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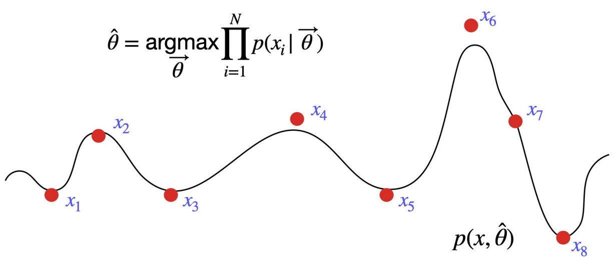

20: Maximum Likelihood Estimation

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February 27, 2023

Ed Discussion: https://edstem.org/us/courses/32220/discussion/2695809

Parameter Estimation

Story so far

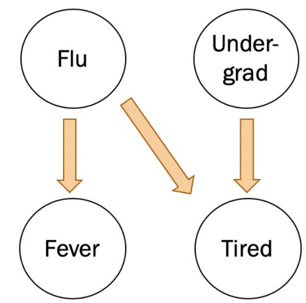
At this point:

If you are provided with a **model** and all the necessary probabilities, you can make predictions!

$Y \sim \text{Poi}(5)$

$X_1, \dots, X_n$ iid
$X_i \sim \text{Ber}(0.2)$,
$X = \sum_{i=1}^{n} X_i$

$P(F_{lu} = 1) = 0.1$ $\quad$ $P(U = 1) = 0.8$



$P(F_{ev} = 1 | F_{lu} = 1) = 0.9$
$P(F_{ev} = 1 | F_{lu} = 0) = 0.05$

$P(T = 1 | F_{lu} = 0, U = 0) = 0.1$
$P(T = 1 | F_{lu} = 0, U = 1) = 0.8$
$P(T = 1 | F_{lu} = 1, U = 0) = 0.9$
$P(T = 1 | F_{lu} = 1, U = 1) = 1.0$

But how do we **infer** the probabilities for a given model?

this is today's focus!

~~What if you want to learn the **structure** of the model, too?~~ Glimpse: Week 10

Machine Learning

you need entire classes to understand machine learning, not just one week!

Some estimators

introduced last Wednesday and Friday

$X_1, X_2, \ldots, X_n$ are n iid random variables, underlying (i.e. unknown)
where X_i drawn from distribution F with $E[X_i] = \mu$, $\text{Var}(X_i) = \sigma^2$.

Sample mean: $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$ unbiased estimate of μ

Sample variance: $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \bar{X})^2$$ unbiased estimate of σ^2

What are parameters?

def Most random variables we've seen thus far are parametric models:

$$\text{Distribution} = \text{model} + \text{parameter } \theta$$

ex The distribution $\text{Ber}(0.2)$ = model is Bernoulli, parameter is $\theta = 0.2$.

For each of the distributions below, what is the parameter θ?

1. $\text{Ber}(p)$ $\qquad \theta = p$
2. $\text{Poi}(\lambda)$
3. $\text{Uni}(\alpha, \beta)$
4. $\mathcal{N}(\mu, \sigma^2)$
5. $Y = mX + b$

What are parameters?

<u>def</u> Most random variables we've seen thus far are **parametric models**:

$$\text{Distribution} = \text{model} + \text{parameter } \theta$$

<u>ex</u> The distribution Ber(0.2) = model is Bernoulli, parameter $\theta = 0.2$.

For each of the distributions below, what is the parameter θ?

1. Ber(p) $\quad\theta = p$
2. Poi(λ) $\quad\theta = \lambda$
3. Uni(α, β) $\quad\theta = (\alpha, \beta)$
4. $\mathcal{N}(\mu, \sigma^2)$ $\quad\theta = (\mu, \sigma^2)$
5. $Y = mX + b$ $\quad\theta = (m, b)$

θ is the parameter of a distribution.
θ can be a vector of parameters!

Why do we care?

In the real world, we don't know the true parameters.

