A Model with Financial Frictions

Viktor Tsyrennikov

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1 Model

1.1 General description

The economy is populated by two infinitely lived agents: a household and a financier. Both agents have access to a linear production technology. However, the financier is more capable and the capital that he owns produces higher output than the household could achieve. At the same time the financier is less patient; so he borrows from the household to finance present consumption. The household willingly lends because his return on capital is lower.

Each period the economy experiences shocks to capital production. These shocks could be interpreted as aggregate productivity shocks, as is done in this note, for comparison with other macroeconomic models. Newly produced capital could be sold or purchased in the market for physical capital in the original paper. This note assumes that the aggregate capital stock is fixed.

Importantly, there are no idiosyncratic shocks in the economy. This implies that financial trade, described next, occurs only because of technological and taste differences.

Financial markets trade shares of capital and a risk free bond. Structure of capital ownership affects the aggregate output in the economy which increases with the financier's fraction of capital ownership. If there were only two states of nature – i.e., two values for capital accumulation shocks – markets would be complete in the absence of financial constraints and the equilibrium would be efficient.[1] But the economy will be subjected to several financial constraints. The first is that the agents cannot short-sell capital. Additionally, the financiers will be required to hold a positive share of capital.

1.2 Notation

Time is indexed by $t = 0, 1, 2, ...$ Agents are indexed by $j \in \{h, f\}$. Variables c_j, y_j, k_j, b_j denote agent j's consumption, production income, capital investment, and bond investment, respectively. The aggregate equivalents are denoted by capitalized letters.

The capital productivity shock and its distribution are denoted by z and p_z, respectively.

[1] Equilibrium efficiency means that there does not exist another equilibrium in which welfare of some agents is higher and noone's welfare is lower.

Vector S denotes the set of the "complete" aggregate state variables as will be explained later:

$$S \equiv \{Y, k_h, k_f, b_h, b_f, z\}, \tag{1}$$

where Y denotes the aggregate output that will be defined later.

1.3 Optimization problem

The financier and the household solve the following optimization problem:

$$V_j(k_j, b_j, S) = \max \left[u_j(c_j) + \beta_j E[V_j(k_j', b_j', S')]\right] \tag{2}$$

subject to

$$c_j + q_k k_j' + q_b b_j' \leqslant (a_j + z + q_k)k_j + b_j, \tag{3a}$$
$$k_j' \geqslant \underline{k}_j. \tag{3b}$$

The second equation is the constraint on equity investment. The standard assumption is $\underline{k}_j = 0$, but later it will be assumed that $\underline{k}_f > \underline{k}_h = 0$.

Observe that both agents are assumed to have a general utility function. That is agents could be risk-averse if u_j is concave or risk-neutral if u_j is affine.

1.4 Market clearing

Market clearing conditions are:

$$k_h + k_f = 1, \tag{4a}$$
$$b_h + b_f = 0, \tag{4b}$$
$$c_h + c_f = a_h k_h + a_f k_f + z \equiv Y. \tag{4c}$$

The assumption that capital investments sum to 1 is innocuous as long as the analyzed utilities are homothetic.

1.5 Solving the model

Let λ_c, λ_k denote the Lagrange multipliers on the budget constraint and the capital lower bound, respectively. Let V_{jx} denote the derivative of the value function V_j with respect to variable x.

The first-order optimality conditions are:

$$\begin{aligned}
c: &\quad 0 = u'(c) - \lambda_c, && (5a)\\
b': &\quad 0 = \beta_j E[V_{jb}(k', b', S')] - \lambda_c q_b, && (5b)\\
k': &\quad 0 = \beta_j E[V_{jk}(k', b', S')] - \lambda_k, && (5c)\\
env: &\quad V_{jb}(k', b', S') = \lambda_c, && (5d)\\
env: &\quad V_{jk}(k', b', S') = \lambda_c(a_j + z) + \lambda_k. && (5e)
\end{aligned}$$

The optimality conditions allow to derive the Euler equations for the two dynamic states:

$$q_k = \beta_j E\left[\frac{u'(c')}{u'(c)}\left(a_j + z' + q_k'\right)\right], \tag{6a}$$

$$q_b = \beta_j E\left[\frac{u'(c')}{u'(c)}\right]. \tag{6b}$$

The two Euler equations imply the following asset returns:

$$R_k = \frac{a_j + z' + q_k'}{q_k}, \tag{7a}$$

$$R_b = \frac{1}{q_b}. \tag{7b}$$

1.6 Net worth

Net worth of agent j is defined by the following:

$$n_j \equiv (a_j + z + q_k)k_j + b_j. \tag{8}$$

Because there is no autonomous income net worth n_j is all that an agent can rely on. If an agent's net worth ever reaches 0 that agent is said to be driven out of the market and his consumption is zero from then on.

