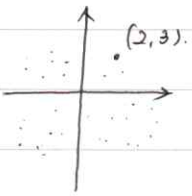
선형대수학 - Ch.3. Euclidean Vector Spaces.



~> 무한히 많은 벡터를 모아놓은 공간.

Vector Space

$\mathbb{R}^n$

Vectors in 2-space, 3-space, and n-space.

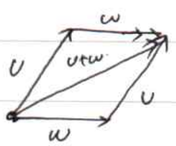
· Vector. $\underline{v} = \overrightarrow{AB}$

equivalent : same direction and size.

$\underline{v} = \underline{w}$

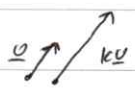
Vector Operations. (using geometric).

· addition.



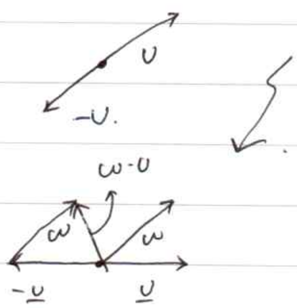
$\underline{v} + \underline{w} = \underline{w} + \underline{v}$

· scalar multiplication.



· subtraction.

$w - v = w + (-v)$.



Vectors in Coordinate Systems.

· (x, y). $\rightarrow$ x축 방향으로 / y축 방향으로...

$\downarrow$ Component.

Def 3) $\underline{v} + \underline{w} = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$.

$k\underline{v} = (kv_1, kv_2, \cdots, kv_n)$.

$-\underline{v} = (-v_1, -v_2, \cdots, -v_n)$.

$\underline{w} - \underline{v} = w + (-v)$.

$= (w_1 - v_1, w_2 - v_2, \cdots, w_n - v_n)$.

Thm 3.1.1)

$\underline{u} + \underline{v} = \underline{v} + \underline{u}$

$(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$.

$\underline{u} + 0 = 0 + \underline{u} = \underline{u}$

$\underline{u} + (-\underline{u}) = 0$.

$k(\underline{u} + \underline{v}) = k\underline{u} + k\underline{v}$

$k(m\underline{u}) = (km)\underline{u}$

$1\underline{u} = \underline{u}$

Thm 3.1.2)

$0\underline{v} = 0$.

$k0 = 0$.

$(-1)\underline{v} = -\underline{v}$.

선형대수학 - Ch 3.2. Norm. Dot Product. and Distance in $\mathbb{R}^n$

Norm of a Vector

$\underline{v} = (v_1, v_2) \cdots \in \mathbb{R}^2, \quad \|v\| = \sqrt{v_1^2 + v_2^2}$

$\underline{v} = (v_1, v_2, \cdots v_n) \cdots \in \mathbb{R}^n, \quad \|v\| = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}$

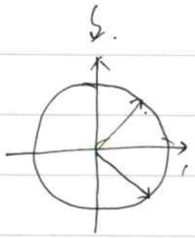
* ex 1) $\|v\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$.

* Thm 3.2.1. (a) $\|v\| \geq 0$

(b) $\|v\| = 0$ then $v = 0$

(c) $\|k\underline{v}\| = |k| \cdot \|\underline{v}\|$

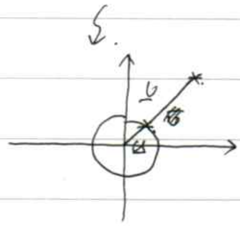
Unit Vector. ··· norm = 1.



··· $\underline{u}$

* Normalization.

$\underline{v} \longrightarrow \underline{u} \quad \cdots \quad \underline{u} = \dfrac{\underline{v}}{\|\underline{v}\|}$



Standard Unit Vectors.

· $\mathbb{R}^2$ ··· $(1, 0), (0, 1)$. → ①축 의 ② 양의 방향.

· $\mathbb{R}^n$ ··· $(1, 0, \cdots, 0)$, $(0, 1, \cdots, 0)$, $\vdots$, $(0, 0, \cdots, 1)$ ∝ standard basis. $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, \cdots, 0)$, $\vdots$, $e_n = (0, 0, \cdots, 1)$

* $\underline{v} = (v_1, v_2, \cdots v_n)$

$= v_1\underline{e_1} + v_2\underline{e_2} + \cdots + v_n\underline{e_n}$ ⇒ linear Combination.

