## 21D – Calculating Fields using Vector Notation

**Equipment:** Paper, pencil, and brain.

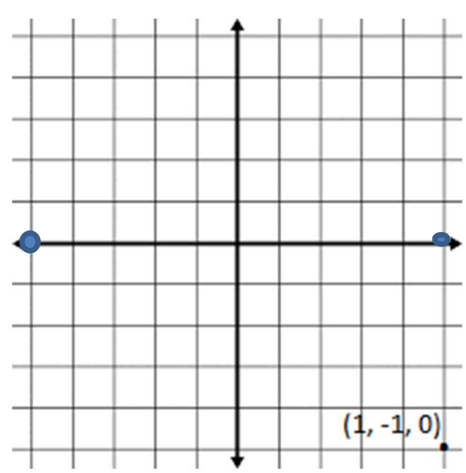
**Key Theoretical Ideas and Text Source:**
Reading: Lecture Notes 2. Young and Freedman Section 21.05.

- The electric field at the point ($x$, $y$, $z$) due to a collection of charges at positions ($x_i$, $y_i$, $z_i$) can be written as:
  $$\vec{E}(x,y,z) = \Sigma_i \frac{q_i}{4\pi\varepsilon_0} \frac{(x-x_i)\hat{\imath} + (y-y_i)\hat{\jmath} + (z-z_i)\hat{k}}{\left((x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right)^{3/2}} \qquad \text{(Eq. 21c)}$$
- For continuous distributions of charge, the $q_i$ above are represented by a differential d$q$', which is the charge in an infinitesimal volume d$V$', where d$q$' = $\rho$d$V$'. (The prime is added to indicate that the variable describes where the charge is.)
- For continuous distributions:
  $$\vec{E}(x,y,z) = \iiint \frac{\rho(x',y',z')}{4\pi\varepsilon_0} \frac{(x-x')\hat{\imath} + (y-y')\hat{\jmath} + (z-z')\hat{k}}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}} dx'dy'dz' \qquad \text{(Eq. 21d).}$$
  - $$\int \frac{\lambda(x')}{4\pi\varepsilon_0} \frac{(x-x')\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{3/2}} dx' \qquad \text{(Eq. 21f)}$$
  - The integral is over the primed coordinates, which denote where the charge is located. This can be a difficult calculation for the general case, but can be simplified for many special cases. Two examples:
    For a sheet of charge in the x-y (z = 0) plane with area charge density $\sigma(x',y')$:
    $$\vec{E}(x,y,z) = \iint \frac{\sigma(x',y')}{4\pi\varepsilon_0} \frac{(x-x')\hat{\imath} + (y-y')\hat{\jmath} + z\hat{k}}{\left((x-x')^2+(y-y')^2+z^2\right)^{3/2}} dx'dy' \qquad \text{(Eq. 21e)}$$
  - For a line of charge along the x-axis with charge linear charge density $\lambda(x')$ on the x axis:
    $$\vec{E}(x,y,z) = \int \frac{\lambda(x')}{4\pi\varepsilon_0} \frac{(x-x')\hat{\imath}+y\hat{\jmath}+z\hat{k}}{\left((x-x')^2+y^2+z^2\right)^{3/2}} dx' \qquad \text{(Eq. 21f)}$$

Questions 1a-g will help you to interpret the relations above and practice with Cartesian vector components. They all deal with the same physical situation, which you will sketch in question 1. In problems 1a-g, substitute numbers for variables when possible and simplify by eliminating terms that drop out or cancel (become zero).

1)

a. Sketch the charges $Q_1$ = +1nC at point $P_1$'=(+1, 0, 0) and $Q_2$ = +1nC at point $P_2$'=(-1, 0, 0) on the x-y plane to the right, along with sketches of the direction for the electric field vectors at points $(\pm1, \pm1, 0)$, $(0, \pm1, 0)$, $(0, 0, 0)$. (You can use the PhET simulation from the last module if you like.)

