Problem 5

Wednesday, August 14, 2024 11:20 AM

(a) Using the same approach as you did in Problem 1, construct a Boolean expression φ_X using variables v_i, where the expression is True if and only if the assignment of truth values to the variables represents a triangle-free vertex cover of X. The relationship between the number of clauses in your expression and the number of edges and triangles in X should match the expression that you gave in Problem 3 (a).

The edges are given as:
1 – 3
1 – 2
2 – 3
4 – 5
4 – 7
5 – 6
6 – 7

Triangle vertices are:
1, 2, 3

Therefore the CNF expression for triangle-free vertex cover of X:
(v1|v3) & (v1|v2) & (v2|v3) & (v4|v5) & (v4|v7) & (v5|v6) & (v6|v7) &
(~v1|~v2|~v3)

There are 8 clauses
There are 7 edges and 1 triangle, therefore it matches the formula from 3a:
number of clauses = m + t

(b) Recall the process described at the top of page 5 of taking any graph G and constructing a Boolean expression φ_G from that graph. In this question, we look at a restricted version of this problem where only 2-regular graphs are considered.

For any 2-regular graph G, let φ_G be a Boolean expression with n variables that is constructed from G, as described at the top of page 5.

Prove by induction that φ_G has at most $\frac{4n}{3}$ clauses.

Base case:
The smallest 2-regular graph has 3 vertices with 3 edges and 1 triangle (4 clauses)
Therefore we can restrict n >= 3.
Let c represent the count of clauses.

$$n = 3 \rightarrow c = 4$$

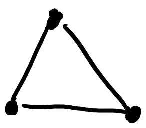
$$\text{bound} = \frac{4n}{3} = \frac{12}{3} = 4$$

$C = \text{bound}$ Therefore the statement holds for base case

As deduced in question 3a, the count of clauses is given by:
c = m + t
M is the count of edges and t is the count of trees.

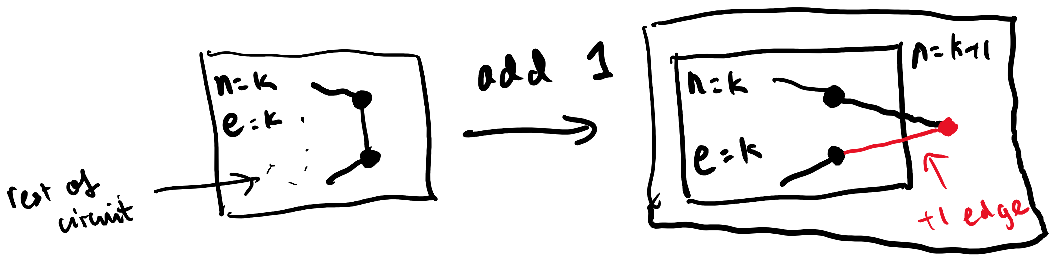
Every component in a 2-regular is a **circuit**. **For any circuit, the number of edges and number of vertexes are equal.**

To prove this inductively:
Base:
N = 3



$\text{edges} = 3 \qquad \therefore n = e$

Inductive step:
Assume for a given circuit, n = k --> e = k



$n = k+1$
$e = k+1$
$\therefore n = e$

Therefore, for any graph that is 2 regular, it will always have n edges for n vertices, because it is a collection of circuits.

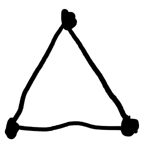
From this we know that the **m** in c = **m**+t formula can only equal **n**. If we have only collections of non-tree circuits, we only have a total clause count equal to n. To maximize count of clauses (c), we need to maximize count of trees (t). Therefore, the worst case scenario (to get as close to breaking the boundary as possible), we need all the components to be trees.

We can now use induction to prove that for any collection of triangles, the number of clauses do not break our boundary condition.

Lets prove inductively that a **triangle only** 2-regular graph will not break the boundary

Base case:
A single triangle graph is

$n = 3$



edges $= 3$

tree $= 1$

$c = 4$

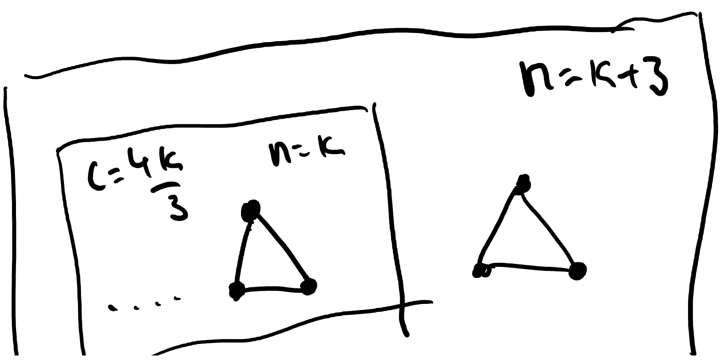
$$\text{bound} = \frac{4n}{3} = \frac{12}{3} = 4 \quad , \quad c = \text{bound}$$

$\therefore$ base case holds

Inductive step:
A graph of n = k vertices meets the boundary 4k/3 clauses

$$n = k \longrightarrow c = \frac{4k}{3}$$

To form another triangle, we need to add a triangle circuit



$$c = \frac{4k}{3} + 3 \text{ edges} + 1$$

$$= \frac{4k+12}{3} = \frac{4(k+3)}{3}$$

for $n = k+3$

$\therefore$ boundary is met

Therefore, we have proved that the worst case scenario for breaking the boundary, which is all triangles in the tree, does not break the boundary condition for any count of triangles.

Therefore by induction we proved that the most clauses we can have is 4n/3 clauses