INTRODUCTION TO OPTIMISATION

Spring 2022

Prerequisites:

- matrix computations
- multivariate calculus

MATRIX COMPUTATIONS

$$\begin{cases} a x + by = b_1 \\ c x + dy = b_2 \end{cases}$$

$$\iff A \begin{pmatrix} x \\ y \end{pmatrix} = b \quad ; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A \in \mathbb{R}^{2\times 2}, \quad b \in \mathbb{R}^2$$

Inner product:
$$\underline{a} \cdot \underline{b} = (\alpha_1 \underline{i} + \alpha_2 \underline{j} + \alpha_3 \underline{k}) \cdot (\beta_1 \underline{i} + \beta_2 \underline{j} + \beta_3 \underline{k})$$
$$= \sum_{i=1}^{3} \alpha_i \beta_i$$

In matrix form:
$$a = (\alpha_1 \ \alpha_2 \ \alpha_3)^T$$
$$b = (\beta_1 \ \beta_2 \ \beta_3)^T$$
$$a^T b = \sum \alpha_i \beta_i$$

GEOMETRY

Equation of a plane: $\mathbb{R}^3$

$$\underline{n} \cdot (\underline{r} - \underline{r}_0) = 0$$

$\underline{n}$ is the normal; $\underline{r}_0$ a point on the plane

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$$

Coordinate form: $c_1 x + c_2 y + c_3 z = d$

$$\left.\begin{aligned} \underline{n} &\hat{=} n = (c_1\ c_2\ c_3)^T \\ \underline{r} &\hat{=} r = (x\ y\ z)^T \\ \underline{r}_0 &\hat{=} r_0 = (x_0\ y_0\ z_0)^T \end{aligned}\right\} \quad n^T r = n^T r_0 = d$$

This notation is agnostic to the dimension: $\mathbb{R}^n$ ok.

What about a line in space?

- There are no coordinate form expressions:

  A line in space is the intersection of two planes.

SURFACE : $z = f(x, y)$

Gradient : $\nabla f = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j}$

Example : $z = ax + by$

$$\nabla f = a \underline{i} + b \underline{j}$$

Fix z ; Constant (intercept changes)

Gradient is the normal to the line.

