Institute of Actuaries of India

Subject CS2A – Risk Modelling and Survival Analysis (Paper A)

May 2023 Examination

INDICATIVE SOLUTION

Introduction
The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) Let Z_n= time between arrival of the n^{th} and $(n – 1)^{th}$ clients.
Then Z_n's are i.i.d. exponential random variables with mean 1 / λ i.e. $E[Z_n]=\frac{1}{\lambda}$
Let T_n = arrival time of the n^{th} passenger = $\sum_{i=1}^{n} z_n$

$$\boldsymbol{E[T_n] = E\left[\sum_{i=1}^{n} z_n\right] = \sum_{i=1}^{n} E[z_n] = \frac{n}{\lambda}}$$

The expected waiting time until the first chopper takes off is

$\boldsymbol{E[T_3]} = \frac{\mathbf{3}}{[\mathbf{1/15}]}$ = 45 minutes

[2]

ii) Let X(t) be the Poisson process with mean λ*t. Note that P(X(t)) = k **=($e^{-\lambda t}$*$(\lambda t)^k$)/k!**,
For k= 1, 2, 3….
We have
P = P[No helicopter takes off in the first 2 hours]
=P[At most 2 passengers in first 120 mins}]
=P[{X(t) <= over (0,120)}]
=P[{X(120) <= 2]
=P[X(120) = 0] + P[X(120) = 1] + P[X(120) = 2]
= $e^{-\frac{120}{15}} + \left(\frac{120}{15}\right)e^{-\frac{120}{15}} + \frac{1}{2}\left(\frac{120}{15}\right)^2 e^{-\left(\frac{120}{15}\right)}$
= 0.01375
= 1.375%

[2]

iii) In order to ensure that the operator flies at least 3 trips in next 3 hours before the weather conditions worsen, there should be at least 9 passengers.

(1)

Hence, the probability that at least 9 clients would arrive is
P = P[{At least 9 passengers arrive in 180 mins}]
= 1 – P[{At most 8 clients arrive in 180 mins }]
= 1 – P[{X(180) <= 8]
= 1 - $\sum_{k=0}^{k=8} P\ [X(180) = k]$
= 1 – P[X(180) = 0] – P[X(180) = 1] – P[X(180) = 2] – P[X(180) = 3] – P[X(180) = 4] – P[X(180) = 5] – P[X(180) = 6] – P[X(180) = 7] – P[X(180) = 8]
= 1 - $\dot{e}^{-\frac{180}{15}}\left[\sum_{0}^{8}\left(\frac{180}{15}\right)^k * \frac{1}{k!}\right]$

(1)

= 1 – 0.15503
= 84.5% (1)

[3]

[7 Marks]

Solution 2:

$M_x(t) = (1\text{-}t/0.2)^{(-15)} = (1\text{-}5t)^{(-15)}$
$M_N(t)= \{0.85/(1\text{-}0.15\exp(t))\}^{3000}$

[0.5]

MGF of aggregate claim:
$M_s(t)= M_N (\log M_x(t))$
= {0.85/(1-0.15exp(log $M_x(t)$))}^3000
= $\{0.85/(1\text{-}0.15\ M_x(t)\}^{3000}$
= {0.85/(1-0.15/(1-5t)$^{(15)}$)} ^ 3000

[0.5]

E(X) = α/ λ =15 /0.2 = 75
V(X) = α/ λ^2
= 15/0.04 =375 [0.5]

E(N) = kq/p = 3000*0.15/0.85 = 529.4118
V(N) = kq/p^2 = 3000*0.15/$(0.85)^2$ = 622.8374
[0.5]

Mean = E(S) = E(X)* E(N) = 75* 529.4118 = 39,705.89
[0.5]

Variance (S) = $\{E(X)\}^2$ * V(N)+ V(X) * E(N)
=75^2*622.8374+ 375 * 529.4118 = 37,01,990

Standard Deviation = $\sqrt{3701990}$ =1924.05 [0.5]

[3 Marks]

Solution 3:

The life expectancy at entry is $e_0 = \sum_{k=1}^{\infty} {}_kp_0$ [0.5]

After four years, the survival rate is 0.45, and
$S_0(4) = \int_4^{\infty} f(t)\,dt = \int_4^{\infty} \mu_e^{-\mu t}\,dt = e^{-4\mu}$
Therefore $e^{-4\mu} = 0.45$
μ = -0.25* ln 0.45 = 0.2
[1]

