Differential Calculus.

(i) Closed interval $[a,b] = \{x / a \le x \le b\}$

(ii) open interval $(a,b) = \{x / a < x < b\}$

(iii) length $= b - a$

- A function is continuous at a point if the graph of the function at that point has no break.

Ex:-

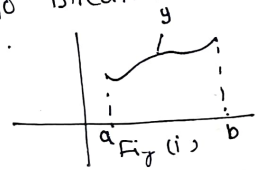


Fig (i)

y is continous at all points in $[a,b]$

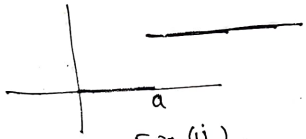


Fig (ii)

the unit step function is not continuous at 'a'.

- A function is differentiable at a point if the graph at that point is such that we can draw only one tangent at the point (ie. point is not fine or sharp)

Ex:-

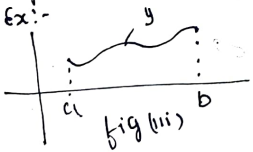


fig (iii)

Function y is diff at all points in (a,b)

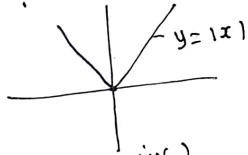


fig (iv)

The function $y = |x|$ is not differentiable at $x = 0$

Rolle's theorem:-

If a real valued function $f(x)$ is such that

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

(iii) $f(a) = f(b)$

Then there exists atleast one point c in (a,b) i.e. $a < c < b$ such that $f'(c) = 0$.

Geometrical meaning of Rolle's theorem:-

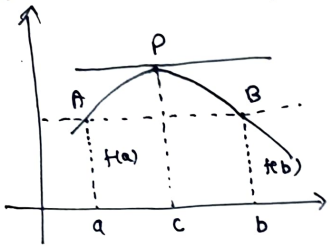
Suppose a curve C in the XY plane having A and B as end points satisfies the following conditions.

(i) The curve is continuous at the points A and B and at every point between A and B

(ii) the curve is smooth at every point between A and B

(iii) the points A and B are at the same perpendicular distance from the X-axis

there exists at least one point 'p' on the curve (between A and B) at which the tangent is parallel to the X-axis.



Verify Rolle's theorem for the following functions.

(i) $f(x) = x^3 - 3x^2 - x + 3$ in $[1,3]$

$f(x)$ is a polynomial and $\therefore$ it is continuous in $[1,3]$ and differentiable in $(1,3)$.

$f(1) = 1 - 3 - 1 + 3 = 0$ i.e. $f(1) = f(3) = 0$

$f(3) = 27 - 3(9) - 3 + 3 = 0$

$\therefore f(x)$ satisfies all conditions of Rolle's theorem.

$f'(x) = 3x^2 - 6x - 1$

let $f'(x) = 0$ i.e. $3x^2 - 6x - 1 = 0$

$$x = \frac{-(-6) \pm \sqrt{36 - 4(3)(-1)}}{2 \times 3} = \frac{6 \pm \sqrt{36+12}}{2 \times 3}$$

$$x = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{16 \times 3}}{6} = \frac{6}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\therefore x = \frac{3 \pm 2\sqrt{3}}{3}$$

i.e. $x = \frac{3}{3} + \frac{2\sqrt{3}}{3}, \quad \frac{3}{3} - \frac{2\sqrt{3}}{3}$

i.e. $x = 1 + \frac{2}{\sqrt{3}}, \quad 1 - \frac{2}{\sqrt{3}}$

clearly $1 < 1 + \frac{2}{\sqrt{3}} < 3$

$\therefore c = 1 + \frac{2}{\sqrt{3}}$

=

(ii) $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$

$f(x)$ is cont in $[0, \pi]$ and diff in $(0, \pi)$

$f(0) = \frac{\sin 0}{e^0} = \frac{0}{e^0} = 0, \quad f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$

$\therefore f(0) = f(\pi) = 0$. $\therefore$ All the conditions of Rolle's theorem are satisfied.

$$f'(x) = \frac{e^x \cdot \cos x - \sin x \, e^x}{(e^x)^2}$$

let $f'(x) = 0$

i.e. $e^x \cos x - e^x \sin x = 0$

i.e. $\cos x = \sin x$

$\therefore x = \frac{\pi}{4}, \pi + \frac{\pi}{4}, 2\pi + \frac{\pi}{4}, \ldots$

Clearly $0 < \pi/4 < \pi$ $\therefore c = \frac{\pi}{4}$.

