

Matrix assignment

January 8, 2024

Questions

1 Matrix

1. If \mathbf{A} is a square matrix satisfying $A'A = I$, write the value of $|A|$.
2. $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, show that $(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I}) = 0$.
3. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

4. If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 1 \\ 5 & 1 & 1 \end{pmatrix}$, find A^{-1} . Hence solve the system of equations

$$x + 3y + 4z = 8$$

$$2x + y + 2z = 5$$

$$\text{and } 5x + y + z = 7$$

5. Find the inverse of the following matrix, using elementary transformations :

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

6. Find $|AB|$, if $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 5 \\ 0 & 0 \end{pmatrix}$.
7. If $\mathbf{A} = \begin{pmatrix} p & 2 \\ 2 & p \end{pmatrix}$ and $|A^3| = 125$, then find the values of p .
8. Show that for the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, $A^3 - 6A^2 + 5A + 11I = 0$
Hence, find \mathbf{A}^{-1} .
9. Using matrix method, solve the following system of equations :
- $$\begin{aligned} 3x - 2y + 3z &= 8 \\ 2x + y - z &= 1 \\ 4x - 3y + 2z &= 4 \end{aligned}$$

2 Differentiation

10. If $y = x|x|$, find $\frac{dy}{dx}$ for $x < 0$.
11. Find the order and degree (if defined) of the differential equation
- $$\frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 = 2x^2 \log \left(\frac{d^2y}{dx^2} \right)$$
12. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants.
13. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.
14. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.
15. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that $\frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$ is
a constant independent of a and b .

16. Solve the differential equation:

$$x \frac{dy}{dx} = y - x \tan \left(\frac{y}{x} \right)$$

17. Solve the differential equation :

$$\frac{dy}{dx} = - \left[\frac{x + y \cos x}{1 + \sin x} \right]$$

18. Differentiate $e^{\sqrt{3x}}$, with respect to x .

19. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$.

20. If $(a + bx) e^{\frac{y}{x}} = x$, then prove that $x^3 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y \right)^2$.

21. Find the differential equation representing the family of curves $y = ae^{2x} + 5$, where a is an arbitrary constant.

22. If $y = \cos(\sqrt{3x})$, then find $\frac{dy}{dx}$.

23. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x .

24. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

3 integration

25. Find:

$$\int \sqrt{3 - 2x - x^2} dx$$

26. Find:

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx$$

27. Find:

$$\int \frac{x-3}{(x-1)^3} e^x dx$$

28. Find:

$$\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$$

29. prove that

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

and hence evaluate

$$\int_0^{\frac{\pi}{2}} \frac{x}{\sin x + \cos x} dx$$

30. Find $\int_1^3 (x^2 + 2 + e^{2x}) dx$ as the limit of sums.

31. Find:

$$\int \frac{x-5}{(x-3)^3} e^x dx$$

32. Find:

$$\int \frac{2 \cos x}{(1 - \sin x)(2 - \cos^2 x)} dx$$

4 Vectors

33. A line passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + \hat{j} - 2\hat{k}$. Find the equation of the line in cartesian form.

34. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between \vec{a} and \vec{b} .

35. Find the volume of a cuboid whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$, $-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.
36. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.
37. Find the vector and cartesian equations of the plane passing through the points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Write the equation of a plane passing through a point $(2, 3, 7)$ and parallel to the plane obtained above. Hence, find the distance between the two parallel planes.
38. Find the cartesian and vector equations of the plane passing through the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.
39. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear.
40. Find $\left| \vec{a} \times \vec{b} \right|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.
41. Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
42. Using method of integration, find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$ and $(3, 1)$.
43. Find the direction cosines of a line which makes equal angles with the coordinate axes.
44. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$ are perpendicular, find the value of λ . Hence find whether the lines are intersecting or not.
45. Find the equation of the line passing through $(2, -1, 2)$ and $(5, 3, 4)$ and of the plane passing through $(2, 0, 3)$, $(1, 1, 5)$ and $(3, 2, 4)$. Also, find their point of intersection.

5 Probability

46. If $P(\text{not } A) = 0.7$, $P(B) = 0.7$ and $P(B \mid A) = 0.5$, then find $P(A \mid B)$.
47. A coin is tossed 5 times. What is the probability of getting
- (i) 3 heads
 - (ii) at most 3 heads
48. Find the probability distribution of X , the number of heads in a simultaneous toss of two coins.
49. There are three coins. One is a two-headed coin, another is a biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin ?
50. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement) both of which are found to be red. Find the probability that the balls are drawn from the second bag.

6 Functions

51. Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.
52. Let $f : N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3, \text{ for some } x \in N\}$. Show that f is invertible. Find its inverse.
53. Examine whether the operation $*$ defined on \mathbf{R} , the set of all real numbers, by $a * b = \sqrt{a^2 + b^2}$ is a binary operation or not, and if it is a binary operation, find whether it is associative or not.

7 Intersection of Conics

54. Using method of integration, find the area of the region enclosed between two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.
55. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$.
56. Find the point on the curve $y^2 = 4x$, which is nearest to the point $(2, -8)$.

8 Optimization

57. A company produces two types of goods, A and B , that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can use at the most 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹40 and that of type B ₹50, find the number of units of each type that the company should produce to maximize profit. Formulate the above LPP and solve it graphically and also find the maximum profit.
58. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.
59. The volume of a cube is increasing at the rate of $8\text{cm}^3/\text{s}$. How fast is the surface area increasing when the length of its edge is 12cm ?

9 Algebra

60. Find the value of $\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$.
61. Solve for x :

$$\tan^{-1}(x + 1) + \tan^{-1}(x - 1) = \tan^{-1}\left(\frac{8}{31}\right)$$