# CS-E4710 Machine Learning: Supervised Methods

Lecture 5: Linear classification

Juho Rousu

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Department of Computer Science
Aalto University

# Course topics

- Part I: Theory
    - Introduction
    - Generalization error analysis & PAC learning
    - Rademacher Complexity & VC dimension
    - Model selection
- Part II: Algorithms and models
    - **Linear models: perceptron, logistic regession**
    - Support vector machines
    - Kernel methods
    - Boosting
    - Neural networks (MLPs)
- Part III: Additional topics
    - Feature learning, selection and sparsity
    - Multi-class classification
    - Preference learning, ranking

# Linear classification

## Linear classification

- Input space $X \subset \mathbb{R}^d$, each $\mathbf{x} \in X$ is a $d$-dimensional real-valued vector, output space: $\mathcal{Y} = \{-1, +1\}$
- Target function or concept $f : X \mapsto \mathcal{Y}$ assigns a (true) label to each example
- Training sample $S = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, with $y_i = f(x_i)$ drawn from an unknown distribution $D$
- Hypothesis class $\mathcal{H} = \{\mathbf{x} \mapsto \operatorname{sgn}\left(\sum_{j=1}^{d} w_j x_j + w_0\right) | \mathbf{w} \in \mathbb{R}^d, w_0 \in \mathbb{R}\}$ consists of functions $h(\mathbf{x}) = \operatorname{sgn}\left(\sum_{j=1}^{d} w_j x_j + w_0\right)$ that map each example in one of the two classes
- $\operatorname{sgn}(a) = \begin{cases} +1, & a \geq 0 \\ -1 & a < 0 \end{cases}$ is the sign function

# Linear classifiers

Linear classifiers

$$h(\mathbf{x}) = \operatorname{sgn}\left(\sum_{j=1}^{d} w_j x_j + w_0\right) = \operatorname{sgn}\left(\mathbf{w}^T\mathbf{x} + w_0\right)$$

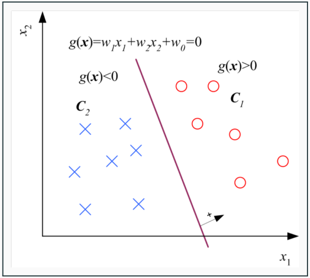
have several attractive properties

- They are fast to evaluate and takes small space to store ($O(d)$ time and space)
- Easy to understand: $|w_j|$ shows the importance of variable $x_j$ and its sign tells if the effect is positive or negative
- Linear models have relatively low complexity (e.g. $VCdim = d + 1$) so they can be reliably estimated from limited data

Good practise is to try a linear model before something more complicated

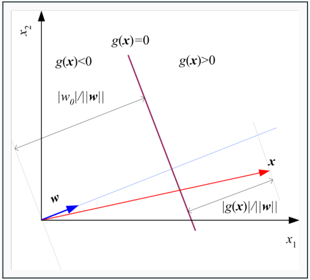
# The geometry of the linear classifier

- The points $\{\mathbf{x} \in X | g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0 = 0\}$ define a hyperplane in $\mathbb{R}^d$, where $d$ is the number of variables in $\mathbf{x}$
- The hyperplane $g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0 = 0$ splits the input space into two half-spaces. The linear classifier predicts $+1$ for points in the halfspace $\{\mathbf{x} \in X | g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0 \geq 0\}$ and $-1$ for points in $\{\mathbf{x} \in X | g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0 < 0\}$



# The geometry of the linear classifier

- $\mathbf{w}$ is the **normal vector** of the hyperplane $\mathbf{w}^T\mathbf{x} + w_0 = 0$
- The distance of the hyperplane from the origin is $|w_0| / \|\mathbf{w}\|$
- If $w_0 < 0$ the hyperplane lies in the direction of $\mathbf{w}$ from origin, otherwise it lies in the direction of $-\mathbf{w}$
- The distance of a point $\mathbf{x}$ from the hyperplane is $|g(\mathbf{x})| / \|\mathbf{w}\|$
- If $g(\mathbf{x}) > 0$, $\mathbf{x}$ lies in the halfspace that is in the direction of $\mathbf{w}$ from the hyperplane, otherwise it lies in the direction of $-\mathbf{w}$ from the hyperplane



