The Signal-to-Noise Ratio for Photon Counting After Photometric Corrections

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ABSTRACT

Photon counting is a mode of processing astronomical observations of low-signal targets that have been observed using an electron-multiplying CCD (EMCCD). In photon counting, the EMCCD amplifies the signal, and a thresholding technique effectively selects for the signal electrons while drastically reducing relative noise sources. Photometric corrections have been developed which result in the extraction of a more accurate signal of electrons, and the Nancy Grace Roman Telescope will utilize a theoretical expression for the signal-to-noise ratio (SNR) given these corrections based on well-calibrated noise parameters to plan observations taken by its coronagraph instrument. I derive here analytic expressions for the SNR for the method of photon counting, before and after these photometric corrections have been applied.

Keywords: photon counting, signal-to-noise ratio, SNR, photometry, CCD, EMCCD, coronagraph, exoplanet

1. INTRODUCTION

Using a coronagraph to observe very low-signal exoplanets is very challenging. The photon signal from the planet must compete with noise sources such as clock-induced charge, dark current, and read noise, where read noise is the most significant source and typically drastically swamps the signal. The Nancy Grace Roman Space Telescope (Roman Telescope for short) will employ an electron-multiplying CCD (EMCCD) for its coronagraph instrument (CGI).[1–4] Utilizing an EMCCD for observation allows the photo-electron signal to be amplified so that the relative read noise is negligible, but other noise sources are also amplified. If the photo-electron signal is small, obtaining the desired SNR may not be possible when processing the EMCCD's output in the conventional, analog way. When the incident photo-electrons on the EMCCD is on the order of 0.1 electrons/pixel/frame or less,[5] photon counting is a method that affords a higher SNR compared to the analog method.

For a short exposure time for a series of frames taken by the EMCCD and a weak photon signal from an observed target, most pixels in the detector will have zero electrons, and occasionally there will be a few pixels with one electron each. Very rarely will a pixel have two or more electrons. So if a high multiplication (high gain) is applied, most pixels will read zero electrons while only a few will have some relatively high number. In photon counting, a threshold number of electrons is applied to each detector pixel. The threshold is usually chosen to be several (e.g., five) standard deviations above the expected read noise (in other words, fives times the read noise). Other sources of noise, like clock-induced charge and dark current, are typically orders of magnitude less than read noise, and they are also subtracted out if dark frames (frames taken in the absence of a target) are subtracted from the observation frames. For a given pixel, if the number of post-gain counts is above the threshold, that pixel is designated as having one electron, and if the number is below the threshold, the pixel is designated as having zero electrons. The frames are then co-added to produce an image that has minimal input from read noise and "extra noise" inherent in the electron multiplication process. A short exposure time is important to minimize the chances of being affected by cosmic rays and charge traps, and a high gain is important to maximize the chances that electrons are clearly below threshold (at 0) or clearly above.

For better accuracy, one should consider the inefficiency inherent in this thresholding. The conversion of a photon to a photo-electron in a detector pixel is a stochastic process, as well as the multiplication in the gain register of the EMCCD, so there will be pixels that are above threshold but actually have more than one

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pre-gain electron. This undercount is known as "coincidence loss".[5] Another possibility, due to the stochastic nature of the multiplication process, is that a pixel had a real electron but did not meet the threshold and thus is overlooked. This effect is known as "threshold loss".[5] A third effect is leakage due to detector effects. For example, because the read noise is Gaussian in nature, thresholding five standard deviations above the mean read noise will not completely cut out electrons due to read noise because of the tail of the distribution that is above the threshold. This is usually a small effect if the threshold is chosen to be high enough above the read noise mean,[5] and this leakage effect is ignored in this work. Taking into account the photometric corrections of coincidence loss and threshold loss gives a more accurate estimate of the mean number of electrons and rejects more of the false counts, and thus the error in our knowledge of the SNR is lower.

The Roman Telescope's CGI will use photon counting and employ photometric corrections so that the error budget requirements are met.[6] Methods of signal extraction from photon-counted observations have been studied in the literature,[7,8] and Bijan Nemati (2020) introduced an algorithm that gets the mean number of electrons for a given detector pixel to within the requisite 0.5% accuracy with respect to the true mean.[5] He does this by iteratively solving for the true mean number of electron counts per pixel from the number of measured counts per pixel by taking into account coincidence loss and threshold loss as applied to the measured number of counts. The algorithm requires a manipulation of the measured number of counts for a given pixel to better correct for false counts, and thus the standard deviation and SNR of the electron counts change as a result, and the error in our knowledge of the signal and SNR is made very small. It turns out that the SNR also increases as a result, which means that less time is required for exposing the CGI to a star for observation while still achieving a target SNR. For a given target star with known low-uncertainty input parameters such as photon flux and noise parameters of the CGI detector, the Roman Telescope's CGI software will predict the optimal observation parameters such as detector exposure time, number of frames of exposure, and gain so that the SNR is maximized given detector saturation constraints and constraints on the allowed ranges for exposure time, number of frames, and gain, which come from detector limitations, considerations of cosmic rays and charge transfer inefficiency, and a limit to the photo-electron flux on the detector as appropriate for photon counting (around 0.1 electrons/pixel/frame, as mentioned earlier). So it is important for observation plans to have an accurate and analytic expression for the theoretical SNR in terms of the noise parameters and photon flux. With the SNR expression in hand and a target SNR to reach, the software in flight would perform a numerical optimization to obtain these optimal observation parameters that maximize the SNR given the bounds on the maximum number of frames, exposure time, and gain. In the following sections, the SNR is derived and presented, before and after these corrections.

