Mass and Energy formula from Relativity

The moving mass is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

The energy is given by

$$E^2 = m_0{}^2\, c^4 + p^2 c^2$$

Case (i): For particles moving with speed less than speed of light

i.e. when $v << c$, then $\frac{v}{c}$ is negligible,

then $m = m_0$ and $E = mc^2$

Case (ii): For particles moving with speed of light i.e. photon

When $v = c$, then $\frac{v}{c} = 1$

then $m = \infty$ (not defined) and $E = pc$

de- Broglie hypothesis

Louis Victor Pierre Raymond, 7th Duc de- Broglie was a French aristocrat and physicist who made groundbreaking contributions to quantum theory. In his 1924 PhD thesis, he postulated the wave nature of electrons and suggested that all matter has wave properties

According to de Broglie, a moving material particle acts as a wave and sometimes wave is associated with moving material particle which controls the particle in every respect.

De Broglie Hypothesis

- We have seen that radiation has dual behavior:
  - Wave-like and particle-like.
- In 1924 de Broglie suggested that the same is true for matter.
- Specifically, he proposed that frequency and wavelength can be associated with an electron's energy and momentum.
  - Here, λ is the **de Broglie wavelength**.
- Recall for photon:

$$\boxed{\begin{aligned}\nu &= \frac{E}{h}\\ \lambda &= \frac{h}{p}\end{aligned}}$$

$$E = pc = h\nu = \frac{hc}{\lambda}$$

de Broglie relations hold for photon.

Consider a particle with kinetic energy **K**. Its momentum is found from

$$\boxed{K = \frac{p^2}{2m} \quad or \quad p = \sqrt{2mK}}$$

Its wavelength is then

$$\boxed{\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}}}$$

de Broglie Wavelength in terms of V

Consider an electron of mass *m* and charge q accelerated through a potential difference of *V* volts

KE of the electrons is equal to the energy of the electron accelerated at a potential of *V* volts

$$\frac{1}{2}mv^2 = qV \rightarrow m^2v^2 = 2mqV \rightarrow p^2 = 2mqV$$

$$p = mv = \sqrt{2mqV}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mqV}}$$

de Broglie wavelength of electron $q = e$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}}$$

$$\lambda = \frac{12.28}{\sqrt{V}} \times 10^{-10}\, m$$

1. Calculate the wavelength of an electron and a bullet of mass 10 gram moving at 100 m/s. Comment on the answers

For electron

$\lambda e = h/mv = 6.63 x 10^{-34} /(9.1 x 10^{-31}) (100)$

$\lambda e = 7.28 \times 10^{-6}$ m

measurable

This is immeasureably small

for bullet

$\lambda = h/mv = 6.6 x 10^{-34} / (0.01)(100)$

$= 6.63 \times 10^{-34}$ m

For ordinary “everyday objects,” we don’t experience that

MATTER CAN BEHAVE AS A WAVE

2. An electron has a rest mass of 9.11×10^{-31} kg. If the electron has a kinetic energy of 1.14×10^{-27} J, what is its de Broglie wavelength? Use a value of 6.63×10^{-34} J·s for the Planck constant. Give your answer in scientific notation to two decimal places.

Planck constant

$$\boxed{\lambda = \frac{h}{p}}$$

momentum

wavelength

$$p = mv$$

$$\boxed{\lambda = \frac{h}{\sqrt{2m\cdot K}}}$$

$$\frac{6.63\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2\times9.11\times10^{-31}\ \text{kg}\times1.14\times10^{-27}\ \text{J}}}$$

$$= 1.4547\ldots\times10^{-5}\ \text{m}$$

$$\boxed{1.45\times10^{-5}\ \text{m}}$$

3. Calculate de Broglie wavelength a neutron having energy of 150 eV.

$$\lambda = \frac{h}{\sqrt{2mK}}$$

K =150eV = 150 x 1.6 x 10^{-19} J = 240 x 10^{-19} J

$$\lambda = \frac{6.63x10^{-34}}{\sqrt{(2)(1.67x10^{-27})(240x10^{-19})}}$$

$$\lambda = 2.39x10^{-12}m = 0.0239\ \text{Å}$$

4. Calculate de Broglie wavelength a proton accelerated through a potential difference of 1KV.

$$\lambda = \frac{h}{\sqrt{2m(qV)}}$$

$$\lambda = \frac{6.6\times10^{-34}}{\sqrt{2\times1.6\times10^{-27}(1.6\times10^{-19}\times1\times10^{3})}}$$

$$\lambda = 0.9 \times 10^{-12}\ m$$

$$\lambda = 0.009\ \text{Å}$$

5. An alpha particle and proton both are accelerated thorough same potential. Which one will have higher de-Broglie's wavelength.

$$\lambda = \frac{h}{\sqrt{2m(qV)}}$$

where, h = Planck's constant and proton and α-particle both are accelerated through the same potential, so V is same.

$$\therefore \quad \lambda \propto \frac{1}{\sqrt{mq}} \quad \text{or} \quad \frac{\lambda_\alpha}{\lambda_p} = \sqrt{\frac{m_p q_p}{m_\alpha q_\alpha}}$$

As, charge on α-particle = 2 × charge on proton

$$q_\alpha = 2q_p \Rightarrow \frac{q_p}{q_\alpha} = \frac{1}{2}$$

Mass of α-particle = 4 × mass of proton

$$m_\alpha = 4 \times m_p \Rightarrow \frac{m_p}{m_\alpha} = \frac{1}{4}$$

$$\therefore \quad \frac{\lambda_\alpha}{\lambda_p} = \sqrt{\frac{1}{4} \cdot \frac{1}{2}} = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \lambda_p = 2\sqrt{2}\lambda_\alpha$$

i.e. proton has greater de-Broglie wavelength than that of α-particle.

