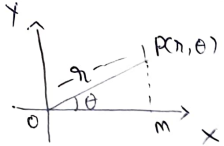
Polar Curves :-

Polar Coordinate System :-

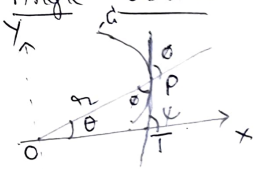


let $P(x,y)$ be a point in two dimensional plane. let $OP = r$ and $\angle OPM = \theta$ then (r,θ) are the polar coordinates of the point P.

$x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

The curve $r = f(\theta)$ is called a polar curve.

Angle between radius vector and tangent :-



Consider a curve C : $r = f(\theta)$. let $P(r,\theta)$ be a point on the curve. Draw a tangent PT to the curve at P. let it makes ψ with X axis. Let $\angle OPT = \phi$, the angle between radius vector and the tangent.

$\psi = \theta + \phi$ (Exterior angle = sum of two interior angles)

$$\frac{dy}{dx} = \tan\psi = \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \quad \text{— (1)}$$

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$ {dividing N and D by $\frac{dr}{d\theta}\cos\theta$}

$$\frac{dy}{dx} = \frac{\frac{\sin\theta}{\cos\theta} + r/\frac{dr}{d\theta}}{1 - \frac{r}{\frac{dr}{d\theta}}\frac{\sin\theta}{\cos\theta}} = \frac{\tan\theta + r\cdot\frac{d\theta}{dr}}{1 - \tan\theta\left(r\frac{d\theta}{dr}\right)} \quad \text{— (2)}$$

From (1) and (2) we get $\tan\phi = r\cdot\frac{d\theta}{dr}$.

① For a curve given in the cartesian form, S.T.

$$\tan\phi = \frac{xy' - y}{x + yy'}$$

Sol:- we have $\psi = \theta + \phi$ $\quad \phi = \psi - \theta$.

and $\tan\psi = \frac{dy}{dx} = y'$.

$$\therefore \tan\phi = \tan(\psi - \theta) = \frac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{r}\Big/\frac{x}{r} = \frac{y}{x}.$$

$$\therefore \tan\phi = \frac{y' - \frac{y}{x}}{1 + y'\cdot\frac{y}{x}} = \frac{\frac{xy' - y}{x}}{\frac{x + yy'}{x}} = \frac{xy' - y}{x + yy'}$$

② S.T. in the curve $r = ae^{\theta\cot\alpha}$ (where α is a const) the radius vector is inclined at a constant angle to the tangent, at every point.

Sol:- $r = ae^{\theta\cot\alpha}$.

$$\therefore \frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cdot\cot\alpha = r\cdot\cot\alpha.$$

$$\tan\phi = r\cdot\frac{d\theta}{dr} = \cancel{r}\cdot\frac{1}{\cancel{r}\cot\alpha} = \tan\alpha$$

$\therefore \phi = \alpha$. which is constant.

③ Find the angle b/w the radius vector and the tangent for the curve $r = a(1-\cos\theta)$. Also find the slope of the curve at $\theta = \frac{\pi}{6}$.

Reduction formula for $\int \tan^n x \, dx$.

let $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \tan^2 x \, dx$.

$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

Put $\tan x = t$ in the first integral.

$\sec^2 x \, dx = dt$.

$$I_n = \int t^{n-2} \, dt - I_{n-2} = \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

If $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$.

Evaluation of $\int_0^{\frac{\pi}{4}} \tan^n x \, dx$

let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$$= \frac{\tan^{n-1} x}{n-1} \Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx = \frac{\tan^{n-1} \frac{\pi}{4} - \tan^{n-1} 0}{(n-1)} - I_{n-2}.$$

But $\tan \frac{\pi}{4} = 1$, $\tan 0 = 0$

$I_n = \frac{1}{n-1} - I_{n-2}$, Similarly $I_{n-2} = \frac{1}{n-3} - I_{n-4}$

$I_{n-4} = \frac{1}{n-5} - I_{n-6}$

The last term will be I_0 or I_1 if n is even or odd

$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x \, dx = \int_0^{\frac{\pi}{4}} dx = x \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx = \log \sec x \Big|_0^{\frac{\pi}{4}} = \log \sec \frac{\pi}{4} - \log \sec 0$

$I_1 = \log \sqrt{2} - \log 1 = \log \sqrt{2}$.

$$\therefore I_n = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - + \cdots \quad I$$

where $I = I_1 = \log \sqrt{2}$ if n is odd, $I = I_0 = \frac{\pi}{4}$ if n is even with appropriate + or − sign.

1) Using Reduction formula evaluate $\int \tan^6 x \, dx$.

We have $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$

$$\therefore I = \int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \int \tan^4 x \, dx$$

$$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - \int \tan^2 x \, dx \right] = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3}$$

$$+ \frac{\tan x}{1} - \int \tan^0 x \, dx \qquad \text{But } \tan^0 x =$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

2) $\int \tan^7 x \, dx$ Ans:- $\frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} - \log \sec x$

3) $\int_0^{\frac{\pi}{4}} \tan^7 x \, dx = \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - I_1$ where $I_1 = \log \sqrt{2}$

$$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \log \sqrt{2} = \frac{2-3+6}{12} - \log \sqrt{2} = \frac{5}{12} - \log \sqrt{2}$$

4) $\int_0^{\frac{\pi}{4}} \tan^8 x \, dx = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - \frac{1}{1} + I_0$ But $I_0 = \frac{\pi}{4}$

$$= \frac{15-21+35-105}{105} + \frac{\pi}{4} = -\frac{76}{105} + \frac{\pi}{4}$$

Reduction formula for $\int \cot^n x \, dx$.

$I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx = \int \cot^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$

$I_n = \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n} x \, dx$. In the first integral put $\cot x = t$ $\quad -\operatorname{cosec}^2 x \, dx = dt$

$I_n = -\int t^{n-2} \, dt - I_n = -\frac{t^{n-1}}{n-1} - I_n = -\frac{\cot^{n-1} x}{n-1} - I_n$

ie $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$.

Evaluation of $\int_{\pi/4}^{\pi/2} \cot^n x \, dx$.

$I_n = \int_{\pi/4}^{\pi/2} \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} \Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \cot^{n-2} x \, dx$.

$= -\frac{\left\{\cot^{n-1} \frac{\pi}{2} - \cot^{n-1} \frac{\pi}{4}\right\}}{(n-1)} - \int_{\pi/4}^{\pi/2} \cot^{n-2} x \, dx$

But $\cot \frac{\pi}{2} = 0$, $\cot \frac{\pi}{4} = 1$

$I_n = -\frac{\{0-1\}}{(n-1)} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$

$\text{lly } I_{n-2} = \frac{1}{n-3} - I_{n-4}$, $I_{n-4} = \frac{1}{n-5} - I_{n-6}$, ...

If n is odd we have to end up with I_1, If n is even we have to end up with I_0..

$\therefore I_n = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - \cdots \quad I$

where $I_0 = \int_{\pi/4}^{\pi/2} \cot^0 x \, dx = \int_{\pi/4}^{\pi/2} dx = x \Big|_{\pi/4}^{\pi/2} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$I_1 = \int_{\pi/4}^{\pi/2} \cot x \, dx = \log \sin x \Big|_{\pi/4}^{\pi/2} = \log \sin \frac{\pi}{2} - \log \sin \frac{\pi}{4}$

$= \log 1 - \log \frac{1}{\sqrt{2}} = 0 - \log \frac{1}{\sqrt{2}} = \log \sqrt{2}$

$\therefore \int_{\pi/4}^{\pi/2} \cot^n x \, dx = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - + \cdots \quad I$

Where $I = I_1 = \frac{\pi}{4}$ if n is odd, $I = I_0 = \log \sqrt{2}$ if n is even with appropriate $+$ or $-$ sign.

