Fourier Series

For the function $f(x)$ defined in $(-\pi,\pi)$ or $(0,2\pi)$ the Fourier Series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$ where

a_0, a_n, b_n are constants called as Fourier coefficients

If $f(x)$ is defined in $(-\pi,\pi)$ and $f(-x) = f(x)$, then the function is said to be an even function

Here $b_n = 0$, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

If $f(x)$ is defined in $(-\pi,\pi)$ and $f(-x) = -f(x)$ then the function is said to be an odd function

Here $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

If $f(x)$ is defined in $(-\pi,\pi)$ and $f(-x)$ neither equal to $f(x)$ nor equal to $-f(x)$ then the function is neither even nor odd. In such cases we have to find all the fourier coefficients. They are given by $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$.

If $f(x)$ is defined in $(0,2\pi)$ then we cannot check for odd or even functions. Here also we

[illegible] fourier coefficients. where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Note (1): $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\}$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\}$.

(2) Bernoulli's rule: If u is a polynomial and v is an integrable function then

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

where dashes denote differentiation and suffixes 1, 2, 3 denote integration.

Here we have to write u, u', u'', in brakets and v_1, v_2, v_3 in brakets. Because to get u' we have to differentiate u entirely, to get u'' we have to differentiate u' entirely and so on. To get v_2, we have to integrate v_1 entirely and so on.

(3) After integration while evaluating the limits we cancel the terms containing sine functions if the limits are 0 to π or $-\pi$ to π or 0 to 2π or 0 to 1 or -1 to 1 or 0 to $2l$ because the

∴ Value of Sine functions is zero at these points

i.e. $\left\{ \cancel{f(x)\sin nx} \right\}_{0\,(0)\,(-\pi)}^{\pi\,(2\pi)\,(\pi)} = 0$

4) $\cos 0 = 1$, $\cos n\pi = (-1)^n$, $\cos 2n\pi = (-1)^{2n} = 1$, $\cos \pi = -1$

$\cos n\pi = \cos(-n\pi)$.

5) The particular numerical series can be deduced from the Fourier Series giving particular values for x like $0, \pi, \pi/2$,

① Find the Fourier Series of the function $f(x) = x^2$ over the interval $(-\pi, \pi)$. Deduce that (a) $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ or $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

(b) $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ or $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \cdots$

Sol:- Since the function is defined in $(-\pi, \pi)$ we can check for odd or even function.

$f(-x) = (-x)^2 = x^2 = f(x)$ ∴ $f(x)$ is an even function

∴ $b_n = 0$ $\quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3\pi}\left[\pi^3 - 0^3\right] = \frac{2\pi^2}{3\pi} = \frac{2\pi^2}{3}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi}\left[\cancel{x^2\left\{\frac{\sin nx}{n}\right\}}^{0} - (2x)\left\{-\frac{\cos nx}{n^2}\right\} + 2\left\{-\frac{\sin nx}{n^3}\right\}\right]_0^{\pi}$

$= \frac{2}{\pi}\left(\frac{2}{n^2}\right)\left\{x\cos nx\right\}_0^{\pi} = \frac{4}{\pi n^2}\left\{\pi\cos n\pi - 0\cos 0\right\}$

$= \frac{4}{\cancel{\pi} n^2}\cancel{\pi}(-1)^n = \frac{4(-1)^n}{n^2}$

Note:- To deduce numerical series from Fourier series we have to give particular values for x like $x = 0, \pi, \pi/2$, To deduce from cosine series we use '0' and π and to deduce from sine series we use $\pi/2$.

The Fourier Series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$f(x) = \frac{\frac{\cancel{2}\pi^2}{3}}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} 0 \sin nx$.

$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$ — ①

for $x = \pi$ $\quad f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2}$

$f(\pi) = (x^2)_{x=\pi} + (x^2)_{x=-\pi}$

$= \pi^2 + (-\pi)^2 = 2\pi^2$. $\quad \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$

i.e. $\frac{3\pi^2 - \pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$ i.e. $\frac{2\pi^2}{3 \times 4} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$

But $(-1)^{2n}=1$ $\therefore$ $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

putting $x=0$ in ①, we get.

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0$$

ie $-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ multiplying by $-\frac{1}{4}$,

we get $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

② Expand $f(x)=x$ in a Fourier Series in the interval $-\pi<x<x$ and Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots$ or $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

$\left[b_n = \frac{2(-1)^{n+1}}{n} \right.$ Hint:- For deduction put $x=\frac{\pi}{2}$ $\left.\right]$

③ Find the Fourier series expansion of the function $f(x)=e^{-ax}$ for $-\pi<x<\pi$.

Sol:- $f(-x) = e^{-a(-x)} = e^{ax} \neq f(x)$ or $\neq -f(x)$

ie. $f(x)$ is neither even nor odd. Hence we have to find out all the constants.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right] = \frac{e^{a\pi}-e^{-a\pi}}{a\pi} = \frac{2\sinh a\pi}{a\pi}$$

$$\left\{ \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \right\}$$

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx\,dx$$

$$\left[\int e^{Ax}\cos Bx\,dx = \frac{e^{Ax}}{A^2+B^2}\{A\cos Bx + B\sin Bx\} \right]$$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\{-a\cos nx + \cancel{n\sin nx}^{\,0}\}\right]_{-\pi}^{\pi}$$

$$a_n = \frac{-a}{(a^2+n^2)\pi}\left\{e^{-ax}\cos nx\right\}_{-\pi}^{\pi}$$

$$= \frac{-a}{(a^2+n^2)\pi}\left\{e^{-a\pi}\cos n\pi - e^{a\pi}\cos(-n\pi)\right\}$$ But $\cos n\pi = \cos(-n\pi)$

$$= \frac{a\cos n\pi\{e^{a\pi}-e^{-a\pi}\}}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx$$

$$\left[\int e^{Ax}\sin Bx\,dx = \frac{e^{Ax}}{A^2+B^2}\{A\sin Bx - B\cos Bx\} \right]$$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}\{\cancel{-a\sin nx}^{\,0} - n\cos nx\}\right]_{-\pi}^{\pi} = \frac{-n\{e^{-ax}\cos nx\}_{-\pi}^{\pi}}{\pi(a^2+n^2)}$$

$$= \frac{2n\sinh a\pi\,(-1)^n}{\pi(a^2+n^2)}$$. Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx.$$

$$f(x) = \frac{\cancel{2}\sinh a\pi}{\cancel{2}\cdot a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx$$

4) obtain the Fourier Series in $(-\pi, \pi)$ for $f(x) = x\cos x$ [VTU Aug 2010]

Sol:- $f(-x) = -x.\cos(-x) = -x\cos x = -f(x)$

$\therefore f(x)$ is an odd function. Hence $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x.\sin nx.\cos x\,dx.$$

$$\left[\sin A\cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}\right]$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\left\{\frac{1}{2}(\sin(nx+x) + \sin(nx-x))\right\}dx$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin\{(n+1)x\}\,dx + \int_0^{\pi} x\sin(n-1)x\,dx\right] \quad -(1)$$

$$b_n = \frac{1}{\pi}\left[x.\left\{\frac{-\cos(n+1)x}{(n+1)}\right\} - \cancel{1\left\{-\frac{\sin(n+1)x}{(n+1)^2}\right\}}^{\,0}\right.$$

$$\left. + x\left\{\frac{-\cos(n-1)x}{(n-1)}\right\} - \cancel{1\left\{\frac{-\sin(n-1)x}{(n-1)^2}\right\}}^{\,0}\right]_0^{\pi} \quad \text{for } n \neq 1$$

$$\left[\text{Because for } n=1,\ \frac{1}{n-1} = \frac{1}{0} = \infty\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{(n+1)}\left(x\cos(n+1)x\right)_0^{\pi} - \frac{1}{(n-1)}\left(x\cos(n-1)x\right)_0^{\pi}\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{(n+1)}\{\pi\cos(n+1)\pi - 0.\cos 0\} - \frac{1}{(n-1)}\{\pi\cos(n-1)\pi - 0.\cos 0\}\right]$$

$$b_n = \frac{1}{\pi}\left[-\frac{1}{(n+1)}.\pi.(-1)^{n+1} - \frac{1}{(n-1)}\pi(-1)^{n-1}\right]$$

$$b_n = \frac{\pi}{\pi}\left[-\frac{(-1)^n(-1)^1}{n+1} - \frac{(-1)^n(-1)^{-1}}{(n-1)}\right] \quad \text{But } (-1)^1 = (-1)^{-1} = -1$$

$$b_n = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = (-1)^n\left[\frac{\cancel{n} - \cancel{1} + \cancel{n} + \cancel{1}}{(n+1)(n-1)}\right]$$

$$\therefore b_n = \frac{2n(-1)^n}{n^2-1} \quad \text{for } n \neq 1$$

$$\text{for } n=1,\ b_1 = \frac{1}{\pi}\left[\int_0^{\pi} x\sin 2x\,dx + \cancel{\int_0^{\pi} x\sin 0\,dx}^{\,0}\right]$$

