
CHAPTER 1

TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS

1 Topological Spaces

1.1 Basic notions, bases, subbases etc.

Definition 1.1. (Topology) A topology τ on a set X is a collection \mathcal{B} of sets called *open sets* such that:

1. \emptyset and X belong to \mathcal{B}
2. For any collection of sets $U_\alpha \in \mathcal{B}$, $\cup_\alpha U_\alpha$ is also in \mathcal{B} (closed under arbitrary unions)
3. For any finite collection $\{U_1, U_2 \dots U_k\}$ of sets of \mathcal{B} , $\cap_{i=1}^k U_i \in \mathcal{B}$ (closed under finite intersections)

Definition 1.2. (Discrete and Indiscrete topology)

1. The topology (\emptyset, X) of a set X is called the *indiscrete topology*.
2. The topology in which every subset of X is an open set is called the *Discrete topology*

Definition 1.3. (Finer and Coarser topologies) Let τ and τ' be two topologies of space X . We say τ' is *finer* than τ if $\tau \subset \tau'$, or every open set in τ is one in τ' . In other words, τ is *coarser* than τ .

Remark 1.4. Of course, not all topologies are comparable, like the above definition suggests. For example, consider the 3 element space $\{x, y, z\}$. $\{\emptyset, (x), (x, y), (x, y, z)\}$ is a topology as can easily be seen. Likewise, $\{\emptyset, (x), (y, z), (x, y, z)\}$ is also a topology, but these two are not comparable, since their symmetric difference is non-empty.

Example 1.5. The finite complement topology: Let X be a set. Consider the set of all $U \in X$ such that U^C is either finite or X . Obviously, \emptyset is in this collection, as is X . Let $\{U_\alpha : \alpha \in A\}$ be an arbitrary collection of such sets. Consider $(\cup_\alpha U_\alpha)^C = \cap_\alpha U_\alpha^C$ which is easily seen to be finite. Hence, closed under arbitrary unions. Consider $\{U_1, U_2 \dots U_k\}$ of such sets. $(\cap_{i=1}^k U_i)^C = \cup_{i=1}^n U_i^C$ which is, at max, finite. Hence, closed under finite intersections. Therefore, this collection is a topology and is called the *Finite Complement Topology*.

Definition 1.6. (Basis for a topology) Let X be a space. \mathcal{C} is called a *basis* for X if the following are satisfied:

1. For every $x \in X$, there exists a C_x in \mathcal{C} so that $x \in C_x$
2. If x, C_1 and C_2 are such that $x \in C_1 \cap C_2 \neq \emptyset$, then there exists $C_{12} \in \mathcal{C}$ such that $x \in C_{12} \subseteq C_1 \cap C_2$

Definition 1.7. (Topology generated by a basis \mathcal{C} of X) The topology $\tau_{\mathcal{C}}$ generated by the basis \mathcal{C} is defined as the collection \mathcal{U} of subsets of X such that if $U \in \mathcal{U}$, and $x \in U$, then there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. It is obvious that \mathcal{C} itself is contained in \mathcal{U} .

Theorem 1.8. *The topology $\tau_{\mathcal{C}}$ generated by \mathcal{C} , a basis, does indeed form a topology.*

Proof. Does X belong to $\tau_{\mathcal{C}}$? let $x \in X$. By virtue of being a basis, there exists a $C_x \subseteq X$ such that $x \in C_x$. Hence, $X \in \tau_{\mathcal{C}}$. Does $\emptyset \in \tau_{\mathcal{C}}$? Vacuously so. Let U_α be open in $\tau_{\mathcal{C}}$. Consider $\cup_\alpha U_\alpha$ and $x \in \cup_\alpha U_\alpha$. This means $x \in U_\delta$ for some δ . That means that there exists C_x so that $x \in C_x \subseteq U_\delta \subseteq \cup_\alpha U_\alpha$. This means $\tau_{\mathcal{C}}$ is closed under arbitrary union. Consider singleton collections, for the moment, of the kind $\{U_1\}$. It is obvious that the intersection of every element in this collection is part of $\tau_{\mathcal{C}}$. Consider a two sized collection $\{U_1, U_2\}$. Let $x \in U_1 \cap U_2$. Obviously, $x \in C_1 \subseteq U_1$ and $x \in C_2 \subseteq U_2$. From the definition of basis, there exists a basis element C so that $x \in C \subseteq C_1 \cap C_2 \subseteq U_1 \cap U_2$ making $\cap_{i=1}^2 U_i$ part of the topology. Assume such is true for all n sized collections. Consider an $n+1$ sized collection $\{U_1, U_2 \cdots U_{n+1}\}$ and consider $x \in (\cap_{i=1}^n U_i) \cap U_{n+1}$. From the $n=2$ case and induction hypothesis, we are done. Hence, closed under finite intersections. $\tau_{\mathcal{C}}$ is indeed a topology. \square

