Module 3

INTRODUCTORY QUANTUM MECHANICS

(As per Revised Curriculum SVU R-2023)

Physical constants:
1. Avogadro's number $N_0 = 6.023 \times 10^{23}$/mol
2. Elementary charge $q = 1.6 \times 10^{-19}$ C
3. Planck's constant $h = 6.63 \times 10^{-34}$ J-s
4. Speed of light in vacuum $c = 3 \times 10^8$ m/s
5. Electron mass/rest mass $= 9.1 \times 10^{-31}$ kg

Classwork:
1. Calculate de' Broglie wavelengths of an electron orbiting in hydrogen atom at a speed of 10^6 m/s.
2. Calculate de' Broglie wavelength of a cricket ball of mass 0.2 kg thrown at a speed of 100 m/s. Compare your results of this and earlier example and comment on the same.
3. What is de' Broglie wavelength of a neutron having energy 1 MeV. Neutron mass $= 1.67 \times 10^{-27}$ kg?
4. By how much potential difference a proton has to be accelerated in order to have the same de' Broglie wavelength as above? Proton charge $q = 1.6 \times 10^{-19}$ C, mass $= 1.67 \times 10^{-27}$ kg.
5. Find kinetic energy of an electron whose de' Broglie wavelength is the same as 1 keV X-ray photon.
6. An electron and a proton have the same kinetic energies. Compare their de' Broglie wavelengths. Given $m_p = 1800\, m_e$ and $q_p = q_e = q$
7. An electron and a muon (μ) are accelerated through the same potential difference. How do their de' Broglie wavelengths compare? Given $m_\mu = 207\, m_e$ and $q_\mu = q_e = q$
8. Calculate uncertainty in the determination of momentum of an electron confined to a quantum well of size 1 nm. What is the percentage uncertainty in the momentum if its average speed is 10^6 m/s?
9. Determine percentage uncertainty in the measurement of momentum of a marble of mass 10 gm confined to a box of dimensions 50 cm if it is moving with a speed of 20 cm/s. What can you say about the measurement?
10. Find minimum energy possessed by an electron in an atom.
11. The lifetime of an excited state of nucleus is usually 1 ps. Estimate uncertainty in energy of a γ-ray emitted by a nucleus.
12. What is the uncertainty in the determination of position of a particle if its momentum is measured to be 2×10^{-24} kg-m/s with an uncertainty of 0.05%
13. Calculate the width of a spectral line if the transition giving rise to this spectral line has occurred during 0.01 µs seconds.
14. Calculate the percentage uncertainty in the measurement of momentum of a neutron having energy 20 MeV confined to a region of width equal to 3 nuclei.
15. The wave function of a particle is given by $\varphi(x) = \sqrt{\dfrac{\pi}{2}}\, x;\ \ 0 \le x \le 1$. Find the probability that the particle can be found between x = 0.45 to x = 0.55.
16. Find the probability that a particle confined to an infinite square well of size "a" can be found within 0.3a to 0.7a in its ground state. Its wave function is given by:

$$\psi(x) = \left(\sqrt{2/a}\right) \sin\left(\frac{n\pi x}{a}\right); \quad -\frac{a}{2} < x < \frac{a}{2}$$

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QM - Solved Examples

① $m = 9.1 \times 10^{-31}$ kg, $v = 10^6$ m/s

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-34} \times 10^6}$$

$$= 7.286 \times 10^{-10} \text{ m}$$

② $m = 0.2$ kg, $v = 100$ m/s

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}}{0.2 \times 100}$$

$$= 3.315 \times 10^{-35} \text{ m}$$

comment : The smallest dimensions which can be experimentally measured are around 10^{-15} m. The wavelength of electron is well-within this measurement limit whereas the wavelength of cricket ball is way beyond the measurement limit as it is infinitesimally small. It is possible to verify wave·nature of electron experimentally while it is impossible to verify wave nature of cricket· ball. Hence we can conclude that the wave nature is significant at the microscopic level but it fades away or becomes insignificant at the macroscopic level.

