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Texts in Applied Mathematics 20

Introduction to Perturbation Methods

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Introduction to Perturbation Methods

With 88 Illustrations

Springer

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To Colette, Matthew and Marianna
A small family with big hearts

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, had led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research level monographs.

Preface

First, let me say hello and welcome to the subject of perturbation methods. For those who may be unfamiliar with the topic, the title can be confusing. The first time I became aware of this was during a family reunion when someone asked what I did as a mathematician. This is not an easy question to answer, but I started by describing how a certain segment of the applied mathematics community was interested in problems that arise from physical problems. Examples such as water waves, sound propagation, and the aerodynamics of airplanes were discussed. The difficulty of solving such problems was also described in exaggerated detail. Next came the part about how one generally ends up using a computer to actually find the solution. At this point I editorialized on the limitations of computer solutions and why it is important to derive, if at all possible, accurate approximations of the solution. This lead naturally to the mentioning of asymptotics and perturbation methods. These terms ended the conversation because I was unprepared for their reactions. They were not sure exactly what asymptotics meant, but they were quite perplexed about perturbation methods. I tried, unsuccessfully, to explain what it means, but it was not until sometime later that I realized the difficulty. For them, as in *Webster's Collegiate Dictionary*, the first two meanings for the word perturb are "to disturb greatly in mind (disquiet); to throw into confusion (disorder)." Although a cynic might suggest this is indeed appropriate for the subject, the intent is exactly the opposite. (For a related comment, see Exercise 3.4.1(d).)

In a nutshell, this book serves as an introduction into how to systematically construct an approximation of the solution of a problem that is

otherwise intractable. The methods all rely on there being a parameter in the problem that is relatively small. Such a situation is relatively common in applications, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics. One of the other cornerstones is scientific computing, and it is interesting that the two subjects have grown up together. However, this is not unexpected given their respective capabilities. When using a computer, one is capable of solving problems that are nonlinear, inhomogeneous, and multidimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that computer solutions do not provide much insight into the physics of the problem (particularly for those who do not have access to the appropriate software or computer), and there is always the question as to whether or not the computed solution is correct. On the other hand, perturbation methods are also capable of dealing with nonlinear, inhomogeneous, and multidimensional problems (although not to the same extent as computer-generated solutions). The principal objective when using perturbation methods, at least as far as the author is concerned, is to provide a reasonably accurate expression for the solution. By doing this one is able to derive an understanding of the physics of the problem. Also, one can use the result, in conjunction with the original problem, to obtain more efficient numerical procedures for computing the solution.

The methods covered in the text vary widely in their applicability. The first chapter introduces the fundamental ideas underlying asymptotic approximations. This includes their use in constructing approximate solutions of transcendental equations as well as differential equations. In the second chapter, matched asymptotic expansions are used to analyze problems with layers. Chapter 3 describes a method for dealing with problems with more than one time scale. In Chapter 4, the WKB method for analyzing linear singular perturbation problems is developed, while in Chapter 5 a method for dealing with materials containing disparate spatial scales (e.g., microscopic versus macroscopic) is discussed. The last chapter examines the topics of multiple solutions and stability.

The mathematical prerequisites for this text include a basic background in differential equations and advanced calculus. In terms of difficulty, the chapters are written so that the first sections are either elementary or intermediate, while the later sections are somewhat more advanced. Also, the ideas developed in each chapter are applied to a spectrum of problems, including ordinary differential equations, partial differential equations, and difference equations. Scattered through the exercises are applications to integral equations, integro-differential equations, differential-difference equations, and delay equations. What will not be found is an in-depth discussion of the theory underlying the methods. This aspect of the subject is important, and references to the more theoretical work in the area are given in each chapter.

The exercises in each section vary in their complexity. In addition to the more standard textbook problems, an attempt has been made to include problems from the research literature. The latter are intended to provide a window into the wide range of areas that use perturbation methods. Solutions to some of the exercises are available from the author's home page located at http://www.math.rpi.edu/~holmes. Also located there is an errata list. Those who may want to make a contribution to one of these files, or have suggestions about the text, can reach the author at holmes@rpi.edu.

I would like to express my gratitude to the many students who took my course in perturbation methods at Rensselaer. They helped me immeasurably in understanding the subject and provided much needed encouragement to write this book. It is a pleasure to acknowledge the suggestions of Jon Bell, Ash Kapila, and Bob O'Malley, who read early versions of the manuscript. I would also like to thank Julian Cole, who first introduced me to perturbation methods and is still, to this day, showing me what the subject is about.

Troy, New York
August, 1994

Mark H. Holmes

Contents

Series Preface **vi**

Preface **vii**

Chapter 1: Introduction to Asymptotic Approximations **1**
1.1 Introduction 1
1.2 Taylor's Theorem and l'Hospital's Rule 3
1.3 Order Symbols 4
1.4 Asymptotic Approximations 7
1.4.1 Asymptotic Expansions 9
1.4.2 Accuracy versus Convergence of an Asymptotic Series 12
1.4.3 Manipulating Asymptotic Expansions 14
1.5 Asymptotic Solution of Algebraic and Transcendental Equations 18
1.6 Introduction to the Asymptotic Solution of Differential Equations 26
1.7 Uniformity 37
1.8 Symbolic Computing 43

Chapter 2: Matched Asymptotic Expansions **47**
2.1 Introduction 47
2.2 Introductory Example 48
2.3 Examples with Multiple Boundary Layers 62

2.4 Interior Layers 68
2.5 Corner Layers 77
2.6 Partial Differential Equations 84
2.7 Difference Equations 98

Chapter 3: Multiple Scales **105**
3.1 Introduction 105
3.2 Introductory Example 106
3.3 Slowly Varying Coefficients 117
3.4 Forced Motion Near Resonance 123
3.5 Boundary Layers 132
3.6 Introduction to Partial Differential Equations 134
3.7 Linear Wave Propagation 139
3.8 Nonlinear Waves 142
3.9 Difference Equations 153

Chapter 4: The WKB and Related Methods **161**
4.1 Introduction 161
4.2 Introductory Example 162
4.3 Turning Points 173
4.4 Wave Propagation and Energy Methods 185
4.5 Wave Propagation and Slender Body Approximations . . . 190
4.6 Ray Methods 197
4.7 Parabolic Approximations 207
4.8 Discrete WKB Method 212

Chapter 5: The Method of Homogenization **223**
5.1 Introduction 223
5.2 Introductory Example 224
5.3 Multidimensional Problem: Periodic Substructure 234
5.4 Porous Flow 241

Chapter 6: Introduction to Bifurcation and Stability **249**
6.1 Introduction 249
6.2 Introductory Example 249
6.3 Analysis of a Bifurcation Point 251
6.4 Linearized Stability 255
6.5 Relaxation Dynamics 264
6.6 An Example Involving a Nonlinear Partial Differential Equation 271
6.7 Bifurcation of Periodic Solutions 281
6.8 Systems of Ordinary Differential Equations 287

Appendix A1: Solution and Properties of Transition Layer Equations **297**
A1.1 Airy Functions 297
A1.2 Confluent Hypergeometric Functions 299
A1.3 Higher-Order Turning Points 302

Appendix A2: Asymptotic Approximations of Integrals **305**
A2.1 Introduction 305
A2.2 Watson's Lemma 306
A2.3 Laplace's Approximation 306
A2.4 Stationary Phase Approximation 306

Appendix A3: Numerical Solution of Nonlinear Boundary-Value Problems **309**
A3.1 Introduction 309
A3.2 Examples 310
A3.3 Computer Code 311

References **313**

Index **331**

1

Introduction to Asymptotic Approximations

1.1 Introduction

In this book, we are interested in using what are known as asymptotic expansions to find approximate solutions of differential equations. Usually our efforts will be directed toward constructing the solution of a problem with only occasional regard for the physical situation it represents. However, to start things off it is worth considering a typical physical problem to illustrate where the mathematical problems originate. A simple example comes from the motion of an object projected radially upward from the surface of the Earth. Letting $x(t)$ denote the height of the object, measured from the surface, from Newton's second law we obtain the following equation of motion:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \qquad \text{for } 0 < t, \tag{1.1}$$

where R is the radius of the Earth and g is the gravitational constant. We will assume the object starts from the surface with a given upward velocity, and so $x(0) = 0$ and $x'(0) = v_0$ where v_0 is positive. The nonlinear nature of the above ordinary differential equation makes finding a closed-form solution difficult, and it is thus natural to try to find some way to simplify the equation. For example, if the object does not get far from the surface, then one might try to argue that x is small compared to R and the denominator in (1.1) can be simplified to R^2. This is the type of argument often made in introductory physics and engineering texts. In this case $x \approx x_0$, where $x_0'' = -g$ for $x_0(0) = 0$ and $x_0'(0) = v_0$. Solving this

FIGURE 1.1. Schematic of the solution $x_0(t)$ given in (1.2). This solution comes from the linearization of (1.1) and corresponds to the motion in a uniform gravitational field.

problem yields

$$x_0(t) = -\frac{1}{2}gt^2 + v_0 t. \tag{1.2}$$

One finds in this case that the object reaches a maximum height of $v_0^2/2g$ and comes back to Earth when $t = 2v_0/g$ (see Fig. 1.1). The difficulty with this reduction is that it is unclear how to determine a correction to the approximate solution in (1.2). This is worth knowing since we would then be able to get a measure of the error made in using (1.2) as an approximation and it would also be possible to see just how the nonlinear nature of the original problem affects the motion of the object.

To make the reduction process more systematic, it is first necessary to scale the variables. To do this, let $\tau = t/t_c$ and $y(\tau) = x(t)/x_c$, where t_c is a characteristic time for the problem and x_c is a characteristic value for the solution. We have a lot of freedom in picking these constants, but they should be representative of the situation under consideration. Based on what is shown in Fig. 1.1, we take $t_c = v_0/g$ and $x_c = v_0^2/g$. Doing this, the problem transforms into the following:

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(1+\varepsilon y)^2}, \qquad \text{for } 0 < \tau, \tag{1.3a}$$

where

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 1. \tag{1.3b}$$

In (1.3a), the parameter $\varepsilon = v_0^2/Rg$ is dimensionless and its value is important because it gives us a measure of how high the projectile gets in comparison to the radius of the Earth. In terms of the function $x_0(t)$, it can be seen from Fig. 1.1 that $\varepsilon/2$ is the ratio of the maximum height of the projectile to the radius of the Earth. Assuming $R = 4000$ mi, then $\varepsilon \approx 1.5 \times 10^{-9} v_0^2$ s^2/ft^2. It would therefore appear that if v_0 is much less than 10^3 ft/s, then (1.2) is a reasonably good approximation to the solution. We can verify this assertion by reconstructing the first-order approximation in (1.2). This can be done by assuming that the dependence of the solution on ε can be determined using a Taylor series expansion about $\varepsilon = 0$. In

other words, for small ε, it is assumed that

$$y \sim y_0(\tau) + \varepsilon y_1(\tau) + \cdots .$$

The first term in this expansion is the scaled version of x_0, and this will be shown later (in Section 1.6). What is important is that with this approach it is possible to estimate how well y_0 approximates the solution of (1.3a,b) by finding y_1. The method of deriving the first-order approximation (y_0) and its correction (y_1) is not difficult, but we first need to put the definition of an asymptotic approximation on a firm foundation. Those interested in investigating the ideas underlying the nondimensionalization of a physical problem, as well as some of the theory underlying dimensional analysis, should consult Lin and Segel (1989).

1.2 Taylor's Theorem and l'Hospital's Rule

As in the preceding example, we will typically end up expanding functions in powers of ε. Given a function $f(\varepsilon)$, one of the most important tools for doing this is Taylor's theorem. This is a well-known result, but for completeness it is stated below.

Theorem 1.1: Given a function $f(\varepsilon)$, suppose its $(n+1)$st derivative $f^{(n+1)}$ is continuous for $\varepsilon_a < \varepsilon < \varepsilon_b$. In this case, if ε_0 and ε are points in the interval $(\varepsilon_a, \varepsilon_b)$, then

$$f(\varepsilon) = f(\varepsilon_0) + (\varepsilon - \varepsilon_0)f'(\varepsilon_0) + \cdots + \frac{1}{n!}(\varepsilon - \varepsilon_0)^n f^{(n)}(\varepsilon_0) + R_{n+1}, \quad (1.4a)$$

where

$$R_{n+1} = \frac{1}{(n+1)!}(\varepsilon - \varepsilon_0)^{n+1} f^{(n+1)}(\xi), \quad (1.4b)$$

and ξ is a point between ε_0 and ε.

This result is useful because if the first $n+1$ terms from the Taylor series are used as an approximation of $f(\varepsilon)$, then it is possible to estimate the error using (1.4b).

The Taylor series probably used most often in this book is the *binomial expansion*, which states

$$(x+y)^\alpha = x^\alpha + \alpha y x^{\alpha-1} + \frac{1}{2}\alpha(\alpha-1)x^{\alpha-2}y^2 + \ldots .$$

This holds for any real α and $y^2 < x^2$. Other expansions that we will often have use for include the Maclaurin series ($\varepsilon_0 = 0$) of the exponential and trigonometric functions.

Another useful result is *l'Hospital's rule*, which concerns the value of the limit of the ratio of two functions.

Theorem 1.2: Suppose $f(\varepsilon)$ and $\phi(\varepsilon)$ are differentiable on the interval $(\varepsilon_0, \varepsilon_b)$ and $\phi'(\varepsilon) \neq 0$ in this interval. Also suppose

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f'(\varepsilon)}{\phi'(\varepsilon)} = A,$$

where $-\infty \leq A \leq \infty$. In this case,

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = A,$$

if either one of the following conditions holds:

1) $f \to 0$ and $\phi \to 0$ as $\varepsilon \downarrow \varepsilon_0$,

   or

2) $\phi \to \infty$ as $\varepsilon \downarrow \varepsilon_0$.

The proofs of these theorems, and some of their consequences, can be found in Rudin (1976).

1.3 Order Symbols

To define an asymptotic approximation, we first need to introduce order, or Landau, symbols.[1] The reason for this is that we will be interested in how functions behave as a parameter, typically ε, becomes small. For example, the function $\phi(\varepsilon) = \varepsilon$ does not converge to zero as fast as $f(\varepsilon) = \varepsilon^2$ when $\varepsilon \to 0$, and we need a notation to denote this fact.

Definitions:

1. $f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ means that there are constants k_0 and ε_1 (independent of ε) such that

   $$|f(\varepsilon)| \leq k_0 |\phi(\varepsilon)| \text{ for } \varepsilon_0 < \varepsilon < \varepsilon_1.$$

   We say that f is "big Oh" of ϕ as $\varepsilon \downarrow \varepsilon_0$.

2. $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ means that for every positive δ there is an ε_2 (independent of ε) such that

   $$|f(\varepsilon)| \leq \delta |\phi(\varepsilon)| \text{ for } \varepsilon_0 < \varepsilon < \varepsilon_2.$$

   We say that f is "little oh" of ϕ as $\varepsilon \downarrow \varepsilon_0$.

[1] Hardy (1954) states that these symbols were introduced by Bachmann (1894), and then Landau (1909) popularized their use. For this reason, they are sometimes called Bachmann-Landau symbols (Lagerstrom, 1988).

These definitions may seem cumbersome, but they usually are not hard to apply. However, there are other ways to determine the correct order. Of particular interest is the case where ϕ is not zero near ε_0 (i.e., $\phi \neq 0$ if $\varepsilon_0 < \varepsilon < \varepsilon_b$ for some $\varepsilon_b > \varepsilon_0$). In this case, we have that $f = O(\phi)$ if the ratio $|f/\phi|$ is bounded for ε near ε_0. Another, perhaps more useful, way to make this determination involves the limit

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)}. \tag{1.5}$$

If this limit exists, and is finite, then $f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$. Similarly, a simple way to decide if $f = o(\phi)$ is to use the fact that if

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 0 \tag{1.6}$$

then $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$. The proofs of these statements follow directly from the definition of a limit and are left to the reader.

Examples (for $\varepsilon \downarrow 0$)

1. Suppose $f = \varepsilon^2$. Also let $\phi_1 = \varepsilon$ and $\phi_2 = -3\varepsilon^2 + 5\varepsilon^6$. In this case,

$$\lim_{\varepsilon \downarrow 0} \frac{f}{\phi_1} = 0 \Rightarrow f = o(\phi_1)$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{f}{\phi_2} = -1/3 \Rightarrow f = O(\phi_2).$$

2. If $f = \varepsilon \sin(1 + 1/\varepsilon)$ and $\phi = \varepsilon$, then $|f/\phi| \leq 1$ for $0 < \varepsilon$. Hence, $f = O(\phi)$.

3. If $f(\varepsilon) = \sin(\varepsilon)$ then, using Taylor's theorem, $f = \varepsilon - \frac{1}{2}\,\varepsilon^2 \sin(\xi)$. Thus, $\lim_{\varepsilon \downarrow 0}(f/\varepsilon) = 1$, and from this it follows that $f = O(\varepsilon)$.

4. If $f = e^{-1/\varepsilon}$ then, using l'Hospital's rule, $f = o(\varepsilon^\alpha)$ for all values of α. We say in this case that f is *transcendentally small* with respect to the power functions ε^α.

Some of the properties of the order symbols are examined in the exercises. Three that are worth mentioning are the following (the symbol $\Leftrightarrow$ stands for the statement "if and only if"):

a) $f = O(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f$ is bounded as $\varepsilon \downarrow \varepsilon_0$.

b) $f = o(1)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f \to 0$ as $\varepsilon \downarrow \varepsilon_0$.

c) $f = o(\phi)$ as $\varepsilon \downarrow \varepsilon_0 \Rightarrow f = O(\phi)$ as $\varepsilon \downarrow \varepsilon_0$ (but not vice versa).

The proofs of these statements are straightforward and are left to the reader.

Two symbols we will use occasionally are $<<$ and $\approx$. When saying that $f(\varepsilon) << \phi(\varepsilon)$ it is meant that $f = o(\phi)$, and the statement that $\varepsilon << 1$, or that ε is small, means $\varepsilon \downarrow 0$. The symbol $\approx$ does not have a precise definition and is used simply to designate an approximate numerical value. An example of this is the statement that $\pi \approx 3.14$.

Exercises

1. (a) What value(s) of α, if any, yield $f = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$? i) $f = \sqrt{1+\varepsilon^2}$, ii) $f = \varepsilon \sin(\varepsilon)$, iii) $f = (1-e^\varepsilon)^{-1}$, iv) $f = \ln(1+\varepsilon)$, v) $f = \varepsilon \ln(\varepsilon)$, vi) $f = \sin(1/\varepsilon)$, vii) $f = \sqrt{x+\varepsilon}$, where $0 \le x \le 1$.

   (b) For the functions listed in (a) what value(s) of α, if any, yield $f = o(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$?

2. (a) Show $f = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0 \Rightarrow f = o(\varepsilon^\beta)$ as $\varepsilon \downarrow 0$ for any $\beta < \alpha$.

   (b) Show that if $f = O(g)$ as $\varepsilon \downarrow \varepsilon_0$ then $|f|^\alpha = O(|g|^\alpha)$ as $\varepsilon \downarrow \varepsilon_0$ for any positive α. Give an example to show that this result is not necessarily true if α is negative.

   (c) Give an example to show that $f = O(g)$ as $\varepsilon \downarrow \varepsilon_0$ does not necessarily mean that $e^f = O(e^g)$ as $\varepsilon \downarrow \varepsilon_0$.

3. (a) Assuming $f_1 = O(\phi_1)$ and $f_2 = O(\phi_2)$ as $\varepsilon \downarrow \varepsilon_0$, show i) $f_1 + f_2 = O(|\phi_1| + |\phi_2|)$ as $\varepsilon \downarrow \varepsilon_0$, and ii) $f_1 f_2 = O(\phi_1 \phi_2)$ as $\varepsilon \downarrow \varepsilon_0$.

   (b) The results in part (a) can be rewritten as $O(f) + O(g) = O(|f| + |g|)$ and $O(f)O(g) = O(fg)$. Using this notation show the following: i) $O(f)o(g) = o(f)o(g) = o(fg)$, ii) $O(O(f)) = O(f)$, and iii) $o(O(f)) = O(o(f)) = o(o(f)) = o(f)$.

4. Occasionally it's useful to state the order of a function more precisely. One way to do this is to say $f = O_s(\phi)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow f = O(\phi)$ and $f \neq o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$.

   (a) What value(s) of α, if any, yield $f = O_s(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$? i) $f = \varepsilon \sin(\varepsilon)$, ii) $f = (1-e^\varepsilon)^{-1}$, iii) $f = \ln(1+\varepsilon)$, iv) $f = \varepsilon \ln(\varepsilon)$, v) $f = \sin(1/\varepsilon)$.

   (b) Suppose the limit in (1.5) exists. In this case, show that $f = O_s(\phi)$ as $\varepsilon \downarrow \varepsilon_0 \Leftrightarrow 0 < \lim_{\varepsilon \downarrow \varepsilon_0} |f/\phi| < \infty$.

5. Suppose $f = o(\phi)$ for small ε, where f and ϕ are continuous.

   (a) Give an example to show that it is not necessarily true that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon \phi d\varepsilon\right).$$

(b) Show that

$$\int_0^\varepsilon f d\varepsilon = o\left(\int_0^\varepsilon |\phi| d\varepsilon\right).$$

1.4 Asymptotic Approximations

Our objective is to construct approximations to the solutions of differential equations. It is therefore important that we state exactly what we mean by an approximation. To introduce this idea, suppose we are interested in finding an approximation of $f(\varepsilon) = \varepsilon^2 + \varepsilon^5$ for ε close to zero. Because $\varepsilon^5 << \varepsilon^2$, a reasonable approximation is $f(\varepsilon) \approx \varepsilon^2$. On the other hand, a lousy approximation is $f(\varepsilon) \approx \frac{2}{3}\varepsilon^2$. This is lousy even though the error $f(\varepsilon) - \frac{2}{3}\varepsilon^2$ goes to zero as $\varepsilon \downarrow 0$. The problem with this "lousy approximation" is that the error is of the same order as the function we are using to approximate $f(\varepsilon)$. This observation gives rise to the following definition.

Definition: Given $f(\varepsilon)$ and $\phi(\varepsilon)$, we say that $\phi(\varepsilon)$ is an *asymptotic approximation* to $f(\varepsilon)$ as $\varepsilon \downarrow \varepsilon_0$ whenever $f = \phi + o(\phi)$ as $\varepsilon \downarrow \varepsilon_0$. In this case, we write $f \sim \phi$ as $\varepsilon \downarrow \varepsilon_0$.

As demonstrated in the above example, the idea underlying this definition is that $\phi(\varepsilon)$ serves as an approximation to $f(\varepsilon)$, for ε close to ε_0, when the error is of higher order than the approximating function. In the case where $\phi(\varepsilon)$ is not zero near ε_0, we can make use of (1.6). In particular, we have that $f \sim \phi$ as $\varepsilon \downarrow \varepsilon_0$ if

$$\lim_{\varepsilon \downarrow \varepsilon_0} \frac{f(\varepsilon)}{\phi(\varepsilon)} = 1. \tag{1.7}$$

Examples

1. Suppose $f = \sin(\varepsilon)$ and $\varepsilon_0 = 0$. We can obtain an approximation of $f(\varepsilon)$ by expanding about $\varepsilon = 0$ using Taylor's theorem. This yields

$$f = \varepsilon - \frac{1}{6}\varepsilon^3 + \frac{1}{120}\varepsilon^5 \cos(\xi).$$

From this, the following asymptotic approximations are obtained:

i) $f \sim \varepsilon$,
ii) $f \sim \varepsilon - \frac{1}{6}\varepsilon^3$,
iii) $f \sim \varepsilon + 2\varepsilon^2$.

In comparing these asymptotic approximations with the Taylor series expansion, it would appear that, for small ε, (ii) is the most accurate and (iii) is the least accurate. However, our definition of an asymptotic approximation says little about comparative accuracy. We correct this weakness later by introducing asymptotic expansions.

FIGURE 1.2. Comparison between the function $f = x + e^{-x/\varepsilon}$ and its asymptotic approximation $f \sim x$. Note that the two functions are essentially indistinguishable everywhere except near $x = 0$. In this plot, $\varepsilon = 10^{-2}$.

2. Suppose $f = x + e^{-x/\varepsilon}$ where $0 < x < 1$ is fixed. In this case $f \sim x$ for small ε. However, it is natural to ask how well this does in approximating the function for $0 < x < 1$. If we plot both together, we obtain the curves shown in Fig. 1.2. It's apparent that the approximation is quite good away from $x = 0$. It is also clear that we do not do so well near $x = 0$. This is true no matter what value of ε we choose since $f(0) = 1$. It should be remembered that with the asymptotic approximation, given a value of x, the approximation is a good one if ε is close to $\varepsilon_0 = 0$. What we are seeing in this example is that exactly how small ε can be depends on x (the closer we are to $x = 0$ the smaller ε has to be). In later sections, we will refer to this situation as a case of when the approximation is not uniformly valid on the interval $0 < x < 1$.

3. Consider the function $f = \sin(\pi x) + \varepsilon^3$ for $0 \le x \le \frac{1}{2}$. For small ε it might seem reasonable to expect that $f \sim \sin(\pi x)$. For this to hold, it is required that $f - \sin(\pi x) = o(\sin(\pi x))$ as $\varepsilon \downarrow 0$. If $x \neq 0$, this is true since $\lim_{\varepsilon \downarrow 0}(\varepsilon^3/\sin(\pi x)) = 0$. However, at $x = 0$ this requirement does not hold since $\sin(0) = 0$. Therefore, $\sin(\pi x)$ is not an asymptotic approximation of f over the entire interval $0 \le x \le \frac{1}{2}$. This problem of using an approximating function whose zeros do not agree with the original function is one that we will come across on numerous occasions. Usually, it is a complication we will not worry a great deal about since the correction is relatively small. For instance, in this example the correction is $O(\varepsilon^3)$, while everywhere else the function is $O(1)$. This is not true for the correction that is needed to fix the approximation in the previous example. There the correction at, or very near, $x = 0$ is $O(1)$, and this is the same order as the value of the function through the rest of the interval. We are therefore not able to ignore the problem at $x = 0$, and we will deal with this in Chapter 2.

1.4.1 Asymptotic Expansions

Two observations that come from the preceding examples are that an asymptotic approximation is not unique, and it also does not say much about the accuracy of the approximation. To address these shortcomings, we need to introduce more structure into the formulation. In the earlier examples, the sequence $1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$ was used in the expansion of the function but other, usually more interesting, sequences also arise. In preparation for this, we have the next definitions, which are due to Poincaré (1886).

Definitions:

1. The functions $\phi_1, \phi_2, \ldots$ form an *asymptotic sequence*, or are *well-ordered*, as $\varepsilon \downarrow \varepsilon_0$ if and only if $\phi_n = o(\phi_m)$ as $\varepsilon \downarrow \varepsilon_0$ for all m and n that satisfy $m < n$.

2. If $\phi_1, \phi_2, \ldots$ is an asymptotic sequence, then $f(\varepsilon)$ has an *asymptotic expansion* to n terms, with respect to this sequence, if and only if

$$f = \sum_{k=1}^{m} a_k \phi_k(\varepsilon) + o(\phi_m) \text{ for } m = 1, \ldots, n \text{ as } \varepsilon \downarrow \varepsilon_0, \tag{1.8}$$

where the a_k's are independent of ε. In this case, we write

$$f \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \cdots + a_n\phi_n(\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0. \tag{1.9}$$

The ϕ_k's are called the scale, or gauge, or basis functions.

To make use of this definition, we need to have some idea of what scale functions are available. We will run across a wide variety in this book, but a couple of our favorites will turn out to be the following:

1. $\phi_1 = (\varepsilon - \varepsilon_0)^\alpha$, $\phi_2 = (\varepsilon - \varepsilon_0)^\beta$, $\phi_3 = (\varepsilon - \varepsilon_0)^\gamma$, $\ldots$ where $\alpha < \beta < \gamma < \ldots$ (for $\varepsilon \downarrow \varepsilon_0$), and

2. $\phi_1 = 1$, $\phi_2 = e^{-1/\varepsilon}$, $\phi_3 = e^{-2/\varepsilon}$, $\ldots$ (for $\varepsilon \downarrow 0$).

The first of these is simply a generalization of the power series functions, and the second sequence is useful when we have to describe exponentially small functions. The verification that these do indeed form asymptotic sequences is left to the reader.

Now comes the harder question. Given a function $f(\varepsilon)$, how do we find an asymptotic expansion of it? The most commonly used methods include employing one of the following: 1) Taylor's theorem, 2) l'Hospital's rule, or 3) an educated guess. The latter usually relies on an intuitive understanding of the problem and, many times, on luck. The other two methods are more routine and are illustrated in the examples ahead.

Taylor's theorem is a particularly useful tool because if the function is smooth enough to let us expand about the point $\varepsilon = \varepsilon_0$, then any one of the resulting Taylor polynomials can be used as an asymptotic expansion (for $\varepsilon \downarrow \varepsilon_0$). Moreover, Taylor's theorem enables us to analyze the error very easily.

Examples (for $\varepsilon << 1$)

1. To find a three-term expansion of e^ε, we proceed as follows:

$$\begin{aligned} e^\varepsilon &= 1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3!}\varepsilon^3 + \dots \\ &\sim 1 + \varepsilon + \frac{1}{2}\varepsilon^2. \end{aligned}$$

2. To find a three-term expansion of $\sin(e^\varepsilon)$, we first use Taylor's theorem to expand $\sin(1+\alpha)$ about $\alpha = 0$ to obtain

$$\sin(1+\alpha) = \sin(1) + \alpha\cos(1) - \frac{1}{2}\alpha^2\sin(1) + \dots.$$

Therefore,

$$\begin{aligned} \sin(e^\varepsilon) &= \sin(1 + \varepsilon + \frac{1}{2}\varepsilon^2 + \frac{1}{3!}\varepsilon^3 + \dots) \\ &= \sin(1) + (\varepsilon + \frac{1}{2}\varepsilon^2 + \dots)\cos(1) - \frac{1}{2}(\varepsilon + \dots)^2\sin(1) + \dots \\ &\sim \sin(1) + \varepsilon\cos(1) + \frac{1}{2}\varepsilon^2(\cos(1) - \sin(1)). \end{aligned}$$

A nice aspect about using Taylor's theorem is that the scale functions do not have to be specified ahead of time. This differs from the next procedure, which requires the specification of the scale functions before constructing the expansion.

To describe the second procedure for constructing an asymptotic expansion, suppose the scale functions ϕ_1, ϕ_2, ... are given and the expansion of the function is $f \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \dots$. From the above definition, this means that $f = a_1\phi_1 + o(\phi_1)$. Assuming that we can divide by ϕ_1, we get $\lim_{\varepsilon\downarrow\varepsilon_0}(f/\phi_1) = a_1$. This gives us the value of a_1 and with this information we can determine a_2 by noting $f = a_1\phi_1 + a_2\phi_2 + o(\phi_2)$. Thus, $\lim_{\varepsilon\downarrow\varepsilon_0}[(f - a_1\phi_1)/\phi_2] = a_2$. This idea can be used to calculate the other coefficients of the expansion, and one obtains the following formulas:

$$a_1 = \lim_{\varepsilon\downarrow\varepsilon_0} \frac{f}{\phi_1}, \tag{1.10a}$$

$$a_2 = \lim_{\varepsilon\downarrow\varepsilon_0} \frac{f - a_1\phi_1}{\phi_2}, \tag{1.10b}$$

$$a_3 = \lim_{\varepsilon \downarrow \varepsilon_0} \frac{f - a_1\phi_1 - a_2\phi_2}{\phi_3}, \ldots. \tag{1.10c}$$

This assumes that the scale functions are nonzero for ε near ε_0 and that each of the limits exists. If this is the case, then formulas (1.10) for the a_k's show that the asymptotic expansion is unique.

Example

Suppose $\phi_1 = 1$, $\phi_2 = \varepsilon$, $\phi_3 = \varepsilon^2$, ... and

$$f(\varepsilon) = \frac{1}{1+\varepsilon} + e^{-1/\varepsilon}.$$

From the limit formulas just given, we have that

$$\begin{aligned}
a_1 &= \lim_{\varepsilon \downarrow 0} \frac{f}{1} = 1, \\
a_2 &= \lim_{\varepsilon \downarrow 0} \frac{f-1}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \left(\frac{-1}{1+\varepsilon} + \frac{1}{\varepsilon} e^{-1/\varepsilon} \right) = -1, \\
a_3 &= \lim_{\varepsilon \downarrow 0} \frac{f-1+\varepsilon}{\varepsilon^2} = \lim_{\varepsilon \downarrow 0} \left(\frac{1}{1+\varepsilon} + \frac{1}{\varepsilon^2} e^{-1/\varepsilon} \right) = 1, \ldots .
\end{aligned}$$

Thus, $f \sim 1 - \varepsilon + \varepsilon^2 + \ldots$. What is interesting here is that the exponential makes absolutely no contribution to this expansion. This is because $e^{-1/\varepsilon} = o(\varepsilon^\alpha)$ for all values of α; in other words, the exponential decays so quickly to zero that any power function considers it to be zero. In this case, we say that this exponential function is *transcendentally small* with respect to these scale functions.

As seen in the previous example, two functions can have the same asymptotic expansion. In particular, using the power series functions $\phi_0 = 1$, $\phi_1 = \varepsilon$, $\phi_2 = \varepsilon^2$, ... one obtains the same expansion as in the previous example for any of the following:

i) $f_1 = \frac{\tanh(1/\varepsilon)}{1+\varepsilon}$,

ii) $f_2 = \frac{1+e^{-1/\varepsilon}}{1+\varepsilon}$,

iii) $f_3 = \frac{1}{1+\varepsilon} + \varepsilon^{100} \operatorname{sech}(-1/\varepsilon)$.

This observation brings up the idea of asymptotic equality, or asymptotic equivalence, with respect to a given sequence $\phi_1, \phi_2, \phi_3, \ldots$. We say that two functions f and g are *asymptotically equal*, to n terms, if $f - g = o(\phi_n)$ as $\varepsilon \downarrow \varepsilon_0$.

1.4.2 Accuracy versus Convergence of an Asymptotic Series

It is not unusual to expect that to improve the accuracy of an asymptotic expansion it is simply necessary to include more terms. This is what happens with Taylor series expansions, where in theory one should be able to obtain as accurate a result as desired by simply adding together enough terms. However, with an asymptotic expansion this is not necessarily true. The reason is that an asymptotic expansion only makes a statement about the series in the limit of $\varepsilon \downarrow \varepsilon_0$, whereas increasing the number of terms is saying something about the series as $n \to \infty$. In fact, an asymptotic expansion need not converge! Moreover, even if it converges it does not have to converge to the function that was expanded! These two observations may seem to be major flaws in our definition of an asymptotic expansion but they are, in fact, attributes that we will take great advantage of throughout this book.

A demonstration that a convergent asymptotic expansion need not converge to the function that was expanded can be found in the last example. It is not hard to show that the asymptotic series converges to the function $(1+\varepsilon)^{-1}$. This is clearly not equal to the original function since it is missing the exponential term.

A well-known example of a divergent asymptotic expansion arises with the Bessel function $J_0(z)$, which is defined as

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2}. \tag{1.11}$$

If we let $f(\varepsilon) = J_0\left(\frac{1}{\varepsilon}\right)$, then it can be shown that an asymptotic expansion

FIGURE 1.3. The relative difference, in absolute value, between the exact value of the function $f(\varepsilon) = J_0(\varepsilon^{-1})$, for $\varepsilon = \frac{1}{15}$, and the series representation given in (1.11). Also shown is the relative difference between the exact value and the asymptotic expansion given in (1.12). The values are given as a function of the number of terms used.

of $f(\varepsilon)$ for small ε is (Abramowitz and Stegun, 1972)

$$f \sim \sqrt{\frac{2\epsilon}{\pi}}\left[\alpha\cos\left(\frac{1}{\varepsilon}-\frac{\pi}{4}\right)+\beta\sin\left(\frac{1}{\varepsilon}-\frac{\pi}{4}\right)\right], \tag{1.12a}$$

where

$$\alpha \sim 1-\frac{1\cdot 3^2\varepsilon^2}{2!8^2}+\frac{1\cdot 3^2\cdot 5^2\cdot 7^2\varepsilon^4}{4!8^4}+\dots \tag{1.12b}$$

and

$$\beta \sim \frac{\varepsilon}{8}-\frac{1\cdot 3^2\cdot 5^2\varepsilon^3}{3!8^3}+\dots. \tag{1.12c}$$

It is not hard to show that the series expansions in (1.12b) and (1.12c) are divergent for all nonzero values of ε (see Exercise 8). To see just how well a divergent series like (1.12) can approximate the function $f(\varepsilon)$, the errors in using (1.11) and (1.12) are shown in Fig. 1.3 for $\varepsilon = 1/15$. What is immediately apparent is that the asymptotic approximation does very well using only one or two terms, while the series needs to include more than twenty terms to achieve the same error. If a smaller value of ε is used, the situation is even more pronounced, eventually getting to the point where it is essentially impossible to calculate the value of the Bessel function using the series representation since so many terms are needed.

These observations concerning convergence should always be kept in mind when using asymptotic expansions. Also, it should be pointed out that asymptotic approximations are most valuable when they can give insights into the structure of the solution, or the physics of the problem under study. If decimal-place accuracy is desired, then numerical methods, or perhaps numerical methods combined with an asymptotic solution, are probably worth considering.

The ideas underlying an asymptotic approximation are so natural that they are found in some of the earliest work in mathematics. The basis of the definitions given here can be traced back to at least Legendre (1825). In his *Traité des Fonctions Elliptiques* he considered approximating a function with a series where the error committed using the first n terms is of the same order as the $(n+1)$st term. He referred to this as a semiconvergent series. This is a poor choice of words since an asymptotic series need not converge. Somewhat earlier than this, Laplace (1812) had found use for asymptotic series to be able to evaluate certain special functions because the standard series expansions converged so slowly. He considered the fact that they did not converge to be "inconvenient," an interesting attitude that generally leads to disastrous results. The most significant contribution to the subject was made by Poincaré (1886). He was able to make sense of the numerous divergent series solutions of differential equations that had been found up to that time by introducing the concept of an asymptotic expansion. His ideas

are the basis of our development. It is interesting that his paper actually marked the modern beginnings of two areas in mathematics. One is the asymptotic solution of a differential equation and the other is the study of divergent series. The latter can also be called the theory of summability of a divergent series, and the books by Ford (1916) and Hardy (1960) are good references for this area. However, it is not surprising if the reader has not heard of this particular subject, which has since died of natural causes. For those who may be interested in the early history of the development of asymptotic approximations, the reviews by McHugh (1971) and Schlissel (1977a) should be consulted.

1.4.3 Manipulating Asymptotic Expansions

It is not hard to show that two asymptotic expansions can be added together term by term. Multiplication is also relatively straightforward although more tedious and limited to asymptotic sequences that can be ordered in particular ways (e.g., see Exercise 6). What is not completely clear is whether or not an asymptotic approximation can be differentiated (assuming the functions involved are smooth). To be more specific, suppose we know that

$$f(x,\varepsilon) \sim a_1(x)\phi_1(x,\varepsilon) + a_2(x)\phi_2(x,\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0. \tag{1.13}$$

What is of interest is whether or not it is true that

$$\frac{d}{dx}f(x,\varepsilon) \sim \frac{d}{dx}a_1(x)\phi_1(x,\varepsilon) + \frac{d}{dx}a_2(x)\phi_2(x,\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0. \tag{1.14}$$

Based on the conjecture that things will go wrong if at all possible, the only possible answer to the above question is that, no, it is not always true. To give an example, let $f(x,\varepsilon) = e^{-x/\varepsilon}\sin(e^{x/\varepsilon})$. For small ε one finds that, for $0 < x < 1$,

$$f \sim 0 + 0\cdot\varepsilon + 0\cdot\varepsilon^2 \ldots . \tag{1.15}$$

However, the function

$$\frac{d}{dx}f(x,\varepsilon) = -\frac{1}{\varepsilon}e^{-x/\varepsilon}\sin(e^{x/\varepsilon}) + \frac{1}{\varepsilon}\cos(e^{x/\varepsilon})$$

does not even have an expansion in terms of the scale functions used in (1.15).

Another problem is that even if ϕ_1 and ϕ_2 are well ordered their derivatives may not be. For example, letting $\phi_1 = 1 + x$ and $\phi_2 = \varepsilon\sin(x/\varepsilon)$, for $0 \le x \le 1$, then $\phi_2 = o(\phi_1)$ for ε small, but $\frac{d}{dx}\phi_1$ and $\frac{d}{dx}\phi_2$ are not well ordered. So, the question that we need to address is when can we differentiate an expansion? Well, the answer to this is that if

$$f(x,\varepsilon) \sim a_1(x)\phi_1(\varepsilon) + a_2(x)\phi_2(\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0, \tag{1.16}$$

and if

$$\frac{d}{dx}f(x,\varepsilon) \sim b_1(x)\phi_1(\varepsilon) + b_2(x)\phi_2(\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0, \tag{1.17}$$

then $b_k = \frac{d}{dx}a_k$ (i.e., the expansion for $\frac{d}{dx}f$ can be obtained by differentiating the expansion of $f(x,\varepsilon)$ term by term). Throughout this book, given (1.16) we will automatically assume that (1.17) also holds.

The other operation we will have use for is integration, and in this case the situation is better. If the expansion of the function is as given in (1.16), and all the functions involved are integrable, then

$$\int_a^b f(x,\varepsilon)dx \sim \int_a^b a_1(x)dx\ \phi_1(\varepsilon) + \int_a^b a_2(x)dx\ \phi_2(\varepsilon) \text{ as } \varepsilon \downarrow \varepsilon_0. \tag{1.18}$$

It goes without saying here that the interval $a \le x \le b$ must be within the set of x's where (1.16) holds.

Exercises

1. Are the following sequences well ordered (for $\varepsilon \downarrow 0$)? If not, arrange them so they are or explain why it is not possible to do so.

   (a) $\phi_n = (1-e^{-\varepsilon})^n$ for $n = 0,1,2,3,\ldots$.

   (b) $\phi_n = [2\sinh(\varepsilon/2)]^{2n}$ for $n = 0,1,2,3,\ldots$.

   (c) $\phi_1 = \frac{1}{\varepsilon}$, $\phi_2 = \frac{1}{\varepsilon^{1/2}}$, $\phi_3 = \frac{1}{\varepsilon^{1/3}}$, $\ldots$.

   (d) $\phi_1 = \varepsilon^5 e^{-3/\varepsilon}$, $\phi_2 = \varepsilon$, $\phi_3 = \varepsilon\ln(\varepsilon)$, $\phi_4 = e^{-\varepsilon}$, $\phi_5 = \frac{\sin(\varepsilon^3)}{\varepsilon}$, $\phi_6 = \frac{1}{\ln(\varepsilon)}$.

   (e) $\phi_1 = e^{\varepsilon} - 1 - \varepsilon$, $\phi_2 = e^{\varepsilon} - 1$, $\phi_3 = e^{\varepsilon}$, $\phi_4 = e^{\varepsilon} - 1 - \varepsilon - \frac{1}{2}\varepsilon^2$.

   (f) $\phi_1 = 1$, $\phi_2 = \varepsilon$, $\phi_3 = \varepsilon^2$, $\phi_4 = \varepsilon\ln(\varepsilon)$, $\phi_5 = \varepsilon^2\ln(\varepsilon)$, $\phi_6 = \varepsilon\ln^2(\varepsilon)$, $\phi_7 = \varepsilon^2\ln^2(\varepsilon)$.

   (g) $\phi_k = \begin{cases} 1 \text{ for } k^{-1} \le \varepsilon \\ 0 \text{ for } 0 \le \varepsilon < k^{-1} \end{cases}$ where $k = 1,2,3,\ldots$.

   (h) $\phi_k(\varepsilon) = g^k(\varepsilon)$ for $k = 1,2,\ldots$, where $g(\varepsilon)$ is continuous and $g(0) = 0$.

   (i) $\phi_k(\varepsilon) = g(\varepsilon^k)$ for $k = 1,2,\ldots$, where $g(x) = x\sin(\frac{1}{x})$ for $k = 1,2,\ldots$.

   (j) $\phi_k(\varepsilon) = g(\varepsilon^k)$ for $k = 1,2,\ldots$, where $g(x) = e^{-1/x}$.

2. Assuming $f \sim a_1\varepsilon^{\alpha} + a_2\varepsilon^{\beta} + \ldots$, find α, β (with $\alpha < \beta$) and nonzero a_1, a_2 for the following functions:

   (a) $f = \frac{1}{1-e^{\varepsilon}}$,

(b) $f = \left[1 + \frac{1}{\cos(\varepsilon)}\right]^{3/2}$,

(c) $f = \sinh(\sqrt{1+\varepsilon x})$, for $0 < x < \infty$,

(d) $f = (1+\varepsilon x)^{1/\varepsilon}$, for $0 < x < \infty$,

(e) $f = \int_0^\varepsilon \sin(x + \varepsilon x^2)dx$,

(f) $f = \sum_{n=1}^\infty \left(\frac{1}{2}\right)^n \sin\left(\frac{\varepsilon}{n}\right)$,

(g) $f = \prod_{k=0}^n (1+\varepsilon k)$, where n is a positive integer,

(h) $f = \int_0^\pi \frac{\cos(x)}{(x+\varepsilon)^{3/2}} dx$.

3. Assume $f(\varepsilon)$ and $\phi(\varepsilon)$ are positive for $\varepsilon > 0$ and $f \sim \phi$ as $\varepsilon \downarrow 0$.

   (a) Show $f^\alpha \sim \phi^\alpha$ as $\varepsilon \downarrow 0$.

   (b) Give an example to show that it is not necessarily true that $e^f \sim e^\phi$ as $\varepsilon \downarrow 0$. What else must be assumed? Is it enough to impose the additional assumption that $\phi = O(1)$ as $\varepsilon \downarrow 0$?

4. This problem considers the values of x where $f(x,\varepsilon) = 0$. It's assumed that the function can be expanded as $f(x,\varepsilon) \sim f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \dots$.

   (a) Suppose x_0 is such that $f(x_0,\varepsilon) = 0$. Explain why $f_0(x_0) = f_1(x_0) = \dots = 0$. Give an example to show that $f_0(x_0) = f_1(x_0) = \dots = 0$ does not necessarily mean that $f(x_0,\varepsilon) = 0$.

   (b) If $\alpha > 1$, then the equations $f(x,\varepsilon) = 0$ and $(f(x,\varepsilon))^\alpha = 0$ have exactly the same solutions. Also, note that using the binomial expansion we get $(f(x,\varepsilon))^\alpha \sim (f_0(x))^\alpha + \alpha\varepsilon f_1(x)(f_0(x))^{\alpha-1} + \dots$. Therefore, in solving $(f(x,\varepsilon))^\alpha = 0$ it would appear to be enough to only have $f_0(x) = 0$ (i.e., it is not necessary that $f_1(x) = 0$). This contradicts the conclusion in (a). What's the problem?

5. In this problem, assume the functions are continuous and nonzero for ε near ε_0.

   (a) Show that if $f \sim \phi$ as $\varepsilon \downarrow \varepsilon_0$ then $\phi \sim f$ as $\varepsilon \downarrow \varepsilon_0$.

   (b) Suppose $f \sim \phi$ and $g \sim \varphi$ as $\varepsilon \downarrow \varepsilon_0$. Give an example to show that it is not necessarily true that $f + g \sim \phi + \varphi$ as $\varepsilon \downarrow \varepsilon_0$.

6. Suppose $f(\varepsilon) \sim a_0\phi_0(\varepsilon) + a_1\phi_1(\varepsilon) + \dots$ and $g(\varepsilon) \sim b_0\phi_0(\varepsilon) + b_1\phi_1(\varepsilon) + \dots$ as $\varepsilon \downarrow \varepsilon_0$, where $\phi_0, \phi_1, \phi_2 \dots$ is an asymptotic sequence as $\varepsilon \downarrow \varepsilon_0$.

   (a) Show that $f + g \sim (a_0 + b_0)\phi_0 + (a_1 + b_1)\phi_1 + \dots$ as $\varepsilon \downarrow \varepsilon_0$.

   (b) Assuming $a_0 b_0 \neq 0$, show $fg \sim a_0 b_0 \phi_0^2$ as $\varepsilon \downarrow \varepsilon_0$. Also, discuss the possibilities for the next term in the expansion.

(c) Suppose that $\phi_i\phi_j = \phi_{i+j}$ for all i, j. In this case, show that

$$fg \sim a_0b_0\phi_0 + (a_0b_1 + a_1b_0)\phi_1 + (a_0b_2 + a_1b_1 + a_2b_0)\phi_2 + \ldots \text{ as } \varepsilon \downarrow \varepsilon_0.$$

(d) Under what conditions on the exponents will the following be an asymptotic sequence satisfying the condition in part (c):
$\phi_0 = (\varepsilon - \varepsilon_0)^\alpha$, $\phi_1 = (\varepsilon - \varepsilon_0)^\beta$, $\phi_2 = (\varepsilon - \varepsilon_0)^\gamma$, ...?

7. This problem considers variations on the definitions given in this section. It is assumed that ϕ_1, ϕ_2, ... form an asymptotic sequence.

(a) Explain why $f \sim a_1\phi_1(\varepsilon) + a_2\phi_2(\varepsilon) + \ldots + a_n\phi_n(\varepsilon)$, where $a_1 \neq 0$, implies that the series

$$\sum_{k=1}^{m} a_k\phi_k(\varepsilon)$$

is an asymptotic approximation of $f(\varepsilon)$ for $m = 1, \ldots, n$. Also, explain why the converse is not true.

(b) Assuming the a_m's are nonzero, explain why the definition of an asymptotic expansion is equivalent to saying that $a_m\phi_m(\varepsilon)$ is an asymptotic approximation of

$$f - \sum_{k=1}^{m-1} a_k\phi_k(\varepsilon)$$

for $m = 1, \ldots, n$ (the sum is taken to be zero when $m = 1$).

(c) Explain why the definition of an asymptotic expansion (to n terms) is equivalent to saying that

$$f = \sum_{k=1}^{n} a_k\phi_k(\varepsilon) + o(\phi_n) \text{ as } \varepsilon \downarrow \varepsilon_0.$$

8. The absolute value of the ε^k term in (1.12b) and (1.12c) is

$$\frac{[(2k-1)!]^2}{k2^{5k-2}[(k-1)!]^3} \text{ for } k = 1, 2, 3, \ldots .$$

With this show that each series in (1.12b) and (1.12c) diverges for $\varepsilon > 0$.

9. The entropy jump $[\![S]\!]$ across a shock wave in a gas is given as (Cole and Cook, 1986)

$$[\![S]\!] = c_v \ln\left(\frac{1 + \frac{\gamma-1}{2}\varepsilon}{1 - \frac{\gamma+1}{2}\varepsilon}(1-\varepsilon)^\gamma\right),$$

where $c_v > 0$, $\gamma > 1$, and $\varepsilon > 0$ is the shock strength.

(a) Find a first-term expansion of the entropy jump for a weak shock (i.e., $\varepsilon << 1$).

(b) What is the order of the second term in the expansion?

10. A well-studied problem in solid mechanics concerns the deformation of an elastic body when compressed by a rigid punch. This gives rise to having to evaluate the following integral (Gladwell, 1980):

$$I = \int_0^\infty N(x)\sin(\lambda x)dx,$$

where

$$N(x) = \frac{2\alpha\sinh(x) - 2x}{1 + \alpha^2 + x^2 + 2\alpha\cosh(x)} - 1,$$

and α, λ are constants with $1 \leq \alpha \leq 3$. The infinite interval complicates finding a numerical value of I, and so, write

$$I = \int_0^{x_0} N(x)\sin(\lambda x)dx + R(x_0).$$

The objective is to take x_0 large enough that the error $R(x_0)$ is relatively small.

(a) Find a two-term expansion of $N(x)$ for large x.

(b) Use the first term in the expansion you found in part (a) to determine a value of x_0 so $|R(x_0)| \leq 10^{-6}$.

(c) Based on the second term in the expansion you found in part (a), does it appear that the value of x_0 you found in part (b) is a reasonable choice?

1.5 Asymptotic Solution of Algebraic and Transcendental Equations

One of the easiest ways to illustrate how an asymptotic expansion can be used to find an approximate solution is to consider algebraic equations. As a simple example, consider the quadratic equation

$$x^2 + 0.002x - 1 = 0. \tag{1.19}$$

The fact that the coefficient of the linear term in x is much smaller than the other coefficients can be used to find an approximate solution. To do this, we start with the related equation

$$x^2 + 2\varepsilon x - 1 = 0, \tag{1.20}$$

where $\varepsilon << 1$. The beginning assumption is that the solution has a particular form of expansion, namely

$$x \sim x_0 + \varepsilon^\alpha x_1 + \dots, \tag{1.21}$$

where $\alpha > 0$ (this inequality is imposed so the expansion is well ordered for small ε). It should be pointed out that (1.21) is nothing more than an educated guess. The motivation for making this assumption comes from the observation that in expanding functions one usually ends up using Taylor's theorem and (1.21) is simply a reflection of that type of expansion. The exponent α is to allow for a little flexibility.

Now, substituting (1.21) into (1.20), one finds that

$$\underset{①}{x_0^2} + \underset{②}{2\varepsilon^\alpha x_0\, x_1} + \dots + \underset{③}{2\varepsilon(x_0 + \varepsilon^\alpha x_1 + \dots)} \underset{④}{-\ 1} = 0. \tag{1.22}$$

The first problem to solve comes from the terms labeled ①, ④. This is because this equation is supposed to hold for small ε and therefore we require it to hold as $\varepsilon \downarrow 0$. This gives us the following:

$O(1)$ $\quad x_0^2 - 1 = 0.$

The solutions are $x_0 = \pm 1$. An important observation here is that this problem has as many solutions as the original equation (i.e., both problems are quadratic).

With the solution just given, we now have a term in (1.22) that is $O(\varepsilon)$, namely term ③. Since the right-hand side is zero, it must be that ③ balances with one of the other terms on the left-hand side of the equation. The only choice is ②, and this means $\alpha = 1$ (see Exercise 4). With this we have the next order problem to solve:

$O(\varepsilon)$ $\quad 2x_0x_1 + 2x_0 = 0$

The solution is $x_1 = -1$.

The process used to find x_0 and x_1 can be continued to systematically construct the other terms of the asymptotic expansion. We will not pursue this here but point out that the approximations we have calculated so far are $x \sim \pm 1 - \varepsilon$. To illustrate the accuracy of this result, one of the exact solutions is plotted in Fig. 1.4 along with its asymptotic approximation. It is clear that they are in very good agreement for small values of ε. Also, if you recall, the original problem was to construct approximations of the solutions of (1.19). In this case $\varepsilon = 10^{-3}$, and so, from the asymptotic approximation we get $x \approx -1.001, 0.999$ while the exact solution gives $x = -1.0010005\cdots, 0.9990005\cdots$.

FIGURE 1.4. Comparison between the positive root of (1.20) and the two-term asymptotic expansion $x \sim 1 + \varepsilon$. It is seen that for small ε the asymptotic approximation is very close to the exact value.

It is appropriate to comment on the notation used in the previous paragraph. We referred to the equation that x_0 satisfies as the $O(1)$ equation. However, strictly speaking, based on the definition of the order symbol, every term in (1.22) is $O(1)$. So what we should have done was first introduce a new notation, such as $O_s(1)$ (see Exercise 1.3.4). When stating that $f = O_s(1)$, we would mean that $f = O(1)$ but $f \neq o(1)$. We could then specify, without ambiguity, exactly what terms go into the x_0 equation. A few textbooks do exactly this to try to reduce the confusion for those who are first learning the subject. However, the truth is that few people actually use this symbolism. This means $O(\cdot)$ has two meanings, one connected with boundedness as expressed in the original definition and the other as an identifier of particular terms in an equation.

As a second example, consider the quadratic equation

$$\varepsilon x^2 + 2x - 1 = 0. \tag{1.23}$$

Note that if $\varepsilon = 0$ then the equation becomes linear; that is, the order of the equation is reduced. This is significant because it fundamentally alters the nature of the equation and can give rise to what is known as a singular problem. Nevertheless, we will approach this problem in the same way as in the previous example. In other words, we will try the regular expansion given in (1.21). Carrying out the calculations, one finds that

$$x \sim \frac{1}{2} - \frac{\varepsilon}{8} + \dots . \tag{1.24}$$

The criticism with this is that there are two solutions of (1.23) and the expansion has only produced one. One remedy is to use (1.24) to factor the quadratic equation (1.23) to find the second solution. However, there is a more direct way to find the solution, and it can be adapted to solving differential equations with a similar complication. To explain what this is, note that the problem is singular in the sense that if $\varepsilon = 0$ then the equation

is linear rather than quadratic. To prevent this from happening, we assume

$$x \sim \varepsilon^{\gamma}(x_0 + \varepsilon^{\alpha} x_1 + \ldots), \tag{1.25}$$

where $\alpha > 0$ (so the expansion is well ordered). Substituting this into (1.23), we have that

$$\underset{①}{\varepsilon^{1+2\gamma}(x_0^2 + 2\varepsilon^{\alpha} x_0 x_1 + \ldots)} + \underset{②}{2\varepsilon^{\gamma}(x_0 + \varepsilon^{\alpha} x_1 + \ldots)} \underset{③}{- 1} = 0. \tag{1.26}$$

The terms on the left-hand side must balance to produce zero, and we need to determine the order of the problem that comes from this balancing. There are three possibilities. One of these occurs when $\gamma = 0$, and in this case the balance is between terms ② and ③. However, this leads to the expansion in (1.24) and we have introduced (1.25) to find the other solution of the problem. So, we have the following other possibilities:

i) ① $\sim$ ③ and ② is higher order $\Rightarrow 1 + 2\gamma = 0 \Rightarrow \gamma = -\frac{1}{2} \Rightarrow$ ①, ③ $= O(1)$ but ② $= O(\varepsilon^{-1/2})$. This violates our assumption that ② is higher order (i.e., ② $<<$ ①), so this case is not possible.

ii) ① $\sim$ ② and ③ is higher order $\Rightarrow 1+2\gamma = \gamma \Rightarrow \gamma = -1 \Rightarrow$ ①, ② $= O(\varepsilon^{-1})$ and ③ $= O(1)$. In this case, the conclusion is consistent with the original assumptions, and so, this is the balancing we are looking for.

With $\gamma = -1$, and (1.26), the problem we need to solve is

$$x_0^2 + 2\varepsilon^{\alpha} x_0 x_1 + \ldots + 2(x_0 + \varepsilon^{\alpha} x_1 + \ldots) - \varepsilon = 0.$$

From this, we have:

$$\begin{aligned} O(1) \quad & x_0^2 + 2x_0 = 0 \;\Rightarrow\; x_0 = 0, -2 \text{ and } \alpha = 1 \\ O(\varepsilon) \quad & 2x_0 x_1 + 2x_1 - 1 = 0 \;\Rightarrow\; x_1 = \frac{1}{2(1 + x_0)}. \end{aligned}$$

Thus, a two-term expansion of the second solution of (1.23) is

$$x \sim \frac{1}{\varepsilon}\left(-2 - \frac{\varepsilon}{2}\right).$$

Note that the case of $x_0 = 0$ found in the $O(1)$ problem corresponds to the solution given in (1.24).

One of the comforts in solving algebraic equations is that we have a very good idea of how many solutions to expect. With transcendental equations, this is harder to determine. Perhaps the simplest way to get a handle on

the number of solutions is to graph the functions for specific values of the parameters. To illustrate this idea, consider the equation

$$x^2 + e^{\varepsilon x} = 5. \tag{1.27}$$

Setting $f(x) = x^2 + e^{\varepsilon x}$ we have that $f > 0$ and $f'' > 0$. Also $f(\pm\infty) = \infty$ and $f(0) = 1$. From this information, we can make a sketch of $f(x)$, which shows that there are exactly two solutions of this equation. To find asymptotic expansions of them, assume

$$x \sim x_0 + \varepsilon^\alpha x_1 + \dots .$$

Substituting this into (1.27), and using Taylor's theorem on the exponential, we obtain

$$x_0^2 + 2x_0x_1\varepsilon^\alpha + \dots + 1 + \varepsilon x_0 + \dots = 5.$$

From the $O(1)$ equation, we get that $x_0 = \pm 2$ and $\alpha = 1$, while the $O(\varepsilon)$ equation yields $x_1 = -1/2$. Hence, a two-term expansion of each solution is $x \sim \pm 2 - \varepsilon/2$.

For the last example, we investigate the equation

$$x + 1 + \varepsilon \operatorname{sech}\left(\frac{x}{\varepsilon}\right) = 0. \tag{1.28}$$

Sketching $f = x + 1$ and $g = -\varepsilon \operatorname{sech}(x/\varepsilon)$ shows that there is one solution of this equation. To find an approximation of it, suppose we proceed in the usual manner and assume

$$x \sim x_0 + \varepsilon^\alpha x_1 + \dots . \tag{1.29}$$

Substituting this into (1.28), and remembering $0 < \operatorname{sech}(z) \leq 1$, it follows that $x_0 = -1$. The complication is that it is not possible to find a value of α so that the other terms in (1.28) balance. In other words, our assumption concerning the structure of the second term in (1.29) is incorrect. Given the character of the hyperbolic function, it is necessary to modify the expansion and we now assume

$$x \sim -1 + \mu(\varepsilon), \tag{1.30}$$

where we are not certain what μ is other than $\mu << 1$ (so the expansion is well ordered). Substituting this into (1.28), we get

$$\mu + \varepsilon \operatorname{sech}(-\varepsilon^{-1} + \mu/\varepsilon) = 0. \tag{1.31}$$

Now, since $\operatorname{sech}(-\varepsilon^{-1} + \mu/\varepsilon) \sim \operatorname{sech}(-\varepsilon^{-1}) \sim 2\exp(-1/\varepsilon)$, we therefore have that $\mu = -2\varepsilon \exp(-1/\varepsilon)$. To construct the third term in the expansion, we would extend (1.30) and write

$$x \sim -1 - 2\varepsilon e^{-1/\varepsilon} + \nu(\varepsilon), \tag{1.32}$$

where $\nu << \varepsilon \exp(-1/\varepsilon)$. To find ν we proceed as before; the details are left as an exercise.

Exercises

1. Find a two-term asymptotic expansion, for small ε, of each solution x of the following equations:

   (a) $x^2 + x - \varepsilon = 0$,

   (b) $x^2 - (2+\varepsilon)x + 1 + \varepsilon = 0$,

   (c) $x^2 + (1 - \varepsilon - \varepsilon^2)x + \varepsilon - 2e^{\varepsilon^2} = 0$,

   (d) $x^2 - 2x + (1-\varepsilon^2)^{25} = 0$,

   (e) $\varepsilon x^3 - 3x + 1 = 0$,

   (f) $\varepsilon^2 x^3 - x + \varepsilon = 0$,

   (g) $x^2 + \sqrt{1+\varepsilon x} = e^{\frac{1}{2+\varepsilon}}$,

   (h) $x^2 + \varepsilon\sqrt{2+x} = \cos(\varepsilon)$,

   (i) $x = \int_0^\pi e^{\varepsilon \sin(x+s)} ds$,

   (j) $x^{2+\varepsilon} = \frac{1}{x+2\varepsilon}$,

   (k) $x^2 - 1 + \varepsilon \tanh\left(\frac{x}{\varepsilon}\right) = 0$,

   (l) $\beta = x - (x+\beta)\frac{x-\alpha}{x+\alpha}e^{x/\varepsilon}$, where α and β are positive constants,

   (m) $\varepsilon e^{x^2} = 1 + \frac{\varepsilon}{1+x^2}$,

   (n) $\left(\frac{1}{x} - 1\right) e^{\frac{1}{x}+\alpha} = \left(\frac{1}{\varepsilon} - 1\right) e^{\frac{1}{\varepsilon}}$, where $\alpha > 0$ is constant,

   (o) $\varepsilon = \left(\frac{3}{2}\int_0^x \sqrt{rp(r)}dr\right)^{2/3}$, where $p(r)$ is smooth, positive, and $p'(0) \neq 0$,

   (p) $xe^{-x} = \varepsilon$.

2. An important, but difficult, task in numerical linear algebra is to calculate the eigenvalues of a matrix. One of the reasons why it's difficult is that the roots of the characteristic equation are very sensitive to the values of the coefficients of the equation. A well-known example, due to Wilkinson (1964), illustrating this is the equation

$$x^{20} - (1+\varepsilon)210x^{19} + 20,615x^{18} + \ldots + 20! = 0,$$

   which can be rewritten as

$$(x-1)(x-2)\cdots(x-20) = 210\varepsilon x^{19}.$$

   (a) Find a two-term expansion for each root of this equation.

   (b) The expansion in part (a) has the form $x \sim x_0 + \varepsilon^\alpha x_1$. Based on this result, how small does ε have to be so $|x - x_0| < 10^{-2}$ for every root? Does it seem fair to say that even a seemingly small error in the accuracy of the coefficients of the equation has a tremendous effect on the value of the roots?

3. Find a two-term asymptotic expansion, for small ε, of the following:

   (a) The point $\mathbf{x}_m = (x_m, y_m)$ at which the function $f(\mathbf{x}) = x^2 + 2\varepsilon \sin(x + e^y) + y^2$ attains its minimum value.

   (b) The perimeter of the two-dimensional golf ball described as $r = 1 + \varepsilon \cos(20\theta)$, where $0 \leq \theta \leq 2\pi$. Explain why ε needs to be fairly small before this two-term expansion can be expected to produce an accurate approximation of the arc length. Also, in geometric terms, explain why this happens.

4. To find the second term of the expansion when solving (1.22), we took $\alpha = 1$. Show that the choice $0 < \alpha < 1$ does not determine the next nonzero term in the expansion. Also, explain why the choice $1 < \alpha$ is not appropriate.

5. To determine the natural frequencies of an elastic string, one is faced with solving the equation $\tan(\lambda) = \lambda$.

   (a) After sketching the two functions in this equation on the same graph explain why there are an infinite number of solutions.

   (b) To find an asymptotic expansion of the large solutions of the equation, assume that $\lambda \sim \varepsilon^{-\alpha}(\lambda_0 + \varepsilon^{\beta}\lambda_1)$. Find ε, α, β, λ_0, and λ_1 (note λ_0 and λ_1 are nonzero and $\beta > 0$).

6. An important problem in celestial mechanics is to determine the position of an object given the geometric parameters of its orbit. For an elliptic orbit, one ends up having to solve Kepler's equation, which is

$$\varepsilon \sin(E) = E - M,$$

   where $M = n(t - T)$ and ε is the eccentricity of the orbit. This equation determines E, the eccentric anomaly, in terms of the time variable t. After finding E, the radial and angular coordinates of the object are calculated using formulas from geometry. Note that in this equation n, T, and ε are positive constants.

   (a) After sketching the functions in Kepler's equation on the same graph, explain why there is at least one solution. Show that if M satisfies $j\pi \leq M \leq (j+1)\pi$ then there is exactly one solution and it satisfies $j\pi \leq E \leq (j+1)\pi$. To do this, you will need to use the fact that $0 \leq \varepsilon < 1$.

   (b) The eccentricity for most of the planets in the solar system is small (e.g., for the Earth $\varepsilon = 0.02$). Assuming $\varepsilon << 1$, find the first three terms in an asymptotic expansion for E.

   (c) Show that your result agrees, through the third term, with the series solution (Bessel, 1824)

$$E = M + 2\sum_{n=1}^{\infty} \frac{1}{n} J_n(n\varepsilon) \sin(nM).$$

It is interesting that Bessel first introduced the functions $J_n(x)$ when solving Kepler's equation. He found that he could solve the problem using a Fourier series, and this led him to an integral representation of $J_n(x)$. This is one of the reasons why these functions were once known as Bessel coefficients.

7. In the study of porous media, one comes across the problem of having to determine the permeability, $k(s)$, of the medium from experimental data (Holmes, 1986). Setting $k(s) = F'(s)$, this problem reduces to solving the following two equations:

$$\begin{aligned} \int_0^1 F^{-1}(c - \varepsilon r)dr &= s, \\ F^{-1}(c) - F^{-1}(c - \varepsilon) &= \beta, \end{aligned}$$

where β is a given positive constant. The unknowns here are the constant c and the function $F(s)$, and they both depend on ε (also, s and β are independent of ε). It's assumed that $k(s)$ is smooth and positive.

(a) Find the first term in the expansion of the permeability for small ε.

(b) Show that the second term in the expansion in part (a) is $O(\varepsilon^3)$.

8. Consider functions $y(x)$ and $Y(x)$ defined, for $0 \leq x \leq 1$, through the equations

$$\int_y^2 \frac{dr}{\operatorname{arctanh}(r-1)} = 1 - x, \text{ and } Y(x) = 3e^{-x/\varepsilon}.$$

In the interval $0 \leq x \leq 1$ there is a point of intersection x_s where $y(x_s) = Y(x_s)$. For $\varepsilon << 1$, find the first two terms in the expansion of x_s. It is worth pointing out that implicitly defined functions, like $y(x)$, arise frequently when using matched asymptotic expansions (these are the subject of Chapter 2).

The definitions of order symbols and asymptotic expansions for complex-valued functions, vector-valued functions, and even matrix functions are obtained by simply replacing the absolute value with the appropriate norm. The following exercises examine how these extensions can be used with matrix perturbation problems.

9. Let $\mathbf{A}$ and $\mathbf{D}$ be (real) $n \times n$ matrices.

(a) Suppose $\mathbf{A}$ is symmetric and has n distinct eigenvalues. Find a two-term expansion of the eigenvalues of the perturbed matrix $\mathbf{A}+\varepsilon\mathbf{D}$

where $\mathbf{D}$ is positive definite. What you are finding is known as a *Rayleigh-Schrödinger* series for the eigenvalues.

(b) Suppose $\mathbf{A}$ is the identity and $\mathbf{D}$ is symmetric. Find a two-term expansion of the eigenvalues for the matrix $\mathbf{A} + \varepsilon\mathbf{D}$.

(c) By considering

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

show that a $O(\varepsilon)$ perturbation of a matrix need not result in a $O(\varepsilon)$ perturbation of the eigenvalues. This example also demonstrates that a smooth perturbation of a matrix need not result in a smooth perturbation of the eigenvalues. References for this material, and its extensions to differential equations, are Kato (1980) and Hubert and Sanchez-Palencia (1989).

10. Find a two-term expansion of $(\mathbf{A} + \varepsilon\mathbf{B})^{-1}$ where $\mathbf{A}$ is an invertible $n \times n$ matrix.

11. Let $\mathbf{C}$ be a $m \times n$ matrix with rank n. In this case, the pseudo-inverse of $\mathbf{C}$ is defined as $\mathbf{C}^\dagger \equiv (\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T$.

    (a) Find a two-term expansion of $(\mathbf{A} + \varepsilon\mathbf{B})^\dagger$ where $\mathbf{A}$ is an $m \times n$ matrix with rank n.

    (b) Show the result in part (a) reduces to the expansion from Exercise 10 when $m = n$ and $\mathbf{A}$ is an invertible $n \times n$ matrix.

    (c) A theorem due to Penrose (1955) states that the solutions of the least-squares problem $\min \|\mathbf{Cx} - \mathbf{b}\|_2$ have the form $\mathbf{x} = \mathbf{C}^\dagger\mathbf{b} + (\mathbf{I} - \mathbf{C}^\dagger\mathbf{C})\mathbf{z}$, where $\mathbf{z}$ is an arbitrary n vector. If $\mathbf{C} = \mathbf{A} + \varepsilon\mathbf{B}$, where $\mathbf{A}$ is an $m \times n$ matrix with rank n, then use the result from part (a) to find a two-term asymptotic expansion of the solution. What does your expansion reduce to in the case where $m = n$ and $\mathbf{A} = \mathbf{I}$?

1.6 Introduction to the Asymptotic Solution of Differential Equations

The ideas used to construct asymptotic approximations of the solutions of algebraic and transcendental equations can also be applied to differential equations. In this section we consider regular perturbation problems, which roughly speaking, are problems that retain their essential character as $\varepsilon \downarrow 0$. For example, an equation that is second order remains second order in the limit of ε going to zero. In later chapters, we will take up the study of nonregular, or singular, perturbation problems.

Example: The Projectile Problem

To illustrate how to find an asymptotic approximation of a solution of a differential equation, we return to the projectile example described in Section 1.1. Specifically, we consider the problem

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(1+\varepsilon y)^2}, \quad \text{for } 0 < \tau, \tag{1.33a}$$

where

$$y(0) = 0 \text{ and } y'(0) = 1. \tag{1.33b}$$

The equation in (1.33a) is weakly nonlinear because it is nonlinear but reduces to one that is linear when $\varepsilon = 0$. To construct an approximation of the solution, we start with a working hypothesis on what is the appropriate form of the asymptotic expansion. For the problem at hand, we assume

$$y(\tau) \sim y_0(\tau) + \varepsilon^\alpha y_1(\tau) + \ldots, \tag{1.34}$$

where $\alpha > 0$. As in (1.21), the exponent α is included to allow for a little flexibility. We will also assume that the expansions for the derivatives of $y(\tau)$ can be obtained by differentiating (1.34).

To reduce the differential equation, recall that, for small z, $(1+z)^{-2} \sim 1 - 2z$. So, substituting (1.34) into (1.33a) yields

$$\begin{aligned} y_0''(\tau) + \varepsilon^\alpha y_1''(\tau) + \ldots &= -\frac{1}{[1+\varepsilon(y_0(\tau)+\ldots)]^2} \\ &\sim -1 + 2\varepsilon y_0(\tau) + \ldots, \end{aligned} \tag{1.35a}$$

and, from (1.33b),

$$y_0(0) + \varepsilon^\alpha y_1(0) + \ldots = 0 \text{ and } y_0'(0) + \varepsilon^\alpha y_1'(0) + \ldots = 1. \tag{1.35b}$$

The procedure for finding the terms in the expansion follows the argument used to solve algebraic equations. By equating like powers of ε, we obtain the following problems:

$O(1)$ $\quad y_0''(\tau) = -1$
$\quad y_0(0) = 0,\ y_0'(0) = 1.$

The solution of this problem is $y_0(\tau) = -\frac{1}{2}\tau^2 + \tau$. Also, because of the second term on the right-hand side of (1.35a), we must take $\alpha = 1$ in (1.34).

$O(\varepsilon)$ $\quad y_1''(\tau) = 2y_0(\tau)$
$\quad y_1(0) = 0,\ y_1'(0) = 0.$

The solution here is $y_1(\tau) = \frac{1}{3}\tau^3 - \frac{1}{12}\tau^4$.

Therefore, we have that

$$y(\tau) \sim \tau\left(1 - \frac{1}{2}\tau\right) + \frac{1}{3}\varepsilon\tau^3\left(1 - \frac{1}{4}\tau\right). \tag{1.36}$$

This approximation applies for $0 \leq \tau \leq \tau_h$, where $\tau_h > 0$ is the point where $y(\tau_h) = 0$ (i.e., the time when the projectile returns to the surface of the Earth). A comparison between the asymptotic expansion and the numerical solution of the problem is given in Fig. 1.5. It is seen that the two-term expansion is essentially indistinguishable from the numerical solution, which indicates the accuracy of the approximation.

The second term (y_1) in (1.36) contains the contribution of the nonlinearity, and the first term (y_0) is the solution one obtains in a uniform gravitational field (it is the scaled version of the solution given in (1.2)). Since $y_1' \geq 0$ for $0 \leq \tau \leq \tau_h$ it follows that the contribution of y_1 grows with time and it increases the flight time (i.e., it makes τ_h larger). This is in agreement with the physics, which says that the gravitational force decreases with height, which should allow the projectile to stay up longer (see Exercise 2). This is a fairly simple example so not a lot can be said about the solution. What this does illustrate, though, is that an asymptotic approximation is capable of furnishing insights into the properties of the solution, which allows us to develop an understanding of the situation that may not be possible otherwise.

Example: A Nonlinear Potential Problem

The ideas developed in the last problem can also be applied to partial differential equations. An interesting example arises in the theory for the diffusion of ions through a solution containing charged molecules (i.e., a

FIGURE 1.5. Comparison between the asymptotic solutions in (1.36) and the numerical solution of (1.33a,b). In these calculations $\varepsilon = 10^{-1}$. There is little difference between the numerical solution and the two-term expansion.

polyelectrolyte solution). If the solution occupies a domain Ω, then the electrostatic potential $\phi(\mathbf{x})$ in the solution satisfies the Poisson-Boltzmann equation

$$\nabla^2\phi = -\sum_{i=1}^{k} \alpha_i z_i e^{-z_i\phi}, \text{ for } \mathbf{x} \in \Omega, \tag{1.37}$$

where the α_i's are positive constants, z_i is the valence of the ionic species, and the sum is over all the ionic species that are present. For example, if a simple salt such as NaCl is present, then in the sum $k = 2$ with $z_1 = -1$ and $z_2 = 1$. There is also an electroneutrality condition for the problem which states that the valences and constants must satisfy

$$\sum_{i=1}^{k} \alpha_i z_i = 0. \tag{1.38}$$

On the boundary $\partial\Omega$ a uniform charge is imposed, and this condition takes the form

$$\partial_n\phi = -\varepsilon, \text{ on } \partial\Omega, \tag{1.39}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ (note $\partial_n = \mathbf{n}\cdot\nabla$ is the normal derivative).

For cultural reasons it's worth pointing out two important special cases of (1.37). The first arises when the sum contains two terms, with $z_1 = -1$ and $z_2 = 1$. Using (1.38) allows (1.37) to be rewritten as $\nabla^2\phi = \alpha\sinh(\phi)$, which is known as the *sinh-Poisson equation* (or the sinh-Gordon equation). The second special case occurs when there is only one term in the sum and $z_1 = 1$. In this situation (1.37) reduces to $\nabla^2\phi = \alpha e^{\phi}$, which is *Liouville's equation.* It is also interesting to note that (1.37) arises in several diverse areas of applied mathematics. For example, it occurs in the modeling of semiconductors (Markowich et al., 1990), gel diffusion (Holmes, 1990), combustion (Kapila, 1983), relativistic field theory (D'Hoker and Jackiw, 1982), and hydrodynamic stability (Stuart, 1971). It has also been dealt with extensively by MacGillivray (1972) in deriving Manning's condensation theory.

The problem we must solve consists of the nonlinear partial differential equation in (1.37) subject to the boundary condition in (1.39). This is a very difficult problem, and a closed-form solution is beyond the range of current mathematical methods. To deal with this, in the Debye-Hückel theory presented in most introductory physical chemistry textbooks, it is assumed that the potential is small enough that the Poisson-Boltzmann equation can be linearized. This reduction is usually done heuristically, but here we want to carry out the calculations more systematically. For our problem, a small potential means ε is small. Given the boundary condition

in (1.39), the appropriate expansion of the potential, for small ε, is

$$\phi \sim \varepsilon(\phi_0(\mathbf{x}) + \varepsilon\phi_1(\mathbf{x}) + \ldots). \tag{1.40}$$

Substituting this into (1.37), we obtain

$$\begin{aligned}
&\varepsilon(\nabla^2\phi_0 + \varepsilon\nabla^2\phi_1 + \ldots) \\
&\quad = -\sum \alpha_i z_i e^{-\varepsilon z_i(\phi_0+\varepsilon\phi_1+\ldots)} \\
&\quad \sim -\sum \alpha_i z_i[1 - \varepsilon z_i(\phi_0 + \varepsilon\phi_1 + \ldots) + \frac{1}{2}z_i^2\varepsilon^2\phi_0^2 + \ldots] \\
&\quad \sim -\sum \alpha_i z_i + \varepsilon\sum \alpha_i z_i^2\phi_0 + \varepsilon^2\sum \alpha_i z_i^2(\phi_1 - \frac{1}{2}z_i\phi_0^2) + \ldots .
\end{aligned} \tag{1.41}$$

From the electroneutrality condition in (1.38), and setting

$$\kappa^2 = \sum_{i=1}^{k} \alpha_i z_i^2,$$

it follows that the $O(\varepsilon)$ problem is

$$\nabla^2\phi_0 = \kappa^2\phi_0, \text{ for } \mathbf{x} \in \Omega, \tag{1.42a}$$

where, from (1.39),

$$\partial_n\phi_0 = -1, \text{ on } \partial\Omega. \tag{1.42b}$$

To solve this problem, the domain Ω needs to be specified; an example follows.

The first-term approximation $\phi \sim \varepsilon\phi_0(\mathbf{x})$ is the basis of the classical Debye-Hückel theory in electrochemistry. One of the advantages of our approach is that it is relatively easy to find the correction to this approximation, and this is accomplished by solving the $O(\varepsilon^2)$ problem. From (1.41), one finds that

$$(\nabla^2 - \kappa^2)\phi_1 = -\lambda\phi_0^2, \text{ for } \mathbf{x} \in \Omega, \tag{1.43a}$$

where $\lambda = \frac{1}{2}\sum \alpha_i z_i^3$ and, from (1.39),

$$\partial_n\phi_1 = 0, \text{ on } \partial\Omega. \tag{1.43b}$$

It is assumed here that $\lambda \neq 0$.

It remains to solve the problems we have derived for the first two terms in the expansion. In electrochemistry, an often-used domain in two dimensions is the region in the plane outside of the unit circle (i.e., the region $1 < r < \infty$). Using polar coordinates, the solution is independent of the

angular variable and both (1.42a) and (1.43a) reduce to ordinary differential equations in the radial coordinate. Carrying out the calculations, one finds that the solution of (1.42a,b) is

$$\phi_0 = -\frac{K_0(\kappa r)}{\kappa K_1(\kappa)},$$

where K_0 and K_1 are modified Bessel functions. With this it is not difficult to solve (1.43) for ϕ_1 using, for example, the method of variation of parameters.

Our approach to regular perturbation expansions is constructive, and in the development little has been said about the theoretical foundation of the subject. For example, we have not made any attempt to prove rigorously that the expansions are asymptotic. This requires estimates of the error, and a nice introduction to this subject can be found in Murdock (1991).

Exercises

1. Find a two-term asymptotic expansion, for small ε, of the solution of the following problems:

   (a) $y'' + \varepsilon y' - y = 1$, where $y(0) = y(1) = 1$.

   (b) $y'' + y + \varepsilon y^3 = 0$, where $y(0) = 0$ and $y(\pi/2) = 1$.

   (c) $y'' + f(\varepsilon y) = 0$, where $y(0) = y(1) = 0$ and $f(s)$ is a smooth positive function.

   (d) $y'' + y + y^3 = 0$, where $y(0) = 0$ and $y(\pi/2) = \varepsilon$.

   (e) $\nabla^2 u = 0$, for $x^2 + y^2 < 1$, where $u = e^{\varepsilon \cos(\theta)}$ when $x^2 + y^2 = 1$.

   (f) $\partial_t u = \partial_x[D(\varepsilon u)\partial_x u]$, for $0 < x < 1$ and $0 < t$, where $u(0,t) = u(1,t) = 0$ and $u(x,0) = \sin(4\pi x)$. Also, $D(s)$ is a smooth positive function with $D'(0) > 0$.

2. This problem concerns aspects of the projectile problem.

   (a) Assuming $\tau_h \sim \tau_0 + \varepsilon\tau_1$, find τ_0, τ_1 from (1.36). Give a physical reason why τ_1 is positive.

   (b) Find a two-term expansion for the time at which the projectile reaches its maximum height. How much higher does the projectile get in the nonuniform gravitational field? (You should find a first-term approximation for this.)

3. In the projectile problem, to account for air resistance one obtains the equation

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2} - \frac{k}{R+x}\frac{dx}{dt},$$

where k is a nonnegative constant. Assume here that $x(0) = 0$ and $x'(0) = v_0$.

(a) What is the nondimensional version of this problem if one uses the same scaling as in Section 1.1?

(b) Find a two-term asymptotic expansion of the solution for small ε. Assume in doing this that $\alpha = kv_0/gR$ is independent of ε.

(c) Does the addition of air resistance increase or decrease the flight time?

4. The eigenvalue problem for the vertical displacement, $y(x)$, of an elastic string with variable density is

$$y'' + \lambda^2 \rho(x, \varepsilon) y = 0, \text{ for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. For small ε assume $\rho \sim 1 + \varepsilon\mu(x)$, where $\mu(x)$ is positive and continuous. In this case, the solution $y(x)$ and eigenvalue λ depend on ε, and the appropriate expansions are $y \sim y_0(x) + \varepsilon y_1(x)$ and $\lambda \sim \lambda_0 + \varepsilon\lambda_1$ (better expansions will be discussed in Section 3.3).

(a) Find y_0 and λ_0.

(b) Find y_1 and λ_1.

5. An interesting problem that can trace its origins back to Bernoulli and Euler is that of a whirling elastic string that is held fixed at each end (something like a jump-rope). The problem comes down to solving for a function $y(x)$ and a frequency of rotation ω that satisfy (Caughey, 1970)

$$y'' + \omega^2 \left(\frac{1}{\gamma(1 + \varepsilon^2 y^2)^{1/2}} + \alpha^2 \right) y = 0, \text{ for } 0 < x < 1,$$

where $y'(0) = y'(1) = 0$. Here α is a constant that satisfies $0 < \alpha < 1$, and γ is a constant that depends on y through the equation

$$\gamma = \frac{1}{\alpha^2} \left(\frac{1}{1 - \alpha^2} - \int_0^1 \frac{dx}{(1 + \varepsilon^2 y^2)^{1/2}} \right).$$

There is one additional condition imposed on the solution, namely that

$$\int_0^1 y(x) dx = 0.$$

What we have here is an eigenvalue problem where the frequency of rotation is the eigenvalue.

(a) Find a two-term expansion of ω for small ε.

(b) Comment on the accuracy of the expansion you found in part (a) for $0 < \alpha < 1$.

(c) Caughey (1970) proves that whirling cannot occur at frequencies below $\omega = \pi$. Is your expansion in part (a) consistent with this result?

6. The equation for the displacement, $u(x)$, of a weakly nonlinear string, on an elastic foundation and with a uniform forcing, is

$$\frac{d}{dx}\left(\frac{u_x}{\sqrt{1+\varepsilon(u_x)^2}}\right) - k^2 u = 1, \text{ for } 0 < x < 1,$$

where $u(0) = u(1) = 0$ and k is a positive constant. Find a two-term expansion of the solution for small ε. You should find, but do not need to solve, the problem for the second term.

7. The equation for the displacement, $u(x)$, of a nonlinear beam, on an elastic foundation and with a small periodic forcing, is

$$u'''' - \kappa u'' + k^2 u = \varepsilon F_0 \sin(\pi x), \text{ for } 0 < x < 1,$$

where $u(0) = u''(0) = u(1) = u''(1) = 0$, k is a positive constant, and F_0 is a nonzero constant. Also,

$$\kappa = \frac{1}{4}\int_0^1 (u_x)^2 dx.$$

Find a two-term expansion of the solution for small ε.

8. Consider the following eigenvalue problem involving a Fredholm integral equation:

$$\int_0^a K(x,s)y(s)ds = \lambda y(x), \text{ for } 0 < x < a.$$

The kernel $K(x,s)$ is known and is assumed to be smooth and positive. Also, the eigenfunction is taken to be positive and normalized so that

$$\int_0^a y^2 ds = a.$$

The eigenvalue λ and the eigenfunction $y(x)$ depend on the value of the parameter a. This exercise investigates how to find this dependence in the case where a is small.

(a) Find the first two terms in the expansions of λ and $y(x)$ for small a.

(b) By changing variables, transform the integral equation into the following (Knessl and Keller, 1991a) equation:

$$\int_0^1 K(a\xi, ar)\phi(r)dr = \frac{\lambda}{a}\phi(\xi), \text{ for } 0 < \xi < 1.$$

What happens to the normalization?

(c) From part (b), find a two-term expansion of λ and $\phi(\xi)$ for small a.

(d) Explain why the expansions in parts (a) and (c) are the same for λ but those for $y(x)$ differ.

9. In quantum mechanics, the perturbation theory for bound states involves the time-independent (normalized) Schrödinger equation

$$\Psi'' - [V_0(x) + \varepsilon V_1(x)]\Psi = -E\Psi, \text{ for } -\infty < x < \infty,$$

where $\Psi(-\infty) = \Psi(\infty) = 0$. In this problem, the eigenvalue E is the energy, V_1 is the perturbing potential, and ε is called the coupling constant. The potentials V_0 and V_1 are given continuous functions. This exercise examines what is known as a logarithmic perturbation expansion to find the corrections to the energy (Imbo and Sukhatme, 1984). To do this, it is assumed that the unperturbed ($\varepsilon = 0$) state is nonzero (more specifically, it is a nondegenerate ground state).

(a) Assuming $\Psi \sim \Psi_0(x)+\varepsilon\Psi_1(x)+\varepsilon^2\Psi_2(x)$ and $E \sim E_0+\varepsilon E_1+\varepsilon^2 E_2$, find what problem the first term in each of these expansions satisfies. In this problem, assume

$$\int_{-\infty}^{\infty} \Psi_0^2 dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} |V_1(x)| dx \quad \text{is finite.}$$

(b) Letting $\Psi = e^{\varphi(x)}$, find the problem $\varphi(x)$ satisfies.

(c) Expand $\varphi(x)$ for small ε and from this find E_1 and E_2 in terms of Ψ_0 and the perturbing potential.

(d) For a harmonic oscillator (i.e., $V_0 = \lambda^2 x^2$ with $\lambda > 0$) with perturbing potential $V_1 = \alpha x e^{-\gamma x^2}$ (where α and γ are positive), show that

$$E \sim \lambda - \frac{1}{4}\left(\frac{\varepsilon\alpha}{\gamma+\lambda}\right)^2 \left(\frac{\lambda}{\lambda+2\gamma}\right)^{1/2}.$$

10. From Newton's second law, the displacement, $x(t)$, of the mass in a mass-spring-dashpot system satisfies $mx'' = F_s + F_d + F_e$, where m is the mass, F_s is the restoring force in the spring, F_d is the damping force, and F_e is the external force (see Fig. 1.6). In this problem, take $F_e = 0$, and assume $x(0) = 0$ and $x'(0) = v_0$.

(a) Suppose there is linear damping, $F_d = -cx'$, and the spring is linear, $F_s = -kx$. Nondimensionalize the problem using the time scale for the undamped ($c = 0$) system. Your answer should contain the parameter $\alpha = ck^{1/2}/m^{3/2}$; explain why α is the ratio of the period of the undamped oscillator to the damping time scale.

FIGURE 1.6.

(b) Assume there is linear damping but that the spring is nonlinear with $F_s = -kx - Kx^3$, where K is a positive constant. The equation in this case is known as Duffing's equation. Nondimensionalize the problem using the same scaling as in part (a). The problem should contain two parameters, α and $\varepsilon = Kv_0^2k^2/m^3$. Explain why the latter is the ratio of the nonlinear to linear component of the spring.

(c) For the scaled problem in part (b), find a two-term expansion of the solution in the case where the spring is weakly nonlinear (i.e., ε is small).

11. Consider the second-order difference equation

$$y_{n+1} - 2y_n + y_{n-1} = (\alpha + \varepsilon f_n)y_n, \text{ for } n = 1, 2, 3, \ldots,$$

where α is a positive constant and $f_n = (1/2)^n$.

(a) Find a two-term expansion of the solution for small ε.

(b) Suppose $y_0 = 1$, $y_1 = 0$, and $\alpha = 2$. Calculate the exact solution, and the two-term expansion you found in part (a), for $n = 2, 3, \ldots, 10$. Comment on the accuracy of the approximation.

12. The transport of calcium in a cell is described by the following nonlinear diffusion problem for the concentration $c(\mathbf{x}, t)$ of calcium in a bounded domain Ω (Holmes and Bell, 1991):

$$\nabla^2 c = \varepsilon[\partial_t c + f(c)], \text{ for } \mathbf{x} \in \Omega \text{ and } t > 0,$$

where

$$\partial_n c = \begin{cases} 0 & \text{for } \mathbf{x} \in \partial\Omega_0 \\ \varepsilon g(t) & \text{for } \mathbf{x} \in \partial\Omega_1 \end{cases}$$

and $c(\mathbf{x}, 0) = 0$. Here ∂_n is the normal derivative and $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$ is the boundary of Ω. Also, f and g are smooth positive functions, and $\partial\Omega$ is regular. Find the ordinary differential equation and initial condition that determine the first term in the expansion of $c(\mathbf{x}, t)$ for small ε.

13. It is not uncommon to come across a problem involving a long, thin object (e.g., fluid flow in a tube, motion of an elastic bar, air flow over an airplane wing, etc.). This exercise examines what happens to the normal derivative in such situations.

    (a) Let the surface of such an object be described in dimensional coordinates as $z^* = \ell f(x^*/L, y^*/L)$ where ℓ and L are length scales associated with the size of the object and $f(x, y)$ is smooth. Because the object is long and thin we take $\ell << L$. i) Write the formula for a normal to the surface and the associated normal derivative. After this, convert the expressions to the dimensionless coordinates $x = x^*/L$, $y = y^*/L$, $z = z^*/L$. ii) Let $\varepsilon = \ell/L$. Find a two-term expansion of the normal and the normal derivative for small ε.

    (b) Suppose the surface is an axisymmetric tube of variable radius that can be described in cylindrical coordinates as $r^* = \ell R(z^*/L)$, where $R(z)$ is smooth. After switching to the scaled coordinates $r^* = \ell r$ and $z^* = Lz$, show that a two-term expansion of the normal derivative, for $\ell << L$, is

$$\partial_{n^*} \sim \frac{1}{\ell}(\partial_r - \varepsilon^2 R'(z)\partial_z).$$

14. The theory of self-gravitating annuli orbiting a central mass is used to study planetary rings, rings around galaxies, and other such phenomena. A problem that arises in this theory is to find the mass density, $\rho(r)$, of the ring. This is determined from the following problem (Christodoulou and Narayan, 1992):

$$\frac{1}{r}\frac{d}{dr}r\left(\frac{d}{dr}\rho^{1/n}\right) + \alpha\rho = \frac{1}{r^3} - \frac{\beta}{r^{2q}}, \text{ for } 1-\varepsilon < r < 1+\varepsilon,$$

    where $\rho(1-\varepsilon) = \rho(1+\varepsilon) = 0$. Also, $\alpha > 0$, $\beta > 1$, $q > 3/2$, and $n \geq 1$ are constants. This problem is concerned with what is known as a *slender annulus approximation*, that is, $\varepsilon << 1$.

