# Robotics I

# 04. Jacobian Matrix
##   (Introduction)

# Review

How do I calculate the end effector pose from joint angles and how do I calculate the joint angles from the end effector pose?

- Kinematics of serial robots
- Forward kinematics
- Denavit-Hartenberg notation
- Inverse kinematics



**Literature**

W. Khalil & E. Dombre - Modeling, Identification & Control of Robots, pp. 35-83
F. Pfeiffer & E. Reithmeier - Robot dynamics, pp. 27-59
W. Khalil & J. F. Kleinfinger - A new Geometric Notation for Open and Closed-Loop Robots
J. J. Craig - Introduction to Robotics Mechanics and Control, pp. 68-141
A. Fuchs - Kinematic calibration of a planar PKM, p. 5, 6

# Total Content
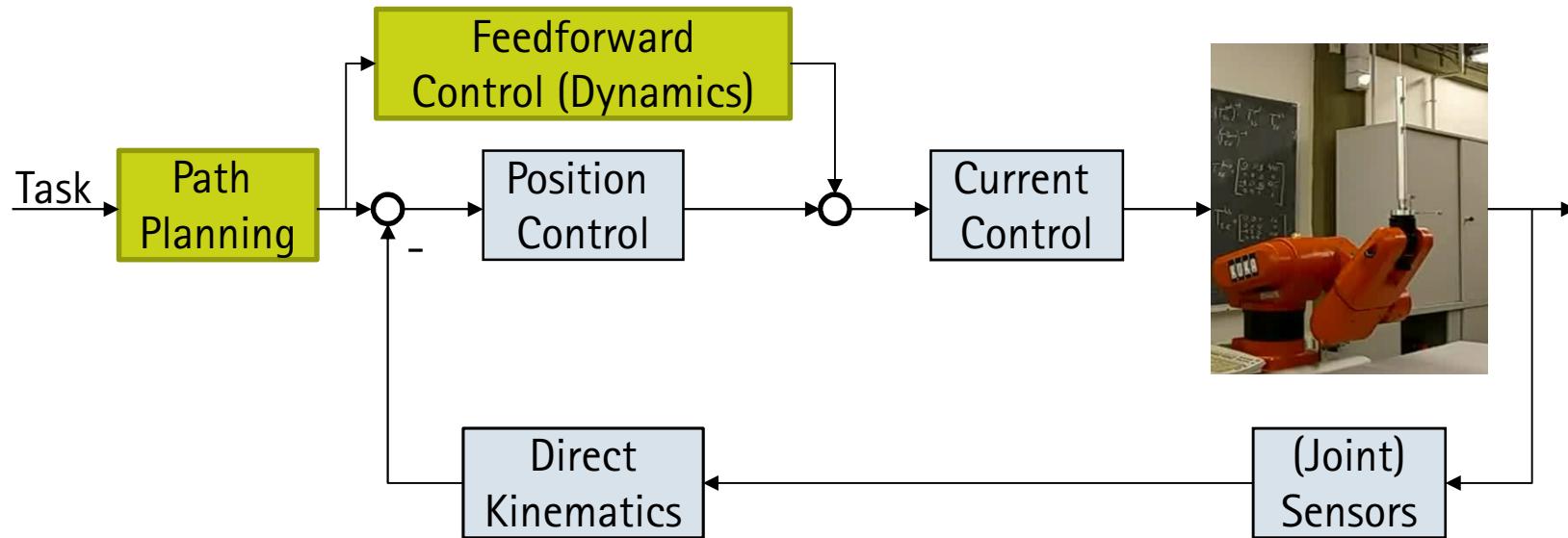
# Total Content
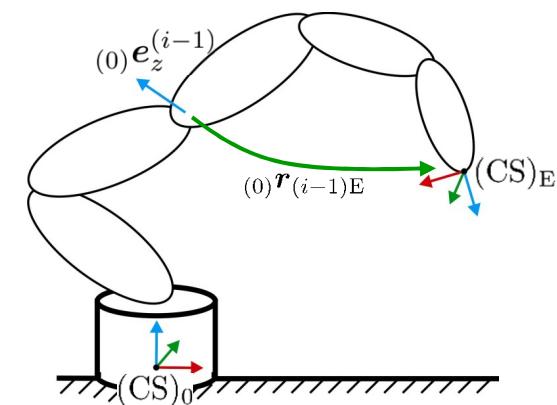
**Connection of essential lecture contents in the form of a control loop:**

The Jacobian matrix relates end-effector and joint-angular velocities and is an important element for analysis and design.

# Jacobian Matrix

**How do I relate end-effector and joint-angular velocities and how do I determine important properties of my robot?**

- Differential kinematics
- Jacobian matrix
- Manipulability
- Static force and moment model
- Singularities

**Literature**
B. Heimann et al. - Mechatronics (Components - Methods - Examples), pp. 190-199
J. J. Craig - Introduction to Robotics Mechanics and Control, pp. 152-182
L. Sciavicco et al - Modeling and Control of Robot Manipulators, pp. 69-117

# Differential Kinematics

Relationship between joint angular velocities/accelerations $\dot{\boldsymbol{q}},\ \ddot{\boldsymbol{q}}$ and velocity/acceleration $\dot{\boldsymbol{x}}_{\mathrm{E}},\ \ddot{\boldsymbol{x}}_{\mathrm{E}}$ of the end effector pose. This includes linear correlation between small changes in end effector and joint coordinates $\Delta\boldsymbol{x}_{\mathrm{E}},\ \Delta\boldsymbol{q}$ :

$$\Delta\boldsymbol{x}_{\mathrm{E}} \approx \dot{\boldsymbol{x}}_{\mathrm{E}}\,\Delta t,\ \ \Delta\boldsymbol{q} \approx \dot{\boldsymbol{q}}\,\Delta t$$

From forward kinematics

$$\boldsymbol{x}_{\mathrm{E}} = \boldsymbol{f}(\boldsymbol{q})$$

working point-dependent differential relationships can be determined

$$\Delta\boldsymbol{x}_{\mathrm{E}} = \boldsymbol{J}(\boldsymbol{q})\,\Delta\boldsymbol{q} \quad \Leftrightarrow \quad \Delta\boldsymbol{q} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\Delta\boldsymbol{x}_{\mathrm{E}}$$

↳ Jacobian matrix ↳ inverse Jacobian matrix
(pay attention to invertibility, later)

➡ Different approaches to calculating the Jacobian matrix

# Differential Kinematics

**Possible applications**
Incremental path calculation

$$\Delta \boldsymbol{q} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\Delta \boldsymbol{x}_{\mathrm{E}}$$

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \Delta \boldsymbol{q}_i = \boldsymbol{q}_i + \boldsymbol{J}^{-1}(\boldsymbol{q}_i)\,\Delta \boldsymbol{x}_{\mathrm{E},\,i} = \boldsymbol{q}_i + \boldsymbol{J}^{-1}(\boldsymbol{q}_i)\,(\boldsymbol{x}_{\mathrm{E},\,i+1} - \boldsymbol{x}_{\mathrm{E},\,i})$$

➡ avoids repeated, time-consuming determination of the inverse kinematics

More on this later...

