24.06.13.
NO. 24.06.11.

선형대수 6.4 – Best Approximations; Least Squares. (최소제곱)

Least square Solutions of Linear Systems.

- Least square problem.

$A\underline{x} = \underline{b}$ → m eq. n unknowns.

* $A\underline{x} = \underline{b}$: inconsistent.

(현실) $A\underline{x} = \underline{b}'$ → $A\underline{x} = \underline{b}' + \underline{z}$ (noise) → 역시 계속이 inconsistent.

"최대한 b와 비슷한 값을 찾아서"

✓ Minimize $\|\underline{b} - A\underline{x}\|$ with respect to Euclidean inner product.

We call such a vector a least square solution $\underline{x}$.

$\underline{b} - A\underline{x}$: least square error vector

$\|\underline{b} - A\underline{x}\|$: least square error.

$A\underline{x} = \underline{b}$ 해 존재한다면, 해에 대해 $\underline{b} - A\underline{x} = 0$ … $\|\underline{b} - A\underline{x}\| = 0$. (norm)

+) least square solution: solution of $\underline{b} - A\underline{x}$

But 해 없다면 $\underline{b} - A\underline{x} \neq 0$ 일지, norm 최소화 하는 x를 least square solution.

inconsistent.

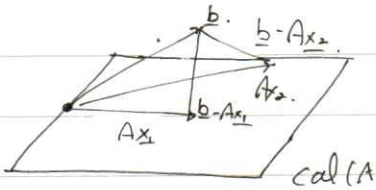
$A\underline{x} = b$ → consistent. then $\underline{b} \subset col(A)$.

↳ A의 column spaces 상 … B는 x에 대해 A의 column spaces. ↓ 존재하면 해.

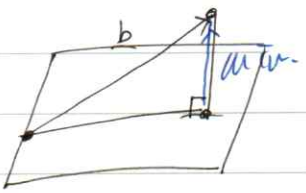
$\|\underline{b} - A\underline{x}\| = 0$.

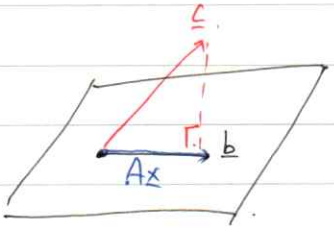
↕

$\underline{b} \notin col(A)$.



→ $\underline{b} - A\underline{x}$ 길이를 최소화 하려면? 직각으로 꽂아야 ∴.

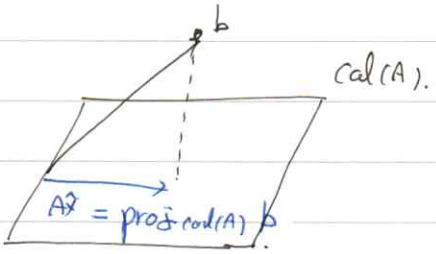




b가 Ax상에 있다면 해가 있다고 있다면 계속 가까운게 직각…

* $A\underline{x} + \underline{e} = \underline{b}$ (e: error...) 그래도 $A\underline{x} = \underline{b}$를 만족한다.

↳ $\min \|\underline{b} - A\underline{x}\| = \hat{x}$ → 최대 최소 최적



*Thm 6.4.1 Best Approximation theorem.

W: finite dim subspace of V.

b: in V.

$\text{proj}_W b$ = best approximation.

∴ ⇒ $\|\underline{b} - \text{proj}_{\underline{W}} \underline{b}\| < \|\underline{b} - \underline{w}\|$

$\underline{w} \neq \text{proj}_{\underline{w}} \underline{b}$

↑

V.

NO.

* Approximations in Real.

$A\underline{x} = \underline{b}'$ $\quad \underline{b}' \in col(A)$. 원래

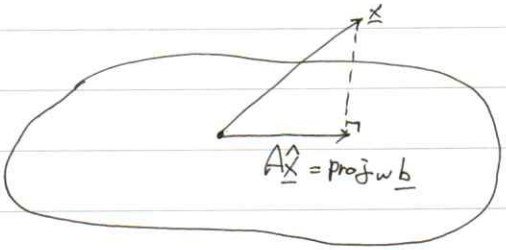
↳ $\underline{b} \notin col(A)$ 원래 + 오차.

$A\underline{x} = \underline{b}\ (= \underline{b}' + \underline{z})$

… 최대한 답을 찾아야하는데?

↳ 현실과 완전히 같지 않아도, 답과 가장 가까운 값을 찾으면.

$\rightarrow \hat{x} = \arg\min_{x} \|\underline{b} - A\underline{x}\|$.



Least Square Error … $\|\underline{b} - A\underline{x}\|$.

Least Square Solution … $A\hat{\underline{x}} = \text{proj}_W \underline{b}$

$A^TA \Rightarrow n \times n$ mat. $\Rightarrow$ invertible.

$-A\hat{\underline{x}} = -\text{proj}_W \underline{b}$ → Proj 표현 ×. 식 조작용.

$\underline{b} - A\hat{\underline{x}} = \underline{b} - \text{proj}_W \underline{b}$

$A^T(\underline{b} - A\hat{\underline{x}}) = A^T(\underline{b} - \text{proj}_W \underline{b}) = \underline{0}$.

$A^TA\underline{x} = A^T\underline{b}$ — Normal Equations.

ex1) $x_1 - x_2 = 4$

$3x_1 + 2x_2 = 1$

$-2x_1 + 4x_2 = 3$.

* 솔루션이 있으면 항상 lss = solution (diff = 0).

$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

↳

$A^TA = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

$X = \begin{bmatrix} 11/95 \\ 143/285 \end{bmatrix}$.

$\|b - Ax\| = \left\| \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 11/95 \\ 143/285 \end{bmatrix} \right\|$.

ex2) $3x_1 + 2x_2 - x_3 = 2.$
$x_1 - 4x_2 + 3x_3 = -2.$
$x_1 + 10x_2 - 7x_3 = 1.$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 2 \\ 1 & -4 & 3 & -2 \\ 1 & 10 & -7 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & \end{array}\right] \times$$

$A^TA\underline{x} = A^T\underline{b}$ → "

↓

augmented matrix.

↓ G-J-E.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/7 & 2/7 \\ 0 & 1 & -5/7 & 13/84 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

"Infinitely Many Least (Square) Solution. →

$$\begin{cases} x_1 = -1/7\, t + 2/7 \\ x_2 = 5/7\, t + 13/84 \\ x_3 = t. \end{cases}$$

"3/6 $\sqrt{6}$ "

↳ 여러 개의 답은 LSS가 존재함.

*Thm 6.4.3. * AA^T: invertible ?

A^TA: invertible ⟷ A: n linearly indep. cols.

*Thm 6.4.4. A: n linearly indep cols.

$\underline{x} = (A^TA)^{-1}A^T\underline{b}$

proj $w\,\underline{b} = A\underline{x}$

$= A(A^TA)^{-1}A^T\underline{b}$

Equivalent.

*Thm 6.4.5. A: invertible (A: nxn).

(r) $\lambda = 0$ is not an eigenvalue of A.

(s) A^TA: invertible.