Ans:

Indeterminate forms.

Expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\times\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ are called indeterminate forms.

L'Hospital rule:- If we get indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)} = \ldots$

ie. we have to consider derivatives of numerator and denominator until we get the limit as a finite value or ∞.

Note:- ① To evaluate the limits in the other indeterminate forms we reduce them to either to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ before using L'Hospital rule.

② We use certain standard limits like $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 0}\frac{\tan x}{x}=1$, $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$.

③ When the given limit can be written as product of two limits with one limit gives a constant and other limit gives an indeterminate form. We evaluate the limits separately.

④ If in the denominator there exists functions of the form $x^k\sin^m x$ or $x^k\cos^m x$ we multiply and divide by suitable x^1 to use standard limits.

Evaluate the following limits.

① $\lim_{x\to 0}\frac{a^x-b^x}{x} = \left(\frac{a^0-b^0}{0} = \frac{1-1}{0} = \frac{0}{0}\right)$

the indeterminate form. Using L'Hospit rule

$= \lim_{x\to 0}\frac{a^x\log a - b^x\log b}{1} = a^0\log a - b^0\log b$

$= \log a - \log b = \log\left(\frac{a}{b}\right)$

② $\lim_{x\to 0}\frac{x\cos x - \log(1+x)}{x^2} = \left(\frac{0}{0}, \text{ indeterminate}\right)$

using L'Hospital rule

$= \lim_{x\to 0}\frac{x(-\sin x)+\cos x - \frac{1}{1+x}}{2x}$.

again we get $\left(=\frac{1-1}{0}=\frac{0}{0}\right)$ indeterminate

using L'Hospital rule again, we get

$= \lim_{x\to 0}\frac{-x\cos x - \sin x - \sin x - \frac{(-1)}{(1+x)^2}}{2}$

$= \frac{-0.\cos 0 - \sin 0 - \sin 0 + \frac{1}{(1+0)^2}}{2} = \frac{1}{2}$

$$\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x - 2\tan x}{1+\cos 4x} = \left(\frac{\sec^2\frac{\pi}{4} - 2\tan\frac{\pi}{4}}{1+\cos 4\frac{\pi}{4}} = \frac{2-2}{1-1} = \frac{0}{0}\right)$$

Using L'Hospital rule

$$= \lim_{x\to\frac{\pi}{4}} \frac{2\sec x\tan x\cdot\sec x - 2\sec^2 x}{-\sin 4x\cdot(4)}$$

$$= \lim_{x\to\frac{\pi}{4}} \frac{2\sec^2 x\,(\tan x -1)}{-4\cdot\sin 4x} = \lim_{x\to\frac{\pi}{4}}\left(-\frac{1}{2}\sec^2 x\right)\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{\sin 4x}$$

$$= -\frac{1}{2}\left(\sec\frac{\pi}{4}\right)^2\cdot\lim_{x\to\frac{\pi}{4}}\frac{\sec^2 x}{4\cos 4x} = -\frac{1}{2}(\sqrt{2})^2\cdot\frac{(\sec\frac{\pi}{4})^2}{4\cos 4\cdot\frac{\pi}{4}}$$

$$= -\frac{1}{2}\cdot 2\cdot\frac{(\sqrt{2})^2}{4\cdot(-1)} = \frac{4}{8} = \frac{1}{2}$$

(4) $$\lim_{x\to0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x} = \lim_{x\to0}\frac{2\sinh x - 2\log(1+x)}{x^2\frac{\sin x}{x}}$$

But $\lim_{x\to0}\frac{\sin x}{x}=1$

$$=2\lim_{x\to0}\frac{\sinh x - \log(1+x)}{x^2} = \left(2\,\frac{\sinh 0 - \log 1}{0} = \frac{0}{0}\right)$$

Using L'Hospital rule.