$f(x) = (x-a)^m (x-b)^n$ in $[a,b]$, $b>a$, $m>1$, $n>1$

$f(x)$ is cont. in $[a,b]$, diff. in (a,b)

$f(a) = (a-a)^m (a-b)^n = 0$, i.e. $f(a) = f(b) = 0$

$f(b) = (b-a)^m (b-b)^n = 0$,

$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m \, n(x-b)^{n-1}$

let $f'(x) = 0$

i.e. $(x-a)^{m-1}(x-b)^{n-1}\{m(x-b) + n(x-a)\} = 0$

i.e. $(x-a)^{m-1} = 0$ or $(x-b)^{n-1} = 0$ or $m(x-b)+n(x-a)=0$

i.e. $x-a=0$, $(x-b)=0$, $mx - bm + nx - an = 0$

$x = a$, $x = b$, $x(m+n) = bm+an$

$\therefore x = \dfrac{mb+na}{m+n}$

Since m and n are +ve

$a < \dfrac{mb+na}{m+n} < b$

$\therefore c = \dfrac{mb+na}{m+n}$

(iv) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ Ans $c = \pi$

(v) $f(x) = x(x+3)e^{-\frac{x}{2}}$ in $[-3,0]$ Ans $c = -2$.

(vi) $f(x) = \log[(x^2+ab)/(a+b)x]$ in $[a,b]$, $b>0$, $a>0$

$f(x) = \log(x^2+ab) - \log(a+b)x$

$f(x) = \log(x^2+ab) - \log(a+b) - \log x$.

Clearly $f(x)$ is cont in $[a,b]$, diff in (a,b)

$f(a) = \log(a^2+ab) - \log(a+b)a = 0$

$f(b) = \log(b^2+ab) - \log(a+b)b = 0$

$f(a) = f(b) = 0$

$f'(x) = \dfrac{2x}{x^2+ab} - 0 - \dfrac{1}{x}$

let $f'(x) = 0$

i.e. $\dfrac{2x^2 - (x^2+ab)}{(x^2+ab)(x)} = 0$

i.e. $2x^2 - x^2 - ab = 0$

$x^2 - ab = 0$

i.e. $x = \sqrt{ab}$ i.e. $x = +\sqrt{ab}, -\sqrt{ab}$

Since a and b are +ve.

$a < \sqrt{ab} < b$

$\therefore c = \sqrt{ab}$

Lagrange's mean value theorem

(First mean value theorem of Differential Calculus)

State and Prove Lagrange's mean value theorem or

State and prove first mean value theorem of Differential Calculus

Statement:- If $f(x)$ is a real valued function such that

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b)

then there exists a constant $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

proof:- let us consider a function

$\phi(x) = f(x) - Kx$ where K is a constant.

clearly $\phi(x)$ is continuous in $[a,b]$ and differentiable in (a,b).

$\phi(a) = f(a) - Ka$, $\phi(b) = f(b) - Kb$

let $\phi(a) = \phi(b)$

i.e. $f(a) - Ka = f(b) - Kb$

i.e. $Kb - Ka = f(b) - f(a)$ $\therefore K = \dfrac{f(b)-f(a)}{b-a}$.

For this value of K, $f(x)$ satisfies all the conditions of Rolle's theorem. Hence By Rolle's theorem we have $\phi'(c) = 0$ for $a < c < b$

$\phi'(x) = f'(x) - K$ $\therefore \phi'(c) = f'(c) - K$

i.e. $f'(c) - K = 0$ i.e. $f'(c) = K = \dfrac{f(b)-f(a)}{b-a}$

Note 1:- we can also write the result as

$f(b) - f(a) = (b-a) f'(c)$

i.e. $f(b) = f(a) + (b-a) f'(c)$.