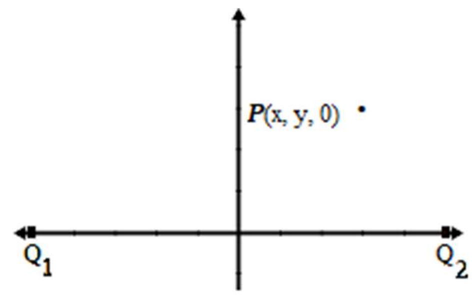


You now have an indication of what the resultant $\vec{E}$ field looks like at each of the specific points. The point of observation will now be located at an arbitrary point ***P****=(x, y, 0)*.

b. *Field vector at any point* ***P****=(x, y, 0)*:

i. On the diagram, sketch the vectors $\vec{E}_1$ *and* $\vec{E}_2$ and the resultant vector $\vec{E}(x, y, 0)$ at the point ***P****=(x, y, 0)*.

ii. For the charges in question 1 – $P_1'$=(-1, 0, 0) and $P_2'$=(1, 0, 0), rewrite equation 21c to find the field at **any** point ***P****=(x, y, 0)* in the *x*-*y* (*z* = 0) plane.

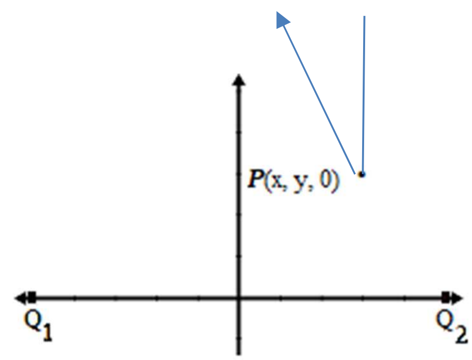



$$\vec{E}(x, y, 0) = \frac{9((x+1)i + (y)j)}{((x+1)^2 + y^2)^{3/2}} + \frac{9((x-1)i + (y)j)}{((x-1)^2 + y^2)^{3/2}}$$

c. *Component of the field in the y – direction at any point* ***P****=(x, y, 0)*:

i. On the diagram, sketch the resultant vector $\vec{E}(x, y, 0)$ and the *y* – component - $\vec{E}_y$. at the point ***P****=(x, y, 0)*.

ii. Write the equation for the *y* - component ($\vec{E}_y$) (Simplify your answer from question 1b, above), for the electric field.

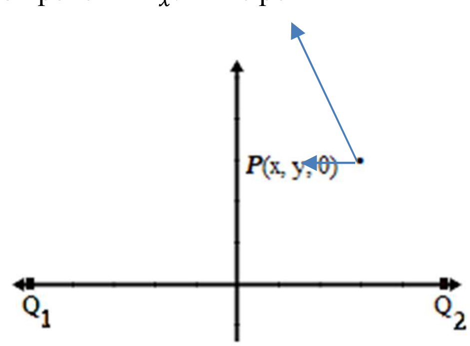



$$E_y(x, y, 0) = \frac{9(y)j}{((x+1)^2+y^2)^\wedge 3/2} + \frac{9(y)j}{((x-1)^2+y^2)^{3/2}}$$

d. *Component of the field in the x – direction at any point* ***P****=(x, y, 0)*:

i. On the diagram, sketch the resultant vector $\vec{E}(x, y, 0)$ and the *x* – component - $\vec{E}_x$. at the point ***P****=(x, y, 0)*.

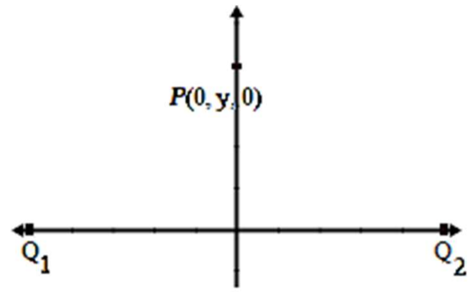
ii. Write the equation for the *x* - component ($\vec{E}_x$) (Simplify your answer to question 1b, above), for the electric field




$$E_x(x, y, 0) = \frac{9(x+1)i}{((x+1)^2+y^2)^{3/2}} + \frac{9(x-1)i}{((x-1)^2+y^2)^{3/2}}$$

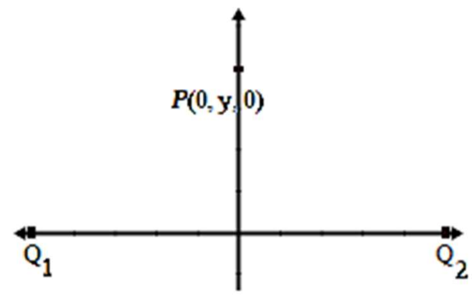
You now have an indication of what the resultant $\vec{E}$ field looks like at the arbitrary point ***P****=(x, y, 0)*. ***The point of observation will now be shifted to P=(0, y, 0).***

e. Find the field in the y – direction at the point ***P****=(0,y,0)*: Write the *y* - component (simplify your answer to question 1b, above), of the field in the *y*-direction ($\vec{E}_y$).