For k >= 4,
${}_kp_0 = \int_k^{\infty} \mu_e^{-\mu t}\,dt = e^{-k\mu} = 0.45^{(k/4)}$
[0.5]

So $e_0 = 0.85 + 0.60 + 0.55 + \sum_{k=4}^{\infty} 0.45^{k/4}$
[0.5]

$= 2 + \frac{0.45}{1-0.45^{(1/4)}}$ (From sum of a geometric series = a/(1-r))
[0.5]

= 2 + 2.4867
= 4.4867 years [1]

[4 Marks]

Solution 4:

i)

Start previous Day	*Start morning 1*	*2*	*3*	*4*
1	0.5	0	0	0.5
2	0.25	0.5	0	0.25
3	0.25	0.25	0.5	0
4	0	0.25	0.25	0.5

[2]

ii) If stationary distribution is $\pi = (\pi_1 \;\; \pi_2 \;\; \pi_3 \;\; \pi_4)$
Then π *A* = π where A is the matrix in (i)
[0.5]

$0.5\,\pi_1 + 0.25\,\pi_2 + 0.25\,\pi_3 = \pi_1$ (I) [0.5]

$0.5\,\pi_2 + 0.25\,\pi_3 + 0.25\,\pi_4 = \pi_2$ (II) [0.5]

$0.5\,\pi_3 + 0.25\,\pi_4 = \pi_3$ (III)
[0.5]

$0.5\ \pi_1 + 0.25\ \pi_2 + 0.5\ \pi_4 = \pi_4$ (IV)

[0.5]

From (III), $\pi_3 = 0.5\ \pi_4$

[0.5]

From (II), $\pi_2 = 0.75\ \pi_4$

[0.5]

From (I), $\pi_1 = 0.625\ \pi_4$

[0.5]

$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 = (0.625 + 0.75 + 0.5 + 1)\ \pi_4$

[0.5]

Solving the above equation,
$\pi_1 = 0.21739$, $\pi_2 = 0.26087$, $\pi_3 = 0.17391$, $\pi_4 = 0.34783$ [0.5]

[Max 5]

iii) Probability of restocking is 0.5 if in π_1 and 0.25 if in π_2
So long term rate = 0.5 * 0.21739 + 0.25 * 0.26087 [1]

= 0.108695 + 0.06522 = 0.17391 per trading day [1]

[2]

iv) Probability of losing a sale is 0.25 if in π_1 [1]

So expected long term lost sales per day = 0.25 * 0.21739 = 0.05435 [1]

[2]

v)

Start previous Day	*Start morning* 2	3	4
2	0.5	0	0.5
3	0.25	0.5	0.25
4	0.25	0.25	0.5

[1]

Let the stationary distribution be expressed as λ
Then $\lambda M = \lambda$ where M is the matrix above
$\lambda_2 = 0.5\ \lambda_2 + 0.25\ \lambda_3 + 0.25\ \lambda_4$ (A) [0.5]

$\lambda_3 = 0.5\ \lambda_3 + 0.25\ \lambda_4$ (B) [0.5]

$\lambda_4 = 0.5\ \lambda_2 + 0.25\ \lambda_3 + 0.5\ \lambda_4$ (C) [0.5]

From (B), $\lambda_3 = 0.5\ \lambda_4$
From (A), $\lambda_2 = 0.75\ \lambda_4$
Solving the equation $\lambda_2 + \lambda_3 + \lambda_4 = 1$, we get
$\lambda_2 = 0.33333$ or 1/3 [0.5]

$\lambda_3 = 0.22222$ or 2/9 [0.5]

$\lambda_4 = 0.22222 / 0.5 = 0.44444$ or 4/9 [0.5]

As no more than two vaccines sell per day, there are no lost sales.

Probability of restocking 0.5 if in λ_2 and 0.25 in λ_3 = 0.5*0.33333 + 0.25*0.22222 = 0.22222 [1]

[5]

vi) Restocking at two or more vaccines would not result in fewer lost sales than restocking at 1, because the probability of selling more than 2 vaccines is zero.

