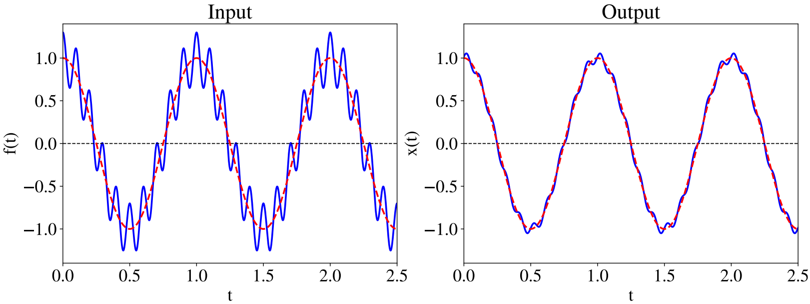
In this work, we study a second-order system with relay time-delayed feedback derived from real-world applications.

# Band-pass filter

A **band-pass filter** is a device (or program) which takes an input signal, passes frequencies within a certain bandwidth centred on a centre frequency, and attenuates signals outside that band.

Band-pass filters are frequently used to clean signals in electrical circuits and digital audio processing.

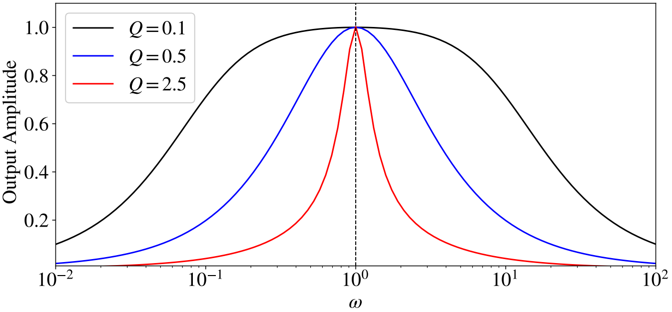


# Band-pass filter system

A simple two-pole band-pass filter can be described by the linear system

$$Q\Omega^{-1}\dot{x} = -x - y + f(t)$$
$$\dot{y} = Q\Omega x$$

with output signal $x(t)$, input signal $f(t)$, centre frequency $\Omega$ and quality factor $Q$ (centre frequency over bandwidth). For $f(t) = \cos(\omega t)$,



# Band-pass filter system with feedback

We introduce negative relay feedback to the band-pass filter system by setting the time-delayed relay output as the input

$$f\left(t\right) = -\mathrm{sgn}\left[x\left(t-\tau\right)\right]$$

to get

$$\begin{aligned} Q\Omega^{-1}\dot{x} &= -x - y - \mathrm{sgn}\left[x(\mathrm{t}-\tau)\right] \\ \dot{y} &= Q\Omega x \end{aligned}$$

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The equivalent second order equation is

$$\Omega^{-2}\ddot{y} + \dot{y}\,(Q\Omega)^{-1} + y = -\mathrm{sgn}\,[\dot{y}(\mathrm{t}-\tau)]$$

# Band-pass filter system with feedback as a map

The feedback **switches** whenever $x(t-1) = 0$. To know when switching events occur in the future, we need to know when zero crossing events occurred in the last unit of time.

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Due to the piecewise linearity, the system can be simulated as a map between switching events.

The **dimension** of the map is $\nu+2$, where $\nu$ is the number of zero crossing events in the past unit of time, at a switching event.

# Integrating the piecewise linear system as a map

Let

$$A = \Omega \begin{pmatrix} -Q^{-1} & -Q^{-1} \\ Q & 0 \end{pmatrix}, \quad G \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + \operatorname{sgn}(x(t-1)) \end{pmatrix}$$

so that $G$ maps the fixed point to the origin. Then

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \Omega Q^{-1} \operatorname{sgn}(x(t-1)) \\ 0 \end{pmatrix}$$

