MS-C2105 - Introduction to optimization **Fabricio Oliveira**
Final Exam **9.4.2019**

The exam consists of 4 problems and is three hours long. The exam is graded on a scale of 0-100 points. The points assigned to each question are indicated in parenthesis within the text. Read the questions carefully and make sure your answers are clear, readable, and well organised. Good luck!

Problem 1 (25pt)

Consider the following linear programming problem:

$$
\begin{aligned}
P_1 : \text{max. } z_1 = & \ 3x_1 + 2x_2 \\
\text{s.t.: } & 4x_1 + 2x_2 \leq 6 \\
& 2x_1 + 2x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

a. (10pt): Solve this problem using the Simplex method. You are required to:

1. Provide the problem in the standard form and the initial basis.
2. Indicate the variables becoming basic/ nonbasic at each iteration and why.
3. Provide the system of equations OR table form for each basis considered in the progress of the algorithm.
4. Indicate the optimal solution (x_1^*, x_2^*) and optimal objective function value z_1^*.
5. Indicate which constraints are active and inactive at the optimum.

Hint: you only need 2 iterations. You can choose to use either the table representation OR the systems of equations.

b. (5pt): Provide a graphical representation of problem P_1 in **(a.)**. You are required to:

1. Indicate the feasible region.
2. Indicate the points (x_1, x_2) that the method visited at each iteration (including the starting point).

c. (10pt): Show how the information from the optimal basis obtained in **(a.)** can be used to solve P_2 by answering the following:

$$
\begin{aligned}
\text{min. } P_2 : z_2 = & 6y_1 + 4y_2 \\
\text{s.t.: } & 4y_1 + 2y_2 \geq 3 \\
& 2y_1 + 2y_2 \geq 2 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

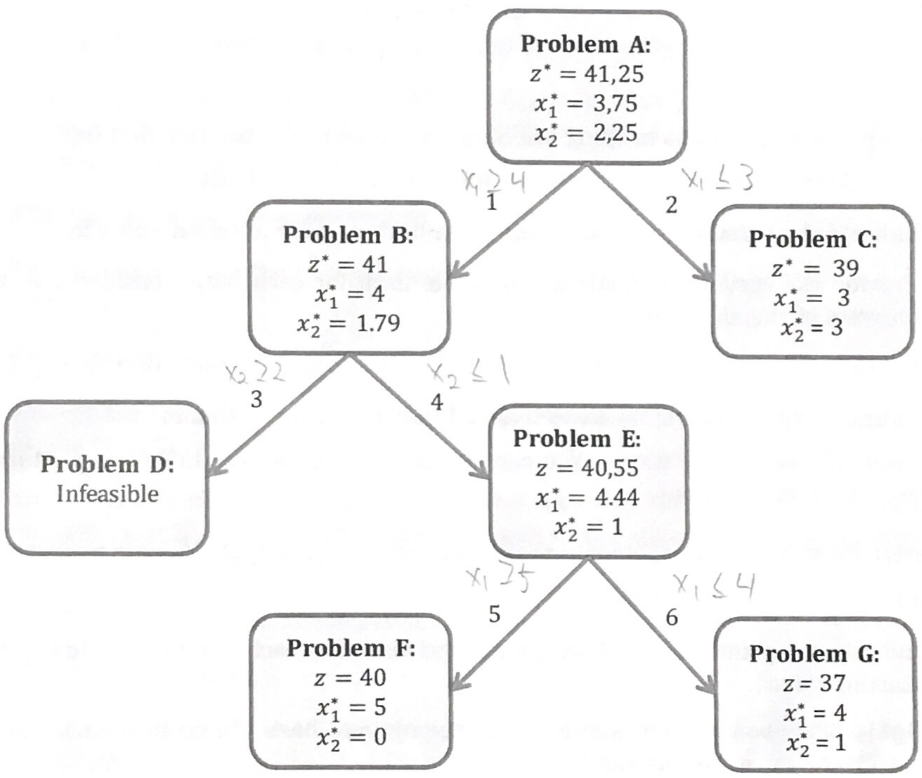
1. What is the relationship between P_1 and P_2? Justify your answer.
2. What result allows you to draw your conclusions? (no proofs required; it suffices to indicate which result and its consequences).
3. What is the optimal point (y_1, y_2) and optimal objective function value z_2^*, according to the information from the optimal basis obtained in **(a.)**.

