Problem Set 5

OSM Lab-Math
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Problem 1 (7.1)
$\forall x, y \in conv(S)$, x, y can be written as $\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{m} \beta_j y_j$, with each $x_i, y_j \in S, \alpha_i, \beta_j > 0$, and $\sum_1^n \alpha_i = \sum_1^m \beta_j = 1$.

$\forall \lambda$ that satisfies $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)y = \lambda \sum_{i=1}^{n} \alpha_i x_i + (1-\lambda) \sum_{j=1}^{m} \beta_j y_j$
Since $\lambda \sum_1^n \alpha_i + (1-\lambda) \sum_1^m \beta_j = 1$, and $x_i, y_j \in S$, by the definition of convex hull, we know $\lambda x + (1-\lambda)y \in conv(S)$. So $conv(S)$ is a convex set.

Problem 2 (7.2)
(i)
Let V be a hyperplane that satisfies $\forall x \in V, < a, x >= b$ ($a \in V, a \neq 0, b \in \mathbb{R}$).
$\forall x, y \in V, \lambda$ s.t. $0 \leq \lambda \leq 1$,
$< a, \lambda x + (1 - \lambda y) >= \lambda < a, x > +(1-\lambda) < a, y >= \lambda b + (1-\lambda) b = b$
$\Rightarrow \lambda x + (1-\lambda)y \in V$, V is convex.

(ii)
Let H be a half space that satisfies $\forall x \in H, < a, x >\leq b$ ($a \in H, a \neq 0, b \in \mathbb{R}$).
$\forall x, y \in H, \lambda$ s.t. $0 \leq \lambda \leq 1$,
$< a, \lambda x + (1 - \lambda y) >= \lambda < a, x > +(1-\lambda) < a, y >\leq \lambda b + (1-\lambda) b = b$
$\Rightarrow \lambda x + (1-\lambda)y \in H$, H is convex.

Problem 3 (7.4)
(i)

$$\begin{aligned}
&||x-p||^2 + ||p-y||^2 + 2 < x-p, p-y > \\
&= < x-p, x-p > + < p-y, p-y > +2 < x-p, p-y > \\
&=(< x-p, x-p > + < x-p, p-y >) + (< p-y, p-y > + < x-p, p-y >) \\
&= < x-p, x-y > + < x-y, p-y >=< x-y, x-y >= ||x-y||^2
\end{aligned}$$

(ii)
Suppose $< x-p, p-y >\geq 0, \forall y \in C$, then
$||x-y||^2 = ||x-p||^2 + ||p-y||^2 + 2 < x-p, p-y >\geq ||x-p||^2$
$\Rightarrow ||x-y|| \geq ||x-p||$

(iii)
Since C is convex, $p, y \in C, \lambda \in [0,1]$, we know $z \in C$.
By (i),

$$\begin{aligned}
||x-z||^2 &= ||x-p||^2 + ||p-z||^2 + 2 < x-p, p-z > \\
&= ||x-p||^2 + 2 < x-p, \lambda y + (1-\lambda)p > +||\lambda y + (1-\lambda)p - p||^2 \\
&= ||x-p||^2 + 2\lambda < x-p, p-y > +\lambda^2 ||y-p||^2
\end{aligned}$$

(iv)
Let p be the projection of x onto C. By (iii), we know
$\frac{||x-z||^2-||x-p||^2}{\lambda} = 2 < x-p, p-y > +\lambda||y-p||^2$, if $\lambda \in (0,1]$
Since p is a projection, by definition, $||x-z|| \geq ||x-p||$, so the left hand side of the equation ≥ 0.
$\Rightarrow < x-p, p-y > \geq -\frac{\lambda||y-p||^2}{2} (\star), \forall \lambda \in (0,1]$
Suppose that $< x-p, p-y > \geq 0$ does not hold for some $y \in C$, then we know $\exists y_0 \in C$ s.t. $< x-p, p-y_0 > = a < 0$. However, as $\lambda \to 0$, $-\frac{\lambda||y_0-p||^2}{2} \to 0$, so $-\frac{\lambda||y_0-p||^2}{2} < a$ must be true for some values of λ. This contradicts the fact that $(\star)$ holds true for all $\lambda \in (0,1]$.
$\Rightarrow < x-p, p-y > \geq 0$

The complete proof:
Suppose p is the projection of x onto C, then by (iv), $< x-p, p-y > \geq 0$.
Suppose $< x-p, p-y > \geq 0$, then by (ii), $||x-y|| > ||x-p||$, so p is a projection of x onto C.