- But we do get to **observe data**: # times coin comes up heads, lifetimes of disk drives produced, # visitors to website per day, offer amount for a used bike

← whenever you see ^ over a parameter, it almost always means an estimate!

def **estimator** $\hat{\theta}$: a **random variable** estimating true parameter θ.

In parameter estimation,

We use the **point estimate** of parameter estimate (best single value):

- Provides an understanding of the process generating the data
- Can make future **predictions** based that model
- Can even run simulations to generate more data

Maximum Likelihood Estimator

Defining the likelihood of data: Bernoulli

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

- X_i was drawn from distribution $F = \text{Ber}(\theta)$ with unknown parameter θ.
- Observed sample:

$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

intuition tells us $\hat{p} = 0.8$.
but is our intuition correct?

How likely is this sample if, say, $\theta = 0.4$?

conditioned on a belief that $\theta = 0.4$! this is technically an event.

$$P(\text{sample}|\underbrace{\theta = 0.4}) = \underbrace{(0.4)^8(0.6)^2}_{\text{Likelihood of data given parameter } \theta = 0.4} = 0.000236$$

Likelihood of data
given parameter $\theta = 0.4$

Is there a better choice for θ?

Defining the likelihood of data

Consider a sample of n iid random variables $X_1, X_2, \dots, X_n$.

- X_i was drawn from a distribution with density function $f(X_i|\theta)$.
  (or mass)
- Sample: $(X_1, X_2, \dots, X_n)$

Likelihood question:

How likely is the sample $(X_1, X_2, \dots, X_n)$ given the parameter θ?

this follows from generic definition when X_i are iid.

Likelihood function, $L(\theta)$: this is the definition of $L(\theta)$ in all scenarios

$$L(\theta) = f(X_1, X_2, \dots, X_n|\theta) = \boxed{\prod_{i=1}^{n} f(X_i|\theta)}$$

This is just a product, since X_i are iid.

Maximum Likelihood Estimator

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

def The **Maximum Likelihood Estimator (MLE)** of θ is the value of θ that maximizes $L(\theta)$. $\longrightarrow$ i.e. maximizes the likelihood of the observed data.

$$\theta_{MLE} = \arg\max_{\theta} \; L(\theta)$$

Maximum Likelihood Estimator

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

def The **Maximum Likelihood Estimator (MLE)** of θ is the value of θ that maximizes $L(\theta)$.

$$\theta_{MLE} = \arg\max_{\theta} \boxed{L(\theta)}$$

Likelihood of your sample

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

For continuous X_i, $f(X_i|\theta)$ is PDF, and for discrete X_i, $f(X_i|\theta)$ is PMF

Maximum Likelihood Estimator

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

def The **Maximum Likelihood Estimator (MLE)** of θ is the value of θ that maximizes $L(\theta)$.

$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

The argument θ
that maximizes $L(\theta)$

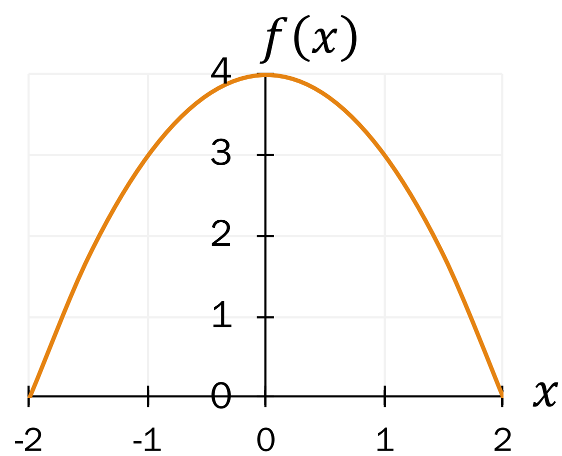
argmax and log likelihood

New function: arg max

$$\arg\max_x f(x)$$

The argument x that maximizes the function $f(x)$.

Let $f(x) = -x^2 + 4$, where $-2 < x < 2$.



