
CHAPTER 1

TOPOLOGICAL SPACES AND CONTINUOUS FUNCTIONS

1 Topological Spaces

1.1 Basic notions, bases, subbases etc.

Definition 1.1. (Topology) A topology τ on a set X is a collection \mathcal{B} of sets called *open sets* such that:

1. \emptyset and X belong to \mathcal{B}
2. For any collection of sets $U_\alpha \in \mathcal{B}$, $\cup_\alpha U_\alpha$ is also in \mathcal{B} (closed under arbitrary unions)
3. For any finite collection $\{U_1, U_2 \dots U_k\}$ of sets of \mathcal{B} , $\cap_{i=1}^k U_i \in \mathcal{B}$ (closed under finite intersections)

Definition 1.2. (Discrete and Indiscrete topology)

1. The topology (\emptyset, X) of a set X is called the *indiscrete topology*.
2. The topology in which every subset of X is an open set is called the *Discrete topology*

Definition 1.3. (Finer and Coarser topologies) Let τ and τ' be two topologies of space X . We say τ' is *finer* than τ if $\tau \subset \tau'$, or every open set in τ is one in τ' . In other words, τ is *coarser* than τ .

Remark 1.4. Of course, not all topologies are comparable, like the above definition suggests. For example, consider the 3 element space $\{x, y, z\}$. $\{\emptyset, (x), (x, y), (x, y, z)\}$ is a topology as can easily be seen. Likewise, $\{\emptyset, (x), (y, z), (x, y, z)\}$ is also a topology, but these two are not comparable, since their symmetric difference is non-empty.

Example 1.5. The finite complement topology: Let X be a set. Consider the set of all $U \in X$ such that U^C is either finite or X . Obviously, \emptyset is in this collection, as is X . Let $\{U_\alpha : \alpha \in A\}$ be an arbitrary collection of such sets. Consider $(\cup_\alpha U_\alpha)^C = \cap_\alpha U_\alpha^C$ which is easily seen to be finite. Hence, closed under arbitrary unions. Consider $\{U_1, U_2 \dots U_k\}$ of such sets. $(\cap_{i=1}^k U_i)^C = \cup_{i=1}^n U_i^C$ which is, at max, finite. Hence, closed under finite intersections. Therefore, this collection is a topology and is called the *Finite Complement Topology*.

Definition 1.6. (Basis for a topology) Let X be a space. \mathcal{C} is called a *basis* for X if the following are satisfied:

1. For every $x \in X$, there exists a C_x in \mathcal{C} so that $x \in C_x$
2. If x, C_1 and C_2 are such that $x \in C_1 \cap C_2 \neq \emptyset$, then there exists $C_{12} \in \mathcal{C}$ such that $x \in C_{12} \subseteq C_1 \cap C_2$

Definition 1.7. (Topology generated by a basis \mathcal{C} of X) The topology $\tau_{\mathcal{C}}$ generated by the basis \mathcal{C} is defined as the collection \mathcal{U} of subsets of X such that if $U \in \mathcal{U}$, and $x \in U$, then there exists $C \in \mathcal{C}$ such that $x \in C \subseteq U$. It is obvious that \mathcal{C} itself is contained in \mathcal{U} .

Theorem 1.8. *The topology $\tau_{\mathcal{C}}$ generated by \mathcal{C} , a basis, does indeed form a topology.*

Proof. Does X belong to $\tau_{\mathcal{C}}$? let $x \in X$. By virtue of being a basis, there exists a $C_x \subseteq X$ such that $x \in C_x$. Hence, $X \in \tau_{\mathcal{C}}$. Does $\emptyset \in \tau_{\mathcal{C}}$? Vacuously so. Let U_{α} be open in $\tau_{\mathcal{C}}$. Consider $\cup_{\alpha} U_{\alpha}$ and $x \in \cup_{\alpha} U_{\alpha}$. This means $x \in U_{\delta}$ for some δ . That means that there exists C_x so that $x \in C_x \subseteq U_{\delta} \subseteq \cup_{\alpha} U_{\alpha}$. This means $\tau_{\mathcal{C}}$ is closed under arbitrary union. Consider singleton collections, for the moment, of the kind $\{U_1\}$. It is obvious that the intersection of every element in this collection is part of $\tau_{\mathcal{C}}$. Consider a two sized collection $\{U_1, U_2\}$. Let $x \in U_1 \cap U_2$. Obviously, $x \in C_1 \subseteq U_1$ and $x \in C_2 \subseteq U_2$. From the definition of basis, there exists a basis element C so that $x \in C \subseteq C_1 \cap C_2 \subseteq U_1 \cap U_2$ making $\cap_{i=1}^2 U_i$ part of the topology. Assume such is true for all n sized collections. Consider an $n+1$ sized collection $\{U_1, U_2 \cdots U_{n+1}\}$ and consider $x \in (\cap_{i=1}^n U_i) \cap U_{n+1}$. From the $n=2$ case and induction hypothesis, we are done. Hence, closed under finite intersections. $\tau_{\mathcal{C}}$ is indeed a topology.

□

Theorem 1.9. *Let \mathcal{C} be a basis for X . The topology generated by \mathcal{C} , that is, $\tau_{\mathcal{C}}$ is precisely the topology obtained by taking all possible, arbitrary union's of \mathcal{C} .*

Proof. Consider an arbitrary union of elements $C_{\alpha} \in \mathcal{C}$, $\cup_{\alpha} C_{\alpha}$. $x \in \cup C \implies \exists \alpha$ such that $x \in C_{\alpha}$. Therefore, $\cup_{\alpha} C_{\alpha}$ is in $\tau_{\mathcal{C}}$. Now, let $U \in \tau_{\mathcal{C}}$. For every $x \in U$, there exists C_x in \mathcal{C} such that $x \in C_x \subseteq U$ (Definition of the generated topology). That means, $U = \cup_{x: x \in U} C_x$ which makes it the union of sets in the collection \mathcal{C} . We are done.

□

Example 1.10. If X is a metric space, the collection $\{B_{\varepsilon}(x) : x \in X, \varepsilon \in \mathbb{R}^+\}$ is a basis for X .

Remark 1.11. Note that, every topology τ is its own basis. It must also be noted that the same topology can be generated by two different bases. Consider for example the set of all open subsets of \mathbb{R} . Also consider the set of all open balls in \mathbb{R} . Or, consider the set of all $\frac{1}{n}$ balls around all rational points of \mathbb{R}

Theorem 1.12. *(Going from a topology to a basis) Suppose (X, τ) is a topological space, and \mathcal{C} is a collection of sets in X (which is a subset of τ) such that for every $x \in X$ and for every $U \in \tau$ such that $x \in U$, $\exists C_x \in \mathcal{C}$ such that $x \in C_x \subseteq U$. Then τ is actually generated by \mathcal{C}*

Proof. Let $x \in X$. That means, $\exists C_x \in \mathcal{C}$ (from hypothesis) such that $x \in C_x \subseteq X$. Moreover, let $x \in C_1$ and C_2 . By hypothesis, all $C \in \mathcal{C}$ are considered open sets, therefore, $C_1 \cap C_2$ is an open set, and $x \in C_1 \cap C_2$. That means $\exists C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. That makes \mathcal{C} a basis. Its now obvious that $\tau = \tau_{\mathcal{C}}$

□

Theorem 1.13. *(Criteria for determining coarseness from basis) Let τ and τ' be two topologies of X . Also suppose τ is generated by \mathcal{C} and τ' , by \mathcal{C}' . Then the following are equivalent:*

1. τ' is finer than τ
2. for each $x \in X$ and each basis element $C \in \mathcal{C}$ containing x , there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$

Proof. \implies) Suppose τ' is finer than τ , that means every open set open in τ is also open in τ' . Let $C \in \mathcal{C}$, which is open in τ (and hence, in τ' too) and let $x \in C$. Since C is open in τ' , and $x \in C$, there exists a basis element of τ' , called C' such that $x \in C' \subseteq C$.

\impliedby) Let it be that for every $x \in X$ and for every $C \in \mathcal{C}$ such that $x \in C$, there exists $C' \in \mathcal{C}'$ such that $x \in C' \subseteq C$. By the definition of topology generated by bases, it is clear that \mathcal{C} is part of the topology of τ' . Consider $U \in \tau$.

U is an arbitrary union of elements of \mathcal{C} (by the alternate characterisation of generated topology), making U an element of τ' . We are done.

(aliter) Let U be such that for every $x \in U$, $\exists C \in \mathcal{C}$ such that $x \in C \subseteq U$. Since we are told that there exists $C' \subseteq C$ so that $x \in C'$, we have that for all $x \in U$, $\exists C'_x \in \mathcal{C}'$ so that $x \in C'_x \subseteq U$, making U part of the topology of τ' .

□

Definition 1.14. (Standard topology of \mathbb{R}) Let \mathcal{B} be the collection of all open intervals of the kind $(a, b) : a, b \in \mathbb{R}$. That this is a basis is obvious. Define the standard topology as that topology generated by this basis.

Definition 1.15. (lower limit topology \mathbb{R}_l) Consider \mathcal{B}' the collection of all sets of the kind $[a, b) : a, b \in \mathbb{R}$. Again, that this is a basis is obvious. Define the lower limit topology as the topology generated by \mathcal{B}' , and denote it as \mathbb{R}_l

Definition 1.16. (K topology \mathbb{R}_K) Let K be the set of all numbers of the kind $1/n$ for $n \in \mathbb{Z}_+$. Denote \mathcal{B}'' as the collection of all open intervals of the kind (a, b) along with the sets of the form $(a, b) - K$. Again, that this is a basis is obvious. The topology generated by \mathcal{B}'' is called the K topology, denoted by \mathbb{R}_K

Theorem 1.17. \mathbb{R}_K and \mathbb{R}_l are strictly finer than \mathbb{R} , but \mathbb{R}_K and \mathbb{R}_l are not comparable.

Proof. Consider $B \in \mathcal{B}$, of the form (a, b) . This is also in \mathcal{B}' , and a smaller set of the kind $[c, d)$ lies inside (a, b) , which means that, if $x \in (a, b)$, there exists $B' \in \mathcal{B}'$ and $B'' \in \mathcal{B}''$ so that $x \in B' \subset (a, b)$ and $x \in B'' \subset (a, b)$. Moreover, since both \mathbb{R}_l and \mathbb{R}_K have elements that are not in \mathbb{R} , it is obviously strictly finer. (Consider $a \in [a, b)$, there is no element of the kind (x, y) that is inside this, that contains a .)

Consider an element $B = [0, 1)$ where we choose $x = 0$. Clear that no element from \mathbb{R}_K can come inside B whilst containing 0. On the other hand, consider 0 inside $(-1, 1) - K$ in \mathbb{R}_K . No interval of the kind $[a, b)$ can contain 0 and itself be contained inside $(-1, 1) - K$. Hence, it is clear that \mathbb{R}_K and \mathbb{R}_l are incomparable.

