A little book about motion

> "Philosophy is written in this grand book – I mean the universe – which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth."
>
> – Galileo Galilei, *The Assayer* (1623), As translated in The Philosophy of the Sixteenth and Seventeenth Centuries (1966) by Richard Henry Popkin, p. 65

Lesson 7: The "suvat" equations

In lessons 2 to 5, I introduced position, velocity, and acceleration and our focus was on motion graphs and the conceptual understanding gained from studying them. A picture is often worth a thousand words, and motion graphs are "pictures of motion" that are similarly very useful. Motion graphs, however, are only one side of the story, and the underlying equations are equally important. In many cases, it's actually more convenient taking an algebraic approach to a motion problem, and in this lesson I'm going to be talking about that. In particular, I'm going to derive what we call the **"suvat" equations**: *A simple set of equations that completely describe motion with constant acceleration*.

This lesson is going to be rather technical as I will be going over a lot of subtle (mathematical) details. These details are super important! They are in fact the beginnings of *integral calculus*, so the seed to understanding much more advanced physics is being planted here.

Area in a velocity vs. time motion graph

Let's get started by looking at a very simple case of a particle moving with zero acceleration. This is a special case of constant acceleration. If the acceleration is zero, then we know the velocity is constant. This, of course, follows from the definition of acceleration, e.g.[1]

$$a = 0 \quad \Rightarrow \quad \frac{\Delta v}{\Delta t} = 0 \quad \Rightarrow \quad \Delta v = 0 \quad \Rightarrow \quad v = \text{constant}$$

When the velocity is constant, it's very easy to find the displacement of the particle during any time interval, since from the definition of velocity,[2]

$$v \equiv \frac{\Delta x}{\Delta t} \quad \Rightarrow \quad \Delta x = v\Delta t$$

For example, if the velocity of a particle is $15\,\mathrm{m/s}$ then during a time interval of $\Delta t = 2\,\mathrm{s}$ (e.g. from $0\,\mathrm{s}$ to $2\,\mathrm{s}$) the particle will of course have moved

$$\Delta x = 15\,\mathrm{m/s} \cdot 2\,\mathrm{s} = 30\,\mathrm{m}$$

[1] You get exactly the same if you use the language of differential calculus, since

$$0 = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$
$$= \text{slope of velocity vs. time graph}$$
$$\Rightarrow v(t) = \text{constant}$$

[2] Again, since v is constant I don't need to worry about using differential calculus here. When things are straight and simple, differential calculus is never needed.

During a time interval of $\Delta t = 3\,\mathrm{s}$ (e.g. from $2\,\mathrm{s}$ to $5\,\mathrm{s}$) it will have displaced itself

$$\Delta x = 15\,\mathrm{m/s} \cdot 3\,\mathrm{s} = 45\,\mathrm{m}$$

Therefore, the overall displacement during the first five seconds is $75\,\mathrm{m}$. It's important to know how to visualise these displacements on the velocity vs. time motion graph as *the area under the velocity vs. time graph*, see figure 1. The total area adds up to $75\,\mathrm{m}$ which is the total displacement of the particle.

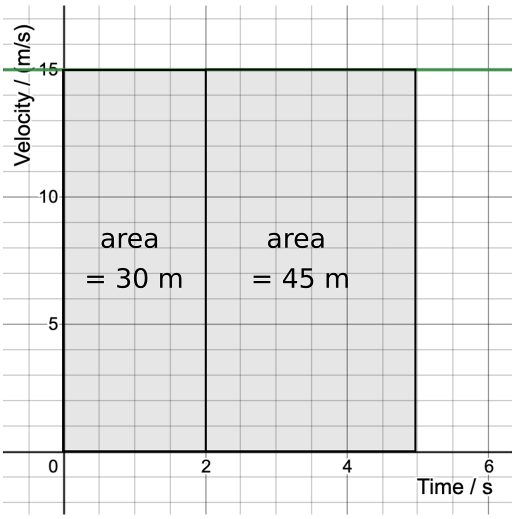


Figure 1: When velocity is constant, it's easy to calculate the displacement, simply multiply the constant velocity by the time interval. This can be visually interpreted as the rectangular area below the constant velocity vs. time graph.

