선형대수학 - Ch.2.3. Properties of Determinants; Cramer's Rule.

- $\det(kA) = k^n \det(A)$.

ex1)
- $\det(A+B) \neq \det(A) + \det(B)$.

$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$

$\left.\begin{matrix} \det(A) = 1 \\ \det(B) = 8 \end{matrix}\right\} \neq \det(A+B) = 23$

Thm 2.3.1)

$A = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 5 & 6 \end{bmatrix}$

Determinant of a Matrix Product.

- E is elementary matrix. E, B are $n \times n$ matrix.

Thm 2.3.2) $\underbrace{\det(AB)}_{k\det(B)} = \underbrace{\det(A)}_{k}\det(B)$.

Thm 2.3.3.) $\det(A) \neq 0 \iff$ invertible.

* if A is invertible then $\left\{ \text{A가 가역이면 Elementary Mat의 결합으로 변환 가능.} \right.$

$\boxed{A = E_r E_{r-1} \cdots E_1}$

$$\boxed{\begin{matrix} \det(A) = \det(E_r) \cdot \det(E_{r-1}) \cdot \cdots \cdot \det(E_1). \\ \neq 0 \quad \Leftarrow \quad \neq 0 \qquad \neq 0 \qquad \neq 0 \end{matrix}} \leftarrow (\det(E) \neq 0).$$

Ex3)

- 해가 무수히 많다. / 없다. → 비가역이다.

Thm 2.3.4)

$$\boxed{\det(AB) = \det(A)\det(B).}$$

- if any matrix is not-invertible → $\boxed{0 = 0.}$

else. $\boxed{\det(AB) = \det(E_1 E_2 \cdots E_r)\det(B).}$

Thm 2.3.5).

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

proof. $AA^{-1} = I.$

$\det(A)\det(A^{-1}) = 1.$

$\det(A^{-1}) = \frac{1}{\det(A)}$

Adjoint.

def 1).

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}^{T}$$

Thm 2.3.6)

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A).$$

proof. $A\operatorname{adj}(A) = \det(A)I.$

$$A\operatorname{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ & & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ & \cdots & C_{nn} \end{bmatrix}$$

$$= a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}.$$

$$= \begin{bmatrix} \det(A) & & & \\ & \det(A) & & 0 \\ & 0 & \ddots & \\ & & & \det(A) \end{bmatrix}$$

$$= \det(A)I.$$

Thm 2.3.7). Cramer's Rule.

- $A\underline{x} = \underline{b}$

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \cdots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

$$A_1 = [b \ a_2 \ a_3 \cdots a_n], \quad A_2 = [a_1 \ b \ a_3 \cdots a_n], \quad A_n = [a_1 \ a_2 \ a_3 \cdots b].$$

Equivalence Theorem.

$A^{-1}Ax = A^{-1}0$
$x = 0$

- A is invertible.
- $Ax = 0$ has only the trivial solution.
- The reduced row echelon form of A is I_n.
- A can be expressed as a product of Elementary matrices.
  $A = E_1 E_2 \cdots E_r$.
- $Ax = b$ is consistent for every $n \times 1$ matrix b.
  $\begin{bmatrix} & \end{bmatrix} x = \begin{bmatrix} \ \end{bmatrix}$ → 어떤 꼴이든 해 존재.
- $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.
- $\det(A) \neq 0$.