□

(How Dugundji goes about developing an idea of basis)

Let \mathcal{T}_X be the topology of a space X . We call $\mathcal{B} \subset \mathcal{T}_X$ a *basis* if every open set in \mathcal{T}_X is a union of elements of \mathcal{B} . (and by definition, since \mathcal{B} is asked to be a subset of \mathcal{T}_X , we see that every $B \in \mathcal{B}$ is also open. So we can safely say every open set in \mathcal{T}_X is an arbitrary union of \mathcal{B} , and conversely every arbitrary union of elements of \mathcal{B} belongs in \mathcal{T}_X)

Theorem: Let $\mathcal{B} \subseteq \mathcal{T}_X$. Then, the following are equivalent:

1. \mathcal{B} is a basis for \mathcal{T}_X
2. For every $G \in \mathcal{T}_X$, and for every $(x \in X, x \in G) \implies x \in B_x \subset G$ for some $B_x \in \mathcal{B}$

Proof. \implies) Let \mathcal{B} be a basis for \mathcal{T}_X . Let $x \in G$ for some $G \in \mathcal{T}_X$. Since $G = \cup_q B_q$ for $B_q \in \mathcal{B}$, we have that there exists atleast one B_q so that $x \in B_q \subseteq G$.

\Leftarrow) Let U be an open set in \mathcal{T}_X . We are told that for every $x \in U$, there is an element of \mathcal{B} so that $x \in B_x \subseteq U$. This implies U is an arbitrary union of elements of \mathcal{B} . This means that \mathcal{B} forms a basis for \mathcal{T}_X .

□

Definition 1.18. (Subbasis) A subbasis S is a collection of subsets of X whose union amounts to X

Definition 1.19. (Topology generated by a subbasis) τ_S , the topology generated by the subbasis S is defined as the set of all arbitrary unions of the set of all finite intersection of elements of S .

Theorem 1.20. τ_S , the topology generated by a subbasis S , is indeed a topology.

Proof. Let S be the set whose union forms X (Subbasis).

Let $Y = \{\gamma : \gamma = \cap_{i=1}^n S_i \text{ for some } n \text{ and } S_i \in S\}$, i.e, Y is the set of all finite intersections of elements of S . Consider the set of all arbitrary union of elements of Y , and call it τ_S . Obviously τ_S contains \emptyset and X . Let $\{U_\alpha\}$ be a collection of open sets in τ_S , i.e, each U_α is an arbitrary union of elements that are finite intersection of elements of S . The union of all U_α , therefore, would still be an arbitrary union of finite intersections of elements of S , hence, is back in τ_S . Now consider $\{U_1, U_2 \cdots U_k\}$, elements of τ_S such that each U_i is an arbitrary union of a collection of sets that are each a finite intersection of elements of S . Consider the finite intersection $\cap_{i=1}^k U_i = \cap_{i=1}^k (\cup_{\alpha} (\cap_{j=1}^{n_{\alpha}} S_{j,\alpha}))$ which is ultimately an arbitrary union of a finite intersection of sets in S , making this part of τ_S . Hence, τ_S is indeed a topology.

□

Theorem 1.21. Let $\{\tau_\alpha : \alpha \in A\}$ be a collection of topologies of X , then there is a unique "smallest" topology τ_S such that τ_S contains all the topologies $\{\tau_\alpha\}$ (kinda like a LUB), and there is a unique "largest" topology τ_L that is contained in all these topologies $\{\tau_\alpha\}$.

Proof. τ_S that contains all topologies $\{\tau_\alpha\}$:

Uniqueness: If two such topologies exists, we can take the intersection of them to result in a smaller topology (intersection is still a topology) that contains $\{\tau_\alpha\}$.

Existence: Consider the topology generated by arbitrarily unioning the sets generated by finitely intersecting all elements of $\{\tau_\alpha : \alpha \in A\}$. Explicitly, we collect $\mathcal{R} = \{U : \exists x \text{ such that } U \in \tau_x\}$ and we take all possible finite intersections of elements in \mathcal{R} and collect that in a set \mathcal{S} . Then, we take all possible arbitrary unions of elements of \mathcal{S} and call it τ_S .

For later use:

- \mathcal{R} = all the open sets in all the topologies.
- \mathcal{S} = finite intersection of all elements of \mathcal{R}
- τ_S = arbitrary union of all elements of \mathcal{S}

Is τ_S a topology? X and \emptyset are obviously there, since X and \emptyset are there in \mathcal{R} , and we take the singleton intersection of these in \mathcal{S} , and finally, we take the singleton union of these to arrive at $X, \emptyset \in \tau_S$. Let $\{G_\delta : \delta \in \Delta\}$ be a collection of sets in τ_S (i.e, a collection of sets that are an arbitrary union of a collection of finite intersection of elements of \mathcal{R}). $\cup_\delta G_\delta$ would also be an arbitrary union of elements of \mathcal{S} making it part of τ_S . Let $\{G_1, G_2, \cdots G_k\}$ be in τ_S . $\cap_{i=1}^k G_i = \cap_{i=1}^k (\cup_{\delta_i} (\cap_{j=1}^{n_{\delta_i}} R_{\delta_i,j}))$ which, from De Morgan's Laws, can easily be seen to be an arbitrary union of a finite intersection of elements from \mathcal{R} . Hence, the intersection is also in τ_S making it a topology that contains all of $\{\tau_\alpha\}$.

τ_L that is contained in all topologies $\{\tau_\alpha\}$

Existence: Consider $\cap_\alpha \tau_\alpha$, which is a topology that is contained in all of τ_α .

Uniqueness: If two topologies are such that they are contained in every τ_α , then they are indeed contained in $\cap_\alpha \tau_\alpha$

□

Theorem 1.22. *Let A be a basis for τ , then τ_A , the topology generated by A (The one generated by taking arbitrary union of elements of A) is precisely the intersection of all topologies $\{\tau_\alpha : \alpha \in G\}$ that contain A*

Proof. τ_A is the topology generated by arbitrary union of elements of A . We are told $A \subseteq \tau_\alpha$ for all $\alpha \in G$. Let $U \in \tau_A$. U is an arbitrary union of elements of A , which, since τ_α is a topology, is also contained in τ_α . Every open set in τ_A is also in τ_α . Hence $\tau_A \subseteq \cap_\alpha \tau_\alpha$.

Note that, τ_A is itself a topology containing A , which means $\tau_A = \cap_\alpha \tau_\alpha$

□

Theorem 1.23. *Let A be a subbasis for X . The topology τ_A generated by A , i.e., by taking arbitrary union of elements that are finite intersection of A , is precisely the intersection of all topologies $\{\tau_\alpha\}$ that contain A .*

Proof. Obviously τ_A contains A , since each element of A is a singleton union of the singleton intersection of itself. Moreover, let U be any open set in τ_A , i.e., U is an arbitrary union of a collection of finite intersections of elements of A . Let τ_α be a topology that contains A . That means a finite intersection of elements of A is still in τ_α , and an arbitrary union of these are still in τ_α . Hence, U is in τ_α , for any α . Hence, $\tau_A \subseteq \tau_\alpha$ for every α , implying $\tau_A \subseteq \cap_\alpha \tau_\alpha$. Since τ_A is part of $\{\tau_\alpha\}$, we see

$$\tau_A = \cap_\alpha \tau_\alpha$$

where $\{\tau_\alpha\}$ is the collection of all topologies that contain A .

□

(How Dugundji goes about developing subbasis)

Theorem: Given any collection of subsets of X , $\Sigma = \{A_\alpha : \alpha \in \mathbb{A}\}$, there exists a unique, smallest topology $\mathcal{T}(\Sigma)$ that contains Σ as a subset. Moreover, this has a characterisation: It is the collection of all the arbitrary unions of finite intersections of elements of Σ .

Proof. Uniqueness: If two such topologies \mathcal{T}_1 and \mathcal{T}_2 exists, then $\mathcal{T}_1 \cap \mathcal{T}_2$ becomes a smaller topology that contains Σ .

Existence: Take all elements of Σ and take all possible finite intersections, and collect them in the set \mathcal{R} . Then take all possible arbitrary union of elements of \mathcal{R} and collect them in a set \mathcal{S} . This would be a topology, since arbitrary union of an arbitrary union of finite intersections is an arbitrary union of finite intersections. Moreover, finite intersections of arbitrary unions of finite intersections are ultimately arbitrary union of finite intersections. We denote this topology by $\mathcal{T}(\Sigma)$.

Consider for a moment the set of all topologies \mathcal{T}_γ that contains Σ . $\cap_\gamma \mathcal{T}_\gamma$, intersection of all topologies that contain Σ , is indeed a topology, as is easy to verify. Moreover, it is the smallest topology that contains Σ ; This much is evident.

Let U be an arbitrary union of a finite intersection of elements of Σ . Since $\cap_\gamma \mathcal{T}_\gamma$ contains Σ , it contains U as well, implying that $\mathcal{T}(\Sigma) \subseteq \cap_\gamma \mathcal{T}_\gamma$. Since $\mathcal{T}(\Sigma)$ is itself a topology that contains Σ , we see that $\mathcal{T}(\Sigma) = \cap_\gamma \mathcal{T}_\gamma$, i.e $\mathcal{T}(\Sigma)$ is the smallest topology that contains Σ . □

Definition:(Topology generated by a family Σ , $\mathcal{T}(\Sigma)$) The topology generated by any family of subsets of X is defined as $\mathcal{T}(\Sigma)$, the collection of all arbitrary union of finite intersections of elements of Σ .

1.2 Order Topology

Definition 1.24. (Simple Order) Let X be a set. Define $C : X \times X \rightarrow \{y, n\}$ such that:

1. If $x \neq y$, either xCy or yCx
2. For no x is it true that xCx
3. if xCy and yCz , then xCz

This type of relation is called *simple order* relation.

Pro Tip: For intuition purposes, replace C with $<$.

Definition 1.25. (Order Topology on a space X .) Let X be a space with more than two points, equipped with a simple order relation $<$. Consider the collection \mathcal{B} of sets of the following kind:

1. (a, b) where $a < b$. i.e, all open intervals in X
2. $[a_0, b)$ for all $b \in X$, where a_0 is the smallest element of X
3. $(a, b_0]$ for all $a \in X$ where b_0 is the largest element in X

Then the order topology τ_O is defined as that topology generated by \mathcal{B} .

Remark 1.26. Note that, if X has no smallest element, type (2) sets won't exist in \mathcal{B} . Likewise, if X has no largest element, type (3) sets won't exist in \mathcal{B} .

Theorem 1.27. \mathcal{B} is actually a basis (hence, τ_O is actually a well defined topology on X)

Proof. Let $x \in X$. If x is not the largest element, nor the smallest, then there exists (a_0, b_0) that contains x

If x is the largest element, then $(a_0, x]$ contains x . If x is the smallest element, then $[x, b_0)$ contains x . Hence, for every $x \in X$, there exists an element B from \mathcal{B} such that $x \in B$.

Note the following, if we intersect two type 1 sets, we get back a smaller type 1 set. i.e, $(a, b) \cap (a', b') := \{x \in X : a < x < b \text{ and } a' < x < b'\}$.

If we intersect two type 2 sets, we get back a type 2 set (obviously)

If we intersect two type 3 sets, we get back a type 3 set (obviously)

If we intersect a type 1 and type 2 set, i.e (a, b) and $[a_0, x)$, we are basically saying $\{t \in X : a < t < b \text{ and } a_0 \leq t < x\}$ We get back a type 1 set.

If we intersect a type 1 and type 3 set, i.e (a, b) and $(x, b_0]$, we get $\{t \in X : a < t < b \text{ and } x < t \leq b_0\}$ which would yield a set of type 1.

If we intersect a type 2 and type 3 set, i.e, $[a_0, x)$ and $(y, b_0]$, we want basically $\{t \in X : a_0 \leq t < x \text{ and } y < t \leq b_0\}$. It is easy to see that this intersection falls into one of the 3 categories.

Hence, \mathcal{B} forms a basis, and hence, τ_B is a topology.

