poisson distribution.

Find the mean and variance of a poisson distribution.

Sol:- For Poisson distribution the P.D.F is

$$P(x) = \frac{e^{-K} K^{x}}{x!}$$ where K is constant.

The distribution is

x	0	1	2	3	. . .
$P(x)$	e^{-K}	$e^{-K}K$	$\frac{e^{-K}K^2}{2!}$	$\frac{e^{-K}K^3}{3!}$	. . .

$$\text{Mean} = \mu = E[X] = \sum x_i \, p(x_i)$$

$$= 0 \cdot e^{-K} + 1 \cdot Ke^{-K} + \cancel{2} \cdot \frac{K^2}{\cancel{2!}} e^{-K} + \cancel{3} \cdot \frac{e^{-K} \cdot K^3}{\cancel{3!}\, 2!} + \dots$$

$$= Ke^{-K} \left\{ 1 + K + \frac{K^2}{2!} + \dots \right\}$$

$$\left[\text{But } 1 + K + \frac{K^2}{2!} + \dots = e^{K} \right]$$

$$= Ke^{-K} \cdot e^{K} = Ke^{-K+K} = Ke^{0} = K \quad \therefore \boxed{\mu = K}$$

$$E[X^2] = \sum x_i^2 \, p(x_i) = \sum (x_i^2 - x_i + x_i) \, p(x_i)$$

$$= \sum \{ x_i (x_i - 1) \} p(x_i) + \sum x_i \, p(x_i)$$

$$= 0 \cancel{(-1)e^{-K}} + 1 \cancel{(0) Ke^{-K}} + \cancel{2}(1) \frac{K^2 e^{-K}}{\cancel{2!}} + 3 \cancel{(2)} \frac{K^3 e^{-K}}{\cancel{3!}} + \dots + \mu$$

$$= K^2 e^{-K} \left\{ 1 + K + \frac{K^2}{2!} + \dots \right\} + K$$

$$E[X^2] = K^2 e^{-K} e^{K} + K = K^2 + K.$$

$$\text{Variance} = \sigma^2 = E[X^2] - \mu^2 = K^2 + K - K^2 = K.$$

$\therefore$ For poisson distribution variance and mean are the same.

S.T. Poisson distribution is the limiting form of Binomial distribution.

For a Binomial distribution the P.D.F is

$$P(x) = {}^{n}C_{x} \, p^{x} q^{n-x} \quad \text{where } p + q = 1.$$

$$P(x) = \frac{n(n-1)(n-2) \dots (n-(x-1))}{x!} p^{x} (1-p)^{n-x}.$$

$$P(x) = \frac{n \cdot n(1-\frac{1}{n}) \cdot n(1-\frac{2}{n}) \dots n(1-\frac{(x-1)}{n}) \cdot p^{x} (1-p)^{n}}{x! \, (1-p)^{x}}.$$

$$P(x) = \frac{n^{x} \cdot p^{x} (1-\frac{1}{n})(1-\frac{2}{n}) \dots (1-\frac{(x-1)}{n}) \cdot (1-p)^{n}}{x! \, (1-p)^{x}}.$$

let $np = K$ i.e. $p = \frac{K}{n}$.

$$P(x) = \frac{K^{x} (1-1/n)(1-2/n) \dots (1-\frac{(x-1)}{n}) (1-\frac{K}{n})^{n}}{x! \, (1-\frac{K}{n})^{x}}$$

We have to take the lim as $p \to 0$, $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n} = 0, \quad \lim_{n\to\infty} \frac{2}{n} = 0, \quad \dots \quad \lim_{n\to\infty} \left(1 - \frac{(x-1)}{n}\right) = 0$$

$\lim_{p\to 0} (1-p)^x = 1^x = 1$

$$\lim_{n\to\infty} \left(1-\frac{k}{n}\right)^n = \lim_{n\to\infty}\left[\left\{1+\frac{1}{(-n/k)}\right\}^{\frac{-n}{k}}\right]^{-k}$$

$= e^{-k}$ Because $\lim_{x\to\infty} (1+f(x))^{1/f(x)} = e.$

$$\therefore \lim_{\substack{n\to\infty \\ p\to 0}} P(x) = \frac{k^x \cdot 1 \cdot 1 \cdots 1 \cdot e^{-k}}{x! \cdot 1} = \frac{e^{-k}k^x}{x!}$$

$=$ P.D.F of poisson distribution.

<u>Note</u>:- Since $\mu = \sigma^2 = k$.

We rewrite the P.D.F of poisson distribution as $P(x) = P(\mu, x) = \dfrac{e^{-\mu}\mu^x}{x!}$

<u>Exponential distribution</u>:-

The continuous probability ~~function~~ distribution for which the P.D.F is

$$E(\alpha, x) = \begin{cases} \alpha e^{-\alpha x} & , 0 \le x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

is called as Exponential distribution. $E(\alpha, x)$ satisfies the conditions (i) $E(\alpha, x) \ge 0$ (ii) $\int_{-\infty}^{\infty} E(\alpha, x) = 1$.

Mean $= \mu = E[X] = \int_{-\infty}^{\infty} x\, E(\alpha, x)\, dx$.

$$= \int_{-\infty}^{0} x \cdot 0\, dx + \int_0^{\infty} x \cdot \alpha e^{-\alpha x}\, dx = \alpha \int_0^{\infty} x e^{-\alpha x}\, dx$$

$$= \alpha \left[x\left(\frac{e^{-\alpha x}}{-\alpha}\right) - 1\left(\frac{e^{-\alpha x}}{(-\alpha)^2}\right)\right]_0^{\infty} = \alpha\left[-\frac{1}{\alpha}\left(xe^{-\alpha x}\right)_0^{\infty} - \frac{1}{\alpha^2}\left(e^{-\alpha x}\right)_0^{\infty}\right]$$

$$= \alpha\left[-\frac{1}{\alpha}(0-0) - \frac{1}{\alpha^2}(e^{-\infty} - e^0)\right] = \alpha\left[-\frac{1}{\alpha^2}(-1)\right] = \frac{\alpha}{\alpha^2} = \frac{1}{\alpha}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 E(\alpha, x)\, dx = \int_{-\infty}^{0} x^2 \cdot 0\, dx + \int_0^{\infty} x^2 \cdot \alpha e^{-\alpha x}\, dx$$

$$= \alpha\left[x^2\left(\frac{e^{-\alpha x}}{-\alpha}\right) - (2x)\left(\frac{e^{-\alpha x}}{(-\alpha)^2}\right) + -2\left(\frac{e^{-\alpha x}}{(-\alpha)^3}\right)\right]_0^{\infty}$$

$$= \alpha\left[-\frac{1}{\alpha}\left(x^2 e^{-\alpha x}\right)_0^{\infty} - \frac{2}{\alpha^2}\left(xe^{-\alpha x}\right)_0^{\infty} - \frac{2}{\alpha^3}\left(e^{-\alpha x}\right)_0^{\infty}\right]$$

$$= \alpha\left[-\frac{1}{\alpha}(0) - 2/\alpha^2 (0) - 2/\alpha^3 (e^{-\infty} - e^0)\right]$$

$$= \alpha\left[\frac{2}{\alpha^3}\right] = \frac{2\alpha}{\alpha^3} = \frac{2}{\alpha^2}.$$

Variance $= \sigma^2 = E[X^2] - \mu^2 = \dfrac{2}{\alpha^2} - \dfrac{1}{\alpha^2} = \dfrac{1}{\alpha^2}$

S.D $= \sigma = \dfrac{1}{\alpha}$.

