Institut für Mechatronische Systeme
imes Leibniz Universität Hannover

Leibniz Universität Hannover

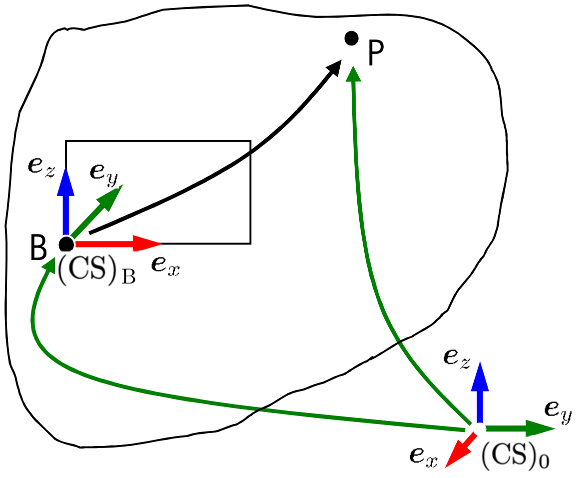
Robotics I

03. Forward and Inverse Kinematics (Introduction)

Review

How do I know where the robot's end-effector is located?

- Joint- and Task Space
- Coordinate Systems
- Rotations I
  - Rotation Matrices
  - Elementary Rotations
  - Composition of Rotations
- Describing Spatial Position and Orientation
- Homogeneous Transformation
- Rotations II
  - Rotation around an Axis
  - Quaternions



$$_{(0)}\boldsymbol{x}_\mathrm{P} = {}^0\boldsymbol{T}_\mathrm{B}\,{}_{(\mathrm{B})}\boldsymbol{x}_\mathrm{P}$$

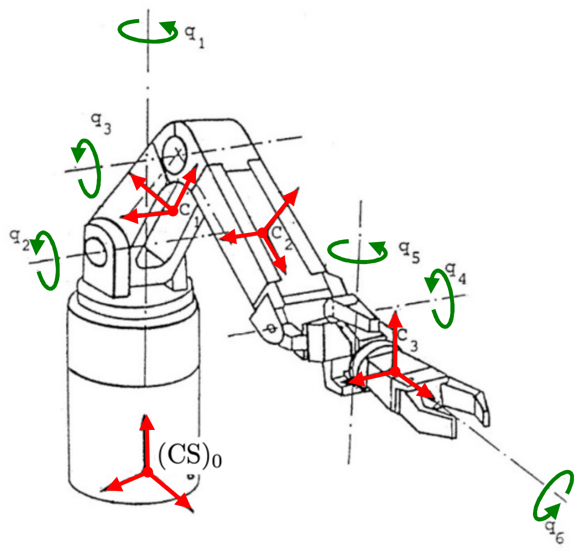
$${}^0\boldsymbol{T}_\mathrm{B} = \left(\begin{array}{ccc|c} & {}^0\boldsymbol{R}_\mathrm{B} & & {}_{(0)}\boldsymbol{r}_\mathrm{B} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Total Content

Control

- Sensors
- Advanced Control Methods
- Multi-Axis Control
- Single-Axis Control
- Dynamics: Newton-Euler and Lagrange
- Path Planning
- Kinematically Redundant Robots
- Jacobian Matrix – Velocities and Forces
- Forward and Inverse Kinematics
- Coordinate Transformations
- Introduction

Kinematics and Dynamics

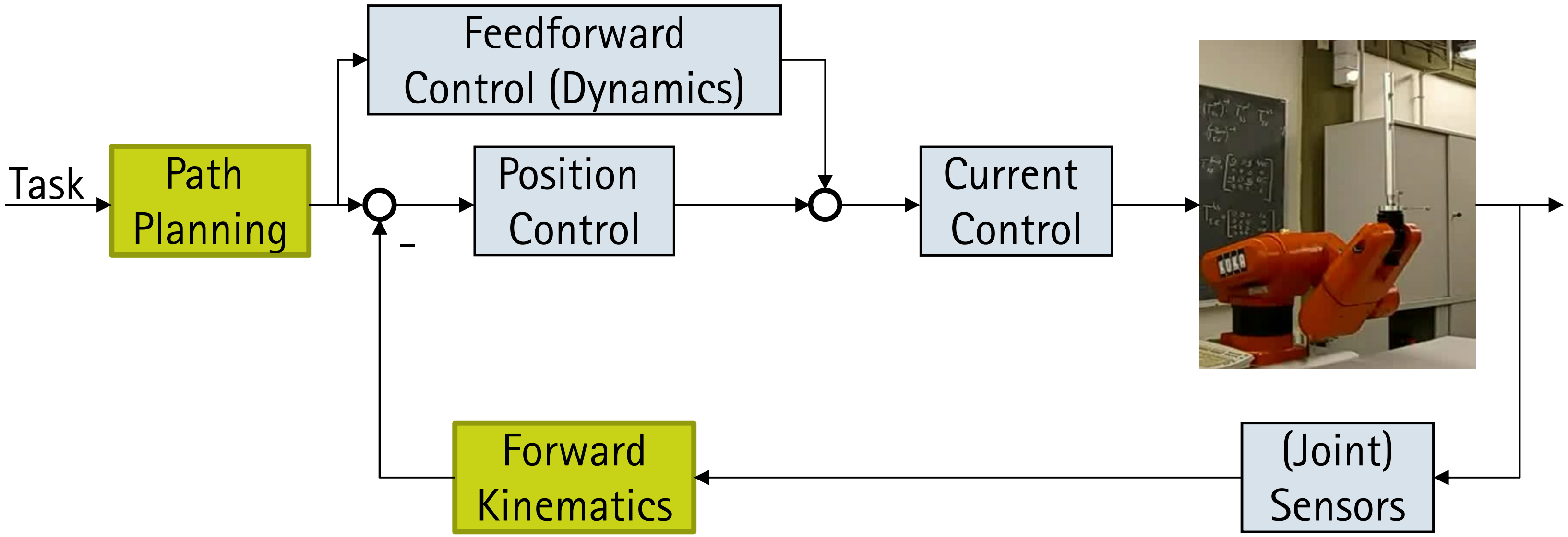


Pfeiffer / Reithmeier - Roboterdynamik

Total Content

Connection of essential lecture topics in the form of a control loop

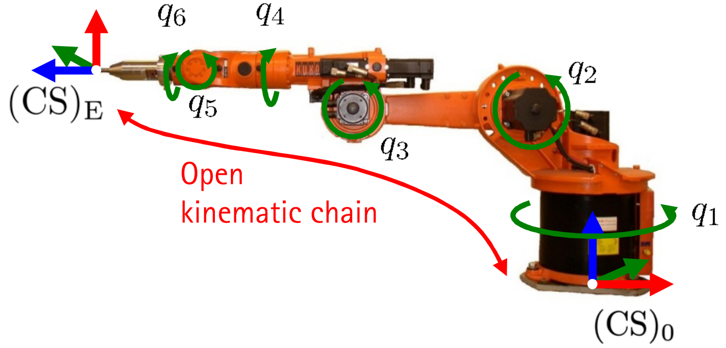
Forward and inverse kinematics describe the relationship between joint coordinates and end-effector pose and form the basis for differential kinematic modeling.



Forward and Inverse Kinematics

How do I calculate the end-effector pose from joint angles, and how do I determine the joint angles from the end-effector pose?