Note 2:-

If length of the interval $[a,b]$ is h

i.e. $b - a = h$, then $b = a+h$.

if we put $\theta = \dfrac{c-a}{b-a}$ then $c = a + \theta h$ where $0 < \theta < 1$

Then the Lagrange's theorem can be written as

If (i) $f(x)$ is cont in $[a, a+h]$, (ii) diff in $(a, a+h)$ then there exists atleast one θ, with $0 < \theta < 1$ such that $f(a+h) = f(a) + h f'(a+\theta h)$

Verify the Lagrange's mean value theorem for the following functions.

(i) $f(x) = (x-1)(x-2)(x-3)$ in $[0,4]$

$f(x)$ is cont. in $[0,4]$ and diff in $(0,4)$

$f'(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$

$f'(x) = x^2 - 3x - 2x + 6 + x^2 - 3x - x + 3 + x^2 - 2x - x + 2$

$f'(x) = 3x^2 - 12x + 11$

let $f'(x) = \dfrac{f(4) - f(0)}{4-0}$

i.e. $f'(x) = \dfrac{(3)(2)(1) - (-1)(-2)(-3)}{4}$

$3x^2 - 12x + 11 = \dfrac{6+6}{4} = \dfrac{\cancel{12}\,3}{\cancel{4}}$

$3x^2 - 12x + 8 = 0$

$x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(8)}}{2\times 3} = \dfrac{12 \pm \sqrt{144-96}}{6}$

$x = \dfrac{12 \pm \sqrt{48}}{6} = \dfrac{12 \pm \sqrt{16\times 3}}{6} = \dfrac{\cancel{12}^{\,2}}{\cancel{6}} \pm \dfrac{\cancel{4}^{\,2}\sqrt{3}}{\cancel{6}\,3}$

$x = 2 \pm \dfrac{2\sqrt{3}}{3} = 2 \pm \dfrac{2}{\sqrt{3}}$.

$x = 2 + \dfrac{2}{\sqrt{3}}, \; 2 - \dfrac{2}{\sqrt{3}}$ Since both values are in between 0 and 4

$c_1 = 2 + \dfrac{2}{\sqrt{3}}, \quad c_2 = 2 - \dfrac{2}{\sqrt{3}}$.

(ii) $f(x) = \cos^2 x$ in $[0, \frac{\pi}{2}]$

$f(x)$ is continuous in $[0, \frac{\pi}{2}]$ and diff in $(0, \frac{\pi}{2})$

$f'(x) = -2\cos x \sin x = -\sin 2x$.

$f(\frac{\pi}{2}) = (\cos \pi/2)^2 = 0, \quad f(0) = (\cos 0)^2 = 1$

let $f'(x) = \dfrac{f(\pi/2) - f(0)}{\frac{\pi}{2} - 0}$

i.e. $-\sin 2x = \dfrac{0-1}{\frac{\pi}{2}}$

$\cancel{-}\sin 2x = \cancel{-}\dfrac{2}{\pi}$ $\quad 2x = \sin^{-1}\left(\dfrac{2}{\pi}\right)$

i.e. $x = \dfrac{1}{2}\sin^{-1}\left(\dfrac{2}{\pi}\right)$

Clearly $0 < x < \dfrac{\pi}{2}$ $\quad \therefore c = \dfrac{1}{2}\sin^{-1}\left(\dfrac{2}{\pi}\right)$.

(iii) $f(x) = \tan^{-1} x$ in $[0,1]$ $\quad c = \sqrt{\dfrac{4}{\pi} - 1}$

(iv) $f(x) = \sin^{-1} x$ in $[0,1]$ $\quad c = \sqrt{1 - \dfrac{4}{\pi^2}}$

(2) If $f(x)$ is a quadratic function, prove that the value of θ in the Lagrange's mean value theorem is equal to $\frac{1}{2}$ for all intervals $[a,b]$.

let $f(x) = Ax^2 + Bx + C$

Consider the interval $[a,b]$

let $b - a = h$ then $b = a+h$.