[we get the above integrals by putting $n=1$ in (1)]

$$b_1 = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)\Big|_0^{\pi} - \cancel{1\left\{-\frac{\sin 2x}{2^2}\right\}_0^{\pi}}^{\,0}\right]$$

$$b_1 = -\frac{1}{2\pi}\left[x\cos 2x\right]_0^{\pi} = -\frac{1}{2\pi}\left[\pi\cos 2\pi - \pi\cos 0\right]$$

$$b_1 = -\frac{\pi}{2\pi}[1-1] = 0$$

The Fourier Series is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + b_1\sin x + \sum_{n=2}^{\infty} b_n\sin nx.$$

$$f(x) = \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

5) Find the Fourier Series of $f(x) = |x|$ in $-\pi < x < \pi$ and hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol: $f(x) = |-x| = |x| = f(x)$ $\therefore$ function is even

hence $b_n = 0$

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} x\,dx$ Because $|x| = x$ if x is +ve i.e. for $x \in (0, \pi)$

$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi} \left[\pi^2 - 0\right] = \frac{\pi^2}{\pi} = \pi.$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx$

$= \frac{2}{\pi} \left[\cancel{x \left\{\frac{\sin nx}{n}\right\}}^{\,0} - 1 \left\{\frac{-\cos nx}{n^2}\right\} \right]_0^{\pi} = \frac{2}{\pi n^2} (\cos nx)_0^{\pi}$

$= \frac{2}{\pi n^2} (\cos n\pi - \cos 0) = \frac{2\{(-1)^n - 1\}}{\pi n^2}$

Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\{(-1)^n - 1\}}{\pi n^2} \cos nx.$

for $x = 0$, $f(0) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\{(-1)^n - 1\}}{n^2} \cos 0$

But $f(0) = |0| = 0$

$(-1)^n - 1 = \begin{cases} -1-1 = -2 & \text{if } n \text{ is odd i.e. } n \text{ is of the form } 2n-1 \\ 0 & \text{if } n \text{ is even.} \end{cases}$

$0 = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-2)}{(2n-1)^2}$

i.e. $-\frac{\pi}{2} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$ multiplying by $-\frac{\pi}{4}$.

$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

6) Expand $f(x) = x(2\pi - x)$ in a fourier series for x such that $0 < x < 2\pi$.

$f(x) = x(2\pi - x) = (2\pi x - x^2)$ which is defined in $(0, 2\pi)$.

$\therefore$ We cannot check for odd or even function.

$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2)\,dx.$

$= \frac{1}{\pi} \left[\frac{\cancel{2}\pi \cdot x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi} \left[\left(\pi \cdot 4\pi^2 - \frac{8\pi^3}{3}\right) - 0\right]$

$= \frac{1}{\pi} \left[4\pi^3 - \frac{8\pi^3}{3}\right] = \frac{1}{\pi} \left[\frac{12\pi^3 - 8\pi^3}{3}\right] = \frac{1}{\pi} \left[\frac{4\pi^3}{3}\right] = \frac{4\pi^2}{3}$

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{2\pi} (2\pi x - x^2) \cos nx\,dx$

$= \frac{1}{\pi} \left[\cancel{(2\pi x - x^2) \left\{\frac{\sin nx}{n}\right\}}^{\,0} - (2\pi - 2x) \left\{\frac{-\cos nx}{n^2}\right\} - \cancel{(-2) \left\{\frac{-\sin nx}{n^3}\right\}}^{\,0} \right]_0^{2\pi}$

$= \frac{1}{\pi} \left(\frac{1}{n^2}\right) \left[(2\pi - 2x) \cos nx\right]_0^{2\pi} = \frac{2}{\pi n^2} \left[(2\pi - 4\pi) \cos 2n\pi - (2\pi - 0) \cos 0\right]$

But $\cos 0 = \cos 2n\pi = 1$

$$\therefore a_n = \frac{1}{\pi n^2}\left[-2\pi - 2\pi\right] = \frac{1}{\pi n^2}(-4\pi) = -\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left\{\frac{-\cos nx}{n}\right\} - \cancel{(2\pi - 2x)\left\{\frac{-\sin nx}{n^2}\right\}}^{\,0} - (-2)\left\{\frac{\cos nx}{n^3}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{(2\pi x - x^2)\cos nx\right\}_0^{2\pi} + \frac{2}{n^3}(\cos nx)_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\left\{(4\pi^2 - 4\pi^2)\cos 2n\pi - 0\cdot\cos 0\right\} + \frac{2}{n^3}\left\{\cos 2n\pi - \cos 0\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(0-0) + \frac{2}{n^3}(1-1)\right] = 0$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2}\right)\cos nx$$

(7) Find the Fourier Series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{(\pi - x)^2}{2^2} = \frac{1}{4}(x-\pi)^2$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(x-\pi)^2\,dx = \frac{1}{4\pi}\left[\frac{(x-\pi)^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(2\pi-\pi)^3 - (-\pi)^3\right] = \frac{1}{12\pi}\left[\pi^3 + \pi^3\right] = \frac{\cancel{2}^{1}\pi^3}{\cancel{12}_{6}\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(x-\pi)^2\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\cancel{(x-\pi)^2\left\{\frac{\sin nx}{n}\right\}}^{\,0} - 2(x-\pi)\left\{\frac{-\cos nx}{n^2}\right\} + \cancel{2\left\{\frac{-\sin nx}{n^3}\right\}}^{\,0}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{2}{n^2}\right)\left\{(x-\pi)\cos nx\right\}\right]_0^{2\pi} = \frac{1}{2\pi n^2}\left[(2\pi-\pi)\cos 2n\pi - (-\pi)\cos 0\right]$$

$$= \frac{1}{2\pi n^2}\left[\pi + \pi\right] = \frac{2\pi}{2\pi n^2} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}(x-\pi)^2\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(x-\pi)^2\left\{-\frac{\cos nx}{n}\right\}_0^{2\pi} - \cancel{2(x-\pi)\left\{\frac{-\sin nx}{n^2}\right\}}^{\,0} + 2\left\{\frac{\cos nx}{n^3}\right\}_0^{2\pi}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}\left\{(x-\pi)^2\cos nx\right\}_0^{2\pi} + \frac{2}{n^3}\left\{\cos nx\right\}_0^{2\pi}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}\left\{(2\pi-\pi)^2\cos 2n\pi - (-\pi)^2\cos 0\right\} + \frac{2}{n^3}\left\{\cos 2n\pi - \cos 0\right\}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{1}{n}\left\{\pi^2 - \pi^2\right\} + \frac{2}{n^3}\left\{1-1\right\}\right] = 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx.$$

Note:- To check for odd or evenness of the splitting function $f(x) = \begin{cases} f_1(x) & \text{in } -\pi < x < 0 \\ f_2(x) & \text{in } 0 < x < \pi \end{cases}$,

we start with one of the expression. After changing x to $-x$, we compare the resultant with other.

(6) Find the Fourier Series of $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x < \pi \end{cases}$

and hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

In $-\pi < x \le 0$, $f(x) = 1 + \frac{2x}{\pi}$

$f(-x) = 1 - \frac{2x}{\pi} = f(x)$ in $0 \le x < \pi$.

Hence $f(x)$ is an even function $\therefore b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{2}{\pi}\cdot\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi - \frac{\pi^2}{\pi}\right) - (0 - 0)\right] = \frac{2}{\pi}\left[\pi - \pi\right] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\cancel{\left(1 - \frac{2x}{\pi}\right)\left\{\frac{\sin nx}{n}\right\}}^{\,0} - \left(-\frac{2}{\pi}\right)\left\{-\frac{\cos nx}{n^2}\right\}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-2}{\pi n^2}\right](\cos nx)_0^{\pi} = -\frac{4}{\pi^2 n^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right]$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ (7)

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left[1 - (-1)^n\right]\cos nx.$$

But $1 - (-1)^n = \begin{cases} 1 - (-1) = 1 + 1 = 2 & \text{if } n \text{ is odd i.e. } 2n-1 \\ 1 - 1 = 0 & \text{if } n \text{ is even i.e. } 2n \end{cases}$

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{2}{(2n-1)^2}\cos(2n-1)x.$$

putting $x = 0$, $f(0) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 0}{(2n-1)^2}$

But $f(0) = \left(1 + \frac{2x}{\pi}\right)_{x=0} = 1$

$\therefore 1 = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ i.e. $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

(4) Find the Fourier expansion of the function

$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } 0 < x < \pi \end{cases}$

In $-\pi < x < 0$, $f(x) = -K$

$f(-x) = -K = -f(x)$ in $0 < x < \pi$

$\therefore f(x)$ is an odd function. Hence $a_0 = 0, a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} K\sin nx\,dx$$

$$= \frac{2K}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{2K}{n\pi}\left[\cos nx\right]_0^{\pi}$$

$$= -\frac{2K}{n\pi}\left[\cos n\pi - \cos 0\right] = \frac{2K\left[1 - (-1)^n\right]}{n\pi}$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

$$f(x) = \sum_{n=1}^{\infty} \frac{2K}{n\pi}\{1-(-1)^n\} \sin nx.$$

⑦ Obtain the Fourier expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x \le \pi \end{cases}$$

and deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Though the function is defined in $(-\pi, \pi)$ it is neither even nor odd. Hence we have to find all the constants.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi x\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi(0-(-\pi)) + \frac{1}{2}(\pi^2 - 0)\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{-\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\cancel{-\pi\left\{\frac{+\sin nx}{n}\right\}_{-\pi}^{0}}^{\,0} + \cancel{x\left\{\frac{\sin nx}{n}\right\}_0^{\pi}}^{\,0} - 1\left\{\frac{-\cos nx}{n^2}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi^2}\left[\cos nx\right]_0^{\pi} = \frac{1}{\pi n^2}\left[\cos n\pi - \cos 0\right] = \frac{(-1)^n - 1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

⑤

$$b_n = \frac{1}{\pi}\left[(-\pi)\left\{-\frac{\cos nx}{n}\right\}_{-\pi}^{0} + x\left\{\frac{-\cos nx}{n}\right\}_0^{\pi} - 1\cancel{\left\{\frac{-\sin nx}{n^2}\right\}_0^{\pi}}^{\,0}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos nx)_{-\pi}^{0} - \frac{1}{n}(x\cos nx)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(\cos 0 - \cos(-n\pi)) - \frac{1}{n}(\pi\cos n\pi - 0\cos 0)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-(-1)^n) - \frac{1}{n}\pi(-1)^n\right]$$ Because $\cos(-n\pi) = \cos n\pi$

$$b_n = \frac{1}{\pi}\cdot\frac{\pi}{n}\left[1 - (-1)^n - (-1)^n\right] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{\pi n^2}\cos nx + \sum_{n=1}^{\infty}\frac{\{1-2(-1)^n\}}{n}\sin nx.$$

for $x=0$, $f(0) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos 0 + \sum_{n=1}^{\infty}\frac{(1-2(-1)^n)}{n}\sin 0$

But $\sin 0 = 0$, $\cos 0 = 1$.