# Differential Kinematics

From differential relationship

$$\Delta \boldsymbol{x}_\mathrm{E} = \boldsymbol{J}(\boldsymbol{q})\,\Delta \boldsymbol{q} \quad \Leftrightarrow \quad \Delta \boldsymbol{q} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\Delta \boldsymbol{x}_\mathrm{E}$$

follows connection between $\dot{\boldsymbol{x}}_\mathrm{E} \approx {}^{\Delta \boldsymbol{x}_\mathrm{E}}\!/\!_{\Delta t}$ and $\dot{\boldsymbol{q}} \approx {}^{\Delta \boldsymbol{q}}\!/\!_{\Delta t}$ :

$$\dot{\boldsymbol{x}}_\mathrm{E} = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} \quad \Leftrightarrow \quad \dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{x}}_\mathrm{E}$$

The same applies to acceleration:

$$\ddot{\boldsymbol{x}}_\mathrm{E} = \boldsymbol{J}(\boldsymbol{q})\,\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}$$
$$\Leftrightarrow$$
$$\ddot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\,\ddot{\boldsymbol{x}}_\mathrm{E} - \boldsymbol{J}^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{J}}(\boldsymbol{q},\,\dot{\boldsymbol{q}})\,\boldsymbol{J}^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{x}}_\mathrm{E}$$

Application in robot control

# Robotics I

## 04. Jacobian Matrix (Analytical/Geometric)

# Jacobian Matrix

**Analytical Jacobian matrix**
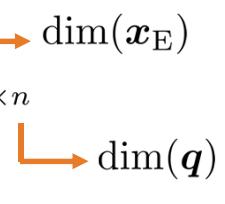
If given $x_{\mathrm{E}} = f(q)$, differential correlations can be described using the analytical Jacobian matrix $J(q)$ :

$$\frac{\mathrm{d}x_{\mathrm{E}}}{\mathrm{d}t} = \frac{\mathrm{d}f(q)}{\mathrm{d}t} = \frac{\partial f(q)}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} = J(q)\,\dot{q} \quad \text{,with} \quad \dot{x}_{\mathrm{E}} = \lim_{\Delta t \to 0} \frac{\Delta x_{\mathrm{E}}}{\Delta t}$$

$$\dot{q} = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t}$$

$$\Rightarrow \quad \dot{x}_{\mathrm{E}} = J(q)\,\dot{q} \quad \Leftrightarrow \quad \dot{q} = J^{-1}(q)\,\dot{x}_{\mathrm{E}}$$

Analytical Jacobian matrix is the derivative of the vectorial function $f(q)$ with respect to $q$ :

$$J(q) = \frac{\partial f(q)}{\partial q} = \begin{pmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \cdots & \frac{\partial f_m}{\partial q_n} \end{pmatrix} \in \mathbb{R}^{m \times n} \quad \begin{array}{l} \longrightarrow \dim(x_{\mathrm{E}}) \\ \\ \longrightarrow \dim(q) \end{array}$$

# Jacobian Matrix

**Analytical Jacobian matrix**

Example: Planar RR robot:

$$\dim(\boldsymbol{x}_{\mathrm{E}}) = \dim(\boldsymbol{q}) = 2$$
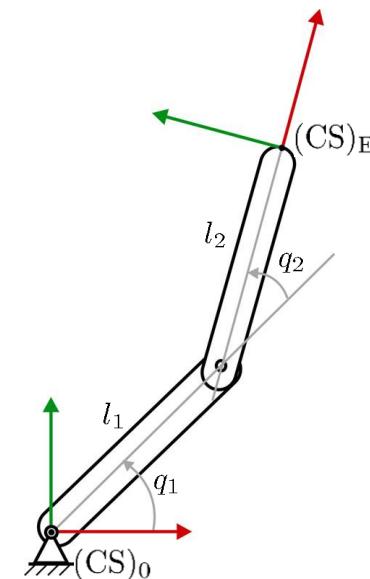
Forward kinematics (previous lecture):

$$_{(0)}x_{\mathrm{E}} = l_1 \, \cos(q_1) + l_2 \, \cos(q_1 + q_2)$$
$$_{(0)}y_{\mathrm{E}} = l_1 \, \sin(q_1) + l_2 \, \sin(q_1 + q_2)$$

Analytical Jacobian matrix:

$$\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} = \ldots$$

Attention: Angle/angular velocities of the end effector not given in $(\mathrm{CS})_0$
 (depending on composed rotations)

# Jacobian Matrix

**Geometric Jacobian matrix**

Components can be determined geometrically:
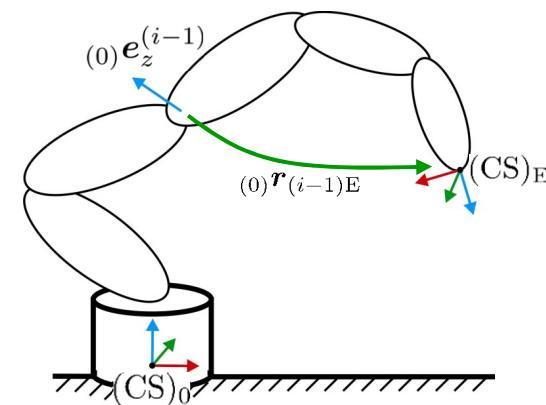
Vectorial relationship between $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{x}}_{\mathrm{E,t}} = (\dot{x}_{\mathrm{E}},\, \dot{y}_{\mathrm{E}},\, \dot{z}_{\mathrm{E}})^{\mathrm{T}}$ as well as $\dot{\boldsymbol{x}}_{\mathrm{E,r}} = (\omega_x,\, \omega_y,\, \omega_z)^{\mathrm{T}}$ in $(\mathrm{CS})_0$ :

$$_{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,g}} = \begin{pmatrix} _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,t}} \\ _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,r}} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_1} & \boldsymbol{j}_{\mathrm{t}_2} & \cdots & \boldsymbol{j}_{\mathrm{t}_n} \\ \boldsymbol{j}_{\mathrm{r}_1} & \boldsymbol{j}_{\mathrm{r}_2} & \cdots & \boldsymbol{j}_{\mathrm{r}_n} \end{pmatrix}}_{\boldsymbol{J}_{\mathrm{g}}(\boldsymbol{q})} \dot{\boldsymbol{q}}$$

The following (coordinate systems according to DH):

Prismatic joints:
$$\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_i} \\ \boldsymbol{j}_{\mathrm{r}_i} \end{pmatrix} = \begin{pmatrix} _{(0)}\boldsymbol{e}_z^{(i-1)} \\ \boldsymbol{0} \end{pmatrix}$$

