CHAPTER 1

TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS

1 Topological Spaces

1.1 Basic notions, bases, subbases etc.

Definition 1.1. (Topology) A *topology* τ on a set X is a collection $\mathscr{B}$ of sets called *open sets* such that:

1. $\varnothing$ and X belong to $\mathscr{B}$
2. For any collection of sets $U_\alpha \in \mathscr{B}$, $\cup_\alpha U_\alpha$ is also in $\mathscr{B}$ (closed under arbitrary unions)
3. For any finite collection $\{U_1, U_2 \cdots U_k\}$ of sets of $\mathscr{B}$, $\cap_{i=1}^k U_i \in \mathscr{B}$ (closed under finite intersections)

Definition 1.2. (Discrete and Indiscrete topology)

1. The topology $(\varnothing, X)$ of a set X is called the *indiscrete topology*.
2. The topology in which every subset of X is an open set is called the *Discrete topology*

Definition 1.3. (Finer and Coarser topologies) Let τ and τ' be two topologies of space X. We say τ' is *finer* than τ if $\tau \subset \tau'$, or every open set in τ is one in τ'. In other words, τ is *coarser* than τ.

Remark 1.4. Of course, not all topologies are comparable, like the above definition suggests. For example, consider the 3 element space $\{x, y, z\}$. $\{\varnothing, (x), (x, y), (x, y, z)\}$ is a topology as can easily be seen. Likewise, $\{\varnothing, (x), (y, z), (x, y, z)\}$ is also a topology, but these two are not comparable, since their symmetric difference is non-empty.

Example 1.5. The finite complement topology: Let X be a set. Consider the set of all $U \in X$ such that U^C is either finite or X. Obviously, $\varnothing$ is in this collection, as is X. Let $\{U_\alpha : \alpha \in A\}$ by an arbitrary collection of such sets. Consider $(\cup_\alpha U_\alpha)^C = \cap_\alpha U_\alpha^C$ which is easily seen to be finite. Hence, closed under arbitrary unions. Consider $\{U_1, U_2 \cdots U_k\}$ of such sets. $(\cap_{i=1}^k U_i)^C = \cup_{i=1}^n U_i^C$ which is, at max, finite. Hence, closed under finite intersections. Therefore, this collection is a topology and is called the *Finite Complement Topology*.

Definition 1.6. (Basis for a topology) Let X be a space. C is called a *basis* for X if the following are satisfied:

1. For every $x \in X$, there exists a C_x in C so that $x \in C_x$
2. If x, C_1 and C_2 are such that $x \in C_1 \cap C_2 \neq \varnothing$, then there exists $C_{12} \in C$ such that $x \in C_{12} \subseteq C_1 \cap C_2$

Definition 1.7. (Topology *generated* by a basis $\mathcal{C}$ of X) The topology $\tau_{\mathcal{C}}$ generated by the basis $\mathcal{C}$ is defined as the collection $\mathcal{U}$ of subsets of X such that if $U \in \mathcal{U}$, and $x \in U$, then there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. It is obvious that $\mathcal{C}$ itself is contained in $\mathcal{U}$.

Theorem 1.8. *The topology $\tau_{\mathcal{C}}$ generated by $\mathcal{C}$, a basis, does indeed form a topology.*

Proof. Does X belong to $\tau_{\mathcal{C}}$? let $x \in X$. By virtue of being a basis, there exists a $C_x \subseteq X$ such that $x \in C_x$. Hence, $X \in \tau_{\mathcal{C}}$. Does $\varnothing \in \tau_{\mathcal{C}}$? Vacuously so. Let U_α be open in $\tau_{\mathcal{C}}$. Consider $\cup_\alpha U_\alpha$ and $x \in \cup_\alpha U_\alpha$. This means $x \in U_\delta$ for some δ. That means that there exists C_x so that $x \in C_x \subseteq U_\delta \subseteq \cup_\alpha U_\alpha$. This means $\tau_{\mathcal{C}}$ is closed under arbitrary union. Consider singleton collections, for the moment, of the kind $\{U_1\}$. It is obvious that the intersection of every element in this collection is part of $\tau_{\mathcal{C}}$. Consider a two sized collection $\{U_1, U_2\}$. Let $x \in U_1 \cap U_2$. Obviously, $x \in C_1 \subseteq U_1$ and $x \in C_2 \subseteq U_2$. From the definition of basis, there exists a basis element C so that $x \in C \subseteq C_1 \cap C_2 \subseteq U_1 \cap U_2$ making $\cap_{i=1}^{2} U_i$ part of the topology. Assume such is true for all n sized collections. Consider an $n+1$ sized collection $\{U_1, U_2 \cdots U_{n+1}\}$ and consider $x \in (\cap_{i=1}^{n} U_i) \cap U_{n+1}$. From the $n = 2$ case and induction hypothesis, we are done. Hence, closed under finite intersections. $\tau_{\mathcal{C}}$ is indeed a topology. $\square$