Problem 4 (7.6)
Denote the set $\{x \in \mathbb{R}^n | f(x) \leq c\}$ by A. $\forall x, y \in A, \lambda \in [0,1]$, since f is a convex function
$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda c + (1-\lambda)c = c$.
$\lambda x + (1-\lambda y) \in A$, so A is convex.

Problem 5 (7.7)
$\forall x, y, t \in [0,1]$, since all f_i's are convex

$$\begin{aligned} f(tx+(1-t)y) &= \sum_{i=1}^{k} \lambda_i f_i(tx+(1-t)y) \\ &\leq \sum_{i=1}^{k} \lambda_i [t f_i(x) + (1-t) f_i(y)] \\ &= t \sum_{i=1}^{k} \lambda_i f_i(x) + (1-t) \sum i = 1^k f_i(y) = t f(x) + (1-t) f(y) \end{aligned}$$

So f is convex.

Problem 6 (7.13)
Suppose that f is not constant, then $\exists x, y, x < y$ s.t $f(x) \neq f(y)$
Assume without loss of generality that $f(x) < f(y)$
$\forall z > y$, write $y = \frac{y-x}{z-x} z + \frac{z-y}{z-x} x$. Note that $\frac{y-x}{z-x} + \frac{z-y}{z-x} = 1$. Since f is convex,
$f(y) \leq \frac{y-x}{z-x} f(z) + \frac{z-y}{z-x} f(x)$
$\Rightarrow f(z) \geq \frac{z-x}{y-x} f(y) - \frac{z-y}{y-x} f(x)$
Note that as $z \to \infty$, the right-hand side of the inequality $\to \infty$, which contradicts

the boundedness assumption.
So f is a constant function.

Problem 7 (7.20)
Let $g(x) = f(x) - f(0)$, so $g(0) = 0$. Since f is convex and concave, g is also convex and concave. Moreover, $\forall x,$ and y_1, y_2 that lie on the line joining 0 and x, we know:
$g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \leq \lambda_1 g(x)$, since g is convex.
$g(y_1) = g(\lambda_1 x + (1 - \lambda_1) \times 0) \geq \lambda_1 g(x)$, since g is concave.
$\Rightarrow g(y_1) = \lambda_1 g(x)$. Similarly, $g(y_2) = \lambda_2 g(x)$

$\forall a, b \in \mathbb{R}, g(ay_1 + by_2) = (a\lambda_1 + b\lambda_2)g(x) = a\lambda_1 g(x) + b\lambda_2 g(x) = a(g(y_1)) + b(g(y_2))$
Since x can be any point in the domain of f, we know hat g is a linear transformation on its whole domain.
$\Rightarrow g$ is affine.
Since f is a translation of g, f is affine.

Problem 8 (7.21)
Suppose that x^* is a local minimizer for the second problem. $\forall x \neq x^*$ in a neighborhood of x^* that satisfies the constraint, we know $f(x) \geq f(x^*)$. Since ϕ is strictly increasing, $\phi \circ f(x) \geq \phi \circ f(x^*)$. So x^* is a local minimizer for the first problem.

Suppose that x^* is a local minimizer for the first problem and that it is not a local minimizer for the second problem. Then in an neighborhood of x^*, $\exists x \neq x^*, f(x) < f(x^*)$, and $\phi \circ f(x) > \phi \circ f(x^*)$. Since ϕ is strictly increasing, the aforementioned statement is impossible. So this is a contradiction. So x^* is also a local minimizer for the second problem.