2. CORRECTIONS TO THE MEAN PHOTON RATE

The conversion of a photon to a photo-electron as it interacts with a detector pixel is a Poisson process, and so is the generation of dark current and clock-induced charge. If the mean expected number of electrons per pixel is λ, the number of electrons n actually produced per pixel is given by the Poisson distribution function,

$$P_p(n|\lambda) = \frac{\lambda^n e^{-\lambda}}{n!}. \tag{1}$$

The number of electrons per pixel is a Poisson variate, and the electron counts per pixel are then amplified as the detector frame of pixels goes through the gain register. The multiplication of the electron counts per pixel is a statistical process which follows the Erlang distribution,[9]

$$P_e(x|g,n) = \frac{x^{n-1}e^{-x/g}}{g^n(n-1)!}, \tag{2}$$

where x is the number of electrons after multiplication given that $n \geq 1$ electrons entered the gain register at a gain value of g. This distribution is accurate for large values of gain, and this is the case with photon counting. If n electrons in a given pixel enter the gain register, the gain is defined by the mean number of electrons after multiplication through the gain register, gn.

In photon counting, N frames are taken, and the mean number of electrons per pixel over all the frames (before multiplication) is λN. And so the mean number of electrons per pixel per frame is simply λ, and remember that only observations for which λ is small (typically 0.1 electrons/pixel/frame or less) are being considered. Now, after multiplication, the pixels that pass the threshold are each designated as one electron count, and those that do not pass the threshold are designated as 0 counts. The effect of thresholding makes the mean number $\langle c \rangle$ of 1-designated counts per pixel per frame related to the mean number λ of electrons per pixel per frame as follows:

$$\langle c \rangle = \lambda \epsilon_c \epsilon_{th}, \tag{3}$$

where ϵ_c corrects for coincidence loss and ϵ_{th} corrects for threshold loss. Nemati (2020) derives these factors in his paper,[5] and they are

$$\epsilon_c = \frac{1 - e^{-\lambda}}{\lambda} \tag{4}$$

and

$$\epsilon_{th} = e^{-\tau/g} \left(1 + \frac{\tau^2 \lambda^2 + 2g\tau\lambda(3+\lambda)}{2g^2(6 + 3\lambda + \lambda^2)} \right). \tag{5}$$

I will derive the probability distribution governing photon-counting and revisit this expression for the mean in the next section.

3. DERIVATION OF PROBABILITY DISTRIBUTION FOR PHOTON COUNTING

For a given pixel in a given frame, the probability of getting x electrons after multiplication given a mean number of pre-gain counts λ is given by

$$P^0(x|\lambda) = \begin{cases} C(\lambda) \cdot \sum_{i=1}^{\infty} P_e(x|g,i) P_p(i|\lambda), & i > 0 \\ C(\lambda) \cdot P_p(0|\lambda), & i = 0, \end{cases} \tag{6}$$

where $C(\lambda)$ is a normalization factor. There is little error in truncating this sum to 3 terms instead of an infinite number of terms since i, the number of pre-gain electrons, is rarely more than 1 for a given pixel for photon-counting conditions ($\lambda << 1$). The Erlang distribution is not applicable in the $i = 0$ case since there is no gain when there are 0 electrons. I call the distribution in which the sum is truncated to 3 terms $P_3^0(x|\lambda)$, with a normalization factor we will call $C_3(\lambda)$. The factor $C_3(\lambda)$ is obtained through normalization:

$$\begin{aligned} 1 &= C_3(\lambda) \int_0^{\infty} dx \; P_3^0(x|\lambda) \\ 1 &= C_3(\lambda) \left[e^{-\lambda} + \int_0^{\infty} dx \sum_{i=1}^{3} P_e(x|i) P_p(i|\lambda) \right] \\ 1 &= C_3(\lambda) \left[e^{-\lambda} + \int_0^{\infty} dx \; \lambda e^{-\lambda} \frac{e^{-x/g}}{g} \left(1 + \frac{\lambda x}{2g} + \frac{\lambda^2 x^2}{12 g^2} \right) \right] \\ C_3(\lambda) &= \frac{6 e^{\lambda}}{\lambda(\lambda(\lambda+3)+6)+6}. \end{aligned} \tag{7}$$

As discussed earlier, if the number of post-gain counts in a pixel is above a threshold τ (say, 5 times the read noise), that pixel is deemed as having 1 electron. The probability ϵ'_{th} of passing the threshold τ is given by

$$\begin{aligned} \epsilon'_{th} &= \int_{\tau}^{\infty} dx \; P_3^0(x|\lambda) = C_3(\lambda) \int_{\tau}^{\infty} dx \; \lambda e^{-\lambda} \frac{e^{-x/g}}{g} \left(1 + \frac{\lambda x}{2g} + \frac{\lambda^2 x^2}{12 g^2} \right) \\ &= \frac{\lambda e^{-\frac{\tau}{g}} \left(2g^2(\lambda(\lambda+3)+6) + 2g\lambda(\lambda+3)\tau + \lambda^2\tau^2 \right)}{2g^2(\lambda(\lambda(\lambda+3)+6)+6)}. \end{aligned} \tag{8}$$

Therefore, the probability P_T of having c electrons after thresholding is given by a binomial distribution:

$$P_T(c) = \begin{cases} 1 - \epsilon'_{th}, & c = 0 \\ \epsilon'_{th}, & c = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

This probability distribution is normalized, so the mean expected value is given by

$$\langle c\rangle = \sum_{0}^{1} c P_T(c) = \epsilon'_{th}, \tag{10}$$

and the standard deviation is given by

$$\sigma = \sqrt{\langle c^2\rangle - \langle c\rangle^2} = \sqrt{\epsilon'_{th}(1-\epsilon'_{th})}. \tag{11}$$

