MATH 4100

FF15

Linear Algebra - MATH-4100

Final Examination - December 18, 2015.

An Orthonormal Basis for $V = \mathcal{P}_6(\mathbb{R})$

Define an inner product on $V = \mathcal{P}(\mathbb{R})$ by

$$\langle p(x), q(x) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} p(x)q(x)dx.$$

Starting with the basis $(1, x, x^2, x^3, x^4, x^5, x^6)$ for V, the Gramm-Schmidt algorithm computes an orthonormal basis, $(e_0(x), e_1(x), e_2(x), e_3(x), e_4(x), e_5(x), e_6(x))$, where

$$\begin{aligned}
e_0(x) &= 1 \\
e_1(x) &= x \\
e_2(x) &= \frac{1}{\sqrt{2}}(x^2 - 1) \\
e_3(x) &= \frac{1}{\sqrt{6}}(x^3 - 3x) \\
e_4(x) &= \frac{1}{2\sqrt{6}}(x^4 - 6x^2 + 3) \\
e_5(x) &= \frac{1}{2\sqrt{30}}(x^5 - 10x^3 + 15x) \\
e_6(x) &= \frac{1}{12\sqrt{5}}(x^6 - 15x^4 + 45x^2 - 15)
\end{aligned}$$

1. $V = \mathcal{P}(\mathbb{R})$, with inner product given above. Define a linear functional, φ on V by

$$\varphi(p(x)) = \frac{p(1) + p(-1)}{2}.$$

Find a ploynomial $q(x)$ such that

$$\varphi(p(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} p(x)q(x)dx$$

for all polynomials $p(x)$ of degree 6 or less.

2. Let $V = \mathbb{R}^3$ and let U be the orthogonal complement of span $(2, 1, -2)$.

   (a) Compute an orthonormal basis of U.

   (b) Find a vector $v \in U$ that is closest to $w = (1, 1, -1)$.

   (c) Compute $\|v - w\|$.

3. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by $T(x, y) = (9x - 3y, 17y - 3x)$. Is T a positive operator? Prove your assertion. Is $\langle T(w, x), (y, z)\rangle$ an inner product on $\mathbb{R}^2$?

4. Suppose $T \in \mathcal{L}(V)$ is invertible.

   (a) If $\lambda \in \mathbb{F}$ and $\lambda \neq= 0$, prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

   (b) Prove that T and T^{-1} have the same eigenvectors.

5. Suppose that $T \in \mathbb{R}^3$ has an upper-triangular matrix with respect to the basis $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 1)$ and $v_3 = (1, 1, 2)$. Find an orthonormal basis of $\mathbb{R}^3$ for which T has an upper-triangular ~~basis~~. matrix

6. We know that if $\dim V = n$ and $(v_1, \ldots, v_n)$ is a basis of V, then every vector $v \in V$ is a linear combination of the basis vectors.

$$v = \sum_{i=1}^{n} a_i v_i,$$

where the scalars a_i are uniquely determined. If $(v_1, \ldots, v_n)$ is an orthonormal basis, then it is easy to determine the values of the scalars. In the equation above, take the inner product of both sides with v_i, The result is $\langle v, v_i \rangle = a_i \langle v_i, v_i \rangle = a_i$ for each i, resulting in

$$v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i,$$

and consequently

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, v_i \rangle|^2.$$

(a) What happens if the v_i are merely orthogonal (that is, $\langle v_i, v_j \rangle = 0$ if $i \neq j$) but of arbitrary (non-zero) norm? Derive a formula for the a_i's (above) and for $\|v\|^2$.

(b) Note that $v_1 = (1, 1, 1)$, $v_2 = (1, -2, 1)$ and $v_3 = (1, 0, -1)$ is an orthogonal basis of $\mathbb{R}^3$. Express $v = (5, -2, 7)$ as a linear combination of v_1, v_2 and v_3.

7. Define $T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^2)$ by $T(x_1, x_2, x_3, x_4, x_5) = (x_2 + 2x_3 + 3x_4 + 4x_5, 4x_1 + 3x_2 + 2x_3 + x_4)$.

(a) With respect to the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^5$, compute $A = \mathrm{Mat}(TT^*)$ and $B = \mathrm{Mat}(T^*T)$.

(b) Compute the rank of A and the rank of B.

(c) Is TT^* invertible? Is T^*T invertible? Why?

(d) Compute an orthonomal basis of eigenvectors of TT^*.

(e) Compute the eigenvalues of TT^*.

(f) Compute the distinct eigenvalues of T^*T.

(g) For each non-zero eigenvalue of T^*T, compute an eigenvector of T^*T.

8. Define $T \in \mathcal{L}(\mathbb{R}^3)$ by $T(x, y, z) = (y, x - z, y)$. For all computations, you may use matrices.

   (a) Show that T is nilpotent. What is the smallest positive integer n such that $T^n = 0$?

   (b) Does $\mathbb{R}^3$ have a basis of eigenvectors of T? Why?

   (c) Compute the singular values of T.

   (d) Compute a positive square root, R of T^*T.

9. Define $T \in \mathbb{R}^2$ by $T(x, y) = (5x + 3y, 4y)$. You may use matrices for computations.

   (a) Compute the singular values of T.

   (b) Compute $R = \sqrt{T^*T}$. (It suffices to compute the matrix of R with respect to the standard basis.)

   (c) Compute the polar form of T.