    (a) Rescale the independent variable so the interval is independent of ε.

    (b) Assuming the density is bounded for small ε, find the first term in an expansion of the mass density.

    (c) Find the second term in the expansion of the mass density.

15. To find the flow of air over an airplane wing, as illustrated in Fig. 1.7, one can reduce the problem to solving the following integral equation for $u(x, \varepsilon)$ (Geer and Keller, 1968):

$$\int_{\alpha(\varepsilon)}^{\beta(\varepsilon)} K(x, \zeta, \varepsilon)u(\zeta, \varepsilon)d\zeta = -2\pi f'(x), \text{ for } \alpha(\varepsilon) < x < \beta(\varepsilon),$$

where

$$K(x,\zeta,\varepsilon) = \frac{f(x) - (x-\zeta)f'(x)}{(x-\zeta)^2 + \varepsilon^2 f^2(x)},$$

and $0 \le \alpha(\varepsilon) < \beta(\varepsilon) \le 1$. This equation is to determine $u(x,\varepsilon)$ as well as the endpoints $\alpha(\varepsilon)$ and $\beta(\varepsilon)$. The function $f(x)$ is known, and it is associated with the shape of the airplane wing (see Fig. 1.7). For this reason, $f(x)$ is assumed to be smooth, it is positive for $0 < x < 1$, and $f(0) = f(1) = 0$. We will also assume $f'(x) \neq 0$ at $x = 0, 1$.

(a) For small ε, explain why it is not appropriate to expand K using the binomial expansion.

(b) In the case where the wing can be described by the ellipse $4\left(x - \frac{1}{2}\right)^2 + \left(\frac{y}{\varepsilon}\right)^2 = 1$, the solution of the integral equation is

$$u(x,\varepsilon) = \frac{\varepsilon}{1-2\varepsilon}\frac{2-4x}{\sqrt{(\beta-x)(x-\alpha)}},$$

where $2\alpha = 1 - \sqrt{1-4\varepsilon^2}$ and $2\beta = 1 + \sqrt{1-4\varepsilon^2}$. For small ε, find the first two terms in the expansions for α, β, and u.

1.7 Uniformity

It is usually the case that the functions expanded in an asymptotic expansion depend on more than just the expansion parameter. This can interfere with the accuracy, or even existence, of an asymptotic expansion. To illustrate the situation, consider the following equation for y:

$$(y-1)(y-x) + \varepsilon y = 0.$$

We are interested here in values of x in the interval $0 < x < 1$. For small ε the appropriate expansion of the solution is $y \sim y_0 + \varepsilon y_1 + \ldots$. Substituting this into the equation and carrying out the calculations, one finds the following expansion for one of the solutions:

$$y \sim 1 - \frac{\varepsilon}{1-x}.$$

FIGURE 1.7.

There is no doubt that this is an asymptotic expansion of the solution; the only question is how accurate it is for values of x that are close to $x = 1$. Given the nature of the $O(\varepsilon)$ term in the expansion, it is expected that it will be necessary to have very small ε's if such x's are used. To say this in a slightly different way, suppose we are going to use only the first-term approximation $y \sim 1$ and we want to determine how small ε should be so the error in this approximation is less than, say, 10^{-4} for all the x's under consideration. Using the second term as a measure of the error, we are asking if it is possible to find an interval $0 < \varepsilon < \varepsilon_2$ so $\varepsilon/(1-x) < 10^{-4}$ for $0 < x < 1$. Clearly this is not possible because $\varepsilon/(1-x)$ is unbounded as a function of x over this interval. For this reason, the expansion is not uniformly valid for $0 < x < 1$. On the other hand, it is uniformly valid if, say, $0 < x < 3/4$.

The following definition is introduced to make precise what is meant by a uniform approximation. It is written for the specific case of small ε because that is the only limit we will be considering in this book when dealing with uniformity.

Definition: Suppose $f(x,\varepsilon)$ and $\phi(x,\varepsilon)$ are continuous functions for x from a set I and ε from an interval $0 < \varepsilon < \varepsilon_1$. In this case, $\phi(x,\varepsilon)$ is a *uniformly valid* asymptotic approximation to $f(x,\varepsilon)$ for x in I as $\varepsilon \downarrow 0$ if given any positive constant δ there is an ε_2 (independent of x and ε) so that

$$|f - \phi| \leq \delta|\phi| \text{ for } x \in I \text{ and } 0 < \varepsilon < \varepsilon_2.$$

The critical point in this definition is that it is possible to find an interval near $\varepsilon = 0$ (specifically, $0 < \varepsilon < \varepsilon_2$) such that the inequality holds for all values of x that are under consideration. This is essentially the idea used to define uniform convergence, and this enables us to make several useful observations.

First, suppose x_0 is a point from I or is a point from the boundary of I. We introduce functions $\phi_0(\varepsilon)$ and $f_0(\varepsilon)$ that are defined as

$$\phi_0(\varepsilon) = \lim_{x \to x_0} \phi(x,\varepsilon) \quad \text{and} \quad f_0(\varepsilon) = \lim_{x \to x_0} f(x,\varepsilon). \tag{1.44}$$

It is assumed that the convergence here is uniform (see Exercise 7). It should also be pointed out that the x's used in these limits are restricted to the set I. Now, if $f \sim \phi$ is uniformly valid on I and if $\phi(x,\varepsilon)$ is bounded, then it must be that

$$\lim_{\varepsilon \downarrow 0} \lim_{x \to x_0} \phi(x,\varepsilon) = \lim_{\varepsilon \downarrow 0} \lim_{x \to x_0} f(x,\varepsilon). \tag{1.45}$$

The proof of this statement is similar to the one relating uniform convergence and continuity and is left as an exercise. To illustrate how the result can be used, recall that in Section 1.4 it was found that, for small

ε, $x + e^{-x/\varepsilon} \sim x$ for $0 < x < 1$. It was also stated that this approximation is not uniform and the problem was at $x = 0$ (see Fig. 1.2). Since $f(x, \varepsilon) = x + e^{-x/\varepsilon}$ and $\phi(x, \varepsilon) = x$, then the above limits are

$$\lim_{\varepsilon \downarrow 0} \lim_{x \to 0} \phi(x, \varepsilon) = 0$$

and

$$\lim_{\varepsilon \downarrow 0} \lim_{x \to 0} f(x, \varepsilon) = 1.$$

It is not hard to show that the limits in (1.44) are uniform for these functions. Therefore, since (1.45) does not hold, it follows that $f \sim \phi$ is not uniformly valid on the interval $0 < x < 1$.

As illustrated in the previous example, the result in (1.45) is useful for showing that an expansion is nonuniform. There are also simple tests for proving that an approximation is uniform. We will state only one of these; to do this assume the set I is bounded and let $\bar{I}$ be the closure of I (recall that the closure consists of the points of I along with the boundary points for I). Also, as stated earlier, we assume the functions are continuous. It is a common mistake in asymptotics to think that if an expansion holds on $\bar{I}$ it must be uniformly valid (this is often stated when constructing what are known as composite expansions using the method of matched asymptotic expansions). An example that illustrates this is given in Exercise 3. What is true is that if $|\phi|$ is a monotonically decreasing function of ε for each x from I, and if the asymptotic approximation holds on $\bar{I}$, then it is uniformly valid on I. Again the proof of this is very similar to the analogous result for uniform convergence and is left to the reader.

There is another way to express the result in (1.45) that incorporates some of the ideas associated with an asymptotic approximation. To explain, suppose x_0 is a point from I or is a point from the boundary of I. Also, let $f(x, \varepsilon) \sim \phi(x, \varepsilon)$ for x from I, and let $f(x_0, \varepsilon) \sim \phi_0(\varepsilon)$. If the approximation is uniform, it is natural to expect that $\phi(x_0, \varepsilon)$ and $\phi_0(\varepsilon)$ are somehow related to one another. This is true and it can be used to determine if an approximation is not uniform. The precise statement is that if

$$\lim_{\varepsilon \downarrow 0} \frac{\phi(x_0, \varepsilon)}{\phi_0(\varepsilon)} \neq 1, \tag{1.46}$$

then $f(x, \varepsilon) \sim \phi(x, \varepsilon)$ is not uniform on I. It is assumed here that ϕ_0 is nonzero, so the quotient in (1.46) makes sense. Some readers may wonder whether it is true that if the approximation is uniform then it must be that $\phi_0(\varepsilon) = \phi(x_0, \varepsilon)$; the answer is no. These functions only need to be *equal asymptotically*, that is,

$$\lim_{\varepsilon \downarrow 0} \frac{\phi(x_0, \varepsilon)}{\phi_0(\varepsilon)} = 1. \tag{1.47}$$

To summarize these results, we have the following theorem.

Theorem 1.3: Assume f, ϕ, and ϕ_0 are continuous for $x \in \bar{I}$ and $0 < \varepsilon < \varepsilon_1$. Also, let I be nonempty and bounded and let $x_0 \in \bar{I}$.
(a) If $f \sim \phi$ for $x \in \bar{I}$, and if $|\phi(x, \varepsilon)|$ is monotonically decreasing in ε (for $0 < \varepsilon < \varepsilon_1$ and $x \in I$), then this approximation is uniformly valid on I (as well as on $\bar{I}$).
(b) Suppose $f \sim \phi$ for $x \in I$ and $f \sim \phi_0(\varepsilon)$ for $x = x_0$. If $\phi_0(\varepsilon)$ is nonzero and (1.46) holds, then $f \sim \phi$ is not uniformly valid on I.

The idea of uniformity can be extended to asymptotic expansions. To explain how, suppose we have a function $f(x, \varepsilon)$ and

$$f(x, \varepsilon) \sim a_1(x)\phi_1(x, \varepsilon) + a_2(x)\phi_2(x, \varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

In other words, we have expanded f in terms of the basis functions ϕ_1 and ϕ_2, and they possibly depend on x (in addition to ε). To be able to say that this expansion is uniformly valid, at the minimum we must have that it remains well ordered for all values of x under consideration. The exact statement is: The expansion is uniformly valid for x from I if given any positive constant δ we can find an ε_2 (independent of x and ε) such that for $0 < \varepsilon < \varepsilon_2$,

$$|f(x, \varepsilon) - a_1(x)\phi_1(x, \varepsilon)| \leq \delta |a_1(x)\phi_1(x, \varepsilon)|$$

and

$$|f(x, \varepsilon) - a_1(x)\phi_1(x, \varepsilon) - a_2(x)\phi_2(x, \varepsilon)| \leq \delta |a_2(x)\phi_2(x, \varepsilon)|,$$

for all values of x under consideration. This idea extends in a straightforward manner to expansions involving an arbitrary number of scale functions.

As a comment in passing, in the above discussion the variable x was taken to be a scalar. Nevertheless, the definitions and theorem that have been given hold (without change) for the case where x is a vector.

As usual, given a function $f(x, \varepsilon)$, one of the main tools that is used to establish uniformity is Taylor's theorem, which is illustrated in the following examples.

Examples

1. If $f = \sin(x + \varepsilon)$, then $f \sim \sin(x)$ is uniformly valid for $\frac{\pi}{4} < x < \frac{3\pi}{4}$. This can be established using Taylor's theorem (about $\varepsilon = 0$), which gives us that
$$f = \sin(x) + \varepsilon \cos(x + \xi).$$
Thus, $\lim_{\varepsilon \downarrow 0}[f - \sin(x)] = 0$ and from this we have that $f \sim \sin(x)$ for $\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$. Also, since $\phi = \sin(x)$ is independent of ε, it follows from Theorem 1.3 that the approximation is uniformly valid over the

stated interval. What may be surprising is that it is not uniform for $0 < x < \frac{\pi}{2}$ even though (1.45) holds. The easiest way to see this is to note $f \sim \varepsilon$ at $x = 0$, and so, $\phi_0(\varepsilon) = \varepsilon$ and $\phi(0, \varepsilon) = 0$. Thus, the limit in (1.46) is not equal to 1 and this means the approximation is nonuniform on this interval.

2. A solution of the equation

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{1.48}$$

is

$$y(x) = e^{\frac{\alpha x}{\varepsilon}} + e^{\frac{\beta x}{\varepsilon}}, \tag{1.49}$$

where $\alpha = -1 + \sqrt{1 - 2\varepsilon}$ and $\beta = -1 - \sqrt{1 - 2\varepsilon}$. To obtain an asymptotic approximation of this solution for small ε, note $\alpha \sim -\varepsilon - \varepsilon^2$ and $\beta \sim -2 + \varepsilon$. Thus, for $0 < x < 1$,

$$y(x) \sim e^{\frac{-\varepsilon - \varepsilon^2}{\varepsilon} x} + e^{\frac{-2+\varepsilon}{\varepsilon} x} \tag{1.50a}$$

$$\sim e^{-x} + e^{-2x/\varepsilon} \tag{1.50b}$$

$$\sim e^{-x}. \tag{1.50c}$$

The last approximation, however, is not uniform for $0 < x < 1$. This is because the exponential that was dropped in going from (1.50b) to (1.50c) is of the same order as (1.50c) when $x = 0$ (one can also use (1.45) to show nonuniformity). If we were to retain this term, that is, if we use the approximation

$$y(x) \sim e^{-x} + e^{-2x/\varepsilon}, \tag{1.51}$$

FIGURE 1.8. Comparison between the exact solution, (1.49), of the differential equation, a nonuniform approximation, (1.50c), of the solution, and a uniform approximation, (1.51), of the solution. The uniform approximation is so close to the exact solution that the two curves are almost indistinguishable. In these calculations, $\varepsilon = 10^{-2}$.

then we have a uniformly valid approximation over the interval $0 < x < 1$ (see Exercise 2). To illustrate the differences between these approximations, they are plotted in Fig. 1.8 in the case of $\varepsilon = 10^{-2}$. Clearly, the uniform approximation gives a very good description of the function over the entire interval. Also, over most of the interval (1.50c) is close to the exact solution. The exception, as expected, is the region near $x = 0$, particularly in the region where $x = O(\varepsilon)$. This region is an example of what is known as a boundary layer and will be investigated extensively in the next chapter.

Exercises

1. For small ε find an asymptotic approximation of each solution y of the following equations and determine if it is uniform (in x) over the indicated interval:

   (a) $y^2 + (1 + \varepsilon - x)y - x = 0$, for $0 < x < 1$;

   (b) $y^3 + (1 - 2x)y^2 + (x^2 - 2x - \varepsilon)y + x^2 = 0$, for $0 < x < 1$.

2. Show that (1.51) is a uniform approximation to (1.49) for $0 < x < 1$.

3. If $f(\varepsilon) = 1$ and $\phi(\varepsilon) = 1 + \frac{x}{\varepsilon}e^{-x/\varepsilon}$, then $f \sim \phi$ for $0 \le x \le 1$ as $\varepsilon \downarrow 0$. Is this a uniform approximation? Does this violate Theorem 1.3?

4. For the function in (1.49), show that $\lim_{\varepsilon\downarrow 0} \lim_{x\downarrow 0} y(x) \neq \lim_{x\downarrow 0} \lim_{\varepsilon\downarrow 0} y(x)$. This observation is an indication that (1.48) is a singular perturbation problem, and this type of problem is the subject of the next chapter.

5. For the following functions, assuming $\varepsilon << 1$, find an asymptotic expansion of the form $f(x, \varepsilon) \sim \phi(x)$ that holds for $-1 < x < 1$. Sketch $f(x, \varepsilon)$ and $\phi(x)$, and then explain why the approximation is not uniform for $-1 < x < 1$.

   (a) $f(x, \varepsilon) = x + \exp((x^2 - 1)/\varepsilon)$.

   (b) $f(x, \varepsilon) = x + \tanh(x/\varepsilon)$.

   (c) $f(x, \varepsilon) = x + \operatorname{sech}(x/\varepsilon)$.

6. For the following functions, assuming $\varepsilon << 1$, find an asymptotic expansion of the form $f(t, \varepsilon) \sim \phi(t)$ that holds for $0 \le t < \infty$. Explain why the approximation is not uniform for $0 \le t < \infty$.

   (a) $f(t, \varepsilon) = \exp(-\varepsilon t)\sin(t)$.

   (b) $f(t, \varepsilon) = \sin(\omega t)$, where $\omega = 1 + \varepsilon$.

7. The equality of the limits in (1.45) was derived using the assumption that the limits in (1.44) are uniform. This means, for example, that

given any $\delta > 0$ it is possible to find η (independent of ε from the interval $0 < \varepsilon < \varepsilon_1$) so that

$$|\phi_0(\varepsilon) - \phi(x, \varepsilon)| < \delta \text{ if } |x - x_0| < \eta.$$

(a) Show that the limit is uniform if $|\phi_0(\varepsilon) - \phi(x, \varepsilon)|$ is monotonically decreasing as a function of ε.

(b) Show that the condition in part (a) holds if $\phi(x, \varepsilon) \geq \phi_0(\varepsilon)$ and

$$\frac{d}{d\varepsilon}(\phi(x, \varepsilon) - \phi_0(\varepsilon)) \geq 0.$$

1.8 Symbolic Computing

Anyone who uses perturbation methods is struck almost immediately by the amount of algebra that is sometimes necessary to find an approximation. Because one of the objectives of the derivation is to have an answer that contains the parameters in the problem, it is not convenient to use a computing language like FORTRAN or C to carry out these calculations since they require all variables (and parameters) to be evaluated. An alternative is to use a symbolic language like Macsyma, Maple, or Mathematica. Symbolic manipulators have been around for some time but only recently have they become widely used. For those who may be unfamiliar with these systems, they are capable of manipulating mathematical expressions without having to evaluate parameters. This ability has a tremendous potential for constructing asymptotic expansions, and so we will illustrate some simple applications. More extensive presentations can be found in Rand and Armbruster (1987) and Heck (1993).

The simplest example is in the use of Taylor's theorem. For example, suppose we want a two-term expansion of the function

$$f(\varepsilon) = \sinh\left(\frac{1}{10\alpha + \cos(\varepsilon)}\right),$$

where α is a constant. If we expect that the expansion of this function is a regular power series, that is,

$$f \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots,$$

then we can obtain the coefficients using a Taylor expansion. To do this, one must enter the formula for f, and in the following command *ep* designates ε and a represents α. In Maple, the command for the Taylor series expansion is then

$$\text{taylor}(f, ep, 3); \tag{1.52}$$

In a fraction of a second the machine dutifully responds with

$$\frac{1}{2}\exp\left(\frac{1}{10a+1}\right)-\frac{1}{2}\exp\left(-\frac{1}{10a+1}\right)$$
$$+\left(\frac{1}{4}\frac{\exp\left(\frac{1}{10a+1}\right)}{(10a+1)^2}+\frac{1}{4}\frac{\exp\left(-\frac{1}{10a+1}\right)}{(10a+1)^2}\right)ep^2+O(ep^3).$$

Clearly, the effort involved here is minimal, and this makes the use of a symbolic manipulator quite attractive. However, at the same time certain cautions are necessary. It is natural to think that since this is so easy then why not go for lots of terms and change the 3 in (1.52) to, say, 1000. There may actually be situations where something like this is necessary, but one would risk creating what is known as *intermediate expression swell.* This is the situation where the expressions being manipulated are so large that the machine is incapable of handling them with the allotted memory even though the final result is relatively simple. In terms of asymptotic approximations, such Herculean feats as deriving 1000 terms of an expansion are generally of little value. In fact, as we have seen earlier, the results may actually get worse. It should also be pointed out that even though these systems can be big time savers there are certain things they do not do easily, such as simplifying certain expressions. For example, the above expansion can be rewritten as

$$\sinh\left(\frac{1}{10a+1}\right)+\frac{\cosh\left(\frac{1}{10a+1}\right)}{2(10a+1)^2}ep^2+O(ep^3).$$

It can be very difficult to get the program to "notice" this fact. Also, the program does not warn the user that this expansion is singular when $10a+1=0$. The point here is that one must not stop thinking about the problem, or its solution, just because the work is being done by a machine.

The expansions for the solutions of many of the algebraic and differential equations studied in this chapter can be solved quite easily using a symbolic manipulator. To illustrate, consider the projectile problem (1.33). The Maple commands that can be used to construct the expansion are given in Table 1.1. The cpu time needed to carry out these commands depends on the machine being used; however, it should not be more than about five or ten seconds. Also, even though the example presented here uses Maple, it is possible to carry out similar calculations using most other symbolic programs.

It is strongly recommended that anyone using asymptotic expansions experiment with a symbolic computing system using a program like the one described here.

TABLE 1.1. Maple commands that can be used to find a two-term expansion of the projectile problem (1.33)

Command	Explanation
`y:=y0(t)+ep*y1(t)+ep*ep*y2(t);`	As in (1.34), this defines the terms in the expansion.
`de:=diff(y,t,t)+1/(1+ep*y)^2;`	As in (1.33a), this defines the differential equation.
`DE:=taylor(de,ep,3);`	As in (1.35a), this expands the equation in powers of *ep*.
`de0:=coeff(DE,ep,0);`	Find the $O(1)$ equation.
`de1:=coeff(DE,ep,1);`	Find the $O(ep)$ equation.
`y0(t):=rhs(dsolve({de0=0,y0(0)=0,D(y0)(0)=1},y0(t)));`	Solve the $O(1)$ problem using the initial conditions given in (1.33b).
`y1(t):=rhs(dsolve({de1=0,y1(0)=0,D(y1)(0)=0},y1(t)));`	Solve the $O(ep)$ problem.

Exercises

1. Given a function $f(\varepsilon)$, suppose its expansion has the form $f(\varepsilon) \sim a_1\varepsilon^{\alpha} + a_2\varepsilon^{\beta} + \dots$.

   (a) Determine a sequence of operations that will find (in order) α, a_1, β, and a_2.

   (b) Using a symbolic manipulator, employ your procedure to try to find the expansions for the functions of Exercise 1.4.2.

2. Consider the boundary-value problem

$$y'' = f(\varepsilon, x, y, y'), \text{ for } a < x < b,$$

   where $y(a) = \alpha$ and $y(b) = \beta$. Assume f is a smooth function and the constants a, b, α, and β are independent of ε. It's also assumed that the problem has a unique solution.

   (a) Construct an algorithm that will find the first two terms in the expansion of y for small ε.

   (b) Use your procedure to find the expansions for problems (a)–(c) of Exercise 1 at the end of Section 1.6.

2

Matched Asymptotic Expansions

2.1 Introduction

The ideas underlying an asymptotic approximation appeared in the early 1800s, when there was considerable interest in developing formulas to evaluate special functions. An example is the expansion of Bessel's function, given in (1.12), that was derived by Poisson in 1823. It was not until later in the century that the concept of an asymptotic solution of a differential equation took form, and the most significant efforts in this direction were connected with celestial mechanics. The subject of this chapter, what is traditionally known as matched asymptotic expansions, appeared somewhat later. Its early history is strongly associated with fluid mechanics, and specifically, aerodynamics. The initial development of the subject is credited to Prandtl (1905), who was concerned with the flow of a fluid past a solid body (such as an airplane wing). The partial differential equations for viscous fluid flow are quite complicated, but he argued that under certain conditions the effects of viscosity are concentrated in a narrow layer near the surface of the body. This happens, for example, with air flow across an airplane wing; a picture of this situation is shown in Fig. 2.1. This observation allowed Prandtl to go through an order of magnitude argument and omit terms he felt to be negligible in the equations. The result was a problem that he was able to solve. This was a brilliant piece of work, but it relied strongly on his physical intuition. For this reason, there were numerous questions about his reduction that went unresolved for decades. For example, it was unclear how to obtain the correction to his approximation,

and it is now thought that Prandtl's derivation of the second term is incorrect (Lagerstrom, 1988). This predicament was resolved when Friedrichs (1942) was able to show how to reduce a boundary layer problem systematically. In analyzing a model problem (Exercise 2.2.1), he used a stretching transformation to match inner and outer solutions, which is the basis of the method discussed in this chapter. This procedure was not new, however, as demonstrated by Gans' (1915) use of some of these ideas to solve problems in optics.

The golden age for matched asymptotic expansions was the 1950s, and it was during this period that the method was refined and applied to a wide variety of physical problems. A short historical development of the method is presented in O'Malley (1991) and a nice discussion of boundary layer methods used in the nineteenth century is given by Van Dyke (1994). The popularity of matched asymptotic expansions was also greatly enhanced with the appearance of two very good books, one by Van Dyke (1964) and another by Cole (1968). The method is now one of the cornerstones of applied mathematics. At the same time, it is still being extended, both in the type of problems it is used to resolve as well as in the theory.

2.2 Introductory Example

The best way to explain the method of matched asymptotic expansions is to use it to solve a problem. The following example takes several pages to complete because it is used to introduce the ideas and terminology. As the procedure becomes more routine, the derivations will become much shorter.

The problem we will study is

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{2.1a}$$

where

$$y(0) = 0 \tag{2.1b}$$

and

$$y(1) = 1. \tag{2.1c}$$

This equation is similar to the one used in Section 1.7 to discuss uniform and nonuniform approximations. The difference is that we will now derive the approximation directly from the problem rather than from a formula for the solution (see Exercise 6).

An indication that this problem is not going to be as straightforward as the differential equations considered in Section 1.6 is that if $\varepsilon = 0$ then the problem is no longer second order. This leads to what is generally known as a *singular perturbation problem*, although singularity can occur for other

FIGURE 2.1. Supersonic air flow, at Mach 1.45, over a wedge. The high-speed flow results in a shock layer in front of the wedge across which the pressure undergoes a rapid transition. Because of its position in the flow, the shock is an example of an interior layer (see Section 2.4). There are also boundary layers present (see Section 2.2). These can be seen near the surface of the wedge, and they are thin regions where the flow drops rapidly to zero (which is the speed of the wedge). From Bleakney et al. (1949).

reasons. In any case, to construct a first-term approximation of the solution for small ε we will proceed in four steps. The fifth step will be concerned with the derivation of the second term in the expansion.

Step 1: Outer Solution

To begin, we will assume that the solution can be expanded in powers of ε. In other words,

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \dots . \tag{2.2}$$

Substituting this into (2.1a), we obtain

$$\varepsilon(y_0'' + \dots) + 2(y_0' + \varepsilon y_1' + \dots) + 2(y_0 + \varepsilon y_1 + \dots) = 0.$$

The $O(1)$ equation is therefore

$$y_0' + y_0 = 0, \tag{2.3}$$

and the general solution of this is

$$y_0(x) = ae^{-x}, \tag{2.4}$$

where a is an arbitrary constant. Looking at the solution in (2.4), we have a dilemma because there is only one arbitrary constant but there are two boundary conditions (2.1b,c). What this means is that the solution in (2.4) and the expansion in (2.2) are incapable of describing the solution over the entire interval $0 \leq x \leq 1$. At the moment, we have no idea which boundary condition, if any, we should require $y_0(x)$ to satisfy, and the determination of this will have to come later. This leads to the question of what to do next. Well, based on what is observed in Example 2 in Section 1.7, it is a reasonable working hypothesis to assume that (2.4) describes the solution over most of the interval but that there is a boundary layer at either $x = 0$, or at $x = 1$, where a different approximation must be used. Assuming for the moment that it's at $x = 0$, then we are probably missing a term like $e^{-\beta x/\varepsilon}$. The derivation of this term is the objective of the next step. Because we are going to end up with approximations of the solution over different regions, we will refer to (2.4) as the first term in the expansion of the *outer solution.*

Step 2: Boundary Layer

Based on the assumption that there is a boundary layer at $x = 0$, we introduce a *boundary-layer coordinate* given as

$$\bar{x} = \frac{x}{\varepsilon^{\alpha}}, \tag{2.5}$$

where $\alpha > 0$. From our earlier discussion, it is expected that $\alpha = 1$, and this will be shown conclusively below. After changing variables from x to $\bar{x}$, we will take $\bar{x}$ to be fixed when expanding the solution in terms of ε. This has the effect of stretching the region near $x = 0$ as ε becomes small. Because of this, (2.5) is sometimes referred to as a *stretching transformation.*

From the change of variables in (2.5), and the chain rule, we have that

$$\frac{d}{dx} = \frac{d\bar{x}}{dx}\frac{d}{d\bar{x}} = \frac{1}{\varepsilon^{\alpha}}\frac{d}{d\bar{x}}.$$

If we let $Y(\bar{x})$ denote the solution of the problem when using this boundary-layer coordinate, then (2.1a) transforms to

$$\varepsilon^{1-2\alpha}\frac{d^2Y}{d\bar{x}^2} + 2\varepsilon^{-\alpha}\frac{dY}{d\bar{x}} + 2Y = 0, \tag{2.6a}$$

where, from (2.1b),

$$Y(0) = 0. \tag{2.6b}$$

The boundary condition at $x = 0$ has been included here because the boundary layer is at the left end of the interval.

The appropriate expansion for the *boundary-layer solution* is now

$$Y(\bar{x}) \sim Y_0(\bar{x}) + \varepsilon^{\gamma} Y_1(\bar{x}) + \ldots, \tag{2.7}$$

where $\gamma > 0$. As stated above, in this expansion $\bar{x}$ is held fixed as ε goes to zero (in the same way that x is held fixed in the outer expansion). Substituting the expansion in (2.7) into (2.6a), we get that

$$\underset{①}{\varepsilon^{1-2\alpha} \frac{d^2}{d\bar{x}^2}(Y_0 + \ldots)} + \underset{②}{2\varepsilon^{-\alpha} \frac{d}{d\bar{x}}(Y_0 + \ldots)} + \underset{③}{2(Y_0 + \ldots)} = 0. \tag{2.8}$$

Just as with the algebraic equations studied in Section 1.4, it is now necessary to determine the correct balancing in (2.8). The balance between terms ② and ③ was considered in Step 1, and so the following possibilities remain (also see Exercise 4):

i) ① $\sim$ ③ and ② is higher order $\Rightarrow 1 - 2\alpha = 0 \Rightarrow \alpha = \frac{1}{2} \Rightarrow$ ①, ③ $= O(1)$ but ② $= O(\varepsilon^{-1/2})$. This violates our original assumption that ② is higher order, and so this balance is not possible.

ii) ① $\sim$ ② and ③ is higher order $\Rightarrow 1 - 2\alpha = -\alpha \Rightarrow \alpha = 1 \Rightarrow$ ①, ② $= O(\varepsilon^{-1})$ and ③ $= O(1)$. In this case, the conclusions are consistent with the original assumptions, and so this is the balancing we are looking for. This is said to be a *distinguished limit* for the equation.

With this, we have the following problem to solve:

$$O\left(\tfrac{1}{\varepsilon}\right) \quad Y_0'' + 2Y_0' = 0, \text{ for } 0 < \bar{x} < \infty, \tag{2.9a}$$

$$Y_0(0) = 0. \tag{2.9b}$$

The general solution of this problem is

$$Y_0(\bar{x}) = A(1 - e^{-2\bar{x}}), \tag{2.10}$$

where A is an arbitrary constant. It should be observed that the boundary-layer equation (2.9a) contains at least one term of the outer-layer equation (2.3). This is important for the successful completion of Step 3.

The boundary-layer expansion in (2.7) is supposed to describe the solution in the immediate vicinity of the endpoint $x = 0$. It is therefore not unreasonable to expect that the outer solution (2.4) applies over the remainder of the interval (this is assuming there are no other layers). This means that the outer solution should satisfy the boundary condition at $x = 1$. From (2.4) and (2.1c), one finds that

$$y_0(x) = e^{1-x}. \tag{2.11}$$

FIGURE 2.2. Sketch of the inner solution (2.10) and the outer solution (2.11). Note the overlap region along the x-axis where both solutions are essentially constant. Since these approximations are supposed to be describing the same continuous function, it must be that these constants are the same. Hence, $A = e^1$.

Step 3: Matching

It remains to determine the constant A in the first-term approximation of the boundary-layer solution (2.10). To do this, the approximations we have constructed so far are summarized in Fig. 2.2. The important point here is that both the inner and outer expansions are approximations of the same function. Therefore, in the transition region between the inner and outer layers we should expect that the two expansions give the same result. This is accomplished by requiring that the value of Y_0 as one comes out of the boundary layer (i.e., as $\bar{x} \to \infty$) is equal to the value of y_0 as one comes into the boundary layer (i.e., as $x \to 0$). Imposing this condition yields $Y_0(\infty) = y_0(0)$. Thus, $A = e^1$ and (2.10) becomes

$$Y_0(\bar{x}) = e^1 - e^{1-2\bar{x}}. \tag{2.12}$$

This completes the derivation of the inner and outer approximations of the solution of (2.1). The next step is to combine them into a single expression (this is Step 4).

Because of the importance of matching, the procedure will now be explained in more detail. The approach to be described will work on more difficult problems than the simple example in (2.1). The idea is to introduce

FIGURE 2.3. Schematic of the domains of validity of the inner and outer expansions as assumed in the matching procedure. The intermediate variable is to be located within the overlap region.

an intermediate variable $x_\eta = x/\eta(\varepsilon)$ that is positioned between the $O(1)$ coordinate of the outer layer and the $O(\varepsilon)$ coordinate of the inner layer. This intermediate variable is to be located in the transition region, or overlap domain, seen in Fig. 2.2, and for this reason it should be expected that $\eta(\varepsilon)$ satisfies $\varepsilon << \eta << 1$. The precise conditions imposed on $\eta(\varepsilon)$ are stated explicitly in the following matching procedure (see Fig. 2.3):

i) Change variables in the outer expansion (from x to x_η) to obtain y_{outer}. It is assumed that there is an $\eta_1(\varepsilon)$ so that y_{outer} still provides a first-term expansion of the solution for any $\eta(\varepsilon)$ that satisfies $\eta_1(\varepsilon) << \eta(\varepsilon) \leq 1$.

ii) Change variables in the inner expansion (from $\bar{x}$ to x_η) to obtain y_{inner}. It is assumed that there is an $\eta_2(\varepsilon)$ so that y_{inner} still provides a first-term expansion of the solution for any $\eta(\varepsilon)$ that satisfies $\varepsilon \leq \eta(\varepsilon) << \eta_2$.

iii) It is assumed that the domains of validity of the expansions for y_{outer} and y_{inner} overlap, that is, $\eta_1 << \eta_2$. In this overlap region, for the expansions to match, it is required that the first terms from y_{outer} and y_{inner} are equal.

The assumptions contained in (i) and (ii) can be proved under fairly mild conditions and are the essence of Kaplun's extension theorem (Lagerstrom, 1988). Actually, one seldom is interested in determining η_1 or η_2 but only that there is an interval for $\eta(\varepsilon)$ so that y_{inner} and y_{outer} match. It is important, however, that the matching not depend on the specific choice of $\eta(\varepsilon)$. For example, if one finds that matching can only occur if $\eta(\varepsilon) = \varepsilon^{1/2}$, then there is no overlap domain and the procedure is not applicable. In comparison to the situation for (i) and (ii), the assumption on the existence of an overlap domain in (iii) is a different matter, and a satisfactory proof has never been given. For this reason, it has become known as Kaplun's hypothesis on the domain of validity (Lagerstrom, 1988).

To use the matching procedure just described, we introduce the intermediate variable x_η, defined as

$$x_\eta = \frac{x}{\varepsilon^\beta}, \tag{2.13}$$

where $0 < \beta < 1$. This interval for β comes from the requirement that the scaling for the intermediate variable lies between the outer scale, $O(1)$, and the inner scale, $O(\varepsilon)$. Actually, it may be that in carrying out the matching of y_{inner} and y_{outer} we have to reduce this interval for β. In any case, to see if the expansions match, note that the inner solution, from (2.7) and (2.10), becomes

$$\begin{aligned} y_{\text{inner}} &\sim A\left(1 - e^{-2x_\eta/\varepsilon^{1-\beta}}\right) + \ldots \\ &\sim A + \ldots, \end{aligned} \tag{2.14}$$

and the outer solution, from (2.2) and (2.11), becomes

$$\begin{aligned} y_{\text{outer}} &\sim e^{1-x_\eta \varepsilon^\beta} + \dots \\ &\sim e^1 + \dots . \end{aligned} \tag{2.15}$$

The expansions in (2.14) and (2.15) are supposed to agree to the first term in the overlap domain and therefore $A = e^1$.

The modification of the intermediate variable procedure to deal with higher-order expansions will not be given here, but in Step 5 a two-term expansion is matched to illustrate the situation. For those who may be interested, Lagerstrom (1988) has a more extensive discussion of the subtleties of matching using an intermediate variable. Also, there are other ways to match; a quite popular one due to Van Dyke (1975) is discussed in Exercise 14. His procedure is relatively simple to use but can occasionally lead to incorrect results (Fraenkel, 1969; Lagerstrom, 1988).

Step 4: Composite Expansion

Our description of the solution consists of two pieces, which we will now combine to form a *composite expansion.* This is done by adding the expansions and then subtracting the part that is common to both. Thus,

$$\begin{aligned} y &\sim y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - y_0(0) \\ &\sim e^{1-x} - e^{1-2x/\varepsilon}. \end{aligned} \tag{2.16}$$

It may seem strange that it is possible to combine expansions over different intervals and still have an asymptotic approximation of the solution. However, note that in the outer region the boundary-layer solution, $Y_0(x/\varepsilon)$, is constant to first order. This constant is $y_0(0)$, and to compensate for its contribution the term $-y_0(0)$ is included in the composite expansion. Similarly, in the boundary-layer region the outer solution, $y_0(x)$, is constant

FIGURE 2.4. Graph of the exact solution of (2.1) and the composite expansion given in (2.16) in the case where $\varepsilon = 10^{-2}$. The two curves are so close that they are almost indistinguishable in the graph.

to first order. However, the $-y_0(0)$ term removes its contribution in this region. The fact that the adjustment in each case involves the constant $y_0(0)$ is not a coincidence since it is the first term in the inner and outer expansions found from matching. Because of this, it is referred to as the common part of the expansions.

The fact that the composite expansion gives a very good approximation to the solution over the entire interval is shown in Fig. 2.4. Note, however, that it satisfies the boundary condition at $x = 0$ exactly but the one at $x = 1$ is only satisfied asymptotically. This is not of particular concern since the expansion also satisfies the differential equation in an asymptotic sense. However, an alternative expansion, which satisfies both boundary conditions, is developed in Exercise 16.

Step 5: Second Term

Generally, to illustrate a method we will only derive the first term in an expansion. However, the second term is important as it gives a measure of the error. The procedure to find the second term is very similar to finding the first so only the highlights will be given here.

By substituting the outer expansion (2.2) into the problem and collecting the $O(\varepsilon)$ terms, one finds that $y_1' + y_1 = -\frac{1}{2}y_0''$ with $y_1(1) = 0$. The solution of this problem is

$$y_1 = \frac{1}{2}(1-x)e^{1-x}.$$

Similarly, from the boundary-layer equation (2.8) we get that $\gamma = 1$ and $Y_1'' + 2Y_1' = -2Y_0$ with $Y_1(0) = 0$. The general solution of this is

$$Y_1 = B(1 - e^{-2\bar{x}}) - \bar{x}e^1(1 + e^{-2\bar{x}}),$$

where B is an arbitrary constant. To match the expansions, we use the intermediate variable given in (2.13). The outer expansion in this case takes the form

$$\begin{aligned} y_{\text{outer}} &\sim e^{1-x_\eta\varepsilon^\beta} + \frac{\varepsilon}{2}(1 - x_\eta\varepsilon^\beta)e^{1-x_\eta\varepsilon^\beta} + \dots \\ &\sim e^1 - \varepsilon^\beta x_\eta e^1 + \frac{\varepsilon}{2}e^1 + \frac{1}{2}\varepsilon^{2\beta}e^1x_\eta^2 + \dots, \end{aligned} \tag{2.17}$$

and, setting $\xi = -2x_\eta/\varepsilon^{1-\beta}$, the boundary-layer expansion becomes

$$\begin{aligned} y_{\text{inner}} &\sim e^1\left(1 - e^\xi\right) + \varepsilon\left[B\left(1 - e^\xi\right) - \frac{x_\eta e^1}{\varepsilon^{1-\beta}}\left(1 + e^\xi\right)\right] + \dots \\ &\sim e^1 - \varepsilon^\beta x_\eta e^1 + B\varepsilon. \end{aligned} \tag{2.18}$$

Matching these we get $B = \frac{1}{2}e^1$. Note, however, that these expansions do not appear to agree since (2.17) contains a $O(\varepsilon^{2\beta})$ term that (2.18) does

not have. To explain why this occurs, note that both expansions produce a $O(\varepsilon^\beta)$ term that does not contain an arbitrary constant. If this term is not identical for both expansions, there is no way the expansions will match. In the outer expansion, this term comes from the $O(1)$ problem, and in the boundary layer it comes from the $O(\varepsilon)$ solution. In a similar manner, one finds that the x_η^2 term in (2.17) also comes from the first term. However, for the boundary layer it comes from the $O(\varepsilon^2)$ problem (the verification of this is left as an exercise). Therefore, the expansions match.

Occasionally it happens that an expansion (inner or outer) produces a term of an order, or form, that the other does not have. A typical example of this occurs when trying to expand in powers of ε. It can happen that to be able to match with an expansion from an adjacent layer it is necessary to include other terms, such as those involving $\ln(\varepsilon)$. This process of having to insert scales into an expansion because of what is happening in another layer is called *switchbacking*. Some of the more famous examples of this involve logarithmic scales, and these are discussed in Lagerstrom (1988).

It is now possible to construct a two-term composite expansion. The basic idea is to add expansions and then subtract the common part. This yields the following result:

$$\begin{aligned} y &\sim y_0 + \varepsilon y_1 + Y_0 + \varepsilon Y_1 - \left(e^1 - x_\eta e^1\sqrt{\varepsilon} + \frac{\varepsilon}{2}e^1\right) \\ &\sim e^{1-x} - (1+x)e^{1-2x/\varepsilon} + \frac{\varepsilon}{2}\left[(1-x)e^{1-x} - e^{1-2x/\varepsilon}\right]. \end{aligned}$$

Note that the common part in this case contains the terms in (2.17) and (2.18) except for the x_η^2 term in (2.17).

Discussion

The importance of matching cannot be overemphasized. Numerous assumptions went into the derivation of the inner and outer approximations, and matching is one of the essential steps that supports these assumptions. If they had not matched, it would have been necessary to go back and determine where the error was made. The possibilities when this happens are almost endless, but the following are useful places to start to look:

1. The boundary layer is at $x = 1$ and not at $x = 0$. In this case, the boundary-layer coordinate is

$$\bar{x} = \frac{x-1}{\varepsilon^\alpha}. \tag{2.19}$$

   For certain problems, it may be necessary to replace the denominator with a function $\mu(\varepsilon)$, where $\mu(\varepsilon)$ is determined from the balancing or matching. See Section 2.3.

2. There are boundary layers at both ends of the interval. See Section 2.3.

3. There is an interior layer. In this case, the stretching transformation is

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha}, \tag{2.20}$$

where x_0 is the location of the layer (it may depend on ε). See Sections 2.4 and 2.5.

4. The form of the expansion is incorrect. For example, the outer expansion may have the form $y \sim \xi(\varepsilon)y_0(x) + \ldots$, where ξ is a function determined from the balancing or matching in the problem. See Exercise 2(b) and Section 2.5.

5. The solution simply does not have a layer structure, and other methods need to be used (see Exercise 3 and Chapter 3).

Occasionally, it is so unclear how to proceed that one may want to try to solve the problem numerically to get an insight into the structure of the solution (see Appendix 3). The difficulty with this is that the presence of boundary or interior layers can make it hard to obtain an accurate numerical solution (see Section 2.7). If this happens, one can try taking values of ε that are not too small so the layer is present but not so localized that it dominates the numerical solution. Another possibility to help guide the analysis occurs when the problem originates from an application and one is able to use physical intuition to determine the approach to take to solve the problem.

Exercises

1. Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\varepsilon y'' = a - y', \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and a is a given positive constant.

(a) After finding the first term of the inner and outer expansions, derive a composite expansion of the solution of this problem.

(b) Derive a two-term composite expansion of the solution of this problem.

2. Find a composite expansion of the following:

(a) $\varepsilon y'' + 2y' + y^3 = 0$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1/2$.

(b) $\varepsilon y'' = f(x) - y'$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$. Also, $f(x)$ is continuous.

(c) $\varepsilon y'' + (1+2x)y' - 2y = 0$, for $0 < x < 1$, where $y(0) = \varepsilon$ and $y(1) = \sin(\varepsilon)$.

(d) $\varepsilon y'' + y' + y = \int_0^1 K(\varepsilon x, s)y(s)ds$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$. Also, $K(x,s) = e^{-s(1+x)}$.

(e) $\varepsilon y'' + y(y'+3) = 0$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = 1$.

(f) $\varepsilon y'' = e^{\varepsilon y'} + y$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.

(g) $\varepsilon y'' - y^3 = -1 - 7x^2$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 2$.

3. A small parameter multiplying the highest derivative does not guarantee that boundary or interior layers are present. This problem presents examples of such situations.

(a) After solving each of the following problems, explain why the method of matched asymptotic expansions cannot be used (in a straightforward manner) to find an asymptotic approximation of the solution.

i) $\varepsilon^2 y'' + \omega^2 y = 0$, for $0 < x < 1$ and $\omega \neq 0$.

ii) $\varepsilon^2 y'' + \varepsilon y' + y = 1$, for $0 < x < \pi$, where $y(0) = y(\pi) = 0$.

iii) $\varepsilon^2 y'' = y'$, for $0 < x < 1$, where $y'(0) = -1$ and $y(1) = 0$.

iv) $\varepsilon^2 y'' - 2xy' - 2y = 0$, for $-1 < x < 1$, where $y(-1) = 1$ and $y(1) = 2$.

(b) For what values of α and β, if any, will the following problem have a boundary layer(s)?

$$\varepsilon^2 y'''' + \omega^2 y = 0, \text{ for } 0 < x < 1,$$

where $y(0) = y(1) = 0$, $y''(0) = \alpha$, $y''(1) = \beta$. Also, ω is a positive constant.

4. (a) For (2.8) consider the balance of $\textcircled{1} >> \textcircled{2}, \textcircled{3}$. This case is not a distinguished limit because the order (α) is not unique. Explain why the solutions from this region are contained in (2.10).

(b) Discuss the case of $\textcircled{2} >> \textcircled{1}, \textcircled{3}$ in conjunction with the outer solution (this also is not a distinguished limit).

5. Suppose for the intermediate scaling one takes $\eta = \varepsilon^\alpha$. Determine what range of α's are possible so that x_η is in the overlap domain. Carry out the matching procedure.

6. The exact solution of (2.1) is

$$y(x) = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}},$$

where $\varepsilon r_\pm = -1 \pm \sqrt{1 - 2\varepsilon}$. Obtain the inner, outer, and composite expansions directly from this formula.

7. Consider the problem

$$\varepsilon y'' + y' + xy = 0, \text{ for } \alpha(\varepsilon) < x < \beta(\varepsilon),$$

where $y(\alpha) = 1$ and $y(\beta) = 0$. One way to deal with this ε-dependent interval is to change coordinates and let $s = (x-\alpha)/(\beta-\alpha)$. This fixes the domain and puts the problem into a Lagrange-like viewpoint.

(a) Find the transformed problem.

(b) Assuming $\alpha \sim \varepsilon\alpha_1 + \ldots$ and $\beta \sim 1+\varepsilon\beta_1$, find a first-term composite expansion of the solution of the transformed problem. Transform back to the variable x and explain why the first-term composite expansion is unaffected by the perturbed domain.

(c) Find the second term in the composite expansion of the solution of the transformed problem. Transform back to the variable x and explain how the two-term composite expansion is affected by the perturbed domain.

8. Consider the problem

$$\varepsilon y'' + p(x)y' + q(x)y = f(x), \text{ for } 0 < x < 1,$$

where $y(0) = \alpha$ and $y(1) = \beta$. Assume $p(x)$, $q(x)$, and $f(x)$ are continuous and $p(x) > 0$ for $0 \le x \le 1$.

(a) Find a composite expansion of the solution.

(b) Suppose $p(x) < 0$ for $0 \le x \le 1$. Show that the transformation $\tilde{x} = 1 - x$ and the result from part (a) can be used to obtain a composite expansion of the solution.

9. Consider the problem

$$4\varepsilon y'' + 6\sqrt{x}y' - 3y = -3, \text{ for } 0 < x < 1,$$

where $y(0) = 0$ and $y(1) = 3$.

(a) Find a composite expansion of the solution.

(b) Find a two-term composite expansion of the solution.

10. The Michaelis-Menten reaction scheme for an enzyme catalyzed reaction is

$$\frac{ds}{dt} = -s + (s + k - 1)c \quad \text{and} \quad \varepsilon\frac{dc}{dt} = s - (s + k)c, \text{ for } t > 0,$$

where $s(0) = 1$ and $c(0) = 0$. Here $s(t)$ is the concentration of substrate, $c(t)$ is the concentration of the chemical produced by the catalyzed reaction, and k is a positive constant. Find the first term in the expansions in (i) the outer layer, (ii) the initial layer ($\tilde{t} = t/\varepsilon$), and (iii) a composite expansion.

11. This problem is concerned with the integral equation

$$\varepsilon y(x) = -q(x) \int_0^x [y(s) - f(s)] s ds, \text{ for } 0 \le x \le 1,$$

where $f(x)$ is smooth and positive.

(a) Taking $q(x) = 1$, find a composite expansion of the solution $y(x)$.

(b) Find a composite expansion of the solution in the case where $q(x)$ is positive and continuous but not necessarily differentiable.

12. A modified version of the Grodsky model for insulin release is (Carson et al., 1983) to find $y = y(t, \lambda)$ that satisfies

$$\varepsilon \frac{dy}{dt} = -y + f(t) + \int_0^\infty y(t, \lambda) e^{-\gamma\lambda} d\lambda, \text{ for } 0 < t < \infty,$$

where $y = g(\lambda)$ when $t = 0$. Also, $\gamma > 1$.

(a) Find a composite expansion of the solution for small ε.

(b) Derive the composite expansion you obtained in part (a) from the exact solution, which is

$$y = [g(\lambda) - \varepsilon\gamma g_0(1 - e^{-t/\varepsilon\gamma})]e^{-t/\varepsilon} + \int_0^t f(\tau) e^{-(\gamma-1)(t-\tau)/\varepsilon\gamma} d\tau,$$

where $g_0 = \int_0^\infty g(\lambda) e^{-\gamma\lambda} d\lambda$. Also, what is the composite expansion when $\gamma = 1$?

13. In the study of calcium transport in a cell, one comes across the following nonlinear diffusion problem for the concentration $c(\mathbf{x}, t)$ of calcium (Holmes and Bell, 1991):

$$\nabla^2 c = \varepsilon[\partial_t c + f(c)], \text{ for } \mathbf{x} \in \Omega \quad \text{and} \quad t > 0,$$

where

$$\partial_n c = \begin{cases} 0 & \text{for } \mathbf{x} \in \Omega_0 \\ \varepsilon g(t) & \text{for } \mathbf{x} \in \Omega_1 \end{cases}$$

and $c(\mathbf{x}, 0) = h(\mathbf{x})$. Here ∂_n is the normal derivative and $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$ is the boundary of Ω. Also, f and g are smooth positive functions and $\partial\Omega$ is regular. Find the nonlinear ordinary differential equation and initial condition that determine the first term in the expansion of $c(\mathbf{x}, t)$ for small ε.

14. Another way to match inner and outer expansions comes from Van Dyke (1975). To explain the procedure, suppose two terms have been calculated in both regions and the boundary layer coordinate is $\bar{x} = x/\varepsilon^\alpha$. One then does the following: i) substitute x/ε^α for $\bar{x}$ into the

inner expansion and expand the result to two terms (with x fixed); ii) substitute $\varepsilon^{\alpha}\bar{x}$ for x into the outer expansion and expand the result to two terms (with $\bar{x}$ fixed). After rewriting the results from (i) and (ii) in terms of x, the matching condition states that the two expansions should agree exactly (to two terms).

(a) Using this matching procedure, find a two-term composite expansion of the solution of (2.1).

(b) Using this matching procedure, find a two-term composite expansion of the solution of

$$\varepsilon y'' = f(x) - y', \text{ for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$, and $f(x)$ is a given smooth function.

15. As seen in Fig. 2.4, in the boundary layer the solution of (2.1) is concave down (i.e., $y'' < 0$). This observation is useful for locating layers and is discussed further in Section 2.4. However, not all boundary layers have strict concavity properties, and this problem contains such an example. The interested reader is referred to Howes (1978) for an extended discussion of this situation.

(a) Find a composite expansion for the solution of

$$\varepsilon^2 y'' = (x - y)(y - 2), \text{ for } 0 < x < 1,$$

where $y(0) = 3$ and $y(1) = 1$.

(b) Explain why the solution of this problem does not have a boundary layer that is strictly concave up or concave down but has one that might be identified as concave-convex.

16. Some consider it bothersome that a composite expansion generally does not satisfy the boundary conditions exactly. One procedure that has been used to correct this situation is to note that the composite expansion for (2.1a), before imposing boundary condition (2.1c), is $y \sim a(e^{-x} - e^{-2x/\varepsilon})$. Substituting this into (2.1c), it is then found that $a = e^1/(1 - e^{-3/\varepsilon})$.

(a) This violates our assumption, as expressed in (2.2), that $y_0(x)$ is independent of ε. However, is the result still an asymptotic approximation of the solution for $0 \le x \le 1$?

(b) Use this idea to find a first-term composite expansion that satisfies the boundary conditions exactly for the solution of the problem

$$\varepsilon y'' = f(x) - y', \text{ for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 1$ and $f(x)$ is a given smooth function.

2.3 Examples with Multiple Boundary Layers

Almost all of the principal ideas underlying the method of matched asymptotic expansions were introduced in the example of Section 2.2. In the remainder of this chapter, these ideas are applied, and in most cases extended, to more complicated problems. The example we now consider illustrates the situation of multiple boundary layers. The problem is

$$\varepsilon^2 y'' + \varepsilon x y' - y = -e^x, \text{ for } 0 < x < 1, \tag{2.21a}$$

where

$$y(0) = 2 \text{ and } y(1) = 1. \tag{2.21b, c}$$

One difference here is that one of the coefficients of the equation depends on x. This is not responsible for the multiple boundary layers but, as we shall see, it does result in different equations for each layer.

Step 1: Outer Expansion

The expansion of the solution in this region is the same as in the last example, namely, $y \sim y_0 + \ldots$. From this and (2.21a) one obtains the first-term approximation

$$y_0 = e^x. \tag{2.22}$$

Clearly, this function is incapable of satisfying either boundary condition, an indication that there are boundary layers at each end.

Steps 2 and 3: Boundary Layers and Matching

For the left endpoint we introduce the boundary-layer coordinate $\bar{x} = x/\varepsilon^\alpha$, in which case (2.21a) becomes

$$\underset{①}{\varepsilon^{2-2\alpha}\frac{d^2Y}{d\bar{x}^2}} + \underset{②}{\varepsilon\bar{x}\frac{dY}{d\bar{x}}} - \underset{③}{Y} = \underset{④}{-e^{\varepsilon^\alpha \bar{x}}}. \tag{2.23}$$

As before, $Y(\bar{x})$ is used to designate the solution in this boundary-layer region. The balance in this layer is between terms ①, ③, ④, and so $\alpha = 1$. The appropriate expansion for Y is $Y \sim Y_0(\bar{x}) + \ldots$, and from (2.23) and (2.21b) we have that

$$Y_0'' - Y_0 = -1, \text{ for } 0 < \bar{x} < \infty, \tag{2.24a}$$

where

$$Y_0(0) = 2. \tag{2.24b}$$

Note that (2.24a) has at least one term in common with the equation for the outer region (which we should expect if there is to be any hope of matching the inner and outer expansions). The general solution of (2.24) is

$$Y_0(\bar{x}) = 1 + Ae^{-\bar{x}} + (1 - A)e^{\bar{x}}. \tag{2.25}$$

This must match with the outer solution given in (2.22). The matching condition is $Y_0(\infty) = y_0(0)$, and so from (2.22) we have that $A = 1$.

To determine the solution in the boundary layer at the other end, we introduce the boundary-layer coordinate

$$\tilde{x} = \frac{x - 1}{\varepsilon^{\beta}}. \tag{2.26}$$

In this region, we will designate the solution as $\tilde{Y}(\tilde{x})$. Introducing (2.26) into (2.21a), one obtains the equation

$$\varepsilon^{2-2\beta}\frac{d^2\tilde{Y}}{d\tilde{x}^2} + (1 + \varepsilon\tilde{x})\varepsilon^{1-\beta}\frac{d\tilde{Y}}{d\tilde{x}} - \tilde{Y} = -e^{1+\varepsilon^{\beta}\tilde{x}}. \tag{2.27}$$

The distinguished limit in this case occurs when $\beta = 1$. So, the expansion $\tilde{Y} \sim \tilde{Y}_0(\tilde{x})$ yields the problem

$$\tilde{Y}_0'' + \tilde{Y}_0' - \tilde{Y}_0 = -e^1, \text{ for } -\infty < \tilde{x} < 0, \tag{2.28a}$$

where

$$\tilde{Y}_0(0) = 1. \tag{2.28b}$$

It is important to notice that this boundary-layer equation has at least one term in common with the equation for the outer region. Solving (2.28), the general solution is found to be

$$\tilde{Y}_0(\tilde{x}) = e^1 + Be^{r_+\tilde{x}} + (1 - e^1 - B)e^{r_-\tilde{x}}, \tag{2.29}$$

where $2r_{\pm} = -1 \pm \sqrt{5}$. The matching requirement here is that $\tilde{Y}_0(-\infty) = y_0(1)$. Hence, from (2.29), $B = 1 - e^1$. As a final comment, note that boundary-layer equation (2.28a) differs from (2.24a), and this is due to the x dependence of the terms in the original problem.

Step 4: Composite Expansion

The last step is to combine the three expansions into a single expression. This is done in the usual way of adding the expansions together and then

FIGURE 2.5. Comparison between the composite expansion (2.30) and the numerical solution of (2.21). The two curves are very close, so over most of the interval they are almost indistinguishable. In this calculation $\varepsilon = 10^{-2}$.

subtracting the common parts. From (2.22), (2.25), and (2.29), a first-term expansion of the solution over the entire interval is

$$\begin{aligned} y &\sim y_0(x) + Y_0(\bar{x}) - Y_0(\infty) + \tilde{Y}_0(\tilde{x}) - \tilde{Y}_0(-\infty) \\ &\sim e^x + e^{-x/\varepsilon} + (1 - e^1)e^{-r(1-x)/\varepsilon}, \end{aligned} \tag{2.30}$$

where $2r = -1 + \sqrt{5}$. This approximation is shown in Fig. 2.5 along with the numerical solution. One is able to clearly see the boundary layers at the endpoints of the interval as well as the accuracy of the asymptotic approximation.

Some interesting complications arise when using matched asymptotic expansions with nonlinear equations. For example, it is not unusual that the solution of a nonlinear equation is defined implicitly. To illustrate this situation, consider the problem

$$\varepsilon y'' + \varepsilon y' - e^y = -2 - x, \text{ for } 0 < x < 1, \tag{2.31}$$

where $y(0) = 0$ and $y(1) = 1$. The outer expansion is $y \sim \ln(x + 2)$, and this holds for $0 < x < 1$. For the boundary layer at $x = 0$ the coordinate is $\bar{x} = x/\sqrt{\varepsilon}$, and one finds that $Y \sim Y_0$, where

$$Y_0'' - e^{Y_0} = -2,$$

for $Y_0(0) = 0$. Multiplying the differential equation by Y_0' and integrating yield

$$\frac{1}{2}\left(\frac{d}{d\bar{x}}Y_0\right)^2 = B - 2Y_0 + e^{Y_0}. \tag{2.32}$$

We can determine the constant B if we look at how the expansions in the previous examples matched. For example, from Fig. 2.2 it is seen that for the boundary-layer solution $Y_0' \to 0$ and $Y_0 \to y_0(0)$ as $\bar{x} \to \infty$. Assuming this monotonic convergence occurs in the present example, then, since

$y_0(0) = \ln(2)$, we get $B = 2[-1 + \ln(2)]$. Now, solving (2.32) for Y_0' we obtain

$$Y_0' = \pm\sqrt{2(B - 2Y_0 + e^{Y_0})}.$$

To determine which sign to take, note that $Y_0(0) < y_0(0)$. Thus, assuming the boundary-layer solution increases from $Y_0(0) = 0$ to $Y_0(\infty) = y_0(0) = \ln(2)$, we take the + sign in the above expression. Doing this, separating variables, and then integrating yields the following result:

$$\int_0^{Y_0} \frac{ds}{[2(B - 2s + e^s)]^{1/2}} = \bar{x}. \tag{2.33}$$

This is the solution of the $O(1)$ problem in the boundary layer, and it defines Y_0 implicitly in terms of $\bar{x}$. It is important to note that the assumptions that were made to derive this result, such as $Y_0' \to 0$ and $Y_0 \to y_0(0)$ as $\bar{x} \to \infty$, hold for this solution. It is also worth mentioning that another approach to matching in a case like this one involves using the phase plane. Extensive use of this procedure can be found in O'Malley (1976, 1991).

It is possible to have multiple layers in a problem and have them occur on the same side of the interval. An example of this arises in the problem

$$\varepsilon^3 y'' + x^3 y' - \varepsilon y = x^3, \text{ for } 0 < x < 1, \tag{2.34a}$$

where

$$y(0) = 1 \text{ and } y(1) = 3. \tag{2.34b}$$

The outer expansion is $y \sim x + 2$, and this holds for $0 < x \leq 1$. To investigate what is happening near $x = 0$, set $\bar{x} = x/\varepsilon^\alpha$; hence

$$\underset{①}{\varepsilon^{3-2\alpha}\frac{d^2}{d\bar{x}^2}Y} + \underset{②}{\varepsilon^{2\alpha}\bar{x}^3\frac{d}{d\bar{x}}Y} - \underset{③}{\varepsilon Y} = \underset{④}{\varepsilon^{3\alpha}\bar{x}^3}. \tag{2.35}$$

There are two distinguished limits (other than the one for the outer region). One comes when ① ~ ③, in which case $\alpha = 1$. This is an inner-inner layer. The other balance occurs when ② ~ ③, in which case $\alpha = \frac{1}{2}$. This is an inner layer. Note that the equation for the inner-inner layer does not have any terms in common with the equation for the outer region but the inner layer equation shares terms with both.

Exercises

1. Find a composite expansion of the following and sketch the solution:
   (a) $\varepsilon y'' + \varepsilon(x+1)^2 y' - y = x - 1$, for $0 < x < 1$, where $y(0) = 0$, and

$y(1) = -1$.

(b) $\varepsilon \frac{d}{dx}\left(E(x)\frac{u'}{1-\varepsilon u'}\right) - u = f(x)$, for $0 < x < 1$, where $u(0) = u(1) = 0$. Also, $E(x)$ and $f(x)$ are known, smooth, positive functions.

(c) $\varepsilon y'' - y' + y^2 = 1$, for $0 < x < 1$, where $y(0) = 1/3$ and $y(1) = 1$.

(d) $\varepsilon y'' - e^x y = f(x)$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.

(e) $\varepsilon y'' + \varepsilon y' - y^2 = -1 - x^2$, for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 2$.

(f) $\varepsilon y'''' - (1 + x)y' = 1 - x^2$, for $0 < x < 1$, where $y(0) = y'(0) = y(1) = y'(1) = 0$.

(g) $\varepsilon^3 y'' = \frac{(y-e^x)^3}{(1+\varepsilon y')^2}$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 4$.

(h) $\varepsilon y'' + y(1 - y)y' - y = 0$, for $0 < x < 1$, where $y(0) = -2$ and $y(1) = -2$.

(i) $\varepsilon y'' + y(y')^2 - y = 0$, for $0 < x < 1$, where $y(0) = 3$ and $y(1) = 1$.

(j) $\varepsilon^2 y'' + \varepsilon(3 - x^2)y' - 4y = 4x$, for $-1 < x < 1$, where $y(-1) = 2$ and $y(1) = 3$.

(k) $\varepsilon(x^2 y')' = \frac{x^2(y-1)}{y}$, for $0 < x < 1$, where $y'(0) = 0$ and $y(1) = 2$.

(l) $(y^\varepsilon)'' - y = -e^x$, for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 3$.

(m) $\varepsilon y'' + y(1 - y^2)y' - ky = 0$, for $0 < x < 1$, where $y(0) = 3$ and $y(1) = 2$. Also, $0 < k < 4$.

2. Consider the boundary-value problem

$$\varepsilon y'' - xy' - \kappa y = -1, \text{ for } -1 < x < 1,$$

where $y(0) = y(1) = 0$. Assuming

$$\kappa = \int_{-1}^{1} y^2 dx,$$

find the first term in an expansion of κ for small ε.

3. The Reynolds equation from the gas lubrication theory for slider bearings is (DiPrima, 1968; Shepherd, 1978)

$$\varepsilon \frac{d}{dx}(H^3 y y') = \frac{d}{dx}(Hy), \text{ for } 0 < x < 1,$$

where $y(0) = y(1) = 1$. Here $H(x)$ is a known, smooth, positive function with $H(0) \neq H(1)$.

(a) Find a composite expansion of the solution for small ε. Note that the boundary-layer solution will be defined implicitly but that it is still possible to match the expansions.

(b) Show that if the boundary layer is assumed to be at the opposite end from what you found in part (a) then the inner and outer expansions do not match.

4. The eigenvalue problem for the vertical displacement, $u(x)$, of an elastic beam that is under tension is

$$\varepsilon^2 u'''' - u'' = \lambda u, \text{ for } 0 < x < 1,$$

where $u = u' = 0$ at $x = 0, 1$. Here λ is the eigenvalue and it depends on ε.

(a) Find the first term in the expansions for $u(x)$ and λ.

(b) Find the second term in the expansions for $u(x)$ and λ.

5. Find a composite expansion of the solution of the following:

$$\varepsilon^2 y'' + 2\varepsilon p(x) y' - q(x) y = f(x), \text{ for } 0 < x < 1,$$

where $y(0) = \alpha$, $y(1) = \beta$. The functions $p(x)$, $q(x)$ and $f(x)$ are continuous and $q(x)$ is positive for $0 \le x \le 1$.

6. In the study of an ionized gas confined to a bounded domain Ω, the potential $\phi(\mathbf{x})$ satisfies

$$-\nabla^2 \phi + h\left(\frac{\phi}{\varepsilon}\right) = \alpha, \text{ for } \mathbf{x} \in \Omega,$$

where conservation of charge requires

$$\int_\Omega h\left(\frac{\phi}{\varepsilon}\right) dV = \beta,$$

and, assuming the exterior of the region is a conductor, $\partial_n \phi = \gamma$ on $\partial\Omega$. The function $h(s)$ is smooth and strictly increasing with $h(0) = 0$. The positive constants α and β are known (and independent of ε) and the constant γ is determined from the conservation of charge equation.

(a) For the case of one dimension, suppose Ω is the interval $0 < x < 1$. What does the problem reduce to? Find γ.

(b) Find the exact solution of the problem in part (a) when $h(s) = s$. Sketch the solution for $\gamma < 0$ and describe the boundary layers that are present.

FIGURE 2.6. Solution to Exercise 6(c) for $\varepsilon = 10^{-3}$.

(c) For the one-dimensional problem in part (a), find the first term in the inner and outer expansions of the solution. In doing this, take $h(s) = s^{2k+1}$, where k is a positive integer, and assume $\beta < \alpha$. For comparison, the numerical solution is shown in Fig. 2.6 for $k = 1$, $\alpha = 3$, $\beta = 1$, and $\varepsilon = 10^{-3}$.

(d) Discuss the steps needed to find a composite expansion involving the terms derived in part (c).

(e) For $\Omega \subset \mathbb{R}^n$, where $n > 1$, find γ.

2.4 Interior Layers

The rapid transitions in the solution that are characteristic of a boundary layer do not have to occur only at the boundary. When this happens, the problems tend to be somewhat harder to solve simply because the location of the layer is usually not known until the expansions have been matched. However, the expansion procedure is essentially the same as in the previous examples. To illustrate how the method works, consider the problem

$$\varepsilon y'' = yy' - y, \text{ for } 0 < x < 1, \tag{2.36a}$$

where

$$y(0) = 1 \tag{2.36b}$$

and

$$y(1) = -1. \tag{2.36c}$$

Step 1: Outer Expansions

The appropriate expansion in this region is the same as usual, in other words,

$$y(x) \sim y_0(x) + \dots . \tag{2.37}$$

From (2.36a) one finds that

$$y_0 y_0' - y_0 = 0, \tag{2.38}$$

and so either $y_0 = 0$ or else

$$y_0 = x + a, \tag{2.39}$$

where a is an arbitrary constant. The fact that we have two possible solutions means that the matching might take somewhat longer than before because we will have to determine which of these solutions matches to the inner expansion.

Steps 2 and 3: Interior Layer and Matching

Generally, when one first begins trying to solve a problem it is not known where the layer(s) is. If we began this problem as we did the previous two and assumed there is a boundary layer at either one of the endpoints, we would find that the expansions do not match. This is a lot of effort for no results, but fortunately there is a simpler way to come to the same conclusion. To illustrate how, suppose it is assumed that there is a boundary layer at $x = 0$ and (2.39) is the outer solution. This situation is shown in Fig. 2.7. If the solution behaves like the other example problems we have examined, then near $x = 0$ it would be expected that $y' < 0$ and $y'' > 0$. This means that $\varepsilon y'' > 0$ but $y(y' - 1)$ is both positive and negative in the boundary layer. Therefore it is impossible to satisfy the differential equation (2.36a) and have a boundary layer as indicated in Fig. 2.7. It is possible to rule out a boundary layer at $x = 1$ in the same way. It should be pointed out that these are only plausibility arguments and they do not prove anything. What they do is guide the analysis and hopefully reduce the work necessary to obtain the solution. However, it is recommended that Exercise 15 from Section 2.2 be looked at to see how this sort of argument might work on a nonconvex (or nonconcave) boundary layer.

Based on the above observations, we investigate the possibility of an

FIGURE 2.7. Schematic of the solution if there is a (convex) boundary layer at $x = 0$ and the linear function in (2.39) is the outer solution.

interior layer. This is done by introducing the interior-layer coordinate

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha}, \tag{2.40}$$

where $0 < x_0 < 1$. The location of the layer, $x = x_0$, is not known and will be determined later. Actually, the possibilities of either $x_0 = 0$ or $x_0 = 1$ could be included here, but we will not do so. Also, note that since $0 < x_0 < 1$ then there are two outer regions, one for $0 < x < x_0$ and another for $x_0 < x < 1$. Now, substituting (2.40) into (2.36a) yields

$$\varepsilon^{1-2\alpha} Y'' = \varepsilon^{-\alpha} Y Y' - Y. \tag{2.41}$$

The distinguished limit here occurs when $\alpha = 1$ (i.e., the first and second terms balance). Also, as in the previous examples, we are using Y to designate the solution in the layer. Expanding the interior-layer solution as

$$Y(\bar{x}) \sim Y_0(\bar{x}) + \dots, \tag{2.42}$$

it follows from (2.41) that

$$Y_0'' = Y_0 Y_0'. \tag{2.43}$$

Integrating this, one obtains

$$Y_0 = B \frac{1 - De^{B\bar{x}}}{1 + De^{B\bar{x}}}, \tag{2.44}$$

where B and D are arbitrary constants. Other solutions of (2.43) are

$$Y_0 = B \tan(C + B\bar{x}/2)$$

and

$$Y_0 = \frac{2}{C - \bar{x}}.$$

The existence of multiple solutions is characteristic of nonlinear equations. This makes the problem interesting, but it also means that the matching procedure is not as straightforward as it was for the linear equations studied earlier. For those linear problems we relied on being able to find the general solution in each region and then determining the constants by matching. For nonlinear problems, the concept of a general solution has little meaning, and because of this it can sometimes be difficult to obtain a solution that is general enough to be able to match to the outer expansion(s).

Of the solutions to (2.43), the one given in (2.44) is capable of matching to the outer expansions as $\bar{x} \to \pm\infty$. Again it should be remembered that the working hypothesis here is that $0 < x_0 < 1$ (see Fig. 2.8). Thus, the outer expansion for $0 < x < x_0$ should satisfy $y(0) = 1$. From this it follows that

$$y_0 = x + 1, \text{ for } 0 \le x < x_0. \tag{2.45}$$

Similarly, the outer region on the other side of the layer should satisfy the boundary condition at $x = 1$, and this yields

$$y_0 = x - 2, \text{ for } x_0 < x \leq 1. \tag{2.46}$$

Now, for (2.44) to be able to match to either (2.45) or (2.46) it is going to be necessary that both B and D in (2.44) are nonzero (in fact, without loss of generality, we will take B to be positive). The requirements imposed in the matching are very similar to those obtained for boundary layers. In particular, we must have that $Y_0(\infty) = y_0(x_0^+)$ and $Y_0(-\infty) = y_0(x_0^-)$. From (2.44) and (2.45) we get

$$B = x_0 + 1,$$

and from (2.44) and (2.46) we have

$$-B = x_0 - 2.$$

Solving these equations, one finds that $B = \frac{3}{2}$ and $x_0 = \frac{1}{2}$.

It remains to determine D in (2.44), and the only requirement on this constant, so far, is that it is nonzero. It might seem strange that we have seemingly satisfied everything for the first-order problem, including matching, yet we still have not completely determined the first term in the expansion. What we are missing is the global structure that is imposed by the problem. This can be missed when patching the solution together like we have done using inner and outer layers and simple power series expansions. One way to resolve this situation can be found by looking a little closer at Fig. 2.8. It suggests the solution has a symmetry with respect to the point where the curve crosses the x-axis, which appears to be at $x = 1/2$. To prove this, suppose the boundary conditions are more general than those given in (2.36b,c). In particular, suppose $y(0) = a$ and $y(1) = b$. The solution can then be written as $y = f(x, a, b)$. Now, if we change variables and let $s = 1 - x$ (which produces a flip around $x = 1/2$) and $z = -y$ (which produces a flip around $y = 0$), then

$$\varepsilon z'' = zz' - z, \text{ for } 0 < s < 1,$$

FIGURE 2.8. Schematic of the linear functions that make up the outer expansion and the interior-layer solution connecting them.

where $z(0) = -b$ and $z(1) = -a$. This is the same problem as in (2.36), and therefore the solution is $z = f(s, -b, -a)$. Transforming back yields $y = -f(1-x, -b, -a)$. The solution is unique, and so $f(x, a, b) = -f(1-x, -b, -a)$. This odd-looking relationship is the symmetry condition we are looking for, and we now take $x = 1/2$, $a = 1$, and $b = -1$. In this case, we find that $f(\frac{1}{2}, 1, -1) = 0$, and therefore $y(\frac{1}{2}) = 0$. With this it follows that $D = 1$ in (2.44), and

$$Y(\bar{x}) \sim -\frac{3}{2}\tanh\left(\frac{3\bar{x}}{4}\right). \tag{2.47}$$

The symmetry argument used to determine the unknown constant is relatively easy to carry out once the symmetry is known. However, finding a symmetry can be difficult, if not impossible. Fortunately, there are other ways to resolve this problem, and one proposed by Grasman and Matkowsky (1977) is described in Exercise 3. Other methods include a phase-plane analysis (O'Malley, 1976, 1991), an extension of the variational approach that eliminates exponentially small eigenvalues of an associated linearized problem (Kath et al., 1987; Ward, 1992), and the use of exponentially small terms in the asymptotic expansions (Lange, 1983). Because all of these have limitations in terms of their applicability, there is still work to be done in resolving this problem.

Step 4: Composite Expansion

It can be more difficult to construct a composite expansion when the outer solutions are discontinuous across the interior layer like they are in this problem. What is done is to find one for the interval $0 \le x \le x_0$ and then another for $x_0 \le x \le 1$. As it turns out, for this example the expansions on either side are the same, and the result is

$$y \sim x + 1 - \frac{3}{1 + e^{-3(2x-1)/4\varepsilon}}, \text{ for } 0 \le x \le 1. \tag{2.48}$$

This composite expansion is graphed in Fig. 2.9 to illustrate the nature of the interior layer. The rapid transition from one outer solution to the other is typical of what is sometimes called a shock solution. Also shown in Fig. 2.9 is the numerical solution, and it is clear that the composite expansion is in close agreement with the numerical solution.

The interior layer of the previous example consists of a monotone transition from one outer solution to another. Other types of interior layers are also seen, including spikes and nonmonotone transition layers. Examples of these are shown in Fig. 2.10. There are also some very unusual interior layers. For example, O'Malley (1976) gives an example where there is a spike at each rational point on the interval $-1 < x < 1$. An analysis of nonmonotone interior-layer problems can be found in DeSanti (1987).

FIGURE 2.9. Comparison between the composite expansion given in (2.48) and the numerical solution of (2.36). In the calculations $\varepsilon = 10^{-2}$.

FIGURE 2.10. Examples of nonmonotone interior layers. In (a) there is a spike generated by the function $y(x) = \operatorname{sech}^2[(x - \frac{1}{2})/\varepsilon]$, and in (b) there is a nonmonotone transition layer generated by the function $y(x) = \frac{1}{2}\tanh[(x - \frac{1}{2})/\varepsilon] + \operatorname{sech}^2[(x - \frac{1}{2})/\varepsilon]$. In both graphs, $\varepsilon = 0.02$.

Exercises

1. Find a first-term expansion of the solution of each of the following:

   (a) $\varepsilon y'' = -\left(x^2 - \frac{1}{4}\right) y'$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = -1$.

   (b) $\varepsilon y'' + 2xy' + (2 + \varepsilon x^2)y = 0$, for $-1 < x < 1$, where $y(-1) = 2$ and $y(1) = -2$.

(c) $\varepsilon y'' = yy' - y^3$, for $0 < x < 1$, where $y(0) = \frac{3}{5}$ and $y(1) = -\frac{2}{3}$.

(d) $\varepsilon y'' + (x-a)(x-b)y' + x(y-1) = 0$, for $0 < x < 1$, where $y(0) = -2$ and $y(1) = 2$. Also, $a = \frac{1}{4}$ and $b = \frac{3}{4}$.

(e) $\varepsilon y'' + y(1-y^2)y' - \frac{1}{2}y = 0$, for $0 < x < 1$, where $y(0) = -1$ and $y(1) = 1$.

(f) $\varepsilon y'' + y(y'+3) = 0$, for $0 < x < 1$, where $y(0) = -1$ and $y(1) = 1$.

2. Consider the problem

$$\varepsilon y'' + y(1-y)y' - xy = 0, \text{ for } 0 < x < 1,$$

where $y(0) = 2$ and $y(1) = -2$. The numerical solution is shown in Fig. 2.11 in the case where $\varepsilon = 10^{-3}$. Based on this information, derive a first-term approximation of the solution (note that one of your layer solutions will contain an undetermined constant).

3. One way to resolve the problem of having an undetermined constant after matching is to use a variational principle. To illustrate this approach, consider the problem

$$\varepsilon y'' + p(x,\varepsilon)y' + q(x,\varepsilon)y = 0, \text{ for } 0 < x < 1,$$

where $y(0) = a$ and $y(1) = b$. Associated with this is the functional

$$I(v) = \int_0^1 L(v,v')dx, \text{ where } L = \frac{1}{2}[\varepsilon(v')^2 - qv^2]e^{\frac{1}{\varepsilon}\int_0^x p(s)ds}.$$

In this variational formulation L is a Lagrangian for the equation.

(a) Show that if $\frac{d}{dr}I(y+ru) = 0$ at $r = 0$ for all smooth functions $u(x)$ satisfying $u(0) = u(1) = 0$, then $y(x)$ is a solution of the above differential equation. In other words, an extremal of the functional is a solution of the differential equation.

(b) Consider the problem

$$\varepsilon y'' - (2x-1)y' + 2y = 0, \text{ for } 0 < x < 1,$$

FIGURE 2.11. Solution of problem in Exercise 2.

where $y(0) = 1$ and $y(1) = -3$. The solution of this problem has a boundary layer at each end of the interval. Find a composite expansion of the solution for $0 \le x \le 1$. Your solution will contain an arbitrary constant which will be designated as k in what follows.

(c) From your result in part (b) derive an expansion for the Lagrangian L.

(d) Explain why the constant k should be such that $\frac{d}{dk}I(y) = 0$. From this determine k.

4. In the Langmuir-Hinshelwood model for the kinetics of a catalyzed reaction, the following problem appears:

$$\varepsilon\frac{dy}{dx} = 1 - \frac{1}{x}F(y), \text{ for } 0 < x < 1,$$

where $F(y) = 2(1-y)(\alpha+y)/y$ and $y(1) = 0$. Also, $0 < \alpha < 1$. In this problem ε is the deactivation rate parameter and $y(x)$ is the concentration of the reactant (Kapila, 1983).

(a) For small ε, find a first-term expansion of the solution in the outer region and in the boundary layer.

(b) Find a composite expansion of the solution for $0 < x \le 1$.

5. This problem examines a differential-difference equation. The specific problem is (Lange and Miura, 1991)

$$\varepsilon^2 y''(x) - y(x) + q(x)y(x-1) = f(x), \text{ for } 0 < x < 3/2,$$

where $y(x) = 0$ for $-1 \le x \le 0$ and $y(3/2) = 1$. The functions $q(x)$ and $f(x)$ are assumed smooth. What is significant here is that the solution is evaluated at $x-1$ in one term of the equation. To answer the following questions, keep in mind that $y(x)$ and $y'(x)$ are continuous for $0 \le x \le 3/2$.

(a) There is a layer at $x = 0^+$, $x = 1^\pm$, and $x = 3/2$. Use this information to find a first-term approximation of the solution. To do this, you should consider $x < 1$ and $x > 1$ separately and then require smoothness at $x = 1$. Also, you will have to find the first two terms in the layer at $x = 1$ to get the expansions to match.

(b) Find the exact solution of the problem in the case where $f(x) = 0$ and compare the result with the expansion from part (a).

6. Consider the problem

$$\varepsilon^2 y'' + 2\varepsilon y' + 2(y - xg)^2 = \varepsilon h(x), \text{ for } 0 < x < 1,$$

where $y(0) = \text{sech}^2(\frac{1}{2\varepsilon})$ and $y(1) = 1 + \text{sech}^2(\frac{1}{2\varepsilon})$. Also, $g(x) = e^{\varepsilon(x-1)}$ and $h(x) = [\varepsilon^2 + (2+\varepsilon^2)(1+\varepsilon x)]g(x)$.

(a) Suppose one were to argue that the exponentially small terms in the boundary conditions can be ignored and the usual power series expansion of the solution can be used. Based on this assumption, find the first two terms of a composite expansion of the solution.

(b) The exact solution of the problem is

$$y(x) = xe^{\varepsilon(x-1)} + \operatorname{sech}^2\left(\frac{2x-1}{2\varepsilon}\right).$$

Discuss this solution in connection with your expansion from part (a).

7. Consider the following problem:

$$\varepsilon y'' + (x-a)(x-b)(4y'-1) = 0, \text{ for } 0 < x < 1,$$

where $y(0) = -2$ and $y(1) = 3$. Also, $0 < a < b < 1$.

(a) Using the plausibility argument used in the discussion for Fig. 2.7, explain why there are two possibilities for the position of the layer. In particular, it may be at $x = 0$ or at $x = b$.

(b) For both cases, calculate the first terms in the expansions. Explain why it is not possible to determine the position for the layer from these expansions.

(c) Find the exact solution. By examining the solution near $x = 0$, explain why the layer is at $x = b$ if $0 < a < \frac{b}{3}$ and at $x = 0$ if $\frac{b}{3} \leq a < b$.

8. In the study of explosions of gaseous mixtures, one finds a model where the (nondimensional) temperature $T(t)$ of the gas satisfies (Kassoy, 1976; Kapila, 1983)

$$T' = \varepsilon(T_\infty - T)^n \exp\left(\frac{T-1}{\varepsilon T}\right),$$

for $T(0) = 1$. Here $T_\infty > 1$ is a constant known as the adiabatic explosion temperature. Also, n is a positive integer (it is the overall reaction order). If we assume a high activation energy, then the parameter ε is small.

(a) What is the steady-state temperature?

(b) Find the first two terms in a regular expansion of the temperature. This expansion satisfies the initial condition and describes the solution in what is known as the ignition period. Explain why the expansion is not uniform in time. However, use your expansion, the result from part (a), and the fact that $T' \geq 0$ to make a rough sketch of what the solution should look like.

(c) It remains to determine the solution in the transition region that connects the solutions in parts (a) and (b). One way to do this is to introduce the layer coordinate

$$\tau = \frac{t - 1 - f(\varepsilon)}{\mu(\varepsilon)},$$

where $\mu(\varepsilon)$ is determined from balancing in the layer and $f(\varepsilon)$ from matching with the expansion from the ignition region. Find a two-term expansion of the solution in this layer (note that the second term will be defined implicitly).

(d) Matching the first terms in the expansion from parts (b) and (c), show that

$$f(\varepsilon) = \exp\left(-(T_\infty - 1)/\varepsilon\right).$$

Matching the second terms is fairly involved; the details can be found in Kapila (1983). Also, it is actually possible to solve the original problem in closed form, although the solution is not simple (Parang and Jischke, 1975).

2.5 Corner Layers

One of the distinguishing features of the problems we have studied in this chapter is the rapid changes in the solution in the layer regions. The problems we will now investigate are slightly different because the rapid changes will be in the slope, or derivatives of the solution, and not in the value of the solution itself. To illustrate this, we consider the following problem:

$$\varepsilon y'' + \left(x - \frac{1}{2}\right) y' - y = 0, \text{ for } 0 < x < 1, \tag{2.49a}$$

where

$$y(0) = 2 \tag{2.49b}$$

and

$$y(1) = 3. \tag{2.49c}$$

Incidentally, the coefficient of the y' term in this equation changes sign in the interval. For this reason, the equation is said to have a *turning point* at $x = 1/2$. By itself this means little, but as we will see it does play a role in this example by preventing boundary layers at the endpoints of the interval. This also happened with the nonlinear problem in the previous section, which ended up having an interior layer.

Step 1: Outer Expansion

The solution in this region is expanded in the usual power series as follows:

$$y(x) \sim y_0(x) + \varepsilon y_1(x) + \dots . \tag{2.50}$$

From (2.49a) one then finds that

$$y_0 = a\left(x - \frac{1}{2}\right), \tag{2.51}$$

where a is an arbitrary constant. As usual, we must satisfy two boundary conditions with only one integration constant.

Step 2: Corner Layer

We begin by determining whether there are boundary layers. These can be ruled out fairly quickly using the plausibility argument presented in the previous section. For example, if there is a boundary layer at $x = 0$ and (2.51) is the outer solution, then we have a situation similar to that shown in Fig. 2.7. In the boundary layer $y'' > 0$, $y' < 0$ and there is a portion of the curve where $y < 0$. This means that $\varepsilon y'' > 0 > -\left(x - \frac{1}{2}\right) y' + y$. It is therefore impossible to satisfy the differential equation (2.49a) and have a boundary layer as indicated in Fig. 2.7. Using a similar argument, and the fact that the coefficient of the y' term changes sign in the interval, one can also argue that there is not a boundary layer at the other end.

It therefore appears that there is an interior layer, and to investigate this we introduce the stretched variable

$$\bar{x} = \frac{x - x_0}{\varepsilon^\alpha}. \tag{2.52}$$

With an interior layer there is an outer solution for $0 \le x < x_0$ and one for $x_0 < x \le 1$. Using boundary conditions (2.49b,c) and the general solution given in (2.51), we have that the outer solution is

$$y \sim \begin{cases} -4\left(x - \frac{1}{2}\right) & \text{for } 0 \le x < x_0 \\ 6\left(x - \frac{1}{2}\right) & \text{for } x_0 < x \le 1 \end{cases}. \tag{2.53}$$

Taking $x_0 = \frac{1}{2}$, we find that the outer solution is continuous, giving us the situation shown in Fig. 2.12. The outer solution is continuous but is not differentiable at $x = \frac{1}{2}$. It is for this reason that we have what is called a *corner region*, or derivative layer, at $x_0 = \frac{1}{2}$.

Now, substituting (2.52) into (2.49a), and letting Y designate the solution in this region, we get

$$\varepsilon^{1-2\alpha} Y'' + \bar{x} Y' - Y = 0. \tag{2.54}$$

FIGURE 2.12. Schematic of corner layer formed by outer solution in (2.53). The solution from the corner-layer region should provide a smooth transition between these linear functions.

The distinguished limit here occurs for $\alpha = \frac{1}{2}$. Unlike what we have assumed in the previous examples, we now take

$$Y \sim y_0(x_0) + \varepsilon^\gamma Y_0 + \dots . \tag{2.55}$$

For this example $y_0(x_0) = 0$. Also, the multiplicative factor ε^γ is needed to be able to match to the outer solution (the constant γ will be determined from the matching). So, substituting this into (2.54) yields

$$Y_0'' + \bar{x}Y_0' - Y_0 = 0, \text{ for } -\infty < \bar{x} < \infty. \tag{2.56}$$

We are in the unfortunate situation of having to solve the full problem to be able to determine what is going on in this layer. If we can solve this problem, one might wonder why it is necessary to bother with an asymptotic approximation in the first place. There is no arguing with this sentiment, and this is one of the situations where perturbation methods show some of their limitations. However, there are several comments that should be made. First, it is sometimes easier to solve a problem on an infinite, or semi-infinite, interval as occurs in an interior or boundary-layer problem. Second, even if we cannot solve the layer problem in closed form, it is usually possible to say a great deal about the structure of the solution.

As it turns out, it is possible to solve (2.56) in closed form by using power series methods. However, a simpler way is to notice that $Y_0 = \bar{x}$ is a solution, and so, using the method of reduction of order, one finds that the general solution is

$$Y_0 = A\bar{x} + B\bar{x}\left[-e^{-\bar{x}^2/2} - \int_0^{\bar{x}} e^{-s^2/2}ds\right]. \tag{2.57}$$

Step 3: Matching

In the examples from the previous two sections, the layer analysis and matching were carried out in a single step. This is not done here because the matching in this problem is slightly different and is worth considering

in more detail. To do the matching, we introduce the intermediate variable

$$x_\eta = \frac{x - (1/2)}{\varepsilon^\kappa}, \tag{2.58}$$

where $0 < \kappa < \frac{1}{2}$. Rewriting the outer solution (2.53) in this variable yields

$$y \sim \begin{cases} -4\varepsilon^\kappa x_\eta & \text{for } x_\eta < 0 \\ 6\varepsilon^\kappa x_\eta & \text{for } 0 < x_\eta \end{cases}. \tag{2.59}$$

Also, using the fact that

$$\int_0^\infty e^{-s^2/2} ds = \sqrt{\frac{\pi}{2}},$$

it follows from (2.57) that

$$Y \sim \begin{cases} \varepsilon^{\gamma+\kappa-1/2} x_\eta \left(A + B\sqrt{\frac{\pi}{2}}\right) & \text{for } x_\eta < 0 \\ \varepsilon^{\gamma+\kappa-1/2} x_\eta \left(A - B\sqrt{\frac{\pi}{2}}\right) & \text{for } 0 < x_\eta \end{cases}. \tag{2.60}$$

To be able to match (2.59) and (2.60), we must have $\gamma = \frac{1}{2}$. In this case,

$$A + B\sqrt{\frac{\pi}{2}} = -4$$

and

$$A - B\sqrt{\frac{\pi}{2}} = 6,$$

from which it follows that $A = 1$ and $B = -5\sqrt{2/\pi}$.

Step 4: Composite Expansion

Even though the situation is slightly more complicated than before, the construction of a composite expansion follows the same rules as in the earlier examples. For example, for $0 \le x \le \frac{1}{2}$,

$$\begin{aligned} y &\sim -4\left(x - \frac{1}{2}\right) + \varepsilon^{1/2}\bar{x}\left\{1 - 5\sqrt{2/\pi}\left[-e^{-\bar{x}^2/2} - \int_0^{\bar{x}} e^{-s^2/2} ds\right]\right\} \\ &\quad +4\varepsilon^{1/3} x_\eta \\ &= \left(x - \frac{1}{2}\right)\left\{1 + 5\sqrt{\frac{2}{\pi}}\left[e^{-(2x-1)^2/(8\varepsilon)} + \int_0^{(2x-1)/(2\sqrt{\varepsilon})} e^{-s^2/2} ds\right]\right\}. \end{aligned} \tag{2.61}$$

One finds that this is also the composite expansion for $\frac{1}{2} \le x \le 1$. Thus, (2.61) is a composite expansion over the entire interval. This function is

FIGURE 2.13. The composite expansion (2.61), the exact solution of (2.49), and the outer expansion (2.53). The differences between the composite and exact solution are so small that the two curves are almost indistinguishable. In these calculations $\varepsilon = 10^{-2}$.

shown in Fig. 2.13 and is clearly in very good agreement with the exact solution.

The equation studied in this section is related to one that has received a great deal of attention because the solution has been found to have some rather interesting properties. To explain the situation, consider

$$\varepsilon y'' - \left(x - \frac{1}{2}\right) y' + ky = 0, \text{ for } 0 < x < 1, \tag{2.62}$$

where k is a constant. This equation has a simple turning point at $x = \frac{1}{2}$. Assuming that $y(0) = y(1) = 1$, then the solution is

$$y(x) = \frac{M\left(-\frac{k}{2}, \frac{1}{2}, \frac{1}{8\varepsilon}(2x-1)^2\right)}{M\left(-\frac{k}{2}, \frac{1}{2}, \frac{1}{8\varepsilon}\right)}, \tag{2.63}$$

where M is a confluent hypergeometric function (see Appendix A1.2). It is assumed here that the value of ε is such that the denominator in (2.63) is nonzero. Using the known asymptotic properties of M, one finds that for small ε and $x \neq 1/2$,

$$y(x) \sim \begin{cases} e^{-x(1-x)/2\varepsilon} & \text{for } k \neq 0, 2, 4, \ldots \\ (2x-1)^k & \text{for } k = 0, 2, 4, \ldots \end{cases}.$$

This shows that there is a boundary layer at each endpoint for all but a discrete set of values for the constant k. When there are boundary layers, the solution in the outer region is transcendentally small and goes to zero as $\varepsilon \downarrow 0$. What is significant is that for $k = 0, 2, 4, \ldots$ this does not happen. This behavior at a discrete set of points for the parameter k is reminiscent of resonance and has become known as the Ackerberg-O'Malley resonance. Those who may be interested in pursuing this topic are referred to the original paper by Ackerberg and O'Malley (1968) and a later study by De Groen (1980).

