# SIMPLEX

Linear optimisation & solution of linear systems :

Sisters , cousins , or just friends ?

# LINEAR SYSTEMS : $Ax = b$ (General Case)

**Theorem** Any $\underset{p \times n}{A}\, x = b$ can be transformed to

$$\left[\begin{array}{cc|c} \underset{r \times r}{I} & \underset{r \times (n-r)}{F} & \underset{r \times 1}{b_1} \\ \underset{(p-r) \times r}{0} & \underset{(p-r) \times (n-r)}{0} & \underset{(p-r) \times 1}{b_2} \end{array}\right]$$

| If | # of solutions |
| --- | --- |
| $r < p$ and $b_2 \neq 0$ | 0 |
| $(r = p$ or $b_2 = 0)$ and $r = n$ | 1 |
| $(r = p$ or $b_2 = 0)$ and $r < n$ | $\infty$ |

Terminology : Pivot variables

Free variables

Rank of $A$ is the number of pivot variables : $r$

This canonical reduced echelon form may include row and column permutations.

EXAMPLE (See also Mathematica Notebook)

Maximise $P = 3x + 4y + z$

$$\begin{cases} 3x + 10y + 5z \leq 120 \\ 5x + 2y + 8z \leq 6 \\ 8x + 10y + 3z \leq 105 \end{cases}$$

$$x, y, z \geq 0$$

Transformed

$$\begin{cases} 3x + 10y + 5z + s_1 & & & = 120 \\ 5x + 2y + 8z & + s_2 & & = 6 \\ 8x + 10y + 3z & & + s_3 & = 105 \end{cases}$$

or

$$Ax = b$$

GAUSS : Reduced Echelon Form

$$
\begin{matrix}
1 & & & -37/219 & 10/219 & 35/219 & -235/73 \\
& 1 & & 49/438 & -31/438 & 1/438 & 1933/146 \\
& & 1 & 17/219 & 25/219 & -22/219 & -40/73
\end{matrix}
$$

A particular solution $(s_i = 0)$ is

$$
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -235/73 \\ 1933/146 \\ -40/73 \end{pmatrix} = x_p \quad \text{(rightmost column, of course)}
$$

It is immediately clear that $x < 0$ and $z < 0$ are not in the ground set.

**Q**: Where is the solution to our minimisation problem?

**A**: Somewhere in the set of all possible solutions.

# SIMPLEX (But not with tableaux!)

$$\begin{array}{ccccccccc}
3 & 10 & 5 & 1 & 0 & 0 & 120 & 12 & \\
5 & 2 & 8 & 0 & 1 & 0 & 6 & 3 & \leftarrow (2) \\
8 & 10 & 3 & 0 & 0 & 1 & 105 & 10.5 & \\
\hline
-3 & -4 & -1 & 0 & 0 & 0 & 0 & & \leftarrow \text{objective function} \\
 & \uparrow & & & & & & & \\
 & (1) & & & & & & &
\end{array}$$

Pivot selection: (1) Column

(2) Row

Row operation: Let us permute the system first.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 5 | 8 | 0 | 1 | 0 | 6 | : 2 |
| 10 | 3 | 5 | 1 | 0 | 0 | 120 | −5 |
| 10 | 8 | 3 | 0 | 0 | 1 | 105 | −5 |
| −4 | −3 | −1 | 0 | 0 | 0 | 0 | 4 |

(**)

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 5/2 | 4 | 0 | 1/2 | 0 | 3 |
| 0 | −22 | −35 | 1 | −5 | 0 | 90 |
| 0 | −7 | −37 | 0 | −5 | 1 | 75 |
| 0 | 7 | 15 | 0 | 2 | 0 | 12 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | | 5/2 | 4 | 1/2 | 3 |
| | 1 | | −22 | −35 | −5 | 90 |
| | | 1 | −7 | −37 | −5 | 75 |
| ↑ | ↑ | ↑ | ↑ | ↑ | ↑ | |
| $y$ | $s_1$ | $s_3$ | $x$ | $z$ | $s_2$ | |

Solution :

$$\begin{pmatrix} 0 & 3 & 0 & 90 & 0 & 75 \end{pmatrix}^T$$

$$P = 4 \cdot 3 = 12$$

SANITY CHECK :

$$\begin{pmatrix} 3 & 10 & 5 & 1 & 0 & 0 \\ 5 & 2 & 8 & 0 & 1 & 0 \\ 8 & 10 & 3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 90 \\ 0 \\ 75 \end{pmatrix} = \begin{pmatrix} 120 \\ 6 \\ 105 \end{pmatrix}$$

HURRAH !

<u>Comment on (**) :</u>

The first two row operations are elimination steps, However, the last one is substitution, therefore one has to normalise first.