Revolute joints:
$$\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_i} \\ \boldsymbol{j}_{\mathrm{r}_i} \end{pmatrix} = \begin{pmatrix} _{(0)}\boldsymbol{e}_z^{(i-1)} \times {}_{(0)}\boldsymbol{r}_{(i-1)\mathrm{E}} \\ _{(0)}\boldsymbol{e}_z^{(i-1)} \end{pmatrix}$$



L. Sciavicco et. al. – Modeling and Control of Robot Manipulators, p. 69 ff.
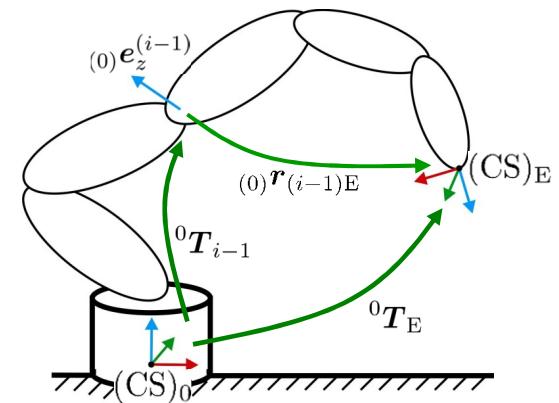
# Jacobian Matrix

Elements of the geometric Jacobian matrix

$$_{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,g}} = \begin{pmatrix} _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,t}} \\ _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,r}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_1} & \boldsymbol{j}_{\mathrm{t}_2} & \cdots & \boldsymbol{j}_{\mathrm{t}_n} \\ \boldsymbol{j}_{\mathrm{r}_1} & \boldsymbol{j}_{\mathrm{r}_2} & \cdots & \boldsymbol{j}_{\mathrm{r}_n} \end{pmatrix} \dot{\boldsymbol{q}}$$

Prismatic joints:
$$\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_i} \\ \boldsymbol{j}_{\mathrm{r}_i} \end{pmatrix} = \begin{pmatrix} _{(0)}\boldsymbol{e}_z^{(i-1)} \\ \boldsymbol{0} \end{pmatrix}$$

Revolute joints:
$$\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_i} \\ \boldsymbol{j}_{\mathrm{r}_i} \end{pmatrix} = \begin{pmatrix} _{(0)}\boldsymbol{e}_z^{(i-1)} \times {}_{(0)}\boldsymbol{r}_{(i-1)\mathrm{E}} \\ _{(0)}\boldsymbol{e}_z^{(i-1)} \end{pmatrix}$$

Elements can be determined using known transformations, for example:

$$_{(0)}\boldsymbol{e}_z^{(i-1)} = \underbrace{{}^0\boldsymbol{R}_1 \, {}^1\boldsymbol{R}_2 \ldots {}^{i-2}\boldsymbol{R}_{i-1}}_{{}^0\boldsymbol{R}_{i-1}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} _{(0)}\boldsymbol{r}_{(i-1)\mathrm{E}} \\ 0 \end{pmatrix} = \underbrace{{}^0\boldsymbol{T}_1 \, {}^1\boldsymbol{T}_2 \ldots {}^{n-1}\boldsymbol{T}_{\mathrm{E}}}_{{}^0\boldsymbol{T}_{\mathrm{E}}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \underbrace{{}^0\boldsymbol{T}_1 \, {}^1\boldsymbol{T}_2 \ldots {}^{i-2}\boldsymbol{T}_{i-1}}_{{}^0\boldsymbol{T}_{i-1}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



Attention
$$_{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,g}} = \left( _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,t}}^{\mathrm{T}}, \; _{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,r}}^{\mathrm{T}} \right)^{\mathrm{T}} \neq \dot{\boldsymbol{x}}_{\mathrm{E}}$$

# Jacobian Matrix

**Relationship between analytical and geometric Jacobian matrix**

Calculation $\dot{\boldsymbol{x}}_{\mathrm{E,r}} = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$ dependent on composed rotations (example Kardan angle):

$$
{}_{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,r}} = \begin{pmatrix} \dot{\phi}_{\mathrm{E}} \\ 0 \\ 0 \end{pmatrix}_{\mathrm{KARD}} + \boldsymbol{R}_x(\phi_{\mathrm{E}}) \begin{pmatrix} 0 \\ \dot{\psi}_{\mathrm{E}} \\ 0 \end{pmatrix}_{\mathrm{KARD}} + \boldsymbol{R}_x(\phi_{\mathrm{E}})\,\boldsymbol{R}_y(\psi_{\mathrm{E}}) \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{\mathrm{E}} \end{pmatrix}_{\mathrm{KARD}}
$$

$$
= \underbrace{\begin{pmatrix} 1 & 0 & \sin(\psi_{\mathrm{E}}) \\ 0 & \cos(\phi_{\mathrm{E}}) & -\sin(\phi_{\mathrm{E}})\cos(\psi_{\mathrm{E}}) \\ 0 & \sin(\phi_{\mathrm{E}}) & \cos(\phi_{\mathrm{E}})\cos(\psi_{\mathrm{E}}) \end{pmatrix}}_{\boldsymbol{\Omega}_{\mathrm{KARD}}} \begin{pmatrix} \dot{\phi}_{\mathrm{E}} \\ \dot{\psi}_{\mathrm{E}} \\ \dot{\theta}_{\mathrm{E}} \end{pmatrix}_{\mathrm{KARD}}
$$

➡ Relationship between analytical and geometric Jacobian matrix:

$$
{}_{(0)}\dot{\boldsymbol{x}}_{\mathrm{E,g}} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{\mathrm{KARD}} \end{pmatrix} \dot{\boldsymbol{x}}_{\mathrm{E}} \quad \Rightarrow \quad \boldsymbol{J}_{\mathrm{g}}(\boldsymbol{q})\,\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{\mathrm{KARD}} \end{pmatrix} \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}} \ \ \forall\, \dot{\boldsymbol{q}}
$$

$$
\Rightarrow \quad \boldsymbol{J}_{\mathrm{g}}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{\mathrm{KARD}} \end{pmatrix} \boldsymbol{J}(\boldsymbol{q})
$$