1) Using Reduction formula Evaluate $\int \cot^7 x \, dx$

We have $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$

let $I = \int \cot^7 x \, dx = -\frac{\cot^6 x}{6} - \int \cot^5 x \, dx$

$= -\frac{\cot^6 x}{6} - \left\{-\frac{\cot^4 x}{4} - \int \cot^3 x \, dx\right\} = -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} + \int \cot^3 x \, dx$

$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \int \cot x \, dx$

But $\int \cot x \, dx = \log \sin x$

$= -\frac{\cot^6 x}{6} + \frac{\cot^4 x}{4} - \frac{\cot^2 x}{2} - \log \sin x$

2) $\int_{\pi/4}^{\pi/2} \cot^6 x \, dx = \frac{1}{5} - \frac{1}{3} + \frac{1}{1} - I_0 \qquad I_0 = \frac{\pi}{4}$

$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{3-5+15}{15} - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}$

3) $\int_{\pi/4}^{\pi/2} \cot^5 x \, dx$. Ans:- $\frac{1}{4} - \log \sqrt{2}$.

Reduction formula for $\int \sec^n x \, dx$.

Let $I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \, \sec^2 x \, dx$.

Integrating by parts method

$$I_n = \sec^{n-2} x \cdot \tan x - \int \tan x \cdot (n-2) \sec^{n-3} x \sec x \tan x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \tan^2 x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \, (1 + \sec^2 x) \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \left[-\int \sec^{n-2} x \, dx + \int \sec^n x \, dx \right]$$

$$I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2} - (n-2) I_n$$

$$(1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$ dividing by $(n-1)$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$$

i.e. $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x \, dx$$

Evaluation of $\int_0^{\pi/4} \sec^n x \, dx$

$$I_n = \int_0^{\pi/4} \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} \Big|_0^{\pi/4} + \frac{(n-2)}{(n-1)} \int_0^{\pi/4} \sec^{n-2} x \, dx$$

$$I_n = \frac{\sec^{n-2} \frac{\pi}{4} \tan \frac{\pi}{4} - \sec^{n-2} 0 \cdot \tan 0}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$$

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \left(\frac{n-2}{n-1} \right) I_{n-2}.$$

1) Evaluate $\int \sec^6 x \, dx$ using Reduction formula.

we have $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \left(\frac{n-2}{n-1} \right) \int \sec^{n-2} x \, dx$$

$$\therefore \int \sec^6 x \, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \int \sec^4 x \, dx$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx \right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15} \sec^2 x \tan x + \frac{8}{15} \tan x$$

2) $\int_0^a (a^2 + x^2)^{\frac{5}{2}} \, dx$.

put $x^2 = a^2 \tan^2 \theta$, $x = a \tan \theta$, $dx = a \sec^2 \theta \, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$; $x = 0$, $\theta = \tan^{-1} 0 = 0$; $x = a$, $\theta = \tan^{-1}\left(\frac{a}{a}\right) = \frac{\pi}{4}$

$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2 (1 + \tan^2 \theta) = a^2 \sec^2 \theta$

$$I = \int_0^{\pi/4} (a^2)^{\frac{5}{2}} (\sec^2 \theta)^{\frac{5}{2}} \cdot a \sec^2 \theta \, d\theta$$

$$= a^6 \int_0^{\pi/4} \sec^7 \theta \, d\theta = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5}{6} I_5 \right]$$

$$= a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5}{6} \left(\frac{(\sqrt{2})^3}{4} + \frac{3}{4} I_3 \right) \right] = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15}{24} I_3 \right]$$

$$= a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15}{24} \left(\frac{\sqrt{2}}{2} + \frac{1}{2} I_1 \right) \right]$$ But $I_1 = \int_0^{\pi/4} \sec \theta \, d\theta$

$$= \log (\sec \theta + \tan \theta)_0^{\pi/4} = \log \left\{ \sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right\} - \log (\sec 0 + \tan 0)$$

$$= \log (\sqrt{2} + 1) - \log (1 + 0) = \log (\sqrt{2} + 1)$$

$$\therefore I = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15\sqrt{2}}{48} + \frac{15}{48} \log (\sqrt{2} + 1) \right]$$

3) $\int_0^{\pi/4} \sec^6 x \, dx$ Ans $\frac{23}{15}$

Reduction formula for $\int \operatorname{cosec}^n x \, dx$.

$$I_n = \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$$

Integrating using integration by parts method

$$I_n = \operatorname{cosec}^{n-2} x \, (-\cot x) - \int (-\cot x)(n-2) \operatorname{cosec}^{n-3} x \, (-\operatorname{cosec} x \cot x) \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \left[\int \operatorname{cosec}^n x \, dx - \int \operatorname{cosec}^{n-2} x \, dx \right]$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$(1 + n - 2) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2} \quad \text{dividing by } (n-1)$$

$$I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{(n-1)} + \left(\frac{n-2}{n-1}\right) I_{n-2}$$

1) using Reduction formula Evaluate $\int \operatorname{cosec}^5 x \, dx$

we have $\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{(n-1)} + \left(\frac{n-2}{n-1}\right) \int \operatorname{cosec}^{n-2} x \, dx$

$$I = \int \operatorname{cosec}^5 x \, dx = -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4} \int \operatorname{cosec}^3 x \, dx$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4} \left[-\frac{\operatorname{cosec} x \cot x}{2} + \frac{1}{2} \int \operatorname{cosec} x \, dx \right]$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8} \operatorname{cosec} x \cot x + \frac{3}{8} \log(\operatorname{cosec} x - \cot x)$$

2) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta \, d\theta$

consider $I = \int \operatorname{cosec}^3 \theta \, d\theta = -\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} \int \operatorname{cosec}\theta \, d\theta$

$$= -\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} (\log \operatorname{cosec}\theta - \cot\theta)$$

$$\therefore \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta \, d\theta = \left[-\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} (\log(\operatorname{cosec}\theta - \cot\theta) \right]_{\pi/3}^{\pi/2}$$

$$= -\frac{\operatorname{cosec}\frac{\pi}{2} \cot\frac{\pi}{2}}{2} + \frac{1}{2} \log(\operatorname{cosec}\frac{\pi}{2} - \cot \pi/2)$$

$$- \left\{ -\frac{\operatorname{cosec}\frac{\pi}{3} \cot\frac{\pi}{3}}{2} + \frac{1}{2} \log(\operatorname{cosec}\frac{\pi}{3} - \cot \pi/3) \right]$$

But $\operatorname{cosec}\frac{\pi}{2} = 1$, $\cot\frac{\pi}{2} = 0$, $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

$$= -\frac{1}{2}\{1 \cdot 0\} + \frac{1}{2} \log(1 - 0) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \frac{1}{2} \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= 0 + 0 + \frac{1}{3} - \frac{1}{2} \log\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3} - \frac{1}{2} \log\left(\frac{1}{\sqrt{3}}\right)$$

$$=$$

3) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 \theta \, d\theta$ Ans: $2\sqrt{3} + \frac{3}{4}\sqrt{3} - \frac{3}{8} \log(2 - \sqrt{3})$

...tion formula for $\int \tan^n x\,dx$.

let $I_n = \int \tan^n x\,dx = \int \tan^{n-2}x \tan^2 x\,dx$.

$= \int \tan^{n-2}x.(\sec^2 x - 1)\,dx = \int \tan^{n-2}x \sec^2 x\,dx - \int \tan^{n-2}x\,dx$

put $\tan x = t$ in the first integral.

$\sec^2 x\,dx = dt$.

$$I_n = \int t^{n-2}.dt - I_{n-2} = \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$$

ie. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$.