Exercises

1. Find a composite expansion of the following:

   (a) $\varepsilon y'' + y(y')^2 - y = 0$, for $0 < x < 1$, where $y(0) = 1$ and $y(1) = 1$.

   (b) $\varepsilon y'' = 9 - (y')^2$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$.

   (c) $\varepsilon y'' + e^x(xy' - y) = x^2$, for $-1 < x < 1$, where $y(-1) = 1$ and $y(1) = -1$.

   (d) $\varepsilon y'' + y(1-y)y' - y = 0$, for $0 < x < 1$, where $y(0) = y(1) = 1/2$.

   (e) $\varepsilon y'' + (2y-1)((y')^3 - 1) = 0$, for $0 < x < 1$, where $y(0) = 3$ and $y(1) = 1$.

2. Consider the problem

$$\varepsilon y'' - (x-a)(x-b)y' - x(y-1) = 0, \text{ for } 0 < x < 1,$$

   where $y(0) = -2$ and $y(1) = 2$. The numerical solution is shown in Fig. 2.14 in the case where $a = 1/4$, $b = 3/4$, and $\varepsilon = 10^{-4}$. Based on this information, derive a first-term approximation of the solution for arbitrary $0 < a < b < 1$.

3. Corner layers can occur within a boundary layer, and as an example of this consider the problem

$$\varepsilon y'' + \tanh(y') - y = -1, \text{ for } 0 < x < 1,$$

   where $y(0) = 3$ and $y(1) = 5$. Find a first-term expansion of the solution.

4. Consider the problem (Lorentz, 1982)

$$\varepsilon y'' + y(1-y^2)y' - y = 0, \text{ for } 0 < x < 1,$$

   where $y(0) = \alpha$ and $y(1) = \beta$. Find a first-term approximation of the solution in the following cases:

FIGURE 2.14. Solution of problem in Exercise 2.

FIGURE 2.15. Solution of problem in Exercise 4.

(a) $\alpha = 1/2$ and $\beta = 2$. The numerical solution of the problem is shown in Fig. 2.15, by the solid curve for $\varepsilon = 0.0002$.

(b) $\alpha = 3/2$ and $\beta = -3/2$. The numerical solution of the problem is shown in Fig. 2.15, by the dashed curve for $\varepsilon = 0.01$.

(c) $\alpha = 3/2$ and $\beta = 2$.

5. Consider the problem

$$\varepsilon y'' + xp(x)y' - q(x)y = 0, \text{ for } -1 < x < 1,$$

where $y(-1) = a$ and $y(1) = b$. The functions $p(x)$ and $q(x)$ are continuous, $p(x) \neq 0$, and $q(x) > 0$ for $-1 < x < 1$.

(a) If $p(0) < 0$, then there is a boundary layer at each end. Find a composite expansion and then show that $y \to 0$ as $\varepsilon \downarrow 0$ for $-1 < x < 1$.

(b) If $p(0) > 0$, then there is a corner layer. Find a composite expansion of the solution in this case.

6. This exercise concerns the shock wave produced by a cylinder that is expanding with a constant radial velocity of εa_0, where a_0 is the velocity of sound in still air (Lighthill, 1949). The velocity potential in this case has the form $\phi(r,t) = a_0^2 t f(\eta)$, where r is the radial distance from the center of the expanding cylinder, $\eta = r/(a_0 t)$, and $f(\eta)$ is determined below. Also, the radius of the cylinder is $r = \varepsilon a_0 t$, and the radius of the shock wave is $r = \alpha a_0 t$, where α is a constant determined below. The problem that determines $f(\eta)$ and α is

$$\left[1 - (\gamma - 1)\left(f - \eta f' + \frac{1}{2}(f')^2\right)\right](f' + \eta f'') = \eta(\eta - f')^2 f'', \text{ for } \varepsilon < \eta < \alpha,$$

where $f'(\varepsilon) = \varepsilon$, $f(\alpha) = 0$, and

$$f'(\alpha) = \frac{2[\alpha - (1/\alpha)]}{\gamma + 1}.$$

In this problem γ is a positive constant called the *adiabatic index*. Show that, for small ε, $\alpha \sim 1 + \frac{3}{8}(\gamma+1)^2\varepsilon^4$.

2.6 Partial Differential Equations

The subject of boundary and interior layers and how they appear in the solutions of partial differential equations is enormous. We will examine two examples that lend themselves to matched asymptotic expansions.

Example 1. Elliptic Problem

The first example concerns finding the function $u(x,y)$ that is the solution of the following boundary-value problem:

$$\varepsilon\nabla^2 u + \alpha\partial_x u + \beta\partial_y u + u = f(x,y), \text{ for } (x,y) \text{ in } \Omega, \tag{2.64a}$$

where

$$u = g(x,y), \text{ for } (x,y) \text{ on } \partial\Omega. \tag{2.64b}$$

The domain Ω is assumed to be bounded and simply connected, and to have a smooth boundary. The coefficients α and β are constants with at least one of them nonzero. The functions $f(x,y)$ and $g(x,y)$ are assumed to be continuous.

To illustrate what a solution of (2.64) looks like, consider the special case where $f(x,y) = a$ and $g(x,y) = b$, where a and b are constants. Taking Ω to be the unit disk and using polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, then the solution of the problem is

$$u(\rho,\varphi) = a + \frac{1}{F(\rho,\varphi)}\sum_{n=-\infty}^{\infty} I_n(\chi\rho)[a_n\sin(n\varphi) + b_n\cos(n\varphi)]/I_n(\chi), \tag{2.65}$$

where $F(\rho,\varphi) = \exp\left(\frac{\rho}{2\varepsilon^2}(\alpha\cos\varphi + \beta\sin\varphi)\right)$, $\chi = \frac{1}{2\varepsilon^2}\sqrt{2-4\varepsilon^2}$, and I_n is a modified Bessel function,

$$a_n = \frac{b-a}{2\pi}\int_0^{2\pi} F(1,\varphi)\sin(n\varphi)d\varphi$$

and

$$b_n = \frac{b-a}{2\pi}\int_0^{2\pi} F(1,\varphi)\cos(n\varphi)d\varphi.$$

This solution is shown in Fig. 2.16. A few observations can be made from this figure that will make the derivation of the asymptotic approximation easier to follow. First, a boundary layer is clearly evident in the solution.

FIGURE 2.16. Plot of the solution (2.65) and its associated contour map when $a = 1$, $b = 0$, and $\varepsilon = 0.05$. The domain for the problem is $x^2 + y^2 < 1$, but to generate the plot the solution was extended to the unit square $0 \le x$, $y \le 1$ by setting $u = 0$ for $x^2 + y^2 \ge 1$. It is apparent there is a boundary layer, and its presence is particularly pronounced in the region of the boundary near $x = y = -1/\sqrt{2}$.

For example, if one starts at $x = y = -1/\sqrt{2}$ and then moves into the domain, the solution undergoes a rapid transition to what appears to be an outer solution. It is also seen that the layer is not present around the entire boundary $\partial\Omega$ but only over a portion of $\partial\Omega$. Moreover, if one follows the solution around the edge of the boundary, it's not easy to identify exactly where the solution switches between the boundary layer and the outer region. However, wherever it is, the transition is relatively smooth. This latter observation will be useful later when deciding on the importance of what are called tangency points.

Before jumping into the derivation of the asymptotic expansion, it is worth noting that when $\varepsilon = 0$ the differential equation in (2.64a) reduces to a first-order hyperbolic equation. This change in type, from an elliptic to a hyperbolic equation, has important consequences for the analysis and helps explain some of the steps that are taken next.

Outer Expansion

The procedure that we will use to find an asymptotic approximation of the solution is very similar to that used for ordinary differential equations. The first step is to find the outer solution and to do this assume

$$u \sim u_0(x, y) + \varepsilon u_1(x, y) + \dots . \tag{2.66}$$

Substituting this into (2.64) yields the following $O(1)$ problem:

$$\alpha\partial_x u_0 + \beta\partial_y u_0 + u_0 = f(x,y), \text{ in } \Omega, \tag{2.67a}$$

where

$$u_0 = g(x,y), \text{ on } \partial\Omega_o. \tag{2.67b}$$

When constructing an asymptotic approximation, it is not always immediately clear which boundary condition, if any, the outer solution should satisfy. In (2.67b) the portion of the boundary where u_0 satisfies the original boundary condition has been identified as $\partial\Omega_o$. This is presently unknown and, in fact, it could turn out to be empty (e.g., see Exercise 2). In any case, to solve (2.67a) we change coordinates from x, y to r, s, where s is the characteristic direction for (2.67a). Specifically, we let $x = \alpha s + \xi(r)$ and $y = \beta s + \eta(r)$. The functions ξ and η can be chosen in a number of different ways, and our choice is based on the desire to have a simple orthogonal coordinate system. In particular, we take

$$x = \alpha s + \beta r \quad \text{and} \quad y = \beta s - \alpha r. \tag{2.68}$$

With this, $\partial_s = \alpha\partial_x + \beta\partial_y$, and so (2.67a) becomes

$$\partial_s u_0 + u_0 = f(\alpha s + \beta r, \beta s - \alpha r). \tag{2.69}$$

This is easy to solve, and the general solution is

$$u_0 = a_0(r)e^{-s} + \int^s e^{\tau - s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)d\tau, \tag{2.70}$$

where $a_0(r)$ is arbitrary. Since we have not yet determined exactly what portion of the boundary condition, if any, the outer solution should satisfy, we are not yet in position to convert back to x, y coordinates.

Before investigating the boundary layer, it is instructive to consider the change of coordinates given in (2.68). Fixing r and letting s increase, one obtains the directed lines shown in Fig. 2.17. Thus starting at boundary point P, as the variable s increases one crosses the domain Ω and arrives at boundary point P_*. In terms of our first-term approximation, we need to know what value u_0 starts with at P and what value it has when the point P_* is reached. However, we only have one integration constant in (2.70), so there is going to be a boundary layer at either P or at P_*. As we will see later, for our problem the outer solution will satisfy the boundary condition at P_* and there will be a boundary layer of width ε at P (see Fig. 2.18). Therefore, the set $\partial\Omega_o$ in (2.67b) consists of those points of $\partial\Omega$ where the characteristic curves leave Ω as s increases (like P_*). Based on these comments, we will assume that the points of the boundary can be separated into three disjoint sets: the points where the characteristics leave

FIGURE 2.17. Schematic drawing of the characteristic curves obtained from the problem for the outer solution. These curves are determined from (2.68) and are directed straight lines with an orientation determined by the direction of increasing s. As drawn, the coefficients α and β are assumed positive. Also note the tangency points T_R and T_L.

Ω (this is $\partial\Omega_o$), the points where they enter Ω (this will be designated as $\partial\Omega_b$), and the tangency points (this is $\partial\Omega_t$). For the domain shown in Fig. 2.17 there are two tangency points (i.e., T_L and T_R). These points can cause real headaches when constructing an asymptotic approximation and will be left until the end. Also, to keep the presentation simple, we will assume that the situation is as pictured in Fig. 2.17 (i.e., the domain is convex). This means that if a characteristic curve enters the domain, then the only other time it intersects the boundary is when it leaves the domain. Moreover, $\partial\Omega_o$ and $\partial\Omega_b$ are assumed to be smooth curves.

Now that we know (or think we know) where the boundary layer is located, we are in a position to complete the specification of the outer solution given in (2.70). To do this, assume $\partial\Omega_o$ can be described as $s = h_o(r)$ for $r_0 < r < r_1$. In this case, using (2.67b), the outer solution in (2.70) becomes

$$u_0 = g(\alpha h_o + \beta r, \beta h_o - \alpha r)e^{h_o - s} - \int_s^{h_o(r)} e^{\tau - s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)d\tau. \quad (2.71)$$

This solution does not apply along $\partial\Omega_b$ or on $\partial\Omega_t$. To be more specific, suppose the curve $s = h_b(r)$ for $r_0 < r < r_1$ describes $\partial\Omega_b$. In this case, (2.71) holds for $r_0 < r < r_1$ and $h_b(r) < s \leq h_o(r)$.

Boundary-Layer Expansion

To find out what goes on in the boundary layer, we introduce the boundary coordinate (see Fig. 2.18)

$$\bar{s} = \frac{s - h_b(r)}{\varepsilon}. \quad (2.72)$$

This coordinate, by necessity, depends on r since the boundary depends on r. This complicates the calculations in making the change of variables from

FIGURE 2.18. Schematic of the solution as a function of the characteristic coordinate s as defined in (2.68). There is a boundary layer of width $O(\varepsilon)$ at the left end of the interval. Note that, in connection with Fig. 2.17, the point $s = h_b(r)$ can be thought of as corresponding to the point P, and $s = h_o(r)$ can be thought of as corresponding to the point P_*.

(r, s) to $(r, \bar{s})$. One finds using the chain rule that the derivatives transform as follows:

$$\partial_s \to \frac{1}{\varepsilon}\partial_{\bar{s}}, \quad \partial_r \to -\frac{h_b'}{\varepsilon}\partial_{\bar{s}} + \partial_r,$$

and

$$\partial_s^2 \to \frac{1}{\varepsilon^2}\partial_{\bar{s}}^2, \quad \partial_r^2 \to \frac{(h_b')^2}{\varepsilon^2}\partial_{\bar{s}}^2 - \frac{h_b''}{\varepsilon}\partial_{\bar{s}} - \frac{2h_b'}{\varepsilon}\partial_{\bar{s}}\partial_r + \partial_r^2.$$

To convert the original problem into the boundary-layer coordinates, we must first change from x, y to r, s coordinates in (2.64a). This is relatively easy since from (2.68) one finds that

$$\partial_x = \frac{1}{\gamma}(\alpha\partial_s + \beta\partial_r) \text{ and } \partial_y = \frac{1}{\gamma}(\beta\partial_s - \alpha\partial_r),$$

where $\gamma = \alpha^2 + \beta^2$. With this, (2.64a) becomes

$$\varepsilon(\partial_s^2 + \partial_r^2)u + \gamma\partial_s u + \gamma u = \gamma f. \tag{2.73}$$

Now, letting $U(r, \bar{s})$ denote the solution in the boundary layer and then substituting the boundary layer coordinates into (2.73) yields

$$[\mu\partial_{\bar{s}}^2 + \gamma\partial_{\bar{s}} + O(\varepsilon)]U = \varepsilon\gamma f, \tag{2.74}$$

where

$$\mu(r) = 1 + (h_b')^2. \tag{2.75}$$

In keeping with our usual assumptions, $h_b(r)$ is taken to be a smooth function. However, note that at the tangency points shown in Fig. 2.17, $h_b'(r) = \infty$. For this reason, these points will have to be dealt with separately after we finish with the boundary layer.

We are now in position to expand the solution in terms of the usual power series expansion, and so let

$$U(r, \bar{s}) \sim U_0(r, \bar{s}) + \dots . \tag{2.76}$$

Substituting this into (2.74) yields the equation

$$\mu\partial_{\bar{s}}^2 U_0 + \gamma\partial_{\bar{s}} U_0 = 0.$$

The general solution of this is

$$U_0(r,\bar{s}) = A(r) + B(r)e^{-\gamma\bar{s}/\mu}, \tag{2.77}$$

where $A(r)$ and $B(r)$ are arbitrary. Now, from the boundary and matching conditions, we must have $U_0(r,0) = g$ and $U_0(r,\infty) = u_0(r,h_b)$. Imposing these on our solution in (2.77) yields

$$U_0(r,\bar{s}) = u_0(r,h_b) + [g(\alpha h_b + \beta r, \beta h_b - \alpha r) - u_0(r,h_b)]e^{-\gamma\bar{s}/\mu}, \tag{2.78}$$

where

$$u_0(r,h_b) = g(\alpha h_o + \beta r, \beta h_o - \alpha r)e^{h_o - h_b} - \int_{h_b}^{h_o} e^{\tau - h_b(r)} f(\alpha\tau + \beta r, \beta\tau - \alpha r)d\tau.$$

Composite Expansion

It is not difficult to put together a composite expansion that will give us a first-term approximation of the solution in the outer- and boundary-layer regions. Adding (2.71) and (2.78) together and then subtracting their common part yields

$$u \sim g(\alpha h_o + \beta r, \beta h_o - \alpha r)e^{h_o - s} - \int_s^{h_o(r)} e^{\tau - s} f(\alpha\tau + \beta r, \beta\tau - \alpha r)d\tau + [g(\alpha h_b + \beta r, \beta h_b - \alpha r) - u_0(r,h_b)]e^{-\gamma\bar{s}/\mu}. \tag{2.79}$$

This approximation holds for $r_0 < r < r_1$ and $h_b(r) \le s \le h_o(r)$. This result may not be pretty, but it does give us a first-term approximation of the solution in the boundary layer and outer domain. What is interesting is that we have been able to patch together the solution of an elliptic problem using solutions to hyperbolic and parabolic problems. Those who may be interested in the theoretical foundation of the approximations constructed here are referred to Levinson (1950), Eckhaus and de Jager (1966), and Il'in (1992).

Given the effort invested in deriving the composite expansion in (2.79), it is natural to wonder just how well it does in approximating the solution of the partial differential equation. To demonstrate its accuracy, suppose $f = 1$, $g = 0$, and Ω is the unit disk. The resulting approximation obtained from (2.79) and the exact solution given in (2.65) are shown in Fig. 2.19 (also see Fig. 2.16). Based on this graph, it seems that we have done reasonably well with our approximation. However, we are not finished, as (2.79) does not hold in the immediate vicinity of the tangency points T_R and T_L, which are shown in Fig. 2.17.

FIGURE 2.19. Comparison between the exact solution (2.65) and the composite expansion given in (2.79) in the case where $\varepsilon = 0.05$. The values of each function are given along the line $x = \rho\cos(\frac{\pi}{4})$, $y = \rho\sin(\frac{\pi}{4})$ where $-1 \le \rho \le 1$.

Parabolic Boundary Layer

To complete the construction of a first-term approximation of the solution of (2.64) it remains to find out what happens near the tangency points shown in Fig. 2.17. We will concentrate on T_L, and to do this we let (r_0, s_0) be its coordinates in the r, s system. Also, suppose the smooth curve $r = q(s)$ describes the boundary $\partial\Omega$ in this region. In this case $r_0 = q(s_0)$ and $q'(s_0) = 0$. It will be assumed here that $q''(s_0) \neq 0$. The boundary-layer coordinates are now

$$\tilde{r} = \frac{r - q(s)}{\varepsilon^{\alpha}} \quad \text{and} \quad \tilde{s} = \frac{s - s_0}{\varepsilon^{\beta}}. \tag{2.80}$$

The transformation formulas for the derivatives are similar to those derived earlier for the boundary layer along $\partial\Omega_b$, so they will not be given. The result is that (2.73) takes the form

$$\begin{aligned}\varepsilon(\varepsilon^{-2\alpha}\partial_{\tilde{r}}^2 + \varepsilon^{-2\beta}\partial_{\tilde{s}}^2 + \ldots)\tilde{U} \\ + \gamma(\varepsilon^{-\beta}\partial_{\tilde{s}} - \varepsilon^{\beta-\alpha}\tilde{s}q_0''\partial_{\tilde{r}} + \ldots)\tilde{U} + \gamma\tilde{U} = \gamma f.\end{aligned}$$

Here $\tilde{U}(\tilde{r}, \tilde{s})$ is the solution in this region, and we have used the Taylor series expansion $q'(s_0 + \varepsilon^{\beta}\tilde{s}) \sim \varepsilon^{\beta}\tilde{s}q_0''$ to obtain the above result. There are at least two balances that need to be considered. One is $\alpha = \beta = 1$ and the other is $2\beta = \alpha = 2/3$. The latter is the one of interest, and assuming

$$\tilde{U}(\tilde{r}, \tilde{s}) \sim \tilde{U}_0(\tilde{r}, \tilde{s}) + \ldots \tag{2.81}$$

one obtains the equation

$$(\partial_{\tilde{r}}^2 - \gamma\tilde{s}q_0''\partial_{\tilde{r}} + \gamma\partial_{\tilde{s}})\tilde{U}_0 = 0, \tag{2.82}$$

for $0 < \tilde{r} < \infty$ and $-\infty < \tilde{s} < \infty$. This is a parabolic equation, and for this reason this region is referred to as a *parabolic boundary layer*. The solution is

required to match to the solutions in the adjacent regions and should satisfy the boundary condition at $\tilde{r} = 0$. The details of this calculation will not be given here but can be found in van Harten (1976). The papers by Cook and Ludford (1971, 1973) should also be consulted, as they have an extensive analysis of such parabolic layers and how they appear in problems where the domain has a corner. The theory necessary to establish the uniform validity of the expansions when corners are present in the boundary, and a historical survey of this problem, can be found in Shih and Kellogg (1987).

Example 2. Parabolic Problem

To illustrate the application of boundary-layer methods to parabolic equations, we consider the problem of solving

$$u_t + uu_x = \varepsilon u_{xx}, \text{ for } -\infty < x < \infty \text{ and } 0 < t, \tag{2.83}$$

where $u(x, 0) = \phi(x)$. It is assumed here that $\phi(x)$ is smooth and bounded except for a jump discontinuity at $x = 0$. Moreover, $\phi'(x) \geq 0$ for $x \neq 0$ and $\phi(0^-) > \phi(0^+)$.

The nonlinear diffusion equation in (2.83) is known as *Burgers' equation.* It has become the prototype problem for studying shock waves and for the use of what are called viscosity methods for finding smooth solutions to such problems. In what follows, we will concentrate on the constructive aspects of finding asymptotic approximations to the solution. The theoretical underpinnings of the procedure can be found in Il'in (1992) and its extension to systems in Goodman and Xin (1992).

Outer Expansion

The first step is to find the outer solution, and to do this we assume $u \sim u_0(x, t) + \ldots$. Substituting this into (2.83) produces the first-order hyperbolic equation

$$\partial_t u_0 + u_0 \partial_x u_0 = 0. \tag{2.84}$$

The characteristics for this equation are the straight lines $x = x_0 + \phi(x_0)t$, and the solution $u_0(x, t)$ is constant along each of these lines (see Fig. 2.20). Therefore, given a point (x, t), then $u_0(x, t) = \phi(x_0)$, where x_0 is determined from the equation $x = x_0 + \phi(x_0)t$. This construction succeeds if the characteristics do not intersect and they cover the upper half-plane. For us the problem is with intersections. Consequently, based on the nonlinear hyperbolic nature of the reduced problem in (2.84), we will assume there is a single (smooth) curve $x = s(t)$ across which the outer solution is discontinuous. The properties of this curve will be determined from the analysis of the transition layer that connects these two outer regions.

FIGURE 2.20. The straight lines are the characteristics for the reduced problem (2.84). Because ϕ has a discontinuity at $x = 0$, and $\phi' \geq 0$ for $x \neq 0$, these lines intersect along the shock curve $x = s(t)$.

Inner Expansion

The interior-layer coordinate is $\bar{x} = [x - s(t)]/\varepsilon^{\alpha}$. Letting $U(\bar{x}, t)$ denote the solution of the problem in this layer, then (2.83) takes the form

$$\partial_t U - \varepsilon^{-\alpha} s'(t) \partial_{\bar{x}} U + \varepsilon^{-\alpha} U \partial_{\bar{x}} U = \varepsilon^{1-2\alpha} \partial_{\bar{x}}^2 U. \tag{2.85}$$

Balancing terms in this equation, one finds $\alpha = 1$. Thus, the appropriate expansion of the solution is $U \sim U_0(\bar{x}, t) + \dots$, and substituting this into (2.85) yields

$$-s'(t)\partial_{\bar{x}} U_0 + U_0 \partial_{\bar{x}} U_0 = \partial_{\bar{x}}^2 U_0.$$

Integrating this, one finds that

$$\partial_{\bar{x}} U_0 = \frac{1}{2} U_0^2 - s'(t) U_0 + A(t). \tag{2.86}$$

The boundary conditions to be used come from matching with the outer solution. They are

$$\lim_{\bar{x} \to -\infty} U_0 = u_0^- \quad \text{and} \quad \lim_{\bar{x} \to \infty} U_0 = u_0^+, \tag{2.87}$$

where

$$u_0^{\pm} = \lim_{x \to s(t)^{\pm}} u_0(x, t). \tag{2.88}$$

From (2.86) it follows that $A(t) = -\frac{1}{2}(u_0^-)^2 + s'(t) u_0^-$ and

$$s'(t) = \frac{1}{2}(u_0^+ + u_0^-). \tag{2.89}$$

The differential equation in (2.89) determines the position of the shock and is known as the Rankine-Hugoniot condition. Its solution requires an initial condition, and because of the assumed location of the discontinuity in $\phi(x)$ we take $s(0) = 0$.

To complete the analysis of the first-order problem in the shock layer, we separate variables in (2.86) and then integrate to obtain, for the case where $u_0^+ < u_0^-$,

$$U_0(\bar{x}, t) = \frac{u_0^+ + b(\bar{x}, t)u_0^-}{1 + b(\bar{x}, t)}, \tag{2.90}$$

where

$$b(\bar{x}, t) = B(t)e^{\bar{x}(u_0^+ - u_0^-)/2}. \tag{2.91}$$

In (2.91), $B(t)$ is an arbitrary nonzero function.

The indeterminacy in $U_0(\bar{x}, t)$, as given by the unspecified function in (2.91), is the same difficulty we ran into when investigating interior layers in Section 2.4. However, for this nonlinear diffusion problem it is possible to determine $B(t)$, up to a multiplicative factor, by examining the next order problem. To state the result, note that given a point $(x, t) = (s(t), t)$ on the shock then $u_0^+ = \phi(x^+)$ and $u_0^- = \phi(x^-)$. Here $x^\pm$ are the initial points on the x-axis for the two characteristics that intersect at $(s(t), t)$ (see Fig. 2.21). In this case, one finds from the $O(\varepsilon)$ problem that (see Exercise 6)

$$B(t) = B_0\sqrt{\frac{1 + t\phi'(x^+)}{1 + t\phi'(x^-)}}. \tag{2.92}$$

The constant B_0 is found from the initial condition.

To determine B_0 in (2.92) one must match the shock-layer solution with the solution from the initial layer located near $x = x_0$. The appropriate coordinate transformation for this layer is $\bar{x} = [x - s(t)]/\varepsilon$ and $\tau = t/\varepsilon$. The steps involved in this procedure are not difficult, and one finds that $B_0 = 1$ (see Exercise 6). With this the first-term approximation of the solution in the shock layer is determined.

To demonstrate the accuracy of the asymptotic approximation, we show in Fig. 2.22 the approximation along with the exact solution at two different values of t and ε. For $\varepsilon = 0.1$ the two solutions are so close that they are essentially indistinguishable in the graph. They are even in reasonable

FIGURE 2.21. Schematic showing the shock and two characteristics that intersect on the shock.

FIGURE 2.22. Graph of the exact solution of Burgers' equation (dashed curves) and the asymptotic approximation (solid curves) in the case where $\phi(x) = 1$ if $x < 0$ and $\phi(x) = 0$ if $x > 0$. The solutions are compared at $t = 10$ and at $t = 20$. In (a) $\varepsilon = 1$, and in (b) $\varepsilon = 1/10$.

agreement when $\varepsilon = 1$, although the differences are more apparent for the smaller value of t. The sharpening of the shock layer as ε decreases is also evident in these figures. This is expected because we found that the width of this layer is $O(\varepsilon)$.

In this section, we have considered elliptic and parabolic problems. Matched asymptotic expansions can also be applied to hyperbolic problems, and several examples are worked out in Kevorkian and Cole (1981). It should also be pointed out that the methods developed in Chapters 3 and 4 are applied extensively to hyperbolic equations.

Exercises

1. A special case of (2.64) is the problem of solving

$$\varepsilon \nabla^2 u + \partial_y u = 1, \text{ in } \Omega,$$

where $u = x + y$ on $\partial\Omega$. Let Ω be the unit disk $x^2 + y^2 < 1$.

(a) Sketch the domain and characteristic curves (see Fig. 2.17). Identify the points T_L and T_R.

(b) What is the composite expansion, as in (2.79), for this problem?

(c) What is the parabolic-layer equation, as in (2.82), in this case?

2. Find a first-term composite expansion, for the outer and boundary layers, of the solution of

$$\varepsilon \nabla^2 u + u = f(x, y) \text{ in } \Omega,$$

where $u = g(x, y)$ on $\partial\Omega$. Let Ω be the unit disk $x^2 + y^2 < 1$.

3. In this exercise, variations of the elliptic boundary-value problem (2.64) are considered.

(a) If the coefficients α and β in (2.64a) are negative, how does the composite expansion in (2.79) change?

(b) If one of the coefficients α, β in (2.64a) is negative and the other is positive, how does the composite expansion in (2.79) change?

4. The equation of one-dimensional heat conduction in a material with a low conductivity is (Plaschko, 1990)

$$u_t = \varepsilon u_{xx} + v(t) u_x, \text{ for } 0 < x \text{ and } 0 < t,$$

where $u(0, t) = g(t)$, $u(\infty, t) = 0$, and $u(x, 0) = h(x)$. Assume that the functions $v(t)$, $g(t)$, and $h(x)$ are smooth with $v(t) > 0$ for $0 \leq t < \infty$, $g(0) = h(0)$, and $h(\infty) = 0$.

(a) Find a first-term composite expansion of the solution.

(b) Find the second term in the composite expansion. Is the expansion in part (a) uniformly valid over the interval $0 \leq t < \infty$? What conditions need to be placed on the functions $h(x)$ and $v(t)$? A method for constructing uniformly valid approximations in a case like this is the subject of Chapter 3.

5. Using the Cole-Hopf transformation, it is possible to solve Burgers' equation (2.83) (Whitham, 1974). In the case where

$$u(x, 0) = \begin{cases} u_1 & \text{for } x < 0 \\ u_2 & \text{for } 0 < x \end{cases},$$

where $u_1 > u_2$ are constants, one finds $u(x, t) = \frac{u_2 + K(x,t) u_1}{1 + K(x,t)}$ where

$$K(x, t) = \frac{\operatorname{erfc}\left(\frac{x - u_1 t}{2\sqrt{\varepsilon t}}\right)}{\operatorname{erfc}\left(-\frac{x - u_2 t}{2\sqrt{\varepsilon t}}\right)} e^{(x - v_0 t)(u_2 - u_1)/2\varepsilon}$$

and $v_0 = \frac{1}{2}(u_1 + u_2)$. Compare this with the first-term approximation derived for (2.83). Make sure to comment on the possible differences for small and large values of t.

6. The function $B(t)$ in (2.91) can be found by matching the second term in the inner and outer expansions, and this exercise outlines the necessary steps.

(a) Show the second term in the outer expansion is

$$u_1(x,t) = \frac{t\phi''(\xi)}{(1+t\phi'(\xi))^2},$$

where the value of ξ is determined from the equation $x = \xi + t\phi(\xi)$.

(b) By changing variables from $\bar{x}$ to $z = (1-b)/(1+b)$, where b is given in (2.91), show that the equation for U_1 becomes

$$(1-z^2)\partial_z^2 U_1 + 2U_1 = \frac{4(s''+r'z)}{r^2(1-z^2)} - \frac{2}{r^2}\left(\frac{rB'}{B} + r'\ln\left(\frac{1-z}{B(1+z)}\right)\right),$$

where $r = \frac{1}{2}(u_0^+ - u_0^-)$.

(c) Solve the equation in part (b), and from this obtain the first two terms in the expansion of U_1 for $z \to 1$ and for $z \to -1$.

[Hint: Because of the nature of this calculation, the use of a symbolic computer system is recommended.]

(d) To match the inner and outer expansions, introduce the intermediate variable $\bar{x}_\eta = [x - s(t)]/\varepsilon^\eta$. With this, show that the outer expansion expands as

$$u \sim \phi(\xi) + \varepsilon^\eta \frac{\bar{x}_\eta \phi'(\xi)}{1+t\phi'(\xi)} + \varepsilon \frac{t\phi''(\xi)}{(1+t\phi'(\xi))^2} + \ldots,$$

where ξ is determined from the equation $s(t) = \xi + t\phi(\xi)$. Do the same for the inner expansion, and by matching the two derive the result in (2.92).

(e) To find the constant B_0 introduce the initial-layer coordinates $\bar{x} = [x - s(t)]/\varepsilon$ and $\tau = t/\varepsilon$. Find the first term in this layer and then match the result with (2.90) to show $B_0 = 1$.

7. This exercise involves modifications of the expansions for Burgers' equation.

(a) Discuss the possibility of obtaining a composite expansion for the solution of (2.83).

(b) The center of the shock wave is where u is halfway between u_0^+ and u_0^-. If this is located at $x = X(t)$, find the first two terms in the expansion of $X(t)$. A discussion of this, and other aspects of the problem, can be found in Lighthill (1956).

(c) Because the position of the shock is determined by the solution, then it should, presumably, depend on ε. How do things change if one allows for the possibility that $s(t) \sim s_0(t) + \varepsilon s_1(t) + \ldots$?

8. Consider the linear diffusion problem

$$u_t + \alpha u_x + \beta u = \varepsilon u_{xx}, \text{ for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x,0) = \phi(x)$. Assume $\phi(x)$ is as before and that α and β are positive constants.

(a) Find the first terms in the inner and outer expansions of the solution.

(b) Comment on the differences between the characteristics of the shock layer for Burgers' equation and the one you found in part (a).

9. Find the first term in the inner and outer expansions of the solution of

$$u_t + f(u)u_x = \varepsilon u_{xx}, \text{ for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x,0) = \phi(x)$. Assume $\phi(x)$ is as before and $f(r)$ is smooth with $f'(r) > 0$.

10. Consider the problem of solving

$$\frac{\varepsilon}{r}\partial_r\left[\frac{1}{r}\partial_r(r^2 w)\right] = \partial_t w + \mu w + \kappa e^{-2t}, \text{ for } 0 \le r < 1,\ 0 < t,$$

where $w(r,0) = 0$, $\partial_r w(0,t) = 0$, and $w(1,t) = 0$. Also, μ and κ are positive constants with $\mu \neq 2$. This problem arises in the study of the rotation of a cylindrical container filled with a dilute suspension (Ungarish, 1993). In this context, $w(r,t)$ is the angular velocity of the suspension and the boundary layer is known as a Stewartson shear layer. For small ε, find a first-term composite expansion of the solution.

11. The equation for the velocity potential $\Phi(x,y)$ for steady air flow over an airplane wing is (Cole and Cook, 1986)

$$(a^2 - \Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (a^2 - \Phi_y^2)\Phi_{yy} = 0,$$

where

$$a^2 = a_\infty^2 + \frac{1}{2}(\gamma - 1)(U^2 - \Phi_x^2 - \Phi_y^2).$$

Here $a_\infty > 0$, $U > 0$, and $\gamma > 1$ are constants. The wing is assumed to be described by the curve $y = \varepsilon f(x)$, for $0 < x < 1$ (see Fig. 2.23). In this case, the boundary conditions are that $\Phi = Ux$ as $x \to -\infty$ and

$$\Phi_y = \begin{cases} \varepsilon f'(x)\Phi_x & \text{when } y = \varepsilon f(x) \text{ and } 0 < x < 1 \\ 0 & \text{when } y = 0 \text{ and } -\infty < x < 0 \text{ or } 1 < x \end{cases}.$$

FIGURE 2.23. Schematic of air flow over an airplane wing as assumed in Exercise 11.

(a) The thickness ε of the wing is small, and this is the basis of what is known as *small disturbance theory.* The appropriate expansion for the potential in this case has the form

$$\Phi \sim Ux + \varepsilon^{\alpha}\phi_1 + \varepsilon^{\beta}\phi_2 + \dots .$$

Find α and then determine what problem ϕ_1 satisfies.

(b) Find ϕ_1 in the case where $M_\infty > 1$, where $M_\infty = U/a_\infty$.

(c) For the case where $M_\infty > 1$, find ϕ_2 and explain why for the expansion to be valid it must be that $\varepsilon << (M_\infty^2 - 1)^{3/2}$.

[Hint: Use characteristic coordinates $\xi = x - (M_\infty^2 - 1)^{1/2}y$, $\eta = x + (M_\infty^2 - 1)^{1/2}y$.]

12. For a semiconductor to function properly, one must be concerned with the level of impurities that diffuse in from the outer surface and occupy vacant locations in its crystalline structure. A problem for the concentration of impurities $c(x,t)$ and vacancies $v(x,t)$ is (King et al., 1992)

$$\left.\begin{aligned} \partial_t c &= \partial_x(v\partial_x c - c\partial_x v) \\ \partial_t v + r\partial_t c &= \varepsilon^2\partial_x^2 v \end{aligned}\right\} \quad \text{for } 0 < x < \infty \text{ and } 0 < t,$$

where $c = 0$ and $v = 1$ when $t = 0$; $c = 1$ and $v = \mu$ when $x = 0$; and $c \to 0$ and $v \to 1$ as $x \to \infty$. Also, $r > 0$ and $\mu > 0$ are constants. For small ε, derive a composite expansion of the solution of this problem.

2.7 Difference Equations

Up until now, when discussing boundary-layer problems we have dealt almost exclusively with differential equations. We will now expand our horizons and investigate what happens with singularly perturbed difference equations. As will be seen, many of the ideas developed in the first part of this chapter will reappear when analyzing difference equations, but there are subtle and interesting differences. The approach used here is based, in part, on the paper of Comstock and Hsiao (1976).

Our starting point is the boundary-value problem

$$\varepsilon y_{n+1} + \alpha_n y_n + \beta_n y_{n-1} = 0, \text{ for } n = 1, 2, \ldots, N-1, \tag{2.93}$$

where

$$y_0 = a, y_N = b. \tag{2.94}$$

What we have here is a second-order linear difference equation with prescribed values at the ends (where $n = 0, N$). In what follows it is assumed N is fixed, and we will investigate how the solution behaves for small ε. It should also be pointed out that we will be assuming that the coefficients α_n are nonzero.

There are a couple of observations about the problem that should be made before starting the derivation of the asymptotic approximation of the solution. First, it is clear that the problem is singular for small ε since the reduced equation $\alpha_n y_n + \beta_n y_{n-1} = 0$ is first order and cannot be expected to satisfy both boundary conditions. The second observation can be made by considering an example. If $\alpha_n = 2$ and $\beta_n = a = b = 1$, then the solution of (2.93),(2.94) is

$$y_n = \chi m_2^n + (1-\chi) m_1^n \quad \text{where} \quad \chi = \frac{1 - m_1^N}{m_2^N - m_1^N}$$

where $m_1 = -\varepsilon^{-1}(1 + \sqrt{1-\varepsilon})$ and $m_2 = -\varepsilon^{-1}(1 - \sqrt{1-\varepsilon})$. For small ε this reduces to

$$y_n \sim \left(-\frac{1}{2}\right)^n + \left[1 - \left(-\frac{1}{2}\right)^N\right] \left(-\frac{\varepsilon}{2}\right)^{N-n}, \text{ for } n = 1, 2, \ldots, N. \tag{2.95}$$

This shows boundary-layer behavior near the end $n = N$ in the sense that if one starts at $n = N$, and then considers the values at $n = N-1$, $N-2$, $\ldots$, the $O(\varepsilon^{N-n})$ term in (2.95) rapidly decays. Moreover, away from the immediate vicinity of the right end, this term is small in comparison to the other term in the expansion.

We now derive an asymptotic approximation of the solution of (2.93), (2.94). The easiest component to obtain is the outer expansion, which is determined by simply assuming an expansion of the form

$$y_n \sim \bar{y}_n + \varepsilon \bar{z}_n + \ldots. \tag{2.96}$$

Substituting this into (2.93) and then equating like powers of ε, one finds that

$$\alpha_n \bar{y}_n + \beta_n \bar{y}_{n-1} = 0. \tag{2.97}$$

Based on the observations made earlier, we expect the boundary layer to be at $n = N$. So, we require $\bar{y}_0 = a$. Solving (2.97) and using this boundary condition, one finds that

$$\bar{y}_n = \kappa_n a, \text{ for } n = 0, 1, 2, 3, \ldots, \tag{2.98}$$

where $\kappa_0 = 1$; otherwise

$$\kappa_n = \prod_{j=1}^{n} \left(-\frac{\beta_j}{\alpha_j} \right). \tag{2.99}$$

Now the question is, how do we deal with the boundary layer at the right end? The first step is to rescale the problem by letting

$$y_n = \varepsilon^{\gamma(n)} Y_n. \tag{2.100}$$

The exponent $\gamma(n)$ is determined by balancing, and to do this we substitute (2.100) into (2.93) to obtain

$$\underset{①}{\varepsilon^{1+\gamma(n+1)} Y_{n+1}} + \underset{②}{\alpha_n \varepsilon^{\gamma(n)} Y_n} + \underset{③}{\beta_n \varepsilon^{\gamma(n-1)} Y_{n-1}} = 0. \tag{2.101}$$

In the outer region, the balancing takes place between terms ② and ③. For the boundary layer, we have two possibilities to investigate:

i) ① $\sim$ ③ and ② is higher order $\Rightarrow \gamma(n+1) = \gamma(n-1) - 1$. Thus, if $n = 2k$ then $\gamma(2k) = \gamma(0) - k$, and if $n = 2k+1$ then $\gamma(2k+1) = \gamma(1) - k$. In the case where $n = 2k$ we have ①, ③ $= O(\varepsilon^{1+\gamma(1)-k})$, ② $= O(\varepsilon^{\gamma(0)-k})$; if $n = 2k+1$ we have ①, ③ $= O(\varepsilon^{\gamma(0)-k})$, ② $= O(\varepsilon^{\gamma(1)-k})$. From this it follows that it is not possible to pick values for $\gamma(0)$ and $\gamma(1)$ that are consistent with our original assumption that ② is higher order. Thus, this balance is not possible.

ii) ① $\sim$ ② and ③ is higher order $\Rightarrow \gamma(n+1) = \gamma(n) - 1 \Rightarrow \gamma(n+1) = \gamma(0) - n \Rightarrow$ ①, ② $= O(\varepsilon^{1+\gamma(0)-n})$ and ③ $= O(\varepsilon^{2+\gamma(0)-n})$. This is consistent with the original assumption, and so this is the balancing we are looking for.

Now that the balancing has been determined, there is one additional observation to make. The solution as written in (2.100) is going to be involved with the boundary condition at $n = N$. For this reason, it is required that $\gamma(N) = 0$. Therefore, $\gamma(n) = N - n$ and (2.101) takes the form

$$Y_{n+1} + \alpha_n Y_n + \varepsilon \beta_n Y_{n-1} = 0, \text{ for } n = N-1, N-2, \ldots, \tag{2.102}$$

where $Y_N = c$. The value of c is determined from the boundary condition at $n = N$, and this will be done once the composite expansion is constructed. The appropriate expansion of the boundary-layer solution is

$$Y_n \sim \bar{Y}_n + \varepsilon \bar{Z}_n + \ldots. \tag{2.103}$$

Introducing this into (2.102) yields the equation

$$\bar{Y}_{n+1} + \alpha_n \bar{Y}_n = 0.$$

The solution of this that also satisfies the boundary condition $Y_N = c$ is

$$\bar{Y}_n = \lambda_{N-n} c, \tag{2.104}$$

where $\lambda_0 = 1$; otherwise

$$\lambda_k = \prod_{j=1}^{k} \left(-\frac{1}{\alpha_{N-j+1}} \right). \tag{2.105}$$

We are going to use (2.98) and (2.104) to construct a composite expansion of the solution. As usual, this is obtained by adding the expansions together, which gives us

$$\begin{aligned} y_n &\sim \bar{y}_n + \varepsilon^{N-n} \bar{Y}_n \\ &= \kappa_n a + \varepsilon^{N-n} \lambda_{N-n} c. \end{aligned} \tag{2.106}$$

The boundary-layer solution does not contribute to the first order in the outer region; however, this is not true of the outer solution's contribution in the boundary layer. We will deal with this by taking c so that the above result satisfies the boundary condition at $n = N$. This gives us $c = b - \kappa_N a$, and so the composite expansion of the solution is

$$y_n \sim \kappa_n a + \varepsilon^{N-n} \lambda_{N-n} (b - \kappa_N a). \tag{2.107}$$

It is possible to prove that this does indeed give us an asymptotic approximation of the solution (see Comstock and Hsiao, 1976). A demonstration of the accuracy of the approximation is given in Fig. 2.24. It is seen that the numerical solutions of the difference equation and the asymptotic expansion are in very close agreement.

There is a very interesting connection between this difference equation and the numerical solution of an associated differential equation. To explain, consider the boundary-value problem

$$\varepsilon y'' + p(x) y' + q(x) y = 0, \text{ for } 0 < x < 1, \tag{2.108}$$

where $y(0) = a$ and $y(1) = b$. It is assumed that the functions $p(x)$ and $q(x)$ are continuous. Now, if $p(x) < 0$, for $0 \le x \le 1$, then there is a boundary layer at $x = 1$ with width $O(\varepsilon)$ (see Exercise 8 of Section 2.2). Given this fact, suppose we want to solve the problem numerically using finite differences. The standard centered difference approximation will be used for the second derivative, but for the first derivative we will consider using either the forward-difference approximation

$$y'(x_n) \approx \frac{y_{n+1} - y_n}{h} \tag{2.109}$$

FIGURE 2.24. Comparison between the asymptotic expansion in (2.107) and the numerical solution of (2.93),(2.94) where $\alpha_n = -(1 + \frac{n}{N})$, $\beta_n = \frac{3}{2} - \frac{n}{N}$, $a = 1$, $b = -1$, $N = 20$, and $\varepsilon = 10^{-2}$.

or the backward-difference approximation

$$y'(x_n) \approx \frac{y_n - y_{n-1}}{h}. \tag{2.110}$$

Using the backward difference, we get from (2.108) that

$$\varepsilon y_{n+1} + (\alpha_n - 2\varepsilon)y_n + (\beta_n + \varepsilon)y_{n-1} = 0, \tag{2.111}$$

where $\alpha_n = hp_n + h^2 q_n$ and $\beta_n = -hp_n$. Because this equation differs from (2.93) only in the addition of higher-order terms in the coefficients, a composite expansion of the solution is still given in (2.107). Thus, the difference equation has a boundary layer in the same location as the associated differential equation. This is good if one expects the numerical solution to have any resemblance to the solution of the original problem. What is interesting is that the forward difference (2.109) results in a difference equation with a boundary layer at $x = 0$ and not at $x = 1$ (see Exercise 1). This observation is strong evidence that one should use (2.110) rather than (2.109) to solve this problem.

Another way to approximate the first derivative is to use a centered difference. This would seem to be a better choice because it is an $O(h^2)$ approximation while the approximations in (2.109) and (2.110) are $O(h)$. However, its major limitation is that it cannot delineate a boundary layer at either end of the interval (see Exercise 4).

Before completing the discussion of the numerical solution of (2.108), it is worth making a comment about the order of the stepsize h. Presumably h should be relatively small for the finite-difference equation to be an accurate approximation of the differential equation. This introduces a second small parameter into the problem, and one has to be careful about its size in comparison to ε. For example, since $\alpha_n = O(h)$ and $\beta_n = O(h)$, it should not be unexpected that one must require $\varepsilon << h$ to guarantee the expansion is well ordered. We will not pursue this topic, but interested readers should consult the articles by Brown and Lorenz (1987), Farrell (1987), and Vulanovic and Farrell (1993).

Exercises

1. (a) Find a composite expansion of the difference equation

$$\omega_n y_{n+1} + \alpha_n y_n + \varepsilon y_{n-1} = 0, \text{ for } n = 1, 2, \ldots, N-1,$$

where $y_0 = a$ and $y_N = b$. Also, the ω_n's are nonzero.

(b) Suppose one uses the forward-difference approximation given in (2.109) to solve (2.108). Show that you get a difference equation like the one in part (a), and write down the resulting composite expansion of the solution.

(c) At which end does the difference equation you found in part (b) have a boundary layer? What condition should be placed on $p(x)$ so that this numerical approximation can be expected to give an accurate approximation of the solution?

2. Find a composite expansion of the difference equation

$$\varepsilon y_{n+1} + \alpha_n y_n + \varepsilon \beta_n y_{n-1} = 0, \text{ for } n = 1, 2, \ldots, N-1,$$

where $y_0 = a$ and $y_N = b$. It is suggested that you first solve the problem in the case where α_n and β_n are independent of n. With this you should be able to find the expansion of the solution to the full problem.

3. This problem investigates the forward and backward stability of certain difference equations. In what follows, the coefficients α_n, β_n, and ω_n are assumed to be nonzero and bounded.

(a) Consider the initial-value problem

$$\varepsilon y_{n+1} + \alpha_n y_n + \beta_n y_{n-1} = 0, \text{ for } n = 1, 2, \ldots,$$

where $y_0 = a$ and $y_1 = b$. Explain why the solution of this problem becomes unbounded as n increases. You can do this, if you wish, by making specific choices for the coefficients α_n and β_n.

(b) Consider the initial-value problem

$$\omega_n y_{n+1} + \alpha_n y_n + \varepsilon y_{n-1} = 0, \text{ for } n = 1, 2, \ldots,$$

where $y_0 = a$ and $y_1 = b$. Explain why the solution of this problem is bounded as n increases. You can do this, if you wish, by making specific choices for the coefficients ω_n and α_n.

(c) Now suppose the problems in parts (a) and (b) are to be solved for $n = 0, -1, -2, -3, \ldots$. Explain why the solution of the problem in (a) is bounded as $n \to -\infty$ but the solution of the problem from (b) is unbounded as $n \to -\infty$. These observations give rise to the statement that the equation in (a) is backwardly stable while the equation in (b) is forwardly stable. These properties are reminiscent of what is found for the heat equation.

4. This problem examines the use of centered-difference approximations to solve the singularly perturbed boundary-value problem in (2.108).

   (a) Find a first-term composite expansion of the solution of the difference equation

   $$(\alpha+\varepsilon)y_{n+1}+2(\beta-\varepsilon)y_n-(\alpha-\varepsilon)y_{n-1}=0, \text{ for } n=1,2,\ldots,N-1,$$

   where $y_0=a$, $y_N=b$, and $\alpha\neq 0$. Are there any boundary layers for this problem?

   (b) Suppose one uses the centered-difference approximation of the first derivative to numerically solve (2.108). Letting $p(x)$ and $q(x)$ be constants, show that you get a difference equation like the one in part (a), and write down the resulting composite expansion of the solution. What terms in the differential equation do not contribute to the composite expansion?

   (c) The solution of (2.108) has a boundary layer at $x=0$ if $p(x)>0$ for $0\leq x\leq 1$ and one at $x=1$ if $p(x)<0$ for $0\leq x\leq 1$. Comment on this and the results from parts (a) and (b).

3
Multiple Scales

3.1 Introduction

When one uses matched asymptotic expansions, the solution is constructed in different regions that are then patched together to form a composite expansion. The method of multiple scales differs from this approach in that it essentially starts with a generalized version of a composite expansion. In doing this, one introduces coordinates for each region (or layer); these new variables are considered to be independent of one another. A consequence of this is that what may start out as an ordinary differential equation is transformed into a partial differential equation. Exactly why this helps to solve the problem, rather than make it harder, will be discussed as the method is developed in this chapter.

The history of multiple scales is more difficult to delineate than, say, boundary-layer theory. This is because the method is so general that many apparently unrelated approximation procedures are special cases of it. One might argue that the procedure got its start in the first half of the 19th century. For example, Stokes (1843) used a type of coordinate expansion in his calculations of fluid flow around an elliptic cylinder. Most of these early efforts were limited, and it was not until the latter half of the 19th century that Poincaré (1886), based on the work of Lindstedt (1882), made more extensive use of the ideas underlying multiple scales in his investigations into the periodic motion of planets. He found that the approximation obtained from a regular expansion accurately described the motion for only a few revolutions of the planet, after which the approximation was way

off. The error was due, in part, to the contributions of the second term of the expansion. He referred to this difficulty as the presence of a secular term. To remedy the situation, he expanded the independent variable with the intention of making the approximation uniformly valid by removing the secular term. This idea is also at the heart of the modern version of the method. What Poincaré was missing was the introduction of multiple independent variables based on the expansion parameter. This step came much later; the most influential work in this regard has been by Kuzmak (1959) and Cole and Kevorkian (1963).

3.2 Introductory Example

As in the last chapter, we will introduce the ideas underlying the method by going through a relatively simple example. The problem to be considered is to find the $y(t)$ that satisfies

$$y'' + \varepsilon y' + y = 0, \quad \text{for } 0 < t, \tag{3.1a}$$

where

$$y(0) = 0 \text{ and } y'(0) = 1. \tag{3.1b}$$

This equation corresponds to an oscillator (i.e., a mass-spring-dashpot) with weak damping, where the time variable has been scaled by the period of the undamped system (see Exercise 1.6.10). This is the classical example used to illustrate multiple scales. Our approach to constructing the approximation comes from the work of Cole and Kevorkian (1963) and Reiss (1971).

Regular Expansion

It appears that this is not a layer type of problem since ε is not multiplying the highest derivative in (3.1a). It seems, therefore, that the solution has a regular power series expansion. So, assuming

$$y(t) \sim y_0(t) + \varepsilon y_1(t) + \ldots, \tag{3.2}$$

one finds from (3.1) that

$$y(t) \sim \sin(t) - \frac{1}{2}\varepsilon t \sin(t). \tag{3.3}$$

In comparison, the exact solution of the problem is

$$y(t) = \frac{1}{\sqrt{1-\varepsilon^2/4}} e^{-\varepsilon t/2} \sin\left(t\sqrt{1-\varepsilon^2/4}\right). \tag{3.4}$$

FIGURE 3.1. Comparison between the regular perturbation expansion in (3.3) and the exact solution given in (3.4) when $\varepsilon = 10^{-1}$. The two curves are reasonably close at the beginning (i.e., up to $t \approx 10$), but they differ significantly for larger values of t.

These functions are plotted in Fig. 3.1. It is apparent that we have not been particularly successful in finding a very accurate approximation of the solution for $0 \leq t < \infty$. This is because the second term in the expansion is as large as the first term once $\varepsilon t \approx 1$. For this reason, the second term is said to be a *secular term*, and for the expansion to be well ordered we must have $\varepsilon t << 1$. The problem with our approximation is that the solution (3.4) decays but the first term in (3.3) does not. The second term tries to compensate for this, and in the process it eventually becomes as large as the first term.

Another way to look at what is happening in the problem is to consider the energy. By multiplying (3.1a) by the velocity y', integrating, and then using the initial conditions (3.1b), one obtains the equation

$$H(y, y') = H(0, 1) - \varepsilon \int_0^t [y'(\tau)]^2 d\tau, \tag{3.5}$$

where

$$H(y, y') \equiv \frac{1}{2}(y')^2 + \frac{1}{2}y^2. \tag{3.6}$$

The function H is the Hamiltonian and consists of the sum of the kinetic and potential energies for the system. Thus, (3.5) shows that the energy equals the amount included at the start minus whatever is lost due to damping. To relate this to the multiple-scale expansion, recall that a system is said to be conservative, or energy is conserved, if H is independent of t. On the other hand, if $\frac{d}{dt}H < 0$, so that energy is lost, then the system is dissipative. Using these ideas, we have that (3.1) is dissipative but the problem for $y_0(t)$ is conservative. Therefore, the first term in the approximation in (3.3) is a conservative approximation to a dissipative problem. This is one of many situations where the multiple-scales method is generally needed.

Multiple-Scale Expansion

This problem has an oscillatory component of the solution that occurs on a time scale that is $O(1)$ and also has a slow variation in the solution that takes place on a time scale $O\left(\frac{1}{\varepsilon}\right)$. To incorporate two time scales into the problem, we introduce the variables

$$t_1 = t \tag{3.7}$$

and

$$t_2 = \varepsilon^\alpha t. \tag{3.8}$$

Based on the earlier discussion of the exact solution, it is expected that $\alpha = 1$. However, we will derive this later when constructing the expansion of the solution. These two time scales will be treated as independent. One consequence of this is that the original time derivative transforms as follows:

$$\frac{d}{dt} \to \frac{dt_1}{dt}\frac{\partial}{\partial t_1} + \frac{dt_2}{dt}\frac{\partial}{\partial t_2} = \frac{\partial}{\partial t_1} + \varepsilon^\alpha \frac{\partial}{\partial t_2}.$$

Substituting this into (3.1) yields

$$\left(\partial_{t_1}^2 + 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \varepsilon^{2\alpha}\partial_{t_2}^2\right) y + \varepsilon\left(\partial_{t_1} + \varepsilon^\alpha \partial_{t_2}\right) y + y = 0, \tag{3.9a}$$

where

$$y = 0 \text{ and } \left(\partial_{t_1} + \varepsilon^\alpha \partial_{t_2}\right) y = 1, \quad \text{for } t_1 = t_2 = 0. \tag{3.9b}$$

To simplify the notation, we use the symbol ∂_{t_1} in place of $\frac{\partial}{\partial t_1}$ (and similarly for ∂_{t_2}). The benefits of introducing these two time variables are not yet apparent. In fact, one might argue that we have made the problem harder since the original ordinary differential equation has been turned into a partial differential equation. The reasons for doing this will become evident in the discussion that follows. It should be pointed out that the solution of (3.9) is not unique and that we need to impose more conditions for uniqueness on the solution. This freedom will enable us to prevent secular terms from appearing in the expansion (at least over the time scales we are using).

We will now use a power series expansion of the form

$$y \sim y_0\left(t_1, t_2\right) + \varepsilon y_1\left(t_1, t_2\right) + \cdots. \tag{3.10}$$

Substituting this into (3.9a) yields the following:

$$\begin{aligned}(\partial_{t_1}^2 &+ 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \cdots)(y_0 + \varepsilon y_1 + \cdots) \\ &+ \varepsilon(\partial_{t_1} + \cdots)(y_0 + \cdots) + y_0 + \varepsilon y_1 + \cdots = 0.\end{aligned} \tag{3.11}$$

In writing equation (3.11), we made an effort to include only the terms that might contribute to the first two terms of the expansion. From this and (3.9b), we obtain the first problem to solve:

$$O(1)\ \left(\partial_{t_1}^2+1\right)y_0=0,$$
$$y_0=0,\quad \partial_{t_1}y_0=1\quad \text{at } t_1=t_2=0.$$

The general solution of this problem is

$$y_0=a_0(t_2)\sin(t_1)+b_0(t_2)\cos(t_1), \tag{3.12a}$$

where

$$a_0(0)=1 \text{ and } b_0(0)=0. \tag{3.12b}$$

For the moment, the coefficients $a_0(t_2)$ and $b_0(t_2)$ are arbitrary functions of t_2 except that they are required to satisfy the initial conditions given in (3.12b).

To go any farther, we need to determine α in the slow time scale (3.8). To do this, note that in the $O(\varepsilon)$ equation obtained from (3.11) we are going to have the term $\partial_{t_1}y_0$. It is not hard to check that this will cause a secular term to appear in the expansion. However, this can be prevented if we make use of the dependence of y_0 on the time scale t_2. In (3.11) the only term available to us that can be used to prevent a secular term is $2\varepsilon^{\alpha}\partial_{t_1}\partial_{t_2}y_0$. Balancing this with $\varepsilon\partial_{t_1}y_0$ gives us $\alpha=1$. With this, and the initial conditions in (3.9b), we get the following problem:

$$O(\varepsilon)\ \left(\partial_{t_1}^2+1\right)y_1=-2\partial_{t_1}\partial_{t_2}y_0-\partial_{t_1}y_0,$$
$$y_1=0 \text{ and } \partial_{t_1}y_1=-\partial_{t_2}y_0\quad \text{when } t_1=t_2=0.$$

From (3.12a), the differential equation for y_1 is

$$\left(\partial_{t_1}^2+1\right)y_1=(2b_0'+b_0)\sin(t_1)-(2a_0'+a_0)\cos(t_1). \tag{3.13}$$

The general solution of this problem is found to be

$$\begin{aligned} y_1 &= a_1(t_2)\sin(t_1)+b_1(t_2)\cos(t_1)\\ &\quad -\frac{1}{2}(2b_0'+b_0)\,t_1\cos(t_1)-\frac{1}{2}(2a_0'+a_0)\,t_1\sin(t_1), \end{aligned} \tag{3.14a}$$

where

$$a_1(0)=b_1(0)=0. \tag{3.14b}$$

In (3.14a) we have the possibility of secular terms in the expansion. However, the functions a_0 and b_0 can be chosen to prevent this. From (3.14a) and (3.12b), we find that

$$\begin{matrix} 2b_0'+b_0=0\\ 2a_0'+a_0=0 \end{matrix}\ \Rightarrow\ \begin{matrix} b_0(t_2)=\beta_0e^{-t_2/2}\\ a_0(t_2)=\alpha_0e^{-t_2/2} \end{matrix}\ \Rightarrow\ \begin{matrix} b_0(t_2)=0\\ a_0(t_2)=e^{-t_2/2} \end{matrix}. \tag{3.15}$$

In the last step, we impose the initial conditions given in (3.12b).

FIGURE 3.2. Comparison between the multiple-scale approximation in (3.16) and the exact solution given in (3.4). The two curves are so close that they are essentially indistinguishable from one another. In these calculations $\varepsilon = 10^{-1}$.

It is worth pointing out that it is not actually necessary to solve (3.13). Secular terms are avoided by simply removing the solutions of the associated homogeneous equation in (3.13). This means we require the coefficients of $\sin(t_1)$ and $\cos(t_1)$ in (3.13) to be zero, which leads to the same conclusions given in (3.15).

Putting our results together, we have found that

$$y \sim e^{-\varepsilon t/2} \sin(t). \tag{3.16}$$

This is a first-term approximation that is valid up to at least $\varepsilon t = O(1)$. This approximation and the exact solution are shown in Fig. 3.2, from which it is clear that the multiple-scale result is quite good. In fact, one can prove that this is a uniformly valid asymptotic approximation of the solution for $0 \le t \le O\left(\frac{1}{\varepsilon}\right)$ (see Exercise 6).

Three Time Scales

What if one wants an approximation that holds up to $\varepsilon^2 t = O(1)$? In this case, the third time scale $t_3 = \varepsilon^2 t$ needs to be introduced. The difference now is that the coefficients a_0 and b_0 in (3.12a) depend on both t_2 and t_3. In particular, to remove secular terms in (3.14a), one obtains

$$a_0(t_2, t_3) = \alpha_0(t_3)\, e^{-t_2/2} \text{ and } b_0(t_2, t_3) = \beta_0(t_3)\, e^{-t_2/2},$$

where $\alpha_0(0) = 1$ and $\beta_0(0) = 0$. To find $\alpha_0(t_3)$ and $\beta_0(t_3)$, it is necessary to solve the $O(\varepsilon^2)$ equation, which is

$$\left(\partial_{t_1}^2 + 1\right) y_2 = -2\partial_{t_1}\partial_{t_2} y_0 - \partial_{t_1} y_0 - \partial_{t_1} y_1 - \left(\partial_{t_2}^2 + 2\partial_{t_1}\partial_{t_3}\right) y_0.$$

Substituting the solutions for y_0 and y_1 given in (3.12a) and (3.14a) shows that to prevent secular terms in the expansion we must require that

$$a_1 = \alpha_1(t_3)\, e^{-t_2/2} + \frac{1}{8}\left(\beta_0 - 8\alpha_0'\right) t_2 e^{-t_2/2} \tag{3.17}$$

and

$$b_1 = \beta_1(t_3)\, e^{-t_2/2} - \frac{1}{8}(\alpha_0 + 8\beta_0')\, t_2 e^{-t_2/2}. \tag{3.18}$$

This is a different situation than before, as we have apparently removed the secular terms yet still have not determined α_0 and β_0. These coefficients can be found if we remember that our objective is to obtain an accurate asymptotic approximation—this means minimizing the error. This is discussed in more detail later, but for the moment note that our expansion is

$$\begin{aligned} y \sim [\alpha_0(t_3) \;+\; \varepsilon(\alpha_1(t_3) + \frac{1}{8}(\beta_0 - 8\alpha_0')t_2)]e^{-t_2/2}\sin(t_1) \\ + \;[\beta_0(t_3) + \varepsilon(\beta_1(t_3) - \frac{1}{8}(\alpha_0 + 8\beta_0')t_2)]e^{-t_2/2}\cos(t_1). \end{aligned}$$

To prevent the $O(\varepsilon t_2)$ terms from interfering with the validity of our expansion, we take $\beta_0 - 8\alpha_0' = 0$ and $\alpha_0 + 8\beta_0' = 0$. From this, and the initial conditions (3.12b), it follows that $\alpha_0 = \cos(t_3/8)$ and $\beta_0 = -\sin(t_3/8)$. Therefore, a first-term approximation valid up to $\varepsilon^2 t = O(1)$ is

$$y \sim e^{-\varepsilon t/2}\sin(\omega t), \tag{3.19}$$

where $\omega = 1 - \frac{1}{8}\varepsilon^2$.

Discussion and Observations

Now that we have gone through the analysis of a problem using multiple scales, it is worth reflecting on what we did. The approach used to construct the expansion in (3.16) was to introduce the two time scales $t_1 = t$ and $t_2 = \varepsilon t$. For this reason, the method is sometimes called *two-timing*, and t_1 is said to be the fast time scale and t_2 the slow scale. This terminology implies that the method is used only with the time variable, which is definitely not the case. As we will see in Section 3.5, it is possible to solve some of the boundary-layer problems in Chapter 2 by using these same ideas.

The time scales that should be used depend on the problem and the interval over which one wants an approximation of the solution. The list of possible scales is endless, but some possibilities include

1) Several time scales: for example, one may need to use the four time scales: $t_1 = t/\varepsilon, t_2 = t, t_3 = \varepsilon t, t_4 = \varepsilon^2 t$.

2) More complex dependence on ε: for example, $t_1 = \left(1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots\right)t$ and $t_2 = \varepsilon t$, where the ω_i's are determined while solving the problem (this is called the *method of strained coordinates* or *Lindstedt's method*). In the previous problem, one would find that $t_1 = \left(1 - \varepsilon^2/8 + \cdots\right)t$.

3) The correct scaling may not be immediately apparent, and one starts off with something like $t_1 = \varepsilon^\alpha t$ and $t_2 = \varepsilon^\beta t$, where $\alpha < \beta$.

4) Nonlinear time dependence: for example, one may have to assume $t_1 = f(t, \varepsilon)$ and $t_2 = \varepsilon t_1$, where the function $f(t, \varepsilon)$ is determined from the problem (this type of scaling is used in the next section).

Multiple scales are introduced to keep the expansion well ordered. This should not be interpreted to mean that they are always to remove unbounded terms, although many times this is what happens (as in the preceding example). In fact, whatever freedom there is in the dependence on t_2 is used to minimize the error in the approximation. We did this in the previous example by trying to reduce the contribution of the next term in the expansion. The coefficients a_0 and b_0 in the first-order approximation in (3.12a) were determined by requiring y_1 to be as small as we could make it over the time scales we were using. This minimization principle is, in fact, the condition imposed in the multiple-scale method.

To illustrate the principle of minimum error, suppose the exact solution is found to be $y(t) = 1+\varepsilon^2 t+\varepsilon^4 \sin(t)$. Also, suppose we use the time scales $t_1 = t, t_2 = \varepsilon t$, and $t_3 = \varepsilon^2 t$. In this case, we can take $y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$, where

i) $y_0 = 1,\ y_1 = 0,\ y_2 = t_1,$

ii) $y_0 = 1,\ y_1 = t_2,\ y_2 = 0,$

iii) $y_0 = 1 + t_3,\ y_1 = 0,\ y_2 = 0,$

or

iv) $y_0 = 1 + t_3,\ y_1 = -t_2,\ y_2 = t_1.$

Each one of these is correct. However, in determining y_0, we want to minimize the value of the error $E_0 = \max \left|y(t) - y_0\left(t, \varepsilon t, \varepsilon^2 t\right)\right|$, where the max is taken over the interval $0 \le t \le O\left(1/\varepsilon^2\right)$ (i.e., over the time interval $0 \le t \le T/\varepsilon^2$ where T is fixed). In the above example, this means $y_0 = 1 + t_3$. Once this result is established, then in determining y_1 we want the error

$$E_1 = \max \left|y(t) - y_0\left(t, \varepsilon t, \varepsilon^2 t\right) - \varepsilon y_1\left(t, \varepsilon t, \varepsilon^2 t\right)\right|$$

to be minimized. One finds that $y_1 = 0$, and the other terms can be obtained using a similar procedure. For many problems, however, it is not an easy matter to actually prove that the error is minimized. There are also questions associated with what norm to use and the completeness of the time scales. Some of these issues are discussed in Smith (1985).

To illustrate another aspect of multiple scales, suppose one has a problem that contains the three time scales $t_1 = t, t_2 = \varepsilon t$, and $t_3 = \varepsilon^2 t$. Also suppose the following expansions have been constructed:

i) $y \sim y_0(t_1, t_2)$,

ii) $y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2)$,

iii) $y \sim y_0(t_1, t_2, t_3)$.

With several different approximations of the same function, it is natural to ask how they compare. Well, the difference in (i) and (iii) is that (i) is guaranteed to hold up to $\varepsilon t = O(1)$ while (iii) holds up to $\varepsilon^2 t = O(1)$. There is no reason to expect, however, that (iii) is more accurate than (i) over the time interval where they both apply, that is, for $0 \leq \varepsilon t \leq O(1)$. On the other hand, (ii) should give a more accurate approximation than either (i) or (iii) as ε becomes small and $0 \leq \varepsilon t \leq O(1)$. Note that the effort to find (ii) and (iii) is likely to be about the same since the $O(\varepsilon^2)$ problem must be examined to be able to find y_1 as well as remove the secular terms associated with the t_3 time scale.

Before we close this section, it is worth commenting on the equation

$$y'' + \omega^2 y = 0, \tag{3.20}$$

where ω is a positive constant. The general solution of this can be written as

$$y(t) = a_0 \sin(\omega t) + b_0 \cos(\omega t), \tag{3.21}$$

or as

$$y(t) = A \cos(\omega t + \theta). \tag{3.22}$$

In many of the examples and exercises in this chapter, equation (3.20) must be solved to obtain the first term in the approximation. Indeed, this is exactly what happened in the earlier example, and it resulted in the solution given in (3.12a). In this example, where the problem given in (3.1) is linear, either form of the general solution can be used without difficulty. However, this usually is not the case when the original problem is nonlinear. It is recommended that whenever using multiple scales on a nonlinear problem and (3.20) appears, the solution be written as in (3.22). With this representation, A and θ are functions of the slower time variables (just as a_0 and b_0 are when using (3.21)). The reason for using the more compact form given in (3.22) is that it can greatly simplify certain calculations, such as finding powers of y^n. Some formulas that prove useful for such situations are given in Exercise 2.

Exercises

1. Find a first-term expansion of the solutions of the following that is valid for large t.

   a) $y'' + \varepsilon(y')^3 + y = 0$, for $0 < t$, where $y(0) = 0$ and $y'(0) = 1$.

   b) $\varepsilon y'' + y' + y = 0$, for $0 < t$, where $y(0) = 0$ and $y'(0) = 1$.

c) $\varepsilon y'' + \varepsilon y' + y = \cos(t)$, for $0 < t$, where $y(0) = 0$ and $y'(0) = 0$.

d) $\varepsilon y'' + \varepsilon\alpha y' + y + \varepsilon y^3 = 0$, for $0 < t$, where $y(0) = 0, y'(0) = 1$, and α is a positive constant.

2. In the following, n is a positive integer.

a) Show that

$$\cos^{2n+1}\varphi = \sum_{j=0}^{n} \alpha_j \cos((2j+1)\varphi),$$

where

$$\alpha_0 = \frac{(2n+1)2n(2n-1)\cdots(n+2)}{n!\ 2^{2n}}.$$

b) Show that

$$\sin^{2n+1}\varphi = \sum_{j=0}^{n} \beta_j \sin((2j+1)\varphi)\,,$$

where $\beta_0 = \alpha_0$.

c) Show that

$$\cos^{2n}\varphi = \sum_{j=0}^{n} \gamma_j \cos(2j\varphi) \text{ and } \sin^{2n}\varphi = \sum_{j=0}^{n} \kappa_j \cos(2j\varphi).$$

3. The equation for the angular displacement $\theta(t)$ of a pendulum is

$$\theta'' + \sin(\theta) = 0, \quad \text{for } 0 < t,$$

where $\theta(0) = \varepsilon$ and $\theta'(0) = 0$. In the case of small amplitude oscillations (i.e., $\varepsilon << 1$), find a first-term expansion of the solution that is valid for large t.

4. To determine the secular terms in the second term of the expansion, we solved the $O(\varepsilon)$ equation to produce (3.14a). There are other ways to obtain this same result, several of which will be discussed in this chapter. One of these methods is the subject of this exercise.

a) Multiply (3.13) by $\sin(t_1)$ and then integrate both sides over the interval $0 < t_1 < T$. It is convenient at this point to introduce the average

$$\langle p(t_1,\tau), q(t_1,\tau)\rangle_\infty \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t_1,\tau)q(t_1,\tau)dt_1.$$

Show that by making certain assumptions about the limiting behavior of the solution one can use this average to produce one of the solutions in (3.15).

b) Using the ideas in part (a), derive the second solution in (3.15).

5. Consider the weakly damped oscillator equation

$$y'' + \varepsilon F(y') + y = 0, \quad \text{for } 0 < t, \quad \text{where } y(0) = 0 \text{ and } y'(0) = 1.$$

It is assumed that $F(s)$ is a continuous, monotonically increasing function with $F(0) = 0$.

a) Show that finding the first term in a multiple-scale expansion of the solution reduces to solving two first-order equations.

b) What does the result in part (a) reduce to when $F(s) = |s|s$?

6. In this problem, you are to prove that (3.16) is a uniformly valid approximation of the solution for $0 \le t \le T/\varepsilon$, where $T > 0$ is fixed. Do this by showing that $|y(t) - y_0(t, \varepsilon t)| = O(\varepsilon)$ for $0 \le t \le T/\varepsilon$.

a) Find the exact solution, and then use Taylor's theorem to prove the result.

b) Let $z(t) = y(t) - y_0(t, \varepsilon t)$. After finding the problem $z(t)$ satisfies, prove the result.

7. In the study of Josephson junctions, the following problem appears (Sanders, 1983):

$$\phi'' + \varepsilon(1 + \gamma \cos\phi)\phi' + \sin\phi = \varepsilon\alpha, \quad \text{for } 0 < t,$$

where $\phi(0) = \phi'(0) = 0$ and γ is a positive constant. Find a first-term approximation of $\phi(t)$ that is valid for large t.

8. The equation for a relativistic damped oscillator is (Struble and Harris, 1964)

$$\frac{d}{dt}\left(\frac{u_t}{\sqrt{1 + \varepsilon\mu\,(u_t)^2}}\right) + 2\alpha\varepsilon u_t + u = 0,$$

where α and μ are positive constants. The initial conditions are $u(0) = 1$ and $u_t(0) = 0$. Find a first-term approximation of $u(t)$ that is valid for large t.

9. In the study of quantum jumps, single-ion trapping experiments are carried out using something called a Paul trap (Cook, 1990). In nondimensional form, the equation governing the ion motion in such a device is

$$\varepsilon\,(i\phi_t + \phi_{xx}) = v(x)\phi\cos\left(\frac{t}{\varepsilon}\right), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $v(x)$ is a specified function. Note that, for small ε, the coefficient of the right-hand side of this equation is rapidly varying. In the theory of quantum jumps, this observation serves as the starting

point for something called a rotating wave approximation. The result is a reduced equation for $\phi(x,t)$ where the coefficients of the equation are independent of t. You are asked in this problem to derive the result using multiple scales.

a) Find a first-term approximation of $\phi(x,t)$ that is valid for $t = O(1)$. Note that you need to derive only the differential equation that determines the first-term approximation.

b) The multivariable version of the quantum jump equation is

$$\varepsilon\left(i\phi_t + \nabla^2\phi\right) = v(\mathbf{x})\phi\cos\left(\frac{t}{\varepsilon}\right).$$

This problem can be reduced in exactly the same way as was done in part (a). Instead, let $\phi(\mathbf{x},t) = \psi(\mathbf{x},t)\exp[-iv(\mathbf{x})\sin(t/\varepsilon)]$. Then find a first-term expansion of $\psi(\mathbf{x},t)$ that is valid for $t = O(1)$. Compare your result with the one found in (a).

10. This problem considers how to apply the multiple-scale approach to the first-order system

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}(\mathbf{y},\varepsilon),$$

where $\mathbf{y}(0) = \mathbf{a}$. It is assumed $\mathbf{f}$ is smooth with $\mathbf{f}(\mathbf{y},0) = \mathbf{0}$. Also, letting $\mathbf{y}^T = (u,v)$, it is assumed that

$$A = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix},$$

where $\alpha^2 < \beta\gamma$. The matrix $\mathbf{A}$ and vector $\mathbf{a}$ do not depend on ε.

a) Show that the weakly damped oscillator equation given in (3.1a) can be written in this form, as can the pendulum equation given in Exercise 3.

b) Introduce the time scales $t_1 = t, t_2 = \varepsilon t$, and then show that the first term in the expansion of the solution has the form

$$\mathbf{y}_0(t_1,t_2) = a_0(t_2)\mathbf{q}(t_1) + \overline{a}_0(t_2)\overline{\mathbf{q}}(t_1),$$

where the overbar ($\overline{}$) indicates a complex conjugate. Here $\mathbf{q}$ is an eigenvector associated with the first-order problem.

c) Suppose the solution is known to be $2\pi/\omega$ periodic in t_1, where $\omega = (\beta\gamma - \alpha^2)^{1/2}$. This information can be used to remove secular terms. To show how, let $\mathbf{p}(t_1)$ be the solution of the (first-order) adjoint equation $\mathbf{p}' = -A^T\mathbf{p}$. From the equation for the $O(\varepsilon)$ term, show that

$$\int_0^{2\pi/\omega} \left(\partial_{t_2}\mathbf{y}_0 - \partial_\varepsilon\mathbf{f}\left(\mathbf{y}_0,0\right)\right)\cdot\mathbf{p}\,dt_1 = 0.$$

Explain why this is a first-order differential equation that determines the coefficient function $a_0(t_2)$. Also, determine the initial condition to be imposed on $a_0(t_2)$.

d) Consider the system of equations

$$\begin{aligned} u' &= u + v - \varepsilon u(2 + \varepsilon uv), \\ v' &= -4u - v + \varepsilon u(2 + \varepsilon uv). \end{aligned}$$

After identifying the matrix A and the function $\mathbf{f}$, use the results of parts (b) and (c) to find a first-term approximation of $\mathbf{y}(t)$ that is valid for large t (assume the solution satisfies the periodicity condition mentioned in (c)).

e) In the study of chemical oscillators, the following system of equations is found (Schnakenberg, 1979):

$$\begin{aligned} u' &= u + v + \varepsilon u(2u + 2v + \varepsilon uv), \\ v' &= -4u - v - \varepsilon u(2u + 2v + \varepsilon uv). \end{aligned}$$

In this case, show that the result in part (c) indicates that the wrong time scale has been introduced. Extend the results of parts (b) and (c) to include this situation, and in the process find a first-term approximation of $\mathbf{y}(t)$ that is valid for large t (assume the solution satisfies the periodicity condition mentioned in part (c)).

3.3 Slowly Varying Coefficients

To illustrate what happens when a more complicated time scale must be used, consider the following problem:

$$y'' + k^2(\varepsilon t)y = 0, \quad \text{for } 0 < t, \tag{3.23a}$$

where

$$y(0) = a \text{ and } y'(0) = b. \tag{3.23b}$$

This equation corresponds to an undamped oscillator with a spring that has a slowly varying restoring force. This is the sort of situation that could arise, for example, if the spring gradually fatigued or changed periodically in time. We will assume that $k(\tau)$ is a smooth positive function for $\tau \geq 0$. It is also assumed that it is not constant.

Before deriving a multiple-scales approximation to the solution of (3.23), it is worth examining what a solution can look like. Therefore, in Fig. 3.3 the numerical solution is shown for a particular choice of the function $k(\varepsilon t)$. It is evident from this graph that there are (at least) two time scales. There is a fast time scale associated with the rapid oscillations in the solution,

FIGURE 3.3. Numerical solution, shown by the dashed curve, of (3.23) when $k(\varepsilon t) = 2 + \cos(\varepsilon t)$, $\varepsilon = 10^{-1}$, $a = 0$, and $b = 1$. Also shown, by the solid curve, is the amplitude of the oscillation determined from the multiple-scales approximation given in (3.31).

and there is a slower time scale involved with the variation of the amplitude of the oscillation. Our objective is to determine exactly what these time scales are and how they depend on the coefficient function $k(\varepsilon t)$.

Based on our success with the weakly damped oscillator in Section 3.2, suppose we assume the two principal time scales for this problem are $t_1 = t$ and $t_2 = \varepsilon t$. Using the same expansion as in (3.10), one finds that

$$y_0 = a_0(t_2)\sin[k(t_2)\,t_1] + b_0(t_2)\cos[k(t_2)\,t_1]\,.$$

To determine a_0 and b_0, we need to solve the $O(\varepsilon)$ equation. One finds that this equation is

$$\begin{aligned}\left(\partial_{t_1}^2 + k^2\right) y_1 \;=\; & -2\left[(a_0 k)' - b_0 k k' t_1\right]\cos[k(t_2)\,t_1] \\ & + 2\left[(b_0 k)' + a_0 k k' t_1\right]\sin[k(t_2)\,t_1]\,.\end{aligned}$$

Looking at the right-hand side of this equation, one sees that terms like $t_1^2\sin[k(t_2)\,t_1]$ and $t_1\sin[k\,(t_2)\,t_1]$ are going to appear in the solution for y_1. Since a_0, b_0, and k are independent of t_1, the only way to prevent this is to take $a_0 = b_0 = 0$. In other words, the only solution we are able to find that does not contain secular terms is the zero solution. Clearly, we have not gotten very far with our simple expansion.

It is necessary to think a little harder on how to choose the time scales, and the first place to begin is with t_1. This time scale should be a measure of the period of oscillation. In this problem, the period depends on $k(\varepsilon t)$. At the moment, it is unclear exactly how to define this time scale, and so we will let

$$t_1 = f(t, \varepsilon) \tag{3.24}$$

and

$$t_2 = \varepsilon t. \tag{3.25}$$

The function $f(t,\varepsilon)$ is required to satisfy the following conditions:

1) $f(t,\varepsilon)$ is nonnegative and increases with t,

2) $\varepsilon t << f$ as $\varepsilon \downarrow 0$ (i.e., t_1 is the fast time scale and t_2 is the slow scale),

3) $f(t,\varepsilon)$ is smooth.

Now, with the multiple-scale transformation and the chain rule, we have

$$\frac{d}{dt} = \frac{\partial f}{\partial t}\frac{\partial}{\partial t_1} + \varepsilon\frac{\partial}{\partial t_2}$$

and

$$\frac{d^2}{dt^2} = f_t^2\frac{\partial^2}{\partial t_1^2} + f_{tt}\frac{\partial}{\partial t_1} + 2\varepsilon f_t\frac{\partial^2}{\partial t_1 \partial t_2} + \varepsilon^2\frac{\partial^2}{\partial t_2^2}.$$

Substituting this into (3.23a) yields

$$\underset{1}{\left(f_t^2\partial_{t_1}^2 + f_{tt}\partial_{t_1} + 2\varepsilon f_t\partial_{t_1}\partial_{t_2} + \varepsilon^2\partial_{t_2}^2\right)y} + \underset{2}{k^2(\varepsilon t)y} = 0. \tag{3.26}$$

The oscillation takes place because of the balance between terms 1 and 2. We therefore let $f_t = k(\varepsilon t)$. Integrating this yields

$$f(t,\varepsilon) = \int_0^t k(\varepsilon\tau)d\tau. \tag{3.27}$$

It is not hard to show that this choice satisfies the three requirements listed earlier. Also, with this $f_{tt} = \varepsilon k'(t_2)$. So, with the expansion

$$y \sim y_0(t_1,t_2) + \varepsilon y_1(t_1,t_2) + \cdots, \tag{3.28}$$

we obtain the following from (3.26) and (3.23b):

$O(1)$ $\left(\partial_{t_1}^2 + 1\right)y_0 = 0,$

$y_0(0,0) = a, k(0)\partial_{t_1}y_0(0,0) = b.$

The general solution of this problem is

$$y_0 = a_0(t_2)\sin(t_1) + b_0(t_2)\cos(t_1), \tag{3.29}$$

where $a_0(0) = b/k(0)$ and $b_0(0) = a$.

$O(\varepsilon)$ $k^2\left(\partial_{t_1}^2 + 1\right)y_1 = -2k\partial_{t_1}\partial_{t_2}y_0 - k'\partial_{t_1}y_0$

$$= -\left(a_0k' + 2ka_0'\right)\cos(t_1) + \left(b_0k' + 2kb_0'\right)\sin(t_1).$$

To prevent secular terms, let $a_0k' + 2ka_0' = b_0k' + 2kb_0' = 0$. Thus,

$$a_0 = \frac{\alpha_0}{\sqrt{k}} \text{ and } b_0 = \frac{\beta_0}{\sqrt{k}}, \tag{3.30}$$

where α_0 and β_0 are constants determined from the initial conditions.

Our first-term approximation is therefore

$$y \sim \frac{1}{\sqrt{k(\varepsilon t)}}\left(\alpha_0 \sin\left(\int_0^t k(\varepsilon\tau)d\tau\right) + \beta_0 \cos\left(\int_0^t k(\varepsilon\tau)d\tau\right)\right). \tag{3.31}$$

The values of α_0 and β_0 are determined from the initial conditions, and one finds that $\beta_0 = y(0)\sqrt{k(0)}$ and $\alpha_0 = y'(0)[k(0)]^{-1/2}$.

As a check on the accuracy of the multiple-scale approximation in (3.31), the amplitude function $A(t) = \alpha_0/\sqrt{k(\varepsilon t)}$ is shown in Fig. 3.3. This is plotted, rather than the entire approximation, because of the highly oscillatory nature of the solution. It is evident that the multiple-scale approximation gives a very accurate description of the solution.

There are other ways to derive the approximation in (3.31). One that is relatively easy to apply is the WKB method (see Exercise 4.2.2). The advantage of the multiple-scales approach is that it works on nonlinear problems. This is evident in the original paper on this problem by Kuzmak (1959), but this method requires us to be able to recognize the appropriate scales and then determine what terms are responsible for secular terms.

Exercises

1. This problem concerns the slowly varying undamped oscillator (3.23).

   a) Suppose that one uses Taylor's theorem to expand $k(\varepsilon t)$ for small ε and then uses a standard multiple-scale transformation as in (3.7), (3.8). How does the first term in this case compare with (3.31)? Over what time interval is this expansion valid?

   b) Show that the fast time scale given in (3.27) satisfies the following:
   i) $f(t, \varepsilon)$ is positive and increases with t,

   ii) $\varepsilon t << f$ as $\varepsilon \downarrow 0$,

   iii) $f(t, \varepsilon)$ is smooth.

   Is it necessary that $f(0, \varepsilon) = 0$?

   c) Suppose that instead of (3.25) one takes $t_2 = \varepsilon t_1$. Does this alter (3.31)?

2. Consider the problem

$$\frac{d}{dt}\left(D(\varepsilon t)\frac{dy}{dt}\right) + y = 0, \quad \text{for } 0 < t,$$

where $y(0) = \alpha$ and $y'(0) = \beta$. The coefficient $D(\tau)$ is a smooth positive function with $D' > 0$. Find a first-term approximation of the solution valid for large t.

3. The equation for the angular displacement $\theta(t)$ of a pendulum whose length is slowly varying is

$$\theta'' + \kappa(\varepsilon t)\sin(\theta) = 0, \quad \text{for } 0 < t,$$

where $\theta(0) = \varepsilon$ and $\theta'(0) = 0$. The function $\kappa(\tau)$ is smooth and positive. Find a first-term expansion of the solution that is valid for large t.

4. Mathieu's equation is

$$y'' + [\lambda + \varepsilon\cos(t)]y = 0, \quad \text{for } 0 < t,$$

where $y(0) = a$ and $y'(0) = b$. Also, λ is a constant. This equation describes the small amplitude oscillations of a pendulum whose length varies periodically with time. If the pendulum's natural frequency is a particular multiple of the frequency of the length variation, then instability can occur. This is indicated in Fig. 3.4, which shows the regions in the ε, λ-plane where the motion is stable (S) and unstable (N). Equations for the boundaries of these regions are derived in the following problems.

a) Assuming λ is independent of ε, use a regular expansion to show that secular terms appear in the second term of the expansion if $\lambda = 1/4$ and in the third term if $\lambda = 1$. In fact, it is possible to show that secular terms appear if $\lambda = n^2/4$, where $n = 0, 1, 2, 3, \ldots$ (you do not need to show this).

b) To investigate what happens when secular terms are possible, suppose $\lambda \sim \frac{1}{4}(1 + 2\varepsilon\lambda_1)$. Find a first-term approximation of the solution

FIGURE 3.4. Stability regions in the ε, λ-plane for Mathieu's equation. An S indicates a stable region and an N an unstable region.

that is valid for large t. From this, show that the solution may be unbounded, depending on the initial conditions, if $|\lambda_1| < 1$. Because of this, the curves $\lambda \sim \frac{1}{4}(1 \pm 2\varepsilon)$ form the stability boundaries in this region.

c) To investigate what happens near $\lambda = 1$, suppose $\lambda \sim 1 + \varepsilon^2\lambda_1$. Find a first-term approximation of the solution that is valid for large t. From this, show that the solution may be unbounded, depending on the initial conditions, if $\frac{-1}{12} < \lambda_1 < \frac{5}{12}$.

5. Consider the following boundary-value problem for the function $u(x)$:

$$\varepsilon^2 u'' + \left[\omega^2 - k^2(x)\right] u = 0, \quad \text{for } 0 < x < \infty,$$

where $u(0) = 1$ and $u \to 0$ as $x \to \infty$. Also, $k(x)$ is smooth and positive, and ω is a positive constant.

a) Find a first-term approximation of the solution if $k(x) > \omega$ for $0 \le x < \infty$.

b) Suppose $k'(x) > 0$ for $0 \le x < \infty$ and $k(x_t) = \omega$, where $0 < x_t < \infty$. Find a first-term approximation of the solution for $0 \le x < x_t$ and another approximation of the solution for $x_t < x < \infty$. The approximations from these two regions need to be matched; how to do this will be discussed in Section 4.3.

6. The equation for what is known as the Rayleigh oscillator is

$$y'' - \varepsilon\left(1 - \frac{1}{3}(y')^2\right) y' + y = 0, \quad \text{for } 0 < t,$$

where $y(0) = 0$ and $y'(0) = 1$. Find a first-term approximation of the solution valid for large t.

7. The eigenvalue problem for the vertical displacement of an elastic string with variable density is

$$y'' + \lambda^2 \rho(x, \varepsilon) y = 0, \quad \text{for } 0 < x < 1, \quad \text{where } y(0) = y(1) = 0.$$

For small ε assume $\rho \sim 1 + \varepsilon\mu(x)$, where $\mu(x)$ is smooth and positive. This problem is based, in part, on the work of Akulenko and Shamayev (1986).

a) Find the first two terms in the expansions of $y(x)$ and λ. After examining the second term for $y(x)$, explain why secular terms are present.

b) Secular terms can be avoided without resorting to multiple scales directly. Do this by making a change of variables similar to that in (3.24). The objective is to transform the problem into one of the form

$$Y'' + \varepsilon h(s, \varepsilon) Y' + \nu^2 Y = 0, \quad \text{for } 0 < s < 1,$$

where $Y = 0$ for $s = 0, 1$. Also, ν is a positive constant (that depends on ε).

c) Find the first two terms in the expansion of Y and ν. Are secular terms present in the second terms?

d) What do the expansions in part (c) reduce to when $\mu(x) = x$? Transform the result back into the x, y-plane.

3.4 Forced Motion Near Resonance

The problem to be considered corresponds to a damped, nonlinear oscillator that is forced at a frequency near resonance. In particular, the problem that is investigated is

$$y'' + \varepsilon\lambda y' + y + \varepsilon\kappa y^3 = \varepsilon\cos(1+\varepsilon\omega)t, \quad \text{for } 0 < t, \tag{3.32a}$$

where

$$y(0) = 0 \text{ and } y'(0) = 0. \tag{3.32b}$$

In the differential equation, the damping ($\varepsilon\lambda y'$), nonlinearity ($\varepsilon\kappa y^3$), and forcing ($\varepsilon\cos(1+\varepsilon\omega)t$) are small. Also, ω, λ, and κ are constants with λ and κ non-negative. This oscillator has been studied extensively and is an example of a forced Duffing equation (see Exercise 1.6.10).

Because of the small forcing, and the zero initial conditions, it is natural to expect the solution to be small. For example, if we were to consider the simpler equation

$$y'' + y = \varepsilon\cos(\Omega t), \tag{3.33}$$

then the solution is, assuming $\Omega \neq \pm 1$,

$$y(t) = \frac{\varepsilon}{1-\Omega^2}[\cos(\Omega t) - \cos(t)]. \tag{3.34}$$

Therefore, the solution is small since it is $O(\varepsilon)$. This observation is not true when the driving frequency $\Omega \approx 1$. To investigate this situation, suppose $\Omega = 1 + \varepsilon\omega$. A particular solution in this case is, for $\omega \neq 0, -2/\varepsilon$,

$$y = \frac{-1}{\omega(2+\varepsilon\omega)}\cos(1+\varepsilon\omega)t; \tag{3.35}$$

and for $\omega = 0$ or $\omega = -2/\varepsilon$,

$$y = \frac{1}{2}\varepsilon t\sin(t). \tag{3.36}$$

In both cases a relatively small, order $O(\varepsilon)$, forcing results in at least an $O(1)$ solution. Also, the behavior of the solution is significantly different for

FIGURE 3.5. Numerical solution, shown with the dashed curve, of the nonlinear oscillator in (3.32) when $\varepsilon = 0.05$, $\lambda = 2$, $\kappa = 4$, and $\omega = 3/8$. The solid curve is the numerical solution of (3.42), (3.43) for the amplitude function $A(\varepsilon t)$ obtained from the multiple-scales approximation (3.45).