# Jacobian Matrix

**Geometric Jacobian matrix**

Example: Planar RR robot

Homogeneous transformation matrices from Denavit-Hartenberg parameters
(previous lecture):
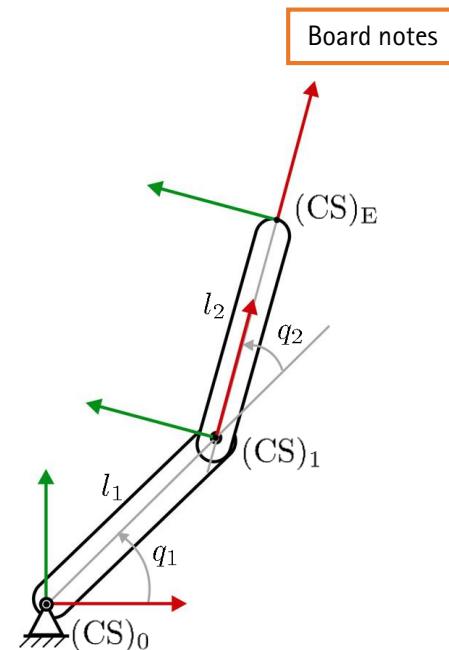
$$^0\boldsymbol{T}_1(q_1) = \begin{pmatrix} \cos(q_1) & -\sin(q_1) & 0 & l_1\cos(q_1) \\ \sin(q_1) & \cos(q_1) & 0 & l_1\sin(q_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^1\boldsymbol{T}_{\mathrm{E}}(q_2) = \begin{pmatrix} \cos(q_2) & -\sin(q_2) & 0 & l_2\cos(q_2) \\ \sin(q_2) & \cos(q_2) & 0 & l_2\sin(q_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Joint $i = 2$

Rotation axis of the 2nd joint

$$_{(0)}\boldsymbol{e}_z^{(1)} = {}^0\boldsymbol{R}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(q_1) & -\sin(q_1) & 0 \\ \sin(q_1) & \cos(q_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

# Jacobian Matrix

**Example: Planar RR robot**
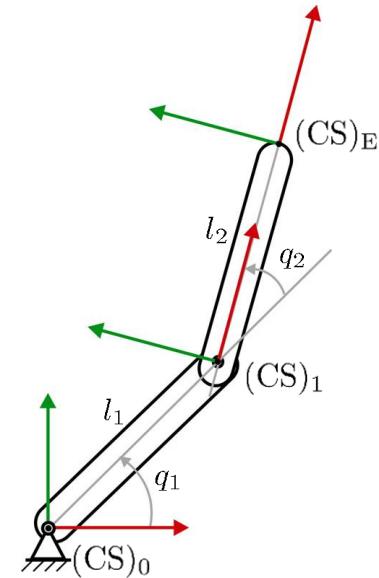
Joint $i = 2$

Direction vector to the end effector

$$
\begin{pmatrix} {}^{(0)}\boldsymbol{r}_{1\mathrm{E}} \\ 0 \end{pmatrix} = {}^0\boldsymbol{T}_{\mathrm{E}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - {}^0\boldsymbol{T}_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

Geometric Jacobian matrix:

$$
= \begin{pmatrix} l_2\cos(q_1+q_2) + l_1\cos(q_1) \\ l_2\sin(q_1+q_2) + l_1\sin(q_1) \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} l_1\cos(q_1) \\ l_1\sin(q_1) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} l_2\cos(q_1+q_2) \\ l_2\sin(q_1+q_2) \\ 0 \\ 0 \end{pmatrix}
$$
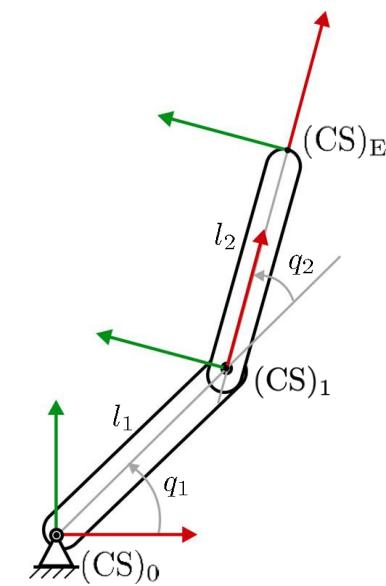
# Jacobian Matrix

**Example: Planar RR robot**

2nd column of the Jacobian matrix

$$
\begin{pmatrix} \boldsymbol{j}_{\mathrm{t}_2} \\ \boldsymbol{j}_{\mathrm{r}_2} \end{pmatrix} = \begin{pmatrix} {}_{(0)}\boldsymbol{e}_z^{(1)} \times {}_{(0)}\boldsymbol{r}_{1\mathrm{E}} \\ {}_{(0)}\boldsymbol{e}_z^{(1)} \end{pmatrix}
$$

$$
= \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} l_2 \cos(q_1 + q_2) \\ l_2 \sin(q_1 + q_2) \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{pmatrix}
$$

$$
= \begin{pmatrix} -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$
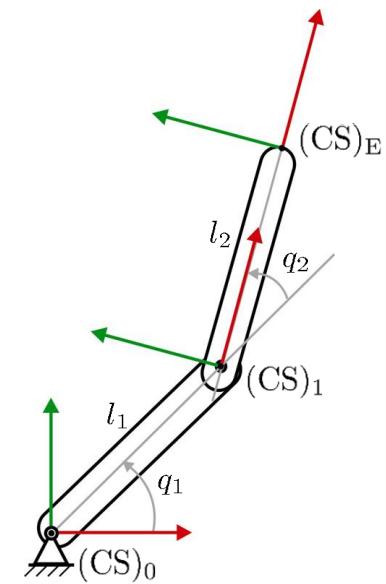
# Jacobian Matrix

**Example: Planar RR robot**

Complete geometric Jacobian matrix

$$\boldsymbol{J}_{\mathrm{g}} = \left( \begin{array}{cc} \boldsymbol{j}_{\mathrm{t}_1} & \boldsymbol{j}_{\mathrm{t}_2} \\ \boldsymbol{j}_{\mathrm{r}_1} & \boldsymbol{j}_{\mathrm{r}_2} \end{array} \right)$$

$$= \left( \begin{array}{cc} -l_2 \sin(q_1 + q_2) - l_1 \sin(q_1) & -l_2 \sin(q_1 + q_2) \\ l_2 \cos(q_1 + q_2) + l_1 \cos(q_1) & l_2 \cos(q_1 + q_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{array} \right)$$

# Robotics I

## 04. Jacobian Matrix
### (Derived Properties)

# Jacobian Matrix

Numerous properties and performance characteristics can be derived from the Jacobian matrix.
These are for example:

- Manipulability
- Static rigidity
- Singularities
- etc.

➡ see the following slides

# Manipulability

**Manipulability of non-redundant robots**

Central evaluation criterion for robots:

Effect of joint movements on the end effector. Good manipulability if small change $\Delta\boldsymbol{q}$ leads to large change $\Delta\boldsymbol{x}_{\mathrm{E}}$.