Theorem 1.9. *Let $\mathcal{C}$ be a basis for X. The topology generated by $\mathcal{C}$, that is, $\tau_{\mathcal{C}}$ is precisely the topology obtained by taking all possible, arbitrary union's of $\mathcal{C}$.*

Proof. Consider an arbitrary union of elements $C_\alpha \in \mathcal{C}$, $\cup_\alpha C_\alpha$. $x \in \cup C \implies \exists \alpha$ such that $x \in C_\alpha$. Therefore, $\cup_\alpha C_\alpha$ is in $\tau_{\mathcal{C}}$. Now, let $U \in \tau_{\mathcal{C}}$. For every $x \in U$, there exists C_x in $\mathcal{C}$ such that $x \in C_x \subseteq U$ (Definition of the generated topology). That means, $U = \cup_{x : x \in U} C_x$ which makes it the union of sets in the collection $\mathcal{C}$. We are done. $\square$

Example 1.10. If X is a metric space, the collection $\{B_\varepsilon(x) : x \in X, \varepsilon \in \mathbb{R}^+\}$ is a basis for X.

Remark 1.11. Note that, every topology τ is its own basis. It must also be noted that the same topology can be generated by two different bases. Consider for example the set of all open subsets of $\mathbb{R}$. Also consider the set of all open balls in $\mathbb{R}$. Or, consider the set of all $\frac{1}{n}$ balls around all rational points of $\mathbb{R}$

Theorem 1.12. *(Going from a topology to a basis) Suppose (X, τ) is a topological space, and $\mathcal{C}$ is a collection of sets in X such that for every $x \in X$ and for every $U \in \tau$, $\exists C_x \in \mathcal{C}$ such that $x \in C_x \subseteq U$. Then τ is actually generated by $\mathcal{C}$*

Proof. Let $x \in X$. That means, $\exists C_x \in \mathcal{C}$ (from hypothesis) such that $x \in C_x \subseteq X$. Moreover, let $x \in C_1$ and C_2. By hypothesis, all $C \in \mathcal{C}$ are conidered open sets, therefore, $C_1 \cap C_2$ is an open set, and $x \in C_1 \cap C_2$. That means $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. That makes $\mathcal{C}$ a basis. Its now obvious that $\tau = \tau_{\mathcal{C}}$ $\square$

Theorem 1.13. *(Criteria for derermining coarseness from basis) Let τ and τ' be two topologies of X. Also suppose τ is generated by $\mathcal{C}$ and τ', by $\mathcal{C}'$. Then the following are equivalent:*

1. *τ' is finer than τ*
2. *for each $x \in X$ and each basis element $C \in \mathcal{C}$ containing x, there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$*

Proof. $\implies$) Suppose τ' is finer than τ, that means every open set open in τ is also open in τ'. Let $C \in \mathcal{C}$, which is open in τ (and hence, in τ' too) and let $x \in C$. Since C is open in τ', and $x \in C$, there exists a basis element of τ', called C' such that $x \in C' \subset C$.

$\impliedby$) Let it be that for every $x \in X$ and for every $C \in \mathcal{C}$ such that $x \in C$, there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$. By the definition of topology generated by bases, it is clear that $\mathcal{C}$ is part of the topology of τ'. Consider $U \in \tau$.

U is an arbitrary union of elements of $\mathcal{C}$ (by the alternate characterisation of generated topology), making U an element of τ'. We are done.

(aliter) Let U be such that for every $x \in U$, $\exists C \in C$ such that $x \in C \subseteq U$. Since we are told that there exists $C' \subseteq C$ so that $x \in C'$, we have that for all $x \in U$, $\exists C'_x \in \mathcal{C}'$ so that $x \in C'_x \subseteq U$, making U part of the topology of τ'. □

Definition 1.14. (Standard topology of $\mathbb{R}$) Let $\mathcal{B}$ be the collection of all open intervals of the kind $(a, b) : a, b \in \mathbb{R}$. That this is a basis is obvious. Define the standard topology as that topology generated by this basis.