[1]

It would, however, result in more restocking charges than restocking at 1.
Therefore, it must result in lower profits than restocking at 1 so is not optimal. [1]

[2]

vii) Costs if restock at zero vaccines: 0.17391C + 0.05435P [0.5]

Costs if restock at one vaccine: 0.22222C

[0.5]

So should change restocking approach if 0.22222C < 0.17391C + 0.05435P
i.e. C < 1.1255P

[1]

[2]

[20 Marks]

Solution 5:

i) Substituting u= 5y $^{(1/3)}$) in the given integral expression:

y =(u/5)^3
➔ dy =1/125* 3 u^2 *du [0.5]

The integral becomes:

$$3/125 \int_{5x^\wedge(\frac{1}{3})}^{\infty} \text{u}^\wedge 2 * \exp(-u)\, du$$

[0.5]

$$= 6/125 \int_{5x^\wedge(\frac{1}{3})}^{\infty} 1^3 * \text{u}^{3-1} * \exp\frac{(-u)}{(2!)}\, du$$

= 6/125* P[U> 5*x$^{(1/3)}$] , where U is Gamma (3,1) [0.5]

$= \frac{6}{125} * P\{Y > 10 * x^\wedge(\frac{1}{3})\}$, where Y is Chi (6)

[1.5 marks for final expression and the justification]

[4]

ii) Mean residual life of Weibull (5,1/3) distribution is :
e(x)= $\int_x^\infty \{1 - F(y)\}dy/$ (1-F(x))

[0.5]

= $\int_x^\infty$ $\exp(-5\text{y}^\wedge(1/3))\, dy/$ exp(-5x^(1/3))

[0.5]

= 6/125* P[Y> 10*x^(1/3)]/ exp(-5x^(1/3)), where Y is Chi (6)

[1 mark for correct expression and mention of Y's distribution]

[2]

iii) When x =1, we have e(x) = 6/125* P(chi square (6) >10)/ e^{-5}
= 6/125* (1-.875348)/ exp (-5)=.888

[0.5]

When x =8, we have e(x) = 6/125* P(chi square (6) >20)* e^{-10}
= 6/125* (1 - 0.997231)/ e^{-10} = 2.928

[0.5]

When x =27, we have e(x) = 6/125* P(chi square (6) >30)* e^{-15}
= 6/125* (1 - 0.999961)/ e^{-15} = 6.168

[0.5]

The mean residual life is an increasing function of x as the value of the function is increasing with increasing x.

[0.5]

[2]

[8 Marks]

Solution 6:

E[min(T,5000) = $\int_0^{5000} s_0(t)\,dt = \int_0^{4800} 1\,dt + \int_{4800}^{5000}\left(1 - \frac{t-4800}{1200}\right)dt$ [1]

= 4800 + 5t - $\frac{t^2}{2400}$

= 4800 + 183.333333

= 4983.33333 [1]

The number of kilowatt-hours used by 50 bulbs is 50 * 0.015 * 4983.333 = 3737.5 KWH

[1]

[3 Marks]

Solution 7:

X ~ Individual claim

Y ~ Commission

N ~ Poi(0.3n)

E[X] = 900/2 = 450

E[X^2] = 3X900^2/(2^2x1) + 450^2 = 810000

[1]

E[S] = 0.3nE[X+Y]

Var(S) = 0.3n[E(X^2)+2E(X) E(Y) + E(Y^2)]

[1]

E[Y] = 100/5 = 20

E[Y^2] = Γ(102) / { Γ(100) X 5^2}

= 10! / {90! X 25}

= 404

Let 'n' be the number of policies.

E[S] = 0.3n(450+20)

= 141n

Var(S) = 0.3(810000+2*450*20+404)

= 2428521.2n

[1]

S ~ N(141n, 248521.2n) *(approximation)*

Profit → P(S<=150n)

➔ P(S<150n) ~P[N(0,1)] < (150n-141n)/sqrt(248521.2n)

[1]

➔ P[N(0,1)] < 0.018sqrt(n)

➔ 0.018sqrt(n) >= 1.64485

➔ n > 8350 [2]

[6 Marks]

Solution 8:

i) When t=0, ψ(0) = $\lim_{t\to 0}(-\ln t)^{\alpha}$ =∞

Therefore, it is a strict generator function.