For small deflections $\Delta\boldsymbol{q} \approx \Delta t\,\dot{\boldsymbol{q}}$ and $\Delta\boldsymbol{x}_{\mathrm{E}} \approx \Delta t\,\dot{\boldsymbol{x}}_{\mathrm{E}}$, therefore the entries of the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ are decisive for the required relationship: $\dot{\boldsymbol{x}}_{\mathrm{E}} = \boldsymbol{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}}$

# Manipulability

**Manipulability of non-redundant robots**

From $\dot{x}_{\mathrm{E}} = J(q)\,\dot{q}$   the following system of equations result via eigenvalues/eigenvectors of $J(q)$ (change of basis):

$$\dot{x}_{\mathrm{E}}^{\star} = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix} \dot{q}^{\star} = \mathrm{diag}(\boldsymbol{\lambda})\,\dot{q}^{\star}$$

Manipulability of individual "degrees of freedom" of the end effector $\dot{x}_{\mathrm{E}}^{\star}$ :

$|\lambda_i|$ large : good manipulability ("good translation")

$|\lambda_i|$ small : poor manipulability ("bad translation")

$|\lambda_i| = 0$   : No manipulability (singularity, loss of rank)

Manipulability of the overall system (<u>homogenization of $J$ may be useful</u>):

$$\mu(q) = \left| \prod_i \lambda_i \right| = |\det(J(q))|$$

# Static Force and Moment Model

**Force interaction with the environment**

Force vector at the end effector:

$$\mathcal{F} = \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{pmatrix} \in \mathbb{R}^6$$

Principle of virtual work:

Work in end effector coordinates stays the same
Work in joint coordinates:

$$\Delta \boldsymbol{x}_{\mathrm{E}}^{\mathrm{T}} \, \mathcal{F} = \Delta \boldsymbol{q}^{\mathrm{T}} \, \boldsymbol{\tau}$$
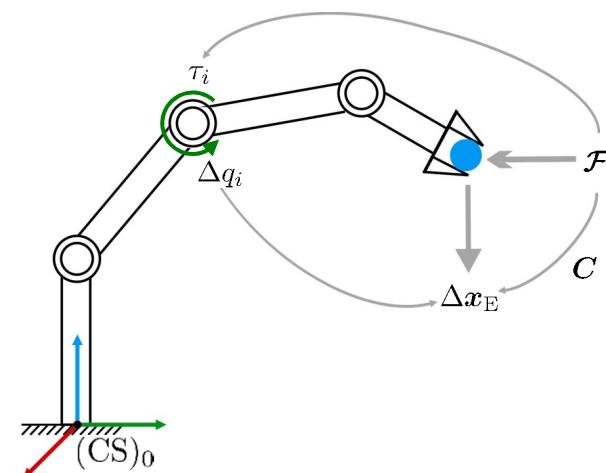
From this follows
(using differential kinematics):

$$(\boldsymbol{J}(\boldsymbol{q}) \, \Delta \boldsymbol{q})^{\mathrm{T}} \, \mathcal{F} = \Delta \boldsymbol{q}^{\mathrm{T}} \, \boldsymbol{\tau} \quad \Leftrightarrow \quad \Delta \boldsymbol{q}^{\mathrm{T}} \, \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q}) \, \mathcal{F} = \Delta \boldsymbol{q}^{\mathrm{T}} \, \boldsymbol{\tau} \ \ \forall \, \Delta \boldsymbol{q}$$

$$\boldsymbol{\tau} = \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q}) \, \mathcal{F} \quad \Leftrightarrow \quad \mathcal{F} = \left( \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q}) \right)^{-1} \, \boldsymbol{\tau}$$

Pay attention to invertibility

# Static Force and Moment Model

**Stiffness analysis**

Description of the actuator joints by spring constants (joints linearly elastic):

$$\boldsymbol{\tau} = \boldsymbol{K}\,\Delta\boldsymbol{q}$$

$\quad\quad\hookrightarrow$ diagonal stiffness matrix $\boldsymbol{K} = \mathrm{diag}(\boldsymbol{k})$

Determination of the resulting end effector displacement:
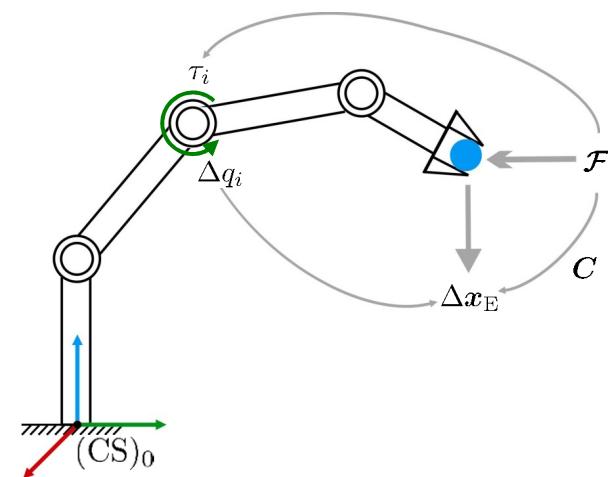
$$\boldsymbol{\tau} = \boldsymbol{K}\,\Delta\boldsymbol{q}$$
$$\Rightarrow\ \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})\,\boldsymbol{\mathcal{F}} = \boldsymbol{K}\,\Delta\boldsymbol{q}$$
$$\Rightarrow\ \boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})\,\boldsymbol{\mathcal{F}} = \boldsymbol{K}\,\boldsymbol{J}^{-1}(\boldsymbol{q})\,\Delta\boldsymbol{x}_{\mathrm{E}}$$

$$\Delta\boldsymbol{x}_{\mathrm{E}} = \underbrace{\boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{K}^{-1}\,\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})}_{\boldsymbol{C}}\,\boldsymbol{\mathcal{F}}$$

$\quad\quad\quad\hookrightarrow$ Symmetrical compliance matrix $\boldsymbol{C} = \boldsymbol{C}^{\mathrm{T}}$

# Static Force and Moment Model

**Main directions of compliance**

Main directions (eigenvectors) of the $C$-matrix point in the direction
of the maximum or minimum end effector deflection

Approach:

$$\Delta x_{\mathrm{E}} = C\,\mathcal{F} \quad \Rightarrow \quad C\,f_{\mathrm{EV}_i} = \lambda_i\,f_{\mathrm{EV}_i}$$
$$\Rightarrow \quad (C - \lambda_i\,E)\,f_{\mathrm{EV}_i} = 0$$
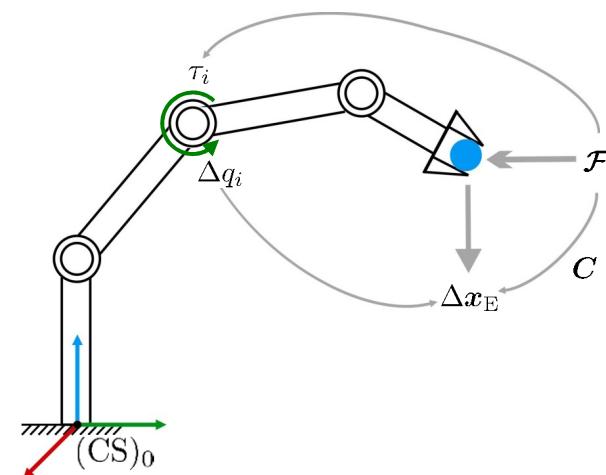
Determining the eigenvalues $\lambda_i$

$$\det(C - \lambda_i\,E) = 0$$

➡️      Largest eigenvalue: "softest" direction
smallest eigenvalue: "stiffest" direction