$$E_y(0, y, 0) = \frac{18(y)j}{(x^2+y^2)^{3/2}}$$

f. At the same point, ***P****(0, y, 0)* find the *Field in the x – direction*: Write the *x* - component (question 1b, above), of the field in the *x*-direction ($\vec{E}_x$).



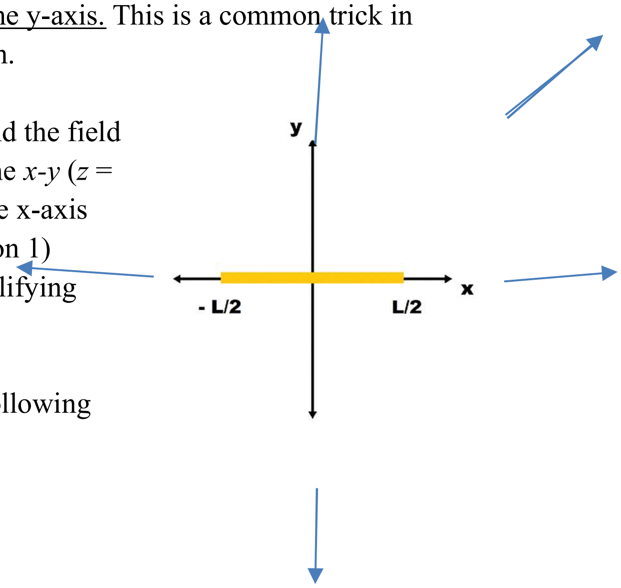

$$E_x(0, y, 0) = \frac{18(0)i}{(x^2+y^2)^{3/2}}$$

g. Show that the *x*-component of the field is zero at the point (0, y, 0).

The top term of eq. f is 0, therefore the x-component of the field must also be 0

<u>Note that you could have avoided doing the actual vector calculation by noting that the *x*-components of the field from the two charges oppose and cancel one another along the y-axis.</u> This is a common trick in field calculations and takes advantage of the symmetry of the problem.

2) Now let's tackle a harder problem. We will use equation 21f to find the field for a continuous distribution of charge. We want to find the field in the *x-y* (*z* = 0) plane due to a line of total charge *Q* distributed uniformly along the x-axis ***P'****=(x', 0, 0)* between points *x'* = +*L*/2 and *x'* = -*L*/2. Just as in question 1) above, we will use the same physical situation for all parts (a-i), simplifying when possible as we proceed.




a. Sketch your guesses for the resultant electric field vector at the following points including (*L*,0,0), (-*L*,0,0), (0,*L*,0), (0,-*L*,0), (*L*,*L*,0).

b. Linear charge density $\lambda$, is defined as the ratio of charge to length.

a) Write the linear charge density, $\lambda$, in terms of *Q* and *L*.

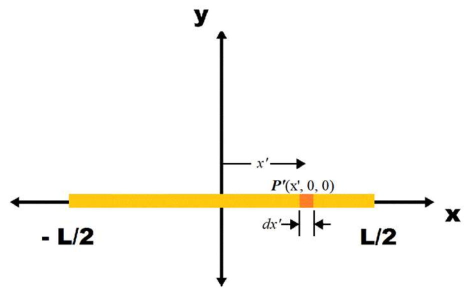
$\boxed{\lambda = \_\_Q/L\_\_\_\_\_\_}$

b) Write the linear charge density, $\lambda$, in terms of increment of charge *dq'* and the corresponding infinitesimal length *dx'*.