Using Pauli matrices, we show that

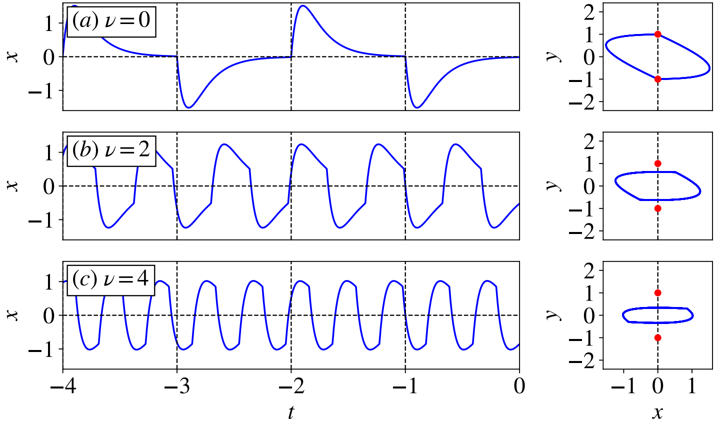
$$e^{At} = e^{-\frac{\Omega}{2Q}t} \cos\left(t \frac{\Omega}{2Q} \sqrt{4Q^2 - 1}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{-\frac{\Omega}{2Q}t} \frac{1}{\sqrt{4Q^2 - 1}} \sin\left(t \frac{\Omega}{2Q} \sqrt{4Q^2 - 1}\right) \begin{pmatrix} -1 & -2 \\ 2Q^2 & 1 \end{pmatrix}$$

so that we can integrate the system between switching events as

$$\begin{pmatrix} x(t+s) \\ y(t+s) \end{pmatrix} = G^{-1} e^{As} G \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

# Periodic solutions to overdamped system $Q < \frac{1}{2}$

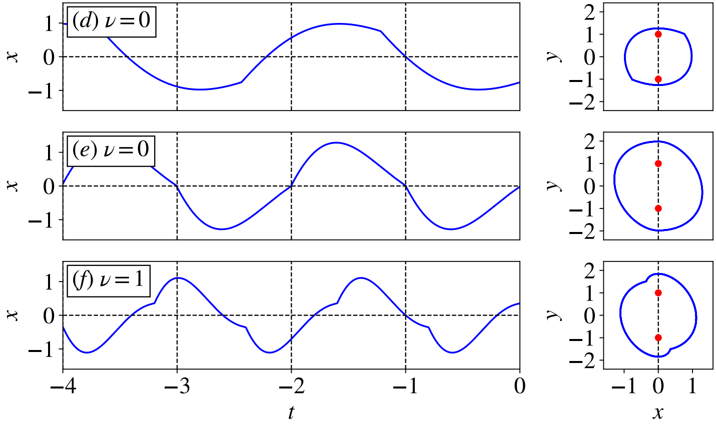
Constant parameters $\Omega$, $Q$, different ICs




$\nu$: The number of zero crossings in the past unit of time, at a switching event.

# Periodic solutions to underdamped system $Q > \frac{1}{2}$

Constant $Q$, increasing $\Omega$, same ICs

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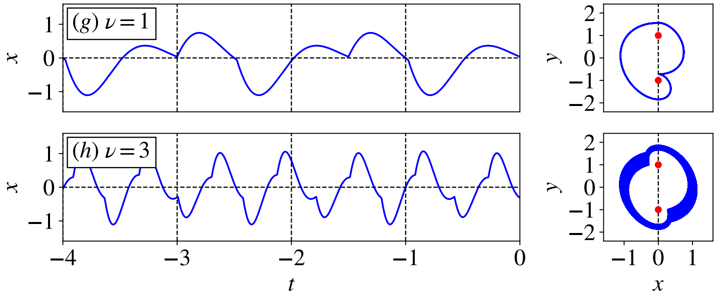
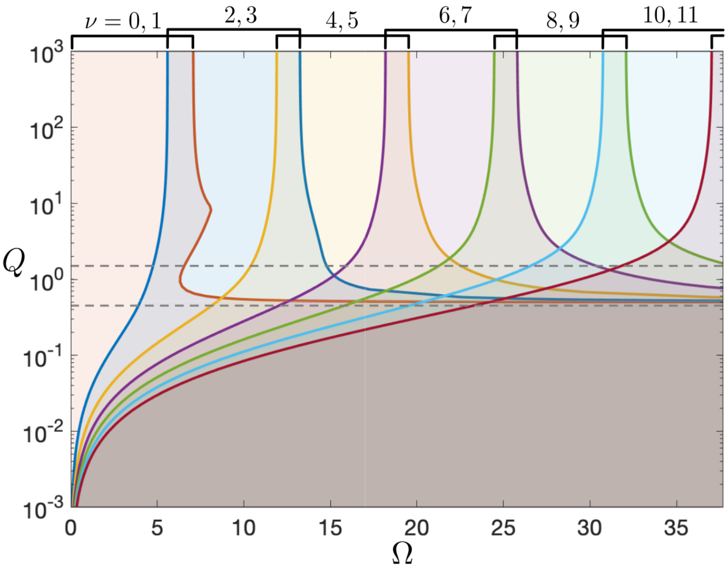



