"basis에 관한 것"

orthogonal and orthonormal Sets.

$S = \{\underline{v_1}, \dots, \underline{v_n}\}$ $\quad {}_nC_2 = \binom{n}{2}$ ... n개 중 2개 뽑아서 orthogonal 체크 필요.

$\langle \underline{v_i}, \underline{v_j} \rangle = 0. \Rightarrow S$: Orthogonal Set

- Orthonormal = Orthogonal (직교) + normal. (정규화.) ⟶ norm = 1 이면 normal이라 칭함.

  ↳ Orthogonal Set's Condition

  $+ \|v_i\| = 1$ for all $i \Rightarrow S$: orthonormal set.

*ex1)

$\underline{v_1} = (0, 1, 0).$
$\underline{v_2} = (1, 0, 1).$ ⟶ $\|v_2\| = \sqrt{2}$
$v_3 = (1, 0, -1)$ $\quad \|v_3\| = \sqrt{2}$

... Not Orthonormal. set.

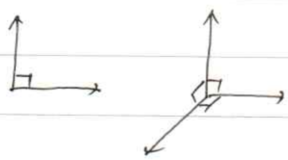
$\langle \underline{v_1}, \underline{v_2} \rangle = 0.$
$\langle \underline{v_2}, \underline{v_3} \rangle = 0.$ ⟶ Orthogonal Set
$\langle \underline{v_3}, \underline{v_1} \rangle = 0$

- $\dfrac{1}{\|\underline{v_i}\|} \underline{v_i}$: Unit Vector "$\underline{v \to u}$: normalization"

*ex2)

$\underline{u_1} = (0, 1, 0).$
$\underline{u_2} = \frac{1}{\sqrt{2}} (1, 0, 1).$
$\underline{u_3} = \frac{1}{\sqrt{2}} (1, 0, -1)$

*thm 6.3.1. $S = \{\underline{v_1}, \dots, \underline{v_n}\}$: Orthogonal set ⇒ linearly indep. "증명"



... n-dim

ex3).

- P_n — $\begin{cases} p = a_0 + a_1x + \cdots + a_nx^n \\ q = b_0 + b_1x + \cdots + b_nx^n \end{cases}$
- ✓ $\langle p, q \rangle = a_0b_0 + a_1b_1 + \cdots + a_nb_n$.
- $S = \{1, x, x^2, \cdots, x^n\}$.

$\boxed{\langle x^i, x^j \rangle = 0}$ ← $(0 \le i \ne j \le n)$.

i차항의 계수 j차항의 계수

→ Orthogonal basis.

- $\|x^i\| = 1 \Rightarrow$ Orthonormal. → Orthonormal basis.

$\sqrt{\langle x^i, x^i \rangle} = \sqrt{1} = 1.$

—(정의에 의해)→

c.f.) $e_1 = (1, 0, \cdots, 0)$.

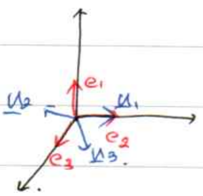
$\vdots$

$e_n = (0, 0, \cdots, 1)$.

→ Orthonormal basis.

(standard basis in Euclid)

+).



Orthonormal

Orthonormal.

??: Dim-n에서 Orthonormal basis는 몇개?
(정답) 무한히 많다.

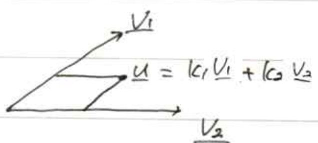
Coordinated Relative to Orthonormal Bases.

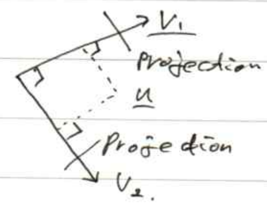
- $S = \{\underline{v_1}, \cdots, \underline{v_n}\}$: basis.

$\underline{u} = c_1\underline{v_1} + c_2\underline{v_2} + \cdots + c_n\underline{v_n}$.

$(\underline{u})_s = (c_1, c_2, \cdots, c_n)$. ⟶ n개 변수 있는 연립방정식 풀어야 찾을 수 있음.

But... 직교하는 basis는 표현이 조금 더 쉽지 않을까?





NO.

* Thm 6.3.2. $S = \{\underline{v}_1, \cdots, \underline{v}_n\}$: Orthogonal basis.

$\underline{u} = \text{Proj}_{\underline{v}_1}\underline{u} + \cdots + \text{Proj}_{\underline{v}_n}\underline{u}$

$= \langle \underline{v}_1, \underline{u} \rangle / \|\underline{v}_1\|^2 \cdot \underline{v}_1 + \langle \underline{v}_2, \underline{u} \rangle / \|\underline{v}_2\|^2 \cdot \underline{v}_2 + \cdots + \langle \underline{v}_n, \underline{u} \rangle / \|v_n\|^2 \cdot v_n$

$(\underline{u})_S = \left(\langle \underline{v}_1, \underline{u} \rangle / \|\underline{v}_1\|^2, \langle \underline{v}_2, \underline{u} \rangle / \|\underline{v}_2\|^2, \cdots, \langle \underline{v}_n, \underline{u} \rangle / \|\underline{v}_n\|^2 \right)$

$\updownarrow$ Orthogonal.

orthonormal.

$* \ \|\underline{v}_i\| = 1 \Rightarrow \|\underline{v}_i\|^2 = 1.$

$(\underline{u})_S = (\langle \underline{v}_1, \underline{u} \rangle, \langle \underline{v}_2, \underline{u} \rangle, \cdots, \langle \underline{v}_n, \underline{u} \rangle)$

* ex5).

