* Thm 4.9.1) $\dim(\text{row}(A)) = \dim(\text{col}(A))$.

* Def 1) $\text{rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A))$.

$\text{nullity}(A) = \dim(\text{null}(A))$.

Column space의 Dim. = Row space의 Dim

* ex 1)

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

ERO.

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Dim(row) = 2.

Free Variables

Dim(col) = 2. (= rank = 2).

⇒ dim (= rank) 는 RREF에서 확인.

$x_3 = r,\ x_4 = s,\ x_5 = t,\ x_6 = u.$

$$\begin{cases} x_1 = 4r + 28s + 37t - 13u. \\ x_2 = 2r + 12s + 16t - 5u. \end{cases}$$

$$\underline{x} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

✓ ✓ ✓ ✓ → Basis.

Basis of Null space

Nullity(A) = 4.

* ex 2). $m \times n$ A ... $\text{rank}(A) \le \min(m, n)$.

n → leading 1 발생 가능한 수 = $\min(m, n)$.

NO.

*Thm 4.9.2) Dimension Theorem.

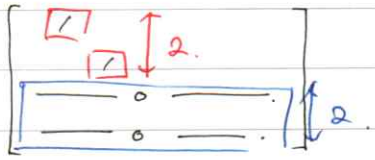
A: n columns.

$\boxed{\text{rank}(A)} + \underline{\text{nullity}(A)} = n.$

leading variables. free variables

all zero.

number of parameters



Thm 4.9.3) $A\underline{x} = \underline{0}$.

(a) $\text{rank}(A) = \#$ of leading variables.

(b) $\text{nullity}(A) = \#$ of parameters.

* ex 4) $A\underline{x} = 0$. (a) $A: 5\times 7$. $\Rightarrow \#$ of params $= 4$.

$\text{rank}(A) = 3$.

(b) $\dim(\text{solution space}) = 2$. $\Rightarrow \text{rank}(A) = 5$.

$\text{nullity}(A) = 2$.

* Thm 4.9.4) $A\underline{x} = \underline{b}$: consistent.

$\downarrow$

$m\times n$ $\quad \text{rank}(A) = r$. $\Rightarrow$ general solution contains $(n-r)$ parameters.

$\underline{x} = \underline{x_h} + \underline{x_p}$.

Fundamental spaces.

row (A) — row(A^T).

col (A) — col(A^T). → 중복.

null (A) — null(A^T).

(def) → left null space of A.

$\therefore \dim(\text{row}(A)) = r$

$\dim(\text{col}(A)) = r$.

$\dim(\text{null}(A)) = n-r$.

$\dim(\text{null}(A^T)) = m-r$.

* Thm 4.9.5). $\text{rank}(A) = \text{rank}(A^T)$. $(\text{rank}(\text{col}(A)) = \text{rank}(\text{row}(A)))$

A^T: m column. $\Rightarrow \text{rank}(A^T) + \text{nullity}(A^T) = m$.

$= \text{rank}(A) + \text{nullity}(A^T)$

Bases for Fundamental spaces.

- $m \times n$ A

$$[A \mid I] \xrightarrow{GJE} \overset{RREF}{[R \mid E]} \updownarrow m-r \text{ row}$$

* A^T에 대해 별도 처리 없이도
↓ 처리할 수 있다.

- Basis for left null space is found from the bottom $m-r$ rows of E.

* ex 5) A in example 1.

$$\left[A \;\middle|\; \begin{matrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{matrix}\right] \rightarrow \left[R \;\middle|\; \begin{matrix} 0 & 0 & -9/2 & 5/2 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{matrix}\right]$$

$m-r = 4-2$,,

→ 2개.

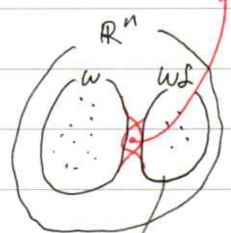
A Geometric Link Between the Fundamental Spaces.

* Def 2). W : S.S of $\mathbb{R}^n$.

Orthogonal complement of W.

: Set of all vectors in $\mathbb{R}^n$

that are orthogonal to every vector in W



0 인것.

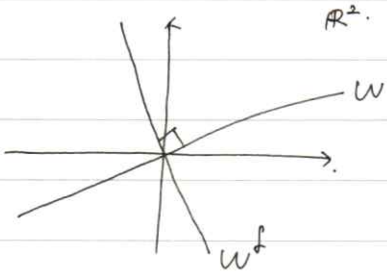
영벡터는

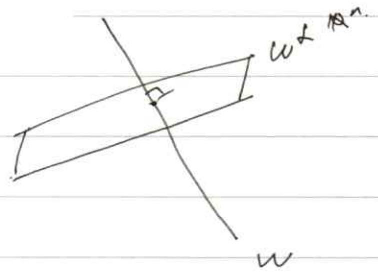
$W^\perp$의 각 원소는 W 구성 원소 모두에 수직.

* Thm 4.9.6). (a) $W^\perp$ = S.S of $\mathbb{R}^n$.

(b) $W \cap W^\perp = \{\underline{0}\}$ → 당연히 겹치지 않을테니..

(c) $(W^\perp)^\perp = W$





* Thm 4.9.7). A : $m \times n$.

(a) $(\text{row}(A))^{\perp} = \text{null}(A)$.

(b) $(\text{col}(A))^{\perp} = \text{null}(A^T)$.

$A\underline{x} = \underline{0}$.

$$\begin{bmatrix} \underline{r_1} \\ \underline{r_2} \\ \vdots \\ \underline{r_n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \underline{r_1} \cdot \underline{x} &= 0. \\ \underline{r_2} \cdot \underline{x} &= 0 \\ &\vdots \\ \underline{r_n} \cdot \underline{x} &= 0. \end{aligned}\right\}$$

내적이 0? cos=0
⇒ 수직.

* Thm 4.9.8) $A_{n \times n}$: invertible.

(a) ~ (q).

(h) Column Vectors of A are linearly indep.

(indep).

$A \xrightarrow[R.]{GJE} [\,\text{Identity}\,]$ ⋯ Identity 이므로 linearly indep.

(i) Row vectors ———— indep.

(j) Column Vectors of A span $\mathbb{R}^n$.

$\mathbb{R}^n$의 벡터들 모두 linearly indep 하므로
$\mathbb{R}^n$ 을 span 한다 할 수 있다.

(k) Row Vectors ————"

(l) Column Vectors of A <u>form a basis</u> for $\mathbb{R}^n$.

(m) Row vectors ————"

(n) A has rank n.

(o). A has nullity 0.

(has no all-zero row).

(p). $(\text{null}(A))^{\perp} = \mathbb{R}^n$

(q) $(\text{row}(A))^{\perp} = \{\underline{0}\}$

null(A) 공간은 $\underline{0}$ 하나이므로.