

Problem Set 5

OSM Lab-Math

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Problem 1 (7.1)

$\forall x, y \in \text{conv}(S)$, x, y can be written as $\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j$, with each $x_i, y_j \in S, \alpha_i, \beta_j > 0$, and $\sum_{i=1}^n \alpha_i = \sum_{j=1}^m \beta_j = 1$.

$\forall \lambda$ that satisfies $0 \leq \lambda \leq 1$, $\lambda x + (1 - \lambda)y = \lambda \sum_{i=1}^n \alpha_i x_i + (1 - \lambda) \sum_{j=1}^m \beta_j y_j$
Since $\lambda \sum_{i=1}^n \alpha_i + (1 - \lambda) \sum_{j=1}^m \beta_j = 1$, and $x_i, y_j \in S$, by the definition of convex hull, we know $\lambda x + (1 - \lambda)y \in \text{conv}(S)$. So $\text{conv}(S)$ is a convex set.

Problem 2 (7.2)

(i)

Let V be a hyperplane that satisfies $\forall x \in V, \langle a, x \rangle = b$ ($a \in V, a \neq 0, b \in \mathbb{R}$).

$\forall x, y \in V, \lambda$ s.t. $0 \leq \lambda \leq 1$,

$$\langle a, \lambda x + (1 - \lambda)y \rangle = \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle = \lambda b + (1 - \lambda)b = b$$

$\Rightarrow \lambda x + (1 - \lambda)y \in V$, V is convex.

(ii)

Let H be a half space that satisfies $\forall x \in H, \langle a, x \rangle \leq b$ ($a \in H, a \neq 0, b \in \mathbb{R}$).

$\forall x, y \in H, \lambda$ s.t. $0 \leq \lambda \leq 1$,

$$\langle a, \lambda x + (1 - \lambda)y \rangle = \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle \leq \lambda b + (1 - \lambda)b = b$$

$\Rightarrow \lambda x + (1 - \lambda)y \in H$, H is convex.

Problem 3 (7.4)

(i)

$$\begin{aligned} & \|x - p\|^2 + \|p - y\|^2 + 2 \langle x - p, p - y \rangle \\ &= \langle x - p, x - p \rangle + \langle p - y, p - y \rangle + 2 \langle x - p, p - y \rangle \\ &= (\langle x - p, x - p \rangle + \langle x - p, p - y \rangle) + (\langle p - y, p - y \rangle + \langle x - p, p - y \rangle) \\ &= \langle x - p, x - y \rangle + \langle x - y, p - y \rangle = \langle x - y, x - y \rangle = \|x - y\|^2 \end{aligned}$$

(ii)

Suppose $\langle x - p, p - y \rangle \geq 0, \forall y \in C$, then

$$\begin{aligned} \|x - y\|^2 &= \|x - p\|^2 + \|p - y\|^2 + 2 \langle x - p, p - y \rangle \geq \|x - p\|^2 \\ \Rightarrow \|x - y\| &\geq \|x - p\| \end{aligned}$$

(iii)

Since C is convex, $p, y \in C, \lambda \in [0, 1]$, we know $z \in C$.

By (i),

$$\begin{aligned} \|x - z\|^2 &= \|x - p\|^2 + \|p - z\|^2 + 2 \langle x - p, p - z \rangle \\ &= \|x - p\|^2 + 2 \langle x - p, \lambda y + (1 - \lambda)p \rangle + \|\lambda y + (1 - \lambda)p - p\|^2 \\ &= \|x - p\|^2 + 2\lambda \langle x - p, p - y \rangle + \lambda^2 \|y - p\|^2 \end{aligned}$$

(iv)

Let p be the projection of x onto C . By (iii), we know

$$\frac{\|x-z\|^2 - \|x-p\|^2}{\lambda} = 2 \langle x-p, p-y \rangle + \lambda \|y-p\|^2, \text{ if } \lambda \in (0, 1]$$

Since p is a projection, by definition, $\|x-z\| \geq \|x-p\|$, so the left hand side of the equation ≥ 0 .

$$\Rightarrow \langle x-p, p-y \rangle \geq -\frac{\lambda \|y-p\|^2}{2} (\star), \forall \lambda \in (0, 1]$$

Suppose that $\langle x-p, p-y \rangle \geq 0$ does not hold for some $y \in C$, then we know