Compare Eq. (10) with the expression for the mean from Nemati's work[5] from Eq. (3), $\lambda\epsilon_c\epsilon_{th}$. For $\lambda << 1$, ϵ'_{th} to fourth order in λ is

$$\epsilon'_{th} \approx \lambda e^{-\frac{\tau}{g}} - \frac{\lambda^2\left(e^{-\frac{\tau}{g}}(g-\tau)\right)}{2g} + \frac{\lambda^3 e^{-\frac{\tau}{g}}\left(2g^2-4g\tau+\tau^2\right)}{12g^2} - \frac{\lambda^4\left(e^{-\frac{\tau}{g}}\left(g^2-g\tau+\tau^2\right)\right)}{12g^2}, \tag{12}$$

and, to fourth order in λ,

$$\lambda\epsilon_c\epsilon_{th} \approx \lambda e^{-\frac{\tau}{g}} - \frac{\lambda^2\left(e^{-\frac{\tau}{g}}(g-\tau)\right)}{2g} + \frac{\lambda^3 e^{-\frac{\tau}{g}}\left(2g^2-4g\tau+\tau^2\right)}{12g^2} - \frac{\lambda^4\left(e^{-\frac{\tau}{g}}\left(g^2-2g\tau+2\tau^2\right)\right)}{24g^2}. \tag{13}$$

These two expressions for the mean are very close, as there is exact agreement to third order in λ. The difference between these two expressions amounts to the fact that Nemati[5] applies the efficiency factors for coincidence loss ϵ_c and thresholding loss ϵ_{th} to the Poisson mean λ to get an estimated mean that works very well.

Eq. (9) is valid for a single frame. For two frames, the distribution is the convolution $P_T * P_T$:

$$(P_T * P_T)(c) = \sum_{k=0}^{\infty} P_T(k)P_T(c-k) = (1-\epsilon'_{th})P_T(c) + \epsilon'_{th}P_T(c-1). \tag{14}$$

For N frames, the distribution is the convolution of P_T with itself N times, which is

$$\sum_{k=0}^{N-1} \binom{N-1}{k} (1-\epsilon'_{th})^{N-1-k}\ \epsilon'^{k}_{th} P_T(c-k). \tag{15}$$

The distribution is already normalized by construction. Only the $k = c$ and $k = c-1$ terms survive in the sum, so one can write more simply the expression for the distribution for N frames:

$$P_T^{*N}(c) = (1-\epsilon'_{th})^{N-c}\ \epsilon'^{c}_{th}\left[\binom{N-1}{c} + \binom{N-1}{c-1}\right]. \tag{16}$$

The mean value for the distribution for N frames can be calculated and is, not surprisingly, N times the mean for one frame,

$$\langle c\rangle_N = N\epsilon'_{th}, \tag{17}$$

and the calculated standard deviation is $\sqrt{N}$ times the standard deviation for one frame:

$$\sigma_N = \sqrt{\langle c^2\rangle_N - \langle c\rangle_N^2} = \sqrt{N\epsilon'_{th}(1-\epsilon'_{th})}. \tag{18}$$

And the mean number of counts for N frames used in Nemati's work, $N\lambda\epsilon_c\epsilon_{th}$, agrees with the exact expression $N\epsilon'_{th}$ to third order in λ.

For N frames taken for an observation, all with the same statistics, note that if one averages the N frames, the righthand side of Eq. (17) is divided by N, and the mean number of counts is given by Eq. (10), and the corresponding standard deviation is given by the square root of the average of the sum of the variance for each frame,

$$\sqrt{\sum \sigma^2/N} = \sqrt{N\epsilon'_{th}(1-\epsilon'_{th})/N} = \sqrt{\epsilon'_{th}(1-\epsilon'_{th})}, \tag{19}$$

the same as Eq. (11).

4. SNR BEFORE PHOTOMETRIC CORRECTIONS

Consider an observation consisting of many frames of the same target. We can model the readout for a single pixel in a single frame, after gain has been applied, as a random variate q:

$$q = b + x + r, \tag{20}$$

where b is the voltage bias, r is the contribution from read noise and is a variate of the normal distribution, and x is a variate of the distribution $P_e(x|g, n)$ as given in Eq. (2). Here, n is a variate of the Poisson distribution $P_p(n|\lambda_{br})$ as given in Eq. (1), and n can be written as the sum of Poisson variates:

$$n = s + i_d t_{fr} + CIC. \tag{21}$$

The relevant photo-electron signal from the observation target that is in the pixel is s, which is expected to be $\langle s \rangle = \phi \eta t_{fr}$, where ϕ is the photon flux (photons/s), t_{fr} is the exposure time for the frame, and η is the pixel's quantum efficiency (electrons/photon). Along with this photo-electron signal, dark current i_d and clock-induced charge CIC contribute to the number of electrons in this pixel.

The mean value of electrons in this pixel is found in practice by averaging over all frames. The expected mean value, λ_{br}, is given by

$$\lambda_{br} = \langle n \rangle = \langle s \rangle + \langle i_d \rangle t_{fr} + \langle CIC \rangle, \tag{22}$$

where the brackets indicate a mean value. So the mean value of q can therefore be written as

$$\langle q \rangle = \langle b \rangle + \langle x \rangle + \langle r \rangle = \langle b \rangle + g\lambda_{br}, \tag{23}$$

where the mean for the Erlang distribution is used and the fact that the read noise mean is $\langle r \rangle = 0$. Other sources may contribute to the number of electrons, such as fixed-pattern noise and electron counts due to astrophysical background. Assuming voltage bias is subtracted out and fixed-pattern noise and astrophysical background are corrected for separately, the measured value of the mean number of electrons for this pixel is obtained by averaging the pixel's value over all the frames and dividing by the gain. Since this process of measuring the mean value for a pixel is is the same for any given pixel, Eq. (22) will be treated as the mean number of electrons per pixel per frame.

