A Model with Financial Frictions

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1 Structure of the model

1.1 General description

The economy is populated by two infinitely lived agents: a household and a financier. Both agents have access to a linear production technology. However, the financier is more capable and the capital that he owns produces higher output than the household could achieve. At the same time the financier is less patient; so he borrows from the household to finance present consumption. The household willingly lends because his return on capital is lower.

Each period the economy experiences shocks to capital production. These shocks could be interpreted as aggregate productivity shocks, as is done in this note, for comparison with other macroeconomic models. Newly produced capital could be sold or purchased in the market for physical capital in the original paper. This note assumes that the aggregate capital stock is fixed.

Importantly, there are no idiosyncratic shocks in the economy. This implies that financial trade, described next, occurs only because of technological and taste differences.

Financial markets trade shares of capital and a risk free bond. Structure of capital ownership affects the aggregate output in the economy which increases with the financier's fraction of capital ownership. If there were only two states of nature – i.e., two values for capital accumulation shocks – markets would be complete in the absence of financial constraints. But the economy will be subjected to several financial constraints. The first is that the agents cannot short-sell capital. Additionally, the financiers will be required to hold a positive share of capital.

1.2 Notation

Time is indexed by $t = 0, 1, 2, ...$ Agents are indexed by $j \in \{h, f\}$. Variables c_j, y_j, k_j, b_j denote agent j's consumption, production income, capital investment, and bond investment, respectively. The aggregate equivalents are denoted by capitalized letters.

The capital productivity shock and its distribution are denoted by z and p_z, respectively.

Vector S denotes the set of the "complete" aggregate state variables as will be explained later:

$$S \equiv \{Y, k_h, k_f, b_h, b_f, z\}, \tag{1}$$

where Y denotes the aggregate output that will be defined later.

1.3 Optimization problem

The financier and the household solve the following optimization problem:

$$V_j(k_j, b_j, S) = \max \left[u_j(c_j) + \beta_j E[V_j(k'_j, b'_j, S')]\right] \tag{2}$$

subject to

$$c_j + q_k k'_j + q_b b'_j \leqslant (a_j + z + q_k)k_j + b_j, \tag{3a}$$

$$k'_j \geqslant \underline{k}_j. \tag{3b}$$

The second equation is the constraint on equity investment. The standard assumption is $\underline{k}_j = 0$, but later it will be assumed that $\underline{k}_f > \underline{k}_h = 0$.

Observe that both agents are assumed to have a general utility function. That is agents could be risk-averse if u_j is concave or risk-neutral if u_j is affine.

1.4 Market clearing

Market clearing conditions are:

$$k_h + k_f = 1, \tag{4a}$$

$$b_h + b_f = 0, \tag{4b}$$

$$c_h + c_f = a_h k_h + a_f k_f + z \equiv Y. \tag{4c}$$

The assumption that capital investments sum to 1 is innocuous as long as the analyzed utilities are homothetic.

1.5 Solving the model

Let λ_c, λ_k denote the Lagrange multipliers on the budget constraint and the capital lower bound, respectively. Let V_{jx} denote the derivative of the value function V_j with respect to variable x.

The first-order optimality conditions are:

$$
\begin{aligned}
c: \quad & 0 = u'(c) - \lambda_c, && \text{(5a)}\\
b': \quad & 0 = \beta_j E[V_{jb}(k', b', S')] - \lambda_c q_b, && \text{(5b)}\\
k': \quad & 0 = \beta_j E[V_{jk}(k', b', S')] - \lambda_k, && \text{(5c)}\\
env: \quad & V_{jb}(k', b', S') = \lambda_c, && \text{(5d)}\\
env: \quad & V_{jk}(k', b', S') = \lambda_c(a_j + z) + \lambda_k. && \text{(5e)}
\end{aligned}
$$

The optimality conditions allow to derive the Euler equations for the two dynamic states:

$$
\begin{aligned}
q_k &= \beta_j E\left[\frac{u'(c')}{u'(c)}\left(a_j + z' + q_k'\right)\right], && \text{(6a)}\\
q_b &= \beta_j E\left[\frac{u'(c')}{u'(c)}\right]. && \text{(6b)}
\end{aligned}
$$

The two Euler equations imply the following asset returns:

$$
\begin{aligned}
R_k &= \frac{a_j + z' + q_k'}{q_k}, && \text{(7a)}\\
R_b &= \frac{1}{q_b}. && \text{(7b)}
\end{aligned}
$$

1.6 Net worth

Net worth of agent j is defined by the following:

$$
n_j \equiv (a_j + q_k)k_j + b_j. \tag{8}
$$

Because there is no autonomous income net worth n_j is all that an agent can rely on. If an agent's net worth ever reaches 0 that agent is said to be driven out of the market and his consumption is zero from then on.

