

## Problem Set 5

OSM Lab-Math

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### Problem 1 (7.1)

$\forall x, y \in \text{conv}(S)$ ,  $x, y$  can be written as  $\sum_{i=1}^n \alpha_i x_i, \sum_{j=1}^m \beta_j y_j$ , with each  $x_i, y_j \in S, \alpha_i, \beta_j > 0$ , and  $\sum_1^n \alpha_i = \sum_1^m \beta_j = 1$ .

$\forall \lambda$  that satisfies  $0 \leq \lambda \leq 1$ ,  $\lambda x + (1 - \lambda)y = \lambda \sum_{i=1}^n \alpha_i x_i + (1 - \lambda) \sum_{j=1}^m \beta_j y_j$   
Since  $\lambda \sum_1^n \alpha_i + (1 - \lambda) \sum_1^m \beta_j = 1$ , and  $x_i, y_j \in S$ , by the definition of convex hull, we know  $\lambda x + (1 - \lambda)y \in \text{conv}(S)$ . So  $\text{conv}(S)$  is a convex set.

### Problem 2 (7.2)

(i)

Let  $V$  be a hyperplane that satisfies  $\forall x \in V, \langle a, x \rangle = b$  ( $a \in V, a \neq 0, b \in \mathbb{R}$ ).

$\forall x, y \in V, \lambda$  s.t.  $0 \leq \lambda \leq 1$ ,

$$\langle a, \lambda x + (1 - \lambda)y \rangle = \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle = \lambda b + (1 - \lambda)b = b$$

$\Rightarrow \lambda x + (1 - \lambda)y \in V$ ,  $V$  is convex.

(ii)

Let  $H$  be a half space that satisfies  $\forall x \in H, \langle a, x \rangle \leq b$  ( $a \in H, a \neq 0, b \in \mathbb{R}$ ).

$\forall x, y \in H, \lambda$  s.t.  $0 \leq \lambda \leq 1$ ,

$$\langle a, \lambda x + (1 - \lambda)y \rangle = \lambda \langle a, x \rangle + (1 - \lambda) \langle a, y \rangle \leq \lambda b + (1 - \lambda)b = b$$

$\Rightarrow \lambda x + (1 - \lambda)y \in H$ ,  $H$  is convex.

### Problem 3 (7.4)

(i)

$$\begin{aligned} & \|x - p\|^2 + \|p - y\|^2 + 2 \langle x - p, p - y \rangle \\ &= \langle x - p, x - p \rangle + \langle p - y, p - y \rangle + 2 \langle x - p, p - y \rangle \\ &= (\langle x - p, x - p \rangle + \langle x - p, p - y \rangle) + (\langle p - y, p - y \rangle + \langle x - p, p - y \rangle) \\ &= \langle x - p, x - y \rangle + \langle x - y, p - y \rangle = \langle x - y, x - y \rangle = \|x - y\|^2 \end{aligned}$$

(ii)

Suppose  $\langle x - p, p - y \rangle \geq 0, \forall y \in C$ , then

$$\begin{aligned} \|x - y\|^2 &= \|x - p\|^2 + \|p - y\|^2 + 2 \langle x - p, p - y \rangle \geq \|x - p\|^2 \\ \Rightarrow \|x - y\| &\geq \|x - p\| \end{aligned}$$

(iii)

Since  $C$  is convex,  $p, y \in C, \lambda \in [0, 1]$ , we know  $z \in C$ .

By (i),

$$\begin{aligned} \|x - z\|^2 &= \|x - p\|^2 + \|p - z\|^2 + 2 \langle x - p, p - z \rangle \\ &= \|x - p\|^2 + 2 \langle x - p, \lambda y + (1 - \lambda)p \rangle + \|\lambda y + (1 - \lambda)p - p\|^2 \\ &= \|x - p\|^2 + 2\lambda \langle x - p, p - y \rangle + \lambda^2 \|y - p\|^2 \end{aligned}$$

(iv)

Let  $p$  be the projection of  $x$  onto  $C$ . By (iii), we know

$$\frac{\|x-z\|^2 - \|x-p\|^2}{\lambda} = 2 \langle x-p, p-y \rangle + \lambda \|y-p\|^2, \text{ if } \lambda \in (0, 1]$$

Since  $p$  is a projection, by definition,  $\|x-z\| \geq \|x-p\|$ , so the left hand side of the equation  $\geq 0$ .

$$\Rightarrow \langle x-p, p-y \rangle \geq -\frac{\lambda \|y-p\|^2}{2} (\star), \forall \lambda \in (0, 1]$$

Suppose that  $\langle x-p, p-y \rangle \geq 0$  does not hold for some  $y \in C$ , then we know

$\exists y_0 \in C$  s.t.  $\langle x-p, p-y_0 \rangle = a < 0$ . However, as  $\lambda \rightarrow 0$ ,  $-\frac{\lambda \|y_0-p\|^2}{2} \rightarrow 0$ , so  $-\frac{\lambda \|y_0-p\|^2}{2} < a$  must be true for some values of  $\lambda$ . This contradicts the fact that  $(\star)$  holds true for all  $\lambda \in (0, 1]$ .

$$\Rightarrow \langle x-p, p-y \rangle \geq 0$$

The complete proof:

Suppose  $p$  is the projection of  $x$  onto  $C$ , then by (iv),  $\langle x-p, p-y \rangle \geq 0$ .