□

Example 1.28. The standard topology \mathbb{R} is basically the order topology on \mathbb{R} . All one needs to see is that there is no least element, and largest element of \mathbb{R} (Of course, $\mathbb{R} \cup \{-\infty, +\infty\}$, the extended real line, *does* have these). Hence, \mathcal{B} is just all elements of the kind (a, b) , which is precisely the basis for the standard topology.

Definition 1.29. (Rays) Given $a \in X$ (a simply ordered set), there are 4 sets associated with a called *rays*:

1. $(a, +\infty) := \{x : a < x\}$
2. $(-\infty, a) := \{x : x < a\}$

3. $[a, +\infty) := \{x : a \leq x\}$
4. $(-\infty, a] := \{x : x \leq a\}$

Theorem 1.30. *The first two rays are open in the order topology, (and the 2nd two rays are closed in the order topology, i.e, its complement is open)*

Proof. Consider $(a, +\infty)$. If there is no maximum element, then this is precisely an arbitrary union of sets of the kind (a, b_α) . If there is a maximal element, then this is $(a, b_0]$. In a similar fashion, the 2nd set is also open.

Consider $[a, +\infty)^C = (-\infty, a)$, and consider $(-\infty, a]^C = (a, \infty)$. Obvious from here.

□

Theorem 1.31. *The collection of all open rays forms a subbasis for the order topology on X*

Proof. We have to show that the topology generated by arbitrarily unioning a collection of finite intersections of rays gives rise to the topology generated by the basis of order topology of X . Let τ_R be the topology generated by the arbitrary union of finite intersection of rays. Let τ be the order topology on X , generated by arbitrary union of the basis elements of X . We want to show $\tau_R \subseteq \tau$ and $\tau \subseteq \tau_R$.

Consider (a, b) , a basis element of τ . $(a, +\infty) \cap (-\infty, b)$ is a finite intersection of rays, and yields the basis (a, b) . $[a_0, b)$ is, simply, $(-\infty, b)$ and $(a, b_0]$ is simply $(a, +\infty)$. Hence, finite intersection of the rays yield the basis elements. Arbitrary union of finite intersection of rays can yield all the possible open sets of the order topology on X . Hence, $\tau \subseteq \tau_R$. Let S be open in τ_R . It is the arbitrary union of a finite intersection of rays. Note that every ray is either an element of the basis for τ or is an arbitrary union elements of basis of τ . An arbitrary union of a finite intersection of an arbitrary union of elements of the basis yields an arbitrary union of a finite intersection of elements of the basis, giving rise to yet another element of the topology τ . Hence, every open set in τ_R is open in τ , hence $\tau_R \subseteq \tau$

□

1.3 The product topology

Definition 1.32. (Product topology on $X \times Y$) Let τ_X and τ_Y be topologies of X and Y respectively. Let \mathcal{B} be the collection of subsets of $X \times Y$ of the kind $U \times V$ for $U \in \tau_X$ and $V \in \tau_Y$. We define the product topology as that which is generated by the basis \mathcal{B} .

Theorem 1.33. *\mathcal{B} of the previous definition is indeed a basis.*

Proof. Let $(x, y) \in X \times Y$. Since $X \in \tau_X$ and $Y \in \tau_Y$, we have that $(x, y) \in X \times Y$ so the 1st condition for basis is satisfied. Let $(x, y) \in U_1 \times V_1$ and $U_2 \times V_2$. Consider $(U_1 \times V_1) \cap (U_2 \times V_2)$. This is same as $U_1 \cap U_2 \times V_1 \cap V_2$ which is again an element of \mathcal{B} . Hence, it is a basis and hence $\tau_{\mathcal{B}}$ is well defined.

□

Theorem 1.34. *If X 's topology \mathcal{T}_X is generated by a basis \mathcal{B}_X and Y 's topology \mathcal{T}_Y is generated by a basis \mathcal{B}_Y , then the product topology of $X \times Y$*

(given by $\{U \times V : U \text{ open in } \mathcal{T}_X \text{ and } V \text{ open in } \mathcal{T}_Y\}$) has a basis $\mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\}$, i.e.,

$$\mathcal{T}_{X \times Y} = \langle \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\} \rangle = \langle_{\text{basis}} \mathcal{B} := \{C \times D : C \in \mathcal{B}_X, D \in \mathcal{B}_Y\} \rangle$$

Proof. Checking that \mathcal{B} is indeed a basis:

Let $(x, y) \in X \times Y$. $x \in X$ means that there exists $B \in \mathcal{B}_X$ so that $x \in B \subseteq X$. Likewise, for y , there is a $C \in \mathcal{B}_Y$ so that $y \in C \subseteq Y$, which means $(x, y) \in B \times C \subseteq X \times Y$. This means that \mathcal{B} obeys the 1st condition for a basis.

Let $(x, y) \in B, B' \in \mathcal{B}$. i.e., $(x, y) \in C_1 \times D_1$, and $C_2 \times D_2$ for $C_1, C_2 \in \mathcal{B}_X$ and $D_1, D_2 \in \mathcal{B}_Y$. That means that $(x, y) \in (C_1 \times D_1) \cap (C_2 \times D_2) = (C_1 \times C_2) \times (D_1 \times D_2)$. $x \in C_1 \cap C_2$ means there is a C_3 so that $x \in C_3 \subseteq C_1 \cap C_2$. Likewise, $y \in D_1 \cap D_2$. Hence, $(x, y) \in (C_3 \times D_3) \subseteq (C_1 \cap C_2) \times (D_1 \cap D_2) = (C_1 \times D_1) \cap (C_2 \times D_2)$. Hence, \mathcal{B} is a basis.

Now consider any open set in $\mathcal{T}_{X \times Y}$. Since $\mathcal{T}_{X \times Y}$ is generated by $U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y$, Every open set in $\mathcal{T}_{X \times Y}$ is an arbitrary union of sets of the kind $U \times V$, i.e., $\cup_{\alpha} (U_{\alpha} \times V_{\alpha})$. But each $U_{\alpha} = \cup_{\delta(\alpha)} B_{\delta}^{\alpha}$ for $B_{\delta}^{\alpha} \in \mathcal{B}_X$. Likewise, each $V_{\alpha} = \cup_{\delta(\alpha)} C_{\delta}^{\alpha}$ for $C_{\delta}^{\alpha} \in \mathcal{B}_Y$. Let (x, y) be any point in X , that is in $\cup_{\alpha} (U_{\alpha} \times V_{\alpha})$ which means that there exists α_0 so that $(x, y) \in U_{\alpha_0} \times V_{\alpha_0} = \cup_{\delta(\alpha_0)} B_{\delta} \times \cup_{\delta(\alpha_0)} C_{\delta}$. This means $x \in B_{\delta_0}$ and $y \in C_{\delta_1}$ for some indices. This means that $(x, y) \in B_{\delta_0} \times C_{\delta_1}$, which is part of the basis \mathcal{B} . Hence, Every open set in $\mathcal{T}_{X \times Y}$ is open in $\langle \mathcal{B} \rangle$ (or $\mathcal{T}_{X \times Y} \subseteq \langle \mathcal{B} \rangle$). Moreover, it is obvious to see that every element of \mathcal{B} is open in $\mathcal{T}_{X \times Y}$. Hence, arbitrary union of elements of \mathcal{B} , which are precisely the open sets of \mathcal{B} , are also open in $\mathcal{T}_{X \times Y}$. Hence, $\mathcal{T}_{X \times Y} = \langle \mathcal{B} \rangle$.

□

Example 1.35. Consider $X = \mathbb{R}, Y = \mathbb{R}$, with the standard topologies (i.e., that which is generated by the basis elements of the kind $\{(a, b); a < b\}$). The **standard topology** on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is defined as the product topology. A basis for this topology is the set of all $U \times V$ for U and V open in \mathbb{R} , but from the preceeding result we can find an even smaller subset that makes a basis: $\{B \times C : B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ which is precisely $\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{R}; a < b, c < d\}$

Definition 1.36. (Projections) Define $\pi_1 : X \times Y \rightarrow X$ by $\pi_1(x, y) = x$ and $\pi_2 : X \times Y \rightarrow Y$ given by $\pi_2(x, y) = y$. These maps are called projection maps.

Theorem 1.37. *The projection maps are onto, and have the property that, if U is an open subset of X , and V is an open subset of Y , then $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ (well, open hypothesis is not needed really)*

Proof. That they are onto functions is obvious. Let U be an open subset of X . Consider $\pi_1^{-1}(U) := \{(x, y) \in X \times Y : \pi_1(x, y) = x \in U\}$. This is precisely $U \times Y$. Similarly, $\pi_2^{-1}(V) = X \times V$. Note that, in the product topology, $U \times Y$ and $X \times V$ are both open sets. Moreover, the intersection of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ is $(U \times Y) \cap (X \times V) = U \times V$

□

Theorem 1.38. *Let \mathcal{T}_X be a topology for X and \mathcal{T}_Y for Y . Let $\mathcal{T}_{X \times Y} := \langle \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\} \rangle$ be the product topology on $X \times Y$. Then the set $\Gamma := \{\pi_1^{-1}(U) : U \in \mathcal{T}_X\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{T}_Y\}$ forms a subbasis for $\mathcal{T}_{X \times Y}$*

Proof. Γ is full of elements of the kind $U \times Y$ and $X \times V$ (every element is of this kind, and every element of this kind is in Γ). Consider any open set in $\mathcal{T}_{X \times Y}$, which is an arbitrary union of elements of the type $U \times V$, or explicitly, an open set looks like $\cup_\alpha (U_\alpha \times V_\alpha)$ which is the same as $\cup_\alpha ((U_\alpha \times Y) \cap (X \times V_\alpha)) = \cup_\alpha \pi_1^{-1}(U_\alpha) \cap \pi_2^{-1}(V_\alpha)$ which is an arbitrary union of an intersection of elements of Γ . Hence, we see that every set open in $\mathcal{T}_{X \times Y}$ is open in $\langle_{\text{subbasis}} \Gamma \rangle$, which means $\mathcal{T}_{X \times Y} \subseteq \langle_{\text{subbasis}} \Gamma \rangle$. Let C be a set in $\langle_{\text{subbasis}} \Gamma \rangle$, i.e, C is an arbitrary union of a finite intersection of sets in Γ . If the finite intersection only contains elements of the kind $U \times Y$ (or $X \times V$), then the intersection would ultimately be of the kind $U_0 \times Y$ for some U_0 open in \mathcal{T}_X (or $X \times V_0$ for some V_0 open in \mathcal{T}_Y), which would then mean $\cup_0 U_0 \times V$ would finally still yield a set of the form $U \times V$, falling again into $\mathcal{T}_{X \times Y}$ (or the other thing).

If suppose the finite intersection contains both elements of the type $U \times Y$ and $X \times V$, then intersecting them yields $U \times V$, and suppose some other set is of the kind $U' \times Y$ (or $X \times V'$), an intersection with them still yields a set of the type $U'' \times V''$ for some open $U'' \in \mathcal{T}_X$ and $V'' \in \mathcal{T}_Y$. This again shows that, arbitrarily unioning them will still make it fall into $\mathcal{T}_{X \times Y}$, which finally gives us $\langle_{\text{subbasis}} \Gamma \rangle = \mathcal{T}_{X \times Y}$

□

1.4 Subspace Topology

Definition 1.39. (Subspace Topology) Let X be a topological space with a topology \mathcal{T}_X and let $Y \subset X$. We define the **subspace topology** on Y as $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$

Theorem 1.40. *The Subspace topology is indeed a topology on Y*

Proof. $\emptyset = Y \cap \emptyset$ and $Y = Y \cap X$. Hence \emptyset and X are both in \mathcal{T}_Y . Let U_α be a collection of sets in \mathcal{T}_Y , i.e. $U_\alpha = G_\alpha \cap Y$ for $G_\alpha \in \mathcal{T}_X$. $\cup_\alpha U_\alpha = \cup_\alpha (G_\alpha \cap Y) = (\cup_\alpha G_\alpha) \cap Y \in \mathcal{T}_Y$. Closed under arbitrary union.