$$=2\lim_{x\to0}\frac{\cosh x - \frac{1}{(1+x)}}{2x}$$ using L'Hospital rule again

$$=2\lim_{x\to0}\frac{\sinh x - \frac{(-1)}{(1+x)^2}}{2} = 2\cdot\frac{\left(\sinh 0 + \frac{1}{1^2}\right)}{2}$$

$$= 2\,\frac{1}{2} = 1$$

5) $$\lim_{x\to0}\frac{\sin x\cdot\sin^{-1}x}{x^2} = \lim_{x\to0}\frac{\sin x}{x}\cdot\frac{\sin^{-1}x}{x}$$

But $\lim_{x\to0}\frac{\sin x}{x}=1$ $$=\lim_{x\to0}\frac{\sin^{-1}x}{x}.$$

$$=\lim_{x\to0}\frac{\frac{1}{\sqrt{1-x^2}}}{1} = \frac{1}{\sqrt{1}} = 1$$

6) $$\lim_{x\to0}\frac{\tan x - x}{x^2\tan x}$$

$$=\lim_{x\to0}\frac{\tan x - x}{x^3\cdot\left(\frac{\tan x}{x}\right)}$$ But $\lim_{x\to0}\frac{\tan x}{x}=1$

$$=\lim_{x\to0}\frac{\tan x - x}{x^3} = \lim_{x\to0}\frac{\sec^2x-1}{3x^2}$$

$$=\lim_{x\to0}\frac{\cancel{2}\sec x\cdot\sec x\cdot\tan x}{3\cdot\cancel{2}x} = \lim_{x\to0}\frac{\sec^2x}{3}\cdot\frac{\tan x}{x}$$

$$=\frac{\sec^2 0}{3}\cdot1 = \frac{1}{3}\cdot1 = \frac{1}{3}$$

7) Find the constants a and b so that

$$\lim_{x\to0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

Consider $$\lim_{x\to0}\frac{x(1+a\cos x)-b\sin x}{x^3} = \frac{0}{0}$$

Using L'Hospital rule.

$$=\lim_{x\to0}\frac{x(-a\sin x)+1+a\cos x - b\cos x}{3x^2}$$

$$\frac{0+1+a-b}{0}$$ which becomes indeterminate form $\frac{0}{0}$ only if $1+a-b=0$ i.e. $a-b=-1$ — ①

using L' Hospital rule again

$$= \lim_{x\to 0} \frac{-a(x\cos x + \sin x) - a\sin x + b\sin x}{6x} = \frac{0}{0}$$

using L'Hospital rule again, we get

$$= \lim_{x\to 0} \frac{-a\{-x\sin x + \cos x + \cos x\} - a\cos x + b\cos x}{6}$$

$$= \frac{-a(-0\sin 0 + \cos 0 + \cos 0) - a\cos 0 + b\cos 0}{6} = \frac{-2a-a+b}{6}$$

But lim should be equal to 1 $\quad \frac{-3a+b}{6} = 1$

i.e. $-3a+b = 6$ — ②

adding ① and ② $\quad -2a = 5 \quad$ i.e. $a = -\frac{5}{2}$

Substituting in ① $\quad -\frac{5}{2} - b = -1 \quad$ i.e. $-\frac{5}{2} + 1 = b$

$\therefore b = \frac{-5+2}{2} = -\frac{3}{2}$ $\quad \boxed{\therefore a = -\frac{5}{2} \quad b = -\frac{3}{2}}$

⑧ $\lim_{x\to 0} \frac{\sin hx - \log(1+x)}{x\sin x}$ $\quad$ Ans: 2

⑨ $\lim_{x\to 0} \frac{\sin x \sin^{-1}x - x^2}{x^6}$ $\quad$ Ans: $\frac{1}{18}$

⑩ Find a and b such that $\lim_{x\to 0} \frac{x(1-a\cos x)+b\sin x}{x^3} = \frac{1}{3}$.