[1]

The inverse function is found by rearranging the equation:

x= ψ(t) = (-ln t)$^{\alpha}$

-ln t= x^(1/α)

t= exp(-x ^(1/ α))

[1]

[2]

ii) $C[u,v] = \psi^{-1}[\psi(u) + \psi(v)]$

$= \psi^{-1}[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}]$

$= \exp\{- ((-\ln u)^{\alpha} + (-\ln v)^{\alpha}) \wedge(\frac{1}{\alpha})\}$ for $\alpha \geq 1$

[1]

The coefficient of lower tail dependence is given by:

$\lambda_L = \lim_{u \to 0+} \frac{c[u,u]}{u}$

[0.5]

$= \lim_{u \to 0+} [\exp\{-((-\ln u)^{\alpha} + (-\ln u)^{\alpha}) \wedge(\frac{1}{\alpha})\}/u]$

[0.5]

$= \lim_{u \to 0+} [\exp\{-(2\wedge(\frac{1}{\alpha})\ (-\ln u)\}/u]$

[0.5]

$= \lim_{u \to 0+} [\exp\{\ (2\wedge(\frac{1}{\alpha})\ (\ln u)\}/u$

[0.5]

$= \lim_{u \to 0+} (u^{2^{\frac{1}{\alpha}}})\ /u$

$= \lim_{u \to 0+} u^{2^{\frac{1}{\alpha}}-1}$

$= 0$ [1]

[4]

[6 Marks]

<u>Solution 9:</u>

i) $X_t = \mu + \alpha (X_{t-1}\ \mu) + e_t$

$X_{t-1} = \mu + \alpha (X_{t-2} - \mu) + e_{t-1}$

Substituting for X_{t-1}, equation (1) becomes

$X_t = \mu + \alpha (\mu + \alpha (X_{t-2} - \mu) + e_{t-1} - \mu) + e_t$

$= \mu + \alpha^2 * (X_{t-2} - \mu) + \alpha e_{t-1} + e_t$

[1]

Repeating this process for one step

$X_t = \mu + \alpha^3 * (X_{t-3} - \mu) + \alpha^2 e_{t-2} + \alpha e_{t-1} + e_t$

[0.5]

In next two steps:

$X_t = \mu + \alpha^4 * (X_{t-4} - \mu) + + \alpha^3 e_{t-3} + \alpha^2 e_{t-2} + \alpha e_{t-1} + e_t$

After t steps the equation iteratively becomes:

$X_t = \mu + \alpha^t * (X_0 - \mu) + \sum_{j=0}^{t-1}(\alpha^j * e_{t-j})$

[1 mark for showing the iteration sufficiently]

[2.5]

ii) $X_t = \mu + \alpha^t * (X_0 - \mu) + \sum_{j=0}^{t-1}(\alpha^j * e_{t-j})$

Taking variance,

$Var(X_t) = var[\mu + \alpha^t * (X_0 - \mu) + \sum_{j=0}^{t-1}(\alpha^j * e_{t-j})$

[0.5]

$= \alpha^{2t} * var(X_0 - \mu) + \sum_{j=0}^{t-1}(\alpha^{2j} * var(e_{t-j}))$

[0.5]

$= \alpha^{2t} * var(X_0) + \sigma^2 * \sum_{j=0}^{t-1}(\alpha^{2j})$

[0.5]

Now using the formula $a + ar + ar^2 + \dots\dots ar^{(n-1)} = a(1- r^n) / (1- r)$ for the n terms of Geometric progression:

$var(X_t) = \alpha^{2t} * var(X_0) + \sigma^2 (1- \alpha^{2t}) / (1- \alpha^2)$

[1]

[2.5]

iii) The process can be written as

$-6 X_t + 13 X_{t-1} - 9 X_{t-2} + 2 X_{t-3} = e_t$

The characteristic equation is

$-6 + 13 z – 9 z^2 + 2 z^3 =0$

Since z =2 is a root, we can write this equation as

$-6 + 13 z – 9 z^2 + 2 z^3 = (z-2)(az^2+ bz +c)$, where a,b and c are constants.

[0.5]

One way to solve this equation is comparing the coefficients on both sides.