Would it be possible for an agent to recover from $n_j = 0$? The answer is no. Suppose that the unlucky agent borrows additional in the bond market to invest in capital in a zero net-value transaction. If it improves the net worth in some future period it must mean that the expected payoff is positive. In other words the zero net-value transaction must offer arbitrage opportunities which should not be possible in an equilibrium.

For the reason given above $n_j \geqslant 0$ is a vacuous constraint. It must be respected for the competitive equilibrium to exist. However, with risk neutral agents it is possible that $n_j = 0$ on some paths of the economy. When the Inada condition, $u'_j(0) = \infty$, is imposed it must be true that $n_j > 0$ in a competitive equilibrium if it exists.

1.7 The state vector

Notice that how net worth n_j has been accumulated, by investing in capital or bonds, is irrelevant. The simplified formulation depends crucially on the implicit assumption that transaction costs are zero in which case settling gross and net asset transactions is equivalent. In other words, any agent can sell all of his assets and repurchase them back again incurring no cost.

This means that to solve the model it is sufficient to know the distribution of net worth, (n_h, n_f), but not the particular portfolios that led to it.

It is even more useful to cast the model solution in terms of net worth shares. Define the wealth share of agent j:

$$w_j \equiv \frac{n_j}{n_h + n_f} = \frac{(a_j + z + q_k)k_j + b_j}{Y + q_k}.$$

where $n_h + n_f = Y + q_k$ is the total wealth in the economy. The wealth shares always sum to one: $w_h + w_f = 1$. Moreover, because $n_j \geqslant 0$ in any equilibrium the range for each wealth share is $[0, 1]$. However, (w_1, w_f) contains less information and it is not longer sufficient: one would not be able to determine the total output in the economy. For this reason it is added to the state vector. The state vector (w_h, Y) is the "natural" state and it replaces the "complete" state vector in what follows.

1.8 Two tricks

The solution to this model will be obtained by iteratively solving the system of the first-order optimality conditions, i.e., by "time iteration". This is system is hiding two pitfalls: consumption non-negativity and occasionally binding constraints. They can be dealt with efficiently using the tricks explained below.

1.8.1 Consumption non-negativity

Consumption non-negativity constraints have been completely ignored. It is a problem only because a numerical algorithm is used to solve the system. Without further pre-caution a numerical solver can try negative consumption levels. To avoid this problem consumption is re-parameterized:

$$c_j = Y/(1 + e^{-x_j}), \tag{9}$$

where Y is the aggregate output in the economy. Consumption c_j is a strictly increasing function of x_j and $c_j \in (0, Y)$ for any x_j.

This re-parametrization achieves more than promised. Not only it imposes consumption non-negativity is also insures that c_j does not exceed the aggregate output in the economy.

1.8.2 Lower bound on capital investment

It was assumed that $k_j \geqslant \underline{k}_j$. It is similar to consumption non-negativity, yet it must be dealt with very differently. The "consumption trick" does not allow the lower or the upper boundaries to be ever reached, while $k_j = \underline{k}_j$ is expected to occur occasionally.

To solve this issue the re-parametrization is especially clever. The new variable x simultaneously models the capital investment k_j and the Lagrange multiplier associated with the constraint.

$$\begin{aligned} k_j &= \underline{k}_j + [\max(x, 0)]^2, \\ \mu_{kj} &= [\max(-x, 0)]^2. \end{aligned}$$

Both k_j and μ_{kj} are differentiable function of x. If the terms were not squared then the problem would not be differentiable. However, in practice, the re-parameterization works without squaring the terms.

This "trick" has been devised by Garcia and Zangwill (1980) and subsequently introduced to economists by Kubler and Schmedders (2003).

1.9 Initial guess

Functions that need to be solved for are:

$$\rho_{ch}(w_h, K), \tag{10a}$$
$$\rho_{kh}(w_h, K), \tag{10b}$$
$$\rho_{kf}(w_h, K), \tag{10c}$$
$$\rho_{bh}(w_h, K), \tag{10d}$$
$$\rho_{qk}(w_h, K), \tag{10e}$$
$$\rho_{qb}(w_h, K). \tag{10f}$$

Several policy functions have been omitted as they can be constructed from the above:

$$\begin{aligned}\rho_{cf}(w_h, Y) &= Y - \rho_{ch}(w_h, Y),\\ \rho_{bf}(w_h, Y) &= -\rho_{bh}(w_h, Y).\end{aligned}$$

The evolution of the aggregate capital and agent 1's wealth share are given by the following:

$$K'(w_h, K) = \rho_{k1}(w_1, K) + \rho_{k2}(w_1, K), \tag{12a}$$
$$w_h' = \frac{(a_h + \rho_{qk}(w_h', Y'))\rho_{kh}(w_h, Y) + \rho_{bh}(w_h, Y)}{Y' + \rho_{qk}(w_h', Y')}. \tag{12b}$$

Importantly, the last equation is an implicit equation in $w_1'(w_1, K)$.

