Institute of Actuaries of India

Subject CS2A – Risk Modelling and Survival Analysis (Paper A)

September 2021 Examination

INDICATIVE SOLUTION

Introduction
The indicative solution has been written by the Examiners with the aim of helping candidates. The solutions given are only indicative. It is realized that there could be other points as valid answers and examiner have given credit for any alternative approach or interpretation which they consider to be reasonable.

Solution 1:

i) Answer: C (2)
ii) Answer: D (3)
iii) Answer: B (2)
iv) Answer: C (4)
v) Answer: A (3)
vi) Answer: D (4)
vii) Answer: C (2)

[20]

Q. 2)

i)
- The rate obtained using a heterogenous data may not represent the group risk adequately and hence may lead to over or underestimation of the insurance risk and hence the premium calculation
- There would be a risk of adverse selection by the group who are more exposed to risk and the product will look expensive to the less risky population
- Will be difficult to sufficiently understand the exact cause of any deviation of the actual experience against the assumption

(Max 2, 1 mark each) (2)

ii)
- We assume that mortality rates progress smoothly with age. Therefore a crude estimate at age x carries information about the rates at adjacent ages, and graduation allows us to use this fact to "improve" the estimate at age x by smoothing.
- This reduces the sampling errors at each age.
- It is desirable that financial quantities progress smoothly with age, as irregularities are hard to justify to clients.

(Max 2, 1 mark each) (2)

iii) The central exposed to risk at age x, is the waiting time in a multiple-state or Poisson model. (1)

If we let ai be the latest of the time of entry into observation, the date of attaining age label x, or the beginning of the investigation, and let bi be the earliest of death, date of losing age label x or the end of the investigation, then central exposed to risk is = Sum over all lives (bi-ai)

(1)

[2]

iv)
- record all dates of birth
- record all dates of entry into observation
- record all dates of exit from observation
- Reason for exist (cause of the cessation of observation) (2)

[8 Marks]

Solution 3:

i) The completed table is

Loss Table	Reinsurer's loss			Insurer's Loss		
Claims	1	2	3	1	2	3

0	0	-10	-20	-60	-50	-40
150	0	-10	17.5	90	100	72.5
200	0	40	30	140	100	110
250	0	90	42.5	190	100	147.5

(4)

ii) If no claim- arrangement 1 gives the best result
If claims is 150- arrangement 3 gives the best result
If claim is 200 or 250 - arrangement 2 gives the best result
So the strategy for Reinsurance depends the claims and not dominated by any one. (2)

iii) The maximum losses for insurer are

1. 190
2. 100
3. 147.5

So lowest loss is for arrangement 2 so the arrangement 2 is the minimax solution. (1)

[7 Marks]

Solution 4:

i) Let f (s) denote the marginal probability density for S and let f (s/λ) denote the conditional probability density for S/ λ. Then

E[E(S/ λ)]= $\sum_{i=1}^{3} p(\lambda_i) \int_0^{\infty} \mathrm{s}\mathrm{f}\left(\frac{\mathrm{s}}{\lambda}\right) \mathrm{ds}$

$=\int_0^{\infty} \sum_{i=1}^{3} p(\lambda_i)\ \mathrm{s}\mathrm{f}\left(\frac{\mathrm{s}}{\lambda}\right) \mathrm{ds}$

But $\sum_{i=1}^{3} p(\lambda_i)\, \mathrm{f}\left(\frac{\mathrm{s}}{\lambda}\right)$ = f(s) by definition

So

E[E(S/ λ)]= $\int_0^{\infty} sf(s)ds = E(S)$ (2)

ii) Using the results for compound distributions, we have:

E(S/λ)= E(N/λ)* E(X/λ)= E(N/λ)* E(X)=4λ

Var(S/λ)= E(N/λ)* Var(X)+ Var(N/λ)* E(X)^2

=λ*16+λ*4*4

=32λ (2)

iii) E(S)=E[E(S/λ)]= E(4λ)=4E(λ)=12 (2)

iv) Note that E(λ)= 3 λ = and

Var(λ)=0.2x2^2 + 0.6X3^2 +0.2x4^2 -9=0.4

Var(S)=Var[E(S/λ)] +E[Var(S/λ)]

=Var(4λ)+E(32λ)

=16xVar(λ) +32E(λ)

=16x0.4+32x3

=102.4 (2)

[8 Marks]

Solution 5:

i) An ARCH(p) model is

$$X_t = \mu + e_t \sqrt{\alpha_0 + \sum_{k=1}^{p} \alpha_k (X_{t-k} - \mu)^2}$$

Where e_t are independent (0,1)