Note: 0 is the point at which function is not defined

$$\therefore f(0) = \frac{(-\pi)_{x=0} + (x)_{x=0}}{2} = -\frac{\pi}{2}.$$

$$(-1)^n - 1 = \begin{cases} -1-1 & \text{if } n \text{ is odd } \{(2n-1)\} \\ 1-1 & \text{if } n \text{ is even } \{\text{ie } 2n\} \end{cases}$$

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{-2}{\pi(2n-1)^2}.$$

$$\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -\frac{2}{\pi}\left(\sum\frac{1}{(2n-1)^2}\right)$$

$$-\frac{\pi}{4} \times -\frac{\pi}{2} = \sum\frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{8} = \sum\frac{1}{(2n-1)^2}$$

⑧ Find the Fourier Series of $f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$

and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Sol:- Since $f(x)$ is defined in $(0, 2\pi)$ we have to find all constants.

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)\Big|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0 + \left(2\pi(2\pi) - \frac{4\pi^2}{2}\right) - \left(2\pi(\pi) - \frac{\pi^2}{2}\right)\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - 2\pi^2 - 2\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\cdot\pi^2 = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\cancel{x\left\{\frac{-\sin nx}{n}\right\}_0^{\pi}}^{\,0} - 1\left\{\frac{-\cos nx}{n^2}\right\}_0^{\pi} + \cancel{(2\pi - x)\left\{\frac{-\sin nx}{n}\right\}_{\pi}^{2\pi}}^{\,0} - (-1)\left\{\frac{-\cos nx}{n^2}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos nx)_0^{\pi} - \frac{1}{n^2}(\cos nx)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n^2}\left\{\cos n\pi - \cos 0 - (\cos 2n\pi - \cos n\pi)\right\} = \frac{1}{\pi n^2}\left\{(-1)^n - 1 - (-1)^{2n} + (-1)^n\right\} = \frac{1}{\pi n^2}\left\{2(-1)^n - 2\right\} = \frac{2}{\pi n^2}\left\{(-1)^n - 1\right\} \quad \left[(-1)^{2n} = 1\right]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos nx}{n}\right\}_0^{\pi} - \cancel{1\left\{\frac{-\sin nx}{n^2}\right\}_0^{\pi}}^{\,0} + (2\pi - x)\left\{\frac{-\cos nx}{n}\right\}_{\pi}^{2\pi} - \cancel{(-1)\left\{\frac{-\sin nx}{n^2}\right\}_{\pi}^{2\pi}}^{\,0}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(x\cos nx)_0^{\pi} - \frac{1}{n}\left\{(2\pi - x)\cos nx\right\}_{\pi}^{2\pi}\right]$$

$$= -\frac{1}{\pi n}\left[\pi\cos n\pi - \cancel{0\cdot\cos 0}^{\,0} + \cancel{(2\pi - 2\pi)\cos 2n\pi}^{\,0} - (2\pi - \pi)\cos n\pi\right]$$

$$= -\frac{1}{\pi n}\left[\pi\cos n\pi - \pi\cos n\pi\right] = 0$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2\{(-1)^n - 1\}}{\pi n^2}\cos nx.$$

But $(-1)^n - 1 = \begin{cases} -1-1 = -2 & \text{if } n \text{ is odd i.e. } 2n-1 \\ 1-1 = 0 & \text{if } n \text{ is even i.e. } 2n \end{cases}$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-2)}{(2n-1)^2}\cos(2n-1)x.$$

Putting $x = 0$, $f(0) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 0}{(2n-1)^2}$

0 is an end point $\therefore f(0) = \frac{(x)_{x=0} + (2\pi - x)_{x=2\pi}}{2}$

$$f(0) = \frac{0 + 0}{2} = 0$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \quad \text{i.e.} \quad -\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\text{i.e.} \quad -\frac{\pi}{2}\times\frac{\pi}{-4} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \quad \text{i.e.} \quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

⑨ Find the Fourier Series of $f(x) = \begin{cases} (x-\pi)^2, & 0 < x < \pi \\ \pi^2, & \pi < x < 2\pi \end{cases}$ and hence deduce that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

Since $f(x)$ is defined in $(0, 2\pi)$ we have to find all constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)^2\,dx + \int_{\pi}^{2\pi}\pi^2\,dx\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{(x-\pi)^3}{3}\Big|_0^{\pi} + \pi^2 x\Big|_{\pi}^{2\pi}\right] = \frac{1}{\pi}\left[\frac{1}{3}\{(\pi-\pi)^3 - (0-\pi)^3\}\right.$$

$$\left. + \pi^2(2\pi-\pi)\right] = \frac{1}{\pi}\left[\frac{1}{3}(0-(-\pi^3)) + \pi^2\pi\right] = \frac{1}{\pi}\left[\frac{\pi^3}{3} + \pi^3\right]$$

$$a_0 = \frac{1}{\pi}\left[\frac{\pi^3+3\pi^3}{3}\right] = \frac{4\pi^{\cancel{3}\,2}}{3\cancel{\pi}} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)^2\cos nx\,dx + \int_{\pi}^{2\pi}\pi^2\cos nx\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\cancelto{0}{(x-\pi)^2\left\{\frac{\sin nx}{n}\right\}_0^{\pi}} - 2(x-\pi)\left\{-\frac{\cos nx}{n^2}\right\}_0^{\pi} + \cancelto{0}{2\left\{-\frac{\sin nx}{n^3}\right\}_0^{\pi}}\right.$$

$$\left. + \cancelto{0}{\pi^2\left\{\frac{\sin nx}{n}\right\}_{\pi}^{2\pi}}\right] = \frac{1}{\pi}\left[\left(\frac{2}{n^2}\right)\right]\left\{(x-\pi)\cos nx\right\}_0^{\pi}$$

$$a_n = \frac{2}{\pi n^2}\left[\cancelto{0}{(\pi-\pi)\cos n\pi} - (-\pi)\cos 0\right] = \frac{2}{\cancel{\pi}\,n^2}\cancel{\pi} = \frac{2}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)^2\sin nx\,dx + \int_{\pi}^{2\pi}\pi^2\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x-\pi)^2\left\{-\frac{\cos nx}{n}\right\}_0^{\pi} - \cancelto{0}{2(x-\pi)\left\{-\frac{\sin nx}{n^2}\right\}_0^{\pi}} + 2\left\{\frac{\cos nx}{n^3}\right\}_0^{\pi}\right.$$

$$\left. + \pi^2\left\{-\frac{\cos nx}{n}\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}\right)\left\{(x-\pi)^2\cos nx\right\}_0^{\pi} + \frac{2}{n^3}(\cos nx)_0^{\pi} - \frac{\pi^2}{n}(\cos nx)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}\right)\{0.\cos n\pi - (-\pi)^2\cos 0\} + \frac{2}{n^2}(\cos n\pi - \cos 0) - \frac{\pi^2}{n}(\cos 2n\pi - \cos n\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^{\cancel{2}}}{n} + \frac{2}{n^3}\{(-1)^n - 1\} - \cancel{\frac{\pi^2}{n}} + \frac{\pi^2}{n}(-1)^n\right]$$ Because $\cos 2n\pi = 1$

$$= \frac{2(-1)^n}{\pi n^3} - \frac{2}{\pi n^3} + \frac{\pi^{\cancel{2}}}{\cancel{\pi}.n}(-1)^n = \left(\frac{\pi}{n} + \frac{2}{\pi n^3}\right)(-1)^n - \frac{2}{\pi n^3}$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$

$$f(x) = \frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2}\pi^2}{3}\right) + \sum_{n=1}^{\infty}\frac{2}{n^2}\cos nx + \sum_{n=1}^{\infty}\left\{\left(\frac{\pi}{n} + \frac{2}{\pi n^3}\right)(-1)^n - \frac{2}{\pi n^3}\right\}(\sin nx)$$

putting $x = \pi$, we get

$$f(\pi) = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}.$$

π is a point of discontinuity

$$\therefore f(\pi) = \frac{(x-\pi)^2_{x=\pi^-} + (\pi^2)_{x=\pi^+}}{2} = \frac{0+\pi^2}{2} = \frac{\pi^2}{2}.$$

$$\frac{\pi^2}{2} = \frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{\pi^2}{2} - \frac{2\pi^2}{3} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{3\pi^2 - 4\pi^2}{6\times 2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$ dividing by (-1)

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

(10) Find the Fourier Series of $f(x) = x^3$ over the interval $(-\pi, \pi)$. Ans:- $a_0 = 0$, $a_n = 0$, $b_n = 2\left(\frac{\pi^2}{n} - \frac{6}{n^3}\right)(-1)^{n+1}$

(11) Find the Fourier Series of $f(x) = x \sin x$ over the interval $(-\pi, \pi)$. Ans:- $a_0 = 2$, $a_1 = -\frac{1}{2}$, $a_n = \frac{2(-1)^{n+1}}{n^2-1}$ for $n \geq 2$.