Distance in $\mathbb{R}^n$.

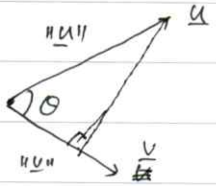
$\underline{u} = (u_1, u_2, \cdots, u_n)$

$\underline{v} = (v_1, v_2, \cdots, v_n)$.

$d(u,v) = \| \underline{v} - \underline{u} \|$

$= \sqrt{(v_1-u_1)^2 + (v_2-u_2)^2 + \cdots + (v_n-u_n)^2}$

Dot Product.



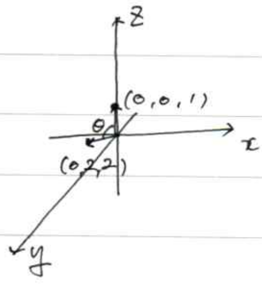
* $0 \le \theta \le 180°$

"방향력의 곱"

*Def 3) $\underline{u} \cdot \underline{v} = \| \underline{u} \| \| \underline{v} \| \cos\theta$.

(= Euclidian Inner Product). "유클리드 공간의 내적"

*Ex 5).



$\theta = 45°$.

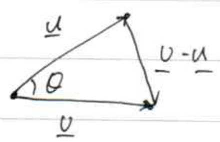
~~‖u‖ ‖v‖ cos45°~~

$\underline{u} \cdot \underline{v} = \| \underline{u} \| \| \underline{v} \| \cos 45°$

$= \sqrt{0+0+1} \sqrt{0+2^2+2^2} \frac{\sqrt{2}}{2}$

$= 2\sqrt{2} \frac{\sqrt{2}}{2}$

Component ~~from~~ Form of the Dot. Product.



$\| \underline{v} - \underline{u} \|^2$

$= \| \underline{u} \|^2 + \| \underline{v} \|^2 - 2 \| \underline{u} \| \| \underline{v} \| \cos\theta$

$\Rightarrow (v_1-u_1)^2 + (v_2-u_2)^2 = u_1^2 + u_2^2 + v_1^2 + v_2^2 - 2\underline{u} \cdot \underline{v}$

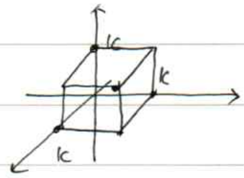
$\therefore \quad \underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2$.

* Ex 7)

$u_1 = (k, 0, 0)$ $\quad d = (k, k, k)$

$u_2 = (0, k, 0)$ $\quad = u_1 + u_2 + u_3$

$u_3 = (0, 0, k)$



$\underline{u}_1 \cdot \underline{d} = \|\underline{u}_1\| \|\underline{d}\| \cos\theta.$

$\cos\theta = \dfrac{\underline{u}_1 \cdot \underline{d}}{\|\underline{u}_1\| \|\underline{d}\|}$

* Thm 3.2.2
3.2.3.

- $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$
- $(\underline{u}) \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$
- $k(u \cdot v) = (ku) \cdot \underline{v}$
- $\underline{v} \cdot \underline{v} \geq 0.$
- $\underline{v} \cdot \underline{v} = 0$ when $\underline{v} = 0$

- $0 \cdot \underline{v} = 0$, $\underline{v} \cdot 0 = 0.$
- $(\underline{u} + \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} + \underline{v} + \underline{w}$
- $\underline{u} \cdot (\underline{v} - \underline{w}) = \underline{u} \cdot \underline{v} - \underline{u} \cdot \underline{w}$
- $(\underline{u} - \underline{v}) \cdot \underline{w} = \underline{u} \cdot \underline{w} - \underline{v} \cdot \underline{w}$
- $k(\underline{u} \cdot \underline{v}) = \underline{u} \cdot (k\underline{v})$

- $\sqrt{v \cdot v} = \|v\| \Rightarrow \sqrt{v \cdot v}^2 = \|v\|^2$

Cauchy-Schwarz Inequality, Angles in $\mathbb{R}^n$.

