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Introduction To Optimization

Simplex standard form
- $\le \Rightarrow +S_i,\ S_i \ge 0$
- $\ge \Rightarrow -S_i,\ S_i \ge 0$
- RHS $< 0 \Rightarrow$ multiply both sides by -1
- $x_i \le 0 \Rightarrow x_i' \ge 0$ where $x_i' = -x_i$
- $x_i \in R \Rightarrow x_i = x_i^+ - x_i^-$
- $\min z = x_1 + x_2 \Rightarrow \max -z = x_1 + x_2$

Simplex Algorithm
$\max z = 3x_1 + 3x_2$
s.t $4x_1 + 2x_2 \le 6$ (1)
$2x_1 + 2x_2 \le 4$ (2)
$x_1, x_2 \ge 0$ (3)

(A)

basic	x_1	x_2	S_1	S_2	Sol
z	-3	-3	0	0	0
S_1	4	2	1	0	6
S_2	2	2	0	1	4

Entering var x_K (PC) negativ coeff, max $|x_K|$
Leaving variable $\min;\ \{b_i/a_{iK} : a_{iK} > 0\}$
$\Rightarrow$ Enter: x_1, leave: S_1

(B)

basic	x_1	x_2	S_1	S_2	Sol
z	0	-3/2	3/4	0	9/2
x_1	1	1/2	1/4	0	3/2
S_2	0	1	-1/2	1	1

$\Rightarrow$ Enter: x_2, leave: S_2

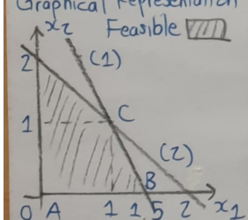
(C)

basic	x_1	x_2	S_1	S_2	Sol
z	0	0	0	3/2	6
x_1	1	0	1/2	-1/2	1
x_2	0	1	-1/2	1	1

dual

Row operations Primal
$R_{other} + m \times R_{pivot} = 0$
$\Rightarrow$ Optimum: $z = 6$
$(x_1, x_2) = (1, 1)$
Active: (1)(2)
Inactive: (3)

Graphical Representation
Feasible



The points visited at each iteration
$A \to B \to C$

Primal & Dual

Primal	Dual
max	min
Ind terms b	obj func c
Obj func c	Ind terms b
i-th row const	i-th col const
i-th col const coeff	i-th row const coeff

Constraints	Variables
$\le$	≥ 0
$\ge$	≤ 0
$=$	$\in R$

Switch if primal min, dual max

Variables	Constraints
≥ 0	$\ge$
≤ 0	$\le$
$\in R$	$=$

Switch if primal min, dual max

$\max z_1 = 3x_1 + 3x_2$	$\min z_2 = 6y_1 + 4y_2$
s.t $4x_1 + 2x_2 \le 6$	s.t $4y_1 + 2y_2 \ge 3$
$2x_1 + 2x_2 \le 4$	$2y_1 + 2y_2 \ge 3$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$

$\min z_2 = \max z_1 = 6$
$(y_1, y_2) = (s_1, s_2) = (0, \frac{3}{2})$
Strong duality
$\min cx,\ Ax \ge b,\ x \ge 0$
$\max wb,\ wA \le c,\ w \ge 0$

Two phase method: Add r_i vars for $\ge$ and $=$ sign
$\min z = 4x_1 + x_2$
s.t $3x_1 + x_2 = 3$
$4x_1 + 3x_2 \ge 6$
$x_1 + 2x_2 \le 4$
$\Rightarrow \min z = r_1 + r_2$
s.t $3x_1 + x_2 + r_1 = 3$
$4x_1 + 3x_2 - S_1 + r_2 = 6$
$x_1 + 2x_2 + S_2 = 4$
$x_1, x_2, S_2, S_1, r_1, r_2 \ge 0$

	x_{12}	r_{12}	s_{12}	Sol
z	0 0	-1 -1	0 0	$=R_2+R_3$
r_1	3 1	1 0	0 0	3
r_2	4 3	0 1	-1 0	6
S_2	1 2	0 0	0 1	4

if original obj min $(-1,-1)$ max $(1,1)$

	r_{12}	Sol
$\Rightarrow$ z	-1 -1	0

basic feasible sol exists
$\Rightarrow$ 2nd phase
Delete col r_1, r_2

	x_{12}	s_{12}	Sol
z	-4 -1	0 0	0

& replace original obj. func

Optimality cond: entering var is non-basic var having most negative coeff in Z row. Optimum reached when all Z-row coeff ≥ 0
Feasibility cond: for both max & min, leaving vars is the basic with smallest non-negative ratio (strictly positive denominator)

Sensitivity Analysis
Optimality Conditions
$r_N = (c_N^T - c_B^T B^{-1} N) \le 0$
$x_B = B^{-1}\bar{b} \ge 0$
1) Change in resources available, term b
Let $\bar{b} = b + \Delta b$
- Optimal cond not affected (r_N not depend on b)
- Feasible cond affected: $B^{-1}(b + \Delta b) \ge 0$ must be true

2) Change in obj. func coeff
- Feasible cond not affected
- Optimal cond affected: $\bar{c}_N - \bar{c}_B^T B^{-1} N \le 0$ must hold

* Changes in resources

	$x_{123\ldots}$	s_3	Sol	Δb_3
z		10	0	10
S_1		-8	24	-8
x_3		-4	8	-4

$\Rightarrow 24 - 8\Delta b_3 \ge 0$, etc
$b_3 = 8 \Rightarrow [b_3 + \Delta b_3] \in [\frac{20}{3}, 10]$

*Changes in obj. func coeff

	x_1	x_2	Sol
z	0	$(5/4\Delta c_1)$	$280+2\Delta c_1$
x_1	1	5/4	2

$\Rightarrow (5/4)\Delta c_1 \ge 0$, etc
$c_1 = 60 \Rightarrow c_1 + \Delta c_1 \in [60, 80]$

Binary Programming
$d \in \{0, 1\}$ (decision var)
$\Rightarrow \max.\ 2x - 10d$ (x can be 0 or 10d)
Mutual ex: $dA + (1-d)B$

Branch-and-bound method
At each node, choose the largest non-integer sol then branch $x_i \le \text{floor}(x_i)$ and $x_i \ge \text{ceil}(x_i)$
Pruning by optimality: if sol are ints $\Rightarrow$ stop
Pruning by infeasibility
Pruning by bound: if bound $\le$ lower bound (min) $\Rightarrow$ stop

$z = 41.25, x_1 = 3.75, x_2 = 2.25$
$x_1 \ge 4$: $z = 41$, $x_1 = 4, x_2 = 1.79$
$x_1 \le 3$: $z = 39$, $x_1 = 3, x_2 = 3$

Newton-Raphson method
Init $\epsilon > 0$, init point x_0, $k = 0$
while $\|d\| > \epsilon$ do
$d = -J(x_k)^{-1} f(x_k)$
$x_{k+1} = x_k + d$
$k += 1$
end while, return x_k

M-method: like 2-phase
$\min z = 4x_1 + x_2 + Mr_1 + Mr_2$
$M = 100$ (min)
$-M = -100$ (max)
if r_{12} non zero $\Rightarrow$ no sol

Local Optimum
- local min: $f'(x_0) = 0, f''(x_0) > 0$
- local max: $f'(x_0) = 0, f''(x_0) < 0$
- inflection: $f'(x_0) = 0, f''(x_0) = 0$

Global Optimum, convex, concave
- $ax + b$: both cv & cc
- $x^2; x^4$: cv, $-x^2, -x^4$: cc
- x^3: cc for $x \le 0$, cv for $x \ge 0$
- e^x: cv
- $\|x\|_p$: cv
- x^p: cv, $p \in (-\infty, 0] \cup [1, +\infty)$; cc, $p \in [0, 1]$
- CV func (CV func) = CV
- $\log(x)$: cc, $x > 0$
- 1/CV = CC, 1/CC = CV

* CV + CV = CV, CC + CC = CC

- Local min, CV $\Rightarrow$ global min
- Local max, CC $\Rightarrow$ global max

Taylor's series
$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0)$

Hessian matrix
$H(x) = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 \end{bmatrix}$
Let $\nabla f(x_1, x_2) = \begin{bmatrix} 10x_1 + 5x_2 \\ 4x_1 + 2x_2 \end{bmatrix}$
$\Rightarrow H(x) = \begin{bmatrix} 10 & 5 \\ 4 & 2 \end{bmatrix}$
Optimality cond: $\nabla f(\bar{x}) = 0$
$\forall x, (x - x_0)^T H(x_0)(x - x_0) > 0$
$\Rightarrow x_0$ is local min
$\forall x, (x - x_0)^T H(x_0)(x - x_0) < 0$
$\Rightarrow x_0$ is local max

Positive def	Negativ def
-All positive eigenvalues	-All negative eigenvalues
-or positive det	-or negativ det

(if only H is symmetric)
Ex: $f(x) = 1/2(x_1^2 + 2x_2^2)$
$\nabla f(x) = [x_1, 2x_2]^T$
$H(x) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow$ 1st order cond
$\Rightarrow \nabla f(x) = 0 \Rightarrow \bar{x} = (0, 0)$
2nd order cond: $H(\bar{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
$Hx = \lambda x \Rightarrow (H - \lambda I)x = 0$
$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$
$\Rightarrow (1 - \lambda)(2 - \lambda) = 0$
$\Rightarrow \lambda_1 = 1, \lambda_2 = 2$
$\Rightarrow H(\bar{x})$ is positive def
$\Rightarrow \bar{x}$ is local optimum
When $H(x)$ is positive def for all $x \Rightarrow f$ is strictly CV
$\Rightarrow \bar{x}$ is global min

Hessian and its det
$H(x) > 0, D(x) > 0 \Rightarrow$ local min
$H(x) < 0, D(x) > 0 \Rightarrow$ local max
$D(x) < 0 \Rightarrow$ saddle point
$D(x) = 0 \Rightarrow$ no conclusion