Evaluation of $\int_0^{\frac{\pi}{4}} \tan^n x\,dx$

let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$

$$= \frac{\tan^{n-1}x}{n-1}\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2}x\,dx = \frac{\tan^{n-1}\frac{\pi}{4} - \tan^{n-1}0}{(n-1)} - I_{n-2}.$$

But $\tan\frac{\pi}{4} = 1$, $\tan 0 = 0$

$I_n = \frac{1}{n-1} - I_{n-2}$, Similarly $I_{n-2} = \frac{1}{n-3} - I_{n-4}$

$I_{n-4} = \frac{1}{n-5} - I_{n-6}$

$\vdots$

The last term will be I_0 or I_1 if n is even or odd

$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x\,dx = \int_0^{\frac{\pi}{4}} dx = x\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{4}$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x\,dx = \log\sec x\Big|_0^{\frac{\pi}{4}} = \log\sec\frac{\pi}{4} - \log\sec 0$

$I_1 = \log\sqrt{2} - \log 1 = \log\sqrt{2}$.

$\therefore I_n = \frac{1}{n-1} - \frac{1}{n-3} + \frac{1}{n-5} - + \cdots \quad I$

where $I = I_1 = \log\sqrt{2}$ if n is odd, $I = I_0 = \frac{\pi}{4}$ if n is even with appropriate + or − sign.

1) using Reduction formula Evaluate $\int \tan^6 x\,dx$.

we have $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$.

$\therefore I = \int \tan^6 x\,dx = \frac{\tan^5 x}{5} - \int \tan^4 x\,dx$

$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - \int \tan^2 x\,dx\right] = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3}$

$+ \frac{\tan x}{1} - \int \tan^0 x\,dx$ But $\tan^0 x = 1$

$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$

2) $\int \tan^7 x\,dx$ Ans:- $\frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} - \log\sec x$.

3) $\int_0^{\frac{\pi}{4}} \tan^7 x\,dx = \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - I_1$ where $I_1 = \log\sqrt{2}$

$= \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \log\sqrt{2} = \frac{2-3+6}{12} - \log\sqrt{2} = \frac{5}{12} - \log\sqrt{2}$

4) $\int_0^{\frac{\pi}{4}} \tan^8 x\,dx = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - \frac{1}{1} + I_0$ But $I_0 = \frac{\pi}{4}$

$= \frac{15-21+35-105}{105} + \frac{\pi}{4} = -\frac{76}{105} + \frac{\pi}{4}$

Reduction formula for $\int \cot^n x\, dx$.

$I_n = \int \cot^n x\, dx = \int \cot^{n-2} x \cot^2 x\, dx = \int \cot^{n-2} x\, (\operatorname{cosec}^2 x - 1)\, dx$

$I_n = \int \cot^{n-2} x \operatorname{cosec}^2 x\, dx - \int \cot^n x\, dx$. In the first integral put $\cot x = t$ $\quad -\operatorname{cosec}^2 x\, dx = dt$

$I_n = -\int t^{n-2}\, dt - I_n = -\dfrac{t^{n-1}}{n-1} - I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_n$

ie. $\int \cot^n x\, dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$.

Evaluation of $\int_{\pi/4}^{\frac{\pi}{2}} \cot^n x\, dx$.

$I_n = \int_{\pi/4}^{\frac{\pi}{2}} \cot^n x\, dx = -\dfrac{\cot^{n-1} x}{n-1}\Big|_{\pi/4}^{\frac{\pi}{2}} - \int_{\pi/4}^{\frac{\pi}{2}} \cot^{n-2} x\, dx$.

$= -\dfrac{\left\{\cot^{n-1}\frac{\pi}{2} - \cot^{n-1}\frac{\pi}{4}\right\}}{(n-1)} - \int_{\pi/4}^{\frac{\pi}{2}} \cot^{n-2} x\, dx$.

But $\cot\frac{\pi}{2} = 0$, $\cot\frac{\pi}{4} = 1$

$I_n = -\dfrac{\{0-1\}}{(n-1)} - I_{n-2} = \dfrac{1}{n-1} - I_{n-2}$

ily $I_{n-2} = \dfrac{1}{n-3} - I_{n-4}$, $I_{n-4} = \dfrac{1}{n-5} - I_{n-6}$, ...

... If n is odd we have to end up with I_1, if n is even we have to end up with I_0.

$\therefore I_n = \dfrac{1}{n-1} - \dfrac{1}{n-3} + \dfrac{1}{n-5} - \cdots\cdots \quad I$

where $I_0 = \int_{\pi/4}^{\frac{\pi}{2}} \cot^0 x\, dx = \int_{\pi/4}^{\pi/2} dx = x\Big|_{\pi/4}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx = \log\sin x\Big|_{\pi/4}^{\frac{\pi}{2}} = \log\sin\frac{\pi}{2} - \log\sin\frac{\pi}{4}$

$= \log 1 - \log\frac{1}{\sqrt{2}} = 0 - \log\frac{1}{\sqrt{2}} = \log\sqrt{2}$

$\therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\, dx = \dfrac{1}{n-1} - \dfrac{1}{n-3} + \dfrac{1}{n-5} - + \cdots \quad I$

where $I = I_1 = \frac{\pi}{4}$ if n is odd, $I = I_0 = \log\sqrt{2}$ if n is even with appropriate + or − sign.

1) Using Reduction formula Evaluate $\int \cot^7 x\, dx$

We have $\int \cot^n x\, dx = -\dfrac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$.

let $I = \int \cot^7 x\, dx = -\dfrac{\cot^6 x}{6} - \int \cot^5 x\, dx$.

$= -\dfrac{\cot^6 x}{6} - \left\{-\dfrac{\cot^4 x}{4} - \int \cot^3 x\, dx\right\} = -\dfrac{\cot^6 x}{6} + \dfrac{\cot^4 x}{4} + \int \cot^3 x\, dx$

$= -\dfrac{\cot^6 x}{6} + \dfrac{\cot^4 x}{4} - \dfrac{\cot^2 x}{2} - \int \cot x\, dx$

But $\int \cot x\, dx = \log\sin x$

$= -\dfrac{\cot^6 x}{6} + \dfrac{\cot^4 x}{4} - \dfrac{\cot^2 x}{2} - \log\sin x$

2) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^6 x\, dx = \frac{1}{5} - \frac{1}{3} + \frac{1}{1} - I_0 \qquad I_0 = \frac{\pi}{4}$

$= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{3-5+15}{15} - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}$

3) $\int_{\pi/4}^{\frac{\pi}{2}} \cot^5 x\, dx$. Ans:- $\frac{1}{4} - \log\sqrt{2}$.

5) Reduction formula for $\int \operatorname{cosec}^n x \, dx$

$$I_n = \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$$

Integrating using integration by parts method

$$I_n = \operatorname{cosec}^{n-2} x \, (-\cot x) - \int (-\cot x)(n-2) \operatorname{cosec}^{n-3} x \, (-\operatorname{cosec} x \cot x) \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) \left[\int \operatorname{cosec}^n x \, dx - \int \operatorname{cosec}^{n-2} x \, dx \right]$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$(1+n-2) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2} \quad \text{dividing by } (n-1)$$

$$I_n = -\frac{\operatorname{cosec}^{n-2} x \cot x}{(n-1)} + \left(\frac{n-2}{n-1}\right) I_{n-2}$$

1) Using Reduction formula Evaluate $\int \operatorname{cosec}^5 x \, dx$

we have $\int \operatorname{cosec}^n x \, dx = -\frac{\operatorname{cosec}^{n-2} x \cot x}{(n-1)} + \left(\frac{n-2}{n-1}\right) \int \operatorname{cosec}^{n-2} x \, dx$

$$I = \int \operatorname{cosec}^5 x \, dx = -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4} \int \operatorname{cosec}^3 x \, dx$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} + \frac{3}{4} \left[-\frac{\operatorname{cosec} x \cot x}{2} + \frac{1}{2} \int \operatorname{cosec} x \, dx \right]$$

$$= -\frac{\operatorname{cosec}^3 x \cot x}{4} - \frac{3}{8} \operatorname{cosec} x \cot x + \frac{3}{8} \log(\operatorname{cosec} x - \cot x)$$

2) $\int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta \, d\theta$.

consider $I = \int \operatorname{cosec}^3 \theta \, d\theta = -\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} \int \operatorname{cosec}\theta \, d\theta$.