$\therefore$ For a uniform distribution, Mean and the Standard deviation are the same.

(Cumulative distribution function)

P.D.F is $f(\alpha, x) = \begin{cases} 0, & -\infty < x < 0 \\ \alpha e^{-\alpha x}, & 0 < x < \infty \end{cases}$

C.D.F is $F(t) = \int_{-\infty}^{t} f(\alpha, x)\,dx$

For $-\infty < t < 0$, $F(t) = \int_{-\infty}^{t} 0 \cdot dx = 0$

For $0 < t < \infty$, $F(t) = \int_{-\infty}^{t} f(\alpha, x)\,dx = \int_{-\infty}^{0} f(\alpha, x)\,dx + \int_{0}^{t} f(\alpha, x)\,dx$

$= \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{t} \alpha e^{-\alpha x}\,dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^t = -\left[e^{-\alpha t} - e^{0}\right]$

$= \left[1 - e^{-\alpha t}\right]$

$\therefore$ C.D.F is $F(t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-\alpha t}, & 0 < t < \infty \end{cases}$

1) The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find the probability that a call (i) ends in less than 3 minutes (ii) takes between 3 and 5 minutes.

Sol: $\mu = 3$ $\therefore \alpha = \frac{1}{\mu} = \frac{1}{3}$.

P.D.F $= f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{3} e^{-x/3}, & 0 < x < \infty \end{cases}$

(i) probability that a call ends in less than 3 minutes $= P(X < 3) = P(-\infty < x < 3)$

$= \int_{-\infty}^{3} f(x)\,dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{3} \frac{1}{3} e^{-x/3}\,dx$

$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^3 = -1\left[e^{-3/3} - e^{0}\right]$

$= 1 - e^{-1} = 1 - \frac{1}{e} = 0.6321.$

(ii) Probability that a call takes between 3 and 5 minutes

$= P(3 < x < 5) = \int_{3}^{5} f(x)\,dx = \int_{3}^{5} \frac{1}{3} e^{-\frac{x}{3}}\,dx$

$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^5 = -\left[e^{-5/3} - e^{-3/3}\right]$

$= e^{-1} - e^{-5/3} = 0.179$

2) The increase in Sales per day in a shop is exponentially distributed with Rs 800 as the Average. If Sales tax is levied at the rate of 6%, find the probability that the increase in sales tax return from that shop will exceed Rs 30 per day.

Sol: let x denote the increase in sales per day

$\mu = \text{Rs } 800 \quad \therefore \alpha = \frac{1}{\mu} = \frac{1}{800}$

let A be the sales for which the sales tax is Rs 30. Since sales tax is 6% of the sales we have $A \times \frac{6}{100} = 30 \quad \therefore A = \frac{\cancel{30} \times 100}{\cancel{6}} = 500$

Since the increase in sales tax has to exceed Rs 30, the increase in sales has to exceed Rs 500. The probability that the increase in sales tax exceeds Rs 30 is given by $P(X > 500) = P(500 < X < \infty)$

$$= \int_{500}^{\infty} f(x)\,dx. \quad \text{But } f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{800} e^{-\frac{x}{800}}, & 0 < x < \infty \end{cases}$$

$$= \int_{500}^{\infty} \frac{1}{800} e^{-\frac{x}{800}}\,dx = \frac{1}{800}\left[\frac{e^{-\frac{x}{800}}}{-\frac{1}{800}}\right]_{500}^{\infty}$$

$$= (-1)\left[e^{-\infty} - e^{-\frac{500}{800}}\right] = e^{-\frac{500}{800}} = 0.5353.$$

3) In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that the shower will last for (i) less than 10 minutes (ii) 10 minutes or more.

Ans:- (i) 0.8647 (ii) 0.1353

D.F: (Cumulative distribution function)

The p.D.F is $f(\alpha, x) = \begin{cases} 0, & -\infty < x < 0 \\ \alpha e^{-\alpha x}, & 0 < x < \infty \end{cases}$

C.D.F is $F(t) = \int_{-\infty}^{t} f(\alpha, x)\, dx$.

For $-\infty < t < 0$, $F(t) = \int_{-\infty}^{t} 0 \cdot dx = 0$

For $0 < t < \infty$, $F(t) = \int_{-\infty}^{t} f(\alpha, x)\, dx = \int_{-\infty}^{0} f(\alpha, x)\, dx + \int_{0}^{t} f(\alpha, x)\, dx$

$= \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{t} \alpha e^{-\alpha x}\, dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^t = -\left[e^{-\alpha t} - e^{0}\right]$

$= \left[1 - e^{-\alpha t}\right]$

$\therefore$ C.D.F is $F(t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-\alpha t}, & 0 < t < \infty \end{cases}$

1) The length of a telephone conversation has an exponential distribution with a mean of 3 minutes. Find the probability that a call (i) ends in less than 3 minutes (ii) takes between 3 and 5 minutes.

Sol: $\mu = 3$ $\therefore \alpha = \frac{1}{\mu} = \frac{1}{3}$.

p.D.F $= f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{3} e^{-x/3}, & 0 < x < \infty \end{cases}$

(i) probability that a call ends in less than 3 minutes $= P(X < 3) = P(-\infty < x < 3)$

$= \int_{-\infty}^{3} f(x)\, dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{3} \frac{1}{3} e^{-x/3}\, dx$

$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3}\right]_0^3 = -1 \left[e^{-3/3} - e^{0}\right]$

$= 1 - e^{-1} = 1 - \frac{1}{e} = 0.6321$.

