Module 4
Posets and Lattice

4.1 Partial ordered relations (Posets) ,Hasse diagram

4.2 Lattice, sublattice

4.3 Types of Lattice ,Boolean Algebra

POSETS(Partially Ordered Sets)

A relation R on a set A is called **partial order if R**

is **REFLEXIVE,**

ANTISYMMETRIC AND

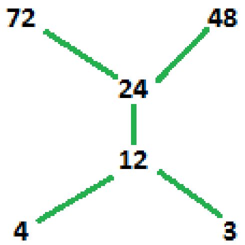
TRANSITIVE

The set A together with the partial order R is

called a POSET (A, R) or A

Examples

- The set N of natural numbers form a poset under the relation '≤' because firstly $x \le x$, secondly, if $x \le y$ and $y \le x$, then we have $x = y$ and lastly if $x \le y$ and $y \le z$, it implies $x \le z$ for all $x, y, z \in N$.
- The set N of natural numbers under divisibility i.e., 'x divides y' forms a poset because x/x for every $x \in N$. Also if x/y and y/x, we have $x = y$. Again if x/y, y/z we have x/z, for every $x, y, z \in N$.

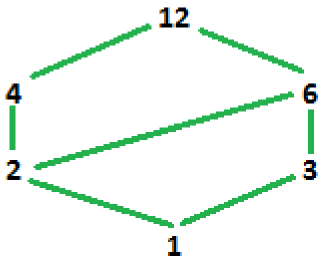


Elements of POSET

- **Maximal Element:** An element a ∈ A is called a maximal element of A if there is no element in c in A such that a ≤ c.

- **Minimal Element:** An element b ∈ A is called a minimal element of A if there is no element in c in A such that c ≤ b.

- **Note: There can be more than one maximal or more than one minimal element.**



HASSE DIAGRAM

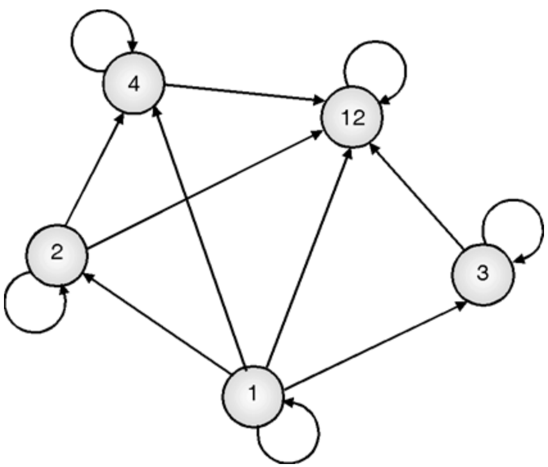
Mathematical diagram used to represent a finite partially ordered set

STEPS:

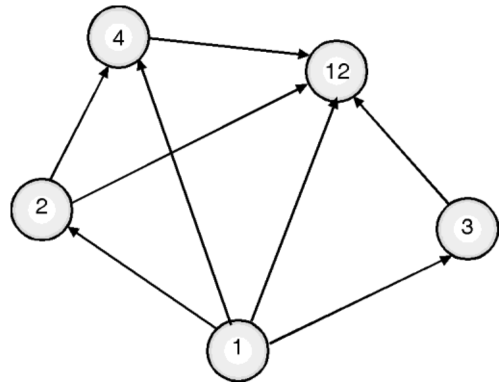
1. Draw the digraph of the given relation
2. Delete all cycles from the graph
3. Eliminate all edges that are implied by transitive relations
4. Draw the digraph of a partial order with all edges pointing upward, so that arrows may be omitted from the edges.
5. Replace the circles representing the vertices by dots.

The resulting diagram of a partial order is called the **Hasse diagram of the partial order of the poset.**

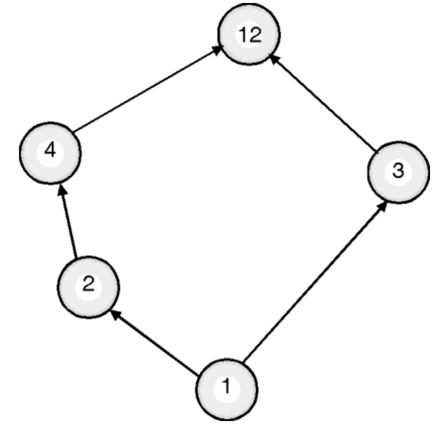
Draw Hasse diagram for the following relations on set
A={1, 2, 3, 4, 12}
R={(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (4, 12), (1, 3), (1, 4), (1, 12), (2, 4), (2, 12), (3,12)}



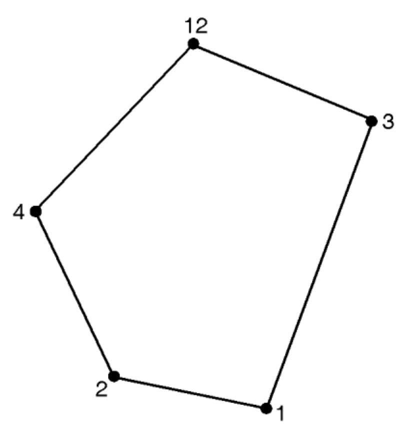
Step 1 : Remove cycles



Step 2 : Remove transition edge. Eliminated transitive edges (1, 4), (2, 12), (1, 12).



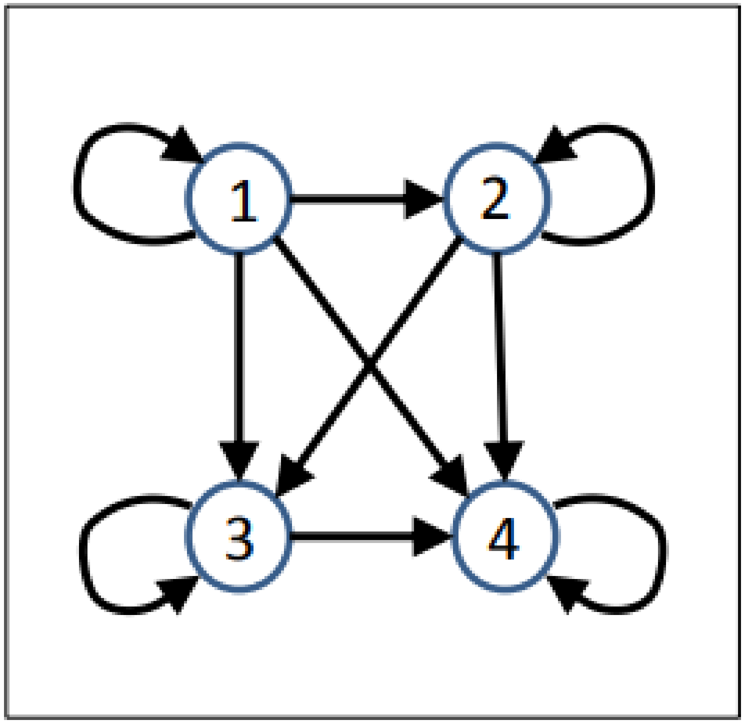
Step 3:
Circles are replaced by dots. Arrows are also removed.