Problem 2 (15pt)

Considering the following IP problem:

$$\begin{aligned} \text{max. } z = {} & 8x_1 + 5x_2 \\ \text{s.t.: } & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45. \end{aligned}$$

After applying the branch-and-bound method, the following tree was obtained.



The arrows represent the derivation of each subproblem from its parent problem. The numbers above the arrows represent the constraints added to the parent problem to generate each subproblem. The variable selection was performed picking the variable with smallest index (e.g., in case x_1 and x_2 are fractional, x_1 is selected). Please answer the following:

a. (6pt): Identify what are the constraints represented by numbers 1, 2, 3, 4, 5, and 6.

b. (4pt): Which node represent the optimal solution for the IP problem? Please justify.

c. (5pt): Consider the possible sequences in which the subproblems B to G have been solved. Is there a sequence in which some of the subproblems do not need to be solved? Justify your answer.

Problem 3 (25pt)

A chemical plant can purchase up to 15 kg of a chemical for \$10/kg. At a cost of \$3/kg, the chemical can be processed into 1kg of product A; or at a cost of \$5/kg, the chemical can be processed into one kg of product B. If x_1 kg of product A are produced, it sells for a price of $\$(30 - x_1)$ per kilo. If x_2 kg of product B are produced, it sells for a price of $\$(50 - 2x_2)$ per kilo. Let y kg be the amount of the chemical purchased. The model that optimise the profit of the chemical plant is given by:

$$
\begin{aligned}
P : \text{max. } z = {} & x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 - 10y \\
\text{s.t.: } & x_1 + x_2 \leq y \\
& y \leq 15 \\
& x_1, x_2, y \geq 0.
\end{aligned}
$$

a. (6pt): Formulate the Karush-Kuhn-Tucker (KKT) optimality conditions for the above problem P.

b. (4pt): Are the KKT conditions sufficient for optimality in this case? If so, would a point satisfying these conditions be a local or global optimum? Please justify your answers.

c. (5pt): Solve the conditions formulated in **(a.)** to obtain a point $x = (x_1, x_2, y)$ satisfying the KKT conditions. **Hint:** start with the cases in which x_1, x_2 and y are all greater than zero and thus $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

d. (5pt): Consider the following situation: a third product, product C, is incorporated as an alternative in the production process. At a cost of \$6/kg, the chemical can be processed into 2kg of product C. Reformulate the model P that optimise the profit of the chemical plant, incorporating the option of producing product C. You are required to identify any elements you may want to include or modify in problem P. C sells for \$ 10/kg

e. (5pt): Consider the following situation: suppose that, due to technical reasons, either product A or product B can be produced, but not both simultaneously. Reformulate the original model P (not the one you formulated in **(d.)**) to include this situation. You are required to identify any elements you may want to include or modify in problem P.

linear constraints

Problem 4 (35pt)

Consider the following function:

$$f(x_1, x_2) = (2 - x_1)^2 + (2 - 2x_1 - x_2)^2$$

a. (5pt): Obtain the optimum/ optima for this function analytically. Please provide:

1. The optimality conditions used to find the candidate point(s).
2. The candidate point(s) obtained.
3. Arguments supporting whether the point(s) is(are) locally or globally optimal.

b. (15pt): Apply a single iteration of the gradient method to find an optimum for this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an optimal step size λ. You are requested to provide:

1. The expression for the gradient step.
2. The calculations for the optimal step size (analytically). **Hint:** should be a value close to 0.1).
3. The new point found.
4. Answer the following: is this point optimal? Please justify without relying on the results from **(a.)**.

c. (15pt): Apply a single iteration of the Newton's method to find the optimal of this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an a step size of $\lambda = 1$. You are requested to provide:

1. The expression for the Newton step.
2. The new point found.
3. Answer the following: is this point optimal? If so, why did the method only took a single iteration? Please justify without relying on the results from **(a.)**.

Hint: Remind that if $g(x) = (f(x))^n$ then the derivative is $g'(x) = nf(x)^{n-1}f'(x)$. Also, you need this result for **(c.)**: $\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix}$.

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, 3.4.2018

A function (non-graphing) calculator is allowed.

1. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & 2x_1 - x_2 & \leq & 2 \\ & -x_1 + x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

   a) Transform the linear problem into the standard form. (1p)
   b) Solve LP problem using the Simplex method. (3p)
   c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Briefly define the following terms:

   a) Slack variable (1p)
   b) Nash equilibrium (1p)
   c) Pareto optimal solution (1p)
   d) Binary variable (1p)
   e) Shadow price (1p)
   f) Convex set (1p)

3. Consider a problem

$$\begin{array}{lrcl} \min & (x_1-3)^2 + (x_2-3)^2 & & \\ \text{s.t.} & x_1^2 + x_2 & \leq & 4 \\ & -x_1 + x_2 & = & 2 \\ & x_2 & \geq & 0, \end{array}$$

   a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
   b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

4. The mighty Hun emperor Mukbar Attila is planning a conquest in Europe. He has narrowed his options down to four vulnerable cities: Paris, London, Rome and Constantinople. M. Attila has an army of 10 000 Huns at his disposal.

   The potential loot of each target (millions of denars) and the troops required for conquest are presented in Table 1.

   Table 1: Target information

	Loot	Required troops
Paris	2 M	2500
London	1 M	1500
Rome	5 M	5000
Constantinople	4 M	3500

   M. Attila does not want to spread his troops too much - therefore, he cannot attack both Rome and London. In addition, the tactically gifted emperor decides that he cannot conquer London without also overtaking Paris.

   a) Formulate the problem as a linear integer optimization problem, when M. Attila wants to maximize the total amount of received loot. (4p)
   b) After some thinking, M. Attila concludes that he wants a minimum loot of 10 million denars. Formulate the resulting goal programming problem, when the loss of 1 million denars is equally acceptable to hiring 500 more soldiers. (2p)
   Neither case needs to be solved.

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

   a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
   b) It is possible to formulate such an optimization problem, which has exactly two solutions.
   c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.

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MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, 25.5.2018

A function (non-graphing) calculator is allowed.

1. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{rrcl} \max & 2x_1 + 3x_2 & & \\ \text{s.t.} & 2x_1 + 3x_2 & \leq & 6 \\ & x_1 - x_2 & \leq & 1 \\ & x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Briefly define the following terms:

a) Slack variable (1p)
b) Nash equilibrium (1p)
c) Pareto optimal solution (1p)
d) Binary variable (1p)
e) Newton's method (1p)
f) Convex set (1p)

3. Consider a problem

$$\begin{array}{lll} \max\ (x_1-3)^2 + x_2^2 & & \\ \text{s.t.} \quad x_1 & \geq & 1 \\ \qquad x_1^2 - 2 & \leq & x_2 \\ \qquad 2x_1 - 2 & = & x_2\ . \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

4. Vesa is a friend of good beer. He is on a beer journey in Belgium and planning exports to home in a beer store. For financial and transportation reasons, he has decided to buy at most three different beer types: his alternatives are blonde, dubbel, and lambic beer types. Each beer type is available in 0.33 and 0.75 liter bottles. The price list of the bottles is presented in Table 1.

Table 1: Price list

	0.33 l	0.75 l
Blonde	€1.5	€3.5
Dubbel	€2	€3.5
Lambic	€4.5	€6

Small bottles are packed into small crates, and big bottles are packed into big crates. Small bottles cannot be packed into big crates or vice versa. One small crate has a space for 24 small bottles and one big crate for 10 big bottles. Vesa is planning his shoppings so that at least half of the bottles are of lambic type, and the number of blonde type bottles must be greater than or equal to the number of dubbel type bottles. Vesa has €150 to spend and he wishes to maximize the amount of bought beer measured in litres. Formulate the problem as a linear integer programming problem when

a) Vesa's car has a space for arbitrarily many crates and crates are free. (3p)
b) Vesa's car has a space for two small crates and one big crate. A small crate costs €5 and a big crate €6. (3p)
Neither case needs to be solved.

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

a) For an optimization problem, there is always either exactly one solution or no solution at all.
b) It is possible to formulate an optimization problem that has exactly two solutions.
c) The solution of an optimization problem is always located in a corner point of the feasible region.

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, 12/2018

A function (non-graphing) calculator is allowed.

1. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{lrcl} \max & x_1 & & \\ \text{s.t.} & 2x_1 - x_2 & \leq & 2 \\ & -x_1 + x_2 & \leq & 1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Briefly define the following terms:

a) Simplex iteration (1p)
b) Nash equilibrium (1p)
c) Portfolio optimization (1p)
d) Subtour (1p)
e) Dual variable (1p)
f) Convex function (1p)

3. Consider a problem

$$\begin{array}{lrcl} \min & (x_1-3)^2 + (x_2-3)^2 & & \\ \text{s.t.} & x_1^2 + x_2 & \leq & 4 \\ & -x_1 + x_2 & = & 2 \\ & x_2 & \geq & 0, \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

4. Consider a problem

$$\begin{array}{rlcl} \max & 9x_1 + 5x_2 & & \\ \text{s.t.} & 4x_1 + 9x_2 & \leq & 35 \\ & x_1 & \leq & 6 \\ & x_1 - 3x_2 & \geq & 1 \\ & 3x_1 + 2x_2 & \leq & 2 \\ & x_1, x_2 & \in & \mathbb{Z}_+ \cup \{0\} \end{array}$$

a) Solve the LP relaxation of the problem graphically. (1p)
b) Determine the solution to the problem with Branch & Bound algorithm. Solve subproblems graphically. (3p)
c) Draw the course of your solution in tree form, and justify with it that the solution you got is the best possible integer solution to the original problem. (2p)

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
b) It is possible to formulate such an optimization problem, which has exactly two solutions.
c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, 4.4.2017

A function (non-graphing) calculator is allowed.

1. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{lrcl} \max & 2x_1 + x_2 & & \\ \text{s.e.} & x_1 & \leq & 2 \\ & x_1 + 2x_2 & \leq & 6 \\ & x_1, x_2 & \geq & 0 \end{array}$$

   a) Transform the linear problem into the standard form. (1p)
   b) Solve LP problem using the Simplex method. (3p)
   c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Briefly define the following terms:

   - slack variable
   - binary choice
   - subtour
   - function f is convex
   - portfolio optimization
   - shadow price

3. Consider a problem

$$\begin{array}{lrcl} \min & (x_1 - 8)^2 + (x_2 - 6)^2 & & \\ \text{s.e.} & x_1^2 + x_2^2 & \leq & 25 \\ & x_1 + 3x_2 & \leq & 15 \\ & x_1, x_2 & \geq & 0, \end{array}$$

   a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
   b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (3p)
   c) What is the solution if the objective function is $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 1)^2$ and the feasible region remains the same?

4. The mighty Hun emperor Mukbar Attila is planning a conquest in Europe. He has narrowed his options down to four vulnerable cities: Paris, London, Rome and Constantinople. M. Attila has an army of 10 000 Huns at his disposal.

   The potential loot of each target (millions of denars) and the troops required for conquest are presented in table 1.

   Table 1: Target information

	Loot	Required troops
Paris	2 M	2500
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Rome	5 M	5000
Constantinople	4 M	3500

   M. Attila does not want to spread his troops too much - therefore, he cannot attack both Rome and London. In addition, the tactically gifted emperor decides that he cannot conquer London without also overtaking Paris.

   a) Formulate the problem as a linear integer optimization problem, when M. Attila wants to maximize the total amount of received loot. (4p)
   b) After some thinking, M. Attila concludes that he wants a minimum loot of 10 million denars. Formulate the resulting goal programming problem, when the loss of 1 million denars is equally acceptable to hiring 500 more soldiers. (2p)
   Neither case needs to be solved.

5. a) Derive a 1-dimensional secant method starting from Newton's method. (2p)
   b) Describe the steps of an n-dimensional gradient method. (4p)

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Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, 4.9.2017

A non-graphical calculator is allowed

1. Consider a problem

$$\begin{array}{lll} \max & 4x_1 + 3x_2 & \\ \text{s.e.} & x_1 + x_2 & \leq \; 3 \\ & -3x_1 + 5 & \geq \; x_2 \\ & 2x_1 - x_2 & \geq \; -1 \\ & x_1, x_2 & \geq \; 0. \end{array}$$

   a) Transform the linear problem into the standard form (1p)
   b) Solve the LP problem using the Simplex method. (3p)
   c) Draw a picture of the feasible region of the problem and show the progress of the Simplex algorithm in the region. (2p)