$$= -\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} (\log \operatorname{cosec}\theta - \cot\theta)$$

$$\therefore \int_{\pi/3}^{\pi/2} \operatorname{cosec}^3 \theta \, d\theta = \left[-\frac{\operatorname{cosec}\theta \cot\theta}{2} + \frac{1}{2} (\log(\operatorname{cosec}\theta - \cot\theta) \right]_{\pi/3}^{\pi/2}$$

$$= -\frac{\operatorname{cosec}\frac{\pi}{2} \cot\frac{\pi}{2}}{2} + \frac{1}{2} \log(\operatorname{cosec}\frac{\pi}{2} - \cot \pi/2)$$

$$- \left\{ -\frac{\operatorname{cosec}\frac{\pi}{3} \cot\frac{\pi}{3}}{2} + \frac{1}{2} \log(\operatorname{cosec}\frac{\pi}{3} - \cot \pi/3) \right]$$

But $\operatorname{cosec}\frac{\pi}{2} = 1$, $\cot\frac{\pi}{2} = 0$, $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$

$$= -\frac{1}{2}\{1 \cdot 0\} + \frac{1}{2} \log(1-0) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \frac{1}{2} \log\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)$$

$$= 0 + 0 + \frac{1}{3} - \frac{1}{2} \log\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3} - \frac{1}{2} \log\left(\frac{1}{\sqrt{3}}\right)$$

$$=$$

3) $\int_{\pi/6}^{\pi/2} \operatorname{cosec}^5 \theta \, d\theta$ Ans: $2\sqrt{3} + \frac{3}{4}\sqrt{3} - \frac{3}{8} \log(2 - \sqrt{3})$

Reduction formula for $\int \sec^n x \, dx$.

let $I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \sec^2 x \, dx$.

Integrating by parts method

$$I_n = \sec^{n-2} x \cdot \tan x - \int \tan x \cdot (n-2) \sec^{n-3} x \sec x \tan x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-3} x \tan^2 x \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (-1 + \sec^2 x) \, dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \left[-\int \sec^{n-2} x \, dx + \int \sec^n x \, dx \right]$$

$$I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2} - (n-2) I_n$$

$$(1 + n - 2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$ dividing by $(n \neq 1)$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$$

i.e. $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{(n-1)} \int \sec^{n-2} x \, dx$$

Evaluation of $\int_0^{\pi/4} \sec^n x \, dx$

$$I_n = \int_0^{\pi/4} \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} \Big|_0^{\pi/4} + \frac{(n-2)}{(n-1)} \int_0^{\pi/4} \sec^{n-2} x \, dx$$

$$I_n = \frac{\sec^{n-2} \frac{\pi}{4} \tan \frac{\pi}{4} - \sec^{n-2} 0 \cdot \tan 0}{n-1} + \frac{(n-2)}{(n-1)} I_{n-2}$$

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \left(\frac{n-2}{n-1}\right) I_{n-2}.$$

1) Evaluate $\int \sec^6 x \, dx$ using Reduction formula.

we have $$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \left(\frac{n-2}{n-1}\right) \int \sec^{n-2} x \, dx$$

$$\therefore \int \sec^6 x \, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \int \sec^4 x \, dx$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx \right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15} \sec^2 x \tan x + \frac{8}{15} \tan x.$$

2) $\int_0^a (a^2 + x^2)^{\frac{5}{2}} dx$.

put $x^2 = a^2 \tan^2 \theta$, $x = a \tan \theta$, $dx = a \sec^2 \theta \, d\theta$

$\theta = \tan^{-1}\left(\frac{x}{a}\right)$; $x = 0$, $\theta = \tan^{-1} 0 = 0$; $x = a$, $\theta = \tan^{-1}\left(\frac{a}{a}\right) = \frac{\pi}{4}$

$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$

$$I = \int_0^{\pi/4} (a^2)^{\frac{5}{2}} (\sec^2 \theta)^{\frac{5}{2}} \cdot a \sec^2 \theta \, d\theta$$

$$= a^6 \int_0^{\pi/4} \sec^7 \theta \, d\theta = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5}{6} I_5 \right]$$

$$= a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5}{6} \left(\frac{(\sqrt{2})^3}{4} + \frac{3}{4} I_3 \right) \right] = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15}{24} I_3 \right]$$

$$= a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15}{24} \left(\frac{\sqrt{2}}{2} + \frac{1}{2} I_1 \right) \right]$$ But $I_1 = \int_0^{\pi/4} \sec \theta \, d\theta$

$$= \log(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = \log\left\{\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right\} - \log(\sec 0 + \tan 0)$$

$$= \log(\sqrt{2} + 1) - \log(1 + 0) = \log(\sqrt{2} + 1)$$

$$\therefore I = a^6 \left[\frac{(\sqrt{2})^5}{6} + \frac{5(\sqrt{2})^3}{24} + \frac{15\sqrt{2}}{48} + \frac{15}{48} \log(\sqrt{2} + 1) \right]$$

3) $\int_0^{\pi/4} \sec^6 x \, dx$ Ans $\frac{28}{15}$

$$\frac{a_1+a_2k}{a_3} = \frac{b_1+b_2k}{b_3} = \frac{c_1+c_2k}{c_3} = \frac{d_1+d_2k}{d_3}$$

for some constant 'k'.

Two lines $L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar if they are lying on the same plane.

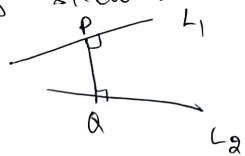
The condition for lines L_1 and L_1 to be coplanar is

$$\begin{vmatrix} (x_2-x_1) & (y_2-y_1) & (z_2-z_1) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

plane containing the lines is

$$\begin{vmatrix} (x-x_1) & (y-y_1) & (z-z_1) \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Lines which are non coplanar are called as skew lines.



For skew lines L_1 and L_2 there is a line PQ which is $\perp$ to both the lines.

P and Q are called as nearest points. To find shortest distance between two lines we use the fact that PQ is $\perp$ to L_1 and also $\perp$ to L_2.

Right circular cone

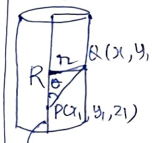
θ - Semivertical angle

To get equation of right circular cone, vertex, semivertical angle and DRs of axis are required.

If $P(x,y,z)$ is a point on the generator then DRs of VP are $(x-x_1), (y-y_1), (z-z_1)$

$$\cos\theta = \frac{a(x-x_1)+b(y-y_1)+c(z-z_1)}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}\sqrt{a^2+b^2+c^2}}$$

Right circular cylinder



r - radius of cross section of cylinder

To get equation of a cylinder, on axis a point $P(x_1,y_1,z_1)$, radius and DRs of axis are required.

Axis DRs a,b,c

First get $\cos\theta$.

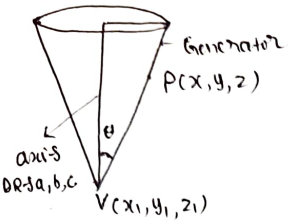
$$\cos\theta = \frac{a(x-x_1)+b(y-y_1)+c(z-z_1)}{\sqrt{a^2+b^2+c^2}\cdot PQ}.$$

$PR = PQ\cos\theta$.

$PQ^2 = PR^2 + RQ^2$

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = \frac{\{a(x-x_1)+b(y-y_1)+c(z-z_1)\}^2}{a^2+b^2+c^2} + r^2$$

Right circular cone



A right circular cone is generated by revolving the line VP about a fixed line such that 'θ' is const. VP is called as generator, $V(x_1, y_1, z_1)$ is the vertex. θ is called as semivertical angle. The fixed line is called as the axis.

let DRs of the axis be a, b, c.