λ=__dq'/dx'_____

c. Express *dq'* in terms of the corresponding infinitesimal length element *dx'*, located at *x'*, and the linear charge density: $\lambda$.

dq'=__$\lambda dx'$_____




Eq. 21c has to be slightly modified to give the increment of field $d\vec{E}$ at point ***P***=(*x, y, z*), due to charge d*q'* at point ***P'***=(*x', y', z'*), yielding,

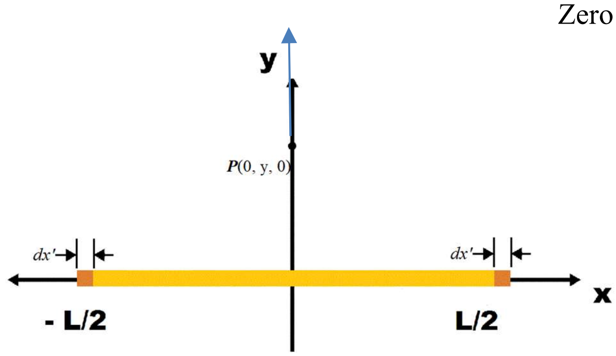
$$d\vec{E}(x,y,z) = d\vec{E}_x + d\vec{E}_y + d\vec{E}_z = \frac{dq'}{4\pi\varepsilon_0}\frac{(x-x')\hat{\imath} + (y-y')\hat{\jmath} + (z-z')\hat{k}}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{3/2}}$$

d. Simplify the above equation for the increment of electric field $d\vec{E}$ at any point in the plane ***P***=(*x, y*, 0) due to an increment of charge d*q'* located at point ***P'***=(*x'*, 0, 0). Use *dq'* from answer 2c above.

$$d\vec{E}(x,y,0) = \frac{k\lambda dx'\left((x-x')i+(y)j\right)}{((x-x')^2+(y)^2)^{3/2}}$$

e. If you want to calculate the total field at any point ***P***=(0, *y*, 0) along the *y* axis, do you need to calculate both the *x* and *y* components? (Is one of the components zero? If one is zero, identify which one is zero, and why it is zero. A sketch can provide a good explanation.)

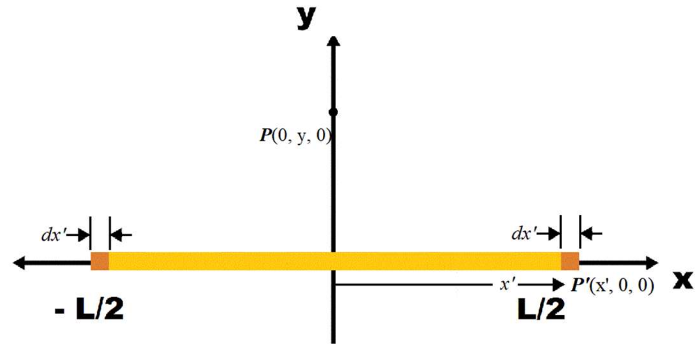
Zero Component : X-component



f. (Ok, so the x-component is zero.) Now write the electric field, in the form of the modified equation shown above at point ***P***=(0, *y*, 0) (in the non-zero direction) due to a charge d*q'* at point ***P'***=(*x'*,0, 0).

$$\boxed{d\vec{E}(0,y,0) = k\lambda dx' \frac{(y-y')j}{\left((x'^2)+(y-y')^2\right)^{3/2}}}$$

g. To find the total field you have to sum over all *dq*'s corresponding to each increment of length *dx*'. Summing of infinitesimals is called integration. Set up the integral over the length of the line of charge, including correct limits and integration variables. (Do not actually solve the integral.).




(Here is the result of your integral if you set it up correctly. $\vec{E} = \frac{1}{4\pi\varepsilon_0 y}\frac{Q}{\left(y^2+\left(\frac{L}{2}\right)^2\right)^{1/2}}\hat{\jmath}$. (Eq. 21g).)

h. Suppose now that the length of the line segment *L* is much, much less than *y*, i.e., *y* >>*L*. If done correctly, Eq. 21g should reduce to a familiar form in the limit (*y* >>*L*).

Go ahead and simplify Eq. 21g for this limit. Hint: Look at the terms shown in the denominator to the right - $\frac{1}{\left(y^2+\left(\frac{L}{2}\right)^2\right)^{1/2}}$. Which one can be ignored? Finally substitute the result back into the equation

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 y}\frac{Q}{\left(y^2+\left(\frac{L}{2}\right)^2\right)^{1/2}}\hat{\jmath}\text{, and solve.}$$

| |
|---|
| $\vec{E} =$ |

i. Is this limit of Eq. 21g physically sensible? Compare it to Coulomb's Law for a point charge Q at the point (0,0,0). Express why the two results coincide!