$\exists y_0 \in C$ s.t. $\langle x-p, p-y_0 \rangle = a < 0$. However, as $\lambda \rightarrow 0$, $-\frac{\lambda \|y_0-p\|^2}{2} \rightarrow 0$, so $-\frac{\lambda \|y_0-p\|^2}{2} < a$ must be true for some values of λ . This contradicts the fact that (\star) holds true for all $\lambda \in (0, 1]$.

$$\Rightarrow \langle x-p, p-y \rangle \geq 0$$

The complete proof:

Suppose p is the projection of x onto C , then by (iv), $\langle x-p, p-y \rangle \geq 0$.

Suppose $\langle x-p, p-y \rangle \geq 0$, then by (ii), $\|x-y\| > \|x-p\|$, so p is a projection of x onto C .

Problem 4 (7.6)

Denote the set $\{x \in \mathbb{R}^n | f(x) \leq c\}$ by A . $\forall x, y \in A, \lambda \in [0, 1]$, since f is a convex function

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda c + (1-\lambda)c = c.$$

$\lambda x + (1-\lambda)y \in A$, so A is convex.

Problem 5 (7.7)

$\forall x, y, t \in [0, 1]$, since all f_i 's are convex

$$\begin{aligned} f(tx + (1-t)y) &= \sum_{i=1}^k \lambda_i f_i(tx + (1-t)y) \\ &\leq \sum_{i=1}^k \lambda_i [tf_i(x) + (1-t)f_i(y)] \\ &= t \sum_{i=1}^k \lambda_i f_i(x) + (1-t) \sum_{i=1}^k \lambda_i f_i(y) = tf(x) + (1-t)f(y) \end{aligned}$$

So f is convex.

Problem 6 (7.13)

Suppose that f is not constant, then $\exists x, y, x < y$ s.t. $f(x) \neq f(y)$

Assume without loss of generality that $f(x) < f(y)$

$\forall z > y$, write $y = \frac{y-x}{z-x}z + \frac{z-y}{z-x}x$. Note that $\frac{y-x}{z-x} + \frac{z-y}{z-x} = 1$. Since f is convex,

$$f(y) \leq \frac{y-x}{z-x}f(z) + \frac{z-y}{z-x}f(x)$$

$$\Rightarrow f(z) \geq \frac{z-x}{y-x}f(y) - \frac{z-y}{y-x}f(x)$$

Note that as $z \rightarrow \infty$, the right-hand side of the inequality $\rightarrow \infty$, which contradicts

the boundedness assumption.
So f is a constant function.

Problem 7 (7.20)

Let $g(x) = f(x) - f(0)$, so $g(0) = 0$. Since f is convex and concave, g is also convex and concave. Moreover, $\forall x$, and y_1, y_2 that lie on the line joining 0 and x , we know:
 $g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \leq \lambda_1 g(x)$, since g is convex.
 $g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \geq \lambda_1 g(x)$, since g is concave.
 $\Rightarrow g(y_1) = \lambda_1 g(x)$. Similarly, $g(y_2) = \lambda_2 g(x)$

$\forall a, b \in \mathbb{R}, g(ay_1 + by_2) = (a\lambda_1 + b\lambda_2)g(x) = a\lambda_1 g(x) + b\lambda_2 g(x) = a(g(y_1)) + b(g(y_2))$
 Since x can be any point in the domain of f , we know that g is a linear transformation on its whole domain.
 $\Rightarrow g$ is affine.
 Since f is a translation of g , f is affine.

Problem 8 (7.21)

Suppose that x^* is a local minimizer for the second problem. $\forall x \neq x^*$ in a neighborhood of x^* that satisfies the constraint, we know $f(x) \geq f(x^*)$. Since ϕ is strictly increasing, $\phi \circ f(x) \geq \phi \circ f(x^*)$. So x^* is a local minimizer for the first problem.

Suppose that x^* is a local minimizer for the first problem and that it is not a local minimizer for the second problem. Then in a neighborhood of x^* , $\exists x \neq x^*, f(x) < f(x^*)$, and $\phi \circ f(x) > \phi \circ f(x^*)$. Since ϕ is strictly increasing, the aforementioned statement is impossible. So this is a contradiction. So x^* is also a local minimizer for the second problem.