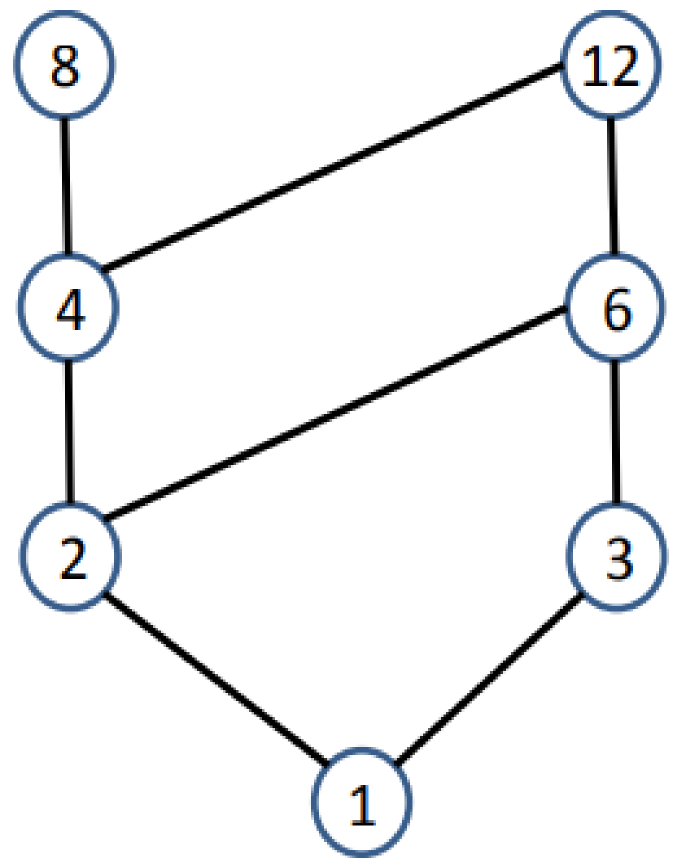
Draw the Hasse diagram of R

R= {a,a),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(c,d),(d,d)}

Consider the poset ({ 1, 2, 3, 4 }, $\leq$)



Consider the poset ({ 1, 2, 3, 4, 6, 8, 12 }, |)



Let A = {a, b, c, d} and R be a relation on A whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i)Prove that R is partial order.

(ii)Draw Hasse diagram of R.

Solution:

(i) R = {(a, a), (a, c), (a, d), (b, b), (b, c), (b, d), (c, d), (c, d), (d, d)} R is reflexive because it contain (a, a), (b, b), (c, c), (d, d).

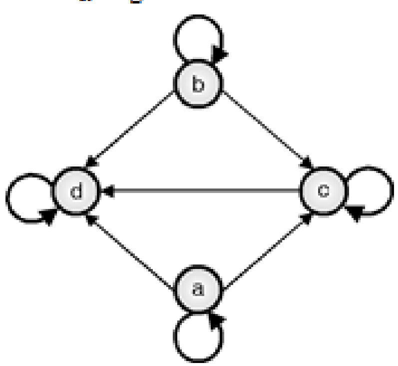
R is antisymmetric because it contain a and b such that if $a \neq b$, then $a \not R b$ or $b \not R a$.

R satisfies this condition, hence R is anti-symmetric.

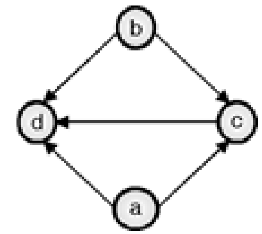
R is transitive since it contain (a, d) and (b, d).

$\therefore$ R is partial order.

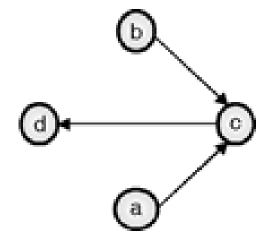
(ii) Diagraph of M_R is given below.



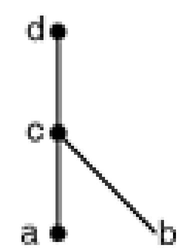
Step 1 : Remove reflexive relation



Step 2 : Remove transitive relation i.e. (a, d) and (b, d).



Step 3 : Circles are replaced by dots and all edges are pointing upward. Arrows are removed.



More Examples

Draw a Hasse diagram for A, R= (divisibility relation) where

(i) A = {1, 2, 3, 4, 5, 6, 7, 8}

R={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(2,4),(2,6),(2,8),(3,6),(4,8),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8)}

(ii) A = {1, 2, 3, 5, 11, 13}

(iii) A = {2, 3, 4, 5, 6, 30, 60}

(iv) A = {1, 2, 3, 6, 12, 24}

(v) A = {1, 2, 4, 8, 16}

(vi) A = {2, 4, 6, 12, 24, 36}

Draw a Hasse diagram for A, (divisibility relation)

where

(i) A = {1, 2, 3, 4, 5, 6, 7, 8};

(ii) A = {1, 2, 3, 5, 11, 13};

(iv) A = {1, 2, 3, 6, 12, 24};

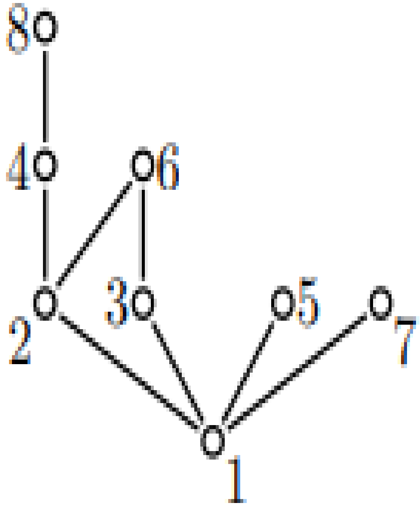
(iii) A = {2, 3, 4, 5, 6, 30, 60};

(v) A = {1, 2, 4, 8, 16, 32, 64};

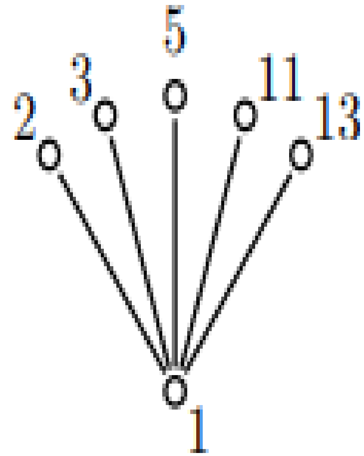
(vi) A = {2, 4, 6, 12, 24, 36}

Solution:

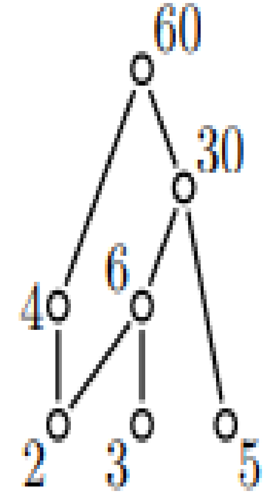
(i):



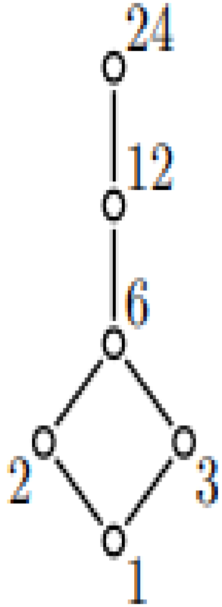
(ii):



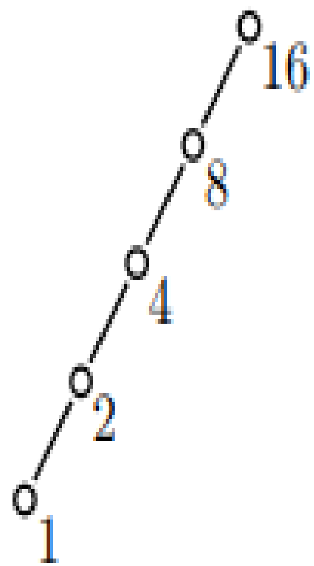
(iii):



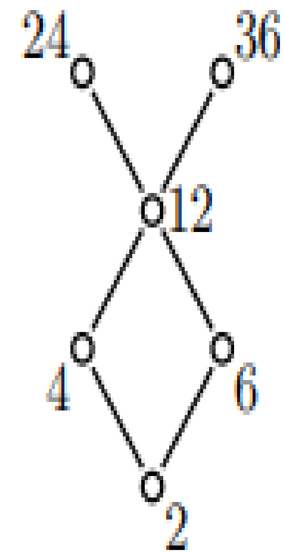
(iv):



