"Open Source Macroeconomics Laboratory Boot Camp Time Series Filtering"

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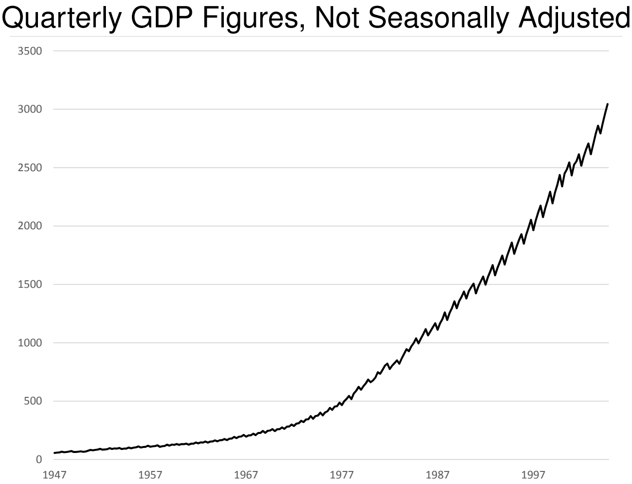
July 26, 2017

Outline

Quantifying Model Behavior

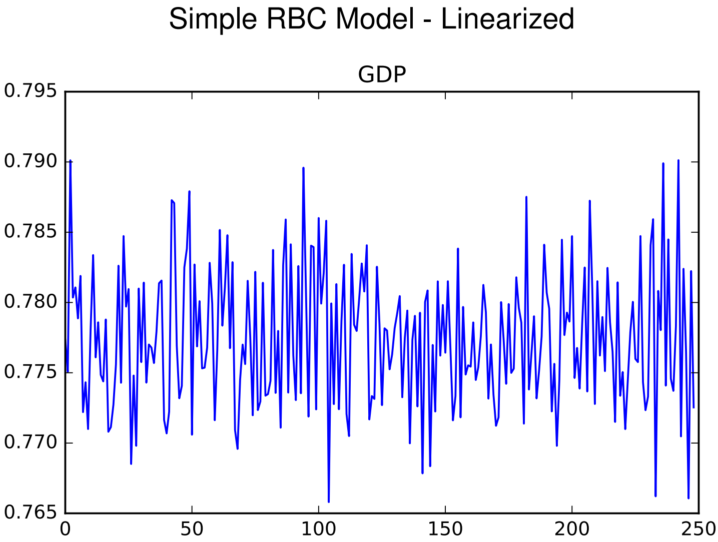
- How do we summarize the behavior of data or a business cycle model?
- How do we compare our model to data from a real economy?
- Models are almost always solved and simulated using a stationary version.
- Real world data is non-stationary.

Real GDP



1947:1 - 2004:4, in billions of dollars

Model GDP


Simple RBC Model - Linearized
GDP
0.795
0.790
0.785
0.780
0.775
0.770
0.765
0
50
100
150
200
250

Quantifying Model Behavior

- Real world data need to be "filtered" to remove a time trend, and to remove seasonal variation.
- Data is often available in "seasonally adjusted" form. Ususally this means that it has been filtered using `X-13ARIMA-SEATS` which was developed and is available for download from the U.S. Bureau of the Census.
- The removal of a time trend is more problematic, since the choice of a detrending filter can change the reported summary statistics.
- We will focus on different ways to remove a time trend and how to compare model data to real world data.

Decompostion of a Time Series

We will write this decomposition using the notation below.

$$x_t = x_t^G + x_t^S + x_t^C \tag{1}$$

where:

- x_t^G is the long-run trend or growth component
- x_t^S is the seasonal component
- x_t^C is the residual or cyclical component

Outline

FRED

Federal Reserve Economic Database (FRED)

`http://research.stlouisfed.org/fred2/`

FRED collects data from both the BEA and BLS and from other sources. As a general rule, if you need to download many related series, such as GDP and all its components, you are better off going to the BEA or BLS websites where this can be accomplished in a single download. FRED has a nice iterface with `Pandas`.