$2z^3 = az^3$

a=2

Comparing the constant term,

-6= -2c

c= 3

[0.5]

Comparing the coefficient of z,

13= c -2b

13=3- 2b

10= -2b

b= -5 [0.5]

So the equation can be written as

$-6 + 13 z – 9 z^2 + 2 z^3 = (z-2)(2z^2-5z +3)$

= (z-2)(2z-3) (z-1)

So the other roots of the equation are λ =1 and λ = 3/2

The process is not stationary since one root is strictly equal to 1 (for the process to be stationary all the roots should be strictly greater than 1). [0.5]

$-6 X_t + 13 X_{t-1} - 9 X_{t-2} + 2 X_{t-3} = e_t$

➔ $-6 X_t + 6 X_{t-1} + 7 X_{t-1} – 7 X_{t-2} - 2 X_{t-2} + 2 X_{t-3} = e_t$

➔ $-6 \nabla X_t + 7 \nabla X_{t-1} -2 \nabla X_{t-2} = e_t$

Characteristic equation then becomes:

➔ -6 + 7z` -2z`2 =0

The roots of this equation are: z` = 2, 3/2

So differencing the process eliminates the root of 1. The two remaining roots (i.e. 2 and 3/2) are strictly greater than 1 in magnitude.

So the differenced process is stationary. Hence the process is I(1).

[2]

[4]

iv) Autocovariance at lag 1, $\gamma 1= cov(X_t, X_{t-1})$

$= cov(\frac{7}{12}X_{t-1} – \frac{1}{12}X_{t-2} + e_t, X_{t-1})$

Since, $cov(et, X_{t-k}) = 0$ for all k>=1

i.e., $\gamma_1 = \frac{7}{12}\gamma_0 - \frac{1}{12}\gamma_1$

Rearranging we get,

$\frac{13}{12}\gamma_1 = \frac{7}{12}\gamma_0$

$\gamma_1 = \frac{7}{13}\gamma_0$

Dividing by γ_0

$\rho_1=\frac{7}{13}$

Autocovariance at lag 2, $\gamma_2= cov(X_t, X_{t-2})$

$= cov(\frac{7}{12}X_{t-1} - \frac{1}{12}X_{t-2} + e_t, X_{t-2})$

ie. $\gamma_2 = \frac{7}{12}\gamma_1 - \frac{1}{12}\gamma_0$

Dividing by γ_0

$\rho_2 = \frac{7}{12}\rho_1 - \frac{1}{12}$

$\rho 2 = \frac{7}{12} * \frac{7}{13} - \frac{1}{12} = \frac{49}{156} - \frac{1}{12} = \frac{36}{156} = \frac{3}{13}$

$\phi_1 = \rho_1 = \frac{7}{13}$

$\phi 2 = (\rho_2 - \rho_1^2)/(1 - \rho_1^2)$

= (3/13 –(7/13)^2) / (1- 49/169)

= (-10)/169 / (120/169)

= - 1/12

[4]

[13 Marks]

Solution 10:

i) The two-factor Lee-Carter model may be written:

$\ln m_{x,t} = a_x + b_x * k_t + \varepsilon_{x,t}$

where:

$m_{x,t}$ is the central mortality rate at age x in year t

[0.5]

a_x describes the general shape of mortality at age x or a_x is the mean of the time- averaged logarithms of the central mortality rate at age x. [0.5]

b_x measures the change in the rates in response to an underlying time trend in the level of mortality of k_t.

[0.5]

k_t reflects the effect of the time trend on mortality at time t, and

[0.5]

$\varepsilon_{x,t}$ are independently distributed normal random variables with means of zero and some variance to be estimated.

[0.5]

The usual constraints imposed in the Lee-Carter Model are that

$$\sum_x b_x = 1$$

and

$$\sum_t k_t = 0$$

[0.5]

[3]

ii)

a) Ignoring error term $\varepsilon_{x,t}$, the mortality rate at age x in projection year t is:

$m_{x,t} = \exp(a_x + b_x * k_t)$

[0.5]

The time trend factor k_t is assumed to decrease linearly from 2.75 at time 0 to -1.25 at time 40. Hence, can be expressed as

$$k_t = 2.75 - 0.1*t$$

[0.5]

Further, parameters a_x and b_x of the model for ages between 60 to 80 (inclusive) are expressed using linear functions as below.
$a_x = 0.105x - 10.95$ &
$b_x = -0.004x + 0.48$

X	a_x	b_x
60	-4.65	0.24
70	-3.6	0.2

[1]

From the above,
$m_{60,t} = \exp(a_{60} + b_{60}*k_t)$
$= \exp(-4.65 + 0.24 *(2.75 – 0.1t))$
$= \exp(-4.65 + 0.66 – 0.024t)$
$= \exp(-3.99 – 0.024t)$

[0.5]

