"행렬에서 역원이 어떻게 되는가?"

Identity ... e. $a * e = e * a = a.$

"항등원" $a * b = b * a \Rightarrow e.$

Properties.

성질들

1. $A+B = B+A$. (Commutative law).
2. $A+(B+C) = (A+B)+C$ (Associative law).
3. $A(BC) = (AB)C$ (Associative law).
4. $A(B+C) = AB+BC$ (Left distribution law).
5. $(B+C)A = BA+CA$ (Right distribution law).

$\Downarrow$

6. $A(B-C) = AB-AC$
7. $(B-C)A = BA-CA.$
8. $a(B+C) = aB+aC.$
9. $a(B-C) = aB-aC.$
10. $(a+b)C = aC+bC.$
11. $(a-b)C = aC-bC.$
12. $a(bC) = (ab)C.$
13. $a(BC) = (aB)C = $ *$B(aC)$.

*Ex. 1)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$(AB)C = A(BC). = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

* Ex 2) $AB \overset{??}{=} BA$

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 9 \\ -3 & 0 \end{bmatrix} \Rightarrow AB \neq BA.$$

∴ 곱셈에 대해, 교환법칙 성립 X.

구체 사례.

1. AB : defined, BA : not defined.
2. AB and BA have different sizes.
3. $AB \neq BA$.

\# Zero Matrices.

· $[0]$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \underline{O}$ (대문자 o).

· $A + \underline{O} = \underline{O} + A = A$. … 덧셈에서 항등원 역할.

* Thm 1.4.2
Page 42, Book.

a. $A + \underline{O} = \underline{O} + A = A$.

b. $A - \underline{O} = A$.

c. $A - A = \underline{O}$. $(= A + (-A))$.

d. $0A = \underline{O}$

e. if $cA = \underline{O}$, then $c = 0$ or $A = \underline{O}$

($\approx ab = 0$ then $a = 0$ or $b = 0$).

* $ab = cb$: $\begin{cases} b \neq 0 \to a = c. \\ b = 0. \end{cases}$

* Ex 3) Failure of the cancellation law.

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$\cdot\ A=\begin{bmatrix}0&1\\0&2\end{bmatrix},\ B=\begin{bmatrix}1&1\\3&4\end{bmatrix},\ C=\begin{bmatrix}2&5\\3&4\end{bmatrix}.$

$$AB=AC=\begin{bmatrix}3&4\\6&8\end{bmatrix},\quad B\neq C.$$

$\therefore$ Cancellation is failed when product between matrices.

* Ex 4). $A=\begin{bmatrix}0&1\\0&2\end{bmatrix},\ B=\begin{bmatrix}3&7\\0&0\end{bmatrix}.\quad AB=\underline{0},\ A\neq\underline{0},\ B\neq\underline{0}.$

Identity Matrices.

· Square Matrix.

$\cdot\ \begin{bmatrix}1&0\\0&1\end{bmatrix}\quad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$ → 주대각 … 1. 나머지 … 0.

· $I_n \Rightarrow$ size n Identity Matrices.

$$AI_3=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

$\begin{cases}a_{11}\ a_{12}\ a_{13}.\\a_{21}\ a_{22}\ a_{23}.\end{cases}$

* Thm 1.4.3).

R: RREF, Square Matrices. $\Rightarrow$ $\begin{cases}\text{least one row of zeros (all zero row).}\\ \quad\text{or}\\ R\text{ is identity Matrix.}\end{cases}$

(증명).

$$\left[\, A \;\vdots\; b \,\right]$$ … Augmented Matrix.

└→ $n \times n$.

→ 해가 무수히 많다.
해가 없다.

$$\begin{bmatrix} & & & & \\ & & & & \\ \boxed{0 \;\; 0 \;\; 0 \;\; 0 \;\; 0} \end{bmatrix}$$

해가 한 개 있다 →

$$\left[\begin{array}{ccccc|c} 1 & & & & & a \\ & 1 & & & & b \\ & & 1 & & & c \\ & & & 1 & & d \\ & & & & 1 & e \end{array}\right]$$

$x_1 = a$.
$x_2 = b$
$x_3 = c$.
$x_4 = d$.
$x_5 = e$.