2. Briefly define the following terms:

   a) Slack variable (1p)
   b) Nash equilibrium (1p)
   c) Pareto optimal solution (1p)
   d) Binary variable (1p)
   e) Shadow price (1p)
   f) Convex set (1p)

3. Consider a problem

$$\begin{array}{lrcr} \min & (x_1 - 5)^2 + (x_2 - 5)^2 & & \\ \text{s.e.} & x_1^2 + x_2^2 & \leq & 25 \\ & x_1 & \leq & 3 \\ & x_1, x_2 & \geq & 0, \end{array}$$

   a) Solve the problem graphically. Draw a picture of the feasible region of the problem and the contour lines of the objective function. (2p)
   b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

4. Consider a problem

$$
\begin{array}{lrcl}
\max & 3x_1 + 2x_2 & & \\
\text{s.e.} & x_1 + 2x_2 & \leq & 3 \\
 & 2x_1 + 2x_2 & \leq & 5 \\
 & x_1, x_2 & \in & \mathbb{Z}_+
\end{array}
$$

a) Solve the LP relaxation of the problem graphically. (2p)
b) Solve the original problem by using the Branch-and-Bound method. Solve the subproblems graphically. (3p)
c) Present your solution in a tree form. With it explain why the solution you found is indeed the optimal integer solution. (1p)

5. a) Derive a 1-dimensional secant method starting from Newton's method. (2p)
b) Describe the steps of an n-dimensional gradient method. (4p)

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Vesa Husgafvel

Exam, 5.4.2016

A function calculator is allowed. A graphing calculator or any other tools are not permitted.

1. Briefly define the following terms of the Simplex method:
   a) pivot element (1p)
   b) exiting variable (1p)
   c) optimality condition (2p)
   d) feasibility condition (2p)

2. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{rrcl} \max & 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 3x_2 & \leq & 5 \\ & 2x_1 - 2x_2 & \leq & 3 \\ & x_1 + 2x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

   a) Transform the linear problem into the standard form. (1p)
   b) Solve LP problem using the Simplex method. (3p)
   c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

3. Consider a problem

$$\begin{array}{rlcl} \max & x_1^2 + (x_2 - 4)^2 & & \\ \text{s.t.} & x_1 + x_2 & = & 2 \\ & x_1^2 + x_2 & \leq & 4 \\ & x_1 & \leq & 1 \\ & x_2 & \geq & 0 \end{array}$$

   a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
   b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and examine if the solution satisfies these conditions. (4p)

4. Vesa is a friend of good beer. He is on a beer journey in Belgium and planning exports to home in a beer store. For financial and transportation reasons, he has decided to buy at most three different beer types: his alternatives are blonde, dubbel, and lambic beer types. Each beer type is available in 0.33 and 0.75 liter bottles. The price list of the bottles is presented in Table 1.

Table 1: Price list

	0.33 l	0.75 l
Blonde	€1.5	€3.5
Dubbel	€2	€3.5
Lambic	€4.5	€6

Small bottles are packed into small crates, and big bottles are packed into big crates. Small bottles cannot be packed into big crates or vice versa. One small crate has a space for 24 small bottles and one big crate for 10 big bottles. Vesa is planning his shoppings so that at least half of the bottles are of lambic type, and the number of blonde type bottles must be greater than or equal to the number of dubbel type bottles. Vesa has €150 to spend and he wishes to maximize the amount of bought beer measured in litres. Formulate the problem as a linear integer programming problem when

a) Vesa's car has a space for arbitrarily many crates and crates are free. (3p)
b) Vesa's car has a space for two small crates and one big crate. A small crate costs €5 and a big crate €6. (3p)
Neither case needs to be solved.