Recall that position is not exactly the same as displacement, e.g. if the particle was $100\,\mathrm{m}$ away initially (at $t = 0\,\mathrm{s}$) from the origin of whatever reference frame is being used, then the particle's position after five seconds would be $175\,\mathrm{m}$. It displaced itself $75\,\mathrm{m}$ from the position $100\,\mathrm{m}$ to the position $175\,\mathrm{m}$. We can easily find an equation for the position of the particle at all times: Assume the initial position of the particle (at $t = 0\,\mathrm{s}$) is x_0 and let x be the position at another time t. It then follows from the definition of velocity that

$$v \equiv \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - 0} \quad \Rightarrow \quad vt = x - x_0 \quad \Rightarrow \quad x = vt + x_0$$

Hence, the position vs. time graph is a straight line with slope v (the constant velocity) and the intercept on the position axis is x_0, the initial position (see figure 2). For our particle example above, this position function would be (having omitted the units for clarity)

$$x(t) = 15t + 100$$

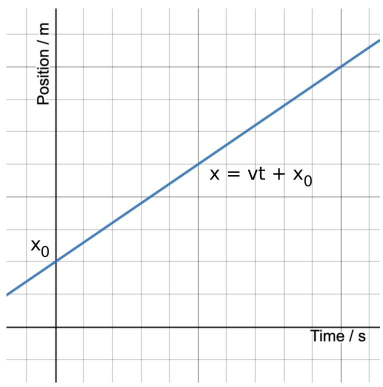


Figure 2: When the velocity is constant (because acceleration is zero), the position vs. time graph is just a straight line. The equation for this position function is $x(t) = vt + x_0$.

Now what would happen if the velocity is constant $15\,\mathrm{m/s}$ for the first two seconds, and then suddenly constant $20\,\mathrm{m/s}$ for the next three seconds? The velocity vs. time graph for this case is shown in figure 3. The displacement over the first two seconds would still be $30\,\mathrm{m}$, but the displacement during the next three seconds would now be $60\,\mathrm{m}$ (not $45\,\mathrm{m}$ as before) because it's moving faster. The area under the velocity vs. time graph during this time interval is now bigger, see figure 4.

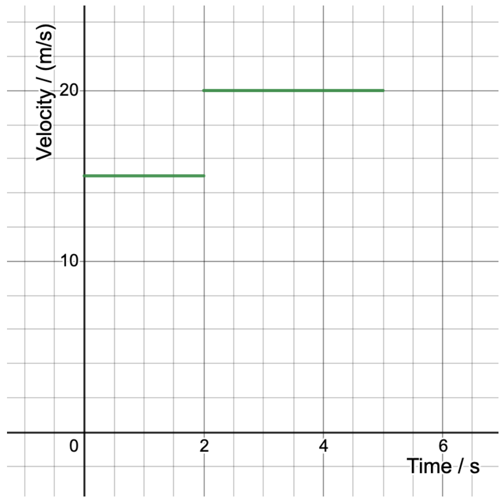


Figure 3: A particle moving at a constant velocity of 15 m/s for the first two seconds and at a constant velocity of 20 m/s for the next three seconds. Don't worry about the unrealistic sudden change at $t = 2$ s, nothing can instantly change its velocity like that. This abstract example is just a stepping stone leading us to the more realistic continuously changing case in figure 7 below.

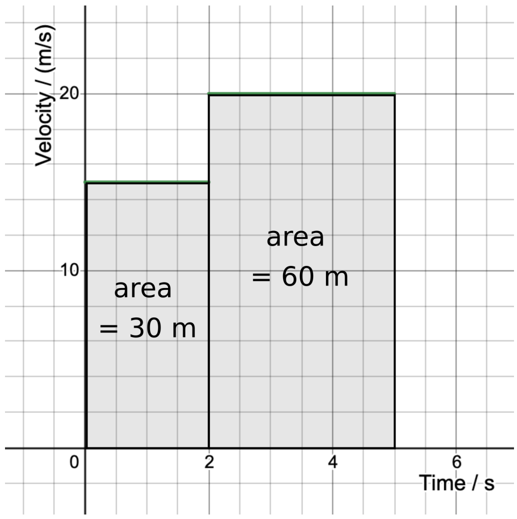


Figure 4: As the velocity increases, the area under the graph also increases.