Determination of the eigenvectors $f_{\mathrm{EV}_i}$ :

$$(C - \lambda_i\,E)\,f_{\mathrm{EV}_i} = 0$$

# Robotics I

# 04. Jacobian Matrix (Singularities)

# Singularities

Jacobian matrix can become singular for certain $\boldsymbol{q}$ (singularities)

- Robot loses at least one end effector degree of freedom (degeneration)
- Loss of rank of the Jacobian matrix, determinant results in zero
- Jacobian matrix not invertible

Singularities can occur both on the workspace boundary
as well as in the middle of the workspace.
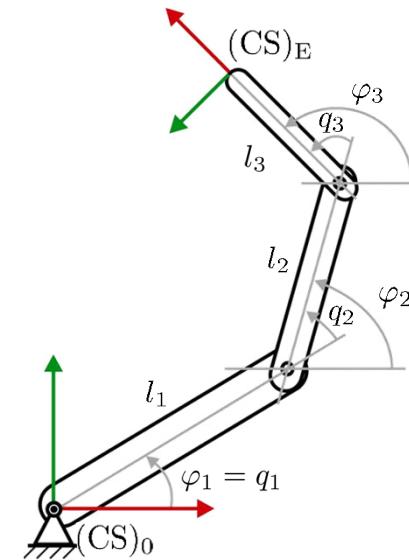
Example of a planar RRR robot

$$\boldsymbol{x}_{\mathrm{E}} = \left({}_{(0)}x_{\mathrm{E}},\ {}_{(0)}y_{\mathrm{E}},\ \theta_{\mathrm{E}}\right)^{\mathrm{T}}$$

$$\boldsymbol{\varphi} = \left(\varphi_1,\ \varphi_2,\ \varphi_3\right)^{\mathrm{T}} = \left(q_1,\ q_1 + q_2,\ q_1 + q_2 + q_3\right)^{\mathrm{T}}$$
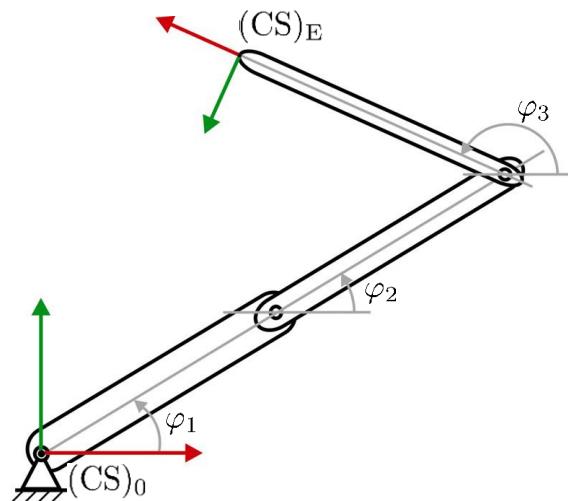
$$l_1 = l_2 = l_3 = 1$$

$$\boldsymbol{J}(\boldsymbol{\varphi}) = \frac{\partial \boldsymbol{x}_{\mathrm{E}}(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}} = \begin{pmatrix} -\sin(\varphi_1) & -\sin(\varphi_2) & -\sin(\varphi_3) \\ \cos(\varphi_1) & \cos(\varphi_2) & \cos(\varphi_3) \\ 0 & 0 & 1 \end{pmatrix}$$
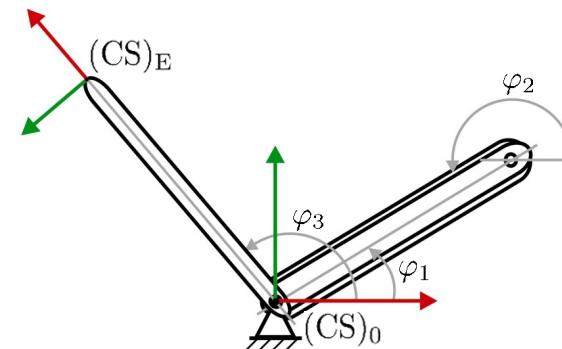
# Singularities

Singularities of the planar RRR robot



$$\varphi_1 = \varphi_2 \qquad\qquad \varphi_1 = \varphi_2 \pm \pi$$

# Singularities

**Practical meanings**

Mathematically, singularities at $q = q_{\text{sing}}$ are characterized as follows:

$$\lim_{q \to q_{\text{sing}}} \det(J(q)) = 0, \quad \lim_{q \to q_{\text{sing}}} \det(J^{-1}(q)) = \infty, \quad J \in \mathbb{R}^{n \times n}$$

At least one row of the Jacobian matrix can be expressed as a linear combination of another row (rank drop):

$$\dot{x}_{\text{E}} = J(q_{\text{sing}})\,\dot{q} \quad \Rightarrow \quad \dot{x}_{\text{E}}^* = \begin{pmatrix} \cdots \\ \vdots & \ddots & \vdots \\ & \mathbf{0}^{\text{T}} & \end{pmatrix} \dot{q} \left. \rule{0pt}{2.5em} \right\}$$

for at least one eigenvalue $\lambda_i$ of $J$ applies:
$$\lambda_i = 0$$

At least one degree of freedom or one combination of degrees of freedom can no longer be varied. All degrees of freedom of the joint space are available for the remaining end effector degrees of freedom

➡ redundant system (see later)