5. a) Derive a 1-dimensional secant method starting from Newton's method. (2p)
b) Describe the steps of an n-dimensional gradient method. (4p)

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo / Markus Mattila

Exam, December 19th 2016

- Briefly define the following terms.

  a) Surplus variable (1p)
  b) Shadow price (1p)
  c) Binary variable (1p)
  d) Function f is convex (1p)
  e) Efficient, or Pareto optimal solution (1p)
  f) Portfolio optimization (1p)

- Use tabular Simplex algorithm in the following problem:

$$\begin{array}{rrcl} \max & 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 3x_2 & \leq & 5 \\ & 2x_1 - 2x_2 & \leq & 3 \\ & x_1 + 2x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

  a) Transform the linear problem into the standard form. (1p)
  b) Solve LP problem using the Simplex method. (3p)
  c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

- Consider a problem

$$\begin{array}{lllc} \min & (x_1 - 5)^2 + (x_2 - 4)^2 & & \\ \text{s.e.} & x_1^2 - 4x_1 - x_2 + 5 & \leq & 0 \\ & 2x_1 + 3x_2 - 12 & = & 0 \\ & -x_1 & \leq & 0 \\ & x_2 - 6 & \leq & 0 \\ & -2x_2 + 3 & \leq & 0. \end{array}$$

  a) Solve the problem graphically. Draw a picture of the feasible region of the problem and draw also the contour lines of the objective function. (2p)
  b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

The exam continues next page

- You are the production manager for car manufacturing at Teddy's Four-Wheel, Inc. Currently, the collection of sold cars consists only of one model, Winnie. The production of a Winnie requires 2 tonnes of steel and 100 hours of work. The company has also an option to extend its operations and to begin to manufacture new sport car model named Bear. A Bear requires 1.5 tonnes of steel and 150 hours of work to be manufactured. In addition, the production of Bers requires an investment to a new production line. The production line costs the same amount as the profit from selling 1000 Bear cars, and the life of the production line would be 10 years. Each week the company has 24 tonnes of steel and 1200 hours of work in use. The profit from a Bear model car is double compared to the profit from selling a Winnie model car.

  a) What kind of weekly production would maximize the profits of the company? Formulate the problem as *linear integer problem*. You do not need to solve the problem. (4p)
  b) You plan to solve the problem with Branch-and-Bound algorithm. Present the general description of the algorithm. (2p)

- Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

  a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
  b) It is possible to formulate such an optimization problem, which has exactly two solutions.
  c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo/Ilmari Pärnänen

Exam, 8.4.2015

1. Briefly define the following terms:
   - binary choice
   - set-covering problem
   - subtour
   - Lagrangian function
   - non-feasible basic solution
   - simplex iteration

2. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{rrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & x_1 + x_2 & \geq & 2 \\ & -x_1 + x_2 & \geq & 1 \\ & x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

   a) Transform the linear problem into the standard form. (1p)
   b) Solve LP problem with Simplex algorithm. (3p)
   c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

3. Consider a problem

$$\begin{array}{lllll} \max & 9x_1 + 5x_2 & & \\ \text{s.t.} & 4x_1 + 9x_2 & \leq & 35 \\ & x_1 & \leq & 6 \\ & x_1 - 3x_2 & \geq & 1 \\ & 3x_1 + 2x_2 & \leq & 2 \\ & x_1, x_2 & \in & \mathrm{Z}_+ \cup \{0\} \end{array}$$

   a) Solve the LP relaxation of the problem graphically. (1p)
   b) Determine the solution to the problem with Branch & Bound algorithm. Solve subproblems graphically. (3p)
   c) Draw the course of your solution in tree form, and justify with it that the solution you got is the best possible integer solution to the original problem. (2p)

4. Mantel Ltd. produces a toy cars, whose final assembly must include four wheels and two seats. The factory producing the parts operates three shifts a day. The followind table provides the amounts produced of each part in the three shifts.

	Unites produced per run	
Shift	Wheels	Seats
1	500	300
2	600	280
3	640	360

Ideally, the number of wheels produced is exactly twice that of the number of seats. However, because production rates vary from shift to shift, exact balance in production may not be even possible. Thus Mantel is interested in determining the number of production runs in each shift that minimizes the imbalance in the production of the parts. The capacity limitations restrict the number of runs to between 4 and 5 for shift 1, 10 and 20 for shift 2, and 3 and 5 for shift 3.

The company has the necessary software to solve the problem, but writing the problem properly to computer causes problems. The manager asks you, a summer trainee who has studied the Introduction to optimization course, to help in the matter. Formulate the manufacturing of the parts as a goal programming model.