1. $\max_x f(x)$?

$$= 4$$

2. $\arg\max_x f(x)$?

$$= 0$$

Argmax properties

$$\arg\max_x f(x)$$

The argument x that maximizes the function $f(x)$.

$$= \arg\max_x \log f(x)$$

(log is an increasing function: $x < y \Leftrightarrow \log x < \log y$)

$$= \arg\max_x (c \log f(x))$$

for any positive constant c

($x < y \Leftrightarrow c \log x < c \log y$)

Finding the argmax with calculus

$$\hat{x} = \arg\max_{x} f(x)$$

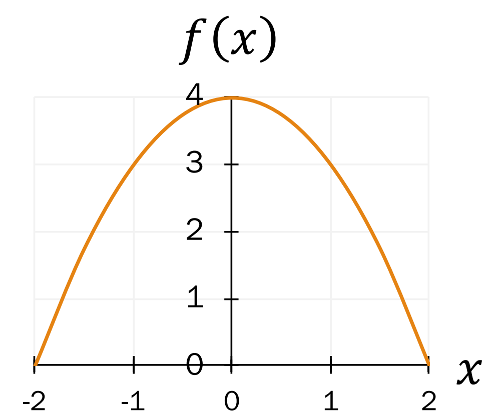
Let $f(x) = -x^2 + 4$,
where $-2 < x < 2$.

Differentiate w.r.t. argmax's argument

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^2 + 4) \quad = 2x$$

Set to 0 and solve

$$2x = 0 \qquad \Rightarrow \qquad \hat{x} = 0$$



Make sure $\hat{x}$ is a maximum

- Check $f(\hat{x} \pm \epsilon) < f(\hat{x})$
- Often ignored in expository derivations
- We'll ignore it here too
(and won't require it in class)

Maximum Likelihood Estimator

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$, drawn from a distribution $f(X_i|\theta)$.

$$L(\theta) = \prod_{i=1}^{n} f(X_i|\theta)$$

θ_{MLE} maximizes the likelihood of our sample, $L(\theta)$:

$$\theta_{MLE} = \arg\max_{\theta} \; L(\theta)$$

θ_{MLE} also maximizes the **log-likelihood function, $LL(\theta)$**:

$$\theta_{MLE} = \arg\max_{\theta} \; LL(\theta)$$

$$LL(\theta) = \log L(\theta) = \log\left(\prod_{i=1}^{n} f(X_i|\theta)\right) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

$LL(\theta)$ is often easier to differentiate than $L(\theta)$.

MLE: Bernoulli

Computing the MLE

$$\theta_{MLE} = \arg\max_{\theta} LL(\theta)$$

General approach for finding θ_{MLE} , the MLE of θ:

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|\theta)$$

2. Differentiate $LL(\theta)$ w.r.t. (each) θ

$$\frac{\partial LL(\theta)}{\partial \theta}$$

To maximize:
$$\frac{\partial LL(\theta)}{\partial \theta} = 0$$

3. Solve resulting equations

(algebra or computer)

4. Make sure derived $\hat{\theta}_{MLE}$ is a maximum
   - Check $LL(\theta_{MLE} \pm \epsilon) < LL(\theta_{MLE})$
   - Often ignored in expository derivations
   - We'll ignore it here too (and won't require it in class)

$LL(\theta)$ is often easier to differentiate than $L(\theta)$.

Maximum Likelihood with Bernoulli

Consider a sample of n iid RVs $X_1, X_2, \dots, X_n$.

What is $\theta_{MLE} = p_{MLE}$?

- Let $X_i \sim \text{Ber}(p)$.

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p)$$

?

$$f(X_i|p) = \begin{cases} p & \text{if } X_i = 1 \\ 1-p & \text{if } X_i = 0 \end{cases}$$

function as expressed is not differentiable! not what we want! :(

2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

3. Solve resulting equations

Maximum Likelihood with Bernoulli

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = p_{MLE}$?

