INSTITUTE OF ACTUARIES OF INDIA

EXAMINATIONS

23rd May 2023

Subject CS2A – Risk Modelling and Survival Analysis (Paper A)

Time allowed: 3 Hours 15 Minutes (14.45 – 18.00 Hours)

Total Marks: 100

INSTRUCTIONS TO THE CANDIDATES

1. ***Please read the instructions inside the cover page of answer booklet and instructions to examinees sent along with hall ticket carefully and follow without exception.***
2. ***Mark allocations are shown in brackets.***
3. ***Attempt all questions beginning your answer to each question on a separate sheet.***
4. ***Please check if you have received complete Question Paper and no page is missing. If so, kindly get new set of Question Paper from the Invigilator.***

AT THE END OF THE EXAMINATION

Please return your answer book and this question paper to the supervisor separately. You are not allowed to carry the question paper in any form with you.

Q. 1) In a historic city in India, an industrialist wants to make money by offering the denizens a first-of-its-kind experience of viewing the historic places in the city from a helicopter. A brand-new 3-seater helicopter is employed for the ride. As it's a 3-seater helicopter, a group of 3 passengers are gathered at the boarding center, before they are allowed to board it. Hence, the helicopter would not fly until all the 3 seats are full. Passengers arrive at the center according to a Poisson process with the rate of $\lambda = 1/15$ per minute.

i) Calculate the expected waiting time until the first helicopter takes off. (2)

ii) What is the probability that helicopter does not take off in the first two hours, assuming that there are no other hindrances? (2)

iii) Today, it has been informed by the authorities that the weather conditions will not be conducive after three hours. However, the operator wants to fly at least 3 rides today before weather condition becomes poor. What is the probability that the operator completes at least 3 rides in three hours? (3)

[7]

Q. 2) The distribution of the number of claims for a motor portfolio is a type 2 negative binomial with parameters k=3,000 and p =0.85. The distribution of the claim size is Gamma with parameters $\alpha = 15$ and $\lambda = 0.2$.

Calculate the Moment Generating Function, Mean and Standard deviation of the aggregate claim distribution. **[3]**

Q. 3) In an actuarial student program, the number of years students stay in the program is distributed as follows:

1 year	0.85
2 years	0.60
3 years	0.55
4 years	0.45

The distribution of the amount of time in the program after 4 years has probability density function $f(t) = \mu * e^{(-\mu * t)}$, with μ selected to match the 0.45 probability of staying in the program 4 years.

Determine the average number of full years that students stay in the program. **[4]**

Q. 4) With the increasing number of G3M2 virus cases in the country, two vaccine manufacturers in the country have quickly developed two different single-vial vaccines. These vaccine manufacturers have their own distribution channel, which are well connected through-out the country. These vaccines need to be stored in a temperature-controlled case and an agency has been entrusted with the responsibility of administering the vaccines to the interested public. Considering that the vaccines have been developed within a very short time, there are very few takers for the vaccine worrying the potential side-effects. Due to limited availability, an agency of a particular vaccine manufacturer is permitted to store only four vaccines.

The number of vials administered during a day by an agency is a random variable with the following discrete distribution:

No. of vials potentially administered in a day	Probability
0	50%
1	25%
2	25%

The probability of administration of greater than one vial in a day remains 50%. If the agency has no vaccines in stock at the end of a day, the agency contacts its supplier to order four more vaccines. The vaccines are delivered the following morning, before the agency opens. The vaccine supplier makes a charge of C for the delivery.

i) Write down the transition matrix for the number of vaccines in stock when the agency opens in a morning, given the number of vaccines when the agency opened the previous day. (2)

ii) Calculate the stationary distribution for the number of vaccines in stock when the agency opens, using your transition matrix in part (i). (5)

iii) Calculate the expected long term average number of restocking orders placed by the agency per day. (2)

iv) Calculate the expected long-term number of orders lost per day. (2)

The agency is unhappy about losing these sales as there is a profit of P on each sale. It therefore considers changing its restocking approach to place an order before it has run out of vaccines. The charge for the delivery remains at C irrespective of how many vaccines are delivered.

v) Evaluate the expected number of restocking orders, and number of lost sales per trading day, if the agency decides to restock if there are fewer than two vaccines remaining in stock at the end of the day. (5)

vi) Explain why restocking when two or more vaccines remain in stock cannot optimize the agency's profits. (2)

The agency wishes to maximize the profit it makes on the vaccines.

vii) Derive a condition in terms of C and P under which the agency should change from only restocking where there are no vaccines in stock, to restocking when there are fewer than two vaccines in stock. (2)

[20]

Q. 5)

i) Show that: $\int_x^\infty \exp\left(-5y^{\frac{1}{3}}\right) dy$ = 6/125 P{Y > 10 $x^{1/3}$}

where Y is a chi-square random variable with six degrees of freedom. (4)

ii) Using (i), deduce an expression involving a chi-square probability for the mean residual life for Weibull(5, 1/3) distribution. (2)

iii) By calculating the values of mean residual life function when x=1 , x=8 and x=27, determine whether the mean residual life of the Weibull (5,1/3) distribution is an increasing or decreasing function of x. (2)

[8]

Q. 6) Light bulbs have the following distribution for the amount of time until burning out:

Time t in hours	**F(t)**
0 – 4800	0
4800 – 6000	(t – 4800)/1200
6000	1

Each bulb uses 0.015 kilowatt-hours of electricity per hour.