To start the iterative solution it is necessary to know only two policy functions: ρ_{ch} and $\rho_q k$:

$$\rho_{ch}(w_h, Y) = w_h Y, \tag{13a}$$
$$\rho_{qk}(w_h, Y) = 0. \tag{13b}$$

These values correspond to the solution that would obtain in the last period of any finite-horizon model.

1.10 Two-dimension approximation

It is well known (from analyses of Krusell and Smith's (1998) model) that the policy functions are close to linear in the dimension of the aggregate states. So, the approximation in the dimension of aggregate output is chosen to be linear. The interpolation in the w_h dimension is performed using cubic splines.

Let (w, Y) be an arbitrary point in the state space. Then compute (univariate) spline interpolation $x(w|Y = \bar{Y}_k)$ of variable x conditional on $\bar{Y}_k$. Index k is chosen so that $Y \in [\bar{Y}_k, \bar{Y}_{k+1}]$. Then set the final interpolation value to:

$$x(w|Y = \bar{Y}_k) + (x(w|Y = \bar{Y}_{k+1}) - x(w|Y = \bar{Y}_k))\frac{Y - \bar{Y}_k}{\bar{Y}_{k+1} - \bar{Y}_k}.$$

2 Numerical Solution

The computed policy functions for the parameters reported in table 1 are shown in figure 1.

Table 1: Model parameterization

Value	Description
0.9500	Discount factor H (household)
0.9300	Discount factor F (financier)
1.0000	CRRA H
1.0000	CRRA F
1.0000	Productivity of H
1.0100	Productivity of F
0.2000	Minimum capital investment of F
0.0500	Disaster state: loss of output
0.0000	Disaster state: probability
2.5000	Volatility of productivity

2.1 Understanding key assumptions

Consider the model in which the agents differ in how they discount future utility but when the markets are complete.

In this case the first-order optimality conditions are:

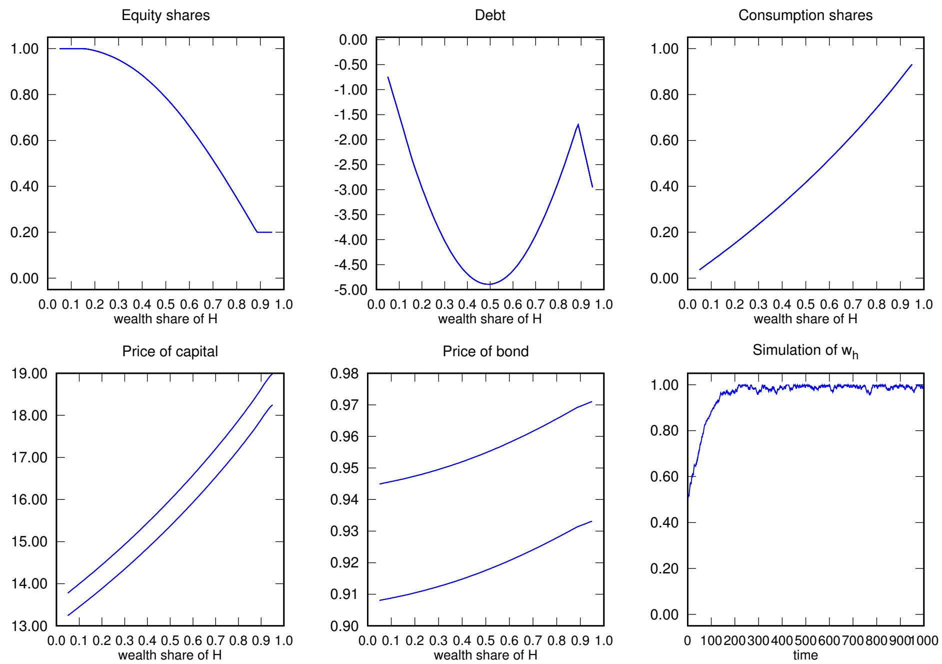
$$\frac{\beta_1^t u_1'(c_{1t}(\sigma))}{\beta_2^t u_2'(c_{2t}(\sigma))} = \frac{\theta_2}{\theta_1}, \tag{14}$$

where θ_j is the Pareto weight of agent j.

