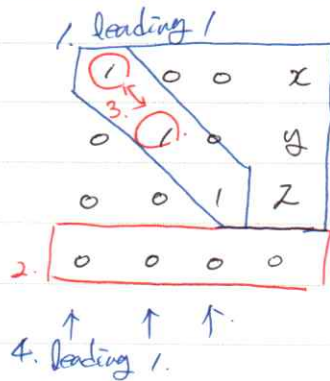


(사다리꼴).
Echelon Form.



1. First non-zero element in every row is 1. (= leading 1).
2. All zero rows are grouped together at the bottom.
3. Leading one ^{at} left must be upper.
4. Each column containing leading 1 has zero everywhere else.

"Reduced Row Echelon Form" (= RREF).
가장행 사다리꼴.

without 4.

"Row Echelon Form" (= REF).
행 사다리꼴.

ex. 1.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \text{ is RREF.}$$

* $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is RREF.

→ leading 1 이 없어서,
조건 위반이 아니요.

(= (bank) is leading 1 exists AND conditions)

ex) Augmented Matrix가
RREF일때.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

leading variable

$0x + 0y + 0z = 1 \rightarrow \text{No solution.}$

ex. $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x + 3z = -1$

$y - 4z = 2$

Free variable.

→ x, y 모두 z로 표현 가능.

$z = t, \quad x + 3t = -1 \quad | \quad x = -3t - 1$

$y + 4t = 2 \quad | \quad y = -4t + 2$

* leading variables는 Free variables로 표현 가능.

Elimination Methods. • Goal: any augmented matrix \Rightarrow RREF.

* ERO: Elementary Row Operation.

- 1) Constant multiple
- 2) Interchange
- 3) Add constant times row to other.

• step 1. Locate the leftmost col that not all zero.

$$\begin{array}{l} \text{S2) } \leftarrow \begin{array}{l} \text{0인가?} \\ \text{여면 교체} \end{array} \left[\begin{array}{cccccc} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array} \right] \\ \quad \quad \quad \leftarrow \begin{array}{l} \text{"3" 모두 0인가?} \end{array} \end{array}$$

• step 2. Interchange the top row if (0,0) is zero.

* Making leading 1.

• step 3. Use ERO ① to make leading 1. (\div).

$$\begin{array}{l} \text{S3) } \leftarrow \begin{array}{l} \text{2}^1 \quad \text{4}^2 \quad \text{-10}^3 \quad \text{6}^4 \quad \text{12}^5 \quad \text{28}^6 \end{array} \\ \text{S4) } \leftarrow \begin{array}{l} \text{2}^0 \quad \text{4}^0 \quad \text{-5}^5 \quad \text{6}^0 \quad \text{-5}^7 \quad \text{-1}^8 \end{array} \end{array}$$

• step 4. Use ERO ③ to make zero below leading 1.

forward phase
backward phase.

• step 5. Repeat step 1 ~ 4 until matrix become REF.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \text{is REF!}$$

• step 6. Make zero above the leading 1's.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

step 1-6: Gauss-Jordan Elimination. (making RREF).

step 1-5: Gaussian Elimination (making REF).

Homogeneous Linear Systems.

* Homogeneous Linear Systems.

$$a_{11}x_1 + \dots + a_{1n}x_n = 0.$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = 0.$$

$\} \rightarrow x_1 = \dots = x_n = 0. \rightarrow \text{trivial solution}$ 자명해.
 $\rightarrow \text{Consistent}$ "일관하다".

↓
 판별: 자명해만 존재?
 무한히 많은 해?

$$\begin{cases} ax+by=0. \\ cx+dy=0. \end{cases} \rightarrow \text{원점을 지난다.} \Rightarrow (0,0) \text{이 자명해.}$$

but $\frac{a}{c} = \frac{b}{d}$ 면 기울기가 같다 \Rightarrow 해가 무수히 많다.

ex 6) Homogeneous Linear System 이 Augmented Matrix로.

역시 (자명해)
 일관하다.

Free Variables in Homogeneous Linear Systems.

(Linear Systems).

1. LS: homogeneous \Rightarrow REF: homogeneous.

2. LS: m equations $\Rightarrow m \geq r$. (All zero rows 발생 가능성 때문).
 REF: r equations

• n unknowns + r leading variables then $n-r$ free variables.

| | x_1 | x_2 | x_3 | x_4 | x_5 | x_6 |
|---|-------|-------|-------|-------|-------|-------|
| ① | 3 | 0 | 4 | 2 | 0 | 0 |
| 0 | 0 | ① | 2 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | ① | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |

← 6# col.

3# 1# row. \Rightarrow leading variables.

$\Rightarrow 6 - 3 = 3$ (free variables).

* Thm 1.2.2. A homogeneous LS with more unknowns than equations has infinitely many solutions.

$$\begin{matrix} m \\ n \end{matrix} \leq \begin{matrix} n \\ r \end{matrix} \rightarrow n - r > 0.$$

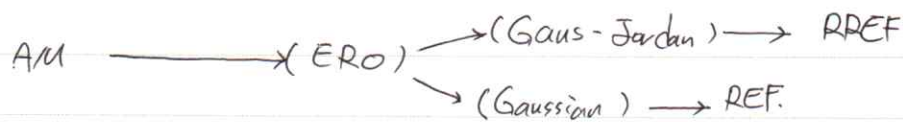
* Ex 7. Back substitution. (아래에서부터 대입).

$$\begin{array}{ccccccc} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$x_6 = \frac{1}{3}$$

$$x_3 = -2x_4 - 1.$$

$$x_1 = -3x_2 + 2x_3 + 2x_5.$$



* EX 8.

a. $\begin{bmatrix} 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No solution.}$

b. $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

\rightarrow Leading 1 \neq Row \neq \rightarrow Unique One Solution.

c. $\begin{bmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Infinitely Many.}$

Some facts about Echelon Forms.

1. Every matrix has a unique RREF. ... ERO 과정에 따라 같은 행, 같은 열을 갖는다.
But 같은 행 같아진다.

2. REFs are not unique

3. The RREF and all REFs have the same # of zeroes,
and the leading 1's always occur in the same positions.

\downarrow
pivoted positions ... \rightarrow \rightarrow \rightarrow

• pivot positions?

 $(1,1), (2,2)$ $(3,3), (4,4)$

• pivot columns

 $: 1, 2, 3, 4.$

• pivot rows

 $: 1, 2, 3, 4.$

$$\begin{pmatrix} 1 & -3 & 7 & 2 & 5 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 1 & 6 & 9 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

* ex. 9.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow$

\leftarrow
 \leftarrow
 \leftarrow