The Big, Bigger, and Biggest Five of Reverse Mathematics

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SLSS24

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Then, we discuss our recent extensions of the Big Five.

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Harvey Friedman & Steve Simpson (courtesy of MFO).

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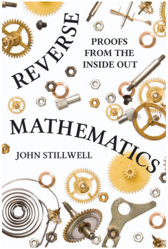

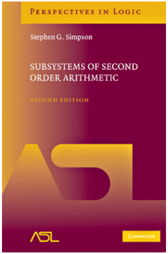

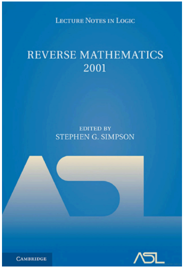

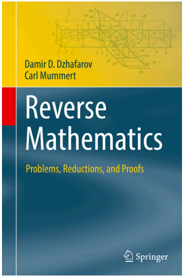

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Having RCA_0 means we have a Turing machine/idealised computer, and some induction to verify our algorithms.

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Hence: RCA_0 proves $A \leftrightarrow T$. One direction is the 'normal' one, the other is the 'reverse' way of doing math.

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Exceptions: theorems from combinatorics in the RM zoo.

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Similar lists for the other Big Five.

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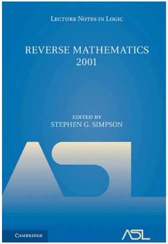

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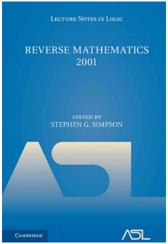

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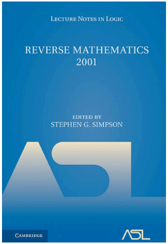

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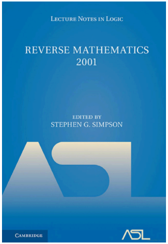

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This coding can get very messy very quickly.

Coding continuous functions

DEFINITION II.6.1 (continuous functions). Within RCA_0, let $\widehat{A}$ and $\widehat{B}$ be complete separable metric spaces. A (code for a) *continuous partial function* ϕ from $\widehat{A}$ to $\widehat{B}$ is a set of quintuples $\Phi \subseteq \mathbb{N} \times A \times \mathbb{Q}^+ \times B \times \mathbb{Q}^+$ which is required to have certain properties. We write $(a,r)\Phi(b,s)$ as an abbreviation for $\exists n\,((n,a,r,b,s) \in \Phi)$. The properties which we require are:

1. if $(a,r)\Phi(b,s)$ and $(a,r)\Phi(b',s')$, then $d(b,b') \leq s + s'$;
2. if $(a,r)\Phi(b,s)$ and $(a',r') < (a,r)$, then $(a',r')\Phi(b,s)$;
3. if $(a,r)\Phi(b,s)$ and $(b,s) < (b',s')$, then $(a,r)\Phi(b',s')$;

where the notation $(a',r') < (a,r)$ means that $d(a,a') + r' < r$.

The idea of the definition is that Φ encodes a partially defined, continuous function ϕ from $\widehat{A}$ to $\widehat{B}$. Recall from the previous section that $\mathrm{B}(a,r)$ denotes the basic open ball centered at a with radius r. Intuitively, $(a,r)\Phi(b,s)$ is a piece of information to the effect that $\phi(x) \in$ the closure of $\mathrm{B}(b,s)$ whenever $x \in \mathrm{B}(a,r)$, provided $\phi(x)$ is defined. This is made precise in the following two paragraphs.

A point $x \in \widehat{A}$ is said to *belong to the domain of* ϕ, abbreviated $x \in \mathrm{dom}(\phi)$, provided the code Φ of ϕ contains sufficient information to evaluate ϕ at x. This means that for all $\epsilon > 0$ there exists $(a,r)\Phi(b,s)$ such that $d(x,a) < r$ and $s < \epsilon$. If $x \in \mathrm{dom}(\phi)$, we define the value $\phi(x)$ to be the unique point $y \in \widehat{B}$ such that $d(y,b) \leq s$ for all $(a,r)\Phi(b,s)$ with $d(x,a) < r$. If $x \in \mathrm{dom}(\phi)$, we can use the code Φ and minimization (theorem II.3.5) to prove within RCA_0 that $\phi(x)$ exists. Then, using condition II.6.1.1, it is easy to prove within RCA_0 that $\phi(x)$ is unique (up to equality of points in $\widehat{B}$, as defined in II.5.1).