BEA

Bureau of Economic Analysis (BEA)
`http://bea.gov`
Two very useful databases here are:

- The NIPA Accounts
  `http://bea.gov/iTable/index_nipa.cfm`
- The Fixed Asset Tables
  `http://bea.gov/iTable/index_FA.cfm`

The former is where you would find data on GDP and its various components. The later is a good source for data on the U.S. capital stock.

BLS

Bureau of Labor Statistics (BLS)

`http://bls.gov/data`

Here you will find detailed data on the labor market including employment, unemployment, labor force participation, hours worked, wages and earnings. You can also find detailed data on the CPI and consumer prices.

Using Pandas

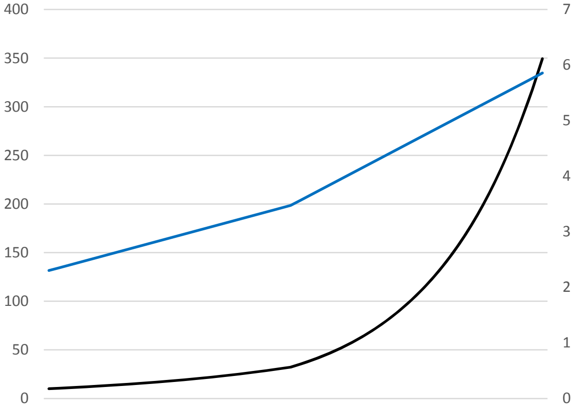
- You can import data directly into Python as a pandas datagframe from many websites, including FRED.
  - `import pandas_datareader.data as web`
  - `start = datetime.datetime(1947, 1, 1)`
  - `end = datetime.datetime(2017, 1, 1)`
  - `Y = web.DataReader("GDPC1", "fred", start, end)`
- Panda will also allow you to export data tables in LaTeX format, which can save a great deal of time.

See the Jupyter notebook, `Periodogram and Filter Example` at the Bootcamp repository.

Outline

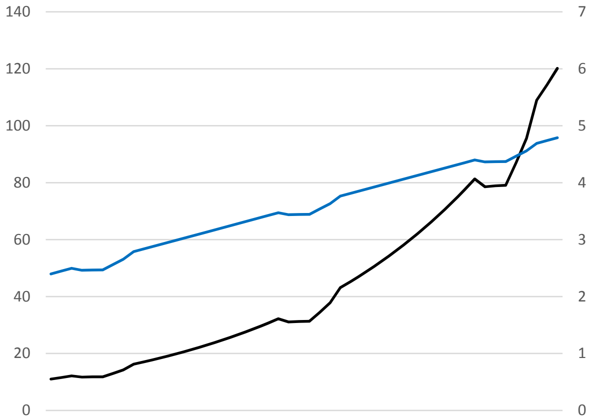
Taking Logs

Our first step for a time-series that is growing over time is to take it's natural logarithm.



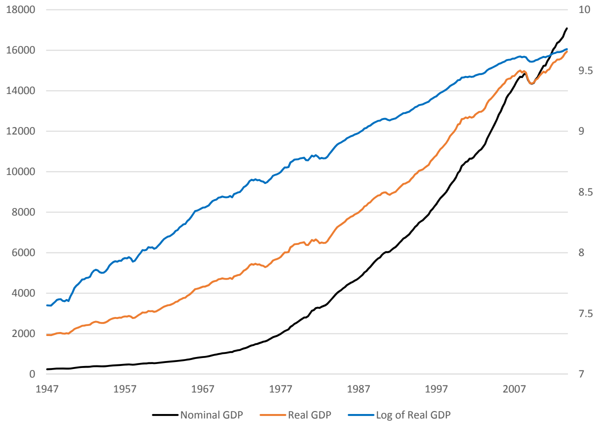
log series in blue with scale on the right

Taking Logs



Taking Logs

We often want to work with constant price or real data, rather than nominal data.