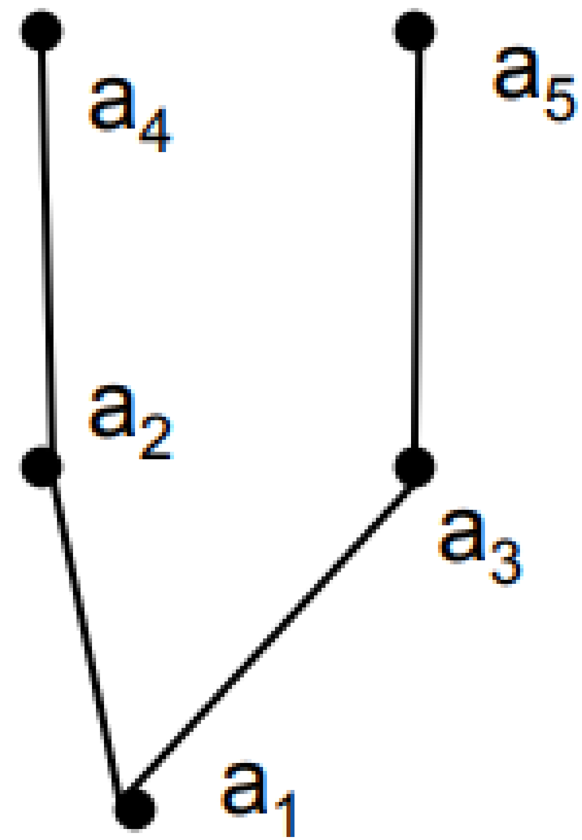
(v):



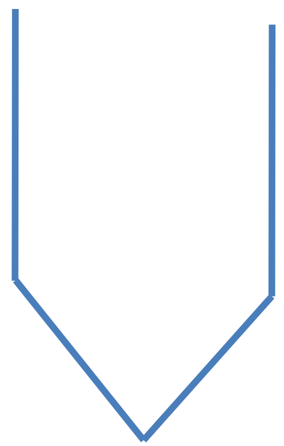
(vi):



Given Hasse diagram to be converted to diagraph



Given Hasse diagram to be converted to diagraph



$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Draw Hasse diagram for D_{12}

D means set of positive integers divisors of 12.

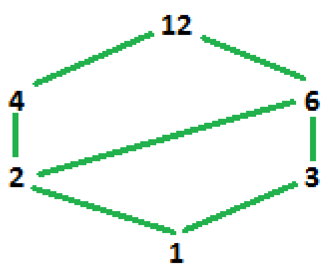
So, $D_{12}=\{1, 2, 3, 4, 6, 12\}$

poset A =

{(1,1),(2,2),(3,3),(4,4),(6,6),(12,12),(1 ,2),

(1 ,3), (1 ,4), (1, 6), (1, 12), (2 ,4), (2. 6).

(2 ,12), (3 ,6), (3, 12), (4 ,12), (6, 1



Draw Hasse Diagram for D_{36}

Draw Hasse diagram for D_{25} ,D_{16} ,D_{80}

The set of numbers that divide 25,16,80

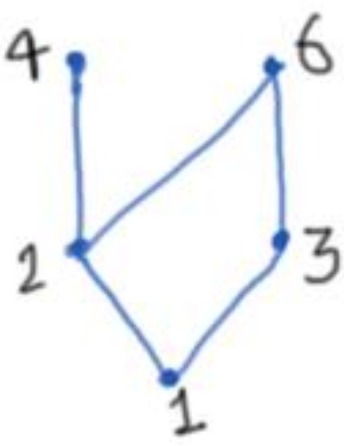
- **Maximal element**: If in a poset, **an element in not related to any other element**, then it is called maximal element.

  Minimal element: If in a poset, **if no element is related to an element**, then it is called minimal element.

  <u>Example:</u>

  [{1,2,3,4,6};A(/)]

  We can draw Hasse diagram as,

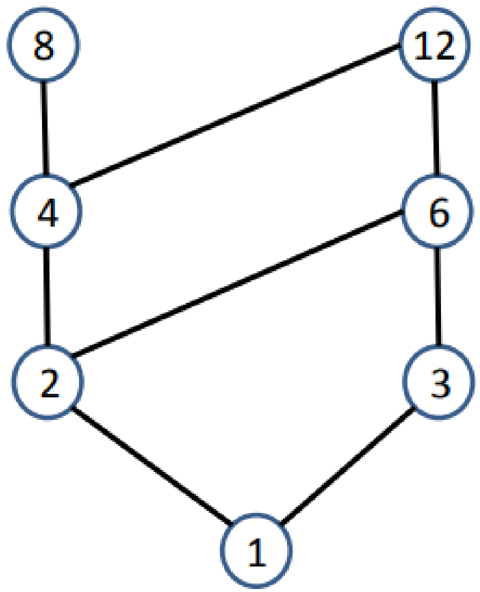


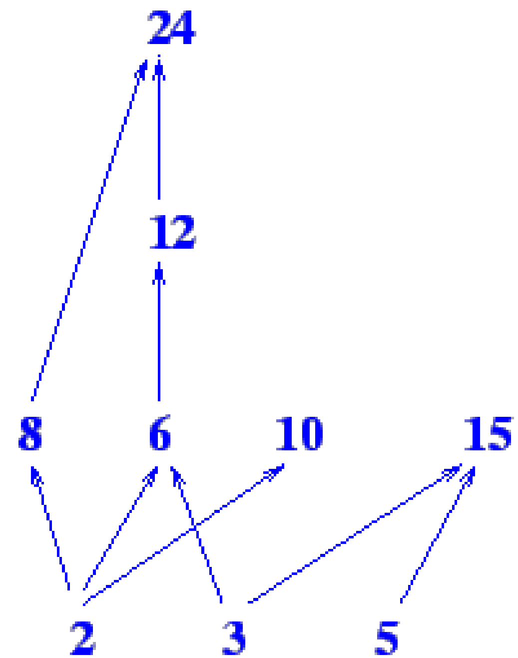
Maximal Element = 4, 6

Minimal Element = 1

Maximal and Minimal Items

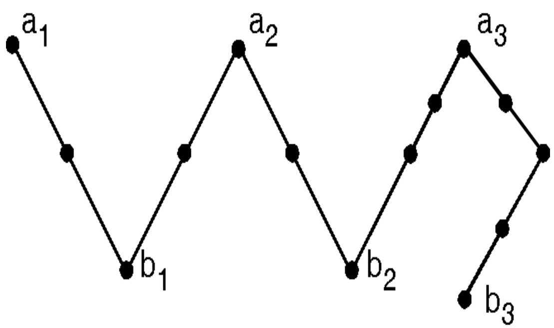
Which items are maximal ? Which are minimal ?

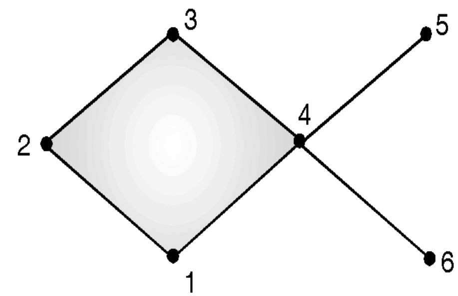


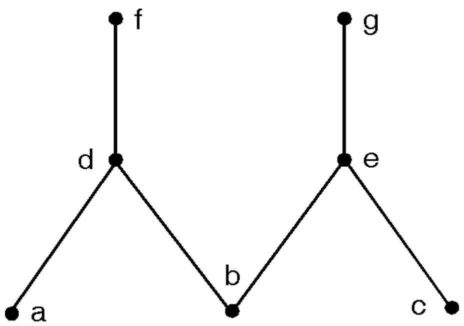


You are *maximal* when there is nobody above you.

Examples







Maximal elements : a1, a2, a3

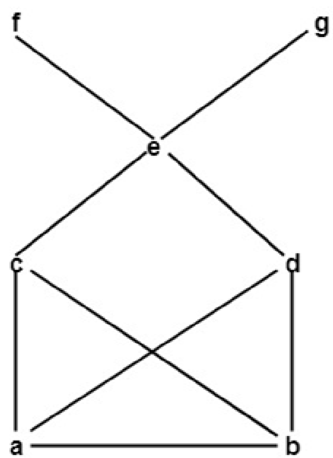
Minimal elements : b1, b2, b3

Maximal elements : 3, 5

Minimal elements : 1, 6

Maximal elements : f, g

Minimal elements : a, b, c



“If there is one <u>unique</u> maximal element a, we call it the <u>maximum</u> element (or the <u>GREATEST</u> element)”

“If there is one <u>unique</u> minimal element a, we call it the <u>minimum</u> element (or the <u>LEAST</u> element)”

Consider the poset {3, 5, 9, 15, 24, 45}, for divisibility relation.