③ Energy = K.E $K = 1.14 eV = 1.6 \times 10^{-13}$ J

$m = 1.67 \times 10^{-27}$ kg

$$\lambda = \frac{h}{\sqrt{2mk}} = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 1.67 \times 10^{-27} \times 1.6 \times 10^{-13}}}$$

$$= 2.87 \times 10^{-14} \, m$$

(4) Here, $\lambda = 2.87 \times 10^{-14} \, m$

$m = 1.67 \times 10^{-27} \, kg, \quad g = 1.6 \times 10^{-19} \, C$

$$\lambda = \frac{h}{\sqrt{2mgV}} \quad \Rightarrow \quad V = \frac{h^2}{2mg\lambda^2}$$

$$\therefore V = \frac{(6.63 \times 10^{-34})^2}{2 \times 1.67 \times 10^{-27} \times 1.6 \times 10^{-19} \times (2.87 \times 10^{-14})^2}$$

$$= 0.998 \times 10^6 \, volt$$

(5) $E_{photon} = 1 \, keV = 1.6 \times 10^{-16} \, J$

For photons, $E_{photon} = \frac{hc}{\lambda_{photon}}$

$$\therefore \lambda_{photon} = \frac{hc}{E_{photon}}$$

given $\lambda_{electron} = \lambda_{photon}$

But $\lambda_{electron} = \frac{h}{\sqrt{2mk}}$ $\qquad$ k : KE of electron

$$\therefore \frac{h}{\sqrt{2mk}} = \frac{hc}{E_{photon}}$$

$$\therefore K = \frac{E^2_{photon}}{2mc^2}$$

$$= \frac{(1.6 \times 10^{-16})^2}{2 \times 9.1 \times 10^{-31} \times (3 \times 10^8)^2}$$

$$= 1.56 \times 10^{-19} \, J$$

or $0.977 \, eV$

⑥ $K_{electron} = K_{proton}$ (given)

$m_p = 1800 \, m_e$ (given)

$\therefore$ ~~$K_{electron}$~~ $\lambda_{electron} = \frac{h}{\sqrt{2m_e K_e}}$

$\lambda_{proton} = \frac{h}{\sqrt{2m_p K_p}}$

$\therefore \lambda_{electron} : \lambda_{proton} = \frac{1}{\sqrt{m_e}} : \frac{1}{\sqrt{m_p}}$

$$= \sqrt{\frac{m_p}{m_e}} = \sqrt{\frac{1800}{1}} = 42.43 : 1$$

⑦ $V_e = V_\mu$ (given) $=$ say V

$m_\mu = 207 \, m_e$ and $q_\mu = q_e$ (given) $=$ say q

$$\lambda_e = \frac{h}{\sqrt{2 q_e m_e V_e}} \quad ; \quad \lambda_\mu = \frac{h}{\sqrt{2 q_\mu m_\mu V_\mu}}$$

$$= \frac{h}{\sqrt{2q \, m_e \, V}} \quad ; \quad = \frac{h}{\sqrt{2q \, m_\mu \, V}}$$

$\therefore \lambda_e : \lambda_u = \dfrac{1}{\sqrt{m_e}} : \dfrac{1}{\sqrt{m_u}}$

$\qquad = \sqrt{\dfrac{m_u}{m_e}} = \sqrt{\dfrac{207}{1}} = 14.39 : 1$

(8) here, take $\Delta x = 1\,nm = 10^{-9}\,m$

$v = 10^6\,m/s$

$\Delta x\, \Delta p = \dfrac{h}{4\pi}$

$\therefore \Delta p = \dfrac{h}{4\pi \Delta x} = \dfrac{6.63 \times 10^{-34}}{4 \times 3.14 \times 10^{-9}}$

$\qquad = 5.28 \times 10^{-26}\ kg\,\dfrac{m}{s}$

$p = mv = 9.1 \times 10^{-31} \times 10^6 = 9.1 \times 10^{-25}\ kg\,\dfrac{m}{s}$