Different parameters

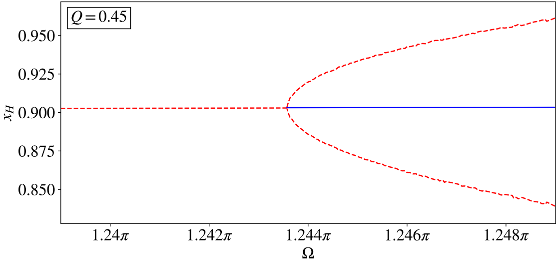
Pair of mirrored asymmetric periodic solution

Aperiodic solution

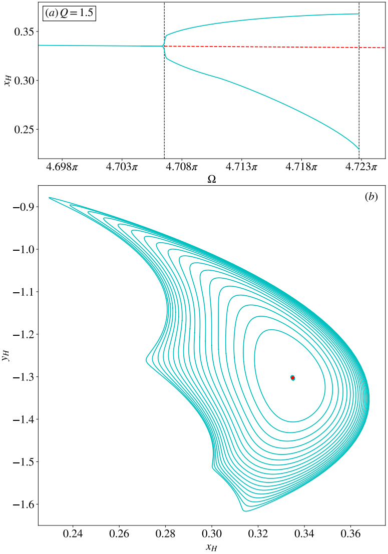
# Stability of symmetric solutions



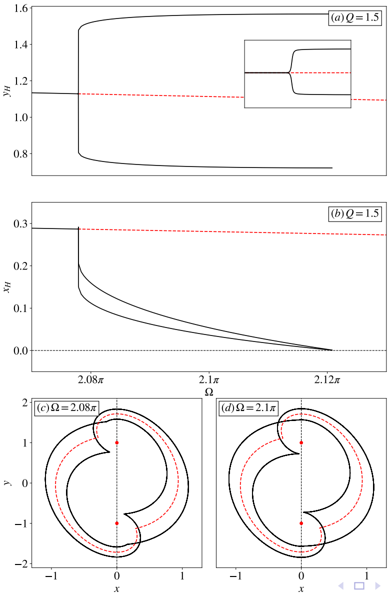
# Subcritical torus bifurcation $\nu = 2$ solution



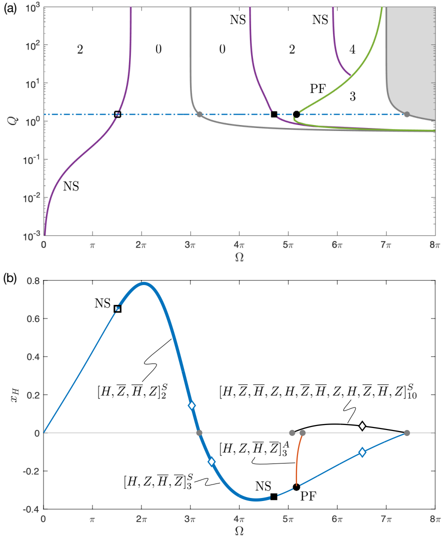
# Supercritical torus bifurcation $\nu = 3$ solution