Removing a Trend

- In addition to removing seasonality, we also wish to remove the trend from many time series.
- The method we use below to measure volatility, persistence and cyclicality require that our data have no time trend.
- Since we are usually working with seasonally adjusted data at this point, the amounts to decomposing the data into a growth component and a cyclical component.

Common Filtering Methods

- Moving Average and Differencing Filters
- Linear Trend or OLS Filter
- Hodrick-Prescott Filter
- Band Pass Filters
    - Baxter-King
    - Christiano-Fitzgerald

- Moving-average filter

$$y_t = \sum_{j=-\infty}^{\infty} b_j x_{t-j} = b(L)x_t$$

- First-difference filter

$$y_t = x_t - x_{t-1}$$

- D-difference filter

$$y_t = x_t - x_{t-d}$$

- Second-difference filter

$$y_t = (x_t - x_{t-1}) - (x_{t-1} - x_{t-2})$$

OLS Filter

The simplest of filters involves fitting set τ to be the "polynomial of best fit" through the time series. The "polynomial of best fit" minimizes the following expression:

$$\min_{\beta} \|y - \beta x\|$$

For a polynomial of degree 1 (a straight line), the solution is given by the well known "normal equations" given by $\hat{\beta} = (X^T X)^{-1} X^T y$, where

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{pmatrix}$$

OLS Filter

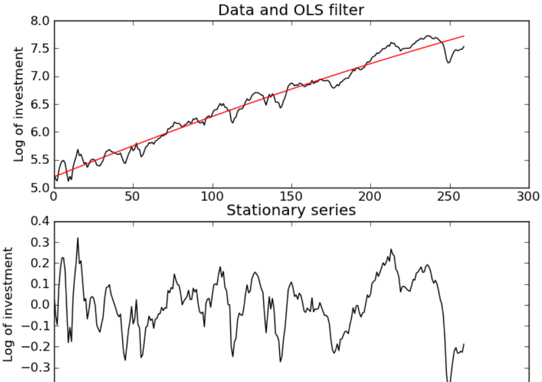
We can generalize this for a polynomial of degree k using the k^{th} degree Vandermonde matrix and the same set of normal equations.

$$X = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^k \\ & \vdots & \ddots & \vdots & \\ 1 & x_n & x_n^2 & \cdots & x_n^k \end{pmatrix}$$

OLS Filter

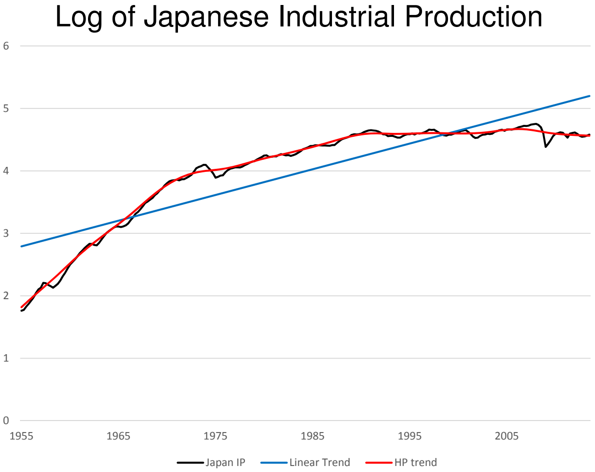
Using a quadratic polynomial filter on logged investment data gives the following trend series (depicted in red) and cyclical series (depicted in the second subpanel)

Log of U.S. Real Investment



OLS Filter

Can lead to odd results, however.