# Learning linear classifiers

# Change of representation

- Consider learning the parameters of the linear discriminant $g(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0$
- For presentation is is convenient to subsume term $w_0$ into the weight vector

$$\mathbf{w} \Leftarrow \begin{bmatrix} \mathbf{w} \\ w_0 \end{bmatrix}$$

and augment all inputs with a constant 1:

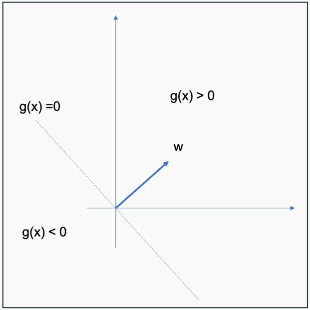
$$\mathbf{x} \Leftarrow \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$$

- The models have the same value for the discriminant:

$$\begin{bmatrix} \mathbf{w} \\ w_0 \end{bmatrix}^T \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \mathbf{w}^T\mathbf{x} + w_0$$

# Geometric interpretation

- Geometrically, the hyperplane defined by the discriminant goes now through origin
- The positive points have an **acute angle** with $\mathbf{w}$: $\mathbf{w}^T\mathbf{x} > 0$
- The negative points have an **obtuse angle** with $\mathbf{w}$: $\mathbf{w}^T\mathbf{x} <= 0$



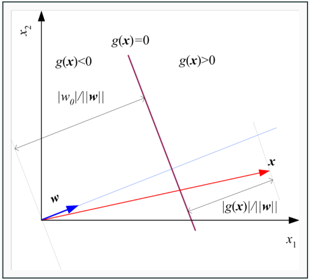
# Checking for prediction errors

- When the labels are $\mathcal{Y} = \{-1, +1\}$ for a training example $(\mathbf{x}, y)$ we have for $g(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$,
  $\operatorname{sgn}(g(\mathbf{x})) = \begin{cases} y & \text{if } \mathbf{x} \text{ is correctly classified} \\ -y & \text{if } \mathbf{x} \text{ is incorrectly classified} \end{cases}$
- Alternative we can just multiply with the correct label to check for misclassification:

$$yg(\mathbf{x}) = \begin{cases} \geq 0 & \text{if } \mathbf{x} \text{ is correctly classified} \\ < 0 & \text{if } \mathbf{x} \text{ is incorrectly classified} \end{cases}$$

# Margin

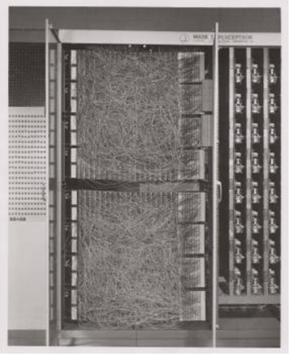
- The geometric margin of an example $\mathbf{x}$ is given by $\gamma(\mathbf{x}) = yg(\mathbf{x})/\left\|\mathbf{w}\right\|$
- It takes into account both the distance $|\mathbf{w}^T\mathbf{x}|/\left\|\mathbf{w}\right\|$ from the hyperplane, and whether $\mathbf{x}$ is on the correct side of the hyperplane
- The unnormalized version of the margin is sometimes called the **functional margin** $\gamma(\mathbf{x}) = yg(\mathbf{x})$
- Often the term **margin** is used for both variants, assuming the context makes clear which one is meant



# Perceptron

# Perceptron

- Perceptron algorithm by Frank Rosenblatt (1956) is perhaps the first machine learning algorithm
- Its purpose was to learn a linear discriminant between two classes
- It was built in hardware and shown to be capable of performing rudimentary pattern recognition tasks
- New York Times in 1958: "the embryo of an electronic computer that [the Navy] expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence." (Source: Wikipedia)



Mark I perceptron ca. 1958 (Picture: Wikipedia)