Ans: $a=\frac{1}{2}$, $b=-\frac{1}{2}$

⑪ $\lim_{x\to \frac{\pi}{2}} \frac{\log\cos x}{\tan x} \left(= \frac{\log\cos\frac{\pi}{2}}{\tan\frac{\pi}{2}} = \frac{\log 0}{\infty} = -\frac{\infty}{\infty} \right)$ $\quad$ 22

using L' Hospital rule.

$$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\cos x}(-\sin x)}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{-\sin x}{\cos x} \times \cos x$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{2}\sin 2x = -\frac{1}{2}\sin 2\frac{\pi}{2} = -\frac{1}{2}\cdot 0 = 0$$

⑫ $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)} = \left(\frac{\log 0}{\log 0} = \frac{-\infty}{-\infty}\right)$

using L' Hospital rule

$$= \lim_{x\to a} \frac{\frac{1}{x-a}}{\frac{1}{e^x-e^a}\cdot e^x} = \lim_{x\to a} \frac{e^x - e^a}{(x-a)\,e^x} = \left(\frac{0}{0}\right)$$

using L' Hospital rule again, we get

$$= \lim_{x\to a} \frac{e^x}{(x-a)e^x + e^x} = \frac{e^a}{(a-a)e^a + e^a} = \frac{e^a}{e^a} = 1$$

⑬ $\lim_{x\to 0} \log_x \tan x = \lim_{x\to 0} \frac{\log_e \tan x}{\log_e x} = \left(\frac{+\infty}{+\infty}\right)$

using L' Hospital rule,

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot\sec^2 x}{\frac{1}{x}} = \lim_{x\to 0} \sec^2 x \cdot \frac{1}{\left(\frac{\tan x}{x}\right)}$$

$$= \sec^2 0 \cdot 1 = 1\cdot 1 = 1$$

denominator

To evaluate limits of the form $0\times\infty$, $\infty-\infty$ convert it into indeterminate form $\frac{0}{0}$ or ∞/∞ then by using L' Hospital rule we find the limit.

14) $\lim_{x\to\infty}(a^{\frac{1}{x}}-1)x$

This lim is $(a^{\frac{1}{\infty}}-1)\infty = (a^0-1)\times\infty = 0\times\infty$.

We can write this as

$$= \lim_{x\to\infty}\frac{(a^{\frac{1}{x}}-1)}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right)$$

Using L' Hospital rule

$$= \lim_{x\to\infty}\frac{a^{\frac{1}{x}}.\log a.\left(-\cancel{\frac{1}{x^2}}\right)}{\left(-\cancel{1/x^2}\right)} \qquad \left[\frac{d}{dx}\{a^{f(x)}\} = a^{f(x)}.\log a.\,f'(x)\right]$$

$$= \lim_{x\to\infty} a^{\frac{1}{x}}.\log a = a^{\frac{1}{\infty}}.\log a = a^0.\log a = \log a$$

15) $\lim_{x\to0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right] = \left(\frac{1}{0}-\frac{1}{0} = \infty-\infty\right)$

$$= \lim_{x\to0}\left[\frac{e^x-1-x}{x(e^x-1)}\right] = \left(\frac{1-1}{0} = \frac{0}{0}\right)$$

Using L' Hospital rule.

$$= \lim_{x\to0}\frac{e^x-1}{x(e^x)+e^x-1} = \left(\frac{0}{0}\right)$$

using L' Hospital rule again, we get

$$= \lim_{x\to0}\frac{e^x}{xe^x+e^x+e^x} = \frac{e^0}{e^0+e^0} = \frac{1}{2}$$

16) $\lim_{x\to0}\left(\frac{1}{x^2}-\frac{1}{\sin^2x}\right) = \infty-\infty$

$$= \lim_{x\to0}\frac{\sin^2x-x^2}{x^2\sin^2x} = \lim_{x\to0}\frac{\sin^2x-x^2}{x^4\left(\frac{\sin x}{x}\right)^2}$$

