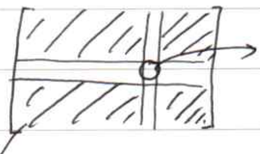
#Chap 2.1. Determinants by Cofactor Expansion.

· $A = [a]$, $A^{-1} = [\frac{1}{a}]I$; $a \overset{?}{=} 0$, $\det(A) = a$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} & \\ & \end{bmatrix}$, $\det(A) = ad - bc$.

*Def 1. · $A = [a_{ij}]$. M_{ij}
↑ $n \times n$.
Minor of entry a_{ij}.



remove column i.
row j.

Remains: Determinant of submatrix.

C_{ij} : Cofactor of entry a_{ij}.

$= (-1)^{i+j} M_{ij}$.

*Thm 2.1.1. Cofactor Expansion.

$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ $C_{11} = M_{11} = a_{22}$ $C_{12} = -M_{12} = -a_{21}$

$C_{21} = -M_{21} = -a_{12}$ $C_{22} = M_{22} = a_{11}$

*Ex 3.

$$A = \begin{array}{c} \begin{matrix} ① & ② & ③ \end{matrix} \\ \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix} \end{array}$$

→ 아무거나 하나 선택.

$\det(A) = 3(-1)^{1+1}\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix}$

$+ 1(-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix}$

$+ 0$

$= -12 + 11$

$= -1$.

0 많으면 Determinant 구하기 쉬울듯.

$*$ Ex5).

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

det.

$$= 1 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= (-2)(-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -6$$

($\neq 0$, invertable)

$*$ Ex6). Lower triangular matrix $\rightarrow$ Triangular Matrix.

Thm 2.1.2.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11} a_{22} (-1)^{2+2} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} a_{44}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & a_{33} & \\ & & & a_{44} \end{vmatrix}$$

Upper triangular — Diagonal. (subset of triangular).

$$\boxed{\therefore \det(A) = \prod_{k=1}^{n} a_{ii}}$$

"Σ의 곱연형"

Useful Technique for 2x2, 3x3 Determinants.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$\Rightarrow \Sigma(\searrow) - \Sigma(\swarrow)$

Ch 2.2 Evaluating Determinants by Row Reduction.

* Determinants는 0이 많으면 편하다.
⇒ 하지만 0이 없는 불친절한 Matrix의 Determinants는?
⇒ Elementary Row Operation으로 처리 … 0 개수↑.

*Thm 2.2.1. · A: all zero row or column → 역행렬 존재할 수 없음 ⇒ Det(A)=0.
⇒ $\mathrm{Det}(A) = 0$.

*Thm 2.2.2. · $\det(A) = \det(A^T)$. … "transpose 해도 + 방향으로 다 지나가니까.."

*Thm 2.2.3. (from ERO).
(a) $k\det(A) = \det(B)$
(b) interchanged ⇒ $\det(A) = -\det(B)$.
(c) adding ⇒ $\det(A) = \det(B)$.

· Page 127, Book.

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$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

* Thm 2.2.4. let E : $n\times n$ elementary matrix.

· $I \xrightarrow{ERO} E$.

(a) $\det(E)=k$.

(b) $\det(E)=-1$.

(c) $\det(E)=1$.

* Ex 1)

$$\begin{vmatrix} 1 & & & \\ & 3 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = 3 \qquad \begin{vmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{vmatrix} = -1 \qquad \begin{vmatrix} 1 & & & 7 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = 1.$$

* Determinant는 [illegible] 방향 처리하므로... Elementary Column Operation 가능.

* Ex 3).

$$A=\begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}, \quad \det(A)=\begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$$

for Gaussian Eliminatio[n]

$$\overset{(b)}{=} -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$\overset{(a).(c)}{=} -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$\overset{(c)(a)}{=} -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} \times(-55).$$

$= (-3)(-55)(1)$

$= 165$..

*EX 4).

Using Column Operations.

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}, \quad \det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix}$$

*EX5)

Using Row Operations.

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}, \quad \det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

* Use ERO and Cofactor Expansion Appropriately.

$$= \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

⋮