# The perceptron algorithm

- The perceptron algorithm a learns a hyperplane separating two classes

$$g(\mathbf{x}) = \mathbf{w}^T\mathbf{x}$$

- It processes incrementally a set of training examples
    - At each step, it finds a training example $\mathbf{x}_i$ that is incorrectly classified by the current model
    - It updates the model by adding the example to the current weight vector together with the label: $\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + y_i\mathbf{x}_i$
    - This process is continued until incorrectly predicted training examples are not found

# The perceptron algorithm

**Input:** Training set $S = \{(\mathbf{x}_i, y_i)\}_{i=1}^m, \mathbf{x} \in \mathbb{R}^d, y \in \{-1, +1\}$

Initialize $w^{(1)} \leftarrow (0, \ldots, 0), t \leftarrow 1$, $stop \leftarrow FALSE$

**repeat**

**if** exists $i$, s.t. $y_i {\mathbf{w}^{(t)}}^T \mathbf{x}_i \leq 0$ **then**

$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + y_i \mathbf{x}_i$

**else**

$stop \leftarrow TRUE$

**end if**

$t \leftarrow t + 1$

**until** *stop*

# Understanding the update rule

- Let us examine the update rule

$$\mathbf{w}^{(t+1)} \leftarrow \mathbf{w}^{(t)} + y_i \mathbf{x}_i$$

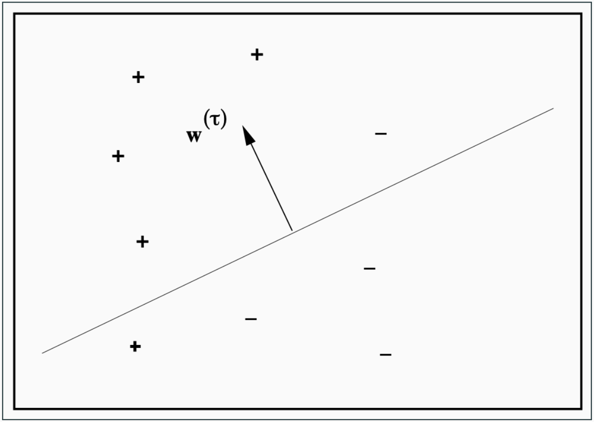
- We can see that the margin of the example $(\mathbf{x}_i, y_i)$ increases after the update

$$y_i g^{(t+1)}(\mathbf{x}_i) = y_i {\mathbf{w}^{(t+1)}}^T \mathbf{x}_i = y_i(\mathbf{w}^{(t)} + y_i \mathbf{x}_i)^T \mathbf{x}_i$$
$$= y_i {\mathbf{w}^{(t)}}^T \mathbf{x}_i + y_i^2 \mathbf{x}_i^T \mathbf{x}_i = y_i g^{(t)}(\mathbf{x}_i) + \|\mathbf{x}_i\|^2$$
$$\geq y_i g^{(t)}(\mathbf{x}_i)$$

- Note that this does not guarantee that $y_i g^{(t+1)}(\mathbf{x}_i) > 0$ after the update, further updates may be required to achieve that

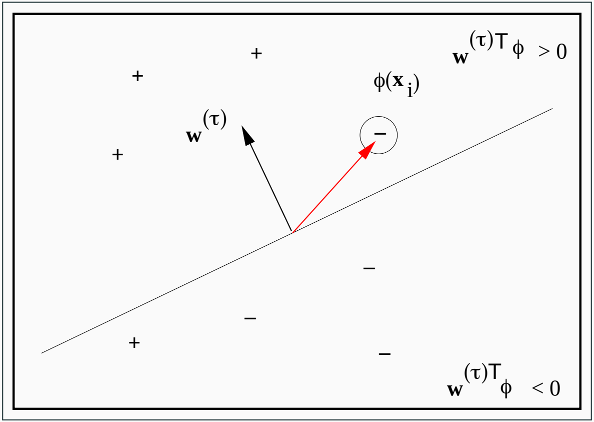
# Perceptron animation

- Assume $\mathbf{w}^{(t)}$ has been found by running the algorithm for $t$ steps
- We notice two misclassified examples