(12) Find the Fourier Series of $f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(13) Find the Fourier Series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \frac{\pi x}{4}, & 0 < x < \pi \end{cases}$ and hence deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

(14) Find the Fourier Series of $f(x) = x^2$ over the interval $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Ans:- $a_0 = \frac{8\pi^2}{3}$, $a_n = \frac{4}{n^2}$, $b_n = -\frac{4\pi}{n}$.

(15) Find the Fourier Series of $f(x) = \begin{cases} -\pi, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$

Ans:- $a_0 = -\frac{\pi}{2}$, $a_n = \frac{\{1-(-1)^n\}}{\pi n^2}$, $b_n = \frac{-(2-(-1)^n)}{n}$

(11) Fourier Series of the functions defined over the intervals $(0, 2l)$ and $(-l, l)$

For both the intervals the Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right).$$

If $f(x)$ is defined in $(-l, l)$ and $f(-x) = f(x)$, then the function is even. Then $b_n = 0$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

If $f(x)$ is defined in $(-l, l)$ and $f(-x) = -f(x)$, then the function is odd. Here $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

If $f(x)$ is defined in $(-l, l)$ and $f(-x) \neq f(x)$ or $\neq -f(x)$ then the function is neither even nor odd. Here we have to find all constants, $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

If $f(x)$ is defined in $(0, 2\pi)$, then we have to find all constants $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad b_n = \frac{1}{l}\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Note:- (1) If the given interval is $(-k, k)$ then we get $l = k$. We have to replace l by k in both Fourier

Series and in the formula for fourier coefficients.

If the given interval is $(0, K)$ then $2l = K$

i.e. $l = \frac{K}{2}$.

(2) To deduce a numerical series we have replace x by l or 0 or $\frac{l}{2}$.

(1) Find the Fourier Series of $f(x) = x - x^2$ over $(-1, 1)$

Sol:- $f(x) = x - x^2$ in $(-1, 1)$ Here $l = 1$.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$a_0 = \left\{\frac{1^2}{2} - \frac{1^3}{3}\right\} - \left\{\frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right\} = \cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}} - \frac{1}{3} = -\frac{2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\cos(n\pi x)\,dx.$$

$$= \left[\cancel{(x - x^2)\left\{\frac{\sin n\pi x}{n\pi}\right\}}^{\to 0}\right]_{-1}^{1} - (1 - 2x)\left\{\frac{-\cos n\pi x}{n^2\pi^2}\right\}_{-1}^{1} + \cancel{(-2)\left\{\frac{-\sin n\pi x}{n^3\pi^3}\right\}}^{\to 0}_{-1}$$

$$= \frac{1}{n^2\pi^2}\left[(1 - 2x)\cos n\pi x\right]_{-1}^{1} = \frac{1}{n^2\pi^2}\left[(1 - 2)\cos n\pi - (1 + 2)\cos(-n\pi)\right]$$

But $\cos n\pi = \cos(-n\pi)$ $\therefore a_n = \frac{(-1 - 3)\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_{-1}^{1}(x - x^2)\sin(n\pi x)\,dx$$

$$= \left[(x - x^2)\left\{\frac{-\cos n\pi x}{n\pi}\right\} - \cancel{(1 - 2x)\left\{\frac{-\sin n\pi x}{n^2\pi^2}\right\}}^{\to 0} + (-2)\left\{\frac{\cos n\pi x}{n^3\pi^3}\right\}\right]_{-1}^{1}$$

$$= \left[-\frac{1}{n\pi}\left\{(x - x^2)\cos n\pi x\right\}_{-1}^{1} - \frac{2}{n^3\pi^3}(\cos n\pi x)_{-1}^{1}\right]$$

$$= -\frac{1}{n\pi}\left[\cancel{(1-1)\cos n\pi}^{\to 0} - (-1-1)\cos(-n\pi)\right] - \frac{2}{n^3\pi^3}\left\{\cancel{\cos n\pi - \cos(-n\pi)}\right\}$$

$$= -\frac{1}{n\pi}(-1)(2)\cos n\pi = \frac{2}{n\pi}(-1)^n$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$

i.e. $f(x) = -\frac{\cancel{2}}{3\times\cancel{2}} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2\pi^2}\cos(n\pi x) + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi x)$

(2) Find the Fourier Series of $f(x) = x^2$ over $(-l, l)$

Sol:- $f(-x) = (-x)^2 = x^2 = f(x)$ $\therefore$ function is even.

$b_n = 0$, $a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{l}\int_0^l x^2\,dx$.

$$= \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2}{l}\cdot\frac{l^3}{3} = \frac{2l^2}{3}.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\int_0^l x^2\cos\left(\frac{n\pi x}{l}\right)dx.$$

$$= \frac{2}{l}\left[\cancel{x^2\left\{\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\}}^{\to 0} - (2x)\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\} + \cancel{2\left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right\}}^{\to 0}\right]_0^l$$

$$= \frac{2}{l}(-2)\cdot\frac{1}{\left(\frac{n^2\pi^2}{l^2}\right)}\left\{x\cos\left(\frac{n\pi x}{l}\right)\right\}_0^l = -\frac{4}{\cancel{l}}\frac{l^{\cancel{2}}}{n^2\pi^2}\left\{l\cos n\pi - 0\cdot\cos 0\right\}$$

$$= -\frac{4l}{n^2\pi^2}l(-1)^n = \frac{4l^2(-1)^n}{n^2\pi^2}.$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \frac{2l^2}{3 \cdot \cancel{2}} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

③ Find the Fourier Series of $f(x) = \begin{cases} 1+2x, & -3 < x \le 0 \\ 1-2x, & 0 < x \le 3 \end{cases}$

Sol:- Function is defined in the interval $(-3,3)$ which is of the form $(-l,l)$ ie. $l=3$.

We can check the function for odd or evenness.

In $-3 < x \le 0$, $f(x) = 1+2x$

$f(-x) = 1-2x = f(x)$ in $0 < x \le 3$.

$\therefore$ $f(x)$ is an even function. Hence $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{3}\int_0^3 (1-2x)\,dx = \frac{2}{3}\left[x - \frac{2x^2}{2}\right]_0^3$$

$$= \frac{2}{3}\left[(3-3^2) - 0\right] = \frac{2}{3}\left[3-9\right] = \frac{2}{3}\cancel{(-6)}^{(-2)} = -4$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (1-2x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[\cancel{(1-2x)\left\{\frac{\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)}\right\}}^{0} - (-2)\left\{-\frac{\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^2}\right\}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-2}{\frac{n^2\pi^2}{3^2}}\right]\left\{\cos\left(\frac{n\pi x}{3}\right)\right\}_0^3 = \frac{-4\cdot 3^2}{\cancel{3}\,n^2\pi^2}\left\{\cos n\pi - \cos 0\right\}$$

$$= \frac{-12}{n^2\pi^2}\left\{(-1)^n - 1\right\} = \frac{12}{n^2\pi^2}\left\{1-(-1)^n\right\}$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ 13

$$f(x) = -\frac{\cancel{4}^{2}}{\cancel{2}} + \sum_{n=1}^{\infty} \frac{12\{1-(-1)^n\}}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)$$

④ Find the Fourier Series of $f(x) = \begin{cases} -1, & -2 < x \le 0 \\ 1, & 0 < x < 2 \end{cases}$

Sol:- $f(x)$ is defined in $(-2,2)$ $\therefore l = 2$

In $-2 < x \le 0$, $f(x) = -1$, $f(-x) = -1 = -f(x)$ in $0 < x < 2$

$\therefore$ $f(x)$ is an odd function. Hence $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{2}\int_0^2 1 \cdot \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \left\{-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\}_0^2 = -\frac{2}{n\pi}\left\{\cos n\pi - \cos 0\right\} = \frac{2\{1-(-1)^n\}}{n\pi}$$

F.S is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \sum_{n=1}^{\infty} \frac{2\{1-(-1)^n\}}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

⑤ Find the Fourier Series of $f(x) = \pi x$ over $(0,2)$

Sol:- The interval is $(0,2) \equiv (0,2l)$ i.e. $2l = 2$ $\therefore l = 1$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{1}\int_0^2 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^2 = \pi\left[\frac{4}{2}\right] = 2\pi.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2 \pi x\cos(n\pi x)\,dx$$

$$= \cancel{(\pi x)\left\{\frac{\sin n\pi x}{n\pi}\right\}_0^2}^{0} - (\pi)\left\{\frac{-\cos n\pi x}{n^2\pi^2}\right\}_0^2.$$

$$a_n = \frac{\pi}{n^2\pi^2}\left\{\cos n\pi x\right\}_0^2 = \frac{1}{n^2\pi}\left\{\cos 2n\pi - \cos 0\right\} = \frac{1-1}{n^2\pi} = 0$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2 \pi x \cdot \sin(n\pi x)\,dx$$

$$b_n = (\pi x)\left\{\frac{-\cos n\pi x}{n\pi}\right\}_0^2 - \pi\left\{\cancel{\frac{-\sin n\pi x}{n^2\pi^2}}^{\,0}\right\}_0^2$$

$$b_n = \frac{-\pi}{n\pi}\left\{x\cos n\pi x\right\}_0^2 = -\frac{1}{n}\left\{2\cos 2n\pi - 0\right\} = \frac{-2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin(n\pi x)$$