(ii) Probability that a call takes between 3 and 5 minutes

$= P(3 < x < 5) = \int_{3}^{5} f(x)\, dx = \int_{3}^{5} \frac{1}{3} e^{-\frac{x}{3}}\, dx$

$= \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3}\right]_3^5 = -\left[e^{-5/3} - e^{-3/3}\right]$

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2) The increase in sales per day in a shop is exponentially distributed with Rs 800 as the Average. If sales tax is levied at the rate of 6%, find the probability that the increase in sales tax return from that shop will exceed Rs 30 per day.

Sol: let x denote the increase in sales per day

mean $= \mu =$ Rs 800 $\quad \therefore \alpha = \frac{1}{\mu} = \frac{1}{800}$

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Since the increase in sales tax has to exceed Rs 30, the increase in sales has to exceed Rs 500. The probability that the increase in sales tax exceeds Rs 30 is given by $P(X > 500) = P(500 < X < \infty)$

$$= \int_{500}^{\infty} f(x)\,dx. \quad \text{But } f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{800} e^{-\frac{x}{800}}, & 0 < x < \infty \end{cases}$$

$$= \int_{500}^{\infty} \frac{1}{800} e^{-\frac{x}{800}}\,dx = \frac{1}{800}\left[\frac{e^{-\frac{x}{800}}}{-\frac{1}{800}}\right]_{500}^{\infty}$$

$$= (-1)\left[e^{-\infty} - e^{-\frac{500}{800}}\right] = e^{-\frac{500}{800}} = 0.5353.$$

3) In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that the shower will last for (i) less than 10 minutes (ii) 10 minutes or more.

Ans:- (i) 0.8647 (ii) 0.1353

S.7. The poisson distribution is the limiting case of Binomial distribution as $n \to \infty$, $p \to 0$.

Sol.: The probability density function of Binomial distribution is

$$b(x) = b(n, p, x) = {}^nC_x \, p^x q^{n-x}, \text{ where } q = 1-p.$$

$$b(x) = \frac{n(n-1)(n-2)\ldots(n-(x-1))}{x!} p^x (1-p)^{n-x}.$$

$$b(x) = \frac{n \cdot n(1-\frac{1}{n}) \cdot n(1-\frac{2}{n}) \ldots n\left(1-\frac{(x-1)}{n}\right) p^x (1-p)^n}{x! \, (1-p)^x}.$$

$$b(x) = \frac{n^x p^x (1-\frac{1}{n})(1-\frac{2}{n}) \ldots \left(1-\frac{(x-1)}{n}\right)(1-p)^n}{x! \, (1-p)^x}.$$

let $\mu = np$ ie. $p = \frac{\mu}{n}$.

$$b(x) = \frac{\mu^x (1-\frac{1}{n})(1-2/n) \ldots \left(1-\frac{(x-1)}{n}\right)(1-\frac{\mu}{n})^n}{x! \, (1-p)^x}.$$

But $\left\{1-\frac{\mu}{n}\right\}^n = \left[\left\{1+\left(\frac{1}{-\frac{n}{\mu}}\right)\right\}^{-\frac{n}{\mu}}\right]^{-\mu}.$

$$\lim_{n\to\infty}\left(1-\frac{\mu}{n}\right)^n = \lim_{n\to\infty}\left[\left\{1+\frac{1}{\left(-\frac{n}{\mu}\right)}\right\}^{\left(-\frac{n}{\mu}\right)}\right]^{-\mu}$$

$$= e^{-\mu}$$

$$\therefore b(x) = \frac{\mu^x (1-\frac{1}{n})(1-\frac{2}{n}) \ldots \left(1-\frac{(x-1)}{n}\right)\left[\left(1+\frac{1}{\left(-\frac{n}{\mu}\right)}\right)^{-\frac{n}{\mu}}\right]^{-\mu}}{x! \, (1-p)^x}.$$

Taking lim as $p \to 0$, $n \to \infty$, we get

$$\lim_{\substack{p\to 0\\ n\to\infty}} b(x) = \frac{\mu^x (1-0)(1-0) \ldots (1-0)\, e^{-\mu}}{x! \, (1-0)^x}.$$

$$= \frac{\mu^x e^{-\mu}}{x!} = p(\mu, x).$$

the probability density function of poisson distribution.

$\therefore$ poisson distribution is the limiting case of Binomial distribution as $p \to 0$, $n \to \infty$.

Exponential distribution.

let $\alpha > 0$, then the continuous probability distribution for which

$$E(\alpha, x) = \begin{cases} \alpha e^{-\alpha x}, & 0 \le x < \infty \\ 0, & \text{elsewhere} \end{cases} = \begin{cases} 0, & -\infty < x \le 0 \\ \alpha e^{-\alpha x}, & 0 \le x < \infty \end{cases}$$

is the Probability density function is called the negative exponential distribution or just the exponential distribution.

In $E(\alpha, x)$, x is called the exponential variate, α is called the parameter of the distribution.

(i) $E(\alpha, x) \ge 0$ (ii) $\int_{-\infty}^{\infty} E(\alpha, x)\,dx = 1$.

S.T. the mean and standard deviation of an exponential distribution are equal.

P.D.F of Exponential distribution is $E(\alpha, x) = \begin{cases} 0, & -\infty < x \le 0 \\ \alpha e^{-\alpha x}, & 0 \le x < \infty \end{cases}$

$$\text{Mean} = \mu = E[X] = \int_{-\infty}^{\infty} x\, E(\alpha, x)\,dx.$$

$$= \int_{-\infty}^{0} x \cdot 0 \cdot dx + \int_{0}^{\infty} x \cdot \alpha e^{-\alpha x}\,dx = \alpha \int_{0}^{\infty} x e^{-\alpha x}\,dx.$$

By using Bernoulli's rule.

