Module 4
Homogeneous Functions

Euler's Throrem and corollaries on Homogeneous Functions

❖ **Homogeneous Functions:**

$u = f(x, y, z)$ is called homogeneous function of degree n,

if $f(xt, yt, zt) = t^n f(x, y, z)$

❖ **Alternative Definition of Homogeneous Functions:**

Alternately, $u = f(x, y)$ is homogeneous if it can be expressed as

$$u = x^n f\left(\frac{y}{x}\right) \text{ (two variable)}$$

and $u = f(x, y, z)$is homogeneous if it can be expressed as

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) \text{ (three variable)}$$

Eg.

1. If $f(x, y) = \frac{x^3+y^3}{x+y}$ then

$$f(xt, yt) = \frac{(xt)^3 + (yt)^3}{xt + yt} = \frac{t^3}{t}\left(\frac{x^3 + y^3}{x + y}\right) = t^2\left(\frac{x^3 + y^3}{x + y}\right) = t^2 f(x, y)$$

$\therefore f(x, y)$ is homogeneous of degree 2.

2. $u = sin^{-1}\left(\frac{x}{y}\right) = f(x, y)$

$$f(xt, yt) = sin^{-1}\left(\frac{tx}{ty}\right) = t^0 . sin^{-1}\left(\frac{x}{y}\right) = t^0 f(x, y)$$

$\therefore u = f(x, y)$ is homogeneous of degree 0.

❖ Euler's Theorem:

- If $u = f(x, y)$ is homogeneous function of two variables of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

- If $u = f(x, y, z)$ is homogeneous function of three variables of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

❖ Corollary 1

- If $u = f(x, y)$ is homogeneous function of two variables of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

- If $u = f(x, y, z)$ is homogeneous function of three variables of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + 2yz\frac{\partial^2 u}{\partial y \partial z} + 2zx\frac{\partial^2 u}{\partial z \partial x} = n(n-1)u$$

❖ NOTE:

If we have certain functions, say $u = \sin^{-1}[\emptyset(x, y)]$ or $\log[\emptyset(x, y)]$

where u is not homogeneous but $\emptyset(x, y)$ is homogeneous function.

i.e. in above examples $f(u) = \sin u$ or e^u will be homogeneous function. Then for such type of functions also we have corollaries of Euler's theorem.

❖ Corollary 2

- If $f(u)$ is homogeneous function of two variables of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

- If $f(u)$ is homogeneous function of three variables of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$$

❖ Corollary 3

If $f(u)$ is homogeneous function of two variables of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)(g'(u) - 1)$$

where, $g(u) = n\frac{f(u)}{f'(u)}$

SOME SOLVED EXAMPLES

1. Verify Euler's theorem for $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

Solution:

To verify Euler's theorem we need to prove that, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

where, n is degree of homogeneous function u.

First we will find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

We have,

$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}$, $\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}$, $\frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$

Then, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\sqrt{x}}{2} + \frac{\sqrt{y}}{2} + \frac{\sqrt{z}}{2} = \frac{1}{2}u \ldots\ldots\ldots (i)$

Let us check that $f(x, y, z)$ is homogenous and hence find the degree of $f(x, y, z)$

Consider $f(xt, yt, zt) = \sqrt{xt} + \sqrt{yt} + \sqrt{zt} = \sqrt{t}f(x, y, z)$

Hence $(xt, yt, zt) = t^{\frac{1}{2}}f(x, y, z)$,

u is homogeneous function of deg ½ … … … (ii)

Then by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u.$

Hence Euler's theorem is verified.

2. If $u = sin^{-1}\left(\frac{x}{y}\right) + cos^{-1}\left(\frac{y}{z}\right) - log\left(\frac{z}{x}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution:

Consider $u = f(x, y, z) = sin^{-1}\left(\frac{x}{y}\right) + cos^{-1}\left(\frac{y}{z}\right) - log\left(\frac{z}{x}\right)$

Then, $f(xt, yt, zt) = sin^{-1}\left(\frac{xt}{yt}\right) + cos^{-1}\left(\frac{yt}{zt}\right) - log\left(\frac{zt}{xt}\right)$

$$= f(x, y, z) = t^0 f(x, y, z)$$

Hence u is homogeneous function of degree zero.

By, Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0$$

3. If $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Solution:

Consider $u = f(x,y) = \frac{\sqrt{x}+\sqrt{y}}{x+y}$

Then, $f(xt,yt) = \frac{\sqrt{xt}+\sqrt{yt}}{xt+yt} = \frac{\sqrt{t}(\sqrt{x}+\sqrt{y})}{t(x+y)} = t^{-\frac{1}{2}}f(x,y)$

Hence u is homogeneous function of degree $-\frac{1}{2}$.

By, Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -\frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right)$$

4. If $u = \frac{x^3y+y^3x}{3x}$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$

Solution:

Consider $u = f(x,y) = \frac{x^3y+y^3x}{3x}$

Then, $f(xt,yt) = \frac{(xt)^3yt+(yt)^3xt}{3xt} = \frac{t^4(x^3y+y^3x)}{t(3x)} = t^3f(x,y)$

Hence u is homogeneous function of degree 3.

Therefore By, Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 3.2u = 6u$$

5. If $u = \frac{x^2y^3z}{x^2+y^2+z^2} + sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right)$ then find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution:

Here u is not homogeneous.

Consider, $u = v + w$

Where, $v = \frac{x^2y^3z}{x^2+y^2+z^2} = f(x,y,z)$ and $w = sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right) = g(x,y,z)$

For v, consider $f(xt,yt,zt) = \frac{x^2t^2y^3t^3zt}{x^2t^2+y^2t^2+z^2t^2} = \frac{t^6}{t^2}\left(\frac{x^2y^3z}{x^2+y^2+z^2}\right) = t^4f(x,y,z)$

v is homogeneous function of degree 4.

By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nv = 4v \quad \dots\dots\dots\dots(i)$$

For w, consider $g(xt,yt,zt) = sin^{-1}\left(\frac{xyt^2+yzt^2}{y^2t^2+z^2t^2}\right) = t^0g(x,y,z)$

w is homogeneous function of degree zero

By Euler's theorem,

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=nw=0 \quad \dots\dots\dots\dots(ii)$$

Adding (i) and (ii), we get

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=4v+0$$

$$x\frac{\partial (v+w)}{\partial x}+y\frac{\partial (v+w)}{\partial y}+z\frac{\partial (v+w)}{\partial z}=4v$$

As $u=v+w$

Hence $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4\frac{x^2y^3z}{x^2+y^2+z^2}$

6. If $u=\frac{x^2+xy}{y\sqrt{x}}+\frac{1}{x^7}sin^{-1}\left(\frac{y^2-xy}{x^2-y^2}\right)$ then find value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} \quad at\ x=1,\ y=2$$

Solution:

Here u is not homogeneous, consider u = v + w

Where, $v=\frac{x^2+xy}{y\sqrt{x}}=f(x,y)\ and\ w=\frac{1}{x^7}sin^{-1}\left(\frac{y^2-xy}{x^2-y^2}\right)=g(x,y)$

For v, consider $f(xt,yt)=\frac{x^2t^2+xyt^2}{yt\sqrt{xt}}=\frac{t^2}{t^{\frac{3}{2}}}\left(\frac{x^2+xy}{y\sqrt{x}}\right)=t^{\frac{1}{2}}f(x,y)$

v is homogeneous function of degree $\frac{1}{2}$

By Euler's theorem and its corollary 1

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=\frac{1}{2}v \quad \dots\dots\dots\dots(i)$$

$$x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=n(n-1)v=\frac{1}{2}\left(\frac{1}{2}-1\right)v=-\frac{1}{4}v \quad \dots\dots\dots\dots(ii)$$