Let $U_1, U_2 \dots U_k$ be a bunch of sets of the kind $G_i \cap Y$. Arbitrary intersection of these yields $\cap_{i=1}^k U_i = \cap_{i=1}^k (G_i \cap Y) = (\cap_{i=1}^k G_i) \cap Y$ which is again of the form $K \cap Y$ for some $K \in \mathcal{T}_X$, which makes this finite intersection a part of \mathcal{T}_Y . Hence, \mathcal{T}_Y is a topology.

□

Lemma 1.41. *Let \mathcal{B}_X be a basis for \mathcal{T}_X on X . Then the basis of \mathcal{T}_Y , the subspace topology of $Y \subseteq X$ is $\mathcal{C} := \{B \cap Y : B \in \mathcal{B}_X\}$*

Proof. Let $y \in Y$, this means $y \in X$, which means $y \in B \in \mathcal{B}_X$. That means $y \in B \cap Y \in \mathcal{C}$. Let $y \in C_1$ and C_2 , which are in \mathcal{C} . That means $y \in (B_1 \cap Y) \cap (B_2 \cap Y) = (B_1 \cap B_2) \cap Y$ which means $y \in (B_3) \cap Y$ so that $B_3 \subseteq B_1 \cap B_2$, implying that $B_3 \cap Y \subseteq B_1 \cap Y \cap B_2 \cap Y$, giving us that if $y \in C_1$ and C_2 in \mathcal{C} , there exists $C_3 (= B_3 \cap Y)$ so that $y \in C_3 \subseteq C_1 \cap C_2$, hence \mathcal{C} forms a basis for \mathcal{T}_Y . Now we need to show that $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$ is the same as $\mathcal{T}_\mathcal{C} := \langle \mathcal{C} \rangle$. Let R be open in \mathcal{T}_Y implying it is of the form $U \cap Y$ for some $U \in \mathcal{T}_X$. But $U = \cup_\alpha B_\alpha$ for basis elements $B_\alpha \in \mathcal{B}$. That gives $R = U \cap Y = (\cup_\alpha B_\alpha) \cap Y = \cup_\alpha (B_\alpha \cap Y)$ which is an arbitrary union of elements of \mathcal{C} . Hence, $\langle \mathcal{C} \rangle \subseteq \mathcal{T}_Y$.

Let G be in $\langle \mathcal{C} \rangle$, i.e. is an arbitrary union of $C_\alpha = B_\alpha \cap Y \in \mathcal{C}$. Hence, $G = \cup_\alpha (B_\alpha \cap Y) = (\cup_\alpha B_\alpha) \cap Y$ making it part of \mathcal{T}_Y (its of the form "some open set" intersected with Y). This means $\mathcal{T}_Y \subseteq \langle \mathcal{C} \rangle$ which ultimately gives $\langle \mathcal{C} \rangle = \mathcal{T}_Y$

□

Lemma 1.42. *Let $Y \subseteq X$ and let \mathcal{T}_Y be the subspace topology on Y inherited from the topology \mathcal{T}_X on X . Then, Y is open in X (i.e. is in \mathcal{T}_X) if and only every set open in Y , i.e. is in \mathcal{T}_Y is open in X .*

Proof. \implies) Let Y be open in X , that means $Y \in \mathcal{T}_X$. $\mathcal{T}_Y := \{U \cap Y : U \in \mathcal{T}_X\}$. Let $G = U \cap Y$ be an open set in \mathcal{T}_Y , for some open set U open in X . Since \mathcal{T}_X is closed under finite intersections, $G = U \cap Y$ is open in \mathcal{T}_X . Hence every set in \mathcal{T}_Y is in \mathcal{T}_X .

\impliedby) If every set in \mathcal{T}_Y is in \mathcal{T}_X , that means that Y is also in \mathcal{T}_X , making Y open in X .

□

Theorem 1.43. *Let X be a space with a topology \mathcal{T}_X and Y with \mathcal{T}_Y . Consider $A \subseteq X$ and $B \subseteq Y$. $A \times B \subseteq X \times Y$. We have that*

$$\mathcal{T}_{A \times B}^S = \mathcal{T}_{A \times B}^P$$

where $\mathcal{T}_{A \times B}^S$ represents the subspace topology of $A \times B$ viewed as a subspace of the product topology $X \times Y$, and $\mathcal{T}_{A \times B}^P$ represents the product topology of $A \times B$, whose individual topologies come from the corresponding subspace topologies.

Proof. Let F be an open set in $\mathcal{T}_{A \times B}^S$, i.e., it is of the form $G \cap (A \times B)$ for some $G \in \mathcal{T}_{X \times Y}$, i.e., G is an arbitrary union $\cup_\alpha (U_\alpha \times V_\alpha)$ ($U_\alpha \in \mathcal{T}_X$ and $V_\alpha \in \mathcal{T}_Y$). Therefore, $F = [\cup_\alpha (U_\alpha \times V_\alpha)] \cap (A \times B) = \cup_\alpha ((U_\alpha \times V_\alpha) \cap (A \times B)) = \cup_\alpha ((U_\alpha \cap A) \times (V_\alpha \cap B))$. Note that $U_\alpha \cap A$ and $V_\alpha \cap B$ are each open in A and B respectively (from the subspace topology inherited from \mathcal{T}_X and \mathcal{T}_Y respectively). Therefore, $U_\alpha \cap A \times V_\alpha \cap B$ is part of the basis for the product topology $A \times B$ (since it's a product of open sets in A and B). Hence, F is an arbitrary union of elements of $\{U \times V : U \in \mathcal{T}_A, V \in \mathcal{T}_B\}$ which means that it is in $\mathcal{T}_{A \times B}^P$. This implies $\mathcal{T}_{A \times B}^S \subseteq \mathcal{T}_{A \times B}^P$.

Let F' be an open set in $\mathcal{T}_{A \times B}^P$, i.e., is an arbitrary union of elements of the kind $U_r \times V_r$ where $U_r \in \mathcal{T}_A$ and $V_r \in \mathcal{T}_B$. $F' = \cup_r (U_r \times V_r) = \cup_r (G_r \cup A) \times (E_r \cup B)$ for $G_r \in \mathcal{T}_X$ and $E_r \in \mathcal{T}_Y$. This is $F' = \cup_r (G_r \times E_r) \cup (A \times B) = \cup_r (\text{basis element of } \mathcal{T}_{X \times Y}) \cap (A \times B) = \cup_r (\text{some basis element of } A \times B)_r$ which is obviously open in $\mathcal{T}_{A \times B}^S$. Hence, $\mathcal{T}_{A \times B}^P \subseteq \mathcal{T}_{A \times B}^S$.

□

Though the subspace topology inherited by $A \times B \subseteq X \times Y$ is precisely the product topology of the corresponding subspace topologies, a similar result, though reasonable to expect for subspaces of order topologies, does not, unfortunately, hold; as the following example reveals:

Example 1.44. (Order topology inherited from superspace is *not* equal to the subspace topology inherited from ordered superspace) Consider $Y = [0, 1] \cup \{2\} \subseteq \mathbb{R}$. The topology on \mathbb{R} is generated by the basis $(a, b) : a, b \in \mathbb{R}$. The subspace topology on Y is generated by the sets $(a, b) \cap Y : a, b \in \mathbb{R}$. Call this $\mathcal{T}_Y^S = \langle \{(a, b) \cap Y : a < b\} \rangle$. The order topology on Y is generated by sets of the kind $(x, y) : x, y \in Y$, $[0, y) : y \in Y$, along with $(x, 2 : x \in Y]$. Note that $\{2\}$ is open in the subspace topology since $\{2\} = (1.5, 2.5) \cap Y$. But $\{2\}$ cannot be open in the order topology inherited by Y since any basis element containing 2 contains some other element of Y as well.

Example 1.45. Let $I = [0, 1]$. Consider $I \times I$ as a subset of $\mathbb{R} \times \mathbb{R}$, which has topology $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^O$ obtained by the simple lexicographic order. The basis elements of $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^O$ is given by $\{(a \cdot b, c \cdot d) : a < c \text{ or } (a = c \text{ and } b < d)\}$ (Since there is no smallest or largest element of $\mathbb{R} \times \mathbb{R}$ in the lexicographic order). The basis for $\mathcal{T}_{I \times I}^S$ is $(a \cdot b, c \cdot d : a, b, c, d \in \mathbb{R}) \cap (I \times I)$. Meanwhile, the basis for $\mathcal{T}_{I \times I}^O$ are sets of the form $\{(p \cdot q, r \cdot s) : p, q, r, s \in I\}$, $[0 \cdot 0, x \cdot y)$, $(x \cdot y, 1 \cdot 1]$. Consider the interval $(0.5 \cdot -5, 0.5 \cdot 5)$ which is open in $\mathbb{R} \times \mathbb{R}$. Take the subspace

element $I \times I \cap (0.5 \cdot -5, 0.5 \cdot 5)$. This would be $[0.5 \cdot 0, 0.5 \cdot 1]$. This is open in the subspace topology, but this is not open in the order topology since no basis element of $\mathcal{T}_{I \times I}^0$ contains $0.5 \cdot 0$ which is also in itself contained in the above interval.

Taking motivation from the previous example, we define the following:

Definition 1.46. (Ordered Square) The set $I \times I \subseteq \mathbb{R} \times \mathbb{R}$, where $I = [0, 1]$ with the lexicographic order topology is called the *ordered square* and is denoted by I_0^2

Lemma 1.47. Let \mathcal{B} be a subbasis for the topology \mathcal{T} on X . Then the subspace topology \mathcal{T}_S of $S \subseteq X$ is generated by the subbasis $\{G \cap S : G \in \mathcal{B}\}$

Proof. Let U be an open set in \mathcal{T}_S . Then it is of the form $K \cap S$ for some open set K in \mathcal{T} , making it an arbitrary union $\cup_\alpha (\cap_{i=1}^{k_\alpha} (E_{\alpha,i}))$ of elements $E_{\alpha,i}$ of \mathcal{B} . This gives $U = (\cup_\alpha (\cap_{i=1}^{k_\alpha} (E_{\alpha,i}))) \cap S = \cup_\alpha ((\cap_{i=1}^{k_\alpha} (E_{\alpha,i})) \cap S) = \cup_\alpha (\cap_{i=1}^{k_\alpha} (E_{\alpha,i} \cap S))$ which is an arbitrary union of a finite intersection of elements from $\{G \cap S : G \in \mathcal{B}\}$. Hence, $\langle_{\text{subbasis}} \{G \cap S : G \in \mathcal{B}\} \rangle \subseteq \mathcal{T}_S$. Any element of $\langle_{\text{subbasis}} \{G \cap S : G \in \mathcal{B}\} \rangle$ is of the form $K = \cup_\alpha (\cap_{i=1}^{k_\alpha} (F_{\alpha,i}))$ for $F_{\alpha,i} = G_{\alpha,i} \cap S$ for $G_{\alpha,i}$ that are in \mathcal{B} . Then we can rewrite K as $\cup_\alpha (\cap_{i=1}^{k_\alpha} (G_{\alpha,i} \cap S)) = \cup_\alpha ([\cap_{i=1}^{k_\alpha} (G_{\alpha,i})]) \cap S$ which is of the form $\gamma \cap S$ where γ is an arbitrary union of a finite intersection of elements of \mathcal{B} , making γ open in \mathcal{T} and ultimately K open in \mathcal{T}_S . Hence, $\langle_{\text{subbasis}} \{G \cap S : G \in \mathcal{B}\} \rangle \subseteq \mathcal{T}_S$ giving us $\langle_{\text{subbasis}} \{G \cap S : G \in \mathcal{B}\} \rangle = \mathcal{T}_S$. □

Definition 1.48. (Convex Sets) Given an ordered set X , a subset Y is called *convex* if for any two points a, b in Y , if $a < b$, then (a, b) is fully contained in Y . Intervals and rays are always convex.