$$= \alpha \left[x\left(\frac{e^{-\alpha x}}{-\alpha}\right) - 1\left(\frac{e^{-\alpha x}}{(-\alpha)^2}\right)\right]_0^{\infty}$$

$$= \alpha \left[-\frac{1}{\alpha}\left(x e^{-\alpha x}\right)_0^{\infty} - \frac{1}{\alpha^2}\left(e^{-\alpha x}\right)_0^{\infty}\right]$$

$$= \alpha \left[-\frac{1}{\alpha}\cancel{(0-0)}^{\,0} - \frac{1}{\alpha^2}\left(e^{-\infty} - e^{0}\right)\right] \quad \text{But } e^{-\infty} = \frac{1}{e^{\infty}} = 0$$

$$= \alpha \left[-\frac{1}{\alpha^2}(-1)\right] = \frac{\alpha}{\alpha^2} \quad \therefore \boxed{\mu = \frac{1}{\alpha}}.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 E(\alpha, x)\,dx = \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_{0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx$$

$$= \alpha \int_0^{\infty} x^2 e^{-\alpha x}\,dx = \alpha \left[\cancel{x^2\left(\frac{e^{-\alpha x}}{-\alpha}\right)}^{\,0} - \cancel{(2x)\left(\frac{e^{-\alpha x}}{(-\alpha)^2}\right)}^{\,0} + 2\left(\frac{e^{-\alpha x}}{(-\alpha)^3}\right)\right]_0^{\infty}$$

$$= \alpha\left(-\frac{2}{\alpha^3}\right)\left[e^{-\alpha x}\right]_0^{\infty} = -\frac{2}{\alpha^2}\left[\cancel{e^{-\infty}} - e^{0}\right] = \frac{2}{\alpha^2}$$

$$\text{Variance} = \sigma^2 = E[X^2] - \mu^2 = \frac{2}{\alpha^2} - \frac{1}{\alpha^2} = \frac{1}{\alpha^2}.$$

$$\text{Standard deviation} = \sigma = \frac{1}{\alpha}$$

Here $\boxed{\mu = \sigma = \frac{1}{\alpha}}$.

Find the Cummulative distribution function of the Exponential distribution.

$$F(t) = P(-\infty < x < t) = \int_{-\infty}^{t} E(\alpha, x)\,dx.$$

$$E(\alpha, x) = \begin{cases} 0, & -\infty < x < 0 \\ \alpha e^{-\alpha x}, & 0 < x < \infty \end{cases}$$

For $-\infty < t < 0$, $F(t) = \int_{-\infty}^{t} E(\alpha, x)\,dx = \int_{-\infty}^{t} 0\,dx = 0$.

For $0 < t < \infty$, $F(t) = \int_{-\infty}^{t} f(\alpha, x)\,dx = \cancel{\int_{-\infty}^{0} 0\,dx}^{\to 0} + \int_{0}^{t} \alpha e^{-\alpha x}\,dx$

$= \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^t = -\left[e^{-\alpha x}\right]_0^t = -\left[e^{-\alpha t} - e^{0}\right] = 1 - e^{-\alpha t}$.

$\therefore$ C.D.F is $F(t) = \begin{cases} 0, & -\infty < t < 0 \\ 1 - e^{-\alpha t}, & 0 \le t < \infty \end{cases}$

1) In a certain Town, the duration of a shower is exponentialy distributed with mean equal to 5 minutes. What is the probability that a shower will last for (i) 10 minutes or more? (ii) less than 10 minutes? (iii) between 10 minutes and 12 minutes.

Sol: $\mu = 5$ $\therefore \alpha = \frac{1}{5} = 0.2$.

P.D.F is $f(x) = f(\alpha, x) = f(0.2, x) = \begin{cases} 0, & -\infty < x < 0 \\ 0.2e^{-0.2x}, & 0 < x < \infty \end{cases}$

(i) Probability that a shower will last for 10 minutes or more $= P(x > 10) = P(10 < x < \infty) = \int_{10}^{\infty} f(\alpha, x)\,dx$

$= \int_{10}^{\infty} 0.2 e^{-0.2x}\,dx = \cancel{0.2}\left[\frac{e^{-0.2x}}{-0.2}\right]_{10}^{\infty} = \frac{e^{-\infty} - e^{-2}}{(-1)}$

$= 0 - \frac{e^{-2}}{-1} = e^{-2} = \frac{1}{e^2} = 0.1353$

(ii) Probability that a shower will last for less that 10 minutes $= P(x < 10) = P(-\infty < x < 10)$

$= \int_{-\infty}^{10} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{10} f(x)\,dx$.

$= \int_{0}^{10} 0.2 e^{-0.2x}\,dx = 0.2\left[\frac{e^{-0.2x}}{-0.2}\right]_0^{10} = \frac{e^{-2} - e^{0}}{-1}$ (ii) (3A)

$= 1 - e^{-2} = 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2} = 0.8647$

(iii) Probability that shower will last between 10 minutes and 12 minutes.

$= P(10 < x < 12) = \int_{10}^{12} f(\alpha, x)\,dx = \int_{10}^{12} 0.2 e^{-0.2x}\,dx$

$= 0.2\left[\frac{e^{-0.2x}}{-0.2}\right]_{10}^{12} = \frac{e^{-2.4} - e^{-2}}{(-1)} = e^{-2} - e^{-2.4}$

$= \frac{1}{e^2} - \frac{1}{e^{2.4}}$

2. The duration of a telephone conversation has found to have an exponential distribution with mean 3 mins. Find the probability that the conversation may lars (i) more that 1 min (ii) less than 3 mins.

Ans: (i) $e^{-\frac{1}{3}}$ (ii) $1 - e^{-1}$

Normal distribution:-

Let μ and σ be two arbitrary real constants such that $-\infty < \mu < \infty$ and $\sigma > 0$. Then the continuous probability distribution for which

$$N(x) = N(\mu, \sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is the P.D.F is called the normal distribution. The corresponding continuous random variable is called the normal variate.

$N(x)$ satisfy the conditions

(i) $N(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} N(x)\,dx = 1$.

Find the mean and standard deviation of normal distribution.

$$\left[\Gamma(k) = \int_0^{\infty} e^{-t} t^{k-1} dt, \quad \Gamma(\tfrac{1}{2}) = 2\int_0^{\infty} e^{-t^2} dt = \sqrt{\pi}\right.$$
$$\left.\Gamma(n+1) = n\Gamma(n)\right]$$

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x\, N(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $\frac{x-\mu}{\sigma} = z$ i.e. $x - \mu = \sigma z$, $x = \mu + \sigma z$

$dx = \sigma\, dz$.