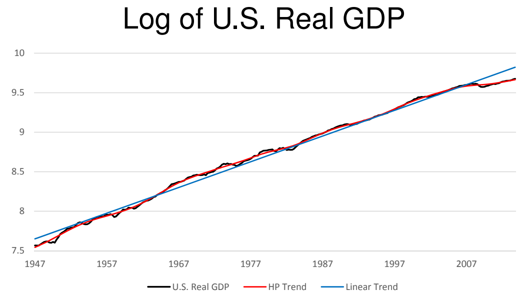
HP Filter

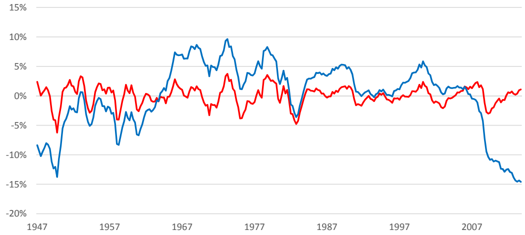
Minimize the following

$$\min_{\{\tau_t\}} \left\{ \sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=1}^{T} [(\tau_t - \tau_{t-1}) - (\tau_{t-1} - \tau_{t-2})]^2 \right\} \quad (2)$$

with $c_t = y_t - \tau_t$ giving the deviation from the trend component (a mean zero process). The parameter λ penalizes changes in the trend component. A higher λ will result in a smoother trend component. In fact, as λ approaches infinity, the optimal $\tau_t - \tau_{t-1}$ tends towards a constant β, so $\tau_t = \tau_0 + \beta_t$, and the filtered series is simply the least squares solution.

HP Filter


Log of U.S. Real GDP
10
9.5
9
8.5
8
7.5
1947
1957
1967
1977
1987
1997
2007
U.S. Real GDP
HP Trend
Linear Trend


15%
10%
5%
0%
-5%
-10%
-15%
-20%
1947
1957
1967
1977
1987
1997
2007

Band-Pass Filters

- Similar to the HP filter; decomposes the data into a trend series and a cyclical series.
- BP filter is that the band-pass filter is two-sided. It removes frequencies outside the chosen band that are higher than the upper cutoff and lower than the lower cutoff.
- HP filter is one-sided and removes only frequencies below a lower cutoff.
- The term “band-pass filter" refers to a whole family of filtering techniques.

Outline

Fourier Transform

- Any time series can be fitted with a weighted sum of an appropriate number of sinusoidal wave functions of various wavelengths.
- With T observations we can sum $T - 1$ wave functions and the resulting time-path will pass through each point.
- The mapping from observations at various points in time to wave functions of various frequencies is called a Fourier transformation.

Discrete Fourier Transform

Consider a sequence of (possibly complex) numbers,
$X = \{x_1, x_2, \ldots, x_T\}$.
The discrete Fourier transform (DFT) of this series is defined as:

$$\hat{x}_k = \sum_{t=1}^{T} x_t e^{-i2\pi kt/T} \tag{3}$$

The DFT is often written as $\mathcal{F}$ so that the series $\hat{X} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_T\}$ can be expressed as the DFT of X by writing $\hat{X} = \mathcal{F}(X)$.
The Inverse DFT, not surprisingly, is written $X = \mathcal{F}^{-1}(\hat{X})$
X is a representation of the series in the “time domain" and $\hat{X}$ is a representation of the same series in the “frequency domain".

Spectral Density

Consider an discrete infinite time series denoted X. We will define the power of X as:

$$P \equiv \lim_{t \to \infty} \frac{1}{2T} \sum_{-T}^{T} x_t^2$$

Consider now a subsample of the series running from 1 to T and denote this as $X^T(t)$

Spectral Density

Define ω as the “angular frequency" measured in radians, so that $\omega = 2\pi k$. The DFT of $X^T(t)$ can be denoted $\hat{X}^T(\omega) = \mathcal{F}\{X^T(t)\}$.
The power spectral density (PSD), spectral density function (SDF) or simply “spectral density" of X is defined in (4).

$$S_x(\omega) \equiv \lim_{T\to\infty} E\{|\hat{X}^T(\omega)|^2\} \tag{4}$$

Since ω is the angular frequency all the information about the spectral density is contained on the interval $[0, 2\pi]$.