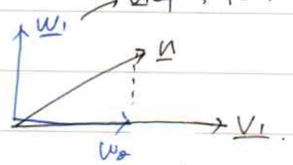
$$\left.\begin{aligned} \underline{v}_1 &= (0, 1, 0). \\ \underline{v}_2 &= (-4/5, 0, 3/5). \\ \underline{v}_3 &= (3/5, 0, 4/5). \\ \underline{u} &= (1, 1, 1). \end{aligned}\right\} \rightarrow (\underline{u})_{\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}} = ?$$

sol) $\underline{u} = \langle \underline{u}, \underline{v}_1 \rangle \underline{v}_1 + \langle \underline{u}, \underline{v}_2 \rangle \underline{v}_2 + \langle \underline{u}, \underline{v}_3 \rangle \underline{v}_3$

$= 1 \cdot \underline{v}_1 - 1/5 \cdot \underline{v}_2 + 7/5 \cdot \underline{v}_3$

$(\underline{u})_S = (1, -1/5, 7/5).$

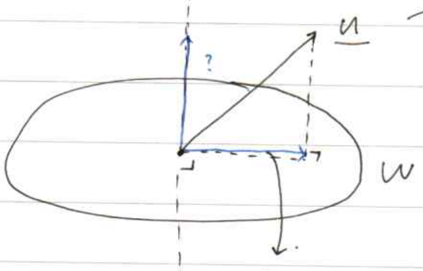
Orthogonal Projection.



→ $\underline{v}_1$과 직교하는 벡터.

$\underline{u} = \underline{w}_1 + \underline{w}_2$

* Thm 6.3.3).



→ 직교 분해 가능.

$\underline{u} = \text{proj}_W \underline{u} + \text{proj}_{W^\perp} \underline{u}$

$\text{Proj}_W \underline{u}$

(복습).

W와 수직인 벡터들만 모아놓음 : null space of W.

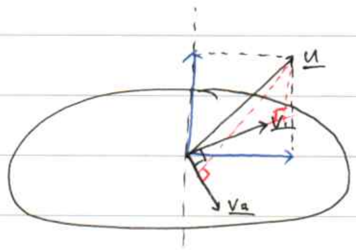
W's Orthogonal Compliment.

$W^\perp$

*Thm (6.3.4). W : subspace of V

$S = \{\underline{v_1}, \underline{v_2}, \cdots, \underline{v_r}\}$: orthogonal basis of W.

$\text{proj}_W \underline{u} =$



$\underline{v_1}, \underline{v_2}$: Orthogonal basis of W

· Orthogonal Projection from u to $\underline{v_1}, \underline{v_2}$.

합.

$\text{Proj}_W \underline{u} = \text{Proj}_{\underline{v_1}} \underline{u} + \text{Proj}_{\underline{v_2}} \underline{u} + \cdots + \text{Proj}_{\underline{v_r}} \underline{u}$

*ex 7) $\mathbb{R}^3 \supset W = \text{span}\{\underline{v_1}, \underline{v_2}\}$.

$\underline{v_1} = (0, 1, 0)$

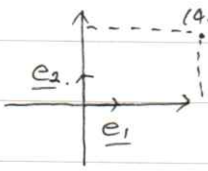
$\underline{v_2} = (-4/5, 0, 3/5)$. } orthonormal.

$\underline{u} = (1, 1, 1)$. } $\text{proj}_W \underline{u} = ?$

$$\text{proj}_W \underline{u} = \text{Proj}_{\underline{v_1}} \underline{u} + \text{Proj}_{\underline{v_2}} \underline{u}$$
$$= \langle \underline{u}, \underline{v_1} \rangle \underline{u} + \langle \underline{u}, \underline{v_2} \rangle \underline{v_2}$$
$$= 1 \cdot (0, 1, 0) + (-1/5)(-4/5, 0, 3/5).$$

∴ Subspace의 Orthonormal basis만 알고 있으면 빠르게 풀린다.

지금까지는 ...

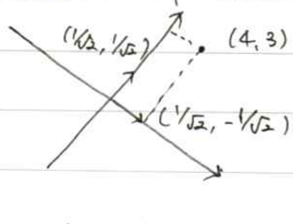


$\underline{e_1}$과 $\underline{e_2}$를 기준으로 잡음.

⇒ (4,3)은 각각 Orthogonal Projection한 결과.

↳ $\text{Proj}_{\underline{e_1}}(4,3) + \text{Proj}_{\underline{e_2}}(4,3)$.

<Standard Basis의 세상>.



→ std basis (4,3)를 새 좌표계에서 좌표 구하려면

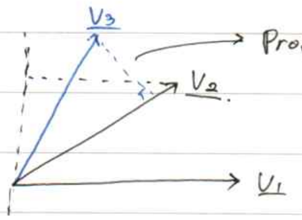
· 정사영해야한다.

· orthogonal. orthonormal에서는 쉽다!

Gram - Schmit Process.

- input : $\{\underline{v_1}, \cdots, \underline{v_r}\}$ any basis for W.
- Output : $\{\underline{u_1}, \cdots, \underline{v_r}\}$ Orthogonal basis for W.

* 찾는 이유? Orthogonal basis의 유용함



→ Projection to orthogonal compliment of $\underline{u_1}$

↳ Orthogonal 하면서 똑같은 공간을 Span 하는구나!

(i).

$(= \underline{u_1})$

(i) $\underline{u_1} = \underline{v_1}$

(ii) $\underline{u_2} = \mathrm{Proj}_{W_1^\perp} \underline{v_2}$

$\Rightarrow W_1 = \mathrm{Span}\{\underline{u_1}\}$.

+) dim ↑.

(iii) $\underline{u_3} = \mathrm{Proj}_{W_2^\perp} \underline{v_3}$

⋮

Orthogonal 하면서 같은 공간을 span할 것.