Clearly $f(x)$ is cont in $[a, a+h]$, diff in $(a, a+h)$

ote By Lagrange's theorem, we have

$f(a+h) = f(a) + h f'(a+\theta h)$ — (1) where $0<\theta<1$

$f(a) = Aa^2 + Ba + h$

$f(a+h) = A(a+h)^2 + B(a+h) + h$

$= A\{a^2 + 2ah + h^2\} + Ba + Bh + h$

$f'(x) = 2Ax + B$, $f'(a+\theta h) = 2A(a+\theta h) + B$

Sub in (1)

$\cancel{Aa^2} + 2Aah + Ah^2 + \cancel{Ba} + Bh + \cancel{h}$

$= \cancel{Aa^2} + \cancel{Ba} + \cancel{h} + h\{2Aa + 2A\theta h + B\}$

$2\cancel{Aah} + Ah^2 + \cancel{Bh} = 2\cancel{Aah} + 2A\theta h^2 + \cancel{Bh}$

$\cancel{A}h^{\cancel{2}} = 2\cancel{A}\theta h^{\cancel{2}}$

$1 = 2\theta$ ie. $\theta = \frac{1}{2}$

(3) If $f'(x) < 0$ at all points of an interval $[a,b]$, P.T. $f(x)$ is strictly a decreasing function.

Sol:- let us consider x_1, x_2 such that $a < x_1 < x_2 < b$

Clearly $f(x)$ in cont in $[x_1, x_2]$, diff in (x_1, x_2)

By Lagrange's theorem we have,

$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ for $x_2 < c < x_1$

But $f'(x) < 0$ for all points

$\therefore f'(c) < 0$

ie. $\frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$

But $x_2 - x_1 > 0$ Because $x_2 > x_1$

$\therefore f(x_2) - f(x_1) < 0$

ie. $f(x_2) < f(x_1)$ for $x_2 > x_1$

$\therefore f(x)$ is a strickly decreasing function.

(4) If $f'(x) > 0$ at all points of the interval $[a,b]$. P.T. $f(x)$ is strictly increasing function.

(5) If $x > 0$, prove the following

(a) $\frac{x}{1+x} < \log(1+x) < x$ (b) $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$

[Note:- The direction of the inequality changes if we multiply by a -ve constant or if we take the reciprocal]

Sol:- Consider $f(x) = \log(1+x)$ in $[0,x]$

Clearly $f(x)$ is continuous in $[0,x]$ and differentiable in $(0,x)$. $\therefore$ By Lagrange's theorem, we have

$f'(c) = \frac{f(x) - f(0)}{x - 0}$ ie. $\frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{where } 0 < c < x.$$

Consider $0 < c < x$ adding 1

$1 < 1+c < 1+x$, Taking reciprocals

$1 > \frac{1}{1+c} > \frac{1}{1+x}$ using result of the theorem

ie. $\frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$, multiplying by x $(x>0)$

$$\frac{x}{1+x} < \log(1+x) < x$$

Taking reciprocals

$\frac{1+x}{x} > \frac{1}{\log(1+x)} > \frac{1}{x}$ adding $-\frac{1}{x}$.

$$\frac{1}{x} + 1 - \frac{1}{x} > \frac{1}{\log(1+x)} - \frac{1}{x} > \frac{1}{x} - \frac{1}{x}$$

ie. $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$

(6) If $a < b$ P.T. $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

deduce that (i) $\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$

(ii) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Sol: Consider $f(x) = \tan^{-1}x$ in $[a,b]$

$f(x)$ is continuous in $[a,b]$, diff in (a,b) (7) K

$\therefore$ By Lagrange's theorem.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

ie. $\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$.

where $a < c < b$, Squaring

$a^2 < c^2 < b^2$, adding 1

$1+a^2 < 1+c^2 < 1+b^2$, taking reciprocals

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$ Sub for $\frac{1}{1+c^2}$

$\times (b-a)$

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2} \quad - (1)$$

putting $b = 2$, $a = 1$ in (1)

$$\frac{2-1}{1+2^2} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1^2}$$

$\frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$, adding $\frac{\pi}{4}$