$\%\ uncertainty = \dfrac{\Delta p}{p} \times 100\,\%$

$\qquad = \dfrac{5.28 \times 10^{-26}}{9.1 \times 10^{-25}} \times 100\,\%$

$\qquad = 5.8\,\%$

(9) $m = 10\,gm = 10^{-2}\,kg$, $v = 20\,cm/s = 0.2\,m/s$

Here, take $\Delta x = 50\,cm = 0.5\,m$

$\Delta x\, \Delta p = \dfrac{h}{4\pi}$ $\qquad \left(\dfrac{h}{4\pi} = 5.28 \times 10^{-35}\ J\text{-}s \right)$

$\qquad\qquad\qquad\qquad$ for all calculations

$$\therefore \quad \Delta p = \frac{5.28 \times 10^{-35}}{\Delta x} = \frac{5.28 \times 10^{-35}}{0.5}$$

$$= 1.056 \times 10^{-34} \, kg \, \frac{m}{s}$$

$$p = mv = 10^{-2} \times 0.2 = 2 \times 10^{-3} \, kg \, \frac{m}{s}$$

$$\% \text{ uncertainty} = \frac{\Delta p}{p} \times 100 \%$$

$$= \frac{1.056 \times 10^{-34}}{2 \times 10^{-3}} \times 100$$

$$= 5.28 \times 10^{-30} \, \%$$

Comment : The uncertainty in the measurement of momentum of electron is significant while the uncertainty in the measurement of momentum of marble is so small that it can practically be taken as zero and the measurement can be called to be accurate. Thus, the uncertainty principle becomes essential at the microscopic level where ⊘ it can be insignificant at the macroscopic level.

(10) If an electron happens to be in an atom, the exact position of the electron is uncertain by an amount equal to the size of their atom. Average atomic sizes are 1Å ie. 10^{-10} m taking this value as uncertainty in the exact determination of position of an electron,

we take $\Delta x \simeq 10^{-10}$ m

$$\Delta x \Delta p = \frac{h}{4\pi} \Rightarrow \Delta p \simeq 5.28 \times 10^{-25} \text{ kg} \frac{m}{s}$$

Now let the momentum of that electron be at least equal to the uncertainty in its own measurement i.e. let $p \simeq \Delta p$

using $K = \frac{p^2}{2m}$, the energy of an electron in an atom is

$$\frac{(5.28 \times 10^{-25})^2}{2 \times 9.1 \times 10^{-31}} = 1.53 \times 10^{-19} \text{ J}$$

or about 0.96 eV.
Indeed this is the minimum value any electron will possess in any atom. Experimently we find minimum energy of electron in H-atom to be around 13.6 eV, which confirms the lower limit set by uncertainty principle

(1) here the lifetime of an excited state is treated as uncertainty in the exact determination of time interval i.e. $\Delta t \simeq 1 \text{ ps} = 10^{-13} \text{ s}$

Use: $\Delta E \Delta t = \frac{h}{4\pi} \Rightarrow$

$$\Delta E = 5.28 \times 10^{-23} \text{ J}$$

(12) $p = 2 \times$
i.e. Δ

$\Delta x \Delta$

(3) Here tak
ΔE
∴ h

This is le
less than-
line or
central

(14) $K = 20 \text{ m}$
width of
avg size
3 nuclei

Neutron i
uncertain
$\Delta x \simeq$

⑫ $p = 2 \times 10^{-24} \ kg \ \frac{m}{s}$, $\Delta p = 0.05 \% \ of \ p$

$$ie \quad \Delta p = 5 \times 10^{-4} \ p = 5 \times 10^{-4} \times 2 \times 10^{-24}$$

$$= 10^{-27} \ kg \ \frac{m}{s}$$

$$\Delta x \Delta p = \frac{h}{4\pi} \Rightarrow \quad \Delta x = 5.28 \times 10^{-7} \ m$$

⑬ Here take $\Delta t = 0.01 \ \mu s = 10^{-8} \ sec$

$$\Delta E \ \Delta r = \frac{h}{4\pi}$$

$$\therefore \quad h \Delta \nu \cdot \Delta t = \frac{h}{4\pi} \Rightarrow \Delta \nu = \frac{1}{4\pi \Delta t}$$

$$\therefore \Delta \nu = \frac{1}{4 \times 3.14 \times 10^{-8}} = 7.96 \ MHz$$

~~This is xx xxxx~~
Yes~~, this~~ This gives the width of the spectral
line or spread in wavelengths w.r.t the
central radiation.