Calculate the expected number of kilowatt-hours used by 50 bulbs in their first 5000 hours. **[3]**

Q. 7) A single policy follows a Poi (0.30)
Individual claim amounts follow a Pareto ($\alpha = 3, \lambda = 900$)
Claim investigation and processing expense (independent of claim) follow Gamma ($\alpha = 100$, $\beta = 5$)
Premium = 150
Portfolio comprises of n policies (independent).
Find number of policies required to be profitable within 95% confidence interval. **[6]**

Q. 8)

i) For Gumbel copula, determine whether the generator function:

$$\psi(t) = (-\ln t)^{\alpha}$$

is a strict generator function for $\alpha >= 1$, and determine the inverse generator function for Gumbel copula. (2)

ii) Derive the coefficient of lower tail dependence for the Gumbel copula for $\alpha \geq 1$. (4)

[6]

Q. 9)

i) Let X_t be an autoregressive process of order 1 i.e. AR(1) given by:

$$X_t = \mu + \alpha (X_{t-1} - \mu) + e_t$$

where $\{e_t : t = 1, 2, \ldots.\}$ is white noise process.

Prove that:

$$X_t = \mu + \alpha^t * (X_0 - \mu) + \sum_{j=0}^{t-1} (\alpha^j * e_{t-j}) \quad (2.5)$$

ii) Prove that:

$$\text{var}(X_t) = \sigma^2 (1 - \alpha^{2t}) \;/\; (1 - \alpha^2) + \alpha^{2t} * \text{var}(X_0)$$

where σ^2 denotes the common variance of the white noise terms $\{e_t : t = 1, 2, \ldots.\}$, and 'var' denotes the variance. (2.5)

iii) $\lambda = 2$ is a root of the characteristic equation of the process:

$$6X_t - 13X_{t-1} + 9X_{t-2} - 2X_{t-3} + e_t$$

Calculate the other roots and classify the process as I(d). (4)

iv) Consider the stationary AR(2) process defined by the equation:

$$X_t = \frac{7}{12}X_{t-1} - \frac{1}{12}X_{t-2} + e_t$$

Determine the values of the ACFs ρ_1 and ρ_2 and the PACFs φ_1 and φ_2. (4)

[13]

Q. 10) For the senior citizen population in a country, a statistician has decided to use the two-factor Lee–Carter model to project future mortality rates and has fitted the model to a set of mortality data.

i) Write down the two-factor Lee–Carter model, clearly defining each of the terms you use. Also list out the constraints that are normally imposed, in order for the model to be uniquely specified. (3)

ii) The parameters a_x and b_x of the model for ages between 60 to 80 (inclusive) are expressed using linear functions as below.

$$a_x = 0.105x - 10.95$$
$$b_x = -0.004x + 0.48$$

Further, it is assumed that the factor k_t decreases linearly from 2.75 at time 0 to -1.25 at time 40.

a) Estimate the central mortality rate for ages 60 and 70 and at times 0, 10, 20, 30 and 40. (5)

b) From the results in ii (a) above, comment on the trend in central mortality rate with time for ages 60 and 70. (1.5)

iii) Describe the disadvantages of Lee-Carter Model. (2.5)

[12]

Q. 11)

i) An insurer believes that claims from a particular type of policy follow an exponential distribution with mean 1000. The insurer wishes to introduce a policy excess so that 25% of losses result in no claim to the insurer. Calculate the size of the excess. (2)

ii) The insurer effects an individual excess of loss reinsurance treaty with a retention limit of 400. Calculate the insurer's expected payment per claim. (3)

iii) Next year, the claim amounts on these policies are expected to increase by 10% but the reinsurance treaty will remain unchanged. Calculate the reinsurer's expected claim pay-out next year on claims in which it is involved. (3)

[8]

Q. 12)

i) Consider the multivariate time series represented by the following pair of equations:

$$X_t = 0.7\,X_{t-1} + 0.4\,Y_{t-1} + e_t^X$$
$$Y_t = 0.1\,X_{t-1} + 0.2\,Y_{t-1} + e_t^Y$$

where e_t^X and e_t^Y are independent white-noise processes. Using eigenvalues, state whether the process is stationary or not. (2)

ii) Consider univariate process:

$$X_n = 0.8\,X_{n-1} - 0.4\,X_{n-2} + e_n$$

where e_n is a white noise process.

Calculate values of autocorrelation function at lag 1, 2 and expression for autocorrelation function at lag k ($k>2$). (4)

iii) Two time series X and Y are defined by the equations:

$$X_t = 0.55\,X_{t-1} + 0.45\,Y_{t-1} + e_t^X$$
$$Y_t = 0.45\,X_{t-1} + 0.55\,Y_{t-1} + e_t^Y$$

where e_t^X and e_t^Y are independent white noise processes.

Show that X and Y are cointegrated, with cointegrating factor (1,-1). (4)

[10]

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