For the Roman Telescope's operations, the mean photon flux should be known very precisely for a given target star's throughput with negligible uncertainty. Before any measurement of a target is done, the quantum efficiency, dark current, and clock-induced charge will be calibrated, and the calibration involves averaging over very many frames, many more than would be needed for a target measurement, and so the error in the knowledge of these parameters is also negligible and does not enter our idealized expressions for noise or SNR that follow.

If N observation frames are taken (whether photon-counting or not), statistically there will be counts due to *noise* electrons. So to accurately capture the profile of electrons due to noise, some number $N_2 \geq N$ of dark frames (frames taken in the absence of a luminous target) must be taken and subtracted from our observation. For the dark frames, the mean number of electrons per pixel per frame is

$$\lambda_{dk} = \langle i_d \rangle t_{fr} + \langle CIC \rangle. \tag{24}$$

So if one averages the "bright" (target observation) frames and averages the "dark" frames and then subtracts the averaged darks from the averaged brights, one would have the mean *signal* per pixel per frame,

$$\lambda_{br} - \lambda_{dk} = \langle s \rangle. \tag{25}$$

The variance will be due to the sum of the variance of the brights and darks. Therefore, we define the ideal SNR per pixel per frame to aim for as

$$SNR = \frac{\lambda_{br} - \lambda_{dk}}{\sqrt{\sigma_{br}^2 + \sigma_{dk}^2}}. \tag{26}$$

In other words, the goal is to get a good estimate for $\lambda_{br} - \lambda_{dk}$ for the numerator and the noise from the corresponding variates whose mean provided the numerator.

The Roman software for the telescope will be given inputs of (good estimates of) photon flux, η, $\langle i_d \rangle$, and $\langle CIC \rangle$, and the software will calculate the optimal observation parameters (exposure time t_{fr}, number of exposure frames N, and gain g) that maximize the SNR given the constraints discussed in the first section. So in order for the software to output these optimal observation parameters, it must have an analytic theoretical expression for the SNR per pixel in terms of the noise parameters and photo-electron signal $\lambda_{br} - \lambda_{dk} = \langle s \rangle$.

For an observation, the goal is to most accurately determine the mean signal per pixel per frame, which is $\lambda_{br} - \lambda_{dk}$. If N observation frames are photon-counted (i.e., summed, subjected to the threshold, and averaged over the N bright frames), statistically there will be 1-designated counts due to noise electrons. So to counter this, $N_2 \geq N$ dark frames must be photon-counted and then subtracted from the observation. However, all that is available for a given pixel location is the number of 1-designated counts over the frames, which is not the same thing as the number of electron counts because of coincidence loss and threshold loss. However, one could naively treat the number of 1-designated counts per pixel divided by the respective number of frames as the number of electrons per pixel per frame. Let N_{br} be the total number of 1-designated counts over all the bright frames for a given pixel, and let N_{dk} be the total number of 1-designated counts over all the dark frames for the pixel. If $c_{br,obs}$ is an observed variate of Eq. (9) representing 1 or 0 for a pixel of a particular frame of the brights, then taking the average for that pixel over all bright frames in practice gives $\langle c_{br,obs} \rangle = N_{br}/N$, and, similarly for the darks, $\langle c_{dk,obs} \rangle = N_{dk}/N_2$. So the SNR for a single observation with N bright frames and N_2 dark frames is

$$SNR_{unc,obs} = \frac{\langle c_{br,obs} \rangle - \langle c_{dk,obs} \rangle}{\sqrt{\sigma(c_{br,obs})^2 + \sigma(c_{dk,obs})^2}}, \tag{27}$$

where the subscript *obs* stands for "observational" and *unc* stands for "uncorrected". (The "corrected" version will be discussed later.) Eq. (10) implies $\langle c_{br,obs} \rangle = \epsilon'_{th}(\lambda_{br})$ and $\langle c_{dk,obs} \rangle = \epsilon'_{th}(\lambda_{dk})$, and Eq. (11) implies $\sigma(c_{br,obs})^2 = \epsilon'_{th}(\lambda_{br})(1 - \epsilon'_{th}(\lambda_{br}))$ and $\sigma(c_{dk,obs})^2 = \epsilon'_{th}(\lambda_{dk})(1 - \epsilon'_{th}(\lambda_{dk}))$. So the corresponding theoretical expression, assuming knowledge of λ_{br} and λ_{dk}, is

$$SNR_{unc,th} = \frac{\epsilon'_{th}(\lambda_{br}) - \epsilon'_{th}(\lambda_{dk})}{\sqrt{\epsilon'_{th}(\lambda_{br})(1 - \epsilon'_{th}(\lambda_{br})) + \epsilon'_{th}(\lambda_{dk})(1 - \epsilon'_{th}(\lambda_{dk}))}}. \tag{28}$$

If one wanted to test Eq. (27) with simulated data to see if it aligns with theoretical expectations, one could compare the numerator of Eq. (28), the signal, to the numerator of Eq. (27) with its uncertainty. For uncertainty $\delta\mu$ of the mean μ, the standard deviation of the mean is used:

$$\delta\mu = \sigma/\sqrt{M}, \tag{29}$$

where M is the number of variates whose mean is μ. So in this case, the number of bright variates is N, and the number of dark variates is N_2, so the uncertainty of the mean $\mu_{unc,obs} \equiv \langle c_{br,obs} \rangle - \langle c_{dk,obs} \rangle$ is found by applying Eq. (29) separately to the brights and darks and adding in quadrature:

$$\delta\mu_{unc,obs} = \sqrt{\sigma(c_{br,obs})^2/N + \sigma(c_{dk,obs})^2/N_2}. \tag{30}$$