⑥ obtain the Fourier series for the function.

$$f(x) = \begin{cases}\pi x & \text{in } 0 \le x \le 1\\ \pi(2-x) & \text{in } 1 \le x \le 2\end{cases}$$ and deduce that $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

Sol:- $f(x)$ is defined in $(0,2)$. It is of the form $(0,2l)$

$\therefore 2l = 2$ i.e. $l = 1$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2\right] = \pi\left[\frac{1}{2} + \left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \pi\left[\frac{1}{2} + 2 - 2 + \frac{1}{2}\right] = \pi\left[\frac{2}{2}\right] = \pi$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx$$

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$$= \pi\left[x\cancel{\left\{\frac{\sin n\pi x}{n\pi}\right\}_0^1}^{\,0} - 1\left\{\frac{-\cos n\pi x}{n^2\pi^2}\right\}_0^1 + (2-x)\cancel{\left\{\frac{\sin n\pi x}{n\pi}\right\}_1^2}^{\,0} - (-1)\left\{\frac{-\cos n\pi x}{n^2\pi^2}\right\}_1^2\right]$$

$$= \pi\left[\frac{1}{n^2\pi^2}(\cos n\pi x)\Big|_0^1 - \frac{1}{n^2\pi^2}(\cos n\pi x)\Big|_1^2\right]$$

$$= \frac{\pi}{n^2\pi^2}\left[\cos n\pi - \cos 0 - (\cos 2n\pi - \cos n\pi)\right]$$ But $\cos 2n\pi = 1$

$$= \frac{1}{n^2\pi}\left[2\cos n\pi - 2\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[x\cdot\left\{\frac{-\cos n\pi x}{n\pi}\right\}_0^1 - 1\cancel{\left\{\frac{-\sin n\pi x}{n^2\pi^2}\right\}_0^1}^{\,0} + (2-x)\left\{\frac{-\cos n\pi x}{n\pi}\right\}_1^2 - (-1)\cancel{\left\{\frac{-\sin n\pi x}{n^2\pi^2}\right\}_1^2}^{\,0}\right]$$

$$= \pi\left[\frac{-1}{n\pi}(x\cos n\pi x)\Big|_0^1 - \frac{1}{n\pi}\left\{(2-x)\cos n\pi x\right\}_1^2\right]$$

$$= \frac{-\pi}{n\pi}\left[(1.\cos n\pi - 0.\cos 0) + (0.\cos 2n\pi - 1.\cos n\pi)\right]$$

$$= \frac{-1}{n}\left[\cos n\pi - \cos n\pi\right] = \frac{-1}{n}\left[(-1)^n - (-1)^n\right] = 0.$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2\left[(-1)^n - 1\right]}{n^2\pi}\cos n\pi x$$

for $x = 0$,

$$f(0) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos 0.$$

But $(-1)^n - 1 = \begin{cases} -1-1 = -2 & \text{if } n \text{ is odd}\\ 1-1 = 0 & \text{if } n \text{ is even}\end{cases}$

$$0 = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-2}{(2n-1)^2}$$

$$-\frac{\pi}{2} = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

⑦ Obtain the Fourier expansion of $f(x) = \begin{cases} l-x, & 0<x<l \\ 0, & l<x<2l \end{cases}$ over the interval $(0, 2l)$. Hence deduce that

(a) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$ (b) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol:- $f(x)$ is defined in $(0, 2l)$.

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\left[\int_0^{l}(l-x)\,dx + \int_l^{2l} 0.\,dx\right]$$

$$= \frac{1}{l}\left[lx - \frac{x^2}{2}\right]_0^l = \frac{1}{l}\left[\left(l^2 - \frac{l^2}{2}\right) - 0\right] = \frac{1}{l}\left[\frac{l^2}{2}\right] = \frac{l}{2}.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[\int_0^{l}(l-x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0.\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[(l-x)\left\{\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\} - (-1)\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{l^2}{n^2\pi^2}\right]\left\{\cos\left(\frac{n\pi x}{l}\right)\right\}_0^l = \frac{l}{n^2\pi^2}\left\{\cos n\pi - \cos 0\right\}$$

$$\therefore a_n = \frac{l}{n^2\pi^2}\left\{1 - (-1)^n\right\}.$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\left[\int_0^{l}(l-x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_l^{2l} 0.\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[(l-x)\left\{\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right\} - (-1)\left\{\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right\}\right]_0^l$$

$$= \frac{1}{l}\left[\frac{-l}{n\pi}\right]\left\{(l-x)\cos\left(\frac{n\pi x}{l}\right)\right\}_0^l = \frac{-1}{n\pi}\left\{0.\cos n\pi - l\cos 0\right\} = \frac{l}{n\pi}$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \frac{l}{4} + \sum_{n=1}^{\infty}\frac{l}{n^2\pi^2}\left\{1-(-1)^n\right\}\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

But $1-(-1)^n = \begin{cases} 1-(-1) = 1+1 = 2 & \text{if } n \text{ is odd i.e. } 2n-1 \\ 1-1 = 0 & \text{if } n \text{ is even i.e. } 2n \end{cases}$

$$\therefore f(x) = \frac{l}{4} + \sum_{n=1}^{\infty}\frac{2l}{(2n-1)^2\pi^2}\cos\left\{\frac{(2n-1)\pi x}{l}\right\} + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

putting $x = 0$, we get

$$f(0) = \frac{l}{4} + \sum_{n=1}^{\infty}\frac{2l}{(2n-1)^2\pi^2}.\cos 0 + \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin 0.$$

But $\sin 0 = 0$. $\cos 0 = 1$

$$f(0) = \frac{(l-x)_{x=0} + 0_{x=2l}}{2} = \frac{l}{2}$$

$$\frac{l}{2} - \frac{l}{4} = \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

i.e. $\left(\frac{2l-l}{4}\right) \times \frac{\pi^2}{2l} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ i.e. $\frac{\cancel{l}\pi^2}{8\cancel{l}} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

putting $x = \frac{l}{2}$ in the Fourier Series, we get

$$f\left(\frac{l}{2}\right) = \frac{l}{4} + \sum_{n=1}^{\infty}\frac{2l}{(2n-1)^2\pi^2}\cos\left(\frac{(2n-1)\pi}{2}\right) + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

But $\cos\frac{(2n-1)\pi}{2} = 0$, $f\left(\frac{l}{2}\right) = l - \frac{l}{2} = \frac{l}{2}$

$$\therefore \frac{l}{2} - \frac{l}{4} = \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \cdots\right]$$

$$\frac{\cancel{l}}{4} \times \frac{\pi}{\cancel{l}} = 1 - \frac{1}{3} + \frac{1}{5} - + \cdots$$

⑧ Find F.S. of $f(x) = \begin{cases} 2-x, & 0<x<4 \\ x-6, & 4<x<8. \end{cases}$

Ans:- $a_0 = 0$, $a_n = 0$, $b_n = \frac{8\{1-(-1)^n\}}{\pi n^2}$

Half Range Fourier Series.

Finding the Fourier Series expansion of function given in $(0,l)$ and $(0,\pi)$ is called Half range Series.

There are two Half range Series

(i) Half range cosine Series (ii) Half range Sine Series.

For the function $f(x)$ defined in $(0,l)$, Half range cosine Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$, $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

For the function $f(x)$ defined in $(0,l)$, Half range Sine Series is $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

Where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$.

For the function $f(x)$ defined in $(0,\pi)$ Half range cosine Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$.

For the function $f(x)$ defined in $(0,\pi)$ Half range Sine Series is

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

Note:- ① If we given interval of the form $(0,k)$, then while finding the Half range expansions, we complete the interval with $(0,l)$ and $\therefore$ we get $l = k$.

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(ii) While find the constants in case of Half range expansions of splitting functions we have to retain terms containing sine functions to take the limits.