⑭ $K = 20 \ Mev = 3.2 \times 10^{-12} \ J$
width of region = 3 nuclei wide
avg size of nuclei $\simeq 10^{-15} \ m$
3 nuclei wide $\simeq 3 \times 10^{-15} \ m$

Neutron is confined to this ~~xy~~ region. Take
uncertainty in exact position of neutron
$$\Delta x \simeq 3 \times 10^{-15} \ m$$

$$\therefore \Delta P = \frac{5.28 \times 10^{-35}}{3 \times 10^{-15}} \simeq 1.76 \times 10^{-20} \text{ kg} \frac{m}{s}$$

$$K = \frac{1}{2} mv^2 = \frac{p^2}{2m}$$

$$P = \sqrt{2mK} \implies P = 2.4 \times 10^{-21} \text{ kg m/sec}$$

$$\% \text{ uncertainty} = \frac{\Delta P}{P} \times 100\% = \frac{1.76 \times 10^{-20}}{} \times 100$$

The momentum is almost accurately measured as neutron is confined to a much wider region (Δx is more) than its usual value

(15) $\phi(x) = \sqrt{\frac{\pi}{2}} x \; ; \; 0 \leq x \leq 1$

probability $P = \int_{0.45}^{0.55} [\phi(x)]^2 dx$

$$= \int_{0.45}^{0.55} \left(\sqrt{\frac{\pi}{2}} x \right)^2 dx$$

$$= \frac{\pi}{2} \left[\frac{x^3}{3} \right]_{0.45}^{0.55}$$

$$= \frac{\pi}{6} \left(0.55^3 - 0.45^3 \right)$$

$$= 0.0394 \simeq 3.94\%.$$

(16) $\phi(x) =$

prob. 1

$$= \int_{0.3a}^{0.7a}$$

for

$\therefore P =$

use :

$$P = \frac{2}{a} \int_0^{a}$$

$$= \frac{1}{a} [x$$

$$= \frac{1}{a} [x]_{0.3}^{0.7}$$

$$= 0.4$$

(16) $\phi(x) = \sqrt{\dfrac{2}{a}} \sin\left(\dfrac{n\pi x}{a}\right)$; $-\dfrac{a}{2} < x < +\dfrac{a}{2}$

prob. $P = \displaystyle\int_{0.3a}^{0.7a} \left[\phi(x)\right]^2 dx$

$= \displaystyle\int_{0.3a}^{0.7a} \left[\sqrt{\dfrac{2}{a}} \sin\left(\dfrac{n\pi x}{a}\right)\right]^2 dx$

for ground state, $n = 1$.

$\therefore P = \dfrac{2}{a} \displaystyle\int_{0.3a}^{0.7a} \sin^2\left(\dfrac{\pi x}{a}\right) dx$

use : $\sin^2 A = \dfrac{1 - \cos 2A}{2}$

$P = \dfrac{2}{a} \displaystyle\int_{0.3a}^{0.7a} \cdot \dfrac{1}{2} dx - \dfrac{2}{a} \cdot \displaystyle\int_{0.3a}^{0.7a} \cdot \dfrac{1}{2} \cdot \cos\dfrac{2\pi x}{a} dx$

$= \dfrac{1}{a} \left[x\right]_{0.3a}^{0.7a} - \dfrac{1}{a} \left[\sin\left(\dfrac{2\pi x}{a}\right)\right]_{0.3a}^{0.7a} \cdot \dfrac{1}{\left(\dfrac{2\pi}{a}\right)}$

~~The 2nd term vanishes or it is multiply~~ ~~~~ ~~and sin(2πx) =0~~

$= \dfrac{1}{a}\left[0.7a - 0.3a\right] - \dfrac{1}{2\pi} \left[\sin(0.7 \times 2\pi) - \sin(0.3 \times 2\pi)\right]$

$P = 0.4 - \dfrac{1}{2\pi} \left[\sin(1.4\pi) - \sin(0.6\pi)\right]$

↑
this term vanishesh

$P = 0.4 = 40\%$