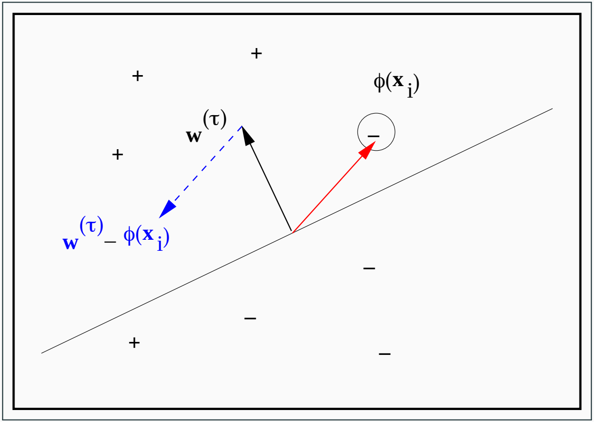
# Perceptron animation

- Select the misclassified example $(\phi(\mathbf{x}_i), -1)$
- Note: $\phi(\mathbf{x}_i)$ is here some transformation of $\mathbf{x}_i$ e.g. with some basis functions but it could be identity $\phi(\mathbf{x}) = \mathbf{x}$



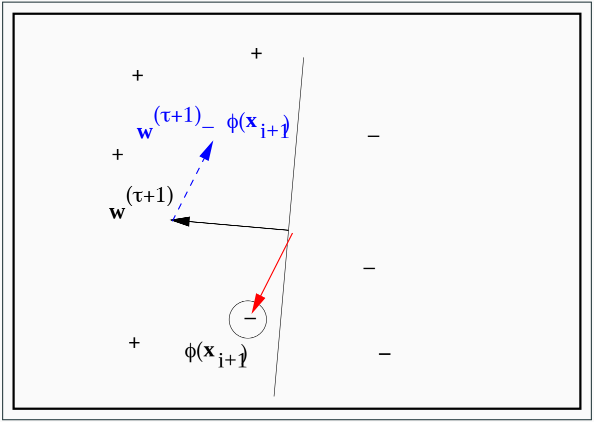
# Perceptron animation

- Update the weight vector: $\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} + y_i \phi(\mathbf{x}_i)$



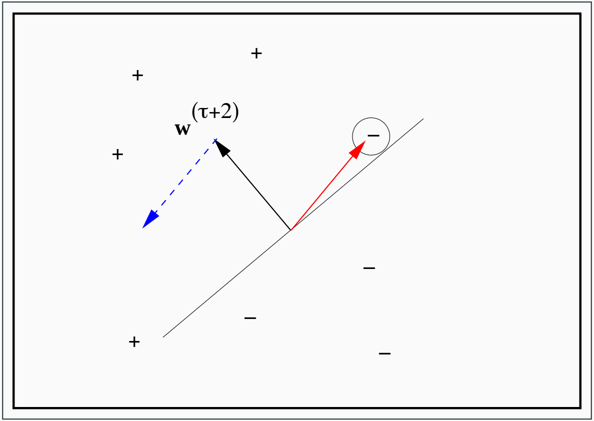
# Perceptron animation

- The update tilts the hyperplane to make the example "more correct", i.e. more negative
- We repeat the process by finding the next misclassified example $\phi(\mathbf{x}_{i+1})$ and update: $\mathbf{w}^{(t+2)} = \mathbf{w}^{(t+1)} + y_{i+1}\phi(\mathbf{x}_{i+1})$