- Let $X_i \sim \text{Ber}(p)$.
- $f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p)$$

$$f(X_i|p) = \begin{cases} p & \text{if } X_i = 1 \\ 1-p & \text{if } X_i = 0 \end{cases}$$

2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

$f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$ where $X_i \in \{0,1\}$

expanded $\begin{cases} X_i = 1? \quad f(X_i=1|p) = p^1(1-p)^{1-1} = p^1(1-p)^0 = p \\ X_i = 0? \quad f(X_i=0|p) = p^0(1-p)^{1-0} = p^0(1-p)^1 = 1-p \end{cases}$

3. Solve resulting equations

✅
- Is differentiable with respect to p
- Valid PMF over discrete domain

Maximum Likelihood with Bernoulli

$\log ab = \log a + \log b$
$\log c^d = d \log c$
properties of log

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = p_{MLE}$?

- Let $X_i \sim \text{Ber}(p)$.
- $f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log f(X_i|p) = \sum_{i=1}^{n} \underbrace{\log(p^{X_i}(1-p)^{1-X_i})}_{\log p^{X_i} + \log (1-p)^{1-X_i}}$$

2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

$$= \sum_{i=1}^{n} [X_i \log p + (1 - X_i) \log(1-p)]$$

$$\log p \sum_{i=1}^{n} X_i + \log(1-p) \sum_{i=1}^{n} 1 - \log(1-p) \sum_{i=1}^{n} X_i$$

3. Solve resulting equations

$$= Y(\log p) + (n - Y)\log(1-p), \text{ where } Y = \sum_{i=1}^{n} X_i$$

Maximum Likelihood with Bernoulli

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = p_{MLE}$?

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2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

$$\frac{\partial LL(\theta)}{\partial p} = Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0$$

3. Solve resulting equations

Maximum Likelihood with Bernoulli

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

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2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

$$\frac{\partial LL(\theta)}{\partial p} = Y\frac{1}{p} + (n-Y)\frac{-1}{1-p} = 0$$

$\frac{Y}{p} = \frac{n-Y}{1-p}$

$Y - \cancel{Yp} = np - \cancel{Yp} \rightarrow p = \frac{Y}{n}$

3. Solve resulting equations

$$p_{MLE} = \frac{1}{n}Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

MLE of the Bernoulli parameter, p_{MLE}, is the unbiased estimate of the mean, $\bar{X}$ (sample mean)

Quick check

- You draw n iid random variables $X_1, X_2, \ldots, X_n$ from the distribution F, yielding the following sample:

$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

- Suppose distribution $F = \text{Ber}(p)$ with unknown parameter p.

1. What is p_{MLE}, the MLE of the parameter p?

   A. 1.0
   B. 0.5
   C. 0.8
   D. 0.2
   E. None/other

$$p_{MLE} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Quick check

- You draw n iid random variables $X_1, X_2, \ldots, X_n$ from the distribution F, yielding the following sample:

$$[0, 0, 1, 1, 1, 1, 1, 1, 1, 1] \qquad (n = 10)$$

- Suppose distribution $F = \text{Ber}(p)$ with unknown parameter p.

1. What is p_{MLE}, the MLE of the parameter p? C. 0.8
2. What is the likelihood $L(\theta)$ of this specific sample?

$f(X_i|p) = p^{X_i}(1-p)^{1-X_i}$ where $X_i \in \{0,1\}$

$$L(\theta) = \prod_{i=1}^{n} f(X_i|p) \quad \text{where } \theta = p$$

$$= p^8(1-p)^2 = 0{,}8^8 \, 0{,}2^2 = 0{,}0067$$

MLE: Poisson and Uniform

Maximum Likelihood with Poisson

Consider a sample of n iid RVs $X_1, X_2, \dots, X_n$.

What is $\theta_{MLE} = \lambda_{MLE}$?

recall that $\log ab = \log a + \log b$

$\log \frac{a}{b} = \log a - \log b$

- Let $X_i \sim \text{Poi}(\lambda)$.
- PMF: $f(X_i|\lambda) = \dfrac{e^{-\lambda}\lambda^{X_i}}{X_i!}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n} (-\lambda \log e + X_i \log \lambda - \log X_i!)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

(using natural log, $\ln e = 1$)

Maximum Likelihood with Poisson

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = \lambda_{MLE}$?