Theorem 1.49. Let X be an ordered set in the order topology \mathcal{T}_X^0 . Let Y be a subset of X that is convex, i.e., for any two points a, b in Y so that $a < b$, we have $(a, b) \subseteq Y$. Then the order topology on Y , \mathcal{T}_Y^0 is the same as the subspace topology \mathcal{T}_Y^S inherited from the order topology \mathcal{T}_X^0 .

Proof.

$$\mathcal{T}_Y^0 := \langle \{(x, y)_Y : x, y \in Y\}, \{[y_L, q)_Y : q \in Y\}, \{(q, y_U]_Y : q \in Y\} \rangle$$

$$\mathcal{T}_Y^S := \langle \{(a, b)_X \cap Y : a, b \in X\}, \{[a_0, b)_X \cap Y : b \in X\}, \{(a, b_0]_X \cap Y : a \in X\} \rangle$$

First we prove that $\mathcal{T}_Y^0 \subseteq \mathcal{T}_Y^S$. Consider the basis elements of the kind $(x, y)_Y : x, y \in Y$ and let $z \in (x, y)_Y$. $(x, y)_Y := \{z \in Y : x < z < y\} = \{z \in X : x < z < y \text{ and } z \in Y\} = (x, y)_X \cap Y$ which is a basis element of \mathcal{T}_Y^S .

Now consider basis elements of the kind $[y_L, y)_Y = \{z \in Y : y_L \leq z < y\} = \{z \in X : y_L \leq z < y \text{ and } z \in Y\} = [y_L, y)_X \cap Y$. if $q \in [y_L, y)_Y$ and $q \neq y_L$, then we can look

at $(y_L, y)_X \cap Y$ which contains q and is readily inside $[y_L, y)_Y$. If $q = y_L$, Consider a smaller element than y_L , say $L \in X$, and consider $(L, y)_X \cap Y = (L, y)_Y = [y_L, y)_Y$, which implies that $q = y_L \in (L, y)_X \cap Y \subseteq [y_L, y)_Y$ (If y_L is the least in X , too, then take the 2nd type basis element of \mathcal{T}_X^S).

Consider the bases elements of the kind $(y, y_U]_Y = (y, y_U]_X \cap Y$. Let q be in this set, and not equal to y_U . Then take $(y, y_U)_X \cap Y$ which is a subset of the above basis element. If $q = y_U$, then in X take a larger element U and consider $(y, U)_X \cap Y = (y, y_U]_Y$ which contains q and is itself contained in $(y, y_U]_Y$. Moreover, if the largest element of X and Y coincide, then take the 3rd type from \mathcal{T}_Y^S .

Therefore, $\mathcal{T}_Y^0 \subseteq \mathcal{T}_Y^S$.

Now we prove that $\mathcal{T}_Y^S \subseteq \mathcal{T}_Y^0$. If our basis element is of the kind $(a, b)_X \cap Y$, and $p \in (a, b)_X \cap Y$, we have a few cases to consider.

1. $a < y$ for every $y \in Y$, and $b > y$ for every $y \in Y$.
 - (a) Suppose no z is in Y so that $a < z < q$, and likewise, no z' is so that $q < z' < b$. That makes q the least, and the largest element of Y . The order topology on Y then would be trivial.
 - (b) Suppose such z exists so that $a < z < q$, but no such z' exists so that $q < z' < b$. That means q is the largest element, and we can pick the basis $(z, q]_Y$ that contains q and is inside $(a, b)_X \cap Y$.
 - (c) Suppose no such z exists but such z' exists, then we see q is the least element, and thus we can look at $[q, z')_Y$ that contains q .
 - (d) If suppose both z and z' exist, then we can simply choose $(z, z')_Y$ containing q .
2. Suppose $z \in Y$ is so that $z < a$ but $b > y$ for every y .
 - (a) Suppose no z' exists in Y so that $q < z' < b$. Then q would be the largest element of Y , and due to convexity of Y , $a \in Y$. This would give us the basis element $(a, q]_Y$ that contains q and is fully contained in $(a, b) \cap Y$.
 - (b) If there exists z' in Y so that $q < z' < b$, simply choose the basis element $(a, z')_Y$ that contains q and is contained in $(a, b)_X \cap Y$.
3. Suppose $z' \in Y$ is so that $b < z'$ but for every $y \in Y$, $a < y$.
 - (a) Suppose no $z'' \in Y$ exists so that $a < z'' < q$, that means that q is the minimal element of Y , and that, from convexity, b is in Y . This means we can pick the basis element $[q, b)_Y$ that contains q and is contained in $(a, b)_X \cap Y$.
 - (b) Suppose there exists $z'' \in Y$ so that $a < z'' < q$. Then we can pick $(z'', b)_Y$ (since b is in Y) as our element that contains q and is contained in $(a, b)_X \cap Y$.

4. Suppose $h \in Y$ so that $b < h$ and $k \in Y$ so that $k < a$, we have $(a, b)_X$ completely inside Y , which gives us $(a, b)_Y$ as our choice of basis element.

If our basis element is of the kind $[a_0, b)_X \cap Y$ and q is in this set, then we have the following cases:

1. If Y has a least element and there exists $z_0 \in Y$ so that $q < z_0 < b$, then we can choose $[y_0, z_0)_Y$ as that which contains q and is contained in $[a_0, b)_X \cap Y$.
2. If Y has a least element and there exists no z in Y so that $q < z < b$, then either its possible that some z' exists in Y so that $b < z'$, but then (q, b) would then fully contained in Y so that we can use $[y_0, b)_Y$ as our basis element. If no such z' exists, then q becomes the largest element in Y because b is larger than any element of Y . That means we can choose $(y_0, q]_Y$ as our basis element that contains q and is contained in $[a_0, b)_X \cap Y$.
3. If Y has no least element, then there exists $p \in Y$ so that $a_0 < p < q$.
 - (a) If suppose there exists no $z \in Y$ so that $q < z < b$, then either its possible that some $z'' \in Y$ so that $b < z''$ which, from convexity condition, makes b fall into Y , and we can choose our interval to be $(p, b)_Y$ accordingly. So assuming there exists no $z'' \in Y$ so that $b < z''$, we see that b is larger than all elements of Y and no element of Y is larger than q . Therefore, q becomes the maximal element of Y , and thus, the set $(p, q]_Y$ can be choosen.
 - (b) If there exists $z \in Y$ so that $q < z < b$, then the interval $(p, z)_Y$ falls into $[a_0, b)_X \cap Y$ and also contains X .

If we take a basis element of the kind $(a, b_0]_X \cap Y$ and q is in this set, then we have the following cases:

1. If Y has a largest element y_0 and there exists z_0 in Y so that $a < z_0 < q$, then we can simply take $(z_0, y_0]_Y$ that contains q , and also is contained in $(a, b_0]_X \cap Y$.
2. If Y has a largest element y_0 but there is no z_0 in Y so that $a < z_0 < q$, then either there is a $t \in Y$ so that $t < a$ which makes a fall inside Y , thus allowing us to make use of $(a, y_0]_Y$. So we suppose a is smaller than all elements in Y . This means that q is indeed the smallest element of Y , so we can make use of $[q, y_0)_Y$ that contains q and falls inside $(a, b_0]_X \cap Y$.
3. If Y has no largest element, then there exists $r \in Y$ so that $q < r < b_0$.
 - (a) If suppose there exists no $z'' \in Y$ so that $a < z'' < q$, then either its possible that some t is there in Y that is smaller than a , thus making a a point of Y so that we can make use of $(a, r)_Y$ that contains q . So we suppose there is no such t , i.e. a is smaller than all elements of Y . This implies q is the minimal element of Y , so that we can make use of $[q, r)_Y$ that contains q and is itself contained in $(a, b_0]_X \cap Y$.

- (b) If there exists $z'' \in Y$ so that $a < z'' < q$, then we can choose $(z'', r)_Y$ as that interval that contains q such that it is itself contained in $(a, b_0]_X \cap Y$.

In all cases, for every basis element of \mathcal{T}_Y^S and for every x in said basis element, there exists a basis element of \mathcal{T}_Y^0 that contains x and is itself contained in \mathcal{T}_Y^0 . Thus $\mathcal{T}_Y^S \subseteq \mathcal{T}_Y^0$ which gives $\mathcal{T}_Y^S = \mathcal{T}_Y^0$.

Proof. (Alt) We know that the order topology is generated by the subbasis of all rays $(a, +\infty)$ and $(-\infty, a)$ for every $a \in X$. Suppose $Y \subseteq X$ is a convex set. Note that, the subspace topology of Y , \mathcal{T}_Y^S is generated by the subbasis $\{(a, +\infty) \cap Y, (-\infty, a) \cap Y : a \in X\}$. Let \mathcal{T}_Y^0 denote the order topology of Y , generated by the subbasis $\{(x, +\infty)_Y, (-\infty, x) : x \in Y\}$ where $(x, +\infty)_Y := \{z \in Y : x < z\}$.

First we prove that $\mathcal{T}_Y^0 \subseteq \mathcal{T}_Y^S$ (which does not require convexity)

Let $U = (x, \infty)_Y = \{z \in Y : x < z\} = (x, \infty)_X \cap Y$ for $x \in Y$. This is easily seen to be a subbasis element of \mathcal{T}_Y^S . Likewise, it is easy to see that $(-\infty, x)_Y$ for $x \in Y$ is also a subbasis element of \mathcal{T}_Y^S . So any arbitrary union of finite intersection of subbases of \mathcal{T}_Y^0 ends up being an arbitrary union of a finite intersection of subbasis elements of \mathcal{T}_Y^S , which makes any set open in \mathcal{T}_Y^0 open in \mathcal{T}_Y^S . Hence $\mathcal{T}_Y^0 \subseteq \mathcal{T}_Y^S$.