$m_{60,0} = \exp(-3.99) = 0.0185$
$m_{60,10} = \exp(-3.99 – 0.24) = \exp(-4.23) = 0.014552$
$m_{60,20} = \exp(-3.99 – 0.48) = \exp(-4.47) = 0.011447$
$m_{60,30} = \exp(-3.99 – 0.72) = \exp(-4.71) = 0.009005$
$m_{60,40} = \exp(-3.99 – 0.96) = \exp(-4.95) = 0.007083$

[1]

Similarly,

$m_{70,t} = \exp(a_{70} + b_{70}*k_t)$
$= \exp(-3.6 + 0.2 *(2.75 – 0.1t))$
$= \exp(-3.6 + 0.55 – 0.02t)$
$= \exp(-3.05 – 0.02t)$

[0.5]

$m_{70,0} = \exp(-3.05) = 0.047359$
$m_{70,10} = \exp(-3.05 – 0.2) = \exp(-3.25) = 0.038774$
$m_{70,20} = \exp(-3.05 – 0.4) = \exp(-3.45) = 0.031746$
$m_{70,30} = \exp(-3.05 – 0.6) = \exp(-3.65) = 0.025991$
$m_{70,40} = \exp(-3.05 – 0.8) = \exp(-3.85) = 0.02128$

[1]
[5]

Alternatively,

Ignoring error term $\varepsilon_{x,t}$, the mortality rate at age x in projection year t is:
$m_{x,t} = \exp(a_x + b_x * k_t)$

[0.5]

The time trend factor k_t is assumed to decrease linearly from 2.75 at time 0 to -1.25 at time 40. Hence, can be expressed as

$$k_t = 2.75 – 0.1*t$$

[0.5]

Further, parameters a_x and b_x of the model for ages between 60 to 80 (inclusive) are expressed using linear functions as below.
$a_x = 0.105x - 10.95$ &
$b_x = -0.004x + 0.48$

$m_{x,t} = \exp(0.105x - 10.95 + (-0.004x + 0.48)* (2.75 – 0.1*t))$
$= \exp(0.105x - 10.95 -0.004*2.75x + 0.004*0.1*x*t + 0.48*2.75 -0.048*t)$
$= \exp(0.105x - 10.95 – 0.011x + 1.32 – 0.048*t +0.0004*x*t)$
$= \exp(0.094x – 9.63 - 0.048*t +0.0004*x*t)$

[2]

$m_{60,0} = \exp(-3.99) = 0.0185$
$m_{60,10} = \exp(-4.23) = 0.014552$
$m_{60,20} = \exp(-4.47) = 0.011447$

$m_{60,30}$ = exp (-4.71) = 0.009005
$m_{60,40}$ = exp (-4.95) = 0.007083

[1]

$m_{70,0}$ = exp (-3.05) = 0.047359
$m_{70,10}$ = exp (-3.25) = 0.038774
$m_{70,20}$ = exp (-3.45) = 0.031746
$m_{70,30}$ = exp (-3.65) = 0.025991
$m_{70,40}$ = exp (-3.85) = 0.02128

[1]
[5]

b) For every year that we project into the future, the mortality rate $m_{60,t}$ is multiplied by a factor of $e^{-0.024}$ (or 0.9763), *i.e.* it decreases by approximately 2.4% *p.a.*

[0.5]

For $m_{70,t}$, we multiply by a factor of $e^{-0.02}$ (or 0.9802) for each year that we project into the future and hence the values of $m_{70,t}$ decrease by approximately 2% *p.a.*

The percentage reduction in $m_{70,t}$ is smaller than that for $m_{60,t}$ since $b_{70} < b_{60}$.

[1]
[1.5]

iii)

- Future estimates of mortality at different ages are heavily dependent on the original estimates of the parameters a_x and b_x, which are assumed to remain constant into the future. These parameters are estimated from past data, and will incorporate any roughness contained in the data. In particular, they may be distorted by past period events which might affect different ages to different degrees. [0.5]
- If the estimated b_x values show variability from age to age, it is possible for the forecast age-specific mortality rates to 'cross over' (such that, for example, projected rates may increase with age at one duration, but decrease with age at the next). [0.5]
- There is a tendency for Lee-Carter forecasts to become increasingly rough over time. [0.5]
- The model assumes that the underlying rates of mortality change are constant over time across all ages, when there is empirical evidence that this is not so. [0.5]
- The Lee-Carter model does not include a cohort term, whereas there is evidence from some countries that certain cohorts exhibit higher mortality improvements than others. [0.5]
- Unless observed rates are used for the forecasting, it can produce 'jump-off' effects (*ie* an implausible jump between the most recent observed mortality rate and the forecast for the first future period). [0.5]

[Max 2.5]
[12 Marks]

Solution 11:

i) If 'X' represents claims from the particular policy of the insurer, we know that X is exponential with mean 1000.
So, parameter of X , λ= 1/1000.