① Find the Half range Fourier cosine Series of the function $f(x) = x - x^2$ in $(0,\pi)$ (Aug 2k)

Sol:- $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(x-x^2)\,dx$

$= \frac{2}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = 2\left[\frac{\pi}{2} - \frac{\pi^2}{3}\right]$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}(x-x^2)\cos nx\,dx$

$= \frac{2}{\pi}\left[\cancelto{0}{(x-x^2)\left\{\frac{\sin nx}{n}\right\}} - (1-2x)\left\{\frac{-\cos nx}{n^2}\right\} + \cancelto{0}{(-2)\left\{\frac{-\sin nx}{n^3}\right\}}\right]_0^{\pi}$

$= \frac{2}{\pi}\left(\frac{1}{n^2}\right)\{(1-2x)\cos nx\}_0^{\pi} = \frac{2}{\pi n^2}\left[(1-2\pi)\cos n\pi - 1.\cos 0\right]$

$= \frac{2}{\pi n^2}\left[(1-2\pi)(-1)^n - 1\right]$

Half range Fourier cosine Series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

$f(x) = \left[\frac{\pi}{2} - \frac{\pi^2}{3}\right] + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[(1-2\pi)(-1)^n - 1\right]\cos nx$

(2) Expand $f(x) = \cos x$ as half range sine series in $(0,\pi)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin nx \cdot \cos x \cdot dx$$

$$\left[\sin A\cos B = \tfrac{1}{2}\{\sin(A+B) + \sin(A-B)\}\right]$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\{\sin(nx+x) + \sin(nx-x)\}\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} \sin(n+1)x\,dx + \int_0^{\pi} \sin(n-1)x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)} - \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-1}{(n+1)}\{\cos(n+1)x\}_0^{\pi} - \frac{1}{(n-1)}\{\cos(n-1)x\}_0^{\pi}\right] = \frac{1}{\pi}\left[-\frac{1}{(n+1)}\{\cos(n+1)\pi - \cos 0\} - \frac{1}{(n-1)}\{\cos(n-1)\pi - \cos 0\}\right]$$

But $\cos(n+1)\pi = \cos(n-1)\pi = (-1)^{n+1}$

$$b_n = \frac{1}{\pi}\{\cos(n+1)\pi - \cos 0\}\left\{-\frac{1}{(n+1)} - \frac{1}{(n-1)}\right\}$$

$$b_n = -\frac{1}{\pi}\{(-1)^{n+1} - 1\}\left\{\frac{n-1+n+1}{(n+1)(n-1)}\right\} = \frac{2n\{1-(-1)^{n+1}\}}{\pi(n^2-1)}$$

Half range Fourier Sine Series is. we get $b_1 = 0$

$$f(x) = \sum_{n=2}^{\infty} b_n \sin nx = \sum_{n=2}^{\infty} \frac{2n\{1-(-1)^{n+1}\}}{\pi(n^2-1)}\sin nx$$

3) Expand the function $f(x) = x(\pi - x)$ over the interval $(0,\pi)$ in half range Fourier Cosine Series and hence deduce that

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

Sol:- $f(x) = x(\pi - x) = \pi x - x^2$

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$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi \cdot \pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{2}{\pi}\left[\frac{3\pi^3 - 2\pi^3}{6}\right] = \frac{2}{\pi}\cdot\frac{\pi^3}{6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,dx.$$

$$= \frac{2}{\pi}\left[\cancel{(\pi x - x^2)\left\{\frac{\sin nx}{n}\right\}}^{\,0} - (\pi - 2x)\left\{\frac{-\cos nx}{n^2}\right\} + \cancel{(-2)\left\{\frac{-\sin nx}{n^3}\right\}}^{\,0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n^2}\{(\pi - 2x)\cos nx\}\right]_0^{\pi} = \frac{2}{\pi n^2}\left[(\pi - 2\pi)\cos n\pi - \pi(\cos 0)\right]$$

$$= \frac{2}{\pi n^2}\cdot\left[-\pi(-1)^n - \pi\right] = \frac{-\cancel{\pi}(2)}{\cancel{\pi} n^2}\left[(-1)^n + 1\right] = \frac{-2\left[1 + (-1)^n\right]}{n^2}$$

Half range Fourier Cosine Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{i.e. } f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{-2\left[1+(-1)^n\right]}{n^2}\cos nx.$$

But $1 + (-1)^n = \begin{cases} 1+1 = 2 & \text{if } n \text{ is even i.e. } n = 2n \\ 1-1 = 0 & \text{if } n \text{ is odd i.e. } n = 2n-1 \end{cases}$

$$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty} \frac{2}{(2n)^2}\cos(2nx) \quad \text{—} \quad (1)$$

putting $x = 0$, we get $f(0) = \frac{\pi^2}{6} - 4\sum_{n=1}^{\infty} \frac{1}{4n^2}\cos 0$

But $f(0) = \frac{\{x(\pi - x)\}_{x=0} + \{x(\pi - x)\}_{x=\pi}}{2} = \frac{0+0}{2} = 0.$

$$0 - \frac{\pi^2}{6} = -\frac{4}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{i.e. } \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Putting $x = \frac{\pi}{2}$ in ①, we get

$$f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{6} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos\left(n \cdot \frac{\pi}{2}\right)}{n^2}$$

But $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{\pi^2}{4}$

$$\frac{\pi^2}{4} - \frac{\pi^2}{6} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

i.e. $\frac{3\pi^2 - 2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

i.e. $\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ Because $(-1)^{n+1} = (-1)^{n-1}$

④ Expand the function $f(x) = \begin{cases} x & \text{in } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{in } \frac{\pi}{2} < x < \pi \end{cases}$

Sol:- $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$

$$= \frac{2}{\pi}\left[x \cdot \left\{\frac{-\cos nx}{n}\right\}_0^{\pi/2} - 1\left\{\frac{-\sin nx}{n^2}\right\}_0^{\pi/2} + (\pi - x)\left\{\frac{-\cos nx}{n}\right\}_{\pi/2}^{\pi} - (-1)\left\{\frac{-\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(x\cos nx\right)_0^{\pi/2} + \frac{1}{n^2}\left(\sin nx\right)_0^{\pi/2} - \frac{1}{n}\left\{(\pi - x)\cos nx\right\}_{\pi/2}^{\pi} - \frac{1}{n^2}\left(\sin nx\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\left(\frac{\pi}{2}\cos n\frac{\pi}{2} - 0 \cdot (\cos 0)\right) + \frac{1}{n^2}\left\{\sin n\frac{\pi}{2} - \sin 0\right\} - \frac{1}{n}\left(\cancel{(\pi - \pi)\cos n\pi}^{\,0} - \frac{\pi}{2}\left(\cos n\frac{\pi}{2}\right)\right) - \frac{1}{n^2}\left(\sin n\pi - \sin n\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos n\frac{\pi}{2} + \frac{\pi}{2n}\cos n\frac{\pi}{2} + \frac{1}{n^2}\left(\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\left(n\frac{\pi}{2}\right)\right] = \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

Half range Fourier Sine Series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\sin nx.$$

⑤ Expand $f(x) = c - x$ in an Half range Fourier cosine series over the interval $(0, c)$ (18)

Sol:- The interval is $(0, c) \equiv (0, l)$ $\therefore l = c$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{c}\int_0^c (c - x)\,dx = \frac{2}{c}\left[cx - \frac{x^2}{2}\right]_0^c$$

$$= \frac{2}{c}\left[c^2 - \frac{c^2}{2}\right] = \frac{2}{c}\left[\frac{c^2}{2}\right] = c$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{c}\int_0^c (c - x)\cos\left(\frac{n\pi x}{c}\right)dx.$$

$$= \frac{2}{c}\left[\cancel{(c - x)\left\{\frac{\sin\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)}\right\}} - (-1)\left\{\frac{-\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^2}\right\}\right]_0^c$$

$$= \frac{2}{c}\,\frac{(-1)}{\frac{n^2\pi^2}{c^2}}\left[\cos\left(\frac{n\pi x}{c}\right)\right]_0^c = \frac{-2c}{n^2\pi^2}\left[\cos\left(\frac{n\pi c}{c}\right) - \cos 0\right]$$

$$= \frac{2c}{n^2\pi^2}\left[1 - (-1)^n\right]$$

Half range Fourier Cosine Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)$ i.e. $f(x) = \frac{c_0}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - (-1)^n}{n^2}\cos\left(\frac{n\pi x}{c}\right)$

⑥ Find the Half range Fourier Cosine Series for.

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2. \end{cases}$$

Sol:- The interval is $(0, 2) = (0, l)$ $\therefore l = 2$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2 - x)\,dx.$$

$$= \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{2} + \left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right) = \frac{1}{2} + 2 - 2 + \frac{1}{2}$$

$$a_0 = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{2}\left[\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \int_0^1 x\cdot\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx.$$

$$= x\left\{\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\}_0^1 - 1\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right\}_0^1 + (2-x)\left\{\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right\}_1^2 - (-1)\left\{\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right\}_1^2$$

$$= \frac{2}{n\pi}\left\{x\sin\left(\frac{n\pi x}{2}\right)\right\}_0^1 + \frac{4}{n^2\pi^2}\left\{\cos\left(\frac{n\pi x}{2}\right)\right\}_0^1 + \frac{2}{n\pi}\left\{(2-x)\sin\left(\frac{n\pi x}{2}\right)\right\}_1^2$$

$$- \frac{4}{n^2\pi^2}\left\{\cos\left(\frac{n\pi x}{2}\right)\right\}_1^2$$

$$= \frac{2}{\pi}\left\{\sin\frac{n\pi}{2} - \sin 0\right\} + \frac{4}{n^2\pi^2}\left\{\cos\frac{n\pi}{2} - \cos 0\right\}$$

$$+ \frac{2}{n\pi}\left\{0.\sin n\pi - \sin\frac{n\pi}{2}\right\} - \frac{4}{n^2\pi^2}\left\{\cos n\pi - \cos\frac{n\pi}{2}\right\}$$

$$= \frac{2}{n\pi}\left\{\sin\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right\} + \frac{4}{n^2\pi^2}\left\{\cos\frac{n\pi}{2} - \cos 0 - \cos n\pi + \cos\frac{n\pi}{2}\right\}$$

$$= \frac{4}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}$$. Half range cosine series

is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left\{2\cos\frac{n\pi}{2} - 1 - (-1)^n\right\}\cos\left(\frac{n\pi x}{2}\right)$$

(7) Expand the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{for } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{for } \frac{1}{2} < x < 1 \end{cases}$$ in a half range Sine Series.

