Monad Ultrapowers on Ultrafilters

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Abstract Based on the first author`s 2014 volume there is a fragment topology, e.g. ultraproducts over which we have ultrafiltre models. We can define a congruence relation on such fragment ultraproduct based on shared equal subfragments. That gives us an ultrapower construction. So we take the power sets on the fragment ultrapower subsets to converge to an ultrapower monad. The paper has some relevance to cardinal arithmetic while ultrafilter model saturations on large cardinal discernability are moot.

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1. N-Types

Let us start with n-types and positive local realizability (Keiser 1967, Nourani 2003):. Given a theory T and a nonnegative integer n, let n(T) be the set of all signature (φ) ⊆ signature (T). Say that ⊆ n(T) is consistent over T if there exist A s.t. A ⊨ T and a1, ..., an ∈|A| such that A ⊨ φ(a1, ..., an) for all φ∈Y. Note that by compactness, if each finite subset of n(T) is consistent over T then so is n(T). formulas φ(x1 ,..., xn) with no free variables other than x1,..., xn, such that sig8 1n A realization of ⊆ n (T) is an n-tuple a1, ..., an ∈ |A|, A s.t. A ⊨ T, and A ⊨ φ(a1,..., an) for all φ ∈ . We then say that is realized in the model A. If F and G and are a pair of adjoint functors, with F left adjoint to G, then the composition G • F is a monad. Partially ordered monads can be composed building upon the underlying monad compositions. In particular we focus on composing the partially ordered powerset monad with the term monad.

2. Omitting types monads The omitting types theorem can be forwarded as follows: From Kiesler 1967:

Theorem 1 Let T be a countable theory. For each i ∈ ω, let pi be an essentially nonprincipal n-types e over T. Then T has a model which omits pi for each i ∈ ω.

Consider term functors on 'direct product algebra category (Nourani 2016). Objects are term functors and morphisms are natural transformations on representation preschiefs(Nourani 2011). Call these nD-type, D for direct product, embedding categories, F-Type categories are the language fragment categories with fragment diagram n-types.(Nourani 2015) forwarded that that there are embedding functors from N-Type to the direct product category realizing a filter for the product algebra trees on nDtypes. (Nourani-Eklund 2016 MAA or AAA Vienna2018) further develops monads on sets, and Term Algebras Monads.

Lemma F.G is a Monad on pair product signature n-type.
Theorem 2 (Nourani 2014)There are embedding functors from F-Type to the direct product category realizing a filter for the product terms on nD-types.

(Nourani-Eklund 2016 MAA, AAA Vienna2018) Nourani-Eklund 2017: volume ON Ultrafilters computing Term Functors (Lambert Publishers 2019). The omitting types theorem can be generalized as follows:
3. A Fragment topology ultrafilter (Nourani 2014-2015, ASL presented term functors on fragment topology constructions are from Nourani 2015 volume on a functorial model theory. F1: L ω 1, K → Set F2 a forgetful functor, Top→ΠK where the singleton element is the discrete product topology on the Keisler fragment K.

Definition Let U be an ultrafilter over I. Two elements f,g of the cartesian product i∈I Ai are said to be Uequivalent, in symbols f =U g, if the set {i : f(i) = g(i)} belongs to U. The U-equivalence class of f is the set fU = {g : f =U g}. The ultraproduct U Ai is defined as the set of U-equivalence classes Ai = {fU : f ∈ Ai}. U i∈I embedding is the mapping d : A → U A such that d(a) is the U-equivalence class of the constant function with value a. It is easily seen that d is injective.

In the above definition, it is easily checked that =U is an equivalence relation on Ai. Given a nonempty set A, the ultrapower of A modulo U is the defined by i∈I as the ultraproduct UA= UAi whereAi =A foreach i∈I.

Definition Consider the fragment topology defined on the preceding that provides us with ultraproducts for ultrafilter models. Let us define a congruence relation on such fragment ultraproduct based on shared equal subfragments. Two fragment ultraproducts are considered sub-fragment Hg congruent when there are Hg-shared subfragments.

Theorem 3 There is a fragment topology, e.g. ultraproducts over witch we have ultrafilter models. based on shared equal subfragments. That provides an ultrapower construction. Taking the power sets on the fragment ultrapower subsets, there is a convergence to an ultrapower monad.

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