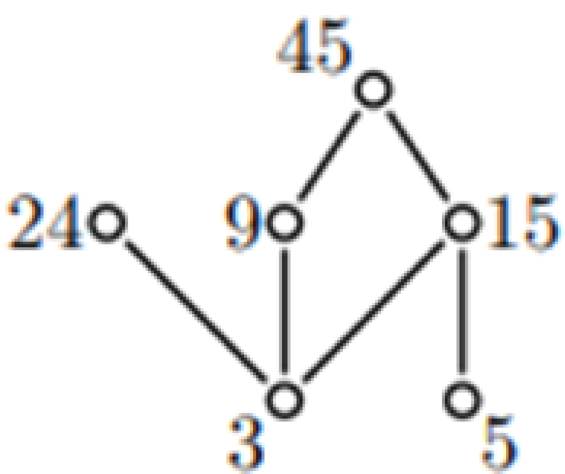
(i) Draw its Hasse diagram.

(ii) Find its maximal, minimal, greatest and least elements when they exist.

(iii) Find maximal, minimal, greatest and least elements of the set M = {3, 9, 15}, when they exist.

Solution:

i) Hasse Diagram



ii) Maximal Elements: 24,45
Minimal Elements: 3,5
Greatest, Least elements do not exist

iii) Maximal Elements: 9,15
Minimal Elements: 3
Greatest element DNE
Least element : 3

Extremal Elements: Minimal

- **Definition**: An element a in a poset $(S, \prec)$ is called minimal if it is not greater than any other element in S.

“If there is one unique minimal element a, we call it the minimum element (or the LEAST element)”

- Greatest Element **I**
- Least Element **O**

You are *greatest* when you are above everyone else.

A POSET HAS ATMOST 1 GREATEST ELEMENT AND ATMOST 1 LEAST ELEMENT

Greatest Element = Universal Upper bound(UB)

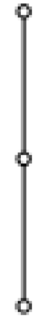
Least Element = Universal Lower bound(LB)

LUB → Least Upper Bound (LUB)

GLB → Greatest Lower bound (GLB)

- **Chain**-A subset of A is called a chain if every two elements in the subset **are related**

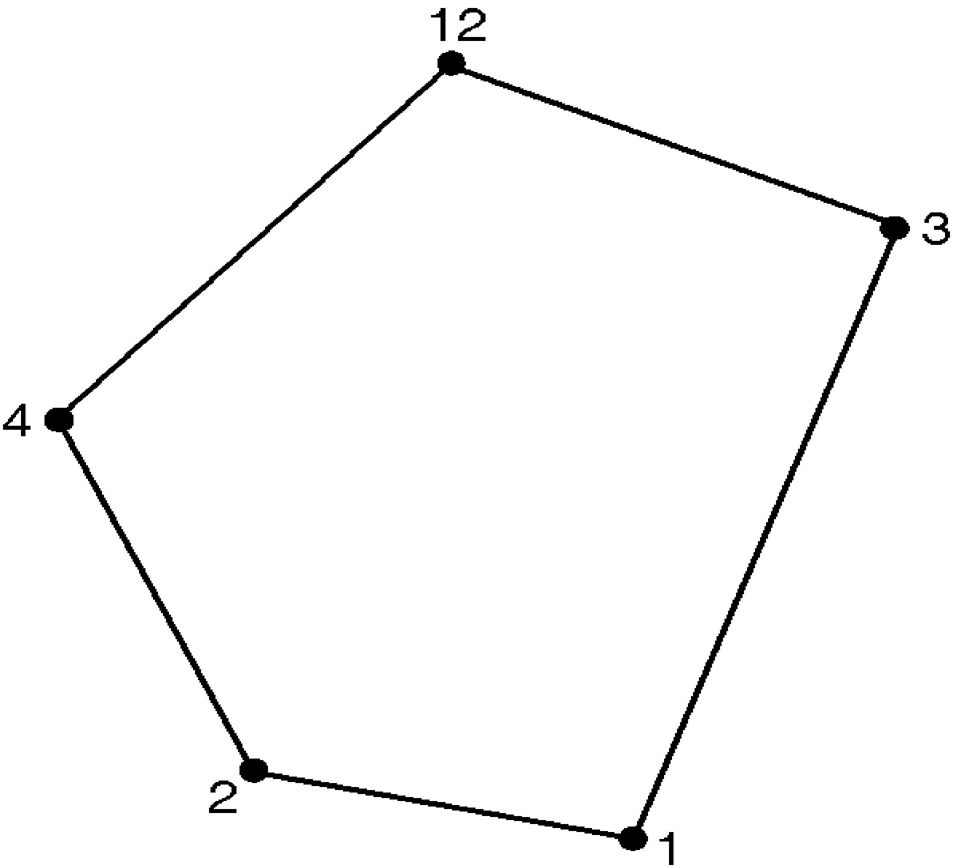
- **Anti chain-** If **no two distinct elements in the subset are related**



(a) chain

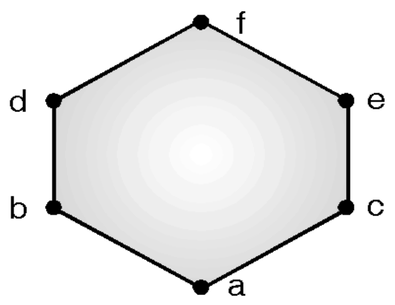
(b) antichain

EXAMPLE

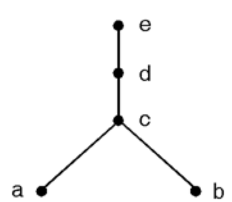


- Chain-{1},{1,2},{1,3},{1,2,4},{1,3,12},{1,2,4,12}…
- Anti chain-{2,3},{4,3}…

Examples



Greatest element **I** = f
Least element **O** = a



Greatest element **I** = e
Least element **O** = none

LATTICES

A LATTICE IS A POSET (L, <=) IN WHICH EVERY SUBSET { a , b } consisting of 2 elements has a **LUB and a GLB**

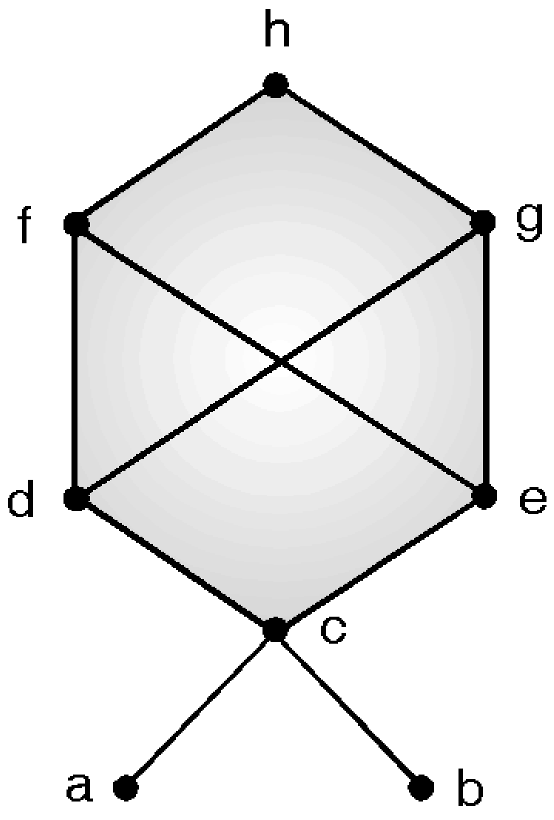
LUB ({ a , b }) by **a V b (<u>join</u> of a and b)**

GLB ({ a , b }) by **a ∧ b (<u>meet</u> of a and b)**

$\wedge$ Meet ,GLB

$\vee$ Join , LUB

Find GLB and LUB for B1and B2



Ex. : A = {a, b, c, d, e, f, g, h}

(i) B1 = {a, b}

(ii) B_2 = {c, d, e}

Solution:

(i) **Upper bounds of set B1 are c, d, e, f, g and h**

LUB - least upper bound is 'c'.

