Linear Algebra.

Elementary Row (Column) operations or transformations

For a given nonzero matrix the following operations are called as Elementary row (column) operations.

(1) Interchange of any two rows (columns)

(2) Multiplication of a row (column) elements by a non zero constant.

(3) Addition of K times the elements of a row (column) to the corresponding elements of some other row (column).

Note:- (i) we use the notations R_j, $j=1,2,3$ to denote row operations and C_j, $j=1,2,3$ to denote column operations.

(ii) The matrix obtained by using a finite no. of elementary transformations is equivalent to the original matrix and we connect them using the equivalence relation '$\sim$'

(iii) In one step we can use a finite no of row or column operations but we cannot use both row and column operations in the same step.

(iv) while using the row operations we write the elements rowwise and while using column operations we write the elements column wise.

In the following example all types of elementary operations are used.

$$A = \begin{bmatrix} 1 & -3 & 4 & 5 \\ 6 & 0 & 7 & 2 \\ 3 & 1 & -3 & -2 \end{bmatrix}$$

(Interchange of R_3 with R_1)

$$\sim \begin{array}{c} \begin{bmatrix} 3 & 1 & -3 & -2 \\ 6 & 0 & 7 & 2 \\ 1 & -3 & 4 & 5 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow R_3 \\ \\ \leftarrow R_1 \end{array}$$

multiply R_2 elements by -2
" R_3 " by 5

$$\sim \begin{bmatrix} 3 & 1 & -3 & -2 \\ -12 & 0 & -14 & -4 \\ 5 & -15 & 20 & 25 \end{bmatrix} \begin{array}{l} \\ \leftarrow -2R_2 \\ \leftarrow 5R_3 \end{array}$$

Add 6 times the elements of R_1 to the corresponding elements of R_2

$$\sim \begin{bmatrix} 3 & 1 & -3 & -2 \\ 6 & 6 & -32 & -16 \\ 5 & -15 & 20 & 25 \end{bmatrix} \begin{array}{l} \\ \leftarrow 6R_1+R_2 \\ \\ \end{array}$$

Interchange column 3 with 2

$$\sim \begin{bmatrix} 3 & -3 & 1 & -2 \\ 6 & -32 & 6 & -16 \\ 5 & 20 & -15 & 25 \end{bmatrix}$$

$\uparrow C_3$ $\uparrow C_2$

Multiply 2nd column by -3 and fourth column by 2

$$\sim \begin{bmatrix} 3 & 9 & 1 & -4 \\ 6 & 96 & 6 & -32 \\ 5 & -60 & -15 & 50 \end{bmatrix}$$

$\uparrow -3C_2$ $\uparrow 2C_4$

Add -5 times the elements of column 1 to elements of column 2

$$\sim \begin{bmatrix} 3 & -6 & 1 & -4 \\ 6 & 66 & 6 & -32 \\ 5 & -85 & -15 & 50 \end{bmatrix}$$

$\uparrow -5C_1+C_2$

Row Echelon form :-

A matrix is said to be in Row Echelon or Echelon form if (i) the zero rows comes after the non zero rows (ii) The first non zero element of any nonzero row (Except I^{st} row) must lie to the right of the first non zero element of its previous row

Ex :-

$$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

are in Echelon form.

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow *$$

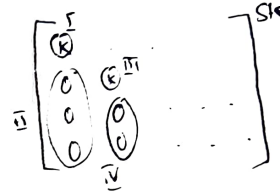
$*$

D and E are not in the Echelon form. The reason is at the place indicated by $*$.

Note :- (1) We can reduce the given matrix to Echelon form using only row operations.

(2) If a matrix is reduced to Echelon form then its rank is equal to the no. of nonzero rows present in that Echelon form.

Procedure to reduce the given matrix to Echelon form



Step (i) Get a nonzero element in Top left hand corner.

(ii) Get zeros below it

(iii) Get a nonzero element in a_{22} position (Second row and second column)

(iv) Get zeros below it.

(v) Continue this process until Echelon form is obtained.

Note(1) :- If it is not possible to get a non zero element in any $a_{11}, a_{22}, a_{33} \ldots$ position, then we have get nonzero element in next position i.e. $a_{12}, a_{23}, a_{34}, \ldots$

(2) Sometimes we try to get non zero element as ± 1 to carry out next operations easily.

(3) Here we have to use only row operations.