- Kinematics of Serial Robots
- Forward Kinematics
- Denavit-Hartenberg Notation
- Inverse Kinematics



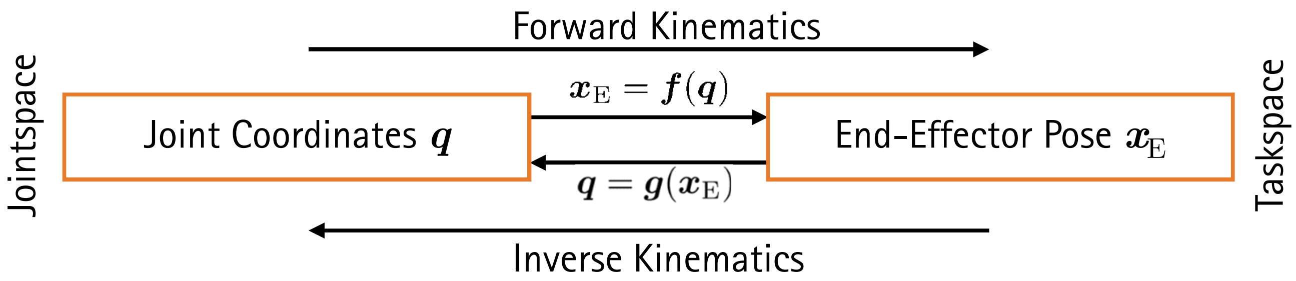
Literature

W. Khalil & E. Dombre – Modeling, Identification & Control of Robots, S. 35–83
F. Pfeiffer & E. Reithmeier – Roboterdynamik, S. 27–59
W. Khalil & J. F. Kleinfinger – A new Geometric Notation for Open and Closed-Loop Robots
J. J. Craig – Introduction to Robotics Mechanics and Control, S. 68–141

Kinematics of Serial Robots

Definition

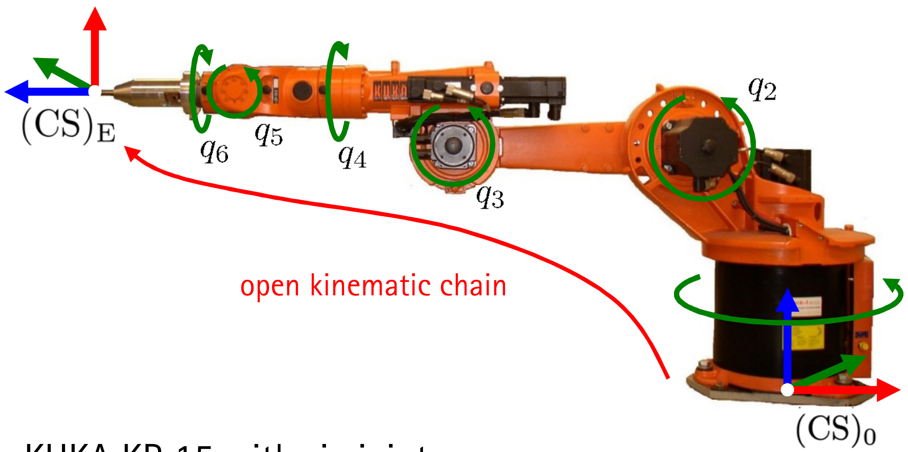
Kinematics of a robot describes the transformation rule that defines the relationship between the joint angles $\boldsymbol{q}$ and the end-effector position $\boldsymbol{x}_\mathrm{E}$. It is determined by the robot's geometry and includes the temporal evolution of movement (velocities and accelerations).



Note: The relationship between the velocities ($\dot{\boldsymbol{q}} \Leftrightarrow \dot{\boldsymbol{x}}_\mathrm{E}$) and accelerations ($\ddot{\boldsymbol{q}} \Leftrightarrow \ddot{\boldsymbol{x}}_\mathrm{E}$) is the content of the next lecture unit.

Forward Kinematics

From each joint position $\boldsymbol{q}$, there results exactly one specific position of the end effector $\boldsymbol{x}_\mathrm{E} = \boldsymbol{f}(\boldsymbol{q})$



KUKA KR 15 with six joints

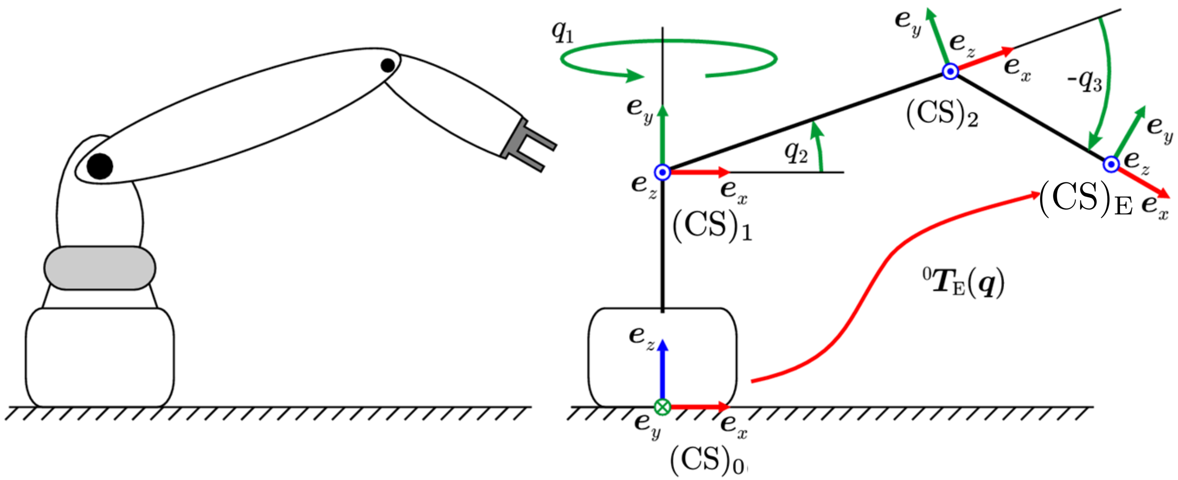
$\boldsymbol{q} = (q_1,\, q_2,\, \ldots,\, q_6)^\mathrm{T}$

$\boldsymbol{x}_\mathrm{E} = \left({}_{(0)}x_\mathrm{E},\, {}_{(0)}y_\mathrm{E},\, {}_{(0)}z_\mathrm{E}, \phi_\mathrm{E},\, \psi_\mathrm{E},\, \theta_\mathrm{E}\right)^\mathrm{T}$

Forward Kinematics

Kinematic Chain

To describe the kinematic chain of a serial robot, link-fixed coordinate systems $(\mathrm{CS})_i$ are introduced. Link i is represented by $(\mathrm{CS})_i$, the end effector by $(\mathrm{CS})_\mathrm{E}$ and the robot base by $(\mathrm{CS})_0$.



The goal is to establish the following model:

$$^0\boldsymbol{T}_\mathrm{E}(\boldsymbol{q}) = {}^0\boldsymbol{T}_1(q_1)\,{}^1\boldsymbol{T}_2(q_2)\,\cdots\,{}^{n-1}\boldsymbol{T}_\mathrm{E}(q_n)$$

Forward Kinematics

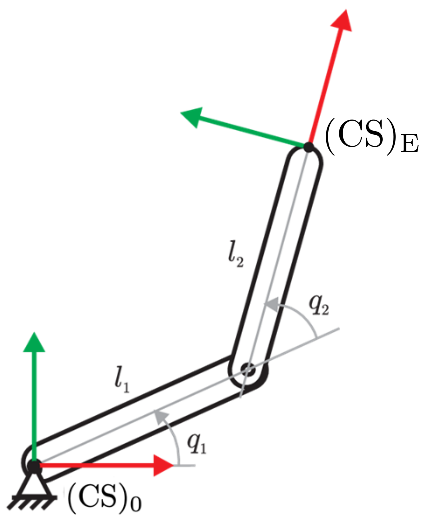
Board notes

Solution of the forward kinematics is quickly and intuitively possible for simple (for example, planar) systems

Example:

$$_{(0)}x_{\mathrm{E}} = \dots$$

$$_{(0)}y_{\mathrm{E}} = \dots$$



However, this approach becomes very complex for robot structures of higher order.