2 Marks for writing the equation correctly and explaining the terms

For an asset with price Z_t above model is used to to model $Ln(Z_t/Z_{t-1})$

- It can be seen that a large departure in X_{t-k} from μ will result in X_t having a larger variance. This will then result in a large volatility for the asset price.
- This behaviour is observed in actual practice as the volatility of the price of a particular asset is much higher following a significant change in the price of the asset.
- Therefore ARCH model is more appropriate to model such asset class. (4)

Consider the time series
Y_t=0.1 +0.4Y_{t-1} +0.9e_{t-1} +e_t
Where e_t is white noise process with variance σ^2

ii) The model is ARIMA(1,0,1) if Y_t is stationary (1)

iii) a) The characteristic polynomial for the AR part is A(z) = 1 − 0.4z the root of which has absolute value greater than 1 so the process is stationary. (1)

b) The characteristic polynomial for the MA part is B(z) = 1 + 0.9z the root of which has absolute value greater than 1 so the process is invertible. (1)

iv) Since the process is stationary we know that E(Y_t) is equal to some constant μ independent of t.

Taking expectations on both sides of the equation defining Y_t gives

E(Y_t) = 0.1 + 0.4E(Y_{t-1})

μ = 0.1 +0.4μ

μ = 0.1/(1 0.4) = 0.1666666

Note that

Cov(Y_t , e_t) = Cov(0.1 + 0.4Y_{t-1} + 0.9e_{t-1} + e_t , e_t)
=0.4Cov(Y_{t-1}, e_t) + 0.9Cov(e_{t-1}, e_t) + Cov(e_t , e_t) = 0 + 0 + σ^2 = σ^2

Similarly

$Cov(Y_t, e_{t-1}) = 0 + 0.4Cov(Y_{t-1}, e_{t-1}) + 0.9Cov(e_{t-1}, e_{t-1}) + Cov(e_t, e_{t-1}) = 0.4\sigma^2 + 0.9\sigma^2 + 0 = 1.3\sigma^2$

So

$\gamma_0 = Cov(Y_t, Y_t) = Cov(Y_t, 0.1 + 0.4Y_{t-1} + 0.9e_{t-1} + e_t)$

$= 0.4\gamma_1 + 0.9 \times 1.3\sigma^2 + \sigma^2 = 0.4\gamma_1 + 2.17\sigma^2$ -----------------(1)

And

$\gamma_1 = Cov(Y_{t-1}, Y_t) = Cov(Y_{t-1}, 0.1 + 0.4Y_{t-1} + 0.9e_{t-1} + e_t)$

$= 0.4\gamma_0 + 0.9\sigma^2$--- (2)

Substituting for γ_1 in (1) gives

$\gamma_0 = 0.4 \times 0.4\gamma_0 + 0.4 \times 0.9\sigma^2 + 2.17\sigma^2 = 0.16\gamma_0 + 2.53\sigma^2$

$\gamma_0 = (2.53 / 0.84)\sigma_{t6}^2 = 3.011905\sigma^2$

Substituting into (2) gives

$\gamma_1 = 0.4 \times 3.011905\sigma^2 + 0.9\sigma^2 = 2.104762\sigma^2$

And in general

$\gamma_s = 0.4\gamma_s - 1$ for $s \geq 2$

So $\gamma_s = 0.4^{s-1} \times 2.104762\,\sigma^2$. (6)

v) We have $(1 - 0.4B)Y_t = 0.1 + 0.9e_{t-1} + e_t$

Hence $Y_t = (1 - 0.4B)^{-1}(0.1 + 0.9e_{t-1} + e_t)$

$$= \sum_{i=0}^{\infty} 0.4^i B^i \,(0.1 + 0.9e_{t-1} + e_t)$$

$$= \frac{0.1}{1 - 0.4} + 0.9\sum_{i=0}^{\infty} 0.4^i e_{t-i-1} + \sum_{i=0}^{\infty} 0.4^i e_{t-i}$$

$$= 0.16667 + e_t + 1.3\sum_{i=1}^{\infty} 0.4^{i-1} e_{t-i}$$

(4)

[17 Marks]

Solution 6:

i) Answer: C (4)

ii) Answer: C (2)

iii) Answer: C (2)

[8 Marks]

Solution 7:

i) Let the constant force of mortality be μ.
Then we have

${}_5p_{30} = \exp\{-\int_0^5 \mu dx\}$ = e ^ (-5μ)

It is given that ${}_5p_{30}$ = L(35) / L(30) = 98193.0 / 98600.4 = 0.995868
Thus, e ^ (-5μ) = 0.995868
⇨ -5 μ = ln(0.995868) = -0.00414
⇨ μ = 0.00083

Using the value of μ calculated, L(32) = L(30) * ${}_2p_{30}$ = L(30) * $\exp\{-\int_0^2 \mu dx$