And the denominator of Eq. (28) would be compared to the denominator of Eq. (27) with its uncertainty. The fractional uncertainty of the standard deviation, $1/\sqrt{2(M-1)}$, is utilized for the uncertainty of the standard deviation, $\delta\sigma$, so that*

$$\delta\sigma = \frac{\sigma}{\sqrt{2(M-1)}}. \tag{31}$$

So in the photon-counting case we are considering, the uncertainty of the standard deviation is found by applying Eq. (31) separately to brights and darks and adding in quadrature:

$$\delta\sigma_{unc,obs} = \sqrt{\frac{\sigma(c_{br,obs})^2}{2(N-1)} + \frac{\sigma(c_{br,obs})^2}{2(N_2-1)}}. \tag{32}$$

*This formulation for the uncertainty in the standard deviation assumes an approximately normal distribution, and this approximation is valid if all possible values of variates (0 through M) falls within 3 standard deviations of the mean for the distribution, and this is true if $M > 9(1 - \epsilon'_{th})/\epsilon'_{th}$ and $M > 9\epsilon'_{th}/(1 - \epsilon'_{th})$. This implies that $M > 176$, and $N = 200$ frames and $N_2 = 300$ frames is used in the simulation discussed below.

This can be done for a single measurement of N bright frames and N_2 dark frames; the comparison can also be done over several trials of the observation, using the average of $\mu_{unc,obs}$, $\delta\mu_{unc,obs}$, $\sigma_{unc,obs} = \sqrt{\sigma(c_{br,obs})^2 + \sigma(c_{dk,obs})^2}$, and $\delta\sigma_{unc,obs}$. The latter is done in the demonstration below.

A simulated photon flux map consisting of 50x50 pixels each with a value of 1 photon/s was simulated and run through the Python module `emccd_detect`[†], which is a EMCCD detector simulator. It created 200 detector frames with a quantum efficiency for each pixel of 0.9 electrons/photon (or e^-/photon), and each frame had an exposure time of 0.05 s. Typical values for read noise (standard deviation of 100 e^-/pixel/frame), dark current (mean value of $8.33 * 10^{-4}$ e^-/s/pixel/frame), and CIC (mean value of 0.01 e^-/pixel/frame) are included, and λ_{br}, as given by Eq. (22), is $0.055e^-$. For darks, 300 frames with the same noise parameters were also simulated, and λ_{dk}, as given by Eq. (23), is $0.01e^-$. I then photon-count these frames (threshold each of the frames and then sum them, separately for brights and darks) using the Python module `PhotonCount`[‡] with a gain of 5000 and a threshold of $500e^-$. This process is done for 500 trials (500 sets of $N = 200$ brights and $N_2 = 300$ darks) for each pixel for the sake of statistics. Each pixel is under identical conditions and independent from the others, so effectively there are $500 * 50 * 50 = 1.25e6$ trials. The script to do this analysis is `noise_script.py`[§], publicly available in the `PhotonCount` module, and one can adjust the input parameters as desired.

I will demonstrate the accuracy of Eq. (28) expression, but first, I examine the probability distribution for N bright frames, Eq. (16) (and the probability distribution is also for dark frames). Fig. 1 was created by `noise_script.py` script, and it shows a histogram of N_{br}, the pixels' values for the summed bright frames, along with the probability distribution, Eq. (16). Since there are 500 trials for each of the 50x50 pixels, there is a total of 1.25e6 entries in the histogram. The histogram counts were normalized for comparison with the probability distribution. The χ^2 value for the fit of the data to the probability distribution is 1.69×10^{-5}, and the critical χ^2 value in this case is 41.3, which indicates the data and the probability distribution are not statistically distinct. The p value is exactly 1.0 to machine precision, which indicates that the null hypothesis should be accepted (i.e., the data follows the distribution very well). The darks were created in the same way, so they also follow the probability distribution well.

I now examine the uncorrected observational SNR per pixel for $N = 200$ bright frames and $N_2 = 300$ dark frames. For the same simulation parameters described above, Eq. (27) is computed for each trial for each of the 50x50 pixels. The uncorrected observational signal (using the numerator of Eq. (27) and Eq. (30)) is averaged over these trials and all pixels, and the result is $0.0395 \pm 0.0160e^-$. The uncorrected theoretical signal (numerator of Eq. (28)) is $0.0395e^-$ and is in agreement. The uncorrected observational noise (using the denominator of Eq. (27) and Eq. (32)) is $0.232 \pm 0.011e^-$, and the uncorrected theoretical noise (denominator of Eq. (28)) is $0.235e^-$ and agrees. Using these uncertainties, the inferred observational SNR range is $(0.0968, 0.251)$, and the average signal over the average noise is $0.0395/0232 = 0.170$. The theoretical SNR is 0.168 and is in agreement with this range.

I will now examine how to get an accurate estimate of the quantity of interest, $\lambda_{br} - \lambda_{dk}$ based on what is observationally available. I will then find the "corrected" SNR.