DRs of VP are $(x-x_1)$, $(y-y_1)$, $(z-z_1)$

$$\therefore \cos\theta = \frac{a(x-x_1)+b(y-y_1)+c(z-z_1)}{\sqrt{a^2+b^2+c^2}\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}$$

ie $(a^2+b^2+c^2)\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\}\cos^2\theta$

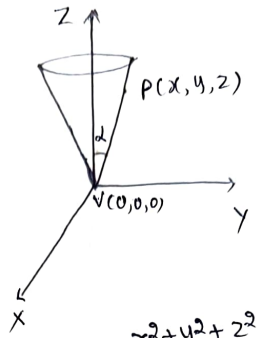
$$= \{a(x-x_1)+b(y-y_1)+c(z-z_1)\}^2$$

which is the equation of a right circular cone.

Note:- To find equation of a cone we require DRs of the axis, vertex and the semivertical angle. If in the problem other data is given, from it we find the above 3 things and then we can find the equation of a cone.

1) Find the equation of a right circular cone having its vertex at the origin, the axis along the z-axis and semivertical angle 'α'.

Sol:- Vertex $V \equiv (0,0,0)$. 'z' axis is the axis. It's DRs are 0, 0, 1



Semivertical angle is α.

let P(x,y,z) be a point on the cone. DRs of VP are x, y, z

$$\therefore \cos\alpha = \frac{0\cdot x + 0\cdot y + 1\cdot z}{\sqrt{0^2+0^2+1^2}\sqrt{x^2+y^2+z^2}}$$

$$\cos^2\alpha = \frac{z^2}{1\,(x^2+y^2+z^2)}$$

$$x^2+y^2+z^2 = \frac{z^2}{\cos^2\alpha}$$

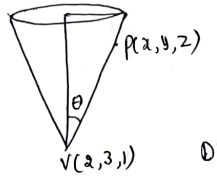
$$x^2+y^2+z^2 = z^2\sec^2\alpha$$

$$x^2+y^2 = z^2\sec^2\alpha - z^2 = z^2(1+\sec^2\alpha)$$

But $1+\tan^2\alpha = \sec^2\alpha$

$$x^2+y^2 = z^2\tan^2\alpha.$$

2) Find the equation of a right circular cone with vertex (2,3,1) axis parallel to the line $-2x = y = 2z$ and one of its generators having DRs 1, −1, 1



axis is parallel to the line $\frac{-2x}{1} = \frac{y}{1} = \frac{2z}{1}$

ie. $\frac{x}{(-\frac{1}{2})} = \frac{y}{1} = \frac{z}{(\frac{1}{2})}$

DRs are $-\frac{1}{2}, 1, \frac{1}{2}$. multiplying by 2.

we get DRs are −1, 2, 1 which are also DRs of axis

θ is not given. It has to be found using DRs of axis and DRs of one of its generator 1, −1, 1

$$\cos\theta = \frac{(-1)(1)+2(-1)+1(1)}{\sqrt{(-1)^2+2^2+1^2}\sqrt{1^2+(-1)^2+1^2}} = \frac{\cancel{1}-2+\cancel{1}}{\sqrt{1+4+1}\sqrt{1+1+1}} = \frac{-2}{\sqrt{6}\sqrt{3}} \quad -(1)$$

let $P(x,y,z)$ be a point on the cone

DRs of VP are $(x-2), (y-3), (z-1)$

$$\therefore \cos\theta = \frac{-1(x-2)+2(y-3)+1(z-1)}{\sqrt{(-1)^2+2^2+1^2}\ \sqrt{(x-2)^2+(y-3)^2+(z-1)^2}}$$

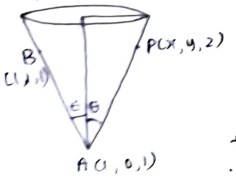
$$\text{ie } \cos^2\theta = \frac{\{-x+2+2y-6+z-1\}^2}{(1+4+1)\ \{(x-2)^2+(y-3)^2+(z-1)^2\}}$$

Substitutating for $\cos\theta$ from ①

$$\frac{4}{6\cdot 3} = \frac{(-x+2y+z-5)^2}{6\ \{x^2-4x+4+y^2-6y+9+z^2-2z+1\}}$$

$$4(x^2+y^2+z^2-4x-6y-2z+14) = 3(-x+2y+z-5)^2$$

3) Find the right circular cone passing through $(1,1,1)$ and having its vertex at $(1,0,1)$, the axis being equally inclined to the coordinate axes.



First we have to find the DRs of the axis let l, m, n be DCs of the axis then $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$.

Since it makes equal angles with the coordinate axes $\alpha=\beta=\gamma$.

$\therefore l=m=n$ But $l^2+m^2+n^2=1$

$l^2+l^2+l^2=1$ ie. $3l^2=1$, $l=\frac{1}{\sqrt{3}}$.

$\therefore$ DCs are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ multiplying by $\sqrt{3}$

we get a set of DRs $1,1,1$ of the axis.

To find θ we use the fact that cone passes through $B(1,1,1)$ DRs of AB are $0,-1,0$

$$\therefore \cos\theta = \frac{1(0)+1(-1)+1(0)}{\sqrt{1^2+1^2+1^2}\ \sqrt{0^2+(-1)^2+0^2}} = \frac{-1}{\sqrt{3}\sqrt{1}} = -\frac{1}{\sqrt{3}}.$$

consider a general point $P(x,y,z)$ on the cone

DRs of AP are $(x-1), y, (z-1)$

$$\therefore \cos\theta = \frac{1(x-1)+1(y)+1(z-1)}{\sqrt{1^2+1^2+1^2}\ \sqrt{(x-1)^2+y^2+(z-1)^2}}$$

$$\text{ie. } -\frac{1}{\sqrt{3}} = \frac{x-1+y+z-1}{\sqrt{3}\ \sqrt{x^2-2x+1+y^2+z^2-2z+1}}$$

Squaring and cross multiplying, we get

$$(-1)^2\{x^2+y^2+z^2-2x-2z+2\} = (x+y+z-2)^2$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-2x-2z+2 = \cancel{x^2}+\cancel{y^2}+\cancel{z^2}+(-2)^2+2xy+2xz-4x+2yz-4y-4z$$

$$4x-2x-2z+2-4-2xy-2xz-2yz+4y+4z=0$$

$$2x+4y+2z-2xy-2xz-4z-2=0 \quad \div 2$$

$$x+2y+z-xy-xz-yz-1=0$$

4) Find the equation of a cone generated by rotating the line $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$ about the line $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

Clearly the lines are intersecting at origin because it is a point common for both the lines.

$\therefore$ Vertex $A \equiv (0,0,0)$

axis is $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

$\therefore$ Its DRs are a, b, c.

DRs of the generator are p, q, r.

$$\therefore \cos\theta = \frac{ap+bq+cr}{\sqrt{a^2+b^2+c^2}\sqrt{p^2+q^2+r^2}} \quad -(1)$$

Consider a general point $P(x,y,z)$ on the cone.

DRs of AP are x, y, z

$$\therefore \cos\theta = \frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}} \quad -(2)$$

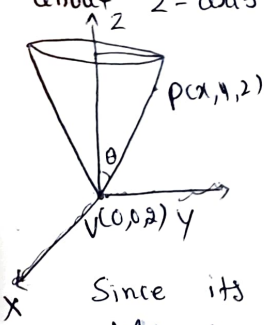
From (1) and (2)

$$\frac{ap+bq+cr}{\cancel{\sqrt{a^2+b^2+c^2}}\sqrt{p^2+q^2+r^2}} = \frac{ax+by+cz}{\cancel{\sqrt{a^2+b^2+c^2}}\sqrt{x^2+y^2+z^2}}$$

Squaring and cross multiplying, we get

$$(ap+bq+cr)^2(x^2+y^2+z^2) = (p^2+q^2+r^2)(ax+by+cz)^2$$

5) Find the equation of the right circular cone obtained by revolving the line $2y+3z-6=0=x$ about z-axis



one of the generator is $2y+3z-6=0=x$.

$x=0$

$2y+3z-6=0$

To get a point on it put $y=0$

$x=0,\ y=0\ \ 3z-6=0,\ z=2$.

