Real Vector Space

Axioms

1. [s] Closed under vector addiction
   $\boldsymbol{u}$ and $\boldsymbol{v}$ are objects in V, then $\boldsymbol{u}+\boldsymbol{v}$ is in V.
2. Comutating under vector addiction
   $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
3. Associativity under vector addiction
   $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{v}+\boldsymbol{u})+\boldsymbol{w}$
4. Identity for vector addiction
   $\boldsymbol{u}+\boldsymbol{0}=\boldsymbol{0}+\boldsymbol{u}$
5. Inverse of $\boldsymbol{u}$ for vector addiction
   $\boldsymbol{u}+(-\boldsymbol{u})=(-\boldsymbol{u})+\boldsymbol{u}$
6. [s] Closed under scalar multiplication
   About scalar k and $\boldsymbol{u}$ that any object in V, $k\boldsymbol{u}$ is in V.
7. [d] $k(\boldsymbol{u}+\boldsymbol{v})=k\boldsymbol{u}+k\boldsymbol{v}$
8. [d] $(k+m)\boldsymbol{u}=k\boldsymbol{u}+m\boldsymbol{u}$
9. [d] Associativity for scalar multiplication
   $k(m\boldsymbol{u})=(km)(\boldsymbol{u})$
10. [d] Identity for scalar multiplication
    $1\boldsymbol{u}=\boldsymbol{u}$

Subspaces

W, subset of vector space V is vector space too?
- Subspace test: check axiom 1 and 6.

$W_1, W_2, ..., W_r \subset V$ (W_i are subspaces of V)
then $W_1 \cap W_2 \cap ... \cap W_r \subset V$ (is subspace of V)

$W=\{\boldsymbol{x} \mid A\boldsymbol{x}=\boldsymbol{0}\}$
- W = Solutiuon space
- Subspace of $\mathbb{R}^n$

Kernel
A is $m \times n$ matrix
$T_A : \mathbb{R}^n \to \mathbb{R}^m$
- 공역의 $\boldsymbol{0}$(trivial solution)으로 매핑되는 정의역의 집합을 Kernel, 핵으로 칭한다.
- Kernel = Solution space, Subspace of $\mathbb{R}$

Spanning Sets

$S=\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_r}\} \subset V$
$W=\{k_1\boldsymbol{v_1}+k_2\boldsymbol{v_2}+...+k_r\boldsymbol{v_r} \mid k_1, k_2, ..., k_r \in R\}$
- W는 V의 Subspace
- V에서 r개의 벡터 뽑아서 선형 결합한 것은 V의 subspace
- S로부터 만들어낼 수 있는 모든 W는 V의 Subspace
- W는 만들 수 있는 가장 작은 subspace V (= 만들 수 있는 다른 모든 V의 subspace는 W를 포함한다.)

Statements #1
- S spans W
- W is spanned by S
- $W=\text{span}\ \{\boldsymbol{v_1}, \boldsymbol{v_2}, ...\boldsymbol{v_r}\}=\text{span}(S)$

Statements #2
- $\text{span}(\boldsymbol{V})$: $\boldsymbol{V}$로 만들 수 있는 모든 선형결합
  $\text{span}(\boldsymbol{V})=(\text{any scalar})\boldsymbol{V}$
- $\text{span}(\boldsymbol{V_1}+\boldsymbol{V_2})=(\text{any scalar \#1})\boldsymbol{V_1}+(\text{any scalar \#2})\boldsymbol{V_2}$

Theorem
- $S=\text{span}\ \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_r}\}; S'=\text{span}\ \{\boldsymbol{s_1}, \boldsymbol{s_2}, ..., \boldsymbol{s_r}\}$가 공집합이 아닌 V 내의 Vector space일때
  $\text{span}\ \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_r}\}=\text{span}\{\boldsymbol{s_1}, \boldsymbol{s_2}, ..., \boldsymbol{s_r}\}$

Linear Independence

- Linearly Independent $\Leftrightarrow k_1\boldsymbol{v_1}+k_2\boldsymbol{v_2}+...+k_r\boldsymbol{v_r}=\boldsymbol{0}$
  for only $k_1=k_2=..=k_r=0$
  (계수 모두가 0일 때 0벡터가 되는 경우를 제외하면 0벡터가 되는 경우가 없어야 함)
- 어떤 구성 요소 한 개가 다른 구성 요소로 표현될 수 없음