Assuming that both u_1 and u_2 are of CRRA class we get:

$$t \cdot [ln(\beta^1) - ln(\beta^2)] - \gamma_1 \ln(c_{1t}(\sigma)) + \gamma^2 \ln(c_{2t}(\sigma)) = \ln(\theta^2/\theta^1). \tag{15}$$

Figure 1: Model solution



Equation 15 is sufficient to analyze the asymptotic properties of the two consumption streams.

Case 1. If $\beta_1 > \beta_2$ and $\gamma_1 = \gamma_2$ then $\lim_{t\to\infty} c_{1t}/c_{2t} = \infty$. That is the more patient agent eventually consumes all output in the economy irrespectively of the relation between γs.

Case 2. If $\beta_1 = \beta_2$ and $\gamma_1 > \gamma_2$ then $c_{1t}/c_{2t} = const, \forall t$. That is the less risk-averse agent will get a larger fraction of output then he would otherwise. This is the benefit of insuring the more risk-averse agent.

The conclusion is that the two modeling choices have very different economic implications. The case with heterogeneous discount factors is extreme in the sense that even small discount-factor heterogeneity dwarfs other economic forces in the limit. The risk-aversion heterogeneity is more natural because the economy remains "stationary." Another modeling choice would be heterogeneity in inter-temporal substitution as in Guvenen (2007).

2.2 Special Case with Full Depreciation

Consider the case when $\delta = 1$ and $z = 0$: that is capital depreciates fully each period and there are no capital accumulation shocks. In this case the key state variables simplify considerably because prices are not needed to compute the wealth shares:

$$w_j = \frac{a_j k_j + b_j}{a^1 k^1 + a^2 k^2}. \tag{16}$$

The total wealth in the economy is $W = a^1 k^1 + a^2 k^2$.

A Appendix

A.1 Endogenous investment

This section describes the formulation like in Brunnermeier and Sannikov (2014). Both agents can produce capital using technology Φ and the new capital be used sold on the market for the price q_k.

The financier and the household solve the following optimization problem:

$$V_j(k_j, b_j, S) = \max\left[u(c_j) + \beta_j E[V_j(k_j', b_j', S')]\right] \tag{17}$$

subject to

$$c_j + q_k m_j k_j + q_b b_j' + i_j k_j = (a_j + z_y)k_j + b_j, \tag{18a}$$

$$k_j' = (1 - \delta + z_k + m_j + \Phi(i_j))k_j, \tag{18b}$$

$$i_j \geqslant 0. \tag{18c}$$

$\Phi(i_j)k_j$ is the amount of capital that is produced from $i_j k_j$ consumption units and $m_j k_j$ units are purchased (or sold if negative) on the capital market.

The first-order optimality conditions for i_j and h_j are:

$$-\lambda_c q_k + \lambda_k = 0,$$
$$-\lambda_c + \lambda_k \Phi'(i_j) + \mu_i = 0,$$

where $\lambda_c, \lambda_k, \mu_i$ are the Largrange multipliers on the constraints in (18). The above system can be solved to obtain:

$$1 = q_k \Phi'(i_j) + \mu_i/\lambda_c.$$

If both agents invest in physical capital, $i_h > 0$ and $i_f > 0$, then:

$$i_f = i_h = \Phi^{-1}(q_k). \tag{20}$$

Observe that the investment level defined by (20) maximizes capital production profit: $i_j = \arg\max_i [q_k \Phi(i) - i]$.

Combining the budget constraint with the capital accumulation equation gives:

$$c_j + q_k k'_j + q_b b'_j = (a_j + z_y - i_j + q_k(1 - \delta + z_k + \Phi(i_j)))k_j + b_j. \tag{21}$$

The above equation implies the return on capital, which depends on the agent, as in the paper:

$$R_j = \frac{a_j + z'_y - i'_j + q'_k(1 - \delta + z'_k + \Phi(i'_j))}{q_k}. \tag{22}$$

Brunnermeier and Sannikov's (2014) specification obtains when $z_y = 0$. The special case analyzed by Rappoport and Walsh (2012) obtains when $z_y = 0, \delta = 0$ and $i_j = \Phi(i_j) = 0$. It is the model with stochastic aggregate capital growth: $K'/K = 1 + z'_k$. The model presented and solved in this note assumes that $z_y \neq 0, z_k = 0$. This simplifies the analysis because the aggregate capital is fixed.