Lower bounds of set B1 is none.

GLB - None

(ii) **Upper bounds of set B2 are f, g, and h.**

LUB – None , There is no least upper bound.

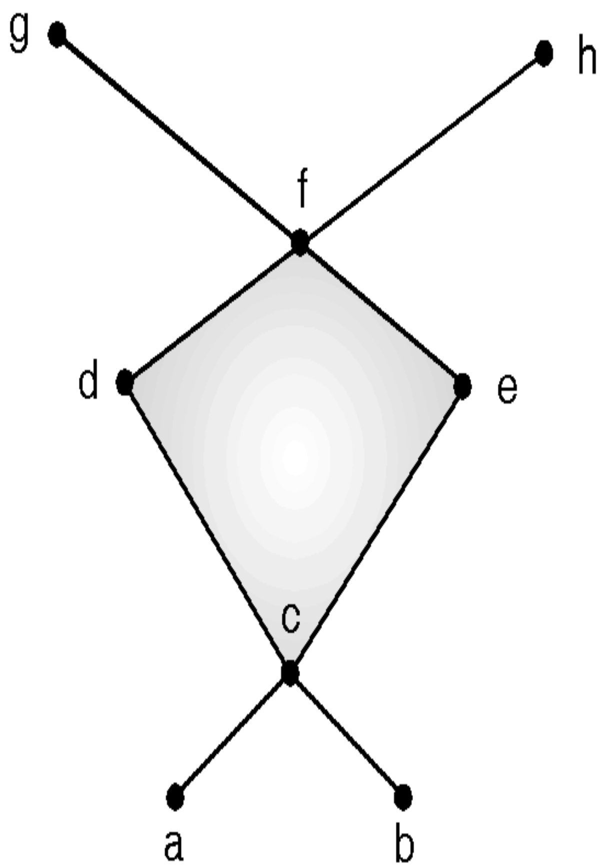
Lower bounds of set B2 are c, a, b

GLB -Greatest lower bound is ‘c’.

$\wedge$ Meet ,GLB
$\vee$ Join , LUB

Let A = {a, b, c, d, e, f, g, h} be the poset whose Hasse diagram is shown in Fig.

Find GLB and LUB of B = {c, d, e}.



Solution:

(1)Upper bounds of B are f, g, h.

Least upper bound (LUB) is f.

(2)Lower bounds of B are c, a, b.

Greatest lower bound (GLB) is ‘c’

Example

Find the greatest lower bound and least upper bound of the set {3, 9, 12} and {1, 2, 4, 5, 10} if they exists in the poset (Z^+, /) . Where / is relation of divisibility.

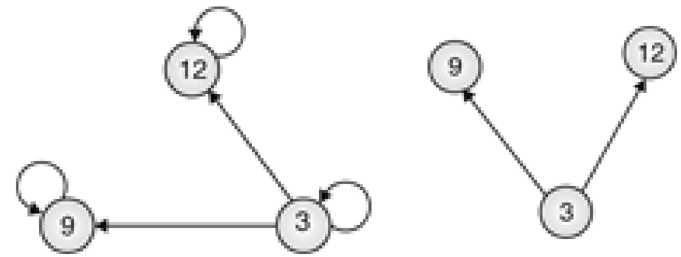
Solution:

(a) A={3, 9, 12}

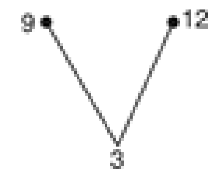
R={(3, 3), (3, 9), (3, 12), (9, 9), (12, 12)}

$$M_R = \begin{array}{c} \\ 3 \\ 9 \\ 12 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 9 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph :



Hasse diagram :



LUB :

	3	9	12
3	3	9	12
9	9	9	-
12	12	-	12

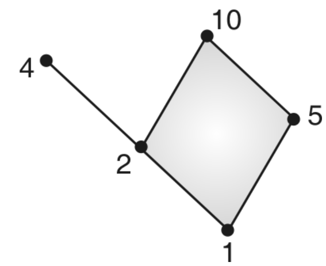
GLB :

	3	9	12
3	3	3	3
9	3	9	3
12	3	3	12

Example

A={1, 2, 4, 5, 10 }
R={(1, 1), (1, 2), (1, 4), (1, 5), (1,10),(2, 2), (2,4), (2, 10), (4, 4), (5, 5), (5, 10), (10, 10)}

Hasse Diagram:

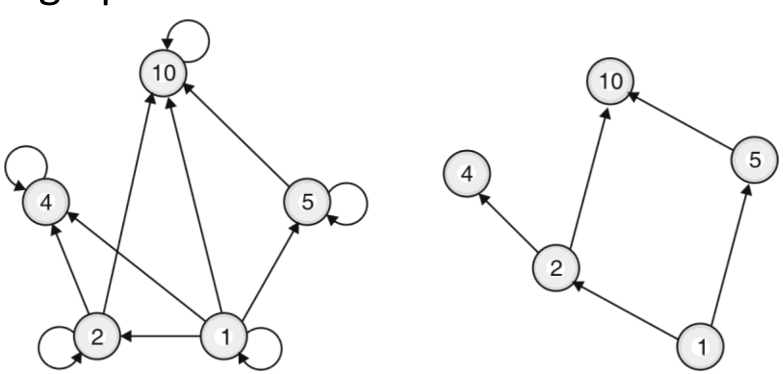


$$
M_R = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 4 & 5 & 10 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

LUB :

$$
\begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 10 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 4 & 5 & 10 \end{array} \\
\begin{bmatrix}
1 & 2 & 4 & 5 & 10 \\
2 & 2 & 4 & 10 & 10 \\
4 & 4 & 4 & - & - \\
5 & 10 & - & 5 & 10 \\
10 & - & 10 & 10 & -
\end{bmatrix}
\end{array}
$$

Digraph:



GLB :

	1	2	4	5	10
1	1	1	1	1	1
2	1	2	2	1	2
4	1	2	4	1	2
5	1	1	1	5	5
10	1	2	2	5	10

Lattice

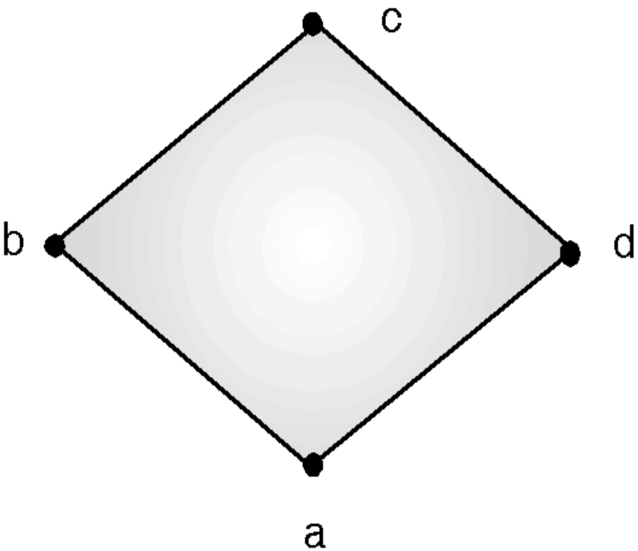
A lattice is a poset (L, $\leq$) in which every subset {a, b} consisting of two elements **has a least upper bound and a greatest lower bound.**

LUB ({a, b}) by $a \vee b$, and call it the **join** of a and b.

GLB ({a, b}) by $a \wedge b$ and call it the **meet** of a and b.