Sol:- The interval is $(0,1) = (0,l)$ $\therefore l = 1$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = 2\int_0^1 f(x)\sin n\pi x\,dx.$$

$$b_n = 2\left[\int_0^{\frac{1}{2}}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{1/2}^1\left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$b_n = 2\left[\left(\frac{1}{4} - x\right)\left\{\frac{-\cos n\pi x}{n\pi}\right\}_0^{\frac{1}{2}} - (-1)\left\{-\frac{\sin n\pi x}{n^2\pi^2}\right\}_0^{\frac{1}{2}}\right.$$

$$\left. + \left(x - \frac{3}{4}\right)\left\{\frac{-\cos n\pi x}{n\pi}\right\}_{\frac{1}{2}}^1 - (1)\left\{\frac{-\sin n\pi x}{n^2\pi^2}\right\}_{1/2}^1\right]$$

$$= 2\left[-\frac{1}{n\pi}\left\{\left(\frac{1}{4} - x\right)\cos n\pi x\right\}_0^{\frac{1}{2}} - \frac{1}{n^2\pi^2}\left\{\sin n\pi x\right\}_0^{\frac{1}{2}}\right.$$

$$\left. - \frac{1}{n\pi}\left\{\left(x - \frac{3}{4}\right)\cos n\pi x\right\}_{1/2}^1 + \frac{1}{n^2\pi^2}\left\{\sin n\pi x\right\}_{1/2}^1\right]$$

$$= 2\left[-\frac{1}{n\pi}\left\{\left(\frac{1}{4} - \frac{1}{2}\right)\cos\frac{n\pi}{2} - \frac{1}{4}\cos 0\right\}\right.$$

$$- \frac{1}{n^2\pi^2}\left\{\sin n\frac{\pi}{2} - \sin 0\right\}$$

$$- \frac{1}{n\pi}\left\{\left(1 - \frac{3}{4}\right)\cos n\pi - \left(\frac{1}{2} - \frac{3}{4}\right)\cos n\frac{\pi}{2}\right\}$$

$$\left. + \frac{1}{n^2\pi^2}\left\{\sin n\pi - \sin n\frac{\pi}{2}\right\}\right]$$

$$= 2\left[-\frac{1}{n\pi}\left\{-\frac{1}{4}\cos n\frac{\pi}{2} - \frac{1}{4} + \frac{1}{4}\cos n\pi + \frac{1}{4}\cos n\frac{\pi}{2}\right\}\right.$$

$$\left. - \frac{1}{n^2\pi^2}\left\{\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right\}\right]$$

$$= 2\left[-\frac{1}{n\pi}\left(\frac{1}{4}\right)\left\{1 - \cos n\pi\right\} - \frac{1}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$b_n = \frac{2}{n\pi}\{1-(-1)^n\} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Half Range Fourier Sine Series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \text{ i.e.}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\{1-(-1)^n\} - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\sin(n\pi x)$$

(8) Expand $f(x) = \sin x$ in a half range Sine Series over the interval $(0, \pi)$.

Ans:- $f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{[1+(-1)^n]}{(n^2-1)}\cos nx$.

(9) Expand $(x-1)^2$ as half-range Fourier Sine and Cosine Series for $0 \le x \le 1$. Hence S.T.

(a) $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ (b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

Ans:-

$a_0 = \frac{2}{3}$, $a_n = \frac{4}{n^2\pi^2}$, $b_n = \frac{2}{n\pi}\left[1 + \frac{2}{n^2\pi^2}\{(-1)^n - 1\}\right]$

For deduction use $x = 0$ and $x = 1$ in cosine Series.

Complex Fourier Series & Exponential Series

(26)

Consider the Fourier Series of $f(x)$ in $(a, a+2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{— (1)}$$

Where $a_0 = \frac{1}{l}\int_a^{a+2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

But we know that $\cos\left(\frac{n\pi x}{l}\right) = \frac{e^{i\left(\frac{n\pi x}{l}\right)} + e^{-i\left(\frac{n\pi x}{l}\right)}}{2}$

$\sin\left(\frac{n\pi x}{l}\right) = \frac{e^{i\left(\frac{n\pi x}{l}\right)} - e^{-i\left(\frac{n\pi x}{l}\right)}}{2i}$, using this in (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\left\{\frac{e^{i\left(\frac{n\pi x}{l}\right)} + e^{-i\left(\frac{n\pi x}{l}\right)}}{2}\right\} + \sum_{n=1}^{\infty} b_n\left\{\frac{e^{i\left(\frac{n\pi x}{l}\right)} - e^{-i\left(\frac{n\pi x}{l}\right)}}{2i}\right\}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{2} + \frac{b_n}{2i}\right\}e^{i\left(\frac{n\pi x}{l}\right)} + \sum_{n=1}^{\infty}\left\{\frac{a_n}{2} - \frac{b_n}{2i}\right\}e^{-i\left(\frac{n\pi x}{l}\right)}$$

let $\frac{a_0}{2} = C_0$, $\frac{a_n}{2} + \frac{b_n}{2i} = \frac{a_n}{2} - i\frac{b_n}{2} = C_n$, $\frac{a_n}{2} - \frac{b_n}{2i} = \frac{a_n}{2} + i\frac{b_n}{2} = C_{-n}$

$$f(x) = C_0 + \sum_{n=1}^{\infty} C_n e^{i\left(\frac{n\pi x}{l}\right)} + \sum_{n=1}^{\infty} C_{-n} e^{-i\left(\frac{n\pi x}{l}\right)}$$

$$f(x) = \sum_{n=0}^{\infty} C_n e^{i\left(\frac{n\pi x}{l}\right)} + \sum_{n=-1}^{-\infty} C_n e^{-i\left(\frac{n\pi x}{l}\right)}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{l}\right)}} \quad \text{—— (1)}$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx, \quad c_n = \frac{a_n}{2} - i\frac{b_n}{2} = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{-i\frac{n\pi x}{l}}\,dx$$

$$\left[\text{Because } f(x)\cos\left(\frac{n\pi x}{l}\right) - i f(x)\sin\left(\frac{n\pi x}{l}\right) = f(x)\left\{\cos\frac{n\pi x}{l} - i\sin\left(\frac{n\pi x}{l}\right)\right\}\right.$$

$$\left. = f(x)\, e^{-i\left(\frac{n\pi x}{l}\right)}\right]$$

$$c_{-n} = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{i\left(\frac{n\pi x}{l}\right)}\,dx.$$

The formulae for c_0, c_n, c_{-n} can be combined to single formula $\boxed{c_n = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{-i\left(\frac{n\pi x}{l}\right)}\,dx}$ —— (2)

where $n = 0, \pm 1, \pm 2, \ldots$

Expression (1) is called as complex Fourier Series of Exponential Series where constant c_n is given by (2).

Note:- (1) $e^{\pm n\pi i} = \cos n\pi \pm i\sin n\pi = \cos n\pi = (-1)^n$.

(2) $e^{\pm 2n\pi i} = \cos 2n\pi \pm i\sin 2n\pi = \cos 2n\pi = (-1)^{2n}$.

(3) After findin c_n, if it is not defined for certain values of n, then we have to find those constants separately.

(1) Find the complex of Exponential Series of the function $f(x) = \cos ax$ in $(-\pi, \pi)$. (21) (21)

Sol:- The complex of Exponential Fourier Series in $(-\pi, \pi)$ is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\left(\frac{n\pi x}{l}\right)}$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx$.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}\cdot\cos ax\,dx \quad \left[\int e^{Ax}\cos Bx\,dx = \frac{e^{Ax}}{A^2+B^2}\{A\cos Bx + B\sin Bx\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{(-in)^2 + a^2}\{-in\cos ax + a\sin ax\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(i^2n^2+a^2)}\left\{-in\left(e^{-inx}\cos nx\right)_{-\pi}^{\pi} + a\left(e^{-inx}\sin ax\right)_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi(a^2-n^2)}\left[-in\left\{e^{-in\pi}\cos a\pi - e^{+in\pi}\cos(-a\pi)\right\} + a\left\{e^{-in\pi}\sin a\pi - e^{in\pi}\sin(-a\pi)\right\}\right]$$

But $e^{-in\pi} = e^{in\pi} = (-1)^n$

$$= \frac{1}{2\pi(a^2-n^2)}\left[(-in)(-1)^n\{\cos a\pi - \cos a\pi\} + a(-1)^n\{\sin a\pi + \sin a\pi\}\right]$$

$$= \frac{1}{\cancel{2}\pi(a^2-n^2)}\left[a(-1)^n\,\cancel{2}\sin a\pi\right]$$ defined for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$$\therefore\ f(x) = \sum_{n=-\infty}^{\infty} \frac{a(-1)^n\sin a\pi}{\pi(a^2-n^2)}\, e^{inx}.$$

② Obtain the complex Fourier Series for $f(x) = ax + bx^2$ over $(-\pi, \pi)$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} (ax+bx^2)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[(ax+bx^2)\left\{\frac{e^{-inx}}{-in}\right\} - (a+2bx)\left\{\frac{e^{-inx}}{i^2n^2}\right\} + 2b\left\{\frac{e^{-inx}}{-i^3n^3}\right\}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{1}{in}\left\{(ax+bx^2)\, e^{-inx}\right\}_{-\pi}^{\pi} - \frac{1}{(-n^2)}\left\{(a+2bx)\, e^{-inx}\right\}_{-\pi}^{\pi} + \frac{2bi}{n^3}\left\{e^{-inx}\right\}_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{i}{n}\left\{(a\pi+b\pi^2)\, e^{-in\pi} - (-a\pi+b\pi^2)\, e^{in\pi}\right\} + \frac{1}{n^2}\left\{(a+2b\pi)\, e^{-in\pi} - (a-2b\pi)\, e^{in\pi}\right\} + \frac{2bi}{n}\left\{e^{-in\pi} - e^{in\pi}\right\}\right]$$

But $e^{-in\pi} = e^{in\pi} = (-1)^n$

$$C_n = \frac{1}{2\pi}\left[\frac{i}{n}(-1)^n\left\{a\pi + \cancel{b\pi^2} + a\pi - \cancel{b\pi^2}\right\} + \frac{(-1)^n}{n^2}\left\{\cancel{a} + 2b\pi - \cancel{a} + 2b\pi\right\} + \frac{2bi}{n^3}\left\{(-1)^n - (-1)^n\right\}\right]$$

$$C_n = \frac{1}{2\pi}\left[\frac{2a\pi i}{n}(-1)^n + \frac{4b\pi(-1)^n}{n^2}\right]$$

$$C_n = \frac{ai(-1)^n}{n} + \frac{2b(-1)^n}{n^2} \quad \text{for } n \neq 0$$

ie. we have to find C_0 separately.