$\omega = 0$ or $\omega = -2/\varepsilon$ than for other values of ω. This is typical of a system that is driven at one of its characteristic frequencies, and it is associated with the phenomenon of resonance. Such behavior is also present in the nonlinear oscillator equation in (3.32a). To demonstrate this, the values of $y(t)$ obtained from the numerical solution of (3.32) are shown in Fig. 3.5. It is seen that the amplitude of the solution is an order of magnitude larger than the forcing.

Another observation that can be made from Fig. 3.5 is that there appear to be at least two time scales present in the problem. One is associated with the rapid oscillation in the solution and the other is connected with the relatively slow increase of the amplitude. Thus, the multiple-scales method is a natural choice for finding an asymptotic approximation to the solution. We will try to give ourselves some flexibility and take $t_1 = t$ and $t_2 = \varepsilon^\alpha t$, where $\alpha > 0$. In this case (3.32a) becomes

$$\left(\partial_{t_1}^2 + 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \varepsilon^{2\alpha}\partial_{t_2}^2\right)y + \varepsilon\lambda(\partial_{t_1} + \varepsilon^\alpha \partial_{t_2})y + y + \varepsilon\kappa y^3 = \varepsilon\cos(t_1 + \varepsilon\omega t_1)\,. \tag{3.37}$$

As stated earlier, because of the zero initial conditions (3.32b) and the small forcing, one would expect the first term in the expansion of the solution to be $O(\varepsilon)$. However, near a resonant frequency the solution can be much larger. It is not clear what amplitude the solution actually reaches, and for this reason we will take

$$y \sim \varepsilon^\beta y_0(t_1, t_2) + \varepsilon^\gamma y_1(t_1, t_2) + \cdots, \tag{3.38}$$

where $\beta < 1$ and $\beta < \gamma$. Substituting (3.38) into (3.37) yields the equation

$$\underset{❶}{(\partial_{t_1}^2 + 2\varepsilon^\alpha \partial_{t_1}\partial_{t_2} + \cdots)(\varepsilon^\beta y_0 + \varepsilon^\gamma y_1 + \cdots)} + \underset{❷}{\varepsilon\lambda(\partial_{t_1} + \cdots)(\varepsilon^\beta y_0 + \cdots)}$$

①

$$+\varepsilon^{\beta} y_0 + \varepsilon^{\gamma} y_1 + \cdots + \varepsilon\kappa(\varepsilon^{3\beta} y_0^3 + \cdots) = \varepsilon\cos(t_1 + \varepsilon\omega t_1). \tag{3.39}$$

$$\bullet 2 \qquad (2) \qquad (3)$$

In writing equation (3.39), we made an effort to include only the terms that might contribute to the first two terms of the expansion. With this, and (3.32b), the following results:

$$O(\varepsilon^{\beta}) \quad (\partial_{t_1}^2 + 1)\, y_0 = 0,$$
$$y_0(0,0) = \partial_{t_1} y_0(0,0) = 0.$$

The general solution of this problem is

$$y_0 = A(t_2)\cos(t_1 + \theta(t_2))\,, \tag{3.40a}$$

where

$$A(0) = 0. \tag{3.40b}$$

The compact form of the solution is used in (3.40a), rather than the form in (3.12a), since it is easier to deal with the nonlinear terms that appear in the next problem. In particular, we will make use of the identity $\cos^3\phi = \frac{1}{4}[3\cos(\phi) + \cos(3\phi)]$.

We need to determine α, β, and γ before going any farther. To do this, note that the higher-order terms in expansion (3.38), such as $\varepsilon^{\gamma} y_1$, are there to deal with the nonzero terms in (3.39). This includes the terms labeled (1), (2), and (3). We can take care of all three if they balance with the terms labeled with a ●2. This gives us $\gamma = 1$, $1 + \beta = \gamma$, and $1 + 3\beta = \gamma$. The remaining observation to make is that all three terms (1), (2), and (3) will produce secular terms in the expansion, and therefore we must choose our slow time scale t_2 to deal with this. This means term ●1 must balance with (1), (2), and (3), which gives us $\alpha + \beta = 1$. From these equations, we find that $\beta = 0$ and $\alpha = \gamma = 1$. With this, the right-hand side of (3.39) is rewritten as $\varepsilon\cos(t_1 + \omega t_2)$. Thus, the next order problem to solve is the following:

$$\begin{aligned} O(\varepsilon) \quad (\partial_{t_1}^2 + 1)\, y_1 &= -2\partial_{t_1}\partial_{t_2} y_0 - \lambda\partial_{t_1} y_0 - \kappa y_0^3 + \cos(t_1 + \omega t_2) \\ &= (2A' + \lambda A)\sin(t_1 + \theta) + 2\theta' A\cos(t_1 + \theta) \\ &\quad -\frac{\kappa}{4} A^3\left[3\cos(t_1 + \theta) + \cos 3\,(t_1 + \theta)\right] \\ &\quad + \cos(t_1 + \omega t_2)\,. \end{aligned} \tag{3.41}$$

We want to determine which terms on the right-hand side of this equation produce secular terms—in other words, which terms are solutions of the

associated homogeneous equation. This can be done by noting

$$\begin{aligned}\cos(t_1+\omega t_2) &= \cos(t_1+\theta-\theta+\omega t_2)\\ &= \cos(t_1+\theta)\cos(\theta-\omega t_2)+\sin(t_1+\theta)\sin(\theta-\omega t_2).\end{aligned}$$

Thus, to remove the $\sin(t_1+\theta)$ and $\cos(t_1+\theta)$ terms in (3.41), it is required that

$$2A'+\lambda A=-\sin(\theta-\omega t_2) \tag{3.42}$$

and

$$2\theta' A-\frac{3\kappa}{4}A^3=-\cos(\theta-\omega t_2)\,. \tag{3.43}$$

From (3.42), (3.43), the initial condition in (3.40b), and assuming $A'(0)>0$, it follows that

$$\theta(0)=-\frac{\pi}{2}. \tag{3.44}$$

To summarize our findings, the first-term expansion of the solution of (3.32) has the form

$$y\sim A(t_2)\cos(t_1+\theta(t_2))\,. \tag{3.45}$$

To find the amplitude A and phase function θ, it remains to solve (3.42), (3.43) along with initial conditions (3.40b), (3.44). Unfortunately, the solution of this nonlinear system of equations is not apparent. In lieu of the exact solution, we can investigate what sort of solution may come from this system. In particular, we will investigate what value the amplitude A approaches if it goes to a steady state as $t\to\infty$. So, assuming $A'=0$, let $A=A_\infty$, where A_∞ is a positive constant. From (3.42), it follows that $\theta=\omega t_2+\theta_\infty$, where θ_∞ is a constant that satisfies $\sin(\theta_\infty)=-\lambda A_\infty$. From this and (3.43), the constant A_∞ is determined from the equation

$$A_\infty^2\left[\lambda^2+\left(2\omega-\frac{3\kappa}{4}A_\infty^2\right)^2\right]=1. \tag{3.46}$$

An example of the possible solutions obtained from (3.46) is shown in Fig. 3.6 as a function of the frequency parameter ω. The interesting parts of the curve occur for $\omega_m<\omega<\omega_M$, where there are three possible steady states for A. The middle one, branch II, is unstable and cannot be reached unless the initial conditions are chosen so that the solution starts exactly at one of these points (the justification of this statement is given in Chapter 6, and Exercise 10 should also be consulted). The other two branches, I and III, are stable, and the solution will approach one of them depending on the initial conditions. This situation is known as *bistability.* Note that if one were to start on branch I and then increase the driving frequency to the point that $\omega>\omega_M$, then the amplitude would jump down to branch III. Similarly, once it is on the lower part of this solution curve and the frequency is decreased below ω_m, the amplitude then jumps up to branch I.

FIGURE 3.6. Schematic of the positive solutions of (3.46) when λ and κ are positive (the exact shape of the curve depends on λ and κ). The curve as shown has three branches: (I) one for $-\infty < \omega \leq \omega_M$, (II) one for $\omega_m < \omega < \omega_M$, and (III) one for $\omega_m \leq \omega < \infty$. Note that $A_M = 1/\lambda$ and $\omega_M = 3\kappa/(8\lambda^2)$.

A comparison between the amplitude function $A(t_2)$ obtained from (3.42), (3.43) and the numerical solution of (3.32) is shown in Fig. 3.5. This type of comparison is made, rather than plotting all the values for the solution that are obtained from (3.45), because of the oscillatory nature of the solution. In any case, it is clearly seen that the agreement is quite good. However, one might argue that if the reduced problem (3.32) must be solved numerically then why not just solve the original problem numerically? It is certainly true that the numerical solution is not limited to small values of the parameter ε. In this regard it is superior to the asymptotic approximation. The asymptotic approximation, on the other hand, is capable of providing quick insight into how the solution depends on the various parameters of the problem. This sort of pro-and-con argument applies to almost all numerical/asymptotic solutions. What is special about the present example is the oscillatory nature of the response. In this case, the asymptotic approximation has the rapid oscillations built into the representation. Thus, the functions that need to be computed numerically are relatively smooth. In this sense, it is easier to solve (3.42), (3.43) numerically than the original equation (3.32).

The analysis here has been for what is sometimes called harmonic, or primary, resonance. There are also subharmonic and superharmonic resonances. To see where these originate, note that the solution in (3.34) is periodic if $\Omega = n$ or if $\Omega = 1/n$, where n is a positive integer with $n \neq 1$. For the first case, the solution has period 2π while the forcing has period $2\pi/n$. This corresponds to what is known as subharmonic oscillation. If $\Omega = 1/n$, then the solution and forcing have period $2\pi n$. This is known as superharmonic, or ultraharmonic, oscillation. With the nonlinear Duffing equation (3.32a), it is possible for there to be a resonant response if Ω is equal to a subharmonic or superharmonic frequency. However, the amplitudes are generally smaller than what is found for harmonic resonance. This is demonstrated in Exercise 2, which examines the use of multiple scales to analyze the subharmonic response.

Exercises

1. In the relativistic mechanics of planetary motion around the Sun, one comes across the problem of solving

$$\frac{d^2u}{d\theta^2} + u = \alpha(1 + \varepsilon u^2), \quad \text{for } 0 < \theta < \infty,$$

where $u(0) = 1$ and $u'(0) = 0$. Here θ is the angular coordinate in the orbital plane; $u(\theta) = 1/r$, where r is the normalized radial distance of the planet from the Sun; and α is a positive constant. Note that if $\varepsilon = 0$ then one obtains the Newtonian description.

a) Find a first-term approximation of the solution that is valid for large θ.

b) Using the result from part (a), find a two-term expansion of the angle $\Delta\theta$ between successive perihelions, that is, the angle between successive maxima in $u(\theta)$.

c) The parameters in the equation are $\varepsilon = 3(h/cr_c)^2$ and $\alpha = r_c/[a(1-e^2)]$, where h is the angular momentum of the planet per unit mass, r_c is a characteristic orbital distance, c is the speed of light, a is the semi-major axis of the elliptical orbit, and e is the eccentricity of the orbit. For the planet Mercury, $h/c = 9.05\times10^3$ km, $r_c = a = 57.91\times10^6$ km, and $e = 0.20563$ (Nobili and Will, 1986). It has been observed that the precession of Mercury's perihelion, defined as $\Delta\phi \equiv \Delta\theta - 2\pi$, after a terrestrial century is $43.11'' \pm 0.45''$ (note that Mercury orbits the sun in 0.24085 years). How does this compare with your theoretical result in (b)?

d) In the 10th edition of Webster's Collegiate Dictionary, the second definition given for the word "perturbation" is "a disturbance of the regular and usu. elliptical course of motion of a celestial body that is produced by some force additional to that which causes its regular motion." Comment on this in conjunction with your result from part (a). Also, how would you change the definition so that it encompasses the ideas in Chapters 1 and 2?

2. This problem investigates the subharmonic resonances of Duffing's equation,

$$y'' + \varepsilon\lambda y' + y + \varepsilon\kappa y^3 = \varepsilon\cos(\Omega t), \quad \text{for } 0 < t.$$

a) Suppose one attempts to use the regular expansion

$$y \sim \varepsilon\left(A\cos(\Omega t) + \varepsilon y_1(t) + \cdots\right).$$

Explain why this is nonuniform if $\Omega = \pm1$ (primary resonance), or if $\Omega = \pm3$ (subharmonic resonance). Are any of these present in the linear problem?

b) Setting $\Omega = 3 + \varepsilon\omega$, find a first-term approximation of the solution that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find the possible steady states (assume $\lambda > 0$).

c) Compare the results from part (b) with those in Fig. 3.6.

3. The following is the equation for van der Pol's oscillator (with small damping):

$$y'' - \varepsilon(1 - y^2)y' + y = 0, \quad \text{for } 0 < t,$$

where $y(0) = \alpha_0$ and $y'(0) = \beta_0$.

a) Find a first-term approximation of the solution that is valid for large t.

b) Sketch the first-term approximation in the phase plane as a function of t when $\alpha_0 = 1$ and $\beta_0 = 0$.

4. The equation for a pendulum with a weak forcing is

$$\theta'' + \frac{1}{\varepsilon}\sin(\varepsilon\theta) = \varepsilon^2 \sin(\omega t), \quad \text{for } 0 < t,$$

where $\theta(0) = \theta'(0) = 0$.

a) For fixed $\omega^2 \neq 1, 1/9, \ldots$ and $\omega^2 \neq 1, 9, \ldots$, find a two-term approximation of $\theta(t)$.

b) For $\omega = 1 + \omega_0\varepsilon^2$, find a first-term approximation of $\theta(t)$ that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find the possible steady states (if any).

5. Suppose a mass is situated between two parallel walls and is connected to the walls by springs (as shown in Fig. 3.7). For a small periodic forcing the equation for the transverse oscillations of the mass is (Stoker, 1950; Arnold and Case, 1982; Frobes, 1991)

$$y'' + y\left[1 - \frac{\lambda}{(1 + y^2)^{1/2}}\right] = \varepsilon \cos(\kappa t + \varepsilon^{2/3}\omega t), \quad \text{for } 0 < t,$$

FIGURE 3.7. Oscillator studied in Exercise 5.

where λ and κ are positive constants with $0 < \lambda < 1$. Also, $y(0) = y'(0) = 0$. Set $\kappa^2 = 1 - \lambda$, and for small ε find a first-term approximation of the solution that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find, and sketch, the possible steady states (if any) for the amplitude.

6. In the study of Raman scattering, one comes across the equation for a forced Morse oscillator with small damping, given as

$$y'' + \varepsilon^{2/3}\alpha y' + (1 - e^{-y})e^{-y} = \varepsilon \cos(1 + \varepsilon^{2/3}\omega)t,$$

where $y(0) = 0$ and $y'(0) = 0$. Also, $\alpha \geq 0$.

a) Find a first-term approximation of $y(t)$ that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find the possible steady states (if any) for the amplitude.

b) Lie and Yuan (1986) used numerical methods to solve this problem. They were interested in how important the value of the damping parameter α is for there to be multiple steady states for the amplitude. They were unable to answer this question because of the excessive computing time it took to solve the problem using the equipment available to them. However, based on their calculations, they hypothesized that multiple steady states for the amplitude are possible even for small values of α. By sketching the graph of A_∞ as a function of ω, for $\alpha \geq 0$ determine whether or not their hypothesis is correct.

7. Figure 3.8 shows an oscillator driven by a belt moving with velocity v. The block moves in the same direction as the belt until the spring has sufficient restoring force to pull it back. The block then retracts until the force in the spring reduces to the point where the frictional force between the block and belt causes the block to start moving again in the same direction as the belt. This back-and-forth motion is an example of what is known as a stick-slip problem. Among other things, this mechanism has been used to model squeaky doors and violin strings (Popp and Stelter, 1990; Gao and Kuhlmann-Wilsdorf, 1990). The equation of motion in this case is

$$y'' + \varepsilon\beta y' + y = f(v - \varepsilon y'), \quad \text{for } 0 < t,$$

where β and v are positive constants. The function $f(v - \varepsilon y')$ accounts for the friction between the mass and the belt. It is an odd function of the relative velocity $V = v - \varepsilon y'$. In this problem, for $V > 0$, we will take $f(V) = \frac{1}{3}aV(V^2 - 3) + b$, where a and b are positive constants. We will also take $y(0) = y'(0) = 0$.

a) For $v^2 > 1 - \beta/a$, find a first-term approximation of the solution that is valid for large t.

FIGURE 3.8. Forced oscillator used to model stick-slip problem in Exercise 7.

b) Find a first-term approximation of the solution that is valid for large t when $\frac{4}{5}(1-\beta/a) < v^2 < 1-\beta/a$. Make sure to point out where the stated conditions on v are used in the derivation.

8. The equation of a nonlinear beam with a small forcing is

$$\partial_x^4 u - \kappa \partial_x^2 u + \partial_t^2 u = \varepsilon f(t)\sin(\pi x), \text{ for } 0 < x < 1 \text{ and } 0 < t,$$

where $u = u'' = 0$ at $x = 0, 1$. Also,

$$\kappa = \frac{1}{4}\int_0^1 (u_x)^2 dx.$$

a) Find a first-term expansion of the solution that is valid for large t when $f(t) = \sin(t)$.

b) If $f(t) = \sin[(\pi + \omega_0\varepsilon)t]$, find a first-term expansion of the solution that is valid for large t.

9. The equation for a weakly damped pendulum whose support is subjected to a vertical excitation is (see Fig. 3.9)

$$\theta'' + \varepsilon\mu\theta' + [1 - f(t,\varepsilon)]\sin\theta = 0,$$

where $\mu > 0$ is constant and $f(t,\varepsilon)$ is the prescribed excitation of the support. We will assume $\theta(0) = \varepsilon a_0$ and $\theta'(0) = \varepsilon b_0$, where a_0 and b_0 are constants independent of ε.

a) For $f(t,\varepsilon) = \varepsilon\alpha\sin^2(t)$, where α is a positive constant, find a first-term expansion of the solution that is valid for large t.

b) For $f(t,\varepsilon) = \alpha\sin(\varepsilon\omega t)$, where α and ω are positive constants, find a first-term expansion of the solution that is valid for large t.

FIGURE 3.9. Forced pendulum studied in Exercise 9.

FIGURE 3.10. Graph for Exercise 10.

10. This problem concerns the forced Duffing problem (3.32) and its asymptotic solution (3.45).

    a) Sketch the positive solutions of (3.46) when $\lambda = 0$. Identify which branch is in phase with the forcing and which branch is out of phase.

    b) One way to obtain a qualitative understanding of the solution of (3.42), (3.43) is to use the phase plane. Set $\phi = \theta - \omega t_2$, and sketch the direction fields in the A, ϕ-plane for $\lambda \neq 0$ and for $\lambda = 0$.

    c) Based on your results from part (b), comment on the possibility of A approaching a constant as $t_2 \to \infty$.

    d) Figure 3.10 shows the numerical solution of the problem (dashed curve) and the numerical solution for $A(\varepsilon t)$ (solid curve). The same parameter values are used as those in Fig. 3.5 except now $\lambda = 0$. Is this result consistent with your results from parts (b) and (c)? Also, why do you think the asymptotic approximation and numerical solution differ?

3.5 Boundary Layers

As stated earlier, the method of multiple scales is not limited to the time variable. To illustrate this, we consider one of the boundary-layer problems from the last chapter. Specifically, we take the following problem from Section 2.2:

$$\varepsilon y'' + 2y' + 2y = 0, \quad \text{for } 0 < x < 1, \tag{3.47a}$$

where

$$y(0) = 0 \text{ and } y(1) = 1. \tag{3.47b}$$

We set $x_1 = x/\varepsilon$ and $x_2 = x$, and equation (3.47a) becomes

$$\left(\partial_{x_1}^2 + 2\varepsilon\partial_{x_1}\partial_{x_2} + \varepsilon^2\partial_{x_2}^2\right)y + 2\left(\partial_{x_1} + \varepsilon\partial_{x_2}\right)y + 2\varepsilon y = 0. \tag{3.48}$$

It is worth noting the intervals that are used for the space scales we have introduced. For the boundary-layer scale, we have that $0 \le x_1 < \infty$, and for the other coordinate we require $0 \le x_2 \le 1$. Now, assuming

$$y \sim y_0(x_1, x_2) + \varepsilon y_1(x_1, x_2) + \cdots, \tag{3.49}$$

one finds, from (3.48), that the general solution for the first term is

$$y_0 = a_0(x_2) + b_0(x_2)e^{-2x_1}. \tag{3.50}$$

To determine the coefficients, we need the $O(\varepsilon)$ equation from (3.48), and we find that

$$(\partial_{x_1}^2 + 2\partial_{x_1})y_1 = -2\partial_{x_1}\partial_{x_2}y_0 - 2\partial_{x_2}y_0 - 2y_0. \tag{3.51}$$

The general solution of this is

$$y_1 = a_1(x_2) + b_1(x_2)e^{-2x_1} - (a_0' + a_0)x_1 + (b_0 - b_0')x_1e^{-2x_1}. \tag{3.52}$$

Since $0 \le x_1 < \infty$, to remove the secular terms we require $a_0' + a_0 = 0$, and to minimize the contribution of the second term, we take $b_0 - b_0' = 0$ (see Exercise 2). Using the fact that $y_0(0,0) = 0$ and $y_0(\infty, 1) = 1$, it follows that the first-term approximation of the solution is

$$y \sim e^{1-x} - e^{1+x-2x/\varepsilon}. \tag{3.53}$$

In comparison, using matched asymptotics, we obtained, using the composite expansion (2.16),

$$y \sim e^{1-x} - e^{1-2x/\varepsilon}.$$

These expressions are unequal, and the difference is in the second exponential in (3.53). However, this term is exponentially small except in the boundary layer. Since the layer has thickness $O(\varepsilon)$, it follows that any contribution the x makes is of second order. In other words, the two expansions are asymptotically equivalent (to first order). As a final comment, note that when using multiple scales, the balancing needed to determine the correct space scale, and the location of the boundary layer, are hidden but are still a very important component of the solution procedure. Those who may want to pursue the connections between matched asymptotic expansions and multiple scales should consult the papers by Wollkind (1977) and Bouthier (1984).

Exercises

1. Consider the problem

$$\varepsilon y'' + p(x)y' + q(x)y = f(x), \quad \text{for } 0 < x < 1,$$

where $y(0) = \alpha$ and $y(1) = \beta$. Here $p(x), q(x)$, and $f(x)$ are smooth, and $p(x) > 0$ for $0 \le x \le 1$.

a) Using matched asymptotic expansions, construct a composite expansion of the solution.

b) Let

$$x_1 = \frac{1}{\varepsilon} \int_0^x p(s)ds$$

and $x_2 = x$. Find a first-term multiple-scale approximation.

c) Discuss the differences between your results in parts (a) and (b). Comment on the specific case where $p(x)$ and $q(x)$ are constants.

2. Set $R(x, \varepsilon) = y(x) - y_0(x/\varepsilon, x)$, where y_0 is defined in (3.50) and $a_0(x)$ and $b_0(x)$ are arbitrary.

a) Find the exact solution to (3.47), and write down the resulting expression for R.

b) Use Taylor's theorem, with remainder, to expand the terms in R for small ε. Use this to explain how a_0 and b_0 should be chosen to minimize the error for $0 \le x \le 1$.

3.6 Introduction to Partial Differential Equations

To begin the study of the application of multiple scales to partial differential equations, we consider the equation for an elastic string with weak damping. Specifically, we will study the problem

$$\partial_x^2 u = \partial_t^2 u + \varepsilon \partial_t u, \quad \text{for } 0 < x < 1 \text{ and } 0 < t, \tag{3.54a}$$

where

$$u = 0 \quad \text{for } x = 0, 1, \tag{3.54b}$$

and

$$u(x, 0) = g(x) \text{ and } \partial_t u(x, 0) = 0. \tag{3.54c}$$

Even though a partial differential equation is involved, (3.54) is similar to the oscillator problem considered in Section 3.2. This is because the input is through the initial conditions (3.54c), and there is weak damping. Consequently, it should not be surprising to find out that a regular expansion results in secular terms. (Another way to reach this conclusion is given in Exercise 2.)

To use multiple scales, we need to determine the appropriate time scales. Given the similarity with the weakly damped oscillator, it seems reasonable to assume that the same time scales can be used in this problem, and so we take $t_1 = t$ and $t_2 = \varepsilon t$. In this case, (3.54) becomes

$$\partial_x^2 u = \left(\partial_{t_1}^2 + 2\varepsilon \partial_{t_1} \partial_{t_2} + \varepsilon^2 \partial_{t_2}^2\right) u + \varepsilon \left(\partial_{t_1} + \varepsilon \partial_{t_2}\right) u, \tag{3.55a}$$

where

$$u = 0 \quad \text{for } x = 0, 1, \tag{3.55b}$$

and

$$u = g(x) \text{ and } (\partial_{t_1} + \varepsilon \partial_{t_2})\, u = 0 \quad \text{at } t_1 = t_2 = 0. \tag{3.55c}$$

As before, the solution is not unique, and we will use this freedom to minimize the contribution of the second term in the expansion. This will prevent the appearance of secular terms.

The appropriate expansion of the solution is

$$u \sim u_0(x, t_1, t_2) + \varepsilon u_1(x, t_1, t_2) + \cdots . \tag{3.56}$$

Introducing this into (3.55) yields the $O(1)$ problem

$$\partial_x^2 u_0 = \partial_{t_1}^2 u_0, \tag{3.57a}$$

where

$$u_0 = 0 \quad \text{for } x = 0, 1, \tag{3.57b}$$

and

$$u_0(x, 0, 0) = g(x) \text{ and } \partial_{t_1} u_0(x, 0, 0) = 0. \tag{3.57c}$$

Using separation of variables, one finds that the general solution of (3.57a,b) is

$$u_0(x, t_1, t_2) = \sum_{n=1}^{\infty} [a_n(t_2) \sin(\lambda_n t_1) + b_n(t_2) \cos(\lambda_n t_1)] \sin \lambda_n x, \tag{3.58}$$

where $\lambda_n = n\pi$. The initial conditions in (3.57c) will be imposed once we determine the coefficients $a_n(t_2)$ and $b_n(t_2)$.

The $O(\varepsilon)$ equation that comes from introducing (3.56) into (3.55a) is

$$\begin{aligned} \partial_x^2 u_1 &= \partial_{t_1}^2 u_1 + 2\partial_{t_1}\partial_{t_2} u_0 + \partial_{t_1} u_0 \\ &= \partial_{t_1}^2 u_1 + \sum_{n=1}^{\infty} A_n(t_1, t_2) \sin \lambda_n x, \end{aligned} \tag{3.59}$$

where

$$A_n = -(2a_n' + a_n)\lambda_n \cos(\lambda_n t_1) + (2b_n' + b_n)\lambda_n \sin(\lambda_n t_1).$$

Given the boundary conditions (3.55b), it is appropriate to expand the solution of (3.59) in a modal expansion like the one that appears in (3.58). Thus, the solution of (3.59) has the form

$$u_1 = \sum_{n=1}^{\infty} v_n(t_1, t_2) \sin \lambda_n x. \tag{3.60}$$

Substituting this into (3.59) and using (3.58), one finds that

$$\partial_{t_1}^2 v_n + \lambda_n^2 v_n = -(2a_n' + a_n)\lambda_n \cos(\lambda_n t_1) + (2b_n' + b_n)\lambda_n \sin(\lambda_n t_1).$$

Since the trigonometric functions on the right-hand side are solutions of the associated homogeneous equation, to prevent secular terms in the expansion we must take

$$2a_n' + a_n = 0 \text{ and } 2b_n' + b_n = 0.$$

Solving these equations, and using the initial conditions (3.57c), one finds that $a_n = 0$ and $b_n = \beta_n e^{-t_2/2}$, where

$$\beta_n = 2\int_0^1 g(x)\sin(\lambda_n x)dx.$$

Therefore, our first-term approximation of the solution is

$$u(x,t) \sim \sum_{n=1}^{\infty} \beta_n e^{-\varepsilon t/2}\cos(\lambda_n t)\sin\lambda_n x. \tag{3.61}$$

It appears from this example that the application of the method of multiple scales to partial differential equations is a relatively straightforward generalization of what was used for ordinary differential equations. It should be understood, however, that this was a very simple example and partial differential equations can easily produce different, and very interesting, uses of multiple scales.

Exercises

1. Consider the following problem:

$$\partial_x^2 u = \partial_t^2 u + \varepsilon u, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = u(1,t) = 0$, $u(x,0) = g(x)$, and $\partial_t u(x,0) = 0$. Using multiple scales, find a first-term approximation of the solution that is valid for large t.

2. This problem concerns the energy associated with the weakly damped string equation in (3.54a).

a) Multiplying (3.54a) by the velocity u_t, show that the equation can be rewritten in the form

$$\partial_t H + \partial_x S = -\Phi.$$

In this case, H is the Hamiltonian, S is the energy flux density, and Φ is the dissipation function (the latter being the only one that depends explicitly on ε in this problem).

b) Suppose a regular expansion of the solution is used. Explain why the first term is a conservative approximation to a dissipative problem.

c) Expand the energy equation in part (a) using multiple scales. Can you interpret the approximation in (3.61) in terms of this result?

3. The problem for a weakly damped elastic string is

$$\partial_x^2 u = \partial_t^2 u + \varepsilon u_t + \varepsilon f(x)\sin(\omega t), \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = u(1,t) = 0$ and $u(x,0) = \partial_t u(x,0) = 0$. Assume $f(x)$ is smooth.

a) When $\omega \neq n\pi$, where n is an integer, find a first-term approximation of the solution that is valid for large t.

b) When $\omega = \pi$, find a first-term approximation of the solution that is valid for large t.

c) Suppose one looks for time-periodic solutions of the form $u(x,t) = v(x)e^{i\omega t}$. Find $v(x)$ and then describe how this periodic solution compares to the large t behavior of the solutions found in parts (a) and (b). Assume that $f(x) = \sin(\pi x)$.

4. Consider the problem

$$\partial_x^2 u = \partial_t u + c(\varepsilon x)u, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = u(1,t) = 0$ and $u(x,0) = f(x)$. Assume $c(s)$ is smooth and positive, and $c'(0) \neq 0$. Find a first-term approximation of the solution that is valid for large t.

5. The equation for a weakly damped beam is

$$\partial_x^4 u + \varepsilon \partial_x^2 u + \partial_t^2 u + \varepsilon \partial_t u = 0, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u = u_{xx} = 0$ at $x = 0, 1, u(x,0) = f(x)$, and $u_t(x,0) = 0$.

a) Find a first-term approximation of the solution that is valid for large t.

b) Find the exact solution, and compare it with the result from part (a).

6. The Kirchhoff equation for the transverse vibrations of a weakly nonlinear string is (Leissa and Sadd, 1994)

$$T(t)\partial_x^2 u = \partial_t^2 u, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where

$$T(t) = 1 + \varepsilon \int_0^1 u_x^2 dx.$$

Also, $u(0,t) = u(1,t) = 0$, $u(x,0) = g(x)$, and $\partial_t u(x,0) = h(x)$. Find a first-term approximation of the solution that is valid for large t.

7. Consider the weakly nonlinear Klein-Gordon equation,

$$\partial_x^2 u = \partial_t^2 u + \alpha u + \varepsilon f(u, u_t), \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = u(1,t) = 0$, $u(x,0) = g(x)$, $\partial_t u(x,0) = h(x)$, and α is a non-negative constant.

a) Using multiple scales, show that a first-term approximation of the solution, valid for large t, has the form (see Exercise 3.1.4)

$$u \sim \sum_{n=1}^{\infty} A_n(\tau) \cos[\kappa_n t + \theta_n(\tau)] \sin(\lambda_n x),$$

where $\kappa_n A_n' = \langle f_n, \sin[\kappa_n t + \theta_n] \rangle_\infty$ and $\kappa_n A_n \theta_n' = \langle f_n, \cos[\kappa_n t + \theta_n] \rangle_\infty$. Also,

$$\langle f_n, q \rangle_\infty = \lim_{t \to \infty} \frac{1}{t} \int_0^t f_n q(s, \tau) ds.$$

b) In a model for wind-induced oscillations of overhead power lines, one finds that $\alpha = 0$ and $f(u, u_t) = -u_t + \beta u_t^3$, where β is a positive constant (Myerscough, 1973). Find what problems A_n and θ_n satisfy (your answer should contain triple sums). Multiple sums are typical of nonlinear multiple-scale problems and clearly limit what can be done analytically. However, occasionally it is possible to reduce them; a good example of this can be found in Keller and Kogelman (1970).

c) Investigate the behavior of the solution when a one-mode approximation is made (i.e., set $A_n = 0$ for $n \geq 2$).

8. This problem examines the piston-driven motion of a gas inside a cylinder. The Lagrangian equations of motion in this case are (Wang and Kassoy, 1990)

$$\left.\begin{aligned} \partial_t \rho + \rho^2 \partial_x u &= 0 \\ \partial_t u + \rho^{\gamma - 1} \partial_x \rho &= 0 \end{aligned}\right\} \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = \varepsilon U(\varepsilon t)$, $u(1,t) = 0$, and at $t = 0$ we have $\rho = 1$ and $u = 0$. Here $\rho(x,t)$ is the density of the gas, $u(x,t)$ is the velocity of the gas, and U is the prescribed velocity of the piston. The parameter ε is the ratio of the piston velocity to the speed of sound, and $\gamma > 1$ is constant. It is assumed that $U(0) = 0$, $U'(0) > 0$, and

$$\int_0^\infty U(\tau) d\tau < 1.$$

a) Using regular expansions for $u(x,t)$ and $\rho(x,t)$, show that secular terms are present in the second terms.

b) Using multiple scales, find a first-term approximation of the solution that is valid for large t.

3.7 Linear Wave Propagation

To investigate how multiple scales can be used for traveling waves, we will again start with the equation for an elastic string with weak damping. The problem is

$$\partial_x^2 u = \partial_t^2 u + \varepsilon \partial_t u, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t, \tag{3.62a}$$

where

$$u(x,0) = F(x) \quad \text{and } \partial_t u(x,0) = 0. \tag{3.62b}$$

To see what happens when the multiple-scales method is not used, we try the regular expansion,

$$u \sim u_0(x,t) + \varepsilon u_1(x,t) + \cdots . \tag{3.63}$$

Substituting this into (3.62) yields the following:

$O(1)$ $\partial_x^2 u_0 = \partial_t^2 u_0,$

$u_0(x,0) = F(x), \partial_{t_1} u_0(x,0) = 0.$

It is convenient at this point to switch to characteristic coordinates and let $\theta_1 = x - t$ and $\theta_2 = x + t$. In this case, $\partial_x = \partial_{\theta_1} + \partial_{\theta_2}, \partial_t = -\partial_{\theta_1} + \partial_{\theta_2}$, and $t = 0$ corresponds to $\theta_1 = \theta_2$. Also, the equation for u_0 reduces to $\partial_{\theta_1}\partial_{\theta_2} u_0 = 0$. Integrating this yields the general solution

$$u_0 = f_0(\theta_1) + g_0(\theta_2),$$

where $f_0(\theta_1)$ and $g_0(\theta_2)$ are determined from the initial conditions. From these conditions, one finds that the solution of the $O(1)$ problem is

$$u_0 = \frac{1}{2}\left[F(\theta_1) + F(\theta_2)\right].$$

$O(\varepsilon)$ $\left(\partial_x^2 - \partial_t^2\right) u_1 = \partial_t u_0$

If we use characteristic coordinates and the $O(1)$ solution, the equation becomes

$$4\partial_{\theta_1}\partial_{\theta_2} u_1 = -\frac{1}{2}F'(\theta_1) + \frac{1}{2}F'(\theta_2).$$

Integrating this, we obtain

$$u_1 = -\frac{1}{8}\theta_2 F(\theta_1) + \frac{1}{8}\theta_1 F(\theta_2) + f_1(\theta_1) + g_1(\theta_2),$$

where $f_1(\theta_1)$ and $g_1(\theta_2)$ are arbitrary functions. Since $-\infty < x < \infty$, this solution contains secular terms; from what we have done so far, there is no way to prevent this from occurring.

Because of the appearance of secular terms in the regular expansion, we will use multiple scales to find an asymptotic approximation of the solution. Unlike the problems examined earlier, this one appears to have secular terms both in space and in time. Consequently, there are several scales we can introduce. For example, we can use one of the following:

i) $t_1 = t$, $t_2 = \varepsilon t$, $x_1 = x$, $x_2 = \varepsilon x$,

ii) $\theta_1 = x - t$, $\theta_2 = x + t$, $\phi_1 = \varepsilon\theta_1$, $\phi_2 = \varepsilon\theta_2$,

iii) $\theta_1 = x - t$, $\theta_2 = x + t$, $t_2 = \varepsilon t$, $x_2 = \varepsilon x$.

These are equivalent, but note that each list contains four variables. Unfortunately, this is usually necessary and means that there is going to be some serious symbol pushing ahead.

We will use the scales given in (iii), which means that

$$\partial_x \to \partial_{\theta_1} + \partial_{\theta_2} + \varepsilon\partial_{x_2} \text{ and } \partial_t \to -\partial_{\theta_1} + \partial_{\theta_2} + \varepsilon\partial_{t_2}.$$

Substituting these into (3.62a), we obtain the following equation:

$$\begin{aligned}\{4\partial_{\theta_1}\partial_{\theta_2} + 2\varepsilon\left[(\partial_{\theta_1} + \partial_{\theta_2})\,\partial_{x_2} \quad - \quad (-\partial_{\theta_1} + \partial_{\theta_2})\,\partial_{t_2}\right] + O(\varepsilon^2)\}\,u \\ = \varepsilon\left[-\partial_{\theta_1} + \partial_{\theta_2} + O(\varepsilon)\right]u.\end{aligned} \tag{3.64a}$$

Also note that the initial conditions at $t = 0$ now correspond to $\theta_1 = \theta_2 = x$ and $t_2 = 0$. In fact, the initial conditions in (3.62b) become

$$u = F(\theta_1) \text{ and } (-\partial_{\theta_1} + \partial_{\theta_2} + \varepsilon\partial_{t_2})\,u = 0 \quad \text{when } \theta_1 = \theta_2 \text{ and } t_2 = 0. \tag{3.64b}$$

The next step is to introduce the expansion for the solution. For this problem, it is simply

$$u \sim u_0(\theta_1, \theta_2, x_2, t_2) + \varepsilon u_1(\theta_1, \theta_2, x_2, t_2) + \cdots. \tag{3.65}$$

This leads to the following problems:

$O(1)$ $4\partial_{\theta_1}\partial_{\theta_2}u_0 = 0,$

$$u_0(\theta_1, \theta_1, x_2, 0) = F(\theta_1), (-\partial_{\theta_1} + \partial_{\theta_2})u_0(\theta_1, \theta_1, x_2, 0) = 0.$$

The general solution of this problem is

$$u_0 = f_0(\theta_1, t_2) + g_0(\theta_2, t_2). \tag{3.66}$$

These functions do not depend on the variable x_2. We are assuming that u_0 does not depend on x_2 because of the differential equation and initial conditions. This hypothesis is made to reduce the effort to solve the problem, and the dependence can always be restored later if necessary.

$$O(\varepsilon) \quad 4\partial_{\theta_1}\partial_{\theta_2}u_1 = 2\left(-\partial_{\theta_1} + \partial_{\theta_2}\right)\partial_{t_2}u_0 + \left(-\partial_{\theta_1} + \partial_{\theta_2}\right)u_0.$$

Substituting the $O(1)$ solution into this equation and solving, one finds that

$$u_1 = -\frac{1}{4}\theta_2\left(2\partial_{t_2} + 1\right)f_0 + \frac{1}{4}\theta_1\left(2\partial_{t_2} + 1\right)g_0 + f_1(\theta_1, t_2) + g_1(\theta_2, t_2).$$

To prevent secular terms, we take

$$\left(2\partial_{t_2} + 1\right)f_0 = 0 \text{ and } \left(2\partial_{t_2} + 1\right)g_0 = 0.$$

Integrating these equations and using the $O(1)$ initial conditions yields

$$f_0 = \frac{1}{2}F(\theta_1)e^{-t_2/2} \text{ and } g_0 = \frac{1}{2}F(\theta_2)e^{-t_2/2}. \tag{3.67}$$

Therefore, a first-term approximation of the solution that is valid for large t is

$$u(x,t) \sim \frac{1}{2}[F(x-t) + F(x+t)]e^{-\varepsilon t/2}. \tag{3.68}$$

Exercises

1. The wave equation with a slowly varying phase velocity is

$$c^2(\varepsilon t)\partial_x^2 u = \partial_t^2 u, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Find a first-term approximation of the solution that is valid for large t. It is assumed that $c(t)$ is a smooth positive function. (Hint: see Section 3.3.)

2. Consider the following advection problem with weak diffusion:

$$\varepsilon\partial_x^2 u = \partial_t u + \partial_x u, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x,0) = f(x)$. Using multiple scales, find a first-term approximation of the solution that is valid for large t.

3. In the study of acoustic wave propagation in a tube, one comes across the horn equation for the pressure $p(x,t)$. This equation can be written as (Rabbitt and Holmes, 1988)

$$p_{xx} + \varepsilon a(\varepsilon x)p_x = p_{tt}, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t.$$

The coefficient $a(\varepsilon x)$ is included to account for a slowly varying cross-section of the tube. Specifically, if the cross-sectional area is $A(\varepsilon x)$, then $a(\varepsilon x) = A'(\varepsilon x)/A(\varepsilon x)$.

a) Using multiple scales, find a first-term approximation of the solution that is valid for large x.

b) Find the second term in the expansion.

4. In the study of quantum motion of an ion in a Paul trap, one is interested in finding what is called a Wigner function $u(x, y, t)$. In the case of a weakly harmonic potential, the equation for this function is (Stenholm, 1992)
$$u_t + u_x - xf(t)u_y = 0.$$
Here $f(t) = a - \varepsilon\cos(t)$, where a is a positive constant. Using multiple scales, find a first-term approximation of the solution that is valid for large t.

3.8 Nonlinear Waves

Traveling wave solutions are found for a wide variety of nonlinear partial differential equations. This differs significantly from the situation for linear problems where wave propagation is limited to particular types of equations. In this section, two examples are analyzed that illustrate some of the situations where nonlinear equations are possible and how to use multiple scales to find them.

Example 1: Nonlinear Wave Equation

Consider the nonlinear wave equation
$$\partial_x^2 u = \partial_t^2 u + K(x, u, \varepsilon), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t, \tag{3.69a}$$
where
$$u(x, 0) = F(x) \text{ and } \partial_t u(x, 0) = G(x). \tag{3.69b}$$
There is little hope of solving this problem in closed form, so our approach is to investigate particular solutions that are in the form of traveling waves. In the example to follow, we will examine the behavior of the solution of the nonlinear Klein-Gordon equation, where
$$K(x, u, \varepsilon) = u + \varepsilon u^3. \tag{3.70}$$
This type of equation comes up in a variety of situations. For example, it describes the motion of a string on an elastic foundation as well as the waves in a cold electron plasma, and it also arises in quantum mechanics.

To explain the approach we will take, consider the linear equation that is obtained when $\varepsilon = 0$. Using the Fourier transform, one finds that the general solution of the linear equation has the form
$$u(x, t) = \int_{-\infty}^{\infty} A(k)e^{i(kx-\omega t)}dk + \int_{-\infty}^{\infty} B(k)e^{i(kx+\omega t)}dk,$$
where $A(k)$ and $B(k)$ are determined from the initial conditions and $\omega = (1 + k^2)^{1/2}$. It is apparent from this that the solution is composed of the

superposition of the plane wave solutions $u_{\pm}(x,t) = e^{i(kx \pm \omega t)}$. It is therefore worth investigating what the nonlinear equation does to one of these plane waves. We will consider $u(x,t) = \cos(kx - \omega t)$, using an expansion where this function is effectively the first term, and we will then determine how the nonlinearity contributes to the second term.

To find an expansion of the solution, the simplest thing to try is one of the form

$$u(x,t) \sim u_0(kx - \omega t) + \varepsilon u_1(x,t) + \cdots . \tag{3.71}$$

However, it is not hard to show that this leads to secular terms. Consequently, we will use multiple scales, and this brings up choices for the scales similar to those we had for the linear wave problem (see Section 3.7). Because we are investigating what effects the nonlinearity has on the right running wave, for the scales we will take $\theta = kx - \omega t$, $x_2 = \varepsilon x$, and $t_2 = \varepsilon t$. In this development, we assume that $\omega = (1 + k^2)^{1/2}$ and $k > 0$. The differential equation in this case becomes

$$[\partial_\theta^2 - 2\varepsilon(k\partial_{x_2} + \omega\partial_{t_2})\partial_\theta + O(\varepsilon^2)]u + u + \varepsilon u^3 = 0. \tag{3.72}$$

The appropriate expansion of the solution is now

$$u(x,t) \sim u_0(\theta, x_2, t_2) + \varepsilon u_1(\theta, x_2, t_2) + \cdots . \tag{3.73}$$

This leads to the following problems:

$O(1)$ $(\partial_\theta^2 + 1)u_0 = 0.$

The general solution of this is

$$u_0 = A(x_2, t_2)\cos[\theta + \phi(x_2, t_2)]. \tag{3.74}$$

As usual, the functions A and ϕ are used to prevent secular terms from appearing in the higher-order terms in the expansion. Note that the initial conditions have not been listed in this problem. These will be discussed after the general form of the expansion has been determined.

$$\begin{aligned} O(\varepsilon) \quad (\partial_\theta^2 + 1)u_1 &= 2\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\partial_\theta u_0 - u_0^3 \\ &= -2\left[\left(k\partial_{x_2} + \omega\partial_{t_2}\right)A\right]\sin(\theta + \phi) - \frac{1}{4}A^3\cos 3(\theta + \phi) \\ &\quad -2\left[\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\phi + \frac{3}{8}A^2\right]A\cos(\theta + \phi) \end{aligned}$$

This is an ordinary differential equation much like what we have seen earlier. To prevent secular terms, it is required that

$$\left(k\partial_{x_2} + \omega\partial_{t_2}\right)A = 0 \tag{3.75}$$

and

$$\left(k\partial_{x_2} + \omega\partial_{t_2}\right)\phi + \frac{3}{8}A^2 = 0. \tag{3.76}$$

These first-order equations can be solved using characteristic coordinates; and thus, let $r = \omega x_2 + kt_2$ and $s = \omega x_2 - kt_2$. With this (3.75), (3.76) become

$$\partial_r A = 0 \text{ and } \partial_r \phi = -\frac{3}{16\omega k} A^2.$$

From this we get $A = A(s)$ and $\phi = -\frac{3}{16\omega k} A^2 r + \phi_0(s)$.

Therefore, we have that a first-term approximation of the solution of the nonlinear Klein-Gordon equation is

$$u \sim A \cos\left[\theta - \frac{3}{16\omega k}(\omega x_2 + kt_2)A^2 + \phi_0\right], \tag{3.77}$$

where $A = A(\omega x_2 - kt_2)$ and $\phi_0 = \phi_0(\omega x_2 - kt_2)$ are arbitrary.

We began this example by asking what the nonlinearity does to the plane wave solution. To answer this, suppose the initial conditions correspond to the plane wave, so

$$u(x,0) = \alpha \cos(kx) \text{ and } u_t(x,0) = \alpha\omega \sin(kx). \tag{3.78}$$

In terms of the multiple scales used to derive (3.77), these translate into the conditions that $u_0(\theta, x_2, 0) = \alpha \cos(\theta)$ and $\partial_\theta u_0(\theta, x_2, 0) = -\alpha \sin(\theta)$. In (3.77) this means $A(\omega x_2) = \alpha$ and $\phi_0(\omega x_2) = \frac{3}{16k} A^2 x_2$. Thus, a first-term approximation of the solution in this case is

$$u(x,t) \sim \alpha \cos\left[kx - \left(\omega + \frac{3\varepsilon\alpha^2}{16\omega}\right) t\right]. \tag{3.79}$$

This shows that the phase velocity

$$v_{\text{ph}} = \left(1 + \frac{3\varepsilon\alpha^2}{16\omega^2}\right) \frac{\omega}{k}$$

is increased by the nonlinearity and that this increase is amplitude dependent. Specifically, the larger the amplitude, the faster it moves. This dependence of the velocity on amplitude is typical of nonlinear waves and does not occur for linear problems.

As it turns out, this example was one of the first wave problems to be analyzed using multiple scales (Luke, 1966). It has since been applied to numerous other problems, such as nonlinear diffusion problems, which brings us to the next example.

Example 2: Nonlinear Diffusion

In the study of wave propagation in chemical and biological systems, one comes across the problem of having to solve

$$u_t = \varepsilon u_{xx} + f(u), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t, \tag{3.80a}$$

where

$$u(x,0) = g(x). \tag{3.80b}$$

There are restrictions on the functions $f(u)$ and $g(x)$, and these will be given later. What is interesting is that the solution of this nonlinear diffusion problem can evolve into a traveling wave of the form $u(x,t) = U(x-vt)$, where v is the velocity of the wave. Such a solution is impossible with linear diffusion, and this is one of the reasons why nonlinear diffusion problems have received a lot of attention over the last few years. We will use multiple scales to find an asymptotic approximation of the solution of (3.80), and our approach will be based on the work of Berman (1978). An introduction to the theory and some of the applications of this equation can be found in Britton (1986) and Murray (1989).

We first need to list the restrictions made on the nonlinearity in (3.80a). In doing so, we specify only what happens for $0 \le u \le 1$. This interval for $u(x,t)$ is the one of physical interest since this function is a nondimensional concentration or density. In what follows we will assume that $f(u)$ is smooth for $0 \le u \le 1$ and positive for $0 < u < 1$. Moreover, $f(0) = f(1) = 0$, $f'(0) > 0$, and $f'(1) < 0$. Perhaps the simplest example, and the most well studied, is the quadratic function $f(u) = u(1-u)$. This gives rise to what is known as Fisher's equation or the Kolmogorov-Petrovsky-Piskunov (KPP) equation. This has been used to model such phenomena as the spatial spread of certain genetic characteristics, the propagation of cosmic rays in the atmosphere, and the evolution of a neutron population in a reactor. It is also worth pointing out that this quadratic function is the same as appears in the logistic equation, which is studied in connection with population dynamics.

The initial condition in (3.80b) must also be consistent with the physics. The requirements on this function are that $0 \le g(x) \le 1$, $g(x) \to 1$ as $x \to -\infty$, and $g(x) \to 0$ as $x \to \infty$. In what follows we will simply take

$$g(x) = \frac{1}{1+e^{\lambda x}}, \tag{3.81}$$

where $\lambda > 0$ is a given constant.

The origin of all these conditions may seem mysterious, and it may be unclear what connections, if any, there are between the choices we are making for $f(u)$ and $g(x)$. To explain what we have done, note that our requirements on $f(u)$ result in two steady-state solutions of the differential equation, namely $u = 0$ and $u = 1$. The function $f(u)$ has also been set up so that $u = 1$ is stable and $u = 0$ is unstable. (The tools needed to show this are developed in Chapter 6.) Therefore, by requiring that $g \to 1$ as $x \to -\infty$, and $g \to 0$ as $x \to \infty$, the initial condition effectively connects these two steady states. The solution of the problem in this case is fairly simple since at any given point on the x-axis $u(x,t)$ simply moves away from $u = 0$ and approaches $u = 1$. What is important is that the speed

at which it approaches $u = 1$ depends nonlinearly on the solution. The resulting movement toward $u = 1$ is what gives rise to the traveling wave. This is a very simplistic explanation of what occurs, but hopefully it will provide some insight into the behavior of the solution.

Our assumption that ε is small in (3.80a) means that there is weak diffusion, and this introduces a second time scale into the problem. To account for this, we introduce the two time scales $t_1 = t$ and $t_2 = \varepsilon t$. The equation in (3.80a) then takes the form

$$(\partial_{t_1} + \varepsilon \partial_{t_2})\, u = \varepsilon u_{xx} + f(u). \tag{3.82}$$

There is nothing unusual about this equation, and so the appropriate expansion for the solution is a regular power series of the form

$$u \sim u_0(x, t_1, t_2) + \varepsilon u_1(x, t_1, t_2). \tag{3.83}$$

The $O(1)$ problem that comes from substituting (3.83) into (3.82) and (3.80b) is

$$\partial_{t_1} u_0 = f(u_0), \tag{3.84a}$$

where

$$u_0(x, 0, 0) = g(x). \tag{3.84b}$$

By separating variables in (3.84a), one finds that

$$\int_{1/2}^{u_0} \frac{dr}{f(r)} = t_1 + \theta(x, t_2), \tag{3.85}$$

where $\theta(x, t_2)$ is arbitrary. Imposing the initial condition (3.84b) yields

$$\theta(x, 0) = \int_{1/2}^{g(x)} \frac{dr}{f(r)}. \tag{3.86}$$

Without knowing exactly what the function $f(u)$ is, we will have to leave $u_0(x, t_1, t_2)$ defined implicitly in (3.85). However, note that we can write the solution in the form $u_0 = U_0(s)$, where $s = t_1 + \theta(x, t_2)$. This form will prove handy when removing secular terms.

To determine the second term in the expansion, note that the $O(\varepsilon)$ equation obtained from (3.82) is

$$\begin{aligned} \partial_{t_1} u_1 &= f'(u_0) u_1 + \partial_x^2 u_0 - \partial_{t_2} u_0 \\ &= f'(u_0) u_1 + f(u_0) \left[\theta_{xx} - \theta_{t_2} + f'(u_0) \theta_x^2 \right]. \end{aligned}$$

Equations (3.85) and (3.84a) have been used in the last step. Fortunately, this equation for u_1 is linear and is not hard to solve. To do this, it is worth pointing out that, from (3.84a), $f'(u_0) = \partial_{t_1} \ln(f(u_0))$. Using this result, one finds that the solution of the above equation for u_1 is

$$u_1 = \left[A(x, t_2) + t_1 \left(\theta_{xx} - \theta_{t_2} \right) + \theta_x^2 \ln(f(u_0)) \right] f(u_0), \tag{3.87}$$

where $A(x, t_2)$ is arbitrary. From a quick inspection of this solution, it would appear that secular terms are going to arise unless we restrict $\theta(x, t_2)$. Note, however, that $u_1 = o(u_0)$ as $t_1 \to \infty$ or as $\theta \to \infty$. An example which illustrates this situation very nicely is the case of when $f(u) = u(1-u)$ and the details are left to the reader. Problems associated with secularity occur when $\theta \to -\infty$. In this limit one finds that the functions u_0 and $f(u_0)$ are comparable and $\ln(f(u_0)) \sim \kappa(t_1 + \theta)$ where $\kappa = f'(0)$ is a positive constant. Thus, $\ln(f(u_0))$ produces a secular term in (3.87) that is of the same order as the other t_1 terms in that expression. Based on these observations, we require

$$\theta_{xx} - \theta_{t_2} + \kappa\theta_x^2 = 0. \tag{3.88}$$

This equation can be solved if we first make the change of variables $w(x, t_2) = e^{\kappa\theta(x,t_2)}$. In this case (3.88) reduces to the linear diffusion equation

$$w_{xx} = w_{t_2}, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t_2. \tag{3.89}$$

The solution of this is

$$w(x, t_2) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} h\big(x + 2r\sqrt{t_2}\big)\, e^{-r^2}\, dr, \tag{3.90}$$

where $h(x) = w(x, 0)$. From (3.86) we get

$$h(x) = \exp\left(\kappa \int_{1/2}^{g(x)} \frac{dr}{f(r)}\right). \tag{3.91}$$

In stating that the solution of (3.89) is given by the formula in (3.90), we are assuming that the function $h(x)$ is well enough behaved that the integral in (3.90) is defined for all x and t_2. The exponential term in (3.90) helps a lot, and because of it we can get away with simply requiring that $h(x) \le Me^{\alpha x^2}$ for $-\infty < x < \infty$, where α, M are constants with $\alpha < 1$. This condition usually is not particularly restrictive. However, in looking at how $h(x)$ is defined in (3.91), it appears that we need to be careful because $f(u)$ is zero when $u = 0$ or $u = 1$ and $g(x)$ approaches these values as $x \to \pm\infty$. However, the conditions we have imposed on $f(u)$ and our assumed form of $g(x)$ in (3.81) make it so this is not an issue. This will be demonstrated later in the case of the quadratic nonlinearity found in Fisher's equation. An examination of other possibilities can be found in Larson (1978).

From the preceding analysis, we have found that a first-term approximation of the solution of (3.80a) is $u \sim u_0(x, t_1, t_2)$, where u_0 is defined implicitly through the equation

$$\int_{1/2}^{u_0} \frac{dr}{f(r)} = t_1 + \frac{1}{\kappa} \ln[w(x, t_2)], \tag{3.92}$$

where $w(x, t_2)$ is given in (3.90) and (3.91).

As stated earlier, nonlinear diffusion problems can give rise to traveling wave solutions. The asymptotic approximation of the solution given in (3.92) can be used to demonstrate this. To show how, we will take $f(u) = u(1-u)$. In this case, we have from (3.85) that $u_0 = U_0(t_1+\theta(x, t_2))$, where

$$U_0(s) = \frac{1}{1+e^{-s}}.$$

This can be rewritten as

$$u_0(x, t_1, t_2) = \frac{w(x, t_2)}{w(x, t_2) + e^{-t_1}}, \tag{3.93}$$

where, from (3.90),

$$\begin{aligned} w(x, t_2) &= e^{-\lambda x}\sqrt{\frac{t_2}{\pi}}\int_{-\infty}^{\infty} e^{-t_2 s(s-2\lambda)}ds \\ &= e^{-\lambda x+\lambda^2 t_2}. \end{aligned}$$

Therefore, $u_0 = U_0(t_1 - \lambda x + \lambda^2 t_2)$, and this gives us a traveling wave with velocity

$$v \sim \frac{1+\lambda^2\varepsilon}{\lambda}. \tag{3.94}$$

What is interesting is that (3.94) shows that the speed of the wave depends on the shape of the initial profile. The steeper it is (so λ increases), the slower it moves.

To illustrate the character of the traveling wave that comes from the nonlinear diffusion equation, the asymptotic solution of Fisher's equation is shown in Fig. 3.11. It is seen that the wave front moves to the right with a velocity of about $1/\lambda$. Also shown in Fig 3.11 is the wave at $t = 5$ and at $t = 10$ as obtained from the numerical solution of the full problem. These curves are included to demonstrate the accuracy of the multiple-scales approximation.

Exercises

1. Suppose one looks for traveling wave solutions of (3.69a) of the form $u(x, t) = f(\theta)$, where $\theta = x - vt$ and K is given in (3.70).

   a) After finding the equation $f(\theta)$ satisfies, show that a regular expansion of $f(\theta)$ contains secular terms.

   b) Expanding both $f(\theta)$ and v, use multiple scales to find a first-term approximation valid for large θ. Explain why the expansion for v is not necessary to remove secular terms.

   c) Suppose the initial conditions are given in (3.78). Find the first-term approximation from part (b), and compare the result to the one given in (3.79).

FIGURE 3.11. Traveling wave solution of Fisher's equation as determined from the asymptotic solution in (3.93). The solution is shown at $t = 0$, $t = 5$, and $t = 10$. For comparison, the numerical solution of the original problem (3.80) is shown with the dashed curves (they are essentially indistinguishable from the asymptotic solution). In these calculations, $\varepsilon = 10^{-2}$ and $\lambda = 1$.

2. The Korteweg-deVries (KdV) equation is

$$\partial_t u + \partial_x u + \alpha u \partial_x u + \beta \partial_x^3 u = 0, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where α and β are positive constants and $u(x,0) = \varepsilon f(x)$ for $0 < \varepsilon << 1$.

a) Setting $\theta = kx - \omega t$, suppose one looks for traveling wave solutions using an expansion of the form

$$u(x,t) \sim \varepsilon[u_0(\theta) + \varepsilon u_1(\theta) + \cdots],$$

where $\omega = k - \beta k^3$ and k is a positive constant. Show that this can lead to secular terms.

b) Multiple scales can be used to prevent the secular terms in the expansion in part (a). Do this, and find a first-term expansion that is valid for large t. What form must $f(x)$ have to be able to generate this traveling wave? The analysis in this problem is simplified greatly if you remember that $f(x)$ is independent of ε.

c) Suppose $u(x,0) = \varepsilon\,[\mu + A(\varepsilon x)\cos\,[\varphi(x,\varepsilon x)]]$, where μ is a nonzero constant and A and φ are smooth bounded functions. Redo part (b), and find a first-term expansion of the traveling wave solution that is valid for large t. What form must $\varphi(x,\varepsilon x)$ have to be able to generate this traveling wave?

3. In the study of waves in shallow water, one comes across the Boussinesq equation

$$u_{tt} = c^2 u_{xx} + \alpha u_{xxxx} + \varepsilon \partial_x^2 u^2, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where c and α are positive constants. Proceeding as in the nonlinear Klein-Gordon example, find a first-term approximation, valid for large t, of the waves traveling to the right.

4. Consider the nonlinear wave equation

$$u_x + u_t = \varepsilon f(u), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = g(x)$.

a) Find an asymptotic approximation of the solution that is valid for large t. Assume $f(u)$ and $g(x)$ are smooth and positive.

b) What is the approximation when $f(u) = u(1-u)$ and $g(x)$ is given in (3.81)? Sketch the approximate solution for a few values of t, and comment on how it compares with the solution of Fisher's equation shown in Fig. 3.11.

5. This problem concerns the nonlinear diffusion equation

$$u_t = \varepsilon u_{xx} + cu_x + f(u), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = g(x)$. Assume $f(u)$ satisfies the same conditions imposed on the nonlinearity in (3.80a).

a) After making the change of variables $r = x + ct$, $\tau = t$, find an asymptotic approximation of the solution that is valid for large τ. Express the result in x, t-coordinates.

b) What does the approximate solution reduce to when $f(u) = u(1-u)$?

6. Suppose the initial condition for the nonlinear diffusion problem in (3.80) is

$$g(x) = \frac{1}{1 + e^{\lambda(x-x_0)(x-x_1)}},$$

where $\lambda > 0$, x_0, and x_1 are constants.

a) For $f(u) = u(1-u)$, find a first-term approximation of the solution.

b) Taking $\lambda = 1$, $\varepsilon = 10^{-2}$, and $x_0 = -x_1 = 5$, sketch the approximate solution for a few values of t. Comment on the differences in the solution with the one shown in Fig. 3.11.

7. In the nonlinear diffusion equation (3.80a), suppose $f(u) = u(1-u^n)$, where n is a positive integer.

a) Find a first-term approximation of the solution that is valid for large t (use the initial condition in (3.81)).

b) What does the approximate solution in part (a) reduce to when $n = 2$? Is the result a traveling wave? Also, compare this result with the solution shown in Fig. 3.11.

8. In the study of wave propagation along the human spine, one comes across the following problem (Murray and Tayler, 1970):

$$\partial_t^2 u = [1 + \varepsilon f(u_x)]\partial_x^2 u, \quad \text{for } 0 < x < \infty \text{ and } 0 < t,$$

where $u(0,t) = g(t)$ and $u(x,0) = \partial_t u(x,0) = 0$. Also, f and g are smooth functions with $f(0) = g(0) = g'(0) = 0$. If we assume that $g(\theta) = 0$ if $\theta \leq 0$, then the solution of this problem is $u = g(t - x)$, when $\varepsilon = 0$.

a) Based on what happens when $\varepsilon = 0$, suppose we assume an expansion of the form

$$u(x,t) \sim u_0(t - x) + \varepsilon u_1(x,t) + \cdots .$$

Show that this leads to secular terms.

b) Using multiple scales, find an asymptotic approximation of the solution that is valid for large x.

c) Suppose there is a positive value of s where $g''(s)f'(-g'(s)) < 0$. Explain why your approximate solution in part (b) holds for $x < x_c$ but may not hold for $x > x_c$, where

$$x_c = \frac{-2}{\varepsilon \min\limits_{0 \leq s}[g''(s)f'(-g'(s))]}.$$

d) The equation for a weakly nonlinear elastic string is

$$\partial_t^2 u = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + \varepsilon(u_x)^2}}\right), \quad \text{for } 0 < x < \infty \text{ and } 0 < t.$$

Assuming the initial and boundary conditions are the same as used in the spine problem, explain why the results from parts (a)–(c) apply to this problem.

9. The equation for the longitudinal motion of a nonlinear elastic bar is

$$\partial_t^2 s = \partial_x^2 s + \varepsilon f(s, s_x), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $f(s, s_x) = 2\partial_x(s\partial_x s)$. The initial conditions are $s(x,0) = F(x)$ and $s_t(x,0) = 0$, where $F(x)$ is a smooth bounded function. In this problem, $s(x,t)$ is the strain in the bar and is related to the displacement $u(x,t)$ through the equation $s = \partial_x u$. This problem is based on the papers of Lardner (1975) and Mortell and Varley (1970).

a) Using multiple scales, find an asymptotic approximation of the strain $s(x,t)$ that is valid for large t.

b) Explain why your approximate solution holds for $t < t_c$ but may not hold for $t > t_c$, where

$$t_c = \frac{2}{\varepsilon \max\limits_{-\infty<x<\infty} |F'(x)|}.$$

c) How do your results in parts (a) and (b) change if $f(s, s_x) = n\lambda\partial_x(s^{n-1}\partial_x s)$, where λ is a nonzero constant and $n \geq 3$ is an integer?

10. The nonlinear progressive wave equation that is used to describe the density fluctuations due to an acoustic disturbance, for uniaxial motion, is (McDonald and Kuperman, 1985)

$$\partial_t\rho + \partial_x\left[\alpha(x)\rho + \frac{1}{2}\beta\rho^2\right] = 0, \quad \text{for } -\infty < x < \infty,$$

where $\rho(x, 0) = 1 + \varepsilon f(x)$. Here $\alpha(x) > 0$ is smooth and $\beta \geq 0$ is constant.

a) Suppose α is constant and we are interested in finding an asymptotic approximation to the traveling wave solutions. This is to be found by assuming an expansion of the form $\rho \sim 1 + \varepsilon\rho_0(x - \lambda t) + \varepsilon^2\rho_1(x, t) + \cdots$, where λ is a constant to be determined from the above equation. Show that this leads to secular terms.

b) For the situation described in part (a), use multiple scales to prevent secularity in the $O(\varepsilon^2)$ term of the expansion.

c) Redo part (b), but assume α is not a constant.

11. Consider the problem

$$\partial_t u + g(u)\partial_x u = 0, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = u_0 + \varepsilon v(x)$ for constant u_0. The function $g(u)$ is assumed smooth and is such that $g(u_0) \neq 0$ and $g'(u_0) > 0$. The function $v(x)$ is also assumed smooth and is zero outside a bounded interval. This differential equation is a scalar conservation law. One interesting feature of the equation is that shock (i.e., discontinuous) solutions will appear in finite time. The effect of this on the multiple-scale expansion developed in this problem can be found in DiPerna and Majda (1985).

a) Show that a regular expansion of the solution leads to the appearance of secular terms.

b) Using multiple scales, find the reduced problem that must be solved to obtain a two-term expansion of the solution.

c) The exact solution of this problem is defined implicitly as $u = u_0 + \varepsilon v(x - tg(u))$. Derive your expansion in part (b) from this equation.

12. In the theory of nonlinear sound propagation in a fluid one comes across the Westervelt equation for the acoustic pressure (Westervelt, 1963)

$$p_{xx} - \mu^2 p_{tt} = -\varepsilon\beta\partial_t^2(p^2),$$

where μ and β are positive constants.

a) For small ε, show that a regular expansion for the solutions of the form

$$p(x,t) \sim p_0(kx - \omega t) + \varepsilon p_1(x,t) + \cdots$$

leads to the appearance of secular terms.

b) Using multiple scales, find the reduced problem that must be solved to obtain a two-term expansion of the solution.

3.9 Difference Equations

An interesting application of multiple scales arises in their use in finding asymptotic solutions of difference equations. This is a relatively new area but one that is important because of the prominence of difference equations in problems in scientific computing, modeling, and other areas. As we usually do, the ideas will be introduced using a relatively simple example. The one we will consider is the following weakly nonlinear, second-order difference equation:

$$y_{n+1} - 2y_n + y_{n-1} = -2\omega^2\left(y_n + \varepsilon y_n^3\right), \quad \text{for } n = 0, \pm 1, \pm 2, \cdots. \tag{3.95}$$

Here ω is a constant that is assumed to satisfy $0 < \omega < 1$. In the case of small ω, this equation is associated with a finite-difference approximation of the nonlinear differential equation $y'' + y + \varepsilon y^3 = 0$. Consequently, it might be expected that the methods we have developed for differential equations can be adapted in a relatively straightforward way to difference equations. This is not exactly true, because difference equations have some unusual features, but it at least gives us a way to start. The approach that is taken here is based on some of the ideas developed in Mickens (1990), Hoppensteadt and Miranker (1977), and Torng (1960).

From what we have seen for differential equations, it is expected that for this problem the solution depends on n and on εn. The latter is the analog of the slow scale and is incorporated into the problem by assuming the solution has an expansion of the form

$$y_n \sim \overline{y}_0(n,s) + \varepsilon\overline{y}_1(n,s) + \cdots, \tag{3.96}$$

where $s = \varepsilon n$. With this,

$$\begin{aligned} y_{n\pm1} &\sim \overline{y}_0(n \pm 1, s \pm \varepsilon) + \varepsilon\overline{y}_1(n \pm 1, s \pm \varepsilon) + \cdots \\ &\sim \overline{y}_0(n \pm 1, s) + \varepsilon[\overline{y}_1(n \pm 1, s) \pm \partial_s\overline{y}_0(n \pm 1, s)] + \cdots. \end{aligned} \tag{3.97}$$

Substituting (3.97) and (3.96) into (3.95), we obtain the following:

$O(1)$ $\overline{y}_0(n+1,s) - 2\,\overline{y}_0(n,s) + \overline{y}_0(n-1,s) = -2\omega^2\overline{y}_0(n,s).$

This is a second-order, homogeneous, linear difference equation, and the general solution is (see Exercise 2)

$$\overline{y}_0 = A(s)\cos[n\phi + \theta(s)], \tag{3.98}$$

where $A(s)$ and $\theta(s)$ are arbitrary functions of the slow scale s and

$$\cos(\phi) = 1 - \omega^2. \tag{3.99}$$

$O(\varepsilon)$
$$\begin{aligned}\overline{y}_1(n+1,s) - 2\overline{y}_1(n,s) + \overline{y}_1(n-1,s) \\ = -2\omega^2\overline{y}_1(n,s) - \partial_s\overline{y}_0(n+1,s) \\ -2\omega^2\overline{y}_0(n,s)^3 + \partial_s\overline{y}_0(n-1,s)\end{aligned} \tag{3.100}$$

To determine how to remove the secular terms, note that a particular solution of the difference equation (see Exercise 2)

$$z_{n+1} - 2z_n + z_{n-1} = -2\omega^2 z_n + f_n, \quad \text{for } n = 1, 2, 3, \cdots,$$

is

$$z_n = \frac{1}{\sin\phi}\sum_{i=1}^{n} f_i \sin[(n-i)\phi].$$

When $f_i = \cos(i\phi)$, the solution can be written as

$$z_n = \frac{1}{2\sin\phi}\left[n\sin(n\phi) - \sum_{i=1}^{n}\sin[(2i-n)\phi]\right]. \tag{3.101}$$

Similarly, if $f_i = \sin(i\phi)$, then a particular solution is

$$z_n = \frac{1}{2\sin\phi}\left[-n\cos(n\phi) + \sum_{i=1}^{n}\cos[(2i-n)\phi]\right]. \tag{3.102}$$

In both (3.101) and (3.102) we are getting terms that grow with n. If either of these is allowed into our expansion (3.96), we will end up with secular terms. To make sure this does not happen, the functions A, θ will be chosen to prevent the $\cos(n\phi + \theta)$ and $\sin(n\phi + \theta)$ terms from appearing in the right-hand side of (3.100). One finds, after using a few trigonometric identities, that this requirement yields the following equations:

$$A' = 0$$

and

$$4A\theta'\sin\phi = 3\omega^2 A^3.$$

From this we get that A is constant and θ is linear in s.

Therefore, a first-term approximation of the solution of (3.95) for small ε, that holds for large n, is

$$y_n \sim A_0 \cos[(\phi + \varepsilon\alpha)n + \theta_0], \tag{3.103a}$$

where

$$\alpha = \frac{3\omega A_0^2}{4\sqrt{2-\omega^2}}, \tag{3.103b}$$

and A_0 and θ_0 are constants (independent of n and ε) and ϕ is defined in (3.99).

To compare our asymptotic approximation (3.103) with the exact solution of (3.95), suppose $y_0 = 0$, $y_1 = 1$, $\omega = 1/2$, and $\varepsilon = 0.01$. The numerical results, which are shown in Fig. 3.12, are clearly very close (at least over the interval shown).

Exercises

1. For small ε, find a first-term approximation of the solution of each of the following that holds for large n.

   a) $y_{n+1} = y_n + \varepsilon(y_n + y_{n+1})$. Also, compare the approximation with the exact solution.

   b) $y_{n+1} - 2y_n + y_{n-1} + 2\omega^2 y_n + \alpha\varepsilon(y_{n+1} - y_{n-1}) = 0$.

   c) $y_{n+1} - 2y_n + y_{n-1} + 2\omega^2(y_n + \varepsilon y_n^3) = \varepsilon\cos[(1+\varepsilon\omega_0)n]$.

2. This exercise concerns the derivation of the general solution of the second-order difference equation

$$y_{n+1} + by_n + cy_{n-1} = f_n \quad \text{for } n = 1, 2, 3, \cdots,$$

FIGURE 3.12. Graph of the exact solution of the difference equation in (3.95) and the asymptotic approximation of the solution given in (3.103).

where b and c are constants that satisfy $b^2 < 4c$.

a) By assuming $y_n = r^n$, show that the general solution of the associated homogeneous equation can be written as $y_n = A\rho^n \cos(n\phi + \theta)$ or as $y_n = \alpha\rho^n \cos(n\phi) + \beta\rho^n \sin(n\phi)$, where $\rho = \sqrt{c}$ and $\cos(\phi) = -b/(2\rho)$.

b) To find a particular solution, assume $y_n = a_n Y_n + b_n Z_n$, where Y_n and Z_n are linearly independent solutions of the associated homogeneous equation. From the difference equation and the imposed constraint $(a_{n+1} - a_n)Y_n + (b_{n+1} - b_n)Z_n = 0$, show that a particular solution is

$$y_n = \frac{1}{\sin\phi} \sum_{i=1}^{n-1} f_i \rho^{n-i-1} \sin[(n-i)\phi] \quad \text{for } n = 2, 3, 4, \cdots .$$

c) Let $w_n = A\rho^n \cos(n\phi + \theta)$, where ρ and θ are defined in part (a). Using the results from parts (a) and (b), comment on the n dependence of the particular solution in comparison to the n dependence of the associated homogeneous equation when (i) $f_n = kw_n + \kappa w_{n-1}$, where k and κ are constants, and (ii) $f_n = w_n^3$.

3. Consider the first-order difference equation

$$y_{n+1} = \alpha y_n + \varepsilon f(y_n, y_{n+1}), \quad \text{for } n = 0, 1, 2, 3, \cdots ,$$

where α is a nonzero constant.

a) Find a first-term approximation of the solution that holds for large n. In the derivation, you will need to assume that the function

$$g(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\alpha^j} f(\alpha^{j-1}x, \alpha^j x)$$

is well defined.

b) To find the numerical solution of the nonlinear differential equation

$$y'(t) = \lambda y + \varepsilon y, \quad \text{for } t > 0, \tag{3.104}$$

where λ is a nonzero constant, one can use Euler's method

$$y_{n+1} = (1 + h\lambda)y_n + \varepsilon h y_n, \tag{3.105}$$

or the backward Euler method

$$(1 - h\lambda)y_{n+1} = y_n + \varepsilon h y_{n+1}. \tag{3.106}$$

Here y_n is the finite-difference approximation of $y(t_n)$, where $t_n = nh$ for a given value of the stepsize h. What does the result from part

(a) reduce to for these two difference equations? Assume $1 \pm h\lambda \neq 0$ and $\lambda + \varepsilon \neq 0$.

c) Find the exact solution of (3.104) and of the two difference equations (3.105), (3.106).

d) Suppose $y(0) = 1$, $\lambda = -2$, $\varepsilon = 10^{-2}$, and $h = 10^{-1}$. On the same axes, plot the exact solution of (3.104), the exact solution of (3.105), and the asymptotic approximation of the solution of (3.105) for $0 \leq t \leq 2$. Also, plot them for $\lambda = -20$. Comment on the relative accuracy of the results.

e) Redo part (d), but use (3.106) instead of (3.105).

4. This problem considers the accuracy of various finite-difference approximations that can be used to solve the nonlinear differential equation numerically:

$$y''(t) + y + \varepsilon y^3 = 0, \quad \text{for } t > 0. \tag{3.107}$$

i) The first equation, which comes directly from using a centered-difference approximation of the derivative, is

$$y_{n+1} - 2y_n + y_{n-1} = -h^2(y_n + \varepsilon y_n^3). \tag{3.108}$$

This equation is basically the same as the one given in (3.95). Note that y_n is the finite-difference approximation of $y(t_n)$, where $t_n = nh$ for some value of the stepsize h.

ii) The second method, which uses the trapezoidal rule in its derivation, results in the difference equation

$$y_{n+1} - 2y_n + y_{n-1} = -\frac{h^2}{2}\left[y_{n+1} + y_{n-1} + \varepsilon\left(y_{n+1}^3 + y_{n-1}^3\right)\right]. \tag{3.109}$$

iii) The third method, which uses Simpson's rule in its derivation, results in the difference equation

$$y_{n+1} - 2y_n + y_{n-1} = -\frac{h^2}{6}\left[y_{n+1} + 4y_n + y_{n-1} + \varepsilon\left(y_{n+1}^3 + 4y_n^3 + y_{n-1}^3\right)\right]. \tag{3.110}$$

So, we have three second-order difference equations from which to pick to solve (3.107).

a) For small ε, find the first term in an expansion of the solution of (3.107) that is valid for large t.

b) For small ε, find first-term expansions of the solutions of (3.108), (3.109), and (3.110) that are valid for large n.

c) By comparing your results in parts (a) and (b), comment on the relative accuracy of the finite-difference approximations. Of interest

here is the solution when h is small. For example, does the phase cause the numerical solution to lead, or follow, the asymptotic solution? Also, what if one were to use the approximation given in (3.98) but the phase θ were found by adding together certain combinations of the phase determined from (i), (ii), and (iii)?

5. In the study of the structural changes in DNA, one finds a model for the motion of a harmonic lattice that involves localized oscillating nonlinear waves (Dauxois et al., 1992). These waves are associated with what are called "breathers." The problem is to find the function $u_n(t)$ that satisfies the equation

$$u_n'' + \omega_d^2\left(u_n - u_n^2\right) = u_{n+1} - 2u_n + u_{n-1}, \quad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

The initial conditions have the form $u_n(0) = \varepsilon f_n$ and $u_n'(0) = \varepsilon g_n$, where f_n and g_n are independent of ε.

a) For small ε, suppose one looks for traveling wave solutions using an expansion of the form

$$u_n \sim \varepsilon[A\cos(kn - \omega t) + \cdots], \tag{3.111}$$

where k and ω are positive constants. From the $O(\varepsilon)$ problem, find the relationship between ω and k so that the solution has the form assumed in (3.111). This equation for ω and k is the dispersion relation. Also, show that the form of expansion assumed in (3.111) leads to secular terms.

b) Use multiple scales to find a first-term expansion of the traveling wave solutions that is valid for large t. What form must f_n and g_n have to be able to generate this traveling wave solution?

6. In a certain problem from population genetics, one is interested in the proportion g_n of a population in the nth generation with a particular genetic trait. The Fisher-Wright-Haldane model for this is (Hoppensteadt, 1982)

$$g_{n+1} = \frac{r_n g_n^2 + s_n g_n(1-g_n)}{r_n g_n^2 + 2s_n g_n(1-g_n) + t_n(1-g_n)^2}, \quad \text{for } n = 1, 2, 3, \cdots.$$

Assume that $r_n = 1 + \varepsilon\rho f(n)$, $s_n = 1 + \varepsilon\sigma f(n)$, and $t_n = 1 + \varepsilon\tau f(n)$, where ρ, σ, and τ are constants and $f(n)$ is a positive function with period N (i.e., $f(n+N) = f(n)$). Also, because of the way g_n is defined, it is required that $0 \le g_n \le 1$. This will happen if the start-off value satisfies $0 \le g_1 \le 1$.

a) Using an expansion similar to the one in (3.96), find a first-term approximation of g_n for small ε that holds for large n. Assume that at least one of the two constants $\alpha = \sigma - \tau$, $\beta = \rho - 2\sigma + \tau$ is nonzero.

b) Using the result from part (a), find a first-term approximation for the possible steady states for g_n. Under what conditions on the constants α and β and the start-off value g_1 will the steady state g_∞ satisfy $0 < g_\infty < 1$?

4

The WKB and Related Methods

4.1 Introduction

In the method of matched asymptotic expansions studied in Chapter 2, the dependence of the solution on the boundary-layer coordinate was determined by solving the boundary-layer problem. In a similar way, when using multiple scales the dependence on the fast time scale was found by solving a differential equation. This does not happen with the WKB method because one begins with the assumption that the dependence is exponential. This is a reasonable expectation since many of the problems we studied in Chapter 2 ended up having an exponential dependence on the boundary-layer coordinate. Also, by making this assumption, we can significantly reduce the work necessary to find an asymptotic approximation of the solution.

The popularity of the WKB method can be traced back to the 1920s and the development of quantum mechanics. In particular, it was used to find approximate solutions of Schrödinger's equation. The name of the method is derived from three individuals who were part of this development, namely, Wentzel, Kramers, and Brillouin. However, as often happens in mathematics, the method was actually derived much earlier. Some refer to it as the method of Liouville and Green, who both published papers on the procedure in 1837. It appears that this too is historically incorrect, since in 1817 Carlini used a version of the approximation in studying elliptical orbits of planets. Given the multiple parenthood of the method, it should not be unexpected that there are other names it goes by, including the phase integral method, the WKBJ method (the J standing for Jefferys),

the geometrical acoustics approximation, and the geometrical optics approximation. The history of the method is surveyed very nicely by Heading (1962) and in somewhat more mathematical detail by Schlissel (1977b). A good, but dated, introduction to its application to quantum mechanics can be found in the book by Borowitz (1967) and to solid mechanics in the review article by Steele (1976). What all this means is that the WKB approximation is probably a very good idea since so many have rediscovered it in a wide variety of disciplines.

4.2 Introductory Example

In the same manner as was done for boundary layers and multiple scales, the ideas underlying the WKB method will be developed by using it to solve an example problem. The problem we begin with is the equation

$$\varepsilon^2 y'' - q(x)y = 0. \tag{4.1}$$

For the moment, we will not restrict the function $q(x)$ other than to assume that it is smooth. Our intention is to construct an approximation of the general solution of this equation. To motivate the approach that will be used, suppose that the coefficient q is constant. In this case, the general solution of (4.1) is

$$y(x) = a_0 e^{-\frac{1}{\varepsilon}x\sqrt{q}} + b_0 e^{\frac{1}{\varepsilon}x\sqrt{q}}. \tag{4.2}$$

The hypothesis made in the WKB method is that the exponential solution in (4.2) can be generalized to provide an approximate solution of (4.1). All that is necessary is to ensure the expansion is general enough so it can handle the variable coefficient in the equation. The specific assumption made when using the WKB method is that the asymptotic expansion of a solution is

$$y \sim e^{\theta(x)/\varepsilon^\alpha}\left[y_0(x) + \varepsilon^\alpha y_1(x) + \cdots\right]. \tag{4.3}$$

One of the distinctive features of the WKB method is that it is fairly specific on how the solution depends on the fast variation, namely, the dependence is assumed to be exponential. This limits the method, but it can make it easier to use than either the multiple-scale or boundary-layer methods.

From (4.3) we get

$$y' \sim \left(\varepsilon^{-\alpha}\theta' y_0 + y_0' + \theta' y_1 + \cdots\right) e^{\theta/\varepsilon^\alpha} \tag{4.4}$$

and

$$y'' \sim \left[\varepsilon^{-2\alpha}\theta_x^2 y_0 + \varepsilon^{-\alpha}\left(\theta_{xx} y_0 + 2\theta_x y_0' + \theta_x^2 y_1\right) + \cdots\right] e^{\theta/\varepsilon^\alpha}. \tag{4.5}$$

These are sufficient for the problem we are considering. However, we will occasionally be considering third- and fourth-order problems, and it is worth

having the formulas in one place so they are easy to find when the time comes. So, for the record, we also have

$$y''' \sim \left[\varepsilon^{-3\alpha}\theta_x^3 y_0 + \varepsilon^{-2\alpha}\theta_x\left(3\theta_x y_0' + 3\theta_{xx}y_0 + \theta_x^2 y_1\right) + \cdots\right] e^{\theta/\varepsilon^\alpha} \tag{4.6}$$

and

$$y^{(4)} \sim \left[\varepsilon^{-4\alpha}\theta_x^4 y_0 + \varepsilon^{-3\alpha}\theta_x^2\left(6\theta_{xx}y_0 + 4\theta_x y_0' + \theta_x^2 y_1\right) + \cdots\right] e^{\theta/\varepsilon^\alpha}. \tag{4.7}$$

The next step is to substitute (4.4) and (4.5) into (4.1). Doing this, one finds that

$$\begin{aligned}\varepsilon^2\left\{\frac{1}{\varepsilon^{2\alpha}}(\theta_x)^2 y_0 \;+\; \frac{1}{\varepsilon^\alpha}\left[\theta'' y_0 + 2\theta' y_0' + (\theta_x)^2 y_1\right] + \cdots\right\} \\ -\; q(x)(y_0 + \varepsilon^\alpha y_1 + \cdots) = 0.\end{aligned} \tag{4.8}$$

Something important happened in this step—the exponential dropped out. This would not have happened if the equation were nonlinear. This point will be brought up again after the example has been completed. Now, balancing the terms in (4.8), one finds that $\alpha = 1$, which leads to the following equation:

$O(1)$ $\;(\theta_x)^2 = q(x).$

This is called the *eikonal equation*, and its solutions are

$$\theta(x) = \pm\int^x \sqrt{q(s)}\; ds. \tag{4.9}$$

To obtain the first term in the expansion, we also need to find $y_0(x)$, which is accomplished by looking at the $O(\varepsilon)$ problem.

$O(\varepsilon)$ $\;\theta'' y_0 + 2\theta' y_0' + (\theta_x)^2 y_1 = q(x) y_1.$

This is the *transport equation*. Since $\theta(x)$ satisfies the eikonal equation, it follows that $\theta'' y_0 + 2\theta' y_0' = 0$. The solution of this is

$$y_0(x) = \frac{c}{\sqrt{\theta_x}}, \tag{4.10}$$

where c is an arbitrary constant.

We have therefore found that a first-term approximation of the general solution of (4.1) is

$$y \sim q(x)^{-1/4}\left(a_0 e^{-\frac{1}{\varepsilon}\int^x \sqrt{q(s)}ds} + b_0 e^{\frac{1}{\varepsilon}\int^x \sqrt{q(s)}ds}\right), \tag{4.11}$$

where a_0 and b_0 are arbitrary, possibly complex, constants. It is apparent from this that the function $q(x)$ must be nonzero. The values of x where

$q(x)$ is zero are called *turning points* and will be discussed in the next section (also see Exercises 5 and 11).

The easiest way to check the accuracy of the WKB approximation is to compare it with the exact solution. This is done in the following two examples.

Examples

1. Suppose $q(x) = -e^{2x}$. In this case (4.11) reduces to

$$\begin{aligned} y &\sim e^{-x/2}\left(a_0 e^{-ie^x/\varepsilon} + b_0 e^{ie^x/\varepsilon}\right) \\ &= e^{-x/2}\left[\alpha_0 \cos(\lambda e^x) + \beta_0 \sin(\lambda e^x)\right], \end{aligned} \tag{4.12}$$

where $\lambda = \varepsilon^{-1}$. To determine the constants in (4.12), suppose the boundary conditions are $y(0) = a$ and $y(1) = b$. One finds that the WKB approximation is then

$$y \sim e^{-x/2}\left(\frac{be^{1/2}\sin\lambda(e^x - 1) - a\sin\lambda(e^x - e)}{\sin\lambda(e-1)}\right). \tag{4.13}$$

For comparison, the exact solution is

$$y(x) = c_0 J_0(\lambda e^x) + c_1 Y_0(\lambda e^x), \tag{4.14a}$$

where

$$c_0 = \frac{1}{d}\left[bY_0(\lambda) - aY_0(\lambda e)\right] \text{ and } c_1 = \frac{1}{d}\left[aJ_0(\lambda e) - bJ_0(\lambda)\right]. \tag{4.14b}$$

Here, J_0 and Y_0 are Bessel functions and $d = J_0(\lambda e)Y_0(\lambda) - Y_0(\lambda e)J_0(\lambda)$. The values obtained from the WKB approximation and the exact solution are shown in Fig. 4.1. It is clear that they are in very good agreement.

2. Another use of the WKB method is to find approximations of large eigenvalues. To illustrate how this is done, suppose the problem is

$$y'' + \lambda^2 e^{2x} y = 0, \quad \text{for } 0 < x < 1, \tag{4.15a}$$

where

$$y(0) = y(1) = 0. \tag{4.15b}$$

The assumption that λ is large leads to the equation given in (4.1), where $\varepsilon = \frac{1}{\lambda}$. Using the WKB approximation given in (4.12), one finds that the zeros of the denominator in (4.13) determine the eigenvalues. Therefore, a first-term approximation of the large eigenvalues is

$$\lambda \sim \frac{n\pi}{e-1} \quad \text{for integer } n. \tag{4.16}$$

FIGURE 4.1. Comparison between the exact solution (4.14) and the WKB approximation given in (4.13). In the calculations, $\varepsilon = 10^{-1}$, $a = 1$, and $b = 0$. The two curves are so close they are essentially indistinguishable from each other.

The exact value of each eigenvalue is determined from the denominator in (4.14b), that is, the eigenvalues are solutions of the equation

$$J_0(\lambda e)Y_0(\lambda) - Y_0(\lambda e)J_0(\lambda) = 0. \tag{4.17}$$

To see just how well (4.16) does in approximating the eigenvalues, the relative error is plotted in Fig. 4.2 for the first twenty eigenvalues. Clearly, the approximation is very good and is much easier to use than (4.17). Moreover, the WKB approximation does well even for the first few eigenvalues.

Second Term of the Expansion

To obtain a measure of the error in using (4.11) to approximate the solution, we will calculate the second term in the expansion. This is not difficult; carrying out the calculation determines that the $O(\varepsilon^2)$ equation is

$$\theta'' y_1 + 2\theta' y_1' + (\theta_x)^2 y_2 + y_0'' = q(x) y_2.$$

FIGURE 4.2. Relative error, in absolute value, between the WKB approximation (4.16) of the eigenvalues and the values obtained by solving (4.17) numerically. Here $\varepsilon = 10^{-1}$.

Using the eikonal equation reduces this to

$$2\theta' y_1' + \theta'' y_1 = -y_0''.$$

Letting $y_1(x) = y_0(x)w(x)$, one finds from the previous equation that

$$w(x) = k + \frac{1}{4}\frac{\theta_{xx}}{(\theta_x)^2} + \frac{1}{8}\int^x \frac{(\theta_{xx})^2}{(\theta_x)^3}\,dx, \tag{4.18}$$

where k is an arbitrary constant.

Now, given a small value of ε, the expansion in (4.11) will be an accurate approximation of the solution if $\varepsilon y_1(x)$ is much smaller than $y_0(x)$. This can be guaranteed if $|\varepsilon w(x)| << 1$. From the formula for $w(x)$ given in (4.18), we can express this requirement in terms of $q(x)$ and its first derivative. In particular, for $x_0 \le x \le x_1$, we will have an accurate approximation if the following holds:

$$\varepsilon\left[|k| + \frac{1}{32}\left|\frac{q'}{q^{3/2}}\right|_\infty \left(4 + \int_{x_0}^{x_1}\left|\frac{q'}{q}\right| dx\right)\right] << 1, \tag{4.19}$$

where $|h(x)|_\infty \equiv \max_{x_0 \le x \le x_1} |h(x)|$. As might be expected, this shows that the expansion becomes nonuniform as a turning point is approached, that is, when one gets close to a point where $q(x) = 0$. An examination of these points is made in Exercise 5. For a more extensive investigation into the theory underlying the WKB method, readers should consult the book by Olver (1974).

The general solutions given in (4.11) and (4.18) contain constants that are determined from boundary conditions. This step is usually not thought provoking, but when using the WKB approximation we have to be a little more careful. The reason is that when attempting to satisfy the boundary conditions it is very possible that the constants depend on ε. It is therefore necessary to make sure this dependence does not interfere with the ordering assumed in the WKB expansion in (4.3). To illustrate this situation, consider the boundary-value problem

$$\varepsilon^2 y'' + e^{2\kappa x} y = 0, \quad \text{for } 0 < x < 1. \tag{4.20}$$

This is a generalization of the examples considered earlier (where $\kappa = 1$). Since $\theta_x = \pm i e^{\kappa x}$, we get, from (4.18),

$$w(x) = k \mp \frac{i\kappa}{8} e^{-\kappa x}. \tag{4.21}$$

A two-term expansion of the general solution of (4.20) is therefore

$$\begin{aligned} y \;\sim\; & e^{-\kappa x/2}\left(a_0 e^{-\frac{i}{\kappa\varepsilon}e^{\kappa x}}\left[1 + \varepsilon\left(a_1 - \frac{i\kappa}{8}e^{-\kappa x}\right)\right]\right. \\ & \left. + b_0 e^{\frac{i}{\kappa\varepsilon}e^{\kappa x}}\left[1 + \varepsilon\left(b_1 + \frac{i\kappa}{8}e^{-\kappa x}\right)\right]\right). \end{aligned} \tag{4.22}$$

One can use this to impose the boundary conditions. As in the example shown in Fig. 4.1, we take $\kappa = 1, y(0) = 1$, and $y(1) = 0$. One finds in this case that in (4.22)

$$a_1 = -\frac{\cos\lambda(e-1) - e^{-1-i(e-1)/\varepsilon}}{8\sin\lambda(e-1)},$$

where $\lambda = \frac{1}{\varepsilon}$. One also finds that b_1 is the complex conjugate of a_1. This shows that for this example the second term in the combined WKB expansion in (4.22) is small except at, or near, the resonant points where $\lambda(e-1) = n\pi$. In particular, if $\varepsilon = 0.1$, then one finds that the maximum value of $|a_1 - \frac{i\kappa}{8}e^{-\kappa x}|$ for $0 \le x \le 1$ is about 0.08. This explains why the curves in Fig. 4.1 are essentially indistinguishable from each other.