Example

Ex. 1



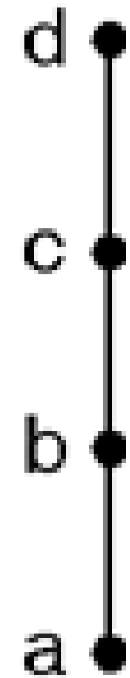
LUB :

∨	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

GLB:

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Ex. 2



LUB :

∨	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

GLB:

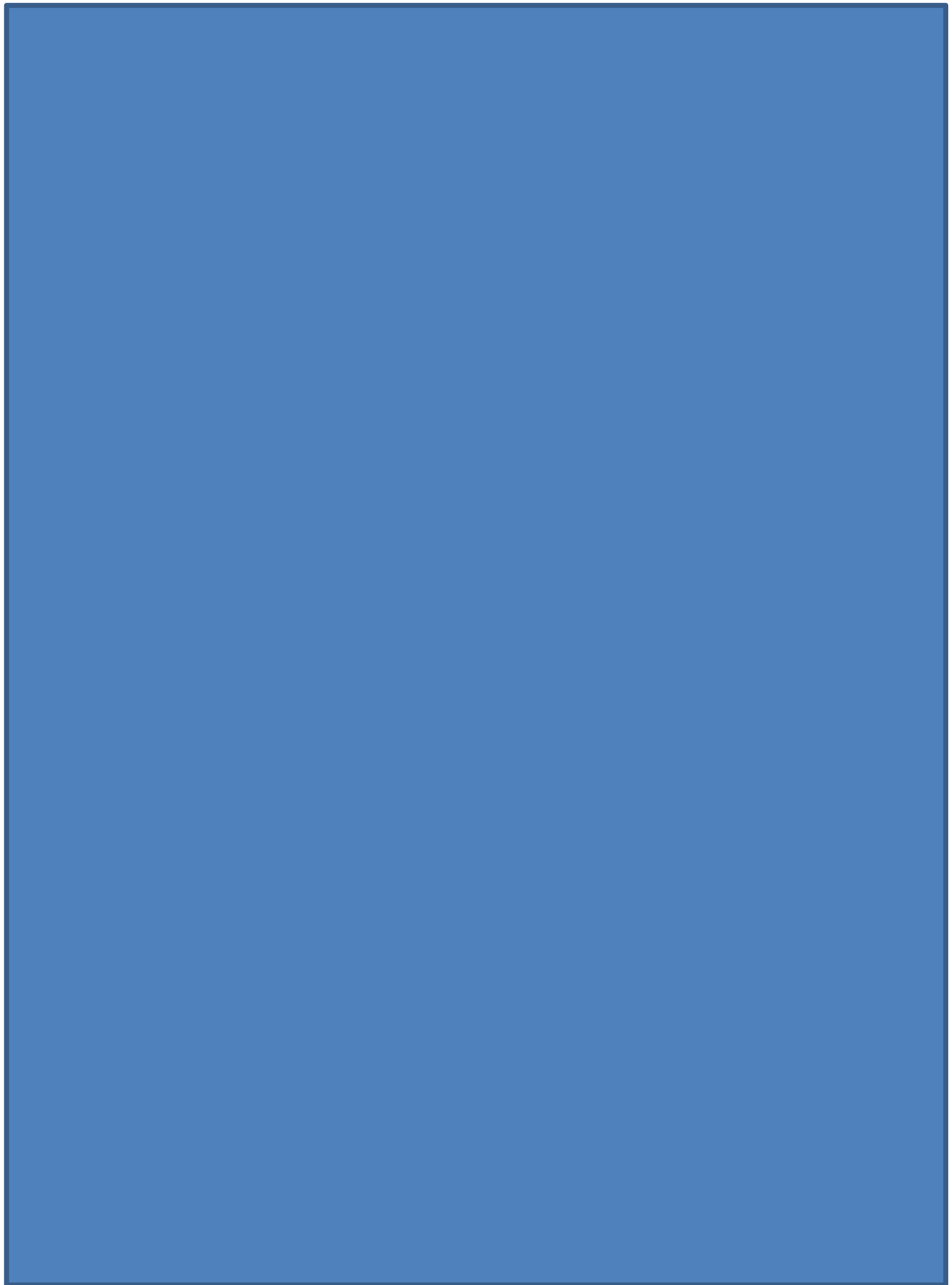
∧	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Lattice

Let n be a positive integer and D_n be the set of all positive divisors of n. Then D_n is a lattice under the relation of divisibility. For instance,

$D_{20} = \{1,2,4,5,10,20\}$ $D_{30} = \{1,2,3,5,6,10,15,20\}$

20 30



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D20 is a Lattice

Properties of Lattices

1. Idempotent Properties

a) a v a = a

b) a Λ a = a

2. Commutative Properties

a) a v b = b v a

b) a Λ b = b Λ a

3. Associative Properties

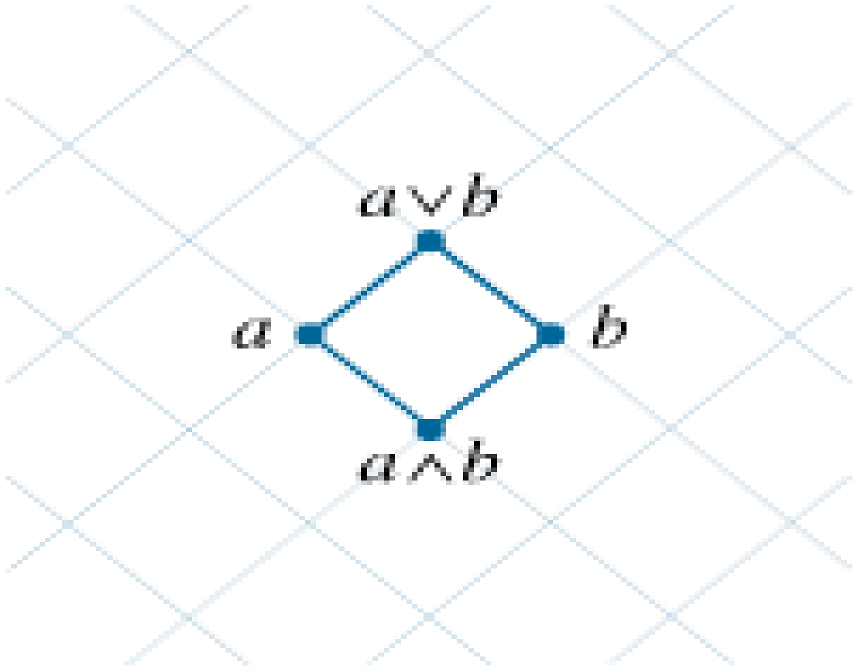
a) a v (b v c)= (a v b) v c

b) a Λ(b Λ c)= (a Λ b) Λ c

4. Absorption Properties

a) a v (a Λ b) = a

b) a Λ (a v b) = a



Sub lattice

- Consider a non-empty subset L_1 of a lattice L. Then L_1 is called a sub-lattice of L if L_1 itself is a lattice i.e., the operation of L i.e.,

$$\mathbf{a \vee b \in L_1 \text{ and } a \wedge b \in L_1}$$

whenever $a \in L_1$ and $b \in L_1$.

ISOMORPHIC LATTICES

Definition : Two lattices L and L' are said to be isomorphic if there is a function $f : L \rightarrow L'$ such that

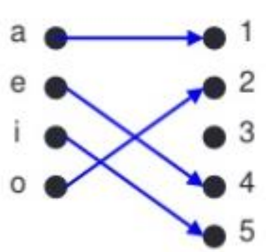
(i) f is one to one

(ii) f is onto (i.e. f is bijection)

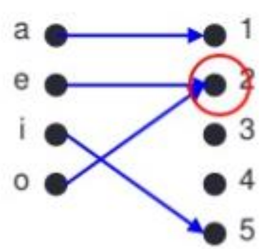
(iii) $f(a \wedge b) = f(a) \wedge f(b)$

(iv) $f(a \vee b) = f(a) \vee f(b)$

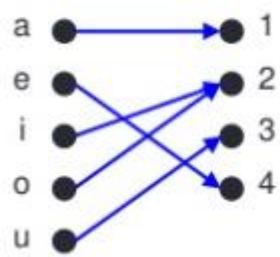
For any elements a, b in L.