The corresponding position vs. time graph is shown in figure 5. It consists of two straight line sections, one for the first two seconds and another for the last three seconds. Writing down the position function for this graph is a bit tricky, but the first section is easy enough. The initial position, when $t = 0$ s, is 100 m, so we get $x = 15t + 100$ (for the first two seconds). But for the next three seconds we need to be careful. It's easy to see that the slope of the line increases to 20 m/s (the larger velocity), but we can't use the initial position of 100 m anymore, because if the particle had been moving with 20 m/s the whole time, then it wouldn't have started at 100 m when $t = 0$ s, instead it would have been at $130 - 20\,\text{m/s} \cdot 2\,\text{s} = 90$ m (extrapolate the line backwards, shown as a dashed line in figure 5). So the equation of the straight line for the last three seconds is $x = 20t + 90$. All in all, one could describe the position vs. time function as the following piecewise function which is graphed in figure 5:

$$x(t) = \begin{cases} 15t + 100 & 0 < t < 2 \\ 20t + 90 & 2 < t < 5 \end{cases}$$

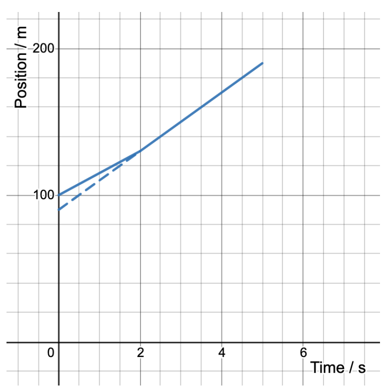


Figure 5: The position vs. time graph of the particle that first moves at a constant velocity of 15 m/s for two seconds and then at a constant velocity of 20 m/s for three seconds.

The first suvat equation, $v = at + v_0$

So far so good. Now let's consider a particle that has a velocity vs. time graph as shown in figure 6. Let's call that a *step graph*. It has a constant velocity for one second, but after each second it increases the constant velocity by 5 m/s . In order to get the total

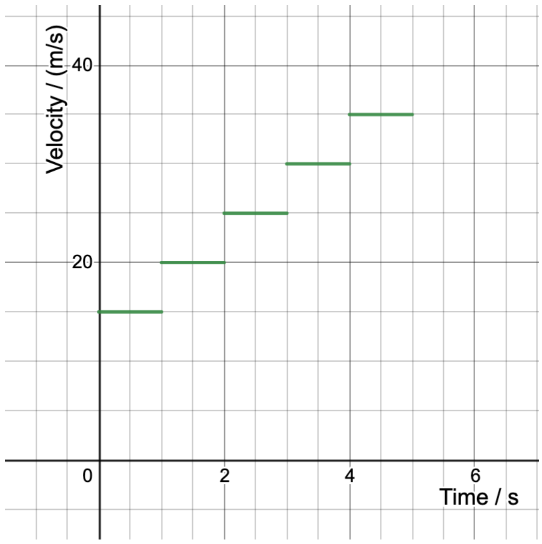


Figure 6: A velocity vs. time step-graph. If you calculate the total area under the graph you should get 125 m.

displacement after five seconds, we need to add the five different rectangular areas below each step. This can be expressed as follows

$$\begin{aligned} \Delta x = \text{total displacement} &= \Delta x_1 + \Delta x_2 + \cdots + \Delta x_5 \\ &= v_1 \Delta t_1 + v_2 \Delta t_2 + \cdots + v_5 \Delta t_5 \\ &= \sum_{i=1}^{5} v_i \Delta t_i \end{aligned}$$

where the notation hopefully makes sense. You can check using figure 6 that this sum in our example is 125 m. Let's go even further now: What if the velocity started at 15 m/s and then *continuously increased at a constant rate of* $5\,\mathrm{m/s^2}$? In other words, what if the motion had a constant acceleration of $5\,\mathrm{m/s^2}$ and the velocity vs. time graph was a smooth, continuous straight line like in figure 7 (not a step graph like the one in figure 6)? How do we calculate the displacement in this case? Since the velocity is changing continuously at a constant rate, it never has a constant value, so how do we add up all the displacement areas?

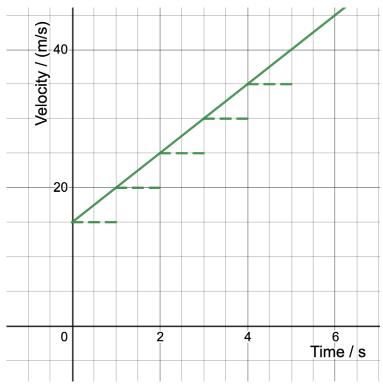


Figure 7: When the velocity is continuously changing (such as the straight line graph shown here) it is never constant, so how can we calculate rectangular areas under the graph? Read on!