= L(30) * exp (-2 * μ)
= 98600.4 * exp (-2 * 0.00083) = 98600.4 * 0.998345
= 98437.24 (4)

ii) Under Uniform distribution of death, the number of deaths are uniform

L(32) = L(30) – 2/5 * {L(30)- L(35)}
= 98600.4 – 2/5 * (98600.4 - 98193.0)
= 98437.44 (2)

iii) Force of mortality = 2* 0.00083 = 0.00166

3p35 = exp (-3 * 0.00166) = 0.995044 (2)

[8 Marks]

Solution 8:

j	t	n	d	w	lambda = d/n	1- lambda	Cumulative survival	
1	2	50	0	2	0 / 50	1 - (0 / 50)	1	1
2	4	48	0	1	0 / 48	1 - (0 / 48)	1	1
3	7	47	3	0	3 / 47	1 - (3 / 47)	44 / 47	0.93617
4	8	44	0	2	0 / 44	1 - (0 / 44)	44 / 47	0.93617
5	10	42	3	1	3 / 42	1 - (3 / 42)	39/42 * 44/47	0.869301
6	11	38	0	2	0 / 38	1 - (0 / 38)	39/42 * 44/47	0.869301
7	14	36	2	0	2 / 36	1 - (2 / 36)	34/36 * 39/42 * 44/47	0.821006
8	15	34	0	3	0 / 34	1 - (0 / 34)	34/36 * 39/42 * 44/47	0.821006
9	17	31	0	1	0 / 31	1 - (0 / 31)	34/36 * 39/42 * 44/47	0.821006
10	18	30	4	0	4 / 30	1 - (4 / 30)	26/30 * 34/36 * 39/42 * 44/47	0.711539

11	20	26	5	1	5 / 26	1 - (5 / 26)	21/26 * 26/30 * 34/36 * 39/42 * 44/47	0.574704
11	24	20	3	0	3 / 20	1 - (3 / 20)	17/20 * 21/26 * 26/30 * 34/36 * 39/42 * 44/47	0.488499

Range	S (x)
1 <= x <7	1
7 <= x <10	0.93617
10 <= x <14	0.869301
14 <= x <18	0.821006
18 <= x <20	0.711539
20 <= x <24	0.574704
x=24	0.488499

[8 Marks]

Solution 9:

i) The chi-squared test is a suitable overall test.

Test statistic = $\frac{\sum(Observed\ number\ of\ death - Expected\ number\ of\ deaths)^2}{Expected\ number\ of\ deaths}$

Age group	Central exposed to risk	Number of Deaths in the sample	Standard mortality rate used	Expected number of deaths	Observed - Expected	[(Observed - Expected) ^ 2] / Expected
20-24	56655	80	0.000937	53.09	26.91	13.64
25-29	61220	78	0.000934	57.18	20.82	7.58
30-34	64908	80	0.001042	67.63	12.37	2.26
35-39	62052	85	0.001358	84.27	0.73	0.01
40-44	58751	120	0.001969	115.68	4.32	0.16
45-49	54900	150	0.003168	173.92	-23.92	3.29
50-54	48679	295	0.00555	270.17	24.83	2.28
55-59	41699	366	0.008925	372.16	-6.16	0.10
						29.32

The test statistic ~ Chi square with m degrees of freedom
Here m is the number of age groups, which in this case is 8. We have not calculated any parameters. Hence, m remains 8

The critical value of the chi-squared distribution at the 5% level of significance
with 8 degrees of freedom is 15.51

Value of test statistic = 29.32
Given, Test statistic > Critical value, we reject the null hypothesis (8)

ii) Thus, standard mortality rates are not a good representation of the actual mortality experience and it is recommended that the company performs experience analysis to set its mortality assumptions. (2)

[10 Marks]

Solution 10:

If X1 and X2 are the aggregate claims in year 1 and 2 respectively.
To avoid ruin we need, X1 < 30 and (X1 + X2) < 60.

P (X1 < 30 and X1 + X2 < 60) = $\int_0^{30} f(x) * P\ (X2 < 60 - x)\ dx$

$\int_0^{30} \frac{1}{25} * \ exp\ \left(-\frac{1}{25}x\right) * \left[1 - exp\left\{-\frac{1}{25} * (60 - x)\right\}\right] dx$

$\int_0^{30} \frac{1}{25} * \ exp\ \left(-\frac{1}{25}x\right) - \frac{1}{25} * exp\left(-\frac{60}{25}\right) dx$
$-exp\ \left(-\frac{30}{25}\right) - \frac{30}{25} * \ exp\left(-\frac{60}{25}\right) + 1$

= 0.589944

[6 Marks]

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