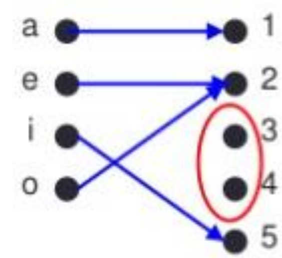
A one-to-one function



A function that is not one-to-one



An onto function



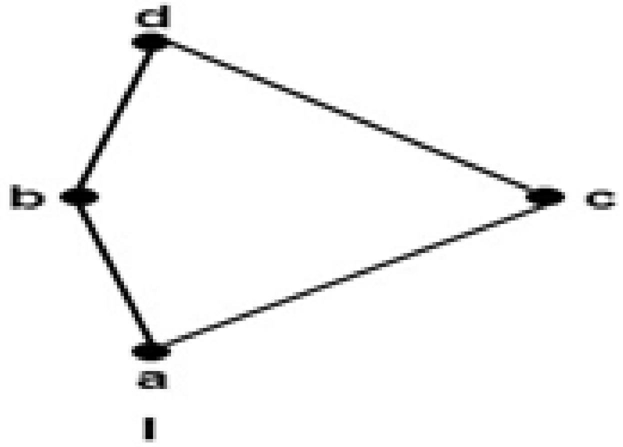
A function that is not onto

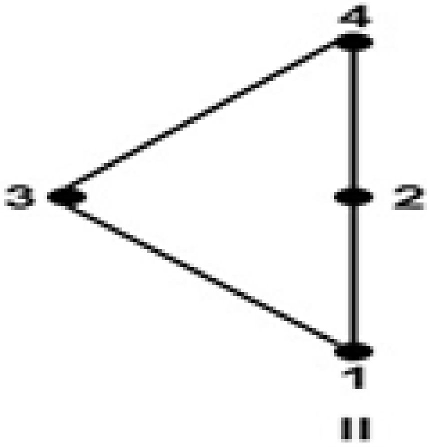
The lattices shown in fig are isomorphic.

Consider the mapping f = {(a, 1), (b, 2), (c, 3), (d, 4)}.

For example f (b ∧ c) = f (a) = 1.

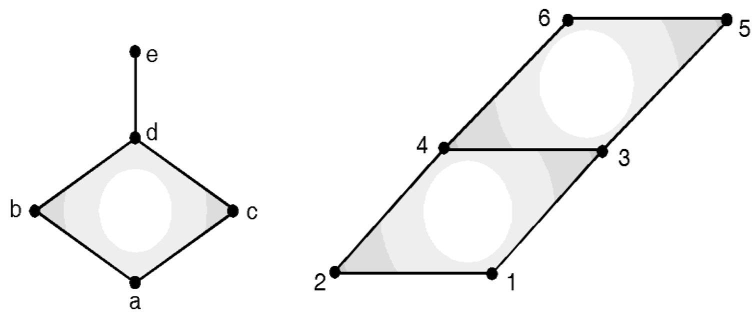
Also, we have f (b) ∧ f(c) = 2 ∧ 3 = 1





Example

Are the two lattices shown in Fig. isomorphic ?



Solution:

The two lattices are not isomorphic, since the two lattices do not have the same number of elements. Hence, the mapping between two lattices cannot be one to one and onto.

Types of Lattices

Bounded Lattice

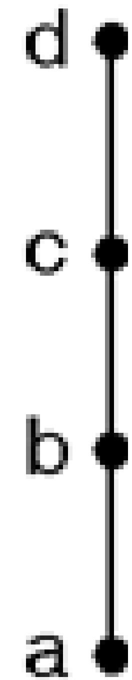
Definition :

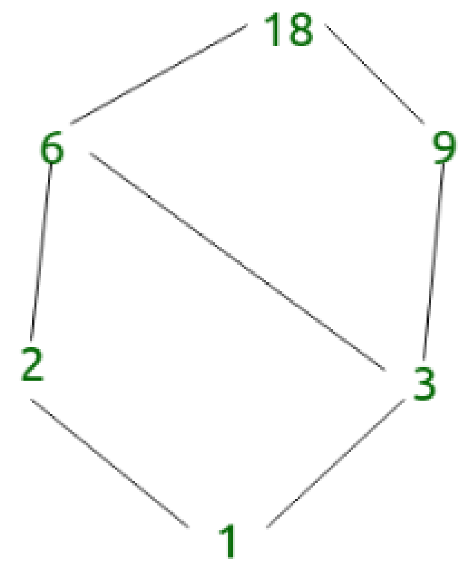
A lattice L is said to be **bounded** if it has a greatest element **I** and a least element **0**.

If L is a bounded lattice, then for all a ϵ A

$$a \vee \mathbf{0} = a, \qquad a \wedge \mathbf{0} = \mathbf{0}$$
$$a \vee \mathbf{I} = \mathbf{I}, \qquad a \wedge \mathbf{I} = a$$

E.g. – D_{18}= {1, 2, 3, 6, 9, 18} is a bounded lattice.





Hasse Diagram of D_{18}

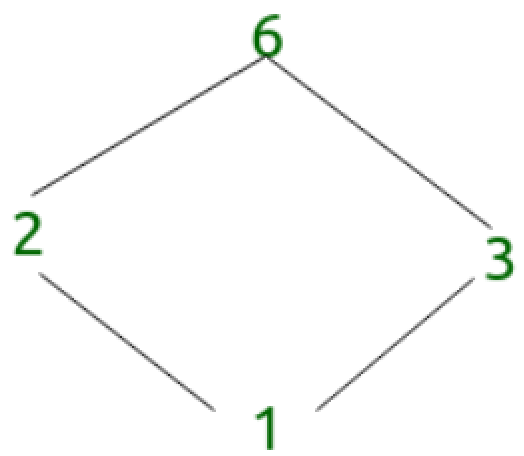
Note: Every Finite lattice is always bounded.

Complemented Lattice

A lattice L is said to be complemented if it is **bounded and if every element in L has a complement**

Each element should have at least one complement.

E.g. – D_6 {1, 2, 3, 6} is a complemented lattice.



Hasse Diagram of D_6

Definition

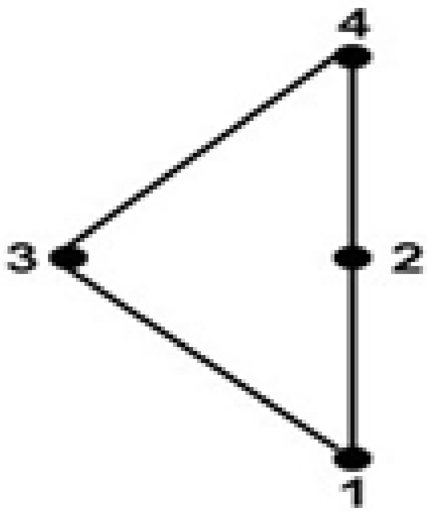
Let L be a bounded lattice with greatest element I

and least element 0, and let $a \in L$.

An element $a' \in L$ is called a complement of a

if. $\mathbf{a \vee a' = I}$ **and** $\mathbf{a \wedge a' = 0}$

$$a \vee a' \quad = I \quad \text{and} \quad a \wedge a' \quad = 0$$

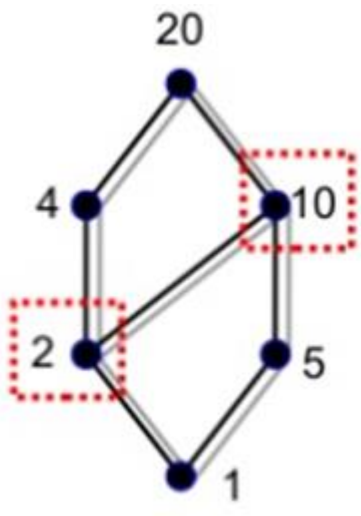


ELEMENT	COMPLIMENT
1	4
2	3
3	2
4	1

Example

$a \vee a' = I$ and $a \wedge a' = 0$

D_{20} is not a complemented lattice



D_{20}

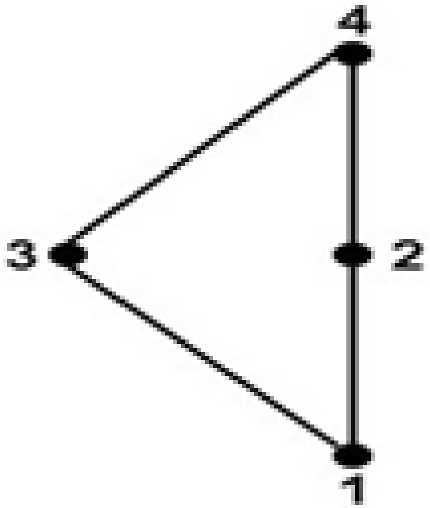
Element	Complement
1	20
2	-
4	5
5	4
10	-
20	1