Periodogram

In practice, we cannot get an infinite series of observations and we need an approximation or estimate of based on finite data. The classic estimate is the periodogram. The periodogram is defined by (5).

$$\begin{aligned} a(k) &= \frac{2}{T}\sum_{t=1}^{T} x_t \cos(kt)dt \\ b(k) &= \frac{2}{T}\sum_{t=1}^{T} x_t \sin(kt)dt \\ P_x(k) &= \sqrt{a(k)^2 + b(k)^2} \end{aligned} \tag{5}$$

Periodogram

The periodogram is normally plotted with ω on the horizontal axis and P_x on the vertical axis.

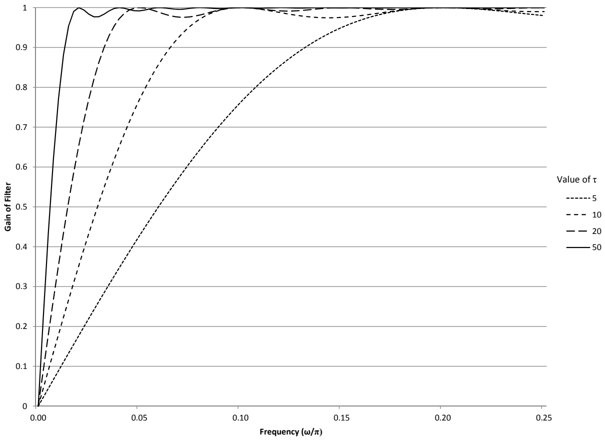
In Python there are several repackaged functions that will generate the periodogram. The most accessible is found in the Scipy package – `scipy.signal.periodogram`

Gain of a Filter

The gain of a filter is the multiplicative effect that it has on data at various frequencies. We plot the gain as we do the periodogram with ω on the horizontal axis and the gain, G_x, on the vertical axis.