But, what about small particles ?

6. Compute the wavelength of an electron ($m = 9.1 \times 10^{-31}$ kg moving at 1×10^{7} [m/s].

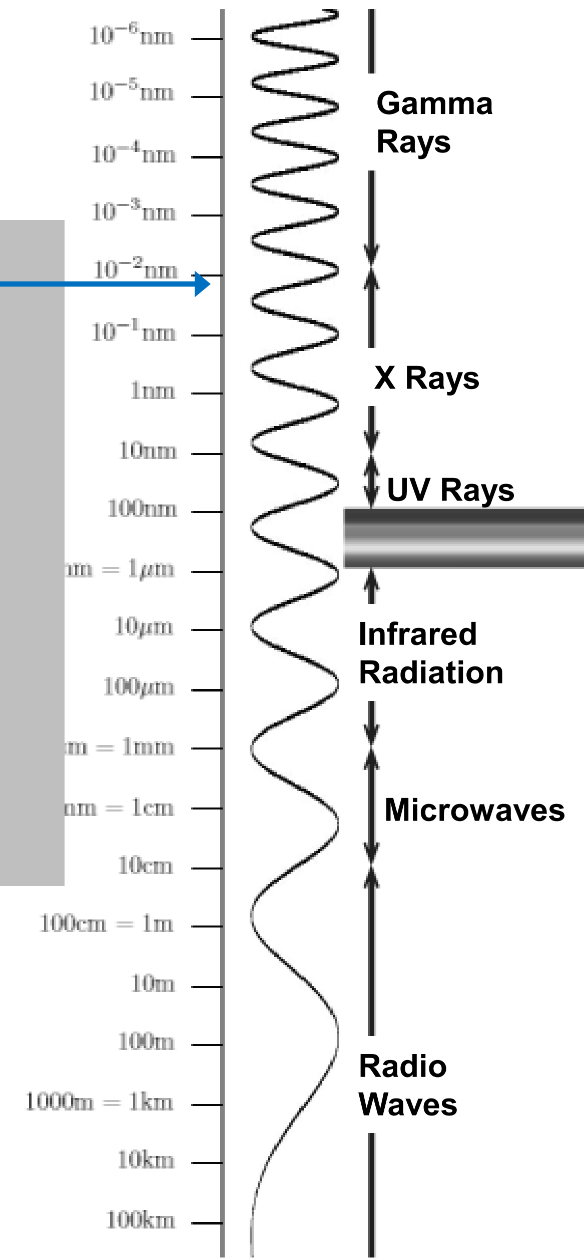
$\lambda = h/mv$

$= 6.6 \times 10^{-34}$ [J s]/(9.1×10^{-31} [kg])(1×10^{7} [m/s])

$=$ **7.3×10^{-11} [m].**

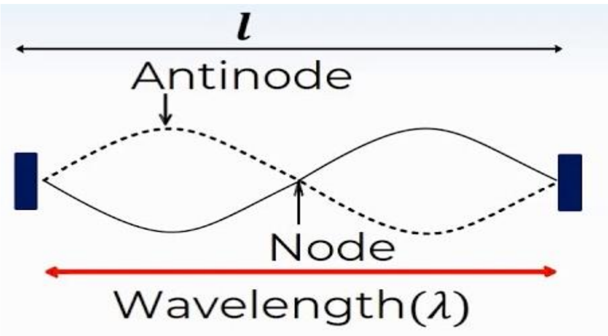
$= 0.73$Å

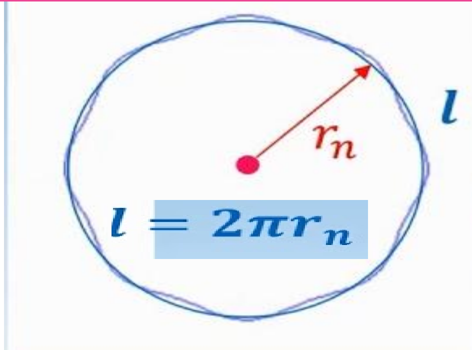
$= 0.073$ nm

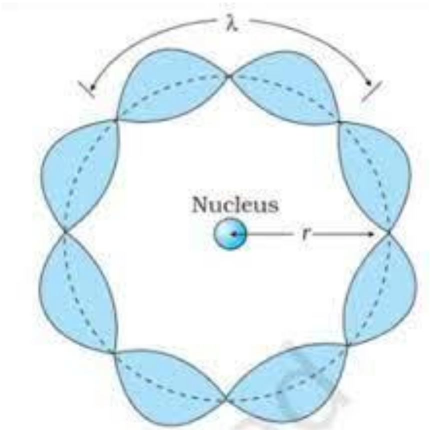


These electrons
have a wavelength in the region
of X-rays

DE BROGLIE'S EXPLANATION OF BOHR'S SECOND POSTULATE OF QUANTIZATION







For an electron moving in n^{th} circular orbit of radius r_n,

$2\pi r_n = n\lambda$

$n = 1, 2, 3 \ldots$

i.e., Circumference of orbit should be integral multiple of de-Broglie wavelength of electron moving in n^{th} orbit.

