Assignment 8 of MATP4820/6610

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Requirement: you must write the solutions by yourself. As different problems may be assigned to MATP4820 and 6610 students, **you must explicitly write whether you are taking MATP4820 or 6610**.

Code submission: you are required submit your source code for any coding question in LMS for verification. In addition, include your code, if any, together with your output in one single PDF file.

Bonus: In addition to the bonus problems (that will be assigned occasionally), you can earn up to 5% bonus credit if you write your homework in LaTex.

Problem 1

Consider the linear program:

$$\begin{aligned} \underset{x_1,x_2}{\text{minimize}} \;\; & -10x_1 - 12x_2 \\ \text{s.t. } & x_1 + 2x_2 \leq 20 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned} \tag{1}$$

1. Plot the feasible region of (1) and find the optimal solution by graph
2. In the lecture, we wrote (1) into an equivalent standard LP. Also, we start from a basic feasible solution and perform one step of the simplex method. Continue on the basic feasible solution obtained in class and find the optimal solution by the simplex method.

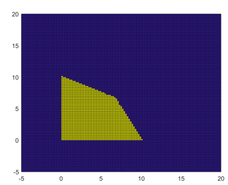
Problem 1.1

To generate the plot I wrote a MATLAB script:

```
x = -5:0.2:20;
[x1, x2] = meshgrid(x);
ineq1 = x1 >= 0;
```

```
ineq2 = x2 >= 0;
ineq3 = x1 + 2*x2 <= 20;
ineq4 = 2*x1 + x2 <= 20;
f1 = double(ineq1);
f2 = double(ineq2);
f3 = double(ineq3);
f4 = double(ineq4);
f = f1.*f2.*f3.*f4;
surf(x1, x2, f);
view(0,90);
```

Figure 1: Problem 1 Feasible Region



Let's write x_1 as the following from the above equalities:

$$x_1 + 2x_2 \leq 20 \implies x_1 \leq 20 - 2x_2$$

To calculate where the two inequalities intersect:

$$\begin{aligned}
x_1 + 2x_2 - 20 &= 2x_1 + x_2 - 20 \\
x_1 + 2x_2 &= 2x_1 + x_2 \\
(20 - 2x_2) + 2x_2 &= 2(20 - 2x_2) + x_2 \\
20 &= 40 - 3x_2 \\
x_2 &= \frac{20}{3} \\
x_1 &= 20 - 2\left(\frac{20}{3}\right) \\
&= \frac{20}{3}
\end{aligned}$$

$$(x_1, x_2) = \left(\frac{20}{3}, \frac{20}{3}\right)$$

We check the four corner points of this space to see which point is the minimizer:

$$\begin{aligned}
&(0,0) : f(0,0) = -10(0) - 12(0) = 0\\
&(0,10) : f(0,10) = -10(0) - 12(10) = -120\\
&(10,0) : f(10,0) = -10(10) - 12(0) = -100\\
&(20/3, 20/3) : f(20/3, 20/3) = -10(20/3) - 12(20/3) = -440/3
\end{aligned}$$

By inspection of the graph we see that the optimal solution is:

$$(x_1^*, x_2^*) = \left(\frac{20}{3}, \frac{20}{3}\right)$$

Problem 1.2

Continuing from in class we now have:

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, B^{-1} = \begin{bmatrix} 0 & 0.5 \\ 1 & -0.5 \end{bmatrix}, N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, c_B = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, c_N = \begin{bmatrix} -12 \\ 0 \end{bmatrix}$$

Update y:

$$\begin{aligned}
y &= B^{-1} c_B\\
&= \begin{bmatrix} 0 & 0.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} -10 \\ 0 \end{bmatrix}\\
&= \begin{bmatrix} 0 \\ -10 \end{bmatrix}
\end{aligned}$$

Update z_n:

$$\begin{aligned}
z_n &= c_N - N^T y\\
&= \begin{bmatrix} -12 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -10 \end{bmatrix}\\
&= \begin{bmatrix} -2 \\ 10 \end{bmatrix}
\end{aligned}$$

We select $q = 2$ as $z_2 = -2 \leq 0$:

$$w = B^{-1} A_2 = \begin{bmatrix} 0 & 0.5 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

We set the following:

$$\begin{aligned} x_2^+ &= \min\left(\frac{(x_B)_1}{w_1}, \frac{(x_B)_2}{w_2}\right) = \min\left(20, \frac{20}{3}\right) = \frac{20}{3} \\ p &= (B)_2 = 3 \end{aligned}$$

We move onto the following updates:

$$\begin{aligned} x_B^+ &= x_B - wx_q^+ \\ &= \begin{bmatrix} 10 \\ 10 \end{bmatrix} - \frac{20}{3}\begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} \\ &= \begin{bmatrix} 20/3 \\ 0 \end{bmatrix} \\ x_N &= \begin{bmatrix} 20/3 \\ 0 \end{bmatrix} \\ x^+ &= \begin{bmatrix} 20/3 \\ 20/3 \\ 0 \\ 0 \end{bmatrix} \\ \mathcal{B}^+ &= \mathcal{B} \cup \{q\} \setminus \{p\} = \{1,3\} \cup \{2\} \setminus \{3\} = \{1,2\} \end{aligned}$$

And we are finished with the second iteration. Beggining the third:

$$x = \begin{bmatrix} 20/3 \\ 20/3 \\ 0 \\ 0 \end{bmatrix}, \mathcal{B} = \{1,2\}, \mathcal{N} = \{3,4\}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$c_N = \begin{bmatrix} -12 \\ 0 \end{bmatrix}, c_B = \begin{bmatrix} -10 \\ -12 \end{bmatrix}, c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We update y, z_n:

$$\begin{aligned} y &= B^{-1}c_B \\ &= \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} -10 \\ -12 \end{bmatrix} \\ &= \begin{bmatrix} -14/3 \\ -8/3 \end{bmatrix} \\ z_n &= c_N - N^T y \end{aligned}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -14/3 \\ -8/3 \end{bmatrix}$$
$$= \begin{bmatrix} 14/3 \\ 8/3 \end{bmatrix}$$

And we can terminate the Simplex algorithm as $z_n \geq \vec{0}$. The optimal point is the same found by analysis of the feasible region.

$$(x_1^*, x_2^*) = (20/3, 20/3)$$

Problem 2 [bonus question]

In the lecture, we talked about how to choose the initial point for the interior point method by solving two constrained quadratic problems, and we gave solutions for both of them. You are required to give a proof for the second one. More specifically, let $\mathbf{A}$ be a matrix with full row rank, i.e., $\mathbf{A}\mathbf{A}^\top$ is nonsingular. Prove that $\tilde{\mathbf{y}} = (\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{A}\mathbf{c}$ and $\tilde{\mathbf{z}} = \mathbf{c} - \mathbf{A}^\top\tilde{\mathbf{y}}$ are the solution to the following problem:

$$\arg\min_{\mathbf{y},\mathbf{z}} \frac{1}{2}\|\mathbf{z}\|_2^2, \text{ s.t. } \mathbf{A}^\top\mathbf{y} + \mathbf{z} = \mathbf{c}.$$