# Singularities

**Practical meanings**

Very high joint velocities occur in the vicinity of singularities ( $\boldsymbol{q} \to \boldsymbol{q}_{\mathrm{sing}}$ )
(to be avoided from an application/robot perspective):
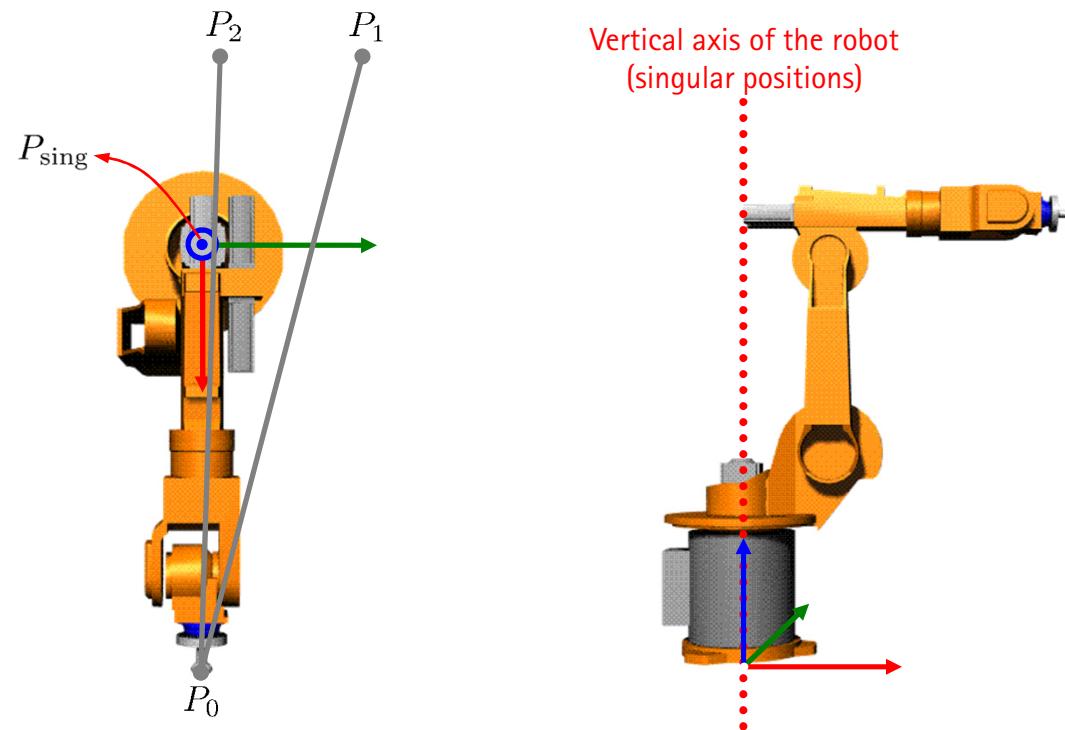
$$\dot{\boldsymbol{q}}^* = \mathrm{diag}\left(\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \cdots, \frac{1}{\lambda_n}\right)\right) \dot{\boldsymbol{x}}_{\mathrm{E}}^* \left.\begin{array}{c} \\ \\ \end{array}\right\}$$

applies to at least one eigenvalue $\lambda_i$ of $\boldsymbol{J}$:

$$\lambda_{\mathrm{sing}} \to 0 \;\Rightarrow\; \lambda_{\mathrm{sing}}^{-1} \to \infty$$

Force application at the end effector results in no deflection in individual spatial directions
(very high/infinite stiffness):
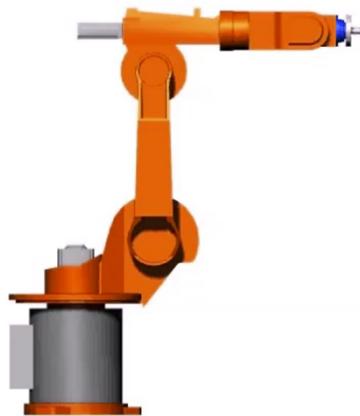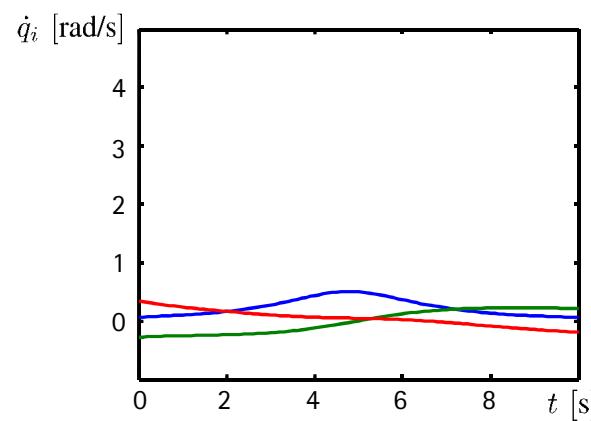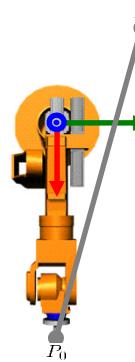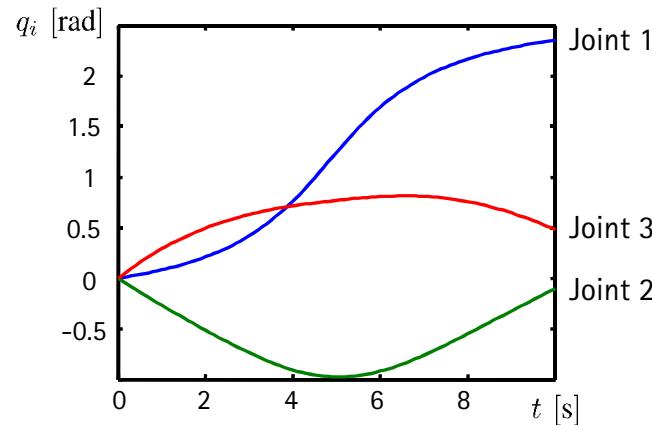
$$\Delta \boldsymbol{x}_{\mathrm{E}}^* = \begin{pmatrix} \cdots \\ \vdots \; \ddots \; \vdots \\ \boldsymbol{0}^{\mathrm{T}} \end{pmatrix} \boldsymbol{\mathcal{F}} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\}$$

at least one eigenvalue of C is zero:

$$\Delta \boldsymbol{x}_{\mathrm{E}} = \boldsymbol{C}\boldsymbol{\mathcal{F}} = (\boldsymbol{J}\boldsymbol{K}^{-1}\boldsymbol{J}^{\mathrm{T}})\boldsymbol{\mathcal{F}}$$

$$\det(\boldsymbol{C}) = \det(\boldsymbol{J}\boldsymbol{K}^{-1}\boldsymbol{J}^{\mathrm{T}})$$

$$= \det(\boldsymbol{J})\det(\boldsymbol{K}^{-1})\det(\boldsymbol{J}^{\mathrm{T}}) = 0$$
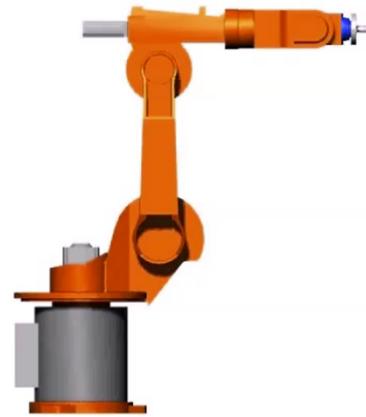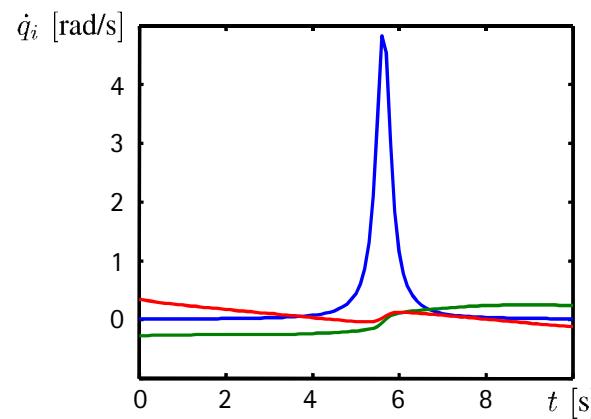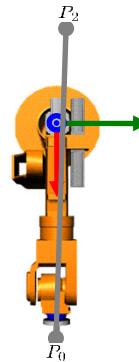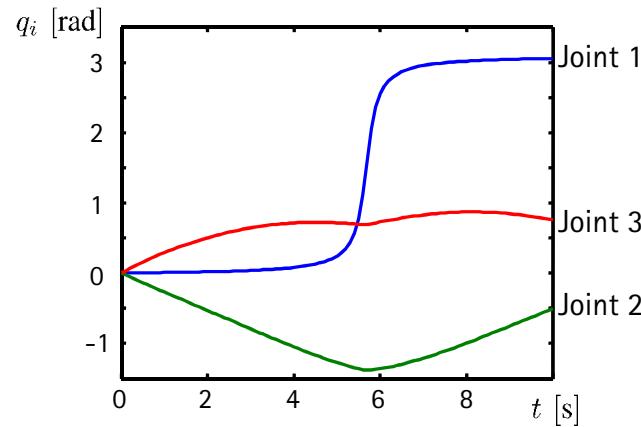
# Singularities