$\therefore (0,0,2)$ is a point on the generator

Since its x and y coordinates are zero it is also a point on z axis which is the axis of the cone $\therefore$ Vertex is $(0,0,2)$

$$\tan\theta = \frac{r}{h} = \frac{a\sin\theta}{a\cos\theta} = \tan\theta$$

let a,b,c be DRs of the generator. then it is ⊥r to Normals to planes $x=0$ and $2y+3z-6=0$ (33)

DRs of Normals are $1,0,0$ and $0,2,3$

$\therefore a+0+0=0$, $0\cdot a+2b+3c=0$.

$\therefore a=0$, $2b+3c=0$ put $c=2$

$b=-\frac{3c}{2} = -\frac{3(2)}{2} = -3$

$\therefore$ DRs are $0,-3,2$.

DRs of axis are $0,0,1$

$$\therefore \cos\theta = \frac{0+0+2}{\sqrt{(-3)^2+2^2}\sqrt{1^2}} = \frac{2}{\sqrt{9+4}} = \frac{2}{\sqrt{13}} \quad -(1)$$

let $P(x,y,z)$ be a point on the cone.

DRs of VP are $x, y, (z-2)$

$$\therefore \cos\theta = \frac{0\cdot x+0\cdot y+1(z-2)}{\sqrt{1}\sqrt{x^2+y^2+(z-2)^2}} \quad -(2)$$

From (1) and (2) $\frac{2}{\sqrt{13}} = \frac{(z-2)}{\sqrt{x^2+y^2+(z-2)^2}}$

Squaring and cross-multiplying

$$4\{x^2+y^2+z^2-4z+4\} = 13(z-2)^2$$

$$4x^2+4y^2+4z^2-16z+16 = 13z^2-52z+52$$

$$4x^2+4y^2+4z^2-13z^2-16z+52z+16-52=0.$$

$$4x^2+4y^2-9z^2+36z-36=0.$$

=

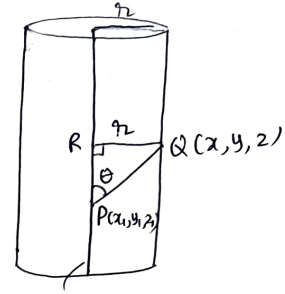
6) Find the equation of right circular cone whose vertex is the origin, whose axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and whose semi vertical angle is 30°

Ans:- $21(x^2+y^2+z^2) = 2(x+2y+3z)^2$

7) Find the equation of the right circular cone generated by rotating (revolving) the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ about the line $\frac{x}{-1} = \frac{y}{1} = \frac{z}{2}$.

Ans:- $2(-x+y+2z)^2 = 7(x^2+y^2+z^2)$

Right circular cylinder.



$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

A right circular cylinder will be generated if a line is rotated about a fixed line such that the distance between them remains same and they are parallel.

The Distance between the lines is called as the radius of the cylinder. Fixed line is called as the axis.

the Line which generates the cylinder is called as the generator. We can find the equation of cylinder if a point on the axis, DRs of the axis, radius are given.

Note:- $\perp^r$ cross section is a circle and its radius is the radius of cylinder and therefore it is also called as radius of cross section.

To find the equation of the cylinder we require point on the axis, DRs of the axis and radius. If they are not given from the given data first we find these things to find equation.

Consider a point $P(x_1, y_1, z_1)$ on the axis. Let the DRs of axis be a, b, c. Radius be 'r' units.
Consider a point $Q(x, y, z)$ on the cylinder.
From Q drop a $\perp^{ln}$ QR to the axis.
then $QR = r$ units. let $\angle QPR = \theta$.

DRs of PR are a, b, c. DRs of PQ are $(x-x_1), (y-y_1), (z-z_1)$

$$\therefore \cos\theta = \frac{a(x-x_1)+b(y-y_1)+c(z-z_1)}{\sqrt{a^2+b^2+c^2}\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}}$$

But $PQ = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$

From Δ^{le} PQR, $\cos\theta = \frac{PR}{PQ}$

$$\therefore PR = PQ.\cos\theta = \cancel{PQ}\frac{\{a(x-x_1)+b(y-y_1)+c(z-z_1)\}}{\sqrt{a^2+b^2+c^2}\cdot\cancel{PQ}}$$

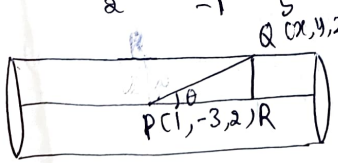
using pythogorus theorem from Δ^{le} PQR, we get

$$(PQ)^2 = (PR)^2 + (RQ)^2$$

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = \frac{\{a(x-x_1)+b(y-y_1)+c(z-z_1)\}^2}{a^2+b^2+c^2} + r^2$$

Which is the equation of the right circular cylinder

1) The radius of a normal section of a right circular cylinder is 2 units and the axis along the line. $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$. Find the equation of cylinder.



Axis is $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$ $\therefore (1,-3,2)$ is a point on the axis let $P(1,-3,2)$. DRs are $2,-1,5$.

let $Q(x,y,z)$ be a point on the surface of the cylinder. Draw $PR \perp^{r}$ to the axis.
$QR = 2$ units. let $\angle QPR = \theta$.

DRs of PQ are $(x-1), (y+3), (z-2)$
DRs of PR are $2, -1, 5$

$$\therefore \cos\theta = \frac{2(x-1)-(y+3)+5(z-2)}{\sqrt{2^2+(-1)^2+5^2}\sqrt{(x-1)^2+(y+3)^2+(z-2)^2}}$$

From Δ^{le} PQR.

$$\cos\theta = \frac{PR}{PQ} \quad \therefore PR = PQ\cos\theta$$

$$PR = \cancel{PQ}\left\{\frac{(2x-2-y-3+5z-10)}{\sqrt{4+1+25}\ \cancel{PQ}}\right\}$$

using Pythogorus theorem.

$$PQ^2 = PR^2 + QR^2$$

$$(x-1)^2+(y+3)^2+(z-2)^2 = \frac{(2x-y+5z-15)^2}{30} + 2^2$$

$$30\{x^2+y^2+z^2-2x+6y-4z+1+9+4\} = 4x^2+y^2+25z^2+225 - 4xy+10yz-60x-10yz+30y-150z+120$$

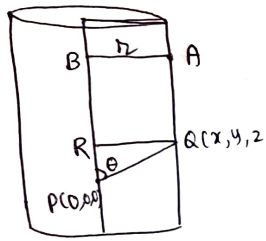
$$30x^2-4x^2+30y^2-y^2+30z^2-25z^2-\cancel{60x}+\cancel{60x}+180y-30y-120z+150z+420-225+4xy-20xz+10yz-120=0$$

$$26x^2+29y^2+5z^2+4xy+10yz-20xz+150y+30z+75=$$

2) Find the equation of the right circular cylinder of radius 2 whose axis passes through the point (1,2,3) and has DRs 2,-3,6.

Ans:- $45x^2+40y^2+13z^2+12xy+36yz-24zx-42x-112y-126z-39=0$

3) Find the equation of the right circular cylinder whose axis passes through the origin, and which has $\frac{x-3}{2}=\frac{y}{3}=\frac{z-4}{6}$ as a generator.



Since axis is parallel to generator DRs of the axis are 2,3,6

∴ Equation of the axis are

$$\frac{x}{2}=\frac{y}{3}=\frac{z}{6}=k$$

Consider a point $A\equiv(3,0,4)$ on the generator.
From A drop ⊥ⁿ AB to the axis.
then $B\equiv(2k,3k,6k)$

DRs of AB are $2k-3,\ 3k,\ 6k-4$.

