Topology Assignment 1

Problem 1: Show that a subset is closed if and only if it contains all of its limit points

Solution: Define x to be a limit point of S if for all neighbourhoods $nbd(x)$, we have $(nbd(x) - \{x\}) \cap S \neq \varnothing$

Define the closure of S, as the smallest closed set that contains S. Denote it $cls_X(S)$

Define

$$L := \{x \in X : \forall nbd(x), nbd(x) \cap S \neq \varnothing\}$$

Lemma: L and $cls_X(S)$ are the same, i.e,

$$\cap_{\alpha : S \subseteq F_\alpha} F_\alpha = \{x \in X : \forall nbd(x), nbd(x) \cap S \neq \varnothing\}$$

Proof. Let x not be in L, i.e, $\exists nbd(x)$ such that $nbd(x) \cap S = \varnothing$ or $nbd(x) \subseteq S^C$ or $S \subseteq nbd(x)^C$. This implies that $cls_X(S) \subseteq nbd(x)^C$ which means that x is not in $cls_X(S)$.

Let x on the other hand, not be in $cls_X(S)$, which means $x \in (cls_X(S))^c$ which means that there exists a neighbourhood of x that does not intersect S, making it not in L. We are done. □

Lemma: $cls_X(S) = S \cup S'$ where S' is the set of all limit points of S.

Proof. If x is a point of $cls_X(S)$, and is in S, then it trivially is in S. say $x \in cls_X(S)$ but isn't a point of S, then for every neighbourhood $nbd(x)$, we have that $nbd(x) - \{x\} \cap S \neq \varnothing$, making x a limit point of S. If x is a limit point of S, of course, it is obviously in $cls_X(S)$, and if it is in S, it obviously is in $cls_X(S)$. Hence, $cls_X(S) = S \cup S'$. □

Lemma: A set is closed if and only if $A = cls_X(A)$

Proof. $\implies$) Let $A = cls_X(A)$. This means obviously that A is closed.

$\impliedby$) Suppose A is closed. Obviously, $A \subseteq \cup_{\alpha : A \subseteq F_\alpha F_\alpha}$, and A itself is a closed set that contains itself, hence $cls_X(A) = A$. □

If A is closed, $A = A \cup A'$ which means that $A' \subseteq A$. If $A' \subseteq A$, then $A \cup A' \subseteq A \cup A = A$.

Problem 2: (a) Constant map is Continuous, (b) Composition of two continuous maps is continuous

Solution: (a) Consider $f : X \to Y$ be $f(x) = c$. Let C be any open set in Y. If it contains c, then $f^{-1}(C) = X$ which is open. If $c \notin C$, then $f^{-1}(C) = \varnothing$ which is also open.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be two continuous functions. Consider $g \circ f : X \to Z$.

Let $G \in \mathscr{T}_Z$ be any open set in Z. $(g \circ f)^{-1}(G) = \{x \in X : g \circ f(x) \in G\} = f^{-1}(g^{-1}(G))$ which is ultimately open in $\mathscr{T}_X$. Hence it is continuous.

Problem 3: Let $\mathscr{T}_D$ be the discrete topology on $\mathbb{R}$ and $\mathscr{T}_E$ be the euclidean topology on $\mathbb{R}$. Does there exist an homeomorphism from $\mathscr{T}_D$ to $\mathscr{T}_E$?

Solution: Let $f : \mathbb{R}, \mathscr{T}_D \to \mathbb{R}, \mathscr{T}_E$ be a homeomorphism. Let $\{U_\alpha\}$ be a basis for $\mathscr{T}_E$, we conjecture that $\{V_\alpha = f^{-1}(U_\alpha)\}$ is a basis for $\mathscr{T}_D$. Let $x \in V_1 \cap V_2$. This is an open set in $\mathscr{T}_D$. $f(V_1 \cap V_2)$ is also an open set in $\mathscr{T}_E$ that contains $f(x)$ which means that there exists U_α such that $f(x) \in U_\alpha \subset f(U_1 \cup U_2)$. This implies $x \in f^{-1}(U_\alpha) \subseteq f^{-1}(U_1 \cup U_2)$. Moreover, let U be open in $\mathscr{T}_D$. $U = f^{-1}(f(U))$. Moreover, since f is continuous and bijective, with a continuous inverse, we have that $f(U)$ is also open in $\mathscr{T}_E$, or rather it is an arbitrary union of basis elements $\cup_\alpha U_\alpha$. This implies $f^{-1}(f(U))$ is an arbitrary union of elements of $\{f^{-1}(U_\alpha)\}$. Hence $\{V_\alpha = f^{-1}(U_\alpha)\}$ is a basis for $\mathscr{T}_D$ (True for any homeomorphism).