- Let $X_i \sim \text{Poi}(\lambda)$.
- PMF: $f(X_i|\lambda) = \dfrac{e^{-\lambda}\lambda^{X_i}}{X_i!}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n}(-\lambda \log e + X_i \log \lambda - \log X_i!)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n}\log(X_i!)$$

(using natural log, $\ln e = 1$)

2. Differentiate $LL(\theta)$ w.r.t. (each) θ, set to 0

$$\frac{\partial LL(\theta)}{\partial \lambda} = ? \quad \frac{d}{d\lambda}(-n\lambda) + \frac{d}{d\lambda}\log\lambda\sum_{i=1}^{n}X_i + \overbrace{\frac{d}{d\lambda}\sum_{i=1}^{n}\log X!}^{0}$$

A. $-n + \dfrac{1}{\lambda}\displaystyle\sum_{i=1}^{n} X_i + n\log\lambda - \sum_{i=1}^{n}\frac{1}{X_i!}\cdot\frac{\partial X_i!}{\partial X_i}$

B. $-n + \dfrac{1}{\lambda}\displaystyle\sum_{i=1}^{n} X_i$

C. None/other/don't know

Maximum Likelihood with Poisson

Consider a sample of n iid RVs $X_1, X_2, \ldots, X_n$.

What is $\theta_{MLE} = \lambda_{MLE}$?

- Let $X_i \sim \text{Poi}(\lambda)$.
- PMF: $f(X_i|\lambda) = \dfrac{e^{-\lambda}\lambda^{X_i}}{X_i!}$

1. Determine formula for $LL(\theta)$

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{e^{-\lambda}\lambda^{X_i}}{X_i!}\right) = \sum_{i=1}^{n}(-\lambda \log e + X_i \log \lambda - \log X_i!)$$

$$= -n\lambda + \log(\lambda)\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \log(X_i!)$$

(using natural log, $\ln e = 1$)

2. Differentiate $LL(\theta)$ w.r.t. (each) θ, set to 0

$$\frac{\partial LL(\theta)}{\partial \lambda} = -n + \frac{1}{\lambda}\sum_{i=1}^{n} X_i = 0$$

$\Rightarrow$ $\frac{1}{\lambda}\sum_{i=1}^{n} X_i = n$

3. Solve resulting equations

$$\lambda_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

MLE of the Poisson parameter, λ_{MLE}, is the unbiased estimate of the mean, $\bar{X}$ (sample mean)

Quick check

1. A particular experiment can be modeled as a Poisson RV with parameter λ, in terms of events/minute.
   Collect data: observe 53 events over the next 10 minutes. What is λ_{MLE}? $\lambda_{MLE} = 5.3$

$$\lambda_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

sample: $(X_1 = x_1, X_2 = x_2, \ldots, X_{10} = x_{10})$

$\sum_{i=1}^{10} x_i = 53 \Rightarrow \lambda_{MLE} = \frac{1}{10} 53 = 5.3$

2. Is the Bernoulli MLE an unbiased estimator of the Bernoulli parameter p? ✅ $X \sim Ber(p)$

$E[p_{MLE}] = p$?

yes! $E[\frac{1}{n}\sum_{i=1}^{n} X_i] = E[\bar{X}] = \mu = p$

3. Is the Poisson MLE an unbiased estimator of the Poisson variance? ✅ $X \sim Poi(\lambda)$ $E[\lambda_{MLE}] = E[\bar{X}] = \lambda = \sigma^2$

4. What does unbiased mean?

Unbiased: If you could repeat your experiment, on average you would get what you are looking for.

