Homework 1

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Problem 1(3.6).
Since $\{B_i\}_{i\in I}$ is a partition of Ω, $\cup_{i\in I}B_i = \Omega$
$A\cap B_i, A\cap B_j, \forall i,j \in I$ are disjoint,
so by finite additivity, we know
$\sum_{i\in I} P(A\cap B_i) = P(\cup_{i\in I}\{A\cap B_i\}) = P(A\cap\{\cup_{i\in I}B_i\})$
$\Rightarrow \sum_{i\in I} P(A\cap B_i) = P(A\cap\Omega) = P(A)$, since $A\subset\Omega$

Problem 1(3.8).
Since $\{E_1, E_2, ..., E_n\}$ is a collection of disjoint events,
$E_1^c, E_2^c, ...E_n^c$ are disjoint, and $\{\cup_{k=1}^n E_k\}^c = \cap_{k=1}^n E_k^c$
So $P(\cup_{k=1}^n E_k) = 1-P(\{\cup_{k=1}^n E_k\}^c) = 1-P(\cap_{k=1}^n E_k^c) = 1-\Pi_{k=1}^n P(E_k^c) = 1-\Pi_{k=1}^n(1-P(E_k))$

Problem 1(3.11).
By Bayes's Rule, $P(\text{s=crime}|\text{s tested }+) = \frac{P(\text{s tested }+|\text{s=crime})\times P(\text{s=crime})}{P(\text{s tested }+)}$
$= \frac{1\times(1/250{,}000{,}000)}{1\times(1/250{,}000{,}000)+(1/3{,}000{,}000)\times(1-1/250{,}000{,}000)} = 0.0119$

Problem 1(3.12).
Without loss of generality, assume that the door chosen is door 1, and the door opened is door 2. Denote the event that door 1 has car by A, the event that it is shown that door 2 has goat by B, and the event that door 3 has goat by C. By Bayes' rule,
$P(A|B) = \frac{P(B|A)\times P(A)}{P(B)} = \frac{(1/2)\times(1/3)}{(1/3)\times(1/2)+(1/3)\times 0+(1/3)\times 1} = \frac{1}{3}$
$P(C|B) = \frac{2}{3}$
So it is better to switch to the other door.
When there are 10 doors, assume that the door chose is 1, and the doors opened are doors 2-9, then
$P(\text{1 has car}|\text{door 2-9 has been chosen by Monty}) = \frac{(1/9)\times(1/10)}{(1/10)\times(1/9)+8\times(1/10)\times(1/9)} = \frac{1}{10}$
$P(\text{door 10, the unchosen door, has car}|\text{door 2-9 has been chosen by Monty}) = \frac{9}{10}$

Problem 1(3.16).
Var$[X]$=$E[(X-\mu)^2] = E[X^2-2X\mu+\mu^2] = E[X^2]-2\mu E[X]+\mu^2$, by linearity of expectation
Since $E[X]=\mu$, Var$[X]$=$E[X^2]-2\mu^2+\mu^2 = E[X^2]-\mu^2$

Problem 1(3.33).
For binomial distribution, $E[B] = np, Var[B] = np(1-p)$
Let the random variable $Y_n = \frac{B}{n}$, then $E[Y_n] = \frac{1}{n}E[B] = p, Var[Y_n] = \frac{1}{n^2}Var[B] = \frac{p(1-p)}{n}$
By Chebyshev's inequality, for a random variable X, $P(|X - E[X]| \geq \epsilon) \leq \frac{Var[X]}{\epsilon^2}$
$\Rightarrow P(|\frac{B}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}$

Problem 1(3.36).
By Central Limit Theorem, we know $\frac{S-np}{\sqrt{np(1-p)}} \rightarrow N(0,1)$
$Z_{5500} = \frac{5500-5000}{\sqrt{6242\times0.801\times(1-0.801)}} = 15.8513$
$\Rightarrow P(S > 5500) \approx 0$