Now we prove that $\mathcal{T}_Y^S \subseteq \mathcal{T}_Y^0$ (here convexity is needed)

Let $U = (x, \infty)_X \cap Y$ for $x \in X$. If $x \in Y$, then this becomes a subbasis element. Suppose x is not in Y . If x is neither an upper bound, nor a lowerbound of Y , then it ends up falling into Y from convexity. Therefore, it either is a lowerbound, in which case U becomes Y itself, or it is an upper bound, in which case U becomes empty. Similarly, suppose $U = (-\infty, x)_X \cap Y$ for $x \in X$. If $x \in Y$, this is precisely a subbasis element. If not, it is either a lower / upper bound, making this again, a subbasis element of \mathcal{T}_Y^0 . Hence $\mathcal{T}_Y^S \subseteq \mathcal{T}_Y^0$ which gives $\mathcal{T}_Y^S = \mathcal{T}_Y^0$. □

Note that, in proving that $\mathcal{T}_Y^0 \subseteq \mathcal{T}_Y^S$, we do not make use of the convexity of Y . This motivates the following theorem:

Theorem 1.50. *Let X be an ordered set with the order topology \mathcal{T}_X^0 . Let Y be a subset of X . The subspace topology \mathcal{T}_Y^S on Y is always finer than (or equal to) the order topology \mathcal{T}_Y^0 on Y inherited from the order on X*

Proof. The first part of the previous theorem □

Lemma 1.51. *If there are two topologies on X such that $\mathcal{T} \subset \mathcal{T}'$ (strict inclusion), then we have that, for any subspace A , the subspace topologies obey $\mathcal{T}_A^S \subseteq \mathcal{T}_A'^S$, where equality can be achieved.*

Proof. We have that $\mathcal{T} \subseteq \mathcal{T}'$. $\mathcal{T}_A^S := \{U \cap A : U \in \mathcal{T}\}$ and $\mathcal{T}_A'^S := \{U' \cap A : U' \in \mathcal{T}'\}$. From here we see that every set open in \mathcal{T}_A^S is also open in $\mathcal{T}_A'^S$ which gives $\mathcal{T}_A^S \subseteq \mathcal{T}_A'^S$. But even if $\mathcal{T} \subsetneq \mathcal{T}'$, it's possible that $\mathcal{T}_A^S = \mathcal{T}_A'^S$. Consider the discrete topology $\mathcal{T}_D = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ on $\{0, 1\}$ and consider the Sierpiński topology $\mathcal{T}_S = \{\emptyset, \{0\}, \{0, 1\}\}$ on $\{0, 1\}$. Let $A = \{0\}$. Then the subspace topology of A from the Discrete topology is the same as the subspace topology on A from the Sierpiński topology.

□

Definition 1.52. (Equivalent bases ($\mathcal{B} \equiv \mathcal{B}'$)) Two bases \mathcal{B} and \mathcal{B}' are said to be equivalent if $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$

Theorem 1.53. $\mathcal{B} \equiv \mathcal{B}'$ if and only if both of the following conditions hold:

1. For all $x \in X$ and $U \in \mathcal{B}$, if $x \in U$, then there exists $U' \in \mathcal{B}'$ so that $x \in U' \subseteq U$
2. For all $y \in X$ and $U' \in \mathcal{B}'$, if $y \in U'$, then there exists $U \in \mathcal{B}$ so that $y \in U \subseteq U'$

Proof. \implies) If $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$, that means that, if we take a basis element of $\mathcal{T}(\mathcal{B}')$, say U' , it is an arbitrary union of elements of \mathcal{B} , and likewise any $U \in \mathcal{B}$ is an arbitrary union of elements of \mathcal{B}' , hence it's easy to see both the properties holding.

\impliedby) Suppose it is such that for every $x \in U \in \mathcal{B}$ we have U'_x so that $x \in U'_x \subseteq U$, then $U = \cup_x U'_x$ making U an arbitrary union of elements of \mathcal{B}' giving us $\mathcal{T}(\mathcal{B}) \subseteq \mathcal{T}(\mathcal{B}')$

Likewise, it is easy to see that if the 2nd condition holds, every element of \mathcal{B}' would be a union of elements of \mathcal{B} , making $\mathcal{T}(\mathcal{B}') \subseteq \mathcal{T}(\mathcal{B})$. Thus $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$ implying that $\mathcal{B} \equiv \mathcal{B}'$.

□

Definition 1.54. (Open Map) A map is $f : X \rightarrow Y$ (where X and Y are topological spaces) is said to be an **open map** if $f(U) \in \mathcal{T}_Y$ for all $U \in \mathcal{T}_X$

Lemma 1.55. (Projections are open) $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ are open maps.

Proof. Let $G = (U_\alpha \times V_\alpha)$ be open in $\mathcal{T}_{X \times Y}$. $\pi_1(G) = \pi_1(\cup_\alpha (U_\alpha \times V_\alpha)) = \cup_\alpha (\pi_1(U_\alpha \times V_\alpha)) = \cup_\alpha U_\alpha$ where U_α is open in \mathcal{T}_X (by definition). Likewise, π_2 is also an open map.

□

Theorem 1.56. *Let $Q = X = X'$ and $R = Y = Y'$, let \mathcal{T} and \mathcal{T}' be two topologies on Q (that, respectively, contain X and X') and let \mathcal{U} and \mathcal{U}' be two topologies on R that respectively contain Y and Y' . Then we have:*

1. *If $\mathcal{T} \subset \mathcal{T}'$ and $\mathcal{U} \subset \mathcal{U}'$, then $\mathcal{T}_{X \times Y} \subseteq \mathcal{T}_{X' \times Y'}$*
2. *If $\mathcal{T}_{X \times Y} \subseteq \mathcal{T}_{X' \times Y'}$, then $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$*

Proof. \implies) We are told that $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$, i.e, any open set in \mathcal{T} is open in \mathcal{T}' and any open set in \mathcal{U} is open in \mathcal{U}' . Consider $\mathcal{T}_{X \times Y} := \langle \{A \times B : A \in \mathcal{T}, B \in \mathcal{U}\} \rangle$ and $\mathcal{T}_{X' \times Y'} := \langle \{E \times F : E \in \mathcal{T}', F \in \mathcal{U}'\} \rangle$. Consider any basis element of $\mathcal{T}_{X \times Y}$, $A \times B$ for $A \in \mathcal{T}$ and $B \in \mathcal{U}$. From hypothesis, $A \in \mathcal{T}'$ and $B \in \mathcal{U}'$ implying that $A \times B$ is also a basis element of $\mathcal{T}_{X' \times Y'}$. This immediately tells us that $\mathcal{T}_{X \times Y} \subseteq \mathcal{T}_{X' \times Y'}$

\impliedby) Suppose that $\mathcal{T}_{X \times Y} \subseteq \mathcal{T}_{X' \times Y'}$. We want to show that $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$. Let $A \times B$ so that $A \in \mathcal{T}$ and $B \in \mathcal{U}$ be any basis element of $\mathcal{T}_{X \times Y}$. We have that $A \times B = \cup_{\alpha} (E_{\alpha} \times F_{\alpha})$ for $E_{\alpha} \in \mathcal{T}'$ and $F_{\alpha} \in \mathcal{U}'$. Let $(x, y) \in A \times B$. This means that $x \in A$ and $y \in B$ and that there is some α so that $(x, y) \in E_{\alpha} \times F_{\alpha}$, or $x \in E_{\alpha} \in \mathcal{T}'$ and $y \in F_{\alpha} \in \mathcal{U}'$. This gives us that, for any open set A in \mathcal{T} and for every point $x \in A$, there exists $E_x \in \mathcal{T}'$ so that $x \in E_x \subseteq A$, which makes A an arbitrary union of elements from \mathcal{T}' . Likewise, any open set $B \in \mathcal{U}$ is the arbitrary union of elements from \mathcal{U}' . That means that $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{U} \subseteq \mathcal{U}'$.

□

Example 1.57. (Random Example) \mathbb{R}^2 generated by the product topology of \mathbb{R} has a basis $\{E \times F : E, F \in \mathcal{T}_{\mathbb{R}}\}$ but since we know sets of the kind (a, b) for a, b rational forms a basis for \mathbb{R} , and that, if \mathcal{C} and \mathcal{D} forms a basis for X and Y , $\{C \times D : C \in \mathcal{C}, D \in \mathcal{D}\}$ forms a basis for $X \times Y$, meaning $\mathbb{R} \times \mathbb{R}$ has a basis of the kind $\{(a, b) \times (c, d) : a, b, c, d \in \mathbb{Q}\}$

Lemma 1.58. (Useful Lemma) *If $A \subseteq X$ and $B \subseteq Y$, then $A \times B \subseteq X \times Y$ is open (in product topology) if and only if A and B are respectively open in X and Y .*

Proof. \impliedby) Trivial

\implies) If $A \times B$ is open in $X \times Y$, it is of the form $\cup_q (E_q \times F_q)$ for $E_q \in \mathcal{T}_X$ and $F_q \in \mathcal{T}_Y$. Let $x \in A$ and $y \in B$ so that $x, y \in A \times B \subseteq X \times Y$. There exists q such that $x \in E_q$ and $y \in F_q$. This means that A and B each are an arbitrary union of open sets of X and Y respectively. We are done.

□

Lemma 1.59. *The dictionary order topology $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^0$ is the same as the product topology $\mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P$ where \mathbb{R}_d is the discrete topology on \mathbb{R} .*

Proof. $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^0 = \langle (x \cdot y, p \cdot q : x, y, p, q \in \mathbb{R}) \rangle$ and $\mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P = \langle A \times B : A \in \mathcal{P}(\mathbb{R}), B \in \mathcal{T}_{\mathbb{R}} \rangle$. Consider a basis element of $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^0$ of the form $(x \cdot y, p \cdot q)$, where, suppose $x < y$ ($x = y$ scenario is trivial). Let $a \cdot b \in (x \cdot y, p \cdot q)$. If $a = y$ or $a = x$, then pick the interval in the Y axis that contains $(a \cdot b)$. Then pick either $\{x\} \times \{\text{that interval}\}$ or $\{y\} \times \{\text{that interval}\}$ as the basis element of $\mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P$, which contains $a \cdot b$ and is itself contained in $(x \cdot y, p \cdot q)$. Suppose $x < a < y$. Then pick $x < \lambda_1 < a < \lambda_2 < y$ and $\delta_1 < b < \delta_2$, and consider $(\lambda_1, \lambda_2) \times (\delta_1, \delta_2)$. This is contained in $(x \cdot y, p \cdot q)$, and this contains $a \cdot b$. This means that $\mathcal{T}_{\mathbb{R} \times \mathbb{R}}^0 \subseteq \mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P$.

Now consider a basis element of $\mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P$ of the form $A \times B$ for any subset A of \mathbb{R} whatsoever, and B open in \mathbb{R} . Let $(x, y) \in A \times B$, with B being an arbitrary union of intervals. That means $y \in (u, v)$ for some interval. Consider $(x \cdot u, x \cdot v)$. It is obvious that $x \cdot y$ is in this set (in lexicographic order) and this set is in itself contained in $A \times B$. Hence $\mathcal{T}_{\mathbb{R}_d \times \mathbb{R}}^P \subseteq \mathcal{T}_{\mathbb{R} \times \mathbb{R}}^0$.

□

2 Closed Sets and Limit points

Definition 2.1. (Closed Set) A set $A \subset X$ of a topological space is *closed* if $X \setminus A$ is open

Lemma 2.2. For a topological space, the following hold:

1. \emptyset and X are both closed
2. Arbitrary intersection of closed sets are closed
3. Finite union of closed sets are closed

Proof. Since X and \emptyset are open, their complements, respectively \emptyset and X , are both closed.

Let $\{C_\alpha\}$ be a collection of closed sets, such that $X \setminus (C_\alpha)$ are open. $\cup_\alpha X \setminus (C_\alpha)$ which is $(\cap_\alpha C_\alpha)^C$ is open, which gives $\cap_\alpha C_\alpha$ to be closed.