General Discussion

The most distinguished characteristic of the WKB approximation is the assumed exponential dependence on the fast variation. If we make this assumption, the first-term approximation of the solution is determined by solving two first-order differential equations. The first one, the eikonal equation, is usually first-order and nonlinear, and it determines the fast variation in the solution. It was not difficult to solve the eikonal equation for this example, but this can be one of the more difficult steps in using the WKB method. The next equation to solve is the transport equation, and this determines the slow variation. Fortunately, the transport equation is linear as are the equations for determining the higher-order terms in the expansion.

The WKB method is relatively easy to use and is capable of providing an approximation with less effort than other methods. However, with this advantage comes limitations. The most significant is the requirement that the equation be linear. This is needed to be able to cancel out the exponential term, as was done to obtain (4.8). For nonlinear problems, the method of multiple scales can be used and is sometimes referred to as the nonlinear WKB (or nonlinear geometric optics) method.

Exercises

1. Use the WKB method to find an approximate solution of the following:

   a) $\varepsilon y'' + 2y' + 2y = 0$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$. Compare your answer with (i) the composite expansion (2.16) obtained using matched asymptotic expansions, and with (ii) the exact solution.

   b) $\varepsilon^2 y'' + \varepsilon x y' - y = -1$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 3$. Compare your answer with the composite expansion obtained using matched asymptotic expansions.

c) $\varepsilon y'' + y' + e^x y = 0$, for $0 < x < 1$, where $y(0) = 0$ and $y(1) = 1$. Compare your answer with the composite expansion obtained using matched asymptotic expansions.

d) $\varepsilon y'' + \left(x + \frac{1}{2}\right) y' + y = 0$, for $0 < x < 1$, where $y(0) = 2$ and $y(1) = 3$.

e) $\varepsilon y^{(iv)} - e^{4x} y = 0$, for $0 < x < 1$, where $y(0) = y(1) = 1$ and $y''(0) = y''(1) = 0$.

2. Consider the following problem:

$$y'' + k^2(\varepsilon t) y = 0, \quad \text{for } 0 < t,$$

where $y(0) = a$ and $y'(0) = b$. Make the change of variables $\tau = \varepsilon t$, and then use the WKB method to construct a first-term approximation of the solution. Compare this result with the one obtained using multiple scales in (3.31).

3. This problem, and the one that follows, considers extensions of the WKB formula for the general solution given in (4.11).

a) Suppose, rather than Eq. (4.1), the equation is

$$\varepsilon^2 y'' - q(x, \varepsilon) y = 0.$$

Assuming $q(x, \varepsilon) \sim q_0(x) + \varepsilon q_1(x)$, show that

$$y \sim q_0(x)^{-1/4} \left(a_0 e^{-\frac{1}{\varepsilon}\kappa(x,\varepsilon)} + b_0 e^{\frac{1}{\varepsilon}\kappa(x,\varepsilon)} \right),$$

where

$$\kappa(x, \varepsilon) = \int^x \sqrt{q_0(s)} \left[1 + \frac{\varepsilon}{2} \frac{q_1(s)}{q_0(s)} \right] ds.$$

In this expansion, is it possible to replace κ with $K(x, \varepsilon) = \int^x \sqrt{q(s, \varepsilon)}\, ds$?

b) Consider the equation

$$\varepsilon^2 y'' + p(x) y' + q(x) y = 0.$$

Assuming $p(x) \neq 0$, show that the WKB approximation of the general solution is

$$y \sim a_0 e^{-\int^x (q/p) ds} + \frac{b_0}{p(x)} e^{\int^x (q/p) ds - \varepsilon^{-2} \int^x p(s) ds}.$$

c) Assuming that $q(x) > 0$, show that the WKB approximation of the general solution of $\varepsilon^2 y'' - q(x) y = f(x)$ is

$$y \sim \int^x G(x, s, \varepsilon) f(s) ds + y_h,$$

where y_h is given in (4.11) and

$$G(x,s,\varepsilon) = \frac{\sinh\left[\frac{1}{\varepsilon}\int_s^x \sqrt{q(r)}\,dr\right]}{\varepsilon[q(x)q(s)]^{1/4}}.$$

What is $G(x,s,\varepsilon)$ when $q(x) < 0$?

d) Find the WKB approximation of the general solution of the equation

$$\varepsilon^2 y'' + \varepsilon^2 p(x)y' - q(x)y = 0,$$

where $q(x)$ is nonzero.

4. Consider the equation

$$\varepsilon^2 y'' + P(x,\varepsilon)y' + Q(x,\varepsilon)y = 0.$$

a) Setting $y = we^{-\kappa}$, where $\kappa = \frac{1}{2}\varepsilon^{-2}\int^x P\,ds$, show that

$$\varepsilon^2 w'' - g(x,\varepsilon)w = 0,$$

where $g = -Q + \frac{1}{4}(2P' + \varepsilon^{-2}P^2)$.

b) Derive the expansions in Exercises 3(a) and 3(b) using (4.11) and the result from part (a) of this exercise.

c) Use the result from part (a) and Exercise 3(a) to find the WKB expansion of the solution of

$$\varepsilon^2 y'' + \varepsilon p(x)y' + q(x)y = 0.$$

5. This problem considers turning points and just how close you can get to one of them and expect an accurate approximation with the WKB formula.

a) In (4.1) suppose $q(x) = ax$, where $a > 0$ is constant. In this case, there is a simple turning point at $x = 0$. Find the first two terms in the WKB approximation, and then show that for the expansion to be well ordered it must be that $\varepsilon^{2/3} << |x|$.

b) In (4.1) suppose that $q(x) \sim q_0(x - x_0)^\alpha$ for x close to x_0, where $\alpha > 0$ and $q_0 \neq 0$. Show that the first-term WKB approximation holds for $\varepsilon^{2/(2+\alpha)} << |x - x_0|$.

6. Suppose that to find an approximate solution of (4.1) one first transforms the equation by letting $y(x) = e^{w(x)}$. Do this, assume $w \sim \varepsilon^{-\alpha}[w_0(x) + \varepsilon^\beta w_1 + \cdots]$, and then find a first-term approximation of the solution of (4.1). Compare this approach with the one based on (4.3). How appropriate is this approach when $y(x)$ is zero?

7. Consider the eigenvalue problem

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - r(x)y = -\lambda^2 q(x)y, \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. Here $p(x), q(x)$, and $r(x)$ are given, smooth, positive functions and $\lambda \geq 0$ is the eigenvalue.

a) Show that if one lets $y(x) = h(x)w(x)$ where $h(x) = [p(x)]^{-1/2}$ then

$$p(x)w'' + [\lambda^2 q(x) - f(x)]w = 0, \quad \text{for } 0 < x < 1,$$

where $f(x) = r - (ph')'/h$.

b) Use a first-term WKB approximation to show that for the large eigenvalues, $\lambda \sim \lambda_n$, where $\lambda_n = \frac{n\pi}{\kappa}$ and

$$\kappa = \int_0^1 \sqrt{\frac{q(x)}{p(x)}}\, dx.$$

What is the corresponding WKB approximation of the eigenfunctions?

c) To find the second term in the expansion of the eigenvalue, one starts with the expansion $\lambda \sim \lambda_n + \frac{1}{n}\lambda_c + \cdots$. Extending your argument in part (b), show that

$$\lambda_c = \frac{1}{2\pi}\int_0^1 [-g(x)g''(x) + (pq)^{-1/2}f(x)]dx,$$

where $g(x) = (p/q)^{1/4}$.

d) When $p = q = 1$, explain why $\lambda \sim \lambda_n$ gives an accurate approximation to all the eigenvalues as long as r_M is small compared to $2\pi^2$, where $r_M = \max_{0\leq x\leq 1} r(x)$.

e) To investigate the accuracy of the WKB result, consider the eigenvalue problem

$$y'' + \lambda^2 q(x)y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$ and $q(x) = \left(\frac{3\pi}{7}\right)^2(x+1)^4$. Calculate the two-term expansion for λ derived in parts (b) and (c), and then compare it with the numerical values found by Bender and Orszag (1978) given below.

λ_1^2	λ_2^2	λ_3^2	λ_4^2	λ_5^2	λ_{10}^2	λ_{20}^2
0.924915	3.89727	8.88444	15.8774	24.8732	99.8654	399.863

8. Find a first-term approximation of

$$\varepsilon^2 y'' + \varepsilon^{3/2} p(x) y' + q(x) y = 0, \quad \text{for } 0 < x < 1,$$

where $q(x)$ is nonzero.

(Hint: Follow the approach that was used to determine (4.3) using (4.2).)

9. The equation for the transverse displacement $u = u(x,t)$ of a nonhomogeneous Euler-Bernoulli beam is

$$\varepsilon \partial_x^2 [D(x) \partial_x^2 u] + \mu(x) \partial_t^2 u = 0, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u = u_{xx} = 0$ at $x = 0, 1$, $u(x,0) = f(x)$, and $u_t(x,0) = 0$. Here $D(x)$ and $\mu(x)$ are smooth, positive functions.

a) Find a first-term WKB approximation of the solution $F(x)$ of the eigenvalue problem

$$\varepsilon \frac{d^2}{dx^2} \left[D(x) \frac{d^2 F}{dx^2} \right] = \lambda^2 \mu(x) F, \quad \text{for } 0 < x < 1,$$

where $F = F'' = 0$ at $x = 0$ or 1. The positive constant λ is the eigenvalue (and it depends on ε).

b) Use the result from part (a) to obtain a first-term approximation of the solution of the beam problem.

10. Bessel's equation is $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$.

a) Find a two-term approximation of the general solution of this equation for $1 << \nu$. You can assume here that $a \le x \le b$, where a and b are positive constants.

b) From your result in (a), the series definition of the Bessel function $J_\nu(x)$, and the Stirling series

$$\Gamma(\nu) \sim \left(\frac{2\pi}{\nu} \right)^{1/2} \left(\frac{\nu}{e} \right)^\nu \left[1 + \frac{1}{12\nu} + \frac{1}{288\nu^2} + O(\nu^{-3}) \right], \quad \text{for } 1 << \nu,$$

find a two-term expansion of $J_\nu(x)$ for large ν.

c) Using the formula $\Gamma(\nu) = \pi \csc(\pi\nu)/\Gamma(1-\nu)$, find a two-term expansion of $J_{-\nu}(x)$ for $1 << \nu$ (assuming ν is not a positive integer). From this show that

$$Y_\nu(x) \sim -\left(\frac{2}{\nu\pi} \right)^{1/2} \left(\frac{2\nu}{ex} \right)^\nu \left[1 + \frac{1}{12\nu}(1 + 3x^2) \right], \quad \text{for } 1 << \nu.$$

11. Consider the initial-value problem

$$\varepsilon^2 y'' + e^{-2t} y = 0, \quad \text{for } 0 < t < \infty,$$

where $y(0) = 0$ and $y'(0) = 1$.

a) Find a first-term WKB approximation of the solution.

b) Determine the second term in the WKB expansion and then discuss the shortcomings of this approximation. Are any of these complications similar to what might be expected near a turning point?

c) Find the exact solution. On the same axes, plot the solution and the first-term expansion from part (a) for $0 \le t \le 10$ (take $\varepsilon = 10^{-2}$). Comment on the differences between the curves in conjunction with the results from part (b).

12. In studying the theory of collective ruin (Peters and Mangel, 1990), one comes across a variable $R(x)$, which is the probability of having resources available for emergencies (i.e., a risk reserve). It satisfies the integro-differential equation

$$\varepsilon\beta(x)R' - R + \lambda \int_0^{x/\varepsilon} R(x - \varepsilon y) e^{-\lambda y} dy = 0, \quad \text{for } 0 < x < \infty,$$

where $R(x) \to 1$ as $x \to \infty$. Also, λ is a positive constant and $\beta(x)$ is a smooth, positive function.

a) Find the exact solution when β is a positive constant. Explain why it's necessary to require $\lambda\beta > 1$.

b) Based on your observations from part (a), find the first term in a WKB expansion of the solution when β is not constant. What conditions must be imposed on λ and $\beta(x)$?

13. In the theory of multiserver queues with a large number of servers and short interarrival times, the following problem appears (Knessl, 1991): Find $P(x, \eta)$ satisfying the first-order differential equation

$$\partial_\eta P(x, \eta) = \mu_0(x + \varepsilon) P(x + \varepsilon, \eta) - g(x, \eta) P(x, \eta), \quad \text{for } 0 < \eta < \infty,$$

where

$$P(x + \varepsilon, 0) = \int_0^\infty \lambda(s) P(x, s) ds.$$

In this problem, $0 < x < 1$ and $g(x, \eta) = \mu_0 x + \lambda(\eta)$, where $\mu_0 > 0$ is constant and $\lambda(\eta) > 0$ is continuous with $\lambda \to \infty$ as $\eta \to \infty$. Find a first-term approximation of the solution of this problem for small ε.

4.3 Turning Points

We will introduce the analysis for turning points using the example from the previous section. The equation to solve is therefore

$$\varepsilon^2 y'' - q(x)y = 0. \tag{4.23}$$

As pointed out in deriving the WKB approximation, we must stay away from the points where $q(x)$ is zero. To explain how to deal with these points, we will initially consider the simplest case and then afterwards investigate how to extend the ideas to more complex situations. So, to begin, suppose there is a single turning point x_t where $q(x) > 0$ if $x > x_t$ and $q(x) < 0$ if $x < x_t$ (see Fig. 4.3). This means that the solution of (4.23) will be oscillatory if $x < x_t$ and exponential if $x > x_t$. The fact that the solution is oscillatory for negative $q(x)$ can be understood if one considers the constant coefficient equation $y'' + \lambda^2 y = 0$, where $\lambda > 0$. The general solution in this case is $y(x) = A_0 \cos(\lambda x + \phi_0)$, which is clearly oscillatory. A similar explanation can be given for the exponential solutions for a positive coefficient.

We can use the WKB approximation on either side of the turning point. This gives the following approximation for the general solution:

$$y \sim \begin{cases} y_L(x), & \text{for } x < x_t \\ y_R(x), & \text{for } x_t < x \end{cases}, \tag{4.24}$$

where

$$y_R = q(x)^{-1/4} \left(a_R e^{-\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds} + b_R e^{\frac{1}{\varepsilon}\int_{x_t}^{x} \sqrt{q(s)}\, ds} \right) \tag{4.25}$$

and

$$y_L = q(x)^{-1/4} \left(a_L e^{\frac{1}{\varepsilon}\int_{x}^{x_t} \sqrt{q(s)}\, ds} + b_L e^{-\frac{1}{\varepsilon}\int_{x}^{x_t} \sqrt{q(s)}\, ds} \right). \tag{4.26}$$

FIGURE 4.3. In the analysis of the turning point, $q(x)$ is assumed to have a simple zero at $x = x_t$ with $q'(x_t) > 0$. This will enable us to use a linear approximation for $q(x)$ near the turning point (see (4.28)).

These expressions come directly from (4.11) except that we have now fixed one of the endpoints in the integrals at the turning point. This particular choice is not mandatory, but it does simplify the formulas in the calculations to follow. It is important to recognize that the coefficients in (4.25) and (4.26) are not independent; we must find out how they are connected by investigating what is happening in a transition layer centered at $x = x_t$. This is very similar to what happened when we were studying interior-layer problems in Chapter 2. When we finish, the approximation of the general solution will only contain two arbitrary constants rather than four as in (4.24).

Solution in Transition Layer

To determine the solution near the turning point, we introduce the transition-layer coordinate

$$\overline{x} = \frac{x - x_t}{\varepsilon^\beta}. \tag{4.27}$$

We know the point x_t, but we will have to determine the value for β from the analysis to follow. Now, to transform the differential equation, we first use Taylor's theorem to conclude

$$q(x_t + \varepsilon^\beta \overline{x}) \sim q(x_t) + \varepsilon^\beta \overline{x} q'(x_t) + \cdots . \tag{4.28}$$

We will assume $q(x)$ has a simple zero at x_t, and so $q'(x_t) \neq 0$. With this, and letting $Y(\overline{x})$ denote the solution in this layer, we get from (4.23) that

$$\varepsilon^{2-2\beta} Y'' - (\varepsilon^\beta \overline{x} q_t' + \cdots) Y = 0, \tag{4.29}$$

where $q_t' = q'(x_t)$. For balancing, we need $2 - 2\beta = \beta$, and thus $\beta = \frac{2}{3}$.

The appropriate expansion of the solution in this region is

$$Y \sim \varepsilon^\gamma Y_0(\overline{x}) + \cdots . \tag{4.30}$$

Introducing this into (4.29), we get the following equation to solve:

$$Y_0'' - \overline{x} q_t' Y_0 = 0, \quad \text{for } -\infty < \overline{x} < \infty. \tag{4.31}$$

This can be transformed into an equation whose solutions are known. Letting $s = (q_t')^{1/3} \overline{x}$, we get *Airy's equation*, which is

$$\frac{d^2}{ds^2} Y_0 - s Y_0 = 0, \quad \text{for } -\infty < s < \infty. \tag{4.32}$$

Because of its importance in applied mathematics, Airy's equation has been studied extensively. It can be solved using power series expansions or the Laplace transform. One finds that the general solution can be written as

$$Y_0 = a Ai(s) + b Bi(s),$$

where $Ai(\cdot)$ and $Bi(\cdot)$ are Airy functions of the first and second kinds, respectively. The definitions and some of the properties of these functions are given in Appendix A1. We are now able to write the general solution of the transition-layer equation (4.31) as

$$Y_0(\overline{x}) = aAi\left[(q_t')^{1/3}\overline{x}\right] + bBi\left[(q_t')^{1/3}\overline{x}\right]. \tag{4.33}$$

From (4.25), (4.26), and (4.33) we have six undetermined constants. However, the solution in (4.33) must match with the outer solutions in (4.25), (4.26). This will lead to connection formulas between the constants, and these will result in two arbitrary constants in the general solution.

Matching

The solution in the transition region must match with the outer solutions given in (4.25) and (4.26). In what follows, keep in mind that $q_t' > 0$. To do the matching, we will use the intermediate variable

$$x_\eta = \frac{x - x_t}{\varepsilon^\eta}, \tag{4.34}$$

where $0 < \eta < 2/3$. Before matching the solutions, note that the terms in the two outer solutions contain the following: For $x > x_t$

$$\begin{aligned} \int_{x_t}^{x} \sqrt{q(s)}\, ds &= \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{q(s)}\, ds \sim \int_{x_t}^{x_t+\varepsilon^\eta x_\eta} \sqrt{(s-x_t)q_t'}\, ds \\ &= \frac{2}{3}\varepsilon r^{3/2} \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} q(x)^{-1/4} &\sim [q_t + (x - x_t)q_t']^{-1/4} \\ &= \varepsilon^{-1/6}(q_t')^{-1/6} r^{-1/4}, \end{aligned} \tag{4.36}$$

where $r = (q_t')^{1/3}\varepsilon^{\eta-2/3}x_\eta$.

Matching for $x > x_t$:

Using the asymptotic expansions for the Airy functions given in Appendix A1, one finds that

$$\begin{aligned} Y &\sim \varepsilon^\gamma Y_0\left(\varepsilon^{\eta-2/3}x_\eta\right) + \cdots \\ &\sim \frac{a\varepsilon^\gamma}{2\sqrt{\pi}r^{1/4}} e^{-\frac{2}{3}r^{3/2}} + \frac{b\varepsilon^\gamma}{\sqrt{\pi}r^{1/4}} e^{\frac{2}{3}r^{3/2}}, \end{aligned} \tag{4.37}$$

and for the WKB solution in (4.25),

$$y_R \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}r^{1/4}}\left(a_R e^{-\frac{2}{3}r^{3/2}} + b_R e^{\frac{2}{3}r^{3/2}}\right). \tag{4.38}$$

For these expressions to match, we must have $\gamma = -\frac{1}{6}$,

$$a_R = \frac{a}{2\sqrt{\pi}}(q_t')^{1/6} \text{ and } b_R = \frac{b}{\sqrt{\pi}}(q_t')^{1/6}. \tag{4.39}$$

Matching for $x < x_t$:

The difference with this case is that $x_\eta < 0$, which introduces complex numbers into (4.26). As before, using the expansions for the Airy functions given in the Appendix A1,

$$\begin{aligned} Y &\sim \varepsilon^\gamma Y_0\left(\varepsilon^{\eta-2/3}x_\eta\right) \\ &\sim \frac{a\varepsilon^\gamma}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2} - \frac{\pi}{4}\right) + \frac{b\varepsilon^\gamma}{\sqrt{\pi}|r|^{1/4}}\cos\left(\frac{2}{3}|r|^{3/2} + \frac{\pi}{4}\right) \\ &= \frac{\varepsilon^\gamma}{2\sqrt{\pi}|r|^{1/4}}\Bigg((ae^{-i\pi/4} + be^{i\pi/4})e^{i\frac{2}{3}|r|^{3/2}} \\ &\quad + (ae^{i\pi/4} + be^{-i\pi/4})e^{-i\frac{2}{3}|r|^{3/2}}\Bigg). \end{aligned} \tag{4.40}$$

In the last step leading to (4.40), the identity $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ was used. As for the WKB expansion, from (4.26),

$$y_L \sim \frac{\varepsilon^{-1/6}}{(q_t')^{1/6}|r|^{1/4}}\left(a_L e^{i\left(\frac{2}{3}|r|^{3/2} - \frac{\pi}{4}\right)} + b_L e^{-i\left(\frac{2}{3}|r|^{3/2} + \frac{\pi}{4}\right)}\right). \tag{4.41}$$

Matching (4.40) and (4.41) yields the following:

$$a_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(a + ib) \tag{4.42}$$

and

$$b_L = \frac{(q_t')^{1/6}}{2\sqrt{\pi}}(ia + b) = i\bar{a}_L, \tag{4.43}$$

where $\bar{a}_L$ is the complex conjugate of a_L. Equations (4.39), (4.42), and (4.43) are known as *connection formulas*. They constitute a system of equations that enable us to solve for four of the constants in terms of the remaining two.

Summary

Solving the connection formulas in (4.39), (4.42) and (4.43) shows that the WKB approximation of the solution of (4.23) is

$$y(x) \sim \begin{cases} \dfrac{1}{|q(x)|^{1/4}} \left(2a_R \cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right) + b_R \cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right)\right), & \text{for } x < x_t \\ \dfrac{1}{q(x)^{1/4}} \left(a_R e^{-\frac{1}{\varepsilon}\kappa(x)} + b_R e^{\frac{1}{\varepsilon}\kappa(x)}\right), & \text{for } x_t < x, \end{cases} \tag{4.44}$$

where

$$\theta(x) = \int_x^{x_t} \sqrt{|q(s)|}\, ds \tag{4.45}$$

and

$$\kappa(x) = \int_{x_t}^{x} \sqrt{q(s)}\, ds. \tag{4.46}$$

It should be remembered that this expansion was derived under the assumption that $x = x_t$ is a simple turning point with $q'(x_t) > 0$. The accuracy of this approximation near the turning point depends on the specific problem. However, one can show that in general one must require $\varepsilon^{2/3} << |x - x_t|$ (see Exercise 5 of Section 4.2). Also, as expected, we have ended up with an expansion for the solution of (4.23) that contains two arbitrary constants (a_R, b_R).

Example

As an example of a turning point problem, consider the following:

$$\varepsilon^2 y'' = x(2-x)y, \quad \text{for } -1 < x < 1, \tag{4.47}$$

where $y(-1) = y(1) = 1$. In this case $q(x) = x(2-x)$, and so there is a simple turning point at $x = 0$ with $q'(0) = 2$. The WKB approximation in (4.44) therefore applies, where, from (4.45) and (4.46),

$$\theta(x) = \frac{1}{2}(1-x)(x^2-2x)^{1/2} - \frac{1}{2}\ln[1 - x + (x^2-2x)^{1/2}]$$

and

$$\kappa(x) = \frac{1}{2}(x-1)(2x-x^2)^{1/2} + \frac{1}{2}\arcsin(x-1) + \frac{\pi}{4}.$$

The constants a_R and b_R are found from the two boundary conditions. For example, since $q(1) = 1$ and $\kappa(1) = \pi/4$, then the condition $y(1) = 1$ leads to the equation $a_R \exp\left(-\frac{\pi}{4\varepsilon}\right) + b_R \exp\left(\frac{\pi}{4\varepsilon}\right) = 1$. The resulting approximation obtained from (4.44) is shown in Fig. 4.4(a) along with the approximation from the transition layer. The singularity in y_R and y_L at

FIGURE 4.4. (a) The transition layer and WKB approximations given in (4.30) and (4.44), respectively, are shown. (b) A comparison between the transition layer and WKB approximations given in (4.30) and (4.44), respectively, and the numerical solution of (4.47). In these calculations, $\varepsilon = \frac{1}{25}$.

$x = 0$ is evident in this plot as is the matching by the Airy functions from the transition layer. To demonstrate the accuracy of the WKB approximation even when a turning point is present, in Fig. 4.4(b) the numerical solution is plotted along with the curves from Fig. 4.4(a). Within their respective domains of applicability, the WBK approximations are essentially indistinguishable from the numerical solution.

One can use (4.44) in a straightforward manner to determine the expansion when $x = x_t$ is a simple turning point with $q'(x_t) < 0$. Carrying out the details, it is found in this case that (see Exercise 1)

$$y(x) \sim \begin{cases} \dfrac{1}{q(x)^{1/4}} \left(\alpha_R e^{-\frac{1}{\varepsilon}\kappa(x)} + \beta_R e^{\frac{1}{\varepsilon}\kappa(x)}\right), & \text{for } x < x_t \\[2ex] \dfrac{1}{|q(x)|^{1/4}} \left(2\alpha_R \cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right) + \beta_R \cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right)\right), & \text{for } x_t < x, \end{cases} \tag{4.48}$$

where $\theta(x)$ and $\kappa(x)$ are given in (4.45) and (4.46). Again, we have a solution that contains two arbitrary constants (α_R, β_R).

We have investigated the case of a single first-order turning point located at $x = x_t$. The solution is in three pieces: the two WKB approximations (y_L, y_R) on either side of the turning point and the transition-layer solution in between. The first-term approximation of the solution in the transition

layer was found to be a solution of Airy's equation (4.31), which is the prototype equation for a simple turning point. In fact, it is possible to find a composite first-term expansion of the solution in terms of Airy functions. This was first derived by Langer (1931); the result is

$$y(x) \sim \frac{\kappa(x)^{1/6}}{|q(x)|^{1/4}} \left[a_0 Ai\left(\left(\frac{3\kappa}{2\varepsilon} \right)^{2/3} \right) + b_0 Bi\left(\left(\frac{3\kappa}{2\varepsilon} \right)^{2/3} \right) \right], \tag{4.49}$$

where $\kappa(x)$ is given in (4.46). The derivation of (4.49) is outlined in Exercise 10. Also, in connection with the WKB approximations given in (4.44),

$$b_R = \frac{a_0}{2\sqrt{\pi}} \left(\frac{2\varepsilon}{3} \right)^{1/6} \text{ and } a_R = \frac{b_0}{\sqrt{\pi}} \left(\frac{2\varepsilon}{3} \right)^{1/6}.$$

Because of the value of having a composite expansion, there have been numerous generalizations of Langer's result. A discussion of some of these can be found in Nayfeh (1973).

In referring to the turning point as simple, it is meant that $q(x)$ has a simple zero at $x = x_t$ (i.e., $q(x_t) = 0$ but $q'(x_t) \neq 0$). Higher-order turning points do arise; for example, a second-order turning point, at $x = 0$, is

$$\varepsilon^2 y'' - x^2 e^x y = 0. \tag{4.50}$$

The reason this is second order is simply that $q(x_t) = q'(x_t) = 0$ but $q''(x_t) \neq 0$. It is also possible to have one of fractional order (e.g., when $q(x) = x^{1/3} e^x$). The prototype equations in these cases are discussed in Appendix A1. Other types do occur, such as logarithmic, although they generally are harder to analyze.

Another complication that can arise is that the position of the turning point may depend on ε. This can lead to *coalescing turning points*; an example of this is found in the equation

$$\varepsilon^2 y'' - (x - \varepsilon)(x + \varepsilon) y = 0. \tag{4.51}$$

This has simple turning points at $x_t = \pm\varepsilon$, but to the first-term they look like one of second order at $x = 0$. Situations such as this are discussed in Steele (1976).

Examples

1. Suppose the problem is

$$\varepsilon^2 y'' + \sin(x) y = 0, \quad \text{for } 0 < x < 2\pi,$$

where $y(0) = a$ and $y(2\pi) = b$. This has three turning points: $x_t = 0$, π, 2π. Because two of these are endpoints, there will be two outer

WKB approximations, one for $0 < x < \pi$ and one for $\pi < x < 2\pi$. Since $\frac{d}{dx}\sin(x) \neq 0$ at $x = \pi$, then the turning point $x_t = \pi$ is simple and is treated in much the same way as the one analyzed above. The solution in the transition layer at $x_t = 0$ will be required to satisfy the boundary condition $y(0) = a$, and the solution in the layer at $x_t = 2\pi$ will satisfy $y(2\pi) = b$.

2. The equation
$$\varepsilon^2 y'' + p(x)y' + q(x)y = 0$$
differs from the one considered above because of the first derivative term. The WKB approximation of the general solution is given in Exercise 3 of Section 4.2. Unlike before, the turning points now occur when $p(x) = 0$.

Exercises

1. By making the change of variable $z = x_t - x$ in (4.23), derive (4.48) from (4.44).

2. Consider the boundary-value problem
$$\varepsilon^2 y'' + x(x+3)^2 y = 0, \quad \text{for } a < x < b,$$
where $y(a) = \alpha$ and $y(b) = \beta$. Find a first-term WKB expansion of the solution for

   a) $a = 0$, $\alpha = 0$, $b = 1$, $\beta = 1$.

   b) $a = -1$, $\alpha = 1$, $b = 0$, $\beta = 0$.

   c) $a = -1$, $\alpha = 0$, $b = 1$, $\beta = 1$.

3. In quantum mechanics one has to solve the time-independent Schrödinger equation given as
$$\varepsilon^2 \psi'' - [V(x) - E]\psi = 0, \quad \text{for } -\infty < x < \infty, \tag{4.52}$$
where the wave function $\psi(x)$ is required to be bounded. Also, the potential $V(x)$ is a given, smooth function, while the energy E is a nonnegative constant that must be solved for along with the wave function.

   a) Using (4.44) and (4.48), find a first-term approximation of the solution in the case of the simple potential well as shown in Fig. 4.5. From this conclude that for there to be a nonzero solution E must satisfy
$$\int_a^b \sqrt{E - V(x)}\, dx = \varepsilon\pi\left(n + \frac{1}{2}\right), \quad \text{for } n = 0, 1, 2, 3, \ldots, . \tag{4.53}$$

FIGURE 4.5. Potential used in Exercise 3.

This is known as the WKB quantization condition.

b) For a harmonic oscillator, the potential $V(x) = \mu x^2$ where $\mu > 0$ is constant. Solve the quantization condition (4.53) in this case. Also describe, in terms of nodes and wavelengths, what the integer n signifies for the solution in the region $a < x < b$. What does n signify for the solution outside this region? It is worth pointing out that for this potential the WKB quantization condition gives the exact values for E.

c) In deriving the WKB solution in part (a), it was assumed that a, b, and E are independent of ε. However, the quantization condition (4.53) means they should depend on ε for there to be a nontrivial solution. How does this dependence affect the derivation of the WKB approximation? Also, what effect, if any, does the dependence of E on the integer n (which may be quite large) have on the accuracy of the approximation?

d) When $V(x) = \mu x^m$, where $m = 2, 4, 6, \ldots$, one can show that $E^\gamma = \mu \varepsilon^m \chi$, where $\gamma = 1 + 2/m$ and χ satisfies the equation

$$\chi^{1/2} \sum_{k=0}^{\infty} a_k \chi^{-k} = 2n + 1, \tag{4.54}$$

where a_k is independent of χ (Dunham, 1932; Bender et al., 1977). For example,

$$a_0 = \frac{2}{\sqrt{\pi}} \frac{\Gamma(1 + 1/m)}{\Gamma(3/2 + 1/m)} \text{ and } a_1 = -\frac{m-1}{6\sqrt{\pi}} \frac{\cot(\pi/m)\Gamma(1/2 + 1/m)}{\Gamma(1/m)}.$$

Setting $N = 2n + 1$ and assuming $m \neq 2$, use (4.54) to find a two-term expansion of E for $N >> 1$. How does the first term compare with your result from part (a)?

e) Below are the numerical values of $E/\varepsilon^{m/\gamma}$ given by Secrest et al. (1962) for $V(x) = x^m$ for various values of m and n. Compare the results from parts (a) and (d) with these values. Make sure to comment on the accuracy as a function of n.

FIGURE 4.6. Potential used in Exercise 4.

$m\backslash n$	0	2	4
4	1.0603621	7.4556979	16.261826

$m\backslash n$	0	2	4
8	1.2258201	10.244947	25.809007

4. This problem concerns the Schrödinger equation in (4.52) and the potential shown in Fig. 4.6. Because $V(x)$ acts like a barrier, this example is commonly used to illustrate the phenomenon of tunneling.

   a) Using (4.44) and (4.48), find a first-term approximation of the solution of (4.52) in the case of the barrier potential shown in Fig. 4.6. Assume E is given and satisfies $0 < E < V_M$, where $V_M = \max V(x)$ and $V(-\infty) = V(\infty) = 0$.

   b) The time-dependent Schrödinger equation has the form

$$-\varepsilon^2 \partial_x^2 \Psi + V\Psi = i\partial_t \Psi.$$

   In regard to the barrier potential, suppose a wave approaches from the left. Because of reflection from the barrier, the solution should then consist of left and right traveling waves in the region $x < a$. However, part of the incident wave will be transmitted through the barrier and result in a right running wave for $x > b$ (this is the phenomenon of tunneling). Assuming $\Psi = \exp(-iEt)\psi(x)$, use the results from part (a) to find first-term approximations of the waves in these two regions.

5. The motion of planetary rings is described using the theory of self-gravitating annuli orbiting a central mass. For circular motion in the plane, with the planet at the origin, one ends up having to find the circumferential velocity $V(r, \theta, t) = v(r)e^{i(\omega t + m\theta)}$. The function $v(r)$ satisfies (Papaloizou and Pringle, 1987)

$$\frac{d}{dr}\left(r\frac{d}{dr}(rv)\right) = m^2(1 - \kappa^2 r^2)v, \quad \text{for } 0 < r < \infty,$$

FIGURE 4.7. Function used in Exercise 7.

where

$$\kappa^2 = \left(\frac{\alpha + \beta m}{m}\right)^2.$$

Here r is the radial coordinate and α and β are positive constants. The parameter m is positive and is a mode number. Find a first-term approximation of the solution for large m.

6. Find a first-term approximation of the solution of

$$\varepsilon^2 y'' + xy' - x(1+x)y = 0, \quad \text{for } -1 < x < 1,$$

where $y(-1) = 1$ and $y(1) = 3e^2$.

(Hint: For large r, $\displaystyle\int_0^r e^{-\alpha s^2} ds \sim \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} - \frac{1}{2\alpha r} e^{-\alpha r^2}$.)

7. Find a first-term approximation of the solution of $\varepsilon^2 y'' - q(x)y = 0$ for $-\infty < x < \infty$, where $q(x)$ is the even function shown in Fig. 4.7. Thus, $q(0) = q'(0) = 0$ but $q''(0) \neq 0$, and $q(\pm a) = 0$ with $q'(\pm a) \neq 0$.

8. Consider the problem

$$\varepsilon y'' + x^2 y' + y = 0, \quad \text{for } -1 < x < 1,$$

where $y(-1) = 1$ and $y(1) = -1$.

a) The WKB/transition-layer analysis based on the coordinate transformation (4.27) does not work on this problem. Explain why. Also explain why failure should be expected based on the balancing in the transition layer and its relationship to the terms responsible for the singularity in the WKB expansion.

b) Letting $r = (x + \varepsilon^\beta x_0)/\varepsilon^\alpha$, find the transition regions that are needed to complete the approximations. Also, state the equations to be solved in each region and the matching conditions to be imposed.

c) Show that it is possible to resolve the difficulty by making the substitution

$$y(x) = u(x)e^{-\frac{1}{6\varepsilon}x^3}.$$

9. For Schrödinger's equation it has been observed that in certain cases it is possible to improve on the WKB quantization condition given in (4.53). In the SWKB method, one introduces what is known as a supersymmetric potential $W(x) \equiv -\varepsilon\psi_0'/\psi_0$, where $\psi_0(x)$ is an eigenfunction corresponding to the lowest energy E_0. Under certain restrictions, it is then possible to transform (4.53) into the following equation:

$$\int_a^b \sqrt{E^* - W^2}\, dx = \varepsilon\pi n, \quad \text{for } n = 0, 1, 2, 3, \ldots, \tag{4.55}$$

where $E^* = E - E_0$. It is significant, and somewhat surprising, that for many of the cases when the problem can be solved in closed form the condition in (4.55) actually yields the exact value for E (Comtet et al., 1985; Dutt et al., 1991). In comparison, (4.53) has been found to be exact for very few cases.

a) Show that $W(x)$ satisfies the equation $\varepsilon W' - W^2 = -V^*$, where $V^*(x) = V(x) - E_0$. From this and the WKB quantization condition (4.53), derive (4.55). To do this, remember that a and b are turning points and assume that $W(a) = -W(b)$.

b) As another approach, introduce the supersymmetric potential into Schrödinger's equation (4.52) and then derive a first-term approximation of the solution. With this, obtain the SWKB quantization condition (4.55).

10. This problem concerns the derivation of the uniformly valid approximation in (4.49).

a) Change variables by letting $s = \varepsilon^{-\alpha} f(x)$ and $y(x) = \Phi(x)Y(s)$. Show that by taking $\Phi(x) = (f')^{-1/2}$ and $f(x) = (3\kappa/2)^{2/3}$, where $\kappa(x)$ is given in (4.46), that (4.1) can be transformed into an equation of the form $Y'' = sY + G(s, \varepsilon)$.

b) From the result in part (a), derive (4.49).

c) Show that (4.49) reduces to the approximation in (4.44) when $x >> x_t$ or when $x_t << x$. Does it reduce to (4.33) when $x \approx x_t$?

11. In the semiclassical description for what are called "shape resonances" (Combes et al., 1984), one finds the eigenvalue problem

$$\varepsilon^2\psi'' - [V(x) - E]\psi = 0, \quad \text{for } 0 < x < \infty,$$

where $\psi(0) = \psi(\infty) = 0$. The potential here is $V(x) = (x-1)^2 \exp(-x^2/4)$. Find a first-term approximation of the solution, and from this derive the quantization condition.

4.4 Wave Propagation and Energy Methods

The WKB method is quite useful for finding an asymptotic approximation of a traveling wave solution of a partial differential equation. To illustrate this, we consider the problem

$$u_{xx} = \mu^2(x)u_{tt} + \alpha(x)u_t + \beta(x)u, \quad \text{for} \begin{cases} 0 < x < \infty \\ 0 < t \end{cases}, \tag{4.56a}$$

where

$$u(0,t) = \cos(\omega t). \tag{4.56b}$$

This is the equation for the displacement of a string that is damped (αu_t) and has an elastic support (βu). Because the string is being forced periodically at the left end, the solution will develop into a wave that travels to the right. To find this solution, we will set $u(x,t) = e^{i\omega t}v(x)$ and then require that the function $v(x)$ be consistent with the observation that the wave moves to the right. Also, for the record, it is assumed that the functions in (4.56a) are smooth with $\mu(x) > 0$ and $\alpha(x)$ and $\beta(x)$ nonnegative.

The equation is linear, but there is no obvious small parameter ε that can serve as the basis of the WKB approximation. To motivate what will be done, suppose $\alpha = \beta = 0$ and μ is constant. In this case, the plane wave solutions of (4.56a) are

$$u(x,t) = e^{i(\omega t - kx)},$$

where $k = \pm\omega\mu$. This shows that for high frequencies the waves have a relatively short wavelength (that is, $|\frac{2\pi}{k}| << 1$). This observation will serve as the basis of our approximation, and in such a circumstance the small parameter is $\varepsilon = \frac{1}{\omega}$. This assumption is what underlies the application of the WKB method to many problems in optics, electromagnetics, and acoustics. For example, in the case of ultrasound, the waves can have frequencies on the order of 10^6 Hz with wavelengths of about 3×10^{-2} m (in air). Even more extreme are the waves for visible light, where the frequencies are in the neighborhood of 10^{15} Hz and the wavelengths are about 5×10^{-7} m.

Based on the above discussion, we will construct an asymptotic approximation of the traveling wave solution of (4.56) in the case of a high frequency. This is done by assuming

$$u(x,t) \sim e^{i[\omega t - \omega^\gamma \theta(x)]}\left[u_0(x) + \frac{1}{\omega^\gamma}u_1(x) + \cdots\right]. \tag{4.57}$$

Substituting this into (4.56a), we obtain

$$\begin{aligned} -\omega^{2\gamma}\theta_x^2(u_0 + \omega^{-\gamma}u_1 + \cdots) + i\omega^\gamma\theta_x(u_0' + \cdots) + \tfrac{d}{dx}(i\omega^\gamma\theta_x u_0 + \cdots) \\ = -\mu^2\omega^2(u_0 + \omega^{-\gamma}u_1 + \cdots) - i\omega\alpha(u_0 + \cdots) + \beta(u_0 + \cdots). \end{aligned}$$

In writing down the above equation, we need to write explicitly only those terms that might contribute to the first term in the expansion. Balancing

the first terms on each side of this equation yields $\gamma = 1$. Thus, we have the following problems:

$O(1)$ $(\theta_x)^2 = \mu^2(x)$.

The solutions of this eikonal equation are $\theta = \pm \int_0^x \mu(s)ds$.

$O(\frac{1}{\omega})$ $\theta'' u_0 + 2\theta' \partial_x u_0 = -\alpha u_0$.

The solution of this transport equation, for $\theta_x = \mu(x)$, is

$$u_0(x) = \frac{a_0}{\sqrt{\mu}} \exp\left(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)} ds\right).$$

Putting the results together and imposing boundary condition (4.56b), we have that the wave, which propagates to the right, has the expansion

$$u(x,t) \sim \sqrt{\frac{\mu(0)}{\mu(x)}} \exp\left(-\frac{1}{2}\int_0^x \frac{\alpha(s)}{\mu(s)} ds\right) \cos\left(\omega t - \omega \int_0^x \mu(s)ds\right). \quad (4.58)$$

Our solution is a traveling wave with an amplitude and phase that depend on x. The damping, as might be expected, causes the wave to be attenuated, while the coefficient $\mu(x)$ contributes both to the phase and amplitude. Interestingly, the approximation is independent of the coefficient $\beta(x)$ in (4.56a).

Energy Methods

To connect what we have done with the energy methods that are also used to study wave problems, we first need to determine an energy equation associated with (4.56a). One way to do this is to multiply (4.56a) by the velocity u_t (also see Exercise 5). Doing this and collecting terms, one obtains an equation of the form

$$\partial_t E + \partial_x S = -\Phi, \quad (4.59)$$

where $E = \frac{1}{2}\mu^2(u_t)^2 + \frac{1}{2}(u_x)^2 + \frac{1}{2}\beta u^2$, $S = -u_t u_x$, and $\Phi = \alpha(u_t)^2$. As is always the case, the problem we are considering has been nondimensionalized. However, if we were to convert back to dimensional coordinates, we would find that E corresponds to the sum of the kinetic and potential energy densities, S is the energy flux, and $\Phi = \alpha(u_t)^2$ is the rate of energy dissipation (or, simply, the dissipation function). We will refer to E, S, and Φ in these physical terms, although it is understood that they are actually nondimensional.

Wave solutions are of particular importance in most areas of application. Consequently, numerous methods have been devised to find approximations

of them. Of interest here are those based on using averages of the energy. The assumption made is that for a high frequency

$$u(x,t) \sim A(x)\cos[\omega t - \varphi(x)], \tag{4.60}$$

where the amplitude A is slowly varying in comparison to the change in the phase φ (i.e., the wave is slowly modulated). It should be noted that the phase velocity v_{ph} for this wave is

$$v_{\text{ph}} \equiv \frac{\omega}{\varphi_x}. \tag{4.61}$$

With this, curves $x = \overline{x}(t)$ in the (x,t)-plane that satisfy $\overline{x}' = v_{\text{ph}}$ are called phase lines. To help in understanding the next result, recall that, from the WKB expansion, the wavelength of the fast variation is $O(\omega^{-1})$.

Claim 1: The eikonal equation for (4.56a) is equivalent to the requirement that the total energy remain constant (to the first term) between any two phase lines that are $O(\omega^{-1})$ apart.

The total energy $E_{\text{tot}}(t)$ between two phase lines is

$$E_{\text{tot}} = \int_{\overline{x}_1(t)}^{\overline{x}_2(t)} E(x,t)dx, \tag{4.62}$$

where $\overline{x}'_1 = \overline{x}'_2 = v_{\text{ph}}$ (and $\overline{x}_1 < \overline{x}_2$). With the assumption in (4.60), we have

$$E \sim \frac{1}{4}A^2(\mu^2\omega^2 + \varphi_x^2)[1 - \cos 2(\omega t - \varphi)], \tag{4.63}$$

$$S \sim \frac{1}{2}\omega\varphi_x A^2[1 - \cos 2(\omega t - \varphi)], \tag{4.64}$$

and

$$\Phi \sim \frac{1}{2}\alpha\omega^2 A^2[1 - \cos 2(\omega t - \varphi)]. \tag{4.65}$$

Now, using the energy equation and the fact that $\overline{x}_2 - \overline{x}_1 = O(\omega^{-1})$ gives us

$$\begin{aligned} \frac{d}{dt}E_{\text{tot}} &= E_2\overline{x}'_2 - E_1\overline{x}'_1 + \int_{\overline{x}_1(t)}^{\overline{x}_2(t)} \partial_t E(x,t)dx \\ &\sim E_2\overline{x}'_2 - S_2 - E_1\overline{x}'_1 + S_1, \end{aligned}$$

where $E_i = E(\overline{x}_i, t)$ and $S_i = S(\overline{x}_i, t)$ for $i = 1, 2$. Using the expansions in (4.63)–(4.65) yields

$$E\overline{x}' - S \sim \frac{\omega A^2}{2\varphi_x}(\omega^2\mu^2 - \varphi_x^2)\sin^2(\omega t - \varphi).$$

Recall that the eikonal equation for this problem is $\omega^2\mu^2 = \varphi_x^2$. From this it is clear that the equation $\frac{d}{dt}E_{\text{tot}} = 0$, to the first term, is equivalent to the eikonal equation.

Claim 2: The transport equation corresponds to the balance of energy over a period.

Given a function $g(t)$, the average to be used is

$$g_{\text{avg}} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} g(t)dt.$$

Averaging the energy equation (4.59) and using (4.63), one obtains

$$\partial_x S_{\text{avg}} = -\Phi_{\text{avg}}. \tag{4.66}$$

Using (4.64) and (4.65) yields

$$\partial_x(\varphi_x A^2) = -\alpha A^2.$$

The solution of this is the same as the solution of the transport equation.

The above results are interesting because they give a physical interpretation of the WKB approximation. However, these claims are not necessarily true for other problems, and so, one must use caution in trying to use them to construct the WKB approximation. It is much better to rely on the original expansion. Also, the above discussion involved the phase velocity, but the group velocity plays an even more fundamental role in energy propagation. The interested reader should consult Whitman (1974) and Lighthill (1965), as well as Exercises 6 and 7.

Exercises

1. For flow of a gas in a long, thin duct, the equation for the velocity potential $\phi(x,t)$ is

$$\partial_x^2\phi + \partial_x(\ln A)\partial_x\phi = \partial_t^2\phi,$$

where $A(x)$ is the cross-sectional area of the duct. This is known as the Webster horn equation (Webster, 1919). Find a first-term approximation of the high-frequency waves. If there are turning points, be sure to identify where they are located.

2. In studying the kinetic theory of waves in plasmas, one comes across having to solve the Fokker-Planck equation,

$$\partial_t u = \partial_x(\alpha(x)u) + \partial_x^2(\beta(x)u), \quad \text{for } 0 < x < \infty,$$

where $u(0,t) = \cos(\omega t)$ and $u \to 0$ as $x \to \infty$. Here $\alpha(x)$ and $\beta(x)$ are positive functions known as the friction and diffusion coefficients, respectively. Find a first-term WKB approximation of the time-periodic solution when ω is large.

3. In modeling the transduction of sound by the human ear, one finds the following equation for the pressure (Holmes, 1981, 1982):

$$\partial_x[H(x)\partial_x p] = K(x)\partial_t^2 p, \quad \text{for } 0 < x < 1,$$

where $p(0,t) = \cos(\omega t)$ and $p(1,t) = 0$. Here $H(x) = A(x) - \delta P(x)/\sqrt{i\omega}$, where $A(x)$, $P(x)$, and $K(x)$ are smooth, positive functions. Also, $\delta > 0$ is constant and ω is the frequency of the sound signal. Find a first-term approximation of the time-periodic solution when ω is large.

4. The equation for the vertical displacement $u(x,t)$ of an inhomogeneous Euler-Bernoulli beam is

$$\partial_x^2[D(x)\partial_x^2 u] + \mu(x)\partial_t^2 u = 0,$$

where $D(x)$ is the bending rigidity and $\mu(x)$ is the mass density. Both of these functions are smooth and positive.

a) Find a first-term approximation of the high-frequency traveling waves. If there are turning points, be sure to identify where they are located.

b) Find E, S, and Φ in the energy equation (4.59) for the beam.

c) Do claims (1) and (2) hold for this equation?

d) For a cantilevered beam, the boundary conditions are $u = u_x = 0$ at $x = 0$ and $u_{xx} = \partial_x(Du_{xx}) = 0$ at $x = 1$. Show that a first-term approximation of the high-frequency modes is

$$\omega \sim k_n^2 \left(\int_0^1 (\mu/D)^{1/4} dx \right)^{-1},$$

where the k_n's are solutions of the equation $\cos(k_n)\cosh(k_n) = -1$.

5. Derive the energy equation (4.59) by multiplying (4.56a) by u_t and then integrating the result over $x_0 \le x \le x_1$. Using integration by parts, and the fact that x_0 and x_1 are arbitrary, obtain (4.59).

6. The group velocity is defined as $v_g \equiv \left(\frac{d}{d\omega}\varphi_x\right)^{-1}$. Is Claim 1 true if "phase lines" is replaced with "group lines"?

7. The Lagrangian L is the difference between the kinetic and potential energies, and so, from (4.56a), $L = \frac{1}{2}\mu^2(u_t)^2 - \frac{1}{2}(u_x)^2 - \frac{1}{2}\beta u^2$. The averaged Lagrangian L is obtained from this by assuming (4.60) and then averaging L over a period. This exercise introduces some of the ideas underlying the method of averaged Lagrangians developed in Whitham (1974).

a) Show that $L \sim \frac{1}{4}A^2(\mu^2\omega^2 - \varphi_x^2)$.

b) For $\alpha = 0$ show that the eikonal and transport equations correspond to the following: i) $\partial_A L = 0$, and ii) $\partial_t(\partial_\omega L) - \partial_x(\partial_{\varphi_x} L) = 0$.

8. The wave equation in three spatial dimensions is $\nabla^2 u = \mu^2 \partial_t^2 u$, where $\mu = \mu(\mathbf{x})$ is smooth and positive.

   a) Show that the energy equation has the form $\partial_t E + \nabla \cdot \mathbf{S} = 0$.

   b) Assume $u(\mathbf{x}, t) \sim A(\mathbf{x}) \cos[\omega t - \varphi(\mathbf{x})]$, and introduce the average

$$\langle f \rangle = \frac{1}{2T} \int_{-T}^{T} f(\mathbf{x}, t) dt.$$

   In this case show that, for $T >> 2\pi/\omega$,

$$\langle E \rangle \sim \frac{1}{4} \left[\omega^2 \mu^2 A^2 + \nabla(Ae^{-i\varphi}) \cdot \nabla(Ae^{i\varphi}) \right]$$

   and

$$\langle \mathbf{S} \rangle \sim -\frac{i\omega}{4} \left[Ae^{-i\varphi} \nabla(Ae^{i\varphi}) - Ae^{i\varphi} \nabla(Ae^{-i\varphi}) \right].$$

4.5 Wave Propagation and Slender Body Approximations

Another important situation where the WKB approximation can be used to obtain an asymptotic approximation of traveling wave solutions arises when the geometry is long and thin. This sort of situation is referred to as *slender body theory.* To illustrate this, consider the following problem:

$$\varepsilon^2 u_{xx} + u_{yy} = \mu^2(x) u_{tt}, \quad \text{for} \quad \begin{cases} 0 < x < \infty \\ -G(x) < y < G(x) \end{cases}, \tag{4.67a}$$

where

$$u(x, y, t) = 0 \quad \text{for } y = \pm G(x), \tag{4.67b}$$

and

$$u(0, y, t) = f(y) \cos(\omega t). \tag{4.67c}$$

This problem corresponds to finding the vertical displacement $u(x, y, t)$ of an elastic membrane that is much longer than it is wide. The membrane is fixed along its lateral sides ($y = \pm G$) and is forced periodically at the left end ($x = 0$). In effect, this forcing is generating waves at $x = 0$ that propagate down the membrane. We are interested in finding these waves and determining their basic properties. For example, once a wave is sent down the membrane, do we ever see it again? You might expect that because the membrane is infinitely long that it just keeps traveling away from the end at $x = 0$. However, as will be seen later, it is actually possible for the wave

to travel a certain distance along the membrane and then turn around and come back to $x = 0$. It is not obvious why this should happen without help from the analysis to follow. In this example, the forcing frequency ω is not restricted, other than being positive.

To find the traveling wave solutions for small ε, we use the WKB approximation

$$u(x, y, t) \sim e^{i[\omega t - \theta(x)/\varepsilon^\alpha]}[u_0(x, y) + \varepsilon^\alpha u_1(x, y) + \cdots]. \tag{4.68}$$

Substituting this into (4.67a) yields

$$\varepsilon^{2-2\alpha}\left[-\theta_x^2 u_0 - i\varepsilon^\alpha(\theta_{xx}u_0 + 2\theta_x\partial_x u_0 + \theta_x^2 u_1) + \cdots\right] + \partial_y^2 u_0 + \varepsilon^\alpha \partial_y^2 u_1 + \cdots$$
$$= -\omega^2\mu^2(u_0 + \varepsilon^\alpha u_1 + \cdots). \tag{4.69}$$

Balancing the terms in this equation, it follows that $\alpha = 1$. This leads to the following problems:

$O(1)$ $\quad \partial_y^2 u_0 + (\omega^2\mu^2 - \theta_x^2)u_0 = 0,$

$\quad u_0 = 0 \quad$ for $y = \pm G(x)$.

This is the eikonal problem. It is actually an eigenvalue problem where θ_x^2 is the eigenvalue and u_0 is an eigenfunction. Setting $\lambda = (\omega^2\mu^2 - \theta_x^2)^{1/2}$, the solution of this can then be written as

$$u_0(x, y) = A(x)\sin[\lambda(y + G)], \tag{4.70}$$

where to satisfy the boundary conditions we must have $2\lambda G = n\pi$. Therefore,

$$\theta_x = \pm\sqrt{\omega^2\mu^2 - \lambda_n^2} \quad \text{for } n = 1, 2, 3, \ldots, \tag{4.71}$$

where $\lambda_n(x) = n\pi/(2G)$. This means that the function θ_x depends on the mode number (n) as well as on the position (x). Also, as long as θ_x is real and nonzero, the wave propagates along the x-axis. However, the higher modes (where n is large enough that $\lambda_n > \omega\mu$) do not propagate since the wavenumber is imaginary.

$O(\varepsilon)$ $\quad \partial_y^2 u_1 + \lambda_n^2 u_1 = i(\theta_{xx}u_0 + 2\theta_x\partial_x u_0)$

$\quad u_1 = 0 \quad$ for $y = \pm G(x)$

This is the transport problem, and we will use it to complete the determination of u_0. Because it is an inhomogeneous version of the eikonal problem, we will have to impose a condition on the right-hand side of the transport equation to guarantee a solution. This situation is covered by the Fredholm Alternative Theorem. To obtain the condition, we multiply

the equation by u_0 and then integrate with respect to y. Doing this, one obtains the following:

$$\int_{-G}^{G} u_0 \left(\partial_y^2 u_1 + \lambda_n^2 u_1\right) dy = i \int_{-G}^{G} u_0(\theta_{xx} u_0 + 2\theta_x \partial_x u_0) dy.$$

Integrating by parts twice with the left integral, and using the eikonal problem, shows that the left-hand side is zero. Thus,

$$0 = \int_{-G}^{G} \partial_x \left(\theta_x u_0^2\right) dy. \tag{4.72}$$

This equation for u_0 is known as a *solvability condition*, and it must be satisfied for the transport problem to have a solution. To solve (4.72), we use Liebnitz's rule to obtain

$$0 = \frac{d}{dx} \int_{-G}^{G} \left(\theta_x u_0^2\right) dy - G' \theta_x u_0^2 \Big|_{y=-G}^{G}.$$

Since $u_0 = 0$ for $y = \pm G$, we have

$$\theta_x \int_{-G}^{G} u_0^2 dy = a,$$

where a is a constant. Thus, from (4.70),

$$A = \frac{a}{\sqrt{\theta_x G(x)}}.$$

Putting the results together, we have found that the WKB approximations for the modes have the form

$$u(x, y, t) \sim \frac{a}{\sqrt{\theta_x G(x)}} e^{i[\omega t \pm \theta(x)/\varepsilon]} \sin[\lambda_n (y + G)], \tag{4.73}$$

where

$$\theta(x) = \int^{x} \sqrt{\omega^2 \mu^2 - \lambda_n^2}\, dx \tag{4.74}$$

and

$$\lambda_n = \frac{n\pi}{2G(x)}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{4.75}$$

It is apparent from this that not every mode is going to produce a traveling wave. This is because $\theta(x)$ is imaginary whenever $\lambda_n > \omega\mu$. For example, as mentioned earlier, the higher modes (where λ_n is large) do not propagate. In any case, to obtain the solution of the original problem, these modes should be added together and the constants a_n then determined by satisfying (4.67c).

Before starting the analysis for the transition-layer regions, it is worth making a few observations. First, the eikonal problem differs significantly from the one obtained in Section 4.4. However, even though it is a linear boundary-value problem, the equation that determines θ_x is nonlinear. In other words, it is still true that the equation that determines $\theta(x)$ is a nonlinear, first-order differential equation. Second, even though the next-order problem differs from what we saw earlier, it is still true that the transport equation, which we referred to as a solvability condition for this problem, is a linear, first-order linear differential equation.

Solution in Transition Region

The turning points in the WKB approximation (4.73) occur when $\theta_x = 0$. Using (4.71), these points satisfy

$$\mu(x_t) = \frac{n\pi}{2\omega G(x_t)}. \tag{4.76}$$

We will consider the case when there is a single turning point $0 < x_t < \infty$. To guarantee there is only one, it is assumed that $\frac{d}{dx}\left(\theta_x^2\right) < 0$ (i.e., θ_x^2 is a decreasing function of x, so if it does go through zero it does not do so again).

The first step in the procedure to find the solution in the transition region is to introduce the transition-layer coordinate

$$\overline{x} = \frac{x - x_t}{\varepsilon^\beta}. \tag{4.77}$$

Letting $U(\overline{x}, y, t)$ denote the solution in this region, we assume that

$$U(\overline{x}, y, t) \sim \varepsilon^\gamma e^{i\omega t}\{U_0(\overline{x}) \sin[\lambda_n(y + G_t)] + \cdots\}, \tag{4.78}$$

where $G_t = G(x_t)$. Substituting (4.77) and (4.78) into the differential equation (4.67a) produces the following result:

$$\varepsilon^{2-2\beta} U_0'' + \theta_x^2 U_0 + \cdots = 0, \tag{4.79}$$

where

$$\begin{aligned} \theta_x^2(x_t + \varepsilon^\beta \overline{x}) &\sim \theta_x^2(x_t) + \varepsilon^\beta \overline{x} \frac{d}{dx}\theta_x^2(x_t) + \cdots \\ &\sim \varepsilon^\beta \overline{x} \frac{d}{dx}\theta_x^2(x_t) + \cdots . \end{aligned} \tag{4.80}$$

For the terms to balance in (4.79), we need $2 - 2\beta = \beta$, and thus $\beta = 2/3$. The equation to be solved is thus

$$U_0'' = \kappa \overline{x} U_0, \quad \text{for } -\infty < \overline{x} < \infty, \tag{4.81}$$

where

$$\kappa = -\frac{d}{dx}\theta_x^2(x_t) = -2\omega^2\mu_t^2\left(\frac{\mu_t'}{\mu_t} + \frac{G_t'}{G_t}\right) > 0. \tag{4.82}$$

This is an Airy equation, and the general solution is

$$U_0 = aAi(\kappa^{1/3}\overline{x}) + bBi(\kappa^{1/3}\overline{x}). \tag{4.83}$$

Matching

What remains is the matching. We can use the WKB approximation of the nth mode on either side of the turning point. So,

$$u(x,y,t) \sim \begin{cases} u_L(x,y,t), & \text{for } 0 \le x < x_t \\ \\ u_R(x,y,t), & \text{for } x_t < x < \infty \end{cases}, \tag{4.84}$$

where

$$u_L = \frac{1}{\sqrt{\theta_x G(x)}}\left\{a_L e^{i[\omega t - \theta(x)/\varepsilon]} + b_L e^{i[\omega t + \theta(x)/\varepsilon]}\right\}\sin[\lambda_n(y+G)] \tag{4.85}$$

and

$$u_R = \frac{1}{\sqrt{|\theta_x| G(x)}} a_R e^{i\omega t - \varphi/\varepsilon}\sin[\lambda_n(y+G)]. \tag{4.86}$$

Also,

$$\theta = \int_x^{x_t}\sqrt{\omega^2\mu^2 - \lambda_n^2}\,ds \text{ and } \varphi = \int_{x_t}^{x}\sqrt{\lambda_n^2 - \omega^2\mu^2}\,ds. \tag{4.87}$$

The endpoints in the above integrals are chosen to make the calculations in the matching somewhat simpler.

Before discussing the matching, note that for the solution to be bounded we must have $b = 0$ in (4.83). It is for this same reason that the b_R term is not included in (4.86). Now, the matching is very similar to what was done in Section 4.2 to derive (4.42) and (4.43). One finds that in (4.78), $\gamma = -1/6$. Using the asymptotic properties of the Airy functions given in Appendix A1, the connection formulas are found to be

$$a = \frac{2\sqrt{\pi}}{\kappa^{1/6}G_t^{1/2}} e^{-i\pi/4} a_L, \tag{4.88}$$

$$b_L = -ia_L, \tag{4.89}$$

and

$$a_R = \frac{\kappa^{1/6}G_t^{1/2}}{2\sqrt{\pi}} a = e^{-i\pi/4} a_L. \tag{4.90}$$

Example

To see more clearly what we have derived, suppose the membrane is tapered and has a uniform density so μ is constant (see Fig. 4.8). Also, suppose the frequency is high enough that the nth mode has a turning point that satisfies $0 < x_t < \infty$. The WKB solution for this mode is

$$u(x,y,t) \sim \frac{\alpha_n}{\sqrt{\theta_x G(x)}} V_n(x) \sin[\lambda_n(y+G)] \cos(\omega t), \tag{4.91}$$

where

$$V_n(x) = \begin{cases} \sin\left[\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right], & \text{for } 0 \le x < x_t \quad (4.92a) \\ e^{-\varphi(x)/\varepsilon}, & \text{for } x_t < x < \infty \quad (4.92b) \end{cases}.$$

The constant α_n in (4.91) is determined from boundary condition (4.67c), and the functions θ and φ are given in (4.87).

We are now in a position to summarize the situation. From (4.85) it is seen that in the region $0 \le x < x_t$ the solution for the mode consists of the superposition of left and right traveling waves. As the right running wave moves toward $x = x_t$, its amplitude grows since both G and θ_x are decreasing. Similarly, the wave that is reflected from the transition region decreases in amplitude as it travels away from the turning point. Together they result in a standing wave, as described in (4.91) and (4.92a). The portion of the disturbance that gets through the transition region ends up decaying exponentially, as shown in (4.92b). This is known as the *evanescent region.*

There are several interesting observations that can be made using our solution. First, if the frequency is decreased, then the position of the turning point, which is a solution of the equation

$$\omega\mu = \frac{n\pi}{2G(x_t)},$$

moves to the left. Suppose ω is decreased to a value $\omega = \omega_c$, which has the turning point $x_t = 0$. In this case ω_c corresponds to a frequency below which

FIGURE 4.8. Tapered membrane that leads to the WKB approximation given in (4.91). It is assumed that for the nth mode there is a turning point at $x = x_t$.

waves do not propagate on the membrane without decaying exponentially. This is called a *cutoff frequency*. In this example, it is referred to as a low-frequency cutoff because frequencies below this value do not propagate. The second observation to be made here is that as the mode number (n) increases, the position of x_t moves to the left. Once n is large enough that this point moves off the membrane (i.e., $x_t \leq 0$), then the wave for that mode will not propagate. In other words, the higher modes do not result in propagating waves; these are known as evanescent modes.

The conclusions stated here clearly depend on the problem that was solved. For example, if the membrane widens instead of tapers, then a completely different response is obtained. Also, in the above example, for each mode there is a possibility of resonance. This occurs when $\sin[\theta(0)/\varepsilon + \pi/4] = 0$, but the analysis of this situation will not be considered.

Exercises

1. The equation for a slender membrane with small damping is

$$\varepsilon^2 u_{xx} + u_{yy} = \mu^2(x) u_{tt} + \varepsilon\alpha(x) u_t, \quad \text{for} \begin{cases} 0 < x < \infty \\ 0 < y < 1, \end{cases},$$

where $u(x,0,t) = u(x,1,t) = 0$, $u(0,y,t) = f(y)\cos(\omega t)$, and $u(x,y,t)$ is bounded as $x \to \infty$. Assume $\mu(x)$ and $\alpha(x)$ are positive and $\mu' < 0$.

a) Use the WKB method to find an approximation of the long-time solution outside of the transition region. You only need to find the general solution for each mode.

b) Assuming the turning point $x_t > 0$, find a first-term approximation in the transition layer and match it to the solutions from part (a). Assume it's a simple turning point.

2. How does the WKB approximation of the mode given in (4.85) change if μ also depends on y?

3. a) Use the results of this section to design a method to determine the width $2G(x)$ of the membrane using information from the point $x = 0$. Assume here that μ is constant.

b) Does your method also work to determine μ if it depends on x?

4. Consider the equation for an elastic string,

$$\partial_y^2 u = \mu^2 \partial_t^2 u, \quad \text{for } -G < y < G,$$

where $u = 0$ for $y = \pm G(x)$. Find the natural frequencies of the string and connect them to the cutoff frequency described earlier.

5. The equation for the velocity potential $\phi(r,\theta,z,t)$ of a gas in a long, narrow tube is

$$\left(\partial_r^2+\frac{1}{r}\partial_r+\frac{1}{r^2}\partial_\theta^2+\varepsilon^2\partial_z^2\right)\phi=\partial_t^2\phi,\quad \text{for } 0\le r\le \overline{r} \text{ and } 0\le\theta\le 2\pi.$$

Suppose the tube is axisymmetric and the cross-sectional radius $r=\overline{r}(z)$ of the tube depends on the longitudinal coordinate z. In this case, the boundary condition along the lateral wall of the tube is (see Exercise 13 of Section 1.6)

$$\left(\partial_r-\varepsilon^2\overline{r}_z\partial_z\right)\phi=0\quad \text{when } r=\overline{r}(z).$$

The parameter ε is the ratio of the characteristic radius of the tube to the tube's length, and it is assumed that ε is small.

a) Use the WKB method to show that the waves propagating down the tube are linear combinations of the modes (Rabbitt and Holmes, 1988)

$$G_{nm}(r,\theta,z,t)=J_m[\gamma_{nm}(z)r]e^{i[\omega t-\varphi(z)/\varepsilon]}\cos(m\theta)$$

and

$$H_{nm}(r,\theta,z,t)=J_m[\gamma_{nm}(z)r]e^{i[\omega t-\varphi(z)/\varepsilon]}\sin(m\theta),$$

where γ_{nm} is determined from the boundary condition in the eikonal problem.

(Hint: Liebnitz's rule states

$$\frac{d}{dz}\int_{\Omega(z)}f(r,\theta,z)dA_{r,\theta}=\int_{\partial\Omega(z)}\overline{r}_z f\,ds+\int_{\Omega(z)}\left(\frac{d}{dz}f\right)dA_{r,\theta}.)$$

b) Assuming $\overline{r}_z\neq 0$, find the turning points of the above modes and the transition-layer equation. Under what conditions will a mode be a propagating wave to the left of the turning point and evanescent to the right? Is this a situation of a low- or high-frequency cutoff?

4.6 Ray Methods

The extension of the WKB method to multidimensional problems is straight-forward although the equations are somewhat harder to solve. To illustrate what is involved, consider the wave equation

$$\nabla^2 u=\mu^2(\mathbf{x})\partial_t^2 u, \tag{4.93}$$

where the vector $\mathbf{x}$ is from $\mathbb{R}^3$ and ∇^2 is the Laplacian. We will be interested in the time-periodic response, and so, let

$$u(\mathbf{x},t)=e^{-i\omega t}v(\mathbf{x}). \tag{4.94}$$

With this (4.93) yields what is known as the reduced wave, or Helmholtz, equation, given as

$$\nabla^2 v + \omega^2 \mu^2(\mathbf{x}) v = 0. \tag{4.95}$$

It should be pointed out that even though the discussion to follow assumes $\mathbf{x} \in \mathbb{R}^3$ when presenting specific examples, the derivation of the asymptotic approximation applies to the general case of when $\mathbf{x} \in \mathbb{R}^n$.

In a similar manner as was done in Section 4.4, for a high frequency, the WKB expansion for the solution of (4.95) is

$$v(\mathbf{x}) \sim e^{i\omega\theta(\mathbf{x})} \left[v_0(\mathbf{x}) + \frac{1}{\omega} v_1(\mathbf{x}) + \cdots \right]. \tag{4.96}$$

The eikonal equation obtained from substituting (4.96) into (4.95) is found to be

$$\nabla\theta \cdot \nabla\theta = \mu^2. \tag{4.97}$$

This is where we start to get hung up, because it is not easy to solve this nonlinear partial differential equation. The usual approach is to introduce characteristic coordinates. This means using curves that are orthogonal to the level surfaces of $\theta(\mathbf{x})$ (see Fig. 4.9). Because of this it is best to briefly review some of the salient features of the mathematical description of surfaces.

We will be considering equations of the form $\theta(\mathbf{x}) = c$, where $\mathbf{x} \in \mathbb{R}^3$. For example, if $\theta(\mathbf{x}) \equiv \mathbf{k} \cdot \mathbf{x}$, where $\mathbf{k}$ is a nonzero constant vector, then $\theta(\mathbf{x}) = c$ is an equation for a plane. Similarly, if $\theta(\mathbf{x}) \equiv \mathbf{x} \cdot \mathbf{x}$, then the level surface $\theta(\mathbf{x}) = c$ is a sphere with radius $\sqrt{c}$ (assuming $c > 0$). To describe the level surfaces in general, suppose we let S_c designate the surface determined by the equation $\theta(\mathbf{x}) = c$. We will parameterize S_c so it is described as $\mathbf{x} = \overline{\mathbf{x}}(\alpha, \beta)$, where the function $\overline{\mathbf{x}}$ satisfies the equation $\theta(\overline{\mathbf{x}}) = c$. To illustrate, if $\theta(\mathbf{x}) \equiv \mathbf{x} \cdot \mathbf{x}$, then we can take α and β to be the spherical coordinate angles, and $\overline{\mathbf{x}}$ is determined from the usual spherical coordinate formulas. In what follows, we will assume the coordinate curves on S_c are orthogonal, that is, $\partial_\alpha \overline{\mathbf{x}} \cdot \partial_\beta \overline{\mathbf{x}} = 0$. Also, recall that $\nabla\theta$ is normal to S_c. From this and the equation $\theta(\mathbf{x}) = c$, it follows that the normal $\nabla\theta$ points in the direction of increasing c.

The level surfaces determined by $\theta(\mathbf{x})$ are called *wave fronts* or *phase fronts*. To explain why these play an important role in the solution process, consider the phase function $\Theta(\mathbf{x}, t) = \omega\theta(\mathbf{x}) - \omega t$, which appears in the expansion of the solution $u(\mathbf{x}, t)$ in (4.94) and (4.96). Suppose we start (at $t = 0$) with the surface S_c over which Θ is constant, namely, $\Theta(\mathbf{x}, 0) = \omega c$. As time increases, the points where $\Theta = \omega c$ change, and therefore the points forming S_c move and form a new surface S_{c+t}. Because this new surface is where $\Theta(\mathbf{x}, t) = \omega c$, then it is determined by the equation $\theta(\mathbf{x}) = c + t$ (see Fig. 4.9). The path each point takes to get from S_c to S_{c+t} is obtained from the eikonal equation (one such path is shown in

Fig. 4.9). To solve this equation, we will use a construction procedure that follows the points that make up the surface S_c. The path each point takes will be parameterized using arclength s along the path. Therefore, we will rewrite the parameterization for the level surfaces so that it includes this dependence on arclength. This gives us

$$\mathbf{x} = \overline{\mathbf{x}}(s, \alpha, \beta).$$

Just how to determine this parameterization is discussed next.

By following the paths, or rays, we can construct the wave at any point in time (this is Huygen's construction procedure). To characterize a ray mathematically, recall that $\nabla\theta$ is normal to the wave front. Now, we want the ray's tangent vector $\overline{\mathbf{x}}_s$ to point in the same direction as $\nabla\theta(\overline{\mathbf{x}})$. Therefore, we will require

$$\frac{d\overline{\mathbf{x}}}{ds} = \lambda\nabla\theta, \tag{4.98}$$

where $\lambda(\overline{\mathbf{x}})$ is a function that is specified below. Since $\theta_s = \nabla\theta \cdot \overline{\mathbf{x}}_s$, we can rewrite the eikonal equation as

$$\frac{d\theta}{ds} = \lambda\mu^2. \tag{4.99}$$

This last equation can be integrated to yield

$$\theta = \theta(0, \alpha, \beta) + \int_0^s \lambda(\overline{\mathbf{x}})[\mu(\overline{\mathbf{x}})]^2 d\sigma. \tag{4.100}$$

The hitch here is that we have not yet completely determined the coordinates that lead to (4.99). To do so requires the solution of (4.98). This system of n equations is generally nonlinear and solved more often than not using numerical methods. However, we still have the freedom of choosing λ; this will be discussed below.

We have not completed the analysis for determining the first term of the WKB approximation. It remains to solve the transport equation. This comes from the $O(\omega)$ problem obtained from substituting (4.96) into (4.95). One finds the following:

$$2\nabla\theta \cdot \nabla v_0 + (\nabla^2\theta)v_0 = 0. \tag{4.101}$$

Since $\frac{d}{ds}v_0 = \nabla v_0 \cdot \overline{\mathbf{x}}_s$, then using (4.98) we can rewrite (4.101) as

$$2\frac{d}{ds}v_0 + \lambda(\nabla^2\theta)v_0 = 0.$$

This is a first-order linear equation that can be integrated to give us the following:

$$v_0(s, \alpha, \beta) = v_0(0, \alpha, \beta)e^{-\frac{1}{2}\int_0^s \lambda\nabla^2\theta d\sigma}. \tag{4.102}$$

FIGURE 4.9. Schematic drawing illustrating the wave fronts (which are defined as the set of points where θ = constant) and the path followed by one of the points making up the wave front.

With (4.100) and (4.102) we have determined θ and v_0 in terms of the rays and their initial values (at $s = 0$).

It therefore appears that the entire problem reduces to solving the ray equations (4.98). These can be rewritten by letting $\lambda = 1/\mu$. Multiplying (4.98) by μ, and then differentiating the result, transforms these equations into the following:

$$\frac{d}{ds}\left(\mu\frac{d}{ds}\overline{\mathbf{x}}\right) = \nabla\mu(\overline{\mathbf{x}}). \tag{4.103}$$

To solve this equation, we need the initial position, $\overline{\mathbf{x}}|_{s=0} = \mathbf{x}_0(\alpha, \beta)$, and the initial direction, $\overline{\mathbf{x}}'|_{s=0} = \mathbf{u}(\alpha, \beta)$, where $\mathbf{u}$ is a given unit vector, of the rays on the surface. The amplitude $v_0(\mathbf{x})$ in this case is (see Exercise 4)

$$v_0(\mathbf{x}) = v_0(\mathbf{x}_0)\sqrt{\frac{\mu(\mathbf{x}_0)J(\mathbf{x}_0)}{\mu(\mathbf{x})J(\mathbf{x})}},$$

where $J(\mathbf{x})$ is the Jacobian of the transformation from ray to Cartesian coordinates. This shows that the WKB approximation breaks down when $\mu(\mathbf{x}) = 0$, which are the turning points for this problem (see section 4.3). It also does not hold when $J = 0$, and the points where this occurs are called *caustics*.

So, the question of whether or not we can solve the ray equations comes down to how the function $\mu(\mathbf{x})$ depends on $\mathbf{x}$. In other words, if μ is simple enough we might be able to solve (4.103) in closed form, but otherwise

it is necessary to find the solution numerically. There are different ways to develop a numerical formulation of the ray-tracing problem. The most direct route using finite differences is discussed in Pereyra et al. (1980). Another approach is to rewrite it as a minimization problem; this has been pursued by Um and Thurber (1987) and Moser et al. (1992).

Examples

1) If μ is constant, so that the medium is homogeneous, then the WKB approximation should reduce to a plane wave solution of the form $v = v_0 e^{i\mathbf{k}\cdot\mathbf{x}}$, where v_0 is constant and $\mathbf{k}$ is a vector of wavenumbers that satisfies $\mathbf{k}\cdot\mathbf{k} = \omega^2\mu^2$. To see if this happens, note that the solution of the ray equations in (4.103) is $\mathbf{x} = \mathbf{x}_0 + s\mathbf{d}$, where $\mathbf{d}$ is a unit vector. Therefore, in a homogeneous medium the rays are just straight lines. Solving for s yields $s = \mathbf{d}\cdot(\mathbf{x}-\mathbf{x}_0)$, and so, from (4.100), $\theta = \theta_0 + \mu\mathbf{d}\cdot\mathbf{x}$. From this we get $\mathbf{k} = \omega\mu\mathbf{d}$, which does indeed satisfy the equation $\mathbf{k}\cdot\mathbf{k} = \omega^2\mu^2$. In this case, $\mathbf{d}$ is the direction vector and determines the direction of propagation of the plane wave.

2) Consider an incident wave of the form

$$u_I \sim e^{i(\omega\theta_I - \omega t)} u_0(\mathbf{x})$$

that strikes an interface across which $\mu(\mathbf{x})$ is discontinuous (see Fig. 4.10). This happens, for example, when sunlight strikes the surface of a lake. The wave is partly reflected, producing a wave u_R, and partly transmitted (or refracted), producing a wave u_T. Letting S designate the interface, then $u(\mathbf{x}, t)$ and its normal derivative must be continuous across the interface. Thus, the following conditions are imposed along S:

$$u_I(\mathbf{x}, t) + u_R(\mathbf{x}, t) = u_T(\mathbf{x}, t) \tag{4.104}$$

and

$$\partial_n[u_I(\mathbf{x}, t) + u_R(\mathbf{x}, t)] = \partial_n u_T(\mathbf{x}, t). \tag{4.105}$$

FIGURE 4.10. Schematic illustrating the reflection and transmission of an incident wave at an interface S across which the function $\mu(\mathbf{x})$ is discontinuous.

It is assumed here that S is smooth and convex. As usual, the expansions for the reflected and transmitted waves are, respectively,

$$u_R \sim e^{i(\omega\theta_R - \omega t)} v_0(\mathbf{x})$$

and

$$u_T \sim e^{i(\omega\theta_T - \omega t)} w_0(\mathbf{x}),$$

Now, for (4.104) and (4.105) to hold at every point on S, it must be that $\theta_I(\mathbf{x}) = \theta_R(\mathbf{x}) = \theta_T(\mathbf{x})$ for $\mathbf{x} \in S$. The interface conditions then also give us that $u_0(\mathbf{x}) + v_0(\mathbf{x}) = w_0(\mathbf{x})$ and $u_0(\mathbf{x})\partial_n\theta_I + v_0(\mathbf{x})\partial_n\theta_R = w_0(\mathbf{x})\partial_n\theta_T$ for $\mathbf{x} \in S$. Thus, $v_0 = Ru_0$ and $w_0 = Tu_0$, where the reflection (R) and transmission (T) coefficients are

$$R(\mathbf{x}) = \frac{\partial_n\theta_I - \partial_n\theta_T}{\partial_n\theta_I + \partial_n\theta_T} \text{ and } T(\mathbf{x}) = \frac{2\partial_n\theta_I}{\partial_n\theta_I + \partial_n\theta_T}.$$

In the derivation of these formulas, we used the relation $\partial_n\theta_R = -\partial_n\theta_I$, which follows from the eikonal equation.

Ray methods have been used to find approximate solutions to some very challenging problems in electromagnetics, acoustics, geophysics, solid mechanics, and other areas. Those who may be interested should consult Keller et al. (1956), Levy and Keller (1959), Born and Wolf (1980), Červený (1985), Farra and Madariaga (1987), and Achenbach et al. (1982). More mathematical issues, such as the behavior near caustics, the Maslov index, and the generalizations of the method, are addressed in Ludwig (1966), Lewis (1968), and Bremmer and Lee (1984).

Exercises

1. Suppose μ is constant on either side of an interface S as shown in Fig. 4.11. Assuming $\mu_+ \neq \mu_-$, then an incident ray when it reaches S gives rise to a reflected ray and a transmitted ray. Let the direction vectors for these plane waves be $\mathbf{d}_i$, $\mathbf{d}_r$, and $\mathbf{d}_t$, respectively, and let $\mathbf{n}$ be the unit normal to S pointing into the μ_+ side. Answer the following using the ideas developed in Examples 1 and 2.

   a) Show that $\mathbf{d}_r$ and $\mathbf{d}_t$ are in the plane determined by $\mathbf{d}_i$ and $\mathbf{n}$ (known as the plane of incidence). This is assuming, of course, that $\mathbf{d}_i$ is not parallel to $\mathbf{n}$.

   b) Show that $\varphi_i = \varphi_r$ and $\mu_- \sin\varphi_t = \mu_+ \sin\varphi_i$. The latter relation is Snell's law. Also, the angles are measured relative to the normal $\mathbf{n}$ as shown in Fig. 4.11.

2. Assuming a high frequency, find a first-term approximation of the time-periodic response of the following:

FIGURE 4.11. Reflection and transmission of an incident wave at an interface S as assumed in Exercise 1.

a) $\nabla^2 u = \partial_t^2 u + \beta(\mathbf{x})\partial_t u$, where $\beta(\mathbf{x})$ is smooth and positive,

b) $\nabla^2 u + \mathbf{a} \cdot \nabla u = \partial_t^2 u$, where $\mathbf{a} = \mathbf{a}(\mathbf{x})$ is smooth.

3. The equation for the vertical displacement $u = u(x, y, t)$ of a Euler-Bernoulli plate is

$$\partial_x^2 M_x + 2\partial_x \partial_y M_{xy} + \partial_y^2 M_y + \mu(x, y)\partial_t^2 u = 0,$$

where $M_x = D(u_{xx} + \nu u_{yy})$, $M_{xy} = (1 - \nu)Du_{xy}$, and $M_y = D(u_{yy} + \nu u_{xx})$. Also, $D = D(x, y)$ and $\mu = \mu(x, y)$ are smooth, positive functions, and ν is a positive constant.

a) For the time-periodic response, in the case of a high frequency, find the eikonal and transport equations.

b) For the propagating waves, investigate the use of characteristic coordinates to solve the eikonal and transport equations. To do this, suppose the solution (i.e., θ and v_0) is prescribed, when $s = 0$, along a curve $\mathbf{x} = \mathbf{x}_0(\alpha)$. The idea here is that by starting (at $s = 0$) on any given point of this curve one then determines the solution along the ray that goes through this point by solving (4.98). For this reason, the solution should be written as $\mathbf{x} = \overline{\mathbf{x}}(s, \alpha)$. In this discussion, explain why the approximation breaks down if $(\nabla\theta \cdot \overline{\mathbf{x}}_{s\alpha})/|\nabla\theta|$ is zero.

4. Let J denote the Jacobian of the transformation from ray to Cartesian coordinates:

$$J = \left| \frac{\partial \mathbf{x}}{\partial(s, \alpha, \beta)} \right|.$$

a) Show that

$$\partial_s J = J\nabla \cdot (\lambda \nabla\theta).$$

b) When $\lambda = 1/\mu$, show that

$$v_0(\mathbf{x}) = v_0(\mathbf{x}_0)\sqrt{\frac{\mu(\mathbf{x}_0)J(\mathbf{x}_0)}{\mu(\mathbf{x})J(\mathbf{x})}},$$

where $\mathbf{x}_0 = \mathbf{x}|_{s=0}$.

5. In studying acoustic wave propagation in a horizontally stratified ocean one comes across the following problem (Ahluwalia and Keller, 1977; Stickler et al., 1984):

$$\nabla^2\phi + \omega^2\mu^2\phi = 0, \quad \text{for } -1 < z < 0,\ 0 < r < \infty,\ 0 \le \overline{\theta} \le 2\pi,$$

where $\phi = 0$ when $z = 0$ and $\partial_z\phi = 0$ when $z = -1$. Also, $\mu = \mu(z)$ is a smooth, positive function of z. It is assumed a sound source is located at $r = 0, z = -1/2$, and this source is independent of the cylindrical coordinate angle $\overline{\theta}$. The initial conditions are that $\theta = 0$ when $s = 0$ and (see Exercise 4)

$$\lim_{s\downarrow 0} J^{1/2} v_0 = \frac{1}{4\pi}\sqrt{\frac{\tan(\varphi)}{\mu(-1/2)}},$$

where φ is the initial angle between the ray and positive z-axis (this angle is assumed to be given). What we are interested in are the direct waves, that is, the sound waves that propagate from the source but that have not yet reached either the upper or lower surfaces.

a) For the direct waves in the region $-1/2 < z$, show that the rays are determined from an equation of the form

$$r = \int_{-1/2}^{z} \frac{\alpha}{\sqrt{\mu^2 - \alpha^2}}\, dz, \quad \text{for } -\frac{1}{2} < z \le z_R,$$

where α is a constant that is determined by μ and the initial angle of the ray, and z_R is determined (in part) by μ and α.

b) Using the result from (a), show that

$$\theta(r, z) = \alpha r + \int_{-1/2}^{z} \sqrt{\mu^2 - \alpha^2}\, dz.$$

c) Show that (see Exercise 4)

$$J = \frac{r}{\mu}\sqrt{\mu^2 - \alpha^2}\int_{-1/2}^{z} \frac{\mu^2}{(\mu^2 - \alpha^2)^{3/2}}\, dz,$$

and from this find v_0.

d) What is a first-term WKB approximation of the direct waves in the region $-1 < z < -1/2$?

6. In studying acoustic wave propagation in an ocean with a slowly varying bottom one comes across the following problem (Burridge and Weinberg, 1977):

$$\varepsilon^2\phi_{xx} + \varepsilon^2\phi_{yy} + \phi_{zz} + k^2\phi = 0, \quad \text{for } 0 < z < y,$$

where $\phi = 0$ on $z = 0$ and $\partial_z \phi = \partial_y \phi$ on $z = y$. Also, $k = \omega\mu$ is a positive (fixed) constant. It is assumed a sound source is located at $x = 0$, $y = 1$. In this problem, we are interested in constructing a first-term WKB approximation of the solution for small ε.

a) Assuming the phase is independent of z, show that the eikonal equation is

$$\theta_x^2 + \theta_y^2 = \kappa_n^2,$$

where $\kappa_n^2 = k^2 - (n\pi/y)^2$ and n is a positive integer.

b) Setting $\overline{\mathbf{x}} = (\overline{x}, \overline{y})$, then show, using (4.98) with $\lambda = 1/\kappa_n$, that $\overline{x}_s^2 + \overline{y}_s^2 = 1$. Thus, we can take $\overline{x}_s = \cos(\xi)$, where ξ is the angle between the ray and x-axis.

c) From the ray equations, show $\kappa_n \overline{x}_s = \alpha$, where $\alpha = (k^2 - (n\pi)^2)^{1/2} \cos(\xi_0)$ and ξ_0 is the angle ξ at the source. Also, show that

$$\overline{x} = \int_1^{\overline{y}} \frac{\alpha}{(\kappa_n^2 - \alpha^2)^{1/2}}\, dy \text{ and } \theta = \int_1^{\overline{y}} \frac{\kappa_n^2}{(\kappa_n^2 - \alpha^2)^{1/2}}\, dy.$$

d) Find, and solve, the transport equation.

7. One can show that the solution of the transport equation (4.101), when $\lambda = 1/\mu$, can be written as (Kline, 1961; Bremmer and Lee, 1984)

$$v_0 = v_0(\mathbf{x}_0) \exp\left(-\frac{1}{2}\int_0^s \left(\frac{1}{R_1} + \frac{1}{R_2}\right) ds\right) \left[\frac{\mu(\mathbf{x}_0)}{\mu(\mathbf{x})}\right]^{1/2},$$

where $R_1(s)$ and $R_2(s)$ are the principal radii of curvature of the wave front at s and $\mathbf{x}_0 = \mathbf{x}|_{s=0}$. Here R_i is taken to be positive if the corresponding normal section bends away from the direction of propagation. Note that focal points, which are where either R_1 or R_2 goes through zero, require special consideration and are discussed in Bremmer and Lee (1984).

a) For $\mu = 1$, show that

$$v_0 = v_0(\mathbf{x}_0) \left[\frac{\rho_1 \rho_2}{(\rho_1 + s)(\rho_2 + s)}\right]^{1/2},$$

where ρ_1 and ρ_2 are the principal radii of curvature of the wave front at $s = 0$. What does this formula reduce to in $\mathbb{R}^2$?

b) Show that the result in part (a) can be written as $v_0 = v_0(\mathbf{x}_0) \cdot [G(s)/G(0)]^{1/2}$, where $G(s)$ is the Gaussian curvature of the wave front.

8. This problem examines the situation where there is spherical symmetry, in which case $\mu = \mu(r)$, where r is the usual radial distance. This is a commonly made assumption when studying optics of lenses or wave propagation through the Earth's atmosphere.

   a) Use the ray equations in (4.103) to show that the vector $\mathbf{p} = \mu\overline{\mathbf{x}} \times \overline{\mathbf{x}}_s$ is independent of s. Explain why this shows that each ray is a plane curve.

   b) Since the ray is a plane curve, we can use polar coordinates $r, \overline{\theta}$ in the plane of the ray to solve (4.103). Assuming the ray can be described as a polar equation of the form $\overline{\theta} = f(r)$, show that the ray equations can be solved to yield

$$\overline{\theta} = \overline{\theta}_0 + \kappa \int_{r_0}^{r} \frac{d\rho}{\rho\sqrt{\mu^2\rho^2 - \kappa^2}},$$

   where $r_0, \overline{\theta}_0$, and κ are constants. Here $(r_0, \overline{\theta}_0)$ can be thought of as the initial point of the ray. In terms of initial data, what does κ correspond to geometrically?

9. For Maxwell's fisheye lens it is assumed that

$$\mu(r) = \frac{1}{1 + r^2},$$

   where r is the usual radial distance. In this problem x, y designate Cartesian coordinates in the plane of the ray (see Exercise 8). Also, the initial point for the ray is at $(x, y, z) = (1, 0, 0)$.

   a) Show that the ray is described by an equation of the form

$$x^2 + (y - \alpha)^2 = 1 + \alpha^2 \text{ and } z = 0.$$

   (Hint: Find

$$\frac{d}{dr} \arcsin\left(\frac{r^2 - 1}{2\alpha r}\right), \quad \text{where } \alpha = (1 - r\kappa^2)^{1/2}/(2\kappa).)$$

   b) Show that the solution of the eikonal equation has the form

$$\theta = \theta_0 + \frac{1}{2}\cos^{-1}\left(\frac{2x}{x^2 + y^2 + z^2 + 1}\right).$$

   c) Find the solution of the transport equation for this problem.

   d) In the case of a general initial point $(x_0, y_0, 0)$, show that the rays are described by an equation of the form $z = 0$ and $(x + \alpha\sin\beta)^2 + (y - \alpha\cos\beta)^2 = \alpha^2 + b^2$, where α, β, and b are constants depending on the initial conditions.