E[estimator] = the truth

Maximum Likelihood with Uniform

Consider a sample of n iid random variables $X_1, X_2, \dots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$f(X_i|\alpha,\beta) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } \alpha \le x_i \le \beta \\ 0 & \text{otherwise} \end{cases}$$

1. Determine formula for $L(\theta)$

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta-\alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \dots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

$LL(\theta) = n \log \frac{1}{\beta-\alpha}$ provided all x_i are such that $\alpha \le x_i \le \beta$

2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0

A. Great, let's do it
B. Differentiation is hard
C. Constraint $\alpha \le x_1, x_2, \dots, x_n \le \beta$ makes differentiation hard

Maximum Likelihood with Uniform: Sample

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Suppose $X_i \sim \text{Uni}(0,1)$.
You observe data:

$$[0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75]$$

Which parameters would give you maximum $L(\theta)$?

A. $\text{Uni}(\alpha = 0 \quad, \beta = 1 \quad)$

B. $\text{Uni}(\alpha = 0.15, \beta = 0.75)$

C. $\text{Uni}(\alpha = 0.15, \beta = 0.70)$

Maximum Likelihood with Uniform: Sample

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\frac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

Suppose underlying $X_i \sim \text{Uni}(0,1)$.

You observe data: $[0.15, 0.20, 0.30, 0.40, 0.65, 0.70, 0.75]$

Which parameters would give you maximum $L(\theta)$?

A. $\text{Uni}(\alpha = 0 \quad, \beta = 1 \quad)$ $\quad (1)^7 = 1$ on behalf of original parameters

B. $\text{Uni}(\alpha = 0.15, \beta = 0.75)$ $\quad \left(\frac{1}{0.6}\right)^7 = 59.5$

C. $\text{Uni}(\alpha = 0.15, \beta = 0.70)$ $\quad \left(\frac{1}{0.55}\right)^6 \cdot 0 = 0$ on behalf of 0.75

Original parameters may not yield maximum likelihood.

Maximum Likelihood with Uniform

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

Let $X_i \sim \text{Uni}(\alpha, \beta)$.

$$L(\theta) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right)^n & \text{if } \alpha \le x_1, x_2, \ldots, x_n \le \beta \\ 0 & \text{otherwise} \end{cases}$$

θ_{MLE}: $\alpha_{MLE} = \min(x_1, x_2, \ldots, x_n)$ $\beta_{MLE} = \max(x_1, x_2, \ldots, x_n)$

Intuition:

- Want interval size $(\beta - \alpha)$ to be as small as possible to maximize likelihood function per datapoint
- Need to make sure all observed data is in interval (if not, then $L(\theta) = 0$)

(demo)

Small samples = problems with MLE

Maximum Likelihood Estimator θ_{MLE} :

$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

- Best explains data we have seen
- Does not attempt to generalize to data not yet observed.

✅ In many cases, $\mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n} X_i$ Sample mean (MLE for Bernoulli p, Poisson λ, Normal μ)

- Unbiased ($E[\mu_{MLE}] = \mu$ regardless of size of sample, n)

⚠️ For some cases, like Uniform: $\alpha_{MLE} \geq \alpha$, ← α of underlying distribution (presumably unknown) $\beta_{MLE} \leq \beta$ ← β, same story

- Biased. Problematic for small sample size
- Example: If $n = 1$ then $\alpha = \beta$, yielding an invalid distribution

Properties of MLE

Maximum Likelihood Estimator θ_{MLE}:

$$\theta_{MLE} = \arg\max_{\theta} L(\theta)$$

- Best explains data we have seen
- Does not attempt to generalize to data not yet observed.

- Often used when sample size n is large relative to parameter space
- Potentially biased (though asymptotically less so, as $n \to \infty$)
- Consistent: $\lim_{n\to\infty} P\left(\left|\hat{\theta} - \theta\right| < \varepsilon\right) = 1$ where $\varepsilon > 0$

  As $n \to \infty$ (i.e., more data), probability that $\hat{\theta}$ significantly differs from θ is zero

MLE: Gaussian

Maximum Likelihood with Normal

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

- Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

$$f(X_i | \underline{\underline{\mu, \sigma^2}}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$$

two parameters!

What is $\theta_{MLE} = (\mu_{MLE}, \sigma^2_{MLE})$?