Line search: Bisection method
initialize: tolerance $\epsilon > 0$
$[a_0, b_0] = [a, b]$, $k = 0$
while $b_k - a_k > \epsilon$ do
$\lambda_k = (b_k + a_k)/2$
Evaluate $f'(\lambda_k)$
if $f'(\lambda_k) = 0$, return λ_k
else if $f'(\lambda_k) > 0$ (min)
else if $f'(\lambda_k) < 0$ (max)
$a_{k+1} = a_k,\ b_{k+1} = \lambda_k$
else $a_{k+1} = \lambda_k,\ b_{k+1} = b_k$
end if
$k += 1$
end while
return $\bar{\lambda} = (a_k + b_k)/2$

Line search: Newton's method
initialize: tolerance $\epsilon > 0$
step size x_0, $k = 0$
while $|f'(x_k)| > \epsilon$ do:
$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$
$k += 1$
end while
return $\bar{x} = x_k$
This is for finding roots of $f'(x)$. $x_{k+1} = x_k - \frac{f(x)}{f'(x)}$ is for roots of $f(x)$
Quadratic approx q of f at x_k: $q(x) = f(x_k) + f'(x_k) \times (x - x_k) + 1/2 f''(x_k)(x - x_k)^2$
For quadratic problems, the approx q is exact $\Rightarrow$ only one iteration is needed

Gradient descent method
initialize: tolerance $\epsilon > 0$
initial point x_0, $k = 0$
while $\|\nabla f(x_k)\| > \epsilon$, do
$d_k = -\nabla f(x_k)$ (min)
$d_k = \nabla f(x_k)$ (max)
$\bar{\lambda} = \text{argmin}_\lambda f(x_k + \lambda d_k)$ min
$\bar{\lambda} = \text{argmax}_\lambda f(x_k + \lambda d_k)$ max
$x_{k+1} = x_k + \bar{\lambda} d_k$
$k = k + 1$
end while
return x_k
- When $\|d\| \le 1$ uses Euclidean norm $\Rightarrow$ steepest descend & gradient method same

Newton's method: deflected steepest descent with $H(x)$
$\Rightarrow$ rely on quadratic approx rather than liner like univar case
Assume $H^{-1}(x_k)$ exists;
Initialize: tolerance $\epsilon > 0$, initial point x_0, iteration $k = 0$
while $\|\nabla f(x_k)\| > \epsilon$ do
$d = -H^{-1}(x_k)\nabla f(x_k)$ min
$d = H^{-1}(x_k)\nabla f(x_k)$ max
$\bar{\lambda} = \text{argmin}_\lambda f(x_k + \lambda d)$ min
$\bar{\lambda} = \text{argmax}_\lambda f(x_k + \lambda d)$ max
$x_{k+1} = x_k + \bar{\lambda} d$
$k += 1$
end while
return x_k
- $\bar{\lambda} = 1 \Rightarrow$ pure Newton's method
- x_0 far from optimal $\Rightarrow$ not converge

KKT conditions - Lagrange
- Lagrangian function
$L(x, \mu) = f(x) + \sum_{i=1}^{e} \mu_i h_i(x)$
$\Rightarrow$ optimize $L(x, \mu)$
$\Rightarrow$ unconstrained prob
$\Rightarrow h_i(x)$ is measure of infea
- KKT general conditions

If $\bar{x}$ is optimal solution
$\Rightarrow (\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies
(1) $\nabla f(\bar{x}) + \sum^{m} \bar{\lambda}_i \nabla g_i(x) + \sum^{e} \bar{\mu}_i \nabla h_i(\bar{x}) = 0$
(2) $g_i(\bar{x}) \le 0$, $i = 1 \to m$
(3) $h_i(\bar{x}) = 0$, $i = 1 \to e$
(4) $\bar{\lambda}_i g_i(\bar{x}) = 0$, $i = 1 \to m$
(5) $\bar{\lambda}_i \ge 0$, $i: 1 \to m$ (min)
$\bar{\lambda}_i \le 0$, $i: 1 \to m$ (max)

Slater's constraint qualification
If CQ holds, KKT conditions become necessary and sufficient for global optimality
CQ conds:
(1) $f(x)$ CV (min); $f(x)$ CC (max)
(2) $g(x)$ CV, $\exists x, g(x) < 0$ min; $g(x)$ CC, $\exists x, g(x) > 0$ max
(3) h: affine functions
$\Rightarrow (\bar{x}, \bar{\lambda})$ satisfies KKT conds and CQ satisfies $\Rightarrow \bar{x}$ is global opt

CQ holds $\Rightarrow (0.5, 1.5)$ is global opt

$\min z = (x_1 - 1)^2 + (x_2 - 2)^2$
$-x_1 + x_2 = 1$, $x_1 + x_2 \le 2$
(1): $\begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 - 2) \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
(2) $x_1 + x_2 - 2 \le 0$ (3) $-x_1 + x_2 - 1 = 0$
(4) $\lambda(x_1 + x_2 - 2) = 0$ (5) $\lambda \ge 0$

9.4.2019

Problem 1 (25pt)

Consider the following linear programming problem:

$$
\begin{aligned}
P_1 : \text{max. } z_1 &= 3x_1 + 2x_2 \\
\text{s.t.: } & 4x_1 + 2x_2 \leq 6 \\
& 2x_1 + 2x_2 \leq 4 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

a. (10pt): Solve this problem using the Simplex method. You are required to:

1. Provide the problem in the standard form and the initial basis.
2. Indicate the variables becoming basic/ nonbasic at each iteration and why.
3. Provide the system of equations OR table form for each basis considered in the progress of the algorithm.
4. Indicate the optimal solution (x_1^*, x_2^*) and optimal objective function value z_1^*.
5. Indicate which constraints are active and inactive at the optimum.

Hint: you only need 2 iterations. You can choose to use either the table representation OR the systems of equations.

b. (5pt): Provide a graphical representation of problem P_1 in **(a.)**. You are required to:

1. Indicate the feasible region.
2. Indicate the points (x_1, x_2) that the method visited at each iteration (including the starting point).

c. (10pt): Show how the information from the optimal basis obtained in **(a.)** can be used to solve P_2 by answering the following:

$$
\begin{aligned}
\text{min. } P_2 : z_2 =& 6y_1 + 4y_2 \\
\text{s.t.: } & 4y_1 + 2y_2 \geq 3 \\
& 2y_1 + 2y_2 \geq 2 \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

1. What is the relationship between P_1 and P_2? Justify your answer.
2. What result allows you to draw your conclusions? (no proofs required; it suffices to indicate which result and its consequences).
3. What is the optimal point (y_1, y_2) and optimal objective function value z_2^*, according to the information from the optimal basis obtained in **(a.)**.

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Introduction To Optimization (MS-C2105)

Simplex method, standard form

- $\le$ sign $\Rightarrow$ add s_i, condition $s_i \ge 0$, change $\le$ into $=$
- $\ge$ sign $\Rightarrow$ if RHS is negative multiply with (-1), add $-s_i$, condition $s_i \ge 0$, change $\ge$ into $=$

Ex: $x_1 + x_2 \le 5$
$\Rightarrow x_1 + x_2 + s_1 = 5$
$s_1 \ge 0$

- $x_1 + x_2 \le -1$
$\Rightarrow -x_1 - x_2 \ge 1$
$\Rightarrow -x_1 - x_2 - s_1 = 1,\ s_1 \ge 0$

- For constraint $x_i \le 0$
Let $x_i' = -x_i \Rightarrow x_i' \ge 0$
Substitute into the obj and constraint functions:
$\left.\begin{array}{l} x_1 + x_2 \le 5 \\ x_2 \le 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 - x_2' \le 5 \\ x_2' \ge 0 \end{array}$

- Free variables: $x_i \in R$
Replace $x_i = x_i^+ - x_i^-$
add constraints $x_i^+ \ge 0,\ x_i^- \ge 0$
$\left.\begin{array}{l} x_1 + x_2 \le 5 \\ x_2 \in R \end{array}\right\} \Rightarrow \begin{array}{l} x_1 + x_2^+ - x_2^- \le 5 \\ x_2^+ \ge 0,\ x_2^- \ge 0 \end{array}$

- In standard form, it maximize the function z (objective)
min $z = x_1 + x_2$
$\Rightarrow$ max $-z = x_1 + x_2$
When optimal sol is found for z, just change its sign
Basic: $-z = -4/3 \Rightarrow z_{min} = 4/3$

* Change into standard form:
min $z = 2x_1 + x_2$
so that $7x_1 - 3x_2 \le 4$
$x_1 + 7x_2 \ge 7$
$x_1 \ge 0, x_2 \in R$
$\Rightarrow$ Standard normal form
max $-z = 2x_1 + x_2^+ - x_2^-$
s.t $7x_1 - 3x_2^+ + 3x_2^- + s_1 = 4$
$x_1 + 2x_2^+ - 2x_2^- - s_2 = 7$
$x_1, x_2^+, x_2^-, s_1, s_2 \ge 0$

- Negative RHS: b_i, multiply by -1
$x_1 - x_2 \ge -7$
$\Rightarrow -x_1 + x_2 \le 7$

Simplex Algorithm

Solve linear programming problem

max $z = 3x_1 + 3x_2$
s.t $4x_1 + 2x_2 \le 6$
$2x_1 + 2x_2 \le 4$
$x_1, x_2 \ge 0$

(A)

basic	x_1	x_2	s_1	s_2	z	Sol	Ratio
z	-3	-3	0	0	1	0	PR
s_1	4	2	1	0	0	6	3/2
s_2	2 PV	2	0	1	0	4	2

Entering variable: x_K (pivot column) negative coeff with largest absolute value
Leaving variable: $\min_i \left\{ \frac{b_i}{a_{iK}} : a_{iK} > 0 \right\}$

$\Rightarrow$ Entering variable: x_1
(B) Leaving variable: s_1

basic	x_1	x_2	s_1	s_2	z	Sol	Row Op
z	0	-3/2	3/4	0	1	9/2	+= $3R_2$
x_1	1	1/2	1/4	0	0	3/2	$R_2/4$
s_2	0	1 PV	-1/2	1	0	1	-= $2R_2$ PR

Ratio: $(3/2)/(1/2) = 3$, $1/1 = 1$
(C) min $(1, 3) = 1$
$\Rightarrow$ Entering var: x_2, leaving var s_2