$\frac{\pi}{4} + \frac{1}{5} < \tan^{-1}2 - \frac{\pi}{4} + \frac{\pi}{4} < \frac{1}{2} + \frac{\pi}{4}$

$$\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{2+\pi}{4}$$

Putting $b = \frac{4}{3}$, $a = 1$ in ①, we get

$$\frac{\frac{4}{3}-1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{4/3-1}{1+1^2}$$

$$\frac{\frac{1}{3}}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1/3}{2} \quad \text{adding } \frac{\pi}{4}$$

$$\frac{1}{3}\cdot\frac{9}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} + \frac{\pi}{4} - \frac{\pi}{4} < \frac{1}{6} + \frac{\pi}{4}$$

$$\frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$$

=

⑦ If $0 < a < b$ P.T. $1 - \frac{a}{b} < \log(b/a) < \frac{b}{a} - 1$

Deduce that $\frac{1}{6} < \log 1.2 < 1/5$.

Hint :- Apply Lagrange's theorem for $f(x) = \log x$ in $[a, b]$

For deduction put $a = 5$, $b = 6$

⑧ Verify L.M.V. theorem for the following functions.

(i) $f(x) = \log x$ in $[1, e]$ Ans:- $c = e-1$

(ii) $f(x) = x^3 - 3x - 1$ in $\left[-\frac{11}{7}, \frac{13}{7}\right]$ Ans:- $c = \frac{1}{7}$

Cauchy's Mean value theorem:-

Suppose two functions $f(x)$ and $g(x)$ satisfy the following three conditions

(i) $f(x)$ and $g(x)$ are continuous in $[a, b]$

(ii) $f(x)$ and $g(x)$ are differentiable in (a, b)

(iii) $g'(x) \neq 0$ in (a, b)

Then there exists at least one point c in the open interval (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

proof:- Consider $\phi(x) = f(x) - Kg(x)$

where K is a constant to be choosen appropriately

Since $f(x)$ and $g(x)$ are cont. in $[a, b]$, diff in (a, b), $\phi(x)$ is also cont in $[a, b]$ and diff in (a, b)

let $\phi(a) = \phi(b)$

i.e. $f(a) - Kg(a) = f(b) - Kg(b)$

$$K g(b) - K g(a) = f(b) - f(a)$$

$$K = \frac{f(b)-f(a)}{g(b)-g(a)}$$

for this value of 'K' $\phi(x)$ satisfies all the conditions of Rolle's theorem. Hence, by Rolle's theorem there exists a constant 'c', $a<c<b$ such that $\phi'(c)=0$

But $\phi'(x) = f'(x) - Kg'(x)$

$\therefore$ $\phi'(c)=0$ is $f'(c) - Kg'(c) = 0$

i.e. $f'(c) = Kg'(c)$

$$\frac{f'(c)}{g'(c)} = K = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Note:- 1) For $b-a=h$, $b=a+h$ then $c=a+\theta h$, $0<\theta<1$ then the result becomes

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)}, \quad 0<\theta<1$$

2) For $g(x)=x$, $g'(x)=1$ $\therefore$ $g'(c)=1$ and $g'(x)\neq 0$ for all $x\in(a,b)$.

Also $g(b)=b$, $g(a)=a$ $\therefore$ The result becomes $f'(c) = \frac{f(b)-f(a)}{b-a}$ which is the result of the Lagrange's theorem. $\therefore$ Lagrange's theorem is the particular case of Cauchy's mean value theorem for $g(x)=x$.

Verify the Cauchy's mean value theorem for the following functions.