Also, we know, de-Broglie wavelength ; $\lambda = \frac{h}{mv}$.

Therefore $2\pi r_n = \frac{h}{mv_n}$

$\Rightarrow m v_n r_n = \frac{nh}{2\pi}$

Phase Velocity

The velocity of single matter wave is called phase velocity

phase velocity

$$v_p = \lambda \nu$$

for a massive particle

$$v_p = \frac{h}{mv}\frac{mc^2}{h} = c\frac{c}{v} > c$$

This is not possible as greater than speed of light

and hence it has no significance

phase velocity does not describe particle motion

Group Velocity

The velocity at which the overall envelope shape of the wave's amplitudes moves through space is called group velocity.

$$v_g = \frac{d\omega}{dk}$$

Group Velocity is equal to the velocity of a particle

$$v_g = v$$

Properties of Matter Waves

- Associated with moving particles.
- Wavelength inversely proportional to mass and velocity.
- Independent of nature of charge.
- Neither electromagnetic nor mechanical waves.
- Associated with probability of finding particle.
- Phase velocity is not significant for the matter waves.
- A velocity called group velocity is significant for the matter waves
- Quantity associated is called wave function Ψ

$\Psi(x, y, z, t) = A + i B$

- $|\Psi|^2$ is real and called probability of finding the particle.

Wave functions

Waves of <u>what</u> ?

"normal" waves

are a disturbance in space

carry energy from one place to another

often (but not always) will (approximately) obey the classical wave equation

"Matter" waves

disturbance is the wave function $\Psi(x, y, z, t)$

probability amplitude Ψ

probability density $p(x, y, z, t) = |\Psi|^2$

Conditions for acceptable wavefunction $\Psi(x, y, z, t)$

1. Ψ must be physically acceptable everywhere

2. Ψ must be **single-valued** everywhere

3. Ψ must **continuous** everywhere

4. Ψ **and its first derivative** must **finite** everywhere.

- For a single-particle system the wavefunction Ψ is given by $\Psi(x, y, z, t) = A + i B$
- Ψ represents the amplitude of the still vaguely (not clear; slightly) defined matter waves.
- Since wavefunctions is complex functions, the physical significance cannot be found from the function itself
- Rather, the physical significance is found in the product of the wavefunction and its complex conjugate, i.e. the absolute square of the wavefunction, which also is called the square of the modulus (also called absolute value).
- Born proposed that the square of the modulus of the wave function is proportional to the probability of finding the particle.

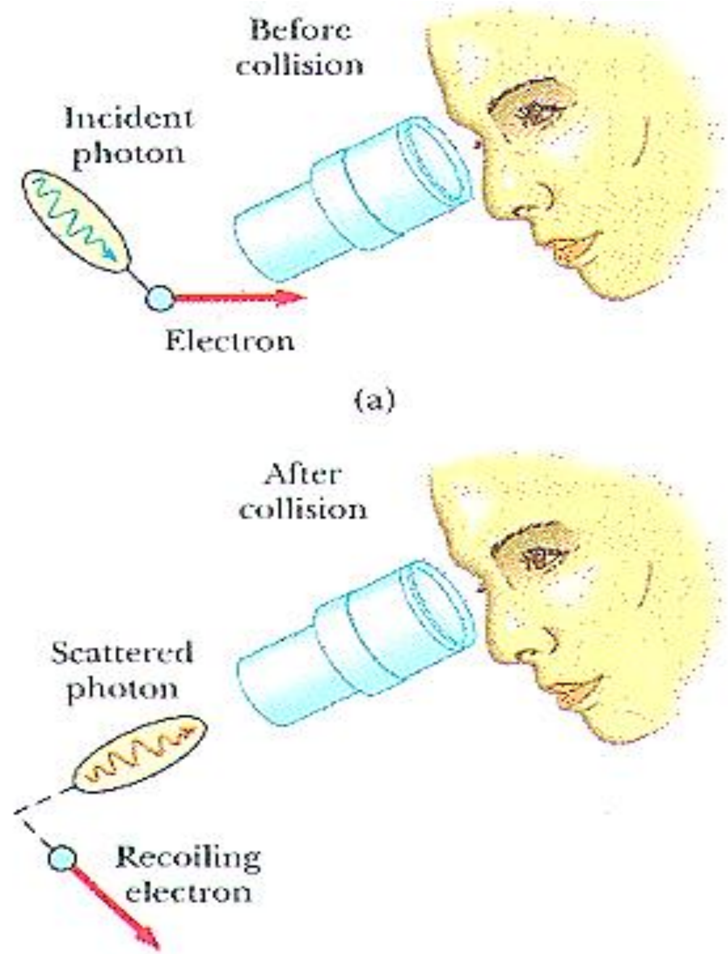
$$P(\vec{r}, t) = \Psi^*(\vec{r}, t)\Psi(\vec{r}, t)$$
$$= |\Psi(\vec{r}, t)|^2$$