Theorem 1.9. *Let \mathcal{C} be a basis for X . The topology generated by \mathcal{C} , that is, $\tau_{\mathcal{C}}$ is precisely the topology obtained by taking all possible, arbitrary union's of \mathcal{C} .*

Proof. Consider an arbitrary union of elements $C_\alpha \in \mathcal{C}$, $\cup_\alpha C_\alpha$. $x \in \cup C \implies \exists \alpha$ such that $x \in C_\alpha$. Therefore, $\cup_\alpha C_\alpha$ is in $\tau_{\mathcal{C}}$. Now, let $U \in \tau_{\mathcal{C}}$. For every $x \in U$, there exists C_x in \mathcal{C} such that $x \in C_x \subseteq U$ (Definition of the generated topology). That means, $U = \cup_{x:x \in U} C_x$ which makes it the union of sets in the collection \mathcal{C} . We are done. \square

Example 1.10. If X is a metric space, the collection $\{B_\varepsilon(x) : x \in X, \varepsilon \in \mathbb{R}^+\}$ is a basis for X .

Remark 1.11. Note that, every topology τ is its own basis. It must also be noted that the same topology can be generated by two different bases. Consider for example the set of all open subsets of \mathbb{R} . Also consider the set of all open balls in \mathbb{R} . Or, consider the set of all $\frac{1}{n}$ balls around all rational points of \mathbb{R}

Theorem 1.12. (Going from a topology to a basis) *Suppose (X, τ) is a topological space, and \mathcal{C} is a collection of sets in X such that for every $x \in X$ and for every $U \in \tau$, $\exists C_x \in \mathcal{C}$ such that $x \in C_x \subseteq U$. Then τ is actually generated by \mathcal{C}*

Proof. Let $x \in X$. That means, $\exists C_x \in \mathcal{C}$ (from hypothesis) such that $x \in C_x \subseteq X$. Moreover, let $x \in C_1$ and C_2 . By hypothesis, all $C \in \mathcal{C}$ are considered open sets, therefore, $C_1 \cap C_2$ is an open set, and $x \in C_1 \cap C_2$. That means $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. That makes \mathcal{C} a basis. Its now obvious that $\tau = \tau_{\mathcal{C}}$ \square

Theorem 1.13. (*Criteria for determining coarseness from basis*) Let τ and τ' be two topologies of X . Also suppose τ is generated by \mathcal{C} and τ' , by \mathcal{C}' . Then the following are equivalent:

1. τ' is finer than τ
2. for each $x \in X$ and each basis element $C \in \mathcal{C}$ containing x , there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$

Proof. \implies) Suppose τ' is finer than τ , that means every open set open in τ is also open in τ' . Let $C \in \mathcal{C}$, which is open in τ (and hence, in τ' too) and let $x \in C$. Since C is open in τ' , and $x \in C$, there exists a basis element of τ' , called C' such that $x \in C' \subseteq C$.

\impliedby) Let it be that for every $x \in X$ and for every $C \in \mathcal{C}$ such that $x \in C$, there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$. By the definition of topology generated by bases, it is clear that \mathcal{C} is part of the topology of τ' . Consider $U \in \tau$.

U is an arbitrary union of elements of \mathcal{C} (by the alternate characterisation of generated topology), making U an element of τ' . We are done.

(aliter) Let U be such that for every $x \in U$, $\exists C \in \mathcal{C}$ such that $x \in C \subseteq U$. Since we are told that there exists $C' \subseteq C$ so that $x \in C'$, we have that for all $x \in U$, $\exists C'_x \in \mathcal{C}'$ so that $x \in C'_x \subseteq U$, making U part of the topology of τ' . \square

Definition 1.14. (Standard topology of \mathbb{R}) Let \mathcal{B} be the collection of all open intervals of the kind $(a, b) : a, b \in \mathbb{R}$. That this is a basis is obvious. Define the standard topology as that topology generated by this basis.

Definition 1.15. (lower limit topology \mathbb{R}_l) Consider \mathcal{B}' the collection of all sets of the kind $[a, b) : a, b \in \mathbb{R}$. Again, that this is a basis is obvious. Define the lower limit topology as the topology generated by \mathcal{B}' , and denote it as \mathbb{R}_l

Definition 1.16. (K topology \mathbb{R}_K) Let K be the set of all numbers of the kind $1/n$ for $n \in \mathbb{Z}_+$. Denote \mathcal{B}'' as the collection of all open intervals of the kind (a, b) along with the sets of the form $(a, b) - K$. Again, that this is a basis is obvious. The topology generated by \mathcal{B}'' is called the K topology, denoted by \mathbb{R}_K

Theorem 1.17. \mathbb{R}_K and \mathbb{R}_l are strictly finer than \mathbb{R} , but \mathbb{R}_K and \mathbb{R}_l are not comparable.