The limits remains the same.

$$= \int_{-\infty}^{\infty} \frac{(\mu + \sigma z)}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} \mu e^{-\frac{z^2}{2}} dz + \int_{-\infty}^{\infty} \sigma z\, e^{-\frac{z^2}{2}} dz\right]$$

We have $\int_{-\infty}^{\infty} f(z)\,dz = \begin{cases} 2\int_0^{\infty} f(z)\,dz & \text{if } f(-z) = f(z) \text{ even function} \\ 0 & \text{if } f(-z) = -f(z) \text{ odd function.} \end{cases}$

$e^{-\frac{z^2}{2}}$ is an even function, $z e^{-\frac{z^2}{2}}$ is an odd function.

$$\text{Mean} = \frac{1}{\sqrt{2\pi}} 2\int_0^{\infty} \mu e^{-\frac{z^2}{2}} dz.$$

put $\frac{z^2}{2} = t^2$, $z^2 = 2t^2$

$z = \sqrt{2}\cdot t$

$dz = \sqrt{2}\,dt$.

$$\text{Mean} = \frac{2\mu}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \cancel{\sqrt{2}}\, dt = \frac{\mu}{\sqrt{\pi}}\left(2\int_0^{\infty} e^{-t^2} dt\right)$$

$$\text{Mean} = \frac{\mu}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}}$$

$$\text{Variance} = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 N(x)\,dx.$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{\sigma^2 z^2}{\cancel{\sigma}\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz$$

But $z^2 e^{-\frac{z^2}{2}}$ is an even function

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left(2\int_0^{\infty} z^2 e^{-\frac{z^2}{2}} dz\right)$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left(2\int_0^\infty z^2 e^{-\frac{z^2}{2}}\,dz\right)$$

put $\frac{z^2}{2} = t$, $z^2 = 2t$, $z = (\sqrt{2})\,t^{1/2}$

$dz = (\sqrt{2})\,\frac{1}{2}\,t^{\frac{1}{2}-1}\,dt$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\,2\int_0^\infty \cancel{2}t^1 e^{-t}\,\frac{\sqrt{2}}{\cancel{2}}\,t^{\frac{1}{2}-1}\,dt$$

$$= \frac{2\sigma^2}{\cancel{\sqrt{2}}\cdot\sqrt{\pi}}\cdot\cancel{\sqrt{2}}\int_0^\infty e^{-t}\,t^{3/2-1}\,dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\,\Gamma(3/2) = \frac{2\sigma^2}{\sqrt{\pi}}\,\Gamma(1/2+1) = \frac{\cancel{2}\sigma^2}{\sqrt{\pi}}\,\frac{1}{\cancel{2}}\,\Gamma(\tfrac{1}{2})$$

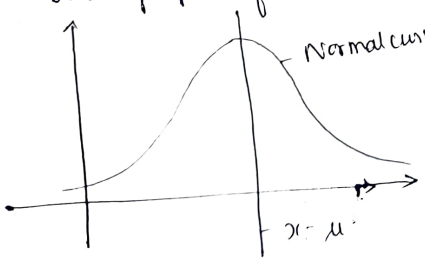
Variance $= \frac{\sigma^2}{\cancel{\sqrt{\pi}}}\cdot\cancel{\sqrt{\pi}} = \sigma^2$.

Standard deviation $= \sqrt{\text{Variance}} = \sigma$

Evaluation of $P(a \le x \le b)$.

$$P(a \le x \le b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx = \int_a^b N(x)\,dx$$

The graph of $N(x)$ is as follows.



The curve is symmetric about $x = \mu$.

put $\frac{x-\mu}{\sigma} = z$

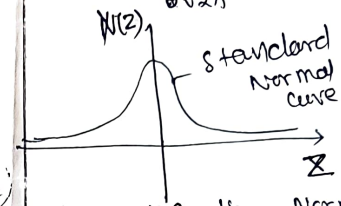
$x - \mu = \sigma z$

$dx = \sigma\,dz$

let the limits of z be z_1 and z_2, then.

$$P(a \le x \le b) = \int_{z_1}^{z_2} \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\,e^{-\frac{z^2}{2}}\,\cancel{\sigma}\,dz = P(z_1 \le z \le z_2)$$

$N(z) = \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\,e^{-\frac{z^2}{2}}$ graph is as follows.



Which is symmetric about $N(z)$ axis.

We can find $P(z_1 \le z \le z_2)$ by using the Normal Probability table. which gives value of $A(z) = \int_0^z \frac{1}{\sqrt{2\pi}}\,e^{-\frac{z^2}{2}}\,dz$.

$A(\infty) = P(0 < z < \infty) = \int_0^\infty \frac{1}{\sqrt{2\pi}}\,e^{-\frac{z^2}{2}}\,dz = 0.5$

$P(z_1 \le z \le z_2) = A(z_2) - A(z_1)$

$P(-z_1 \le z \le -z_2) = P(z_2 \le z \le z_1) = A(z_1) - A(z_2)$

$P(-z_1 \le z \le z_2) = P(-z_1 \le z \le 0) + P(0 \le z \le z_2)$

$= P(0 \le z \le z_1) + P(0 \le z \le z_2)$

$= A(z_1) + A(z_2) = A(z_2) + A(z_1)$

$P(z \ge z_1) = P(z_1 \le z < \infty) = A(\infty) - A(z_1) = 0.5 - A(z_1)$

$P(z \le z_1) = P(-\infty < z \le z_1) = A(z_1) + A(\infty)$

$= A(z_1) + 0.5$

① For the normal distribution with mean 2 and Standard deviation 4, Evaluate the following Probabilities.

(i) $P\{x \geq 5\}$ (ii) $P\{|x| < 4\}$ (iii) $P\{|x| > 3\}$

(iv) $P(-6 < x < 3)$.

Solution:- let $\frac{x-\mu}{\sigma} = z$. Here $\mu = 2$, $\sigma = 4$.

$\therefore \frac{x-2}{4} = z$ ie. $x-2 = 4z$, $x = 4z+2$

(i) $P\{x \geq 5\} = P\{4z+2 \geq 5\} = P\{4z \geq 5-2\}$

$= P\{\frac{4z}{4} \geq \frac{3}{4}\} = P\{z \geq 0.75\} = P(0.75 \leq z \leq \infty)$

$= A(\infty) - A(0.75) = 0.5 - 0.2734$ $[A(0.75) = 0.2734]$

$= 0.2266$.

(ii) $P\{|x| < 4\} = P\{-4 < x < 4\} = P\{-4 < 4z+2 < 4\}$

$= P\{-4-2 < 4z < 4-2\} = P\{-6 < 4z < 2\} = P\{\frac{-6}{4} < \frac{4z}{4} < \frac{2}{4}\}$

$= P(-1.5 < z < 0.5) = A(0.5) + A(1.5) = 0.1915 + 0.4332$

$= 0.6247$

$[A(0.5) = 0.1915$, $A(1.5) = 0.4332]$

(iii) $P\{|x| > 3\} = 1 - P\{|x| \leq 3\} = 1 - P\{-3 \leq x \leq 3\}$

$= 1 - P\{-3 \leq 4z+2 \leq 3\} = 1 - P\{-5 \leq 4z \leq 1\}$

$= 1 - P\{-1.25 \leq z \leq 0.25\} = 1 - [A(0.25) + A(1.25)]$

But $A(0.25) = 0.0987$, $A(1.25) = 0.3944$

$= 1 - 0.0987 - 0.3944 =$

(iv) $P(-6 < x < 3) = P(-6 < 4z+2 < 3) = P(-8 < 4z < 1)$

$= P(-2 < z < 0.25)$. $A(0.25) + A(2.00)$

$= 0.0987 + 0.4772 = 0.5759$

② In an examination taken by 500 candidates the average and S.D of marks obtained are 40% and 10% respectively. Assuming Normal distribution, find (i) how many have scored above 60% (ii) how many will pass if 50% is fixed the minimum for passing and (iii) what should be the minimum for 350 candidates to pass.