10. More often than not, for an inhomogeneous material it is necessary to solve the ray equations numerically. One alternative to this is to consider the medium to be weakly inhomogeneous, that is, to assume $\mu(\mathbf{x}) = 1 + \varepsilon\mu_1(\mathbf{x})$.

    a) Find the first two terms in the expansion of $\overline{\mathbf{x}}$ for small ε. To do this, assume the initial values $\overline{\mathbf{x}}(0)$ and $\overline{\mathbf{x}}'(0)$ are independent of ε.

    b) Suppose $\mu_1(\mathbf{x}) = \sin(\mathbf{k}\cdot\mathbf{x})$, where $\mathbf{k}$ is a constant vector. Are there limitations on s for the expansion to be valid? Would multiple scales help here to extend the range of the expansion of $\overline{\mathbf{x}}$?

    c) Light rays are bent as they pass through the atmosphere; this makes determining the exact location of a star difficult. One approach that has been taken to account for the variation of the index of refraction in the atmosphere is to let $\mu(r) = 1 + \varepsilon e^{-\kappa r}$, where r is the (nondimensional) radial distance measured from the surface of the Earth (Park, 1990). One finds that ε is small; in fact, $\varepsilon \approx 0.032$ (Green, 1985). Using this index of refraction, what are the first two terms in the expansion of $\overline{\mathbf{x}}$ for small ε?

4.7 Parabolic Approximations

In using the WKB method to solve wave propagation problems, one assumes there is a relatively high frequency. This applies to many problems, but there are situations when this assumption is not applicable, or the procedure involved in ray tracing is too prohibitive to be of practical use. An interesting example of the latter situation arises in the use of sound waves in the ocean to measure the effects of global warming (or cooling). This operation gives rise to what is known as an acoustic thermometer. It uses the fact that a 1°C increase in the ocean temperature increases the speed of sound by about 4.6 m/sec. The change in ocean temperature due to climatic changes is small. To be able to measure it, one must have the sound signal propagate over a long distance. A recent experiment established the feasibility of doing this by placing underwater sound sources near Heard Island, which is located in the southern Indian Ocean (Spindel, 1991; Baggeroer and Munk, 1992). Measuring devices to detect the signals were placed off the coasts of Seattle and Nova Scotia, which means the waves traveled up to 18,000 km through the ocean! To achieve such long-range sound propagation in sea water, it is necessary to employ a low-frequency signal; in this experiment tones centered at 57 Hz were used. What we would like to do here is develop an approximation method that will work for such situations. The method to be considered is known as the *parabolic wave approximation.* This appears to have first been used by Leontovich and Fock (1946) to study long-wave radio wave propagation in the troposphere. The popularity of the method, however, is due to Tappert (1977).

The ideas developed below are based on his paper as well as the paper of Siegmann et al. (1985).

To explain how this method works, consider the wave equation

$$\nabla^2 u = \mu^2(\mathbf{x})\partial_t^2 u, \tag{4.106}$$

where $\mathbf{x} \in \mathbb{R}^3$. We will be interested in the time-periodic response, and so, let $u(\mathbf{x}, t) = e^{-i\omega t}v(\mathbf{x})$. With this, (4.106) yields the reduced wave, or Helmholtz, equation,

$$\nabla^2 v + \omega^2\mu^2(\mathbf{x})v = 0. \tag{4.107}$$

To simplify the situation further, we will assume the problem is cylindrically symmetric. In this case, using cylindrical coordinates, (4.107) takes the form

$$v_{rr} + \frac{1}{r}v_r + v_{zz} + \omega^2\mu^2(r, z)v = 0. \tag{4.108}$$

As stated earlier, the parabolic approximation is widely used on acoustic wave propagation problems. With this in mind, we will assume there is a sound source located at $r = r_0$. Thus, the domain we will consider is $r_0 < r < \infty$ and $0 < z < 1$, where v is assumed to be prescribed at $r = r_0$, $v_z = 0$ along $z = 0$, and $v = 0$ along the upper surface $z = 1$. Because the stimulus generates waves that will propagate away from the source, then we will also impose the Sommerfeld radiation condition

$$\lim_{r\to\infty}(v_r - i\omega\mu v) = 0. \tag{4.109}$$

This condition means there is no energy coming in from infinity. In other words, the waves propagate out from $r = r_0$ and not toward $r = r_0$.

To gain some insight into the behavior of the solutions of (4.108), suppose for the moment that $\mu = \mu_0$ is constant and v is independent of z. In this case, the solution of (4.108) that satisfies (4.109) is

$$v = \alpha_0 H_0^{(1)}(kr), \tag{4.110}$$

where $H_0^{(1)}(r)$ is the zeroth-order Hankel function of the first kind and $k = \omega\mu_0$. For large r this function is oscillatory. In fact, one can show that the solution in (4.110) has the form (Abramowitz and Stegun, 1972)

$$v \sim \alpha_0\left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)}, \quad \text{for } kr >> 1. \tag{4.111}$$

This observation does not require a high frequency, and the parabolic approximation makes good use of this fact.

Before deriving the parabolic approximation using multiple scales, we outline the argument that was used in the original derivation of the method. The fundamental assumption made is that the solution of (4.108) can be written as

$$v(r, z) = w(r, z)h(r), \tag{4.112}$$

where $h(r)$ represents a rapidly varying portion of the solution and $w(r,z)$ is its modulation. If we substitute this into (4.108), it is found that

$$\left[w_{rr} + \left(\frac{1}{r} + \frac{2}{h}h_r\right) w_r + w_{zz} + \omega^2\mu^2 w\right] h + \left(h_{rr} + \frac{1}{r}h_r\right) w = 0. \quad (4.113)$$

The choice for the function $h(r)$ is based on the solution of (4.108) when $\mu = \mu_0$. In this approximation, we are primarily interested in the far field, that is, the behavior of the solution for large r. Therefore, we will use the first-term expansion in (4.111) as our choice for $h(r)$. Dropping terms of order $O\left(\frac{1}{r}\right)$ and smaller, we obtain

$$w_{rr} + 2ikw_r + w_{zz} + (\omega^2\mu^2 - k^2)w = 0. \quad (4.114)$$

The last step in the reduction is based on the assumption that $w(r,z)$ is supposed to describe the slow modulation of the wave. It seems plausible in this case that the w_{rr} term in Eq. (4.114) is negligible compared to the others and can therefore be omitted. This is called the *paraxial approximation*, and the result is the standard two-dimensional parabolic equation (PE) given as

$$w_r = \frac{i}{2k}(\omega^2\mu^2 - k^2)w + \frac{i}{2k}w_{zz}. \quad (4.115)$$

There is no doubt that the derivation of this parabolic equation leaves a lot to be desired, and this issue will be addressed below. However, before doing so, note that the value of k has not been specified. This is known as the *separation constant*, and it is reasonable to assume that it is a reference or characteristic value of $\omega\mu$.

The PE in (4.115) can be obtained directly using multiple scales. The first step is to reconsider the scaling of the physical problem. If one nondimensionalizes the radial coordinate using a length scale associated with a typical wavelength of the wave, then (4.108) takes the form (see Exercise 1)

$$v_{rr} + \frac{1}{r}v_r + \varepsilon v_{zz} + \mu^2 v = 0, \quad \text{for } 0 < r < \infty \text{ and } 0 < z < 1, \quad (4.116)$$

where

$$\mu = 1 + \varepsilon\mu_1(\varepsilon r, z). \quad (4.117)$$

The boundary conditions we will use for this problem are the radiation condition (4.109) along with the following:

$$v(0,z) = f(z) \quad \text{for } 0 < z < 1 \quad (4.118a)$$

and

$$v(r,1) = 0, v_z(r,0) = 0 \quad \text{for } 0 < r < \infty. \quad (4.118b)$$

Here $f(z)$ is a given function. In (4.117) we made an important assumption about how $\mu(\mathbf{x})$ depends on the spatial variables. It is assumed that μ is a

small perturbation from a constant and the perturbation is a slowly varying function of r. This leads to the PE in (4.115), but other variations are of interest, and one is examined in Exercise 3.

To find a first-term approximation to the solution of (4.116) and (4.118), we use multiple scales and introduce the slowly varying radial coordinate $R = \varepsilon r$. In this case, the radial derivative transforms as $\partial_r \to \partial_r + \varepsilon\partial_R$, and so (4.116) becomes

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + 1 + 2\varepsilon\mu_1 + 2\varepsilon\partial_r\partial_R + \frac{\varepsilon}{r}\partial_R + \varepsilon\partial_z^2 + \cdots\right) v = 0. \tag{4.119}$$

The appropriate expansion of the solution in this case is

$$v \sim v_0(r, R, z) + \varepsilon v_1(r, R, z) + \cdots . \tag{4.120}$$

Introducing this into (4.119), one finds that the $O(1)$ equation is

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + 1\right) v_0 = 0.$$

The general solution of this is $v_0 = V(R, z)H_0^{(1)}(r)$. With this, the $O(\varepsilon)$ equation obtained from (4.119) is

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + 1\right) v_1 = -2V_R\partial_r H_0^{(1)} - \frac{1}{r}V_R H_0^{(1)} - V_{zz}H_0^{(1)} - 2\mu_1 V H_0^{(1)}. \tag{4.121}$$

Using variation of parameters, one finds that the general solution of this equation is

$$\begin{aligned} v_1 &= aH_0^{(1)}(r) + bH_0^{(2)}(r) \\ &\quad + \alpha\int_0^r \left[H_0^{(1)}(s)H_0^{(2)}(r) - H_0^{(1)}(r)H_0^{(2)}(s)\right] sF(s, R, z)ds, \end{aligned} \tag{4.122}$$

where F designates the right-hand side of (4.121), $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind, and α is a constant. The Hankel functions in (4.122) generate secular terms (this is shown in Exercise 2). To prevent them from appearing, we require that $V(r, z)$ satisfies (see Exercise 2b)

$$\partial_R V = \frac{i}{2}\partial_z^2 V + i\mu_1 V, \quad \text{for } 0 < R < \infty \text{ and } 0 < z < 1. \tag{4.123}$$

This is, in effect, the PE given in (4.115) for $k = 1$. The boundary conditions that go along with this equation are given in (4.118). After solving this problem, the first-order approximation of the solution of (4.116) and (4.118) is

$$v \sim V(\varepsilon r, z)H_0^{(1)}(r), \tag{4.124}$$

which is valid for $0 \leq z \leq 1$ and $0 \leq r \leq O(\varepsilon^{-1})$.

The PE we have derived here is sometimes referred to as the standard small-angle PE. There are other PEs applicable to other situations. The reader is referred to the papers of Siegmann et al. (1985) and Tappert (1977) for discussions of these.

Exercises

1. In dimensional variables (4.108) takes the form

$$\frac{\partial^2}{\partial(r^*)^2}v^* + \frac{1}{r^*}\frac{\partial}{\partial r^*}v^* + \frac{\partial^2}{\partial(z^*)^2}v^* + (\omega^*)^2(\mu^*)^2 v^* = 0.$$

Assume that $\mu^* = \overset{*}{\mu_0}(1 + \varepsilon\mu_1)$, where $\overset{*}{\mu_0}$ is constant and $\mu_1 = \mu_1(\sqrt{\varepsilon}r^*/h, z^*/h)$. Also, the region in which this equation holds is $0 < r^* < \infty$ and $0 < z^* < h$. The asterisks indicate dimensional variables.

a) Find the scaling that leads to (4.116)–(4.118).

b) It was stated that (4.116)–(4.118) are obtained if one nondimensionalizes the radial coordinate using a length scale r_c associated with a typical wavelength of the wave. Comment on this. Also comment on what assumption is made on the height of the channel compared to r_c.

2. For large r the Hankel functions have the expansions (Abramowitz and Stegun, 1972)

$$H_\nu^{(1)}(r) \sim \left(\frac{2}{\pi r}\right)^{1/2} e^{i(r-\pi/4-\pi\nu/2)}$$

and

$$H_\nu^{(2)}(r) \sim \left(\frac{2}{\pi r}\right)^{1/2} e^{-i(r-\pi/4-\pi\nu/2)}.$$

a) Use these expansions to show that, for $1 << r$,

$$\int_0^r sH_0^{(1)}(s)H_0^{(2)}(s)ds \sim \frac{2r}{\pi} \text{ and } \int_0^r sH_1^{(1)}(s)H_0^{(2)}(s)ds \sim -\frac{2ir}{\pi}.$$

b) By using the ideas developed in part (a), and the fact that $\frac{d}{ds}H_0^{(j)}(s) = -H_1^{(j)}(s)$, show that to prevent secular terms from appearing in the expansion for v it is necessary that (4.123) holds.

3. The assumption made on the function μ in Exercise 1 is fairly restrictive. Instead, suppose it is assumed that $\mu^* = \mu_0^*\mu(\varepsilon r, z)$.

a) Find the equation $v(r, z)$ satisfies, and explain the differences between the assumption made here and the one used in Exercise 1.

b) Using the ideas developed in Section 3.3, find a first-term approximation for v that is valid for large r.

4. Suppose (4.116) is rewritten as

$$v_{rr} + \frac{\varepsilon}{R} v_r + \varepsilon v_{zz} + [1 + \varepsilon \mu_1(\varepsilon r, z)]^2 v = 0.$$

a) Find a first-term approximation for v that is valid for large r.

b) How does the approximation you found in part (a) differ from the one given in (4.124)? In answering this, include a comment or two on which approximation should be more accurate.

4.8 Discrete WKB Method

The ideas underlying the WKB method can be extended without much difficulty to difference equations. To show how, and in view of the first example considered in Section 4.2, our starting point is the second-order difference equation

$$y_{n+1} - 2y_n + y_{n-1} = q_n y_n, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots, \tag{4.125}$$

where $q_n = q(\varepsilon n)$. This equation arises, for example, from a finite-difference approximation of (4.1). It also comes from the equation for an elastic string on which point masses are placed at uniform intervals. The discrete WKB approximation that follows is based, in part, on the ideas introduced by Dingle and Morgan (1967a, b) and Wilmott (1985). A somewhat different approach can be found in Braun (1979) and Geronimo and Smith (1992).

The form of the discrete WKB approximation we will use here is

$$y_n \sim e^{\theta(\varepsilon n)/\varepsilon^\alpha} [\overline{y}_0(\varepsilon n) + \varepsilon^\beta \overline{y}_1(\varepsilon n) + \cdots]. \tag{4.126}$$

The functions $\theta(\nu)$, $\overline{y}_0(\nu)$, $\overline{y}_1(\nu), \ldots$ are assumed smooth. In preparation for substituting this expansion into (4.125), we first apply Taylor's theorem to obtain

$$\theta(\varepsilon(n \pm 1)) \sim \theta(\varepsilon n) \pm \varepsilon \theta'(\varepsilon n) + \frac{1}{2}\varepsilon^2 \theta''(\varepsilon n) \pm \cdots.$$

Doing the same for $\overline{y}_0$ and $\overline{y}_1$, we have, from (4.126), that

$$y_{n\pm 1} \sim \exp\left[\varepsilon^{-\alpha}\left(\theta_n \pm \varepsilon \theta_n' + \frac{1}{2}\varepsilon^2 \theta_n''\right)\right] \left[\overline{y}_{0_n} \pm \varepsilon \overline{y}'_{0_n} + \varepsilon^\beta \overline{y}_{1_n} + \cdots\right]. \tag{4.127}$$

In this equation, $\theta_n = \theta(\varepsilon n), \overline{y}_{0_n} = \overline{y}_0(\varepsilon n)$, and so forth.

For the terms in (4.125) to balance, we need to take $\alpha = \beta = 1$. In this case, (4.125) takes the form

$$\begin{aligned} e^{\theta'_n}(1 &+ \frac{1}{2}\varepsilon\theta''_n + \cdots)[\overline{y}_{0_n} + \varepsilon\overline{y}'_{0_n} + \varepsilon\overline{y}_{1_n} + \cdots] \\ &+ e^{-\theta'_n}(1 + \frac{1}{2}\varepsilon\theta''_n + \cdots)[\overline{y}_{0_n} - \varepsilon\overline{y}'_{0_n} + \varepsilon\overline{y}_{1_n} + \cdots] \\ &= (2 + q_n)[\overline{y}_{0_n} + \varepsilon\overline{y}_{1_n} + \cdots]. \end{aligned} \tag{4.128}$$

This leads to the following:

$O(1)$ $e^{\theta'_n} + e^{-\theta'_n} = 2 + q_n$.

This is the eikonal equation. Multiplying by $e^{\theta'_n}$ and then solving the resulting quadratic yields

$$\theta'_n = \ln\left[\frac{1}{2}\left(2 + q_n \pm [q_n(q_n+4)]^{1/2}\right)\right]. \tag{4.129}$$

It is a simple matter to integrate this last equation to obtain $\theta_n(x)$, and so, our next task is to find the equation for $\overline{y}_{0_n}$.

$O(\varepsilon)$ $\left(e^{\theta'_n} - e^{-\theta'_n}\right)\overline{y}'_{0_n} + \frac{1}{2}\theta''_n\overline{y}_{0_n}\left(e^{\theta'_n} + e^{-\theta'_n}\right) = 0$

This is the transport equation, and it can be solved by noting that it can be rewritten as

$$\overline{y}'_{0_n} + \frac{1}{2}\left[\ln\left(e^{\theta'_n} - e^{-\theta'_n}\right)\right]'\overline{y}_{0_n} = 0.$$

Integrating this yields $\overline{y}_{0_n}\left(e^{\theta'_n} - e^{-\theta'_n}\right)^{1/2} = A_0$, where A_0 is a constant. Thus, from (4.129), we have

$$\overline{y}_{0_n} = \frac{A}{[q_n(q_n+4)]^{1/4}}, \tag{4.130}$$

where A is constant.

Adding the two solutions together, a first-term discrete WKB approximation of the general solution of (4.125) in the case of small ε is

$$y_n \sim \frac{1}{[q_n(q_n+4)]^{1/4}}\left(a_0 e^{\theta_+/\varepsilon} + b_0 e^{\theta_-/\varepsilon}\right), \tag{4.131}$$

where

$$\theta_\pm(\varepsilon n) = \int^{\varepsilon n} \ln\left[\frac{1}{2}\left(2 + q(\nu) \pm \sqrt{q(\nu)(q(\nu)+4)}\right)\right] d\nu. \tag{4.132}$$

Using the properties of inverse functions, we can simplify the above expression for $\theta_\pm(x)$ to the following:

$$\theta_\pm(\varepsilon n) = \begin{cases} \pm i \displaystyle\int^{\varepsilon n} \cos^{-1}\left[1 + \tfrac{1}{2}q(\nu)\right] d\nu & \text{if } -4 < q < 0 \quad (4.133a) \\ \pm \displaystyle\int^{\varepsilon n} \cosh^{-1}\left|1 + \tfrac{1}{2}q(\nu)\right| d\nu & \text{if } 0 < q \quad (4.133b) \\ i\pi\varepsilon n \pm \displaystyle\int^{\varepsilon n} \cosh^{-1}\left|1 + \tfrac{1}{2}q(\nu)\right| d\nu & \text{if } q < -4. \quad (4.133c) \end{cases}$$

The similarities between this result and the WKB approximation given in (4.11) are evident. However, there are differences, and perhaps one of the more important ones concerns the turning points. From (4.11) these occur when $q(x) = 0$. For the discrete WKB formula in (4.131), these include the values of n where $q_n = 0, -4$. However, there is also a problem with (4.131) if $q(\nu)$, or $q(\nu) + 4$, changes sign between $\nu = \varepsilon n$ and $\nu = \varepsilon(n+1)$. This will be referred to as a *turning point interval.* The resolution of the solution near a turning point is carried out in a manner similar to what was done for differential equations. This is demonstrated later for a simple turning point. Note that a theoretical analysis of the discrete WKB method can be found in Costin and Costin (1996).

Example

To illustrate the behavior of the solution when a turning point is present, consider the difference equation

$$y_{n+1} - 2y_n + y_{n-1} = \frac{2n - N}{2N} y_n, \quad \text{for } n = -N+1, \ldots, N-1, \quad (4.134)$$

where $y_{-N} = y_N = 1$. In this problem $\varepsilon = 1/N$ and $q(\nu) = (2\nu - 1)/2$. Also, note that if N is even then there is a turning point at $n = N/2$ and a turning point interval when N is odd. We will assume N is even. The solution when $N = 100$ is shown in Fig. 4.12. The oscillatory nature of the solution for $n < 50$ and the exponential character of the solution for $n > 50$ are evident in this figure. These are contained in our WKB approximation (4.131), which, for this problem, reduces to

$$y_n \sim \begin{cases} d_n\left(a_L e^{-iN\theta_n} + b_L e^{iN\theta_n}\right), & \text{for } -N \le n < N/2 \\ d_n\left(a_R e^{N\kappa_n} + b_R e^{-N\kappa_n}\right), & \text{for } N/2 < n \le N \end{cases}, \quad (4.135)$$

where $d_n = \left[\left(\frac{n}{N} - \frac{1}{2}\right)\left(\frac{n}{N} + \frac{7}{2}\right)\right]^{-1/4}$. From (4.133a) and (4.133b) one finds that

$$\kappa_n = 2\sqrt{1 - \eta^2} - 2\eta \cos^{-1}(\eta) \quad (4.136)$$

FIGURE 4.12. Solution of the difference equation (4.134) when $N = 100$. To draw this graph, the individual points have been connected by a smooth curve. This was done, rather than showing the individual data points, because of the number of points included in the plot.

and

$$\theta_n = 2\eta \cosh^{-1}(\eta) - 2\sqrt{\eta^2 - 1}, \tag{4.137}$$

where $\eta = \frac{1}{4}\left(3 + \frac{2n}{N}\right)$. The solution in (4.135) must satisfy the conditions at $n = \pm N$. That still leaves two undetermined constants, which are found by matching (4.135) with the solution from the turning point region.

Turning Points

We will now investigate the behavior of the solution of (4.125) for n near n_0, where $\nu_0 = \varepsilon n_0$ is a simple turning point. In what follows it is assumed that $q(\nu_0) = 0$ with $q'(\nu_0) > 0$. It is also assumed that $q(\nu) + 4 > 0$. In this case, the WKB approximation in (4.131) can be written as

$$y_n \sim \begin{cases} [q_n(q_n+4)]^{-1/4}\left(a_L e^{-i\theta_n/\varepsilon} + b_L e^{i\theta_n/\varepsilon}\right), & \text{for } n < n_0 \\ [q_n(q_n+4)]^{-1/4}\left(a_R e^{\kappa_n/\varepsilon} + b_R e^{-\kappa_n/\varepsilon}\right), & \text{for } n > n_0 \end{cases}, \tag{4.138}$$

where

$$\theta_n = \int_{\varepsilon n}^{\varepsilon n_0} \cos^{-1}\left[1 + \frac{1}{2}q(\nu)\right] d\nu \tag{4.139}$$

and

$$\kappa_n = \int_{\varepsilon n_0}^{\varepsilon n} \cosh^{-1}\left|1 + \frac{1}{2}q(\nu)\right| d\nu. \tag{4.140}$$

To obtain an approximation of the solution near the turning point, we take εn_0 to be fixed and change indices from n to k in such a manner that $n = n_0 + k$. Assuming $\varepsilon|k| << 1$, then from Taylor's theorem,

$$q_n = q(\varepsilon(n_0 + k)) \sim \varepsilon k q'(\varepsilon n_0).$$

The solution in this transition region will be designated as Y_k, and so, $Y_k \equiv y_{n_0+k}$. Now, the appropriate expansion near the turning point is

$$Y_k \sim \varepsilon^{\gamma}\overline{Y}_k + \cdots . \tag{4.141}$$

Substituting this into (4.125), one finds that

$$\overline{Y}_{k+1} - 2\overline{Y}_k + \overline{Y}_{k-1} = \varepsilon k Q_0' \overline{Y}_k, \tag{4.142}$$

where $Q_0' = q'(\varepsilon n_0)$. The general solution of this second order difference equation involves Bessel functions (see Exercise 11). However, we will approach the problem directly and a standard method for solving difference equations like this one is to introduce the transformation (Carrier, et al., 1966)

$$\overline{Y}_k = \int_C e^{ikz} f(z) dz, \tag{4.143}$$

where the contour C is in the complex plane. Both C and the function $f(z)$ are determined from (4.142). Substituting this into (4.142) yields

$$2\int_C e^{ikz}[\cos(z) - 1] f(z) dz = \mu \int_C k e^{ikz} f(z) dz, \tag{4.144}$$

where $\mu = \varepsilon Q_0'$. The next step is to use integration by parts on the integral on the right-hand side of (4.144) to obtain

$$\int_C e^{ikz}\{2[\cos(z) - 1] f(z) - i\mu f'(z)\} dz = 0. \tag{4.145}$$

In this step it has been assumed that the contour has been chosen so the contribution from the endpoints is zero. Later, once C has been determined, we will have to come back to check that this does indeed hold. What we have accomplished by integrating by parts is to get the terms in the braces $\{\}$ in (4.145) to be independent of k. So, for (4.145) to be satisfied for all k, we require that

$$2[\cos(z) - 1] f(z) - i\mu f'(z) = 0. \tag{4.146}$$

Solving this equation yields

$$f(z) = B_0 \exp\left[-\frac{2i}{\mu}(\sin(z) - z)\right], \tag{4.147}$$

where B_0 is a constant. With this, and the original transformation given in (4.143), we have

$$\overline{Y}_k = B_0 \int_C e^{ikz - 2i(\sin(z) - z)/\mu} dz. \tag{4.148}$$

FIGURE 4.13. Three possible contours that can be used in the WKB approximation (4.149). Note C_1 has vertical asymptotes $\mathrm{Re}(z) = -\pi, 0$, while C_2 has asymptotes $\mathrm{Re}(z) = \pm\pi$, and C_3 has asymptotes $\mathrm{Re}(z) = 0, \pi$.

We must now specify the contour C so that all this makes sense. Actually, we are looking for two contours because the general solution of (4.142) consists of the superposition of two independent solutions. The requirements on these contours are that they produce a convergent integral in (4.148), that (4.145) is obtained after integration by parts, and that the solution from this region matches with the outer solutions. The first two requirements are easily satisfied if we make use of the exponential decay of the integrand. There are many possibilities, and three are shown in Fig. 4.13. From this and (4.148) we have that the WKB approximation of the general solution of (4.142) can be written as

$$\overline{Y}_k = \alpha_0 \int_{C_1} e^{ikz-2i(\sin(z)-z)/\mu} dz + \beta_0 \int_{C_3} e^{ikz-2i(\sin(z)-z)/\mu} dz, \quad (4.149)$$

where α_0 and β_0 are arbitrary constants.

It remains to match (4.149) with the outer solutions given in (4.138). Determining the behavior of the functions in (4.138) near the turning point is relatively easy and follows the steps used in Section 4.3. Finding out how $\overline{Y}_k$ behaves as $k \to \pm\infty$, however, requires extensive use of the method of steepest descents (Murray, 1984). The details of the calculations are outlined in Exercise 10. One finds that the connection formulas relating the coefficients in the three regions are

$$b_L = b_R + \frac{1}{2} i a_R, \qquad a_L = \frac{1}{2} a_R + i b_R, \quad (4.150)$$

$$\alpha_0 = (2\pi Q_0')^{-1/2} b_L, \qquad \beta_0 = -i(2\pi Q_0')^{-1/2} a_L, \quad (4.151)$$

and $\gamma = -1/2$.

Putting our results together we, have from (4.138) and (4.150) that the WKB approximation of the solution of (4.125) is

$$y_n \sim \begin{cases} A_n \left(a_R \cos\left(\frac{1}{\varepsilon}\theta_n + \frac{\pi}{4}\right) + 2b_R \cos\left(\frac{1}{\varepsilon}\theta_n - \frac{\pi}{4}\right)\right), & \text{for } n < n_0 \\ A_n \left(a_R e^{\frac{1}{\varepsilon}\kappa_n} + b_R e^{-\frac{1}{\varepsilon}\kappa_n}\right), & \text{for } n > n_0 \end{cases}, \quad (4.152)$$

FIGURE 4.14. Plot of the WKB approximation (4.152) for the solution of the difference equation (4.134) when $N = 100$. For comparison, the numerical solution is also shown. As in Fig. 4.12, because of the number of points included in the plot, the individual points have been connected by a smooth curve.

where $A_n = [|q_n|(q_n + 4)]^{-1/4}, \theta_n$ is given in (4.139), and κ_n in (4.140).

To illustrate the accuracy of this approximation, we return to the example in (4.134). The coefficients a_R and b_R in (4.152) are determined from the boundary conditions $y_{-N} = y_N = 1$. The resulting approximation is shown in Fig. 4.14 along with the numerical solution. It is seen that in their respective regions of applicability the discrete WKB approximations are almost indistinguishable from the numerical solution.

Exercises

1. This problem concerns the equation

$$p_{n+1}y_{n+1} + q_n y_n + p_n y_{n-1} = 0, \quad \text{for } n = 0, 1, 2, \ldots ,$$

where $q_n = q(\varepsilon n)$ and $p_n = p(\varepsilon n)$. It is assumed here that $p(\nu) > 0$, and $p(\nu)$ and $q(\nu)$ are smooth functions of ν.

a) Find a first-term approximation to the solution of this equation.

b) Identify the turning points, and simplify your answer from part (a) if a turning point is not present (as is done in (4.133) and (4.135)).

c) A nonzero solution is said to be oscillatory if given any n there is a $k \geq n$ so that $y_k y_{k+1} \leq 0$. In Patula (1979) it is proved that if $q(\nu) \geq 0$ then all nontrivial solutions of the above difference equation are oscillatory. Is your approximation consistent with this result? For $q(\nu) < 0$, Hooker and Patula (1981) show that all nontrivial solutions are oscillatory if $(4 - \delta)p_n^2 \geq q_n q_{n+1}$ for some $\delta > 0$. Is your approximation consistent with this result?

2. Find the general solution of the turning point equation when

a) εn_0 is a simple turning point where $q_n = 0$ and $\mu < 0$,

b) εn_0 is a simple turning point where $q_n = -4$ and $\mu > 0$,

c) εn_0 is a simple turning point where $q_n = -4$ and $\mu < 0$.

3. The development of the discrete WKB method assumed that the variable coefficient in (4.125) could be described with a continuous function $q(\nu)$. When starting out with a difference equation, there will inevitably be numerous choices that can be made for $q(\nu)$. It is therefore a natural question to ask what effect, if any, this choice will have on the approximation. To investigate this, suppose the difference equation is

$$y_{n+1} - 2y_n + y_{n-1} = 2y_n, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots .$$

a) By assuming $y_n = r^n$, find the general solution of this equation.

b) Now the question is, what should we choose for $q(\nu)$? For example, we can take one of the following: i) $q(\nu) = 2$, ii) $q(\nu) = 2 + \sin(\nu\pi)$, or iii) $q(\nu) = 2 + \sin(20\nu\pi)$. Discuss the effect of these choices on the accuracy of the WKB approximation. You might do this by either investigating the second term in the expansion or else examining the effects on the value of the function $\theta_\pm$ as given in (4.132).

c) Based on your findings in part (b), discuss the case when $q(\nu) = 2 + 4\sin(\nu\pi)$.

4. Find a discrete WKB approximation of the solution of the following. If turning points, or turning point intervals, are present, identify where they are located.

a) $y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2} = q_n y_n$.

b) $\left(1 + \frac{1}{2}p_n\right) y_{n+1} + (q_n - 2)y_n + \left(1 - \frac{1}{2}p_n\right) y_{n-1} = 0$.

c) $\left(1 + \frac{1}{2}\varepsilon^{1/2}p_n\right) y_{n+1} + (q_n - 2)y_n + \left(1 - \frac{1}{2}\varepsilon^{1/2}p_n\right) y_{n-1} = 0$.

(Hint: Assume $y_n \sim \exp(\varepsilon^{-\alpha}\theta_0 + \varepsilon^{-\beta}\theta_1)(y_0 + \varepsilon^\gamma y_1 + \cdots)$.)

5. Consider the eigenvalue problem

$$y_{n+1} - 2y_n + y_{n-1} + \lambda^2\varepsilon^{-2}q_n y_n = 0, \quad \text{for } n = 1, 2, \ldots, N,$$

where $y_0 = y_{N+1} = 0$. Also, $q_n = q(\varepsilon n)$, where $\varepsilon = 1/(N+1)$, and $q(\nu)$ is a smooth, positive function for $0 \le \nu \le 1$.

a) Assuming $\lambda \sim \varepsilon^\gamma(\lambda_0 + \varepsilon^\alpha\lambda_1 + \cdots)$, use the discrete WKB approximation to determine the first term in the expansion for the eigenvalue λ (i.e., find λ_0).

b) When $q(\nu) = \kappa^2$, where κ is a positive constant, the eigenvalues for the problem are $\lambda = \frac{2}{\kappa}(N+1)\sin\left(\frac{j\pi}{2(N+1)}\right)$ for $j = 1, 2, \ldots, N$.

How does the WKB approximation of the eigenvalues compare with this exact result?

c) The eigenvalue problem comes from a finite-difference approximation of the differential equation

$$y'' + \lambda^2 q(x) y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. In this case, y_n is the approximation of $y(x_n)$, where $x_n = nh$ and $h = 1/(N+1)$. In (4.16) the WKB approximation of the large eigenvalues for this problem is given for $q(x) = e^{2x}$. Compare this result with the values obtained from the discrete WKB approximation obtained in part (a) for $q(\nu) = e^{2\nu}$.

6. Consider the second-order difference equation

$$y_{n+1} + y_{n-1} = q_n y_n, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots, \tag{4.153}$$

where $q_n = q(\varepsilon n)$.

a) Find a WKB approximation of the general solution of this equation for small ε.

b) By setting $z_n = \alpha_n y_n$, show how the equation

$$a_{n+1} z_{n+1} + b_n z_n + c_{n-1} z_{n-1} = 0, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots,$$

can be transformed into the one in (4.153). It is assumed here that $a_n = a(\varepsilon n)$ and $c_n = c(\varepsilon n)$ are nonzero. From this, and the result from part (a), write down a WKB approximation of the general solution of this equation.

7. The Bessel function $J_n(x)$ satisfies the second-order difference equation

$$y_{n+1} + y_{n-1} = \frac{2n}{x} y_n, \quad \text{for } n = 1, 2, 3, \ldots.$$

a) For large x find a WKB approximation of the general solution of this equation.

b) Your solution in part (a) should contain two arbitrary constants. If, for large x,

$$y_0 \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \text{ and } y_1 \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right),$$

then find the resulting expansion for y_n.

c) For large x, one can show that

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right).$$

How does this compare with your result in part (b)?

8. Consider the difference equation

$$q_n z_{n+1} - (1 + 2q_n - \varepsilon\beta)z_n + (q_n - \varepsilon\beta)z_{n-1} = 0, \quad \text{for } n = 1, 2, 3, \dots,$$

where $q(\nu) = \nu$ and β is a positive constant.

a) For small ε, find a WKB approximation of the general solution of this equation.

b) If $z_1 \sim \Gamma(\beta)\varepsilon^{\beta-1}e^{1/\varepsilon}$ and $z_2 \sim z_1/\varepsilon$, then find the resulting expansion for z_n.

c) Compare the result from part (b) with the expansion of Kummer's function $M(n, \beta, x)$ for large x. What difference equation does $Z_n = M(n, \beta, x)$ satisfy?

9. In combinatorics it is often of interest to find the asymptotic behavior of a sequence that is defined by a recursion equation. An example of this arises with Stirling numbers of the second kind, $S(n, k)$. Given a set of n elements, $S(n, k)$ is the number of ways to partition it into k nonempty subsets. From this definition, one finds that these numbers satisfy the recursion equation

$$S(n + 1, k) = S(n, k - 1) + kS(n, k), \quad \text{for } k = 1, 2, \dots, n,$$

where $S(n, 0) = 0$, $S(n, n) = 1$, and $S(n, k) = 0$ for $k > n$. One can show that the solution of this can be written in series form as follows:

$$S(n, k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

This exercise derives an approximation of $S(n, k)$ for large n and is based on the paper of Knessl and Keller (1991b).

a) For large n, and fixed k, show that $S(n + 1, k) \sim kS(n, k)$. From this, conclude $S(n, k) \sim k^n/k!$.

b) To find an approximation for $1 << k < n$, let $k = \kappa/\varepsilon$, where $\varepsilon = 1/n$. By letting $S(n, k) = \varepsilon^\gamma R(n, \kappa)$, show

$$R(n + 1, \kappa) = R(n, \kappa - \varepsilon) + \kappa R(n, \kappa).$$

Using a WKB expansion, find a first-term expansion of the solution of this equation for small ε. To determine the solution uniquely, it is necessary to match with the expansion in part (a). You do not need to do this, but the steps involved can be found in Knessl and Keller (1991b).

10. This exercise concerns the transition-layer solution (4.149) and its matching with the WKB approximation in (4.138).

a) In regard to the two contour integrals in (4.149), use Cauchy's Theorem to relate them to the integral over C_2 (see Fig. 4.13).

b) Consider the limit $k \to \infty$. Setting $g(z) = \frac{2i}{\mu k}\left(\sin(z) - \left(1 + \frac{1}{2}\mu k\right)z\right)$, find the stationary points for $g(z)$ and the paths of steepest descent for the contour integrals over C_2 and C_3. Assuming $k \to \infty$, with $\varepsilon k << 1$, use the method of steepest descents to show that $\overline{Y}_k \sim (\pi^2\mu/k)^{1/4}[i(\alpha_0 + \beta_0)\Lambda + \frac{1}{2}(\beta_0 - \alpha_0)\Lambda^{-1}]$, where $\Lambda = \exp\left(\frac{2}{3}\mu^{1/2}k^{3/2}\right)$.

c) Now suppose $k \to -\infty$, with $|\varepsilon k| << 1$. Setting $h(z) = \frac{2i}{\mu\lambda}(\sin(z) - (1 - \frac{1}{2}\mu\lambda)z)$, where $\lambda = -k$, use the method of steepest descents to show that $\overline{Y}_k \sim (\pi^2\mu/\lambda)^{1/4}(-\alpha_0\Theta + \beta_0\Theta^{-1})$, where $\Theta = \exp\left[i\left(\frac{2}{3}\mu^{1/2}\lambda^{3/2} - \pi/4\right)\right]$.

d) Find first-term approximations of the WKB approximations in (4.138) for n near n_0.

e) By matching the results from parts (b)–(d), derive the connection formulas in (4.150) and (4.151).

11. This problem develops another approach to matching the transition layer solution with the WKB expansion in (4.138).

a) Show that the general solution of (4.142) is

$$\overline{Y}_k = AJ_{k+\frac{2}{\mu}}\left(\frac{2}{\mu}\right) + BY_{k+\frac{2}{\mu}}\left(\frac{2}{\mu}\right)$$

where $\mu = \varepsilon Q_0'$.

b) Using the general solution in part (a), derive the corresponding connection formulas. What assumptions do you need to make about εk?

5

The Method of Homogenization

5.1 Introduction

It is common in engineering and scientific problems to have to deal with materials formed from multiple constituents. Examples include laminated plates, fiber reinforced composites, fluid-filled porous solids, and bubbly fluids. Solving a mathematical problem that includes such variations in the structure can be very difficult. It is therefore natural to try to find simpler equations that effectively smooth out whatever substructure variations there may be. An example of this situation occurs when describing the motion of a fluid or solid. One usually does not consider them as composites of discrete interacting molecules. Instead, one uses a continuum approximation that assumes the material to be continuously distributed. Using this approximation, material parameters, such as the mass density, are assumed to represent an average.

In this chapter, we investigate one approach that can be used to smooth out substructure variations that arise with spatially heterogeneous materials. In this sense, our objective differs from what we have done in the other chapters. Namely, beginning with a problem that includes the structural variations we are going to derive a simpler problem that serves as a first-term approximation. The substructural geometric and material variations will not be present in the reduced problem, but they will be used to determine the coefficients of the problem. The procedure we will be developing is called *homogenization*. It goes by other names, including effective media theory, bulk property theory, and the two-space scale method.

FIGURE 5.1. Examples illustrating the coefficient D in Eq. (5.1a) and how it can vary over a length scale that is small in comparison to the length of the interval. In (a) the function is a piecewise constant function, and in (b) the dependence is continuous.

5.2 Introductory Example

To introduce the method of homogenization, we will examine the boundary-value problem

$$\frac{d}{dx}\left(D\frac{du}{dx}\right) = f(x), \quad \text{for } 0 < x < 1, \tag{5.1a}$$

where

$$u(0) = a \text{ and } u(1) = b. \tag{5.1b}$$

Of interest here is when the function D includes a relatively slow variation in x as well as a faster variation over a length scale that is $O(\varepsilon)$. Two examples of this are illustrated in Fig. 5.1. The function in Fig. 5.1(a) is an example of the type of variation that might be found for a laminated structure, since D has jump discontinuities but is continuous over intervals that are $O(\varepsilon)$. The function in Fig. 5.1(b) has a variation similar to the one in Fig. 5.1(a) but instead is continuous. We will concentrate on examples of the latter type, but the discontinuous problem is not that much harder (see Exercise 8). It is worth pointing out that a problem with rapidly varying coefficients, as in (5.1), is not easy to solve even numerically. This makes the smoothing process underlying homogenization quite valuable when dealing with such situations.

We need to express the sort of variation seen in Fig. 5.1(b) mathematically, and this assumption is incorporated into the problem by assuming

$D = D(x, x/\varepsilon)$. As an example, setting $y = x/\varepsilon$, we could have

$$D(x, y) = \frac{1}{1 + \alpha x + \beta g(x)\cos(y)}. \tag{5.2}$$

This function happens to be periodic in the fast variable (y), but this is not required for the derivation that follows. It is of interest to note that this function is plotted in Fig. 5.1(b) for $\alpha = -0.1$, $\beta = 0.1$, $g(x) = e^{4x(x-1)}$, and $\varepsilon = 0.01$.

The question we are going to ask is whether or not it is possible to smooth out $D(x, x/\varepsilon)$ and replace it with an averaged value that is independent of ε. One might try to argue that this is possible because the variations of D are over such a small length scale that a first-term approximation that uses an averaged D is reasonable. If so, then the question arises as to what average should be used. One possibility is simply to average over the fast variable. Doing this and using a first-term approximation, we get

$$D_{\text{avg}}(x) = \lim_{y\to\infty} \frac{1}{y}\int_0^y D(x, r)\, dr. \tag{5.3}$$

For example, if $D(x, y)$ is given in (5.2), then one finds that

$$D_{\text{avg}}(x) = \frac{1}{\sqrt{(1 + \alpha x)^2 - [\beta g(x)]^2}}. \tag{5.4}$$

As we will see below, it is possible to approximate D with an average, but D_{avg} is not the one that should be used. Any ideas what it might be?

A special terminology that has developed in homogenization needs to be explained before we get any farther into the subject. The fast length scale, which is $y = x/\varepsilon$ in the preceding example, is sometimes referred to as the *microscale* or the scale of the substructure. Similarly, the variable x is the slow scale or the *macroscale*. This terminology is used because homogenization has its roots in the theory of composite materials. However, one should not automatically associate the procedure with microscopic versus macroscopic variations. What is needed, among other things, are variations over disproportionate length scales. This may be mean atomic versus molecular length scales or even planetary versus galactic scales.

It is also important to state clearly the assumptions made on the coefficient $D(x, y)$. For $0 \le x \le 1$ and $0 < y < \infty$, this function is assumed to be smooth and satisfies

$$0 < D_m(x) \le D(x, y) \le D_M(x), \tag{5.5}$$

where $D_m(x)$ and $D_M(x)$ are continuous for $0 \le x \le 1$. The forcing function $f(x)$ in (5.1a) is also taken to be continuous. In this case, the solution of (5.1) is well defined.

Given the disparity of the two length scales in this problem, it is natural to use multiple scales to find an approximation of the solution. Normally we would introduce the scales $x_1 = x/\varepsilon$ and $x_2 = x$. However, to try to keep the notation relatively simple, we will only introduce new notation for the scale $y = x/\varepsilon$ and designate the slow scale simply as x. In this case, the derivative transforms as

$$\frac{d}{dx} \to \varepsilon^{-1}\partial_y + \partial_x.$$

The differential equation (5.1a) now takes the form

$$(\partial_y + \varepsilon\partial_x)[D(x,y)(\partial_y + \varepsilon\partial_x)u] = \varepsilon^2 f(x). \tag{5.6}$$

A regular multiple-scale expansion is appropriate for this problem, and so we take

$$u \sim u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots. \tag{5.7}$$

Because the solution of the problem is well behaved, we will assume that each term in the expansion is smooth and a bounded function of y. Now, substituting our expansion in (5.7) into (5.6) leads to the following equation:

$O(1)$ $\partial_y[D(x,y)\partial_y u_0] = 0.$

The general solution of this equation is

$$u_0(x,y) = c_1(x) + c_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)}, \tag{5.8}$$

where y_0 is any given fixed value of y. It turns out that $c_0 = 0$, because the above integral is not a bounded function of y. This can be shown using the upper bound in (5.5). If $y > y_0$, then

$$\int_{y_0}^{y} \frac{ds}{D_M(x)} \le \int_{y_0}^{y} \frac{ds}{D(x,s)},$$

and so

$$\frac{y - y_0}{D_M(x)} \le \int_{y_0}^{y} \frac{ds}{D(x,s)}. \tag{5.9}$$

The left-hand side of this last inequality becomes infinite as $y \to \infty$. Therefore, if u_0 is to be bounded, it must be that $c_0 = 0$. In what follows we will write this dependence simply as $u_0 = u_0(x)$ and forego the use of c_1.

Before proceeding to the $O(\varepsilon)$ equation, we need to make another observation about the integral in (5.8) and (5.9). Using the inequality $D_m(x) \le D(x,y)$, we can extend the result in (5.9) to the following:

$$\frac{y - y_0}{D_M(x)} \le \int_{y_0}^{y} \frac{ds}{D(x,s)} \le \frac{y - y_0}{D_m(x)}. \tag{5.10}$$

Therefore, the integral is unbounded but its growth is confined by linear functions in y as $y \to \infty$. This information is needed below when dealing with the secular terms that appear in the expansion of the solution. We are now ready to go on to the next order equation.

$O(\varepsilon)$ $\partial_y[D(x,y)\partial_y u_1] = -\partial_x u_0 \partial_y D.$

The general solution of this equation is

$$u_1(x,y) = b_1(x) + b_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - y\partial_x u_0. \tag{5.11}$$

Again the integral becomes unbounded as y increases, but so does the last term. Moreover, from (5.10), they are both $O(y)$. To prevent this from occurring, we will require that these two terms cancel each other as y increases. This is done by imposing the condition

$$\lim_{y\to\infty} \frac{1}{y}\left[b_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - y\partial_x u_0\right] = 0. \tag{5.12}$$

This can be rewritten as

$$\partial_x u_0(x) = \langle D^{-1}\rangle_\infty b_0(x), \tag{5.13}$$

where

$$\langle D^{-1}\rangle_\infty = \lim_{y\to\infty} \frac{1}{y}\int_{y_0}^{y} \frac{ds}{D(x,s)}. \tag{5.14}$$

Several mathematical questions need to be considered in connection with this integral, but we will continue the derivation of the asymptotic approximation and come back to these questions later. Our immediate objective is to find $u_0(x)$, and this takes us to the next, and final, equation.

$O(\varepsilon^2)$ $\partial_y[D(x,y)\partial_y u_2] = f(x) - b_0' - \partial_y(D\partial_x u_1).$

The general solution of this is

$$u_2 = d_1(x) + d_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - \int_{y_0}^{y} \partial_x u_1(x,s)ds + (f - b_0')\int_{y_0}^{y} \frac{sds}{D(x,s)}.$$

The last integral is $O(y^2)$ for large y. No other terms in the expression can cancel this growth. Therefore it is necessary to require

$$b_0'(x) = f(x). \tag{5.15}$$

This is the equation we have been looking for.

After all the above work, we are now in position to determine the problem that defines the first-term approximation of the solution of (5.1). From

(5.13) and (5.15) we have that the approximation is the solution of the boundary-value problem

$$\frac{d}{dx}\left(\overline{D}\frac{d}{dx}u_0\right) = f(x), \quad \text{for } 0 < x < 1, \tag{5.16a}$$

where

$$u_0(0) = a \text{ and } u_0(1) = b. \tag{5.16b}$$

The coefficient here is

$$\overline{D}(x) = \frac{1}{\langle D^{-1}\rangle_\infty}. \tag{5.17}$$

It may seem like we have accomplished very little since (5.16) is so similar to the original problem given in (5.1). However, what is significant is that the equation no longer contains the fast scale. The result is a *homogenized differential equation* with what is called a *homogenized*, or *effective*, *coefficient*. The fast variation still contributes, albeit indirectly, through this averaged coefficient. The average that appears in (5.17) is the *harmonic mean* of D (this average was first introduced by Archytas, circa 400 B.C., who was one of the last warrior mathematicians). The relationship of $\overline{D}(x)$ to the arithmetic and geometric means is given in Exercise 7.

Examples

1. If $D(x, y)$ is given in (5.2), then from (5.17),

$$\langle D^{-1}\rangle_\infty = \lim_{y\to\infty} \frac{1}{y}\int_{y_0}^{y}(1 + \alpha x + \beta g(x)\cos(s))ds = 1 + \alpha x.$$

Suppose $f(x) = 0$, $a = 0$, and $b = 1$. Also, let $g(x) = 1$ and $\alpha = 0$. In this case, the homogenized coefficient is $\overline{D} = 1$ and the solution of the homogenized problem in (5.16) is

$$u_0(x) = x. \tag{5.18}$$

The problem, using the $D(x, y)$ given in (5.2), can be solved exactly; the solution is

$$u(x) = \frac{x + \beta\varepsilon\sin(x/\varepsilon)}{1 + \beta\varepsilon\sin(1/\varepsilon)}. \tag{5.19}$$

A comparison between these two functions is given in Fig. 5.2(a). The exact solution does contain a $O(\varepsilon)$ variation in x, but the amplitude of the variation is also $O(\varepsilon)$. Consequently, as ε decreases, the exact solution is a rapidly oscillatory function, but the amplitude of the oscillations is going to zero with ε. In contrast, the coefficient D has an $O(\varepsilon)$ variation in x, but the amplitude of the variation is $O(1)$ (see Fig. 5.2(b)).

FIGURE 5.2. (a) Comparison between the first-term homogenization approximation given in (5.18) and the exact solution, (5.19), for two values of ε. For $\varepsilon = 10^{-2}$, the asymptotic approximation is essentially indistinguishable from the exact solution. (b) Shown are the values of (i) $D(x, x/\varepsilon)$ obtained from (5.2) for $\varepsilon = 10^{-2}$, (ii) $\overline{D}(x) = 1$, and (iii) D_{avg} as given in (5.20). In these calculations, $\alpha = 0$, $\beta = 3/4$, and $g(x) = 1$.

2. Earlier the question was asked if a good approximation to the problem would result if the arithmetic average given in (5.3) were used. In the case where D is given in (5.2), the arithmetic average is

$$D_{\text{avg}} = \frac{1}{\sqrt{(1+\alpha x)^2 - [\beta g(x)]^2}}. \tag{5.20}$$

This shows just how bad our first guess was, since $\overline{D}(x) = (1 + \alpha x)^{-1}$. The larger the amplitude (βg) over the fast scale, the worse the approximation in (5.20) becomes.

3. As another example, suppose

$$D(x, y) = \frac{\sqrt{1+y}}{1+\sqrt{1+y}}. \tag{5.21}$$

In this case

$$\int_0^y \frac{ds}{D(x,s)} = y + 2\sqrt{1+y} - 2,$$

and so $\langle D^{-1}\rangle_\infty = 1$. The first term from the homogenization procedure is just fine. However, note that the approximation for u_1 is

$$u_1 = b_1(x) + 2b_0(x)\left(\sqrt{1+y} - 1\right). \tag{5.22}$$

It is clear that this function is not bounded. The problem here is that our assumption of a regular power series, as given in (5.7), is incorrect in this case. For this problem, we should use an expansion in powers of $\varepsilon^{1/2}$. As it turns out, this modification does not affect the first-term approximation (see Exercise 3).

Properties of the Averaging Procedure

Because of the central role the harmonic mean plays in the homogenization procedure, it is worth spending some time going over its properties. As stated earlier, it is assumed that $D(x, y)$ is smooth and satisfies

$$0 < D_m(x) \leq D(x, y) \leq D_M(x). \tag{5.23}$$

In this case, it is not hard to show that $\overline{D}(x)$ is a smooth, positive function that is independent of the lower endpoint y_0. One can also show that (see Exercise 4)

$$0 < D_m(x) \leq \overline{D}(x) \leq D_M(x). \tag{5.24}$$

The connection of the harmonic mean with the arithmetic and geometric means is given in Exercise 7.

Of particular interest for us is how to determine $\overline{D}(x)$. Besides using the definition given in (5.14) and (5.17), there are also the following two special cases:

i) If $\lim_{y\to\infty} D(x, y) = D_\infty(x)$, then from l'Hospital's rule applied to (5.14) it follows that

$$\overline{D}(x) = D_\infty(x). \tag{5.25}$$

ii) If $D(x, y)$ is periodic in y, then there is a positive y_p so that $D(x, y+y_p) = D(x, y)$ for all x and y under consideration. In this case $\overline{D}(x) = 1/d(x)$, where

$$d(x) = \frac{1}{y_p}\int_0^{y_p} \frac{ds}{D(x, y_0 + s)}. \tag{5.26}$$

The point y_0 here is arbitrary other than needing to be a point within the domain for y. From this result, we see that $\overline{D}$ is directly related to the average of D over a period in y. This sort of averaging will play a prominent role in the multidimensional problems examined in the later sections.

Summary

The method of homogenization is in many ways simply an extension of multiple scales. One difference is that it appears to be necessary to consider the equation for the third term in the expansion even though two scales are used. In our example, this was not a problem, but in multidimensional situations it can be very difficult to find the correct secularity condition.

This is the primary reason that many who use the method assume that the substructure is periodic. Another characteristic of homogenization is that it generally produces a problem in the slow variable where the coefficients are determined from averages over the fast scale. This situation is discussed in detail in Section 5.4.

The procedure can be applied without a great deal of modification to more complicated problems. For example, if we had started with the heat equation

$$\partial_x(D\partial_x u) = \partial_t u + f(x), \tag{5.27}$$

then we would have ended up with the homogenized equation

$$\partial_x(\overline{D}\partial_x u_0) = \partial_t u_0 + f(x). \tag{5.28}$$

It is assumed here that, as before, $D = D(x, y)$. The complication in this case is that the time scale $\tau = t/\varepsilon^2$ appears, and this represents the diffusion time associated with the microscale.

Another twist occurs when the original interval is something like $-1 < x < 1$. Now the average in (5.17) includes the two limits $y \to \pm\infty$, and these limits should be equal for $\langle D^{-1}\rangle_\infty$ to be uniquely determined. If not, then there is a material interface that must be dealt with (i.e., a surface where the material has a discontinuity in its properties).

As stated earlier, the homogenization procedure works just as well on problems where the coefficients are discontinuous. An example of this is given in Exercise 8. A more interesting situation arises when the equations change from layer to layer (rather than just the coefficients). This occurs, for example, for materials made up of a fluid and solid (like a fluid-filled sponge) or a bubbly fluid. Problems like these are investigated in Section 5.4.

Where the method runs into difficulty is with nonlinear equations. For example, if $D = D(x, y, u)$ in (5.1a) or in (5.27), then the method's usefulness is not at all clear because the effective coefficient now depends on the unknown u_0. However, not all nonlinear equations cause complications, as Exercise 2 demonstrates.

The theory underlying homogenization is extensive. Little has been said about the theory in our development, but a good introduction can be found in Sanchez-Palencia (1980). The theoretical aspects are also important when incorporating the ideas of homogenization into a numerical algorithm. A discussion of using the method with finite elements to solve problems involving fiber-reinforced composites is given in Guedes and Kikuchi (1990).

Exercises

1. For each of the following modifications of (5.1), determine the homogenized problem:

   a) $f = f(x, \varepsilon)$, where f has a regular expansion in ε,

b) $D = D(x, y, \varepsilon)$, where D has a regular expansion in ε,

c) $D = D(x, g(x)/\varepsilon)$, where $g(x)$ is smooth and $g' > 0$.

2. Consider the problem

$$\partial_x(D\partial_x u) + g(u) = \partial_t u + f(x, x/\varepsilon), \quad \text{for } 0 < x < 1,$$

where $u = 0$ when $x = 0, 1$ or $t = 0$. Assume $D = D(x, x/\varepsilon)$.

a) Find the homogenized problem for the steady state. Make sure to point out what assumptions you need to impose on f so the procedure can be used.

b) Find the homogenized problem for the time-dependent problem.

3. Find the exact solution when D is given in (5.21). Also, derive the homogenized problem and find its solution.

4. Assume $D(x, y)$ is continuous.

a) Show that $\overline{D}(x)$ is independent of y_0.

b) If $D(x, y)$ satisfies (5.23), then show that $\overline{D}(x)$ satisfies (5.24).

5. What differences, if any, are there between $\overline{D}(x)$ and D_{avg} for the special cases in (5.25) and (5.26)?

6. Suppose $D(x, y)$ is periodic in y. Assuming the terms in the expansion (5.7) are periodic in y (with the same period as D), derive the homogenized problem. Note that the $O(\varepsilon^2)$ equation does not have to be solved in this case as the homogenized equation can be obtained from a solvability condition.

7. The arithmetic mean of $D(x, y)$, with respect to y, is $A(D) = \lim_{y\to\infty} \frac{1}{y}\int_{y_0}^{y} D(x, s)ds$, and the geometric mean is

$$G(D) = \lim_{y\to\infty} \exp\left[\frac{1}{y}\int_{y_0}^{y} \ln(D(x, s))ds\right].$$

Show that $\overline{D}(x) \le G(D) \le A(D)$.

8. Consider a layered material where the equation for the ith layer is

$$\frac{d}{dx}\left(D_i \frac{du}{dx}\right) = f(x), \quad \text{for } x_{i-1} < x < x_i.$$

The layers are such that $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = 1$, where $x_{i+1} - x_i = O(\varepsilon)$ and $\varepsilon = 1/n$. Each layer has its own coefficient $D_i(x)$ as illustrated in Fig. 5.1(a) (in the figure the D_i's are constants,

but this is not assumed in this problem). At the interface between layers, it is required that

$$u(x_i^-) = u(x_i^+) \text{ and } \lim_{x \uparrow x_i}(D_i u_x) = \lim_{x \downarrow x_i}(D_{i+1} u_x).$$

Also, each $D_i(x)$ satisfies $0 < D_m(x) \leq D_i(x) \leq D_M(x)$.

a) Let $y = (x - \overline{x})/\varepsilon$, where $\overline{x}$ is any fixed point satisfying $0 < \overline{x} < 1$. Assuming $\overline{x}$ is in the i_0th layer, so $x_{i_0-1} \leq \overline{x} \leq x_{i_0}$, then the neighboring layer interfaces are located at y_{i_0}, $y_{i_0 \pm 1}$, $y_{i_0 \pm 2}, \ldots$. Setting

$$S_j(x) = \sum_{k=1}^{j} \frac{y_{i_0+k} - y_{i_0+k-1}}{D_{i_0+k}},$$

explain why

$$\frac{y_{i_0+j} - y_{i_0}}{D_M(x)} \leq S_j(x) \leq \frac{y_{i_0+j} - y_{i_0}}{D_m(x)}.$$

This inequality is the discrete version of (5.10).

b) Using multiple scales, show that in the (i_0+j)th layer the first term in the expansion is $u_0 = a_j(x) + y b_j(x)$. Use the interface conditions to conclude that a_j is independent of j and $b_j = 0$. Thus, we can write $u_0 = u_0(x)$.

c) Show that in the $(i_0 + j)$th layer the second term in the expansion is $u_1 = c_j(x) + y d_j(x)$. Use the interface conditions to conclude that

$$\partial_x u_0 = \langle\langle D^{-1} \rangle\rangle_\infty D_{i_0+1}[d_1(x) + \partial_x u_0],$$

where

$$\langle\langle D^{-1} \rangle\rangle_\infty \equiv \lim_{j \to \infty} \frac{1}{y_{i_0+j}} S_j(x).$$

Comment on what needs to be assumed for the limit of $j \to -\infty$.

d) From the third term, show that (5.16) still holds but the averaging is as defined in part (c).

e) Suppose the system is biphasic. In particular, if i is even the layers have width α and $D_i = D_\alpha$, while if i is odd the layers have width β and $D_i = D_\beta$ (where D_α and D_β are constants). Explain why $\phi_\alpha = \alpha/(\alpha+\beta)$ might be called the volume fraction for the α material and $\phi_\beta = \beta/(\alpha + \beta)$ the volume fraction for the β material. Express the effective coefficient for D in terms of these volume fractions. When is the effective coefficient the harmonic mean of D_α and D_β?

f) In materials engineering of composites, a rule of mixtures is used that if applied to the biphasic system in part (e) states that the effective coefficient is $\overline{D} = \phi_\alpha D_\alpha + \phi_\beta D_\beta$ (Tsai, 1988). How well does this compare with your answer in (e)?

5.3 Multidimensional Problem: Periodic Substructure

The method of homogenization produces some interesting results in multidimensional problems. To illustrate the ideas involved with using the procedure, consider the inhomogeneous Dirichlet problem

$$\nabla \cdot (D\nabla u) = f(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{5.29a}$$

$$u = g(\mathbf{x}), \quad \text{for } \mathbf{x} \in \partial\Omega. \tag{5.29b}$$

As before, the coefficient $D = D(\mathbf{x}, \mathbf{x}/\varepsilon)$ is assumed to be positive and smooth. What is different is that we also assume that it is periodic in the fast coordinate. Specifically, there is a vector $\mathbf{y}_p$ with positive entries so that $D(\mathbf{x}, \mathbf{y}+\mathbf{y}_p) = D(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y}$ under consideration. The consequences of this periodicity will be discussed later. It is also necessary to say something about the domain Ω. In what follows it is assumed to be an open connected set with a smooth enough boundary $\partial\Omega$ so that the solution of the above problem is well defined.

Implications of Periodicity

The assumption of periodicity plays a pivotal role in the homogenization procedure that is developed later. It is therefore worthwhile to spend a moment or two to consider some of its consequences. In this discussion, we will consider the problem to be in $\mathbb{R}^2$.

An example of the function D that appears in (5.29a) is $D = 6+\cos(2y_1 - 3y_2)$. In this case, the period vector is $\mathbf{y}_p = (\pi, 2\pi/3)$. The periodicity of this function means that if we know its values in the rectangle $\alpha_0 \le y_1 \le \alpha_0+\pi$, $\beta_0 \le y_2 \le \beta_0+2\pi/3$, where α_0 and β_0 are arbitrary, then we can determine it anywhere (see Fig. 5.3). This observation motivates our introduction of a *cell*. This is denoted by Ω_0 and is a rectangle consisting of the points that satisfy $\alpha_0 \le y_1 \le \alpha_0 + p_1$ and $\beta_0 \le y_2 \le \beta_0 + p_2$, where $\mathbf{y}_p = (p_1, p_2)$ is a period vector. In this definition the numbers α_0 and β_0 are arbitrary, but they are assumed to be given and must be consistent with Ω. For example, if Ω is the square $0 < x_1 < 1$ and $0 < x_2 < 1$, then we must have $\alpha_0, \beta_0 \ge 0$. Also note that in $\mathbb{R}^3$ a cell Ω_0 will have the form of a parallelepiped.

The period $\mathbf{y}_p$ can depend on the slow variable $\mathbf{x}$. For example, if $D = 6 + \cos(y_1 e^{x_2} + 4y_2)$, then $\mathbf{y}_p = (2\pi e^{-x_2}, \pi/2)$. In the derivation of the homogenization problem that follows, it is assumed that $\mathbf{y}_p$ is independent of $\mathbf{x}$.

An important consequence of periodicity arises when determining what values a function can take on the boundary of a cell. To explain, consider opposing boundary points $\mathbf{y}_\ell$ and $\mathbf{y}_r$ as illustrated in Fig. 5.3(b). A one-dimensional version of this situation is shown in Fig. 5.4. If a function, say

FIGURE 5.3. a) Cells in the plane. They are rectangular regions with sides of length p_1 and p_2. b) A single cell illustrating opposing boundary points $\mathbf{y}_\ell$ and $\mathbf{y}_r$.

FIGURE 5.4. Graphs of two functions that are periodic with period $y_p = 1$. In the top curve the periodic extension, from $0 \leq y \leq 1$ to $1 \leq y \leq 2$, results in a function that is not differentiable at $y = 1$. In contrast, the extension for the bottom curve is differentiable. So, for opposing boundary points we require $\partial_y w(y_\ell) = \partial_y w(y_r)$.

$w = w(\mathbf{y})$, is to be periodic, it is necessary that $w(\mathbf{y}_r) = w(\mathbf{y}_\ell)$. Moreover, if it is to be smooth, it must also be that $\nabla_y w(\mathbf{y}_r) = \nabla_y w(\mathbf{y}_\ell)$. In fact, if it is to be C^2, we must have $\partial_{y_i}\partial_{y_j} w(\mathbf{y}_r) = \partial_{y_i}\partial_{y_j} w(\mathbf{y}_\ell)$ $\forall i, j$. These conditions must hold at all opposing boundary points of a cell.

The primary reason for assuming the problem is periodic can be demonstrated with the simple equation $w''(y) = \kappa$. The general solution of this is $w(y) = \frac{1}{2}\kappa y^2 + c_0 y + c_1$. Now, if $w(y)$ is periodic, it must be that $\kappa = 0$. This conclusion can also be obtained without solving the differential equation. All we have to do is to integrate the differential equation over a period $(0 \leq y \leq y_p)$ and use the periodicity to conclude that $\kappa = 0$. This labor-saving observation is used in the homogenization procedure to avoid having to solve the $O(\varepsilon^2)$ problem.

Homogenization Procedure

The procedure to be used is essentially the same as in the previous section. So, letting $\mathbf{y} = \mathbf{x}/\varepsilon$, then the derivatives transform as $\nabla \to \nabla_x + \varepsilon^{-1}\nabla_y$, where the subscript indicates the variables involved in the differentiation. In this case (5.29a) takes the form

$$(\nabla_y + \varepsilon\nabla_x) \cdot [D(\nabla_y + \varepsilon\nabla_x)u] = \varepsilon^2 f(\mathbf{x}). \tag{5.30}$$

A regular multiple-scale expansion is appropriate here, and so

$$u \sim u_0(\mathbf{x}, \mathbf{y}) + \varepsilon u_1(\mathbf{x}, \mathbf{y}) + \varepsilon^2 u_2(\mathbf{x}, \mathbf{y}) + \cdots . \tag{5.31}$$

Because of the assumption on D, we will assume that the terms in this expansion are periodic in $\mathbf{y}$ with period $\mathbf{y}_p$.

With the expansion given in (5.31), the following equations are obtained:

$O(1)$ $\nabla_y \cdot (D\nabla_y u_0) = 0.$

The general solution of this equation, which is periodic in $\mathbf{y}$, is $u_0 = u_0(\mathbf{x})$.

$O(\varepsilon)$ $\nabla_y \cdot (D\nabla_y u_1) = -\nabla_y D \cdot \nabla_x u_0.$

Because u_1 is periodic in $\mathbf{y}$, it suffices to solve this equation only on a cell Ω_0 and then simply extend the solution using periodicity. Two other important observations are that u_0 is independent of $\mathbf{y}$ and the problem for u_1 is linear. Therefore, the solution can be written as

$$u_1(\mathbf{x}, \mathbf{y}) = \mathbf{a} \cdot \nabla_x u_0 + c(\mathbf{x}), \tag{5.32}$$

where the components a_i of the vector $\mathbf{a}$ satisfy the equation

$$\nabla_y \cdot (D\nabla_y a_i) = -\partial_{y_i} D, \quad \text{in } \Omega_0.$$

It is not possible to determine the a_i's until the coefficient function D is specified. Also, the function $c(\mathbf{x})$ in (5.32) is arbitrary except for being independent of $\mathbf{y}$.

$O(\varepsilon^2)$ $\nabla_y \cdot (D\nabla_y u_2) = -\nabla_y \cdot (D\nabla_x u_1) - \nabla_x \cdot (D\nabla_y u_1) - \nabla_x \cdot (D\nabla_x u_0) + f(\mathbf{x}).$

To complete the determination of the homogenized problem, we introduce the average of a function $v(\mathbf{x}, \mathbf{y})$ over a cell Ω_0. This is defined as

$$\langle v \rangle_p \equiv \frac{1}{|\Omega_0|} \int_{\Omega_0} v(\mathbf{x}, \mathbf{y}) dV_y, \tag{5.33}$$

where $|\Omega_0|$ is the volume (or the area if in $\mathbb{R}^2$) of the cell. We are going to average the $O(\varepsilon^2)$ equation. Beginning with the left-hand side, we have, using the divergence theorem,

$$\begin{aligned}\langle \nabla_y \cdot (D\nabla_y u_2)\rangle_p &= \frac{1}{|\Omega_0|}\int_{\partial\Omega_0} D\mathbf{n}\cdot\nabla_y u_2 dS_y \\ &= 0.\end{aligned}$$

The fact that the above integral around the boundary is zero comes from the periodicity. Using the same argument, one finds that $\langle \nabla_y \cdot (D\nabla_x u_1)\rangle_p = 0$. Also, from (5.32),

$$\begin{aligned}\langle D\partial_{y_i} u_1\rangle_p &= \langle D\partial_{y_i}(\mathbf{a}\cdot\nabla_x u_0)\rangle_p \\ &= \langle D\partial_{y_i}\mathbf{a}\rangle_p \cdot \nabla_x u_0. \qquad (5.34)\end{aligned}$$

The other terms in the $O(\varepsilon^2)$ equation average to produce the following: $\langle f\rangle_p = f$ and $\langle \nabla_x \cdot (D\nabla_x u_0)\rangle_p = \nabla_x \cdot (\langle D\rangle_p \nabla_x u_0)$.

By averaging the $O(\varepsilon^2)$ equation over a cell, the homogenized problem for the first-term approximation of the solution is obtained. In $\mathbb{R}^2$ this yields the following:

$$\partial_{x_1}(D_{11}\partial_{x_1}u_0 + D_{12}\partial_{x_2}u_0) + \partial_{x_2}(D_{21}\partial_{x_1}u_0 + D_{22}\partial_{x_2}u_0) = f(\mathbf{x}), \quad \text{for } \mathbf{x}\in\Omega, \qquad (5.35a)$$

where

$$u_0 = g(\mathbf{x}), \quad \text{for } \mathbf{x}\in\partial\Omega. \qquad (5.35b)$$

It is possible to write the homogenized equation in (5.35a) in vector form as

$$\nabla_x \cdot (\mathbf{D}\nabla_x u_0) = f(\mathbf{x}), \quad \text{for } \mathbf{x}\in\Omega.$$

The effective, or homogenized, coefficients in (5.35a) are given as

$$D_{ij} = \langle D\rangle_p \delta_{ij} + \langle D\partial_{y_i} a_j\rangle_p. \qquad (5.36)$$

The functions a_j are smooth periodic solutions of the cell problem

$$\nabla_y \cdot (D\nabla_y a_j) = -\partial_{y_j} D, \quad \text{in } \Omega_0. \qquad (5.37)$$

At this point, there are two options. We can take the point of view that (5.35) is now the problem to solve and essentially ignore where it comes from. In this case, we do not know the coefficients even if we know D. This is not an uncommon situation, and the homogenization procedure is used in this case to justify using a particular set of equations to model a physical system. Basically, what must be done is to design an experiment, or series of experiments, that can be used to measure the D_{ij}'s.

Another option open to us is to try to determine exactly how the homogenized coefficients depend on the substructure. This means solving the cell problems in (5.37). The most common way to do this is numerically (e.g., Auriault et al. (1985) and Guedes and Kikuchi (1990)), but it is also possible to find examples where analytic solutions can be obtained.

Example

Suppose, in $\mathbb{R}^2$, the cell Ω_0 is the rectangular region $0 \le y_1 \le a$, $0 \le y_2 \le b$. To determine the homogenized coefficients in (5.36), it is necessary to solve the cell problem. In this example, we will take

$$D(x_1, x_2, y_1, y_2) = D_0(x_1, x_2)e^{\alpha(y_1)+\beta(y_2)}, \tag{5.38}$$

where $\alpha(y_1)$ and $\beta(y_2)$ are periodic. For example, one might have $\alpha(y_1) = 5\sin(2\pi y_1/a)$ and $\beta(y_2) = \cos(6\pi y_2/b)$. The form of the coefficient given in (5.38) results in a cell problem that is solvable in closed form because D is separable in terms of its dependence on the components of $\mathbf{y}$. With (5.38), the solution for u_1 given in (5.32) is

$$u_1(\mathbf{x}, \mathbf{y}) = c(\mathbf{x}) + \mathbf{a} \cdot \nabla_x u_0,$$

where

$$a_1 = -y_1 + \kappa_1 \int_0^{y_1} e^{-\alpha(s)} ds \text{ and } a_2 = -y_2 + \kappa_2 \int_0^{y_2} e^{-\beta(s)} ds.$$

The constants κ_1 and κ_2 are determined from the requirement that u_1 is periodic. To have $a_1(0) = a_1(a)$ and $a_2(0) = a_2(b)$, one finds that

$$\kappa_1 = a\left(\int_0^a e^{-\alpha(s)} ds\right)^{-1} \text{ and } \kappa_2 = b\left(\int_0^b e^{-\beta(s)} ds\right)^{-1}.$$

With this it follows that $\partial_{y_1} a_2 = \partial_{y_2} a_1 = 0$, and so, from (5.36), we have that $D_{12} = D_{21} = 0$. Also,

$$\begin{aligned} \langle D\partial_{y_1} a_1\rangle_p &= \frac{1}{ab}\int_0^a \int_0^b (-1 + \kappa_1 e^{-\alpha}) D_0 e^{\alpha+\beta} dy_1 dy_2 \\ &= -\langle D\rangle_p + D_0 \left(\frac{1}{b}\int_0^b e^{\beta(s)} ds\right)\left(\frac{1}{a}\int_0^a e^{-\alpha(s)} ds\right)^{-1}, \end{aligned}$$

and

$$\langle D\partial_{y_2} a_2\rangle_p = -\langle D\rangle_p + D_0 \left(\frac{1}{a}\int_0^a e^{\alpha(s)} ds\right)\left(\frac{1}{b}\int_0^b e^{-\beta(s)} ds\right)^{-1}.$$

Therefore, the homogenized differential equation (5.35a) is

$$\partial_{x_1}(D_1 \partial_{x_1} u_0) + \partial_{x_2}(D_2 \partial_{x_2} u_0) = f(\mathbf{x}), \tag{5.39}$$

where $D_i(\mathbf{x}) = \lambda_i D_0(\mathbf{x})$ for

$$\lambda_1 = \left(\frac{1}{b}\int_0^b e^{\beta(s)} ds\right)\left(\frac{1}{a}\int_0^a e^{-\alpha(s)} ds\right)^{-1} \tag{5.40}$$

and

$$\lambda_2 = \left(\frac{1}{a}\int_0^a e^{\alpha(s)}ds\right)\left(\frac{1}{b}\int_0^b e^{-\beta(s)}ds\right)^{-1}. \tag{5.41}$$

Interestingly, in this multidimensional example we do not get the harmonic mean. Rather, for each D_i we get the harmonic mean of one of the subscale components of D multiplied by the arithmetic mean in the other subscale component.

There are several observations that should be made at this point. First, if we consider (5.29a) as a steady-state heat equation, then we have started out with a material that is inhomogeneous and isotropic. What comes out of the homogenization procedure is a steady-state heat equation for an inhomogeneous and anisotropic material. This is saying that a material that is isotropic on the microscale can be anisotropic on the macroscale. The degree of anisotropy depends on the coefficient D. Also, note that the contribution of the substructure is entirely through the homogenized coefficients D_{ij}. To find these terms, it is necessary to determine the solution of the cell equation given in (5.37). This cell problem needs to be solved only once, as it does not depend on the boundary conditions on $\partial\Omega$ or the forcing function. This situation that the cell problem is not explicit in the homogenized equation makes it similar to what are called hidden variables in thermo-viscoelasticity (Fung, 1965). There are also numerous mathematical questions associated with the above analysis. For one thing, it is not obvious that the homogenized problem is well posed. Related to this is the question of whether or not the coefficients D_{ij} are positive. The reader is referred to Sanchez-Palencia (1980), where some of these questions are addressed.

Exercises

1. Suppose $f(\mathbf{x})$ is (5.29a) is replaced with $f(\mathbf{x}, \mathbf{x}/\varepsilon, u)$, where $f(\mathbf{x}, \mathbf{y}, u)$ is periodic in $\mathbf{y}$ with the same period as $D(\mathbf{x}, \mathbf{y})$.

   a) Find the homogenized differential equation. The term involving f will be determined below.

   b) If $f(\mathbf{x}, \mathbf{y}, u) = f_0(\mathbf{x}, \mathbf{y}) + f_1(\mathbf{x}, \mathbf{y})u^k$, where k is a positive integer, then what is the resulting homogenized equation?

   c) What is the homogenized equation if $f(\mathbf{x}, \mathbf{y}, u) = e^{\alpha(\mathbf{x},\mathbf{y})u}$? From this, and the result from (b), comment on the method of homogenization and nonlinear equations.

2. The homogenized equation (5.35a) should equal the original (5.29a) if D is independent of $\mathbf{y}$. For example, this happens in (5.39)–(5.41) if $\alpha = \beta = 0$. Is this true in general?

3. In $\mathbb{R}^2$ suppose $D(x_1, x_2, y_1, y_2) = D(x_2, x_1, y_2, y_1)$. Because of this symmetry, you might expect that $D_{11} = D_{22}$ and $D_{12} = D_{21}$. Is this the case?

4. Find the homogenized equation when the period $\mathbf{y}_p$ depends on $\mathbf{x}$. You can assume that the problem is in $\mathbb{R}^2$.

5. In $\mathbb{R}^2$ suppose D is independent of y_2, that is, $D = D(x_1, x_2, y_1)$. What do (5.35)–(5.37) reduce to in this case?

6. Suppose the Green's function can be found for the cell problem. Explain why the solution of (5.37) in this case can be written as $a_i = G[\partial_{s_i} D]$. Also, show that the homogenized coefficients can now be written as

$$D_{ij} = \langle D \rangle_p \delta_{ij} + \langle (\partial_{y_i} D) G[\partial_{s_j} D] \rangle_p.$$

7. Consider the wave equation

$$\partial_x (D \partial_x u) = \partial_t^2 u, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x, 0) = f(x)$ and $\partial_t u(x, 0) = 0$. In this problem, assume $D = D(x/\varepsilon)$, where $D(y)$ is smooth, positive, and periodic.

a) Introducing the scales x, $y = x/\varepsilon$, $t_1 = t/\varepsilon$, and $t_2 = \varepsilon t$, show that the first term in the expansion has the form

$$u_0(x, y, t_1, t_2) = \sum A_n(x, t_2) F_n(y) \sin(\lambda_n t_1).$$

The substructure modes F_n can be determined once the coefficient D has been specified. This form of the solution is known as a Bloch expansion (Odeh and Keller, 1964; Santosa and Symes, 1991).

b) Use a similar modal expansion to determine u_1. Is it necessary to restrict D to guarantee that secular terms will not be present in u_1?

c) Using a modal expansion, find u_0 from the $O(\varepsilon^2)$ problem.

d) The first-term approximation for the solution of the original problem depends on the substructure scale. Suppose we decide that it is sufficient to have an approximation obtained by averaging the solution over a spatial interval that is between the microscale $O(\varepsilon)$ and the macroscale $O(1)$. Explain why this leads to the following approximation:

$$u(x, t) \sim \frac{1}{y_p} \int_0^{y_p} u_0(x, y, t/\varepsilon, \varepsilon t)\, dy.$$

Carry out this average using your result from part (c) to obtain an approximation of the solution.

FIGURE 5.5. A magnified view of the substructure of the porous medium. It is assumed that the substructure length scales for the fluid (R_f) and solid (R_s) regions are much smaller than the macroscale.

5.4 Porous Flow

One of the more interesting applications of homogenization is in the study of materials made up of multiple constituents. To illustrate the ideas, we will consider the flow of a viscous fluid through a porous solid. This particular problem is important in a wide variety of disciplines, such as in geophysics when studying the motion of a fluid, like water or oil, through sand. An example of the substructure geometry that is being considered is given in Fig. 5.5. The essential assumption concerning the geometry is that the length scale ℓ for the pores and other substructure components is small compared to the macroscopic length scale L. This disparity in length scales is what provides us with our expansion parameter $\varepsilon = \ell/L$.

The solid region is taken to be rigid, and the fluid is assumed to be incompressible and viscous. Letting R_f designate the fluid region, then in R_f the following equations hold:

$$\varepsilon^2 \nabla^2 \mathbf{v} = \nabla p, \tag{5.42}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{5.43}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the fluid velocity vector and $p(\mathbf{x}, t)$ is the fluid pressure. At the fluid–solid interface, ∂R_{fs}, the no-slip condition applies. Thus, at this interface

$$\mathbf{v}\,\big|_{\partial R_{fs}} = \mathbf{0}\,. \tag{5.44}$$

What makes the problem difficult is the complexity of the region R_f. A closed-form solution is essentially unachievable, and even a numerical solution would be nearly impossible to calculate except for the simplest substructure geometries. Therefore, a procedure like homogenization is a particularly valuable tool for such a problem. It should be kept in mind that our objective is not to solve the problem. Rather, we will derive a set of equations that hold over a much simpler domain. The contributions of the substructure will come through the coefficients of the homogenized equation.

Reduction Using Homogenization

To describe the substructure near $\mathbf{x}_0$, we use the variable $\mathbf{y} = (\mathbf{x} - \mathbf{x}_0)/\varepsilon$. With this, (5.42) and (5.43) take the form

$$\varepsilon(\nabla_y + \varepsilon\nabla_x)^2\mathbf{v} = (\nabla_y + \varepsilon\nabla_x)p, \tag{5.45}$$

$$(\nabla_y + \varepsilon\nabla_x)\cdot\mathbf{v} = 0. \tag{5.46}$$

For small ε the appropriate expansion of the fluid velocity is

$$\mathbf{v} \sim \mathbf{v}_0(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{v}_1(\mathbf{x},\mathbf{y}) + \cdots, \tag{5.47}$$

and for the pressure

$$p \sim p_0(\mathbf{x},\mathbf{y}) + \varepsilon p_1(\mathbf{x},\mathbf{y}) + \cdots. \tag{5.48}$$

Substituting these expansions into (5.45) and (5.46), one obtains the following problems:

$$O(1) \quad \nabla_y p_0 = \mathbf{0}, \tag{5.49}$$

$$\nabla_y \cdot \mathbf{v}_0 = 0. \tag{5.50}$$

From (5.44), the interfacial boundary condition is

$$\mathbf{v}_0\,\big|_{\partial R_{fs}} = \mathbf{0}\,. \tag{5.51}$$

It follows from (5.49) that

$$p_0 = p_0(\mathbf{x}). \tag{5.52}$$

It remains to solve (5.50), which will be done in conjunction with solving the $O(\varepsilon)$ equation for the pressure.

$$O(\varepsilon) \quad \nabla_y p_1 + \nabla_x p_0 = \nabla_y^2\mathbf{v}_0, \tag{5.53}$$

$$\nabla_y \cdot \mathbf{v}_1 + \nabla_x \cdot \mathbf{v}_0 = 0. \tag{5.54}$$

Unlike the other examples we have examined in this chapter, the first term in the expansion for the velocity depends on the substructure variable $\mathbf{y}$. This dependence is determined by solving (5.50), (5.51), and (5.53) for p_1 and $\mathbf{v}_0$ in terms of p_0. This is no small task, as these equations constitute a forced Stokes problem that must be solved in the substructure regions occupied by the fluid. What helps us significantly is that $\nabla_x p_0$ is independent of $\mathbf{y}$. With this, and the linearity of the problem, we can write p_1 and each component of $\mathbf{v}_0$ as linear combinations of the components of $\nabla_x p_0$. Therefore, we have

$$\mathbf{v}_0 = \mathbf{B}\nabla_x p_0 \tag{5.55}$$

and

$$p_1 = \mathbf{g} \cdot \nabla_x p_0, \tag{5.56}$$

where the matrix $\mathbf{B}(\mathbf{x}, \mathbf{y})$ and the vector $\mathbf{g}(\mathbf{x}, \mathbf{y})$ are determined by the substructure. Their exact forms are not needed to complete the homogenization procedure; a discussion of their properties can be found in Exercise 2.

Averaging

The next, and final, step in the homogenization process is to average the velocity over a length scale that is between the microscopic and macroscopic scales. This step was not used in the problems considered earlier in this chapter because the first terms in the expansions were always independent of the substructure variable $\mathbf{y}$. This is not the case for the porous flow problem.

To simplify the averaging, we will now assume the substructure is periodic with period cell Ω_0. In conjunction with this, we also assume the terms of the expansions in (5.47) and (5.48) are periodic with the same cell. In this case, the average of a dependent variable $\zeta(\mathbf{x})$ at $\mathbf{x}_0$ is defined as

$$\langle \zeta \rangle \equiv \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \zeta(\mathbf{s}) dV_s, \tag{5.57}$$

where $B_\varepsilon = \{\mathbf{s} : |\mathbf{s} - \mathbf{x}_0| < \sqrt{\varepsilon}\}$ and $|B_\varepsilon|$ is the volume of B_ε. Letting $\zeta_0(\mathbf{x}_0, \mathbf{y})$ be a first-term, multiple-scale approximation of ζ for small ε, then

$$\begin{aligned} \langle \zeta \rangle &\sim \frac{(\text{number of cells in } B_\varepsilon)}{|B_\varepsilon|} \int_{B_\varepsilon \cap \Omega_0} \zeta_0\left(\mathbf{x}_0, \frac{\mathbf{s} - \mathbf{x}_0}{\varepsilon}\right) dV_s \\ &= \frac{(\text{number of cells in } B_\varepsilon)\varepsilon^3}{|B_\varepsilon|} \int_{\Omega_f} \zeta_0(\mathbf{x}_0, \mathbf{y}) dV_y \\ &\sim \frac{1}{|\Omega_0|} \int_{\Omega_f} \zeta_0(\mathbf{x}_0, \mathbf{y}) dV_y, \end{aligned} \tag{5.58}$$

where $|\Omega_0|$ is the volume of Ω_0 and Ω_f is the region occupied by the fluid in Ω_0. A variable of particular importance associated with this average is the volume fraction ϕ^f of the fluid. This comes from the average of the characteristic function of the fluid, that is,

$$\zeta = \begin{cases} 1 & \text{if } \mathbf{x} \in R_f \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $\langle \zeta \rangle \sim \phi^f$, where

$$\phi^f(\mathbf{x}_0) = \frac{|\Omega_f|}{|\Omega_0|} \tag{5.59}$$

and $|\Omega_f|$ is the volume of Ω_f. In what follows it is assumed that Ω_0 is constant, but Ω_f may depend on $\mathbf{x}$.

Before going any farther, it is necessary to state some of the properties of the boundary of the fluid domain Ω_f in a period cell. In particular, $\partial\Omega_f$ consists of two parts. One is the fluid–solid interface within Ω_0, which is denoted as $\partial\Omega_{fs}$, and the other is the portion that lies on $\partial\Omega_0$ (Fig. 5.5). It is assumed that these surfaces are smooth enough that the divergence theorem can be used.

To carry out the averaging, we will use the Reynolds transport theorem, which states

$$\partial_{x_i} \int_{\Omega_f} \zeta(\mathbf{x},\mathbf{y}) dV_y = \int_{\Omega_f} \partial_{x_i}\zeta(\mathbf{x},\mathbf{y}) dV_y + \int_{\partial\Omega_f} \zeta \left(\frac{\partial \mathbf{r}}{\partial x_i}\right) \cdot \mathbf{n}\, dS_y, \tag{5.60}$$

where $\mathbf{r} = \mathbf{r}(\mathbf{x},\mathbf{y})$ describes $\partial\Omega_{fs}$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega_f$. Note that the surface integral in (5.60) does not contain any points on $\partial\Omega_0$ since this surface has been assumed to be independent of $\mathbf{x}$. It follows from (5.60) and (5.59) that

$$\partial_{x_i}\phi^f = \frac{1}{|\Omega_0|} \int_{\partial\Omega_{fs}} \left(\frac{\partial \mathbf{r}}{\partial x_i}\right) \cdot \mathbf{n}\, dS_y. \tag{5.61}$$

Because p_0 is independent of $\mathbf{y}$, the easiest equation to average is (5.55). One finds

$$\langle \mathbf{v}_0 \rangle = -\mathbf{L}\nabla_x p_0, \tag{5.62}$$

where

$$\mathbf{L}(\mathbf{x}) = -\frac{1}{|\Omega_0|} \int_{\Omega_f} \mathbf{B}\, dV_y. \tag{5.63}$$

The homogenized equation for the velocity comes from (5.50). Using the divergence theorem, Reynolds transport theorem (5.60), and the interfacial condition (5.51), the following is obtained:

$$\begin{aligned} 0 &= \int_{\Omega_f} (\nabla_y \cdot \mathbf{v}_1 + \nabla_x \cdot \mathbf{v}_0) dV_y \\ &= -\int_{\partial\Omega_f} \mathbf{v}_1 \cdot \mathbf{n}\, dS_y + \nabla_x \cdot \int_{\Omega_f} \mathbf{v}_0\, dV_y - \sum_{i=1}^{3} \int_{\partial\Omega_f} v_{0_i} \left(\frac{\partial \mathbf{r}}{\partial x_i}\right) \cdot \mathbf{n}\, dS_y \\ &= \nabla_x \cdot \int_{\Omega_f} \mathbf{v}_0\, dV_y. \end{aligned}$$

The first-term approximation of the averaged velocity therefore satisfies the equation

$$\nabla_x \cdot \langle \mathbf{v}_0 \rangle = 0. \tag{5.64}$$

Homogenized Problem

Suppose the porous material occupies a (macroscopic) domain Ω made up of solid (R_s) and fluid (R_f) regions. From the homogenization procedure, we have derived a fluid velocity $\mathbf{v}_h(\mathbf{x})$ and pressure $p_h(\mathbf{x})$ that are defined throughout Ω. From (5.62) and (5.64), these variables satisfy the homogenized equations

$$\mathbf{v}_h = -\mathbf{L}\nabla_x p_h \tag{5.65}$$

and

$$\nabla_x \cdot \mathbf{v}_h = 0. \tag{5.66}$$

Note that (5.66) is the standard continuity equation for an incompressible fluid. The relationship in (5.65) is known as Darcy's law, and the matrix $\mathbf{L}$ is called the permeability tensor (Scheidegger, 1974). It is this homogenized coefficient tensor that contains the contributions of the substructure.

The ideas developed for the porous flow problem can also be used to analyze other composite materials. Examples can be found in the exercises as well as in Sanchez-Palencia (1980). His approach, like ours, assumes periodicity of the substructure. However, it is not necessary to make such an assumption; examples of this are given in Section 5.2. An extension to multidimensional problems can be found in Burridge and Keller (1981).

Exercises

1. There are ways to smooth out the substructure to describe the flow in a porous solid that do not use homogenization. One of these is mixture theory and for the case of a rigid solid, the mixture equations are (Holmes, 1984, 1985)

$$\nabla_x \boldsymbol{\sigma}_M = -\boldsymbol{\pi}_M$$

and

$$\nabla_x \cdot (\phi^f \mathbf{v}_M) = 0.$$

Here $\boldsymbol{\sigma}_M$ is the stress in the fluid, $\mathbf{v}_M$ is the fluid velocity, and the vector $\boldsymbol{\pi}_M$ is assumed to have the form $\boldsymbol{\pi}_M = p_M \nabla_x \phi^f - \mathbf{K}\,\mathbf{v}_M$.

a) Determine the mixture quantities ($\boldsymbol{\sigma}_M, \mathbf{v}_M, p_M$, and $\mathbf{K}$) in terms of the entries that make up the homogenized equations (5.65) and (5.66).

b) The averaging used in (5.58) is known as a phase average since it is based on the total volume of the cell. An alternative is the intrinsic phase average, which uses the true volume occupied by the constituent (i.e., $|\Omega_f|$ is used in (5.58) rather than $|\Omega_0|$). Based on the results of part (a), comment on these averages and the mixture formulation.

2. This problem examines some of the properties of the elements of the matrix $\mathbf{B}$ in (5.55) and the matrix $\mathbf{L}$ in (5.63). It is assumed here that the substructure is periodic.

   a) Let $\mathbf{q}_j(\mathbf{x},\mathbf{y})$ be the jth column vector from the matrix $\mathbf{B}$. Show that $\nabla_y^2\mathbf{q}_j = \nabla_y g_j - \mathbf{e}_j$ and $\nabla_y \cdot \mathbf{q}_j = 0$ in Ω_f. What are the boundary conditions? Here $\mathbf{g}(\mathbf{x},\mathbf{y})$ is the vector in (5.56) and $\mathbf{e}_j$ is the jth unit coordinate vector.

   b) Let $\mathbf{q}_j = (\alpha_j, \beta_j, \gamma_j)$ and note, from part (a), that $\mathbf{e}_j = \nabla_y g_j - \nabla_y^2\mathbf{q}_j$. For $j = 2$, take the dot product of this equation with $(0, \beta_1, 0)$ and then integrate over Ω_f to find a formula for L_{21}. Similarly, using $(\alpha_2, 0, 0)$ and $j = 1$, find L_{12}. With these formulas, and using the continuity equation, show that $L_{12} = L_{21}$. This argument is easily extended to prove that $\mathbf{L}$ is symmetric.

   c) Darcy's law is usually applied to macroscopically isotropic porous materials. In such situations, the matrix $\mathbf{L}$ is replaced with a scalar. Under what conditions is $\mathbf{L}$, as defined in (5.63), a scalar multiple of the identity?

