Elementary Matrices.

· Elementary row operations.

역과정..?

① c times → 1/c times.
② interchange → interchange.
③ another + c times → another + (-c) times.

* Page 55, Book, table 1.

* Def 1.

· $A \xrightarrow{ERO} B$ then $B \xrightarrow{R\text{-}ERO} A$ "행 동치"

=> Matrices A and B are said to be "row equivalent" if either (hence each) can be obtained from the other by a sequence of elementary row operations.

* Def 2.

· A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

· ex) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ is elementary matrix.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $x=3$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. is elementary matrix.

* Thm 1.5.1

If, the elementary matrix E ($m \times m$) results from performing a certain row operation on I_m ($m \times m$) and A is an $m \times n$ matrix, then the produce $\boxed{EA}$ is the matrix that results when this same row operation is performed on A.

*Ex 2) ex).

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \rightarrow I_3 + = 3E_1$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}.$$

$$(EA)_3 = A_3 + 3A_1$$

∴ E가 elementary Row Operation 의 거시적 역할 "행연산의 거시자"

* Ex 3) Inverse Elementary Row Operations.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{+5 \text{ times}} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \xrightarrow{-5 \text{ times}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

ERO　　Inverse ERO.

정리.

* Thm 15.2). Every elementary matrix is invertible, and the inverse is also an elementary matrix

"동치"

Equivalance Theorem. (필요충분조건). If A is an n×n matrix, below is all true or all false.

- (a) A is invertable.
- (b) $Ax = 0$ has only the trivial solution.
- ☆(c) The reduced row echelon form of A is I_n.
- (d) A is expressible as a product of elementary matrices.

(b). $AX=0$, A is invertible.
then $A^{-1}AX = A^{-1}0$.
$X = 0$. → trivial solution.

(d) $A = \overbrace{E_k E_{k-1} \cdots E_1}$

k… A의 형태에 따라 달라짐.

하지만 항상 E의 곱으로 표현 가능.

Inverting Matrices.

$$E_k \cdots E_2 E_1 \quad A = I_n$$

$$(E_k \cdots E_2 E_1)^{-1} (E_k \cdots E_2 E_1) = (E_k \cdots E_2 E_1)^{-1} I_n.$$

$$\therefore \boxed{\begin{cases} A = (E_k \cdots E_2 E_1)^{-1} I_n. \\ A^{-1} = (E_k \cdots E_2 E_1) I_n. \end{cases}}$$

" 가우스-조던 elimination 시의 ERO는 역행렬 구하는데 관련 있다 "

정리

Inversion Algorithm.

* Ex 4). $\begin{cases} [A] \xrightarrow{GJ} [I]. \\ [A|I] \xrightarrow{GJ} [I|A^{-1}]. \end{cases}$

$$\begin{array}{cc} (A) & (I) \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

↓

$$\begin{array}{cc} (I) & (A^{-1}) \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Ex 5). Showing Matrix is not invertible
Using Inversion Algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Left matrix
Cannot be I ⇒ not invertible.

More on Linear Systems and Invertible Matrices.

*Thm 1.6.1 A system of linear equations has { zero. / one. / infinitely many } solutions.

*Thm 1.6.2. A: an invertible matrix of size n×n.
$\underline{b}$: n×1 column vector.
⇒ $A\underline{x} = \underline{b}$ has exactly one solution.
$\underline{x} = A^{-1}\underline{b}$

*ex2). $A\underline{x} = \underline{b}_1$, $A\underline{x} = \underline{b}_2$) … $\left[\begin{array}{c|c} A & \begin{matrix}4\\5\\9\end{matrix}\end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \checkmark \\ 0 & 1 & 0 & \checkmark \\ 0 & 0 & 1 & \checkmark \end{array}\right]$.

$$\left[\begin{array}{c|c|c} A & \begin{matrix}4\\5\\9\end{matrix} & \begin{matrix}1\\6\\-6\end{matrix}\end{array}\right]$$ ← Augmented Augmented Matrix.

↓

$$\left[\begin{array}{c|c|c} I & \begin{matrix}1\\0\\1\end{matrix} & \begin{matrix}2\\1\\-1\end{matrix}\end{array}\right]$$

* Thm 1.6.3. A : square Matrix.

(a) If B is a square matrix satisfying $\underline{BA} = I$ then $B = A^{-1}$.

(b) $\underline{AB} = I$ then $B = A^{-1}$.

* Thm 1.6.4. A : n×n Matrix. The following are equivalent.
(Thm 1.5.3 + α).

(a) A is invertible.
(b)
(c)
(d).

(e) $A\underline{x} = \underline{b}$ is consistent for every n×1 matrix $\underline{b}$

= at least one solution.

(f) has exactly one solution for every n×1 matrix b.

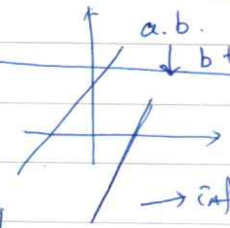
α Thm 1.6.2.

⇒ · A^{-1} … $A\underline{x} = \underline{b}$. $\underline{x} = A^{-1}\underline{b}$. → (f) ✓.
↓
(e) ✓.

· (e) ⇒ (a) ?

for some $\underline{b}$ … $A\underline{x} = \underline{b}$ { one solution.
$\boxed{\text{for every b.}}$ ← ~~infinitely many~~ solution.

a. b.
b += b_1



→ infinitely many.

every b 라는 표현…
b를 바꾸면 infinitely many에서 no solution이 됨.
⇒ $\boxed{\text{every b is one b.}}$

* Thm 1.6.5. Let A and B be square matrices of the same size.
If AB is invertible,
Then A and B must also be invertible. $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$.

* A fundamental problem.

"fixed"
Let A be a fixed $m \times n$ matrix.
Find all $m \times 1$ matrix b such that $A\underline{x} = \underline{b}$ is consistent.

$Ax = b$ 해가 존재하도록 b를 결정해라.

$\Downarrow$.

*Ex 3.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= b_1 \\ x_1 \quad\ + x_3 &= b_2 \\ 2x_1 + x_2 + 3x_3 &= b_3 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 1 & 0 & 1 & b_2 \\ 2 & 1 & 3 & b_3 \end{array}\right].$$

$\Downarrow$ GJ.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_1 - b_2 \\ \hline 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right].$$

infinitely many $\leftarrow$

$$b_3 = b_2 + b_1 \Rightarrow \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \end{bmatrix}.$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= b_1 \\ 2x_1 + 5x_2 + 3x_3 &= b_2 \\ x_1 \quad\quad + 8x_3 &= b_3 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 5 & 3 & b_2 \\ 1 & 0 & 8 & b_3 \end{array}\right]$$

$\Downarrow$ GJ.

$$\begin{array}{c} x_1 \rightarrow \\ x_2 \rightarrow \\ x_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -40b_1 + 16b_2 + 9b_3 \\ 0 & 1 & 0 & 13b_1 - 5b_2 - 3b_3 \\ 0 & 0 & 1 & 5b_1 - 2b_2 - b_3 \end{array}\right]$$