Infinite Series

Series is the Sum of Infinite no. of terms. It is denoted as $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots$

Positive Series: If all the terms are +ve then it is called as positive series.

Nature of the Series:- For the +ve series $\sum_{n=1}^{\infty} u_n$ Define

$S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3 + \cdots$, $S_n = u_1 + u_2 + \cdots + u_n = \sum_{i=1}^{n} u_i$

The +ve Series is said to be convergent if $\lim_{n\to\infty} S_n = K$; divergent if $\lim_{n\to\infty} S_n = \infty$.

Examples:-

① $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} = e$ is Convergent.

② $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Note:- ① Addition or deletion of few terms will not affect the nature of the series.

② The nature of the Series is not affected by multiplying all the elements by a non zero constant. k.

③ If $\sum u_n$ is Conv then $\lim_{n\to\infty} u_n = 0$. But the converse need not be true. ie. If $\lim_{n\to\infty} u_n = 0$ then $\sum u_n$ need not be convergent.

④ If $\lim_{n\to\infty} u_n \neq 0$ then the Series cannot be Convergent

⑤ The nth term of the arithmitic progression with 'a' as the first element and d as the common difference is $a + (n-1)d$.

Ex:	1, 2, 3	n	1, 3, 5, 7	(2n-1)
	2, 3, 4,	(n+1)	3, 5, 7, 9,	(2n+1)
	2, 4, 6, 8,	2n	4, 6, 8,	(2n+2)

Cauchy's Integral test

If $\sum u_n = \sum f(n)$ is a positive series, $f(n)$ is a decreasing function then we find $I = \int_1^\infty f(x)\,dx$

Series is convergent for $I = K$, divergent for $I = \pm\infty$.

Note:- $\sum u_n$ means $\sum_{n=1}^{\infty} u_n$. If u_n is not defined for $n=1$ then we take $n = 2$ or 3 from which u_n is defined. In such cases the lower limit of I has to be changed from 1 to that constant.

P-Series:-

Using Cauchy's Integral test S.T. the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \le 1$

Consider the p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \sum f(n)$$

$$\therefore f(n) = \frac{1}{n^p}$$

$$I = \int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx.$$

Case (i) let $p = 1$

$$I = \int_1^\infty \frac{1}{x}\,dx = \log x \Big|_1^\infty$$

$$= \log\infty - \log 1 = \infty$$

$\therefore$ Series is divergent for $p = 1$

Case (ii) let $p \neq 1$

$$I = \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx.$$

$$= \frac{x^{-p+1}}{-p+1}\Big|_1^\infty = \frac{1}{-p+1}\left\{\lim_{x\to\infty} x^{-p+1} - 1^{-p+1}\right\}$$

But $1^{-p+1} = 1$

$$I = \frac{\lim_{x\to\infty} x^{1-p} - 1}{1-p}$$

But $\lim_{x\to\infty} x^{+ve} = \infty$

and $\lim_{x\to\infty} x^{-ve} = 0$

$$I = \begin{cases} \frac{\infty - 1}{1-p} = \infty & \text{if } 1-p > 0 \\ \frac{0-1}{1-p} & \text{if } 1-p < 0 \end{cases}$$

$$I = \begin{cases} \infty & \text{if } 1 > p \\ \text{finite} & \text{if } 1 < p \end{cases}$$

$\therefore$ Series is div if $p < 1$

conv for $p > 1$

Geometric Series - (r-Series)

(3)

$\sum_{n=1}^{\infty} r^n$ where $r>0$ is called as Geometric Series.

Series is convergent for $r<1$ and divergent for $r \geq 1$.

Proof:- Consider $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} r^n$

$$S_n = r + r^2 + r^3 + \cdots + r^n = \begin{cases} \frac{1-r^n}{1-r} & \text{if } r<1 \\ \frac{r^n-1}{r-1} & \text{if } r>1 \end{cases}$$

$$\text{But } \lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r>1 \\ 0 & \text{if } r<1 \end{cases}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \begin{cases} \frac{1-r^n}{1-r}, & r<1 \\ \frac{r^n-1}{r-1}, & r>1 \end{cases} = \begin{cases} \frac{1-0}{1-r}, & r<1 \\ \frac{\infty-1}{r-1}, & r>1 \end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} \text{finite} & \text{for } r<1 \\ \infty & \text{for } r>1 \end{cases}$$

Series is convergent for $r<1$, div for $r>1$

for $r=1$, $S_n = 1+1^2+1^3+\cdots+1^n = n$.

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty$ $\therefore$ divergent for $r=1$

Ex: $\sum \left(\frac{\sqrt{3}}{\sqrt{2}}\right)^n$ is divergent, $\sum \left(\frac{5}{12}\right)^n$ is convergent.

Comparison test:-

(Quotient test):- If $\sum u_n$ and $\sum v_n$ are two +ve series such that $\lim_{n\to\infty} \frac{u_n}{v_n} = k \neq 0$. then both the series behaves like.

Ex: In use this test we simplify $\sum u_n = \sum (u_n)$ (expression in n) that $\sum v_n$ is a p-series or r-series.

Determine the nature of the following series.

(4) K

(1) $\sum_{n=1}^{\infty} \left\{\frac{1+3^n}{1+5^n}\right\}^{\frac{1}{2}} = \sum \frac{(3^n)^{1/2}}{(5^n)^{1/2}} \left(\frac{\frac{1}{3^n}+1}{\frac{1}{5^n}+1}\right)^{1/2} = \sum \left(\sqrt{\frac{3}{5}}\right)^n \left(\frac{\frac{1}{3^n}+1}{\frac{1}{5^n}+1}\right)^{\frac{1}{2}}.$

$= \sum_{n=1}^{\infty} u_n \qquad u_n = \left(\sqrt{\frac{3}{5}}\right)^n \left\{\frac{\frac{1}{3^n}+1}{\frac{1}{5^n}+1}\right\}^{1/2}$

Consider the series $\sum_{n=1}^{\infty} v_n = \sum \left(\sqrt{\frac{3}{5}}\right)^n$ which is convergent.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\cancel{\left(\sqrt{\frac{3}{5}}\right)^n} \left\{\frac{\frac{1}{3^n}+1}{\frac{1}{5^n}+1}\right\}^{1/2}}{\cancel{\left(\sqrt{\frac{3}{5}}\right)^n}} = \left(\frac{0+1}{0+1}\right)^{\frac{1}{2}} = 1 \neq 0$

$\therefore$ By comparison test $\sum u_n$ is also convergent.