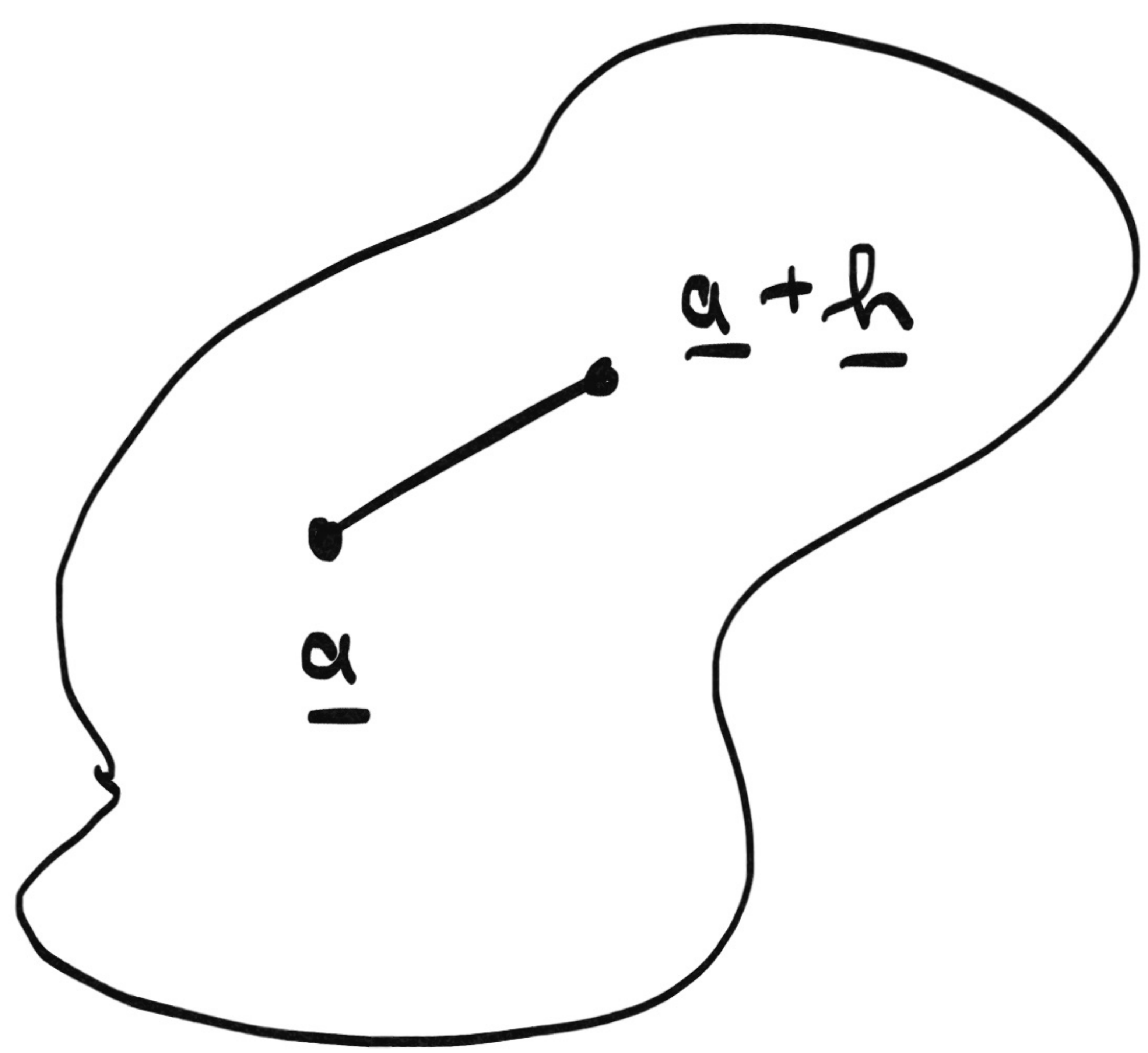
TANGENT PLANE

$z = f(x, y)$; Point (a, b)

$$z = f(a, b) + \frac{\partial f}{\partial x}(x - a) + \frac{\partial f}{\partial y}(y - b)$$

$\Rightarrow$ normal : $\underline{n} = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} - \underline{k}$

TAYLOR



$$f(\underline{a} + \underline{h}) \simeq \sum_{j=0}^{m} \frac{(\underline{h} \cdot \nabla)^j f(\underline{a})}{j!}$$

$m = 1$: Tangent plane!

2^{nd} ORDER (with abusing the notation)

$$f(\underline{x}+\underline{h}) \simeq f(\underline{x}) + \underline{h} \cdot \nabla f(\underline{x}) + \frac{1}{2} \underline{h}\, H_f(\underline{x})\, \underline{h}^T$$

$$H_f(\underline{x}) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} f(\underline{x}) & \frac{\partial^2}{\partial x_2 \partial x_1} f(\underline{x}) & \cdots & \frac{\partial^2}{\partial x_n \partial x_1} f(\underline{x}) \\ \frac{\partial^2}{\partial x_1 \partial x_2} f(\underline{x}) & \frac{\partial^2}{\partial x_2^2} f(\underline{x}) & \cdots & \\ \vdots & & \ddots & \\ \frac{\partial^2}{\partial x_1 \partial x_n} f(\underline{x}) & & & \frac{\partial^2}{\partial x_n^2} f(\underline{x}) \end{pmatrix}$$

The Hessian : Real & Symmetic

It is positive definite if all eigenvalues are positive ; $\lambda_i > 0$

Negative definite : $\lambda_i < 0$

Indefinite, otherwise

LAGRANGE : Constrained Optimisation

For instance: maximise $f(x,y)$ subject to
$g(x,y) = C$

Lagrange function:

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$\nabla L(x,y,\lambda) = 0$ are the critical points.

λ is the Lagrange multiplier.

Geometric interpretation:

$$\nabla f(x,y) \parallel \nabla g(x,y)$$

for the problem to have a solution.

NOTATION (YET ANOTHER LANGUAGE)

Problem P : $\min f(x)$

$$\begin{aligned} \text{s.t.}: \quad & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_j(x) = 0, \quad j = 1, \dots, \ell \\ & x \in \underline{X} \end{aligned}$$

$x \in \mathbb{R}^n$: decision variables

$f : \mathbb{R}^n \to \mathbb{R}$: objective function

$\underline{X} \subseteq \mathbb{R}^n$: ground set (physical constraints)

$g_i, h_j : \mathbb{R}^n \to \mathbb{R}$: constraint functions

$g_i \leq 0$: inequality constraints

$h_j = 0$: equality constraints