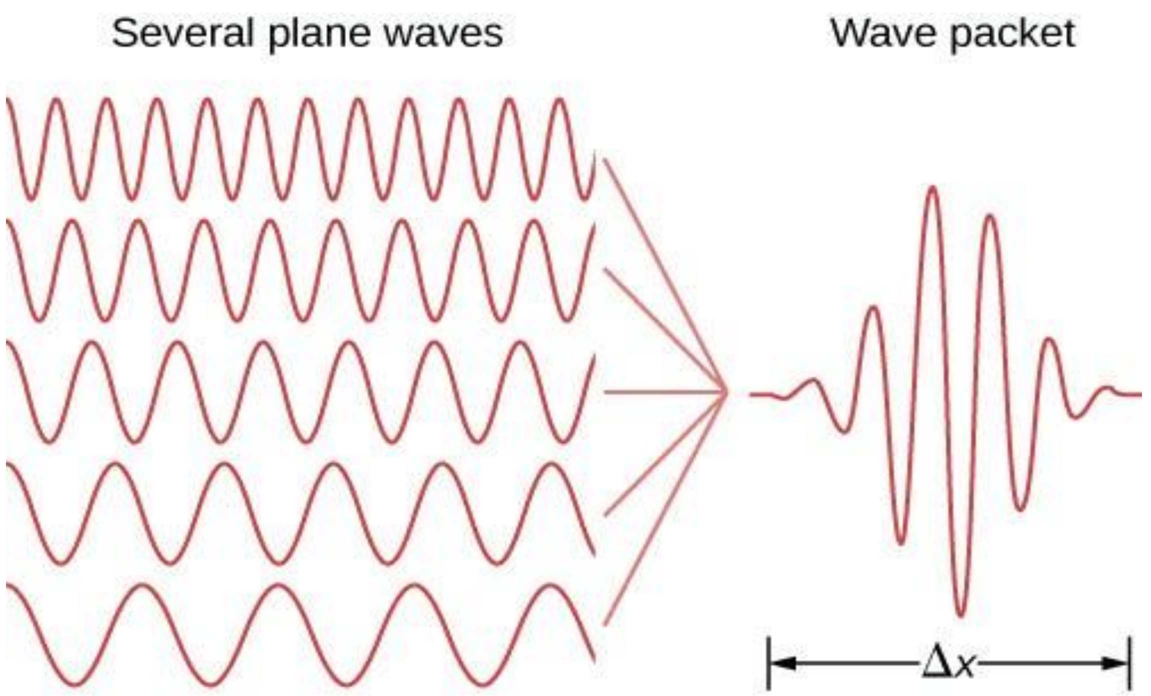
Normalization condition for a wave function ψ

The normalization of a wave function is given by the condition that the integral of the probability density, $|\psi(x,t)|^2$, over all space is equal to one. In one dimension, this is expressed as $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$, which means the probability of finding the particle somewhere is 100%.

- **The condition:** $\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$
- **In three dimensions:** $\int_{-\infty}^{\infty} |\psi(\vec{r},t)|^2 d^3\vec{r} = 1$