(i) $f(x)=e^x$, $g(x)=e^{-x}$ in $[a,b]$

Since $f(x)$ and $g(x)$ are exponential functions they are continuous in $[a,b]$, differentiable in (a,b). $f'(x)=e^x$, $g'(x)=-e^{-x}\neq 0$ for all $x\in(a,b)$.

let $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{e^x}{-e^{-x}} = \frac{e^b-e^a}{e^{-b}-e^{-a}} \quad \text{i.e.} \quad \frac{e^x}{-\frac{1}{e^x}} = \frac{e^b-e^a}{\frac{1}{e^b}-\frac{1}{e^a}}$$

$$\text{i.e.} \quad -e^x\cdot e^x = \frac{e^b-e^a}{\frac{(e^a-e^b)}{e^b e^a}}$$

$$\text{i.e.} \quad -e^{2x} = e^a\cdot e^b\,\frac{(e^b-e^a)}{-(e^b-e^a)}$$

$$\text{i.e.} \quad -e^{2x} = -e^{a+b} \quad \text{i.e.} \quad e^{2x}=e^{a+b}$$

$\therefore$ $2x=a+b$ i.e. $x=\frac{a+b}{2}$ which is the average of a and b. and $\therefore$ $a<x<b$

Hence $c=\frac{a+b}{2}$.

(ii) $f(x)=\sin x$, $g(x)=\cos x$ in $[a,b]$, $0\le a<b\le\frac{\pi}{2}$.

Since $f(x)$ and $g(x)$ are trignometric functions

they are continuous in $[a,b]$ and diff in (a,b)

$f'(x) = \cos x$, $g'(x) = -\sin x \neq 0$ in (a,b)

Because at most $a=0$, $b=\frac{\pi}{2}$ and $g'(x)=0$ at $x=0$.

let $\frac{f'(x)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\cos x}{-\sin x} = \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$\left[\begin{array}{l} \sin C - \sin D \\ = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ \cos C - \cos D \\ = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \end{array}\right]$$

$$\therefore -\cot x = \frac{\cancel{2}\cos\left(\frac{b+a}{2}\right)\cancel{\sin\left(\frac{b-a}{2}\right)}}{-\cancel{2}\sin\left(\frac{b+a}{2}\right)\cancel{\sin\left(\frac{b-a}{2}\right)}}$$

$$+\cot x = +\cot\left(\frac{b+a}{2}\right) \quad \text{ie. } x = \frac{b+a}{2}$$

$$\therefore c = \frac{b+a}{2}$$

(iii) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[\frac{1}{4}, 1]$

$f(x)$ and $g(x)$ are continuous in $[1/4, 1]$ and differentiable in $(\frac{1}{4}, 1)$

$f'(x) = \frac{1}{2\sqrt{x}}$. $g(x) = x^{-1/2}$ $\therefore g'(x) = -\frac{1}{2}x^{-1/2-1}$

ie. $g'(x) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2}\cdot x^{-1/2}\cdot x^{-1} = \frac{-1}{2x\sqrt{x}}$.

and $g'(x) \neq 0$ for all $x \in (\frac{1}{4}, 1)$

let $\frac{f'(x)}{g'(x)} = \frac{f(1) - f(1/4)}{g(1) - g(\frac{1}{4})}$

$f(1) = \sqrt{1} = 1$, $f(1/4) = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

$g(1) = \frac{1}{\sqrt{1}} = 1$, $g(\frac{1}{4}) = \frac{1}{\sqrt{1/4}} = \frac{1}{1/2} = 2$.

$$\therefore \frac{\frac{1}{2\sqrt{x}}}{-\frac{1}{2x\sqrt{x}}} = \frac{1-\frac{1}{2}}{1-2}$$

ie. $\frac{1}{\cancel{2}\cancel{\sqrt{x}}} \times -\cancel{2}x\cancel{\sqrt{x}} = \frac{\frac{1}{2}}{-1}$

ie. $-x = -\frac{1}{2}$ ie. $x = 1/2$

Since $\frac{1}{4} < \frac{1}{2} < 1$ $\quad c = \frac{1}{2}$

(iv) $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1, e]$

Ans: $c = \frac{e}{e-1}$

2. If $f(x)$ is continuous in an interval $[a,b]$ and diff in the interval (a,b), prove that there exists a c with $a<c<b$ such that $2c[f(a)-f(b)] = f'(c)(a^2-b^2)$