Would it be possible for an agent to recover from $n_j = 0$? The answer is no. Suppose that the unlucky agent borrows in the bond market to invest in capital in a zero net-value transaction. If it improves the net worth in some future period it must mean that the expected payoff is positive. In other words the zero net-value transaction must offer arbitrage opportunities which cannot be possible in an equilibrium.

For the reason given above $n_j \geqslant 0$ is a vacuous constraint. It must be respected for the competitive equilibrium to exist. However, with risk neutral agents it is possible that $n_j = 0$ on some paths of the economy. When the Inada condition, $u_j'(0) = \infty$, is imposed it must be true that $n_j > 0$ in a competitive equilibrium if it exists. For the existence of (recursive) competitive equilibria see Kubler and Schmedders [2003] and Brumm et al. [forthcoming 2017].

1.7 The state vector

The "complete" state vector is often larger than necessary. Notice that to solve the model it is sufficient to know the distribution of net worth, (n_h, n_f), but not the particular portfolios that led to it. This alternative state vector encodes information more efficiently which is especially important when the number of traded assets is large. The sufficiency of n_js depends crucially on the implicit assumption that transaction costs are zero in which case settling gross and net asset transactions is equivalent. In other words, any agent can sell all of his assets and repurchase them back again incurring no cost. If transaction costs where non-zero it would be necessary to track changes of investment in each security.

The state vector (n_h, n_f) can be further transformed into net worth shares. Define the wealth share of agent j:

$$w_j \equiv \frac{n_j}{n_h + n_f} = \frac{(a_j + z + q_k)k_j + b_j}{Y + q_k}.$$

where $n_h + n_f = Y + q_k$ is the total wealth in the economy. The wealth shares always sum to one: $w_h + w_f = 1$. Moreover, because $n_j \geqslant 0$ in any equilibrium the range for each wealth share is $[0, 1]$. However, (w_1, w_f) contains less information and it is not longer sufficient: one would not be able to determine the total output in the economy. For this reason it is added to the state vector. The state vector (w_h, Y) is the "natural" state and it replaces the "complete" state vector in what follows.

1.8 Two tricks

The solution to this model will be obtained by iteratively solving the system of the first-order optimality conditions, i.e., by "time iteration". This is system is hiding two pitfalls: consumption non-negativity and occasionally binding constraints. They can be dealt with efficiently using the tricks explained below.

1.8.1 Consumption non-negativity

Consumption non-negativity constraints have been completely ignored. It is a problem only because a numerical algorithm is used to solve the system. Without further pre-caution a numerical solver can try negative consumption levels. To avoid this problem consumption is re-parameterized:

$$c_j = Y/(1 + e^{-x_j}), \tag{9}$$

where Y is the aggregate output in the economy. Consumption c_j is a strictly increasing function of x_j and $c_j \in (0, Y)$ for any x_j.

This re-parametrization achieves more than promised. Not only it imposes consumption non-negativity is also insures that c_j does not exceed the aggregate output in the economy.

1.8.2 Lower bound on capital investment

It was assumed that $k_j \geqslant \underline{k}_j$. It is similar to consumption non-negativity, yet it must be dealt with very differently. The "consumption trick" does not allow the lower or the upper boundaries to be ever reached, while $k_j = \underline{k}_j$ is expected to occur occasionally.

To solve this issue the re-parametrization is especially clever. The new variable x simultaneously models the capital investment k_j and the Lagrange multiplier associated with the constraint.

$$\begin{aligned} k_j' &= \underline{k}_j + [\max(x, 0)]^2, \\ \mu_{kj} &= [\max(-x, 0)]^2. \end{aligned}$$

This formulation, at the cost of introducing the new choice variable, is such that the complementarity conditions are automatically satisfied. Further, both k_j and μ_{kj} are differentiable function of x. If the terms were not squared then the problem would not be differentiable. However, in practice, the re-parameterization works without squaring the terms.

This "trick" has been devised by Garcia and Zangwill (1980) and subsequently introduced to economists by Kubler and Schmedders [2003].