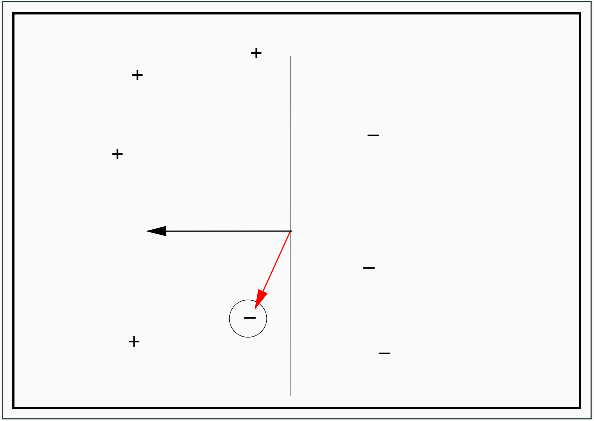
# Perceptron animation

- Next iteration



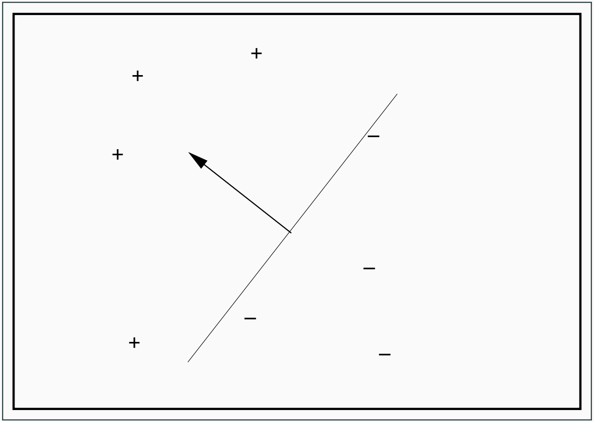
# Perceptron animation

- Next iteration



# Perceptron animation

- Finally we have found a hyperplane that correctly classify the training points
- We can stop the iteration and output the final weight vector



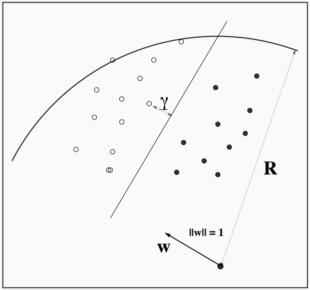
# Convergence of the perceptron algorithm

- The perceptron algorithm can be shown to eventually converge to a consistent hyperplane if the two classes are **linearly separable**, that is, if there exists a hyperplane that separates the two classes
- Theorem (Novikoff):
  - Let $S = \{(\mathbf{x}_i, y_i)\}_{i=1}^m$ be a linearly separable training set.
  - Let $R = \max_{\mathbf{x}_i \in S} \|\mathbf{x}_i\|$.
  - Let there exist a vector $\mathbf{w}_*$ that satisfies $\|\mathbf{w}_*\| = 1$ and $y_i \mathbf{w}_*^T \mathbf{x}_i + b_{opt} \geq \gamma$ for $i = 1 \ldots, m$.
  - Then the perceptron algorithm will stop after at most $t \leq (\frac{2R}{\gamma})^2$ iterations and output a weight vector $\mathbf{w}^{(t)}$ for which $y_i \mathbf{w}^{(t)} \mathbf{x}_i \geq 0$ for all $i = 1 \ldots, m$

# Convergence of the perceptron algorithm

The number of iterations in the bound $t \leq (\frac{2R}{\gamma})^2$ depend on:

- $\gamma$: The largest achievable geometric margin so that all training examples have at least that margin
- $R$: The smallest radius of the $d$-dimensional ball that encloses the training data
- Intuitively: how large the margin in is relative to the distances of the training points



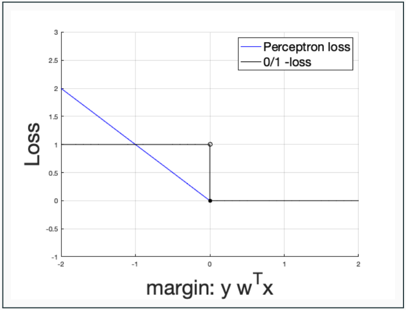

However, Perceptron algorithm does not stop on a non-separable training set, since there will always be a misclassified example that causes an update

# The loss function of the Perceptron algorithm

It can be shown that the Perceptron algorithm is using the following loss:

$$L_{Perceptron}(y, \mathbf{w}^T\mathbf{x}) = \max(0, -y\mathbf{w}^T\mathbf{x})$$

- $y\mathbf{w}^T\mathbf{x}$ is the margin
- if $y\mathbf{w}^T\mathbf{x} < 0$, a loss of $-y\mathbf{w}^T\mathbf{x}$ is incurred, otherwise no loss is incurred