Definition 1.15. (lower limit topology $\mathbb{R}_l$) Conider $\mathcal{B}'$ the collection of all sets of the kind $[a, b) : a, b \in \mathbb{R}$. Again, that this is a basis is obvious. Define the lower limit topology as the topology generated by B', and denote it as $\mathbb{R}_l$

Definition 1.16. (K topology $\mathbb{R}_K$) Let K be the set of all numbers of the kind $1/n$ for $n \in \mathbb{Z}_+$. Denote $\mathcal{B}''$ as the collection of all open intervals of the kind (a, b) along with the sets of the form $(a, b) - K$. Again, that this is a basis is obvious. The topology generated by $\mathbb{B}''$ is called the K topology, denoted by $\mathbb{R}_K$

Theorem 1.17. *$\mathbb{R}_K$ and $\mathbb{R}_l$ are strictly finer than $\mathbb{R}$, but $\mathbb{R}_K$ and $\mathbb{R}_l$ are not comparable.*

Proof. Consider $B \in \mathcal{B}$, of the form (a, b). This is also in $\mathcal{B}'$, and a smaller set of the kind $[c, d)$ lies inside (a, b), which means that, if $x \in (a, b)$, there exists $B' \in \mathcal{B}'$ and $B'' \in \mathcal{B}''$ so that $x \in B' \subset (a, b)$ and $x \in \mathcal{B}'' \subset (a, b)$. Moreover, since both $\mathbb{R}_l$ and $\mathbb{R}_K$ have elements that are not in $\mathbb{R}$, it is obviously strictly finer. (Consider $a \in [a, b)$, there is no element of the kind (x, y) that is inside this, that contains a.)

Consider an element $B = [0, 1)$ where we choose $x = 0$. Clear that no element from $\mathbb{R}_K$ can come inside B whilst containing 0. On the other hand, consider 0 inside $(-1, 1) - K$ in $\mathbb{R}_K$. No interval of the kind $[a, b)$ can contain 0 and itself be contained inside $(-1, 1) - K$. Hence, it is clear that $\mathbb{R}_K$ and $\mathbb{R}_l$ are incomparable. □

Definition 1.18. (Subbasis) A subbasis S is a collection of subsets of X whose union amounts to X

Definition 1.19. (Topology generated by a subbasis) τ_S, the topology generated by the subbasis S is defined as the set of all arbitrary unions of the set of all finite intersection of elements of S.

Theorem 1.20. *τ_S, the topology generated by a subbasis S, is indeed a topology.*

Proof. Let S be the set whose union forms X (Subbasis).

Let $Y = \{\gamma : \gamma = \cap_{i=1}^{n} S_i$ for some n and $S_i \in S\}$, i.e, Y is the set of all finite intersections of elements of S. Consider the set of all arbitrary union of elements of Y, and call it τ_S. Obviously τ_S cotains $\varnothing$ and X. Let $\{U_\alpha\}$ be a collection of open sets in τ_S, i.e, each U_α is an arbitrary union of elements that are finite intersection of elements of S. The union of all U_α, therefore, would still be an arbitrary union of finite intersections of elements of S, hence, is back in τ_S. Now consider $\{U_1, U_2 \cdots U_k\}$, elements of τ_S such that each U_i is an arbitrary union of a collection of sets that are each a finite intersection of elements of S. Consider the finite intersection $\cap_{i=1}^{n} U_i = \cap_{i=1}^{n}(\cup_\alpha(\cap_{j=1}^{n_\alpha} S_{j,\alpha}))$ which is ultimately an arbitrary union of a finite intersection of sets in S, making this part of τ_S. Hence, τ_S is indeed a topology. $\square$

Theorem 1.21. *Let $\{\tau_\alpha : \alpha \in A\}$ be a collection of topologies of X, then there is a unique "smallest" topology τ_S such that τ_S contains all the topologies $\{\tau_\alpha\}$ (kinda like a LUB), and there is a unique "largest: topology τ_L that is contained in all these topologies $\{\tau_\alpha\}$.*

Proof. **τ_S that contains all topologies $\{\tau_\alpha\}$:**

Uniqueness: If two such topologies exists, we can take the intersection of them to result in a smaller topology (intersection is still a topology) that contains $\{\tau_\alpha\}$.