Sol: x be the normal variable denoting the % of marks scored by a student

$z = \frac{x-\mu}{\sigma}$ If $\mu = 40$, $\sigma = 10$

$z = \frac{x-40}{10}$ ie. $x-40 = 10z$, $x = 10z+40$.

The probability that a student is scored above 60% $= P(x > 60) = P(10z+40 > 60)$

$= P(10z > 20) = P(z > 2) = P(2 < z < \infty)$

$= A(\infty) - A(2) = 0.5 - 0.4772 = 0.0228$.

No. of students scored above 60% out of 500

$= 500 \times 0.0228 = 11.4 \approx 11$

The probability that a student will score more than 50% $= P(x > 50)$

$= P(102+40>50) = P(102>10) = P(z>1)$

$= P(1<z<\infty) = A(\infty) - A(1) = 0.5 - 0.3413$
$= 0.1587$.

No. of students who pass with 50% as the minimum for passing is $= 0.1587 \times 500 = 79.35 \approx 79$.

Let m% be the minimum for passing if 350 of the 500 candidates are to pass. Then we should have $500 \times P(x \geq m) = 350$

$P(x \geq m) = \frac{350}{500} = 0.7$

i.e. $P(102+40 \geq m) = 0.7$.

$P\left(z \geq \frac{m-40}{10}\right) = 0.7$.

$P\left(\frac{m-40}{10} \leq z < \infty\right) = 0.7$.

i.e. $A(\infty) - A\left(\frac{m-40}{10}\right) = 0.7$.

i.e. $-A\left(\frac{m-40}{10}\right) = 0.7 - A(\infty)$

i.e. $-A\left(\frac{m-40}{10}\right) = 0.7 - 0.5 = 0.2$

But $A(0.55) = 0.2$.

$\therefore -A\left(\frac{m-40}{10}\right) = A(0.55)$

i.e. $-\left(\frac{m-40}{10}\right) = 0.55$, $-m+40 = 5.5$

$-m = -40+5.5$

$-m = -34.5$, $m = 34.5\%$.

(3) In a normal distribution, 7% are under 35 and 89% are under 63. Find the mean and the standard deviation, given that $A(1.23) = 0.39$ and $A(1.48) = 0.43$ in the usual notation. (45)

Sol:- $P(x<35) = 7\% = 0.07$, $P(x<63) = 89\% = 0.89$

let $\frac{x-\mu}{\sigma} = z$ i.e. $x = \sigma z + \mu$.

(i) $P(x<35)$

$= P(\sigma z + \mu < 35)$

$= P(\sigma z < 35-\mu)$

$= P\left(z < \frac{35-\mu}{\sigma}\right)$

$= P\left(-\infty < z < \frac{35-\mu}{\sigma}\right)$

$= A\left(\frac{35-\mu}{\sigma}\right) + A(\infty)$

$= A\left(\frac{35-\mu}{\sigma}\right) + A(\infty)$

But $P(x<35) = 0.07$

$\therefore A\left(\frac{35-\mu}{\sigma}\right) + A(\infty) = 0.07$

$A\left(\frac{35-\mu}{\sigma}\right) = 0.07 - 0.5$

$A\left(\frac{35-\mu}{\sigma}\right) = -0.43$

$A\left(\frac{35-\mu}{\sigma}\right) = -A(1.48)$

(ii) $P(x<63)$

$= P(\sigma z + \mu < 63)$

$= P(\sigma z < 63-\mu)$

$= P\left(z < \frac{63-\mu}{\sigma}\right)$

$= P\left(-\infty < z < \frac{63-\mu}{\sigma}\right)$

$= A\left(\frac{63-\mu}{\sigma}\right) + A(\infty)$

But $P(x<63) = 0.89$

$A\left(\frac{63-\mu}{\sigma}\right) + 0.5 = 0.89$

$A\left(\frac{63-\mu}{\sigma}\right) = 0.89 - 0.5$

$A\left(\frac{63-\mu}{\sigma}\right) = 0.39$

$A\left(\frac{63-\mu}{\sigma}\right) = A(1.23)$

$\frac{35-\mu}{\sigma} = -1.48$

$35-\mu = -1.48\,\sigma$

$35 = \mu - 1.48\,\sigma$ — ①

$\frac{63-\mu}{\sigma} = 1.23$

$63-\mu = 1.23\,\sigma$.

$63 = 1.23\,\sigma + \mu$ — ②

① - ② gives. $35-63 = (-1.48-1.23)\,\sigma$

$-28 = -2.71\,\sigma$.

$\sigma = \frac{-28}{-2.71} = 10.38$.

Substituting in ①

$35 = \mu - 1.48(10.38)$

$\mu = 35 + 1.48(10.38) \simeq 50.29$.

④ Fit a normal distribution for the following frequency distribution.

x_i	2	4	6	8	10
f_i	1	4	6	4	1

$\Sigma f_i = 1+4+6+4+1 = 16$.

$\Sigma f_i x_i = (1\times2)+(4\times4)+(6\times6)+(4\times8)+(1\times10) = 96$.

$\Sigma f_i x_i^2 = (1\times2^2)+(4\times4^2)+(6\times6^2)+(4\times8^2)+(1\times10^2) = 640$.

mean $\bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{96}{16} = 6. = \mu$

Variance $= \sigma^2 = \frac{\Sigma f_i x_i^2}{\Sigma f_i} - \bar{x}^2 = \frac{640}{16} - 36 = 4$.

$\therefore$ S.D $= \sigma = 2$.

The probability density function of normal distribution $= N(x) = N(\mu, \sigma, x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$N(x) = N(6, 2, x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-6}{2}\right)^2}$

The equation of the normal curve that fits the data is

$y = (\Sigma f_i)\, N(6, 2, x)$

$= 16 \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-6}{2}\right)^2}$

Prabhakar Kumar Dananjaya

(3) A box contains three white balls and two red balls. If two balls are drawn in succession, find the probability that the first removed ball is white and second is red?

$P(W) = \frac{3}{5}$ $\quad P(R/W) = \frac{2}{4}$

$P(W \cap R) = P(W)P(R/W) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}$.