Before we tackle this problem of finding the area, let's first quickly see how we can work out the equation of the smooth velocity function. If the acceleration has a constant (non-zero) value, then it's easy to find the change in velocity, because from the definition of acceleration we get

$$a \equiv \frac{\Delta v}{\Delta t} \quad \Rightarrow \quad \Delta v = a \Delta t$$

Assuming the initial velocity of the particle (when $t = 0$), is v_0 and letting v be the position at another time t, it then follows from the definition of acceleration that

$$a \equiv \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - 0} \quad \Rightarrow \quad at = v - v_0 \quad \Rightarrow \quad v = at + v_0$$

Hence, the velocity vs. time graph is a straight line with slope a (the constant acceleration) and the intercept on the velocity axis is v_0, the initial velocity. This equation is in fact our **first "suvat" equation**. The velocity function of the particle starting at 15 m/s and accelerating at a constant rate of 5 m/s^2 is (see figure 7).

$$v(t) = 5t + 15$$

OK, back to finding the displacement area under the smooth velocity vs. time graph.

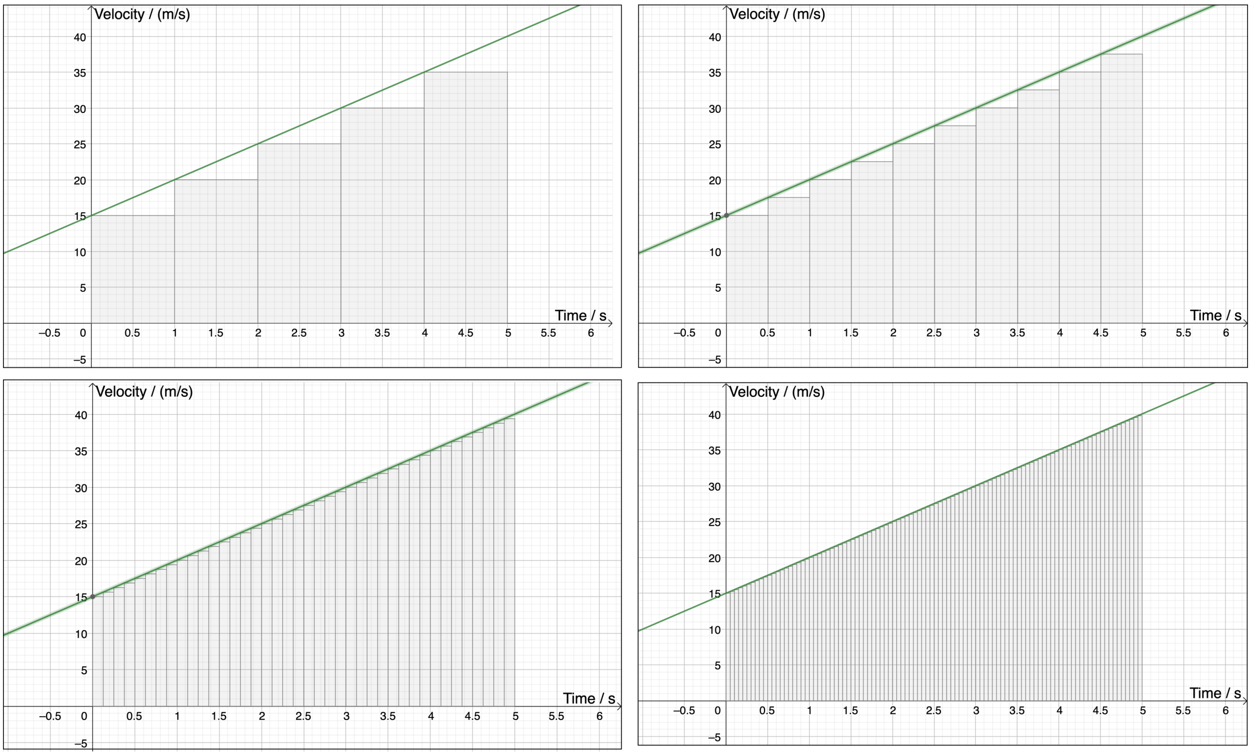


Figure 8: A smooth straight velocity vs. time graph can be approximated to a step graph and the area under the smooth graph is approximately the area of the rectangles under the step graph. This approximation gets better and better as the width of the rectangles (that is, the time intervals during which the velocity is assumed to be constant) get smaller and smaller. Reading from upper-left to lower-right, the areas are respectively 125 (5 rectangles), 132.25 (10 rectangles), 135.9375 (40 rectangles), and 136.875 (100 rectangles). These numbers approach the correct exact answer which is 137.5.