Extension
- $\boldsymbol{f_1}=\sin^2 x; \boldsymbol{f_2}=\cos^2 x; \boldsymbol{f_3}=5$
  $\Rightarrow 5(\boldsymbol{f_1}+\boldsymbol{f_2})=\boldsymbol{f_3} \Rightarrow$ Linearly dependent

Coordinates and Basis

- Finite dimensional vector space can be spanned by a finite set of vector.
- Infinite dimensional vector space cannot be spanned by a finite set of vector.

Definition
- Finite dimensional vector V ($\supset S=\{\boldsymbol{V_1}, \boldsymbol{V_2}, ..., \boldsymbol{V_n}\}$)
- S is basis if
  1. S spans V
  2. S is linearly independent

  ($\Rightarrow S$ = basis of vector space V)

Examples
- $\mathbb{R}^n$ = standard basis vector: $e_1, e_2, ..., e_n$
- $P_n \supset S=\{1, x, ..., x^n\}$
  - $a_0+a_1x+a_2x^2+...+a_nx^n$
    - P_n을 span
    - Linearly independent
- How to check $v_1=(1,2,1), v_2=(2,9,0), v_3=(3,3,4)$ form a basis for $\mathbb{R}^3$
  - Spans $\mathbb{R}^3$?
  - Linearly independent?

Theorem
- S: Basis of vector space V
- $\boldsymbol{v}=c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+...+c_n\boldsymbol{v_n}$
  $\boldsymbol{V}$ is uniquely expressed by $S(=\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n})$
- $\boldsymbol{V}=c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+...+c_n\boldsymbol{v_n}$

Definition
- $S=\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$: Basis of vector space V and
  $\boldsymbol{v}=c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}+...+c_n\boldsymbol{v_n}$
- then $c_1, c_2, ..., c_n$ is coordinates of $\boldsymbol{v}$ relative to S

Definition and Example
- $(\boldsymbol{P})_s=(C_0, C_1, ..., C_n)$: $(\boldsymbol{P})_s$ = Relative from P to s

Dimension

Theorems
- All bases for a (each) finite dimensional vector space have same number of vectors
- Let V: finite dimensional vector space; $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$: basis of V then
  - linearly dependent ($\text{len}(\boldsymbol{V})>n$)
  - not spans V ($\text{len}(\boldsymbol{V})<n$)

Definition and Examples
- Dimension of a finite dimensional vector space = $\dim(V)$ = Number of vectors in a basis
- $\dim(\mathbb{R}^n)=n; \dim(P_n)=n+1; \dim(M_{mn})=mn$

Theorems
- $S \subset V$
  1. S is linearly independent set and $\boldsymbol{v} \notin \text{span}(S)$ then
     $S \cup \{\boldsymbol{v}\}$ is linearly independent
  2. $\boldsymbol{v} \in S$ and $\boldsymbol{v} \in \text{span}(S-\{\boldsymbol{v}\})$ then
     $\text{span}(S)=\text{span}(S-\{\boldsymbol{v}\})$
- V: n-dimensional vector space; $|S|=n$; $S \subset V$
  S: basis $\Leftrightarrow$ (1) S spans V (2) S: linearly independent
  * (1) (2) 둘 중 한 개가 보장되면 나머지 한 개도 보장됨
- S: finite set in finite dimensional vector space V
  1. S spans V but is not a basis
     $\Rightarrow S$ can be reduced to a basis by remaining some vectors from S
  2. S is linearly independent, not span V(= not a basis for V)
     $\Rightarrow S$ can be enlarged to a basis by adding some independent vectors
- W is subspace of a finite-dimensional vector space V then
  1. W is finite-dimensional
  2. $\dim(W) \le \dim(V)$
  3. $W=V$ only if $\dim(W)=\dim(V)$