(4) A box contains 10 white balls and 3 red balls while another bag contains 3 white balls and 5 red balls. Two balls are drawn at random from the first bag and put in the second bag and then a ball is drawn at random from the second bag. What is the probability that it is a white ball?

$P(R_2) = \frac{{}^3C_2}{{}^{13}C_2} = \frac{1}{26}$, $P(W_2) = \frac{{}^{13}C_2}{{}^{13}C_2} =$, $P(R_1 W_1) = \frac{{}^{10}C_1 \times {}^3C_1}{{}^{13}C_2}$.

$P(\cancel{W_1 R_2}) = \cancel{P}\ P_1 + P_2 + P_3 = \frac{59}{130}$

(5) If A and B are independent, S.T. (i) A and $\bar{B}$ (ii) $\bar{A}$ and B (iii) $\bar{A}$ and $\bar{B}$ are independent.

(6) Three students X, Y, Z write an examination. Their chances of passing are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that (i) all of them pass (ii) atleast one of them passes (iii) atleast two of them pass.

$\frac{1}{24}$ $\quad \frac{2}{3}$

(7) A person X can hit a target three times in five shots, a second person Y three in five shots and a third person Z three in four shots. They all fire one shot simultaneously. Find the probability that (a) atleast two shots hit (b) exactly two shots hit.

$\frac{63}{100}$, $\frac{9}{20}$.

(8) A certain problem is given to three students for solving. The probabilities of their solving the problem are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. Find the probability that the problem is solved. Ans:- $\frac{3}{4}$.

(9) S.T. $P(\bar{A}/E) = 1 - P(A/E)$, $P((A \cup B)/C) = P(A/C) + P(B/C) - P((A \cap B)/C)$

(10) Given $P(A) = x$, $P(B) = y$, $P(A \cap B) = z$ find (i) $P(A \cap \bar{B})$ (ii) $P(A/B)$ (iii) $P(\bar{A}/B)$. $P(A/\bar{B})$, $P(B/A)$, $P(\bar{A}/\bar{B})$

(11) For any events A, B, C prove the following.

$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/A \cap B)$

$P((A \cap \bar{B})/C) \neq P\{(A \cap B)/C\} = P(A/C)$

(12) Seven students of whom 4 boys and 3 are girls are selected at random; one after the other for an interview. Find the probability that the selection is alternate boys and girls?

BGBGBGB. $\frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{35}$

~~Person~~ fair Two friends A and B throw a pair of dice alternatively. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.

(i) Probability of A throwing $6 = p_1 = \frac{5}{36}$. (1,5), (2,4), (3,3), (4,2), (5,1)

A not throwing $6 = \bar{p}_1 = \frac{31}{36}$.

(ii) probability of B throwing $7 = p_2 = \frac{6}{36} = \frac{1}{6}$ (1,6), (6,1), (2,5), (5,2), (3,4), (4,3)

$$\bar{p}_2 = \frac{30}{36} = \frac{5}{6}.$$

$$p_1 + \bar{p}_1 \bar{p}_2 p_1 + (\bar{p}_1 \bar{p}_2)^2 p_1 + (\bar{p}_1 \bar{p}_2)^3 p_1 + \cdots$$

$$= p_1 \{ 1 + x + x^2 + x^3 + \cdots \} \quad \text{where } x = \bar{p}_1 \bar{p}_2$$

$$= p_1 (1 - \bar{p}_1 \bar{p}_2)^{-1} = \frac{p_1}{1 - \bar{p}_1 \bar{p}_2} = \frac{\frac{5}{36}}{1 - \frac{31}{36} \cdot \frac{5}{6}} = \frac{30}{216 - 155} = \frac{30}{61}$$

Bayes theorem:-

$$P(A_i/E) = \frac{P(E/A_i)\, P(A_i)}{\sum_{i=1}^{n} P(E/A_i)\, P(A_i)}.$$

Conditional probability

If $P(E) > 0$, conditional probability of A, given E is

$$P(A/E) = \frac{O(A \cap E)}{O(E)} = \frac{P(A \cap E)}{P(E)}.$$

Multiplication theorem

$$P(E \cap F) = P(E/F)\, P(F) = P(F/E) \cdot P(E).$$

Mutually independent events

occurence of the event E does not depend on occurence of the event F and vice versa

$$P(E/F) = P(E), \quad P(F/E) = P(F) \qquad P(E \cap F) = P(E)P(F)$$

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1)\, P(E_2) \cdots P(E_n)$$

1) Two different digits are selected at random from 1 to 9. (i) If the sum is odd, what is the probability that 2 is one of the digit selected? (ii) If 2 is one of the digit selected what is the probability that the sum is odd?

2) A die is tossed. If the number shown is even, what is the probability that is a prime number? If the number shown is prim What is the probability that it is odd?

Ans:- 1/3, 2/3.

$$m_a(t) = \sum p(x_i) + \left\{\sum (x_i - a)\, p(x_i)\right\} t + \frac{1}{2!}\left\{\sum (x_i - a)^2 p(x_i)\right\} t^2$$

$$+ \frac{1}{3!}\left(\sum (x_i - a)^3 p(x_i)\right) t^3 + \cdots$$

But $\sum p(x_i) = 1$, $\mu_r(a) = \sum (x_i - a)^r p(x_i)$, $r = 0, 1, 2, \ldots$

$$m_a(t) = 1 + \mu_1(a)\, t + \frac{\mu_2(a)}{2!} t^2 + \frac{\mu_3(a)}{3!} t^3 + \cdots$$

$$\therefore m_a(t) = \sum_{r=0}^{\infty} \mu_r(a) \frac{t^r}{r!}$$

Consider $m_a(t) = 1 + \mu_1(a) \cdot t + \frac{\mu_2(a)}{2!} t^2 + \cdots$

$$+ \mu_{r-1}(a) \frac{t^{r-1}}{(r-1)!} + \frac{\mu_r(a) t^r}{r!} + \frac{\mu_{r+1}(a) t^{r+1}}{(r+1)!} + \cdots$$

Differentiating 'r' times with 't', we get

$$\frac{d^r}{dt^r}\{m_a(t)\} = \mu_r(a) \cdot \frac{r!}{r!} + \mu_{r+1}(a) \frac{(r+1)(r) \cdots t}{(r+1)!} + \cdots$$

Taking the lim as $t \to \infty$, we get

$$\left[\frac{d^r}{dt^r}\{m_a(t)\}\right] = \mu_r(a)$$

Note: $m_a(t) = \sum e^{t(x_i - a)} p(x_i) = \sum \{e^{tx_i - at}\} p(x_i)$

$$= \sum e^{tx_i} e^{-at} p(x_i) = e^{-at} \sum e^{tx_i} p(x_i) = e^{-at} m_0(t)$$

Where $m_0(t)$ is the moment generating function about the origin.