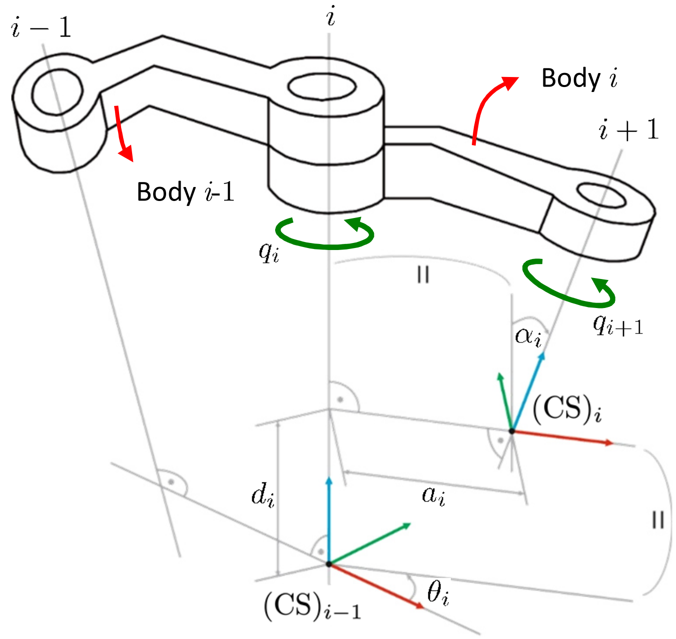
Robotics I

03. Forward and Inverse Kinematics (Denavit-Hartenberg Notation)

Denavit-Hartenberg Notation (referring to Paul)

A special form of homogeneous transformation ${}^{i\text{-}1}\boldsymbol{T}_i(\boldsymbol{q}_i)$ to describe robot kinematics. The position of the fixed coordinate systems is defined as follows:

- The origin of $(\mathrm{CS})_i$ lies at the intersection of the common normals of joints i and $i{+}1$ with the axis of joint $i{+}1$
- Orientation of $(\mathrm{CS})_i$:
  - $\boldsymbol{z}_i$ along the axis of joint $i{+}1$
  - $\boldsymbol{x}_i$ along the common normal between $\boldsymbol{z}_i$ and $\boldsymbol{z}_{i\text{-}1}$, directed to $(\mathrm{CS})_i$
  - $\boldsymbol{y}_i$ defined by the right-hand rule



Caution

The Denavit-Hartenberg notation is ambiguous.

Denavit-Hartenberg Notation

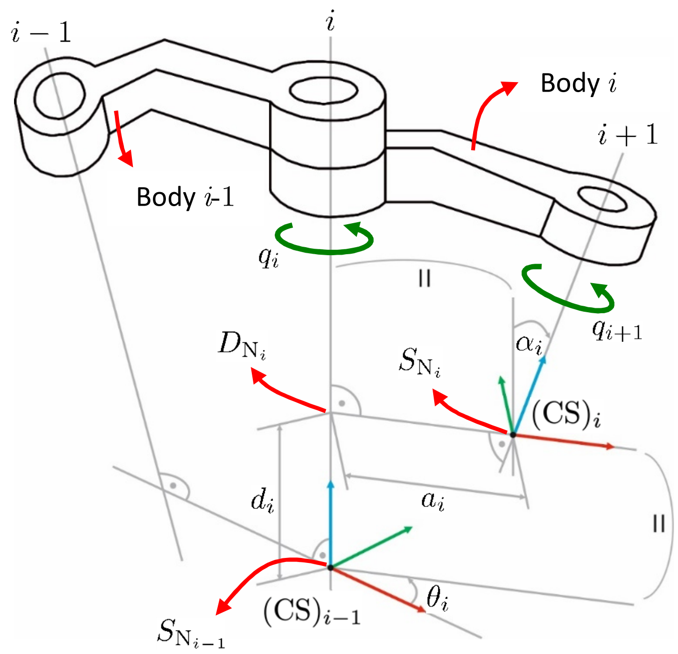
Denavit-Hartenberg Parameters
The position of two consecutive coordinate systems is defined by four Denavit-Hartenberg parameters:

θ_i : Rotation angle of link i around $\boldsymbol{z}_{i\text{-}1}$ with the angle between $\boldsymbol{x}_{i\text{-}1}$ and $\boldsymbol{x}_i$
(for revolute joints $\theta_i = q_i$)

d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$
(for prismatic joints $d_i = q_i$)

a_i : Distance along the common normal between skewed axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$, positive in direction of $\boldsymbol{x}_i$

α_i : Rotation angle around $\boldsymbol{x}_i$ with the angle between $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$



Denavit-Hartenberg Matrices

The Denavit-Hartenberg parameters of link i results in Denavit-Hartenberg Matrix $\boldsymbol{A}_i$, composed of elementary rotations and translations:

$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$

$$= \underbrace{\begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathrm{r}_z}(\theta_i)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)}$$

$$= \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Denavit-Hartenberg Notation

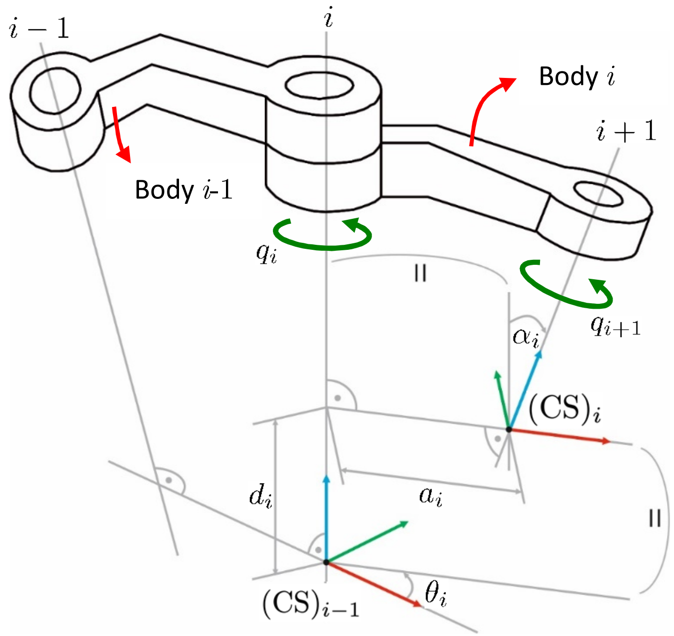
$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$

θ_i : Rotation angle of link i around $\boldsymbol{z}_{i\text{-}1}$ with the angle between $\boldsymbol{x}_{i\text{-}1}$ and $\boldsymbol{x}_i$ (for revolute joints $\theta_i = q_i$)

d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$ (for prismatic joints $d_i = q_i$)

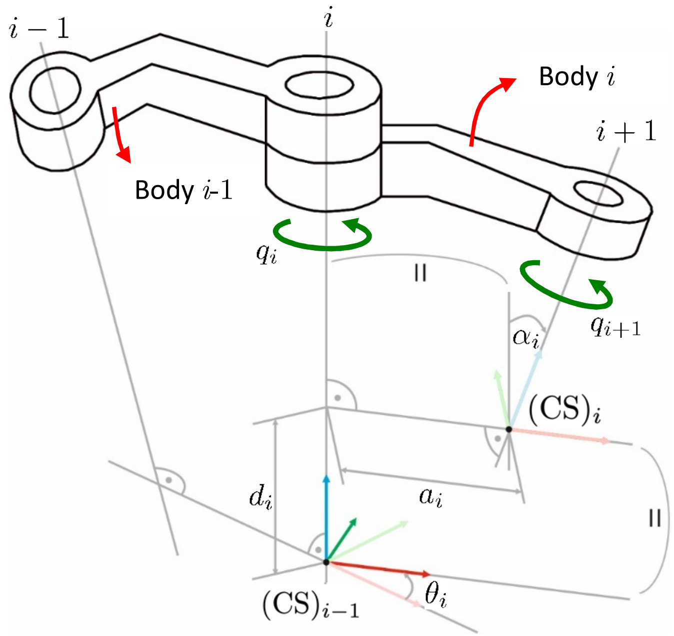
a_i : Distance along the common normal between skewed axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$, positive in direction of $\boldsymbol{x}_i$

