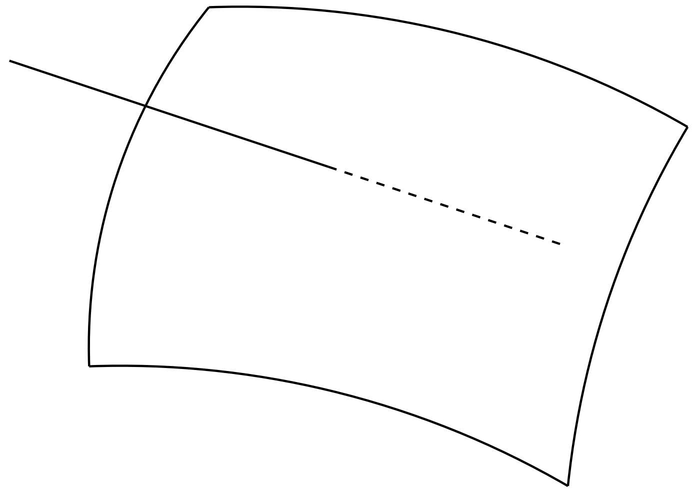

A connected algebraic set with two irreducible components.

COROLLARY 2.32. *The radical of an ideal* $\mathfrak{a}$ *in* $k[X_1, \dots, X_n]$ *is a finite intersection of prime ideals,* $\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$. *If there are no inclusions among the* $\mathfrak{p}_i$*, then the* $\mathfrak{p}_i$ *are uniquely determined up to order (and they are exactly the minimal prime ideals containing* $\mathfrak{a}$*).*

PROOF. Write $V(\mathfrak{a})$ as a union of its irreducible components, $V(\mathfrak{a}) = \bigcup_{i=1}^{n} V_i$, and let $\mathfrak{p}_i = I(V_i)$. Then $\mathrm{rad}(\mathfrak{a}) = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$ because they are both radical ideals and

$$V(\mathrm{rad}(\mathfrak{a})) = V(\mathfrak{a}) = \bigcup V(\mathfrak{p}_i) \overset{2.10\mathrm{b}}{=} V(\bigcap\nolimits_i \mathfrak{p}).$$

The uniqueness similarly follows from the proposition. □

Remarks

An irreducible topological space is connected, but a connected topological space need not be irreducible. For example, the union of two surfaces in 3-space intersecting along a curve is reducible, but connected.

2.33. An algebraic subset V of $\mathbb{A}^n$ is disconnected if and only if there exist radical ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that V is the disjoint union of $V(\mathfrak{a})$ and $V(\mathfrak{b})$, so

$$\begin{cases} V = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) & \iff \quad \mathfrak{a} \cap \mathfrak{b} = I(V) \\ \emptyset = V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b}) & \iff \quad \mathfrak{a} + \mathfrak{b} = k[X_1, \dots, X_n]. \end{cases}$$

Then

$$k[V] \simeq \frac{k[X_1, \dots, X_n]}{\mathfrak{a}} \times \frac{k[X_1, \dots, X_n]}{\mathfrak{b}}$$

by Theorem 1.1.

2.34. A Hausdorff space is noetherian if and only if it is finite, in which case its irreducible components are the one-point sets.

2.35. In $k[X_1, \dots, X_n]$, a principal ideal (f) is radical if and only if f is square-free, in which case f is a product of distinct irreducible polynomials, $f = f_1 \dots f_r$, and $(f) = (f_1) \cap \dots \cap (f_r)$.