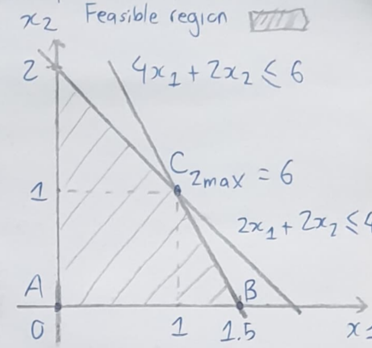
basic	x_1	x_2	s_1	s_2	z	Sol	Row Op
z	0	0	$\boxed{0}$	$\boxed{3/2}$ Dual	1	6	+= $3/2R_3$
x_1	1	0	1/2	-1/2	0	$\boxed{1}$	-= $1/2R_3$
x_2	0	1	-1/2	1	0	$\boxed{1}$ Primal	R_3

To do row operations, only use the pivot row in the form
$R_{other} + m \times R_{pivot} = 0$ ✓
not $m R_{other} + R_{pivot} = 0$ ✗

Optimal solutions
$(x_1, x_2) = (1, 1)$, $z_{max} = 6$
We have $4 \times 1 + 2 \times 1 = 6$
$2 \times 1 + 2 \times 1 = 4$
$1 > 0, 1 > 0 \Rightarrow$ at optimum, constraint $4x_1 + 2x_2 \le 6$, $2x_1 + 2x_2 \le 4$ are active and $x_1, x_2 \ge 0$ are inactive

- If constraint line pass through optimum $\Rightarrow$ active constraint
- If ~ not pass through optimum $\Rightarrow$ inactive constraint

Graphical Representation and the point visited at each iteration



The points the method visit at each iteration is $A \to B \to C$

Primal and Dual

Primal max	Dual min
Independent terms b	Obj. func coeff c
obj func coeff c	Independent b
i-th row of constraint	i-th col of constraint
i-th col of constraint coeff	i-th row of constraint coeff
Constraints $\le$, $\ge$ (switch side if primal min dual max), $=$	Variables ≥ 0, ≤ 0, $\in R$
Variables ≥ 0, ≤ 0, $\in R$	Constraints $\ge$, $\le$ (switch side if primal min dual max), $=$

Example

max $z_1 = 3x_1 + 3x_2$	min $z_2 = 6y_1 + 4y_2$
s.t $4x_1 + 2x_2 \le 6$	s.t $4y_1 + 2y_2 \ge 3$
$2x_1 + 2x_2 \le 4$	$2y_1 + 2y_2 \ge 3$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$

$\Rightarrow$ min z_2 = max $z_1 = 6$
$\Rightarrow (y_1, y_2) = (s_1, s_2)$ in primal $= (0, \frac{3}{2})$
because it is strong duality.
Implication of strong duality
min cx, constraint $Ax \ge b$, $x \ge 0$
max wb, constraint $wA \le c$, $w \ge 0$

If original problem has $\ge$ or $=$ constraints
$\Rightarrow$ Use M-method or Two-phase method
Add r_i artificial vars for $\ge$ and $=$

min $z = 4x_1 + x_2$	Two phase:
s.t $3x_1 + x_2 = 3$	min $z = r_1 + r_2$
$4x_1 + 3x_2 \ge 6$	s.t: $3x_1 + x_2 + r_1 = 3$
$x_1 + 2x_2 \le 4$	$4x_1 + 3x_2 - s_1 + r_2 = 6$
$x_1, x_2 \ge 0$	$x_1 + 2x_2 + s_3 = 4$

For M method: $z = 4x_1 + x_2 + Mr_1 + Mr_2$

1st phase $\longrightarrow$

	x_1	x_2	r_1	r_2	s_1	s_2	Sol
z	0	0	-1	-1	0	0	0 $= R_2 + R_3 \dots$

If original obj is min, (1, 1) if max

$\Rightarrow$

	x_{12} r_1 r_2 s_{12}	Sol
z	-1 -1 basic feasible sol exists $\Leftarrow$	0

$\Rightarrow$ 2nd phase

	x_{12} s_{12}	Sol
z	-4 -1 0 0	0

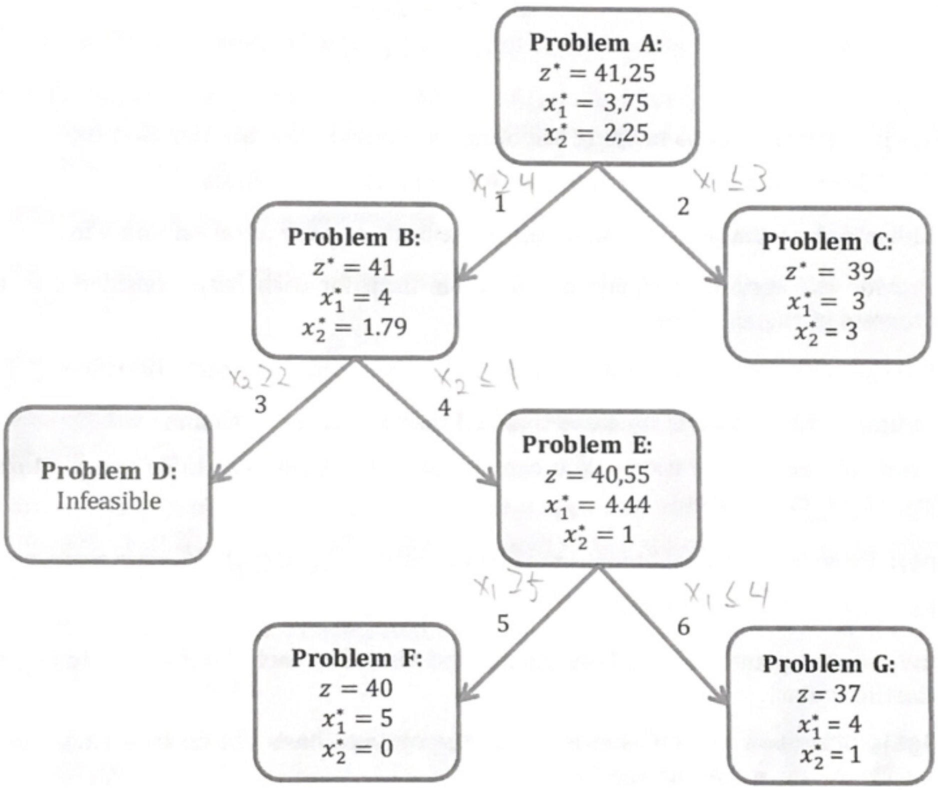
Delete col r_1, r_2 & replace origin obj. func

Problem 2 (15pt)

Considering the following IP problem:

$$\begin{aligned} \max.\ z &= 8x_1 + 5x_2 \\ \text{s.t.: } x_1 + x_2 &\leq 6 \\ 9x_1 + 5x_2 &\leq 45. \end{aligned}$$

After applying the branch-and-bound method, the following tree was obtained.



The arrows represent the derivation of each subproblem from its parent problem. The numbers above the arrows represent the constraints added to the parent problem to generate each subproblem. The variable selection was performed picking the variable with smallest index (e.g., in case x_1 and x_2 are fractional, x_1 is selected). Please answer the following:

a. (6pt): Identify what are the constraints represented by numbers 1, 2, 3, 4, 5, and 6.

b. (4pt): Which node represent the optimal solution for the IP problem? Please justify.

c. (5pt): Consider the possible sequences in which the subproblems B to G have been solved. Is there a sequence in which some of the subproblems do not need to be solved? Justify your answer.

Problem 2:

a) The constraints are:

(1) $x_1 \geq 4$ (2) $x_1 \leq 3$ (3) $x_2 \geq 2$ (4) $x_2 \leq 1$ (5) $x_1 \geq 5$ (6) $x_1 \leq 4$

b) The node of problem F represents the optimal solution for the IP problem

Reason: x_1 and x_2 must be integer solutions => Only problem C, F, G satisfies

Out of the three subproblems, z is maximized at 40 at problem F.

z value in C and G are fathomed by z value in F

c. The sequence is A -> B -> C -> D -> E -> F and the subproblem G need not to be solved. Reason: at C, we set upper bound = 39. Every z of parent node are always larger than the z of children nodes in maximization problem (reverse for minimization) => at problem B, we still need to solve as there is probably z = 40 with integer solutions. Now at F, it got 40 as the solution => upper bound is set as 40. Now, since at F, z = 40 is the highest possible integer solution at the moment, and we know that sub-branching at G would not find a better solution, since z of G will be smaller than z of E, and z of F is already maximum => subproblem G need not to be solved.

Problem 3 (25pt)

A chemical plant can purchase up to 15 kg of a chemical for \$10/kg. At a cost of \$3/kg, the chemical can be processed into 1kg of product A; or at a cost of \$5/kg, the chemical can be processed into one kg of product B. If x_1 kg of product A are produced, it sells for a price of \$$(30 - x_1)$ per kilo. If x_2 kg of product B are produced, it sells for a price of \$$(50 - 2x_2)$ per kilo. Let y kg be the amount of the chemical purchased. The model that optimise the profit of the chemical plant is given by:

$$
\begin{aligned}
P: \max.\ z = \ & x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 - 10y \\
\text{s.t.: } & x_1 + x_2 \leq y \\
& y \leq 15 \\
& x_1, x_2, y \geq 0.
\end{aligned}
$$

a. (6pt): Formulate the Karush-Kuhn-Tucker (KKT) optimality conditions for the above problem P.

b. (4pt): Are the KKT conditions sufficient for optimality in this case? If so, would a point satisfying these conditions be a local or global optimum? Please justify your answers.

c. (5pt): Solve the conditions formulated in **(a.)** to obtain a point $x = (x_1, x_2, y)$ satisfying the KKT conditions. **Hint:** start with the cases in which x_1, x_2 and y are all greater than zero and thus $\lambda_3 = \lambda_4 = \lambda_5 = 0$.

d. (5pt): Consider the following situation: a third product, product C, is incorporated as an alternative in the production process. At a cost of \$6/kg, the chemical can be processed into 2kg of product C. Reformulate the model P that optimise the profit of the chemical plant, incorporating the option of producing product C. You are required to identify any elements you may want to include or modify in problem P. C sells for \$ 10/kg

e. (5pt): Consider the following situation: suppose that, due to technical reasons, either product A or product B can be produced, but not both simultaneously. Reformulate the original model P (not the one you formulated in **(d.)**) to include this situation. You are required to identify any elements you may want to include or modify in problem P.

