(사다리꼴)

Echelon Forms.

1. leading 1

$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3.

2.

4. leading 1.

1. First non-zero element in everyrow is 1. (= leading 1).
2. All zero rows are grouped together at the bottom.
3. Leading above ^at^ lefter must be upper.
4. Each column containing leading 1 has zero everywhere else.

"Reduced Row Echelon Forms" (= RREF).
기약행 사다리꼴.

without 4.

"Row Echelon Forms" (= REF).
행 사다리꼴.

* e.x.1.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$ is RREF.

*)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ is RREF. → leading 1 이 없어서, 조건 판별이 의미 없음.

(= (bool) is leading 1 exists AND conditions)

ex) Augmented Matrix가 RREF일 때.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$0x+0y+0z=1$ → No solution.

leading variable

ex). $$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x+3z=-1$

$y-4z=2$

→ x, y 모두 z로 표현 가능.

$z=t$, $x+3t=-1$, $y+4t=2$. | $x=-3t-1$, $y=-4t+2$ ..

Free variable.

* leading variable들은 free variable로 표현 가능.

Elimination Methods. • Goal: any augmented matrix ⇒ RREF.

* ERO : Elementary Row Operation.
  1) Constant multiple.
  2) Interchange
  3). Add constant times row to other.

• step 1. Locate the leftmost col that not all zero.

S2) 0인가? 0이면 교체

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

S1) 모두 0인가?

• step 2. Interchange the top row if (0,0) is zero.

* Making leading 1.

• step 3. Use ERO ① to make leading 1. $(* = \alpha)$.

$\times -2.$

$$\begin{bmatrix} \cancel{2}\,1 & \cancel{4}\,2 & \cancel{-10}\,-5 & \cancel{6}\,3 & \cancel{12}\,6 & \cancel{28}\,14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ \cancel{2}\,0 & \cancel{4}\,0 & \cancel{-5}\,5 & \cancel{6}\,0 & \cancel{-5}\,-17 & \cancel{-1}\,-29 \end{bmatrix} \begin{matrix} S3) \\ \\ S4) \end{matrix}$$

• step 4. Use ERO ③ to make zero below leading 1.

forward phase
backward phase.

• step 5. Repeat step 1 ~ 4 until matrix become REF.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \text{ is REF!}$$

• step 6. Make zero above the leading 1's.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

step 1 – 6 : Gauss - Jordan Elimination. (Making RREF).
step 1 – 5 : Gaussian Elimination (Making REF).

Homogeneous Linear Systems.

* Homogeneous Linear Systems.

$$\left.\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = 0. \\ \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0. \end{matrix}\right\} \to x_1 = \cdots = x_n = 0.$$

→ trivial solution 자명해.
→ Consistant "일치한다".

관건: 자명해만 존재? 무한히 많은 해?

$$\left.\begin{matrix} ax+by=0. \\ cx+dy=0. \end{matrix}\right\} \to$$ 원점을 지나는 직선. ⇒ $(0,0)$이 자명해.

but $\frac{a}{c} = \frac{b}{d}$ 면 기울기가 같다 ⇒ 해가 무수히 많다.

ex6) Homogeneous Linear System 의 Augmented Matrix라면..
역시 (자명해 / 일치한다) 경우밖에 없다!

Free Variables in Homogeneous Linear Systems.

(Linear Systems).

1. LS : homogeneous ⇒ RREF : homogeneous.
2. LS : m equations, RREF : r equations ⇒ $m \geq r$. (All zero row 발생 가능성 때문).

- n unknowns + r leading variables then $n-r$ free variables.

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

3개 nz row. ⇒ leading Variables.

6개 col.

⇒ $6-3=3$ (free variables).

Page 19, book.

* Thm 1.2.2. A homogeneous LS with more unknowns than equations has infinitely many solutions.

$$\begin{array}{ccc} m & \le & n \\ \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} \\ r & & r \end{array} \rightarrow n - r > 0.$$

* Ex 7. Back substitution. (아래에서부터 대입).

$$\begin{matrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$x_6 = \frac{1}{3}$

$\boxed{x_3 = -2x_4 - 1.}$

$x_1 = -3x_2 + 2\boxed{x_3} - 2x_5.$

AM $\longrightarrow$ (ERO) $\to$ (Gaus-Jordan) $\longrightarrow$ RREF

(ERO) $\to$ (Gaussian) $\longrightarrow$ REF.

* Ex8.

a.
$$\begin{array}{ccccc} 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{array} \longrightarrow \text{No solution.}$$

b.
$$\begin{array}{ccccc} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ \boxed{0 \quad 0 \quad 0} & & & 1 & 0 \end{array}$$

↳ Leading 1 개수 = Row 개수 $\longrightarrow$ Unique One Solution.

c.
$$\begin{array}{ccccc} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array} \longrightarrow \text{Infinitely Many.}$$

Some facts about Echelon Forms.

1. Every matrix has a unique RREF. … ERO 과정에 따라 중간은 변경, 상이할 수 있다.
But 결론은 항상 같아야한다.

2. REFs are not unique

3. The RREF and all REFs have the $\boxed{\text{same \# of zeros,}}$ 개수는 항상 같다. (all 0 row)
and the leading 1's always occur in the $\boxed{\text{same positions}}$. 위치는 항상 같다. (leading 1)

pivot positions …기준 위치.

· pivot positions?

(1,1), (2,2)

(3,3), (4,4).

$$\begin{matrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

· pivot columns

: 1, 2, 3, 4.

· pivot rows

: 1, 2, 3, 4.

* ex 9.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$$

$$\begin{matrix} \uparrow & & \uparrow & & \uparrow & \end{matrix}$$