1.9 Initial guess

Functions that need to be solved for are:

$$
\begin{align}
&\rho_{ch}(w_h, K), \tag{10a}\\
&\rho_{kh}(w_h, K), \tag{10b}\\
&\rho_{kf}(w_h, K), \tag{10c}\\
&\rho_{bh}(w_h, K), \tag{10d}\\
&\rho_{qk}(w_h, K), \tag{10e}\\
&\rho_{qb}(w_h, K). \tag{10f}
\end{align}
$$

Several policy functions have been omitted as they can be constructed from the above:

$$
\begin{aligned}
\rho_{cf}(w_h, Y) &= Y - \rho_{ch}(w_h, Y),\\
\rho_{bf}(w_h, Y) &= -\rho_{bh}(w_h, Y).
\end{aligned}
$$

The evolution of the aggregate capital and agent 1's wealth share are given by the following:

$$
\begin{align}
K'(w_h, K) &= \rho_{k1}(w_1, K) + \rho_{k2}(w_1, K), \tag{12a}\\
w_h' &= \frac{(a_h + \rho_{qk}(w_h', Y'))\rho_{kh}(w_h, Y) + \rho_{bh}(w_h, Y)}{Y' + \rho_{qk}(w_h', Y')}. \tag{12b}
\end{align}
$$

Importantly, the last equation is an implicit equation in $w_1'(w_1, K)$.

To start the iterative solution it is necessary to know only two policy functions: ρ_{ch} and $\rho_q k$:

$$
\begin{align}
\rho_{ch}(w_h, Y) &= w_h Y, \tag{13a}\\
\rho_{qk}(w_h, Y) &= 0. \tag{13b}
\end{align}
$$

These initial values correspond to the solution that would obtain in the last period of any finite-horizon model. This is useful because the value functions can be computed on each iteration step. Having computed value functions opens a possibility to solve models with recursive preferences or to perform welfare analysis.[2] In the model in which agents have recursive preferences the first-order optimality conditions will also involve derivatives of the value functions (marginal values of wealth). The steps would be as follows:

[2]See Atkeson [1991] for an overview of the use of recursive preferences in macroeconomics and finance.

1. A. Set the initial guess for policies as in (13).
   B. Set the value function $V_j(w_h, Y) = u_j(w_j Y)$.

2. A. Solve the system of optimality conditions (5) to update the policy functions.
   B. Update the value function as follows:

$$V_j^{n+1}(w_h, Y) = u_j(\rho_{cj}^{n+1}(w_h, Y)) + \beta_j E[V_j^n(\rho_{wh}^{n+1}(w_h, Y, z'), Y')].$$

1.10 Two-dimensional approximation

It is well known (from analyses of Krusell and Smith's (1998) model) that the policy functions are close to linear in the dimension of the aggregate states. So, the approximation in the dimension of aggregate output is chosen to be linear. The interpolation in the w_h dimension is performed using cubic splines.

Let (w, Y) be an arbitrary point in the state space. Then compute (univariate) spline interpolation $x(w|Y = \bar{Y}_k)$ of variable x conditional on $\bar{Y}_k$. Index k is chosen so that $Y \in [\bar{Y}_k, \bar{Y}_{k+1}]$. Then set the final interpolation value to:

$$x(w|Y = \bar{Y}_k) + (x(w|Y = \bar{Y}_{k+1}) - x(w|Y = \bar{Y}_k))\frac{Y - \bar{Y}_k}{\bar{Y}_{k+1} - \bar{Y}_k}.$$

2 Numerical Solution

The computed policy functions for the parameters reported in table 1 are shown in figure 1.