5. SNR AFTER PHOTOMETRIC CORRECTIONS

If N bright frames and N_2 dark frames are taken for a single observation, the mean number of 1-designated counts per pixel is given by Eq. (10), and, as stated earlier, the expression is equal to the analogous expression from Nemati's[5] work to third order in λ_{br}:

$$\langle c_{obs,br} \rangle = \epsilon'_{th}(\lambda_{br}) \approx \lambda_{br}\epsilon_c(\lambda_{br})\epsilon_{th}(\lambda_{br}), \tag{33}$$

and an analogous expression relating $\langle c_{obs,dk} \rangle$ and λ_{dk} can be written as well. The goal is to get a more accurate approximation to the desired SNR in Eq. (26). The observable in the previous equation is $\langle c_{obs,br} \rangle$, but the desired quantity, λ_{br}, is in terms of this mean of the observed counts and not in terms of the variate $c_{obs,br}$ itself.

[†]Publicly available here: https://github.com/wfirst-cgi/emccd_detect.

[‡]Publicly available here: https://github.com/wfirst-cgi/PhotonCount.

[§]https://github.com/wfirst-cgi/PhotonCount/blob/main/noise_script.py

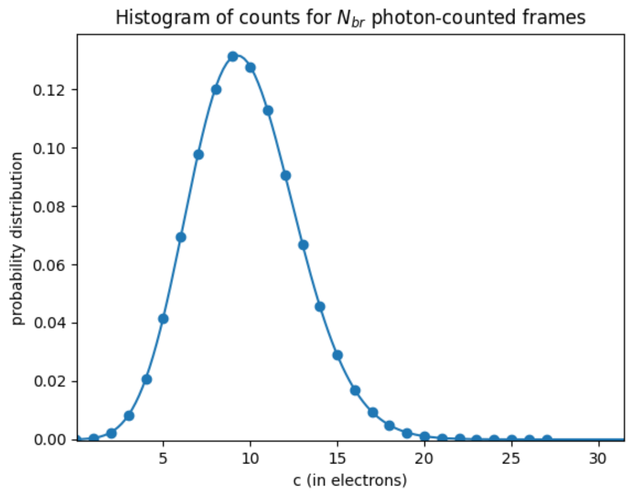

Figure 1. This plot shows a histogram of N_{br}, the pixels' electron count values (horizontal axis), generated with `PhotonCount` (represented by the dots), along with the theoretical probability distribution, Eq. (16) (represented by the curve). An input photon flux map was fed through `emccd_detect` to obtain simulated detector frames, 200 of them, and they were then photon-counted using `PhotonCount`. This process was done 500 times for each of the 50x50 pixels for the sake of robust statistics, and a χ^2 fit was performed. Noise sources are included. See the text for specific values for input parameters and details on the fit. The mean value is $N\epsilon'_{th}(\lambda_{br}) = 9.72e^-$, a little to the right of the peak. The χ^2 value for the fit is 1.69×10^{-5}, and the critical χ^2 value in this case is 41.3, which indicates the data and the probability distribution are not statistically distinct. The p value is exactly 1.0 to machine precision, which indicates that the null hypothesis should be accepted (i.e., the data follows the distribution well).

However, $\langle c_{obs,br}\rangle$ in practice is N_{br}/N, and N_{br} can be treated as a random variable of Eq. (16). So taking one observation to obtain N_{br} gives an estimate of λ_{br} if one solves Eq. (33) for it in terms of N_{br}. To obtain an associated standard deviation with which one can express the denominator of the SNR, multiple trials of N bright frames would be necessary. And in fact, with multiple trials, one can then observationally obtain $\langle N_{br}\rangle_N$, which is in fact theoretically a function of λ_{br} according to Eq. (17). And comparing Eqs. (17) and (10), one obtains

$$\langle N_{br}\rangle_N = N\langle c_{obs,br}\rangle = N\epsilon'_{th}(\lambda_{br}) \approx N\lambda_{br}\epsilon_c(\lambda_{br})\epsilon_{th}(\lambda_{br}). \tag{34}$$

Eq. (34) can be solved for λ_{br} to correct for the photometric effects of coincidence loss and threshold loss. The expression from Nemati's work will be used below for solving for λ_{br} and the SNR since the expression is used in `PhotonCount` and will be used by the Roman Telescope, and the result is accurate, as will be demonstrated. Also, one can do the analogous process for multiple trials of N_2 dark frames to extract λ_{dk} so that one can determine what is actually desired, which is $\lambda_{br} - \lambda_{dk}$. I will go through the analysis for just the brights in what follows, but the process is exactly the same for the darks.

In Eq. (34), ϵ_c and ϵ_{th} are functions of λ_{br}, and there is no exact analytic solution for λ_{br}, and we would like an analytic expression for expressing the corrected SNR analytically. An analytic expression is also useful for the speed of an algorithm that finds λ_{br} for every single pixel. One can get an approximate analytic expression for λ_{br} using Newton's method, which is an iterative method which gets closer to the solution with each iteration. Nemati came up with this solution and tested it,[5] and two iterations of Newton's method is sufficient for a very accurate solution.

Newton's method requires an initial guess as an input, and the initial guess used is the solution to Eq. (34) assuming the expression for $\lambda_{br}\epsilon_c\epsilon_{th}$ is truncated to first order in λ_{br}, as appears in Eq. (13). The result is

$$\langle N_{br}\rangle_N \approx N\lambda_{br}e^{\frac{-\tau}{g}} \;\rightarrow\; \lambda_{guess} = -\ln\left(1 - \frac{\langle N_{br}\rangle_N/N}{e^{-\tau/g}}\right). \tag{35}$$

Newton's method estimates the next iteration in terms of the previous:

$$\lambda_{n+1} = \lambda_n - \frac{f(\lambda_n)}{f'(\lambda_n)}, \tag{36}$$

where λ_n is the nth iteration for the root of the equation

$$f(\lambda_{br}) = N\lambda_{br}\epsilon_c(\lambda_{br})\epsilon_{th}(\lambda_{br}) - \langle N_{br}\rangle_N. \tag{37}$$