Suppose  $\langle x-p, p-y \rangle \geq 0$ , then by (ii),  $\|x-y\| > \|x-p\|$ , so  $p$  is a projection of  $x$  onto  $C$ .

#### Problem 4 (7.6)

Denote the set  $\{x \in \mathbb{R}^n | f(x) \leq c\}$  by  $A$ .  $\forall x, y \in A, \lambda \in [0, 1]$ , since  $f$  is a convex function

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda c + (1-\lambda)c = c.$$

$\lambda x + (1-\lambda)y \in A$ , so  $A$  is convex.

#### Problem 5 (7.7)

$\forall x, y, t \in [0, 1]$ , since all  $f_i$ 's are convex

$$\begin{aligned} f(tx + (1-t)y) &= \sum_{i=1}^k \lambda_i f_i(tx + (1-t)y) \\ &\leq \sum_{i=1}^k \lambda_i [tf_i(x) + (1-t)f_i(y)] \\ &= t \sum_{i=1}^k \lambda_i f_i(x) + (1-t) \sum_{i=1}^k \lambda_i f_i(y) = tf(x) + (1-t)f(y) \end{aligned}$$

So  $f$  is convex.

#### Problem 6 (7.13)

Suppose that  $f$  is not constant, then  $\exists x, y, x < y$  s.t.  $f(x) \neq f(y)$

Assume without loss of generality that  $f(x) < f(y)$

$\forall z > y$ , write  $y = \frac{y-x}{z-x}z + \frac{z-y}{z-x}x$ . Note that  $\frac{y-x}{z-x} + \frac{z-y}{z-x} = 1$ . Since  $f$  is convex,

$$f(y) \leq \frac{y-x}{z-x}f(z) + \frac{z-y}{z-x}f(x)$$

$$\Rightarrow f(z) \geq \frac{z-x}{y-x}f(y) - \frac{z-y}{y-x}f(x)$$

Note that as  $z \rightarrow \infty$ , the right-hand side of the inequality  $\rightarrow \infty$ , which contradicts

the boundedness assumption.  
So  $f$  is a constant function.

**Problem 7 (7.20)**

Let  $g(x) = f(x) - f(0)$ , so  $g(0) = 0$ . Since  $f$  is convex and concave,  $g$  is also convex and concave. Moreover,  $\forall x$ , and  $y_1, y_2$  that lie on the line joining 0 and  $x$ , we know:  
 $g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \leq \lambda_1 g(x)$ , since  $g$  is convex.  
 $g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \geq \lambda_1 g(x)$ , since  $g$  is concave.  
 $\Rightarrow g(y_1) = \lambda_1 g(x)$ . Similarly,  $g(y_2) = \lambda_2 g(x)$

$\forall a, b \in \mathbb{R}, g(ay_1 + by_2) = (a\lambda_1 + b\lambda_2)g(x) = a\lambda_1 g(x) + b\lambda_2 g(x) = a(g(y_1)) + b(g(y_2))$   
 Since  $x$  can be any point in the domain of  $f$ , we know that  $g$  is a linear transformation on its whole domain.  
 $\Rightarrow g$  is affine.  
 Since  $f$  is a translation of  $g$ ,  $f$  is affine.

**Problem 8 (7.21)**

Suppose that  $x^*$  is a local minimizer for the second problem.  $\forall x \neq x^*$  in a neighborhood of  $x^*$  that satisfies the constraint, we know  $f(x) \geq f(x^*)$ . Since  $\phi$  is strictly increasing,  $\phi \circ f(x) \geq \phi \circ f(x^*)$ . So  $x^*$  is a local minimizer for the first problem.

Suppose that  $x^*$  is a local minimizer for the first problem and that it is not a local minimizer for the second problem. Then in a neighborhood of  $x^*$ ,  $\exists x \neq x^*, f(x) < f(x^*)$ , and  $\phi \circ f(x) > \phi \circ f(x^*)$ . Since  $\phi$  is strictly increasing, the aforementioned statement is impossible. So this is a contradiction. So  $x^*$  is also a local minimizer for the second problem.