Let $C_1, C_2 \dots C_k$ be closed. $(\cup_i C_i)^c = \cap_i C_i^C$ which is open. Hence $\cup_i C_i$ is closed.

□

Theorem 2.3. Let $Y \subseteq X$ where X is some topological space, and Y has the subspace topology. Then, $A \subseteq Y$ is closed in Y if and only if A is the intersection of a closed set of X with Y .

Proof. \implies) Let A be closed in Y . That means $S = Y \setminus A$ is open in Y . $S = Y \cap G$ for some open set G open in X . $Y \setminus S = Y \cap (S^c) = Y \cap (Y \cap G)^c$ which gives $S = Y \cap (Y^c \cup G^c) = Y \cap G^c$. This gives $A = Y \cap G^c$, whence we are done.

\impliedby) Suppose $A = Y \cap F$ for some closed set F , closed in X . $F = G^c$ for some open set G in X . $Y \setminus A = Y \cap A^c = Y \cap (Y \cap G^c)^c = Y \cap (Y^c \cup G)$ which gives $Y \setminus A = Y \cap G$, which makes $Y \setminus A$ open in Y , making A closed in Y .

□

2.1 Closures and interiors

Definition 2.4. (Closure of a set A) The closure of a set A in a topological space X is defined as the intersection of all closed sets that contain A . This is denoted \overline{A} or $cls_X(A)$ (subscript telling with respect to which space the closure is taken) and is obviously closed.

Definition 2.5. (Interior of a set A) The interior of a set A is defined as the union of all the open sets contained in A . It is denoted A° or $int_X(A)$.

Lemma 2.6. A is closed if and only if $A = \overline{A}$ and A is open if and only if $A^\circ = A$.

Proof. 1) \implies) A being closed implies it is the smallest closed set that contains itself. Hence $A = \overline{A}$.

\impliedby) If $A = \overline{A}$, obviously A is closed.

2) \implies) If A is open, it contains itself and is larger than any open set that is contained in A . Therefore, $A = A^\circ$.

\impliedby) If $A = A^\circ$, then A is obviously open.

□

Some random stuff on relative closures and interiors:

Theorem 2.7. Let $Y \subseteq X$ and $A \subseteq Y \subseteq X$. Then we have the following formulas:

1. $cls_Y(A) = Y \cap cls_X(A)$
2. $int_Y(A) = int_X(A \cup Y^c) \cap Y$

Proof. 1) $cls_Y(A) = F \cap Y$ for some closed set F closed in X , this much is obvious. It is also clear that F must contain A , which implies $cls_X(A) \subseteq F$, implying that $cls_X(A) \cap Y \subseteq cls_Y(A)$. Also note that, $cls_Y(A) \subseteq cls_X(A) \cap Y$ since $cls_Y(A)$ is supposed to be the smallest closed set in Y that contains A , and $cls_X(A) \cap Y$ is a closed set that contains A . Therefore, $cls_X(A) \cap Y = cls_Y(A)$.

2) $int_Y(A) = \cup_{G \text{ open in } X, G \cap Y \subseteq A} (G \cap Y) = \cup_{G \text{ open in } X, G \subseteq A \cup Y^c} (G \cap Y)$ which gives $int_X(A \cup Y^c) \cap Y = int_Y(A)$.

□

The situation for metric spaces: We define closure for metric spaces as

$$\overline{A} := \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset\}$$

This is equivalent to the usual limit point definition, which is, in turn, equivalent to the smallest closed set definition (in a metric space). Taking motivation from that, we give the following alternate definition for closure:

Definition 2.8. (Closure of a set A)

$$\overline{A} := \{x \in X : \forall U \in \mathcal{T}_X \text{ such that } x \in U, U \cap A \neq \emptyset\}$$

basically states that the closure is the collection of points of X such that, for every neighbourhood U of said point, U intersects A .

Theorem 2.9. *The two definitions are equivalent, i.e., the set*

$$B := \{x \in X : \forall nbd(x), nbd(x) \cap A \neq \emptyset\}$$

and the set

$$C := \cap_{F \text{ closed in } X: A \subseteq F} F$$

are the same.

Proof. First, suppose $x \notin \overline{A}$ which means $x \in (\overline{A})^c$ which is an open set that does not contain A , i.e., has no intersection with A . So for this x , there exists a neighbourhood (open) $(\overline{A})^c$ that does not intersect with A , i.e., $x \notin B$.

Now suppose $x \notin B$, i.e., there exists a neighbourhood $nbd(x)$ that has no intersection with A , i.e., $nbd(x) \subseteq A^c$. So $A \subseteq nbd(x)^c$, which means $\overline{A} \subseteq nbd(x)^c$, or $nbd(x) \subseteq (\overline{A})^c$ which implies x is not inside \overline{A} .

□

Lemma 2.10. *If suppose topology on X is given by a basis \mathcal{B} . Then $x \in \overline{A}$ if and only if for every basis element B containing x , $B \cap A \neq \emptyset$*

Proof. If $x \in \overline{A}$, we know that for every neighbourhood of x , it intersects with A . Since every basis element of x is a neighbourhood, it is true for this also, i.e, every basis element containing x intersects with A . Suppose, on the converse, that every basis element containing x intersects with A . Let $nb(x)$ be a neighbourhood of x . By definition, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq nb(x)$, which means that $nb(x) \cap A \neq \emptyset$. We are done.

□

Theorem 2.11.

Proof.

□

2.2 Limit Points

Definition 2.12. (Limit Point) Let A be a subset of a topological space X . If $x \in X$, we say x is a *limit point* of A if $\forall nb(x), nb(x) \setminus \{x\} \cap A \neq \emptyset$. Said differently, x is a limit point of A if in every neighbourhood $nb(x)$, there exists a point of A that is not x .

Theorem 2.13. Let F' be the set of all limit points of F . Then $\overline{F} = F \cup F'$

Proof. Consider $F \cup F'$, i.e, the set of all points of F , as well as the set of all limit points of F . Note that, if $x \in F$, then $x \in \overline{F}$ since every neighbourhood of x obviously intersects F . Let x be a limit point, i.e, every neighbourhood of x intersects F . Yet again, we see, by definition, $x \in \overline{F}$. Hence, $F \cup F' \subseteq \overline{F}$. Let $x \in \overline{F}$. If x is not in F , that means that every neighbourhood $nb(x)$ intersects F , but more can be said, i.e, $nb(x) \setminus \{x\}$ intersects F , making x a limit point of F . Therefore, either an element of \overline{F} is a point of F or a limit point of F , implying that $\overline{F} = F \cup F'$

□

Theorem 2.14. (Cool Theorem)

$$(A^\circ)^C = \overline{(A^C)} \text{ and } (A^C)^\circ = \overline{A^C}$$

Proof. $A^\circ \subseteq A$ which gives $A^C \subseteq (A^\circ)^C$ which means $\overline{A^C} \subseteq (A^\circ)^C$.

We know $A^C \subseteq \overline{A^C}$ which gives $(\overline{A^C})^C \subseteq A$ which gives $(\overline{A^C})^C \subseteq A^\circ$ or equivalently $(A^\circ)^C \subseteq \overline{(A^C)}$

This implies

$$(A^\circ)^C = \overline{(A^C)}$$

For part (2), we can simply replace in (1), A with A^c , i.e, $((A^c)^\circ)^c = \overline{(A^c)}$, which means

$$(A^c)^\circ = \overline{A}^c$$

□

Definition 2.15. (Derived Set) Sometimes the set of all limit points of a set F , denoted F' , is called the derived set of F .

Theorem 2.16. *A set A is closed if and only if it contains all of its limit points*

Proof. \implies) If A is closed, then $\overline{A} = A \cup A' = A$, which means $A' \subseteq A$, or A contains all its limit points.

\impliedby) Suppose A contain all its limit points, then $A' \subseteq A$ implying that $\overline{A} = A \cup A' = A$.

□

2.3 Hausdorff Spaces

Definition 2.17. (Hausdorff Space) A topological space X is said to be *Hausdorff* if for each pair of distinct points x_1, x_2 , there exists neighbourhoods U_1 and U_2 of x_1 and x_2 respectively such that $U_1 \cap U_2 = \emptyset$.

Theorem 2.18. *Every finite set in a Hausdorff space is closed*

Proof. A set is closed if and only if it contains all of its limit points. Let $S = \{x_1, x_2, \dots, x_p\}$ be our given set. Say x is a limit point of S , which is not in S . That means that there exists pairs $(nbd(x_1), nbd_1(x))$, $(nbd(x_2), nbd_2(x))$, \dots and $(nbd(x_p), nbd_p(x))$ such that $nbd(x_i) \cap nbd_i(x) = \emptyset$ (Hausdorff property). This implies that $Q = \cap_{i=1}^p nbd_i(x)$ is a neighbourhood of x that intersects with none of $nbd(x_i)$, which means Q is a neighbourhood of x that does not intersect with S . This contradicts the assumption that x is a limit point of S that is not in S . This implies x is either in S or is not a limit point of S . Hence, S contains all of its limit points.

□

Example 2.19. Let $X = \{a, b, c\}$ and $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. This topology is non-Hausdorff, since, given a and b , two distinct points, we can't find U_1 and U_2 such that $a \in U_1$, $b \in U_2$ and $U_1 \cap U_2 = \emptyset$

If we consider the topology $\mathcal{T}_2 = \{\emptyset, \{a, b\}, \{a, b, c\}\}$ (again, a non-Hausdorff topology), we note that b is a limit point of the point set $\{a\}$, but b does not belong in $\{a\}$ (obviously).

We can, hence, have non-Hausdorff topologies so that finite point sets need not contain their limit points. Moreover, if we consider the indiscrete topology, every element is a limit point of every set.

Let X be the topology $\{\emptyset, \{a, b\}, \{b\}, \{b, c\}, \{a, b, c\}\}$. Consider the set $S = \{b, c\}$. a is a limit point of S since every neighbourhood of a contains an element of $\{b, c\}$ that is not a .

For metric spaces, finite point sets have no limit points in X at all. But this is a very strong condition, and we have seen the existence of finite point sets that do have a limit point.

But if our space is Hausdorff (actually, weaker, a T_1 space), finite point sets must necessarily have no limit points, as the next theorem (and the definition) outlines:

Definition 2.20. (T_1 axiom) A space X in which finite point sets are closed is called a T_1 space.

Example 2.21. (We can have T_1 spaces that are non-Hausdorff). Consider \mathbb{R} and the topology finite complement topology, i.e, a set is open if and only if its complement is empty, \mathbb{R} or finite. We note that, by definition, closed sets are the finite sets (and \emptyset and \mathbb{R}), which means that it is T_1 . Let x_1 and x_2 be any two points. Let $x_1 \in U_1$ and $x_2 \in U_2$. Since only finite points are not in U_1 and U_2 , they will necessarily be non disjoint. Hence, this space can never be Hausdorff. (In fact this space is called a *Hyperconnected space*)

Theorem 2.22. Let X be a T_1 space and A a subset of X . If A is a subset of X , then x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .

Proof. \implies) Suppose x is a limit point of A , i.e, for every neighbourhood $nb(x)$, we have $(nb(x) - \{x\}) \cap A \neq \emptyset$. Suppose also that $(nb(x)) \cap A$ is finite for some neighbourhood of x , enumerate it as $\{x_1, x_2, \dots, x_k\}$. Since every finite point set is closed, $nb(x) \setminus \{x_1, x_2, \dots, x_k\} = nb(x) \cap \{x_1, x_2, \dots, x_k\}^c$ is open, and contains x . But note that this neighbourhood does not intersect A at all, because we removed all the points that belonged in the intersection of $nb(x)$ and A . This contradicts the assumption that x is a limit point.