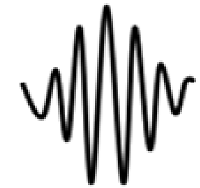
Heisenberg's Uncertainty Principle





Several plane waves
Wave packet
Δx

larger Δx



smaller Δx

for large Δx

λ **is also large**

$$\lambda = \frac{h}{p}$$

therefore, p is small

smaller Δp

for small Δx

λ **is also small**

$$\lambda = \frac{h}{p}$$

therefore, p is large

larger Δp

$$\Delta x \; \Delta p = \text{Constant}$$

uncertainty in position

uncertainty of momentum

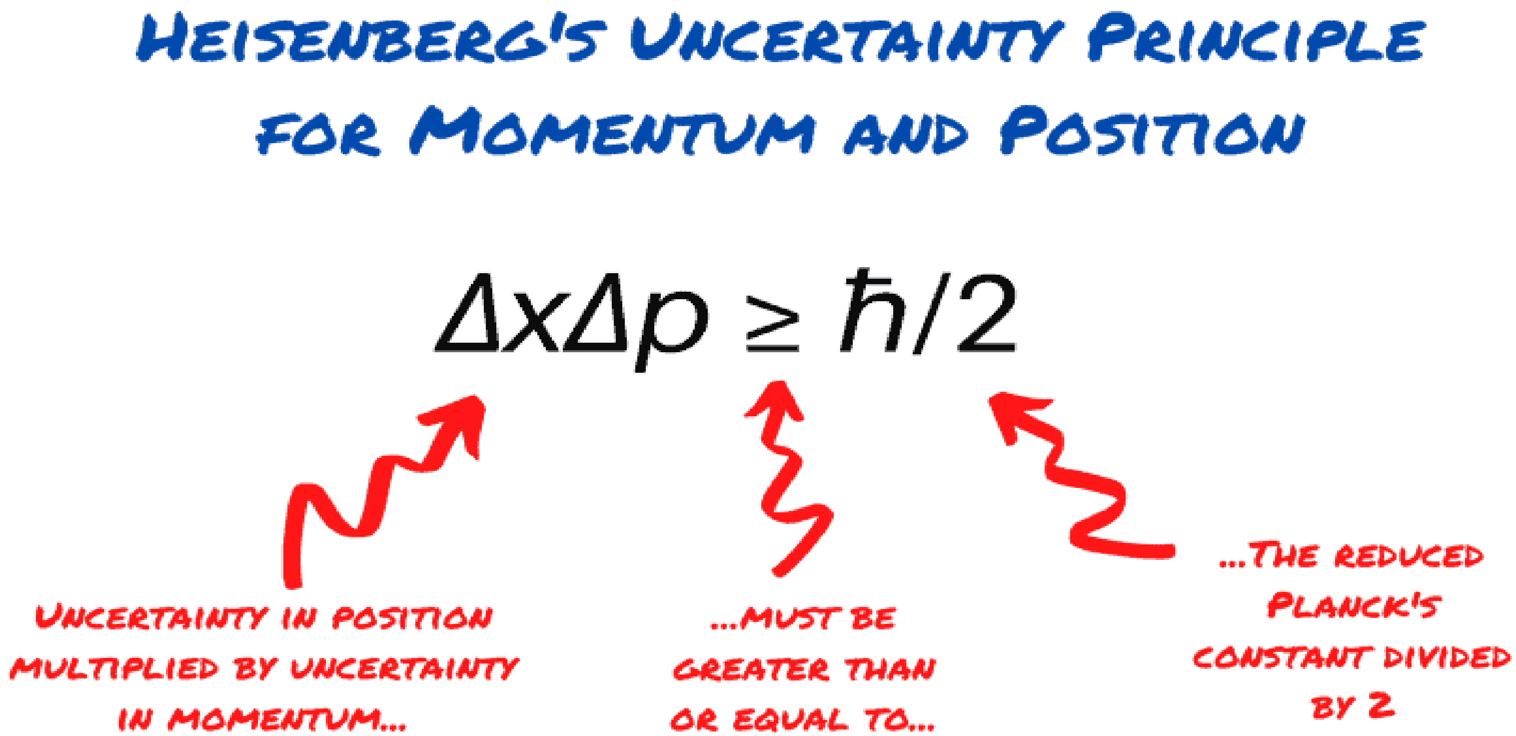
Planck's constant

$$\Delta x \; \Delta p \geq \frac{h}{4\pi}$$

uncertainty in position

uncertainty of momentum

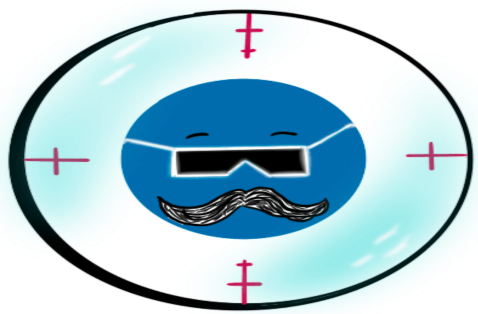
It is not possible to precisely specify a particle's position and momentum at the same time.



What Does Uncertainty Principle Tell Us?

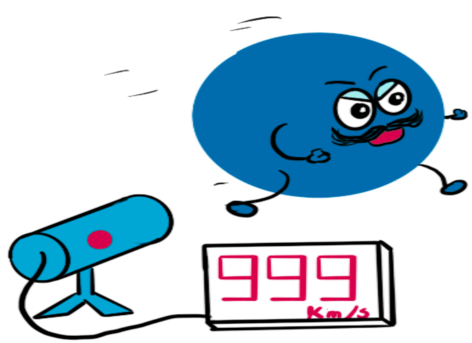
$$\Delta x \, \Delta P \geqslant \frac{h}{4\pi} \leftarrow \text{Fixed}$$

Position Uncertainty Momentum uncertainty



$\Delta x \downarrow \quad \Delta P \uparrow$

If we can pinpoint **the position** of an electron, we cannot know its **momentum**



$\Delta x \uparrow \quad \Delta P \downarrow$

If we know the **momentum** of an electron, we cannot locate its **position**

According to classical mechanics, the ***momentum*** and ***position*** of a moving particle can be determined simultaneously with any desired accuracy. But Heisenberg pointed out that when a particle is considered as a wave, it is not possible to locate the particle ***precisely*** and measure its momentum accurately at ***the same time***. This statement is known ***as Heisenberg's uncertainty principle*** and is a consequence ***of wave-particle duality***

Hence Heisenberg's Uncertainty principle states that it is impossible to measure precisely and simultaneously the position of a particle along a particular direction (say x) and its momentum in the same direction, p_x with unlimited accuracy.

Position-Momentum Uncertainty.