Change of Basis

- $(\boldsymbol{V})_s \neq (\boldsymbol{V}')s$
  $(\boldsymbol{V})_s=(C_1, C_2, ..., C_n) \in \mathbb{R}^n$
  how to $S \to S'$; $(\boldsymbol{V})_s \to (\boldsymbol{V}')_s$?
- $B(=(\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_n})) \to B'(=(\boldsymbol{u_1'}, \boldsymbol{u_2'}, ..., \boldsymbol{u_n'})) \Rightarrow [\boldsymbol{v}]_B \to [\boldsymbol{v}]_{B'}$
  $[\boldsymbol{v}]_{B'}=P[\boldsymbol{V}]_B$ where $P=\left[[\boldsymbol{u_1}]_{B'}, [\boldsymbol{u_2}]_{B'}, ..., [\boldsymbol{u_n}]_{B'}\right]$
  * P is transition matrix from B to B'

Invertibility
- $(P_{B'\to B})(P_{B\to B'})=(P_{B'\to B'})=I$

Efficient Method for Computing Transition Matrices
1. $[B'^{\text{new}} \mid B^{\text{old}}]$
2. Elementary Row operation
3. $[I \mid P_{B\to B'}]$

Theorem
- $B=\{\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_n}\}$
  $S=\{\text{standard matrices}\}$
  then $P_{B\to S}=[\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_n}]$

Row Space, Column Space, and Null Space

Definitions
1. $A_{m\times n}=\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\\ \vdots\\ \boldsymbol{r_m}\end{bmatrix}=[\boldsymbol{c_1}\ \ \boldsymbol{c_2}\ \ \cdots\ \ \boldsymbol{c_n}]$
   $\boldsymbol{r_i}$: row vectors, $\boldsymbol{c_i}$: column vectors
2. Row space of A: $\text{row}(A)=\text{span}\{\boldsymbol{r_1}, \boldsymbol{r_2}, ..., \boldsymbol{r_m}\}$
   Column space of A: $\text{col}(A)=\text{span}\{\boldsymbol{c_1}, \boldsymbol{c_2}, ..., \boldsymbol{c_n}\}$
   Null space of A: $\text{null}(A)=\text{Solution Space}=\{\boldsymbol{x} \mid A\boldsymbol{x}=\boldsymbol{0}\}$

Theorem
- $A\boldsymbol{x}=\boldsymbol{b}$: Consistent $\Leftrightarrow b \in \text{col}(A)$
  $\to A\boldsymbol{x}=[\boldsymbol{c_1}\ \ \boldsymbol{c_2}\ \ \cdots\ \ \boldsymbol{c_n}]\begin{bmatrix}\boldsymbol{x_1}\\ \boldsymbol{x_2}\\ \vdots\\ \boldsymbol{x_n}\end{bmatrix}$
  $=x_1\boldsymbol{c_1}+x_2\boldsymbol{c_2}+...+x_n\boldsymbol{c_n}=\boldsymbol{b} \in \text{col(A)}=\text{span}\{\boldsymbol{c_1}, \boldsymbol{c_2}, ..., \boldsymbol{c_n}\}$

Example
- $A\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_6\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}$
  $\to \begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_6\end{bmatrix}=r\begin{bmatrix}a_1\\ a_2\\ \vdots\\ a_6\end{bmatrix}+s\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_6\end{bmatrix}+t\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_6\end{bmatrix}+\begin{bmatrix}d_1\\ d_2\\ \vdots\\ d_6\end{bmatrix}$
  - $r[\ ]+s[\ ]+t[\ ]$: General solution of homogeneous system
  - $[\ ]$: Constant; particular solution of non-homogeneous system

Theorem
- Row equivalent matrices have the same row space, null space
  - (참조) Elementary row operation을 해도 null space(= solution space)는 변화가 없다. $\Rightarrow$ 해가 유지된다.
- But not about column space
  $A=\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\ 00 & \end{bmatrix}$ then $\text{col}(A)=\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\text{col}(B)=\begin{bmatrix}1\\ 0\end{bmatrix}$

Theorem
- Elementary row operation do not change dependency relationships between column vectors
  - Elementary row operations $\neq$ column operation

Why null space is null space?
- $A\boldsymbol{x}=\boldsymbol{0} \Rightarrow [\boldsymbol{c_1}\ \ \boldsymbol{c_2}\ \ \cdots\ \ \boldsymbol{c_n}]\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\begin{bmatrix}\boldsymbol{r_1}\cdot\boldsymbol{x}\\ \boldsymbol{r_2}\cdot\boldsymbol{x}\\ \vdots\\ \boldsymbol{r_n}\cdot\boldsymbol{x}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}$
  - 두 벡터의 도트곱이 0이다. $\Leftrightarrow$ 두 벡터가 수직이다.