(4) Many operations will give same result we select simplest among them.

Find the rank of the matrix $A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ using elementary row operations.

Sol:- Here we reduce to Row Echelon form.

$$A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 4 & 0 & 2 & 1 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix} \begin{matrix} \leftarrow R_2 \\ \leftarrow R_1 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 2 & 1 & 3 \\ 0 & +2 & -2 & 0 \end{bmatrix} \begin{matrix} \\ \leftarrow -2R_1 + R_2 \\ \leftarrow -R_1 + R_3 \\ \leftarrow -R_1 + R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -6 & -7 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_2 + R_3 \\ \leftarrow R_2 + R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & -2 & -4 & -7 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow -2R_3 + R_4 \end{matrix}$$

There are four nonzero rows $\therefore$ Rank of the matrix A is 4 i.e. $\rho(A) = 4$

(2) Reduce the following matrices to Echelon form and hence find the ranks.

$$B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & +14 & 2 \end{bmatrix} \begin{matrix} \\ \leftarrow -2R_1 + R_2 \\ \leftarrow -3R_1 + R_3 \\ \leftarrow -R_1 + R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow -2R_2 + R_3 \\ \leftarrow 2R_2 + R_4 \end{matrix}$$

$\therefore \rho(B) = 2$.

(3)

$$G = \begin{bmatrix} 2 & 4 & 7 & 7 \\ 3 & 1 & -2 & -2 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix} \begin{matrix} \leftarrow R_3 \\ \\ \leftarrow R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{bmatrix} \begin{matrix} \\ \leftarrow -3R_1 + R_2 \\ \leftarrow -2R_1 + R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 0 & 20 \end{bmatrix} \leftarrow R_2 + R_3$$

$\rho(C) = 3$

(4) $D = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

$\rho(D) = 2$

(5) $E = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

$\rho(E) = 2$

Normal form :-

A matrix is said to be in normal form if the leading submatrix is a unit matrix and the remaining elements are zeros

Note:- If a matrix is reduced to normal form then the rank is equal to the order of the unit matrix.

Ex:- $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \end{bmatrix}$

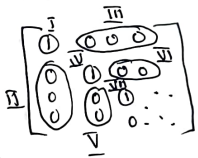
$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 \end{bmatrix}$

are in normal form.

Matrix $D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is not in the normal form because the unit matrix I_2 is not a leading submatrix.

Reduction of a matrix to normal form



Note:- ① Here we are using both row and column operations.

② To get zeros below a_{11} element we use only row operations involving R_1, to get zeros below a_{22} element we use row operations involving R_2 and so on.

③ To get zeros besides a_{11} element we use column operations involving C_1 and to get zeros besides a_{22} element we use column operations involving C_2 and so on.

Find the rank of the following matrices by reducing them to the normal form.

① $A = \begin{bmatrix} 0 & 6 & 3 & 2 \\ 2 & 2 & 4 & 3 \\ 1 & 3 & 6 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 6 & 3 \\ 2 & 2 & 4 & 3 \\ 0 & 6 & 3 & 2 \end{bmatrix} \begin{matrix} \leftarrow R_3 \\ \\ \leftarrow R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 3 & 6 & 3 \\ 0 & -4 & -8 & -3 \\ 0 & 6 & 3 & 2 \end{bmatrix} \leftarrow -2R_1+R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -8 & -3 \\ 0 & 6 & 3 & 2 \end{bmatrix}$

$-3C_1+C_2$ $\quad -6C_1+C_3$ $\quad -3C_1+C_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -8 & -3 \\ 0 & 4 & 3 & 2 \end{bmatrix}$

$-C_4+C_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 4 & 3 & 2 \end{bmatrix} \leftarrow -R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & -29 & -10 \end{bmatrix} \leftarrow -4R_2+R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -29 & -10 \end{bmatrix}$

$-8C_2+C_3$ $\quad -3C_3+C_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \end{bmatrix}$

$-\frac{1}{29}C_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$10C_3+C_4$

$\sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$

$\rho(A) = 3$

② $B = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix} \begin{matrix} \\ \leftarrow -2R_1+R_2 \\ \leftarrow -3R_1+R_3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -5 & -10 \\ 0 & -6 & -2 & -4 \end{bmatrix}$