Proof. Consider $B \in \mathcal{B}$, of the form (a, b) . This is also in \mathcal{B}' , and a smaller set of the kind $[c, d)$ lies inside (a, b) , which means that, if $x \in (a, b)$, there exists $B' \in \mathcal{B}'$ and $B'' \in \mathcal{B}''$ so that $x \in B' \subseteq (a, b)$ and $x \in B'' \subseteq (a, b)$. Moreover, since both \mathbb{R}_l and \mathbb{R}_K have elements that are not in \mathbb{R} , it is obviously strictly finer. (Consider $a \in [a, b)$, there is no element of the kind (x, y) that is inside this, that contains a .)

Consider an element $B = [0, 1)$ where we choose $x = 0$. Clear that no element from \mathbb{R}_K can come inside B whilst containing 0. On the other hand, consider 0 inside $(-1, 1) - K$ in \mathbb{R}_K . No interval of the kind $[a, b)$ can contain 0 and itself be contained inside $(-1, 1) - K$. Hence, it is clear that \mathbb{R}_K and \mathbb{R}_l are incomparable. \square

(How Dugundji goes about developing an idea of basis)

Let \mathcal{T}_X be the topology of a space X . We call $\mathcal{B} \subset \mathcal{T}_X$ a *basis* if every open set in \mathcal{T}_X is a union of elements of \mathcal{B} . (and by definition, since \mathcal{B} is asked to be a subset of \mathcal{T}_X , we see that every $B \in \mathcal{B}$ is also open. So we can safely say every open set in \mathcal{T}_X is an arbitrary union of \mathcal{B} , and conversely every arbitrary union of elements of \mathcal{B} belongs in \mathcal{T}_X)

Theorem: Let $\mathcal{B} \subseteq \mathcal{T}_X$. Then, the following are equivalent:

1. \mathcal{B} is a basis for \mathcal{T}_X
2. For every $G \in \mathcal{T}_X$, and for every $(x \in X, x \in G) \implies x \in B_x \subset G$ for some $B_x \in \mathcal{B}$

Proof. \implies) Let \mathcal{B} be a basis for \mathcal{T}_X . Let $x \in G$ for some $G \in \mathcal{T}_X$. Since $G = \cup_q B_q$ for $B_q \in \mathcal{B}$, we have that there exists atleast one B_q so that $x \in B_q \subseteq G$.

\impliedby) Let U be an open set in \mathcal{T}_X . We are told that for every $x \in U$, there is an element of \mathcal{B} so that $x \in B_x \subseteq U$. This implies U is an arbitrary union of elements of \mathcal{B} . This means that \mathcal{B} forms a basis for \mathcal{T}_X . □

Definition 1.18. (Subbasis) A subbasis S is a collection of subsets of X whose union amounts to X

Definition 1.19. (Topology generated by a subbasis) τ_S , the topology generated by the subbasis S is defined as the set of all arbitrary unions of the set of all finite intersection of elements of S .

Theorem 1.20. τ_S , the topology generated by a subbasis S , is indeed a topology.

Proof. Let S be the set whose union forms X (Subbasis).

Let $Y = \{\gamma : \gamma = \cap_{i=1}^n S_i \text{ for some } n \text{ and } S_i \in S\}$, i.e, Y is the set of all finite intersections of elements of S . Consider the set of all arbitrary union of elements of Y , and call it τ_S . Obviously τ_S contains \emptyset and X . Let $\{U_\alpha\}$ be a collection of open sets in τ_S , i.e, each U_α is an arbitrary union of elements that are finite intersection of elements of S . The union of all U_α , therefore, would still be an arbitrary union of finite intersections of elements of S , hence, is back in τ_S . Now consider $\{U_1, U_2 \cdots U_k\}$, elements of τ_S such that each U_i is an arbitrary union of a collection of sets that are each a finite intersection of elements of S . Consider the finite intersection $\cap_{i=1}^n U_i = \cap_{i=1}^n (\cup_\alpha (\cap_{j=1}^{n_\alpha} S_{j,\alpha}))$ which is ultimately an arbitrary union of a finite intersection of sets in S , making this part of τ_S . Hence, τ_S is indeed a topology. □

Theorem 1.21. Let $\{\tau_\alpha : \alpha \in A\}$ be a collection of topologies of X , then there is a unique "smallest" topology τ_S such that τ_S contains all the topologies $\{\tau_\alpha\}$ (kinda like a LUB), and there is a unique "largest" topology τ_L that is contained in all these topologies $\{\tau_\alpha\}$.