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{\sqrt{n}}{\sqrt{n+1}}\right) = \sum u_n$

$= \sum_{n=1}^{\infty} \left(\frac{\sqrt{n+1} - n}{\sqrt{n}\sqrt{n+1}}\right) = \sum_{n=1}^{\infty} \frac{\sqrt{n}\sqrt{1+\frac{1}{n}} - n}{\sqrt{n}.\sqrt{n}\sqrt{1+\frac{1}{n}}}$

$= \sum_{n=1}^{\infty} \frac{\cancel{n}\left(\frac{1}{\sqrt{n}}\sqrt{1+\frac{1}{n}} - 1\right)}{\cancel{n}\sqrt{1+\frac{1}{n}}} = \sum u_n.$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt{n}}\sqrt{1+\frac{1}{n}} - 1}{\sqrt{1+\frac{1}{n}}}\right) = \frac{0-1}{\sqrt{1}} = -1 \neq 0.$

$\therefore \sum u_n$ is divergent.

(3) $\sum (\sqrt{n^4+1} - \sqrt{n^4-1}) = \sum u_n.$

multiplying and dividing by rationalizing factor

$= \sum \frac{(\sqrt{n^4+1} - \sqrt{n^4-1}) \times (\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})} = \sum \frac{(n^4+1) - (n^4-1)}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$

$$\sum \frac{n^4+1-n^4+1}{\sqrt{n^4}\sqrt{\left(1+\frac{1}{n^4}\right)}+\sqrt{n^4}\sqrt{\left(1-\frac{1}{n^4}\right)}} = \sum \frac{2}{(n^4)^{1/2}\left\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right\}}$$

$$= \sum \frac{2}{n^2\left\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right\}} = \sum u_n \text{ let } \sum v_n = \sum \frac{1}{n^2} \text{ which is conv.}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{2}{n^2\left\{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right\}}}{\frac{1}{n^2}} = \frac{2}{\sqrt{1}+\sqrt{1}} = \frac{2}{\sqrt{2}} \neq 0$$

$\therefore$ By quotient test (Comparison) $\sum u_n$ is also convergent

(4) $$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$$

$$= \sum \frac{\sqrt{(n+1)}-1}{(n+2)^3-1} = \sum u_n = \sum \frac{\sqrt{n}\sqrt{\left(1+\frac{1}{n}\right)}-1}{n^3\left(1+\frac{2}{n}\right)^3-1}$$

$$= \sum \frac{\sqrt{n}\left\{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right\}}{n^3\left\{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right\}} = \sum \frac{1}{n^{5/2}}\left\{\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}}\right\}$$

let $\sum v_n = \sum \frac{1}{n^{5/2}}$ which is convergent.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{5/2}}\left\{\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}}\right\}}{\frac{1}{n^{5/2}}} = \frac{\sqrt{1+0}-0}{(1+0)^3-0}$$

$= \frac{\sqrt{1}}{1} = 1 \neq 0$ $\therefore$ By quotient test $\sum u_n$ is also convergent.

(5) $$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p} = \sum u_n \qquad f(n) = \frac{1}{n(\log n)^p}.$$

$$I = \int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\log x)^p}\,dx.$$ put $\log x = t$. $\frac{1}{x}dx = dt$.

$t = \log x$ $\quad$ for $x = 2$, $t = \log 2$, $\quad x \to \infty, t \to \log\infty = \infty$

(6)

$$I = \int_{\log 2}^{\infty} \frac{1}{t^p}\, dt$$

Case (i) :- for $p = 1$, $\quad I = \int_{\log 2}^{\infty} \frac{1}{t}\, dt = \log t \Big|_{\log 2}^{\infty}$

$= \log\infty - \log(\log 2) = \infty - K = \infty$ $\quad$ ∴ Series is divergent.

Case (ii) :- for $p \neq 1$, $\quad I = \int_{\log 2}^{\infty} t^{-p}\, dt = \frac{t^{-p+1}}{-p+1}\Big|_{\log 2}^{\infty}$

$$= \frac{1}{1-p}\left\{\lim_{t\to\infty} t^{1-p} - (\log 2)^{1-p}\right\} = \frac{1}{1-p}\left\{\lim_{t\to\infty} t^{1-p} - K\right\}$$

$$= \begin{cases} \frac{\infty - K}{1-p} = \infty & \text{if } 1-p > 0 \text{ i.e. } 1 > p \\ \frac{0-K}{1-p} = \text{finite} & \text{if } 1-p < 0 \text{ i.e. } 1 < p \end{cases}$$

∴ Convergent for $p \geq 1$, divergent for $p < 1$

Exercise :-

① $\sum \frac{\sqrt{n}}{n^2+1}$ Conv. $\quad$ ② $\sum \frac{1}{\sqrt{2n^3+2n+3}}$ conv.

③ $\sum \left(\frac{3^{n+1}-1}{2^{n+1}+2}\right)^{1/3}$ divergent.

Cauchy's root test :-

Consider +ve series $\sum u_n$. Here we evaluate.

$\lim_{n\to\infty} (u_n)^{1/n} = l$. $\quad$ For $l < 1$ Conv, $l > 1$ div, $l = 1$ test fails.

Note: ① $\lim_{n\to\infty} n^{1/n} = 1$, ② $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

③ $\lim_{n\to\infty} \left(1+\frac{1}{f(n)}\right)^{f(n)} = e$

nd the nature of the following Series.

$\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

ere $f(n) = \frac{1}{n \log n}$ which is a decreasing function.