$-C_1+C_2$ $\quad -C_1+C_3$ $\quad -2C_1+C_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & -6 & -2 & -4 \end{bmatrix} \leftarrow -R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 28 & 56 \end{bmatrix} \leftarrow 6R_2+R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 28 & 56 \end{bmatrix}$

$-5C_2+C_3$ $\quad -10C_2+C_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \leftarrow \frac{1}{2}R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & 0 \end{bmatrix}$

$-2C_3+C_4$

$\rho(B) = 3$

③ $C = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & -1 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & -1 & 2 \end{bmatrix} \begin{matrix} \leftarrow R_1 \\ \\ \leftarrow R_3 \\ \\ \end{matrix}$

$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} \leftarrow -4R_1+R_2 \\ \leftarrow -2R_1+R_3 \\ \leftarrow -R_1+R_4 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}$

C_1+C_2 $\quad -3C_1+C_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} \begin{matrix} \\ \leftarrow R_2 \\ \\ \leftarrow R_4 \end{matrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} \leftarrow -R_2$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -16 \end{bmatrix} \leftarrow -6R_2+R_4$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -6 & -16 \end{bmatrix}$

$-C_2+C_3$ $\quad -C_2+C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow R_4 \\ \leftarrow R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -16 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow -\frac{1}{6}C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow 16C_3 + C_4$

$\rho(C) = 3.$

(4) $A = \begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

$\rho(D) = 3.$

(5) $E = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$\rho(E) = 2.$

Homogeneous system of equations.

The set of equations
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0$$

is called as Homogeneous system of equations.

$x_1 = 0,\ x_2 = 0,\ \dots,\ x_n = 0$ is a solution.

It is called as trivial or zero solution.

A solution other than this, i.e. a solution in which atleast one value is different from zero is called as non-trivial or non-zero solution.

We can represent the system by $AX = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The system will have a non trivial solution if $\rho(A) <$ no. of unknowns.

Note (i):- To find rank of A we reduce A to the echelon form.

(ii) To get the nontrivial solution we solve the matrix equation $AX = 0$, where we consider the echelon form of A.

(iii) If A is square matrix, then the condition for the system to have nontrivial solution is equivalent to $|A| = 0$

1) Solve the following system of equations.

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix} \begin{matrix} \\ \leftarrow -2R_1 + R_2 \\ \leftarrow -R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow -2R_2 + R_3$$

$\rho(A) = 2 < 3$ (no. of unknowns)

The system has non-trivial solution.

To get the solution consider

$$AX = 0$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 3y - 2z = 0$$
$$-7y + 8z = 0$$

$\therefore -7y = -8z,\ y = \frac{8}{7}z$

let $z = k$, then $y = \frac{8}{7}k$.

$x = 2z - 3y = 2k - 3\left(\frac{8}{7}\right)k$

$x = \frac{14k - 24k}{7} = \frac{-10k}{7}$

$\therefore x = \frac{-10k}{7},\ y = \frac{8}{7}k,\ z = k$

2) Solve the system

$$3x - y - z + 2u = 0$$
$$3x + y - z - 2u = 0$$
$$12x + 3y - 4z - 6u = 0$$

$$A = \begin{bmatrix} 3 & -1 & -1 & 2 \\ 3 & 1 & -1 & -2 \\ 12 & 3 & -4 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 2 & 0 & -4 \\ 0 & 7 & 0 & -14 \end{bmatrix} \begin{matrix} \\ \leftarrow -R_1+R_2 \\ \leftarrow -4R_1+R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 7 & 0 & -14 \end{bmatrix} \begin{matrix} \\ \leftarrow \frac{1}{2}R_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow -7R_2+R_3 \end{matrix}$$

$\rho(A) = 2 < 4$ (no. of unknowns)

System has a non-zero solution.

To get it, consider $AX = 0$

$$\begin{bmatrix} 3 & -1 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x - y - z + 2u = 0$$
$$y - 2u = 0. \quad \text{put } u = k_1$$

$$y = 2u = 2k_1$$

$$3x = y + z - 2u \quad \text{put } z = k_2$$

$$3x = \cancel{2k_1} + k_2 - \cancel{2k_1}, \quad \therefore x = \frac{k_2}{3}.$$

Solution is $\boxed{x = \frac{k_2}{3},\ y = 2k_1,\ z = k_2,\ u = k_1}$

3) Solve the system

$$x + y - z + u = 0$$
$$x - y + 2z - u = 0$$
$$3x + y + u = 0$$

Ans: $x = -\frac{1}{2}k_1,\ z = k_1$

$y = \frac{3}{2}k_1 - k_2,\ u = k_2$

4) Find the values of λ for which the system $3x + y - \lambda z = 0$, $4x - 2y - 3z = 0$, $2\lambda x + 4y + \lambda z = 0$ has a non trivial solution. Find the solution in each case.