Proof. τ_S that contains all topologies $\{\tau_\alpha\}$:

Uniqueness: If two such topologies exists, we can take the intersection of them to result

in a smaller topology (intersection is still a topology) that contains $\{\tau_\alpha\}$.

Existence: Consider the topology generated by arbitrarily unioning the sets generated by finitely intersecting all elements of $\{\tau_\alpha : \alpha \in A\}$. Explicitly, we collect $\mathcal{R} = \{U : \exists x \text{ such that } U \in \tau_x\}$ and we take all possible finite intersections of elements in \mathcal{R} and collect that in a set \mathcal{S} . Then, we take all possible arbitrary unions of elements of \mathcal{S} and call it τ_S .

For later use:

- \mathcal{R} = all the open sets in all the topologies.
- \mathcal{S} = finite intersection of all elements of \mathcal{R}
- τ_S = arbitrary union of all elements of \mathcal{S}

Is τ_S a topology? X and \emptyset are obviously there, since X and \emptyset are there in \mathcal{R} , and we take the singleton intersection of these in \mathcal{S} , and finally, we take the singleton union of these to arrive at $X, \emptyset \in \tau_S$. Let $\{G_\delta : \delta \in \Delta\}$ be a collection of sets in τ_S (i.e, a collection of sets that are an arbitrary union of a collection of finite intersection of elements of \mathcal{R}). $\cup_\delta G_\delta$ would also be an arbitrary union of elements of \mathcal{S} making it part of τ_S . Let $\{G_1, G_2, \dots, G_k\}$ be in τ_S . $\cap_{i=1}^k G_i = \cap_{i=1}^k (\cup_{\delta_i} (\cap_{j=1}^{n_{\delta_i}} R_{\delta_i, j}))$ which, from De Morgan's Laws, can easily be seen to be an arbitrary union of a finite intersection of elements from \mathcal{R} . Hence, the intersection is also in τ_S making it a topology that contains all of $\{\tau_\alpha\}$.

τ_L that is contained in all topologies $\{\tau_\alpha\}$

Existence: Consider $\cap_\alpha \tau_\alpha$, which is a topology that is contained in all of τ_α .

Uniqueness: If two topologies are such that they are contained in every τ_α , then they are indeed contained in $\cap_\alpha \tau_\alpha$ □

Theorem 1.22. *Let A be a basis for τ , then τ_A , the topology generated by A (The one generated by taking arbitrary union of elements of A) is precisely the intersection of all topologies $\{\tau_\alpha : \alpha \in G\}$ that contain A*

Proof. τ_A is the topology generated by arbitrary union of elements of A . We are told $A \subseteq \tau_\alpha$ for all $\alpha \in G$. Let $U \in \tau_A$. U is an arbitrary union of elements of A , which, since τ_α is a topology, is also contained in τ_α . Every open set in τ_A is also in τ_α . Hence $\tau_A \subseteq \cap_\alpha \tau_\alpha$.

Note that, τ_A is itself a topology containing A , which means $\tau_A = \cap_\alpha \tau_\alpha$ □

Theorem 1.23. *Let A be a subbasis for X . The topology τ_A generated by A , i.e, by taking arbitrary union of elements that are finite intersection of A , is precisely the intersection of all topologies $\{\tau_\alpha\}$ that contain A .*

Proof. Obviously τ_A contains A , since each element of A is a singleton union of the singleton intersection of itself. Moreover, let U be any open set in τ_A , i.e, U is an arbitrary union of a collection of finite intersections of elements of A . Let τ_α be a topology that contains A . That means a finite intersection of elements of A is still in τ_α , and an arbitrary union of these are still in τ_α . Hence, U is in τ_α , for any α . Hence, $\tau_A \subseteq \tau_\alpha$ for every α , implying $\tau_A \subseteq \cap_\alpha \tau_\alpha$. Since τ_A is part of $\{\tau_\alpha\}$, we see

$$\tau_A = \cap_\alpha \tau_\alpha$$

where $\{\tau_\alpha\}$ is the collection of all topologies that contain A . \square

(How Dugundji goes about developing subbasis)

Theorem: Given any collection of subsets of X , $\Sigma = \{A_\alpha : \alpha \in \mathbb{A}\}$, there exists a unique, smallest topology $\mathcal{T}(\Sigma)$ that contains Σ as a subset. Moreover, this has a characterisation: It is the collection of all the arbitrary unions of finite intersections of elements of Σ .

Proof. Uniqueness: If two such topologies \mathcal{T}_1 and \mathcal{T}_2 exists, then $\mathcal{T}_1 \cap \mathcal{T}_2$ becomes a smaller topology that contains Σ .