Types of Lattices-Distributive Lattice

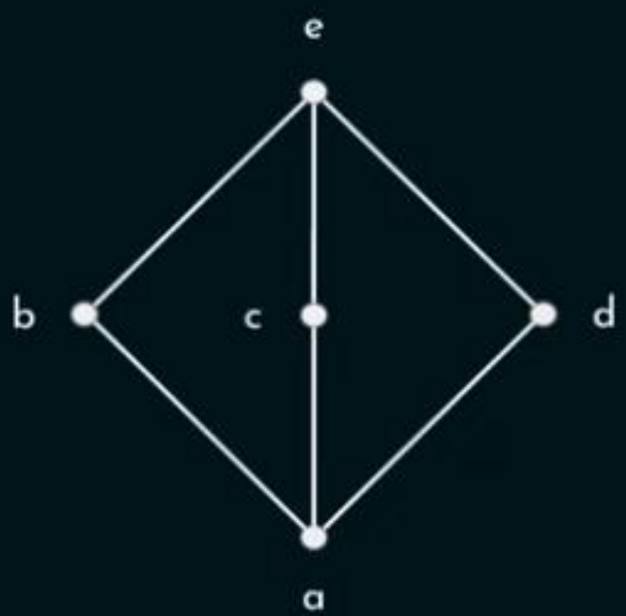
A lattice L is called distributive if for any elements a, b and c in L we have the following distributive properties.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

****A LATTICE IS DISTRIBUTIVE IF EVERY ELEMENT IN L HAS ATMOST ONE COMPLIMENT**



It satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3, and 4.



For a lattice to be distributive lattice, the following properties must be satisfied.

$\forall a,b,c \in L$

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Let's consider the elements b, c, and d.

(i) $\text{LHS} = b \vee (c \wedge d)$

$= b \vee a$

$= b$

(ii) $\text{RHS} = (b \vee c) \wedge (b \vee d)$

$= e \wedge e$

$= e$

Therefore, the above lattice is **not a distributive lattice.**

Example

$a \vee \mathbf{0} = a$, $a \wedge \mathbf{0} = \mathbf{0}$

$a \vee \mathbf{I} = \mathbf{I}$, $a \wedge \mathbf{I} = a$

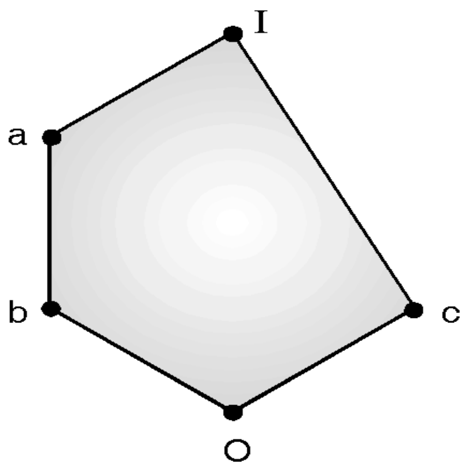


Fig.(a)

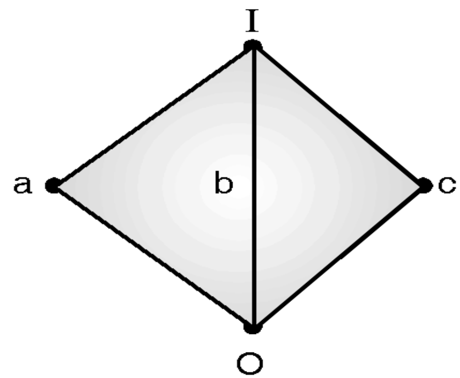


Fig.(b)

(a)

$a \wedge (b \vee c) = a \wedge I = a$

while $(a \wedge b) \vee (a \wedge c) = b \vee 0 = b$

So Fig. (a) is non-distributive

(b)

$a \wedge (b \vee c) = a \wedge I = a$

while $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

So Fig. (b) is non-distributive.

Boolean Algebra

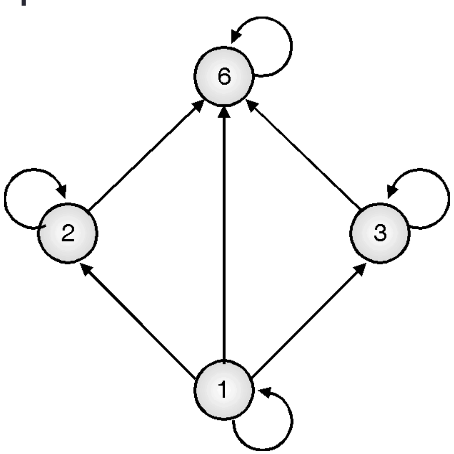
A *boolean algebra* is a lattice which contains

1. 2^n elements for any integer $n \geq 0$.
2. A greatest element and a least element.
3. and which is both complemented and distributive.

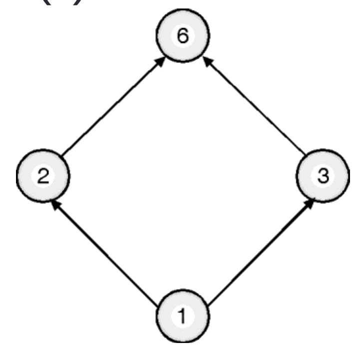
Ex. Determine whether the following posets are Boolean algebras. Justify your answer.
A = {1, 2, 3, 6} with divisibility
Solution:
Given set A is {1, 2, 3, 6} and the Partial order relation of divisibility on set A is
R={(1, 1), (1, 2), (1, 3), (1, 6), (2, 2), (2, 6), (3, 3), (3, 6), (6, 6)}
Matrix of the above relation is,

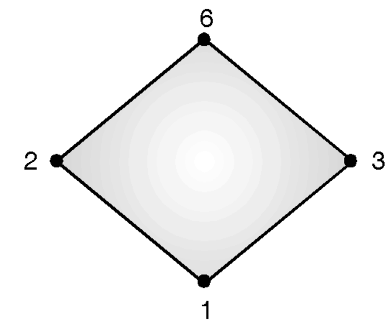
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraph is as follows :



To convert above digraph into Hasse diagram.
(i)Remove cycles
(ii)Remove transitive edge (1, 6)





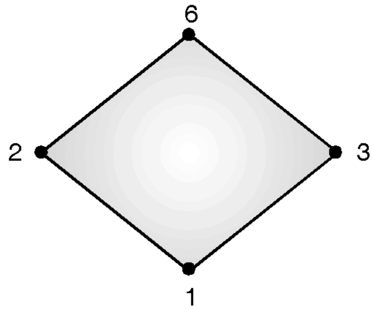
LUB :

∨	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

GLB :

∧	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

Every pair of elements in A has a GLB and a LUB. Therefore A is a lattice.



A has a least element (0) = 1

and a greatest element (I) = 6

Number of elements in A is $4=2^2$

$2 \vee 3 = 6$;

$2 \wedge 3 = 1$

$\therefore$ Complement of 2 is 3.

or complement of 3 is 2

A is a complemented lattice,

Also we can show that the operations $\vee$, $\wedge$ are distributive

$2 \vee (2 \wedge 3) = (2 \vee 2) \wedge (2 \vee 3)$

$2 \vee 1 = 2 \wedge 6$

$2 = 2$

$\therefore$ A is a distributive lattice.

A satisfies all requirements of Boolean Algebra.

$\therefore$ A under divisibility is a Boolean Algebra.