α_i : Rotation angle around $\boldsymbol{x}_i$ with the angle between $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$



Denavit-Hartenberg Notation

$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$



θ_i : Rotation angle of link i around $\boldsymbol{z}_{i\text{-}1}$ with the angle between $\boldsymbol{x}_{i\text{-}1}$ and $\boldsymbol{x}_i$ (for revolute joints $\theta_i = q_i$)

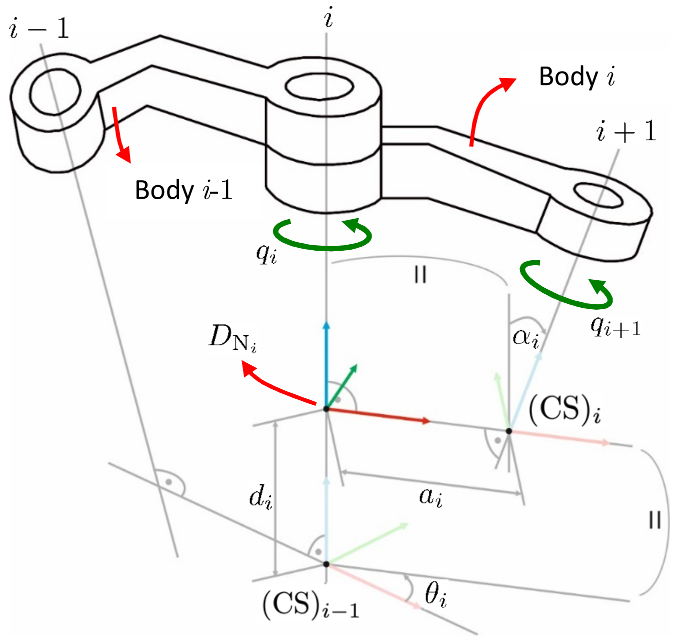
d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$ (for prismatic joints $d_i = q_i$)

a_i : Distance along the common normal between skewed axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$, positive in direction of $\boldsymbol{x}_i$

α_i : Rotation angle around $\boldsymbol{x}_i$ with the angle between $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$

Denavit-Hartenberg Notation

$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$



- θ_i : Rotation angle of link i around $z_{i\text{-}1}$ with the angle between $x_{i\text{-}1}$ and x_i (for revolute joints $\theta_i = q_i$)
- d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$ (for prismatic joints $d_i = q_i$)
- a_i : Distance along the common normal between skewed axes $z_{i\text{-}1}$ and z_i, positive in direction of x_i
- α_i : Rotation angle around x_i with the angle between $z_{i\text{-}1}$ and z_i

Denavit-Hartenberg Notation

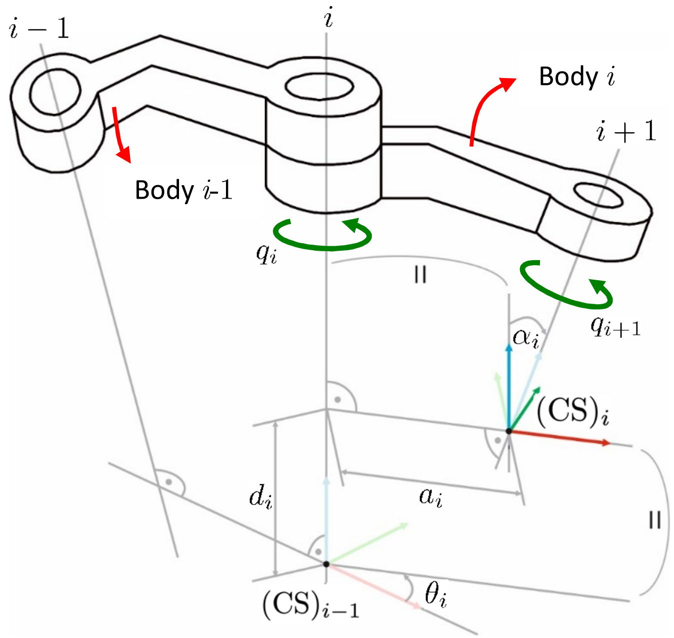
$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$

θ_i : Rotation angle of link i around $\boldsymbol{z}_{i\text{-}1}$ with the angle between $\boldsymbol{x}_{i\text{-}1}$ and $\boldsymbol{x}_i$ (for revolute joints $\theta_i = q_i$)

d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$ (for prismatic joints $d_i = q_i$)

a_i : Distance along the common normal between skewed axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$, positive in direction of $\boldsymbol{x}_i$

α_i : Rotation angle around $\boldsymbol{x}_i$ with the angle between $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$



Denavit-Hartenberg Notation

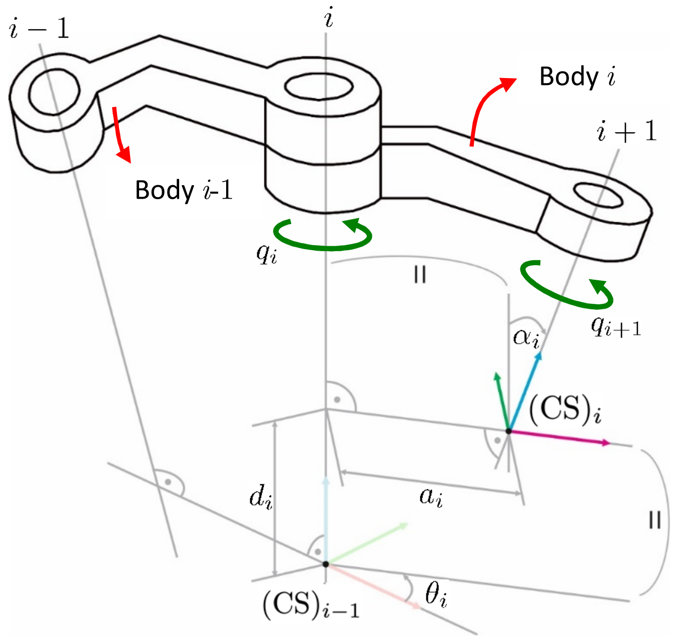
$$\boldsymbol{A}_i(q_i) = {}^{i-1}\boldsymbol{T}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$

θ_i : Rotation angle of link i around $\boldsymbol{z}_{i\text{-}1}$ with the angle between $\boldsymbol{x}_{i\text{-}1}$ and $\boldsymbol{x}_i$ (for revolute joints $\theta_i = q_i$)

d_i : Distance between two normal intersections of $S_{\mathrm{N}(i\text{-}1)}$ and $D_{\mathrm{N}i}$, positive in direction of $\boldsymbol{z}_{i\text{-}1}$ (for prismatic joints $d_i = q_i$)

a_i : Distance along the common normal between skewed axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$, positive in direction of $\boldsymbol{x}_i$

α_i : Rotation angle around $\boldsymbol{x}_i$ with the angle between $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$