Rank, Nullity and Fundamental Spaces

Theorems and Definitions
- $\dim(\text{row}(A))=\dim(\text{col}(A))$
- $\text{rank}(A)=\dim(\text{row}(A))=\dim(\text{col}(A))$
- $\text{nullity}(A)=\dim(\text{null}(A))$
- A: n columns $\Rightarrow$ rank(A) + nullity(A) = n
  rank(A): #leading variables
  nullity(A): #free variables (= all zero row, number of params)

Example
- $R=\begin{bmatrix}1 & 0 & -4 & -28 & -37 & 13\\ 0 & 1 & -2 & -12 & -16 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$
  - $\dim(\text{row}(R))=2; \dim(\text{col}(R))=2; \text{rank}=2$
    * rank는 reduced row echelon form에서 확인
  - $\boldsymbol{x}=r\begin{bmatrix}4\\ 2\\ 1\\ 0\\ 0\\ 0\end{bmatrix}+s\begin{bmatrix}28\\ 12\\ 0\\ 1\\ 0\\ 0\end{bmatrix}+t\begin{bmatrix}37\\ 16\\ 0\\ 0\\ 1\\ 0\end{bmatrix}+u\begin{bmatrix}-13\\ -5\\ 0\\ 0\\ 0\\ 1\end{bmatrix} \Rightarrow \text{basis}$
    Basis of null space; Nullity = 4

Theorem
- $(A\boldsymbol{x})_{m\times n}=\boldsymbol{b}$: consistent, rank(A) = r
  $\Rightarrow$ general solution contains $(n-r)$ parameters.
  $\boldsymbol{x}=\boldsymbol{x_n}+\boldsymbol{x_p}$

Fundamental Spaces
- $\text{row}(A)\ \ (=\text{col}(A^T))$
- $\text{col}(A)\ \ (=\text{row}(A^T))$
- $\text{null}(A);\ \ \text{null}(A^T)=\text{left null space of A}$
- $\dim(\text{row}(A))=r$
  $\dim(\text{col}(A))=r$

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$\dim(\text{null}(A)) = n - r$
$\dim(\text{null}(A^T)) = n - r$

- $\text{rank}(A) = \text{rank}(A^T)$
- A^T: m column
  $\Rightarrow \text{rank}(A^T) + \text{nullity}(A^T) = \text{rank}(A) + \text{rank}(A^T) = m$

Bases for Fundamental Spaces

- $m \times n$ size A
  $[A|I] \overset{\text{GJE}}{\to} [\text{RREF}|E] \updownarrow m - r$ row
- Basis for left null space is found from the bottom $m - r$ rows of E
  * A^T에 대해 별도 처리 없이도 처리할 수 있다.

Definition: A Geometric Link Between the Fundamental Spaces

- W: Subspace of $\mathbb{R}^n$
- Orthogonal Complement of W: Set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W
  1. $W^{\mathcal{L}} =$ subspace of $\mathbb{R}^n$
  2. $W \cap W^{\mathcal{L}} = \{\mathbf{0}\}$
  3. $(W^{\mathcal{L}})^{\mathcal{L}} = W$

Equivalent Statements

Theorem 1.6.4 (Page 64)

a. A is invertible.
b. $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.
c. The reduced row echelon form of A is I_n.
d. A is expressible as a product of elementary matrices.
e. $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $n \times 1$ matrix $\boldsymbol{b}$.
f. $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $n \times 1$ matrix $\boldsymbol{b}$.