Sol:- $A = \begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}$

$$\sim \begin{bmatrix} 3 & 1 & -\lambda \\ 0 & -\frac{10}{3} & \frac{4\lambda-9}{3} \\ 0 & \frac{-2\lambda+12}{3} & \frac{2\lambda^2+3\lambda}{3} \end{bmatrix} \begin{matrix} \\ \leftarrow -\frac{4}{3}R_1+R_2 \\ \leftarrow -\frac{2}{3}\lambda R_1+R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 1 & -\lambda \\ 0 & 10 & -4\lambda+9 \\ 0 & -2\lambda+12 & 2\lambda^2+3\lambda \end{bmatrix} \begin{matrix} \\ \leftarrow -3R_2 \\ \leftarrow 3R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 3 & 1 & -\lambda \\ 0 & 10 & -4\lambda+9 \\ 0 & 0 & \frac{6\lambda^2+48\lambda-54}{5} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \frac{1}{5}(\lambda-6)R_2+R_3 \end{matrix}$$

The system will have a non-trivial solution only if $\rho(A) = 2 < 3$

For this $\frac{6\lambda^2+48\lambda-54}{5} = 0$

i.e. $6\lambda^2 + 48\lambda - 54 = 0 \quad \div 6$

$$\lambda^2 + 8\lambda - 9 = 0$$

$$\lambda^2 + 9\lambda - \lambda - 9 = 0 \qquad \begin{matrix} -9 \\ \wedge \\ 9 \quad -1 \end{matrix}$$

$$\lambda(\lambda+9) - 1(\lambda+9) = 0$$

$$(\lambda+9)(\lambda-1) = 0 \quad \therefore \lambda = 1, -9.$$

For $\lambda = 1$, solution is $AX = 0$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 10 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x + y - z = 0$, $\quad 10y + 5z = 0$

$$y = -\frac{5}{10}z = -\frac{z}{2}$$

let $z = k$, $y = -\frac{k}{2}$

$$3x = -y + z = \frac{k}{2} + k = \frac{3k}{2} \quad \therefore x = \frac{3k}{2 \cdot 3}$$

$$\boxed{x = \frac{k}{2},\ y = -\frac{k}{2},\ z = k}.$$

For $\lambda = -9$, solution is $AX = 0$

$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & 10 & 45 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x + y + 9z = 0$, $\quad 10y + 45z = 0$

$$y = -\frac{45}{10}z = -\frac{9}{2}z$$

let $z = k$, $y = -\frac{9}{2}k$

$3x + y + 9z = 0$

$x = \frac{1}{3}(-y - 9z) = \frac{1}{3}\left(-\left(\frac{-9}{2}\right)k - 9k\right) = \frac{1}{3}\left(\frac{9-18}{2}\right)k$

$x = \frac{-\cancel{9}\,3}{\cancel{6}\,2}k \quad \therefore \boxed{x = -\frac{3}{2}k,\ y = -\frac{9}{2}k,\ z = k}$

Non-Homogeneous Systems of Linear equations

Consider the system of m simultaneous linear equations in 'n' unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots &+ a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots &+ a_{2n}x_n = b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots &+ a_{mn}x_n = b_m
\end{aligned}
$$

Here a's and b's are constants. ~~Atleast one~~ $b \neq 0$

This system will form a system of Non-Homogeneous equations.

We can represent this system by the matrix equation $AX = B$.

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

is called the coefficient matrix.

The system is said to be consistent if it has a solution, otherwise it is said to be inconsistent.

We solve the system of non-homogeneous equations using various methods.

The system will be consistent if $\rho(A,B) = \rho(A)$ where $[A,B]$ is called as the augmented matrix.

Note:- ① To find ranks of (A,B) and A we reduce the augmented matrix $[A,B]$ to Echelon form. To get rank of A we consider the Echelon form of (A,B) leaving the last column.