Existence: Take all elements of Σ and take all possible finite intersections, and collect them in the set \mathcal{R} . Then take all possible arbitrary union of elements of \mathcal{R} and collect them in a set \mathcal{S} . This would be a topology, since arbitrary union of an arbitrary union of finite intersections is an arbitrary union of finite intersections. Moreover, finite intersections of arbitrary unions of finite intersections are ultimately arbitrary union of finite intersections. We denote this topology by $\mathcal{T}(\Sigma)$.

Consider for a moment the set of all topologies \mathcal{T}_γ that contains Σ . $\cap_\gamma \mathcal{T}_\gamma$, intersection of all topologies that contain Σ , is indeed a topology, as is easy to verify. Moreover, it is the smallest topology that contains Σ ; This much is evident.

Let U be an arbitrary union of a finite intersection of elements of Σ . Since $\cap_\gamma \mathcal{T}_\gamma$ contains Σ , it contains U as well, implying that $\mathcal{T}(\Sigma) \subseteq \cap_\gamma \mathcal{T}_\gamma$. Since $\mathcal{T}(\Sigma)$ is itself a topology that contains Σ , we see that $\mathcal{T}(\Sigma) = \cap_\gamma \mathcal{T}_\gamma$, i.e $\mathcal{T}(\Sigma)$ is the smallest topology that contains Σ . \square

Definition:(Topology generated by a family Σ , $\mathcal{T}(\Sigma)$) The topology generated by any family of subsets of X is defined as $\mathcal{T}(\Sigma)$, the collection of all arbitrary union of finite intersections of elements of Σ .

1.2 Order Topology

Definition 1.24. (Simple Order) Let X be a set. Define $C : X \times X \rightarrow \{y, n\}$ such that:

1. If $x \neq y$, either xCy or yCx
2. For no x is it true that xCx
3. if xCy and yCz , then xCz

This type of relation is called *simple order* relation.

Pro Tip: For intuition purposes, replace C with $<$.

Definition 1.25. (Order Topology on a space X .) Let X be a space with more than two points, equipped with a simple order relation $<$. Consider the collection \mathcal{B} of sets of the following kind:

1. (a, b) where $a < b$. i.e, all open intervals in X
2. $[a_0, b)$ for all $b \in X$, where a_0 is the smallest element of X
3. $(a, b_0]$ for all $a \in X$ where b_0 is the largest element in X

Then the order topology τ_O is defined as that topology generated by \mathcal{B} .

Remark 1.26. Note that, if X has no smallest element, type (2) sets won't exist in \mathcal{B} . Likewise, if X has no largest element, type (3) sets won't exist in \mathcal{B} .

Theorem 1.27. \mathcal{B} is actually a basis (hence, τ_O is actually a well defined topology on X)

Proof. Let $x \in X$. If x is not the largest element, nor the smallest, then there exists (a_0, b_0) that contains x

If x is the largest element, then $(a_0, x]$ contains x . If x is the smallest element, then $[x, b_0)$ contains x . Hence, for every $x \in X$, there exists an element B from \mathcal{B} such that $x \in B$.

Note the following, if we intersect two type 1 sets, we get back a smaller type 1 set. i.e, $(a, b) \cap (a', b') := \{x \in X : a < x < b \text{ and } a' < x < b'\}$.

If we intersect two type 2 sets, we get back a type 2 set (obviously)

If we intersect two type 3 sets, we get back a type 3 set (obviously)

If we intersect a type 1 and type 2 set, i.e (a, b) and $[a_0, x)$, we are basically saying $\{t \in X : a < t < b \text{ and } a_0 \leq t < x\}$ We get back a type 1 set.

If we intersect a type 1 and type 3 set, i.e (a, b) and $(x, b_0]$, we get $\{t \in X : a < t < b \text{ and } x < t \leq b_0\}$ which would yield a set of type 1.

If we intersect a type 2 and type 3 set, i.e, $[a_0, x)$ and $(y, b_0]$, we want basically $\{t \in X : a_0 \leq t < x \text{ and } y < t \leq b_0\}$. It is easy to see that this intersection falls into one of the 3 categories.

Hence, \mathcal{B} forms a basis, and hence, τ_B is a topology. \square

Example 1.28. The standard topology \mathbb{R} is basically the order topology on \mathbb{R} . All one needs to see is that there is no least element, and largest element of \mathbb{R} (Of course, $\mathbb{R} \cup \{-\infty, +\infty\}$, the extended real line, *does* have these). Hence, \mathcal{B} is just all elements of the kind (a, b) , which is precisely the basis for the standard topology.

Definition 1.29. (Rays) Given $a \in X$ (a simply ordered set), there are 4 sets associated with a called *rays*:

1. $(a, +\infty) := \{x : a < x\}$
2. $(-\infty, a) := \{x : x < a\}$
3. $[a, +\infty) := \{x : a \leq x\}$

4. $(-\infty, a] := \{x : x \leq a\}$

Theorem 1.30. *The first two rays are open in the order topology, (and the 2nd two rays are closed in the order topology, i.e, its complement is open)*

Proof. Consider $(a, +\infty)$. If there is no maximum element, then this is precisely an arbitrary union of sets of the kind (a, b_α) . If there is a maximal element, then this is $(a, b_0]$. In a similar fashion, the 2nd set is also open.