The problem we are facing here is very similar to the problem we faced in lesson 4 where we wanted to make sense of an instantaneous velocity. In that case, the problem was solved when Newton (and others) introduced the concept of a “limit”. We looked at what the limit of average velocities approached as the time interval under consideration got smaller and smaller. Here we do exactly the same: We imagine the motion being made up of small time intervals during which the velocity is constant (even though it isn’t) and then we add up all the corresponding rectangular displacement areas. In other words, *the perfectly straight and smooth velocity vs. time graph will be approximated to a step graph*. And when the steps get smaller and smaller (corresponding to smaller and smaller time intervals), then *the area under the step graph will come closer and closer to the exact area under the smooth graph*, see figure 8.

When these approximations to the area approach the same number, we *define* this number to be the area under the smooth graph and we call it the **integral** of the velocity function. The notation is as follows

$$\int v(t)\,\mathrm{d}t \equiv \lim_{\Delta t \to 0} \sum_i v_i \Delta t$$
$$= \text{the area under a smooth velocity vs. time graph}$$

This is the crucial step in understanding **integral calculus**: *The area under any graph can be found by approximating it to a sum of rectangles of smaller and smaller width.*[3] A lot can be said about this field of mathematics and you will learn that another time, but in this course we will only consider very simple applications. For example, our particle's velocity increases at a constant rate of $5\,\mathrm{m/s^2}$, so it's easy to see that the area under the straight line is in fact just a trapezoid! Hence, during the first five seconds, the total displacement of the particle moving according to figure 8 is simply

[3] The integral symbol (introduced by Leibniz) $\int$ represents the letter 's' for sum because we are indeed adding up many terms of the form $v\Delta t$.

$$\text{total displacement} = \text{area of trapezoid} = \frac{1}{2}(5)(15+40) = 137.5\,\mathrm{m}$$

We can check this against our previous approximation using figure 6: When we calculated the approximate area in figure 6 (the same as the upper-left graph in figure 8), we were missing the five triangular "gaps" under the graph. Each of those triangles had an area of 2.5 m, so

$$125\,\mathrm{m} + 5 \cdot 2.5\,\mathrm{m} = 137.5\,\mathrm{m}$$

The second suvat equation, $x = \frac{1}{2}at^2 + v_0 t + x_0$

The final step in all this is to now work out an equation for the position vs. time graph when the acceleration is constant. This will be our second "suvat" equation, and we obtain it in the following way: Assume the velocity is increasing at a constant rate (constant, non-zero acceleration), so the first suvat equation is valid

$$v = at + v_0 \tag{1}$$

The displacement of the particle is the area under the velocity vs. time graph, and since that area is a trapezoid (see figure 9), the displacement after t seconds is[4]

[4] The area formula of a trapezoid is

$$A = \frac{1}{2}h(a+b)$$

where a and b are the lengths of the two parallel sides and h is the height. If you don't like trapezoids, then view the area as a triangle and a rectangle put together. You get the same result of course.

$$\Delta x = \frac{1}{2}t(v_0 + v)$$

Inserting our first suvat equation (1) into this, we get

$$\Delta x = \frac{1}{2}t\left(v_0 + (at + v_0)\right) = \frac{1}{2}at^2 + v_0 t$$

which is a quadratic equation as we know it should be from our previous lessons! This is our **second "suvat" equation**. We often write it out as a *position* function, by simply remembering that $\Delta x = x - x_0$:

$$x - x_0 = \frac{1}{2}at^2 + v_0 t \quad \Rightarrow \quad x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

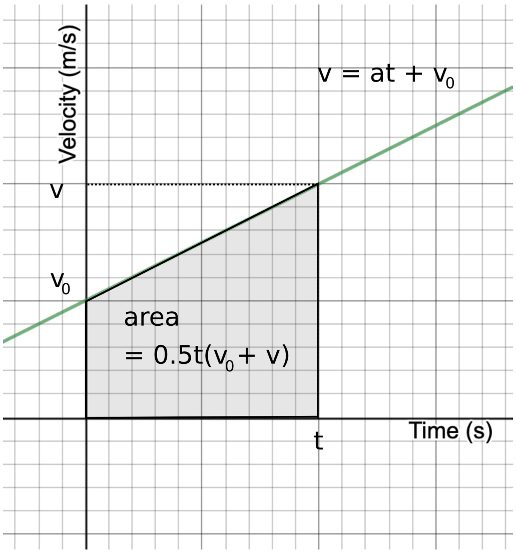


Figure 9: The area under a velocity vs. time graph is the displacement. Here we use that knowledge to find the second "suvat" equation.