5. Consider a problem

$$\begin{array}{lrcl} \min & (x_1 - a)^2 + (x_2 - b)^2 & & \\ \text{s.t.} & -4x_1 + 4x_2 & \leq & 12 \\ & 5x_1 + 2x_2 & \leq & 20 \\ & x_1, x_2 & \geq & 0, \end{array}$$

where i) $a = 2, b = 2$ and ii) $a = 4, b = 6$.

a) Solve cases i) and ii) graphically. Draw a picture of the feasible region of the problem and draw also the contour lines of the objective function. (2p)
b) In both cases, present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo/Markus Mattila

Exam, 8.4.2015

1. Briefly define the following terms:
   - slack variable
   - binary choice
   - subtour
   - function f is convex
   - portfolio optimization
   - shadow price

2. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{lrcl} \max & 3x_1 - 2x_2 & & \\ \text{s.e.} & x_1 + x_2 & \geq & 2 \\ & 3x_1 - x_2 & \leq & 8 \\ & -x_1 + 4x_2 & \leq & 12 \\ & x_1, x_2 & \geq & 0 \end{array}$$

   a) Transform the linear problem into the standard form. (1p)
   b) Solve LP problem using the M method. (3p)
   c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

3. Consider a problem

$$\begin{array}{lllr} \min & (x_1 - 5)^2 + (x_2 - 3)^2 & & \\ \text{s.e.} & x_2^2 - x_1 & \leq & 0 \\ & 2x_2 - x_1 & = & 0 \\ & x_1, x_2 & \geq & 0. \end{array}$$

   a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
   b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

4. Mantel Ltd. produces a toy cars, whose final assembly must include four wheels and two seats. The factory producing the parts operates three shifts a day. The followind table provides the amounts produced of each part in the three shifts.

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The company has the necessary software to solve the problem, but writing the problem properly to computer causes problems. The manager asks you, a summer trainee who has studied the Introduction to optimization course, to help in the matter. Formulate the manufacturing of the parts as a goal programming model.

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
b) It is possible to formulate such an optimization problem, which has exactly two solutions.
c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.

AALTO UNIVERSITY SCHOOL OF SCIENCE
Department of Mathematics and Systems Analysis

MS-C2105 Introduction to optimization Harri Ehtamo/Markus Mattila

Exam, December 14th 2015

1. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{rrcr} \max & 2x_1 + x_2 & & \\ \text{s.t.} & 2x_1 - x_2 & \leq & 6 \\ & x_1 + 2x_2 & \leq & 10 \\ & x_1 - x_2 & \geq & -3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem with Simplex algorithm. (3p)
c) Draw a picture of the feasible region of the problem and how Simplex algorithm progresses in the region. (2p)

2. Consider a problem

$$\begin{array}{lllr} \min & (x_1 - 5)^2 + (x_2 - 4)^2 & & \\ \text{s.e.} & x_1^2 - 4x_1 - x_2 + 5 & \leq & 0 \\ & 2x_1 + 3x_2 - 12 & = & 0 \\ & -x_1 & \leq & 0 \\ & x_2 - 6 & \leq & 0 \\ & -2x_2 + 3 & \leq & 0. \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and draw also the contour lines of the objective function. (2p)
b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

The exam continues next page

3. You are the production manager for car manufacturing at Teddy's Four-Wheel, Inc. Currently, the collection of sold cars consists only of one model, Winnie. The production of a Winnie requires 2 tonnes of steel and 100 hours of work. The company has also an option to extend its operations and to begin to manufacture new sport car model named Bear. A Bear requires 1.5 tonnes of steel and 150 hours of work to be manufactured. In addition, the production of Bers requires an investment to a new production line. The production line costs the same amount as the profit from selling 1000 Bear cars, and the life of the production line would be 10 years. Each week the company has 24 tonnes of steel and 1200 hours of work in use. The profit from a Bear model car is double compared to the profit from selling a Winnie model car.

   a) What kind of weekly production would maximize the profits of the company? Formulate the problem as *linear integer problem*. You do not need to solve the problem. (4p)
   b) You plan to solve the problem with Branch-and-Bound algorithm. Present the general description of the algorithm. (2p)

4. Briefly define the following terms.

   a) Corner point (1p)
   b) Shadow price (1p)
   c) Binary variable (1p)
   d) Lagrangian function (1p)
   e) Efficient, or Pareto optimal solution (1p)
   f) Subtour (1p)

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

   a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
   b) It is possible to formulate such an optimization problem, which has exactly two solutions.
   c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.