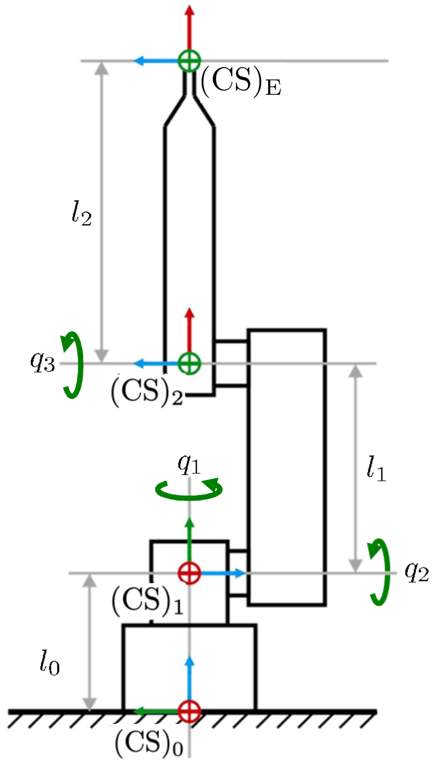


Denavit-Hartenberg Notation

Special Cases

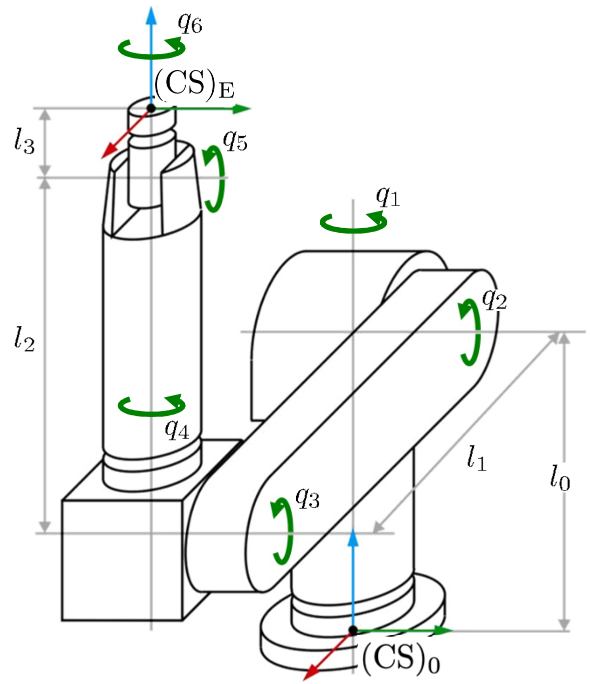
The following special cases lead to simplifications but also to ambiguities:

- Parallel joint axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$ (∞-many normals):
  Origin of $(\mathrm{CS})_i$ freely selectable, reasonable choice: $d_i = 0$
  (see q_2/q_3)
- Joint axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$ intersect:
  (Axes $\boldsymbol{z}_{i\text{-}1}$ and $\boldsymbol{z}_i$ are not skewed):
  Origin of $(\mathrm{CS})_i$ at the intersection, $a_i = 0$ (see q_1/q_2)
- Position of $(\mathrm{CS})_0$:
  $\boldsymbol{z}_0$ is determined by the first joint axis,
  Origin of $(\mathrm{CS})_0$ not unique, since no joint i-1 exists,
  Origin freely selectable, reasonable choice: at the base
- Position of $(\mathrm{CS})_n = (\mathrm{CS})_{\mathrm{E}}$:
  $\boldsymbol{z}_{\mathrm{E}}$ is not unique, since no joint $n+1$ exists,
  common choice: $\boldsymbol{z}_{\mathrm{E}}$ points out of the end effector, not always possible



Procedure for a Given Structure (Assumption: Robot in Defined Zero Position)

For n joint axes, n generalized coordinates (q_1,... , q_n) result, along with $n+1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.



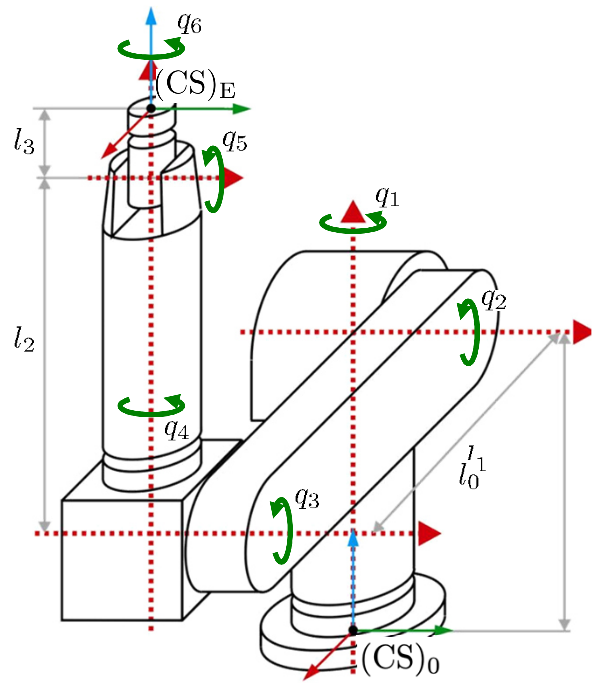
1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i+1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i+1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.

Denavit-Hartenberg Notation

Procedure for a Given Structure

For n joint axes, n generalised coordinates (q_1,… , q_n) result, along with $n+1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.

1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i+1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i+1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.

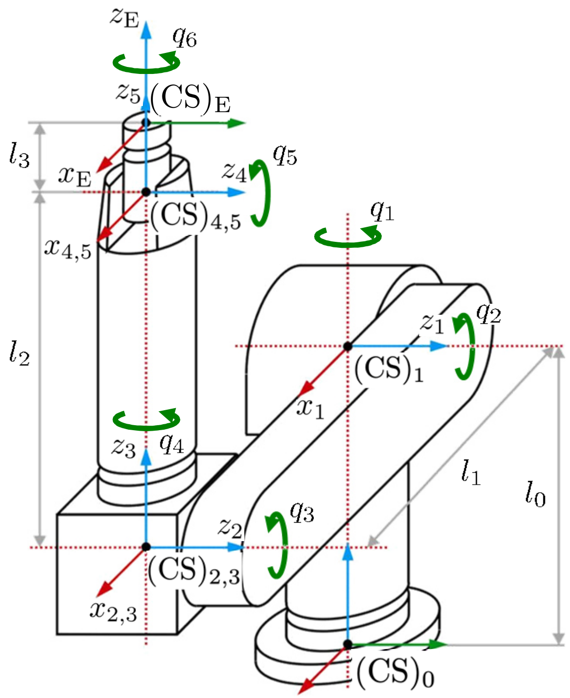


Denavit-Hartenberg Notation

Procedure for a Given Structure

For n joint axes, n generalised coordinates (q_1,… , q_n) result, along with $n{+}1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.

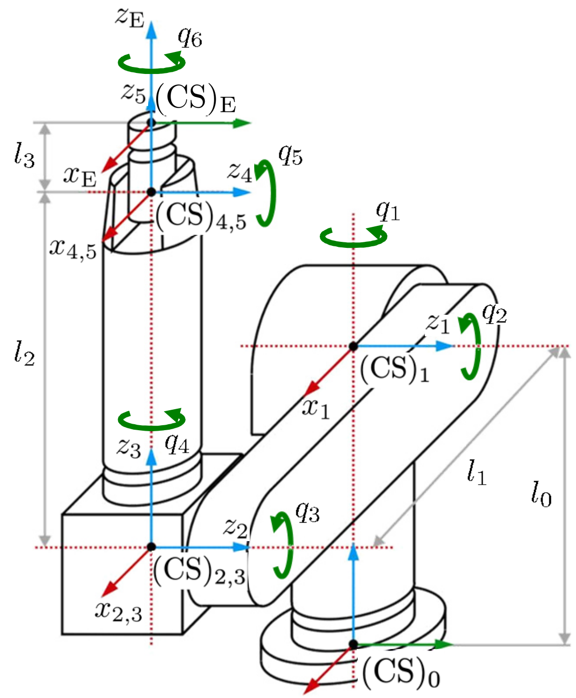
1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i{+}1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i{+}1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.



Procedure for a Given Structure

For n joint axes, n generalised coordinates (q_1,... , q_n) result, along with $n+1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.

1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i+1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i+1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.

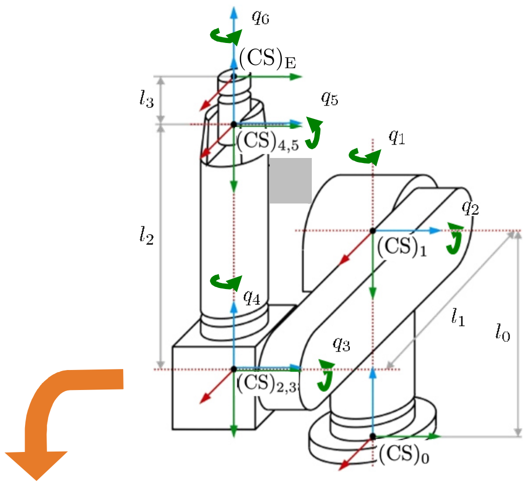


Denavit-Hartenberg Notation

Procedure for a Given Structure

For n joint axes, n generalised coordinates (q_1,... , q_n) result, along with $n+1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.