If Δx is uncertainty in position measurement along x-direction and Δp_x is uncertainty in momentum in same direction then the product of these two is of order ***of*** $\hbar = \frac{h}{2\Pi}$ i.e.,

$$\Delta x . \Delta p \approx \hbar / 2$$

where h is plank's constant.

Similarly,

$$\Delta y . \Delta p_y \approx \hbar / 2$$

$$\Delta z . \Delta p_z \approx \hbar / 2$$

From first equation, we find that ***smaller*** the value ***of*** Δx, ***more*** will be the value ***of*** Δp_x i.e. more precisely we locate the particle, less precisely we can determine the momentum and vice-versa.

Each of the relations given above, ***represent position-momentum uncertainty.***

Time-energy Uncertainty

Uncertainty principle is universal and it holds good for any pair of canonical conjugate variables (the ones whose product has dimensions of ***action***(joule-second) like

position and momentum,

energy and time,

angle and angular momentum etc.

Therefore. if ΔE is uncertainty in measurement of energy of a system and Δt is corresponding uncertainty in measurement of time, then

$$\Delta E \,.\Delta t \approx \hbar/2$$

This relation is known as energy-time uncertainty relation.

Angular momentum and angular displacement uncertainty.

Heisenberg's uncertainty principle puts limits on the errors ΔJ and $\Delta\theta$ in the simultaneous measurements of angular momentum J and angular displacement θ of a particle.

i.e. $$\Delta J \,.\, \Delta\theta \approx \hbar/2$$

Implications

- It is impossible to know *both* the position and momentum exactly, i.e., $\Delta x = 0$ and $\Delta p = 0$ is not possible
- These uncertainties are inherent in the physical world and have nothing to do with the skill of the observer

$$\hbar = 1.054 \times 10^{-34}\,[\mathrm{J} \cdot \mathrm{s}]$$

- Because *h* is so small, these uncertainties are not observable in normal everyday situations

Example of Baseball

- A pitcher throws a 0.1 kg baseball at speed of 40 m/s
- So momentum is 0.1 x 40 = 4 kg m/s
- Suppose the momentum is measured to an accuracy of 1 percent , i.e.,

  $\Delta p = 0.01\, p = 4 \times 10^{-2}$ kg m/s
- The uncertainty in position is then

$$\Delta x \geq \frac{h}{4\pi \Delta p} = 1.3 \times 10^{-33} \mathrm{m}$$

- **No wonder one does not observe the effects of the uncertainty principle in everyday life!**

Example of Electron

- Same situation, but baseball replaced by an electron which has mass 9.11 x 10^{-31} kg traveling at 40 m/s

- So momentum = 3.6 x 10^{-29} kg m/s

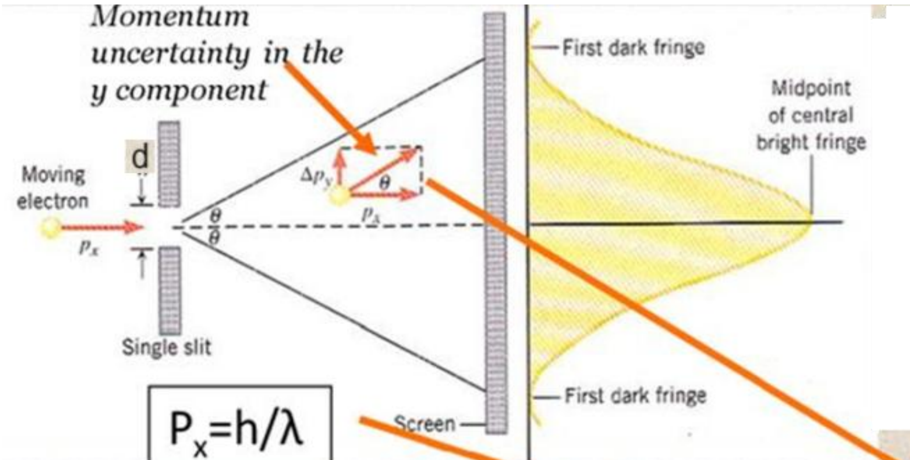
  and its uncertainty = 3.6 x 10^{-31} kg m/s

- The uncertainty in position is then

$$\Delta x \geq \frac{h}{4\pi \Delta p} = 1.4 \times 10^{-4}\mathrm{m}$$

Experimental illustration of Uncertainty Principle: Single slit diffraction.

To see more clearly into the nature of uncertainty, we consider electrons passing through a slit:



$P_x = h/\lambda$

*We apply the condition of minima from **single slit diffraction**,*

$$d \sin\theta = n\lambda$$

and postulate that λ is the de Broglie wavelength.

$$\sin\theta = \frac{\lambda}{d}$$

$\because$ for small θ $\quad \sin\theta \approx \tan\theta$

$$\tan\theta = \frac{\Delta p_y}{p_x} = \frac{\Delta p_y}{h/\lambda}$$

Since the electron can pass the slit through anywhere over the width d the uncertainty in the *y* position of the electron is $\Delta y = d$.

$$\therefore \quad \frac{\Delta p_y}{h/\lambda} = \frac{\lambda}{d} \Rightarrow \Delta p_y d = h$$

$$\Delta p \Delta y \approx h$$

2. Heisenberg's Gamma Ray Microscope

To find the order of limitations, in the measurements of position and momentum of an electron, Heisenberg suggested an experiment consisting of a high power gamma ray microscope. In this experiment, photons from ***source S*** collide with the electron in a beam.