For example, the position function of the particle we have been using as an example (initial position 100 m, initial velocity 15 m/s, and constant acceleration 5 m/s^2) is

$$x(t) = \frac{1}{2}(5)t^2 + 15t + 100$$

Let's summarise! First our numerical example: If a particle has a constant acceleration of 5 m/s^2, an initial velocity of 15 m/s, and an initial position of 100 m, then the motion is completely described by the three equations:

$$a(t) = 5$$
$$v(t) = 5t + 15$$
$$x(t) = \frac{1}{2}5t^2 + 15t + 100$$

with corresponding motion graphs shown in figure 10.

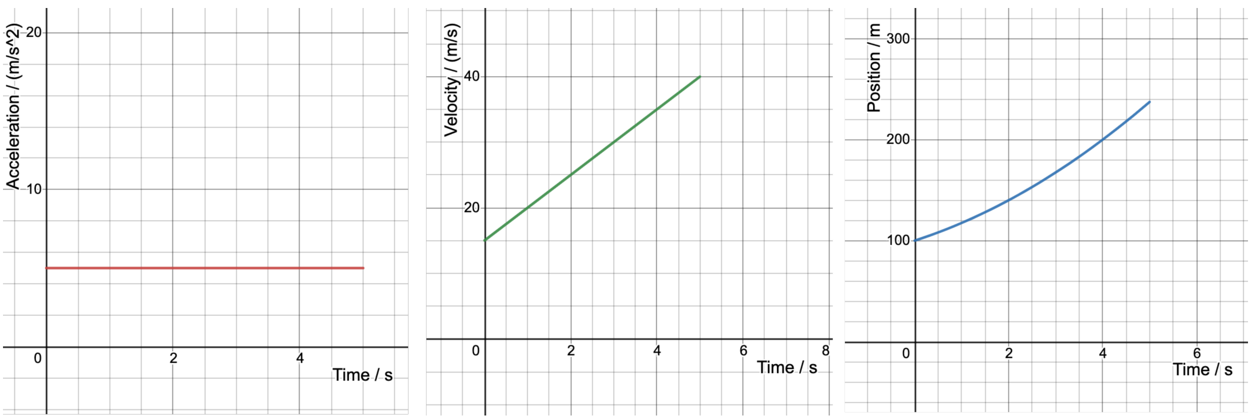


Figure 10: The three motion graphs describing the motion of a particle moving with a constant acceleration of 5 m/s^2, an initial velocity of 15 m/s, and an initial position of 100 m.

The total displacement during the first five seconds can be found by calculating the total area under the velocity vs. time graph and in this case it's just a simple trapezoid shape:

$$\begin{aligned}\text{displacement during first five seconds} &= \Delta x \\ &= \text{area under v vs. t} = 137.5\,\text{m}\end{aligned}$$

You can check this against the position motion graph: 100 m + 137.5 m = 237.5 m, which is indeed the position after 5 seconds.

As a final point to note – try to convince yourself of this on your own – *the total change in velocity during a given time interval is the area under the acceleration vs. time graph*, so for this example the total change in velocity is

$$\Delta v = \text{area under a vs. t graph} = 5\,\frac{\text{m}}{\text{s}^{\cancel{2}}} \cdot 5\,\cancel{\text{s}} = 25\,\text{m/s}$$

which of course agrees with the particle having changed its velocity from 15 m/s to 40 m/s.

Summary

Here is a general summary: When the acceleration of a particle, a, is constant (and this includes the case of zero acceleration), the motion of the particle is completely described by the two suvat equations

$$\begin{aligned} v(t) &= at + v_0 \\ x(t) &= \frac{1}{2}at^2 + v_0 t + x_0 \end{aligned}$$

and

- the displacement during any time interval can be calculated as the total area under the velocity vs. time graph,
- the change in velocity during any time interval can be calculated as the total area under the acceleration vs. time graph.