The complete KKT conditions

For the sake of completeness, we state the KKT conditions for general problems.

Theorem 4 (KKT general conditions)

Let P be min. $\{f(x) : g(x) \leq 0, h(x) = 0\}$ with differentiable $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^l$. If $\overline{x}$ is optimal for P, then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ satisfies

$$\nabla f(\overline{x}) + \sum_{i=1}^{m} \overline{\lambda}_i \nabla g_i(\overline{x}) + \sum_{i=1}^{l} \overline{\mu}_i \nabla h_i(\overline{x}) = 0$$

$$g_i(\overline{x}) \leq 0, \ i = 1, \dots, m$$

$$h_i(\overline{x}) = 0, \ i = 1, \dots, l$$

$$\overline{\lambda}_i g_i(\overline{x}) = 0, \ i = 1, \dots, m$$

$$\overline{\lambda}_i \geq 0, \ i = 1, \dots, m.$$

Problem 3:

a) P: max. $z = x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 - 10y$

s.t: $x_1 + x_2 \leq y$, $y \leq 15$

$x_1, x_2, y \geq 0$

$\Rightarrow f(x_1, x_2, y) = x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 - 10y$

$g_1(x_1, x_2, y) = x_1 + x_2 - y$

$g_2(x_1, x_2, y) = y - 15$

$g_3(x_1, x_2, y) = -x_1$

$g_4(x_1, x_2, y) = -x_2$

$g_5(x_1, x_2, y) = -y$

The KKT optimality conditions for the problem P:

(1) $\nabla f(x_1, x_2, y) + \sum_{i=1}^{5} \lambda_i \nabla g_i(x_1, x_2, y) = 0$

(2) $g_i(x_1, x_2, y) \leq 0$, $i = 1, \dots, 5$

(3) $\lambda_i g_i(x_1, x_2, y) = 0$, $i = 1, \dots, 5$

(4) $\lambda_i \leq 0$, $i = 1, \dots, 5$

$\Rightarrow$ (1) $\begin{bmatrix} 27 - 2x_1 \\ 45 - 4x_2 \\ -10 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \lambda_5 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(2) $x_1 + x_2 - y \leq 0$, $y - 15 \leq 0$, $-x_1 \leq 0$, $-x_2 \leq 0$, $-y \leq 0$

(3) $\lambda_1(x_1 + x_2 - y) = 0$, $\lambda_2(y - 15) = 0$

$-\lambda_3 x_1 = 0$, $-\lambda_4 x_2 = 0$, $-\lambda_5 y = 0$

(4) $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \leq 0$

Sufficiency of optimality conditions

If Slater's constraint qualification (CQ) holds, the KKT conditions become necessaru and sufficient for global optimality. Slater's CQ conditions are

1. f convex (concave for max.) function
2. g convex functions with strict interior (i.e. , exists x such that $g(x) < 0$)
3. h affine functions.

Theorem 5 (Necessary and sufficient optimality conditions)

Consider the problem P : min. $\{f(x) : g(x) \leq 0, h(x) = 0\}$ with $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ such that Slater's CQ are met. Then $\overline{x}$ is globally optimal for P if and only if $(\overline{x}, \overline{\lambda})$ satisfies the KKT conditions.

b) Slater's CQ conditions

- Function $f(x_1, x_2, y)$ is concave (maximize)

  $f(x_1, x_2, y) = 27x_1 - x_1^2 + 45x_2 - 2x_2^2 - 10y$

  We have $-x_1^2$ and $-x_2^2$ are concave, $27x_1$, $45x_2$ and $-10y$ are both concave and convex but we treat as concave

  $\Rightarrow f(x_1, x_2, y)$ is concave
- g convex functions with strict interior

  All g functions are affine $\Rightarrow$ they are all convex and we can easily see there are infinite solution so that $g_i < 0$, $i = 1 \dots 5$

As Slater's CQ hold, KKT conditions are also sufficient for global optimality

$\Rightarrow$ If a point satisfies these conditions KKT $\Rightarrow$ it is a global optimum

c) First case: $x_1, x_2, y > 0 \Rightarrow \lambda_3 = \lambda_4 = \lambda_5 = 0$

$$\Rightarrow \begin{cases} 27 - 2x_1 + \lambda_1 = 0 \\ 45 - 4x_2 + \lambda_1 = 0 \\ -10 + \lambda_1 + \lambda_2 = 0 \\ \lambda_1(x_1 + x_2 - y) = 0 \\ \lambda_2(y - 15) = 0 \end{cases} \quad \text{Let } \Rightarrow \lambda_1 = \lambda_2 = 0 \Rightarrow \begin{cases} 27 - 2x_1 = 0 \\ 45 - 4x_2 = 0 \\ -10 = 0 \text{ (contradiction)} \end{cases}$$

$$\text{Let } \lambda_1, \lambda_2 \neq 0 \Rightarrow \begin{cases} 27 - 2x_1 + \lambda_1 = 0 \\ 45 - 4x_2 + \lambda_1 = 0 \\ -10 + \lambda_1 + \lambda_2 = 0 \\ x_1 + x_2 + y = 0 \\ y - 15 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 7 \\ x_2 = 8 \\ \lambda_1 = -13 \\ \lambda_2 = -3 \\ y = 15 \end{cases}$$

$\Rightarrow$ The point $x = (7, 8, 15)$ satisfies the KKT conditions

d) y is kg chemical, x_1 is kg of product A, x_2 is kg of product B
Third product C is incorporated as an alternative. Let's call x_3 the kg of product C and d as the decision to incorporate C into production or not. Also, let S_3 the selling price for each kg of product C. We have 6\$ to produce 2kg of product C $\Rightarrow$ 3\$ to produce 1 kg of C
$\Rightarrow$ The full model is modified as:

$$P: \max z = x_1(30 - x_1) - 3x_1 + x_2(50 - 2x_2) - 5x_2 + \underbrace{d x_3 S_3 - 3 d x_3}_{\text{selling - producing price for product C}} - 15y$$

$$\underbrace{x_1 + x_2 + d x_3}_{\text{product C added}} \leq y$$

$$y \leq 15 \text{ (unchanged)}$$

$$x_1, x_2, y, \underbrace{x_3}_{\text{kg of product C should not be negative}} \geq 0$$

$d = \{0, 1\}$ $\rightarrow$ decision variable of whether to incorporate product C or not

e) We will use the decision variable d. If $d = 0$, product A is produced and not B. If $d = 1$, product B will be produced but not product A
$\Rightarrow$ The full model is modified as follows:

$$P: \max z = (1-d)x_1(30 - x_1) - 3(1-d)x_1 + d x_2(50 - 2x_2) - 5 d x_2 - 15y$$

$$(1-d)x_1 + d x_2 \leq y$$

$$y \leq 15$$

$$x_1, x_2, y \geq 0$$

$$d = \{0, 1\}$$

Problem 4 (35pt)

Consider the following function:

$$f(x_1, x_2) = (2 - x_1)^2 + (2 - 2x_1 - x_2)^2$$

a. (5pt): Obtain the optimum/ optima for this function analytically. Please provide:

1. The optimality conditions used to find the candidate point(s).
2. The candidate point(s) obtained.
3. Arguments supporting whether the point(s) is(are) locally or globally optimal.

1. Optimality conditions used to find the candidate point

- if $f'(x_0) = 0$ and $f''(x_0) > 0$ then x_0 is a local minimum.
- if $f'(x_0) = 0$ and $f''(x_0) < 0$ then x_0 is a local maximum.
- if $f'(x_0) = 0$ and $f''(x_0) = 0$ then x_0 is a inflection point.

2. candidate points obtained

Problem 4: Let $f(x_1, x_2) = (2 - x_1)^2 + (2 - 2x_1 - x_2)^2$

2. The candidate points

$$\nabla f(x_1, x_2) = \begin{bmatrix} 10x_1 + 4x_2 - 12 \\ 4x_1 + 2x_2 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

Problem 1: Finding minima and maxima of functions

Find the minima and/or maxima of the following functions.

a) $f(x_1, x_2) = x_1^3(x_1 - 4) + (x_2 - 5)^2$

b) $f(x_1, x_2, x_3) = (1 - x_2)(1 - x_3) + x_1^2 - 1$

Hint. Use the Hessian.

Solution

a) The first-order conditions are satisfied by stationary points

$$\nabla f(x_1, x_2) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1^3 - 12x_1^2 \\ 2x_2 - 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (x_1, x_2) = (3, 5) \text{ and } (0, 5).$$

Form the Hessian matrix

$$H(x_1, x_2) = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 \\ \partial^2 f/\partial x_1 \partial x_2 & \partial^2 f/\partial x_2^2 \end{bmatrix} = \begin{bmatrix} 12x_1^2 - 24x_1 & 0 \\ 0 & 2 \end{bmatrix}$$

Calculate the values of Hessian matrices for the stationary points

$$H(3,5) = \begin{bmatrix} 36 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad H(0,5) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Solve the eigenvalues of Hessian matrix

$$\det(H(3,5) - \lambda \mathbf{I}) = \det\left(\begin{bmatrix} 36 - \lambda_1 & 0 \\ 0 & 2 - \lambda_2 \end{bmatrix}\right) = 0$$

$$\Rightarrow (36 - \lambda_1)(2 - \lambda_2) = 0 \;\Rightarrow\; \lambda_1 = 36 \text{ and } \lambda_2 = 2$$

and

$$\det(H(0,5) - \lambda \mathbf{I}) = \det\left(\begin{bmatrix} -\lambda_1 & 0 \\ 0 & 2 - \lambda_2 \end{bmatrix}\right) = 0$$

3. The obtained candidate point is the global minimum

3. Optimality of the obtained point

$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 10 & \\ & 2 \end{bmatrix} \Rightarrow$ The obtained point $(2, -2)$ is a local minimum as $10 > 0$ and $2 > 0$

We have $(2 - x_1)^2$ is convex and $(2 - x_1 - x_2)^2$ is convex

$\Rightarrow$ Their linear combination $f(x_1, x_2)$ is also convex

$\Rightarrow (x_1, x_2) = (2, -2)$ is the global minimum

3. <Should use the above method, fix this later>

b. (15pt): Apply a single iteration of the gradient method to find an optimum for this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an optimal step size λ. You are requested to provide:

1. The expression for the gradient step.
2. The calculations for the optimal step size (analytically). **Hint:** should be a value close to 0.1).
3. The new point found.
4. Answer the following: is this point optimal? Please justify without relying on the results from **(a.)**.