Two paths with approach to the vertical axis of the robot

# Singularities
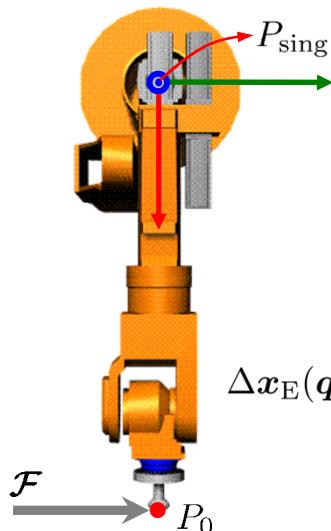
# Singularities

Static force analysis at the points $P_0$ and $P_{\text{sing}}$ (singularity)
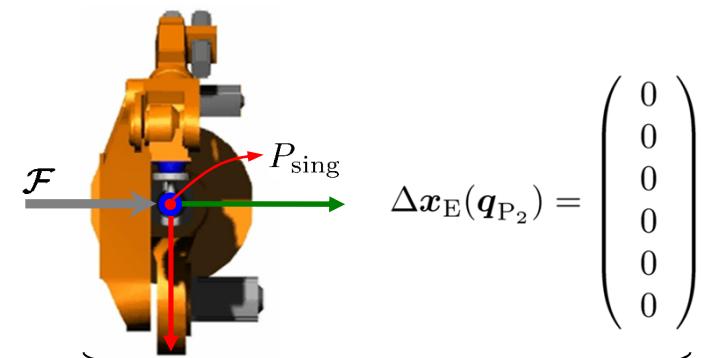
Example KUKA KR 15:

$$\boldsymbol{\mathcal{F}} = (0, F_y, 0, 0, 0, 0)^{\text{T}}, \quad \boldsymbol{K} = \text{diag}(k_1, k_2, \ldots, k_6)$$

$$\Delta \boldsymbol{x}_{\text{E}} = \underbrace{\boldsymbol{J}(\boldsymbol{q}) \, \boldsymbol{K}^{-1} \, \boldsymbol{J}^{\text{T}}(\boldsymbol{q})}_{\boldsymbol{C}} \, \boldsymbol{\mathcal{F}}$$



$$\Delta \boldsymbol{x}_{\text{E}}(\boldsymbol{q}_{\text{P}_1}) = \begin{pmatrix} 0 \\ 128{,}82 \\ 0 \\ 0 \\ 0 \\ 0{,}11 \end{pmatrix}$$

$$\Delta \boldsymbol{x}_{\text{E}}(\boldsymbol{q}_{\text{P}_2}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
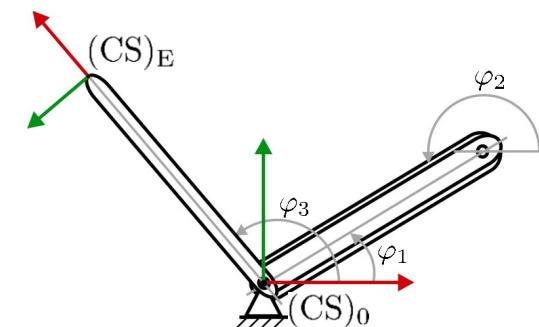
special case here, generally applies:

$$\Delta \boldsymbol{x}_{\text{E}}^* = \begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \boldsymbol{0}^{\text{T}} & \end{pmatrix} \boldsymbol{\mathcal{F}}$$

# Singularities

**Effects**
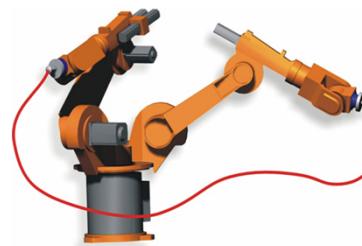
- Loss of degrees of freedom at the end effector
- All drives are available for the remaining degrees of freedom (→ redundant system)
- Despite the low end effector velocity, very high joint velocities occur near the singularity
- High stiffness in individual spatial directions, change in force/moment action on the end effector has practically no effect on actuator forces/moments

➡ Singular configurations should be considered and avoided when planning a robot movement.
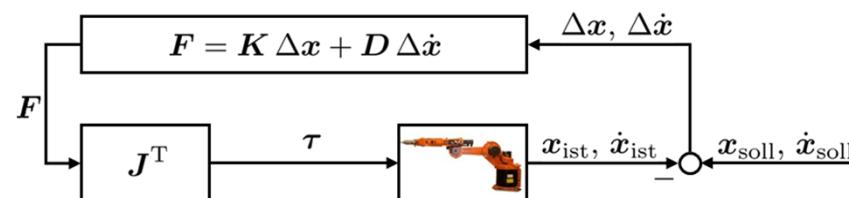
# Singularities

**Measures to avoid singularities**

- Trajectory optimization (later lecture)



- Force control or impedance control (later lecture)

$$F = K\,\Delta x + D\,\Delta \dot{x}$$

with $\Delta x,\ \Delta \dot{x}$

$F$, $J^{\mathrm{T}}$, $\tau$, $x_{\mathrm{ist}},\ \dot{x}_{\mathrm{ist}}$, $x_{\mathrm{soll}},\ \dot{x}_{\mathrm{soll}}$

- Kinematic redundancy (following lecture section)

# Questions for Self-Monitoring

**Jacobian matrix – velocities and forces**
How do I relate end-effector and joint-angular velocities and how do I deduce important properties of my robot?

1. How are the analytical and geometric Jacobian matrices of a general serial kinematic robot calculated? What is the main difference between the two Jacobian matrices?

2. Name two performance characteristics that can be calculated based on the Jacobian matrix!

3. Explain the term singularity and qualitatively draw two singular configurations of a planar RR robot with two rotary drives!