Chebyshev's Inequality :-

For a specified $a > 0$, $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$

where μ is the mean and σ is the standard deviation.

Proof :- For a discrete probability distribution

$$\sigma^2 = E[(X - \mu)^2] = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$$

$$\sigma^2 = \sum_{i=1}^{k-1} (x_i - \mu)^2 p(x_i) + \sum_{i=k}^{n} (x_i - \mu)^2 p(x_i)$$

Since each term of both the series are +ve,

we can write it as $\sigma^2 \geq \sum_{i=k}^{n} (x_i - \mu)^2 p(x_i)$

Choose k such that $x_k = \mu + a$ i.e. $(x_k - \mu)^2 = a^2$

Since $x_i > x_k$ for $i = k+1, k+2, \ldots$

$$(x_i - \mu)^2 \geq (x_k - \mu)^2 = a^2$$

$$\therefore \sigma^2 \geq a^2 \sum_{i=k}^{n} p(x_i)$$ Taking the lim as $n \to \infty$

$$\sigma^2 \geq a^2 \sum_{i=k}^{\infty} p(x_i)$$ But $\sum_{i=k}^{\infty} p(x_i) = P(X \geq x_k)$

$\sigma^2 \geq a^2 P(X \geq \mu + a)$ Because $x_k = \mu + a$.

$\sigma^2 \geq a^2 P(X - \mu \geq a)$ Since $a > 0$

$\sigma^2 \geq a^2 P(|X - \mu| \geq a)$ i.e. $P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$

$P(X<1) = P(-2)+P(-1)+P(0) = 0.1+0.1+0.2 = 0.4$

$P(-1<X\le 2) = P(0)+P(1)+P(2) = 0.2+0.2+0.3 = 0.7$

$P(X>-1) = 1-P(X\le -1) = 1-P(-2)-P(-1)$

$= 1-0.1-0.1 = 1-0.2 = 0.8$

(3) Find $E[X]$, $E[X^2]$ and σ^2 for the probability function $p(x)$ defined by the following table.

x_i :-	1	2	3	n
$p(x_i)$:-	K	$2K$	$3K$	nK

where K is an appropriate constant.

Sol: we have $\sum p(x_i) = 1$

i.e. $K+2K+3K+\dots+nK = 1$

i.e. $K(1+2+3\dots+n) = 1$ i.e. $K\cdot\frac{n(n+1)}{2} = 1$

i.e. $K = \frac{2}{n(n+1)}$

$E[X] = \sum x_i p(x_i) = 1(K)+2(2K)+3(3K)+\dots+n(nK)$

$= K\{1^2+2^2+3^2+\dots+n^2\} = K\cdot\frac{n(n+1)(2n+1)}{6}$

$\therefore E[X] = \frac{2}{\cancel{n(n+1)}}\cdot\frac{\cancel{n(n+1)}(2n+1)}{6} = \frac{2n+1}{3}$

$E[X^2] = \sum x_i^2 p(x_i) = 1^2(K)+2^2(2K)+\dots+n^2(nK)$

$= K\{1^3+2^3+\dots+n^3\} = K\cdot\frac{n^2(n+1)^2}{4}$

$E[X^2] = \frac{2}{\cancel{n(n+1)}}\cdot\frac{n^{\cancel{2}}(n+1)^{\cancel{2}}}{4} = \frac{n(n+1)}{2}$

$\sigma^2 = E[X^2]-\mu^2$

$\sigma^2 = \frac{n(n+1)}{2}-\frac{(2n+1)^2}{9} = \frac{9(n^2+n)-2(4n^2+4n+1)}{18}$

$\sigma^2 = \frac{9n^2+9n-8n^2-8n-2}{18} = \frac{n^2+n-2}{18}$

(ii) Consider the probability distribution given by the table.

x_i :-	0	1	2	3	4	5	6
$p(x_i)$:-	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

Determine $P(X<4)$, $P(X\le 5)$ and $P(3<X\le 6)$

Ans: 0.3265, 0.4898, 0.6735.

Moment Generating function

For a specified real number 'a' and a real parameter 't' the expectation of $e^{t(x-a)}$ in the distribution X is called the moment generating function of the distribution about 'a' and it is denoted by $M_a(t)$

i.e. $M_a(t) = \sum e^{t(x_i-a)} p(x_i)$.

$M_a(t) = \sum\left\{1+t(x_i-a)+\frac{t^2(x_i-a)^2}{2!}+\dots\right\} p(x_i)$

$= \sum\left\{p(x_i)+t(x_i-a)p(x_i)+\frac{t^2}{2!}(x_i-a)^2 p(x_i)+\dots\right\}$

occupied by the remaining 6 letters in 6! different ways. $\therefore O(C) = 6!$

$\therefore P(C) = \frac{O(C)}{O(S)} = \frac{6!}{8!} = \frac{6!}{8 \cdot 7 \cdot 6!} = \frac{1}{56}$

D - Event of having five letters between A and T. The chances are as follows.

A	1	2	3	4	5	T	6
T	1	2	3	4	5	A	6
[illegible]	A	1	2	3	4	5	T
[illegible]	T	1	2	3	4	5	A

Each one in 6! ways.

$\therefore O(D) = 4(6!)$

$\therefore P(D) = \frac{O(D)}{O(S)} = \frac{4(6!)}{8!} = \frac{4 \cdot 6!}{8 \cdot 7 \cdot 6!} = \frac{1}{14}$

(15) A five-digit number is formed using the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number is (i) even (ii) odd (iii) divisible by 5 (iv) greater than 40000

Ans:- $\frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5}$.

(16) Ten pairs of shoes are in a closet. Four shoes are selected at random. Find the probability that there is atleast one pair among the four selected.

Sol: Total number of shoes = 20.

We have to select 2 pairs i.e. 4 shoes at a time. This can be done in $20C_4$ ways.

$\therefore O(S) = 20C_4$.

E - Event of selecting atleast one pair (i.e. one pair and two un-paired shoes, and two pairs)

We can select one pair ($10C_1$) and two up-paired shoes i.e. ($18C_2$) in $10C_1 \times 18C_2$ ways.

Two pairs in $10C_2$ ways.

$\therefore O(E) = 10C_1 \times 18C_2 + 10C_2 = 45 \times 34 + 45$

$\therefore P(E) = \frac{O(E)}{O(S)} = \frac{45(34+1)}{20C_4} = \frac{105}{323}$

(16)