The QF is ~~nege~~ negetive semi-definate if $(-1)^n D_n \geq 0$ and atleast one term is zero.

- In all the other cases it is Indefinite.

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Q Reduce the quadratic form $Q = 3x_1^2+5x_2^2+3x^3-2x_1x_2-2x_2x_3+2x_3x_1$ to a diagonal canonical form and find its nature, rank, index and signature.

Soln

$3x_1^2+5x_2^2+3x^3-2x_1x_2-2x_2x_3+2x_3x_1$

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ → char eq. $|A-\lambda I| = 0$

$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36$

$\Rightarrow \lambda = 2, 3, 6$ → eigen values

Ev for $\lambda = 2 \Rightarrow [A-\lambda I]x = 0$

$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{0} = \frac{x_3}{2}$ ∴ eigen vector $x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = 3 \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$\Rightarrow \frac{x_1}{-1} = \frac{-x_2}{1} = \frac{x_3}{-1}$ ∴ ev $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda = 6$ $\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ → by using cross multiplication -

$\frac{x_1}{-2} = \frac{-x_2}{-4} = \frac{-x_3}{-2}$ ∴ $x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

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$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$

→ To find canonical form → first normalise

$$\bar{x}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \bar{x}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad x_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Normalised eigen vector matrix $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/6 \end{bmatrix}$

$\boxed{X = PY}$

$Q = X^T A X = (PY)^T A [PY]$

$= Y^T (P^T A P) Y$

$P^T A P = [P^T][A][P]$

$= \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$Y^T (P^T A P) Y \Rightarrow$

$[y_1\ y_2\ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$\underline{Q = 2y_1 + 3y_2^2 + 6y_3^2} \Rightarrow$ canonical form.

$= \cancel{\lambda_1 y}\ \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad B \neq 0$

Rank $= 3 = r$

Index $= 3 = P$ $\quad D_1 = 2$

Signature $= 2p - r$ $\quad D_2 = 6$

$= 6 - 3 = 3$ $\quad D_3 = 36$

∴ positive definate

Q $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ $\quad R_3 \longrightarrow R_3 - 3R_1$ $\quad$ {Row-reduction}

$R_2 \rightarrow R_2 - 2R_1$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$ $\quad R_3 \rightarrow R_3 - 2R_2$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ ∴ no of non zero rows $= \underline{2 = \text{rank}}$