1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i+1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i+1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.



i	θ_i	d_i	a_i	α_i
1	q_1	l_0	0	$-\pi/2$
2	q_2	0	l_1	0
3	q_3	0	0	$\pi/2$
4	q_4	l_2	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	l_3	0	0

Denavit-Hartenberg Notation

Procedure for a Given Structure

For n joint axes, n generalised coordinates (q_1,... , q_n) result, along with $n+1$ coordinate systems and n "sets" of Denavit-Hartenberg parameters.

1. Identify rotational and prismatic axes and numerate them from 1 to n.
2. Position $(\mathrm{CS})_i$ on the rotational or prismatic axis $i+1$:
   - $\boldsymbol{z}_i$ points to direction of axis $i+1$
   - Determine the normal between the axes and align $\boldsymbol{x}_i$ in the direction of the normal.
   - "Define the coordinate axis $\boldsymbol{y}_i$ according to the right-hand rule.
3. Read off the Denavit-Hartenberg parameters.
4. Determine the Denavit-Hartenberg matrices and compute the overall transformation.

i	θ_i	d_i	a_i	α_i
1	q_1	l_0	0	$-\pi/2$
2	q_2	0	l_1	0
3	q_3	0	0	$\pi/2$
4	q_4	l_2	0	$-\pi/2$
5	q_5	0	0	$\pi/2$
6	q_6	l_3	0	0

$$\boldsymbol{A}_i(q_i) = \boldsymbol{T}_{\mathrm{r}_z}(\theta_i)\,\boldsymbol{T}_{\mathrm{t}}(0,0,d_i)\,\boldsymbol{T}_{\mathrm{t}}(a_i,0,0)\,\boldsymbol{T}_{\mathrm{r}_x}(\alpha_i)$$

$$\Rightarrow\ {}^0\boldsymbol{T}_{\mathrm{E}}(\boldsymbol{q}) = {}^0\boldsymbol{T}_1(q_1)\cdots{}^5\boldsymbol{T}_{\mathrm{E}}(q_n) = \boldsymbol{A}_1(q_1)\cdots\boldsymbol{A}_6(q_n)$$

Denavit-Hartenberg Notation

Forward Kinematics using Denavit-Hartenberg
The pose of the end effector $\boldsymbol{x}_\mathrm{E}$ as a function of the joint coordinates $\boldsymbol{q}$ from a kinematic chain based on Denavit-Hartenberg matrices:

$$^0\boldsymbol{T}_\mathrm{E}(\boldsymbol{q}) = {}^0\boldsymbol{T}_1(q_1)\, {}^1\boldsymbol{T}_2(q_2) \cdots {}^{n-1}\boldsymbol{T}_\mathrm{E}(q_n) = \boldsymbol{A}_1(q_1)\, \boldsymbol{A}_2(q_2) \cdots \boldsymbol{A}_\mathrm{n}(q_n)$$

Position of the end effector:

$$_{(0)}x_\mathrm{E} = {}^0t_{\mathrm{E},(1,4)},\;\; _{(0)}y_\mathrm{E} = {}^0t_{\mathrm{E},(2,4)},\;\; _{(0)}z_\mathrm{E} = {}^0t_{\mathrm{E},(3,4)}$$

$^0t_{\mathrm{E},(i,j)}$: i-th element of j-th column of $^0\boldsymbol{T}_\mathrm{E}$

Denavit-Hartenberg Notation

Forward Kinematics using Denavit-Hartenberg

The pose of the end effector $\boldsymbol{x}_{\mathrm{E}}$ as a function of the joint coordinates $\boldsymbol{q}$ from a kinematic chain based on Denavit-Hartenberg matrices:

$$^{0}\boldsymbol{T}_{\mathrm{E}}(\boldsymbol{q}) = {}^{0}\boldsymbol{T}_{1}(q_1)\,{}^{1}\boldsymbol{T}_{2}(q_2)\,\cdots\,{}^{n-1}\boldsymbol{T}_{\mathrm{E}}(q_n) = \boldsymbol{A}_1(q_1)\,\boldsymbol{A}_2(q_2)\,\cdots\,\boldsymbol{A}_{\mathrm{n}}(q_n)$$

Orientation of the end effector (dependent on the selected composite rotations)

Example Kardan angles:

$$\boldsymbol{R}_{\mathrm{KARD}} = \boldsymbol{R}_x(\alpha)\,\boldsymbol{R}_y(\beta)\,\boldsymbol{R}_z(\gamma) = \begin{pmatrix} \mathrm{c}_\beta\,\mathrm{c}_\gamma & -\mathrm{c}_\beta\,\mathrm{s}_\gamma & \mathrm{s}_\beta \\ \mathrm{c}_\alpha\,\mathrm{s}_\gamma + \mathrm{s}_\alpha\,\mathrm{s}_\beta\,\mathrm{c}_\gamma & \mathrm{c}_\alpha\,\mathrm{c}_\gamma - \mathrm{s}_\alpha\,\mathrm{s}_\beta\,\mathrm{s}_\gamma & -\mathrm{s}_\alpha\,\mathrm{c}_\beta \\ \mathrm{s}_\alpha\,\mathrm{s}_\gamma - \mathrm{c}_\alpha\,\mathrm{s}_\beta\,\mathrm{c}_\gamma & \mathrm{s}_\alpha\,\mathrm{c}_\gamma + \mathrm{c}_\alpha\,\mathrm{s}_\beta\,\mathrm{s}_\gamma & \mathrm{c}_\alpha\,\mathrm{c}_\beta \end{pmatrix}$$

$$\mathrm{c}_i = \cos(i),\quad \mathrm{s}_i = \sin(i),\quad \mathrm{c}_{ij} = \cos(i+j),\quad \mathrm{s}_{ij} = \sin(i+j)$$

$$^{0}\boldsymbol{T}_{\mathrm{E}} = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{^{0}\boldsymbol{R}_{\mathrm{E}}} & {}_{(0)}\boldsymbol{r}_{\mathrm{E}} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{aligned}
\phi_{\mathrm{E}} &= \alpha = \operatorname{arctan2}\left(-{}^{0}t_{\mathrm{E},(2,3)},\ {}^{0}t_{\mathrm{E},(3,3)}\right), \\
\psi_{\mathrm{E}} &= \beta = \operatorname{arctan2}\left({}^{0}t_{\mathrm{E},(1,3)},\ {}^{0}t_{\mathrm{E},(1,1)}\,c_\gamma - {}^{0}t_{\mathrm{E},(1,2)}\,s_\gamma\right), \\
\theta_{\mathrm{E}} &= \gamma = \operatorname{arctan2}\left(-{}^{0}t_{\mathrm{E},(1,2)},\ {}^{0}t_{\mathrm{E},(1,1)}\right)
\end{aligned}$$

Robotics I

03. Forward and Inverse Kinematics (Inverse Kinematics)

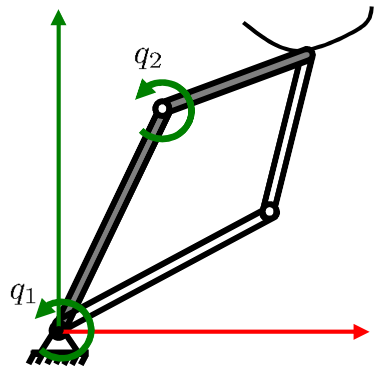
Inverse Kinematics

The joint angles $\boldsymbol{q}$ for a given pose of the end effector $\boldsymbol{x}_\mathrm{E}$ are to be found.

