Lindlad, Noise and possible links with PMMs

Collaboration on Noise, Open quantum systems and more

November 2

1 Introduction

This note provides an extremely abridged, and perhaps overly practical, introduction to open quantum systems. The goal is to provide the basis for understanding and implementing an open quantum system using the Lindblad master equation. And then hopefully link this with PMMs.

2 Open?

In quantum mechanics, a system is considered *closed* if it is isolated from its environment. In contrast, an *open quantum system* interacts with its environment, which can cause the system to lose coherence and entanglement.

All this boils down to whether or not energy is conserved. In a closed system, energy is conserved, while in an open system, energy is not conserved.

3 Density Matrices

As opposed to the wavefunction (a so-called pure state) in closed (Hermitian) quantum mechanics, the density matrix uniquely describes the state of a quantum system in an open system, as well as in closed systems.

With pure states, the state of a quantum system is deterministic and leads to probabilities of outcomes of measurements. In contrast, what if the state itself is uncertain and therefore probabilistic? This is what density matrices describe.[1]

The important properties are summarized here:

- The density matrix of a pure state $|\psi\rangle$ is $\rho = |\psi\rangle\langle\psi|$.
- ρ is Hermitian, positive semi-definite (all eigenvalues non-negative), and has unit trace:

$$\rho = \rho^\dagger, \quad \lambda(\rho) \geq 0, \quad \mathrm{Tr}(\rho) = 1.$$

- Given a probabilistic mixture of states with probabilities $\{p_i\}$ and states $\{|\psi_i\rangle\}$,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| .$$

[1] A natural question might be: "What about probabilistic mixtures of density matrices?" Luckily, such systems are also just described by density matrices.

- For a $d \times d$ density matrix ρ, there exists an orthonormal basis $\{|b_i\rangle\}$ such that

$$\rho = \sum_i \lambda_i |b_i\rangle \langle b_i| ,$$

where λ_i are probabilities.

- The expectation value of an observable X is

$$\langle X \rangle = \mathrm{Tr}(\rho X) = \sum_i \lambda_i \langle b_i| X |b_i\rangle .$$

- The time evolution in a closed system with Hamiltonian H is

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}.$$

3.1 One Qubit Example

The codes for these various examples are included as a separate Python code.. Consider a single qubit system that has a 50% chance of being in $|0\rangle$, a 25% chance of being in $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and a 25% chance of being in $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{4}|+\rangle\langle +| + \frac{1}{4}|-\rangle\langle -| = \begin{bmatrix} 3/4 & 0 \\ 0 & 1/4 \end{bmatrix}.$$

3.2 Two Qubit Example

Consider a system of two qubits that has a 90% chance of being in the Bell state

$$|B_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The remaining 10% is distributed among $|01\rangle$, $|10\rangle$, and $|B_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ in the ratio 3:2:1. Then

$$\rho = \frac{9}{10}|B_+\rangle\langle B_+| + \frac{1}{20}|01\rangle\langle 01| + \frac{1}{30}|10\rangle\langle 10| + \frac{1}{60}|B_-\rangle\langle B_-| .$$

4 Subsystems and Partial Traces

An open quantum system is a subsystem of a closed system composed of system + environment. To describe only the system, we trace out the environment:

$$\rho_A = \mathrm{Tr}_B(\rho).$$

For a pure state $|\psi\rangle = |a\rangle \otimes |b\rangle$, the reduced density matrix is

$$\rho_A = \sum_j (I_A \otimes \langle j|)\rho(I_A \otimes |j\rangle).$$

4.1 Two Qubit Example

From the two-qubit example,

$$\rho_1 = \mathrm{Tr}_2(\rho) = \begin{bmatrix} 61/120 & 0 \\ 0 & 59/120 \end{bmatrix}, \qquad \rho_2 = \mathrm{Tr}_1(\rho) = \begin{bmatrix} 59/120 & 0 \\ 0 & 61/120 \end{bmatrix}.$$

5 von Neumann Equation

For the total density matrix ρ_T of system plus environment with total Hamiltonian H_T,

$$\frac{d\rho_T}{dt} = -i[H_T, \rho_T].$$

6 Lindblad Master Equation

Writing $H_T = H + H_E + H_I$, where H is the system Hamiltonian, H_E the environment Hamiltonian, and H_I their interaction, tracing out the environment and assuming weak coupling yields:

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right),$$

where ρ is the reduced density matrix, $\gamma_k \geq 0$ are decay rates, and L_k are jump operators.

The Heisenberg picture form is

$$\frac{dX}{dt} = i[H, X] + \sum_k \gamma_k \left(L_k^\dagger X L_k - \frac{1}{2}\{L_k^\dagger L_k, X\} \right),$$

and the identity operator satisfies $\frac{dI}{dt} = 0$.

6.1 Liouvillian Superoperator

Define the superoperator $\mathcal{L}$ by

$$\mathcal{L}[\rho] = -i[H, \rho] + \sum_k \gamma_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right),$$

so that

$$\frac{d\rho}{dt} = \mathcal{L}[\rho].$$

6.2 Vectorization: The Fock–Liouville Space

Define $|A\rangle\rangle$ as the column-stacked vectorization of an operator A. The inner product is

$$\langle\langle A|B\rangle\rangle = \mathrm{Tr}\left(A^\dagger B\right),$$

and

$$\frac{d}{dt}\, |\langle\rho\rangle\rangle = \hat{\mathcal{L}}\, |\langle\rho\rangle\rangle\,,$$

with solution

$$|\!\langle \rho(t) \rangle\!\rangle = e^{\hat{\mathcal{L}} t} |\!\langle \rho(0) \rangle\!\rangle .$$

Diagonalizing $\hat{\mathcal{L}}$ gives eigenvalues λ_i and eigenvectors $|\!\langle r_i \rangle\!\rangle$ and $\langle\!\langle l_i |$, allowing

$$|\!\langle \rho(t) \rangle\!\rangle = \sum_i e^{\lambda_i t} \langle\!\langle l_i | \rho(0) \rangle\!\rangle \, |\!\langle r_i \rangle\!\rangle .$$

6.3 Steady State

The steady state $|\!\langle \rho_{\mathrm{ss}} \rangle\!\rangle$ satisfies

$$\hat{\mathcal{L}} \, |\!\langle \rho_{\mathrm{ss}} \rangle\!\rangle = 0.$$

6.4 Harmonic Oscillator Coupled to a Bath

For $H = \omega a^\dagger a$ and jump operators $L_1 = \sqrt{\gamma(\tau+1)}\, a$, $L_2 = \sqrt{\gamma\tau}\, a^\dagger$, the Lindblad equation reads:

$$\frac{d\rho}{dt} = -i[\omega a^\dagger a, \rho] + \gamma(\tau+1)\left[a\rho a^\dagger - \frac{1}{2}\{a^\dagger a, \rho\}\right] + \gamma\tau\left[a^\dagger \rho a - \frac{1}{2}\{a a^\dagger, \rho\}\right].$$

References

1. D. Manzano, *A Short Introduction to the Lindblad Master Equation*, arXiv:1906.04478, 2020.

2. H. Yuen, *Lecture 2: Mixed States and Density Matrices*, 2022.