But $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^2 = 1^2 = 1$

$$= \lim_{x\to0}\frac{\sin^2x-x^2}{x^4} = \left(\frac{0}{0}\right)$$

using L'Hospital rule, we get

$$= \lim_{x\to0}\frac{2\sin x.\cos x-2x}{4x^3} = \lim_{x\to0}\frac{\sin 2x-2x}{4x^3} = \left(\frac{0}{0}\right)$$

$$= \lim_{x\to0}\frac{2\cos 2x-1}{12x^2} = \lim_{x\to0}\frac{-4\sin 2x}{24x}$$

$$= \lim_{x\to0}\left(-\frac{1}{6}\right)\frac{\sin 2x}{\frac{1}{2}(2x)} \qquad \text{But } \lim_{x\to0}\frac{\sin 2x}{2x} = 1$$

$$= -\frac{2}{6} = -\frac{1}{3}$$

17) $\lim_{x\to\frac{\pi}{2}}(1-\sin x)$ Ans: 0

18) $\lim_{x\to1}\left[\frac{2}{x^2-1}-\frac{1}{x-1}\right]$ Ans:- $-\frac{1}{2}$

19) $\lim_{x\to\frac{\pi}{2}}\left[x\tan x-\frac{\pi}{2}\sec x\right]$ Ans:- -1

20) $\lim_{x\to0}\left(\frac{2}{x}-\cot\left(\frac{x}{2}\right)\right)$ Ans:- 0

For the indeterminate forms of the kind $\infty^0, 0^0, 1^\infty$, we take log on both sides to reduce it into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then we use L'Hospital rule.

21) $\lim_{x\to\frac{\pi}{2}} (\sec x)^{\cot x}$

let $S = \lim_{x\to\frac{\pi}{2}} (\sec x)^{\cot x}$. , Taking log on both sides

$\log S = \lim_{x\to\frac{\pi}{2}} \log(\sec x)^{\cot x}$.

$\log S = \lim_{x\to\frac{\pi}{2}} \cot x \cdot \log(\sec x)$

$\log S = \lim_{x\to\frac{\pi}{2}} \frac{\log(\sec x)}{\tan x}$ $\left(= \frac{\log \sec \pi/2}{\tan\frac{\pi}{2}} = \frac{\infty}{\infty}\right)$

using L'Hospital rule, we get

$\log S = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sec x}\cdot \sec x\cdot\tan x}{\sec^2 x} = \left(\frac{\infty}{\infty}\right)$

using L'Hospital rule again

$\log S = \lim_{x\to\frac{\pi}{2}} \frac{\sec^2 x}{2\sec x\cdot\sec x\cdot\tan x} = \frac{1}{2\tan\frac{\pi}{2}} = \frac{1}{\infty} = 0$

ie. $\log S = 0$ $\therefore$ $S = e^0 = 1$

$[\log_e 2 = k \Rightarrow 2 = e^k]$

denominator

22) $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$

let $S = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$ Taking log on both sides

$\log S = \lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x+c^x}{3}\right)}{x} = \left(\frac{\log 1}{0} = \frac{0}{0}\right)$

using L'Hospital rule, we get

$\log S = \lim_{x\to 0} \frac{\frac{1}{\left(\frac{a^x+b^x+c^x}{3}\right)}\frac{1}{3}(a^x\log a + b^x\log b + c^x\log c)}{1}$

$= \lim_{x\to 0} \frac{a^x\log a + b^x\log b + c^x\log c}{a^x+b^x+c^x} = \frac{a^0\log a + b^0\log b + c^0\log c}{a^0+b^0+c^0}$

$= \frac{\log a + \log b + \log c}{1+1+1}$ ie. $\log S = \frac{\log(abc)}{3}$

ie. $3\log S = \log abc$ ie. $\log S^3 = \log(abc)$

ie. $S^3 = abc$ ie. $S = \sqrt[3]{abc}$

23) $\lim_{x\to 0} (1+\sin x)^{\cot x} = (1^\infty)$

let $L = \lim_{x\to 0} (1+\sin x)^{\cot x}$, Taking log on both sides

$\log L = \lim_{x\to 0} \cot x \log(1+\sin x)$

$\log L = \lim_{x\to 0} \frac{\log(1+\sin x)}{\tan x} = \left(\frac{\log 1}{\tan 0} = \frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\frac{1}{1+\sin x}\cdot \cos x}{\sec^2 x} = \lim_{x\to 0} \frac{\cos^3 x}{1+\sin x} = \frac{\cos 0}{1+\sin 0}$$