\impliedby) If for every $nb(x)$, $nb(x) \cap A$ is infinite, then x is trivially a limit point of A .

□

Definition 2.23. (T_1 axiom (Alt)) A space X is said to be T_1 if for every $x, y \in X$, there exists U_x and U_y such that $x \in U_x$, $y \in U_y$, but $x \notin U_y$ and $y \notin U_x$.

Theorem 2.24. The above two definitions are equivalent. i.e., T_1 can be defined either as the space where all finite point sets are closed, or that for any two points $x, y \in X$, there exists U_x and U_y such that $x \in U_x \setminus (U_y)$ and $y \in U_y \setminus (U_x)$

Proof. \implies) Let X be such that every finite point set is closed. Note that, in a T_1 space, a limit point must be such that every neighbourhood of the limit point must contain infinitely many points of the space. Consider two distinct points x, y in X . Look at $S = \{x, y\}$. This set is closed from T_1 , but also has no limit points since every limit point must intersect S infinite times. Therefore, x is not a limit point of S , which means that there exists U_x such that $x \in U_x$ but $y \notin U_x$. Likewise, there exists U_y such that $y \in U_y$ but $x \notin U_y$ (if there wont exist such neighbourhoods, then one of them would become a limit point, which is contradictory to one of the corollaries of T_1).

\impliedby) Suppose for every $x, y \in X$, there exists U_x and U_y such that $x \in U_x \setminus (U_y)$ and $y \in U_y \setminus (U_x)$. If $n = 1$, $\{x_1\}$ is a closed set since if any other point were to be taken, there would exist a neighbourhood of that point that x_1 does not belong to. Moreover, x_1 itself cannot be a limit point, since $nb(x) - \{x\} \cap S = \emptyset$. For $n = 2$, $\{x_1, x_2\}$ set is closed set in a similar fashion. Neither x_1 , nor x_2 can be limit points since there is a neighbourhood of one not containing the other (from hypothesis). Assume this to be true for $n = n$, i.e, every n sized set is closed. Consider $\{x_1, x_2 \cdots x_n, x_{n+1}\} = \{x_1, x_2 \cdots x_n\} \cup \{x_{n+1}\}$, the finite union of two closed sets, which is closed. Hence, all finite point sets are closed.

□

Definition 2.25. (Convergent sequence) A sequence $x_n \in X$ a topological space is said to converge to $x \in X$ if for every neighbourhood $nb(x)$, there exists $n_{nb(x)}$ such that $\forall n \geq n_{nb(x)}$, we have $x_n \in U$.

Theorem 2.26. *In a Hausdorff space, sequences converge to a unique point*

Proof. Let x and y be a limit of x_n . Consider U and V such that $x \in U$ $y \in V$ and $U \cap V = \emptyset$. There exists n_U such that $\forall n \geq n_U$, $x_n \in U$, which means that only finite points of $x_n \in V$, which is a contradiction.

□

Lemma 2.27. 1. *Order topology is always Hausdorff*

2. *Product topology of two Hausdorff spaces is Hausdorff*

3. *Subspace topology of a Hausdorff space is Hausdorff*

Proof. 1) Let $x, y \in X$. WLOG assume $x < y$. Consider $x \in (-\infty, y)$ and $y \in (x, \infty)$. If they have an intersection, it would be $(-\infty, y) \cap (x, \infty) := \{z \in X : x < z < y\} = (x, y)$ which is open. If this (x, y) is non empty, then pick that element $q \in (x, y)$, and consider $(-\infty, q)$ and (q, ∞) .

2) Consider $X \times Y$, where X and Y are Hausdorff topologies. Let $(x, y), (x', y') \in X \times Y$. There exists $A, A' \subseteq X$ such that $x \in A, x' \in A'$ and $A \cap A' = \emptyset$. Likewise for y and y' there exists a corresponding $B, B' \subseteq Y$. Hence, $(x, y) \in A \times B$ and $(x', y') \in A' \times B'$ such that $A \times B \cap A' \times B'$ is empty. These are neighbourhoods, hence, $X \times Y$ is Hausdorff.

3) Let $Y \subseteq X$ where X is Hausdorff. Consider $z, z' \in Y$. Since z and z' are also in X , $\exists U, U' \in \mathcal{T}_X$ such that $z \in U, z' \in U'$ and $U \cap U' = \emptyset$. Therefore, $z \in U \cap Y, z' \in U' \cap Y$ and $(U \cap Y) \cap (U' \cap Y) = \emptyset$. Hence subspace topology is Hausdorff.

□

Theorem 2.28. (For later purposes) If Y is dense in X and X is an ordered set with the order topology, then $\mathcal{T}_Y^S = \mathcal{T}_Y^0$

Proof. Say $Y \subseteq X$ is dense in X , which means $\overline{Y} = X$. Consider X with the order topology \mathcal{T}_X given by the basis elements $\{(a, b), [a_0, b), (a, b_0] : a, b \in X\}$

□

Lemma 2.29. Let X and Y be topological spaces. $A \subseteq X$ and $B \subseteq Y$. Then

$$cls(A \times B) = cls(A) \times cls(B)$$

Proof. $X \times Y - (cls(A) \times cls(B)) = \text{open set}$ (it comes to be the union of two open sets). Hence, $cls(A) \times cls(B)$ is closed and it contains $A \times B$, hence $cls(A \times B) \subseteq cls(A) \times cls(B)$.

□

3 Quotient Topology

Let X be a topological space. Let \sim be an equivalence relation on X . Define X/\sim to be the set of all equivalence classes on X , coming from \sim . Let $\pi : X \rightarrow X/\sim$ be given by $x \mapsto [x]_\sim$. Define a set $V \subseteq X/\sim$ to be open in X/\sim if and only if $\pi^{-1}(V)$ is open in X . This forms a unique topology on X/\sim called the quotient topology.

Lemma 3.1. The above prescription indeed forms a topology.

Proof. X/\sim and \emptyset are trivially in the topology. Let $\{V_\alpha : \alpha \in A\}$ be a collection of open sets in X/\sim . $\pi^{-1}(\cup_\alpha V_\alpha) = \cup_\alpha \pi^{-1}(V_\alpha)$ is open by definition. It is also, similarly, easy to see that a finite intersection of open sets in X/\sim is also open.

□

Definition 3.2. (Quotient Map) Let X and Y be topological spaces. $q : X \rightarrow Y$ is said to be a quotient map if $V \in \mathcal{T}_Y \iff q^{-1}(V) \in \mathcal{T}_X$.

Lemma 3.3. Let $f : X \rightarrow Y$ where X and Y are topological spaces. Say \sim is an equivalence relation on X . Suppose $f(x) = f(y) \iff x \sim y$. Then $\tilde{f} : X/\sim \rightarrow Y$ given by $[x] \mapsto f(x)$ is continuous, if and only if f is continuous. Basically:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \tilde{f} & \\ X/\sim & & \end{array}$$

Proof. \implies) Say $f : X \rightarrow Y$ is continuous. Look at $V \subseteq Y$ that is open in Y . Look at $\tilde{f}^{-1}(V) \subseteq X/\sim$. This is $\cup_{y \in V}(\tilde{f}^{-1}(y))$ which gives $\cup_{y \in V}(\{[x] : f(x) = y\}) = \cup_{y \in V}(\{f^{-1}(y)\})$ which is a set in X/\sim . To show that $\cup_{y \in V}(\{f^{-1}(y)\})$ is open in X/\sim , we need to show that $\pi^{-1}(\cup_{y \in V}(\{f^{-1}(y)\})) = \cup_{y \in Y} \pi^{-1}(\{f^{-1}(y)\}) = \cup_{y \in Y}(f^{-1}(y)) = f^{-1}(U)$ is open in X , which is easily seen to be true. Hence, \tilde{f} is continuous.

\impliedby) Say \tilde{f} is continuous. Look at $g = \tilde{f} \circ \pi : X \rightarrow Y$. $g(x) = \tilde{f}([x]) = f(x)$, which means $g \equiv f$. Since $g = f$ is the composition of two continuous functions, f is continuous.

□

Theorem 3.4. Let $q : X \rightarrow Y$ be a quotient map, then, there exists an equivalence relation \sim such that $X/\sim \cong Y$ under the quotient topology. The equivalence classes being the fibres of q . Basically this:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X/\sim \\ q \downarrow & \nearrow f(\cong) & \\ Y & & \end{array}$$

Proof. Define $f : X/\sim \rightarrow Y$ given by $f([x]) := q(x)$. This is obviously well defined since on equivalence classes, q is identical. Let $y \in Y$. Since q is surjective, $y = q(x)$, which means $y = f([x])$. Hence, f is surjective. Consider $f([x]) = q(x) = f([y]) = q(y)$. Since the equivalence classes are determined by $q(x)$, we have $x \sim y$, which means $[x] = [y]$. Hence, f is bijective.

Since q is a quotient map, it is continuous. This gives us that f is also continuous. We have to show that f is an open map. Consider U open in X/\sim , which means $\pi^{-1}(U)$ is open in X . We want to show that $f(U) \in Y$ is also open in Y . This amounts to showing that $q^{-1}(f(U))$ is open in X . $f(U) = \{f([x]) = q(x) : [x] \in U\}$.

Incoherent ramblings begin: $f(\cup_{\alpha} U_{\alpha}) = \{f(x) : \exists \alpha \text{ such that } x \in U_{\alpha}\}$. Note that $z \in \cup_{\alpha} f(U_{\alpha})$ means that $z = f(x)$ for some $x \in U_{\alpha}$. Moreover, if $z \in f(\cup_{\alpha} U_{\alpha})$, that means that $z = f(x)$ for some $x \in \text{some } U_{\alpha}$. These two sets are hence the same. **End of Incoherent rambling**

Look at $g : Y \rightarrow X/\sim$, given by $g(y = q(x) \in Y) = [x]$. This is well defined since if $y = q(x')$, by definition $[x] = [x']$. Consider $g \circ f : X/\sim \rightarrow X/\sim$. $g \circ f([x]) = g(q(x)) = [x]$. Hence, g is the inverse of f . Let V be open in X/\sim . Look at $g^{-1}(V) = \{z \in Y : g(z) = g(q(x)) = [x] \in V\}$. $\cup_{[x] \in V} g^{-1}(\{[x]\}) = g^{-1}(V)$. If we look at $g^{-1}(\{[x]\}) = \{y \in Y : g(y) = [x]\}$. Since $g(y) = [z]$ where $q(z) = y$, we have $\{y \in Y : g(y) = [x]\} = \{q(x)\}$. $g^{-1}(V) = \cup_{[x] \in V} g^{-1}(\{[x]\}) = \cup_{[x] \in V} \{q(x)\}$. We want to check if $q^{-1}(g^{-1}(V))$ is open, which amounts to checking $\cup_{[x] \in V} q^{-1}(\{q(x)\}) = \cup_{[x] \in V} ([x]) = \cup_{y \in V} q^{-1}(y)$ (since equivalence classes are the fibres). This gives us $q^{-1}(V)$ which is open since q is continuous. Hence, g is also continuous, therefore, f is a bijective continuous map, whose inverse is also continuous, aka, a homeomorphism. Therefore, $X/\sim \cong Y$.

□

Note that, in general, $\pi : X \rightarrow X/\sim$ with the quotient topology need not be open. It is of course continuous.