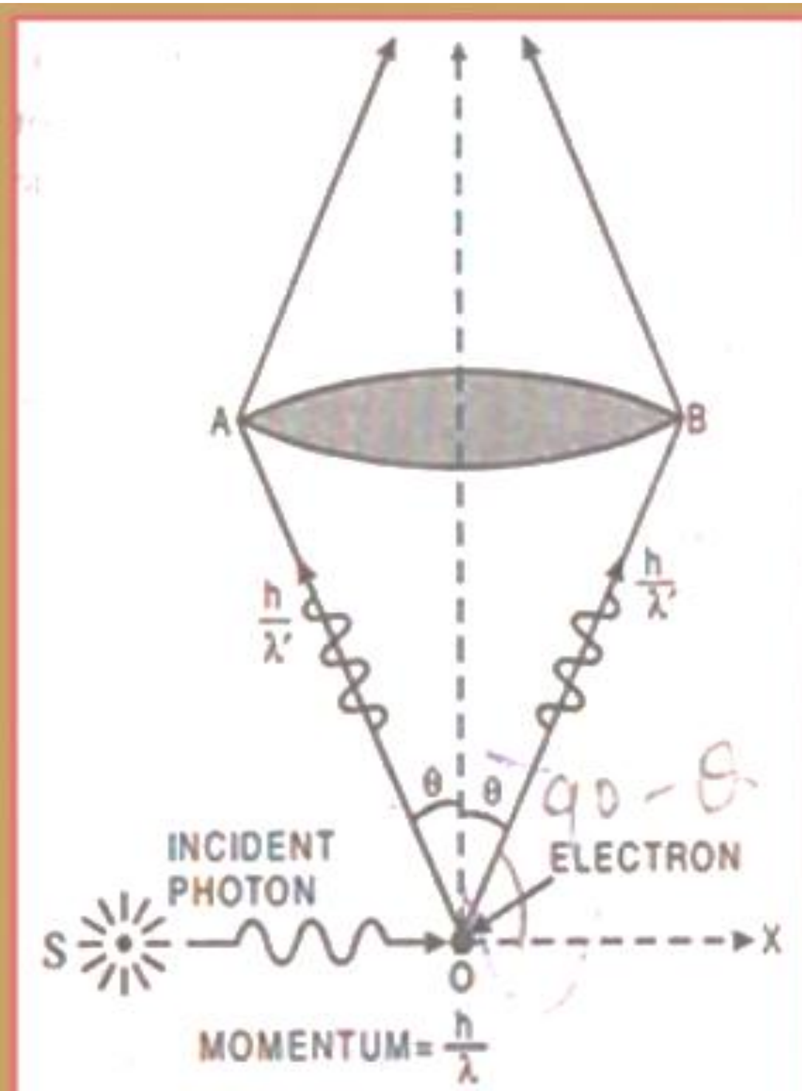
As shown in fig ., some of these photons are scattered into the microscope and enable the observer to see ***the flash*** of light which in turn enables to find position and momentum of the electron at same time.

The minimum separation between two points which are seen as separate through microscope is given by

$$\Delta x = \frac{\lambda'}{2 \sin \theta}$$

where λ'=wavelength of scattered photon entering the microscope

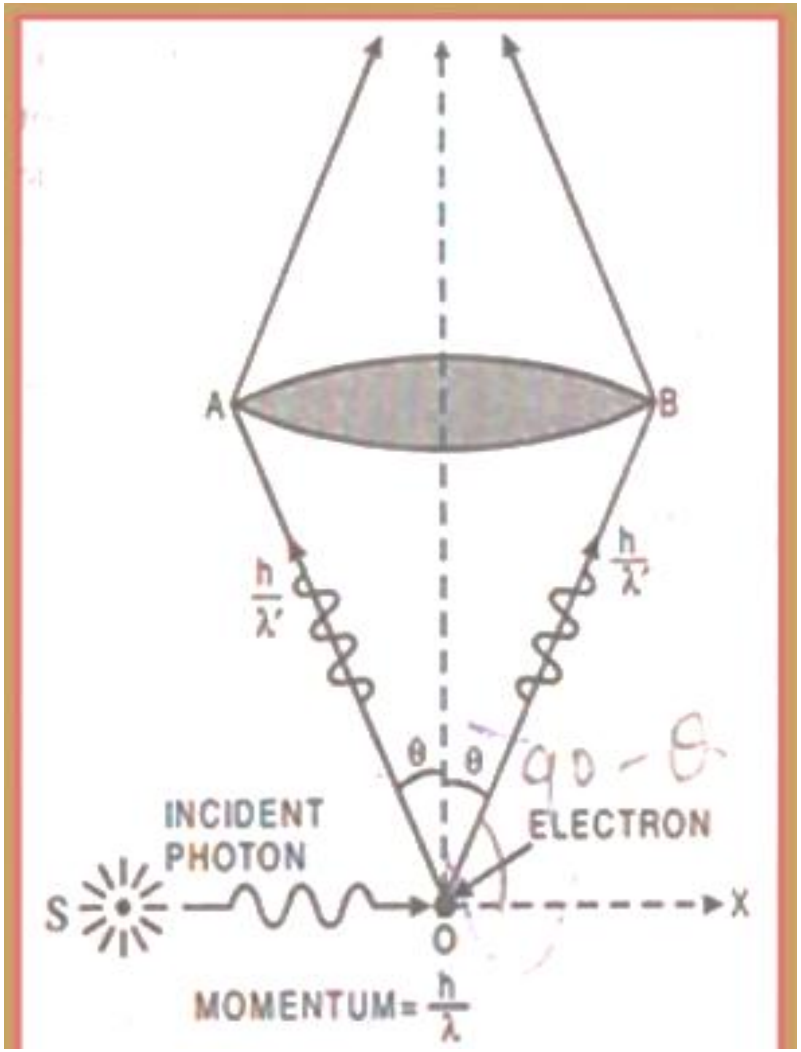
θ=semi angle of cone of light from the electron into the microscope.