Theorem 2.3.8 (Page 141)

g. $\det(A) \neq 0$

Theorem 4.9.8 (Page 283)

h. The column vectors of A are linearly independent.
  $A \overset{\text{GJE}}{\to} [I]$ ... Identity이므로 linearly independent.
i. The row vectors of A are linearly independent.
j. The column vectors of A span $\mathbb{R}^n$.
  $\mathbb{R}^n$의 벡터들인데 linearly independent하므로 $\mathbb{R}^n$을 span한다 할 수 있다.
k. The row vectors of A span $\mathbb{R}^n$.
l. The column vectors of A form a basis for $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. A has rank n.
o. A has nullity 0.
  (has no all-zero row)
p. The orthogonal complement of the null space of A is $\mathbb{R}^n$.
  null(A) 공간은 $\{\mathbf{0}\}$ 하나이므로.
q. The orthogonal complement of the row space of A is $\mathbf{0}$.
  null(A) 공간은 $\{\mathbf{0}\}$ 하나이므로.

Eigenvalues and Eigen Vectors

- A: $n \times n$ matrix (A: Standard matrix of an operation)
- $\boldsymbol{x} \neq 0; \boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x}$: eigenvector (λ에 상응)
  if $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\lambda \Rightarrow \lambda$: eigenvalue

Computing

- $A\boldsymbol{x} = \lambda\boldsymbol{x} \to \lambda\boldsymbol{x} - A\boldsymbol{x} = \mathbf{0}$
  $(\lambda I - A)\boldsymbol{x} = \mathbf{0} \to$ Consistent

Theorem

- $\det(\lambda I - A) = 0$: Characteristic equation of A.
- Diagonal values are eigenvalues
  when A is triangular / diagonal matrix.

Theorem

A : $n \times n$ matrix

1. λ : eigenvalue of A
2. λ : solution of $\det(\lambda I - A) = 0$
3. $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has non trivial solution
4. $\exists \boldsymbol{x} \neq \mathbf{0}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

Theorem: Finding Eigenvectors and Bases for Eigenspaces

$\{\boldsymbol{x} \neq \mathbf{0} \mid (\lambda I - A)\boldsymbol{x} = \mathbf{0}\}$

1. Null space of $\lambda I - A$.
2. The kernel of $T_{\lambda I - A} : \mathbb{R}^n \to \mathbb{R}^n$
3. Set vectors for which $A\boldsymbol{x} = \lambda\boldsymbol{x}$

Theorem: Eigenvalues and Invertibility

A : invertible $\Leftrightarrow \lambda = 0$ is not an eigenvalue of A

- 행렬 A가 eigenvalue를 가지면 가역할 수 없다.
  가역하면 eigenvalue를 갖지 못한다.

Diagonalization

Matrix Diagonalization Problem

$P^{-1}AP$

- $\det(A) = \det(P^{-1}AP) = \det(P^{-1}\det(A)\det(P))$

Definitions and Theorems

1. If A, B: square matrices
   $\Rightarrow B$ is similar to A if $\exists$ invertible matrix P such that $B = P^{-1}AP$.
2. A: diagonalizable
   if it is similar to some diagonal($P^{-1}AP$) matrix.
   P is said to diagonalize A.
3. A: $n \times n$ matrix
   (a) diagonalizable
   (b) A has n linearly independent eigenvectors.
4. (a) $\lambda_1, \lambda_2, ..., \lambda_k$: distinct eigenvalues of A
   (b) $\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_k}$: corresponding eigenvalues is diagonalizable.

Procedure for Diagonalizing a Matrix

Step 1. Check if $\exists n$ linearly independent eigenvalues.
$\lambda I - A\boldsymbol{x} = \mathbf{0}$: Solution space $\Rightarrow \boldsymbol{P_1}, \boldsymbol{P_2}, ..., \boldsymbol{P_n}$.
Step 2. $P = [\boldsymbol{P_1} \mid \boldsymbol{P_2} \mid ... \mid \boldsymbol{P_n}] \leftarrow n \times n$ size
Step 3. $D = P^{-1}AP$ where $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

Theorem: Eigenvalue of Power of a Matrix

- $A\boldsymbol{x} = \lambda\boldsymbol{x}$
  $A^2\boldsymbol{x} = k^{\text{scalar}}\boldsymbol{x} = AA\boldsymbol{x} = A(\lambda\boldsymbol{x}) = \lambda A\boldsymbol{x} = \lambda^2\boldsymbol{x}$
  * eigenvector: stayed same