Gradient (descent) method

Algorithm Gradient descent method

initialise. tolerance $\epsilon > 0$, initial point x_0, iteration count $k = 0$.
while $||\nabla f(x_k)|| > \epsilon$ **do**
    $d_k = -\nabla f(x_k)$.
    $\overline{\lambda} = \text{argmin}_{\lambda \in \mathbb{R}} \{f(x_k + \lambda d_k)\}$.
    $x_{k+1} = x_k + \overline{\lambda} d_k$.
    $k \leftarrow k + 1$.
end while
return x_k.

1. Initialize : tolerance $\varepsilon = 0.01$, initial point $x_0 = (0,0)$, count $= 0$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 10x_1 + 4x_2 - 12 \\ 4x_1 + 2x_2 - 4 \end{bmatrix}$$

- First (single) iteration, $k = 0$

$\|\nabla f(0,0)\| = \|(-12, -4)\| > \epsilon = 0.01$

$\Rightarrow d_0 = -\nabla f(0,0) = [12, 4]^T$

$\Rightarrow \bar{\lambda} = \text{argmin}_\lambda f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 12 \\ 4 \end{bmatrix}\right) = \text{argmin}_\lambda f(12\lambda, 4\lambda)$

$\Rightarrow$ Expression for gradient step : $\bar{\lambda} = 928\lambda^2 - 160\lambda + 8$

2. Optimal step size :

$\bar{\lambda} = \text{argmin}_\lambda (928\lambda^2 - 160\lambda + 8)$

$\nabla f(12\lambda, 4\lambda) = 1856\lambda - 160 = 0 \Rightarrow \bar{\lambda} = 5/58$

$\nabla^2 f(12\lambda, 4\lambda) = 1856 > 0 \Rightarrow \bar{\lambda} = 5/58$ is the optimal step size

3. The new point found

$$x_1 = x_0 + \lambda d_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{5}{58}\begin{bmatrix} 12 \\ 4 \end{bmatrix} = \begin{bmatrix} 30/29 \\ 10/29 \end{bmatrix} \approx \begin{bmatrix} 1.0344 \\ 0.3448 \end{bmatrix}$$

4. We have: $\nabla f\left(\frac{30}{29}, \frac{10}{29}\right) = \begin{bmatrix} -8/29 \\ 24/29 \end{bmatrix} \Rightarrow \left\|\nabla f\left(\frac{30}{29}, \frac{10}{29}\right)\right\| = \frac{8\sqrt{10}}{29} > \epsilon = 0.01$

The new point $x_1 = \left[\frac{30}{29}, \frac{10}{29}\right]$ is not optimal. First $\|\nabla f(x_1)\| > \epsilon$

Secondly, if it's optimal, then $\nabla f(x_1) = 0$

But $\nabla f(x_1) \neq 0 \Rightarrow x_1$ is not an optimal point yet

c. (15pt): Apply a single iteration of the Newton's method to find the optimal of this function. Use $x_0 = (0, 0)$ as a starting point, a tolerance of $\epsilon = 0.01$, and an a step size of $\lambda = 1$. You are requested to provide:

1. The expression for the Newton step.
2. The new point found.
3. Answer the following: is this point optimal? If so, why did the method only took a single iteration? Please justify without relying on the results from **(a.)**.

Hint: Remind that if $g(x) = (f(x))^n$ then the derivative is $g'(x) = nf(x)^{n-1}f'(x)$. Also, you need this result for **(c.)**: $\begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix}$.

The method only took a single iteration because Newton's method uses curvature information (the second derivative) to take a more direct route compared to the gradient descent. The geometric interpretation of Newton's method is that at each iteration, it amounts to the fitting of a parabola to the graph of f(x) at the trial value x_k, having the same slope and curvature as the graph at that point, and then proceeding to the maximum or minimum of that parabola (in higher dimensions, this may also be a saddle point), see below. Note that if f(x) happens to be a quadratic function, then the exact extremum is found in one step.

Newton's method

Algorithm Newton's method

initialise. tolerance $\epsilon > 0$, initial point x_0, iteration count $k = 0$.
while $||\nabla f(x_k)|| > \epsilon$ **do**
  $d = -H^{-1}(x_k)\nabla f(x_k)$.
  $\overline{\lambda} = \text{argmin}_{\lambda \in \mathbb{R}} \{f(x_k + \lambda d)\}$.
  $x_{k+1} = x_k + \overline{\lambda} d$.
  $k \leftarrow k + 1$.
end while
return x_k.

c.

1. Initialize : tolerance $\varepsilon = 0.01$, initial point $x_0 = (0,0)$, count $= 0$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 10x_1 + 4x_2 - 12 \\ 4x_1 + 2x_2 - 4 \end{bmatrix}$$

○ First (single) iteration, $k = 0$

$$\|\nabla f(0,0)\| = \|(-12,-4)\| > \varepsilon = 0.01$$

$$\nabla^2 f(x) = H(x) = \begin{bmatrix} \partial^2 f(x)/\partial x_1^2 & \partial^2 f(x)/\partial x_1 \partial x_2 \\ \partial^2 f(x)/\partial x_2 \partial x_1 & \partial^2 f(x)/\partial x_2^2 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}$$

$$\Rightarrow H^{-1}(x) = \begin{bmatrix} 1/2 & -1 \\ -1 & 5/2 \end{bmatrix}$$

$$\Rightarrow d = -H^{-1}(x_0)\nabla f(0,0) = \begin{bmatrix} -1/2 & 1 \\ 1 & -5/2 \end{bmatrix} \begin{bmatrix} -12 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \bar{\lambda} = \underset{\lambda}{\text{argmin}} \{f(x_0 + \lambda d)\} = \underset{\lambda}{\text{argmin}} \{f(2\lambda, -2\lambda)\}$$

$$= \underset{\lambda}{\text{argmin}} \{(2-2\lambda)^2 + (2 - 4\lambda + 2\lambda)^2\} = \text{argmin} \{2(2-2\lambda)^2\}$$

$$\Rightarrow \bar{\lambda} = 1$$

2. The new point found

$$x_1 = x_0 + \lambda d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

3. The new point ~~op~~ is optimal. First: $\|\nabla f(x_1)\| = 0 \Rightarrow x_1$ is optimal

Briefly explains these terms

a) Slack variable and surplus variable:

- **slack variable**: a variable that is added to an inequality constraint to transform it into an equality. Introducing a slack variable replaces an inequality constraint with an equality constraint and a non-negativity constraint on the slack variable

- surplus variable: a surplus variable or negative slack variable is a variable that is subtracted to an inequality constraint to transform it into an equality. It measures the amount by which the LHS exceeds the RHS.

b) Nash equilibrium: Nash Equilibrium is a game theory. concept that determines the optimal solution in a non-cooperative game in which each player lacks any incentive to change his/her initial strategy. Under the Nash equilibrium, a player does not gain anything from deviating from their initially chosen strategy

c) Pareto or efficient optimal solution: Pareto efficiency or Pareto optimality is a situation where no individual or preference criterion can be better off without making at least one individual or preference criterion worse off or without any loss thereof.

d) binary variable, binary choice:

- Binary variables are variables which only take two values. For example, Male or Female, True or False and Yes or No.

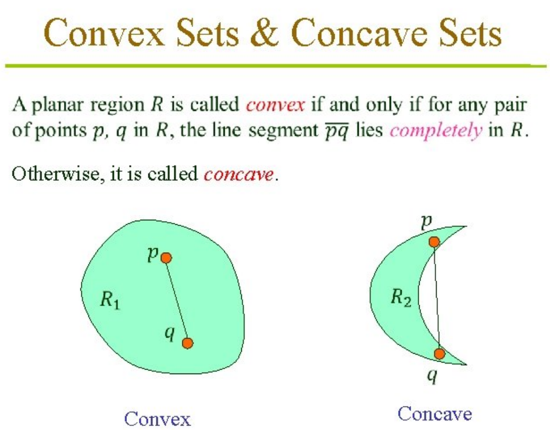
- In decision making, a binary choice means there are only two alternatives (e.g., yes or no, do or don't do)

e) shadow price:

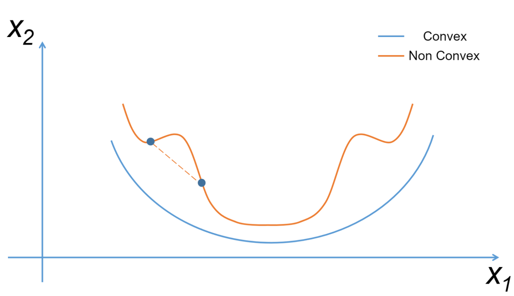
A shadow price is a monetary value assigned to currently unknowable or difficult-to-calculate costs in the absence of correct market prices. It is based on the willingness to pay principle – the most accurate measure of the value of a good or service is what people are willing to give up in order to get it.

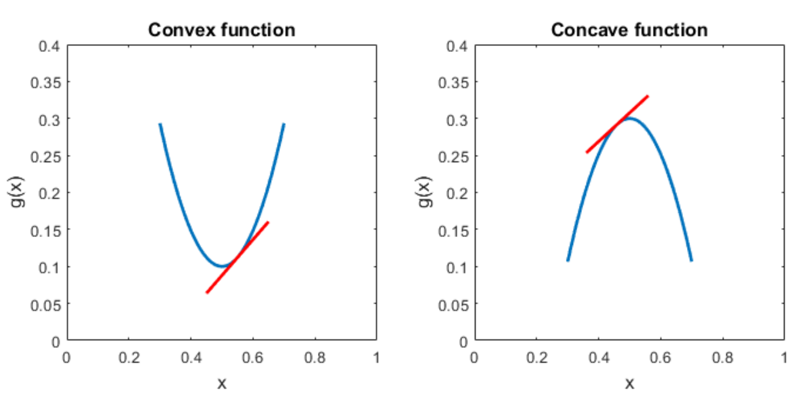
f) convex set and convex function

A convex set is a set of points such that, given any two points A, B in that set, the line AB joining them lies entirely within that set. A convex set; no line can be drawn connecting two points that does not remain completely inside the set.