$I = \int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x \log x}\,dx$.

put $\log x = t$. $\frac{1}{x}\,dx = dt$.

$t = \log x$ when $x = 2$, $t = \log 2$.
$x = \infty$, $t = \log \infty = \infty$.

$\therefore I = \int_{\log 2}^{\infty} \frac{dt}{t} = \log t \Big|_{\log 2}^{\infty}$

$= \log \infty - \log(\log 2) = \infty - K = \infty$

$\therefore$ By Cauchy's integral test the Series is divergent.

(ii) $\sum n e^{-n^2}$

Here $f(n) = n e^{-n^2}$.

$I = \int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-x^2}\,dx$.

put $x^2 = t$, $2x\,dx = dt$.

$t = x^2$

When $x = 1$, $t = 1^2 = 1$
$x = \infty$, $t = \infty^2 = \infty$

$\therefore I = \int_1^{\infty} e^{-t} \frac{dt}{2}$

$= \frac{1}{2}\left\{\frac{e^{-t}}{-1}\right\}_1^{\infty} = -\frac{1}{2}\{e^{-\infty} - e^{-1}\}$

$= -\frac{1}{2}\left\{\frac{1}{e^{\infty}} - \frac{1}{e}\right\} = -\frac{1}{2}\left\{0 - \frac{1}{e}\right\} = \frac{1}{2e}$

$\therefore$ By Cauchy's integral test the Series is convergent.

Test for the convergence of the following Series.

① $\sum \left(1 + \frac{1}{n}\right)^{n^2} = \sum u_n$.

$u_n = \left(1 + \frac{1}{n}\right)^{n^2}$.

$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{\left(1 + \frac{1}{n}\right)^{n^2}\right\}^{\frac{1}{n}}$.

$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{\frac{n^2}{n}} = e > 1$

$\therefore$ By root test Series is divergent

② $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}} = \sum u_n$

$u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right\}^{\frac{1}{n}}$

$= \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{\frac{-n^{3/2}}{n}} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-\sqrt{n}}$

$\left\{\because n^{3/2}/n = n^{3/2} \cdot n^{-1} = n^{3/2 - 1} = n^{1/2} = \sqrt{n}\right\}$

$= \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$

$\therefore$ By root test Series is conv.

③ $\sum \frac{(\sqrt{5} - 1)^n}{n^2 + 1} = \sum u_n$

$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left\{\frac{(\sqrt{5} - 1)^n}{n^2 + 1}\right\}^{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{(\sqrt{5} - 1)^{n/n}}{(n^2)^{1/n} \left(1 + \frac{1}{n^2}\right)^{1/n}}$

$= \lim_{n\to\infty} \frac{\sqrt{5} - 1}{(n^{1/n})^2 \left(1 + \frac{1}{n^2}\right)^{1/n}} = \frac{\sqrt{5} - 1}{1^2 (1 + 0)^0}$

$= \frac{\sqrt{5} - 1}{1\,(1)} = \sqrt{5} - 1 > 1$

$\therefore$ Series is div.

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\dots$$

$$=\sum\left\{\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right\}^{-n}=\sum u_n$$

$$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\left[\left\{\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right\}^{-n}\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\left(\frac{n+1}{n}\right)^{n}-1\right)\right]^{-\frac{\cancel{n}}{\cancel{n}}1}$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left\{\left(1+\frac{1}{n}\right)^{n}-1\right\}\right]^{-1}$$

$$=\left[(1+0)(e-1)\right]^{-1}=(e-1)^{-1}$$

$$=\frac{1}{e-1}<1$$

$\therefore$ Series is Conv.

=

⑤ $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\dots$

Ans: Conv.

⑥ $\sum\frac{n^3}{4^n}$. Ans. Conv.

Find the nature of the following Series for $x>0$.

① $\frac{1}{2}+\left(\frac{2}{3}\right)x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\dots$

The first term is not in the order. Ignoring the first term.

$$=\sum\left\{\frac{(n+1)}{(n+2)}\right\}^n x^n=\sum u_n.$$

$$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{\frac{n}{n}}\cdot x^{n/n}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)x$$

$$=\lim_{n\to\infty}\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(1+\frac{2}{n})}x=\frac{(1+0)}{(1+0)}x=x.$$

By Cauchy's root test Series is conv. for $x<1$, div for $x>1$

For $x=1$, test fails.

For this value we have to find the nature separately.

for $x=1$, the series becomes

$$\sum\left\{\frac{n+1}{n+2}\right\}^n=\sum\frac{\cancel{n}^{n}(1+\frac{1}{n})^n}{\cancel{n}^{n}(1+\frac{2}{n})^n}$$

$$\lim_{n\to\infty} u_n=\lim_{n\to\infty}\frac{(1+\frac{1}{n})^n}{(1+\frac{2}{n})^n}$$

$$=\lim_{n\to\infty}\frac{(1+\frac{1}{n})^n}{\left\{\left(1+\frac{1}{n/2}\right)^{n/2}\right\}^2}$$

$$=\frac{e}{e^2}=\frac{1}{e}<1 \text{ Conv.}$$

$\therefore$ Series is conv for $x\le 1$

div for $x>1$

=

② $\sum\frac{[(2n+1)x]^n}{n^{n+1}}$, $x>0$.

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}}=\lim_{n\to\infty}\left\{\frac{(2n+1)^n x^n}{n^n\, n}\right\}^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{(2n+1)^{\frac{n}{n}}\cdot x^{\frac{n}{n}}}{n^{\frac{n}{n}}\cdot n^{1/n}}=\lim_{n\to\infty}\frac{\cancel{n}(2+\frac{1}{n})\cdot x}{\cancel{n}\, n^{1/n}}$$

$$=\frac{(2+0)x}{1}=2x.$$

By root test conv for $2x<1$ ie. $x<\frac{1}{2}$

div for $2x>1$ ie. $x>\frac{1}{2}$

For $2x=1$ ie. $x=\frac{1}{2}$ test fails.