$\log L = \frac{1}{1}$ $\therefore$ $\underline{L = e^1 = e}$

24) $\lim_{x\to \frac{\pi}{2}} (\sin x)^{\tan x}$ Ans: 1

25) $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$ Ans: e^a

26) $\lim_{x\to a} \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

let $L = \lim_{x\to a} \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$, Taking log weget

$$\log L = \lim_{x\to a} \log\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$= \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right)\cdot \log\left(2-\frac{x}{a}\right) = \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)}$$

$$= \left(\frac{\log 2-\frac{a}{a}}{\cot\left(\frac{\pi a}{2a}\right)} = \frac{\log 1}{\cot\frac{\pi}{2}} = \frac{0}{0}\right)$$

using L'Hospital rule, weget.

$$= \lim_{x\to a} \frac{\frac{1}{\left(2-\frac{x}{a}\right)}\left(-\frac{1}{a}\right)}{\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)\left(\frac{\pi}{2a}\right)} = -\frac{1}{\cancel{a}}\cdot\frac{1}{\left(2-\frac{a}{a}\right)} \times \frac{2\cancel{a}}{\pi}\left(-\sin\frac{\pi a}{2a}\right)$$

$$= -\frac{1}{(2-1)} \times \left(\frac{-1}{\pi}\right)\cdot \sin\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

ie. $\log L = \frac{2}{\pi}$ $\therefore$ $\underline{L = e^{2/\pi}}$

27) $\lim_{x\to\infty} \left[\frac{ax+1}{ax-1}\right]^x$ (VTU Aug 1999)

let $L = \lim_{x\to\infty} \left[\frac{\cancel{x}(a+\frac{1}{x})}{\cancel{x}(a-1/x)}\right]^x = \left(\left(\frac{a}{a}\right)^\infty = 1^\infty\right)$

Taking log on both sides

$$\log L = \lim_{x\to\infty} x \log\left[\frac{a+\frac{1}{x}}{a-1/x}\right]$$

$$= \lim_{x\to\infty} \frac{\log(a+\frac{1}{x}) - \log(a-1/x)}{(1/x)} = \left(\frac{0}{0}\right)$$

using L'Hospital rule, weget

$$\log L = \lim_{x\to\infty} \frac{\frac{1}{(a+\frac{1}{x})}\left(-\frac{1}{x^2}\right) - \frac{1}{(a-1/x)}(+1/x^2)}{(-1/x^2)}$$

$$= \lim_{x\to\infty} \frac{\cancel{\left(-\frac{1}{x^2}\right)}\left[\frac{1}{a+\frac{1}{x}} + \frac{1}{(a-1/x)}\right]}{\cancel{(-1/x^2)}} = \frac{1}{a}+\frac{1}{a} = \frac{2}{a}$$

ie. $\log L = \frac{2}{a}$ $\therefore$ $\underline{L = e^{2/a}}$

(28) $\lim_{x\to 0} (1-x^2)\tan\left(\frac{\pi x}{2}\right)$ (VTU march 2000)

Problem is wrong. $x\to 1$ instead of $x\to 0$

$$L = \lim_{x\to 1} (1-x^2)\tan\left(\frac{\pi x}{2}\right) = (0\times\infty)$$

$$= \lim_{x\to 1} \frac{(1-x^2)}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1} \frac{-2x}{-\sec^2\left(\frac{\pi x}{2}\right)\times\frac{\pi}{2}}$$

$$= \frac{-2\cdot 2/\pi}{-\sec^2\frac{\pi}{2}} = \frac{4/\pi}{\infty} = \underline{0}$$