We know that $\mathscr{T}_E$ has countable basis, i,e, is 2nd countable. Therefore, the collection $\{f^{-1}(U_i)\}$ forms a basis for $\mathscr{T}_D$. But note that (fact from analysis) any infinite discrete metric space is not 2nd countable (since it is not separable / lindelof). To see this, consider all the singleton elements. This is an open cover of $\mathbb{R}$, but contains no countable subcover. Hence, such a homeomorphism cannot exist.

Problem 4: Cofinite topology and countable complement topology:

Let X be a set. Let U be open if and only if U^C is finite, $\varnothing$ or X. Of course $\varnothing$ and X are in the topology. Consider U_α be a collection of open sets. $\cup_\alpha U_\alpha$. its complement is $\cap_\alpha U_\alpha^C$. If their complements are all X, then the intersection is X. If their complements are all empty, then the intersection is empty. If their complements are all finite, then the intersection is also finite, topology is closed under arbitrary unions. Consider $U_1 \cap U_2 \cap \cdots U_k = \cap_{i=1}^k (U_i)$. Its complement is $\cup_{i=1}^k (U_i^c)$. If each complement is finite, its finite union is still finite. If even one of their complements is X, then the finite union is also X. Hence, closed under finite intersections. Hence it defines a topology.

If we replace finiteness with countable: X and $\varnothing$ are still in $\mathscr{T}$. Suppose that U_α are such that each of their complements are X or $\varnothing$ or countable. $(\cup_\alpha U_\alpha)^c = \cap_\alpha U_\alpha^c$ which is ultimately either $X, \varnothing$ or countable (atmost countable). Let $U_1, U_2 \cdots U_l$ be a finite collection whose complements are atmost countable sets. Their intersection's complement is $\cup_{i=1}^l U_l^c$ which is a finite union of atmost countable sets, which is atmost countable. If even one of the complements is X, the complement of the intersection would be X as well. Hence, if we replace finite with atmost countable, we still have a topology.

Problem 5: Let $X = \mathbb{Z}$ and $S_{a,b} := a + b\mathbb{Z} = \{a + bz : z \in \mathbb{Z}\}$

(a) If $x \in \mathbb{Z}$, then $x \in S_{0,x}$ obviously. Let $x \in S_{a,b}$ and $S_{c,d}$. $x = a + qb = c + q'd$. $S_{a,b}$ and $S_{c,d}$ are said to meet at x. The arithmetic progression jumps every b elements for $S_{a,b}$, and jumps every d elements for $S_{c,d}$. Hence, $S_{x,lcm(b,d)}$ contains x and is contained in both $S_{a,b}$ and $S_{c,d}$. Hence, the two conditions for bases are verified.

(b) The topology generated by $\mathcal{B}$ is all possible unions of elements of $\mathcal{B}$. Consider $S_{a,b}^C$. This can be written as the union: $\cup_{j=1}^{b-1} S_{a+j,b}$. Hence, every $S_{a,b}$ is clopen.

(c) Consider $\{-1,1\}^c$. Let there be only finite primes, i,e, $p_1, p_2 \cdots p_q$. Note that, every element in $\mathbb{Z}$ that is not -1 or 1, is a multiple of one of $p_1, p_2 \cdots p_q$. Therefore, every non unit (-1,1) element can be covered by S_{0,p_i}. Hence, $\{-1,1\}^C = \cup_{i=1}^{q}(S_{0,p_i})$ Which implies $\{-1,1\} = \cap_{i=1}^{q}(S_{0,p_i}^C)$. Since S_{0,p_i} are open, we conclude that $\{-1,1\}$ is open. But every open set in this topology must necessarily be infinite or empty, since a union of elements of $S_{a,b}$ ought to be infinite. Hence, we conclude that there must be infinite primes.