Sol:- Consider $f(x)$, $g(x) = x^2$ in $[a,b]$ where $a>0, b>0$

$f(x)$ and $g(x)$ are continuous in $[a,b]$

$g'(x) = 2x$ which equals to zero only at $x=0$

Since $a>0, b>0$ $g'(x) \neq 0$ for all $x \in (a,b)$

By Cauchy's mean value theorem there exists c $a<c<b$ such that $\frac{f'(c)}{g'(x)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

i.e. $\frac{f'(c)}{2c} = \frac{f(b)-f(a)}{b^2-a^2}$

i.e. $\frac{f'(c)}{2c} = \frac{-(f(a)-f(b))}{-(a^2-b^2)}$

i.e. $2c[f(a)-f(b)] = f'(c)[a^2-b^2]$

=

If $0 \le x < 1$, P.T. $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$

Consider $f(x) = \log(1+x)$, $g(x) = \sin^{-1}x$ in $[0,x]$

Clearly $f(x)$ and $g(x)$ are continuous in $[a,b]$ and differentiable in (a,b).

$f'(x) = \frac{1}{1+x}$, $g'(x) = \frac{1}{\sqrt{1-x^2}} \neq 0$ for all $x \in [0,x]$

By Cauchy's mean value theorem there exists a constant c, $a<c<b$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)-f(0)}{g(x)-g(0)}$$

i.e. $\frac{\frac{1}{1+c}}{\frac{1}{\sqrt{1-c^2}}} = \frac{\log(1+x) - \log 1}{\sin^{-1}x - \sin^{-1}(0)}$

$\frac{1}{\cancel{\sqrt{1+c}}} \times \frac{\sqrt{(1-c)}\,\cancel{\sqrt{1+c}}}{1} = \frac{\log(1+x)}{\sin^{-1}x}$.

i.e. $\frac{\sqrt{1-c}}{\sqrt{1+c}} = \frac{\log(1+x)}{\sin^{-1}x}$ — (1)

we have

$0 < c < x$

$0 > -c > -x$

$1 > 1-c > 1-x$

$\sqrt{1} > \sqrt{1-c} > \sqrt{1-x}$

i.e. $\sqrt{1-x} < \sqrt{1-c} < 1$ — (2)

$0 < c < x$

$1 < 1+c < 1+x$

$\sqrt{1} < \sqrt{1+c} < \sqrt{1+x}$

$1 > \frac{1}{\sqrt{1+c}} > \frac{1}{\sqrt{1+x}}$

i.e. $\frac{1}{\sqrt{1+x}} < \frac{1}{\sqrt{1+c}} < 1$ — (3)

From (2) and (3), we get

$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\sqrt{1-c}}{\sqrt{1+c}} < 1$

from (1)

$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$

=

(March 2000 VTU)

Apply mean value theorem to S.T.

$x > \log(1+x) > x - \frac{1}{2}x^2$.

By Lagrange's mean value theorem we can S.T. If $f'(x) > 0$ for all $x \in (a,b)$ then the function $f(x)$ is strickly increasing.

Consider $f(x) = x - \log(1+x)$

$$f'(x) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x}$$

Clearly $f'(x) > 0$ for all $x > 0$

$\therefore$ $f(x)$ is an increasing function.

for $x > 0$, $f(x) > f(0)$

i.e. $x - \log(1+x) > 0 - \log 1$

i.e. $x - \log(1+x) > 0$

i.e. $x > \log(1+x)$ — (1)

Consider $f(x) = \log(1+x) - x + \frac{x^2}{2}$

$$f'(x) = \frac{1}{1+x} - 1 + \frac{2x}{2} = \frac{\cancel{1} - \cancel{1} - x + x(1+x)}{1+x}$$

$$= \frac{-x + x + x^2}{1+x} = \frac{x^2}{1+x} > 0 \text{ for all } x > 0$$

$\therefore$ $f(x)$ is an increasing function.

$f(x) > f(0)$ for all $x > 0$

i.e. $\log(1+x) - x + \frac{x^2}{2} > \log 1 - 0 + 0$

i.e. $\log(1+x) - x + \frac{x^2}{2} > 0$

$\log(1+x) > x - \frac{x^2}{2}$ — (2)

Combining (1) and (2), we get

$x > \log(1+x) > x - \frac{x^2}{2}$