$2x = 1$ ie. $x = \frac{1}{2}$ we get

$$\frac{(2n+1)^n \left(\frac{1}{2}\right)^n}{n^n \, n} \qquad \sum \frac{\cancel{2n}\left(1+\frac{1}{2n}\right)^n \frac{1}{2^n}}{\cancel{2^n} \, n}$$

$$\sum \frac{1}{n} \left\{\left\{1+\frac{1}{2n}\right\}^{2n}\right\}^{\frac{1}{2}} = \sum U_n$$

Consider the series $\sum V_n = \sum \frac{1}{n}$ which is divergent

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n}} \left\{\left(1+\frac{1}{2n}\right)^{2n}\right\}^{\frac{1}{2}}}{\cancel{\frac{1}{n}}}$$

$= e^{\frac{1}{2}} \neq 0$ $\therefore$ By quotient test $\sum U_n$ is also divergent.

Series is con for $x < \frac{1}{2}$

div for $x \geq \frac{1}{2}$.

(3) $\sum \left(\frac{n-2}{n}\right)^{n^2} x^n$, $0 < x < e^2$

$$\lim_{n\to\infty} (U_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left\{1-\frac{2}{n}\right\}^{n^2} x^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left\{1+\frac{1}{\left(-\frac{n}{2}\right)}\right\}^{\frac{n^2}{n}} x^{\cancel{n}/\cancel{n}}.$$

$$= \lim_{n\to\infty} \left[\left\{1+\frac{1}{\left(-\frac{n}{2}\right)}\right\}^{\left(-\frac{n}{2}\right)}\right]^{-2} \cdot x.$$

$$= e^{-2} x = \frac{x}{e^2}$$

Since $0 < x < e^2$, $\frac{x}{e^2} < 1$

Series is convergent for $0 < x < e^2$

(4) $\sum \frac{(x+2)^n}{3^n \cdot n}$, $x > 0$

Conv for $x < 1$ div for $x \geq 1$

(5) $\sum \left(\frac{nx}{n+1}\right)^n$, $x > 0$

Conv for $x < 1$, div for $x \geq 1$

D'Alembert's Ratio test

Consider the +ve series $\sum U_n$

Here we evaluate $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$

For $l < 1$ conv, $l > 1$ div

For $l = 1$ test fails

Note 1:- U_{n+1} is $(n+1)$th term which comes immediately after nth term. To get U_{n+1} from U_n we add coefficient of n to corresponding expression of n in U_n

Note 2:- Whenever factorial is present in U_n or many constants in U_n we use ratio test.

Note 3:- Whenever this test fails we use Raabe's test.

Raabe's test:-

Consider the +ve series $\sum U_n$

Here we evaluate

$$\lim_{n\to\infty} n\left\{\frac{U_n}{U_{n+1}} - 1\right\} = l.$$

For $l > 1$ Series is conv

For $l < 1$ Series is div

For $l = 1$ test fails

Test for convergence the following Series.

① $\frac{1}{3}+\frac{1\cdot 2}{3\cdot 5}+\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\cdots$

$$= \sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots (2n+1)} = \sum U_n$$

$$\therefore U_n = \frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots (2n+1)}$$

$$U_{n+1} = \frac{1\cdot 2\cdot 3\cdots n\cdot (n+1)}{3\cdot 5\cdot 7\cdots (2n+1)(2n+3)}$$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n}$$

$$= \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3\cdots n}(n+1)}{\cancel{3\cdot 5\cdot 7\cdots (2n+1)}(2n+3)} \times \frac{\cancel{3\cdot 5\cdot 7\cdots (2n+1)}}{\cancel{1\cdot 2\cdot 3\cdots n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(2+\frac{3}{n})} = \frac{1}{2} < 1$$

$\therefore$ By Ratio test Series is conv.

② $\sum \frac{(n+1)!}{3^n}$

$$U_n = \frac{(n+1)!}{3^n}, \quad U_{n+1} = \frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+2)!}{3^{n+1}} \times \frac{3^n}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)!}(n+2)\cdot \cancel{3^n}}{\cancel{3^n}\cdot 3\cancel{(n+1)!}} = \frac{\infty}{3} = \infty > 1$$

$\therefore$ By Ratio test Series is divergent

③ $1+\frac{1}{1!}+\frac{2^2}{2!}+\frac{3^3}{3!}+\cdots$

neglecting the first term

$$= \sum \frac{n^n}{n!}$$

$$U_n = \frac{n^n}{n!}, \quad U_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n\,\cancel{(n+1)}\cdot \cancel{n!}}{\cancel{n!}\,\cancel{(n+1)}\cdot n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1$$

$\therefore$ By ratio test Series is div.

④ $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\frac{2\cdot 5\cdot 8\cdot 11}{1\cdot 5\cdot 9\cdot 13}+\cdots$

$$= \sum \frac{2\cdot 5\cdot 8\cdots (3n-1)}{1\cdot 5\cdot 9\cdots (4n-3)} = \sum U_n$$

$$\left\{\begin{array}{l} 2\cdot 5\cdot 8\cdots, \quad a+(n-1)d = 2+(n-1)3 = 3n-1 \\ 1\cdot 5\cdot 9\cdots, \quad a+(n-1)d = 1+(n-1)4 = 4n-3 \end{array}\right\}$$

$$U_n = \frac{2\cdot 5\cdot 8\cdots (3n-1)}{1\cdot 5\cdot 9\cdots (4n-3)}$$

$$U_{n+1} = \frac{2\cdot 5\cdot 8\cdots (3n-1)(3n+2)}{1\cdot 5\cdot 9\cdots (4n-3)(4n+1)}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{3n+2}{4n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(3+\frac{2}{n})}{\cancel{n}(4+\frac{1}{n})} = \frac{3}{4} < 1$$

$\therefore$ Series is conv.