Consider $[a, +\infty)^C = (-\infty, a)$, and consider $(-\infty, a]^C = (a, \infty)$. Obvious from here. \square

Theorem 1.31. *The collection of all open rays forms a subbasis for the order topology on X*

Proof. We have to show that the topology generated by arbitrarily unioning a collection of finite intersections of rays gives rise to the topology generated by the basis of order topology of X . Let τ_R be the topology generated by the arbitrary union of finite intersection of rays. Let τ be the order topology on X , generated by arbitrary union of the basis elements of X . We want to show $\tau_R \subseteq \tau$ and $\tau \subseteq \tau_R$.

Consider (a, b) , a basis element of τ . $(a, +\infty) \cap (-\infty, b)$ is a finite intersection of rays, and yields the basis (a, b) . $[a_0, b)$ is, simply, $(-\infty, b)$ and $(a, b_0]$ is simply $(a, +\infty)$. Hence, finite intersection of the rays yield the basis elements. Arbitrary union of finite intersection of rays can yield all the possible open sets of the order topology on X . Hence, $\tau \subseteq \tau_R$. Let S be open in τ_R . It is the arbitrary union of a finite intersection of rays. Note that every ray is either an element of the basis for τ or is an arbitrary union elements of basis of τ . An arbitrary union of a finite intersection of an arbitrary union of elements of the basis yields an arbitrary union of a finite intersection of elements of the basis, giving rise to yet another element of the topology τ . Hence, every open set in τ_R is open in τ , hence $\tau_R \subseteq \tau$ \square

1.3 The product topology

Definition 1.32. (Product topology on $X \times Y$) Let τ_X and τ_Y be topologies of X and Y respectively. Let \mathcal{B} be the collection of subsets of $X \times Y$ of the kind $U \times V$ for $U \in \tau_X$ and $V \in \tau_Y$. We define the product topology as that which is generated by the basis \mathcal{B} .

Theorem 1.33. *\mathcal{B} of the previous definition is indeed a basis.*

Proof. Let $(x, y) \in X \times Y$. Since $X \in \tau_X$ and $Y \in \tau_Y$, we have that $(x, y) \in X \times Y$ so the 1st condition for basis is satisfied. Let $(x, y) \in U_1 \times V_1$ and $U_2 \times V_2$. Consider $(U_1 \times V_1) \cap (U_2 \times V_2)$. This is same as $U_1 \cap U_2 \times V_1 \cap V_2$ which is again an element of \mathcal{B} . Hence, it is a basis and hence $\tau_{\mathcal{B}}$ is well defined. \square

Theorem 1.34. *If X 's topology \mathcal{T}_X is generated by a basis \mathcal{B}_X and Y 's topology \mathcal{T}_Y is generated by a basis \mathcal{B}_Y , then the product topology of $X \times Y$*

(given by $\{U \times V : U \text{ open in } \mathcal{T}_X \text{ and } V \text{ open in } \mathcal{T}_Y\}$) has a basis $\mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\}$, i.e,

$$\mathcal{T}_{X \times Y} = \langle \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\} \rangle = \langle_{basis} \mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\} \rangle$$

Proof. Checking that \mathcal{B} is indeed a basis:

Let $(x, y) \in X \times Y$. $x \in X$ means that there exists $B \in \mathcal{B}_X$ so that $x \in B \subseteq X$. Likewise, for y , there is a $C \in \mathcal{B}_Y$ so that $y \in C \subseteq Y$, which means $(x, y) \in B \times C \subseteq X \times Y$. This means that \mathcal{B} obeys the 1st condition for a basis.

Let $(x, y) \in B, B' \in \mathcal{B}$. i.e, $(x, y) \in C_1 \times D_1$, and $C_2 \times D_2$ for $C_1, C_2 \in \mathcal{B}_X$ and $D_1, D_2 \in \mathcal{B}_Y$. That means that $(x, y) \in (C_1 \times D_1) \cap (C_2 \times D_2) = (C_1 \times C_2) \times (D_1 \times D_2)$. $x \in C_1 \cap C_2$ means there is a C_3 so that $x \in C_3 \subseteq C_1 \cap C_2$. Likewise, $y \in D_3 \subseteq D_1 \cap D_2$. Hence, $(x, y) \in (C_3 \times D_3) \subseteq (C_1 \cap C_2) \times (D_1 \cap D_2) = (C_1 \times D_1) \cap (C_2 \times D_2)$. Hence, \mathcal{B} is a basis.