2.1 Understanding key assumptions

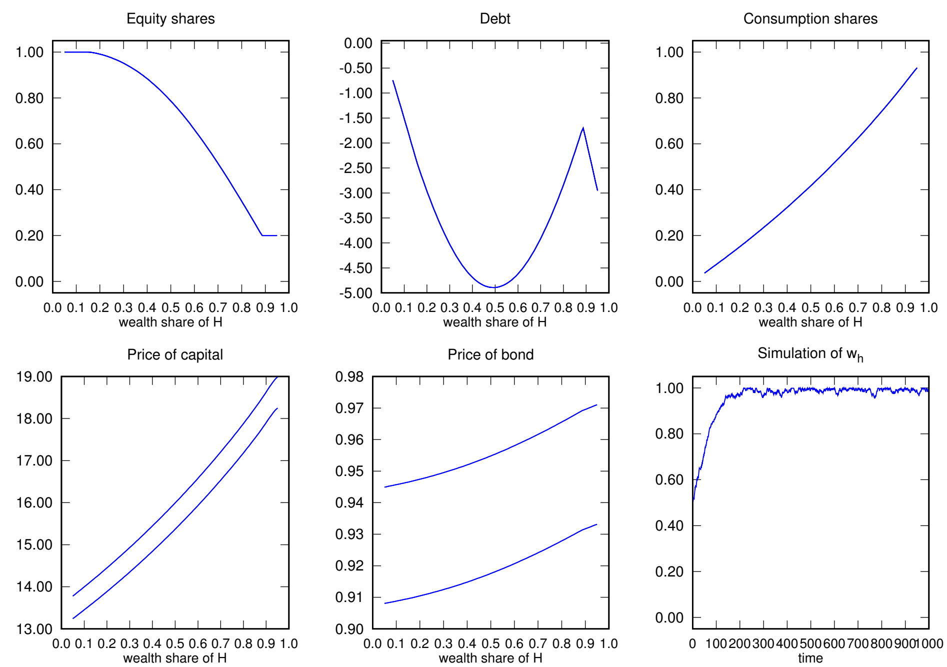
The assumption of heterogeneous discount factors is crucial as it forces the financier towards the financial constraint. Without this the assumption the financial constraint may be inactive. To understand the implications of the discount factor heterogeneity consider a model in which markets are complete (there are no financial constraints and the number of independent securities equals the number of states). In this case the ratio of marginal utilities is constant:

$$\frac{\beta_1^t u_1'(c_{1t}(\sigma))}{\beta_2^t u_2'(c_{2t}(\sigma))} = \frac{\theta_2}{\theta_1}, \tag{14}$$

Table 1: Model parameterization

Value	Description
0.9500	Discount factor H (household)
0.9300	Discount factor F (financier)
1.0000	CRRA H
1.0000	CRRA F
1.0000	Productivity of H
1.0100	Productivity of F
0.2000	Minimum capital investment of F
0.0500	Disaster state: loss of output
0.0000	Disaster state: probability
2.5000	Volatility of productivity

Figure 1: Model solution



where θ_j is the Pareto weight of agent j. Assuming that both u_1 and u_2 are of CRRA class the above can be written as:

$$t \cdot [ln(\beta^1) - ln(\beta^2)] - \gamma_1 \ln(c_{1t}(\sigma)) + \gamma^2 \ln(c_{2t}(\sigma)) = \ln(\theta^2/\theta^1). \tag{15}$$

Equation 15 is sufficient to analyze the asymptotic properties of the two consumption streams.

Case 1. If $\beta_1 > \beta_2$ and $\gamma_1 = \gamma_2$ then $\lim_{t\to\infty} c_{1t}/c_{2t} = \infty$. That is the more patient agent eventually consumes all output in the economy irrespectively of the relation between γs.

Case 2. If $\beta_1 = \beta_2$ and $\gamma_1 > \gamma_2$ then $c_{1t}/c_{2t} = const, \forall t$. That is the less risk-averse agent will get a larger fraction of output then he would otherwise. This is the benefit of insuring the more risk-averse agent.

The conclusion is that the two modeling choices have very different economic implications. The case with heterogeneous discount factors is extreme in the sense that even small discount-factor heterogeneity dwarfs other economic forces in the limit. The risk-aversion heterogeneity is more natural because the economy remains "stationary." Another modeling choice would be heterogeneity in inter-temporal substitution as in Guvenen (2007).

2.2 Other model improvements

An different form of borrowing limit that can be related to bank stress-testing is presented in Stepanchuk and Tsyrennikov [2015]. This paper also have additional details about applying the Garcia-Zangwill trick.

A novel approach to modeling the banking sector was developed in Coimbra and Rey [2017]. The authors assume that the banking sector consists of a continuum of two-period risk-neutral intermediaries each of which is subject to a different leverage constraint. As macroeconomic conditions change the set of constrained intermediaries changes and this effect feeds back into the rate of return.

Atkeson [1991] and Clementi and Hopenhayn [2006] model an intermediary as a single risk-neutral agent. They feature such desirable frictions as moral hazard, limited enforcement, and asymmetric information. Additionally, they solve for the optimal contract between the intermediary and the production sector, unlike other papers.

TBC

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A Appendix

A.1 Endogenous investment

This section describes the formulation like in Brunnermeier and Sannikov (2014). Both agents can produce capital using technology Φ and the new capital be used sold on the market for the price q_k.

