prob8

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(a) Prove, using the Pumping lemma for Regular Languages, that there is no Universal Regular Expression.

The universal regular language (URE) takes in the form of R$x where R is the regular expression and x a string which is accepted by R. This must hold for all R and input x that are accepted by R

To show that URE is not a regular language:

Assume that URE is a regular language. Therefore for any R that accepts an input x, R$x belongs to the URE Language.
The alphabet is : {a,b,∪,∗,(,),ε,∅,$}

Therefore, we can use pumping lemma for a string accepted by URE. If all cases of pumping lemma results in a string that is no longer part of URE, then we know URE is not regular language.

Pumping lemma
If URE language is regular, then there must exist a pumping length p such that any string s in URE language of length |s| >= N can be split into w = xyz

1. |xy| <= N
2. |y| > 0,
3. For all i >= 0, xy^iz is also a part of URE L

Let's consider the regular expression that describes a string that is static in length, such as a string of b's to some arbitrary length N.

$S = R\$x$ where $R: \underbrace{bbb\ldots b}_{N \text{ length}}$

$\therefore x: \underbrace{bbb\ldots b}_{N \text{ length}}$

So U accepts $w = bbb\ldots b \$ bbb\ldots b$

both same length N

then we have 2 possible scenarios as $|xy| \leq N$ and $y \neq \varepsilon$

$\underbrace{bbb\ldots b}_{xy \text{ must be restricted to this range}} \$ \; bb\ldots b$

Scenario 1:

$y = \underbrace{bbb\ldots b}_{= N \text{ length}}$ => if we pump y then we get:

$w_2 = \underbrace{bbb\ldots b}_{2N \text{ length}} \$ \underbrace{bbb\ldots b}_{N \text{ length}}$

For this scenario w_2 is not accepted by U as R does not match x as R is looking for b's of length 2N while x is only length N

Scenario 2 :

$y = \underbrace{bbb\ldots b}_{k \text{ length where } k < N}$ => pumping y gives us :

$w_3 = \underbrace{bbb\ldots b}_{N+k \text{ length}} \$ \underbrace{bbb\ldots b}_{N \text{ length}}$

For this scenario, w_3 is not accepted as R is looking for b's of length N+k while x is b's of N length

As pumping lemma has failed for all possible y's in a string accepted by U, URE is not regular.

Therefore URE U regular expression does not exist

(b) Prove, using the Pumping lemma for Context-Free Languages, that there is no Universal Context-Free Grammar.

This question has the same set up as the previous question, but in the context of Context-Free Language rather than regular language

Let k be the number of nonterminal symbols in Universal Context-Free Grammar U.

For any U, consider the following Context-Free Grammar G:

$S \to T_1$

$T_1 \to a$

$T_1 \to T_2$

$T_2 \to aa$

$T_2 \to T_3$

$T_3 \to aaa$

$\vdots$

$T_N \to a^N$

Where any terminal Tn can be expanded to a^n or Tn+1 until the final terminal TN -> a^N

And pick the x string accepted by this as a^N

In the language U, we can pick the string:

$w = S \to T_1 ; \underbrace{T_1 \to a ; T_1 \to T_2 ; \ldots ; T_N \to a^N}_{\text{N repetitions of } T_n \to a^n \, ; \, T_n \to T_{n+1} \quad n \geq 1} \$ a^N$

It's important to understand that each terminal Tn+1 relies on Tn for n = 1. Once we do some pumping lemma, we can see that removing any terminals within G will break the input string a^N (which is the largest string possible)

Take $N > 2^{k-1}/3$

Consider any u,v,w,y,z such that
- vy =! empty string
- |vxy| <= 2^k
- w = uvxyz

We must test whether:
uxz, uvxyz, uv^ixy^iz for i=0 -> infinity are all in U?

Case 1:
v and y are within the N repetitions of Tn -> a^n; Tn -> Tn+1

$$w = S \to T_1 ; \underbrace{T_1 \to a ; T_1 \to T_2 ; \ldots}_{v} \underbrace{; T_N \to a^N \,\$}_{y} \, a^N$$

If we consider removing the v and y, and consider the string uxz: disconnect between T_2 and T_n

$$S \to T_1 ; T_1 \to a ; T_1 \to T_2 ; T_n \to a^n ; \ldots T_N \to a^N ; a^N$$

Note: T_2 was just an example of where the disconnect can be it can be anywhere

This is no longer part of the language because there is now a disconnect between the terminals. Without the connection, S -> T1 cannot be expanded to TN -> a^N, making a^N invalid

$$uxz \notin U$$

Case 2:
v is within the N repetitions of Tn -> a^n; Tn -> Tn+1, and y is within the a^N string

$$w = S \to T_1 ; T_1 \to \underbrace{a ; T_1 \to T_2 ; \ldots}_{v} ; T_N \to a^N \,\$\, \underbrace{a^N}_{y \leq a^N}$$

If we pump v and y to produce some variation of uv^ixy^iz, then:
- The repetition of v causes a repetition of the set of context grammar rules within G of the word.
- The repetition of y causes a^N to be expanded to a^N+n

The repetition of the rules do not accommodate the new larger a^N+n string, therefore the string uv^ixy^iz is not accepted by U

$$uv^2xy^2z \notin U$$

In every possible case, we have found an i such that uv^ixy^i is not accepted by U, which violates the conclusion of Pumping Lemma for CFLs.

Therefore UCFG is not context-free, meaning U does not exist by definition that it must be a context free grammar.