- In mathematics, a real-valued function is called convex if the line segment between any two points on the graph of the function does not lie below the graph between the two poin





g) Newton's method

In numerical analysis, **Newton's method**, also known as the **Newton–Raphson method**, named after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeroes) of a real-valued function. The most basic version starts with a single-variable function f defined for a real variable x, the function's derivative f', and an initial guess x_0 for a root of f. If the function satisfies sufficient assumptions and the initial guess is close, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a better approximation of the root than x_0. Geometrically, $(x_1, 0)$ is the intersection of the x-axis and the tangent of the graph of f at $(x_0, f(x_0))$: that is, the improved guess is the unique root of the linear approximation at the initial point. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

h) simplex iteration

The SIMPLEX METHOD is an iterative method for solving linear programming problems. For each simplex iteration, it moves along the edge of the convex set of the feasible region of the linear programming problem

i) portfolio optimization

Portfolio optimization is the method of selecting the best portfolio which gives back the most profitable rate of return for each unit of risk taken by the investors. A portfolio is the asset distribution or in other words pool of investment options of an investor. The best portfolio for an investor depends upon various options like risk appetite, expected rate of return, and other cost minimization

j) subtour

subtour is one of the sequence of the connections in the graph but does not necessarily ensures full connectivity

Travelling salesman problem

Let $x_{ij} = 1$ if city j is visited directly after city i, $x_{ij} = 0$ otherwise. Let $N = \{1, \ldots, n\}$. We assume that x_{ii} is not defined for $i \in N$.

A naive model for the TSP could be:

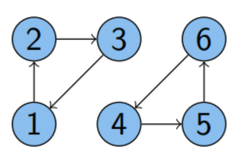
$$(\text{TSP}) : \min_{x} \sum_{i \in N} \sum_{j \in N} C_{ij} x_{ij}$$

$$\text{s.t.: } \sum_{j \in N \setminus \{i\}} x_{ij} = 1, \forall i \in N$$

$$\sum_{i \in N \setminus \{j\}} x_{ij} = 1, \forall j \in N$$

$$x_{ij} \in \{0, 1\}, \forall i, \forall j \in N : i \neq j$$

- This is exactly the assignment problem.
- Also, solutions do not prevent subtours.



Travelling salesman problem

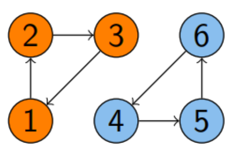
Preventing subtours: constraints that ensure full connectivity.

- Cutset constraints:

$$\sum_{i \in S} \sum_{j \in N \setminus S} x_{ij} \geq 1, \forall S \subset N, S \neq \emptyset$$

- Subtour elimination constraints:

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \forall S \subset N, 2 \leq |S| \leq n - 1$$



$S_1 = \{1, 2, 3\}$

Example for $S_1 = \{1, 2, 3\}$:

Cutset:

$x_{14} + x_{24} + x_{34} + x_{15} + x_{25} + x_{35} + x_{16} + x_{26} + x_{36} \geq 1$

Subtour elim.: $x_{12} + x_{13} + x_{21} + x_{23} + x_{31} + x_{32} \leq 2$

k) dual variable

In Duality Theory, a Dual variable is defined for each of the Primal constraints and conversely for each of the Primal decision variables, a Dual constraint is constructed.

l) pivot element of Simplex method

The pivot column in the Simplex method is determined by the largest reduced cost coefficient corresponding to a basic variable. In other words, it is the largest ratio of right side parameters with the positive coefficients in the pivot column. Pivot element decides the entering and exiting variable of Simplex method. The column of pivot element is the entering variable and the row of the pivot element is the exiting variable

m) exiting variable of Simplex method

The exiting basic variable in simplex method is the basic variable that. Goes to zero first as the entering basic variable in increased => Exiting variable goes from basic to non basic sets of variables

n) entering variable of Simplex method

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the Z-row => Entering variable goes from non-basic to basic sets of variables

o) optimality condition of Simplex method

The optimum is reached at the iteration where all the Z-row
coefficient of the non-basic variables are non-negative (non-positive).

p) feasibility condition of Simplex method

For both maximization and minimization problems the leaving variable is the basic associated with the smallest non-negative ratio (with strictly positive denominator).

q) corner point

The corner points are the vertices of the feasible (convex) region limited by the constraints

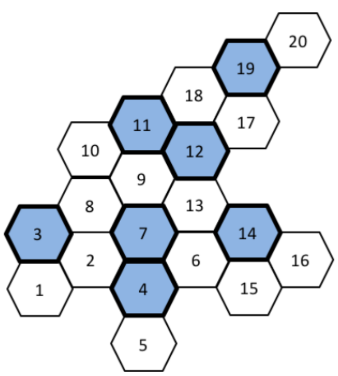
r) non-feasible basic solution
In the presence of an optimum solution, there exists a basic feasible solution that is also an optimum solution. An infeasible basic solution violates at least one of the constraints of the LP problem
s) Lagrangian function
L(x, λ) = f(x) + λ(b − g(x)). xi) . In general, the Lagrangian is the sum of the original objective function and a term that involves the functional constraint and a 'Lagrange multiplier' λ. In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables).
t) set-covering problem

Problem statement:

- A set of $M = \{1, \ldots, m\}$ regions must be served by opening *service centres* (e.g., hospitals, schools, police stations);
- A centre can be opened at $N = \{1, \ldots, n\}$ possible locations;
- If a centre is opened at location $j \in N$, then it serves a subset $S_j \subseteq M$ of regions and has opening cost C_j.

Objective: decide where to open the facilities so that all regions are served and the total opening cost is minimised.

The set covering problem: covering example



- Each location represents a candidate place for a centre;
- Once opened, the centre can only serve immediate neighbours.
- We have $M = \{1, \ldots, 20\}$ and $N = \{3, 4, 7, 11, 12, 14, 19\}$.

In this case: $S_3 = \{1, 2, 3, 8\}, S_4 = \{2, 4, 5, 6, 7\}, \ldots$.

—---

1. Consider a problem

$$
\begin{array}{rll}
\max & 4x_1 + 3x_2 & \\
\text{s.e.} & x_1 + x_2 & \leq 3 \\
 & -3x_1 + 5 & \geq x_2 \\
 & 2x_1 - x_2 & \geq -1 \\
 & x_1, x_2 & \geq 0.
\end{array}
$$

a) Transform the linear problem into the standard form (1p)
b) Solve the LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show the progress of the Simplex algorithm in the region. (2p)

1. Consider: $\max\ 4x_1 + 3x_2$

$$\text{s.t}\quad x_1 + x_2 \leq 3$$
$$-3x_1 + 5 \geq x_2$$
$$2x_1 - x_2 \geq -1$$
$$x_1, x_2 \geq 0$$

a) Transform into standard form

Standard form: $\max\ z = 4x_1 + 3x_2$

$$\text{s.t}\quad x_1 + x_2 + S_1 = 3$$
$$3x_1 + x_2 + S_2 = 5$$
$$-2x_1 + x_2 + S_3 = 1$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

b) Solve the LP problem by Simplex method

A

basic	x_1	x_2	S_1	S_2	S_3	Sol	Ratio
z	-4	-3	0	0	0	0	
S_1	1	1	1	0	0	3	$3/1$
S_2	3	1	0	1	0	5	$5/3$ ← Pivot row
S_3	-2	1	0	0	1	1	

↑ Pivot column

Entering variable: x_1, leaving variable: S_2

B

basic	x_1	x_2	S_1	S_2	S_3	Sol	Row operation	Ratio
z	0	$-5/3$	0	$4/3$	0	$20/3$	$+4R_3$	
S_1	0	$2/3$	1	$-1/3$	0	$4/3$	$-R_3$	2 ← Pivot row
x_1	1	$1/3$	0	$1/3$	0	$5/3$		5
S_3	0	$5/3$	0	$2/3$	1	$13/3$	$+2R_3$	2.6

↑ Pivot column

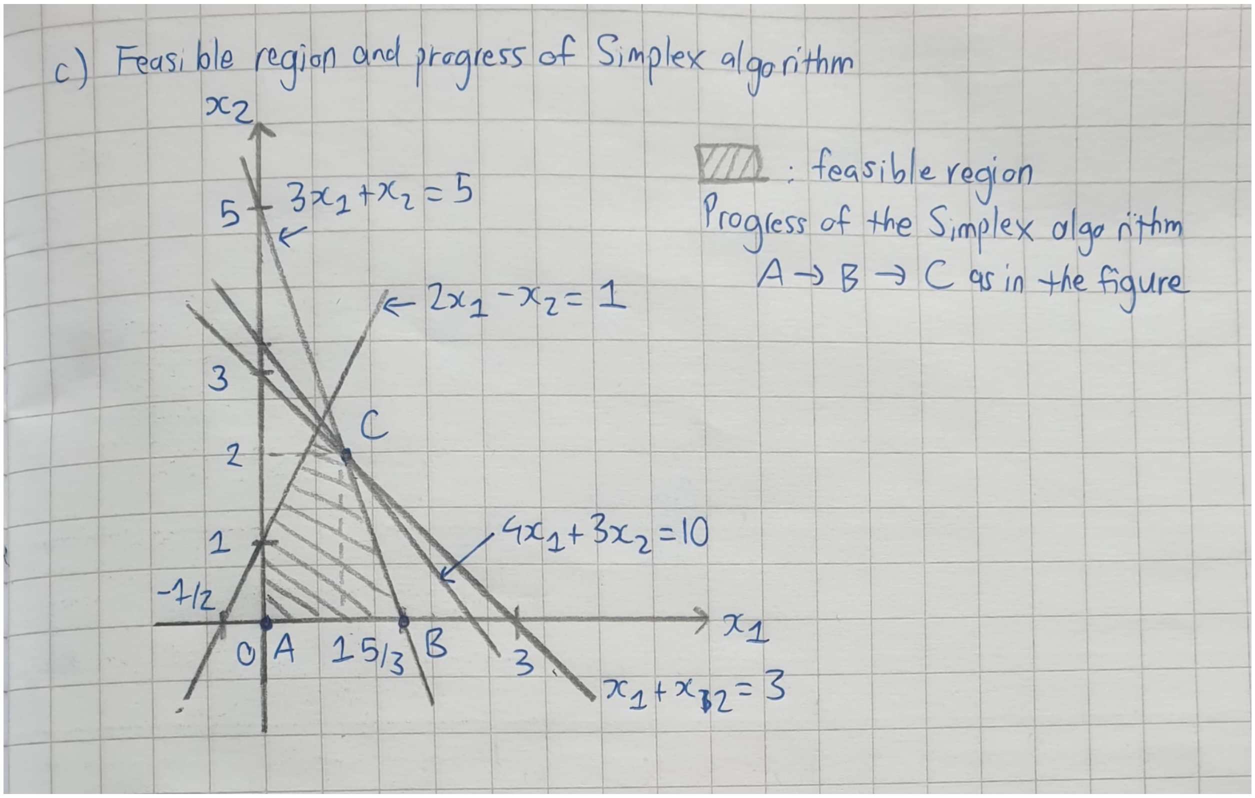
Entering variable: x_2, leaving variable: S_1