Existence: Consider the topology generated by arbitrarily unioning the sets generated by finitely intersecting all elements of $\{\tau_\alpha : \alpha \in A\}$. Explicitly, we collect $\mathcal{R} = \{U : \exists x$ such that $U \in \tau_x\}$ and we take all possible finite intersections of elements in $\mathcal{R}$ and collect that in a set $\mathcal{S}$. Then, we take all possible arbitrary unions of elements of $\mathcal{S}$ and call it τ_S.

For later use:

- $\mathcal{R}$ = all the open sets in all the topologies.
- $\mathcal{S}$ = finite intersection of all elements of $\mathcal{R}$
- τ_S = arbitrary union of all elements of $\mathcal{S}$

Is τ_S a topology? X and $\varnothing$ are obviously there, since X and $\varnothing$ are there in $\mathcal{R}$, and we take the singleton intersection of these in $\mathcal{S}$, and finally, we take the singleton union of these to arrive at $X, \varnothing \in \tau_S$. Let $\{G_\delta : \delta \in \Delta\}$ be a collection of sets in τ_S (i.e, a collection of sets that are an arbitrary union of a collection of finite intersection of elements of $\mathcal{R}$). $\cup_\delta G_\delta$ would also be an arbitrary union of elements of $\mathcal{S}$ making it part of τ_S. Let $\{G_1, G_2, \cdots G_k\}$ be in τ_S. $\cap_{i=1}^{k} G_i = \cap_{i=1}^{k}(\cup_{\delta_i}(\cap_{j=1}^{n_{\delta_i}} R_{\delta_{i,j}}))$ which, from De Morgan's Laws, can easily be seen to be an arbitrary union of a finite intersection of elements from $\mathcal{R}$. Hence, the intersection is also in τ_S making it a topology that contains all of $\{\tau_\alpha\}$.

τ_L that is contained in all topologies $\{\tau_\alpha\}$

Existence: Consider $\cap_\alpha \tau_\alpha$, which is a topology that is contained in all of τ_α.

Uniqueness: If two topologies are such that they are contained in every τ_α, then they are indeed contained in $\cap_\alpha \tau_\alpha$ □

Theorem 1.22. *Let A be a basis for τ, then τ_A, the topology generated by A (The one generated by taking arbitrary union of elements of A) is precisely the intersection of all topologies $\{\tau_\alpha : \alpha \in G\}$ that contain A*

Proof. τ_A is the topology generated by arbitrary union of elements of A. We are told $A \subseteq \tau_\alpha$ for all $\alpha \in G$. Let $U \in \tau_A$. U is an arbitrary union of elements of A, which, since τ_α is a topology, is also contained in τ_α. Every open set in τ_A is also in τ_α. Hence $\tau_A \subseteq \cap_\alpha \tau_\alpha$.

Note that, τ_A is iself a topology containing A, which means $\tau_A = \cap_\alpha \tau_\alpha$ □

Theorem 1.23. *Let A be a subbasis for X. The topology τ_A generated by A, i,e, by taking arbitrary union of elements that are finite intersection of A, is precisely the intersection of all topologies $\{\tau_\alpha\}$ that contain A.*

Proof. Obviousy τ_A contains A, since each element of A is a singleton union of the signleton intersection of itself. Moreover, let U be any open set in τ_A, i.e, U is an arbitrary union of a collection of finite intersections of elements of A. Let τ_α be a topology that contains A. That means a finite intersection of elements of A is still in τ_α, and an arbitrary union of these are still in τ_α. Hence, U is in τ_α, for any α. Hence, $\tau_A \subseteq \tau_\alpha$ for every α, implying $\tau_A \subseteq \cap_\alpha \tau_\alpha$. Since τ_A is part of $\{\tau_\alpha\}$, we see

$$\tau_A = \cap_\alpha \tau_\alpha$$

where $\{\tau_\alpha\}$ is the collection of all topologies that contain A. □

1.2 Order Topology

Definition 1.24. (Simple Order) Let X be a set. Define $C : X \times X \to \{y, n\}$ such that:

1. If $x \neq y$, either xCy or yCx
2. For no x is it true that xCx
3. if xCy and yCz, then xCz

This type of relation is called *simple order* relation.