$\cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \|\underline{v}\|} \rightarrow -1 \leq \cos\theta \leq 1$

$-1 \leq \dfrac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \|\underline{v}\|} \leq 1 \xrightarrow{\|\underline{u}\| \|\underline{v}\|} -\|\underline{u}\| \|\underline{v}\| \leq \underline{u} \cdot \underline{v} \leq \|\underline{u}\| \|\underline{v}\|$

$\xrightarrow{\cos\theta.} -1 \leq \cos\theta \leq 1$

$-\|\underline{u}\| \|\underline{v}\| \leq \|\underline{u}\| \|\underline{v}\| \cos\theta \leq \|\underline{u}\| \|\underline{v}\|$

$-\|\underline{u}\| \|\underline{v}\| \leq \underline{u} \cdot \underline{v} \leq \|\underline{u}\| \|\underline{v}\|$

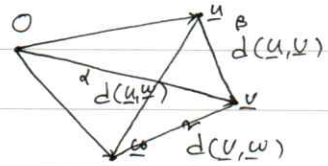
$\Downarrow$

$|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \|\underline{v}\|$

Geometry in $\mathbb{R}^n$

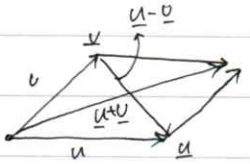
*Thm 3.2.5)

- $\|\underline{u}+\underline{v}\| \le \|\underline{u}\| + \|\underline{v}\|$
- $d(u,v) \le d(u,w) + d(w,v)$



$\longrightarrow$ 삼각형조건. ⋯ $\alpha \le \beta + \gamma$.

Thm 3.2.6)



$$(\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) + (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) = 2(\|\underline{u}\|^2 + \|v\|^2)$$

$$\|u\|^2 + \|v\|^2 + 2\underline{u}\cdot\underline{v}$$
$$+ \|u\|^2 + \|v\|^2 - 2\underline{u}\cdot\underline{v} \quad =$$

Thm 3.2.7)

$$\|\underline{u}+\underline{v}\|^2 = (\underline{u}+\underline{v})(\underline{u}+\underline{v}) = \|\underline{u}\|^2 + 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2.$$
$$\|\underline{u}-\underline{v}\|^2 = (\underline{u}-\underline{v})(\underline{u}-\underline{v}) = \|\underline{u}\|^2 - 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2.$$

$$4(\underline{u}\cdot\underline{v}) = \|\underline{u}+\underline{v}\|^2 - \|\underline{u}-\underline{v}\|^2.$$
$$\underline{u}\cdot\underline{v} = \frac{1}{4}\|\underline{u}+\underline{v}\|^2 - \frac{1}{4}\|\underline{u}-\underline{v}\|^2.$$

Dot Products as Matrix multiplication.

$$\cdot\ \underline{u}\cdot\underline{v} = \underline{u}^T\underline{v} = \underline{v}^T u$$
$$= \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$$
$$= \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$$

* 내적은 row vector와 column vector의 행렬곱으로 처리할 수 있다.

A : n×m Matrix.
u : n×1 column vector.
v : n×1 column vector.

$$(A\underline{u})\cdot\underline{v} = \underline{v}^T(A\underline{u})$$
$$= (\underline{v}^T A)\underline{u}$$
$$= (A^T\underline{v})^T\underline{u}$$

$$\therefore A\underline{u}\cdot\underline{v} = u\cdot A^T v$$
$$\underline{u}\cdot A\underline{v} = A^T\underline{u}\cdot\underline{v}$$

Dot Product View of Matrix Multiplication.

$$AB = \underset{m \times r}{\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}} \underset{r \times n}{\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}} = \underset{m \times n.}{\begin{bmatrix} & & \end{bmatrix}}$$

$$= \begin{bmatrix} r_1 \cdot c_1 & r_2 \cdot c_2 & \cdots & r_1 \cdot c_n \\ r_2 \cdot c_1 & r_2 \cdot c_2 & \cdots & r_2 \cdot c_n \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m \cdot c_2 & \cdots & r_m c_n \end{bmatrix}$$