Let L be the size of the excess. The insurer wants to set L so that P(X<L) = 0.25.

Using the given loss distribution,
➔ P(X<L)= 1- $e^{-\lambda L}$ = 1- $e^{-1/1000L}$

[0.5]

➔ 1- $e^{-1/1000L}$ = 0.25
$e^{-1/1000L}$ = 0.75
-1/1000L = ln(0.75)
= - 0.28768
L = 287.68

[1.5]
[2]

ii) Let X denotes the individual claim amount random variable. X follows exp(1/1000)
Let Y denotes the amount of claim paid by the insurer. Then:

Y = X ; if X <= 400
=400 ; if X > 400

[0.5]

So
$\int_0^{400} \mathrm{x}\, \mathrm{f(x)}\, dx + \int_{400}^{\infty} \mathrm{x}\, \mathrm{f(x)}\, dx$

$=\int_0^{400} \mathrm{x} * 1/1000 * \exp\left(-\frac{1}{1000}x\right) dx + \int_{400}^{\infty} 400 * 1/1000 * \exp\left(-\frac{1}{1000}x\right) dx$

[0.5]

= 1/1000 * ([x* exp (-1/1000x /-1/1000)] (from 0 to 400)

$-1/100 * \int_0^{400} \frac{\exp\left(-\frac{1}{1000}x\right)}{1} * -1000\ dx +$

$400 * 1000 * \int_{400}^{\infty} * \exp\left(-\frac{1}{1000}\right) dx$

=-400*exp(-400/1000)+ $\int_0^{400} * \exp\left(-\frac{x}{1000}\right) dx + \int_{400}^{\infty} 400 * 1/1000 * \exp\left(-\frac{1}{1000}x\right) dx$

= -400* exp(-0.4) – 1000* (exp(-0.4) -1) + 400/1000* -1000{ exp(-∞) – exp(-0.4)
=-400*exp(-0.4)+1000*(1-exp(-0.4))+400*(0 – exp(-0.4))
= 1000(1- exp(-0.4))
= 1000* (1- .67032) = 329.68

[1.5]
[Max 3]

iii) Let Z’ denotes the reinsurer’s claim payment random variable next year.
Y = X ; if 1.1X <= 400 ie. If X<= 400/1.1
=1.1 *400 ; if 1.1 X > 400 is if X> 400/1.1

[0.5]

E(Z’)=$\int_{\frac{400}{1.1}}^{\infty} (1.1\mathrm{x} - 400)\, \mathrm{f(x)}\, dx$

$=\int_{\frac{400}{1.1}}^{\infty} (1.1\mathrm{x} - 400) * \frac{1}{1000} * \exp\left(-\frac{\mathrm{x}}{1000}\right) dx$

[1]

$= \frac{1.1}{1000} \int_{\frac{400}{1.1}}^{\infty} \mathrm{x} * \exp\left(-\frac{\mathrm{x}}{1000}\right) dx$ +0.4*1.1/1000* $\int_{\frac{400}{1.1}}^{\infty} \exp\left(-\frac{\mathrm{x}}{1000}\right) dx$

[0.5]

$= \frac{1.1}{1000}$ * -1000* x * $\exp\left(-\frac{\mathrm{x}}{1000}\right)$ { from $\frac{1.1}{1000}$ to ∞} + 1.1 $\int_{\frac{400}{1.1}}^{\infty} \exp\left(-\frac{\mathrm{x}}{1000}\right) dx$ - 0.4 * $\frac{1.1}{1000}$* (-1000)* $\exp\left(-\frac{\mathrm{x}}{1000}\right)$ { from $\frac{1.1}{1000}$ to ∞}

= 400 * (exp($\frac{-0.4}{1.1}$) -1100* (0- exp ($\frac{-0.4}{1.1}$)) -400 * (0- exp($\frac{-0.4}{1.1}$)

[0.5]

= 400* exp(-0.36364) +1100 * exp(-0.36364) +400* (0 – exp(0.36364)