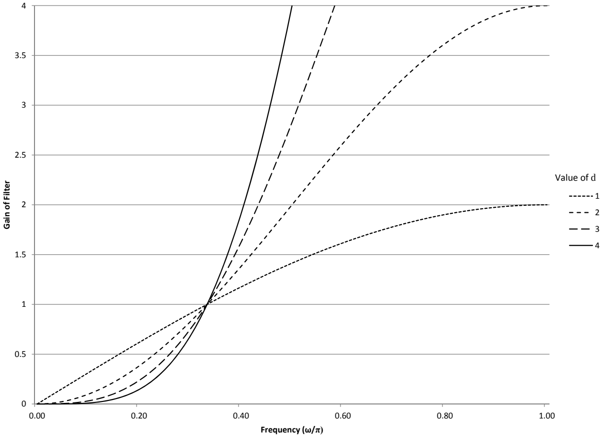
Moving-Average Gain

Gain of Moving-Average Filters



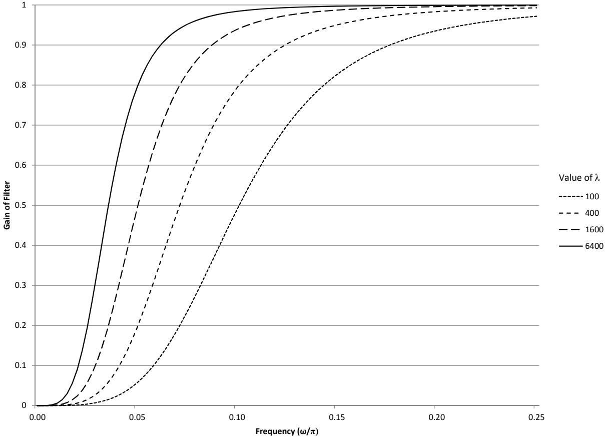
D-Difference Gain

Gain of D-Difference Filters



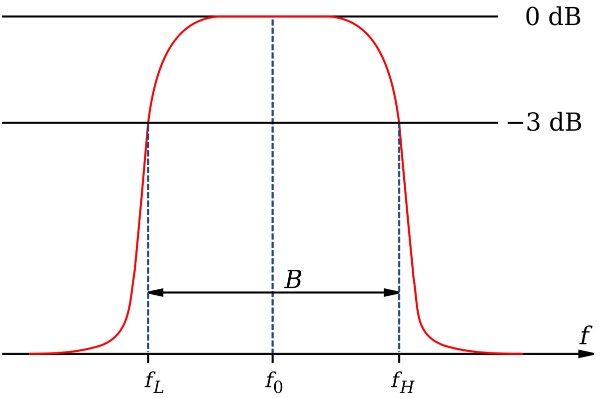
HF Filter Gain

Gain of HP Filter



BP Filter Gain

Gain of BP Filter



Outline

Data Moments

We are interested in quantifying at least three key characteristics of various macroceonomic time series in both the real world data and in our model. We do this by calculating statistical moments of the filtered data.

- volatility - standard deviation of cyclical component
- cyclicality - correlation of cyclical component with the cylical component for GDP
- persistence - autocorrelation of the cyclical component

Stylized Facts Depend on the Filter

Canova (1998) ‘Detrending and Business Cycle Facts’ in *Journal of Monetary Economics*

> *"This paper examines the business cycle properties of a small set of a real US macroeconomic time series using a variety of detrending methods. It is shown that both quantitatively and qualitatively ‘stylized facts’ of US business cycles vary widely across detrending methods and that alternative detrending filters extract different types of information from the data."*

Stylized Facts Depend on the Filter

Hamilton (2017) ‘Why You Should Never Use the Hodrick-Prescott Filter’, *working paper*

- The HP filter produces series with spurious dynamic relations that have no basis in the underlying data-generating process.
- Filtered values at the end of the sample are very different from those in the middle, and are also characterized by spurious dynamics.
- A statistical formalization of the problem typically produces values for the smoothing parameter vastly at odds with common practice, e.g., a value for λ far below 1600 for quarterly data.
- There’s a better alternative. A regression of the variable at date $t+h$ on the four most recent values as of date t offers a robust approach to detrending that achieves all the objectives sought by users of the HP filter with none of its drawbacks.

A Robust Approach

Do not rely on a single filter

- Models are usually written down in a nonstationary form.
- We then transform variables to generate a stationary version of the model.
- We can solve and simulate the stationary model.
- Once we do this, we can retransform variables to generate non-stationary data.
- Filter both the model and real world data with the same set of filters.

An Example

Suppose our production function in the Brock and Mirman model were $Y_t = K_t^{\alpha} e^{(1-\alpha)(gt+z_t)}$.

- $w_t = (1-\alpha)Y_t$
- $r_t = \alpha \frac{Y_t}{K_t}$
- $c_t = w_t + r_t K_t - K_{t+1}$
- $c_t^{-1} - \beta c_{t+1}^{-1} r_{t+1}$

K, t, Y_t, w_t and c_t will all be growing without bound and the model is not stationary.

Stationarized

Transform $\hat{x}_t = \frac{x_t}{e^{gt}}$ for $x = \{K, Y, w, c\}$

- $\hat{Y}_t = \hat{K}_t^{\alpha} e^{(1-\alpha)z_t}$
- $\hat{w}_t = (1-\alpha)\hat{Y}_t$
- $r_t = \alpha \frac{\hat{Y}_t}{\hat{K}_t}$
- $\hat{c}_t = \hat{w}_t + r_t \hat{K}_t - e^g \hat{K}_{t+1}$
- $\hat{c}_t^{-1} - \beta(e^g \hat{c}_{t+1})^{-1} r_{t+1}$

The model is now stationary.

Solve and Simulate

Steps:

- Find the steady state and the (approximate) policy function.
- Simulate the model using $\hat{K}_{t+1} = \Phi(\hat{K}_t, z_t)$.
- Find all non-state variable using definitions from previous slid:
- Convert from variables from stationary to non-stationary using $x_t = e^{gt} x_t$ for $x = \{K, Y, w, c\}$
- Filter this data using your filter of choice and compare the statistical moments with those from real world data detrended with the same filter.