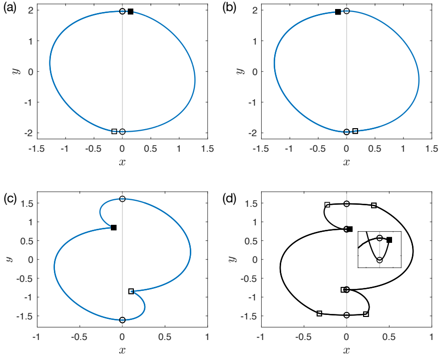


# Pitchfork bifurcation $\nu = 1$ solution

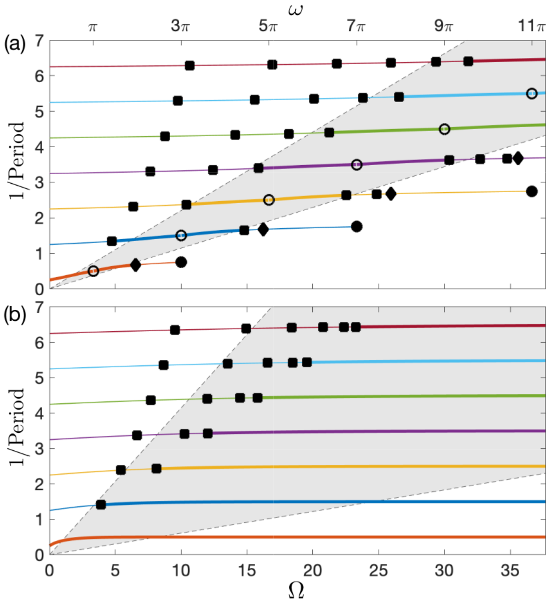


# BCSN bifurcation of $\nu = 3$ solution



# Stability and existence of symmetric solutions



# Stability analysis of the symmetric solution

We construct a map between zero crossings where the $\nu + 1$-dimensional state vector is given by

$$\mathbf{s}_n = (y_{Z,n}, T_{1,n}, T_{2,n}, \ldots, T_{\nu,n})^T.$$

and the Jacobian is given by

$$\mathbf{D}\mathcal{M}_\nu = \begin{pmatrix} a & b & b & \ldots & b & b & b \\ c & d & d & \ldots & d & d & d \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$$

where $a, b, c, d$ are functions of $\nu, \Omega, Q$.

# Stability analysis of the symmetric solution

We define a few functions to aid our analysis...

$$\mu = \frac{\Omega}{2Q} \qquad \omega = \Omega \, \frac{\sqrt{4Q^2 - 1}}{2Q}$$

$$z^* = (\nu + 1)\, T^* - 1 \qquad \delta^* = 1 - \nu\, T^*$$

where $T^*$ (the interval betwen subsequent zero crossing and switching events) is the smallest positive root greater than $\frac{1}{\nu+1}$ of

$$\tan\left(\omega[(\nu+1)\, T^* - 1]\right) = \frac{\sin(\omega T^*)}{e^{\mu T^*} + \cos(\omega T^*)}$$

The y-coordinate of the zero crossing event (the fixed point of the map) is

$$y_Z^* = -1 + \frac{2}{e^{\mu z^*}\, \cos(\omega z^*) + e^{-\mu\delta^*}\, \cos(\omega\delta^*)}.$$

# Stability analysis of the symmetric solution

This allows us to write the coefficients

$$a = -e^{-\mu T^*}\left[\cos(\omega T^*) + \frac{\mu}{\omega}\sin(\omega T^*)\right]$$

$$b = -(\mu^2+\omega^2)(y_Z^*+1)\frac{e^{-\mu T^*}}{\omega}\sin(\omega T^*)$$

$$c = \frac{1}{(y_Z^*+1)}\frac{e^{-\mu T^*}}{\omega}\sin(\omega T^*)$$

$$d = -1 - \left[\cos(\omega T^*) - \frac{\mu}{\omega}\sin(\omega T^*)\right]e^{-\mu T^*}$$

# Stability analysis of the symmetric solution

The Jacobian given by

$$\mathbf{D}\mathcal{M}_\nu = \begin{pmatrix} a & b & b & \dots & b & b & b \\ c & d & d & \dots & d & d & d \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

has characteristic equation

$$0 = \left[(a-1)\,d - b\,c\right] \frac{1-\lambda^\nu}{1-\lambda} + d - (a+d)\,\lambda^\nu + \lambda^{\nu+1}.$$

which can be solved numerically to find the eigenvalues of the map.

# Conclusion

We studied a system featuring switched feedback and linear flow derived from a bandpass filter.

Due to the piecewise linearity, the system can be simulated and analysed through maps.

The stability of periodic solutions depends on smooth bifurcations of maps, while their existence depends on non-smooth bifurcations of maps.

Much of our study of this system was completed analytically.

# Acknowledgements


Illing, Ryan & Amann, *Dynamics of a time-delayed relay system*; (ArXived June, 2023)

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