Δx is measure of accuracy in position determination because any two points with separation less than this are not distinguished. From above equation, it is also clear that **smaller the wavelength smaller is uncertainty in position measurements.**

In order to observe the electron, the photon must enter the microscope within **angle 2θ** after interacting with the electron. In scattering, momentum is transferred from photon to electron as in case of Compton effect.

Therefore, the component of momentum of electron along X-rays will have an uncertainty Δp_x equal to the difference in change of two extreme values of component of momentum of scattered photon along X-direction.



But the change in component of momentum of scattered photon along X-direction has extreme values

$$\left[\frac{h}{\lambda}-\frac{h}{\lambda'}\cos(90^0-\theta)\right] \text{ and } \left[\frac{h}{\lambda}-\frac{h}{\lambda'}\cos(90^0+\theta)\right]$$

where λ' is wavelength of scattered photon

$$\Delta p_x = \left[\frac{h}{\lambda}+\frac{h}{\lambda'}\sin\theta\right] - \left[\frac{h}{\lambda}-\frac{h}{\lambda'}\cos(90^0-\theta)\right]$$

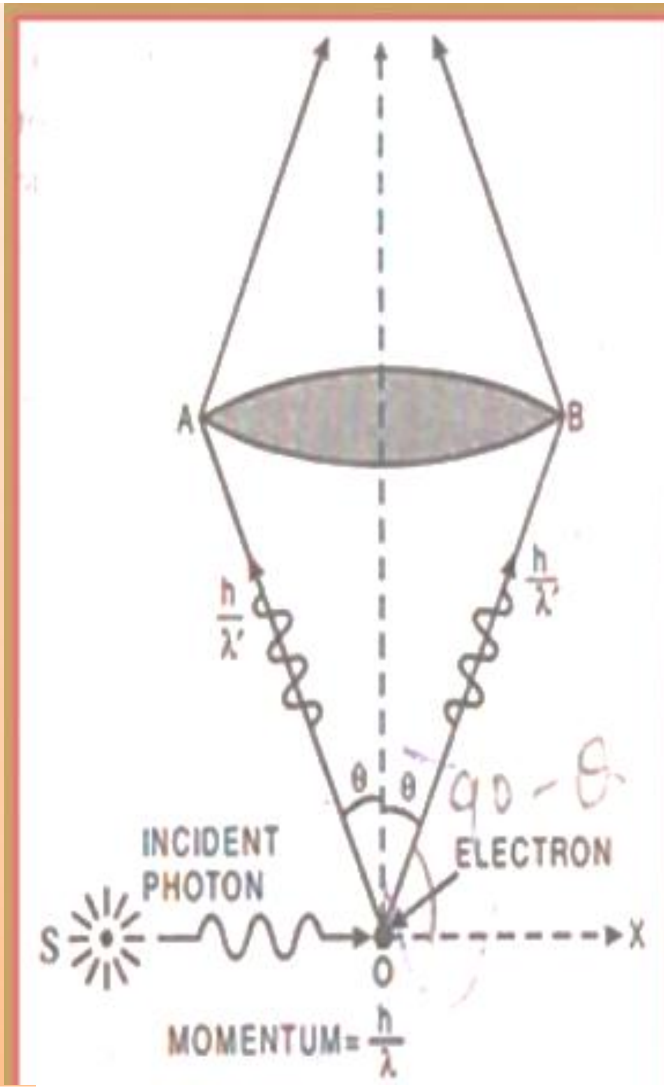
$$= \left[\frac{2h}{\lambda'}\sin\theta\right] - \left[\frac{h}{\lambda}-\frac{h}{\lambda'}\sin\theta\right]$$

$$= \frac{2h}{\lambda'}\sin\theta$$

From above equations

$$\Delta x \,.\, \Delta p_x \approx \frac{\lambda'}{2\sin\theta}\cdot\frac{2h}{\lambda'}\sin\theta$$

or $\quad \Delta x \,.\, \Delta p_x \approx h$



Applications of Heisenberg uncertainty Principle

Non-existence of electron in the nucleus

Refer Class notes