Computing Powers of a Matrix

- $D = P^{-1}AP$; A 행렬 Diagonalization $\Rightarrow$ 대각행렬 D 획득
- $A^{10} =?$
  $D^{10} = P^{-1}AP \cdot P^{-1}AP ... P^{-1}AP$
  $= P^{-1}A^{10}P \Rightarrow A^{10} = PD^{10}P^{-1}$

Theorem: Geomatric and Algebric Multiplicity[중복도]

using example 1

	$\lambda = 1$	$\lambda = 2$ (double root)
multiplicity	1	2
dimension	1	2

algebraic multiplicity
$\leftrightarrow$
geometric multiplicity
$\leftrightarrow$

using example 2

	$\lambda = 1$	$\lambda = 2$ (double root)
multiplicity	1	2
dimension	1	1

1. Geometric multiplicity $\leq$ Algebraic multiplicity
2. diagonalizable $\Leftrightarrow$ geo.mul. = alg.mul. (같아야 linearly independent)

Inner Product Spaces

Definition

- V: real vector space
  $< \boldsymbol{u}, \boldsymbol{v} >$: inner product $\Rightarrow V$: inner product space

4가지 조건 만족 시 연산자 사용 가능

1. $< \boldsymbol{u}, \boldsymbol{v} >=< \boldsymbol{v}, \boldsymbol{u} >$
2. $< \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} >=< \boldsymbol{u}, \boldsymbol{w} > + < \boldsymbol{v}, \boldsymbol{w} >$
3. $< k\boldsymbol{u}, \boldsymbol{v} >= k < \boldsymbol{u}, \boldsymbol{v} >$
4. $< \boldsymbol{u}, \boldsymbol{v} > \geq 0$; Equality $\Leftrightarrow \boldsymbol{v} = 0$

- $\mathbb{R}^n; < \boldsymbol{u}, \boldsymbol{v} >= \boldsymbol{u} \cdot \boldsymbol{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$
- Dot product is inner product in Euclidean space (= Standard inner product)
- V가 $< \boldsymbol{u}, \boldsymbol{v} >$에 의해 정의 혹은 $< \boldsymbol{u}, \boldsymbol{v} >$의 inner product space = V

Definition

- V: real inner product space
- $\|\boldsymbol{v}\| = \sqrt{(< \boldsymbol{u}, \boldsymbol{v} >)}$
- $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(< \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} >)})$
- $\|\boldsymbol{u}\| = 1 \Rightarrow \boldsymbol{u}$: unit vector

Theorem

1. $\| \boldsymbol{v} \| \geq 0$; Equality $\Leftrightarrow \boldsymbol{v} = \mathbf{0}$
2. $\| k\boldsymbol{v}\| = k \| \boldsymbol{v} \|$
3. $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$
4. $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$; Equality $\Leftrightarrow \boldsymbol{u} = \boldsymbol{v}$

- $\mathbb{R}^n; < \boldsymbol{u}, \boldsymbol{v} >= w_1u_1v_1 + w_2u_2v_2 + ... + w_nu_nv_n$
  $\to$ Weighted Euclidean inner product

Inner Products Generated by Matrices

- Matrix inner product $< \boldsymbol{u}, \boldsymbol{v} >= A\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{v})^T A\boldsymbol{u} = \boldsymbol{v}^T A^T A\boldsymbol{u}$
- dot product
  $\to$ inner product
  $\to$ weighted inner product
  $\to$ inner product generated by matrix

Theorem: Dot곱 Axiom의 Inner Product로의 확장

1. $< \mathbf{0}, \boldsymbol{v} >=< \boldsymbol{v}, \mathbf{0} >= 0$
2. $< \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} >=< \boldsymbol{u}, \boldsymbol{v} > + < \boldsymbol{u}, \boldsymbol{w} >$
3. $< \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w} >=< \boldsymbol{u}, \boldsymbol{v} > - < \boldsymbol{u}, \boldsymbol{w} >$
4. $< \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w} >=< \boldsymbol{u}, \boldsymbol{w} > - < \boldsymbol{v}, \boldsymbol{w} >$
5. $k < \boldsymbol{u}, \boldsymbol{v} >=< \boldsymbol{u}, k\boldsymbol{v} >$