AB ⊥ⁿ to axis ∴ $2(2k-3)+3(3k)+6(6k-4)=0$

i.e. $4k-6+9k+36k-24=0$

$49k-30=0$ ∴ $k=\frac{30}{49}$

∴ $B\equiv\left(\frac{60}{49},\frac{90}{49},\frac{180}{49}\right)$

radius = AB = $\sqrt{\left(\frac{60}{49}-3\right)^2+\left(\frac{90}{49}\right)^2+\left(\frac{180}{49}-6\right)^2}$

$$(AB)^2=\left(\frac{60-147}{49}\right)^2+\left(\frac{90}{49}\right)^2+\left(\frac{180-294}{49}\right)^2$$

$$r^2=\frac{(87)^2}{49^2}+\frac{(90)^2}{(49)^2}+\frac{(-114)^2}{(49)^2}=\frac{7569+8100+12996}{(49)^2}$$

$$r^2=\frac{28665}{(49)^2}\qquad\therefore r^2=RQ^2$$

Consider $Q(x,y,z)$ on the cylinder.
DRs of PQ are x,y,z, DRs of PR 2,3,6

$$\therefore\ \cos\theta=\frac{2x+3y+6z}{\sqrt{2^2+3^2+6^2}\,\sqrt{x^2+y^2+z^2}}$$

From △le PQR

$\cos\theta=\frac{PR}{PQ}$ i.e. $PR=PQ\cos\theta$

$$PR=PQ\,\frac{(2x+3y+6z)}{\sqrt{49}\,\cancel{\sqrt{x^2+y^2+z^2}}}$$

But $\sqrt{x^2+y^2+z^2}=PQ$

$PQ^2=PR^2+RQ^2$

$$x^2+y^2+z^2=\frac{(2x+3y+6z)^2}{49}+\frac{\cancel{28665}\ 585}{(49)^2}$$

$$49(x^2+y^2+z^2)=(2x+3y+6z)^2+585$$

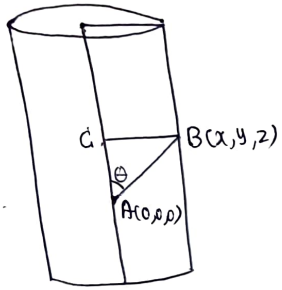
4) Find the equation of the right circular cylinder, given that the radius of its normal section is 4 and the axis passes through the points A(1,-2,3) and B(3,-1,1)

Hint:- DRs of axis are 2, 1, -2

$9(x^2+y^2+z^2-2x+4y-6z-2) = (2x+y-2z+6)^2$

5.T. the equation of the right circular cylinder of radius R whose axis is $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ is

$(cy-bz)^2 + (az-cx)^2 + (bx-ay)^2 = R^2(a^2+b^2+c^2)$.



$BC = R$

$A \equiv (0,0,0)$ is a point on the axis.

DRs of axis are a, b, c.

let $B(x,y,z)$ be a point on cylinder.

DRs of AB are x, y, z.

$$\cos\theta = \frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}\sqrt{x^2+y^2+z^2}}$$

From $\triangle^{le}$ ABC, $\cos\theta = \frac{AC}{AB}$

$$\therefore AC = AB\cos\theta = \cancel{AB}\left\{\frac{ax+by+cz}{\sqrt{a^2+b^2+c^2}\cdot\cancel{(AB)}}\right\}$$

$$AB^2 = AC^2 + CB^2$$

$$x^2+y^2+z^2 = \frac{(ax+by+cz)^2}{a^2+b^2+c^2} + R^2$$

multiplying by $(a^2+b^2+c^2)$

$$(a^2+b^2+c^2)(x^2+y^2+z^2) = (ax+by+cz)^2 + R^2(a^2+b^2+c^2)$$

$$\cancel{a^2x^2} + a^2y^2 + \cancel{a^2z^2} + b^2x^2 + \cancel{b^2y^2} + b^2z^2 + c^2x^2 + c^2y^2 + \cancel{c^2z^2}$$

$$= \cancel{a^2x^2} + \cancel{b^2y^2} + \cancel{c^2z^2} + 2abxy + 2bcyz + 2cazx + R^2(a^2+b^2+c^2)$$

$$a^2y^2 - 2abxy + b^2x^2 + a^2z^2 - 2cazx + c^2x^2 + b^2z^2 - 2bcyz$$

$$+ c^2y^2 = R^2(a^2+b^2+c^2)$$

$$(ay-bx)^2 + (az-cx)^2 + (bz-cy)^2 = R^2(a^2+b^2+c^2)$$

$$(-1)^2(cy-bz)^2 + (az-cx)^2 + (-1)^2(bx-ay)^2 = R^2(a^2+b^2+c^2)$$

$$(cy-bz)^2 + (az-cx)^2 + (bx-ay)^2 = R^2(a^2+b^2+c^2)$$

6) P.T. the equation of the right circular cylinder whose axis is $\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$ and which passes through the point $(0,0,3)$ is $10x^2+13y^2+5z^2-6yz-12zx-4xy-36x-18y+30z-135=0$.

$r = a(1+\cos\theta)$ $\qquad \frac{dr}{d\theta} = a(1-\sin\theta)$

$\tan\psi = r\frac{d\theta}{dr} = \frac{r}{1-\sin\theta} = \frac{\cancel{a}(1+\cos\theta)}{\cancel{a}\sin\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$

[$\sin^2\theta = 1-\cos 2\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$]

$\tan\psi = \frac{\sin\theta/2}{\cos\theta/2} = \tan\left(\frac{\theta}{2}\right) \qquad \psi = \frac{\theta}{2}$

ψ is the angle b/w the radius vector and the tangent at any point $P(r,\theta)$ on the curve.

The slope is $\tan\psi = \frac{dy}{dx}$, $\tan\psi = \theta + \phi$

$\psi = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$

$\frac{dy}{dx} = \tan\psi = \tan\left(\frac{3}{2}\theta\right)$

at $\theta = \frac{\pi}{6}$, slope $= \tan\left(\frac{3}{2}\cdot\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

Ex: 5.1 at every point (r,θ), the tangent to the curve $r^n = a^n\sin n\theta$ makes an angle $(n+1)\theta$ with the initial line (x axis)

Here we have to find $\psi = \theta + \phi$

$r^n = a^n\sin n\theta$ $\qquad$ Diff w.r.t θ

$nr^{n-1}\frac{dr}{d\theta} = a^n n\cos n\theta \qquad \frac{dr}{d\theta} = \frac{a^n\cos n\theta}{r^{n-1}}$

$\tan\phi = r\frac{d\theta}{dr} = \frac{1}{\frac{dr}{d\theta}} = \frac{r}{\frac{a^n\cos n\theta}{r^{n-1}}} = \frac{r^{n-1}\,r}{a^n\cos n\theta}$

$\tan\phi = \frac{r^n}{a^n\cos n\theta} = \frac{a^n\sin n\theta}{a^n\cos n\theta} = \tan n\theta$

$\phi = n\theta \qquad \psi = \theta + \phi = \theta + n\theta = (n+1)\theta$

Ex: Find the angle b/w the radius vector and the tangent of the curve $\frac{l}{r} = 1 + e\cos\theta$

$\frac{l}{r} = 1 + e\cos\theta$ $\qquad$ Diff w.r.t θ

$-\frac{l}{r^2}\frac{dr}{d\theta} = -e\sin\theta \qquad \frac{dr}{d\theta} = \frac{er^2\sin\theta}{l}$

$\tan\phi = \frac{r}{\frac{dr}{d\theta}} = \frac{r}{\frac{er^2\sin\theta}{l}} = \frac{l}{er\sin\theta} = \frac{\frac{l}{r}}{e\sin\theta}$

$\tan\phi = \frac{1+e\cos\theta}{e\sin\theta} \qquad \phi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$

Angle b/w polar curves:

Angle b/w two polar curves is defined in terms the angle b/w the tangents and the radius vector at the point of intersection of the curves.

let ϕ_1 be the angle b/w the radius vector and tangent on the first curve and the ϕ_2 be the angle b/w the radius vector and the tangent on the second curve

then $\phi = |\phi_1 - \phi_2|$ is the angle b/w the curves.