Problem 2(a).
Suppose we toss 3 coins. Let A be the event that coin 1 and coin 2 have the same sides up, B be the event that coin 2 and coin 3 have the same sides up, C be the event that coin 1 and coin 3 have the same sides up.
Then $P(A) = P(B) = P(C) = \frac{1}{2}, P(A\cap B) = P(B\cap C) = P(A\cap C) = \frac{1}{4} = P(A)P(B) = P(B)P(C) = P(A)P(C)$
However, $P(A\cap B\cap C) = \frac{1}{4} \neq P(A)P(B)P(C)$

Problem 2(b).
Let $P(d) = \frac{1}{8}, \forall d \in \{1,2,3,...,8\}$
Let $A = \{1,2,3,4\}, B = \{1,2,5,6\}, C = \{1,3,7,8\}$
So $P(A) = P(B) = P(C) = \frac{1}{2}$
$P(A\cap B) = \frac{1}{4} = P(A)P(B), P(A\cap C) = \frac{1}{4} = P(A)P(C), P(A\cap B\cap C) = \frac{1}{8} = P(A)P(B)P(C)$
$P(B\cap C) = \frac{1}{8} \neq P(B)P(C)$

Problem 3.
Benford's Law states that $P(d) = log_{10}(1+\frac{1}{d}), d \in \{1,2,...,9\}$
1. $\forall d \in \{1,2,...,9\}, 0 < log_{10}(1+\frac{1}{d}) < 1$
2. $\sum_{d=1}^{9} P(d) = \sum_{d=1}^{9} log_{10}(1+\frac{1}{d}) = log_{10}\Pi_{d=1}^{9}(\frac{d+1}{d}) = log_{10}10 = 1$
We can also impose finite additivity, so Benford's Law is a well-defined discrete probability distribution.

Problem 4(a).
$P(\text{tail appears for the first time at the nth flip}) = (\frac{1}{2})^n$
So $E[X] = \sum_{n=1}^{\infty}(\frac{1}{2})^n 2^n = \sum_{n=1}^{\infty} 1 = \infty$

Problem 4(b).
$E[\ln X] = \sum_{n=1}^{\infty}(\frac{1}{2})^n \ln(2^n) = \ln 2\sum_{n=1}^{\infty} n(\frac{1}{2})^n$
$\sum_{n=1}^{\infty} n(\frac{1}{2})^n = \frac{1/2}{(1-1/2)^2} = 2$, so $E[2\ln X] = 2\ln 2$

Problem 5.
Suppose the interest rate is x/unit currency in both countries, for a specified period of time.

Suppose the US investor invests one unit in USD, she is expected to get (1+x) USD after this period of time.
Suppose the US investor invests one unit in CHF, she is expected to get $(1+x) \times 1.25 \times 0.5 + (1+x) \times \frac{1}{1.25} \times 0.5 = 1.025(1+x)$ USD after this period of time.
So the US investor should invest in CHF. The reasoning is the same for the Swiss investor, so she should invest in USD.

Problem 6(a).
Let X be a random variable such that $P(X=x) = \frac{3}{2x^{\frac{5}{2}}}, \forall x \geq 1, P(X=x) = 0, \forall x < 1$.
$E[X] = \int_1^\infty xP(x) = 3 < \infty$
$E[X^2] = \int_1^\infty x^2 P(x) = \infty$

Problem 6(b).
Let X be a standard normal variable. Let Y be a random variable such that $P(Y = X - \frac{1}{3}) = \frac{2}{3}, P(Y = X+1) = \frac{1}{3}$.
Then $P(X > Y) = \frac{2}{3}$, and $E[X] = 0, E[Y] = \frac{2}{3} \times (E[X] - \frac{1}{3}) + \frac{1}{3} \times (E[X]+1) > 0$

Problem 6(c).
Let X be a random variable such that $P(X=1) = \frac{1}{2}, P(X=-1) = \frac{1}{2}$
Let Y be a random variable such that $P(Y = \frac{1}{2}) = \frac{1}{2}, P(X = -\frac{1}{2}) = \frac{1}{2}$
Let Z be the random variable such that $P(Z=0) = 1$
Then $P(X > Y) > 0, P(Y > Z) > 0, P(X > Z) > 0$, So $P(X > Y)P(Y > Z)P(X > Z) > 0$, and $E[X] = E[Y] = E[Z] = 0$.