Jointspace | Joint coordinates $\boldsymbol{q}$ $\xleftarrow{\boldsymbol{q}=\boldsymbol{g}(\boldsymbol{x}_\mathrm{E})}$ End-effector coordinates $\boldsymbol{x}_\mathrm{E}$ | Task space

$\longleftarrow$ inverse Kinematics

Inverse kinematics for "regular" robots $(\mathrm{dim}(\boldsymbol{q})=\mathrm{dim}(\boldsymbol{x}_\mathrm{E})$, with $\mathrm{dim}(\boldsymbol{x}_\mathrm{E})$ being the independent degrees of freedom of the end effector):
Solvable except for symmetries,
except for singular configurations (to be discussed later)



Inverse Kinematics

Analytical Calculations

Given (by forward kinematics):

$$ {}^0\boldsymbol{T}_\mathrm{E} = \left(\begin{array}{ccc|c} \mathrm{c}_{12} & -\mathrm{s}_{12} & 0 & l_1\,\mathrm{c}_1 + l_2\,\mathrm{c}_{12} \\ \mathrm{s}_{12} & \mathrm{c}_{12} & 0 & l_1\,\mathrm{s}_1 + l_2\,\mathrm{s}_{12} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \stackrel{!}{=} \left(\begin{array}{ccc|c} \mathrm{c}_{\theta_\mathrm{E}} & -\mathrm{s}_{\theta_\mathrm{E}} & 0 & {}_{(0)}x_\mathrm{E} \\ \mathrm{s}_{\theta_\mathrm{E}} & \mathrm{c}_{\theta_\mathrm{E}} & 0 & {}_{(0)}y_\mathrm{E} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) $$

➡ four non-linear equations for two unknown quantities

$$ {}_{(0)}x_\mathrm{E} = l_1\,\mathrm{c}_1 + l_2\,\mathrm{c}_{12}, \quad {}_{(0)}y_\mathrm{E} = l_1\,\mathrm{s}_1 + l_2\,\mathrm{s}_{12} $$

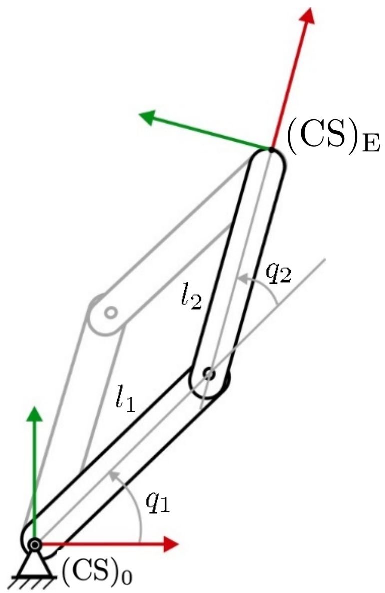
$$ \mathrm{c}_{\theta_\mathrm{E}} = \mathrm{c}_{12}, \quad \mathrm{s}_{\theta_\mathrm{E}} = \mathrm{s}_{12} $$

➡ θ_E given by ${}_{(0)}\boldsymbol{x}_\mathrm{E}$ and ${}_{(0)}\boldsymbol{y}_\mathrm{E}$

Elbow up (+) and down (-)

$$ \Rightarrow \; q_2 = \operatorname{arctan2}\left(\pm\sqrt{1 - \left(\frac{{}_{(0)}x_\mathrm{E}^2 + {}_{(0)}y_\mathrm{E}^2 - l_1^2 - l_2^2}{2\,l_1\,l_2}\right)^2},\; \frac{{}_{(0)}x_\mathrm{E}^2 + {}_{(0)}y_\mathrm{E}^2 - l_1^2 - l_2^2}{2\,l_1\,l_2}\right) $$

$$ \Rightarrow \; q_1 = \operatorname{arctan2}\left({}_{(0)}y_\mathrm{E},\; {}_{(0)}x_\mathrm{E}\right) - \operatorname{arctan2}\left(l_1 + l_2\,\mathrm{c}_2,\; l_2\,\mathrm{s}_2\right) $$



Solution approach in J. J. Craig – Introduction to Robotics Mechanics and Control, S. 122–125

Inverse Kinematics

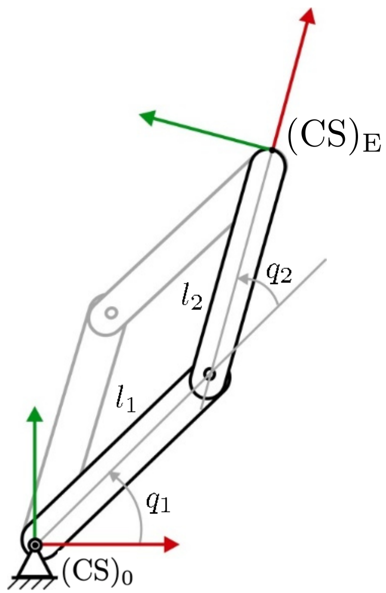
Analytical Calculations

➡ four non-linear equations for two unknowns

$$_{(0)}x_{\mathrm{E}} = l_1\,\mathrm{c}_1 + l_2\,\mathrm{c}_{12}, \quad _{(0)}y_{\mathrm{E}} = l_1\,\mathrm{s}_1 + l_2\,\mathrm{s}_{12}, \quad (\mathrm{c}_{\theta_{\mathrm{E}}} = \mathrm{c}_{12}, \quad \mathrm{s}_{\theta_{\mathrm{E}}} = \mathrm{s}_{12})$$

Squaring and adding

$$\begin{aligned}
_{(0)}x_{\mathrm{E}}^2 +_{(0)} y_{\mathrm{E}}^2 =& l_1^2\, \cos^2(q_1) + 2\,l_1 l_2\, \cos(q_1)\cos(q_1+q_2) + l_2^2\, \cos^2(q_1+q_2) \\
&+ l_1^2\, \sin^2(q_1) + 2\,l_1 l_2\, \sin(q_1)\sin(q_1+q_2) + l_2^2\, \sin^2(q_1+q_2) \\
=& l_1^2 + l_2^2 + 2\,l_1 l_2 \Big(\cos(q_1)\cos(q_1+q_2) + \sin(q_1)\sin(q_1+q_2)\Big) \\
=& l_1^2 + l_2^2 + 2\,l_1 l_2\, \cos(q_2) \\
\Rightarrow q_2 =& \pm\arccos \frac{_{(0)}x_{\mathrm{E}}^2 +_{(0)} y_{\mathrm{E}}^2 - l_1^2 - l_2^2}{2\,l_1 l_2} = \pm\arccos A
\end{aligned}$$



Note

1. Sign corresponds to elbow configuration
2. If $A > 1$, $(_{(0)}x_{\mathrm{E}},\ _{(0)}y_{\mathrm{E}})$ outside of workspace

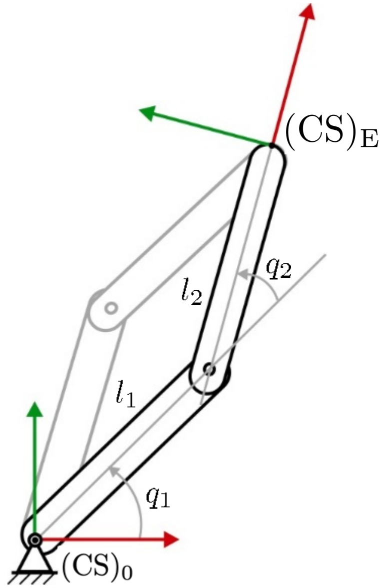
Solution approach in J. J. Craig – Introduction to Robotics Mechanics and Control, S. 122–125