The financier and the household solve the following optimization problem:

$$V_j(k_j, b_j, S) = \max \left[u(c_j) + \beta_j E[V_j(k'_j, b'_j, S')]\right] \tag{16}$$

subject to

$$c_j + q_k m_j k_j + q_b b'_j + i_j k_j = (a_j + z_y)k_j + b_j, \tag{17a}$$

$$k'_j = (1 - \delta + z_k + m_j + \Phi(i_j))k_j, \tag{17b}$$

$$i_j \geqslant 0. \tag{17c}$$

$\Phi(i_j)k_j$ is the amount of capital that is produced from $i_j k_j$ consumption units and $m_j k_j$ units are purchased (or sold if negative) on the capital market.

The first-order optimality conditions for i_j and h_j are:

$$-\lambda_c q_k + \lambda_k = 0,$$
$$-\lambda_c + \lambda_k \Phi'(i_j) + \mu_i = 0,$$

where $\lambda_c, \lambda_k, \mu_i$ are the Largrange multipliers on the constraints in (17). The above system can be solved to obtain:

$$1 = q_k \Phi'(i_j) + \mu_i / \lambda_c.$$

If both agents invest in physical capital, $i_h > 0$ and $i_f > 0$, then:

$$i_f = i_h = \Phi^{-1}(q_k). \tag{19}$$

Observe that the investment level defined by (19) maximizes capital production profit: $i_j = \arg\max_i [q_k \Phi(i) - i]$.

Combining the budget constraint with the capital accumulation equation gives:

$$c_j + q_k k'_j + q_b b'_j = (a_j + z_y - i_j + q_k(1 - \delta + z_k + \Phi(i_j)))k_j + b_j. \tag{20}$$

The above equation implies the return on capital, which depends on the agent, as in the paper:

$$R_j = \frac{a_j + z'_y - i'_j + q'_k(1 - \delta + z'_k + \Phi(i'_j))}{q_k}. \tag{21}$$

Brunnermeier and Sannikov's (2014) specification obtains when $z_y = 0$. The special case analyzed by Rappoport and Walsh (2012) obtains when $z_y = 0, \delta = 0$ and $i_j = \Phi(i_j) = 0$. It is the model with stochastic aggregate capital growth: $K'/K = 1 + z'_k$. The model presented and solved in this note assumes that $z_y \neq 0, z_k = 0$. This simplifies the analysis because the aggregate capital is fixed.

B Test Questions

B.1 Model with growth

Consider the model explained in this note with the alternative law of motion for capital:

$$k'_j = (1 + z')k_j, \tag{22}$$

where z' is a positive random variable with the support bounded away from -1. For example, $1 + z' \in \{1 - e, 1 + e\}, e \in (0, 1)$.

1. Assuming that $u_h(c) = u_f(c) = c^{1-\gamma}/(1 - \gamma)$ formulate the optimization problem of each agent. Observe that the state variable k_j plays two roles: first, as part of a portfolio, and, second, as part of the capital law of motion.

2. Show that the problem can be redefined in terms of $\tilde{k}_j = k_j/(k_h + k_f)$. This means that one can use the same approach as for the model with fixed level of capital.

3. Can the same be done when the agents have different coefficients of risk aversion (γ)?

4. Assume that the agents are equally productive, $a_h = a_f$, and consider the model with heterogeneous risk aversion and deterministic capital growth: $0 < \gamma_f < \gamma_h$ and $1 + z'$ is constant and larger than 1. Using the fact that $u'_h(c_{ht})/u'_f(c_{ft}) = const, \forall t$ determine what happens to $\hat{c}_{jt} = c_{jt}/(aK_t)$ as $t \to \infty$.

B.2 Finite horizon with risk-neutral agents

Consider a setting in which both agents are risk-neutral, $u_j(c) = c$, and the economy lasts 3 periods. To make results quantitatively more interesting the last, third, period is interpreted as the "rest of life". Hence, each agent's welfare function is:

$$c_{0j} + \beta_j c_{1j}/(1 - \beta_j).$$

As before the agents can invest in capital and a risk-free bond:

$$c_{1j} = n_{1j} = a_j(1 + z_1)k_{1j} + b_{1j}, \tag{23a}$$

$$c_{0j} + q_{0k}k_{1j} + q_{0b}b_{1j} = n_{0j}. \tag{23b}$$

1. Solve the model for all $(n_{0h}, n_{0f}) \in \Delta^2$ and $z \in \{-0.10, +0.10\}$ numerically using Python.

2. Show that leverage of each agent is given by:

$$leverage = q_{0k} k_{1j} / (q_{0k} k_{1j} + q_{0b} b_{1j})$$

and re-solve the model with the leverage constraint:

$$q_{0k} k_{1j} / (q_{0k} k_{1j} + q_{0b} b_{1j}) \leqslant L \in \{1, 2, 5\}.$$