3. This exercise investigates a problem that arises in the study of the diffusion of gases in polymers. In what is known as a dual-mode sorption model, the polymer is a composite with a substructure much like the one shown in Fig. 5.5 except now the fluid regions are called microvoids. The concentration of gas in the two regions is governed by the following equations (Sangani, 1986):

$$\partial_t C_s = D_s \nabla^2 C_s \text{ in } R_s \text{ and } \partial_t C_f = D_f \nabla^2 C_f \text{ in } R_f,$$

   where C_s and C_f are the concentrations in the respective regions. At the interface ∂R_s (or, equivalently, at ∂R_f) it is required that

$$(D_s \nabla C_s - D_f \nabla C_f) \cdot \mathbf{n} = 0 \text{ and } C_f = \frac{\alpha C_s}{1 + \varepsilon^2 \beta C_s}.$$

   Here $\mathbf{n}$ is a normal to the interface, and D_s, D_f, α, and β are positive constants (that are independent of ε). It is assumed that the substructure is periodic and does not vary with $\mathbf{x}$.

   a) Using multiple scales, show that the first terms in the expansions of C_s and C_f are independent of $\mathbf{y}$.

   b) Show that the solution of the $O(\varepsilon^2)$ problem can be written as $C_{s_1}(\mathbf{x},\mathbf{y},t) = A_s(\mathbf{x},t) - \mathbf{w}_s \cdot \nabla_x C_{s_0}$, where $\mathbf{w}_s(\mathbf{x},\mathbf{y})$ and $A_s(\mathbf{x},t)$ are independent of the concentrations. You should also find a similar expression for $C_{f_1}(\mathbf{x},\mathbf{y},t)$.

   c) From the $O(\varepsilon^3)$ problem, show that $\partial_t C_{s_0} = \nabla_x \cdot (\mathbf{D}_s \nabla_x C_{s_0})$,

where the diffusivity tensor $\mathbf{D}_s$ is given as

$$\mathbf{D}_s = \frac{1}{|\Omega_s| + \alpha|\Omega_f|}\left[\chi\mathbf{I} - D_s\int_{\Omega_s}(\nabla_y\mathbf{w}_s)^T dV_y - D_f\int_{\Omega_f}(\nabla_y\mathbf{w}_f)^T dV_y\right]$$

for $\chi = D_s|\Omega_s| + \alpha D_f|\Omega_f|$.

d) What do the results from parts (a)–(c) reduce to in the one-dimensional case? Assume that the period cell is $0 < y < 1$, where the "solid" occupies the interval $0 < y < r$ and the "fluid" region is $r < y < 1$.

4. An interesting problem in electrochemistry concerns the diffusion of ions through a solution containing charged molecules. In reference to Fig. 5.5, the molecules form the solid region, and the solution makes up the fluid region. In R_f the following hold (Holmes, 1990):

$$\varepsilon^2\nabla^2\phi = z\alpha S e^{z\phi(\mathbf{x},t)}$$

and

$$\partial_t S + zS\partial_t\phi = \nabla^2 S + z\nabla S\cdot\nabla\phi,$$

where $\varepsilon\partial_n\phi = \sigma$ and $\partial_n S = 0$ on ∂R_f. Here $\phi(\mathbf{x},t)$ is the electrostatic potential in the solution, and $S(\mathbf{x},t)$ is related to the concentration $C(\mathbf{x},t)$ of diffusing ions through the relation $S(\mathbf{x},t) = C(\mathbf{x},t)\exp(-z\phi(\mathbf{x},t))$. This problem arises in the study of the physiology of tissue and in the conformational properties of polynucleotides like DNA. One of the questions of interest is how the geometry and charge density of the charged molecules affect the diffusion of ions.

a) Using the multiple-scale expansions $\phi(\mathbf{x},t) \sim \phi_0(\mathbf{x},\mathbf{y},t) + \varepsilon\phi_1(\mathbf{x},\mathbf{y},t) + \cdots$ and $S(\mathbf{x},t) \sim 1 + \varepsilon S_1(\mathbf{x},\mathbf{y},t) + \cdots$, show that S_1 is independent of $\mathbf{y}$ and find the problem ϕ_0 satisfies.

b) Show that the solution of the $O(\varepsilon^2)$ problem can be written as $S_2(\mathbf{x},\mathbf{y},t) = R_2(\mathbf{x},t) - \mathbf{w}\cdot\nabla_x S_1$ and $\phi_1(\mathbf{x},\mathbf{y},t) = S_1(\mathbf{x},t)f(\mathbf{y})$, where $R_2(\mathbf{x},t)$, $\mathbf{w}(\mathbf{y})$, and $f(\mathbf{y})$ are independent of S_1.

c) From the $O(\varepsilon^3)$ problem, and assuming the substructure is periodic, show that

$$\partial_t S_1\int_{\Omega_f} e^{z\phi_0}dV_y = \nabla_x\cdot\left[\nabla_x S_1\int_{\Omega_f}(\mathbf{I} - \nabla_y\mathbf{w})e^{z\phi_0}dV_y\right].$$

d) Of particular interest is the concentration $C = S(\mathbf{x},t)e^{-z\phi(\mathbf{x},t)}$. Using the results from parts (a)–(c), show that $\langle C\rangle \sim \langle C\rangle_0 + \varepsilon\langle C\rangle_1$, where $\langle C\rangle_0$ is independent of $\mathbf{x}_0$ and t, and

$$\langle C\rangle_1 = S_1(\mathbf{x}_0,t)\frac{\phi^f}{|\Omega_f|}\int_{\Omega_f}(1 + f(\mathbf{y}))e^{z\phi_0(\mathbf{y})}dV_y.$$

e) Using the two-term expansion in part (d), show that the average concentration satisfies the linear diffusion equation $\partial_t C = \nabla_x \cdot (\mathbf{D}\nabla_x C)$, where the diffusivity tensor $\mathbf{D}$ is given as

$$\mathbf{D} = \mathbf{I} - \frac{\alpha z}{\sigma|\partial\Omega_f|}\int_{\Omega_f} e^{z\phi}(\nabla_y \mathbf{w})^T dV_y.$$

Also, state what problem the potential $\phi(\mathbf{y})$ satisfies.

6

Introduction to Bifurcation and Stability

6.1 Introduction

On several occasions when working out examples in the earlier chapters, we came across problems that have more than one solution. Such situations are not uncommon when studying nonlinear problems, and now we will examine them in detail. The first step is to determine when multiple solutions appear. Once the solutions are found, the next step is to determine if they are stable. Thus, we will focus our attention on what is known as *linear stability theory.* In terms of perturbation methods, almost all the tools we need have already been developed in the earlier chapters. For example, the analysis of steady-state bifurcation uses only regular expansions (Chapter 1), and the stability arguments will use regular and multiple-scale expansions (Chapter 3). On certain examples, such as when studying relaxation dynamics, we will use matched asymptotic expansions (Chapter 2).

6.2 Introductory Example

Before presenting the general formulation, it is worthwhile to examine briefly a relatively simple example. One that has a number of interesting characteristics is the nonlinear oscillator equation

$$y'' + 2\beta y' - \lambda y + y^3 = 0, \quad \text{for } 0 < t. \tag{6.1}$$

FIGURE 6.1. Bifurcation diagram for the steady-state solutions of (6.1).

This is a Duffing type equation, and we have come across these cubicly nonlinear oscillators before. We begin by investigating the steady states. From (6.1) we see that they satisfy

$$\lambda y - y^3 = 0. \tag{6.2}$$

The number of solutions of this equation depends on the parameter λ. Namely, $y = 0$ is a solution for all values of λ, and $y = \pm\sqrt{\lambda}$ are solutions if $\lambda \geq 0$. These steady states are shown in Fig. 6.1.

It is convenient to introduce some terminology that will be used throughout this chapter. The point $(\lambda_b, y_b) = (0, 0)$ where the solutions intersect is called a *bifurcation point* and λ is the *bifurcation parameter*. Also, because of the way the solution curves come together, this is known as a pitchfork bifurcation (presumably, the possibility of calling it a trident bifurcation was also considered but it lost out to pitchfork). In any case, we will say that the solutions $y = \pm\sqrt{\lambda}$ bifurcate from the solution $y = 0$, and we will refer to each of these curves as a branch of the solution.

Given the existence of multiple steady states, it is natural to ask if any of them is achievable. In other words, will the solution of (6.1) approach any one of the three steady states as $t \to \infty$? A partial answer to this question can be inferred from Fig. 6.2, where the numerical solution of (6.1) is shown for different initial values. When starting at $(y, y') = (0, 1)$, the solution spirals into the steady state $y_s = 1$, while the initial point $(y, y') = (2, 0)$ results in the solution spiraling into the steady state $y_s = -1$. Even in cases where a particular steady state y_s appears to be achievable, we would like to know what initial points in the (y, y')-plane result in a particular y_s (this set of points is known as the *basin of attraction* for y_s). These are some of the questions we will address in this chapter, but the first topic we need to consider is how to find the bifurcation points.

FIGURE 6.2. Numerical solution of (6.1) using the initial conditions: (i) $y(0) = 0$, $y'(0) = 1$, and (ii) $y(0) = 2, y'(0) = 0$. In these calculations, $\beta = 1/4$ and $\lambda = 1$. The possible steady states in this case are $0, \pm 1$.

6.3 Analysis of a Bifurcation Point

The steady-state equation (6.2) for the oscillator equation in (6.1) is simple enough that all the solutions can be found in closed form. However, this is not possible for an equation such as

$$y'' + 2\beta y' + \lambda y + \sin^2(y) = 0, \quad \text{for } 0 < t. \tag{6.3}$$

For situations like this, we will employ asymptotic approximations for finding the steady states. To describe our method, suppose we want to find the solutions $y = y_s(\lambda)$ of the equation

$$F(\lambda, y) = 0, \tag{6.4}$$

where F is a smooth function of λ and y. The situations when there is a unique solution of this equation are well established and are described by the Implicit Function Theorem. To state this result, suppose we have found a point (λ_0, y_0) at which (6.4) is satisfied. In this case, if $F \in C^1$ and $F_y(\lambda_0, y_0) \neq 0$, then for λ near λ_0:

1) There is a unique solution $y = y_s(\lambda)$ of (6.4) that satisfies $y_s(\lambda_0) = y_0$.

2) If $F \in C^k$, where k is a positive integer, then $y_s \in C^k$.

This means that at a bifurcation point (λ_b, y_b) it must be that $F(\lambda_b, y_b) = 0$ and $F_y(\lambda_b, y_b) = 0$. However, by themselves these two conditions are not enough to determine a bifurcation point, as will be demonstrated later. It should also be pointed out that a more extensive discussion of the application of the Implicit Function Theorem in bifurcation problems can be found in Crandall and Rabinowitz (1980).

Given a known solution $y = y_s(\lambda)$, we want to determine the bifurcation points (see Fig. 6.3). In what follows, we will say that (λ_b, y_b) is a *bifurcation point* if every disk $B_r(\lambda_b, y_b)$, with radius $r > 0$ and center (λ_b, y_b), contains

FIGURE 6.3. Schematic illustrating the ideas underlying the definition of a bifurcation point. It is based on using a known solution $y = y_s(\lambda)$ of (6.4).

a solution $y(\lambda)$ different from $y_s(\lambda)$. This definition is motivated by the example in Fig. 6.1, but our concept of what should be called a bifurcation point will expand as we progress through this material.

The approach we will be using is known as the *Lyapunov-Schmidt method*, and it constructs the solution(s) near a bifurcation value $\lambda = \lambda_b$. This means we are going to look at what is happening to the solution when $\varepsilon = \lambda - \lambda_b$ is small. To do this, we expand $F(\lambda, y)$ in a Taylor series about the point (λ_b, y_b) as follows:

$$\begin{aligned} F(\lambda, y) &= F(\lambda_b, y_b) + (\lambda - \lambda_b)F_\lambda(\lambda_b, y_b) + (y - y_b)F_y(\lambda_b, y_b) \\ &\quad + \frac{1}{2}\left[(\lambda - \lambda_b)^2 F_{\lambda\lambda} + 2(\lambda - \lambda_b)(y - y_b)F_{\lambda y} + (y - y_b)^2 F_{yy}\right] \\ &\quad + \cdots . \end{aligned} \tag{6.5}$$

This expression can be simplified somewhat by using the fact that y_s is a solution. Since $F[\lambda, y_s(\lambda)] = 0$, then differentiating this with respect to λ yields

$$F_\lambda(\lambda, y_s) + F_y(\lambda, y_s)y_s' = 0, \tag{6.6}$$

and, differentiating again,

$$F_{\lambda\lambda} + 2F_{\lambda y}y_s' + F_{yy}(y_s')^2 + F_y y_s'' = 0. \tag{6.7}$$

To simplify the presentation, we will assume y_s does not depend on λ. In this case, it follows from (6.6) and (6.7) that $F_\lambda = F_{\lambda\lambda} = 0$ when $y = y_s$. By continuing this differentiation process, we can also show $F_{\lambda\lambda\lambda} = F_{\lambda\lambda\lambda\lambda} = \cdots = 0$ when $y = y_s$. Since $y_b = y_s(\lambda_b)$, we can therefore simplify the expansion of $F(\lambda, y)$ given in (6.5) to the following:

$$\begin{aligned} F(\lambda, y) &= \frac{1}{2}(y - y_b)[2F_y(\lambda_b, y_b) + 2\varepsilon F_{\lambda y}(\lambda_b, y_b) \\ &\quad + (y - y_b)F_{yy}(\lambda_b, y_b) + \cdots], \end{aligned} \tag{6.8}$$

where

$$\varepsilon = \lambda - \lambda_b. \tag{6.9}$$

Our objective here is to find other solutions of (6.4) besides $y = y_s$. To do this, we will investigate the possibility of solutions that have expansions of the form

$$y \sim y_b + \varepsilon^\alpha y_1, \tag{6.10}$$

where $\alpha > 0$ and $y_1 \neq 0$ are constants. This expansion differs somewhat from others we have considered in that ε can be both positive and negative. This is an issue because α does not have to be an integer. What we do require is that the solution be real-valued, and this condition will be imposed once α and y_1 are determined.

The next step is to introduce (6.10) into the expansion of $F(\lambda, y)$ given in (6.8). After this is done, the steady-state equation (6.4) reduces to the following:

$$F_y(\lambda_b, y_b) + \frac{1}{2}\left[2\varepsilon F_{\lambda y}(\lambda_b, y_b) + \varepsilon^\alpha F_{yy}(\lambda_b, y_b) y_1\right] + \cdots = 0. \tag{6.11}$$

This is exactly the type of equation that was considered in Chapter 1, and from this we can make the following observations:

a) The $O(1)$ terms in (6.11) must balance, and so, for there to be a solution, it must be that $F_y(\lambda_b, y_b) = 0$. This means that if $F_y(\lambda_b, y_b) \neq 0$ then (λ_b, y_b) is not a bifurcation point (at least in the form we have assumed). This conclusion is in agreement with what is stated in the Implicit Function Theorem.

b) On the other hand, the condition $F_y(\lambda_b, y_b) = 0$ by itself is not enough to guarantee a bifurcation point.

Illustrations of these comments are contained in the following examples.

Examples

1) For the steady states of (6.3), $F(\lambda, y) = \lambda y + \sin^2(y)$. One solution of this is $y_s = 0$, and in this case $F_y(\lambda, 0) = \lambda$. This says that the only possible bifurcation point associated with this solution occurs when $\lambda = 0$. We are definitely interested in determining the bifurcating solutions; how to do this will be discussed later.

2) As another example, suppose $F = y^2$. Taking the known solution to be $y_s = 0$, then $F_y(\lambda, 0) = 0$. However, there is no bifurcation point anywhere along the known solution. This demonstrates why the condition $F_y(\lambda_b, y_b) = 0$ is insufficient to determine the bifurcation points.

We have yet to determine the second term in the expansion of the solution in (6.10). To do this, suppose

$$F(\lambda, y) \equiv \lambda y + f(y), \tag{6.12}$$

where $f(y)$ is smooth. In this example, we will assume $f(0) = 0$, so $y_s = 0$ is a solution of (6.4). We will also assume $f'(0) = 0$. In this case $(\lambda_b, y_b) = (0, 0)$ is a possible bifurcation point since $F_y(0, 0) = 0$. These conditions apply to the nonlinear oscillator equation (6.1) as well as to (6.3). Now, expanding $f(y)$ using Taylor's theorem, the steady-state equation in (6.11) becomes

$$\varepsilon^{1+\alpha} + \frac{1}{n!} y_1^{n-1} \varepsilon^{n\alpha} f^{(n)}(0) + \cdots = 0. \tag{6.13}$$

Here n is the integer where $f(0) = f'(0) = f''(0) = \cdots = f^{(n-1)}(0) = 0$ but $f^{(n)}(0) \neq 0$. Balancing the terms in (6.13), we get $1 + \alpha = n\alpha$. Thus,

$$\alpha = \frac{1}{n-1} \text{ and } y_1^{n-1} = \frac{-n!}{f^{(n)}(0)}.$$

The conclusions are

i) for n even,

$$y \sim y_b + \left(\frac{-n!}{f^{(n)}(0)} (\lambda - \lambda_b) \right)^{\frac{1}{n-1}}; \tag{6.14}$$

ii) for n odd,

$$y \sim \begin{cases} y_b \pm \left(\frac{-n!}{f^{(n)}(0)} (\lambda - \lambda_b) \right)^{\frac{1}{n-1}}, & \text{for } \lambda \geq \lambda_b \text{ if } f^{(n)}(0) < 0 \\ y_b \pm \left(\frac{n!}{f^{(n)}(0)} (\lambda_b - \lambda) \right)^{\frac{1}{n-1}}, & \text{for } \lambda \leq \lambda_b \text{ if } f^{(n)}(0) > 0. \end{cases} \tag{6.15}$$

TABLE 6.1. Schematic of the structure of the bifurcating solutions as determined from (6.14) and (6.15).

	$f^{(n)}(0) < 0$	$f^{(n)}(0) > 0$	Type of Bifurcation
n odd	supercritical	subcritical	Pitchfork
n even			Transcritical

FIGURE 6.4. Bifurcation diagrams illustrating secondary bifurcation. In each case, $y_s = 0$ is a solution. In (a) there are two transcritical bifurcations, one at $(0,0)$ and another at (λ_b, y_b). In (b) there is a supercritical pitchfork bifurcation at $(0,0)$ followed by a subcritical saddle-node at $(1, y_t)$.

Sketches of these solutions are given in Table 6.1. For the pitchfork bifurcation, *supercritical* means that the branches that bifurcate from $y_s = 0$ are to the right of the bifurcation value $\lambda = 0$, and *subcritical* means they are to the left. It should be pointed out that some authors use these terms to include certain stability conditions on the solutions, but we will not do that here.

There are two other types of bifurcation points that appear frequently. One is a Hopf bifurcation, which will be discussed later (see Section 6.7). The other is a *saddle-node*, or *tangent, bifurcation.* In this case, no solutions bifurcate from one side of the bifurcation point and (at least) two bifurcate on the other side. An example of this occurs with the function $F \equiv \lambda - y^2$, which has $(0,0)$ as a saddle-node bifurcation point. In fact, it is a supercritical saddle-node since the branches occur for $\lambda \geq 0$.

It needs to be emphasized that the analysis presented here is local in the sense that it only gives us the solutions in the immediate neighborhood of the bifurcation point. It is very possible that after a branch $y = y_{ss}(\lambda)$ bifurcates from $y_s(\lambda)$, another solution bifurcates from $y_{ss}(\lambda)$, a situation called *secondary bifurcation.* An example of this occurs with the function $F = y(y - \lambda)(y - e^{-\lambda})$. As illustrated in Fig. 6.4(a), the branch that bifurcates transcritically from $y_s = 0$ at $(0,0)$ has a transcritical bifurcation at $\lambda_b > 0$. For a similar reason, in the case of a supercritical pitchfork bifurcation one should not expect the branches to be defined for all $\lambda > \lambda_b$, as might seem to be implied from the bifurcation diagram in Fig. 6.1. An example of this situation occurs with the function $F = y[\lambda + y^2(2y - 3)]$, and the bifurcation diagram for this is shown in Fig. 6.4(b). There is a supercritical pitchfork bifurcation at $(0,0)$, but the only interval where there are three solutions is $0 < \lambda < 1$.

6.4 Linearized Stability

Once the equilibrium solutions are determined, the next question to answer is whether or not they are achievable. In other words, if we start the solution

close to a steady state, will it then converge to that steady state, as $t \to \infty$ or will it at least stay close to the steady state as $t \to \infty$? To illustrate how we will answer this question, we return to the damped oscillator equation

$$y'' + 2\beta y' + F(\lambda, y) = 0, \quad \text{for } 0 < t, \tag{6.16a}$$

where $\beta > 0$. Assuming $y = y_s$ is a steady state, then the initial conditions we will use to investigate the stability of this solution are

$$y(0) = y_s + \alpha_0 \delta \text{ and } y'(0) = \beta_0 \delta. \tag{6.16b}$$

The idea here is that we are starting the solution close to the steady state (see Fig. 6.5), and so we assume δ is small. In this context, α_0 and β_0 are arbitrary constants. We will also assume, in this discussion, that the value of λ is not such that (λ, y_s) is a bifurcation point. In other words, we will assume that $F_y(\lambda, y_s) \neq 0$.

The appropriate expansion of the solution for small δ is

$$y(t) \sim y_s + \delta y_1(t) + \cdots . \tag{6.17}$$

Substituting this into (6.16a) and using Taylor's theorem, we obtain

$$\delta y_1'' + \cdots + 2\beta\delta y_1' + \cdots + F(\lambda, y_s) + \delta y_1 F_y(\lambda, y_s) + \cdots = 0. \tag{6.18}$$

Now, since y_s is an equilibrium solution, $F(\lambda, y_s) = 0$. Thus, the $O(\delta)$ problem we get from (6.18) and (6.16b) is

$$y_1'' + 2\beta y_1' + F_y(\lambda, y_s) y_1 = 0,$$

where

$$y_1(0) = \alpha_0 \text{ and } y_1'(0) = \beta_0.$$

The general solution of the equation depends on the value of $F_y(\lambda, y_s)$. One finds that

$$y_1(t) = \begin{cases} a_0 e^{r_+ t} + a_1 e^{r_- t} & \text{if } F_y(\lambda, y_s) \neq \beta^2 \\ (a_0 + a_1 t) e^{r_+ t} & \text{if } F_y(\lambda, y_s) = \beta^2 \end{cases}, \tag{6.19}$$

FIGURE 6.5. The initial conditions used in the linearized stability analysis are taken to be from within a $O(\delta)$ region around the equilibrium solution.

where a_0 and a_1 are constants determined by the initial conditions and

$$r_{\pm} = -\beta \pm \sqrt{\beta^2 - F_y(\lambda, y_s)}. \tag{6.20}$$

Here we are only interested here in whether or not the equilibrium is stable. From our expansion in (6.17), and our solution (6.19), this can be determined from the signs of the exponents $r_{\pm}$. The easiest one to determine is r_- since $\mathrm{Re}(r_-) < 0$ (here $\mathrm{Re}(r)$ designates the real part of r). So, if $\mathrm{Re}(r_+) < 0$, then $y_1(t) \to 0$ as $t \to \infty$. From this condition we make the following conclusions:

a) If $F_y(\lambda, y_s) > 0$, then $y_1(t) \to 0$ as $t \to \infty$ irrespective of the values of the coefficients a_0 and a_1. In this case, the equilibrium solution is said to be *asymptotically stable* (to small perturbations).

b) If $F_y(\lambda, y_s) < 0$, then $y_1(t)$ remains bounded as $t \to \infty$ only as long as $a_0 = 0$ (assuming $F_y(\lambda, y_s) \neq \beta^2$). This happens only if the initial conditions in (6.16b) satisfy $\beta_0 = r_- \alpha_0$. Because of this, the steady state is said to be *unstable.*

It should be observed that the cases of unstable and asymptotically stable are separated by those points that are possible bifurcation points. Also, our conclusions are based on the assumption that the expansion remains well ordered as $t \to \infty$. In certain circumstances, when this is not the case, it is necessary to use a method such as multiple scales to determine stability and instability.

Examples:

1) If $F(\lambda, y) = \lambda y$, then $y = 0$ is an equilibrium solution and $F_y(\lambda, 0) = \lambda$. For this problem, the exact solution has the form given in (6.19). Therefore, we do not have to limit the initial conditions to a small $O(\delta)$ neighborhood of the equilibrium point to determine stability. We conclude that the solution is globally asymptotically stable if $\lambda > 0$ and is unstable if $\lambda < 0$. By *globally asymptotically stable* it is meant that any initial conditions will result in $y(t) \to y_s$ as $t \to \infty$.

2) If $F(\lambda, y) = -\lambda y + y^3$, then the bifurcation diagram is as shown in Fig. 6.1. Since $F_y(\lambda, 0) = -\lambda$ and $F_y(\lambda, \pm\sqrt{\lambda}) = 2\lambda$, then the stability or instability of each branch is as indicated in Fig. 6.6. Note that there is an *exchange of stability* at $(0, 0)$.

Now that we have worked through an example and introduced the concept of stability, it is best to define it more precisely. The easiest way to do this is to rewrite the equation as a first-order system,

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \quad \text{for } t > 0, \quad \text{where } \mathbf{y}(0) = \mathbf{y}_0. \tag{6.21}$$

FIGURE 6.6. The bifurcation diagram for the steady-state solutions of (6.1) and their stability. There is an exchange of stability at the bifurcation point.

For example, with the nonlinear oscillator in (6.16a) we could take $v = y'$ and then set $\mathbf{y}^T = (y, v)$ and $\mathbf{f}^T = (v, -2\beta v - F(\lambda, y))$, where T designates transpose. In the following definitions $\mathbf{y}_s$ is a steady-state solution and $\mathbf{y}(t)$ is the solution of (6.21).

1) $\mathbf{y}_s$ is *stable* if given any $\xi > 0$ there is a $\delta > 0$ so that

$$| \mathbf{y}_s - \mathbf{y}_0 | < \delta \Rightarrow | \mathbf{y}(t) - \mathbf{y}_s | < \xi \quad \text{for all } t \geq 0.$$

2) $\mathbf{y}_s$ is *attractive* if there is an $\eta > 0$ so that

$$| \mathbf{y}_s - \mathbf{y}_0 | < \eta \Rightarrow \lim_{t \to \infty} | \mathbf{y}(t) - \mathbf{y}_s | = 0.$$

3) $\mathbf{y}_s$ is *asymptotically stable* if it is stable and attractive.

In essence, what it means for $\mathbf{y}_s$ to be stable is that the solution of (6.21) remains close to $\mathbf{y}_s$ if we start close to $\mathbf{y}_s$. Attractive means simply that any solution starting close to $\mathbf{y}_s$ approaches $\mathbf{y}_s$ as $t \to \infty$. The fact that both of these definitions require closeness to $\mathbf{y}_s$ is what enables us to employ perturbation methods to analyze the stability properties of the solution. However, it is possible for $\mathbf{y}_s$ to be attractive but not stable, although this will not occur in the examples we will consider. Those who may be interested in an introduction to the theory underlying some of the methods we will be using should consult the book by Hale and Koçak (1991). An introduction with more of an applied flavor can be found in Hagedorn (1982) and Leipholz (1987).

The above discussion of stability was based on the assumption that the damping parameter β in (6.1) and (6.16a) is positive. The case where $\beta < 0$, something known as *negative damping*, will not be considered here. However, the undamped case, which occurs when $\beta = 0$, is worth investigating because it is something that is quite common. In this case, the exponents in (6.19) are imaginary, and the resulting solution is therefore periodic. This means that the expansion in (6.17) remains well ordered, or at least seems to from what we have derived. The reason for the hesitation is that from the Taylor expansion of the function $F(\lambda, y)$, the function $y_1(t)$ is going to

appear in the higher-order problems. As demonstrated in Chapter 3, this may generate secular terms. Since we are trying to determine the long-time behavior of the solution, this development is important. To deal with this, one can use multiple scales, which may also be needed in problems with weak damping. An example of this can be found in Exercises 1(g) and 1(h).

Exercises

1. For the following, describe the steady states, classify the bifurcation points, and determine the stability of each branch. Also, sketch the bifurcation diagram.

   a) $y'' + y' + y(y - \lambda^3 + 3\lambda^2 - 4) = 0$.

   b) $y'' + y' + 9y^2 + 4\lambda^2 = 1$.

   c) $y'' + y' + y[2y^2 + \lambda(\lambda - 2)] = 0$.

   d) $y'' + y' + y(\lambda - 3y + 2y^2) = 0$.

   e) $y'' + y' + \lambda + \alpha y - y^3 = 0$, where α is a given constant ($-\infty < \alpha < \infty$).

   f) $y'' + y' + e^{\lambda y} - 1 - (y - 1)^2 = 0$.

   g) $y'' + yy' + \lambda y + y^3 = 0$.

   h) $y'' + (y')^n + \lambda y - y^3 = 0$, where n is a positive integer.

2. In the study of chemical kinetics, one finds the following equation:

$$\frac{dc}{dt} = \frac{1}{\kappa}(1 - c) - c(1 + \beta - c)^2,$$

   where $c(t)$ satisfies $0 \le c \le 1$ and κ and β are positive constants with $0 < \beta < \frac{1}{8}$. It might be of interest to know that this equation comes from the theory for a continuously fed, well-stirred tank reactor that involves the cubic autocatalytic reaction $C + 2B \to 3B$ (Gray and Scott, 1990). In this setting $c(t)$ is the (scaled) concentration of C.

   a) Sketch the steady states as a function of κ.

   b) As a function of κ, classify the bifurcation points and determine the stability of each branch.

3. In the theory for the buckling of an initially straight rod subjected to an axial load λ, the following problem arises (Euler, 1744):

$$\theta'' + \lambda \sin\theta = 0, \quad \text{for } 0 < x < 1,$$

   where $\theta'(0) = \theta'(1) = 0$. The variable $\theta(x)$ is the angle the tangent to the rod makes with the horizontal at the spatial position x. This variable is related to the vertical displacement $w(x)$ of the rod through the relation $w' = \sin(\theta)$, where $w(0) = w(1) = 0$.

a) Find the solutions that bifurcate from the equilibrium solution $\theta_s = 0$. The values of λ where bifurcation occurs are called buckling loads.

b) In mechanics, the principle of minimum potential energy states that a system will move into a state with the minimum potential energy. For this problem, the potential energy is proportional to

$$V = \int_0^1 [\theta_x^2 + 2\lambda(\cos\theta - 1)]dx.$$

Use this principle to determine the preferred configuration of the rod near the first bifurcation point.

c) Setting $\lambda_n = (n\pi)^2$, find a first-term approximation of V for the solutions that bifurcate from (λ_n, θ_s). Comparing these with the value of V when $\theta_s = 0$, comment on what configuration the rod might take when $\lambda_n < \lambda < \lambda_{n+1}$. Also, explain why this may not actually be the configuration.

4. Consider the following nonlinear eigenvalue problem:

$$y'' + \alpha y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = y(1) = 0$. Also,

$$\alpha = \lambda - \int_0^1 y^2(x)dx,$$

where λ is a constant.

a) Find the (exact) solutions of this problem, and classify the bifurcation points using λ as the bifurcation parameter.

b) Use the principle of minimum energy (see Exercise 3) to determine the stability of the solutions. To do this, let

$$V = \int_0^1 (y_x^2 - \alpha y^2)dx.$$

FIGURE 6.7. Bead on a rotating hoop, as described in Exercise 5.

5. Consider a wire hoop that is suspended from the ceiling (see Fig. 6.7). Suppose the hoop is rotating and there is a bead that is able to slide around the hoop. The equation of motion for the bead, in nondimensional form, is

$$\theta'' + \kappa\theta' + \sin(\theta) = \omega^2 \sin(\theta)\cos(\theta), \quad \text{for } 0 < t,$$

where κ and ω are positive constants. Here κ comes from the friction of the bead on the wire and ω comes from the frequency of rotation. Also, $\theta(t)$ is the angular displacement of the bead measured from the bottom of the hoop and you can assume that $-\pi \leq \theta \leq \pi$.

a) Find the steady states and determine if they are stable. Also, sketch the bifurcation diagram and classify the bifurcation points.

b) What happens to the equilibrium position of the bead as $\omega \to \infty$?

6. The simple pendulum has two steady states: one is stable ($\theta_s = 0$), and the other is unstable ($\theta_s = \pm\pi$). It is interesting that if the support point of the pendulum is driven periodically, then it is possible for the inverted position to be stable also. (Did you ever try to balance an inverted broom on your hand?) When the support is given a vertical periodic forcing, the equation of motion in terms of the polar angle $\phi(t)$ is (see Fig. 6.8)

$$\phi'' - (\kappa - \varepsilon\cos t)\sin\phi = 0, \quad \text{for } 0 < t.$$

Here κ is a positive constant (inversely proportional to the square of the driving frequency Ω) and ε is a positive constant associated with the amplitude of the forcing.

a) What are the steady states? For the inverted case, show that the linearized stability argument leads to having to solve Mathieu's equation, given as

$$\varphi'' - [\kappa - \varepsilon\cos(t)]\varphi = 0, \quad \text{for } 0 < t.$$

What are the initial conditions?

FIGURE 6.8. Inverted pendulum, as studied in Exercise 6.

b) From this point on, we will assume ε is small. By expanding in terms of ε, show that the inverted position is unstable if κ is independent of ε.

c) Now assume $\kappa \sim \kappa_0\varepsilon + \kappa_1\varepsilon^2 + \cdots$. Find conditions on κ_0 and κ_1 so that the inverted position is stable (it does not have to be asymptotically stable). Remember that the initial conditions are independent of ε.

d) Shown in Fig. 6.9 are the experimentally measured values for the frequency of oscillation of the inverted pendulum as a function of the amplitude of the forcing (Ness, 1967). Is your result from part (c) consistent with these values? Make sure to comment on (i) the approximate linearity of each data set, (ii) the change in slope when Ω is changed, and (iii) the increase in ω with Ω. Also, explain why the data indicate that there are at least two time scales in the problem. Note that t is nondimensional and that in physical coordinates $t = 2\pi\Omega t^*$, where t^* is measured in seconds and Ω is measured in cps.

7. Have you ever ridden in a swing? To get the swing higher, you move a small amount up and down in the seat with a period that is related to the period of the swing. The swing in this case can be modeled as a pendulum. Also, because the mass at the end is moving up and down it is a pendulum with a variable length. The equation in this case is

$$\theta'' + 2\frac{\ell'(t)}{\ell(t)}\theta' + \sin(\theta) = 0, \quad \text{for } 0 < t.$$

Here $\ell(t) = 1 + \varepsilon\cos(\omega t)$ is the length of the swing, where ε and ω are the amplitude and frequency, respectively, of the up and down motion in the seat.

a) What are the steady states? For $\theta_s = 0$ show that the linearized

FIGURE 6.9. Experimentally determined values of the frequency (ω) of oscillation of the inverted pendulum as a function of the amplitude (ε) of the forcing (Ness, 1967).

stability argument leads to having to solve

$$v'' + 2\frac{\ell'(t)}{\ell(t)}v' + v = 0, \quad \text{for } 0 < t.$$

What are the initial conditions?

b) Show that if the swinger does nothing (so, $\omega = 0$) then the steady state $\theta_s = 0$ is stable.

c) What the swinger must do is find a frequency ω that makes the steady state unstable (so the amplitude grows and the swing goes higher). Find such a frequency when ε is small. Do this by constructing a first-term expansion of v that is valid for large t.

8. If the results in Fig. 6.6 are correct, then experimentally one should see a very sharp change in the solution in the neighborhood of the bifurcation point. However, in reality it is more usual that the change is rapid but reasonably smooth because of imperfections in the material. Accounting for this is the objective of *imperfect bifurcation.* To illustrate the ideas underlying this, consider the problem for the buckling of a slightly crooked column (Reiss, 1977). The equation is

$$\theta'' + \lambda \sin\theta = \delta g(x), \quad \text{for } 0 < x < 1,$$

where $\theta'(0) = \theta'(1) = 0$ and $0 \le \delta << 1$. The function $g(x)$ in this case represents the imperfection due to the column not being exactly straight. In Exercise 3 it is found that the smallest bifurcation value is $\lambda_1 = \pi^2$, which is known as the Euler buckling load.

a) Assuming $\delta << (\lambda - \pi^2)^{3/2}$, find a first-term approximation for small ε of the solutions that bifurcate from $(\lambda_b, \theta_b) = (\pi^2, 0)$. The answer here should agree with the result from Exercise 3(a). The assumption on δ means that the approximation will not apply in the immediate vicinity of (λ_b, θ_b). To find out what is going on there, proceed to part (b).

b) For $\varepsilon = \varepsilon_0 \delta^{2/3}$, where ε_0 is a constant, find a first-term approximation for small δ of the solutions that bifurcate from $(\pi^2, 0)$.

c) In part (b) you derived a cubic equation that determines the amplitude of the solution. What condition on $g(x)$ guarantees that this equation has at least one real root? Designate this real root as $A(\varepsilon_0)$.

d) By examining the behavior of $A(\varepsilon_0)$ as $\varepsilon_0 \to \pm\infty$, explain how this solution connects two of the branches you found in part (a). Your solution should resemble part of the graph shown in Fig. 6.10. The other curves come from the other real roots of the cubic (but you do not need to show this). Note that one consequence of this particular form of imperfection is the loss of symmetry in the bifurcation diagram.

FIGURE 6.10. Bifurcation diagram for Exercise 8.

6.5 Relaxation Dynamics

One of the more interesting aspects of equilibrium points is their effect on the dynamic response of the solution. This can be quite dramatic. As an example of such a situation, we will consider the initial-value problem

$$\varepsilon \frac{dy}{dt} = y - \frac{1}{3}y^3 - f(t), \quad \text{for } 0 < t, \tag{6.22}$$

where $y(0) = \alpha$. This equation does not arise from any particular physical problem. The reason for using it as our first example is that it gives a nice illustration of what is known as "fast dynamics" and bistability. After this, more physically relevant problems, like the van der Pol oscillator, are easier to understand.

To explain what is distinctive about (6.22), suppose $f(t)$ is constant. In fact, let $f(t) = \lambda$. In this case, as shown in Fig. 6.11, the steady-state solution has two (saddle-node) bifurcation points, one at $(\lambda_b, y_b) = (-2/3, -1)$ and another at $(\lambda_b, y_b) = (2/3, 1)$. The upper and lower branches are asymptotically stable, but the middle branch (which connects the two bifurcation points) is unstable. Now, if we happen to pick the initial condition to be within the basin of attraction of the upper branch, then the solution will approach this branch very rapidly. This is because the ε multiplying the highest derivative in (6.22) will cause the solution to change on the time scale $\bar{t} = t/\varepsilon$. Consequently, there will be an initial time interval, of

FIGURE 6.11. Steady-state solutions obtained from (6.22) when $f(t) = \lambda$.

length $O(\varepsilon)$, during which the solution will approach the equilibrium solution exponentially. This is what is referred to as "fast dynamics." The second point to be made here is that this behavior will occur even if $f(t)$ depends on t. This will happen as long as $f(t)$ does not change too quickly, that is, if it changes slowly over time intervals of length $O(\varepsilon)$. Assuming this is the case, then the question arises as to what the solution does if $f(t)$ causes the solution to pass near one of the bifurcation points. Answering this question and making more sense out of this discussion are some of the objectives of what follows.

To make the problem definite, suppose we want to solve

$$\varepsilon \frac{dy}{dt} = y - \frac{1}{3}y^3 - t, \quad \text{for } 0 < t, \tag{6.23a}$$

where

$$y(0) = 2. \tag{6.23b}$$

The numerical solution of this problem is shown in Fig. 6.12 for two values of ε. There is an initial layer where the solution drops from $y(0) = 2$ down to the upper branch of the equilibrium solution. (This layer occupies such a small time interval that the solution curve is almost indistinguishable from the vertical axis.) There is also a sharp drop to the lower branch once the solution passes the bifurcation point. This part of the curve is something like what occurs in cartoons, where the coyote (or some other poor soul) runs off a cliff but it takes a moment or two before he drops to the new equilibrium level. We are now going to analyze this situation using matched asymptotic expansions (see Chapter 2).

Outer Expansion

The easiest expansion to determine is what we will call the outer solution. This is obtained from a regular expansion of the form

$$y(t) \sim y_0(t) + \varepsilon y_1(t) + \cdots . \tag{6.24}$$

FIGURE 6.12. Numerical solution of (6.23) for two values of ε. Also shown is the curve obtained from the outer solution y_0.

Substituting this into (6.23a), one finds that

$$y_0 - \frac{1}{3}y_0^3 = t. \tag{6.25}$$

The solutions of this equation generate the curve shown in Fig. 6.11 (where t replaces λ). Which branch of this curve we should use will be determined from matching. As we will see, (6.25) determines the solution everywhere except near $t = 0$ and near $t = 2/3$.

Initial-Layer Expansion

The solution in (6.25) does not satisfy the initial condition. This is dealt with in the usual manner of introducing the initial-layer coordinate $\bar{t} = t/\varepsilon$. The appropriate expansion of the solution $y = Y(\bar{t})$ in this region is $Y(\bar{t}) \sim Y_0(\bar{t}) + \cdots$. Problems like this were studied extensively in Chapter 2, so only the results of this procedure will be given. One finds that

$$Y \sim 2\frac{\sqrt{12 - 3e^{-2\bar{t}}}}{4 - e^{-2\bar{t}}}. \tag{6.26}$$

By matching this expansion with the one from the outer region, we can determine which branch must be used from (6.25). In particular, since $Y_0(\infty) = 3^{1/2}$, $y_0(t)$ is determined by the upper branch. More precisely, the upper branch determines $y_0(t)$ but only as long as $0 < t < 2/3$ (which is in agreement with the numerical solutions shown in Fig. 6.12).

So, we have found an approximation of the solution up to the bifurcation point. As we will see later, there is a layer right near the bifurcation point (that allows our solution to turn the corner) and then another layer that enables the solution to move from the upper to the lower branch.

Corner-Layer Expansion

Now, as is evident from Fig. 6.11, the upper branch stops at $t = 2/3$. To determine what the solution does after this point, we introduce the coordinate

$$\tilde{t} = \frac{t - 2/3}{\varepsilon^\alpha}. \tag{6.27}$$

The solution in this region will be denoted as $y = \tilde{Y}(\tilde{t})$. We are interested in the behavior in the immediate vicinity of the bifurcation point (where $y_p = 1$), and so the appropriate expansion is

$$\tilde{Y} \sim 1 + \varepsilon^\gamma \tilde{Y}_1 + \cdots. \tag{6.28}$$

Substituting (6.28) and (6.27) into (6.23a), we get

$$\varepsilon^{1-\alpha+\gamma}\frac{d}{d\tilde{t}}\tilde{Y}_1 + \cdots = -\varepsilon^{2\gamma}\tilde{Y}_1^2 + \cdots - \varepsilon^\alpha \tilde{t}.$$

Balancing the terms in this equation gives us $\alpha = 2/3$ and $\gamma = 1/3$. The resulting equation for $\tilde{Y}_1$ is

$$\frac{d}{d\tilde{t}}\tilde{Y}_1 + \tilde{Y}_1^2 + \tilde{t} = 0.$$

This is a Riccati equation, which may be transformed into a linear problem (Holmes and Stein, 1976). This is accomplished by making the substitution $\tilde{Y}_1 = w'/w$. The result is an Airy equation for $w(\tilde{t})$, which yields (see Appendix A1)

$$\tilde{Y}_1 = -\frac{a_0 Ai'(-\tilde{t}) + a_1 Bi'(-\tilde{t})}{a_0 Ai(-\tilde{t}) + a_1 Bi(-\tilde{t})}. \tag{6.29}$$

To determine the constants a_0 and a_1, it is necessary to match (6.29) with the outer solution by considering the limits of $t \uparrow 2/3$ and $\tilde{t} \downarrow -\infty$. With this in mind, we introduce the intermediate variable $t_\eta = (t-2/3)/\varepsilon^\eta$, where $0 < \eta < 2/3$. From (6.29), we get

$$\tilde{Y} \sim \begin{cases} 1 + \varepsilon^{\eta/2}(-t_\eta)^{1/2} & \text{if } a_1 = 0 \\ 1 - \varepsilon^{\eta/2}(-t_\eta)^{1/2} & \text{if } a_1 \neq 0 \end{cases}$$

and, from (6.25), $y \sim 1 + \varepsilon^{\eta/2}(-t_\eta)^{1/2}$ (see Exercise 1). Thus, $a_1 = 0$, and the transition-layer expansion is therefore

$$\tilde{Y} \sim 1 - \varepsilon^{1/3}\frac{Ai'(-\tilde{t})}{Ai(-\tilde{t})}. \tag{6.30}$$

Interior-Layer Expansion

The major difficulty with the result in (6.30) is that the Airy function is oscillatory for negative values of its argument. This can be seen in the graph of $Ai(x)$ given in Appendix A1. The first zero $x = x_0$ of this function occurs when $x_0 = -2.33811\ldots$. This means that the expansion given in (6.30) is essentially meaningless when $-\tilde{t} = -2.33811\ldots$, and we have a situation reminiscent of what occurs at a turning point for the WKB expansion (see Section 4.2). In terms of the original time variable, this singularity occurs when $t = t_0$, where $t_0 = 2/3 + \varepsilon^{2/3}\tilde{t}_0$ for $\tilde{t}_0 = 2.3381\ldots$. To determine what is happening near this point, we introduce the layer coordinate

$$t^* = \frac{t - t_0}{\varepsilon^\kappa}. \tag{6.31}$$

In this layer, we will designate the solution as $Y^*(t^*)$.

Now, the appropriate expansion is $Y^* \sim Y_0^* + \cdots$. Introducing this into (6.23a) yields

$$\varepsilon^{1-\kappa}\frac{d}{dt^*}Y_0^* + \cdots = Y_0^* + \cdots - \frac{1}{3}{Y_0^*}^3 + \cdots - \left(\frac{2}{3} + \varepsilon^{2/3}\tilde{t}_0 + \varepsilon^\kappa t^*\right).$$

Balancing terms yields $\kappa = 1$, and the $O(1)$ equation is given as

$$\frac{d}{dt^*}Y_0^* = Y_0^* - \frac{1}{3}{Y_0^*}^3 - \frac{2}{3}, \quad \text{for } -\infty < t^* < \infty. \tag{6.32}$$

The solution of this differential equation is defined implicitly through the equation

$$\frac{1}{1-Y_0^*} - t^* = A_0 + \frac{1}{3}\ln\left|\frac{2+Y_0^*}{1-Y_0^*}\right|, \tag{6.33}$$

where A_0 is an arbitrary constant. By looking at the direction field for (6.32), it can be seen that $Y_0^*(\infty) = -2$ and $Y_0^*(-\infty) = 1$. So, as $t^* \uparrow \infty$ this function will match with the lower branch of y_0 as $t \downarrow 2/3$. Moreover, it will match as $t^* \downarrow -\infty$ to the transition-layer expansion (6.30). The details of the matching, using intermediate variables, are considered in Exercise 1. This leaves the value of A_0 undetermined. A discussion of possible ways to find A_0 can be found in Section 2.4; however, exactly what value it has is an open question.

We have completed the analysis of the problem. However, it is worth commenting on the steps we had to take to obtain the solution. The major effort was to determine how the solution moves from one branch to the other. This takes place in two steps. The first takes the solution past the bifurcation point and lets the solution drop a short distance below $y_b = 1$. Once this happened, the balancing in the equation changed, and this was the reason for the interior layer. The drop in the interior-layer solution occurs very quickly, over a time scale that is $O(\varepsilon)$, and this connects the upper branch with the lower.

Exercises

1. To match the solution from the interior layer with the ones from the outer and corner regions, introduce the intermediate variable

$$\tau = \frac{t - (2/3 + \varepsilon^{2/3}\tilde{t}_0)}{\varepsilon^\nu},$$

where $0 < \nu < 1$.

a) The behavior of Y^* can be determined from (6.33) by assuming $Y_0^* \sim 1 + g(\tau, \varepsilon)$, where $g << 1$. For $\tau < 0$, show that $Y_0^* \sim 1 + \varepsilon^{\nu-1}/\tau$. Similarly, assuming $y_0 \sim 1 + G(\tau, \varepsilon)$, show that $y_0 \sim 1 + \varepsilon^{\nu-1}/\tau$.

b) For $\tau > 0$, show that $Y_0^* \sim -2 + \exp(-3\varepsilon^{\nu-1}\tau)$. Explain why this shows that the interior-layer solution matches with the outer solution.

2. Suppose

$$\varepsilon\frac{dy}{dt} = \frac{1}{3}y^3 - y - t, \quad \text{for } 0 < t,$$

where $y(0) = 0$. Find a first-term approximation of the solution for $0 \le t < \infty$.

3. The van der Pol equation is

$$\varepsilon y'' - (1 - y^2)y' + y = 0, \quad \text{for } t > 0.$$

Assume that $y(0) = \sqrt{3}$ and $y'(0) = 0$.

a) Letting $y = -v'$, show that the equation can be written as the first-order system

$$\begin{aligned} v' &= -y, \\ \varepsilon y' &= v + y - \frac{1}{3}y^3. \end{aligned}$$

What are the initial conditions? The numerical solution of this system is shown in Fig. 6.13 for $\varepsilon = 10^{-2}$. The solution is periodic, and the solid curve in the graph is the trajectory through one cycle. Note that this path consists of alternate fast and slow components, which together form what is known as a relaxation oscillation.

b) Find the first term in the outer expansion of y and v. From this, determine where the first corner layer is located (give the values of t, y, and v).

c) Find the first term in the corner-layer expansion of y and v. With this, determine the location of the first transition layer. The analysis of the transition layers is rather involved and can be found in Kevorkian and Cole (1981) and MacGillivray (1990).

d) In each transition layer, the solution very quickly switches from one branch of a cubic to another. Use this observation to find a first-term approximation of the period of the oscillation. For comparison, the period found numerically is $T = 1.91$ when $\varepsilon = 10^{-2}$ and $T = 1.68$ when $\varepsilon = 10^{-3}$. Use this same idea to sketch the first-term approximation of $y(t)$ for $0 \le t \le 3T$.

4. Rayleigh (1883) undertook a study of what he called "maintained vibrations" that occur in organ pipes, singing flames, finger glasses, and

FIGURE 6.13.

other such systems. The equation he used to model such vibrations was

$$\varepsilon y'' - \left[1 - \frac{1}{3}(y')^2\right] y' + y = 0, \quad \text{for } t > 0.$$

Assume that $y(0) = 0$ and $y'(0) = -\sqrt{3}$.

a) Letting $v = y'$, write the problem as a first-order system. Make sure to give the initial conditions.

b) Find the first term in the outer expansion of y and v. From this, determine where the first corner layer is located (give the values of t, y, and v).

c) Find the first term in the corner-layer expansion of y and v. With this, determine the location of the first transition layer.

d) In the transition layers, the solution very quickly switches from one branch of a cubic to another. Use this observation to find a first-term approximation of the period of the oscillation. Use this same idea to sketch the first-term approximation of $y(t)$ for $0 \le t \le 3T$.

5. In the study of pattern formation, a model for the activation of a gene by a chemical signal is (Lewis et al., 1977)

$$g' = \lambda - g + \frac{kg^2}{1 + g^2}, \quad \text{for } 0 < t,$$

where $\lambda \ge 0$ and $k > 2$ are constants (to simplify things take $k = 4$). Also note that because $g(t)$ designates a concentration it is required to be nonnegative.

a) Sketch the bifurcation diagram. Also, find and then classify the bifurcation point.

b) Determine which of the three steady-state branches are stable and which are unstable.

c) Suppose λ increases slowly in time. In particular, let $\lambda = \varepsilon t$ where $0 < \varepsilon << 1$. What problem does g satisfy if you change variables to $\tau = \varepsilon t$ and assume $g(0) = 0$? From this, find a first-term approximation of the solution for $0 \le \tau < \infty$.

d) For the problem in part (c), give a qualitative description of the solution for $0 \le t < \infty$. Why do you think this equation is used as a model of a "biological switching mechanism"? Also, discuss the limiting value of the solution (as $t \to \infty$) when $\lambda = \varepsilon t e^{-\alpha \varepsilon t}$, where α is a positive constant.

6. A model for the bistable iodate-arsenous acid reaction gives rise to the equation (Ganapathisubramanian and Showalter, 1986)

$$\frac{dy}{dt} = y(1 + \alpha y)(\beta - y) + \lambda(1 - y),$$

where $\lambda \geq 0$. In this equation $y(t)$ designates a concentration and therefore must be nonnegative. Also, λ is the bifurcation parameter and α and β are assumed to be the fixed constants $\alpha = 5$ and $\beta = 15$.

a) Sketch the bifurcation diagram and classify the two bifurcation points. Designate the values of λ where bifurcation occurs as λ_0, λ_1 where $\lambda_0 < \lambda_1$.

b) Determine which steady states are stable and which are unstable (assume $\lambda \neq \lambda_0, \lambda_1$).

c) Suppose λ increases slowly in time. In particular, let $\lambda = \varepsilon t$ where $0 < \varepsilon << 1$. (i) What problem does y satisfy if you change variables to $\tau = \varepsilon t$ and assume $y(0) = 15$? (ii) Find a first-term approximation of the solution for $0 \leq \tau < \infty$.

7. This problem examines the behavior of the nonlinear oscillator

$$y'' + \varepsilon\kappa y' + \lambda y - 3y^2 + 2y^3 = \varepsilon F \cos(\omega t),$$

where κ, λ, and ω are positive constants.

a) Consider the case of no forcing (i.e., $F = 0$). Sketch the bifurcation diagram, classify the two bifurcation points, and determine which steady states are stable and which are unstable.

b) Letting y_s designate a steady-state solution of the unforced problem, suppose the forced problem begins so that $y(0) = y_s$ and $y'(0) = 0$. In this case, given the amplitude of the forcing function, it might be expected that the forcing contributes to the $O(\varepsilon)$ term in the expansion of the solution. For resonance, however, one would expect it to contribute earlier, perhaps to the $O(1)$ or the $O(\varepsilon^{1/2})$ term (see Section 3.4). For what values of the driving frequency ω do you expect there to be a resonant response?

c) Suppose $y_s = \frac{1}{4}(3+\sqrt{9-8\lambda})$, where $8\lambda < 9$ and $\omega^2 = \lambda - 6y_s + 6y_s^2$. Find a two-term approximation of $y(t)$ that is valid for large t. If you are not able to solve the problem that determines the t_2 dependence, then find the possible steady states (if any) for the amplitude.

6.6 An Example Involving a Nonlinear Partial Differential Equation

Many of the ideas discussed in the previous sections can be applied to partial differential equations. One of the more well known examples of bifurcation involving a partial differential equation arises with what is known as buckling. If you take a yardstick and compress it along its axis, what you find is that the yardstick essentially does nothing until you press hard

enough, in which case it "pops" into a bowed shape. This is called the first buckling mode and is an example of bifurcation from the zero state. This mode is very stable. As an experiment, you might try to hold the yardstick to prevent it from entering this bowed configuration and see if you can get it into the second buckling mode, which resembles the curve $\sin(2\pi x)$ for $0 \le x \le 1$. You should find that the second mode is fairly difficult to maintain, as a relatively small perturbation will cause the yardstick to pop back into the first mode.

For an introductory example, the equation for a yardstick is a little too cumbersome (see Exercise 3). Therefore, to introduce the ideas underlying bifurcation for partial differential equations, we will consider the following simpler problem:

$$u_{tt} + u_t = u_{xx} + \lambda u + f(u), \quad \text{for } 0 < x < 1, \tag{6.34a}$$

where

$$u = 0 \text{ when } x = 0, 1. \tag{6.34b}$$

The initial conditions will be discussed later. What we have is a (damped) nonlinear wave equation where the nonlinearity is contained in the function $f(u)$. To be able to make any headway in solving the problem, we need to be fairly clear on what form $f(u)$ can have. The type of nonlinearity appearing in Duffing's equation is a reasonable first example, so we will assume $f(0) = f'(0) = f''(0) = 0$ but $f'''(0) \neq 0$. This includes such functions as $f = \pm u^3$. Note that $f(0) = 0$ means $u = 0$ is a steady-state solution. This is a very common situation in physical problems and is therefore not an unreasonable assumption. The harder condition to explain is why we are assuming $f(u)$ does not include quadratic terms near $u = 0$. The short answer is that it is harder to derive the expansion in this case, although the ideas involved are essentially the same as the case we will consider. This will be evident when working out Exercise 4. Anyway, we have a problem to solve and that is what we now set out to do.

Steady-State Solutions

To determine the steady states, we need to solve the problem

$$u_{xx} + \lambda u + f(u) = 0, \quad \text{for } 0 < x < 1, \tag{6.35a}$$

where

$$u = 0 \text{ when } x = 0, 1. \tag{6.35b}$$

This is a nonlinear eigenvalue problem where we are looking for what values of λ result in a nonzero solution. As in Section 6.2, we will look for the solutions that bifurcate from $u_s = 0$. This is done by letting

$$\varepsilon = \lambda - \lambda_b, \tag{6.36}$$

where λ_b is the value of λ where the bifurcation takes place. In this case, the appropriate expansion for small ε is

$$u \sim u_s + \varepsilon^\alpha u_1(x) + \varepsilon^\beta u_2(x) + \cdots . \tag{6.37}$$

We are interested in finding nonzero solutions of (6.35), and so we will require that u_1 is not identically zero and $0 < \alpha < \beta$. Substituting this into (6.35a) and using Taylor's theorem yields

$$\varepsilon^\alpha(\partial_x^2 u_1 + \lambda_b u_1) + \varepsilon^\beta(\partial_x^2 u_2 + \lambda_b u_2) + \varepsilon^{1+\alpha} u_1 + \frac{1}{6}\varepsilon^{3\alpha} u_1^3 f'''(0) + \cdots = 0. \tag{6.38}$$

$$\textcircled{1} \qquad \textcircled{2} \qquad \textcircled{3}$$

This leads to the following problems:

$$O(\varepsilon^\alpha) \quad \partial_x^2 u_1 + \lambda_b u_1 = 0, \quad \text{for } 0 < x < 1$$
$$u_1(0) = u_1(1) = 0$$

What we have here is a linear eigenvalue problem, and the bifurcation value λ_b is the eigenvalue. Solving this, one finds

$$\lambda_b = (n\pi)^2, \quad \text{for } n = 1, 2, 3, \ldots , \tag{6.39}$$

and

$$u_1 = A_n \sin(n\pi x). \tag{6.40}$$

We have not yet completely determined u_1 since the coefficient A_n in (6.40) is arbitrary. Also, the value of α is not known. Both of these are determined from the next order problem. To determine what this problem is, note that in (6.38) the $O(\varepsilon^\beta)$ term must deal with the contributions of u_1 found in either term $\textcircled{2}$ or $\textcircled{3}$. The balancing of $\textcircled{1}$ with $\textcircled{2}$ (and $\textcircled{3} << \textcircled{1}, \textcircled{2}$), or $\textcircled{1}$ with $\textcircled{3}$ (and $\textcircled{2} << \textcircled{1}, \textcircled{3}$), leads to the conclusion that $A_n = 0$. Therefore, for a nonzero solution the terms $\textcircled{1}$, $\textcircled{2}$, and $\textcircled{3}$ must balance. From this we obtain that $\beta = 1 + \alpha = 3\alpha$. Thus, $\alpha = 1/2$ and $\beta = 3/2$.

$$O(\varepsilon^{3/2}) \quad \partial_x^2 u_2 + \lambda_b u_2 = -u_1 - \tfrac{1}{6} f'''(0) u_1^3, \quad \text{for } 0 < x < 1,$$
$$u_2(0) = u_2(1) = 0$$

This is a nonlinear version of the $O(\varepsilon^\alpha)$ eigenvalue problem. We therefore have a situation similar to what we came across with the WKB method in Section 4.5. A solvability condition is going to come from this problem (as required by the Fredholm Alternative Theorem). To find this condition, multiply the differential equation by u_1, integrate both sides over the interval $0 < x < 1$, and then integrate by parts. The result is

$$\int_0^1 \left[u_1^2 + \frac{1}{6} f'''(0) u_1^4 \right] dx = 0. \tag{6.41}$$

Using the solution in (6.40), we get from (6.41) that

$$A_n^2 = -\frac{8}{f'''(0)}. \tag{6.42}$$

From the above analysis, we conclude that there are an infinite number of bifurcation points associated with the steady state $u_s = 0$. These are the eigenvalues associated with a linear boundary-value problem, and they are $\lambda_b = (n\pi)^2$. The functions bifurcating from these points are

$$u \sim \pm\sqrt{-\frac{8(\lambda - \lambda_b)}{f'''(0)}}\sin(n\pi x), \quad \text{for } n = 1, 2, 3, \ldots . \tag{6.43}$$

Thus, there is a supercritical pitchfork bifurcation if $f'''(0) < 0$ and a subcritical pitchfork bifurcation if $f'''(0) > 0$ (see Fig. 6.14). It should be emphasized that our approximation in (6.43) of the bifurcating solutions is valid only for λ close to λ_b.

Linearized Stability Analysis

The next step is to decide whether or not the steady-state solutions we have found are stable. As before, we will do this by solving the problem using the initial conditions

$$u(x, 0) = u_s(x) + \delta g(x) \text{ and } u_t(x, 0) = \delta h(x). \tag{6.44}$$

We are starting the solution out close to the equilibrium solution $u = u_s(x)$, and for this reason the parameter δ is small. In what follows, we will assume that the value of λ does not correspond to a bifurcation value, that is, $\lambda \neq (n\pi)^2$. Also, the functions $g(x)$ and $h(x)$ in (6.44) are arbitrary (but smooth).

The appropriate expansion of the solution is

$$u(x, t) \sim u_s(x) + \delta v_1(x, t) + \cdots . \tag{6.45}$$

FIGURE 6.14. Bifurcation diagram for the nonlinear wave problem when $f'''(0) < 0$. What is sketched here is $A = \varepsilon^{1/2}A_n$, where A_n is given in (6.42). In this case, there is a supercritical pitchfork bifurcation at each $\lambda_b = (n\pi)^2$, for $n = 1, 2, 3, \ldots$.

Substituting this into (6.34), (6.44) yields the following problem:

$$\partial_t^2 v_1 + \partial_t v_1 = \partial_x^2 v_1 + [\lambda + f'(u_s)]v_1, \quad \text{for } 0 < x < 1, \tag{6.46a}$$

where

$$v_1 = 0 \quad \text{at } x = 0, 1 \tag{6.46b}$$

and

$$v_1(x,0) = g(x), \quad \partial_t v_1(x,0) = h(x). \tag{6.46c}$$

This problem is linear and can be solved using separation of variables. One finds that

$$v_1(x,t) = \sum_{m=1}^{\infty} (a_m e^{r_+ t} + b_m e^{r_- t}) F_m(x), \tag{6.47}$$

where $r_\pm = \frac{1}{2}\left(-1 \pm \sqrt{1 + 4\kappa_m}\right)$ (we are assuming here that $\kappa_m \neq -1/4$). The function $F_m(x)$ satisfies

$$F_m'' + [\lambda + f'(u_s) - \kappa_m]F_m = 0, \quad \text{for } 0 < x < 1, \tag{6.48}$$

where $F_m(0) = F_m(1) = 0$. It is interesting that this is another eigenvalue problem where the eigenvalue is the mth separation constant κ_m. To find κ_m, we must solve (6.48) but, since the coefficient of this equation depends on x, a closed-form solution is not at hand. However, we are primarily interested in whether or not the steady state is stable, and for that to happen we need, from (6.47), $\mathrm{Re}(r_+) < 0$. From the formula for r_+, this is equivalent to the condition that $\kappa_m < 0$ for all values of m.

Stability of Zero Solution

We will first consider the particular case of $u_s = 0$. Since $f'(0) = 0$, then the solution of (6.48) is

$$F_m(x) = \sin(m\pi x) \tag{6.49}$$

and

$$\kappa_m = \lambda - (m\pi)^2. \tag{6.50}$$

The coefficients a_m and b_m in (6.47) are determined by satisfying the initial conditions (6.46c). Assuming

$$g(x) = \sum_{m=1}^{\infty} g_m \sin(m\pi x) \text{ and } h(x) = \sum_{m=1}^{\infty} h_m \sin(m\pi x),$$

then

$$a_m + b_m = g_m \text{ and } r_+ a_m + r_- b_m = h_m. \tag{6.51}$$

It is a simple matter to solve these equations for a_m and b_m.

The stability of the steady-state solution $u_s = 0$ depends on the exponents $r_\pm$ in (6.47). These depend on the mode number m. Using the result given in (6.50), we have the following cases:

1) If $\lambda < \pi^2$, then $r_- < r_+ < 0$. Therefore, $v_1 \to 0$ as $t \to \infty$, and this means the steady-state solution $u_s = 0$ is asymptotically stable.

2) If $\pi^2 < \lambda < 4\pi^2$, then $r_- < r_+ < 0$ if $m \geq 2$, but $r_- < 0 < r_+$ if $m = 1$. In this case, the steady state $u_s = 0$ is unstable. However, note that if the initial conditions are such that $h_1 = r_- g_1$, then $v_1 \to 0$ as $t \to \infty$.

3) If $n^2\pi^2 < \lambda < (n+1)^2\pi^2$, for $n = 1, 2, 3, \ldots$, then the steady state $u_s = 0$ is unstable. In particular, it is stable only to initial disturbances that satisfy $h_m = r_- g_m$ for $m = 1, 2, \ldots, n-1$.

Stability of the Branches That Bifurcate from the Zero Solution

The next question is whether or not the solutions that bifurcate from $u_s = 0$ are stable. We do not have formulas for the nonzero steady states except right near the bifurcation point. It is therefore going to be necessary to incorporate our ε expansions into the stability argument. To do this, recall that we found that bifurcation occurs when $\lambda = (n\pi)^2$, and the expansion for the nonzero steady state is

$$u_s \sim \varepsilon^{1/2} A_n \sin(n\pi x). \tag{6.52}$$

We are going to find a first-term approximation to the solution of the linearized stability problem (6.46) for small ε. With this objective in mind, we first remember that $\lambda = \lambda_b + \varepsilon$ and $f'(0) = f''(0) = 0$, but $f'''(0) \neq 0$. Thus, from Taylor's theorem and (6.42),

$$\begin{aligned} \lambda + f'(u_s) &\sim \lambda_b + [1 + \frac{1}{2} f'''(0) A_n^2 \sin^2(n\pi x)]\varepsilon \\ &= \lambda_b + [1 - 4\sin^2(n\pi x)]\varepsilon. \end{aligned}$$

Our objective is to solve (6.48); the unknowns for this equation are F_m and κ_m. These quantities can depend on ε, and to account for this we expand them as follows:

$$F_m \sim \overline{F}_0 + \varepsilon \overline{F}_1 + \cdots \text{ and } \kappa_m \sim \overline{\kappa}_0 + \varepsilon \overline{\kappa}_1 + \cdots. \tag{6.53}$$

Substituting these into (6.48), the following problems appear:

$$O(1) \quad \overline{F}_0'' + (\lambda_b - \overline{\kappa}_0)\overline{F}_0 = 0, \quad \text{for } 0 < x < 1,$$
$$\overline{F}_0(0) = \overline{F}_0(1) = 0.$$

Solving this eigenvalue problem yields $\overline{F}_0 = \sin(m\pi x)$ and $\overline{\kappa}_0 = (n^2 - m^2)\pi^2$. This information is sufficient to answer the question about stability except when $n = m$. For this we need the next order problem.

$$O(\varepsilon) \quad \overline{F}_1'' + (\lambda_b - \overline{\kappa}_0)\overline{F}_1 = -[1 - 4\sin^2(n\pi x) - \overline{\kappa}_1]\overline{F}_0, \quad \text{for } 0 < x < 1,$$
$$\overline{F}_1(0) = \overline{F}_1(1) = 0.$$

As happened earlier, we are facing an inhomogeneous version of the eigenvalue problem. Multiplying by $\overline{F}_0$ and integrating yields the following result:

$$\overline{\kappa}_1 = \frac{\displaystyle\int_0^1 \left[1 - 4\sin^2(n\pi x)\right] \overline{F}_0^2 dx}{\displaystyle\int_0^1 \overline{F}_0^2 dx} = \begin{cases} -1, & \text{if } m \neq n \\ -2, & \text{if } m = n \end{cases}. \tag{6.54}$$

As stated above, a bifurcating solution will be stable if $\kappa_m < 0$. To determine if this is the case, suppose, for the sake of argument, that we are considering the third bifurcation point, so $n = 3$. What we need to determine is whether or not it is true that $\kappa_m < 0$ for every value of m. The answer is no because we have found that, for $m \neq n$, $\kappa_m \sim (n^2 - m^2)\pi^2$. In particular, if $m = 2$ or if $m = 1$, then $\kappa_m > 0$. This conclusion holds for the other bifurcation points except, possibly, $n = 1$. For this value of n we have $\kappa_m \sim -2\varepsilon$ when $m = 1$ and $\kappa_m \sim -(m^2 - 1)\pi^2$ for $m \geq 2$. Hence this branch will be stable if $\varepsilon > 0$, that is, $\lambda > \lambda_b$.

Our expansions for the two branches are

$$u \sim \pm\sqrt{-\frac{8(\lambda - \lambda_b)}{f'''(0)}}\sin(\pi x). \tag{6.55}$$

From this we have that if $f'''(0) > 0$ then $\varepsilon < 0$ (i.e., it is a subcritical bifurcation), and if $f'''(0) < 0$ then $\varepsilon > 0$ (i.e., it is supercritical). Therefore, the solutions branching from the first bifurcation point ($n = 1$) are stable if $f'''(0) < 0$ and unstable if $f'''(0) > 0$. To illustrate this situation, using the given boundary conditions, the solutions that bifurcate from the zero solution at $\lambda = \pi^2$ are unstable for the equation

$$u_{tt} + u_t = u_{xx} + \lambda u + u^3, \tag{6.56}$$

but they are stable for

$$u_{tt} + u_t = u_{xx} + \lambda u - u^3. \tag{6.57}$$

It should again be pointed out that our analysis is local. We can only say that the solutions that branch from $u_s = 0$ at $\lambda = \pi^2$ are stable if λ is close to π^2. For larger values of λ, we have no idea what is going on from what we have done. It is possible in certain cases to say something about the global structure of the bifurcating solutions; those who are interested should consult Crandall and Rabinowitz (1980). The approach used in global bifurcation is usually nonconstructive, that is, it does not actually determine the solution. About the only way this can be done is using numerical methods. For that aspect of the subject, the books by Hassard et al. (1981) and Keller et al. (1987) are recommended.

Exercises

1. In the theory of Rayleigh-Bénard convection, one studies the motion of a thin fluid layer that is heated from below. A model for this is the Ginzburg-Landau equation

$$u_t = u_{xx} + \lambda u - u^3, \quad \text{for } 0 < x < \pi \text{ and } 0 < t,$$

where λ is associated with the Rayleigh number for the flow (DiPrima et al., 1971). The boundary conditions are $u = 0$ at $x = 0, \pi$.

a) Find a first-term approximation of the steady-state solutions that bifurcate from $u_s = 0$.

b) Carry out a linearized stability analysis of the steady states.

2. Consider the nonlinear diffusion problem

$$\partial_t u = \partial_x^2 u + \lambda u(1-u), \quad \text{for } 0 < x < 1,$$

where $u = 0$ at $x = 0, 1$. This is known as Fisher's equation, or the Kolmogorov-Petrovsky-Piskunov (KPP) equation, and it has been used to model such phenomena as the spacial spread of certain genetic characteristics.

a) Find a first-term approximation of the steady-state solutions that bifurcate from $u_s = 0$.

(Hint: for n even, $\alpha = 1/2$, $\beta = 1$, and $\gamma = 3/2$.)

b) For what values of λ is the steady state $u_s = 0$ stable or unstable? Assume λ is not a bifurcation value.

3. The equation for the transverse displacement $u = u(x,t)$ of a nonlinear beam subject to an axial load λ is (Eringen, 1952; Reiss and Matkowsky, 1971)

$$\partial_x^4 u + \alpha(t)\partial_x^2 u + \partial_t^2 u + \partial_t u = 0, \quad \text{for } 0 < x < 1,$$

where

$$\alpha(t) = \lambda - \frac{1}{4}\int_0^1 u_x^2 dx.$$

The boundary conditions are $u = u_{xx} = 0$ at $x = 0, 1$.

a) Find the steady-state solutions. The zero solution is called the unbuckled state and the others buckled states. The values of λ where the buckled states appear are known as critical buckling loads.

b) For what values of λ is each steady state stable or unstable?

4. This problem considers the effects of changing the nonlinearity in the wave equation (6.34a). The first modification has a relatively minor effect but the second is more substantial.

   a) Explain how the results of this section change when $f(0) = f''(0) = 0$ but $f'(0) \neq 0$ and $f'''(0) \neq 0$. A simple example of this is $f(u) = u + u^3$.

   b) Carry out the analysis of this section when $f(0) = f'(0) = 0$ but $f''(0) \neq 0$. A simple example of this is $f(u) = \pm u^2$.

   (Hint: if n is odd, then $u \sim -3(\lambda - \lambda_n)\lambda_n \sin(\lambda_n x)/(2f''(0))$.)

5. The potential energy for the nonlinear string equation in (6.34) is

$$V(u) = \frac{1}{2}\int_0^1 [u_x^2 - \lambda u^2 - 2F(u)]dx,$$

   where $F'(u) = f(u)$. In this problem take $f(u) = \kappa u^3$, where κ is a nonzero constant, and let $u_n(x)$ denote the steady state in (6.43). Sketch $V(u_n)$ for λ near λ_n (as is done for A in Fig. 6.14). Based on this, is it reasonable to conjecture that the stable solution has the smallest potential energy? Explain your reasoning.

6. The Kuramoto-Sivashinsky equation is

$$\partial_t u + \partial_x^4 u + \lambda \partial_x^2 u + \frac{1}{2}(\partial_x u)^2 = 0, \quad \text{for } -\infty < x < \infty,$$

   where $u(x,t)$ is required to be periodic in x. This equation arises in the study of the propagation of flame fronts, in the instabilities in reaction–diffusion systems, and in drift waves in plasmas (Mitani, 1984).

   a) Setting $u(x,t) = -\kappa t + v(x,t)$, where κ is a constant, find the problem $v(x,t)$ satisfies.

   b) For the problem in part (a), under what conditions on λ and κ is $v_s = 0$ an asymptotically stable steady-state solution?

   c) For the problem in part (a), determine the steady-state solution(s) that bifurcate from $v_s = 0$ at the point where v_s changes stability. To do this, take λ to be the bifurcation parameter. Also, note that the requirement of periodicity results in κ depending on λ.

   d) Determine if the steady-state solution(s) found in part (c) are asymptotically stable. In doing this, make sure to comment on the periodicity requirements of the initial conditions you use.

7. This problem reconsiders the Fisher equation (see Exercise 2)

$$\partial_t u = \partial_x^2 u + \lambda u(1-u), \quad \text{for } -\infty < x < \infty.$$

We are now interested in traveling wave solutions of the form $u(x,t) = f(x - \alpha t)$, where $-\infty < \alpha < \infty$.

a) After finding the equation $f(z)$ satisfies, determine the stationary states for f and their stability.

b) For physical reasons $u(x,t)$ must be nonnegative. Because of this, show that traveling waves of the form we have assumed will only propagate when $\alpha > 0$.

c) Based on what you found in parts (a) and (b), can you say that these traveling waves are able to take a region near an unstable steady state to a stable state? For example, suppose the initial profile is $u(x,0) = \frac{1}{2}(1 + \tanh(kx))$, where k is a constant.

8. The Schlögl model in nonequilibrium phase transitions involves the reaction–diffusion equation (Schlögl, 1972)

$$\partial_t u = \partial_x^2 u + f(u,\lambda), \quad \text{for } 0 < x < 1,$$

where $f(u,\lambda) = -u^3 + 2u^2 - u + \lambda$ and $u_x(0,t) = u_x(1,t) = 0$. The constant λ is positive. The solution $u(x,t)$ is the density of a chemical in solution and is therefore assumed to be nonnegative.

a) Find the homogeneous steady states (i.e., u_s constant) and determine their stability.

b) For $0 < \lambda < \frac{4}{27}$, show that any steady state that bifurcates from one of the constant states is unstable.

c) If $0 < \lambda < \frac{4}{27}$, then one can write $f(u,\lambda) = -(u - u_1)(u - u_2)(u - u_3)$ where the u_i's depend on λ. Assuming $-\infty < x < \infty$, find the traveling wave solutions of the equation that have the form $u(x,t) = u_0 + \alpha \tanh[\beta(x - \chi t)]$.

d) Using the results from parts (a) and (c), explain why certain of the traveling waves can take a region near an unstable steady state to one of the stable states. (Hint: do this by examining the sign of χ and the behavior as $x \to \pm\infty$.)

9. In the study of phase transitions, in nonlinear viscoelasticity, one comes across the following problem:

$$\partial_t^2 u = \partial_x \sigma(u_x) + \kappa u_{xxt}, \quad \text{for } 0 < x < 1 \text{ and } 0 < t,$$

where $u(0,t) = 0$ and $\sigma(u_x) + \kappa u_{xt} = \lambda$ at $x = 1$. Here $u(x,t)$ is the displacement of the material, $\sigma(u_x)$ is the elastic stress, $\kappa > 0$ is a constant associated with the viscoelastic properties of the material, and $\lambda \geq 0$ is a constant. A constitutive law is needed to specify σ, and in this problem we will let $\sigma(u_x) \equiv u_x(u_x^2 - 9u_x + 15)$.

a) Sketch σ as a function of u_x and determine the steady states for this problem.

b) Determine whether the steady states found in part (a) are stable or unstable.

10. Consider the nonlinear Klein-Gordon equation

$$u_{tt} = u_{xx} - \lambda g(u), \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where $u(x,t)$ is bounded as $x \to \pm\infty$ and $u(x, t+2\pi) = u(x,t)$. Also, assume $g(u)$ is smooth with $g(0) = 0$ and $g'(0) = 1$.

a) One steady-state solution of this problem is $u_s = 0$. For what values of λ is it stable?

b) To obtain the sine-Gordon equation, one takes $g(u) = \sin(u)$. In this case, show that

$$u_B(x,t) = 4\arctan\left(\frac{\beta\sin(nt)}{n\cosh(\beta x)}\right),$$

where n is a positive integer and $\beta = (\lambda - n^2)^{1/2}$ is a solution of the problem (assuming $\lambda \geq n^2$). Sketch u_B as a function of x for $n = 1$ and $n = 10$ (in each case take $\beta = 1$). Because these solutions are localized in space and periodic in time, they are known as "breathers" and they arise in nonlinear optics and quantum mechanics. Note that they bifurcate from the trivial solution analyzed in part (a).

c) To determine if the breathers are stable, show that one must solve

$$v_{tt} = v_{xx} - \lambda g'(u_B)v, \quad \text{for } -\infty < x < \infty \text{ and } 0 < t,$$

where v is bounded as $x \to \pm\infty$ and $v(x, t+2\pi) = v(x,t)$.

d) For values of λ near where a breather bifurcates from $u_s = 0$, use the method of multiple scales to determine if the breather is stable or unstable.

6.7 Bifurcation of Periodic Solutions

One of the more interesting responses that can come from a nonlinear problem is a periodic motion known as a *self-sustained oscillation.* To explain what this is, we begin with van der Pol's equation, given as

$$y'' - \lambda(1 - y^2)y' + y = 0, \quad \text{for } 0 < t. \tag{6.58}$$

The nonlinearity in the equation is in the damping. What is important is that the coefficient can be both positive and negative, depending on the

value of the solution. Assuming $\lambda > 0$, then for small values of $y(t)$ the coefficient of the damping term is negative, a situation that can lead to unbounded solutions. However, for large values of $y(t)$ the coefficient is positive, which should help stabilize the problem. As we will see later, this give and take in the damping will lead to a self-sustained oscillation.

Equation (6.58) was studied by van der Pol (1926) in connection with the triode circuits that were used in some of the early radios. It is possible to build one of these circuits relatively cheaply, and those who may be interested in trying this should consult Horwitz and Hill (1989) or Keener (1983). This equation is the classic example in the subject of relaxation oscillations. It has applicability in mechanics, electronics, and other fields, and an overview of the subject can be found in Grasman (1987).

The first step in analyzing (6.58) is to consider the steady states. The only one we get is $y_s = 0$. To determine if it is stable, we use the initial conditions

$$y(0) = \alpha_0\delta \text{ and } y'(0) = \beta_0\delta. \tag{6.59}$$

Now, the appropriate expansion of the solution for small δ is $y \sim y_s + \delta y_1(t) + \cdots$. Substituting this into (6.58), one finds that

$$y_1(t) = a_0 e^{r_+ t} + a_1 e^{r_- t}, \tag{6.60}$$

where a_0 and a_1 are constants and $2r_\pm = \lambda \pm \sqrt{\lambda^2 - 4}$. The values of these exponents are shown in Fig. 6.15, which shows that $\mathrm{Re}(r_\pm) < 0$ if $\lambda < 0$ and $\mathrm{Re}(r_\pm) > 0$ if $\lambda > 0$. From this it follows that $y_s = 0$ is asymptotically stable if $\lambda < 0$ and unstable if $\lambda > 0$.

The question now is, what happens to the solution when $\lambda > 0$? A hint about the answer can be found in the way the zero solution becomes unstable. As λ increases in (6.60), and goes from negative to positive, the exponents $r_\pm$ are complex conjugates and their real parts change sign. This differs from what we found earlier when a steady-state solution went unstable. For example, in (6.19) the solution becomes unstable because one (and not both) of the exponents goes from negative to positive. Moreover, the exponents are real.

To see exactly what is happening as λ goes from negative to positive, let

$$\varepsilon = \lambda - \lambda_b, \tag{6.61}$$

where $\lambda_b = 0$. In this case (6.58) becomes

$$y'' - \varepsilon(1 - y^2)y' + y = 0, \quad \text{for } 0 < t. \tag{6.62}$$

The equation is weakly damped for small ε, and so, based on our experience from Chapter 4, we will use the time scales $t_1 = t$ and $t_2 = \varepsilon t$. Introducing these into (6.62) yields

$$(\partial_{t_1}^2 + 2\varepsilon\partial_{t_1}\partial_{t_2} + \cdots)y - \varepsilon(1 - y^2)(\partial_{t_1} + \varepsilon\partial_{t_2})y + y = 0. \tag{6.63}$$

FIGURE 6.15. The values of the exponents $r_\pm$ in (6.60). If $\lambda \leq -2$, then both are negative and satisfy $r_\pm \leq -1$. When $\lambda \geq 2$, both exponents are positive and satisfy $r_\pm \geq 1$. When $-2 < \lambda < 2$, the exponents are complex conjugates with $|r_\pm| = 1$. The point at which $r_\pm = \pm i$ occurs when $\lambda = 0$.

The appropriate expansion for the solution is

$$y \sim y_0(t_1, t_2) + \varepsilon y_1(t_1, t_2) + \cdots . \tag{6.64}$$

Introducing this into (6.63) gives the following equations:

$O(1)$ $(\partial_{t_1}^2 + 1)y_0 = 0.$

The general solution of this equation is

$$y_0 = A(t_2)\cos[t_1 + \phi(t_2)]. \tag{6.65}$$

We will assume, without loss of generality, that $A \geq 0$.

$O(\varepsilon)$ $(\partial_{t_1}^2 + 1)y_1 = (1 - y_0^2)\partial_{t_1} y_0 - 2\partial_{t_1}\partial_{t_2} y_0.$

Setting $\eta = t_1 + \phi(t_2)$, then

$$\begin{aligned}
&(1 - y_0^2)\partial_{t_1} y_0 - 2\partial_{t_1}\partial_{t_2} y_0 \\
&\quad = -[1 - A^2(1 - \sin^2(\eta)]A\sin\eta + 2(A'\sin\eta + A\phi'\cos\eta) \\
&\quad = [2A' - A(1 - \frac{1}{4}A^2)]\sin\eta + 2A\phi'\cos\eta + \frac{1}{4}A^3\sin 3\eta.
\end{aligned}$$

To prevent secular terms in the expansion, we set $A\phi' = 0$ and

$$2A' = A(1 - \frac{1}{4}A^2). \tag{6.66}$$

Thus, ϕ is constant and the solutions of (6.66) are

$$A(t_2) = \begin{cases} 2\sqrt{\frac{c}{c+4e^{-t_2}}} \\ 2 \end{cases}, \tag{6.67}$$

where c is a constant determined from the initial conditions.

To summarize, we have found that a first-term approximation of the solution of the initial-value problem, for λ close to the bifurcation value $\lambda_b = 0$, is

$$y(t) \sim 2\sqrt{\frac{c}{c + 4e^{-\lambda t}}}\ \cos(t + \phi_0) \tag{6.68}$$

and

$$y'(t) \sim -2\sqrt{\frac{c}{c + 4e^{-\lambda t}}}\ \sin(t + \phi_0), \tag{6.69}$$

where c and ϕ_0 are constants determined from the initial conditions (see Exercise 6). It is assumed here that the initial conditions are such that $y^2(0) + y_t^2(0) \neq 4$. From this we conclude that as $t \to \infty$ the solution $y(t) \to 0$ if $\lambda < 0$. This is expected because we have shown that the zero solution is an asymptotically stable steady state when λ is negative. However, when $\lambda > 0$, the limiting solution is $y(t) = 2\cos(t + \phi_0)$. This is the self-sustained oscillation mentioned earlier. To describe the situation in more detail it is convenient to use the phase plane. From (6.68), (6.69) we have $y^2 + y_t^2 \sim 4c/(c + e^{-\lambda t})$. Therefore, if $0 < \lambda << 1$, then $y^2 + y_t^2 \to 4$ as $t \to \infty$. In this case, the limit $(y, y') = (2\cos(t + \phi_0), -2\sin(t + \phi_0))$ is a periodic solution and there are no other periodic solutions near it. In other words, it is a limit cycle.

The limit cycle and the asymptotic approximation (6.68), (6.69) are shown in Fig. 6.16(a) when $y(0) = 1.0$ and $y'(0) = 0$. For comparison, the numerical solution of the problem is also shown, and the two are seen to be in close agreement. The circular nature of the limit cycle is lost as λ increases. The solution in this case becomes more like a relaxation oscillation, and the analysis of this situation can be carried out using the ideas developed in Section 6.5 (see Exercise 6.5.3).

The multiple-scale analysis has shown that if we start close to the limit cycle (closeness here means we are close to it in the phase plane) then the solution approaches the limit cycle as $t \to \infty$. For this reason, the limit cycle is said to be *asymptotically stable* or to be an *asymptotically stable orbit.*

We have found that the steady state $y_s = 0$ loses stability at $\lambda_b = 0$ and a stable limit cycle appears for $0 < \lambda - \lambda_b << 1$. In the vernacular of bifurcation theory, this is classified as a degenerate supercritical *Hopf bifurcation* (Strogatz, 1994). Hopf bifurcations will be investigated in more detail in the next section. For the moment, it is worth concentrating on what happens when the two exponents $r_\pm$ in (6.60) cross the imaginary axis. They are complex conjugates, and they cross over with "nonzero velocity," in other words, $\frac{d}{d\lambda}\mathrm{Re}[r_\pm] \neq 0$ for $\lambda = \lambda_b$. The latter is known as a *transversality condition.*

Besides being supercritical, the bifurcation is said to be degenerate. This means that the steady state $y_s = 0$, for $\lambda < 0$, is not continuous with

FIGURE 6.16. Solution of the van der Pol equation showing the approach to the limit cycle when starting at $y(0) = 1$, $y'(0) = 0$ and taking $\lambda = 0.1$. In (a) the trajectories obtained from the asymptotic approximation in (6.68), (6.69) and from the numerical solution of (6.58) are shown. Also included is the limit cycle obtained from the asymptotic solution. In (b), the values of $y(t)$ obtained from the numerical and asymptotic solutions are shown. The two are so close that the curves are essentially indistinguishable.

the limit cycle that is present for $\lambda > 0$. This discontinuity affects the form of the expansion. If the limit cycle is continuous with the steady-state solution, then an example of an appropriate multiple-scale expansion is

$$y \sim y_s + \varepsilon^{\alpha} y_0(t_1, t_2) + \varepsilon^{\beta} y_1(t_1, t_2) + \cdots, \tag{6.70}$$

where $0 < \alpha < \beta$. In writing this, it is assumed that y_s is independent of λ and that two time scales are sufficient to determine the behavior of the solution for large t.

Exercises

1. This problem deals with the nonlinear oscillator

$$y'' + (y^2 - \lambda)y' + (y + 1)(y^2 + y - \lambda) = 0, \quad \text{for } 0 < t.$$

a) Find the steady states and determine for what values of λ they are

stable. Sketch the bifurcation diagram and classify the three bifurcation points.

b) Find a first-term approximation of the limit cycle that appears at the Hopf bifurcation point.

2. Consider the modified Rayleigh equation

$$y'' - \lambda[1 - \alpha(y')^{2n}]y' + y = 0, \quad \text{for } 0 < t,$$

where α is a positive constant and n is a positive integer.

a) Find the steady state and determine for what values of λ it is stable.

b) Find a first-term approximation of the limit cycle that appears at the Hopf bifurcation point.

3. The logistic equation with delay, which is used in population modeling (Hutchinson, 1948), is

$$y'(t) = ry(t)\left[1 - \frac{y(t-T)}{y_0}\right], \quad \text{for } 0 < t,$$

where $y(t) = a$ for $-\infty < t \leq 0$. In this equation $T > 0$ is the delay and also the bifurcation parameter. Also, r and y_0 are positive constants and $y(t) \geq 0$.

a) Find the steady states of the differential equation and determine their stability.

b) Determine what happens near the point where the steady state changes stability.

4. A Volterra delay equation that comes up in population modeling is (Cushing, 1977)

$$y'(t) = \lambda y(t)\left[y_0 - \int_{-\infty}^{t} K(t-\tau)y(\tau)d\tau\right],$$

where $K(t) = te^{-t}$. In this equation, λ and y_0 are positive constants and $y(t) \geq 0$.

a) Find the steady states and determine their stability.

b) Determine what happens near the point where the nonzero steady state changes stability.

5. Suppose one uses the multiple-scale expansion in (6.70) to solve the van der Pol equation in (6.62). In this case $y_s = 0$.

a) Show that the general solution of the $O(\varepsilon^{\alpha})$ equation is the same as given in (6.65). With this, then show that one choice for the balancing is $\beta = 2$ and $\alpha = 1$.

b) Show that removing secular terms results in the conclusion that $A = A_0 e^{t_2/2}$.

c) The first-term approximation is oscillatory but with a growing amplitude. Explain why this violates the assumptions made in the balancing used to determine β in (a). What is the proper balance for this equation?

6. This problem examines some other aspects of the derivation of (6.68), (6.69).

a) The multiple-scale expansion assumes that ε is nonnegative. Show that (6.68) and (6.69) hold in the case of negative ε.

b) Depending on the value of the constant c in (6.68) and (6.69), it would appear that there is a possibility that the amplitude has a singularity. Assuming $y(0) = a_0$ and $y'(0) = 0$, show that this does not happen.

c) Describe the solution when $\lambda = 0$. Explain why it is not a limit cycle.

6.8 Systems of Ordinary Differential Equations

The ideas developed for scalar problems can be extended without much difficulty to systems of equations. To explain how, suppose the system is

$$\frac{d}{dt}\mathbf{y} = \mathbf{f}(\lambda, \mathbf{y}), \tag{6.71}$$

where $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{f} \in \mathbb{R}^n$. As before, λ is a scalar that is used in the bifurcation analysis. Let $\mathbf{y}_s(\lambda)$ be a steady-state solution of (6.71), and suppose $(\lambda_b, \mathbf{y}_b)$, where $\mathbf{y}_b = \mathbf{y}_s(\lambda_b)$, is a possible bifurcation point. To find the solutions that bifurcate from this point, set $\varepsilon = \lambda - \lambda_b$. Now, if $f_k(\lambda, \mathbf{y})$ is the kth component of $\mathbf{f}$, then, using Taylor's theorem and setting $\mathbf{z} = \mathbf{y} - \mathbf{y}_b$,

$$\begin{aligned} f_k(\lambda, \mathbf{y}) \sim f_k(\lambda_b, \mathbf{y}_b) \; &+ \; \varepsilon \partial_\lambda f_k(\lambda_b, \mathbf{y}_b) + \mathbf{z} \cdot \nabla_y f_k(\lambda_b, \mathbf{y}_b) \\ &+ \; \frac{1}{2}\left[\varepsilon^2 \partial_\lambda^2 f_k(\lambda_b, \mathbf{y}_b) + 2\varepsilon \mathbf{z} \cdot \nabla_y \partial_\lambda f_k(\lambda_b, \mathbf{y}_b)\right. \\ &\quad \left. + \, \mathbf{z}^T \mathbf{H}_k \mathbf{z}\right] + \cdots . \end{aligned} \tag{6.72}$$

In the last term in (6.72), $\mathbf{H}_k$ is the Hessian of f_k and is the $n \times n$ symmetric matrix with i, j-component

$$(\mathbf{H}_k)_{ij} = \partial_{y_i} \partial_{y_j} f_k(\lambda_b, \mathbf{y}_b).$$

To simplify the discussion, we assume $\mathbf{y}_s$ is independent of λ. In this case (6.72) reduces to

$$f_k \sim \mathbf{z} \cdot \nabla_y f_k(\lambda_b, \mathbf{y}_b) + \frac{1}{2}\left[2\varepsilon \mathbf{z} \cdot \nabla_y \partial_\lambda f_k(\lambda_b, \mathbf{y}_b) + \mathbf{z}^T \mathbf{H}_k \mathbf{z}\right] + \cdots . \quad (6.73)$$

Expanding the solution as

$$\mathbf{y} \sim \mathbf{y}_b + \varepsilon^\alpha \mathbf{y}_1 + \cdots , \quad (6.74)$$

and substituting this into (6.73), yields

$$f_k \sim \varepsilon^\alpha \mathbf{y}_1 \cdot \nabla_y f_k(\lambda_b, \mathbf{y}_b) + \frac{1}{2}\varepsilon^\alpha \left[2\varepsilon \mathbf{y}_1 \cdot \nabla_y \partial_\lambda f_k(\lambda_b, \mathbf{y}_b) + \varepsilon^\alpha \mathbf{y}_1^T \mathbf{H}_k \mathbf{y}_1\right] + \cdots . \quad (6.75)$$

Therefore, from the steady-state equation $\mathbf{f}(\lambda, \mathbf{y}) = \mathbf{0}$, the $O(\varepsilon^\alpha)$ problem is

$$\mathbf{L}\mathbf{y}_1 = \mathbf{0}, \quad (6.76)$$

where $\mathbf{L}$ is the $n \times n$ matrix with i, j-component $L_{ij} = \partial_{y_i} f_j(\lambda_b, \mathbf{y}_b)$. This is the Jacobian matrix of $\mathbf{f}$ at $(\lambda_b, \mathbf{y}_b)$. An alternate notation for this matrix is $\mathbf{f}_\mathbf{y}$ and is known as the gradient of $\mathbf{f}$ with respect to $\mathbf{y}$ (in continuum mechanics the notation $\nabla \mathbf{f}$ is sometimes used).