⑤ $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\cdots$

$$= \sum \left(\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots (2n+1)}\right)^2 = \sum U_n$$

$$U_n = \left\{\frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdot (2n+1)}\right\}^2, \quad U_{n+1} = \left\{\frac{1\cdot 2\cdot 3\cdots n(n+1)}{3\cdot 5\cdot 7\cdot (2n+1)(2n+3)}\right\}^2$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+3)^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^2}(2+\frac{3}{n})^2} = \frac{(1+0)^2}{(2+0)^2} = \frac{1}{4}$$

$\therefore$ By Ratio test Series is conv.

$$\frac{2}{3} + \frac{2\cdot 4}{3\cdot 5} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)} = \sum u_n.$$

$$\therefore u_n = \frac{2\cdot 4\cdot 6 \cdots 2n}{3\cdot 5\cdot 7 \cdots (2n+1)}, \quad u_{n+1} = \frac{2\cdot 4\cdot 6 \cdots 2n(2n+2)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2n+2}{2n+3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(2+\frac{2}{n}\right)}{\cancel{n}\left(2+\frac{3}{n}\right)} = \frac{2}{2} = 1$$

$\therefore$ Ratio test fails. We have to use Raabe's test.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} n\left\{\frac{2n+3}{2n+2} - 1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{2n+3-2n-2}{2n+2}\right\} = \lim_{n\to\infty} \frac{\cancel{n}(1)}{\cancel{n}\left(2+\frac{2}{n}\right)}$$

$= \frac{1}{2} < 1$ $\therefore$ By Raabe's test Series is divergent.

=

(7) $1 + 2\cdot\frac{1}{2} + \frac{2\cdot 4}{1\cdot 3}\cdot\frac{1}{3} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot\frac{1}{4} + \cdots$

neglecting the first term we get

$$\frac{2}{1}\cdot\frac{1}{2} + \frac{2\cdot 4}{1\cdot 3}\cdot\frac{1}{3} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\cdot\frac{1}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)}\cdot\frac{1}{(n+1)}$$

$$u_n = \frac{2\cdot 4\cdot 6 \cdots 2n}{1\cdot 3\cdot 5 \cdots (2n-1)}\cdot\frac{1}{n+1}$$

$$u_{n+1} = \frac{2\cdot 4\cdot 6 \cdots 2n(2n+2)}{1\cdot 3\cdot 5 \cdots (2n-1)(2n+1)}\cdot\frac{1}{n+2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(2n+2)}{(2n+1)}\cdot\frac{1}{(n+2)}\cdot\frac{n+1}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(2+\frac{2}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)}\cdot\frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{2}{n}\right)}$$

$= \frac{2\cdot 1}{2\cdot 1} = 1$ $\therefore$ Ratio test fails.

$\therefore$ We use Raabe's test.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{(2n+1)(n+2)}{(2n+2)(n+1)} - 1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{(2n+1)(n+2) - (2n+2)(n+1)}{(2n+2)(n+1)}\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{2n^2+4n+n+2 - (2n^2+2n+2n+2)}{2n^2+2n+2n+2}\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{2n^2}+5n+\cancel{2}-\cancel{2n^2}-4n-\cancel{2}}{2n^2+4n+2}\right\}$$

$$= \lim_{n\to\infty} \cancel{n}\left\{\frac{\cancel{n}}{\cancel{n^2}\left(2+\frac{4}{n}+\frac{2}{n^2}\right)}\right\}$$

$= \frac{1}{2} < 1$

$\therefore$ By Raabe's test Series is div.

=

(8) $1 + \frac{2^2}{3\cdot 4} + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} + \cdots$

neglecting the first term

$$\frac{2^2}{3\cdot 4} + \frac{2^2\cdot 4^2}{3\cdot 4\cdot 5\cdot 6} + \cdots$$

$$\begin{bmatrix} 3\cdot 5\cdot 7 \cdots & 2n+1 \\ 4\cdot 6\cdot 8 \cdots & 2n+2 \end{bmatrix}$$

$$= \sum \frac{2^2\cdot 4^2 \cdots (2n)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+1)(2n+2)} = \sum u_n$$

$$u_{n+1} = \frac{2^2\cdot 4^2 \cdots (2n)^2\cdot(2n+2)^2}{3\cdot 4\cdot 5\cdot 6 \cdots (2n+1)(2n+2)(2n+3)(2n+4)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+3)(2n+4)}$$

$$= \lim_{n\to\infty} \frac{n^2\left(2+2/n\right)^2}{n\left(2+\frac{3}{n}\right)n\left(2+\frac{4}{n}\right)} = \frac{2^2}{2\cdot 2} = 1$$

$\therefore$ Ratio test fails.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{4n^2}+\cancel{14n}+6n+12-\cancel{4n^2}-\cancel{8n}-4}{4n^2+4n+1}\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{6n+8}{4n^2+4n+1}\right\}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\,\cancel{n}\left(6+8/n\right)}{\cancel{n^2}\left(4+\frac{4}{n}+\frac{4}{n^2}\right)} = \frac{6}{4} > 1$$

By Raabe's test Series is conv

Find the nature of the series for $x>0$

$$\frac{x}{1+\sqrt{1}}+\frac{x^2}{2+\sqrt{2}}+\frac{x^3}{3+\sqrt{3}}+\cdots$$

$$= \sum \frac{x^n}{n+\sqrt{n}} = \sum u_n.$$

$$u_n = \frac{x^n}{n+\sqrt{n}}, \quad u_{n+1} = \frac{x^{n+1}}{(n+1)+\sqrt{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{(n+1)+\sqrt{n+1}} \times \frac{n+\sqrt{n}}{x^n}$$

$$= \lim_{n\to\infty}\frac{\cancel{x^n}\, x \cdot \cancel{n}\left(1+\frac{1}{\sqrt{n}}\right)}{\cancel{n}\left\{\left(1+\frac{1}{n}\right)+\sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)\right\}\cdot \cancel{x^n}}$$

$$= \frac{1}{(1+0)+0(1)}\, x = x.$$

By Ratio test Series is conv for $x<1$, div for $x>1$

Test fails for $x=1$.