C

basic	x_1	x_2	S_1	S_2	S_3	Sol	Row Operation
z	0	0	$5/2$	$1/2$	0	10	$+5/3 R_2$
x_2	0	1	$3/2$	$-1/2$	0	2	
x_1	1	0	$-1/2$	$1/2$	0	1	$-1/3 R_2$
S_3	0	0	$-5/2$	$3/2$	1	1	$-5/3 R_2$

The coefficients of z-row are all positive $\Rightarrow$ Simplex algorithm stops

$\Rightarrow Z_{max} = 10$ at $(x_1, x_2) = (1, 2)$ and $(S_1, S_2, S_3) = (0, 0, 1)$



1. Use tabular Simplex algorithm in the following problem:

$$
\begin{array}{lrcl}
\max & x_1 & & \\
\text{s.t.} & 2x_1 - x_2 & \leq & 2 \\
 & -x_1 + x_2 & \leq & 1 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

1. Use tabular Simplex algorithm in the following problem:

$$
\begin{array}{lrcl}
\max & 2x_1 + 3x_2 & & \\
\text{s.t.} & 2x_1 + 3x_2 & \leq & 6 \\
 & x_1 - x_2 & \leq & 1 \\
 & x_2 & \leq & 1 \\
 & x_1, x_2 & \geq & 0
\end{array}
$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

1. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{lrcl} \max & 2x_1 + x_2 & & \\ \text{s.e.} & x_1 & \leq & 2 \\ & x_1 + 2x_2 & \leq & 6 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Use tabular Simplex algorithm in the following problem:

$$\begin{array}{lrcl} \max & 2x_1 + 3x_2 & & \\ \text{s.t.} & x_1 + 3x_2 & \leq & 5 \\ & 2x_1 - 2x_2 & \leq & 3 \\ & x_1 + 2x_2 & \leq & 4 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the Simplex method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{lrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & x_1 + x_2 & \geq & 2 \\ & -x_1 + x_2 & \geq & 1 \\ & x_2 & \leq & 3 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem with Simplex algorithm. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

2. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{lrcl} \max & 3x_1 - 2x_2 & & \\ \text{s.e.} & x_1 + x_2 & \geq & 2 \\ & 3x_1 - x_2 & \leq & 8 \\ & -x_1 + 4x_2 & \leq & 12 \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem using the M method. (3p)
c) Draw a picture of the feasible region of the problem and show how Simplex algorithm progresses in the region. (2p)

1. Use tabular Simplex algorithm in the following problem.

$$\begin{array}{lrcl} \max & 2x_1 + x_2 & & \\ \text{s.t.} & 2x_1 - x_2 & \leq & 6 \\ & x_1 + 2x_2 & \leq & 10 \\ & x_1 - x_2 & \geq & \text{-3} \\ & x_1, x_2 & \geq & 0 \end{array}$$

a) Transform the linear problem into the standard form. (1p)
b) Solve LP problem with Simplex algorithm. (3p)
c) Draw a picture of the feasible region of the problem and how Simplex algorithm progresses in the region. (2p)

---

3. Consider a problem

$$\begin{array}{lrcl} \min & (x_1 - 3)^2 + (x_2 - 3)^2 & & \\ \text{s.t.} & x_1^2 + x_2 & \leq & 4 \\ & -x_1 + x_2 & = & 2 \\ & x_2 & \geq & 0, \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

5. Consider a problem

$$\begin{array}{lrcl} \min & (x_1-a)^2+(x_2-b)^2 & & \\ \text{s.t.} & -4x_1+4x_2 & \leq & 12 \\ & 5x_1+2x_2 & \leq & 20 \\ & x_1,x_2 & \geq & 0, \end{array}$$

where i) $a=2, b=2$ and ii) $a=4, b=6$.

a) Solve cases i) and ii) graphically. Draw a picture of the feasible region of the problem and draw also the contour lines of the objective function. (2p)
b) In both cases, present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

3. Consider a problem

$$\begin{array}{lrcl} & \max\ (x_1-3)^2+x_2^2 & & \\ \text{s.t.} & x_1 & \geq & 1 \\ & x_1^2-2 & \leq & x_2 \\ & 2x_1-2 & = & x_2\ . \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

3. Consider a problem

$$\begin{array}{lrcl} \min & (x_1-5)^2+(x_2-3)^2 & & \\ \text{s.e.} & x_2^2-x_1 & \leq & 0 \\ & 2x_2-x_1 & = & 0 \\ & x_1,x_2 & \geq & 0. \end{array}$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

3. Consider a problem

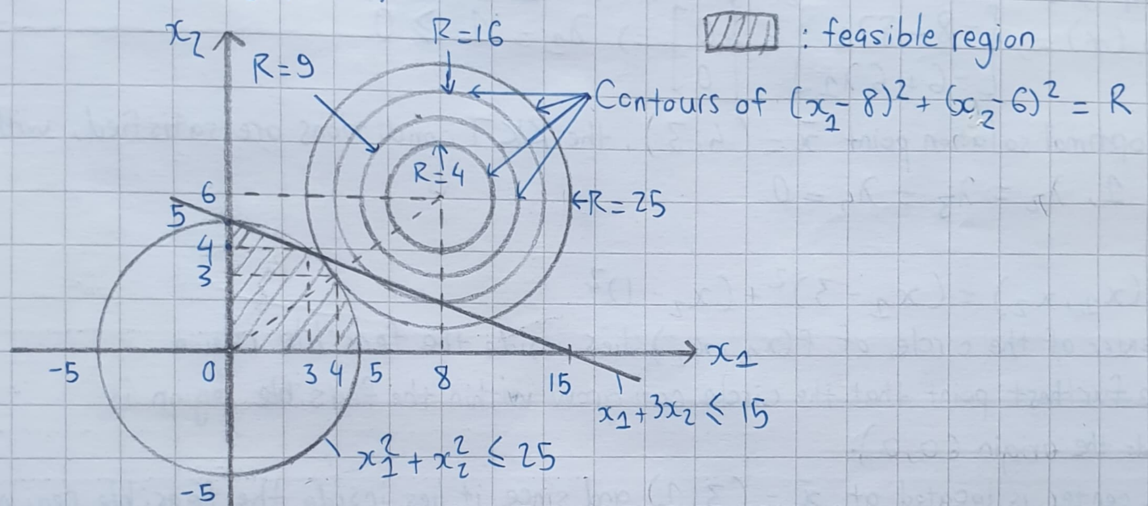
$$
\begin{array}{lrcl}
\min & (x_1 - 8)^2 + (x_2 - 6)^2 & & \\
\text{s.e.} & x_1^2 + x_2^2 & \leq & 25 \\
 & x_1 + 3x_2 & \leq & 15 \\
 & x_1, x_2 & \geq & 0,
\end{array}
$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (3p)
c) What is the solution if the objective function is $f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 1)^2$ and the feasible region remains the same?

3, Consider the problem : $\min (x_1 - 8)^2 + (x_2 - 6)^2$

$$
\begin{array}{ll}
\text{s.t} & x_1^2 + x_2^2 \leq 25 \\
 & x_1 + 3x_2 \leq 15 \\
 & x_1, x_2 \geq 0
\end{array}
$$

a) Solve the problem graphically



Graphically, the lowest R such that the contour of $(x-8)^2 + (y-6)^2$ touches the feasible region is $R = 25$

$\Rightarrow \min f(x_1, x_2) = 25$ at $(x_1, x_2) = (4, 3)$ (answer)

b. The KKT conditions are

$$f(\bar{x}) = (x_1-8)^2+(x_2-6)^2 \Rightarrow \nabla f(\bar{x}) = \begin{bmatrix} 2(x_1-8) \\ 2(x_2-6) \end{bmatrix}$$

$$g_1(\bar{x}) = x_1^2 + x_2^2 - 25 \Rightarrow \nabla g_1(\bar{x}) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$g_2(\bar{x}) = x_1 + 3x_2 - 15 \Rightarrow \nabla g_2(\bar{x}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$g_3(\bar{x}) = -x_1 \Rightarrow \nabla g_3(\bar{x}) = [-1, 0]^T, \quad g_4(\bar{x}) = -x_2 \Rightarrow \nabla g_4(\bar{x}) = [0,-1]^T$$

(1) $\nabla f(\bar{x}) + \sum \lambda_i g_i(x) = 0$

$$\Rightarrow \begin{bmatrix} 2(x_1-8) \\ 2(x_2-6) \end{bmatrix} + \lambda_1 \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Replace $\bar{x} = (4,3)$

$$\Rightarrow \begin{bmatrix} -8+8\lambda_1+\lambda_2-\lambda_3 \\ -6+6\lambda_1+3\lambda_2-\lambda_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (*)$$