Pro Tip: For intuition purposes, replace C with $<$.

Definition 1.25. (Order Topology on a space X.) Let X be a space with more than two points, equipped with a simple order relation $<$. Consider the collection $\mathcal{B}$ of sets of the following kind:

1. (a, b) where $a < b$. i.e, all open intervals in X

2. $[a_0, b)$ for all $b \in X$, where a_0 is the smallest element of X

3. $(a, b_0]$ for all $a \in X$ where b_0 is the largest element in X

Then the order topology τ_O is defined as that topology generated by $\mathcal{B}$.

Remark 1.26. Note that, if X has no smallest element, type (2) sets won't exist in $\mathcal{B}$. Likewise, if X has no largest element, type (3) sets won't exist in $\mathcal{B}$.

Theorem 1.27. *$\mathcal{B}$ is actually a basis (hence, τ_O is actually a well defined topology on X)*

Proof. Let $x \in X$. If x is not the largest element, nor the smallest, then there exists (a_0, b_0) that contains x

If x is the largest element, then $(a_0, x]$ contains x. If x is the smallest element, then $[x, b_0)$ contains x. Hence, for every $x \in X$, there exists an element B from $\mathcal{B}$ such that $x \in B$.

Note the following, if we intersect two type 1 sets, we get back a smaller type 1 set. i.e, $(a, b) \cap (a', b') := \{x \in X : a < x < b \text{ and } a' < x < b'\}$.
If we intersect two type 2 sets, we get back a type 2 set (obviously)
If we intersect two type 3 sets, we get back a type 3 set (obviously)
If we intersect a type 1 and type 2 set, i.e (a, b) and $[a_0, x)$, we are basically saying $\{t \in X : a < t < b \text{ and } a_0 \leq t < x\}$ We get back a type 1 set.
If we intersect a type 1 and type 3 set, i.e (a, b) and $(x, b_0]$, we get $\{t \in X : a < t < b \text{ and } x < t \leq b_0\}$ which would yield a set of type 1.
If we intersect a type 2 and type 3 set, i.e, $[a_0, x)$ and $(y, b_0]$, we want basically $\{t \in X : a_0 \leq t < x \text{ and } y < t \leq b_0\}$. It is easy to see that this intersection falls into one of the 3 categories.

Hence, $\mathcal{B}$ forms a basis, and hence, τ_B is a topology. □

Example 1.28. The standard topology $\mathbb{R}$ is basically the order topology on $\mathbb{R}$. All one needs to see is that there is no least element, and largest element of $\mathbb{R}$ (Of course, $\mathbb{R} \cup \{-\infty, +\infty\}$, the extended real line, *does* have these). Hence, $\mathcal{B}$ is just all elements of the kind (a, b), which is precisely the basis for the standard topology.

Definition 1.29. (Rays) Given $a \in X$ (a simply ordered set), there are 4 sets associated with a called *rays*:

1. $(a, +\infty) := \{x : a < x\}$

2. $(-\infty, a) := \{x : x < a\}$

3. $[a, +\infty) := \{x : a \geq x\}$

4. $(-\infty, a] := \{x : x \leq x\}$

Theorem 1.30. *The first two rays are open in the order topology, (and the 2nd two rays are closed in the order topology, i.e, its complement is open)*

Proof. Consider $(a, +\infty)$. If there is no maximum element, then this is precisely an arbitrary union of sets of the kind (a, b_α). If there is a maximal element, then this is $(a, b_0]$. In a similar fashion, the 2nd set is also open.

Consider $[a, +\infty)^C = (-\infty, a)$, and consider $(-\infty, a]^C = (a, \infty)$. Obvious from here. □

Theorem 1.31. *The collection of all open* rays *forms a subbasis for the order topology on* X

Proof. We have to show that the topology generated by arbitrarily unioning a collection of finite intersections of rays gives rise to the topology generated by the bais of order topology of X. Let τ_R be the topology generated by the arbitrary union of finite intersection of rays. Let τ be the order topology on X, generated by arbitrary union of the basis elements of X. We want to show $\tau_R \subseteq \tau$ and $\tau \subseteq \tau_R$.