Now consider any open set in $\mathcal{T}_{X \times Y}$. Since $\mathcal{T}_{X \times Y}$ is generated by $U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y$, Every open set in $\mathcal{T}_{X \times Y}$ is an arbitrary union of sets of the kind $U \times V$, i.e, $\cup_\alpha (U_\alpha \times V_\alpha)$. But each $U_\alpha = \cup_{\delta(\alpha)} B_\delta^\alpha$ for $B_\delta^\alpha \in \mathcal{B}_X$. Likewise, each $V_\alpha = \cup_{\delta(\alpha)} C_\delta^\alpha$ for $C_\delta^\alpha \in \mathcal{B}_Y$. Let (x, y) be any point in X , that is in $\cup_\alpha (U_\alpha \times V_\alpha)$ which means that there exists α_0 so that $(x, y) \in U_{\alpha_0} \times V_{\alpha_0} = \cup_{\delta(\alpha_0)} B_\delta \times \cup_{\delta(\alpha_0)} C_\delta$. This means $x \in B_{\delta_0}$ and $y \in C_{\delta_1}$ for some indices. This means that $(x, y) \in B_{\delta_0} \times C_{\delta_1}$, which is part of the basis \mathcal{B} . Hence, Every open set in $\mathcal{T}_{X \times Y}$ is open in $\langle \mathcal{B} \rangle$ (or $\mathcal{T}_{X \times Y} \subseteq \langle \mathcal{B} \rangle$). Moreover, it is obvious to see that every element of \mathcal{B} is open in $\mathcal{T}_{X \times Y}$. Hence, arbitrary union of elements of \mathcal{B} , which are precisely the open sets of \mathcal{B} , are also open in $\mathcal{T}_{X \times Y}$. Hence, $\mathcal{T}_{X \times Y} = \langle \mathcal{B} \rangle$. \square

Example 1.35. Consider $X = \mathbb{R}, Y = \mathbb{R}$, with the standard topologies (i.e, that which is generated by the basis elements of the kind $\{(a, b); a < b\}$). The **standard topology** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is defined as the product topology. A basis for this topology is the set of all $U \times V$ for U and V open in \mathbb{R} , but from the preceeding result we can find an even smaller subset that makes a basis: $\{B \times C : B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ which is precisely $\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}; a < b, c < d\}$

Definition 1.36. (Projections) Define $\pi_1 : X \times Y \rightarrow X$ by $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \rightarrow Y$ given by $\pi_2(x, y) = y$. These maps are called projection maps.

Theorem 1.37. *The projection maps are onto, and have the property that, if U is an open subset of X , and V is an open subset of Y , then $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ (well, open hypothesis is not needed really)*

Proof. That they are onto functions is obvious. Let U be an open subset of X . Consider $\pi_1^{-1}(U) := \{(x, y) \in X \times Y : \pi_1(x, y) = x \in U\}$. This is precisely $U \times Y$. Similarly, $\pi_2^{-1}(V) = X \times V$. Note that, in the product topology, $U \times Y$ and $X \times V$ are both open sets. Moreover, the intersection of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ is $(U \times Y) \cap (X \times V) = U \times V$ \square

Theorem 1.38. *Let \mathcal{T}_X be a topology for X and \mathcal{T}_Y for Y . Let $\mathcal{T}_{X \times Y} := \langle \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\} \rangle$ be the product topology on $X \times Y$. Then the set $\Gamma := \{\pi_1^{-1}(U) : U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\}$ forms a subbasis for $\mathcal{T}_{X \times Y}$*

Proof. Γ is full of elements of the kind $U \times Y$ and $X \times V$ (every element is of this kind, and every element of this kind is in Γ). Consider any open set in $\mathcal{T}_{X \times Y}$, which is an arbitrary union of elements of the type $U \times V$, or explicitly, an open set looks like $\cup_\alpha (U_\alpha \times V_\alpha)$ which

is the same as $\cup_{\alpha}((U_{\alpha} \times Y) \cap (X \times V_{\alpha})) = \cup_{\alpha}\pi_1^{-1}(U_{\alpha}) \cap \pi_2^{-1}(V_{\alpha})$ which is an arbitrary union of an intersection of elements of Γ . Hence, we see that every set open in $\mathcal{T}_{X \times Y}$ is open in $\langle_{\text{subbasis}} \Gamma \rangle$, which means $\mathcal{T}_{X \times Y} \subseteq \langle_{\text{subbasis}} \Gamma \rangle$. Let C be a set in $\langle_{\text{subbasis}} \Gamma \rangle$, i.e, C is an arbitrary union of a finite intersection of sets in Γ . If the finite intersection only contains elements of the kind $U \times Y$ (or $X \times V$), then the intersection would ultimately be of the kind $U_0 \times Y$ for some U_0 open in \mathcal{T}_X (or $X \times V_0$ for some V_0 open in \mathcal{T}_Y), which would then mean $\cup_0 U_0 \times V$ would finally still yield a set of the form $U \times V$, falling again into $\mathcal{T}_{X \times Y}$ (or the other thing).