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} (ax+bx^2)\, e^0\, dx = \frac{1}{2\pi}\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{a\pi^2}{2} + \frac{b\pi^3}{3} - \left(\frac{a\pi^2}{2} - \frac{b\pi^3}{3}\right)\right] = \frac{1}{\cancel{2\pi}}\left[\frac{\cancel{2}b\pi^3}{3}\right]$$

Complex Fourier Series is

$$f(x) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n\, e^{i(nx)}$$

$$f(x) = \frac{b\pi^2}{3} + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty}\left\{\frac{ai(-1)^n}{n} + \frac{2b(-1)^n}{n^2}\right\} e^{inx}$$

③ Obtain complex Fourier Series of $f(x) = e^{ax}$ over the interval $(-l, l)$.

Sol:- $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-i\left(\frac{n\pi x}{l}\right)}\, dx = \frac{1}{2l}\int_{-l}^{l} e^{ax}\, e^{-i\left(\frac{n\pi x}{l}\right)}\, dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(\frac{al - in\pi}{l}\right)x}\, dx \qquad \left[ax - \frac{in\pi x}{l} = \frac{alx - in\pi x}{l} = \left(\frac{al - in\pi}{l}\right)x \right]$$

$$= \frac{1}{2l}\left[\frac{e^{\left(\frac{al-in\pi}{l}\right)x}}{\left(\frac{al-in\pi}{l}\right)}\right]_{-l}^{l} = \frac{1}{2\cancel{l}} \cdot \frac{\cancel{l}}{(al-in\pi)}\left\{e^{\left(\frac{al-in\pi}{l}\right)x}\right\}_{-l}^{l}$$

$$= \frac{1}{al-in\pi} \times \frac{al+in\pi}{al+in\pi}\left[e^{\left(\frac{al-in\pi}{\cancel{l}}\right)\cancel{l}} - e^{\left(\frac{al-in\pi}{\cancel{l}}\right)(-\cancel{l})}\right]$$

$$\frac{al+in\pi}{a^2l^2-i^2n^2\pi^2}\left[e^{al}e^{-in\pi}-e^{-al}e^{in\pi}\right]$$

But $e^{-in\pi}=e^{in\pi}=(-1)^n$

$$=\frac{(al+in\pi)}{a^2l^2+n^2\pi^2}(-1)^n\cdot\left[e^{al}-e^{-al}\right]=\frac{2(-1)^n(al+in\pi)\sinh al}{a^2l^2+n^2\pi^2}.$$

$$f(x)=\sum_{n=-\infty}^{\infty}f(x)\,e^{i\left(\frac{n\pi x}{l}\right)}$$

$$=\sum_{n=-\infty}^{\infty}\frac{2(-1)^n(al+in\pi)\sinh al}{a^2l^2+n^2\pi^2}\,e^{i\left(\frac{n\pi x}{l}\right)}$$

=

4) obtain the Exponential Fourier Series of

$$f(x)=\begin{cases}K & \text{for } 0<x<l\\ -K & \text{for } l<x<2l\end{cases}$$

where K is a +ve constant.

Hence deduce the corresponding trignometric Fourier Series.

Sol:- $C_n=\frac{1}{2l}\int_0^{2l}f(x)\,e^{-i\left(\frac{n\pi x}{l}\right)}dx$.

$$=\frac{1}{2l}\left[\int_0^{l}K\,e^{-i\left(\frac{n\pi x}{l}\right)}dx+\int_l^{2l}(-K)\,e^{-i\left(\frac{n\pi x}{l}\right)}dx\right]$$

$$=\frac{1}{2l}\left[K\cdot\left\{\frac{e^{-\frac{in\pi x}{l}}}{-\frac{in\pi}{l}}\right\}_0^{l}-K\left\{\frac{e^{-\frac{in\pi x}{l}}}{-\frac{in\pi}{l}}\right\}_l^{2l}\right]$$

$$=\frac{K}{2l}\left(\frac{-l}{in\pi}\right)\left[\left(e^{-\frac{in\pi x}{l}}\right)_0^{l}-\left(e^{-\frac{in\pi x}{l}}\right)_l^{2l}\right]$$

$$=\frac{Ki}{2n\pi}\left[e^{-in\pi}-e^{0}-\left(e^{-i2n\pi}-e^{-in\pi}\right)\right]$$

$$=\frac{Ki}{2n\pi}\left[(-1)^n-1-1+(-1)^n\right]=\frac{\cancel{2}Ki\left[(-1)^n-1\right]}{\cancel{2}n\pi}\quad \text{for } n\neq 0.$$

for $n=0$, $C_n=\frac{1}{2l}\left[\int_0^{l}K\,dx+\int_l^{2l}(-K)\,dx\right]$

$$=\frac{K}{2l}\left[x\big|_0^{l}-x\big|_l^{2l}\right]=\frac{K}{2l}\left[l-0-(2l-l)\right]=0.$$

$$f(x)=C_0+\sum_{\substack{n=-\infty\\(n\neq 0)}}^{\infty}C_n\,e^{i\left(\frac{n\pi x}{l}\right)}$$

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{Ki}{n\pi}\left[(-1)^n-1\right]\,e^{i\left(\frac{n\pi x}{l}\right)}$$

To deduce trignometric Fourier Series.

consider $C_0=\frac{a_0}{2}$, $C_n=\frac{a_n}{2}-i\frac{b_n}{2}$, $C_{-n}=\frac{a_n}{2}+i\frac{b_n}{2}$

$$\frac{a_0}{2}=0,\quad \frac{a_n}{2}-i\frac{b_n}{2}=\frac{Ki\left[(-1)^n-1\right]}{n\pi}\quad -(1)$$

$$\frac{a_n}{2}+i\frac{b_n}{2}=\frac{Ki\left[(-1)^{-n}-1\right]}{-n\pi}\quad -(2)$$

Adding (1) and (2) $\cancel{2}\frac{a_n}{\cancel{2}}=0$ Because $(-1)^n=(-1)^{-n}$

(1) − (2) gives $-\frac{2ib_n}{2}=\frac{2Ki\left[(-1)^n-1\right]}{n\pi}$

$$b_n = \frac{2K\left[1-(-1)^n\right]}{n\pi}$$

Fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

$$f(x) = \sum_{n=1}^{\infty} \frac{2K\left[1-(-1)^n\right]}{n\pi} \sin\left(\frac{n\pi x}{l}\right)$$

⑤ Find the complex Fourier Series of the function.

$$f(x) = \begin{cases} \pi x & \text{for } 0 < x < 1 \\ \pi(2-x) & \text{for } 1 < x < 2. \end{cases}$$

Ans:- $C_0 = \frac{\pi}{2}$, $C_n = \frac{(-1)^n - 1}{n^2\pi}$, $n \neq 0$

⑥ Find the Exponential Fourier Series of the function $f(x) = e^{ax}$ over $(-\pi, \pi)$

$$C_n = \left(\frac{\sinh a\pi}{\pi}\right)(-1)^n \frac{(a+in)}{a^2+n^2}$$

Harmonic Analysis :-

Find the Fourier Series by considering a set of values of x and the corresponding values of the function is called as Harmonic Analysis.

If m is the mean value of $\phi(x)$ over the interval (a,b) then $m = \frac{1}{(b-a)} \int_a^b \phi(x)\,dx = [[\phi(x)]]$.

Consider the Fourier Series of the function $f(x)$ defined over $(a, a+2l)$

then $a_0 = \frac{1}{l} \int_a^{a+2l} f(x)\,dx = 2\left[\frac{1}{2l} \int_a^{a+2l} f(x)\,dx\right] = 2\left[[f(x)]\right]$

$$a_n = 2\left[\left[f(x)\cos\left(\frac{n\pi x}{l}\right)\right]\right]$$

$$b_n = 2\left[\left[f(x) \cdot \sin\left(\frac{n\pi x}{l}\right)\right]\right]$$

Note:- ① While taking Averages we will not consider the last value of x and $f(x)$ because the value of the function at x_0 and x_n are same i.e. $f(x_0) = f(x_n)$.

② While considering 'n' also we will not count the last value of x i.e. x_n