Consider (a, b), a basis element of τ. $(a, +\infty) \cap (-\infty, b)$ is a finite intersection of rays, and yields the basis (a, b). $[a_0, b)$ is, simply, $(-\infty, b)$ and $(a, b_0]$ is simply $(a, +\infty)$. Hence, finite intersection of the rays yield the basis elements. Arbitrary union of finite intersection of rays can yield all the possible open sets of the order topology on X. Hence, $\tau \subseteq \tau_R$. Let S be open in τ_R. It is the arbitrary union of a finite intersection of rays. Note that every ray is either an element of the basis for τ or is an arbitrary union elements of basis of τ. An arbitrary union of a finite intersection of an arbitrary union of elements of the basis yields an arbitrary union of a finite intersection of elements of the basis, giving rise to yet another element of the topology τ. Hence, every open et in τ_R is open in τ, hence $\tau_R \subseteq \tau$ □

1.3 The product topology

Definition 1.32. (Product topology on $X \times Y$) Let τ_X and τ_Y be topologies of X and Y respectively. Let $\mathcal{B}$ be the collection of subsets of $X \times Y$ of the kind $U \times V$ for $U \in \tau_X$ and $V \in \tau_Y$. We define the product topology as that which is generated by the basis $\mathcal{B}$.

Theorem 1.33. $\mathcal{B}$ *of the previous definition is indeed a basis.*

Proof. Let $(x, y) \in X \times Y$. Since $X \in \tau_X$ and $Y \in \tau_Y$, we have that $(x, y) \in X \times Y$ so the 1st condition for basis is satisfied. Let $(x, y) \in U_1 \times V_1$ and $U_2 \times V_2$. Consider $(U_1 \times V_1) \cap (U_2 \times V_2)$. This is same as $U_1 \cap U_2 \times V_1 \cap V_2$ which is again an element of $\mathcal{B}$. Hence, it is a basis and hence $\tau_{\mathcal{B}}$ is well defined. □

Theorem 1.34. *If X's topology $\mathscr{T}_X$ is generated by a basis $\mathcal{B}_X$ and Y's topology $\mathscr{T}_Y$ is generated by a basis $\mathcal{B}_Y$, then the product topology of $X \times Y$*

(given by$\langle\{U \times V : U$ open in$\mathscr{T}_X$ and V open in $\mathscr{T}_Y\}\rangle$) has a basis $\mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\}$, i.e,

$$\mathscr{T}_{X\times Y} = \langle\{U \times V : U \in \mathscr{T}_X, V \in \mathscr{T}_Y\}\rangle = \langle_{basis}\mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\}\rangle$$

Proof. Checking that $\mathcal{B}$ is indeed a basis:

Let $(x,y) \in X \times Y$. $x \in X$ means that there exists $B \in \mathcal{B}_X$ so that $x \in B \subseteq X$. Likewise, for y, there is a $C \in \mathcal{B}_Y$ so that $y \in C \subseteq Y$, which means $(x,y) \in B \times C \subseteq X \times Y$. This means that $\mathcal{B}$ obeys the 1st condition for a basis.

Let $(x,y) \in B, B' \in \mathcal{B}$. i.e, $(x,y) \in C_1 \times D_1$, and $C_2 \times D_2$ for $C_1, C_2 \in \mathcal{B}_X$ and $D_1, D_2 \in \mathcal{B}_Y$. That means that $(x,y) \in (C_1 \times D_1) \cap (C_2 \times D_2) = (C_1 \times C_2) \times (D_1 \times D_2)$. $x \in C_1 \cap C_2$ means there is a C_3 so that $x \in C_3 \subseteq C_1 \cap C_2$. Likewise, $y \in D_3 \subseteq D_1 \cap D_2$. Hence, $(x,y) \in (C_3 \times D_3) \subseteq (C_1 \cap C_2) \times (D_1 \times D_2) = (C_1 \times D_1) \cap (C_2 \times D_2)$. Hence, $\mathcal{B}$ is a basis.