② If $\rho(A,B) = \rho(A) =$ no. of unknowns then the system has got a unique solution.

③ If $\rho(A,B) = \rho(A) <$ no. of unknowns then the system has got infinitely many solutions.

④ If the system is consistent then we find the solution using the matrix equation $A'X = B'$ where A' and B' are reduced forms of A and B respectively.

⑤ This method is called as Gauss elimination method or Matrix method.

① Check for consistency.

$x+y+z=4$, $2x+5y-2z=3$, $x+7y-7z=5$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

$$(A,B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{array}\right] \begin{array}{l} \\ \leftarrow -2R_1+R_2 \\ \leftarrow -R_1+R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow -2R_2+R_3 \end{array}$$

$\rho(A,B) = 3$, $\rho(A) = 2$

$\rho(A,B) \neq \rho(A)$

$\therefore$ The system is not consistent.

② Check for consistency and solve if consistent.

$2x-3y+z=-5$

$3x+2y-2z=9$

$x+y+3z=0$

$$(A,B) = \left[\begin{array}{ccc|c} 2 & -3 & 1 & -5 \\ 3 & 2 & -2 & 9 \\ 1 & 1 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 3 & 2 & -2 & 9 \\ 2 & -3 & 1 & -5 \end{array}\right] \begin{array}{l} \leftarrow R_3 \\ \\ \leftarrow R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -11 & 9 \\ 0 & -5 & -5 & -5 \end{array}\right] \begin{array}{l} \\ \leftarrow -3R_1+R_2 \\ \leftarrow -2R_1+R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -5 & 9 \\ 0 & 0 & 50 & -50 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow -5R_2+R_3 \end{array}$$

$\rho(A,B) = \rho(A) = 3$

$\therefore$ System is consistent.

To get the solution consider $A'X = B'$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -11 \\ 0 & 0 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ -50 \end{bmatrix}$$

$x+y+3z=0$

$-y-11z=9$

$50z=-50$

$\therefore z = \frac{-50}{50} = -1$

$-y-11z=9$

$-y+11=9$

$-y=9-11=-2 \therefore y=2$

$x+y+3z=0$

$x+2-3=0 \quad x=1$

$\boxed{x=1,\ y=2,\ z=-1}$

3) Solve by using Gauss elimination method

$x-3y-8z+10=0$

$3x+y-4z=0$

$2x+5y+6z=13.$

Sol:- In the first equation taking 10 to R.H.S we get -10.

$$(A,B)=\left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{array}\right]\begin{array}{l} \\ \leftarrow -3R_1+R_2 \\ \leftarrow -2R_1+R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 11 & 22 & 33 \end{array}\right]\begin{array}{l} \\ \leftarrow \frac{1}{10}R_2 \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} \\ \\ \leftarrow -11R_2+R_3 \end{array}$$

$\rho(A,B)=\rho(A)=2$

$\therefore$ System is consistent.

To get the solution consider $A'X=B'$

$$\begin{bmatrix} 1 & -3 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -10 \\ 3 \\ 0 \end{bmatrix}$$

$x-3y-8z=-10$

$y+2z=3$

put $z=k$

$y=3-2z=3-2k$

$x=-10+3y+8z$

$x=-10+9-6k+8k$

$x=2k-1$

$\boxed{\therefore x=2k-1,\ y=3-2k,\ z=k}$

4) S.T. The following system is consistent and solve the system.

$x+2y+2z=1$

$2x+y+z=2$

$3x+2y+2z=3$

$y+z=0$

$$(A,B)=\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]\begin{array}{l} \\ \leftarrow -2R_1+R_2 \\ \leftarrow -3R_1+R_3 \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right]\begin{array}{l} \\ \leftarrow R_4 \\ \leftarrow R_2 \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} \\ \\ \leftarrow 4R_2+R_3 \\ \leftarrow 3R_2+R_4 \end{array}$$

$\rho(A,B)=\rho(A)=2$

$\therefore$ System is consistent.

Consider $A'X=B'$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x+2y+2z=1$

$y+z=0$

let $z=k$, $y=-z=-k$.

$x=1-2y-2z=1+2k-2k=1$

$\therefore \boxed{x=1,\ y=-k,\ z=k}$

5) Solve by Gauss elimination method.