Board notes

Analytical Calculations

- four non-linear equations for two unknowns

$$ {}_{(0)}x_{\mathrm{E}} = l_1\,\mathrm{c}_1 + l_2\,\mathrm{c}_{12}, \quad {}_{(0)}y_{\mathrm{E}} = l_1\,\mathrm{s}_1 + l_2\,\mathrm{s}_{12}, \quad (\mathrm{c}_{\theta_{\mathrm{E}}} = \mathrm{c}_{12}, \quad \mathrm{s}_{\theta_{\mathrm{E}}} = \mathrm{s}_{12}) $$



Application of the addition theorems:

$$ \begin{aligned} {}_{(0)}x_{\mathrm{E}} &= l_1\ \cos(q_1) + l_2\ \cos(q_1)\cos(q_2) - l_2\ \sin(q_1)\sin(q_2) \\ &= \big(l_1 + l_2\ \cos(q_2)\big)\cos(q_1) - l_2\ \sin(q_1)\sin(q_2) \\ {}_{(0)}y_{\mathrm{E}} &= l_1\ \sin(q_1) + l_2\ \sin(q_1)\cos(q_2) + l_2\ \cos(q_1)\sin(q_2) \\ &= \big(l_1 + l_2\ \cos(q_2)\big)\sin(q_1) + l_2\ \cos(q_1)\sin(q_2) \end{aligned} $$

Solution approach in A. Fuchs – Kinematische Kalibrierung einer planaren PKM, S. 5, 6

Geometric Determination

Determination of joint angles from a geometric perspective using the law of cosines (to determine q_2)

$$ {}_{(0)}x_{\mathrm{E}}^2 + {}_{(0)}y_{\mathrm{E}}^2 = l_1^2 + l_2^2 - 2\, l_1\, l_2\, \cos(\pi \pm q_2) $$

Elbow up (+) and down (-)

results in

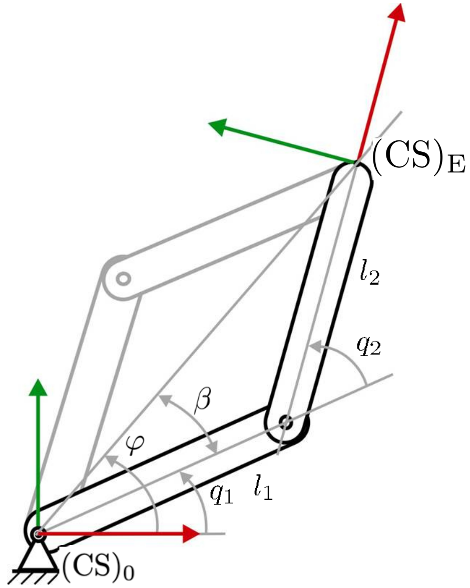
$$ q_2 = \pm \arccos\left(\frac{{}_{(0)}x_{\mathrm{E}}^2 + {}_{(0)}y_{\mathrm{E}}^2 - l_1^2 - l_2^2}{2\, l_1\, l_2} \right) $$

Then, similarly to q_2 (to determine q_1)

$$ \varphi = \operatorname{arctan2}\left({}_{(0)}y_{\mathrm{E}},\ {}_{(0)}x_{\mathrm{E}}\right) $$

$$ \beta = \arccos\left(\frac{{}_{(0)}x_{\mathrm{E}}^2 + {}_{(0)}y_{\mathrm{E}}^2 + l_1^2 - l_2^2}{2\, l_1 \sqrt{{}_{(0)}x_{\mathrm{E}}^2 + {}_{(0)}y_{\mathrm{E}}^2}} \right) $$

$$ q_1 = \varphi \pm \beta $$



Solution approach in J. J. Craig – Introduction to Robotics Mechanics and Control, S. 126, 127

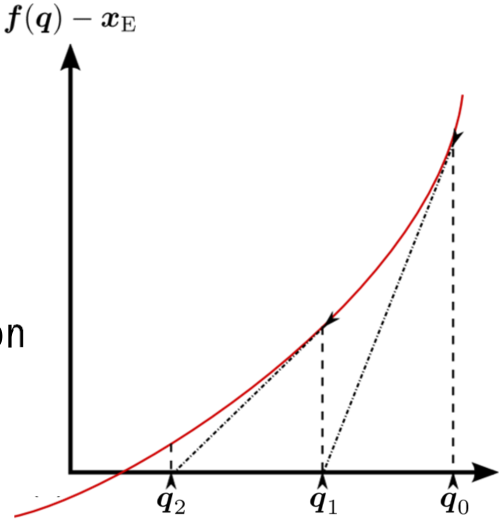
Numerical Calculation

After rearranging of $\boldsymbol{x}_{\mathrm{E}} = \boldsymbol{f}(\boldsymbol{q})$ you obtain

$$\boldsymbol{f}(\boldsymbol{q}) - \boldsymbol{x}_{\mathrm{E}} = \boldsymbol{0}$$

Solution iteratively using the Newton-Raphson method

- Linearization of the function at the starting point
- Zero crossing of the tangent as an improved approximation
- New starting point for the next iteration step
- Termination when the improvement is sufficiently small



Disadvantages

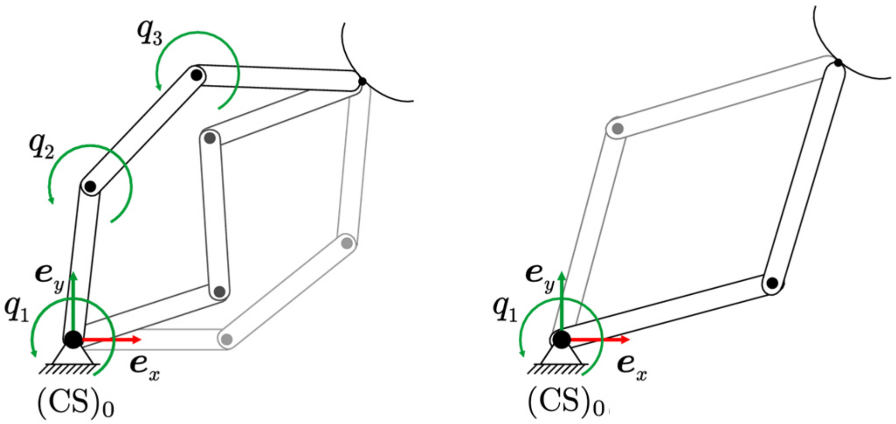
- Complex and slow algorithm
- Convergence not guaranteed

Kinematics for $\dim(\boldsymbol{q}) \neq \dim(\boldsymbol{x}_\mathrm{E})$

Solutions for forward and inverse kinematics for $\dim(\boldsymbol{q}) \neq \dim(\boldsymbol{x}_\mathrm{E})$:

- $\dim(\boldsymbol{q}) > \dim(\boldsymbol{x}_\mathrm{E})$: Kinematically redundant robots (left)
  $\boldsymbol{q} = \boldsymbol{g}(\boldsymbol{x}_\mathrm{E})$ results in infinitely many solutions (see later lectures)

- $\dim(\boldsymbol{q}) < \dim(\boldsymbol{x}_\mathrm{E})$: Underactuated Robots (right)
  $\boldsymbol{x}_\mathrm{E} = \boldsymbol{f}(\boldsymbol{q})$ generally not solvable because of missing joint information



Self-assessment Questions

Forward and Inverse Kinematics

How do I calculate the end-effector pose from joint angles, and how do I determine the joint angles from the end-effector pose?

1. Determine the forward kinematics of a planar RPR robot (two rotational and one linear actuator) as a function of the joint coordinates q_1, q_2 und q_3! Use the 'intuitive' approach and the Denavit-Hartenberg notation!
2. What special cases can occur when using the Denavit-Hartenberg notation, and how do you choose the placement of the coordinate systems in these cases?
3. Determine the inverse kinematics of a planar RPR robot (two rotational and one linear actuator) as a function of the end-effector position $_{(0)}x_{\mathrm{E}}$ and $_{(0)}y_{\mathrm{E}}$, as well as the end-effector orientation (around the z-axis) $_{(0)}\gamma_{\mathrm{E}}$!