1. Determine formula for $LL(\theta)$
2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0
3. Solve resulting equations

$$LL(\theta) = \sum_{i=1}^{n} \log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}\right) = \sum_{i=1}^{n} \left[-\log\left(\sqrt{2\pi}\sigma\right) - (X_i-\mu)^2/(2\sigma^2)\right]$$

(using natural log)

$$= -\sum_{i=1}^{n} \log\left(\sqrt{2\pi}\sigma\right) - \sum_{i=1}^{n} \left[(X_i-\mu)^2/(2\sigma^2)\right]$$

Maximum Likelihood with Normal

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

- Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

$$f(X_i|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$$

What is $\theta_{MLE} = (\mu_{MLE}, \sigma^2_{MLE})$?

1. Determine formula for $LL(\theta)$
2. Differentiate $LL(\theta)$ wrt (each) θ, set to 0
3. Solve resulting equations

$$LL(\theta) = -\sum_{i=1}^{n} \log(\sqrt{2\pi}\sigma) - \sum_{i=1}^{n} [(X_i-\mu)^2/(2\sigma^2)]$$

with respect to μ

$$\frac{\partial LL(\theta)}{\partial \mu} = \sum_{i=1}^{n} [2(X_i-\mu)/(2\sigma^2)]$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\mu) = 0$$

Maximum Likelihood with Normal

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1. Determine formula for $LL(\theta)$
2. Differentiate $LL(\theta)$ w.r.t. (each) θ, set to 0
3. Solve resulting equations

$$LL(\theta) = -\sum_{i=1}^{n} \log\left(\sqrt{2\pi}\sigma\right) - \sum_{i=1}^{n} \left[(X_i - \mu)^2/(2\sigma^2)\right]$$

with respect to μ

$$\frac{\partial LL(\theta)}{\partial \mu} = \sum_{i=1}^{n} \left[2(X_i - \mu)/(2\sigma^2)\right]$$

$$= \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) = 0$$

with respect to σ

$$\frac{\partial LL(\theta)}{\partial \sigma} = -\sum_{i=1}^{n} \frac{1}{\sigma} + \sum_{i=1}^{n} 2(X_i - \mu)^2/(2\sigma^3)$$

$$= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (X_i - \mu)^2 = 0$$

Maximum Likelihood with Normal

Consider a sample of n iid random variables $X_1, X_2, \dots, X_n$.

- Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$.

$$f(X_i | \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i - \mu)^2/(2\sigma^2)}$$

What is $\theta_{MLE} = (\mu_{MLE}, \sigma^2_{MLE})$?

3. Solve resulting equations

Two equations, two unknowns:

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) = 0 \qquad -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (X_i - \mu)^2 = 0$$

First, solve for μ_{MLE}:

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} X_i - \frac{1}{\sigma^2} \sum_{i=1}^{n} \mu = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} X_i = n\mu \quad \Rightarrow \quad \mu_{MLE} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

unbiased

another sample mean!

Maximum Likelihood with Normal

Consider a sample of n iid random variables $X_1, X_2, \ldots, X_n$.

- Let $X_i \sim \mathcal{N}(\mu, \sigma^2)$. $\quad f(X_i|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(X_i-\mu)^2/(2\sigma^2)}$

What is $\theta_{MLE} = (\mu_{MLE}, \sigma^2_{MLE})$?

3. Solve resulting equations

Two equations, two unknowns: $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu) = 0 \qquad -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)^2 = 0$

First, solve for μ_{MLE}: $\frac{1}{\sigma^2}\sum_{i=1}^{n}X_i - \frac{1}{\sigma^2}\sum_{i=1}^{n}\mu = 0 \Rightarrow \sum_{i=1}^{n}X_i = n\mu \Rightarrow \mu_{MLE} = \frac{1}{n}\sum_{i=1}^{n}X_i$

unbiased

Next, solve for σ_{MLE}: $\frac{1}{\sigma^3}\sum_{i=1}^{n}(X_i-\mu)^2 = \frac{n}{\sigma} \Rightarrow \sum_{i=1}^{n}(X_i-\mu)^2 = \sigma^2 n \Rightarrow \sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_{MLE})^2$

biased!!