Inverse of a Matrix.

" $a \cdot a^{-1} = a^{-1} \cdot a = 1$. "

- A, B is square matrix with same size.
  $AB = BA = I$.

⇕

- A (or B) is invertable (가역하다).
  non-singular (비특이하다).
- If is not invertable, then singular

*Ex 5.)

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A, B is invertable.

*Ex 6).

let　　　then.

$$BA = B\,[\,c_1 \;\; c_2 \;\; 0\,] = [\,Bc_1 \;\; Bc_2 \;\; 0\,]$$

→ Identity Matrix는 절대 될 수 없다. (all zero column의 존재)
+ all zero row.

* Thm 1.4.4.) B, C. inverses of A.

$\Rightarrow B = C.$ (= inverse is always unique).

$AB = BA = I.$

$\Rightarrow B = A^{-1}$

* Thm 1.4.5).

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\begin{cases} ad-bc = 0 \rightarrow \text{not invertible.} \\ ad-bc \neq 0 \rightarrow A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{cases}$

* det (A) = determinant of A.

$= ad - bc.$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

a, d switch. b ×-1. c ×-1

* Ex 7).

$A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{12-5}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix}.$

$= \dfrac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix}.$

* Ex 8).

$\begin{cases} ax+by = u. \\ cx+dy = v. \end{cases} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$ "넘긴다"

... 역원을 이용해서.

" 곱의 교환법칙 성립X.
위치에 조심할것! "

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix}.$

$\begin{cases} \boxed{ad-bc = 0.} \\ ad-bc \neq 0. \end{cases}$?? $\leftarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} u \\ v \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$

*. 해가 없거나
무수히 많다. { 왜??

$\begin{matrix} ax+by = u \\ cx+dy = v \end{matrix} \rightarrow \dfrac{a}{c} = \dfrac{b}{d} \neq \dfrac{u}{v}.$

기울기가 같다 ⇒ { 해가 무수히 많다.
해가 없다.

* Thm 1.4.6.

$$(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}.$$

이유.

$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(AB)(AB)^{-1} = (AB)(B^{-1})(A^{-1}).$$
$$I = A\underbrace{BB^{-1}}_{I}A^{-1}$$
$$= AIA^{-1} = AA^{-1}.$$

거듭제곱.

Powers of a Matrix.

def.
$$\begin{cases} A^0 = I \\ A^n = \underbrace{AA\cdots A}_{n.} \\ A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n.} \\ A^rA^s = A^{r+s}. \\ (A^r)^s = A^{rs}. \end{cases}$$

* Thm 1.4.7. If A is invertable and n is non-negative integer.

a. A^{-1} ... Invertable.
   $(A^{-1})^{-1} = A.$

b. $A^n = (A^n)^{-1} = A^{-n} = (A^{-1})^n.$
   A^n ... Invertable.

c. kA ... Invertable. ... any non-zero scalar k.
   $(kA)^{-1} = k^{-1}A^{-1}.$

Matrix Polynomial. "행렬도 Matrix Polynomial로 표현 가능하다" "다항식 표현"

$$\begin{cases} p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m \end{cases}$$

$$p(A) = A^2 - 2A - 5I$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Thm 1.4.8.

(a) $(A^T)^T = A$.

(b) $(A+B)^T = A^T + B^T$

(c) $(A-C)^T = A^T - B^T$.

(d) $(kA)^T = kA^T \longleftrightarrow (kA^{-1}) = \frac{1}{k} A^{-1}$.

(e) $(AB)^T = B^T A^T \approx (AB)^{-1} = B^{-1} A^{-1}$.

* Thm 1.4.9. $(A^T)^{-1} = (A^{-1})^T$