Matrix Diagonalization Problem.

$P^{-1}AP$ $\quad \boxed{A \to P^{-1}AP}$ $\to$ Diagonal Mat. similarity transformation.

- $\text{Det}(A) = \text{Det}(P^{-1}AP) = \text{Det}(P^{-1})\,\text{Det}(A)\,\text{Det}(P)$
  $= \cancel{1/\text{Det}(P)} \cdot \text{Det}(A) \cdot \cancel{\text{Det}(P)}$

- Invertibility.
- Rank, Nullity.
- Trace (대각 원소들의 합).
- Characteristic equation.
- eigenvalues. eigenspace dim.

*** Def 1)** If A, B : square matrices

$\Rightarrow$ B is similar to $\boxed{A}$ if $\exists$ invertible matrix P s.t. $\boxed{B = P^{-1}AP}$.

*** Def 2)** A : diagonalizable

if it is similar to some diagonal mat. $\to P^{-1}AP$.

P is said to diagonalize A.

*** Thm 5.2.1)** A : $n \times n$ matrice.

(a) diagonalizable $\quad D = P^{-1}AP,\ \exists P$

(b) A has n linearly independent eigenvectors.

*** Thm 5.2.2)** (a) $\lambda_1, \cdots \lambda_k$: distinct eigenvalues of A. $\to$ k개의 eigenvalue, 서로 다름.

$\underline{v}_1, \cdots \underline{v}_k$: corresponding eigenvectors.

$\Rightarrow \{\underline{v}_1, \cdots, \underline{v}_k\}$: linearly indep.

(b) $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Procedure for Diagonalizing a Matrix.

step 1. check if $\exists n$ linearly independent eigenvalues.

$\downarrow$ n개의 lin-indep. 한 eigenvectors.

$(\lambda I - A)\underline{x} = \underline{0}$: solution Space. $\rightarrow$ $\underline{P_1}, \underline{P_2}, \cdots, \underline{P_n}$.

Step 2. $P = [\, \underline{P_1} \,|\, \underline{P_2} \,|\, \cdots \,|\, \underline{P_n} \,]$ $\leftarrow n \times n$.

step 3. $D = P^{-1} A P$ where $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

*ex1)

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

double root.

sol) $(\lambda - 1)(\lambda - 2)^2 = 0.$ $\Rightarrow \lambda = 1,\ \lambda = 2.$

$\lambda = 2 \rightarrow$ eigenvector. $\underline{P_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \underline{P_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$ $\leftarrow$ dim=2.

(eigenspace's bases)

$\lambda = 1 \rightarrow \underline{P_3} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$ $\leftarrow$ dim=1.

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \rightsquigarrow P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}.$$

$$= \begin{bmatrix} 2 & & \\ & 2 & \\ & & 1 \end{bmatrix}$$

*ex2)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}.$$

double root.

sol) $\det(\lambda I - A) = (\lambda - 1)(\lambda - 2)^2 = 0.$ $\lambda = 1,\ \lambda = 2$

- $\lambda = 1$ $\underline{P_1} = \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}.$ $\leftarrow$ dim=1
- $\lambda = 2$ $P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\leftarrow$ dim=1.

) 3개의 lin-indep eigen vector가 없다!

NO.

*ex3)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

sol) $\lambda = 4$, $\lambda = 2 \pm \sqrt{3}$. → 세개 distinct한 value… invertible?

$$P^{-1}AP = \begin{bmatrix} 4 & & \\ & 2+\sqrt{3} & \\ & & 2-\sqrt{3} \end{bmatrix}$$

*ex4)

$$A = \begin{bmatrix} -1 & 2 & 4 & 0 \\ & 3 & 1 & 7 \\ & & 5 & 8 \\ & & & -2 \end{bmatrix}$$

upper triangle

$\lambda = -1$, $\lambda = 3$, $\lambda = 5$, $\lambda = -2$.

$$P^{-1}AP = \begin{bmatrix} -1 & & & \\ & 3 & & \\ & & 5 & \\ & & & 2 \end{bmatrix}$$

Eigenvalue of Power of a Matrix.

Thm 5.2.3)

scalar.

$\cdot\ A\underline{x} = \lambda \underline{x}$

$$\begin{aligned} A^2\underline{x} &= k\underline{x} \\ &= A \cdot A\underline{x} \\ &= A(\lambda \underline{x}) \\ &= \lambda A\underline{x} \\ &= \lambda^2 \underline{x}, \end{aligned}$$

$\Rightarrow A \rightarrow \lambda$ $\quad \underline{P_1}, \dots, \underline{P_k}.$

$A^2 \rightarrow \lambda^2$ $\quad \underline{P_1}, \dots, \underline{P_k}.$

$\vdots$

$\boxed{A^n \rightarrow \lambda^n.}$

+) eigen vector: "Stayed Same"

* ex 5).

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}. \qquad A^7 ?$$

sol) (A).

$\lambda = 1$, $\lambda = 2$. $\qquad$ A^7: $\lambda = 1^7$, $\lambda = 2^7 = 128$.

$$P_1 = \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xleftrightarrow{\text{same.}} P_1 = \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Computing Powers of a Matrix.

$D = P^{-1}AP$ .. A행렬 Diagonalization ⇒ 대각행렬 D 획득..

$A^{10} = ?$ $\qquad$ diagonal D^{10}? -- 대각행렬의 거듭제곱은 구하기 쉬움.

D^{10} 사용가능.

$$\boxed{D^{10}} = \underbrace{P^{-1}AP \cdot P^{-1}AP \cdots P^{-1}AP}_{10회.}$$

$$= P^{-1}A^{10}P. \longrightarrow \boxed{A^{10} = PD^{10}P^{-1}}$$

쉽다. ~~빠르다!~~

* ex 6).

$A^{13} = ?$

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

~~double root.~~

$\lambda = 2$, $\lambda = 1$, $\lambda = -2$.

$$P = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$A^{13} = PD^{13}P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & & \\ & 2^{13} & \\ & & 1 \end{bmatrix} P^{-1}$$

Geomatric and Algebric Multiplicity → "중복도"

using ex1) $(\lambda-1)(\lambda-2)^2=0$

	$\lambda=1$	$\lambda=2$. (double root).
multiplicity	1	2
dim	1	2

$\xleftrightarrow{\text{alg.}}$

$\xleftrightarrow{\text{geo.}}$

항상 일치는 x.

using ex2). $(\lambda-1)(\lambda-2)^2=0$.

	$\lambda=1$	$\lambda=2$. (double root).
	1	2.
	1	1

*thm 5.2.4. (a) geo. mul. $\leq$ alg. mul.

(b) diagonalizable $\Longleftrightarrow$ geo. mul. = alg. mul.

"같아야 lin-indep. 충족"