Note: If the angles are in terms of $\tan^{-1}$ then $\tan|\phi_1 - \phi_2| = |\tan(\phi_1 - \phi_2)| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$

The orthogonal curves are the curves having an angle $\frac{\pi}{2}$ b/w them. The condition is.

$|\phi_1 - \phi_2| = \frac{\pi}{2}$ or 90° or $\tan\phi_1 \cdot \tan\phi_2 = -1$

Find the angle b/w the following pairs of curves.

(i) $r = a(1-\cos\theta)$, $r = 2a\cos\theta$.

$$\frac{dr}{d\theta} = a(\sin\theta)$$

$$\tan\phi_1 = \frac{r}{\frac{dr}{d\theta}} = \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$\tan\phi_1 = \frac{2\sin^2\theta/2}{2\sin\theta/2 \cdot \cos\theta/2}$$

$$\tan\phi_1 = \frac{\sin\theta/2}{\cos\theta/2} = \tan\theta/2$$

$$\phi_1 = \frac{\theta}{2}$$

$$\frac{dr}{d\theta} = -2a\sin\theta.$$

$$\tan\phi_2 = \frac{r}{\frac{dr}{d\theta}}$$

$$\tan\phi_2 = \frac{2a\cos\theta}{-2a\sin\theta}$$

$$\tan\phi_2 = -\cot\theta$$

$$\tan\phi_2 = \tan\left(\frac{\pi}{2}+\theta\right)$$

$$\therefore \phi_2 = \frac{\pi}{2}+\theta.$$

$$\phi = |\phi_1 - \phi_2| = \left|\frac{\theta}{2} - \left(\frac{\pi}{2}-\theta\right)\right| = \left|-\frac{\pi}{2} + \theta/2\right|$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2}$$

(2) $r = a\log\theta$

$$r = \frac{a}{\log\theta}.$$

$$\frac{dr}{d\theta} = a \cdot \frac{1}{\theta}$$

$$\tan\phi_1 = \frac{r}{\frac{dr}{d\theta}} = \frac{a\log\theta}{\frac{a}{\theta}}$$

$$\tan\phi_1 = \theta\log\theta.$$

$$\frac{dr}{d\theta} = -\frac{a}{(\log\theta)^2} \cdot \frac{1}{\theta}.$$

$$\tan\phi_2 = \frac{r}{\frac{dr}{d\theta}} = \frac{\frac{a}{\log\theta}}{-\frac{a}{\theta(\log\theta)^2}}$$

$$\tan\phi_2 = -\theta\log\theta.$$

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$$

$$\tan|\phi_1 - \phi_2| = \left|\frac{\theta\log\theta - (-\theta\log\theta)}{1+(\theta\log\theta)(-\theta\log\theta)}\right| = \left|\frac{2\theta\log\theta}{1-\theta^2(\log\theta)^2}\right|$$

The point of intersection is point common to both the curves. $a\log\theta = \frac{a}{\log\theta}$ i.e. $(\log\theta)^2 = 1$

i.e. $\log\theta = 1$ i.e. $\theta = e^1 = e$.

$$\therefore |\phi_1 - \phi_2| = \tan^{-1}\left(\frac{2e\log e}{1-e^2(\log e)^2}\right) = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$$

P.T. the following pairs of curves intersect orthogonally.

(1) $r = a\sec^2\left(\frac{\theta}{2}\right)$, $r = b\,\mathrm{cosec}^2(\theta/2)$.

$$\frac{dr}{d\theta} = a\sec\left(\frac{\theta}{2}\right)\sec\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)\cdot\frac{1}{2}$$

$$\frac{dr}{d\theta} = a\tan\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\theta}{2}\right)$$

$$\tan\phi_1 = \frac{r}{\frac{dr}{d\theta}} = \frac{a\sec^2(\theta/2)}{a\tan(\theta/2)\sec^2(\theta/2)}$$

$$\tan\phi_1 = \cot(\theta/2)$$

$$\frac{dr}{d\theta} = -b\frac{1}{2}\cos\left(\frac{\theta}{2}\right)\cdot\mathrm{cosec}\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right)$$

$$\frac{dr}{d\theta} = -b\cot(\theta/2)\,\mathrm{cosec}^2(\theta/2)$$

$$\tan\phi_2 = \frac{r}{\frac{dr}{d\theta}} = \frac{b\,\mathrm{cosec}^2(\theta/2)}{-b\cot(\theta/2)\,\mathrm{cosec}^2(\theta/2)}$$

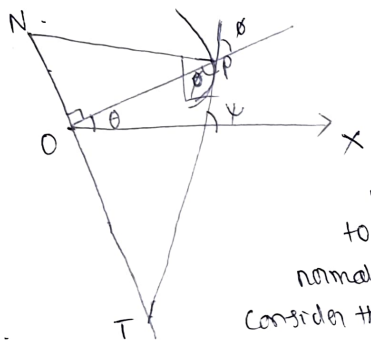
$$\tan\phi_2 = -\frac{1}{\cot(\theta/2)}$$

$\tan\phi_1 \cdot \tan\phi_2 = \cot\left(\frac{\theta}{2}\right)\left(\frac{-1}{\cot(\theta/2)}\right) = -1$

$\therefore$ The Curves are orthogonal

H.W: $r^n = a^n \cos n\theta$, $r^n = b^n \sin n\theta$

Polar Subtangent and polar Subnormal.



Consider a polar curve $r = f(\theta)$. let $P(r,\theta)$ be a point on the curve. let ϕ be the angle b/w radius vector and the tangent to the curve. Draw PN normal to the tangent. Consider the line NT which is the extension to meet the normal at N and Normal at the point T. The line segment ON is called sub normal and OT is the subnormal.

From Δ^{le} OPT, $\tan\phi = \frac{OT}{OP}$ But $OP = r$.

ie $\tan\phi = \frac{OT}{r}$ $\therefore$ Subtangent $= OT = r\tan\phi$

$\therefore$ Subtangent $= r\cdot\left(r\frac{d\theta}{dr}\right) = r^2\frac{d\theta}{dr}$ or $\frac{r^2}{\frac{dr}{d\theta}}$.

$\angle OPN = 90 - \phi = \frac{\pi}{2} - \phi$.

From Δ^{le} OPN, $\tan\left(\frac{\pi}{2}-\phi\right) = \frac{ON}{OP}$ But $OP = r$.

ie. $\cot\phi = \frac{ON}{r}$ ie. $ON = r\cdot\cot\phi$

Subnormal $= ON = r\cdot\frac{1}{\tan\phi} = r\cdot\frac{1}{r\cdot\frac{d\theta}{dr}} = \frac{dr}{d\theta}$.

① Find the polar Subtangent and Sub normal for the cardiod $r = a(1-\cos\theta)$.

Sol:- $\frac{dr}{d\theta} = a\sin\theta$.

polar Subtangent $= \frac{r^2}{\frac{dr}{d\theta}} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta} = \frac{a\{2\sin^2\theta/2\}^2}{2\sin\frac{\theta}{2}\cos\theta/2}$

$= \frac{2a^2\sin^3(\theta/2)}{2\sin(\theta/2)\cos\theta/2} = 2a\cdot\sin^2(\theta/2)\cdot\frac{\sin\theta/2}{\cos\theta/2}$

$= 2a\cdot\sin^2(\theta/2)\cdot\tan(\theta/2)$

polar Sub Normal $= \frac{dr}{d\theta} = a\sin\theta$

② S.T. for the curve $r = ae^{b\theta^2}$, the product of the polar subnormal and the polar subtangent is r^2.

$r = ae^{b\theta^2}$. $\frac{dr}{d\theta} = ae^{b\theta^2}\cdot 2b\theta = 2abr\theta$

polar Subtangent $= \frac{r^2}{\frac{dr}{d\theta}} = \frac{r^2}{2br\theta} = \frac{r}{2b\theta}$. | Subnormal $= \frac{dr}{d\theta} = 2br\theta$

$\therefore$ product P.N $\times$ P.T $= \frac{r}{2b\theta}\cdot 2br\theta = r^2$