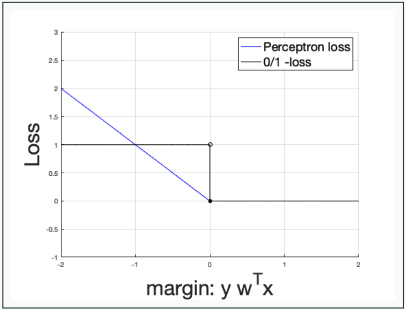


# Convexity of Perceptron loss

A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is convex if for all $x, y$, and $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y).$$

- Geometrical interpretation: the graph of a convex function lies below the line segment from $(x, f(x))$ to $(y, f(y))$
- It is easy to see that Perceptron loss is convex but zero-one loss is not convex



# Convexity of Perceptron loss

- The convexity of the Perceptron loss has an important consequence: every local minimum is also the global minimum
- In principle we can minimize it with incremental updates that gradually decrease the loss
- In contrast, finding a hyperplane that minimizes the zero-one loss is computationally hard (NP-hard to minimize training error)
- However, we need better algorithms than the Perceptron, which terminate when we are close to the optimum

# Logistic regression

# Logistic regression

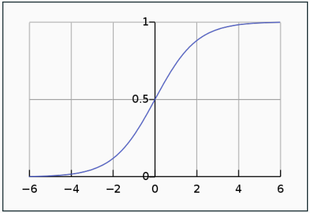
Logistic regression is a classification technique (despite the name)

- it gets its name from the logistic function

$$\phi_{logistic}(z) = \frac{1}{1 + \exp(-z)} = \frac{\exp(z)}{1 + \exp(z)}$$

  that maps a real valued input $z$ onto the interval $0 < \phi_{logistic}(z) < 1$

- The function is an example of **sigmoid** ("S" shaped) functions



# Logistic function: a probabilistic interpretation

- The logistic function $\phi_{logistic}(z)$ is the inverse of **logit function**
- The logit function is the logarithm of **odds ratio** of probability $p$ of and event happening vs. the probability of the event not happening, $1 - p$;

$$z = logit(p) = \log \frac{p}{1-p} = \log p - \log(1-p)$$

- Thus the logistic function

$$\phi_{logistic}(z) = logit^{-1}(z) = \frac{1}{1 + \exp(-z)}$$

answer the question "what is the probability $p$ that gives the log odds ratio of $z$"

# Logistic regression

- Logistic regression model assumes a underlying conditional probability:

$$Pr(y|\mathbf{x}) = \frac{\exp(+\frac{1}{2}y\mathbf{w}^T\mathbf{x})}{\exp(+\frac{1}{2}y\mathbf{w}^T\mathbf{x}) + \exp(-\frac{1}{2}y\mathbf{w}^T\mathbf{x})}$$

where the denominator normalizes the right-hand side to be between zero and one.

- Dividing the numerator and denominator by $\exp(+\frac{1}{2}y\mathbf{w}^T\mathbf{x})$ reveals the logistic function

$$Pr(y|\mathbf{x}) = \phi_{logistic}(y\mathbf{w}^T\mathbf{x}) = \frac{1}{1 + \exp(-y\mathbf{w}^T\mathbf{x})}$$

- The margin $z = y\mathbf{w}^T\mathbf{x}$ is thus interpreted as the log odds ratio of label $y$ vs. label $-y$ given input $\mathbf{x}$:

$$y\mathbf{w}^T\mathbf{x} = \log \frac{Pr(y|\mathbf{x})}{Pr(-y|\mathbf{x})}$$

# Logistic loss

- Consider the maximization of the likelihood of the observed input-output in the training data:

$$\mathbf{w}^* = \mathrm{argmax}_{\mathbf{w}} \prod_{i=1}^{m} P(y_i|\mathbf{x}_i) = \mathrm{argmax}_{\mathbf{w}} \prod_{i=1}^{m} \frac{1}{1 + \exp(-y\mathbf{w}^T\mathbf{x})}$$

- Since the logarithm is monotonically increasing function, we can take the logarithm to obtain an equivalent objective:

$$\sum_{i=1}^{m} \log Pr(y_i|\mathbf{x}_i) = -\sum_{i=1}^{m} \log(1 + \exp(-y_i\mathbf{w}^T\mathbf{x}_i))$$