For w, consider $g(xt,yt)=\frac{1}{x^7t^7}sin^{-1}\left(\frac{y^2t^2-xyt^2}{x^2t^2-y^2t^2}\right)=t^{-7}\frac{1}{x^7}sin^{-1}\left(\frac{y^2-xy}{x^2-y^2}\right)=t^{-7}g(x,y)$

w is homogeneous function of degree -7

By Euler's theorem and its corollary 1

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=nw=-7w \quad \dots\dots\dots\dots(iii)$$

$$x^2\frac{\partial^2 w}{\partial x^2}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2}=n(n-1)w=-7(-7-1)w=56w \quad \dots\dots\dots\dots(iv)$$

Adding(ii) ,(iv) and (i) ,(iii) together we get

$$x^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 w}{\partial x^2}\right)+2xy\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial y}\right)+y^2\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\right)+x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)$$

$$=-\frac{1}{4}v\ +56w+\frac{1}{2}v\ -7w$$

As $u=v+w$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{4}v\ +\frac{1}{2}v\ +56w-7w=\frac{1}{4}v+49w$$

$at\ x=1\ and\ y=2,\ \ v=\frac{3}{2}\ and\ w=sin^{-1}\left(-\frac{2}{3}\right)$

Hence,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{8}+49sin^{-1}\left(-\frac{2}{3}\right)$$

7. If $u=\frac{x^2y^2z^2}{x^2+y^2+z^2}+\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$

Solution:

Here u is not homogeneous, consider $u=v+w$

Where, $v=\frac{x^2y^2z^2}{x^2+y^2+z^2}=g(x,y,z)\ and\ \ w=\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$

For v, consider $g(xt,yt,\ zt)=\frac{x^2t^2y^2t^2z^2t^2}{x^2t^2+y^2t^2+z^2t^2}=\frac{t^6}{t^2}g(x,y,z)=t^4g(x,y,\ z)$

v is homogeneous function of deg 4

By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=nv=4v\ \ \dots\dots\dots\dots(i)$$

clearly w is not homogeneous, Let $f(w)=\cos w=\ \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}=h(x,y,z)$

consider $h(xt,yt,\ zt)=\frac{xt+yt+zt}{\sqrt{xt}+\sqrt{yt}+\sqrt{zt}}=t^{\frac{1}{2}}h(x,y,\ z)$

hence $f(w)=cosw$ is homogeneous function of deg ½

By corollary 2 of Euler's theorem,

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=n\frac{f(w)}{f\prime(w)}=\frac{1}{2}\frac{\cos w}{(-\sin w)}=-\frac{1}{2}\cot w\ \ \dots\dots\dots\dots(ii)$$

Adding(i) and (ii),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4v-\frac{1}{2}\cot w$$

$$= 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) - \frac{1}{2}\cot\left(\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right)$$

8. If $u = cosec^{-1}\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$$

Solution:

Since u is not homogeneous, consider $f(u) = cosec(u) = \sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}} = h(x,y)$

$$h(xt,yt) = \sqrt{\frac{\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)t^{\frac{1}{2}}}{\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)t^{\frac{1}{3}}}} = t^{\frac{1}{12}}h(x,y)$$

Thus $f(u) = cosec(u)$is homogeneous function of degree $\frac{1}{12}$

Hence by corollary 3 of Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$$where,\ g(u) = n\frac{f(u)}{f'(u)} = \frac{1}{12}\frac{cosec\,u}{(-cosec\,u\cot u)} = -\frac{1}{12}\tan u$$

$$\text{And } [g'(u) - 1] = -\frac{1}{12}\sec^2 u - 1$$

$$= -\frac{1}{12}(1 + \tan^2 u) - 1 = -\frac{13}{12} - \frac{\tan^2 u}{12}$$

Hence, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

$$= -\frac{1}{12}\tan u\left[-\frac{1}{12}(13 + \tan^2 u)\right]$$

$$= \frac{1}{144}\tan u\,[13 + \tan^2 u]$$

9. If $u = \tan^{-1}(x^2 + 2y^2)$ then prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$

Solution:

Since u is not homogeneous,

Consider $f(u)=\tan(u)=x^2+2y^2=h((x,y)$

$h(xt,yt)=x^2t^2+2y^2t^2=t^2h(x,y)$

Thus $f(u)=\tan(u)$ is homogeneous function of degree 2

Then by corollary 2 of Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f\prime(u)}=2\frac{tanu}{\sec^2 u}=2\sin u\cos u=\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where, $g(u)=n\frac{f(u)}{f\prime(u)}=sin2u$

And $[g'(u)-1]=[cos2u(2)-1]$

Hence by corollary 3 of Euler's Theorem

$$\begin{aligned}x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}&=g(u)[g'(u)-1]\\&=\sin 2u\,[2cos2u-1]\\&=2\sin 2u\cos 2u-sin2u\\&=\sin 4u-sin2u\end{aligned}$$

10. If $x=e^u\tan v$, $y=e^u\sec v$ then find $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)$

Solution:

First we have to express u and v as functions of x and y

Consider $y^2-x^2=e^{2u}\sec^2 v-e^{2u}\tan^2 v=e^{2u}$

thus $u=\frac{1}{2}\log(y^2-x^2)$

And divide to get $\frac{x}{y}=\frac{e^u\tan v}{e^u\sec v}=\sin v$ thus $v=\sin^{-1}\left(\frac{x}{y}\right)$

Now we check for v,

v is homogeneous function of degree zero.

By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv=0$$

Then required product $\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)=0$

11. If $u=log\left(\frac{x^3+y^3}{x^2+y^2}\right)$ then find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

Solution:

u is not homogeneous

Consider $f(u)=e^u=\frac{x^3+y^3}{x^2+y^2}=h(x,y)$

$h(xt,yt)=\frac{t^3}{t^2}\left(\frac{x^3+y^3}{x^2+y^2}\right)=t\,h(x,y)$

Hence $f(u)=e^u$ is homogeneous function of degree 1

Hence by corollary 2 of Euler's theorem,

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=1\frac{e^u}{e^u}=1$

SOME PRACTICE PROBLEMS

1. If $u = \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^n$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.
2. If $u=\sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.
3. If $u=(x/y)^{(y/x)}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.
4. Verify Euler's Theorem
   i) $u=3x^2yz+5xy^2z+4xyz^2$
   ii) $u=\frac{x}{y}+\frac{y}{x}$
   iii) $u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$
   iv) $u=\frac{x^2+y^2}{x+y}$
   v) $u = x^2\tan^{-1}\left(\frac{y}{x}\right)$
   vi) $u=\frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$
   vii) $u=\frac{x(x^3-y^3)}{x^3+y^3}$
   viii) $u=\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$
5. If $u = \log(x^2+y^2)+\frac{x^2+y^2}{x+y}-2\log(x+y)$, Find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.
6. If $u=x^3\sin^{-1}\left(\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3x^3\sin^{-1}\left(\frac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right)$
7. If $u=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$, prove that $x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial x}+2u=0$.
8. If $u = \frac{f(\theta)}{r}$ where $x=r\cos\theta,\ y= r\sin\theta$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+u=0$.
9. If $u=f(v)$, where v is a homogenous function of x, y of degree n, then prove that,

   $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,v\,f'(v)$. Hence deduce that if $u=\log v$, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n$.
10. If $u=\frac{x^2y^2}{x^2+y^2}+\cos\left(\frac{xy}{x^2+y^2}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2y^2}{x^2+y^2}$.
11. If $u=\frac{x+y}{x^2+y^2}$ Find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.
12. $u = \sin^{-1}\left(\frac{x}{y}\right)+\cos^{-1}\left(\frac{y}{z}\right)+\tan^{-1}\left(\frac{z}{x}\right)$ Find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