We write $\phi(x)$ *is defined* to mean that $x \in \mathrm{dom}(\phi)$. We say that ϕ is *totally defined on* $\widehat{A}$ if $\phi(x)$ is defined for all $x \in \widehat{A}$. We write $\phi\colon \widehat{A} \to \widehat{B}$ to mean that ϕ is a continuous, totally defined function from $\widehat{A}$ to $\widehat{B}$.

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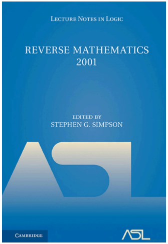

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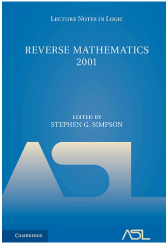

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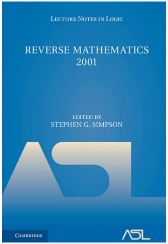

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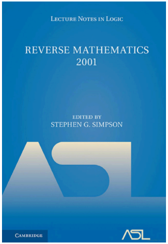

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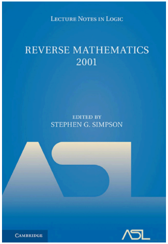

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The base theory RCA_0^ω

RCA_0^ω makes use of the language of finite types: $n \in \mathbb{N}$ or n^0, $f \in \mathbb{N}^\mathbb{N}$ or f^1, $Y : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ or Y^2, et cetera.

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Real numbers and '$=_\mathbb{R}$' defined as in RCA_0; $\mathbb{R}\to\mathbb{R}$-functions are $\mathbb{N}^\mathbb{N}\to\mathbb{N}^\mathbb{N}$-functions extensional relative to '$=_\mathbb{R}$'.

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Regulated means: the left and right limits exist (Bourbaki, Darboux).

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- A cadlag function on $[0, 1]$ is bounded.
- A cadlag function on $[0, 1]$ is Riemann integrable.
- A bounded Baire 1 function on $[0, 1]$ has a supremum.
- A usco function on $[0, 1]$ with a sup attains a max.
- A bounded quasi-continuous function on $[0, 1]$ has a sup.
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Regulated means: the left and right limits exist (Bourbaki, Darboux).
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WILD: there are $2^{\mathfrak{c}}$ non-measurable quasi-continuous functions and $2^{\mathfrak{c}}$ non-Borel bounded and measurable quasi-continuous functions.

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Same for ACA_0, ATR_0, and Π^1_1-CA_0 (see our JML23 paper).

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Extending the language (from second- to third-order) yields a massive extension of the Big Five.

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We call these systems 'Big' since they boast many equivalences.

Big in RM: the uncountability of $\mathbb{R}$

Central to real analysis are D_f and C_f, the sets of discontinuity and continuity points for $f : \mathbb{R} \to \mathbb{R}$.

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By contrast, for regulated f, D_f is height-countable in a weak system.

Definition

A set $A \subset \mathbb{R}$ is height-countable if there a height function $H : \mathbb{R} \to \mathbb{N}$ for A, i.e. $\{x \in A : H(x) < n\}$ is finite for all $n \in \mathbb{N}$.

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Similar results for BV-functions (sometimes involving extra axioms).

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Recently, Dag Normann and I have obtained a plethora of equivalences (over RCA_0^ω or extensions) between:

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New 'Big' systems: uncountability of $\mathbb{R}$, Baire category theorem, Jordan decomposition theorem, and pigeon hole principle.

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- the special role of the Axiom of Choice,
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Thanks!

Questions?

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