- The right-hand side is the **logistic loss**:

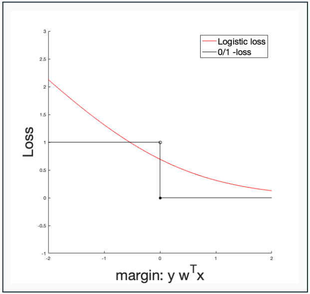
$$L_{logistic}(y, \mathbf{w}^T\mathbf{x}) = \log(1 + \exp(-y\mathbf{w}^T\mathbf{x}))$$

- Minimizing the logistic loss correspond maximizing the likelihood of the training data

# Geometric interpretation of Logistic loss

$$L_{logistic}(y, \mathbf{w}^T\mathbf{x}) = \log(1 + \exp(-y\mathbf{w}^T\mathbf{x}))$$

- Logistic loss is convex and differentiable
- It is a monotonically decreasing function of the margin $y\mathbf{w}^T\mathbf{x}$
- The loss changes fast when the margin is highly negative $\implies$ penalization of examples far in the incorrect halfspace
- It changes slowly for highly positive margins $\implies$ does not give extra bonus for being very far in the correct halfspace



# Logistic regression optimization problem

- To train a logistic regression model, we need to find the $\mathbf{w}$ that minimizes the average logistic loss $J(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} L_{logistic}(y_i, \mathbf{w}^T\mathbf{x}_i)$ over the training set:

$$\min \quad J(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} \log(1 + \exp(-y_i\mathbf{w}^T\mathbf{x}_i))$$

$$w.r.t \text{ parameters } \mathbf{w} \in \mathbb{R}^d$$

- The function to be minimized is continuous and differentiable
- However, it is a non-linear function so it is not easy to find the optimum directly (e.g. unlike in linear regression)
- We will use **stochastic gradient descent** to incrementally step towards the direction where the objective decreases fastest, the **negative gradient**

# Gradient

- The gradient is the vector of partial derivatives of the objective function $J(\mathbf{w})$ with respect to all parameters $w_j$

$$\nabla J(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} \nabla J_i(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} \left[\tfrac{\partial}{\partial w_1} J_i(\mathbf{w}), \ldots, \tfrac{\partial}{\partial w_d} J_i(\mathbf{w})\right]^T$$

- Compute the gradient by using the regular rules for differentiation. For the logistic loss we have

$$\begin{aligned}\frac{\partial}{\partial w_j} J_i(\mathbf{w}) &= \frac{\partial}{\partial w_j} \log(1 + \exp(-y_i \mathbf{w}^T x_i)) = \frac{\exp(-y_i \mathbf{w}^T x_i)}{1 + \exp(-y_i \mathbf{w}^T x_i)} \cdot (-y_i x_{ij}) \\ &= -\frac{1}{1 + \exp(y_i \mathbf{w}^T x_i)} y_i x_{ij} = -\phi_{logistic}(-y_i \mathbf{w}^T x_i) y_i x_{ij}\end{aligned}$$

# Stochastic gradient descent

- We collect the partial derivatives with respect to a single training example into a vector:

$$\nabla J_i(\mathbf{w}) = \begin{bmatrix} -(\phi_{logistic}(-y_i \mathbf{w}^T \mathbf{x}_i) y_i) \cdot x_{i1} \\ \vdots \\ -(\phi_{logistic}(-y_i \mathbf{w}^T \mathbf{x}_i) y_i) \cdot x_{ij} \\ \vdots \\ -(\phi_{logistic}(-y_i \mathbf{w}^T \mathbf{x}_i) y_i) \cdot x_{id} \end{bmatrix} = -\phi_{logistic}(-y_i \mathbf{w}^T \mathbf{x}_i) y_i \cdot \mathbf{x}_i$$

- The vector $-\nabla J_i(\mathbf{w})$ gives the update direction that fastest decreases the loss on training example $(\mathbf{x}_i, y_i)$