13. If $u = xy\, f\left(\frac{y}{x}\right) + yz\, g\left(\frac{y}{z}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$

14. If $u = x^2 f\left(\frac{y}{x}\right) + y^2 g\left(\frac{x}{y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$

15. If $u = \frac{x^3 y^3}{x^3 + y^3}$ Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

16. If $u = \frac{x-y-z}{x^2 + y^2 + z^2}$ Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u = 0.$

17. If $u = \frac{x^2 y + y^2 x}{x^2 + y^2}$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$

18. If $u = x^3 e^{-(y/x)}$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u.$

19. If If $u = \log\left(\frac{\sqrt{x^2 + y^2}}{x + y}\right)$, then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.

20. If $y = x\cos u$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

21. If $u = x^2 \sin^{-1}\frac{y}{x} - y^2 \cos^{-1}\frac{x}{y}$, find $x^2\frac{\partial^2 u}{\partial x^2} + 2\,xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$

22. If $u = x \sin^{-1}\frac{y}{x} + y \tan^{-1}\frac{y}{x}$, find $x^2\frac{\partial^2 u}{\partial x^2} + 2\,xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

23. If $u = x^3 \sin^{-1}\frac{y}{x} + x^4 \tan^{-1}\frac{y}{x}$, find $x^2\frac{\partial^2 u}{\partial x^2} + 2\,xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

$at\ x = 1,\ y = 1.$

24. If $u = x^n f\left(\frac{y}{x}\right) + y^n f\left(\frac{x}{y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n^2 u.$

25. If $u = \frac{x^4 + y^4}{x^2 y^2}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$at\ x = 1, y = 2.$

26. If find value of $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ $at\ x = 1,\ y = 1.$

when $z = x^6 \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + xy}\right) + \frac{x^4 + y^4}{x^2 y^2}$.

27. If $u = \frac{(x^2 + y^2)^m}{2m(2m-1)} + xf\left(\frac{y}{x}\right) + yg\left(\frac{x}{y}\right)$ find value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

28. If $u = x^3\left(\tan^{-1}\frac{y}{x} + \frac{y}{x} e^{-y/x}\right) + y^{-3}\left(\sin^{-1}\frac{y}{x} + \frac{x}{y}\log\frac{x}{y}\right)$, prove that

$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u.$

29. If $u = \sin^{-1}\left(\frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}\tan u.$

30. If If $u = \frac{1}{3}\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$,find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

31. If $u = log(x^3 + y^3 - x^2y - xy^2)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3 \text{ and } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -3.$$

32. If $u = \tan^{-1}\left(\frac{x^3+y^3}{2x+3y}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$

33. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2sinucos3u = sin2u(1 - 4sin^2u) = sin4u - sin2u.$$

34. If $u = sin^{-1}\left(\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}\right)$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{400}(\tan^2 u - 19).$$

35. If $u = \sinh^{-1}\left(\frac{x^3+y^3}{x^2+y^2}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\tanh^3 u.$

36. If $u = e^{\frac{x}{y}} + log(x^3 + y^3 - x^2y + xy^2)$ *then find*

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

37. If $u = \log r$ & $r^2 = x^2 + y,^2$ prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 1 = 0.$

38. If $u = \sin^{-1}(x^2+y^2)^{\frac{1}{5}}$, show that $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = \frac{2}{25}\tan u(2\tan^2 u - 3).$

39. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that

(*i*) $x\,u_x + y\,u_y = \frac{\tan u}{2}$ (*ii*) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = -\frac{\sin u\cos 2u}{4\cos^3 u}.$

40. If $u = \log\frac{x+y}{\sqrt{x^2+y^2}} + sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{sin\,w\,\cos 2w}{4cos^3 w}$$, where $w = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

41. If $u = \sec^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, Find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$

42. If $u = log\left(\frac{x^3+y^3}{x^2+y^2}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1.$