Plugging in Eqs. (4) and (5) into Eq. (37) results in $f(\lambda_{br})$, and $f'(\lambda_{br})$ is

$$\begin{aligned} f'(\lambda_{br}) = \frac{e^{-\tau/g-\lambda_{br}}N}{(2g^2(6+3\lambda_{br}+\lambda_{br}^2)^2}\ &(2g^2(6+3\lambda_{br}+\lambda_{br}^2)^2+\tau^2\lambda_{br}(-12+3\lambda_{br}+3\lambda_{br}^2+\lambda_{br}^3+3e^{\lambda_{br}}(4+\lambda_{br})) \\ &+2g\tau(-18+6\lambda_{br}+15\lambda_{br}^2+6\lambda_{br}^3+\lambda_{br}^4+6e^{\lambda_{br}}(3+2\lambda_{br}))). \end{aligned} \tag{38}$$

Starting with λ_{guess} and iterating twice results in a long analytic but unwieldy expression for λ_2. To get an manageable expression for the corrected mean and standard deviation, I had to expand λ_2 for $\langle N_{br}\rangle_N/N \ll 1$, and I found that truncating to third order was sufficient. This expansion is valid since $\lambda_{br} \ll 1$ for effective photon counting (as described previously), and from the rightmost part of Eq. (34), it follows that $\langle N_{br}\rangle_N/N < \lambda_{br}$ since the factors ϵ_c and ϵ_{th} must be less than 1 (see Eqs. (4) and (5)). To third order, λ_2 for the brights is

$$\lambda_{2,br} = \frac{e^{\tau/g}\langle N_{br}\rangle_N}{N} + \frac{e^{2\tau/g}\langle N_{br}\rangle_N^2(g-\tau)}{2gN^2} + \frac{e^{3\tau/g}\langle N_{br}\rangle_N^3(4g^2-8g\tau+5\tau^2)}{12g^2N^3}. \tag{39}$$

One can see that $\lambda_{2,br}$ is smaller than 1 even in the minimal case of $N = 1$ since $\tau/g < 1$ (typically $\sim 1/10$, as it is in the case of the aforementioned simulations) and $\langle N_{br}\rangle_N$ would only be 1 at most in that case. Of course, for accurate photon counting, N will certainly be bigger than 1.

Let $L_{2,br}$ be defined as the righthand side of Eq. (39) for just one trial, so that

$$L_{2,br} = \frac{e^{\tau/g}N_{br}}{N} + \frac{e^{2\tau/g}N_{br}^2(g-\tau)}{2gN^2} + \frac{e^{3\tau/g}N_{br}^3(4g^2-8g\tau+5\tau^2)}{12g^2N^3}. \tag{40}$$

After all, `PhotonCount` estimates $\lambda_{2,br}$ by solving for it from a single trial, so we define this estimate as $L_{2,br}$, and I will average this estimate for the signal for the SNR. So the "corrected" SNR per pixel per frame is

$$SNR_{corr} = \frac{\langle L_{2,br}\rangle_N - \langle L_{2,dk}\rangle_{N_2}}{\sqrt{\sigma_N(L_{2,br})^2 + \sigma_{N_2}(L_{2,dk})^2}} \tag{41}$$

where *corr* stands for "corrected". This equation will stand for both the observational corrected and theoretical corrected SNR. For a given pixel, one would calculate this observationally by finding N_{br} (and N_{dk}) for each trial and then computing $L_{2,br}$ (and $L_{2,dk}$) with Eq. (40), and then one would find $\lambda_{2,br} = \langle L_{2,br}\rangle$ (and $\lambda_{2,dk} = \langle L_{2,dk}\rangle$) by averaging over all the trials. And the standard deviation of $L_{2,br}$ and $L_{2,dk}$ over all the trials would provide the denominator of Eq. (41).

I now turn to the theoretical corrected SNR per pixel per frame, which assumes true knowledge of λ_{br} and λ_{dk}. First, $\langle N_{br}\rangle_N$, $\langle N_{br}^2\rangle_N$, and $\langle N_{br}^3\rangle_N$ need to be computed. Eq. (17) already provides $\langle N_{br}\rangle_N$. One can compute (using, for example, *Mathematica*) $\langle N_{br}^2\rangle_N$ and $\langle N_{br}^3\rangle_N$:

$$\langle N_{br}^2\rangle_N = \sum_0^N c^2 P_T^{*N}(c) = N\epsilon'_{th}(\lambda_{br})(1+(N-1)\epsilon'_{th}(\lambda_{br}))\,, \tag{42}$$

$$\begin{aligned} \langle N_{br}^3\rangle_N = \sum_0^N c^3 P_T^{*N}(c) = &\frac{\epsilon'_{th}(\lambda_{br})}{1-\epsilon'_{th}(\lambda_{br})}\Bigg((1-\epsilon'_{th}(\lambda_{br}))^N\, {}_4F_3\left(2,2,2,1-N;\ 1,1,1;\ 1+\frac{1}{\epsilon'_{th}(\lambda_{br})-1}\right)+ \\ &(N-1)(1-\epsilon'_{th}(\lambda_{br}))^2((N-2)\epsilon'_{th}(\lambda_{br})((N-3)\epsilon'_{th}(\lambda_{br})+3)+1)\Bigg), \end{aligned} \tag{43}$$

where ${}_4F_3$ is a generalized hypergeometric function.