We are interested in the case where there are nonzero solutions of (6.76). For this to happen, $\mathbf{L}$ must be singular. In other words, λ_b must be such that

$$\det(\mathbf{L}) = 0. \quad (6.77)$$

This equation determines the possible bifurcation values λ_b, but the values of α and $\mathbf{y}_1$ must be found from the higher-order problems just as was done for the scalar problems in Section 6.2. Also note that (6.77) is the multidimensional version of the requirement that $F_y(\lambda_b, y_b) = 0$ at a bifurcation point.

The stability of a steady state $\mathbf{y}_s$ is determined by solving (6.71) with the initial condition

$$\mathbf{y}(0) = \mathbf{y}_s + \delta \boldsymbol{\alpha}_0. \quad (6.78)$$

For small δ the expansion of the solution is $\mathbf{y}(t) \sim \mathbf{y}_s + \delta \mathbf{v}(t) + \cdots$. Substituting this into (6.71), we get

$$\frac{d}{dt}\mathbf{v} = \mathbf{L}\mathbf{v}. \quad (6.79)$$

To solve this, let $\mathbf{v}(t) = \mathbf{V}e^{rt}$, where $\mathbf{V}$ is a constant vector. Inserting this into (6.79) yields the polynomial equation

$$\det(\mathbf{L} - r\mathbf{I}) = 0, \quad (6.80)$$

where $\mathbf{I}$ is the identity matrix. Therefore, the exponents in the solution are eigenvalues of the matrix $\mathbf{L}$. If all the eigenvalues of $\mathbf{L}$ satisfy $\mathrm{Re}(r) < 0$,

then the steady state is asymptotically stable, and it is unstable if any are such that $\mathrm{Re}(r) > 0$. The imaginary axis is therefore the boundary that separates asymptotically stable from unstable. If $\mathbf{L}$ does not have any eigenvalues with zero real part, then the steady state is said be *hyperbolic*.

The following example illustrates, and in some ways extends, the ideas discussed above. The problem is from $\mathbb{R}^2$, and there are a number of alternative approaches to analyzing it. One of these is the subject of Exercise 3.

Example

Consider the system

$$\begin{aligned} y' &= v - y[v^2 + y^2 - \lambda(1-\lambda)], && (6.81)\\ v' &= -y - v[v^2 + y^2 - \lambda(1-\lambda)]. && (6.82) \end{aligned}$$

The steady state is $y = v = 0$. Now, setting $f(y,v) = v - y[v^2 + y^2 - \lambda(1-\lambda)]$ and $g(y,v) = -y - v[v^2 + y^2 - \lambda(1-\lambda)]$, then

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} f_y(0,0) & f_v(0,0) \\ g_y(0,0) & g_v(0,0) \end{pmatrix} \\ &= \begin{pmatrix} \lambda(1-\lambda) & 1 \\ -1 & \lambda(1-\lambda) \end{pmatrix}. \end{aligned}$$

Thus, $\det(\mathbf{L} - r\mathbf{I}) = 0$ gives the eigenvalues $r_\pm = \lambda(1-\lambda) \pm i$. This says that the steady state is stable if $\lambda < 0$ or if $\lambda > 1$ and is unstable for $0 < \lambda < 1$.

To determine what happens in the above example near $\lambda_b = 0$, set $\varepsilon = \lambda - \lambda_b$. We will use multiple scales, and so let $t_1 = t$ and $t_2 = \varepsilon t$. The appropriate expansion for the solution in this case is

$$\mathbf{y} \sim \mathbf{y}_s + \varepsilon^\alpha \mathbf{v}_1(t_1,t_2) + \varepsilon^\beta \mathbf{v}_2(t_1,t_2) + \cdots, \tag{6.83}$$

where $\mathbf{y}_s = \mathbf{0}$ and $\mathbf{v}_i = (y_i, v_i)$. Substituting this expansion into (6.81), (6.82) yields

$$\begin{aligned} (\partial_{t_1} + \varepsilon\partial_{t_2})\left(\varepsilon^\alpha y_1 + \varepsilon^\beta y_2 + \cdots\right) &= \varepsilon^\alpha v_1 + \varepsilon^\beta v_2 \\ &\quad -\varepsilon^\alpha y_1\left[\varepsilon^{2\alpha}\left(v_1^2 + y_1^2\right) - \varepsilon + \varepsilon^2\right] + \cdots, \\ (\partial_{t_1} + \varepsilon\partial_{t_2})\left(\varepsilon^\alpha v_1 + \varepsilon^\beta v_2 + \cdots\right) &= -\varepsilon^\alpha y_1 - \varepsilon^\beta y_2 \\ &\quad -\varepsilon^\alpha v_1\left[\varepsilon^{2\alpha}\left(v_1^2 + y_1^2\right) - \varepsilon + \varepsilon^2\right] + \cdots. \end{aligned}$$

From this we get the following:

$O(\varepsilon^\alpha)$ $\partial_{t_1} y_1 = v_1$

$$\partial_{t_1} v_1 = -y_1.$$

The general solution of this system is $y_1 = A(t_2)\sin[t_1 + \theta(t_2)]$ and $v_1 = A(t_2)\cos[t_1 + \theta(t_2)]$. Also, from balancing, we have $\beta = \alpha+1$ and $2\alpha = 1$.

$$O(\varepsilon^\beta) \quad \partial_{t_1} y_2 = v_2 - y_1[(v_1^2 + y_1^2) - 1] - \partial_{t_2} y_1$$
$$\partial_{t_1} v_2 = -y_2 - v_1[(v_1^2 + y_1^2) - 1] - \partial_{t_2} v_1.$$

One can solve the first equation for v_2 and reduce this system to a single second-order equation for y_2. Doing so, one finds that to remove the secular terms, $\theta = \theta_0$ is constant and $A(t_2)$ satisfies $A' = A(A^2 - 1)$. Solving this, we obtain the nonzero solutions

$$A(t_2) = \frac{1}{\sqrt{1 + ce^{-t_2}}}, \tag{6.84}$$

where c is a constant determined from the initial conditions.

Therefore, for nonzero initial data, the above analysis shows that as $t \to \infty$,

$$\mathbf{y} \sim \begin{cases} \mathbf{0} & \text{if } \lambda < 0 \\ \sqrt{\lambda}(\sin(t + \theta_0), \cos(t + \theta_0)) & \text{if } 0 < \lambda << 1 \end{cases}. \tag{6.85}$$

The zero solution is expected when $\lambda < 0$ since it is stable in that region. For positive values of λ we get a periodic solution, and this means there is a supercritical Hopf bifurcation at the point $(\lambda_b, \mathbf{y}_b) = (0, \mathbf{0})$. Using the same arguments, one can show that there is a subcritical Hopf bifurcation at $\lambda = 1$. The bifurcation diagram, for the full range of λ values, is shown in Fig. 6.17.

It is worth pointing out which properties of the eigenvalues of $\mathbf{L}$ played an important role in the appearance, and disappearance, of the periodic solution. Recall that we found $r_\pm = \lambda(1 - \lambda) \pm i$. Therefore,

FIGURE 6.17. Surface formed from the limit cycles of (6.81), (6.82). An approximation of this surface when $0 \le \lambda << 1$ can be obtained directly from (6.85), and the general equation for the surface is derived in Exercise 3.

i) A single pair of eigenvalues cross the imaginary axis to change the stability of the steady state. They moved into the right half-plane at $\lambda = 0$ and moved back into the left half-plane when $\lambda = 1$.

ii) When the eigenvalues cross the imaginary axis, they satisfy the transversality condition. This condition states that $\frac{d}{d\lambda}\mathrm{Re}(r) \neq 0$ when $\mathrm{Re}(r) = 0$. In particular, in the preceding example, $\frac{d}{d\lambda}\mathrm{Re}(r) = 1$ when $\lambda = 0$ and $\frac{d}{d\lambda}\mathrm{Re}(r) = -1$ when $\lambda = 1$.

iii) When the eigenvalues cross the imaginary axis, they are simple.

These conditions are the basis of what is known as the Poincaré–Andronov–Hopf bifurcation theorem. Those who may be interested in pursuing an investigation of the theory underlying this material should consult the books by Hale and Koçak (1991), Guckenheimer and Holmes (1983), and Marsden and McCracken (1976).

Exercises

1. For (6.71), assuming $n = 2$, show the following:

   a) A steady state $\mathbf{y}_s$ is asymptotically stable if $\mathrm{tr}(\mathbf{L}) < 0$ and $\det(\mathbf{L}) > 0$.

   b) At a Hopf bifurcation point, $\mathrm{tr}(\mathbf{L}) = 0$ and $\det(\mathbf{L}) > 0$.

   c) At a bifurcation point where a steady-state solution changes stability, $\det(\mathbf{L}) = 0$ and $\mathrm{tr}(\mathbf{L}) < 0$.

2. The polynomial $p(r) = r^n + a_1 r^{n-1} + \cdots + a_n$ is a Hurwitz polynomial if all of its roots have negative real parts. The Routh-Hurwitz criterion states that $p(r)$ is a Hurwitz polynomial if and only if the principal minors of the matrix H are all positive, where (Leipholz, 1987)

$$H = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & \cdots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & a_n \end{pmatrix}.$$

   In this matrix $H_{ii} = a_i$ and $a_k = 0$ if $k > n$.

   a) Suppose $\mathbf{y} = \mathbf{y}_s(\lambda)$ is a steady-state solution of (6.71). Explain why it is asymptotically stable if $p(r) = \det(r\mathbf{I} - \mathbf{L})$ is a Hurwitz polynomial.

b) When $n = 3$, show that a steady state is asymptotically stable if $a_1 > 0$, $a_1 a_2 - a_3 > 0$, and $a_3 > 0$.

c) When $n = 4$, show that a steady state is asymptotically stable if $a_1 > 0$, $a_3 > 0$, $a_4 > 0$, and $a_3(a_1 a_2 - a_3) - a_4 a_1^2 > 0$.

3. This problem considers the system given in (6.81), (6.82).

a) Letting $y = r\cos(\theta)$ and $v = r\sin(\theta)$, find the differential equations satisfied by $r(t)$ and $\theta(t)$.

b) Sketch the integral curves for $r(t)$, and from these comment on the limit of the solution as $t \to \infty$.

c) Carry out a multiple-scale analysis to determine what happens near the bifurcation point corresponding to $\lambda = 0$.

d) Find the exact solution and compare its behavior with what you found in parts (b) and (c).

4. In modeling the populations of a predator and its prey, one finds the equations (Collings and Wollkind, 1990)

$$\begin{aligned} y' &= y(1-y) - \frac{\alpha y v}{\beta + y}, \\ v' &= \lambda v \left(1 - \frac{v}{y}\right), \end{aligned}$$

where α, β, and λ are positive constants and $\beta < 1$. As usual, λ is the bifurcation parameter. Also, since $y(t)$ and $v(t)$ represent population densities of prey and predator, respectively, they are assumed to be positive.

a) Find the steady state and show it is stable if $\lambda > \frac{(\alpha-\mu)(1-\alpha-\beta+\mu)}{(1-\alpha+\beta+\mu)}$, where $\mu = [(1-\alpha-\beta)^2 + 4\beta]^{1/2}$.

b) Determine what happens near the point where the steady state changes stability.

5. The Sel'kov model for the enzymatic driven reactions that result in the conversion of ADP into ATP is

$$\begin{aligned} y' &= 1 - y v^{\gamma}, \\ v' &= \lambda v (y v^{\gamma - 1} - 1), \end{aligned}$$

where $\gamma > 1$ and $\lambda > 0$ are constants. In these equations, y and v designate the concentrations of ADP and ATP, respectively.

a) Find the steady state and determine for what values of λ it is stable.

b) Determine what happens near the point where the steady state changes stability.

6. In the description of the dynamics of excitons in semiconductor physics, one finds the following problem (Lee, 1992):

$$\begin{aligned} n' &= \gamma - \alpha n^2 x, \\ x' &= \alpha n^2 x - \frac{x}{1+x}, \end{aligned}$$

where α and γ are constants that satisfy $0 < \gamma < 1$ and $\alpha > 0$. Also, $x(t)$ and $n(t)$ are densities and are therefore nonnegative.

a) Find the steady state and determine for what values of γ and α it is stable.

b) Determine what happens near the point where the steady state loses stability.

7. In the study of chemical reactions that involve thermal feedback, one finds the following problem (Gray and Scott, 1990):

$$\begin{aligned} c' &= \mu - \kappa c e^T, \\ T' &= c e^T - T, \end{aligned}$$

where $c(t)$ is the concentration of chemical and $T(t)$ is the temperature rise due to self-heating. Also, μ and κ are positive constants.

a) Find the steady state and determine for what values of μ and κ it is stable. Explain why the steady state is stable if $\kappa > e^{-2}$.

b) Suppose the value of κ is fixed and satisfies $0 < \kappa < e^{-2}$. Using μ as the bifurcation parameter, sketch the bifurcation diagrams for the steady states of c and T. Letting μ_ℓ and μ_r, where $\mu_\ell < \mu_r$, identify the two values of μ where c changes stability. Classify these two bifurcation points.

c) Determine what happens near $\mu = \mu_\ell$. Also, comment on what you expect to occur near $\mu = \mu_r$.

d) Use the smallness of κ to find the first three terms in expansions for μ_ℓ and μ_r.

8. A classical model in the study of oscillatory systems is the Brusselator. The equations in this case are (Gray et al., 1988):

$$\begin{aligned} x' &= \mu - (1+\alpha)x + x^2 y, \\ y' &= x - x^2 y. \end{aligned}$$

Here μ and α are positive constants.

a) Find the steady state and determine its stability when $\alpha \geq 1$.

b) Suppose $0 < \alpha < 1$ and μ is the bifurcation parameter. Determine the stability of the steady state and find out what happens near the point where the steady state loses stability.

FIGURE 6.18. Elastic pendulum studied in Exercise 9.

9. An elastic pendulum is a pendulum where the rod contains an elastic spring as shown in Fig. 6.18. The (nondimensional) equations of motion for this system are (Minorsky, 1947; Heinbockel and Struble, 1963)

$$\begin{aligned}(1+z)\theta'' + 2z'\theta' + \sin\theta &= 0,\\ z'' + \kappa^2 z + 1 - \cos\theta - (1+z)\theta_t^2 &= 0,\end{aligned}$$

where $\theta(t)$ is the usual angular coordinate and $z(t)$ is the stretch in the spring. Also, κ^2 is a positive constant, and for physical reasons it is required that $-1 < z$.

a) What is the steady state for this system?

b) Suppose the initial conditions are $\theta(0) = \theta'(0) = 0$, $z(0) = \varepsilon\alpha_0$, and $z'(0) = 0$, where α_0 is a given nonzero constant. Describe what these conditions correspond to in physical (or geometric) terms. Use this to find the exact solution of the problem and describe the motion.

c) To investigate the stability of the solution you found in part (b), suppose $\theta(0) = \varepsilon a_0$, $\theta'(0) = \varepsilon b_0$, $z(0) = \varepsilon\alpha_0$, and $z'(0) = 0$. For small ε, find an asymptotic approximation of the solution that is valid for large t. You should find the explicit t_1 dependence, but you only need to find the equations that determine the t_2 dependence.

10. It is possible for traveling wave solutions to bifurcate from a steady state. An example of this occurs with the system of equations

$$\begin{aligned}\partial_t u &= [\lambda - f(r)]u - v + \partial_x^2 u,\\ \partial_t v &= u + [\lambda - f(r)]v + \partial_x^2 v,\end{aligned}$$

where $r^2 = u^2 + v^2$ and $f(r)$ is smooth with $f(0) = f'(0) = 0$ but $f''(0) > 0$. This is a prototype problem that comes up in the study of pattern formation (Howard and Kopell, 1977; Murray, 1989). In this context u, v represent deviations of chemical concentrations from equilibrium and can be positive or negative.

a) For what values of λ is the steady state $u = v = 0$ stable? Explain why one would classify the point where there is a change in the stability as a Hopf bifurcation point.

b) Letting $u = r\cos\theta$ and $v = r\sin\theta$, show that the problem becomes

$$\begin{aligned} \partial_t r &= [\lambda - f(r)]r - r\theta_x^2 + \partial_x^2 r, \\ \partial_t \theta &= 1 + r^{-2}\partial_x(r^2\theta_x). \end{aligned}$$

With this, find the traveling wave solutions that depend on $\theta = kx - \omega t$ and $r = r_0$.

c) The multiple-scale analysis of the traveling wave will be replaced here with a simpler approach. In particular, assume $\theta \sim kx - \omega t + \varepsilon^\alpha \phi(x,t)$ and $r \sim r_0 + \varepsilon^\alpha \rho(x,t)$, where $\phi = \phi_0 e^{(st+i\xi x)}$ and $\rho = \rho_0 e^{(st+i\xi x)}$. For stability, we need that $\mathrm{Re}(s) < 0$ for all values of the constant ξ. Show that this results in the conclusion that the stable traveling waves are those that satisfy $0 < \lambda - f(r_0) < \frac{1}{4} r_0 f'(r_0)$.

11. Consider the system of equations

$$\begin{aligned} y' &= v - y(v^2 + y^2 - \lambda), \\ v' &= -y - v(v^2 + y^2 - \mu). \end{aligned}$$

a) In what region in the (λ, μ)-plane is the steady-state solution $y = v = 0$ stable? What portion of the boundary of this region gives rise to a Hopf bifurcation?

b) For points along the boundary where a Hopf bifurcation occurs, find the solution bifurcating from the steady-state solution. You do not have to consider the cases when $\mu = \pm 1$.

Appendix A1

Solution and Properties of Transition Layer Equations

A1.1 Airy Functions

This section concerns the solutions of Airy's equation. A more extensive presentation can be found in Abramowitz and Stegun (1972), and the derivation of some of these formulas can be found in Olver (1974).

Differential Equation

$$y'' = xy, \quad \text{for} - \infty < x < \infty.$$

General Solution

$$y(x) = \alpha_0 Ai(x) + \beta_0 Bi(x),$$

where

$$\begin{aligned} Ai(x) &\equiv \frac{1}{3^{2/3}\pi} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\kappa+1}{3}\right) \sin\left[\frac{2\pi}{3}(\kappa+1)\right]}{\kappa!} \left(3^{1/3}x\right)^k \\ &= Ai(0)\left(1 + \frac{1}{6}x^3 + \cdots\right) + Ai'(0)\left(x + \frac{1}{12}x^4 + \cdots\right) \end{aligned}$$

and

$$\begin{aligned} Bi(x) &\equiv e^{\pi i/6} Ai\left(xe^{2\pi i/3}\right) + e^{-\pi i/6} Ai\left(xe^{-2\pi i/3}\right) \\ &= Bi(0)\left(1 + \frac{1}{6}x^3 + \cdots\right) + Bi'(0)\left(x + \frac{1}{12}x^4 + \cdots\right). \end{aligned}$$

Particular Values

$$Ai(0) = \frac{\Gamma\left(\frac{1}{3}\right)}{2\pi 3^{1/6}} \qquad Ai'(0) = -\frac{3^{1/6}\Gamma\left(\frac{2}{3}\right)}{2\pi}$$

$$Bi(0) = \sqrt{3}Ai(0) \qquad Bi'(0) = -\sqrt{3}Ai'(0)$$

$$\int_0^\infty Ai(x)dx = \frac{1}{3} \qquad \int_{-\infty}^0 Ai(x)dx = \frac{2}{3} \qquad \int_{-\infty}^0 Bi(x)dx = 0$$

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}.$$

Graph of Airy Functions

Asymptotic Approximations

Setting $\zeta = \frac{2}{3}|x|^{3/2}$ and $\eta = \frac{5}{72\zeta}$, then

$$Ai(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\zeta - \dfrac{\pi}{4}\right) + \eta(x)\sin\left(\zeta - \dfrac{\pi}{4}\right)\right] & \text{as } x \to -\infty \\ \dfrac{1}{2\sqrt{\pi}x^{1/4}}e^{-\zeta}[1 - \eta(x)] & \text{as } x \to +\infty \end{cases}$$

and

$$Bi(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}|x|^{1/4}}\left[\cos\left(\zeta + \dfrac{\pi}{4}\right) + \eta(x)\sin\left(\zeta + \dfrac{\pi}{4}\right)\right] & \text{as } x \to -\infty \\ \dfrac{1}{\sqrt{\pi}x^{1/4}}e^{\zeta}[1 + \eta(x)] & \text{as } x \to +\infty \end{cases}$$

Also, setting $\nu = \frac{7}{72\zeta}$, then

$$Ai'(x) \sim \begin{cases} \dfrac{-1}{\sqrt{\pi}}|x|^{1/4}\left[\cos\left(\zeta + \dfrac{\pi}{4}\right) - \nu(x)\sin\left(\zeta + \dfrac{\pi}{4}\right)\right] & \text{as } x \to -\infty \\ \dfrac{-1}{2\sqrt{\pi}}x^{1/4}e^{-\zeta}[1 + \nu(x)] & \text{as } x \to +\infty \end{cases}$$

and

$$Bi'(x) \sim \begin{cases} \dfrac{1}{\sqrt{\pi}}|x|^{1/4}\left[\cos\left(\zeta - \dfrac{\pi}{4}\right) - \nu(x)\sin\left(\zeta - \dfrac{\pi}{4}\right)\right] & \text{as } x \to -\infty \\ \dfrac{1}{\sqrt{\pi}}x^{1/4}e^{\zeta}[1 - \nu(x)] & \text{as } x \to +\infty. \end{cases}$$

Connection with Bessel Functions

Setting $\zeta = \frac{2}{3}|x|^{3/2}$, then

$$Ai(x) = \begin{cases} \dfrac{1}{3}|x|^{1/2}\left[J_{\frac{1}{3}}(\zeta) + J_{-\frac{1}{3}}(\zeta)\right] & \text{for } x \le 0 \\ \dfrac{1}{3}x^{1/2}\left[I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)\right] & \text{for } x \ge 0 \end{cases}$$

and

$$Bi(x) = \begin{cases} \left(\dfrac{1}{3}|x|\right)^{1/2}\left[-J_{\frac{1}{3}}(\zeta) + J_{-\frac{1}{3}}(\zeta)\right] & \text{for } x \le 0 \\ \left(\dfrac{1}{3}x\right)^{1/2}\left[I_{-\frac{1}{3}}(\zeta) + I_{\frac{1}{3}}(\zeta)\right] & \text{for } x \ge 0. \end{cases}$$

A1.2 Confluent Hypergeometric Functions

This section concerns the properties of the solutions of a differential equation that arises frequently when solving turning-point problems. It is related to the hypergeometric equation. Much of the material presented here is adopted from Abramowitz and Stegun (1972), Buchholz (1969), and Slater (1960).

Differential Equation

$$y'' + \alpha x y' + \beta y = 0, \quad \text{for } -\infty < x < \infty,$$

where α and β are nonzero constants.

General Solution

$$y(x) = \alpha_0 M\left(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) + \beta_0 x M\left(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right),$$

where $M(a, b, z)$ is a confluent hypergeometric function and is known as Kummer's function (it is also denoted as ${}_1F_1(a, b, z)$). Note that

$M(a, b, -\frac{1}{2}\alpha x^2)$ is an even function of x, and so the above solution consists of the sum of an even and an odd function.

The series definition of Kummer's function is

$$M(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{1}{k!} z^k,$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ and $(a)_0 = 1$. This series is absolutely convergent for all values of a, b, and z except for $b = 0, 1, -2, -3, \ldots$.

Particular Values

$$M(a, b, 0) = 1, \quad M(0, b, z) = 1, \quad \text{and } M(a, a, z) = e^z$$

Special Cases

If $\alpha = \beta$, then

$$M\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = e^{-\alpha x^2/2}$$

and

$$M\left(1, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{1}{x}\int_0^x e^{\alpha(s^2-x^2)/2} ds.$$

If $\alpha = -\beta$, then

$$M\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = e^{-\alpha x^2/2} - \alpha x \int_0^x e^{-\alpha r^2/2} dr$$

and

$$M\left(0, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = 1.$$

If $\beta = 2\alpha$, then

$$M\left(1, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = 1 - \alpha x \int_0^x e^{\alpha(s^2-x^2)/2} ds$$

and

$$M\left(\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = e^{-\alpha x^2/2}.$$

If $\alpha = 2\beta$, then

$$M\left(\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \Gamma\left(\frac{3}{4}\right)\left(-\frac{1}{8}\alpha x^2\right)^{1/4} e^{-\alpha x^2/4} I_{-1/4}\left(-\frac{1}{4}\alpha x^2\right)$$

and

$$M\left(\frac{3}{4}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \Gamma\left(\frac{5}{4}\right)\left(-\frac{1}{8}\alpha x^2\right)^{-1/4} e^{-\alpha x^2/4} I_{1/4}\left(-\frac{1}{4}\alpha x^2\right).$$

Polynomials

When the first argument of $M(a,b,z)$ is a nonpositive integer, the function reduces to an expression involving a Laguerre polynomial. In particular, if $n = 0,1,2,3,\ldots$, then

$$M\left(-n, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{\sqrt{\pi}n!}{2\Gamma(\frac{3}{2}+n)} \sum_{k=0}^{n} \binom{n+\frac{1}{2}}{n-k} \frac{1}{k!} \left(\frac{1}{2}\alpha x^2\right)^k$$

and

$$M\left(-n, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) = \frac{\sqrt{\pi}n!}{\Gamma(\frac{1}{2}+n)} \sum_{k=0}^{n} \binom{n-\frac{1}{2}}{n-k} \frac{1}{k!} \left(\frac{1}{2}\alpha x^2\right)^k .$$

Asymptotic Approximations

The asymptotic expansions for large $|x|$ depend on the sign of α and also on whether or not the first argument of the function is a nonpositive integer (see Polynomials above).

I. For $x^2 \to \infty$,

$$M\left(\frac{\alpha+\beta}{2\alpha}, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) \sim \begin{cases} \dfrac{\sqrt{\pi}}{2\Gamma\left(\frac{\alpha+\beta}{2\alpha}\right)} \left(-\dfrac{1}{2}\alpha x^2\right)^{\frac{\beta-2\alpha}{2\alpha}} e^{-\frac{1}{2}\alpha x^2} & \text{if } \alpha < 0 \\[2ex] \dfrac{\sqrt{\pi}}{2\Gamma\left(\frac{2\alpha-\beta}{2\alpha}\right)} \left(\dfrac{1}{2}\alpha x^2\right)^{-\frac{\alpha+\beta}{2\alpha}} & \text{if } \alpha > 0. \end{cases}$$

The approximation for $\alpha < 0$ does not hold when $\frac{\alpha+\beta}{2\alpha} = -n$, where $n = 0,1,2,3,\ldots$, and these cases are included in III below. Also, the approximation for $\alpha > 0$ does not hold when $\frac{2\alpha-\beta}{2\alpha}$ is a nonpositive integer. The case for $\beta = 2\alpha$ is included in Special Cases, and Slater (1960) should be consulted for the others.

II. For $x^2 \to \infty$,

$$M\left(\frac{\beta}{2\alpha}, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) \sim \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{\beta}{2\alpha}\right)} \left(-\dfrac{1}{2}\alpha x^2\right)^{\frac{-\alpha+\beta}{2\alpha}} e^{-\frac{1}{2}\alpha x^2} & \text{if } \alpha < 0 \\[2ex] \dfrac{\sqrt{\pi}}{\Gamma\left(\frac{\alpha-\beta}{2\alpha}\right)} \left(\dfrac{1}{2}\alpha x^2\right)^{-\frac{\beta}{2\alpha}} & \text{if } \alpha > 0. \end{cases}$$

The approximation for $\alpha < 0$ does not hold when $\frac{\beta}{2\alpha} = -n$, where $n = 1,2,3,\ldots$, and these cases are included in III below. Also, the approximation for $\alpha > 0$ does not hold when $\frac{\alpha-\beta}{2\alpha}$ is a nonpositive integer.

The case for $\beta = \alpha$ is included in Special Cases, and Slater (1960) should be consulted for the others.

III. The nonpositive integer cases are

$$M\left(-n, \frac{3}{2}, -\frac{1}{2}\alpha x^2\right) \sim \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{2}+n)} \left(\frac{1}{2}\alpha x^2\right)^n$$

and

$$M\left(-n, \frac{1}{2}, -\frac{1}{2}\alpha x^2\right) \sim \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}+n)} \left(\frac{1}{2}\alpha x^2\right)^n .$$

Plots

A1.3 Higher-Order Turning Points

Analysis of higher-order turning points can be found in Willner and Rubenfeld (1976).

Differential Equation

$$y'' = x^\gamma y, \quad \text{for } -\infty < x < \infty \quad \text{and } \gamma \geq 0.$$

General Solution

$$y(x) = \alpha_0 y_1(x) + \beta_0 y_2(x),$$

where, setting $\nu = 1/(\gamma+2)$ and $\kappa = (\gamma+2)/2$,

$$y_1(x) = \begin{cases} |x|^{1/2}\left[J_\nu\left(2\nu|x|^\kappa\right) + J_{-\nu}\left(2\nu|x|^\kappa\right)\right] & \text{for } x \le 0 \\ x^{1/2}\left[I_{-\nu}\left(2\nu x^\kappa\right) - I_\nu\left(2\nu x^\kappa\right)\right] & \text{for } x \ge 0 \end{cases}$$

and

$$y_2(x) = \begin{cases} |x|^{1/2}\left[-J_\nu\left(2\nu|x|^\kappa\right) + J_{-\nu}\left(2\nu|x|^\kappa\right)\right] & \text{for } x \le 0 \\ x^{1/2}\left[I_{-\nu}\left(2\nu x^\kappa\right) + I_\nu\left(2\nu x^\kappa\right)\right] & \text{for } x \ge 0. \end{cases}$$

Asymptotic Approximations

Setting $\xi = \frac{2}{\gamma+2}|x|^{\frac{2+\gamma}{2}}$, then

$$y_1(x) \sim \begin{cases} \dfrac{2\cos(\nu\pi/2)}{\sqrt{\pi\nu}}|x|^{-\gamma/4}\cos\left(\xi - \dfrac{\pi}{4}\right), & \text{as } x \to -\infty \\ \dfrac{\sin(\nu\pi)}{\sqrt{\pi\nu}}x^{-\gamma/4}e^{-\xi} & \text{as } x \to +\infty \end{cases}$$

and

$$y_2(x) \sim \begin{cases} \dfrac{2\sin(\nu\pi/2)}{\sqrt{\pi\nu}}|x|^{-\gamma/4}\cos\left(\xi + \dfrac{\pi}{4}\right) & \text{as } x \to -\infty \\ \dfrac{1}{\sqrt{\pi\nu}}x^{-\gamma/4}e^{\xi} & \text{as } x \to +\infty. \end{cases}$$

Appendix A2

Asymptotic Approximations of Integrals

A2.1 Introduction

This appendix summarizes some formulas for approximating integrals. Specifically, approximations for integrals of the form

$$\int_a^b f(t)e^{-xg(t)}dt \quad \text{and} \quad \int_a^b f(t)e^{ixg(t)}dt$$

are given for the case of large x. Those who may be interested in a more extensive development of this material should consult Murray (1984) or Olver (1974).

In this appendix, the following are assumed:

i) $-\infty \leq a < b \leq \infty$,

ii) $f(t)$ and $g(t)$ are continuous and $g(t)$ is real-valued,

iii) as $t \to t_0$, where $a \leq t_0 \leq b$,

$$f(t) \sim f_0(t-t_0)^\alpha + f_1(t-t_0)^\beta, \quad \text{where } -1 < \alpha < \beta \quad \text{and} \quad f_0 \neq 0,$$

and

$$g(t) \sim g_0 + g_1(t-t_0)^\lambda, \quad \text{where } \lambda > 0 \quad \text{and} \quad g_1 \neq 0,$$

iv) a, b, f, and g are independent of x.

Each of the following approximations will require one or two additional conditions.

A2.2 Watson's Lemma

One of the simplest approximations arises when $a = 0$ and $g(t) = t$. In this case,

$$\int_0^b f(t)e^{-xt}dt \sim \frac{f_0\Gamma(1+\alpha)}{x^{1+\alpha}} + \frac{f_1\Gamma(1+\beta)}{x^{1+\beta}} \quad \text{as } x \to \infty.$$

A2.3 Laplace's Approximation

If the minimum of $g(t)$, for $a \le t \le b$, occurs only at $t = t_0$, where $a < t_0 < b$, then

$$\int_a^b f(t)e^{-xg(t)}dt \sim \frac{2f_0}{\lambda}\Gamma\left(\frac{1+\alpha}{\lambda}\right)\left(\frac{1}{xg_1}\right)^{\frac{1+\alpha}{\lambda}} e^{-xg_0} \quad \text{as } x \to \infty.$$

If the minimum of $g(t)$ for $a \le t \le b$ occurs only at $t = t_0$, where $t_0 = a$ or $t_0 = b$, then

$$\int_a^b f(t)e^{-xg(t)}dt \sim \frac{f_0}{\lambda}\Gamma\left(\frac{1+\alpha}{\lambda}\right)\left(\frac{1}{xg_1}\right)^{\frac{1+\alpha}{\lambda}} e^{-xg_0} \quad \text{as } x \to \infty.$$

When the minimum occurs at $t_0 = a$ then we must require $a > -\infty$, and when it occurs at $t_0 = b$ then we require $b < \infty$ (i.e., the minimum occurs at a finite point). Also, note that the only difference between the above two approximations is a factor of 2.

A particularly useful special case is when $\alpha = 0$ and $\lambda = 2$ (and $g(t)$ is smooth). So, for example, when $t_0 = a$ the above approximation simplifies to the following:

$$\int_a^b f(t)e^{-xg(t)}dt \sim f(a)\left(\frac{\pi}{2xg''(a)}\right)^{\frac{1}{2}} e^{-xg(a)} \quad \text{as } x \to \infty.$$

A2.4 Stationary Phase Approximation

In what follows, it is assumed $g(t)$ is smooth and has only one stationary point. Specifically, in the interval $a \le t \le b, g'(t) = 0$ only at $t = t_0$ and this is a finite point (i.e., $-\infty < t_0 < \infty$).

Suppose $t_0 = a$ or $t_0 = b$. If in the expansion for $f(t)$ we have $\alpha = 0$, then

$$\int_a^b f(t)e^{ixg(t)}dt$$

$$\sim \frac{f(t_0)}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)\left(\frac{1}{x|g_1|}\right)^{\frac{1}{\lambda}} e^{i\left(xg_0 \pm \frac{\pi}{2\lambda}\right)} \quad \text{as } x \to \infty \left.\right\} \begin{array}{ll} + & \text{if } g_1 > 0 \\ - & \text{if } g_1 < 0. \end{array}$$

In the special case when $\lambda = 2$, this approximation can be written as

$$\int_a^b f(t)e^{ixg(t)}dt$$

$$\sim f(t_0)\sqrt{\frac{\pi}{2x|g''(t_0)|}}e^{i\left(xg(t_0)\pm\frac{\pi}{4}\right)} \text{ as } x \to \infty \left.\right\} \begin{array}{ll} + & \text{if } g''(t_0) > 0 \\ - & \text{if } g''(t_0) < 0. \end{array}$$

Suppose $a < t_0 < b$. If $\alpha = 0$, then

$$\int_a^b f(t)e^{ixg(t)}dt$$

$$\sim \frac{2f(t_0)}{\lambda}\Gamma\left(\frac{1}{\lambda}\right)\left(\frac{1}{x|g_1|}\right)^{\frac{1}{\lambda}} e^{i\left(xg_0\pm\frac{\pi}{2\lambda}\right)} \text{ as } x \to \infty \left.\right\} \begin{array}{ll} + & \text{if } g_1 > 0 \\ - & \text{if } g_1 < 0. \end{array}$$

Appendix A3

Numerical Solution of Nonlinear Boundary-Value Problems

A3.1 Introduction

Many times, both asymptotic and numerical methods are used to examine the solution of a problem. For example, if the structure of the solution is unclear, one can try to solve the problem numerically for specific values of the parameters to gain some insight. There are many programs available that will calculate the numerical solution of differential equations under a wide variety of conditions. For those who may not have access to such software, this appendix contains a FORTRAN program for solving the boundary-value problem

$$y'' = f(x, y, y'), \quad \text{for } x_0 < x < x_1,$$

where $y(x_0) = y_0$ and $y(x_1) = y_1$. The function f in this equation can also depend on ε. It should be pointed out that no pretense is made about the sophistication of the algorithm. It uses standard second-order differences for approximating the derivatives, and Newton's method to solve the resulting system (see either Burden and Faires (1993) or Keller (1992)). It is possible to do much better than this, particularly for singular perturbation problems, but this code will suffice for many applications. Those who may be interested in pursuing some of the ideas underlying finding the numerical solution of problems with layers should consult Section 2.7.

For input the program requires (in the parameter statements) values for

$$x0 \equiv x_0, \quad y0 \equiv y_0, \quad x1 \equiv x_1, \quad y1 \equiv y_1,$$

and

$$ep \equiv \varepsilon, \quad nx \equiv \text{integer number of grid points along the } x\text{-axis.}$$

The code also needs (in the function statements) the formulas for $f(x, y, z)$, $f_y(x, y, z)$, and $f_z(x, y, z)$, where z is used in place of y'. In the program $fy(x, y, z)$ designates $f_y(x, y, z)$ and $fz(x, y, z)$ designates $f_z(x, y, z)$. Although it probably will not be necessary, the value of the stopping error (denoted as error in the code and set to 0.00001) can be changed.

A3.2 Examples

1) The particular functions and parameter values in the following code are for the problem

$$\varepsilon y'' = e^{\varepsilon y'} + y, \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = -1$. The solution is calculated for $\varepsilon = 0.0001$ and $nx = 1000$. Also, $f(x, y, z) = (e^{\varepsilon z} + y)/\varepsilon$, $fy(x, y, z) = 1/\varepsilon$, and $fz(x, y, z) = e^{\varepsilon z}$. The output from the code is displayed in Fig. A.3.1. In this case, it took three Newton iterations and finished with a Cauchy error of 2.677×10^{-15} (using the sup norm).

2) To solve the nonlinear partial differential equation $u_t = u_{xx} + F(u)$, one can first discretize in time. Using, say, the trapezoidal method and letting $U_n(x)$ denote the finite-difference approximation of $u(x, t_n)$, then

$$U''_{n+1} = \frac{2}{k} U_{n+1} - F(U_{n+1}) + G(U_n),$$

where $G(U_n) = -U''_n - \frac{2}{k} U_n - F(U_n)$ and k is the step size (i.e., $k = t_{n+1} - t_n$). It is straightforward to use the program listed below to solve this nonlinear boundary-value problem for U_{n+1}.

FIGURE A3.1. Numerical solution obtained for example problem.

3) Consider the linear problem

$$\varepsilon y'' + p(x,\varepsilon)y' + q(x,\varepsilon)y = g(x,\varepsilon), \quad \text{for } x_0 < x < x_1,$$

where $y(x_0) = y_0$ and $y(x_1) = y_1$. Also, p, q, and g are assumed to be continuous with $q(x,\varepsilon) < 0$ for $x_0 \le x \le x_1$. In this case, the numerical scheme is stable if the number of grid points along the x-axis satisfies

$$nx \ge \frac{x_1 - x_0}{2\varepsilon} \max_{x_0 \le x \le x_1} |p(x,\varepsilon)|.$$

A proof of this statement, and its extension to the nonlinear system, can be found in Keller (1992).

A3.3 Computer Code

```
c ***************************************************************
c *
c *   FORTRAN code to solve the BVP:
c *                                  y'' = f(x,y,y') for x0 < x < x1
c *                                  y(x0) = y0       y(x1) = y1
c *  Input:
c *       Values (in the two parameter statements):
c *                 x0, y0, x1, y1, ep, nx
c *       Functions (following the dimension statement):
c *                 f = f(x,y,z), fy = diff(f,y), fz = diff(f,z)
c *
c ***************************************************************
      implicit real*8(a-h,o-z)
      parameter(x0=0.0,y0=1.0,x1=1.0,y1=-1.0)
      parameter(ep=0.0001,nx=1000)
      dimension y(0:nx+1),a(nx+1),c(nx+1),u(nx+1),v(nx+1)
      f(x,y,z)=(exp(ep*z)+y)/ep
      fy(x,y,z)=1/ep
      fz(x,y,z)=exp(ep*z)
      error = 0.00001
      dx=(x1-x0)/(nx+1)
      dxx=dx*dx
c
c     Start with linear solution
c
      do 4 ix=0,nx+1
    4       y(ix)=y0+(y1-y0)*ix/(nx+1)
c
```

```
c     Newton iteration for solving ODE
c
   8  do 10 j = 1, nx
         x = x0 + j*dx
         z = (y(j + 1) - y(j - 1))/(2*dx)
         a(j) = 2 + dxx*fy(x, y(j), z)
         c(j) = -1 - 0.5*dx*fz(x, y(j), z)
  10     v(j) = -(2*y(j) - y(j + 1) - y(j - 1) + dxx*f(x, y(j), z))
      v(1) = v(1)/a(1)
      u(1) = -(2 + c(1))/a(1)
      do 20 j = 2, nx
         xl = a(j) - c(j)*u(j - 1)
         v(j) = (v(j) - c(j)*v(j - 1))/xl
  20     u(j) = -(2 + c(j))/x1
      vv = v(nx)
      y(nx) = y(nx) + vv
      err = abs(vv)
      do 40 jj = nx - 1, 1, -1
         vv = v(jj) - u(jj)*vv
         err = max(err, abs(vv))
  40     y(jj) = y(jj) + vv
      if(err.gt.error) to to 8
c
c      Print the values of x and y
c
      write (6,100) (x0 + ix*dx, y(ix), ix = 0, nx + 1)
 100  format(2e14.5)
      stop
      end
```

References

Abramowitz, M. and Stegun, I.A. (1972). *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables.* Dover, New York.

Ackerberg, R.C. and O'Malley, R.E. (1970). "Boundary layer problems exhibiting resonance." *Studies Appl. Math* **49**, 277–295.

Ahluwalia, D.S. and Keller, J.B. (1977). "Exact and asymptotic representations of the sound field in a stratified ocean," in *Wave Propagation and Underwater Acoustics* (J.B. Keller and J.S. Papadakis, eds.). Springer-Verlag, Berlin, pp. 14–85.

Akulenko, L.D. and Shamayev, A.S. (1986). "Approximate solution of some perturbed boundary value problems." *PMM* **50**, 146–154.

Albano, A.M., Abraham, N.B., and Chyba, D.E. (1984). "Bifurcations, propagating solutions, and phase transitions in a nonlinear chemical reaction with diffusion." *Am. J. Phys.* **53**, 161–167.

Arnold, T.W. and Case, W. (1982). "Nonlinear effects in a simple mechanical system." *Am. J. Phys.* **50**, 220–224.

Auriault, J.-L., Borne, L., and Chambon, R. (1985). "Dynamics of porous saturated media, checking of the generalized law of Darcy." *J. Acoust. Soc. Am.* **77**, 1641–1650.

Bachmann, P. (1894). *Die analytische Zahlentheorie.* Teubner, Leipzig.

Baggeroer, A. and Munk, W. (1992). "The Heard Island feasibility test." *Physics Today* Sept., 22–30.

Bender, C.M., Olaussen, K., and Wang, P.S. (1977). "Numerological analysis of the WKB approximation in large order." *Phys. Rev. D* **16**, 1740–1748.

Bender, C.M. and Orszag, S. (1978). *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill, New York.

Berman, V.S. (1978). "On the asymptotic solution of a nonstationary problem on the propagation of a chemical reaction front." *Dokl Akad Nauk SSSR* **242** (translated in *Soviet Math Dokl* **19**, 1076–1078 1978).

Bleakney, W., Weimer, D.K., and Fletcher, C.H. (1949). "The shock tube: A facility for investigations in fluid dynamics." *Rev. Sci. Instrum.* **20**, 807–815.

Born, M. and Wolf, E. (1980). *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light*, 6th ed. Pergamon Press, Oxford.

Borowitz, S. (1967). *Fundamentals of Quantum Mechanics.* W.A. Benjamin, New York.

Bouthier, M. (1984). "Comparison of the matched asymptotic expansions method and the two-variable technique." *Q.J. Appl. Math.* **41**, 407–422.

Bramson, M. (1983). *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves.* Memoirs of the American Mathematical Society, American Mathematical Society, Providence, RI.

Braun, P.A. (1979). "WKB method for three-term recursion relations and quasienergies of an anharmonic oscillator." *Theor. Math. Phys.* **37**, 1070–1081.

Bremmer, H. and Lee, S.W. (1984). "Propagation of a geometrical optics field in an isotropic inhomogeneous medium." *Radio. Sci.* **19**, 243–257.

Britton, N.F. (1986). *Reaction-Diffusion Equations and Their Applications to Biology.* Academic Press, London.

Brown, D.L. and Lorenz, J. (1987). "A high-order method for stiff boundary value problems with turning points." *SIAM J. Sci. Stat. Comput.* **8**, 790–805.

Buchholz, H. (1969). *The Confluent Hypergeometric Function with Special Emphasis on Its Applications.* Springer-Verlag, New York.

Burden, R.L. and Faires, J.D. (1993). *Numerical Analysis*, 5th ed. PWS-KENT Publishing, Boston.

Burridge, R. and Keller, J.B. (1981). "Poroelasticity equations derived from microstruture." *J. Acoust. Soc. Am.* **70**, 1140–1146.

Burridge, R. and Weinberg, H. (1977). "Horizontal rays and vertical modes," in *Wave Propagation and Underwater Acoustics* (J.B. Keller and J.S. Papadakis, eds.). Springer-Verlag, Berlin, pp. 86–152.

Carlini, F. (1817). *Ricerche sulla convergenza della serie che serva alla soluzione del problema di Keplero*, Milan.

Carrier, G.F., Krook, M., and Pearson, C.E. (1966). *Functions of a Complex Variable: Theory and Technique.* McGraw-Hill, New York.

Carrier, G.F. and Lewis, J.A. (1953). "Boundary layer problems in applied mathematics," in *Advances in Applied Mechanics, Vol. III* (R. von Mises and T. von Kármán, eds.). Academic Press, New York, 1–19.

Carson, E.R., Cobelli, C., and Finkelstein, L. (1983). *The Mathematical Modeling of Metabolic and Endocrine Systems.* John Wiley, New York.

Caughey, T.K. (1970). "Large amplitude whirling of an elastic string — a nonlinear eigenvalue problem." *SIAM J. Appl. Math.* **18**, 210–237.

Cerveny, V. (1985). "The application of ray tracing to the numerical modeling of seismic wavefields in complex structures," in *Seismic Shear Waves. Part A: Theory* (G.P. Dohr, ed.). Pergamon Press, London, pp. 1–124.

Christodoulou, D.M. and Narayan, R. (1992). "The stability of accretion tori. IV. Fission and Fragmentation of slender self-gravitating annuli." *Astrophys. J.* **388**, 451–466.

Cole, J.D. (1968). *Perturbation Methods in Applied Mathematics.* Blaisdell, Waltham, MA.

Cole, J.D. and Cook, L.P. (1986). *Transonic Aerodynamics.* Elsevier Science Publishers, Amsterdam.

Cole, J.D. and Kevorkian, J. (1963). "Uniformly valid asymptotic approximations for certain differential equations," in *Nonlinear Differential Equations and Nonlinear Mechanics* (J.P. LaSalle and S. Lefschetz, eds.). Academic Press, New York, pp. 113–120.

Collings, J.B. and Wollkind, D.J. (1990). "A global analysis of a temperature-dependent model system for a mite predator–prey interaction." *SIAM J. Appl. Math.* **50**, 1348–1372.

Combes, J.M., Duclos, P., and Seiler, R. (1984). "On the Shape Resonance," in *Resonances — Models and Phenomena* (S. Albeverio, L.S. Ferreira, and L. Streit, eds.). Springer-Verlag, Berlin, pp. 64–77.

Comstock, C. and Hsiao, G.C. (1976). "Singular perturbations for difference equations." *Rocky Mtn. J. Math.* **6**, 561–567.

Comtet, A., Bandrauk, A.D., and Cambell, D.K. (1985). "Exactness of semiclassical bound state energies for supersymmetric quantum mechanics." *Phys. Lett. B* **150**, 159–162.

Cook, L.P. and Ludford, G.S.S. (1971). "The behavior as $\varepsilon \to +0$ of solutions to $\varepsilon\nabla^2 w = \partial w/\partial y$ in $|y| \leq 1$ for discontinuous boundary data." *SIAM J. Math. Anal.* **2**, 567–594.

Cook, L.P. and Ludford, G.S.S. (1973). "The behavior as $\varepsilon \to 0^+$ of solutions to $\varepsilon\nabla^2 w = (\partial/\partial y)w$ on the rectangle $0 \leq x \leq 1, |y| \leq 1$." *SIAM J. Math. Anal.* **4**, 161–184.

Cook, R.J. (1990). "Quantum jumps," in *Progress in Optics, Vol. XXVIII* (E. Wolf, ed.). North-Holland, Amsterdam, pp. 361–416.

Costin, O. and Costin, R. (1996). "Rigorous WKB for finite-order linear recurrence relations with smooth coefficients," *SIAM J. Math. Anal.* **27**, 110–134.

Crandall, M.G. and Rabinowitz, P.H. (1980). "Mathematical theory of bifurcation," in *Bifurcation Phenomena in Mathematical Physics and Related Topics* (C. Bardos and D. Bessis, eds.). D. Reidel Publ. Co., Boston, pp. 3–46.

Cushing, J.M. (1977). *Integrodifferential Equations and Delay Models in Population Dynamics.* Lecture Notes in Biomathematics, Springer-Verlag, Berlin.

D'Hoker, E. and Jackiw, R. (1982). "Classical and quantal Liouville field theory." *Phys. Rev. D* **26**, 3517–3542.

Dauxois, T., Peyrard, M., and Willis, C.R. (1992). "Localized breather-like solution in a discrete Klein-Gordon model and application to DNA." *Physica D* **57**, 267–282.

DeGroen, P.P.N. (1980). "The nature of resonance in a singular perturbation problem of turning point type." *SIAM J. Math. Anal.* **11**, 1–22.

Derjaguin, B.V., Push, V.E., and Tolstoi, D.M. (1957). "A theory of stick-slip sliding of solids," in *Proceedings of the Conference on Lubrication and Wear.* The Institution of Mechanical Engineers, London, 257–268.

DeSanti, A.J. (1987). "Nonmonotone interior layer theory for some singularly perturbed quasilinear boundary value problems with turning points." *SIAM J. Math. Anal.* **18**, 321–331.

Dingle, R.B. and Morgan, G.J. (1967a). "WKB methods for difference equations I." *Appl. Sci. Res.* **18**, 221–237.

Dingle, R.B. and Morgan, G.J. (1967b). "WKB methods for difference equations II." *Appl. Sci. Res.* **18**, 238–245.

DiPerna, R.J. and Majda, A. (1985). "The validity of nonlinear geometric optics for weak solutions of conservation laws." *Comm. Math. Phys.* **98**, 313–347.

DiPrima, R.C. (1968). "Asymptotic methods for an infinitely long slider squeeze-film bearing." *J. Lub. Tech.* **90**, 173–183.

DiPrima, R.C., Eckhaus, W., and Segel, L.A. (1971). "Non-linear wave-number interaction in near-critical two-dimensional flows." *J. Fluid Mech.* **49**, 705–744.

Dunham, J.L. (1932). "The Wentzel-Brillouin-Kramers method of solving the wave equation." *Phys. Rev.* **41**, 713–720.

Dutt, R., Khare, A., and Sukhatme, U.P. (1991). "Supersymmetry-inspired WKB approximation in quantum mechanics." *Am. J. Phys.* **59**, 723–727.

Eckhaus, W. and de Jager, E.M. (1966). "Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type." *Arch. Rational. Mech. Anal.* **23**, 26–86.

Eringen, A.C. (1952). "On the non-linear vibration of elastic bars." *Q.J. Appl. Math.* **IX**, 361–369.

Farra, V. and Madariaga, R. (1987). "Seismic waveform modeling in heterogeneous media by ray perturbation theory." *J. Geophys. Res.* **92**, 2697–2712.

Farrell, P.A. (1987). "Sufficient conditions for uniform convergence of a class of difference schemes for a singularly perturbed problem." *IMA J. Num. Anal.* **7**, 459–472.

Fisher, R.A. (1937). "The wave of advance of advantageous genes." *Ann. Eugenics* **7**, 353–369.

Forbes, L.K. (1991). "Forced transverse oscillations in a simple spring-mass system." *SIAM J. Appl. Math.* **51**, 1380–1396.

Ford, W.B. (1916). *Studies on Divergent Series and Summability and the Asymptotic Developments of Functions Defined by Maclaurin Series.* Univ. of Mich., Ann Arbor (reprinted by Chelsea Publ., New York, 1960).

Fraenkel, L.E. (1969). "On the method of matched asymptotic expansions. Part I: A matching principle." *Proc. Camb. Philos. Soc.* **65**, 209–231.

Friedrichs, K.O. (1941). "The mathematical structure of the boundary layer problem." in *Fluid Mechanics* (R. von Mises and K.O. Friedrichs, eds.). Brown Univ., Providence, RI (reprinted by Springer-Verlag, New York, 1971). pp. 171–174.

Fung, Y.C. (1965). *Foundations of Solid Mechanics.* Prentice-Hall, Englewood Cliffs, NJ.

Ganapathisubramanian, N. and Showalter, K. (1986). "Relaxation behavior in a bistable chemical system near the critical point and hysteresis limits." *J. Chem. Phys.* **84**, 5427–5436.

Gans, R. (1915). "Fortpflanzung des Lichts durch ein Inhomogenes Medium." *Ann. Phys. (Leipzig)* **47**, 709–738.

Gao, C. and Kuhlmann-Wilsdorf, D. (1990). "On stick-slip and velocity dependence of friction at low speeds." *J. Tribology* **112**, 354–360.

Geer, J.F. and Keller, J.B. (1968). "Uniform asymptotic solutions for potential flow around a thin airfoil and the electrostatic potential about a thin conductor." *SIAM J. Appl. Math.* **16**, 75–101.

Geronimo, J.S. and Smith, D.T. (1992). "WKB (Liouville-Green) analysis of second order difference equations and applications." *J. Approx. Theory* **69**, 269–301.

Gladwell, G.M.L. (1980). *Contact Problems in the Classical Theory of Elasticity.* Sijthoff & Noordhoff, Germantown, MD.

Goodman, J. and Xin, Z. (1992). "Viscous limits for piecewise smooth solutions to systems of conservation laws." *Arch. Rational Mech. Anal.* **121**, 235–265.

Grasman, J. (1987). *Asymptotic Methods for Relaxation Oscillations and Applications.* Applied Mathematical Sciences Vol. 63, Springer-Verlag, New York.

Grasman, J. and Matkowsky, B.J. (1977). "A variational approach to singularly perturbed boundary value problems for ordinary and partial differential equations with turning points." *SIAM J. Appl. Math.* **32**, 588–597.

Gray, P. and Scott, S.K. (1990). *Chemical Oscillations and Instabilities: Non-linear Chemical Kinetics.* Oxford University Press, Oxford.

Gray, P., Scott, S.K., and Merkin, J.H. (1988). "The Brusselator model of oscillatory reactions." *J. Chem. Soc., Faraday Trans.* **84**, 993–1012.

Green, R.M. (1985). *Spherical Astronomy.* Cambridge Univ. Press, Cambridge.

Guckenheimer, J. and Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Applied Mathetical Sciences, Vol. 42, Springer-Verlag, New York.

Guedes, J.M. and Kikuchi, N. (1990). "Preprocessing and postprocessing for materials based on the homogenization method with adaptive finite element methods." *Computer Meth. Appl. Mech. Eng.* **83**, 143–198.

Hagedorn, P. (1982). *Non-linear Oscillations.* Oxford University Press, Oxford.

Hale, J.K. and Koçak, H. (1991). *Dynamics and Bifurcations.* Springer-Verlag, New York.

Hardy, G.H. (1954). *Orders of Infinity.* 2nd ed. Cambridge Univ. Press, Cambridge.

Hassard, B.D., Kazarinoff, N.D., and Wan, Y.-H. (1981). *Theory and Applications of Hopf Bifurcation.* London Mathematical Society. Lecture note series, Cambridge University Press, New York.

Heading, J. (1962). *An Introduction to Phase-Integral Methods.* Methuen & Co, London.

Heck, A. (1993). *Introduction to Maple.* Springer-Verlag, New York.

Heinbockel, J.H. and Struble, R.A. (1963). "Resonant oscillations of an extensible pendulum." *Z. angew Math. Phys.* **14**, 262–269.

Holmes, M.H. (1981). "A hydroelastic model of the cochlea: An analysis for low frequencies." in *Mathematical Modeling of the Hearing Process* Lecture Notes in Biomathematics (M. Holmes and L. Rubenfeld, eds.), Springer-Verlag, New York, pp. 55–69.

Holmes, M.H. (1982). "A mathematical model of the dynamics of the inner ear." *J. Fluid Mech.* **116**, 59–75.

Holmes, M.H. (1984). "Comparison theorems and similarity solution approximations for a nonlinear diffusion equation arising in the study of soft tissue." *SIAM J. Appl. Math.* **44**, 545–556.

Holmes, M.H. (1985). "A theoretical analysis for determining the nonlinear permeability of soft tissue from a permeation experiment." *Bull. Math. Biol.* **47**, 669–683.

Holmes, M.H. (1986). "Finite deformation of soft tissue: analysis of a mixture model in uni-axial compression." *J. Biomech. Eng.* **108**, 372–381.

Holmes, M.H. (1990). "Nonlinear ionic diffusion in polyelectrolyte gels: an analysis based on homogenization." *SIAM J. Appl. Math.* **50**, 839–852.

Holmes, M.H. and Bell, J. (1991). "Auditory transduction: a model for the role of intracellular calcium in short-term adaptation." *Mathematical and Computer Modeling* **15**, 35–55.

Holmes, M.H. and Stein, F.M. (1976). "Sturmian theory and transformations for the Riccati equation." *Port. Math.* **35**, 65–73.

Hooker, J.W. and Patula, W.T. (1981). "Riccati type transformations for second-order linear difference equations." *J. Math. Anal. Appl.* **82**, 451–462.

Hoppensteadt, F.C. (1982). *Mathematical Methods in Population Biology.* Cambridge Univ. Press, Cambridge.

Hoppensteadt, F.C. and Miranker, W.L. (1977). "Multitime methods for systems of difference equations." *Studies in Appl. Math.* **56**, 273–289.

Horowitz, P. and Hill, W. (1989). *The Art of Electronics*, 2nd ed. Cambridge Univ. Press, Cambridge.

Howes, F.A. (1978). *Boundary-Interior Layer Interactions in Nonlinear Singular Perturbation Theory.* Memoirs of the American Mathematical Society, American Math. Society, Providence, RI.

Hubert, J.S. and Sanchez-Palencia, E. (1989). *Vibration and Coupling of Continuous Systems.* Springer-Verlag, Berlin.

Hutchinson, G.E. (1948). "Circular causal systems in ecology." in *Annals of the New York Academy of Sciences* (R.W. Miner, ed.). New York Academy of Sciences, New York, pp. 221–246.

Il'in, A.M. (1992). *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems.* Translations of Mathematical Monographs vol. 102, American Math. Society, Providence, RI.

Imbo, T. and Sukhatme, U. (1984). "Logarithmic perturbation expansions in nonrelativistic quantum mechanics." *Am. J. Phys.* **52**, 140–146.

Kapila, A.K. (1983). *Asymptotic Treatment of Chemically Reacting Systems.* Pitman, Boston.

Kassoy, D.R. (1976). "Extremely rapid transient phenomena in combustion, ignition and explosion." in *Asymptotic Methods and Singular Perturbations* (R.E. O'Malley, ed.). American Math. Society, Providence, RI, 61–72.

Kath, W.L., Knessl, C., and Matkowsky, B.J. (1987). "A variational approach to nonlinear singularly perturbed boundary-value problems." *Studies in Appl. Math.* **77**, 61–88.

Kato, T. (1980). *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin.

Keener, J.P. (1983). "Analog circuitry for the van der Pol and FitzHugh-Nagumo equations." *IEEE Trans. Systems, Man, Cybernetics* **SMC-13**, 1010–1014.

Keller, H.B. (1992). *Numerical Methods for Two-Point Boundary Value Problems.* Dover Publications, New York.

Keller, H.B., Nandakumaran, A.K., and Ramaswamy, M. (1987). *Lectures on Numerical Methods in Bifurcation Problems.* Springer-Verlag, Berlin.

Keller, J.B. and Kogelman, S. (1970). "Asymptotic solutions of initial value problems for nonlinear partial differential equations." *SIAM J. Appl. Math.* **18**, 748–758.

Keller, J.B., Lewis, R.M., and Seckler, B.D. (1956). "Asymptotic solution of some diffraction problems." *Comm. Pure Appl. Math.* **IX**, 207–265.

Kevorkian, J. and Cole, J.D. (1981). *Perturbation Methods in Applied Mathematics.* Springer-Verlag, New York.

King, J.R., Meere, M.G., and Rogers, T.G. (1992). "Asymptotic analysis of a non-linear model for substitutional diffusion in semiconductors." *Z. angew Math. Phys.* **43**, 505–525.

Kline, M. (1961). "A note on the expansion coefficient of geometrical optics." *Comm. Pure Appl. Math.* **XIV**, 473–479.

Knessl, C. (1991). "The WKB approximation to the G/M/m queue." *SIAM J. Appl. Math.* **51**, 1119–1133.

Knessl, C. and Keller, J.B. (1991a). "Asymptotic properties of eigenvalues of integral equations." *SIAM J. Appl. Math.* **51**, 214–232.

Knessl, C. and Keller, J.B. (1991b). "Stirling number asymptotics from recursion equations using the ray method." *Studies Appl. Math.* **84**, 43–56.

Kuzmak, G.E. (1959). "Asymptotic solutions of nonlinear second order differential equations with variable coefficients." *J. Appl. Math. Mechs. (PMM)* **23**, 730–744.

Lagerstrom, P.A. (1988). *Matched Asymptotic Expansions: Ideas and Techniques.* Springer-Verlag, New York.

Landau, E. (1909). *Handbuch der Lehre von der Verteilung der Primzahlen.* Teubner, Berlin.

Lange, C.G. (1983). "On spurious solutions of singular perturbation problems." *Studies in Appl. Math.* **68**, 227–257.

Lange, C.G. and Miura, R.M. (1991). "Singular perturbation analysis of boundary value problems for differential-difference equations. IV. A nonlinear example with layer behavior." *Studies Appl. Math.* **84**, 231–273.

Langer, R.E. (1931). "On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order." *Trans. Math. Soc.* **33**, 23–64.

Langer, R.E. (1935). "On the asymptotic solutions of ordinary differential equations, with reference to the Stokes' phenomenon about a singular point." *Trans. Math. Soc.* **37**, 397–416.

Lardner, R.W. (1975). "The formation of shock waves in Krylov-Bogoliubov solutions of hyperbolic partial differential equations." *J. Sound and Vib.* **39**, 489–502.

Larson, D.A. (1978). "Transient bounds and time-asymptotic behavior of solutions to nonlinear equations of Fisher type." *SIAM J. Appl. Math.* **34**, 93–103.

Lee, K.K. (1992). *Lectures on Dynamical Systems, Structural Stability and Their Applications.* World Scientific, Singapore.

Leipholz, H. (1987). *Stability Theory*, 2nd ed. John Wiley, Chichester.

Leissa, A.W. and Saad, A.M. (1994). "Large amplitude vibrations of strings." *J. Appl. Mech.* **61**, 296–301.

Leontovich, M. and Fock, V. (1946). *J. Phys. USSR* **10**, 13–24.

Levinson, N. (1950). "The first boundary value problem for $\varepsilon\Delta u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$ for small ε." *Annals Math* **51**, 428–445.

Levy, B.R. and Keller, J.B. (1959). "Diffraction by a smooth object." *Comm Pure Appl. Math.* **XII**, 159–209.

Lewis, J., Slack, J.M.W., and Wolpert, L. (1977). "Thresholds in development." *J. Theor. Biol.* **65**, 579–590.

Lewis, R.M. (1966). "Asymptotic theory of wave-propagation." *Arch. Rational Mech. Anal.* **20**, 191–250.

Lie, G.C. and Yuan, J.-M. (1986). "Bistable and chaotic behavior in a damped driven Morse oscillator: A classical approach." *J. Chem. Phys.* **84**, 5486–5493.

Lighthill, M.J. (1949). "A technique for rendering approximate solutions to physical problems uniformly valid." *Philos. Mag.* **XL**, 1179–1201.

Lighthill, M.J. (1956). "Viscosity effects in sound waves of finite amplitude." in *Surveys in Mechanics* (G.K. Batchelor and R.M. Davies, eds.). Cambridge Univ. Press, Cambridge, pp. 250–351.

Lighthill, M.J. (1965). "Group velocity." *J. Inst. Math. Applic.* **1**, 1–28.

Lin, C.-C. and Segel, L.A. (1989). *Mathematics Applied to Deterministic Problems in the Natural Sciences*, SIAM Classics in Applied Mathematics, SIAM Publications, Philadelphia, PA.

Lindstedt, A. (1882). "Ueber die integration einer für die störungstheorie wichtigen differentialgleichung." *Astron. Nachr.* **103**, 211–220.

Lorentz, J. (1982). "Nonlinear boundary value problems with turning points and properties of difference schemes." in *Theory and Applications of Singular Perturbations* (W. Eckhaus and E.M. de Jager, eds.). Springer-Verlag, Berlin, pp. 150–169.

Ludwig, D. (1966). "Uniform asymptotic expansions at a caustic." *Comm. Pure Appl. Math.* **20**, 215–250.

Luke, J.C. (1966). "A perturbation method for nonlinear dispersive wave problems." *Proc. R. Soc. London* **A292**, 403–412.

MacGillivray, A.D. (1972). "Analytic description of the condensation phenomenon near the limit of infinite dilution based on the Poisson–Boltzmann equation." *J. Chem. Phys.* **56**, 83–85.

MacGillivray, A.D. (1990). "Justification of matching with the transition expansion of van der Pol's equation." *SIAM J. Math. Anal.* **21**, 221–240.

Markowich, P.A., Ringhofer, C.A., and Schmeiser, C. (1990). *Semiconductor Equations.* Springer-Verlag, Wien.

Marsden, J. and McCracken, M. (1976). *Hopf Bifurcation and Its Applications.* Springer-Verlag, New York.

McDonald, B.E. and Kuperman, W.A. (1985). "Time domain solution of the parabolic equation including nonlinearity." *Comp. Math.* **11**, 843–851.

McHugh, J.A.M. (1971). "An historical survey of ordinary linear differential equations with a large parameter and turning points." *Arch. History Exact Sci.* **7**, 277–324.

Mickens, R.E. (1990). *Difference Equations: Theory and Applications*, 2nd ed. Van Nostrand Reinhold, New York.

Minorsky, N. (1947). *Introduction to Non-linear Mechanics: Topological Methods, Analytical Methods, Non-linear Resonance, Relaxation Oscillations.* J.W. Edwards, Ann Arbor, MI.

Mitani, T. (1984). "Stable solution of nonlinear flame shape equation." *Combustion Sci .Tech.* **36**, 235–247.

Mortell, M.P. and Varley, E. (1970). "Finite amplitude waves in bounded media: nonlinear free vibrations of an elastic panel." *Proc. R. Soc. London* **A318**, 169–196.

Moser, T.J., Nolet, G., and Snieder, R. (1992). "Ray bending revisited." *Bull. Seism. Soc. Am.* **82**, 259–288.

Murdock, J.A. (1991). *Perturbations: Theory and Methods.* Wiley-Interscience, New York.

Murray, J.D. (1984). *Asymptotic Analysis.* Springer-Verlag, New York.

Murray, J.D. (1989). *Mathematical Biology.* Springer-Verlag, Berlin.

Murray, J.D. and Tayler, A.B. (1970). "An asymptotic solution of a class of nonlinear wave equations: a model for the human torso under impulsive stress." *SIAM J. Appl. Math.* **18**, 792–809.

Myerscough, C.J. (1973). "A simple model of the growth of wind-induced oscillations in overhead lines." *J. Sound and Vib.* **28**, 699–713.

Nayfeh, A.H. (1973). *Perturbation Methods.* Wiley, New York.

Ness, D.J. (1967). "Small oscillations of a stabilized, inverted pendulum." *Am. J. Phys.* **35**, 964–967.

Nobili, A.M. and Will, C.M. (1986). "The real value of Mercury's perihelion advance." *Nature* **320**, 39–41.

O'Malley, R.E. (1976). "Phase-plane solutions to some singular perturbation problems." *J. Math. Anal. and Appl.* **54**, 449–466.

O'Malley, R.E. (1991). *Singular Perturbation Methods for Ordinary Differential Equations.* Applied Mathematical Sciences Vol. 89. Springer-Verlag, New York.

Odeh, F. and Keller, J.B. (1964). "Partial differential equations with periodic coefficients and Bloch waves in crystals." *J. Math. Phys.* **5**, 1499–1504.

Olver, F.W.J. (1974). *Introduction to Asymptotics and Special Functions.* Academic Press, New York.

Papaloizou, J.C.B. and Pringle, J.E. (1987). "The dynamical stability of differentially rotating discs — III." *Mon. Not. R. Astron. Soc.* **225**, 267–283.

Parang, M. and Jischke, M.C. (1975). "Adiabatic ignition of homogeneous systems." *AIAA J.* **13**, 405–408.

Park, D. (1990). *Classical Dynamics and Its Quantum Analogues.* 2nd ed. Springer-Verlag, Berlin.

Patula, W.T. (1979). "Growth and oscillation properties of second order linear difference equations." *SIAM J. Math. Anal.* **10**, 55–61.

Penrose, R. (1955). "A generalized inverse for matrices." *Proc. Cambridge Philos. Soc.* **51**, 406–413.

Pereyra, V., Lee, W.H.K., and Keller, H.B. (1980). "Solving two-point seismic-ray tracing problems in a heterogeneous medium. Part 1. A general adaptive finite difference method." *Bull. Seism. Soc. Am.* **70**, 79–99.

Peters, C.S. and Mangel, M. (1990). "New methods for the problem of collective ruin." *SIAM J. Appl. Math.* **50**, 1442–1456.

Plaschko, P. (1990). "Matched asymptotic approximations to solutions of a class of singular parabolic differential equations." *Z. angew Math. Mech.* **70**, 63–64.

Poincaré. H. (1886). "Sur les intégrales irrégulières des équations linéaires." *Acta Math.* **8**, 295–344.

Poincaré. H. (1892). *Les Méthodes Nouvelles de la Méchanique Céleste II* (reprinted by Dover Publ. New York, 1957).

Poisson, S.D. (1823). "Sur la distribution de la chaleur dans les corps solides." *J. l'École R. Polytech.* **12**, 249–403.

Popp, K. and Stelter, P. (1990). "Stick-slip vibrations and chaos." *Philos. Trans. R. Soc. London* **A332**, 89–105.

Prandtl, L. (1905). "Über Flüssigkeitsbewegung bei sehr kleiner Reibung." in *Verhandlungen des III. Internationalen Mathematiker-Kongresses, Heidelberg 1904.* Teubner, Leipzig, 484–491 (translation in NACA Technical Memo 452, 1928).

Rabbitt, R.D. and Holmes, M.H. (1988). "Three-dimensional acoustic waves in the ear canal and their interaction with the tympanic membrane." *J. Acoust. Soc. Am.* **83**, 1064–1080.

Rand, R.H. and Armbruster, D. (1987). *Perturbation Methods, Bifurcation Theory and Computer Algebra.* Springer-Verlag, New York.

Rayleigh, J.W.S. (1883). "On maintained vibrations." *Philos. Mag.* **XV**, 229–235.

Reiss, E.L. (1971). "On multivariable asymptotic expansions." *SIAM Rev.* **13**, 189–196.

Reiss, E.L. (1977). "Imperfect bifurcation." in *Applications of Bifurcation Theory* (P.H. Rabinowitz, ed.). Academic Press, New York, pp. 37–71.

Reiss, E.L. and Matkowsky, B.J. (1971). "Nonlinear dynamic buckling of a compressed elastic column." *Q.J. Appl. Math.* **29**, 245–260.

Rudin, W. (1976). *Principles of Mathematical Analysis.* 3rd ed. McGraw-Hill, New York.

Sanchez-Palencia, E. (1980). *Non-Homogeneous Media and Vibration Theory.* Lecture Notes in Physics Vol. 127, Springer-Verlag, New York.

Sanders, J.A. (1983). "The driven Josephson equation: an exercise in asymptotics." in *Asymptotic Analysis II — Surveys and New Trends* (F. Verhulst, ed.). Springer-Verlag, New York, pp. 288–318.

Sangani, A.S. (1986). "An application of an homogenization method to a model of diffusion in glassy polymers." *J. Polymer Sci.* **24**, 563–575.

Santosa, F. and Symes, W.W. (1991). "A dispersive effective medium for wave propagation in periodic composites." *SIAM J. Appl. Math.* **51**, 984–1005.

Scheidegger, A.E. (1974). *The Physics of Flow Through Porous Media.* 3rd ed. Univ. of Toronto Press, Toronto.

Schlissel, A. (1977). "The development of asymptotic solutions of linear ordinary differential equations, 1817–1920." *Arch. History Exact Sci.* **6**, 307–378.

Schlissel, A. (1977). "The initial development of the WKB solutions of linear second order ordinary differential equations and their use in the connection problem." *Historia Math.* **4**, 183–204.

Schlögl, F. (1972). "Chemical reaction models for non-equilibrium phase transitions." *Z. Phys.* **253**, 147.

Schnakenberg, J. (1979). "Simple chemical reaction systems with limit cycle behaviour." *J. Theor. Biol.* **81**, 389–400.

Schoombie, S.W. (1992). "A discrete multiple scales analysis of a discrete version of the Korteweg-deVries equation." *J. Comp. Phys.* **101**, 55–70.

Secrest, D., Cashion, K., and Hirschfelder, J.O. (1962). "Power-series solutions for energy eigenfunctions." *J. Chem. Phys.* **37**, 830–835.

Shepherd, J.J. (1978). "On the asymptotic solution of the Reynolds equation." *SIAM J. Appl. Math.* **34**, 774–791.

Shih, S.-D. and Kellogg, R.B. (1987). "Asymptotic analysis of a singular perturbation problem." *SIAM J. Math. Anal.* **18**, 1467–1511.

Siegmann, W.L., Kriegsmann, G.A., and Lee, D. (1985). "A wide-angle three-dimensional parabolic wave equation." *J. Acoust. Soc. Am.* **78**, 659–664.

Slater, L.J. (1960). *Confluent Hypergeometric Functions.* Cambridge Univ. Press, Cambridge.

Smith, D.R. (1985). *Singular-Perturbation Theory: An Introduction with Applications.* Cambridge Univ. Press, Cambridge.

Spindel, R.C. (1991). "Heard Island global warming test." *Environ. Sci. and Technol.* **25**, 210–212.

Steele, C.R. (1976). "Application of the WKB method in solid mechanics." in *Mechanics Today* (S. Nemat-Nasser, ed.). Pergamon Press, New York, pp. 243–295.

Stenholm, S. (1992). "Quantum motion in a Paul trap." *J. Modern Optics* **39**, 279–290.

Stickler, D. C., Tavantzis, J., and Ammicht, E. (1984). "Calculation of the second term in the geometrical acoustics expansion." *J. Acoust. Soc. Am.* **75**, 1071–1076.

Stoker, J.J. (1950). *Nonlinear Vibrations in Mechanical and Electrical Systems.* Wiley-Interscience, New York.

Stokes, G.G. (1843). "On some cases of fluid motion." *Trans. Camb. Philos. Soc.* **8**, 105–165.

Strogatz, S.H. (1994). *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering.* Addison-Wesley Publishing Co., Reading, MA.

Struble, R.A. and Harris, T.C. (1964). "Motion of a relativistic damped oscillator." *J. Math. Phys.* **5**, 138–141.

Stuart, J.T. (1971). "Stability problems in fluids," in *Mathematical Problems in the Geophysical Sciences* (W. H. Reid, ed.). American Mathematical Society, Providence, RI, pp. 139–155.

Tappert, F.D. (1977). "The parabolic approximation method." in *Wave Propagation and Underwater Acoustics* (J.B. Keller and J.S. Papadakis, eds.). Springer-Verlag, New York, pp. 224–287.

Torng, H.C. (1960). "Second-order non-linear difference equations containing small parameters." *J. Franklin Inst.* **269**, 97–104.

Tsai, S.W. (1988). *Composites Design.* 4th ed., Think Composites, Dayton, OH.

Um, J. and Thurber, C. (1987). "A fast algorithm for two-point seismic ray tracing." *Bull. Seism. Soc. Am.* **77**, 972–986.

Ungarish, M. (1993). *Hydrodynamics of Suspensions: Fundamentals of Centrifugal and Gravity Separation.* Springer-Verlag, Berlin.

van der Burgh, A. (1968). "On the asymptotic solutions of the differential equations of the elastic pendulum." *J. Mécanique* **7**, 507–520.

van der Pol, B. (1926). "On relaxation oscillations." *Philos. Mag.* **2**, 978–992.

Van Dyke, M. (1964). *Perturbation Methods in Fluid Mechanics.* Academic Press, New York. Annotated edition from Parabolic Press, Stanford, CA (1975).

Van Dyke, M. (1994). "Nineteenth-century roots of the boundary-layer idea." *SIAM Rev.* **36**, 415–424.

van Harten, A. (1976). "On an elliptic singular perturbation problem." in *Ordinary and Partial Differential Equations* (W.N. Everitt and B.D. Sleeman, eds.). Springer-Verlag, Berlin, pp. 485–495.

Vatta, F. (1979). "On the stick-slip phenomenon." *Mech. Res. Comm.* **6**, 203–208.

Vulanovic, R. and Farrell, P.A. (1993). "Continuous and numerical analysis of a multiple boundary turning point problem." *SIAM J. Numer. Anal.* **30**, 1400–1418.

Wang, M. and Kassoy, D.R. (1990). "Dynamic response of an inert gas to slow piston acceleration." *J. Acoust. Soc. Am.* **87**, 1466–1471.

Ward, M.J. (1992). "Eliminating indeterminancy in singularly perturbed boundary value problems with translation invariant potentials." *Studies in Appl. Math.* **87**, 95–134.

Webster, A.G. (1919). "Acoustical impedance, and the theory of horns and the phonograph." *Proc. Nat. Acad. Sci.* **5**, 275–282.

Westervelt, P.J. (1963). "Parametric acoustic array." *J. Acoust. Soc. Am.* **35**, 535–537.

Whitham, G.B. (1974). *Linear and Nonlinear Waves.* Wiley, New York.

Wilkinson, J.H. (1964). *Rounding Errors in Algebraic Processes.* Prentice-Hall, Englewood Cliffs, NJ.

Willner, B. and Rubenfeld, L.A. (1976). "Uniform asymptotic solutions for a linear ordinary differential equation with one m-th order turning point: analytic theory." *Comm. Pure Appl. Math.* **XXIX**, 343–367.

Wilmott, P. (1985). "A note on the WKB method for difference equations." *IMA J. Appl. Math.* **34**, 295–302.

Wollkind, D.J. (1977). "Singular perturbation techniques: a comparison of the method of matched asymptotic expansions with that of multiple scales." *SIAM Rev.* **19**, 502–516.

Index

$<<$ symbol, 6
$\cong$ symbol, 6
$\sim$ symbol, 7, 9

Acherberg-O'Malley resonance, 81
Acoustic thermometer, 207
Adiabatic explosion temperature, 76
Air flow, supersonic, 49
Airy functions, 175, 297–298
  expansions of, 298–299
  graph of, 298
Airy's equation, 174, 297
Arithmetic mean, 232
Asymptotic approximations
  of algebraic equations, 18–22
  defined, 7
  of differential equations, 26–31
  differentiating, 14
  ideas underlying, 13, 47
  of integrals, 305–307
  of transcendental equations, 21–22
  uniform, 37–42
Asymptotic expansions, 1
  coefficients of, 10
  concept introduced, 13–14
  defined, 9
  differentiating, 14–15
  divergent, 12
  finding, 9–11
  integrating, 15
  manipulating, 14–15
  matched, see Matched asymptotic expansions
  nonuniform, 39
  of eigenvalues, 25–26, 67, 122, 164, 170, 180, 219–220
  well ordered, 19
Asymptotic sequences, 9
Asymptotic series, accuracy versus convergence of, 12–14
Asymptotic stability, 257, 258
Asymptotically equal functions, 11
Asymptotically stable orbit, 284
Asymptotically stable solutions, 258
Attraction, basin of, 250
Attractive solutions, 258
Averaging in method of homogenization, 243–244
Averaging procedure, properties of, 230

Bachmann–Landau symbols, 4
Balancing terms, 19
Basin of attraction, 250
Basis functions, 9
Beam equation, 171, 189
  damped, 137
  nonlinear, 131, 137, 278
Bessel's equation, 171
Bifurcating solutions, schematic of structure of, 254
Bifurcation
  degenerate, 284–285
  global, 277
  imperfect, 263
  of periodic solutions, 281–285
  pitchfork, 250
  steady-state, 249
Bifurcation and stability
  introduction to, 249–291
  introductory example, 249–251
Bifurcation parameter, 250
Bifurcation point, 250, 251
  analysis of, 251–255
Big oh, 4
  two meanings of, 20
Binomial expansion, 3
Biological switching mechanism, 270
Bistability, 126, 264
Bloch expansion, 240
Boundary-layer coordinate, 50, 63, 87
Boundary layers, 50–51
  matching and, 52, 64–65, 89
  multiple, 62–65
  multiple scales and, 132–133
  parabolic, 90–91
Boundary-value problems
  FORTRAN program for solving, 309–312
  nonlinear, numerical solution of, 309–312
Boussinesq equation, 149
Breathers, 158, 281
Brusselator, 293
Buckled states, 278
Buckling, 271–272
Buckling loads, 260
Burgers' equation, 91
  solutions of, 95

Caustics, 200
Cells, 234–235
  averages of functions over, 236–237
  in plane, 235
Characteristic coordinates, introducing, 198
Coefficients, slowly varying, 117–120
Cole-Hopf transformation, 95
Composite expansion, 54–55, 56, 61, 63–64, 72, 80, 89, 101
Computing, symbolic, 43–45
Confluent hypergeometric functions, 81, 299–302
Connection formulas, 176
Convergent asymptotic expansions, 12
Corner-layer expansion, 266–267
Corner layers, 77–81
Corner region, 78
Cutoff frequency, 196

Darcy's law, 245
Debye-Hückel theory, 30
Degenerate bifurcation, 284–285
Difference equations
  boundary layers and, 98–102
  multiple scales and, 153–155
  WKB and, 212–218
Differential-difference equations, 75
Differential equations
  approximate solutions of, 1
  asymptotic approximations of, 26–31
  homogenized, 228
  ordinary, systems of, 287–291
Diffusion, nonlinear, 144–148
Diffusivity tensor, 248
Dirichlet problem, inhomogeneous, 234
Discrete WKB method, 212–218
Disproportionate length scales, 225
Dissipation function, 136, 186
Distinguished limit, 51

Divergent asymptotic expansions, 12
Divergent series, summability of, 14
Dual-mode sorption model, 246
Duffing's equation, 35, 249
  forced, 123, 128

Effective coefficients, 228
Eigenvalues of matrix, 23
Eikonal equation, 163
Elliptic equations, 84–91
Energy, total, 187
Energy equation, 136, 186, 190
Energy methods, 186–188
  wave propagation and, 185–188
Euler buckling load, 263
Exchange of stability, 257
Expansion, composite, 54–55
Exponential functions, transcendentally small, 11
Exponentially small functions, 9
Expression swell, intermediate, 44
Extension theorem, Kaplun's, 53

Fast dynamics, 264–265
Fast length scale, 225
Fisher-Wright-Haldane model, 158
Fisher's equation, 145, 278
Flow, porous, 241–245
Fokker-Planck equation, 188
Forced motion near resonance, 123–127
Fredholm Alternative Theorem, 191, 273
Fredholm integral equation, 33
Functions
  asymptotically equal, 11, 39
  averages of, over cells, 236–237
  transcendentally small, 5
  well-ordered, 9

Gauge functions, 9
Geometric mean, 232
Ginzburg-Landau equation, 278
Global asymptotic stability, 257
Global bifurcation, 277

Hamiltonian function, 107, 136
Harmonic mean, 228, 230
Harmonic resonance, 127
Helmholtz equation, 198
Hessian, 287
Hidden variables, 239
Higher-order turning points, 302–303
Homogenization, 223
  method of, 223–245
    averaging in, 243–244
    introductory example, 224–231
    multidimensional problem, 234–239
    with nonlinear equations, 231
    reduction using, 242–243
Homogenization procedure, 236–237
Homogenized coefficients, 228
  substructure and, 237
Homogenized differential equations, 228
Hopf bifurcation, 255, 284
Hurwitz polynomial, 291
Hypergeometric equation, 299
Hypergeometric functions, confluent, 81, 299–302

Ignition period, 76
Imperfect bifurcation, 263
Implicit Function Theorem, 251
Inhomogeneous Dirichlet problem, 234
Initial-layer expansion, 266
Inner-inner layer, 65
Inner solution, 53
Instability, 257
Integrals
  asymptotic approximations of, 305–307
Interior layer, 68–73
  matching and, 69–72
Interior-layer coordinate, 70, 92
Interior-layer expansion, 267–268
Intermediate variables, 53
Intrinsic phase average, 245

Kaplun's extension theorem, 53
Kaplun's hypothesis on domain of validity, 53
Kepler's equation, 24–25
Klein-Gordon equation, nonlinear, 138, 142, 281
Kolmogorov-Petrovsky-Piskunov (KPP) equation, 145, 278
Korteweg-deVries (KdV) equation, 149
Kummer's function, 299–300
Kuramoto-Sivashinsky equation, 279

Lagrangians, averaged, 189
Landau symbols, 4
Langmuir-Hinshelwood model, 75
Laplace's approximation, 306
Layer analysis, matching and, 79–80
Least-squares problem, 26
Length scales, disproportionate, 225
l'Hospital's rule, 4
Liebnitz's rule, 197
Lindstedt's method, 111
Linear stability theory, 249
Linear wave propagation, 139–141
Linearized stability, 255–259
Linearized stability analysis, 274–275
Liouville's equation, 29
Little oh, 4
Logistic equation with delay, 286
Lyapunov-Schmidt method, 252

Maclaurin series, 3
Macroscale, 225
Maintained vibrations, 269–270
Maple commands for projectile problem, 45
Matched asymptotic expansions, 39, 47–102
  introductory example, 48–57
Matching
  boundary layers and, 52, 64–65
  corner layer analysis and, 79–80
  importance of, 56
  interior layer and, 69–72
  overlap domain and, 53
  transition region and, 175–176
Materials from multiple constituents, 223
Mathieu's equation, 121, 261
Matrix
  eigenvalues of, 23
  pseudo-inverse of, 26
Matrix perturbation problems, 25–26
Maxwell's fisheye lens, 206
Mercury's perihelion, 128
Michaelis-Menten reaction scheme, 59
Microscale, 225
Microvoids, 246
Mixture theory, 245
Mixtures, rule of, 233
Morse oscillator, forced, 130
Multiple boundary layers, 62–65
Multiple constituents, materials from, 223
Multiple-scale expansion, 108–110
Multiple scales, 105–155
  boundary layers and, 132–133
  characteristic coordinates and, 140, 143
  difference equations and, 153–155
  first-order systems and, 116–117
  history of, 105
  introductory example, 106–113
  minimum error, principle of, 112
  three time scales, 110
  for traveling waves, 139–141
Multiple solutions of nonlinear equations, 70

Negative damping, 258
No-slip condition, 241
Nonlinear boundary-value problems, numerical solution of, 309–312
Nonlinear diffusion, 144–148
Nonlinear geometric optics method, 167
Nonlinear wave equation, 142–144

Nonlinear waves, 142–148
Nonlinear WKB method, 167
Nonuniform asymptotic expansions, 39
Numerical solution of nonlinear boundary-value problems, 309–312

Order symbols, 4–6
Ordinary differential equations, systems of, 287–291
Outer expansion, 55
Outer scale, 53
Outer solution, 49–50, 62, 68, 78, 85, 91, 265

Parabolic boundary layers, 90–91
Parabolic equations (PE), 91–94
  standard two-dimensional, 209–211
Parabolic wave approximations, 207–211
Paraxial approximation, 209
Paul trap, 115, 142
PE, see Parabolic equations
Pendulum, 114, 121, 262–263
  elastic, 294
  forced, 129, 131
  inverted, 261–262
Periodic solutions, bifurcation of, 281–285
Periodic substructure, 231, 234–239
Permeability tensor, 245
Perturbation problems
  matrix, 25–26
  regular, 26
  singular, 48
Phase average, 245
Phase fronts, 198
Pitchfork bifurcation, 250
Poincaré-Andronov-Hopf bifurcation theorem, 291
Poisson-Boltzmann equation, 29
Porous flow, 25, 241–245
Power series functions, generalization of, 9
Primary resonance, 127
Projectile problem, 1–3, 27–28
  Maple commands for, 45
Pseudo-inverse of matrix, 26

Quadratic equations, 18–22
Quantization condition, 181
Quantum jump equation, 116

Rankin-Hugoniot condition, 92
Ray, 199
Ray methods, WKB method and, 197–202
Ray-tracing problem, 201
Rayleigh oscillator, equation for, 122, 270
Rayleigh-Schrödinger series, 26
Reduced wave equation, 198
Reduction using method of homogenization, 242–243
Relaxation dynamics, 264–268
Relaxation oscillation, 269
Resonance, forced motion near, 123–127
Reynolds equation, 66
Reynolds transport theorem, 244
Riccati equation, 267
Rotating wave approximation, 116
Routh-Hurwitz criterion, 291

Saddle-node bifurcation, 255
Scale functions, 9
Scales, multiple, see Multiple scales
Scaling variables, 2, 34, 36, 211
Schlögl model, 280
Schrödinger equation, 34, 180, 182
Secondary bifurcation, 255
Secular term, 106, 107
  avoiding, 110
Self-sustained oscillation, 281
Sel'kov model, 292
Semiconvergent series, 13
Shock solution, 72
Singular perturbation problems, 48
sinh-Poisson equation, 29
Slender annulus approximation, 36
Slender body approximations

scaling and, 36
wave propagation and, 190–196
Slow scale, 225
Slowly varying coefficients, 117–120
Small disturbance theory, 36–37, 98
Solvability condition, 192, 273
Sommerfeld radiation condition, 208
Spikes, 72, 73
Stability
asymptotic, 257
of branches bifurcating from zero solution, 276–277
exchange of, 257
global asymptotic, 257
linearized, 255–259
of zero solution, 275–276
Stability analysis, linearized, 274–275
Stability theory, linear, 249
Stable solutions, 258
Stationary phase approximation, 306–307
Steady-state bifurcation, 249
Steady-state solutions, 272–274
Stick-slip problem, 130–131
Stirling numbers, 221
Stirling series, 171
Stokes problem, forced, 242
Strained coordinates, method of, 111
Stretched variables, 78
Stretching transformation, 50
String equation
damped, 134, 137, 139
energy, 136, 279
nonlinear, 137, 151
Subcritical bifurcation, 255
Subharmonic oscillation, 127
Substructure
homogenized coefficients and, 237
periodic, 231, 234–239
Supercritical bifurcation, 255
Superharmonic oscillation, 127
Supersonic air flow, 49
Switchbacking, 56
Switching mechanism, biological, 270
SWKB method, 184
Symbolic computing, 43–45

Tangency points, 87
Tangent bifurcation, 255
Taylor polynomials, 10
Taylor's theorem, 3
Terms, balancing, 19
Thermometer, acoustic, 207
Time scales
fast, 117
possible, 111–112
slow, 109
three, 110–111
two, 108
Transcendental equations, asymptotic approximations of, 21–22
Transcendentally small, 5, 11
Transformation, stretching, 50
Transition-layer coordinate, 193
Transition-layer equations, solution and properties of, 297–303
Transition layers
nonmonotone, 72, 73
solutions in, 174–175
Transition region
between inner and outer layers, 52–53
matching and, 175–176
solution in, 193–194
Transport equation, 163
Transversality condition, 284
Traveling waves, multiple scales for, 139–141
Triode circuits, 282
Tunneling, phenomenon of, 182
Turning point interval, 214
Turning points, 77, 164, 173–180, 215–218
higher-order, 302–303
Two-timing method, 111

Ultraharmonic oscillation, 127
Unbuckled state, 278
Uniform asymptotic approximations, 37–42

Uniformly valid asymptotic approximations, 38

Van der Pol equation, 129, 269, 281
Variables
  hidden, 239
  intermediate, 53
  scalar, 40
  scaling, 2
  stretched, 78
  vector, 40
Vibrations, maintained, 269–270
Viscoelasticity, nonlinear, 280–281
Volterra delay equation, 286

Watson's lemma, 306
Wave approximations
  parabolic, 207–211
  rotating, 116
Wave fronts, 198
Wave propagation
  energy methods and, 185–188
  slender body approximations and, 190–196
Wave solutions, 186–187
Waves
  nonlinear, 142–148
  traveling, multiple scales for, 139–141
Webster horn equation, 188
Well-ordered functions, 9
Wentzel-Kramers-Brillouin method, see WKB method
Westervelt equation, 153
Wigner function, 142
WKB method, 120, 161–218
  application of, 185
  characteristic of, 167
  discrete, 212–218
  history of, 162
  introductory example, 162–167
  nonlinear, 167
  physical interpretation of, 188
  quantization condition, 181
  ray methods and, 197–202
WKBJ method, 161