# Stochastic gradient descent

- Evaluating the full gradient

$$\nabla J(\mathbf{w}) = \frac{1}{m}\sum_{i=1}^{m} \nabla J_i(\mathbf{w}) = -\frac{1}{m}\sum_{i=1}^{m} \phi_{logistic}(-y_i \mathbf{w}^T \mathbf{x}_i) y_i \cdot \mathbf{x}_i$$

  is costly since we need to process all training examples
- Stochastic gradient descent instead uses a series of smaller updates that depend on single randomly drawn training example $(\mathbf{x}_i, y_i)$ at a time
- The update direction is taken as $-\nabla J_i(\mathbf{w})$
- Its expectation is the full negative gradient:

$$-\mathbb{E}_{i=1\ldots,m}\left[\,\nabla J_i(\mathbf{w})\,\right] = -\nabla J(\mathbf{w})$$

- Thus on average, the updates match that of using the full gradient

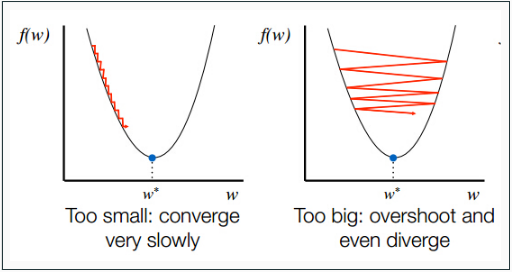
# Stochastic gradient descent algorithm

Initialize $\mathbf{w} = 0$
**repeat**
  Draw a training example $(x_i, y_i)$ uniformly at random
  Compute the update direction corresponding to the training example:
  $\Delta\mathbf{w} = -\nabla J_i(\mathbf{w})$
  Determine a stepsize $\eta$
  Update $\mathbf{w} = \mathbf{w} - \eta\nabla J_i(\mathbf{w})$
**until** stopping criterion statisfied
Output $\mathbf{w}$

# Stepsize selection

Consider the SGD update: $\mathbf{w} = \mathbf{w} - \eta \nabla J_i(\mathbf{w})$

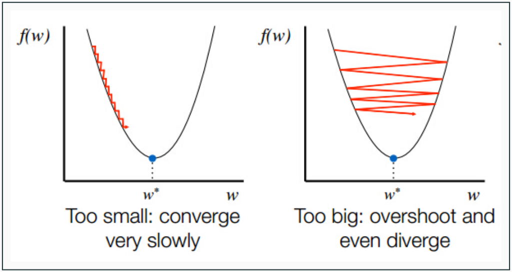
- The stepsize parameter $\eta$, also called the **learning rate** is a critical one for convergence to the optimum value
- One uses small constant stepsize, the initial convergence may be unnecessarily slow
- Too large stepsize may cause the method to continually overshoot the optimum.




Source: https://dunglai.github.io/2017/12/21/gradient-descent/

# Diminishing stepsize

- Initially larger but diminishing stepsize is one option:
$$\eta^{(t)} = \frac{1}{\alpha t}$$
for some $\alpha > 0$, where $t$ is the iteration counter
- Caution: In practice, finding a good value for parameter $\alpha$ requires experimenting with several values




Source: https://dunglai.github.io/2017/12/21/gradient-descent/

## Stopping criterion

When should we stop the algorithm? Some possible choices:

1. Set a maximum number of iterations, after which the algorithm terminates
    - This needs to be separately calibrated for each dataset to avoid premature termination
2. Gradient of the objective: If we are at a optimum point $\mathbf{w}^*$ of $J(\mathbf{w})$, the gradient vanishes $\nabla J(\mathbf{w}^*) = 0$, so we can stop $\|J(\mathbf{w})\| < \gamma$ where $\gamma$ is some user-defined parameter
3. It is usually sufficient to train until the **zero-one error** on training data does not change anymore
    - This usually happens before the logistic loss converges

# Summary

- Linear classification model are and important class of machine learning models, they are used as standalone models and appear as building blocks of more complicated, non-liner models
- Perceptron is a simple algorithm to train linear classifiers on linearly separable data
- Logistic regression is a classification method that can be interpreted as maximizing odds ratios of conditional class probabilities
- Stochastic gradient descent is an efficient optimization method for large data that is nowadays very widely used