Gram-Schmidt Process

Orthogonal and Orthonormal Sets

$S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}; \binom{n}{2} \cdots n$개 중 2개 뽑아서 orthogonal 체크 필요

- $< \boldsymbol{v_i}, \boldsymbol{v_j} >= 0 \Rightarrow S$: orthogonal set
- orthogonal set condition +
  $\| v_i \| = 1$ for all $i \Rightarrow$ S: orthonormal set

Theorem

- $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$: Orthogonal Set $\Rightarrow$ linearly independent

Coordinated Relative to Orthonormal Bases

- $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$: basis
  $\boldsymbol{u} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + ... + c_n\boldsymbol{v_n}$
  $(\boldsymbol{u})_s = (c_1, c_2, ..., c_n) \to n$개 변수 있는 연립방적식 풀어야 찾을 수 있음
  +) 직교하는 basis는 표현이 조금 더 쉽지 않을까?

Theorem

- $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$: Orthogonal basis
- $\boldsymbol{u} = \text{proj}_{\boldsymbol{r_1}}\boldsymbol{u} + \text{proj}_{\boldsymbol{r_2}}\boldsymbol{u} + ... + \text{proj}_{\boldsymbol{r_n}}\boldsymbol{u}$
  $=< \boldsymbol{v_1}, \boldsymbol{u} > / \|\boldsymbol{v_1}\|^2 \cdot \boldsymbol{v_1} + < \boldsymbol{v_2}, \boldsymbol{u} > / \|\boldsymbol{v_2}\|^2 \cdot \boldsymbol{v_2} + ... + < \boldsymbol{v_n}, \boldsymbol{u} > / \|\boldsymbol{v_n}\|^2 \cdot \boldsymbol{v_n}$
- Orthogonal:
  $(\boldsymbol{u})_s = (< \boldsymbol{v_1}, \boldsymbol{u} > / \|\boldsymbol{v_1}\|^2, < \boldsymbol{v_2}, \boldsymbol{u} > / \|\boldsymbol{v_2}\|^2, ..., < \boldsymbol{v_n}, \boldsymbol{u} > / \|\boldsymbol{v_n}\|^2)$
- Orthonormal:
  $(\boldsymbol{u})_s = (< \boldsymbol{v_1}, \boldsymbol{u} >, < \boldsymbol{v_2}, \boldsymbol{u} >, ..., < \boldsymbol{v_n}, \boldsymbol{u} >)$

Orthogonal Projection

- $\boldsymbol{u} = \boldsymbol{w_1} + \boldsymbol{w_2}$
- $\boldsymbol{u} = \text{proj}_w\boldsymbol{u} + \text{proj}_{w^\perp}\boldsymbol{u}$
- W와 수직인 벡터들의 모음:
  - null space of W
  - W's Orthogonal compliment
  - $W^\perp$
- Orthogonal Projection from $\boldsymbol{u}$ to $\boldsymbol{v_1}$, $\boldsymbol{v_2}$
  $\text{proj}_w\boldsymbol{u} = \text{proj}_{\boldsymbol{v_1}}\boldsymbol{u} + \text{proj}_{\boldsymbol{v_2}}\boldsymbol{u} + ... + \text{proj}_{\boldsymbol{v_r}}\boldsymbol{u}$

Gram-Schmidt Process

- input: $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_r}\}$ any basis for W.
- output: $\{\boldsymbol{v_2}, \boldsymbol{v_2}, ..., \boldsymbol{v_r}\}$ orthogonal basis for W.
- 찾는 이유: 유용성; Projection to Orthogonal Compliment of $\boldsymbol{v_1}$
  $\Rightarrow$ Orthogonal 하면서 똑같은 공간을 span하는구나!

1. $\boldsymbol{u_1} = \boldsymbol{v_1}$
2. $\boldsymbol{u_2} = \text{proj}_{w_1^\perp}\boldsymbol{v_2} \Rightarrow W_1 = \text{span}\{\boldsymbol{u_1}\}$
3. $\cdots$
4. $\boldsymbol{u_i} = \text{proj}_{w_{i-1}^\perp}\boldsymbol{v_i} \Rightarrow W_i = \text{span}\{\boldsymbol{u_i}\}$