(2)
$$\begin{cases} \lambda_1(x_1^2+x_2^2-25)=0 \\ \lambda_2(x_1+3x_2-15)=0 \\ -\lambda_3 x_1 = 0 \\ -\lambda_4 x_2 = 0 \end{cases} \Rightarrow (x_1,x_2) = (4,3) \begin{cases} \lambda_1 \times 0 = 0 \\ \lambda_2 \times (-2) = 0 \\ -\lambda_3 \times 4 = 0 \\ -\lambda_4 \times 3 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_2 = 0 \\ \lambda_3 = 0 \\ \lambda_4 = 0 \end{cases}$$

(3) $\lambda_i \geq 0$ (Satisfies for $\lambda_2, \lambda_3, \lambda_4$) $\Rightarrow \lambda_1 \geq 0$

From $(*)$ $\Rightarrow \begin{bmatrix} -8+8\lambda_1 \\ -6+6\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_1 = 1 \geq 0$

$\Rightarrow$ With optimal solution point $\bar{x} = (4,3)$, the KKT conditions are satisfied, with $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = 0$

c. If $f(x_1,x_2) = (x_1-3)^2 + (x_2-1)^2$

$\Rightarrow$ Center of the circle of $f(x_1,x_2)$ lies inside the feasible region

~~$\Rightarrow$ The furthest point that the circle can grow within the feasible region is towards the origin (0,0)~~

The center is located at $\bar{x} = (3,1)$ and since it lies inside the feasible region, the radius can simply be 0

$\Rightarrow$ The solution is $\bar{x} = (3,1)$ with $f(x_1,x_2)_{min} = 0$ (answer)

3. Consider a problem

$$
\begin{array}{lrcl}
\min & (x_1-5)^2+(x_2-5)^2 & & \\
\text{s.e.} & x_1^2+x_2^2 & \leq & 25 \\
 & x_1 & \leq & 3 \\
 & x_1, x_2 & \geq & 0,
\end{array}
$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and the contour lines of the objective function. (2p)
b) Show that the necessary Karush-Kuhn-Tucker (KKT) conditions are satisfied in the optimal solution point. (4p)

3. Consider a problem

$$
\begin{array}{lll}
\max & x_1^2+(x_2-4)^2 & \\
\text{s.t.} & x_1+x_2 & = 2 \\
 & x_1^2+x_2 & \leq 4 \\
 & x_1 & \leq 1 \\
 & x_2 & \geq 0
\end{array}
$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and also draw the contour lines of the objective function. (2p)
b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and examine if the solution satisfies these conditions. (4p)

- Consider a problem

$$
\begin{array}{llcl}
\min & (x_1-5)^2+(x_2-4)^2 & & \\
\text{s.e.} & x_1^2-4x_1-x_2+5 & \leq & 0 \\
 & 2x_1+3x_2-12 & = & 0 \\
 & -x_1 & \leq & 0 \\
 & x_2-6 & \leq & 0 \\
 & -2x_2+3 & \leq & 0.
\end{array}
$$

a) Solve the problem graphically. Draw a picture of the feasible region of the problem and draw also the contour lines of the objective function. (2p)
b) Present the necessary Karush-Kuhn-Tucker (KKT) conditions and see if the solution satisfies these conditions. (4p)

---

4. Consider a problem

$$
\begin{array}{lrcl}
\max & 3x_1 + 2x_2 & & \\
\text{s.e.} & x_1 + 2x_2 & \leq & 3 \\
 & 2x_1 + 2x_2 & \leq & 5 \\
 & x_1, x_2 & \in & \mathbb{Z}_+
\end{array}
$$

a) Solve the LP relaxation of the problem graphically. (2p)
b) Solve the original problem by using the Branch-and-Bound method. Solve the subproblems graphically. (3p)
c) Present your solution in a tree form. With it explain why the solution you found is indeed the optimal integer solution. (1p)

4. Consider a problem

$$
\begin{array}{lll}
\max & 9x_1 + 5x_2 & \\
\text{s.t.} & 4x_1 + 9x_2 & \leq \ 35 \\
 & x_1 & \leq \ 6 \\
 & x_1 - 3x_2 & \geq \ 1 \\
 & 3x_1 + 2x_2 & \leq \ 2 \\
 & x_1, x_2 & \in \ \mathbb{Z}_+ \cup \{0\}
\end{array}
$$

a) Solve the LP relaxation of the problem graphically. (1p)
b) Determine the solution to the problem with Branch & Bound algorithm. Solve subproblems graphically. (3p)
c) Draw the course of your solution in tree form, and justify with it that the solution you got is the best possible integer solution to the original problem. (2p)

3. Consider a problem

$$
\begin{array}{lll}
\max & 9x_1 + 5x_2 & \\
\text{s.t.} & 4x_1 + 9x_2 & \leq \ 35 \\
 & x_1 & \leq \ 6 \\
 & x_1 - 3x_2 & \geq \ 1 \\
 & 3x_1 + 2x_2 & \leq \ 2 \\
 & x_1, x_2 & \in \ Z_+ \cup \{0\}
\end{array}
$$

a) Solve the LP relaxation of the problem graphically. (1p)
b) Determine the solution to the problem with Branch & Bound algorithm. Solve subproblems graphically. (3p)
c) Draw the course of your solution in tree form, and justify with it that the solution you got is the best possible integer solution to the original problem. (2p)

4. The mighty Hun emperor Mukbar Attila is planning a conquest in Europe. He has narrowed his options down to four vulnerable cities: Paris, London, Rome and Constantinople. M. Attila has an army of 10 000 Huns at his disposal.

   The potential loot of each target (millions of denars) and the troops required for conquest are presented in Table 1.

   Table 1: Target information

	Loot	Required troops
Paris	2 M	2500
London	1 M	1500
Rome	5 M	5000
Constantinople	4 M	3500

   M. Attila does not want to spread his troops too much - therefore, he cannot attack both Rome and London. In addition, the tactically gifted emperor decides that he cannot conquer London without also overtaking Paris.

   a) Formulate the problem as a linear integer optimization problem, when M. Attila wants to maximize the total amount of received loot. (4p)
   b) After some thinking, M. Attila concludes that he wants a minimum loot of 10 million denars. Formulate the resulting goal programming problem, when the loss of 1 million denars is equally acceptable to hiring 500 more soldiers. (2p)
   Neither case needs to be solved.

4. Vesa is a friend of good beer. He is on a beer journey in Belgium and planning exports to home in a beer store. For financial and transportation reasons, he has decided to buy at most three different beer types: his alternatives are blonde, dubbel, and lambic beer types. Each beer type is available in 0.33 and 0.75 liter bottles. The price list of the bottles is presented in Table 1.

Table 1: Price list

	0.33 l	0.75 l
Blonde	€1.5	€3.5
Dubbel	€2	€3.5
Lambic	€4.5	€6

Small bottles are packed into small crates, and big bottles are packed into big crates. Small bottles cannot be packed into big crates or vice versa. One small crate has a space for 24 small bottles and one big crate for 10 big bottles. Vesa is planning his shoppings so that at least half of the bottles are of lambic type, and the number of blonde type bottles must be greater than or equal to the number of dubbel type bottles. Vesa has €150 to spend and he wishes to maximize the amount of bought beer measured in litres. Formulate the problem as a linear integer programming problem when

a) Vesa's car has a space for arbitrarily many crates and crates are free. (3p)
b) Vesa's car has a space for two small crates and one big crate. A small crate costs €5 and a big crate €6. (3p)
Neither case needs to be solved.

- You are the production manager for car manufacturing at Teddy's Four-Wheel, Inc. Currently, the collection of sold cars consists only of one model, Winnie. The production of a Winnie requires 2 tonnes of steel and 100 hours of work. The company has also an option to extend its operations and to begin to manufacture new sport car model named Bear. A Bear requires 1.5 tonnes of steel and 150 hours of work to be manufactured. In addition, the production of Bers requires an investment to a new production line. The production line costs the same amount as the profit from selling 1000 Bear cars, and the life of the production line would be 10 years. Each week the company has 24 tonnes of steel and 1200 hours of work in use. The profit from a Bear model car is double compared to the profit from selling a Winnie model car.

  a) What kind of weekly production would maximize the profits of the company? Formulate the problem as *linear integer problem*. You do not need to solve the problem. (4p)
  b) You plan to solve the problem with Branch-and-Bound algorithm. Present the general description of the algorithm. (2p)

4. Mantel Ltd. produces a toy cars, whose final assembly must include four wheels and two seats. The factory producing the parts operates three shifts a day. The followind table provides the amounts produced of each part in the three shifts.

	Unites produced per run	
Shift	Wheels	Seats
1	500	300
2	600	280
3	640	360

Ideally, the number of wheels produced is exactly twice that of the number of seats. However, because production rates vary from shift to shift, exact balance in production may not be even possible. Thus Mantel is interested in determining the number of production runs in each shift that minimizes the imbalance in the production of the parts. The capacity limitations restrict the number of runs to between 4 and 5 for shift 1, 10 and 20 for shift 2, and 3 and 5 for shift 3.

The company has the necessary software to solve the problem, but writing the problem properly to computer causes problems. The manager asks you, a summer trainee who has studied the Introduction to optimization course, to help in the matter. Formulate the manufacturing of the parts as a goal programming model.

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5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

a) For a linear programming (LP) problem, there is always either exactly one solution or no solution at all.
b) It is possible to formulate such an optimization problem, which has exactly two solutions.
c) Consider a linear programming (LP) problem. If the solution of the primal problem is unbounded, then the dual problem has no feasible solution.

5. Which of the following claims are true and which are false? Justify your choice. For a correct answer you gain +1 point and for right justification +1 point. For a false answer you get -2 points and an empty answer is worth of 0 points.

a) For an optimization problem, there is always either exactly one solution or no solution at all.
b) It is possible to formulate an optimization problem that has exactly two solutions.
c) The solution of an optimization problem is always located in a corner point of the feasible region.

5. a) Derive a 1-dimensional secant method starting from Newton's method. (2p)
b) Describe the steps of an n-dimensional gradient method. (4p)