= 1100* exp(-0.36364)

= 1100* 0.695144 =764.6583

[0.5]

[3]

[8 Marks]

Solution 12:

i) Calculate the eigenvalues of the matrix:

$$\begin{pmatrix} 0.7 & 0.4 \\ 0.1 & 0.2 \end{pmatrix}$$

[0.5]

i.e. we need to determine the value of λ for which

$$det\begin{pmatrix} 0.7-\lambda & 0.4 \\ 0.1 & 0.2-\lambda \end{pmatrix} = 0$$

[0.5]

➔ (0.7- λ) * (0.2- λ)- 0.04 =0

➔ 0.10- 0.9 λ + λ^2 =0

λ = 0.77 or 0.129

[0.5]

Since both the eigenvalues are strictly less than 1 in magnitude, the process is stationary.

[0.5]

[2]

ii) The autocovariance at lag 1 is :

$\gamma_1 = cov(X_n, X_{n-1})$

$= cov(0.8 X_{n-1} - 0.4 X_{n-2} + e_n, X_{n-1})$

$= 0.8cov(X_{n-1}, X_{n-1}) - 0.4cov(X_{n-2}, X_{n-1}) + cov(e_n, X_{n-1})$

[0.5]

$\gamma_1 = 0.8\gamma_0 - 0.4\gamma_1$

ie. $1.4\gamma_1 = 0.8\gamma_0$

$\rho_1 = \gamma_1/\gamma_0 = 8/14$

[0.5]

The autocovariance at lag 2 is:

$\gamma_2 = cov(X_n, X_{n-2})$

$\gamma_2 = 0.8\gamma_1 - 0.4\gamma_0$

[0.5]

$\rho_2 = \gamma_2/\gamma_0$

[0.5]

So, $\rho_2 = 0.8\rho_1 - 0.4$

$\rho_2 =$ 8/10 * 8/14 – 4/10

= 4/5 *4/ 7- 2/5

= 16/35 -2/5

= 2/35

[0.5]

In general, for k>2,

$\gamma_k = cov(X_n, X_{n-k})$

$\gamma_k = 0.8\gamma_{k-1} - 0.4\gamma_{k-2}$

[0.5]

$\rho_k = \gamma_k/\gamma 0$

[0.5]

Therefore,

$\rho_k = 0.8\ \rho_{k-1} - 0.4\rho_{k-2}$

[0.5]

[4]

iii) To show that the processes X and Y are I(1):

From the first equation:

$Y_{t-1} = 1/\ 0.45 * [\ X_t - 0.55\ X_{t-1} - e_t^X\ ]$

[0.5]

Using this in the second equation,

$1/\ 0.45 * [\ X_{t+1} - 0.55\ X_t - e_{t+1}^X\ ] = 0.45\ X_{t-1} + 0.55/\ .45 * [\ X_t - 0.55\ X_{t-1} - e_t^X\ ] + e_t^Y$

[0.5]

This simplifies to:

$X_{t+1} = 1.1\ X_t - 0.1\ X_{t-1} + e_{t+1}^X - .55\ e_t^X + 0.45\ e_t^Y$

Equivalently,

$X_t = 1.1\ X_{t-1} - 0.1\ X_{t-2} + e_t^X - .55\ e_{-1}^X + 0.45\ e_{t-1}^Y$

[0.5]

The characteristic polynomial of the AR part of this equation is:

$1 - 1.1\ \lambda + 0.1\ \lambda^2 = 0$

➔ $\lambda = 10$ or 1

The roots of this equation are 10 and 1. Since one root is equal to 1, X is not stationary.

[0.5]

Differencing once will eliminate the root of 1. Since the only root is strictly greater than 1 in magnitude X is I (1).

[0.5]

The process of Y is may be expressed as a linear form of I(1) process X and e_t^Y, therefore, Y is also I (1).

[0.5]

Now to verify if: X-Y is stationary:

$X_t - Y_t = 0.1\ X_{t-1} - 0.1\ Y_{t-1} + e_t^X - e_t^Y$

Setting $W_t = X_t - Y_t$,

The process is AR (1) since the root of its characteristic equation is 10 and this is greater than 1 in magnitude.

[0.5]

Therefore, W_t is stationary, $X_t - Y_t$ is also thus stationary and X and Y are cointegrated, with cointegrating factor (1,-1).

[0.5]

[4]

[10 Marks]

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