$5x+3y+7z=4$

$3x+26y+2z=9$

$7x+2y+10z=5$

$$(A,B)=\left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]\begin{array}{l} \leftarrow -2R_2+R_1 \\ \\ \\ \end{array}$$

$$\sim\left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 0 & -121 & 11 & -33 \\ 0 & -341 & 31 & -93 \end{array}\right]\begin{array}{l} \\ \leftarrow 3R_1+R_2 \\ \leftarrow 7R_1+R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 0 & -11 & 1 & -3 \\ 0 & -11 & 1 & -3 \end{array}\right]\begin{array}{l} \\ \leftarrow \frac{1}{11}R_2 \\ \leftarrow \frac{1}{31}R_3 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} -1 & -49 & 3 & -14 \\ 0 & -11 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} \\ \\ \leftarrow -R_2+R_3 \end{array}$$

$\rho(A,B)=\rho(A)=2$ $\therefore$ Consistent.

$A'X=B'$

$$\begin{bmatrix} -1 & -49 & 3 \\ 0 & -11 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -14 \\ -3 \\ 0 \end{bmatrix}$$

$-x-49y+3z=-14$

$-11y+z=-3$

let $z=k$

$-11y=-3-z=-3-k$

$\therefore y=-\frac{1}{11}(-3-k)=\frac{1}{11}(3+k)$

$-x = -14 + 49y - 32$

$-x = -14 + \frac{49}{11}(3+k) - 3(k)$

$-x = \frac{-154 + 147 + 49k - 33k}{11}$

$-x = \frac{-7 + 16k}{11}$

$\therefore x = \frac{7 - 16k}{11}$

$x = \frac{7-16k}{11}, \; y = \frac{3+k}{11}, \; z = k$

(6) Find the values of λ and μ for which the system

$x+y+z = 6$
$x+2y+3z = 10$
$x+2y+\lambda z = \mu$

has (i) unique solution (ii) Infinitely many solutions and (iii) no solution.

Sol:- $(A,B) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & (\lambda-1) & (\mu-6) \end{array}\right] \begin{array}{l} \\ \leftarrow -R_1+R_2 \\ \leftarrow -R_1+R_3 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & (\lambda-3) & (\mu-10) \end{array}\right] \begin{array}{l} \\ \\ \leftarrow -R_2+R_3 \end{array}$

System will have unique solution if $\rho(A,B) = \rho(A)$ = no. of unknowns = 3.

$\rho(A,B) = \rho(A) = 3$

if $\lambda - 3 \neq 0$ i.e. $\lambda \neq 3$.

$\therefore$ System has a unique solution for $\lambda \neq 3$.

System will have infinitely many solutions if $\rho(A,B) = \rho(A) = 2$

if $\lambda - 3 = 0$ and $\mu - 10 = 0$

if $\lambda = 3$ and $\mu = 10$

System will have no solution if $\rho(A,B) \neq \rho(A)$

i.e. if $\rho(A) = 2$ and $\rho(A,B) = 3$

i.e. if $\lambda - 3 = 0$ and $\mu - 10 \neq 0$

i.e. if $\lambda = 3$ and $\mu \neq 10$

(7) S.T. the equations

$-2x+y+z = a$,
$x-2y+z = b$
$x+y-2z = c$

have a solution only if $a+b+c = 0$. Find all possible solutions when $a=1, b=1, c=-2$

Ans:- $x = k-1, y = k-1, z = k$

(8) S.T. the system $x+y+z=4$, $2x+y-z=1$, $x-y+2z=2$ is consistent. Solve the system.

Ans:- $x = \frac{3}{7}$, $y = \frac{13}{7}$, $z = \frac{12}{7}$.

(9) S.T. the following system of equations is consistent and solve the system.

$x+2y+2z=1$, $2x+y+z=2$, $3x+2y+2z=3$, $y+z=0$.

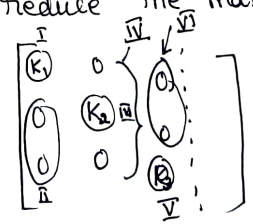
Ans:- $x=1, y=-k, z=k$

Gauss - Jordan Method.

we use this method to find the unique solution of a non homogeneous system of equations consisting of same no. of equations in the same no. of unknowns.

i.e. Coefficient matrix A is a square matrix.

Here we start with the augmented matrix (A,B). using only the row operations we reduce the matrix A to diagonal matrix.



Finaly using the matrix equation $AX = B$ we get the solution.