If suppose the finite intersection contains both elements of the type $U \times Y$ and $X \times V$, then intersecting them yields $U \times V$, and suppose some other set is of the kind $U' \times Y$ (or $X \times V'$), an intersection with them still yields a set of the type $U'' \times V''$ for some open $U'' \in \mathcal{T}_X$ and $V'' \in \mathcal{T}_Y$. This again shows that, arbitrarily unioning them will still make it fall into $\mathcal{T}_{X \times Y}$, which finally gives us $\langle_{\text{subbasis}} \Gamma \rangle = \mathcal{T}_{X \times Y}$ \square

1.4 Subspace Topology

Definition 1.39. (Subspace Topology) Let X be a topological space with a topology \mathcal{T}_X and let $Y \subset X$. We define the **subspace topology** on Y as $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$

Theorem 1.40. *The Subspace topology is indeed a topology on Y*

Proof. $\emptyset = Y \cap \emptyset$ and $Y = Y \cap X$. Hence \emptyset and X are both in \mathcal{T}_Y . Let U_{α} be a collection of sets in \mathcal{T}_Y , i.e $U_{\alpha} = G_{\alpha} \cap Y$ for $G_{\alpha} \in \mathcal{T}_X$. $\cup_{\alpha} U_{\alpha} = \cup_{\alpha} (G_{\alpha} \cap Y) = (\cup_{\alpha} G_{\alpha}) \cap Y \in \mathcal{T}_Y$. Closed under arbitrary union.

Let $U_1, U_2 \dots U_k$ be a bunch of sets of the kind $G_i \cap Y$. Arbitrary intersection of these yields $\cap_{i=1}^k U_i = \cap_{i=1}^k (G_i \cap Y) = (\cap_{i=1}^k G_i) \cap Y$ which is again of the form $K \cap Y$ for some $K \in \mathcal{T}_X$, which makes this finite intersection a part of \mathcal{T}_Y . Hence, \mathcal{T}_Y is a topology. \square

Lemma 1.41. *Let \mathcal{B}_X be a basis for \mathcal{T}_X on X . Then the basis of \mathcal{T}_Y , the subspace topology of $Y \subseteq X$ is $\mathcal{C} := \{B \cap Y : B \in \mathcal{B}_X\}$*

Proof. Let $y \in Y$, this means $y \in X$, which means $y \in B \in \mathcal{B}_X$. That means $y \in B \cap Y \in \mathcal{C}$. Let $y \in C_1$ and C_2 , which are in \mathcal{C} . That means $y \in (B_1 \cap Y) \cap (B_2 \cap Y) = (B_1 \cap B_2) \cap Y$ which means $y \in (B_3) \cap Y$ so that $B_3 \subseteq B_1 \cap B_2$, implying that $B_3 \cap Y \subseteq B_1 \cap Y \cap B_2 \cap Y$, giving us that if $y \in C_1$ and C_2 in \mathcal{C} , there exists $C_3 (= B_3 \cap Y)$ so that $y \in C_3 \subseteq C_1 \cap C_2$, hence \mathcal{C} forms a basis for \mathcal{T}_Y . Now we need to show that $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$ is the same as $\mathcal{T}_{\mathcal{C}} := \langle \mathcal{C} \rangle$. Let R be open in \mathcal{T}_Y implying it is of the form $U \cap Y$ for some $U \in \mathcal{T}_X$. But $U = \cup_{\alpha} B_{\alpha}$ for basis elements $B_{\alpha} \in \mathcal{B}$. That gives $R = U \cap Y = (\cup_{\alpha} B_{\alpha}) \cap Y = \cup_{\alpha} (B_{\alpha} \cap Y)$ which is an arbitrary union of elements of \mathcal{C} . Hence, $\langle \mathcal{C} \rangle \subseteq \mathcal{T}_Y$.

Let G be in $\langle \mathcal{C} \rangle$, i.e, is an arbitrary union of $C_{\alpha} = B_{\alpha} \cap Y \in \mathcal{C}$. Hence, $G = \cup_{\alpha} (B_{\alpha} \cap Y) = (\cup_{\alpha} B_{\alpha}) \cap Y$ making it part of \mathcal{T}_Y (its of the form "some open set" intersected with Y). This means $\mathcal{T}_Y \subseteq \langle \mathcal{C} \rangle$ which ultimately gives $\langle \mathcal{C} \rangle = \mathcal{T}_Y$ \square