for $x=1$ we get $\sum \frac{1^n}{n+\sqrt{n}} = \sum \frac{1}{n+\sqrt{n}}$

We can use Raabe's test or Quotient test

$$= \sum \frac{1}{n\left(1+\frac{1}{\sqrt{n}}\right)} = \sum u_n$$

consider $\sum v_n = \sum \frac{1}{n}$ which is conv

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{\cancel{n}\left(1+\frac{1}{\sqrt{n}}\right)}}{\frac{1}{\cancel{n}}} = 1 \neq 0$$

$\therefore \sum u_n$ is also conv by quotient test

$\therefore$ Series is conv for $x<1$, div for $x \geq 1$

(2) $\sum \frac{(3x+5)^n}{(n+1)!}$

$$u_n = \frac{(3x+5)^n}{(n+1)!}, \quad u_{n+1} = \frac{(3x+5)^{n+1}}{(n+2)!}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(3x+5)^{n+1}}{(n+2)!}\times\frac{(n+1)!}{(3x+5)^n}$$

$$= \lim_{n\to\infty}\frac{\cancel{(3x+5)^n}(3x+5)\cdot\cancel{(n+1)!}}{\cancel{(n+1)!}\,(n+2)\,\cancel{(3x+5)^n}}$$

$$= \frac{3x+5}{\infty} = 0 < 1$$

$\therefore$ By Ratio test conv for all $x>0$

(3) $\sum \frac{1}{x^n+x^{-n}}$

$$u_n = \frac{1}{x^n+x^{-n}}, \quad u_{n+1} = \frac{1}{x^{n+1}+x^{-n-1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1}{x^{n+1}+x^{-n-1}}\times\frac{x^n+x^{-n}}{1}$$

$$= \lim_{n\to\infty}\frac{\cancel{x^n}\left(1+x^{-2n}\right)}{\cancel{x^n}\left(x+x^{-2n-1}\right)}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{x^{2n}}\right)}{\left(x+\frac{1}{x^{2n+1}}\right)} = \frac{1}{x} < 1 \text{ if } x>1$$

$\therefore$ Series is conv

For $x<1$, $\lim_{n\to\infty} x^n = 0$

$$\therefore u_n = \frac{1}{x^n + x^{-n}} = \frac{1}{x^n + \frac{1}{x^n}} = \frac{x^n}{x^{2n}+1}$$

$$u_{n+1} = \frac{x^{n+1}}{x^{2n+2}+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{x^{2n+2}+1} \times \frac{x^{2n}+1}{x^n}$$

$$= \lim_{n\to\infty} \frac{x \cdot \cancel{x^n} \cdot (x^{2n}+1)}{\cancel{x^n}(x^{2n+2}+1)}$$

$$= \frac{x(0+1)}{(0+1)} = x < 1$$

$\therefore$ conv.

For $x=1$, $\sum \frac{1}{1^n+1^{-n}} = \sum \frac{1}{2} = \sum u_n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2} \neq 0$$

$\therefore$ Series is div.

(4) $\sum \frac{(n!)^2}{(2n)!} x^{2n}$.

$$u_n = \frac{(n!)^2}{(2n)!} x^{2n}, \quad u_{n+1} = \frac{\{(n+1)!\}^2}{(2n+2)!} x^{(2n+2)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\cancel{(n!)^2}(n+1)^2 \cancel{x^{2n}} \cdot x^2}{\cancel{(2n)!}(2n+1)(2n+2)} \times \frac{\cancel{2n!}}{\cancel{(n!)^2}\cancel{x^{2n}}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2 \cdot x^2}{\cancel{n}(2+\frac{1}{n})\cancel{n}(2+\frac{2}{n})} = \frac{x^2}{4}$$

By Ratio test Series is conv for $\frac{x^2}{4} < 1$ ie $x^2 < 4$

div for $\frac{x^2}{4} > 1$ ie. $x^2 > 4$

for $\frac{x^2}{4} = 1$ ie. $x^2 = 4$ test fails.

for $x^2 = 4$ we get $\sum \frac{(n!)^2}{(2n)!} (4)^n$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^2 \cdot 4}{(2n+1)(2n+2)}$$

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} n\left\{\frac{(2n+1)(2n+2)}{4(n+1)^2} - 1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{4n^2}+4n+2n+2-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{-2n-2}{4n^2+8n+4}\right\}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(-2-\frac{2}{n})}{\cancel{n^2}(4+\frac{8}{n}+\frac{4}{n^2})}$$

$$= -\frac{2}{4} = -\frac{1}{2} < 1$$

$\therefore$ By Raabe's test Series is div.

(5) $1 + \frac{a}{b}x + \frac{a(a+1)}{b(b+1)}x^2 + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^3 + \dots$

neglecting the first term.

$$= \sum \frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)} x^n$$

$$u_n = \frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)(b+2)\dots(b+n-1)} x^n$$

$$u_{n+1} = \frac{a(a+1)(a+2)\dots(a+n-1)(a+n)}{b(b+1)(b+2)\dots(b+n-1)(b+n)} x^{n+1}$$

$$\lim_{n\to\infty} \frac{(a+n)}{(b+n)} x = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(\frac{a}{n}+1)}{\cancel{n}(\frac{b}{n}+1)} x = x.$$

For $x<1$ Series is conv

$x>1$. series is div by Ratio test

for $x=1$ test fails

we have to use Raabe's test.

For $x=1$, $\frac{u_{n+1}}{u_n} = \frac{a+n}{b+n}$.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{b+n}{a+n}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{b+\cancel{n}-a-\cancel{n}}{a+n}\right\}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(b-a)}{\cancel{n}(\frac{a}{n}+1)} = \underline{\underline{b-a}}$$

By Raabe's test conv for $b-a>1$

ie $b>a+1$

div for $b-a<1$ ie. $b<a+1$

Test fails for $b=a+1$.