Now consider any open set in $\mathscr{T}_{X\times Y}$. Since $\mathscr{T}_{X\times Y}$ is generated by $U \times V : U \in \mathscr{T}_X, V \in \mathscr{T}_Y$, Every open set in $\mathscr{T}_{X\times Y}$ is an arbitrary union of sets of the kind $U \times V$, i.e, $\cup_\alpha (U_\alpha \times V_\alpha)$. But each $U_\alpha = \cup_{\delta(\alpha)} B^\alpha_\delta$ for $B^\alpha_\delta \in \mathcal{B}_X$. Likewise, each $V_\alpha = \cup_{\delta(\alpha)} C^\alpha_\delta$ for $C^\alpha_\delta \in \mathcal{B}_Y$. Let (x,y) be any point in X, that is in $\cup_\alpha (U_\alpha \times V_\alpha)$ which means that there exists α_0 so that $(x,y) \in U_{\alpha_0} \times V_{\alpha_0} = \cup_{\delta(\alpha_0)} B_\delta \times \cup_{\delta(\alpha_0)} C_\delta$. This means $x \in B_{\delta_0}$ and $y \in C_{\delta_1}$ for some indices. This means that $(x,y) \in B_{\delta_0} \times C_{\delta_1}$, which is part of the basis $\mathcal{B}$. Hence, Every open set in $\mathscr{T}_{X\times Y}$ is open in $\langle \mathcal{B} \rangle$ (or $\mathscr{T}_{X\times Y} \subseteq \langle \mathcal{B} \rangle$). Moreover, it is obvious to see that every element of $\mathcal{B}$ is open in $\mathscr{T}_{X\times Y}$. Hence, arbitrary union of elements of $\mathcal{B}$, which are precisely the open sets of $\mathcal{B}$, are also open in $\mathscr{T}_{X\times Y}$. Hence, $\mathscr{T}_{X\times Y} = \langle \mathcal{B} \rangle$. □

Example 1.35. Consider $X = \mathbb{R}, Y = \mathbb{R}$, with the standard topologies (i.e, that which is generated by the basis elements of the kind $\{(a,b); a < b\}$). The ***standard topology*** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is defined as the product topology. A basis for this topology is the set of all $U \times V$ for U and V open in $\mathbb{R}$, but from the preceeding result we can find an even smaller subset that makes a basis: $\{B \times C : B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ which is precisely $\{(a,b) \times (c,d) : a,b,c,d \in \mathbb{R}; a < b, c < d\}$

Definition 1.36. (Projections) Define $\pi_1 : X \times Y \to X$ by $\pi_1(x,y) = x$ and $\pi_2 : X \times Y \to Y$ given by $\pi_2(x,y) = y$. These maps are called projection maps.

Theorem 1.37. *The projection maps are onto, and have the property that, if U is an open subset of X, and V is an open subset of Y, then $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ (well, open hypothesis is not needed really)*

Proof. That they are onto functions is obvious. Let U be an open subset of X. Consider $\pi_1^{-1}(U) := \{(x,y) \in X \times Y : \pi_1(x,y) = x \in U\}$. This is precisely $U \times Y$. Similarly, $\pi_2^{-1}(V) = X \times V$. Note that, in the product topology, $U \times Y$ and $X \times V$ are both open sets. Moreover, the intersection of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ is $(U \times Y) \cap (X \times V) = U \times V$ □

Theorem 1.38. *Let $\mathscr{T}_X$ be a topology for X and $\mathscr{T}_Y$ for Y. Let $\mathscr{T}_{X\times Y} := \langle \{U \times V : U \in \mathscr{T}_X, V \in \mathscr{T}_Y\} \rangle$ be the product topology on $X \times Y$. Then the set $\Gamma := \{\pi_1^{-1}(U) : U \in \mathscr{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathscr{T}_Y\}$ forms a subbasis for $\mathscr{T}_{X\times Y}$*

Proof. Γ is full of elements of the kind $U \times Y$ and $X \times V$ (every element is of this kind, and every element of this kind is in Γ). Consider any open set in $\mathscr{T}_{X\times Y}$, which is an arbitrary union of elements of the type $U \times V$, or explicitly, an open set looks like $\cup_\alpha (U_\alpha \times V_\alpha)$ which is the same as $\cup_\alpha ((U_\alpha \times Y) \cap (X \times V_\alpha)) = \cup_\alpha \pi_1^{-1}(U_\alpha) \cap \pi_2^{-1}(V_\alpha)$ which is an arbitrary union of an intersection of elements of Γ. Hence, we see that every set open in $\mathscr{T}_{X\times Y}$ is open in $\langle_{subbasis}\Gamma\rangle$, which means $\mathscr{T}_{X\times Y} \subseteq \langle_{subbasis}\Gamma\rangle$. Let C be a set in $\langle_{subbasis}\Gamma\rangle$, i.e, C is an arbitrary union of a finite intersection of sets in Γ □