The variance of $L_{2,br}$ is needed for the denominator of Eq. (41), and one can use error propagation to get it. The variance of a function $f(x)$ of one variable is given by

$$\sigma(f(x))^2 = \left(\frac{\partial f}{\partial x}(\langle x\rangle)\right)^2 \sigma(x)^2. \tag{44}$$

Applying this, it follows that

$$\sigma(L_{2,br})^2_N = \left(\frac{e^{\tau/g}}{N} + 2\frac{e^{2\tau/g}(g-\tau)}{2gN^2}\langle N_{br}\rangle + 3\frac{e^{3\tau/g}(4g^2-8g\tau+5\tau^2)}{12g^2N^3}\langle N_{br}\rangle^2\right)^2 \sigma(N_{br})^2. \tag{45}$$

Using the previous equations, one has expressions for all the pieces needed to calculate the theoretical corrected SNR and compare it with the observational corrected SNR. I simulated 500 trials for each of the 50x50 pixels under the same conditions as described in Sect. 4 and found the observational corrected SNR as described below Eq. (41). The numerator (signal) and denominator (noise) of the SNR are examined separately. Eqs. (29) and (31) provide the uncertainty of the signal and noise respectively, where M in those equations is now equal to 500, the number of trials per pixel for both brights and darks, and $\mu_{corr,obs} = \langle L_{2,br}\rangle_N - \langle L_{2,dk}\rangle_{N_2}$. Then, instead of examining a single instance of the signal and noise that was obtained over 500 trials, I average $\mu_{corr,obs}$, $\delta\mu_{corr,obs}$, $\sigma_{corr,obs} = \sqrt{\sigma_N(L_{2,br})^2 + \sigma_{N_2}(L_{2,dk})^2}$, and $\delta\sigma_{corr,obs}$ over the 50x50 pixels, since the pixels are under identical conditions and are independent.

For the observational corrected signal, I find $0.0451 \pm 0.0008e^-$, and the theoretical corrected signal is $0.0451e^-$ and agrees. For the observational corrected noise, I find $0.0187 \pm 0.0006e^-$, and the theoretical corrected noise is $0.0187e^-$ and agrees. Using these uncertainties, the inferred observational SNR range is $(2.29, 2.54)$, and the average signal over the average noise is 2.42. The theoretical SNR is 2.42 and is in agreement. Note that $\lambda_{br} - \lambda_{dk} = 0.055e^- - 0.01e^- = 0.045e^-$, which agrees with the observational corrected signal, and this demonstrates that Nemati's expression for the mean (approximation used in Eq. (34)) is good.

The corrected SNR, using the example parameters I used, is increased by a factor of 14 over the uncorrected SNR. This increase from the uncorrected to the corrected SNR also means that the exposure time for a certain target SNR value is smaller for the corrected case, and time is a precious commodity for the Roman CGI. As one might naturally expect, as the number of frames (bright or dark) increases, the theoretical corrected signal approaches the Poisson mean (λ_{br} or λ_{dk}), and the theoretical corrected noise approaches 0. So increasing the number of observation frames makes the signal more accurate and the SNR higher.

The table below summarizes the results for the "uncorrected" and "corrected" analysis.

	Observational	**Theoretical**
uncorrected signal	$0.0395 \pm 0.0160e^-$	$0.0395e^-$
uncorrected noise	$0.232 \pm 0.011e^-$	$0.235e^-$
uncorrected SNR	$(0.0968, 0.251)$, average: 0.170	0.168
corrected signal	$0.0451 \pm 0.0008e^-$	$0.0451e^-$
corrected noise	$0.0187 \pm 0.0006e^-$	$0.0187e^-$
corrected SNR	$(2.29, 2.54)$, average: 2.42	2.42

Table 1. Summary of the "uncorrected" and "corrected" simulations and theoretical expectations. Simulation parameters used: $\lambda_{br} = 0.055e^-$, $\lambda_{dk} = 0.01e^-$, $N = 200$, $N_2 = 300$, $g = 5000$, and $\tau = 500e^-$. The same simulation parameters were applied for each frame's 50x50 pixels, and the number of trials of photon-counting for these N bright frames and N_2 dark frames was 500.

6. CONCLUSION

In this paper, I have derived the theoretical mean and standard deviation of the number of photo-electron counts per pixel per frame for a photon-counted stack of frames. The process of designating photo-electron signals per pixel per frame above a certain threshold follows a binomial distribution, and the mean and standard deviation follow straightforwardly. I then show how applying photometric corrections based on the measured number of 1-designated counts for a given pixel using Newton's method corrects the mean and standard deviation, and I calculate the theoretical expressions for these. I also provide `noise_script.py`, available to the public[¶], which utilizes the publicly available Python modules `emccd_detect` and `PhotonCounting` (links in a previous footnote) to simulate photon-counted detector frames. The script also demonstrates how the simulations agree with the theoretical expressions, and the results of the script are provided in this paper.

These expressions are useful for coronagraph missions that utilize photon counting for observations with low signal flux. These derived expressions are especially useful for the analysis done for the Roman Telescope's CGI's observational planning. The error budget demands a low level of uncertainty in the knowledge of the mean number of photo-electron counts for a given pixel, and an algorithm developed by Bijan Nemati[5] meets this requirement and is employed in `PhotonCount`. The corresponding SNR that takes this algorithm's corrections into account was derived in this work. For a given set of low-error input parameters for the target star and noise properties of the CGI's detector, Roman's software will return the optimal detector exposure time, number of frames, and gain so that the SNR is maximized subject to detector and photon-counting constraints, and these analytic expressions for the corrected mean number of photo-electrons and standard deviation are necessary for the SNR calculation and observational planning. The error in the knowledge of the true SNR is minimized with our corrected SNR, and it is also larger than the expected SNR without applying corrections, which affords lower exposure times for reaching a target SNR.

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