$\therefore$ the series for $b=a+1$ and $x=1$ is

(6) $$\sum \frac{a(a+1)(a+2)\cdots(a+n-1)}{(a+1)(a+2)\cdots(a+n)} = \sum \frac{a}{a+n}$$

$$= \sum \frac{a}{n(\frac{a}{n}+1)} = \sum a_n$$

Consider $\sum v_n = \sum \frac{1}{n}$. which is div

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{a}{\cancel{n}(\frac{a}{n}+1)}}{\frac{1}{\cancel{n}}} = a \neq 0$$

$\therefore$ By quotient test $\sum u_n$ is also div.

$$= 1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+\cdots$$

$$= \sum \frac{3\cdot 6\cdot 9\cdots 3n}{7\cdot 10\cdot 13\cdots(3n+4)} x^n.$$

$\{7, 10, 13,$ $\quad a+(n-1)d = 7+(n-1)3 = 3n+4$

$$u_n = \frac{3\cdot 6\cdot 9\cdots 3n}{7\cdot 10\cdot 13\cdots(3n+4)} x^n$$

$$u_{n+1} = \frac{3\cdot 6\cdot 9\cdots 3n(3n+3)}{7\cdot 10\cdot 13\cdots(3n+4)(3n+7)} x^{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3n+3}{3n+7}\cdot x$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(3+\frac{3}{n})}{\cancel{n}(3+7/n)} x = \frac{3x}{3} = x$$

By ratio test conv for $x<1$
div for $x>1$

For $x=1$ test fails

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{3n+7}{3n+3}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}\right\}$$

$$= \lim_{n\to\infty} \frac{n(4)}{n(3+3/n)} = \frac{4}{3} > 1$$

$\therefore$ By Raabe's test series is conv

$\therefore$ conv for $x \le 1$, div for $x>1$

$$= \frac{x}{1\cdot 2}+\frac{x^2}{3\cdot 4}+\frac{x^3}{5\cdot 6}+\cdots$$

$$= \sum \frac{x^n}{(2n-1)\cdot(2n)} = \sum u_n$$

$$u_n = \frac{x^n}{(2n-1)(2n)}, \quad u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\cancel{x^n}\cdot x}{(2n+1)(2n+2)} \frac{(2n-1)(2n)}{\cancel{x^n}}$$

$$= \lim_{n\to\infty} x \frac{\cancel{n}(2-\frac{1}{n})2\cancel{n}}{\cancel{n}(2+\frac{1}{n})\cdot\cancel{n}(2+\frac{2}{n})} = \frac{4x}{4} = x$$

By ratio test conv for $x<1$, div for $x>1$

For $x=1$ test fails.

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{(2n+1)(2n+2)}{(2n-1)(2n)}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{4n^2}+4n+2n+2-\cancel{4n^2}+2n}{4n^2-2n}\right\}$$

$$= \lim_{n\to\infty} \frac{n(6n+2)}{4n^2-2n} = \lim_{n\to\infty} \frac{\cancel{n}(6+\frac{2}{n})}{\cancel{n}(4-\frac{2}{n})} = \frac{6}{4} > 1$$

Conv. $\therefore$ conv for $x \le 1$, div for $x>1$

$$1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^4}{8}+\frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10}\cdot\frac{x^6}{12}+\cdots$$

neglecting the first term.

1, 5, 9 . . . n^{th} term $a+(n-1)d = 1+(n-1)4 = 4n-3$.

3, 7, 11 . . . $(n-1)^{th}$ term $a+(n-2)d = 3+(n-2)4 = 4n-5$

2, 6, 10 . . . n^{th} term $2+(n-1)d = 2+(n-1)4 = 4n-2$.

4, 8, 12 . . . $(n-1)^{th}$ term $a+(n-2)d = 4+(n-2)4 = 4n-4$

$$\therefore = \sum \frac{1\cdot3\cdot5\cdot7\cdot9\cdots(4n-5)(4n-3)}{2\cdot4\cdot6\cdot8\cdot10\cdots(4n-4)(4n-2)}\frac{x^{2n}}{4n} = \sum u_n$$

$$\therefore u_{n+1} = \frac{1\cdot3\cdot5\cdot7\cdot9\cdots(4n-5)(4n-3)(4n-1)(4n+1)x^{2n+2}}{2\cdot4\cdot6\cdot8\cdot10\cdots(4n-4)(4n-2)4n(4n+2)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{(4n-1)(4n+1)}{4n(4n+2)}\frac{\cancel{x^{2n}}\cdot x^2}{\cancel{x^{2n}}}$$

$$= \lim_{n\to\infty}\frac{\cancel{n}(4-\frac{1}{n})\cancel{4}(4+\frac{1}{n})x}{\cancel{4}\cancel{n}(4+\frac{2}{n})} = \frac{4\cdot4\cdot x}{4\cdot4} = x$$

By Ratio test Series is conv for $x<1$, div for $x>1$

For $x=1$, Ratio test fails.

For $x=1$, $\frac{u_{n+1}}{u_n} = \frac{(4n-1)(4n+1)}{4n(4n+2)}$

$$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}}-1\right\} = \lim_{n\to\infty} n\left\{\frac{4n(4n+2)}{(4n-1)(4n+1)}-1\right\}$$

$$= \lim_{n\to\infty} n\left\{\frac{\cancel{16n^2}+8n-\cancel{16n^2}+1}{16n^2-1}\right\}$$

$$= \lim_{n\to\infty}\frac{\cancel{n}\cdot\cancel{n}(8-\frac{1}{n})}{\cancel{n^2}(16-\frac{1}{n^2})} = \frac{8}{16} = \frac{1}{2} < 1$$

By Raabs test Series is div.

$\therefore$ conv for $x<1$, div for $x\geq1$

Procedure to select a test for a +ve Series.

(1) If the function $f(n)$ is integrable use integral test {usually when we get $\frac{1}{n^p}$, log function or exponential function}

(2) After simplication of u_n a part is a p-Series or r-Series use quotient test.

(3) If the nth term is not tending to zero as $n\to\infty$ use the result that if $\lim_{n\to\infty} u_n \neq 0$ then Series is div.

(4) If the nth root is reducible to standard limit use cauchy's root test.