Problem 7(a).
The statement is true. The cumulative distribution function of Y is
$\Phi(y) = P(XZ < y) = P(XZ < y|Z=1)P(Z=1) + P(XZ < y|Z=-1)P(Z=-1) = \frac{1}{2}P(X < y) + \frac{1}{2}P(-X < y) = \frac{1}{2}P(X < y) + \frac{1}{2}P(X > -y) = P(X < y) = \Phi(x)$, since the Normal Distribution is symmetric.
So Y and X are the same distribution $\Rightarrow Y \sim N(0,1)$

Problem 7(b).
The statement is true.
$Y = XZ \Rightarrow |Y| = |XZ| = |X||Z| = |X|$, since Z is either 1 or -1.
$P(|X| = |Y|) = 1$

Problem 7(c).
The statement is true.
For example, $P(Y=1|X=1) = \frac{1}{2} \neq P(Y=1)$,

Problem 7(d).
The statement is true.
$Cov[X,Y] = E[XY] - E[X]E[Y] = E[XY] = E[X^2 Z]$
Since X, Z are independent, $E[X^2 Z] = E[X^2]E[Z] (*)$
Since $E[Z] = \frac{1}{2}(1 + (-1)) = 0, (*) = 0$

Problem 7(e).
The statement is false.
As seen in previous parts, X, Y are both normally distributed variables and their covariance is equal to zero. However, they are dependent.

Problem 8.
We know $m \in [0,1], M \in [0,1]$, so for $x \in [0,1]$ $P(m < x) = 1 - P(m \geq x) = 1 - P(X_1 \geq x, X_2 \geq x, ...X_n \geq x) = 1 - \Pi_1^n P(X_i \geq x)$, since the variables are independent.
Since $P(X_i \geq x) = 1 - x$, $P(m < x) = 1 - (1-x)^n$

$P(M < x) = P(X_1 < x, X_2 < x, ...X_n < x) = \Pi_1^n P(X_i < x)$, since the variables are independent.
Since $P(X_i < x) = x$, $P(M < x) = x^n$

$P(m = x) = n(1-x)^{n-1}$
$P(M = x) = nx^{n-1}$
$E[m] = \int_0^1 xn(1-x)^{n-1}dx = \frac{1}{n+1}$
$E[M] = \int_0^1 xnx^{n-1}dx = \frac{n}{n+1}$

Problem 9(a).
Denote the number of good states by X, then $E[X] = 1000 \times \frac{1}{2} = 500, Var[X] = 1000 \times \frac{1}{2} \times \frac{1}{2} = 250$
By Chebyshev Inequality and Central Limit Theorem, $\frac{X-500}{5\sqrt{10}} \sim$ N(0,1)
So $P(X$ differs from 500 by at most 2%$) = 1 - 2P(X > 510)$
$Z = \frac{510-500}{5\sqrt{10}} = 0.6324$
$\Rightarrow P(X$ differs from 500 by at most 2%$) = 1 - 2 \times 0.2636 = 1 - 0.9681 = 0.4728$

Problem 9(b).
Let Y be the proportion of good states. By Central Limit Theorem, $Y \sim N(\frac{1}{2}, \frac{1}{4n})$. By Chebyshev's inequality, $P(|Y - 0.5| \geq 0.5 \times 0.01) \leq \frac{1}{(0.005)^2 4n}$. So $\frac{1}{(0.005)^2 4n} \leq 0.01$
$\Rightarrow n \geq 1,000,000$

Problem 10.
Since $e^{\theta X}$ is a differentiable convex function, $E[e^{\theta X}] \geq e^{E[\theta X]}$, by Jensen's Inequality.
$\Rightarrow (e^{E[X]})^\theta = e^{E[\theta X]} \leq 1$
Since $E[X] < 0, e^{E[X]} < 1$,for the inequality above to hold, $\theta > 0$