(5). If factorial or many constants are present in n^{th} term use. Ratio test if it fails use Raabe's test

Alternating Series:-

Series of the form

$a_1 - a_2 + a_3 - a_4 + \cdots$

$= \sum(-1)^{n-1}a_n$ where $a_n>0$ is called an alternating Series.

An alternating Series is either convergent or oscillatory.

A Series is said to be oscillatory if $\lim_{n\to\infty} S_n$ is more than one value.

Ex: $\sum(-1)^{n-1} = 1-1+1-1+1-1+\cdots$

$$S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.}\end{cases}$$

$$\lim_{n\to\infty} S_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ the Series is oscillatory.

Leibnitz's test :-

An alternating Series $\sum(-1)^{n-1}a_n$ or $\sum(-1)^{n+1}a_n$ or $\sum(-1)^n a_n$ will be convergent if the following two conditions are satisfied

① $a_{n+1} \le a_n$ for all $n \ge m$.

② $\lim_{n\to\infty} a_n = 0$.

Note:- ① To check for first condition we take reciprocal if numerator is just a constant.

② If numerator is also a function in n we consider $\frac{a_{n+1}}{a_n}$ or $a_{n+1} - a_n$.

Find the nature of the following series.

① $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

$= \sum(-1)^{n-1}\frac{1}{n^2} = \sum(-1)^{n-1}a_n$.

$a_n = \frac{1}{n^2}$, $a_{n+1} = \frac{1}{(n+1)^2}$.

W.K.T. $(n+1)^2 > n^2$

Taking reciprocal

$\frac{1}{(n+1)^2} < \frac{1}{n^2}$ i.e. $a_{n+1} < a_n$ for all n.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2} = \frac{1}{\infty} = 0$

$\therefore$ By Leibnitz's test Series is conv.

② $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$

$= \sum(-1)^{n-1}\frac{1}{n\sqrt{n}} = \sum(-1)^{n-1}a_n$

$a_n = \frac{1}{n\sqrt{n}}$, $a_{n+1} = \frac{1}{(n+1)\sqrt{n+1}}$.

W.K.T. $n+1 > n$, $\sqrt{n+1} > \sqrt{n}$

multiplying we get

$(n+1)\sqrt{n+1} > n\sqrt{n}$, Taking reciprocal

$\frac{1}{(n+1)\sqrt{n+1}} < \frac{1}{n\sqrt{n}}$ i.e. $a_{n+1} < a_n$ for all n.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n\sqrt{n}} = \frac{1}{\infty} = 0$

$\therefore$ By Leibnitz's test Series is conv.

③ $\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \cdots$

$= \sum(-1)^{n-1}\frac{(2n+3)}{2n} = \sum(-1)^{n-1}a_n$.

$a_n = \frac{2n+3}{2n}$, $a_{n+1} = \frac{2n+5}{2n+2}$.

$\frac{a_{n+1}}{a_n} = \frac{2n+5}{2n+2} \times \frac{2n}{2n+3}$

$\frac{a_{n+1}}{a_n} = \frac{4n^2+10n}{4n^2+6n+10n+6}$

$\frac{a_{n+1}}{a_n} = \frac{4n^2+10n}{4n^2+10n+6} < 1$ for $n \ge 1$

$\therefore a_{n+1} < a_n$. But $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n+3}{2n}$

$= \lim_{n\to\infty}\frac{n(2+3/n)}{2n} = 1 \ne 0$

$\therefore$ Series is oscillatory.

(4) $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - + \ldots$

$= \sum (-1)^{n-1} \frac{(1+2+3+\ldots+n)}{(n+1)^3} = \sum (-1)^{n-1} \frac{n(n+1)}{2(n+1)^3}$

$\sum (-1)^{n-1} \frac{n}{2(n+1)^2} = \sum (-1)^{n-1} a_n$

[Because $1+2+3+\ldots+n = \frac{n(n+1)}{2}$]

$a_n = \frac{n}{2(n+1)^2}$, $a_{n+1} = \frac{n+1}{2(n+2)^2}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{2(n+2)^2} \times \frac{2(n+1)^2}{n}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(n+2)^2 n} = \frac{n^3+3n^2+3n+1}{n^3+4n^2+4n}$

$\frac{a_{n+1}}{a_n} = \frac{n^3+3n^2+3n+1}{n^3+3n^2+3n+n^2+n}$

1 is less than n^2+n for all $n \geq 1$

$\therefore \frac{a_{n+1}}{a_n} < 1$ i.e. $a_{n+1} < a_n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2(n+1)^2}$

$= \lim_{n\to\infty} \frac{n}{2n^2(1+\frac{1}{n})^2} = \frac{1}{\infty} = 0$

By Leibniz's test Series is conv.

(5) $\sum \frac{\cos n\pi}{n^2+1}$ But $\cos n\pi = (-1)^n$

$= \sum \frac{(-1)^n}{n^2+1} = \sum (-1)^n a_n$

$a_n = \frac{1}{n^2+1}$, $a_{n+1} = \frac{1}{(n+1)^2+1}$

clearly $(n+1)^2+1 > n^2+1$

$\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$ i.e. $a_{n+1} < a_n$ for all $n \geq 1$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$

$\therefore$ By leibniz's test Series is conv

(6) $\sum \frac{(-1)^{n-1} n}{n^2+1}$ conv.

(7) $\sum \frac{(-1)^{n-1} n}{2n-1}$ oscillatory

(8) $\sum \frac{(-1)^{n-1}}{1+\log n}$ conv.

ABSOLUTE AND CONDITIONAL CONVERGENCE

Consider the Series $\sum u_n$ which is either +ve or alternating or -ve Series.

(i) The Series is said to be absolutely conv if $\sum |u_n|$ is conv

(ii) The Series is said to be conditionaly conv if $\sum |u_n|$ is div but $\sum u_n$ is conv.

procedure :-

1. First consider $\sum |u_n|$ which is a +ve Series.
2. Using the respective test find the nature. If it is conv the Series is abs. conv.
3. If $\sum |u_n|$ is div then using leibniz's test find the nature of $\sum u_n$ to check for conditional convergence