An observant student might notice that the name "suvat" has something to do with the variables contained in the equations: We have 'v', 'a', and 't' of course meaning velocity, acceleration and time, but what about 's' and 'u'? Well we often use the symbol 'u' to stand for the initial velocity v_0, so often the first suvat equation is written as

$$v = at + u \tag{2}$$

Secondly, 's' is an old notation for displacement (from latin: spatium), so the second suvat equation is often written as (using 'u' for initial velocity and remembering displacement is $\Delta x = x - x_0$)

$$s = \frac{1}{2}at^2 + ut \tag{3}$$

And there you have them! The famous "suvat" equations covered in all high-school physics courses around the world. These two equations, (2) and (3), can be combined to give another two very useful equations. One comes from eliminating time from the equations:

$$\text{eliminate time} \quad \Rightarrow \quad 2as = v^2 - u^2$$

and the other one comes from eliminating the acceleration (this equation has in fact already been mentioned – can you see where?):

$$\text{eliminate acceleration} \quad \Rightarrow \quad s = \frac{1}{2}t(u+v)$$

Finally, let's briefly talk about what we do if the acceleration is *not* constant. In that case we don't have a simple straight line velocity graph so it's difficult to work out an expression for the displacement (the area under the graph). In these cases you have to either rely on numerical approximations obtained by looking at step graphs like we did above (this is actually what really goes on behind the screen when we use computers to solve complex physics problems) or you have to know how to use the rules of integral calculus. Something to look forward to learning another day!

Lesson 7: Questions and activities

1. Go back to lesson 5 on page 34 and write down the three equations of motion for the motion graphs for our cart gently rolling down a slope. Compare these equations with the mathematical models $x = At^2$ and $v = Bt$ from lessons 3 and 4.
2. Go back to lesson 5, question 4 on page 37 and write down the three equations of motion for the tennis balls thrown straight up in the air.
3. When the acceleration is zero, what do the suvat equations reduce to? Draw the motion graphs for this special case.
4. Let's revisit lesson 6, question 3 (c): Use the suvat equations to calculate
   (a) how long it takes for the block to come to rest.
   (b) how far it slides before coming to rest.
5. Use the suvat equations (or the other two useful equations) to answer all the following questions:
   (a) What is the speed of a stone 3.0 s after it falls from rest with a constant acceleration of 10 m/s^2?
   (b) Determine the distance covered by a cart on a track while it accelerates at 4.0 m/s^2 for 0.50 s from an initial speed of 1.0 m/s.
   (c) A ball rolls from rest down an inclined plane with a uniform acceleration of 4.0 m/s^2.
      i. What is its speed after 8.0 s?
      ii. How long will it take to reach a speed of 36 m/s?

iii. How long does it take to travel a distance of 200 m, and what is its speed after that time?

iv. How far does it travel during the fourth second of its motion?

(d) What is the displacement of a cyclist while he accelerates from 1.5 m/s to 2.5 m/s in 2.0 s? What assumption have you made?

(e) Calculate the final speed of a runner who accelerates at 0.5 m/s^2 from an initial speed of 3.0 m/s while she covered 16 m.

6. A hot-air balloonist, rising vertically with a constant velocity of magnitude 5.00 m/s, releases a sandbag at an instant when the balloon is 40.0 m above the ground. Let the ground be position 0 m. After it is released, the sandbag is in free fall. Use $g = 10\,\text{m/s}^2$ and neglect air resistance.

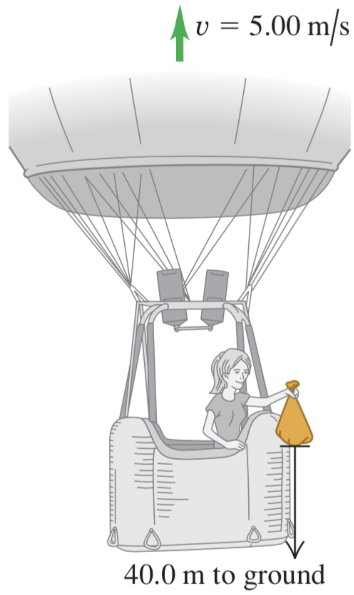


Figure 11: Question 6.

(a) Compute the position and velocity of the sandbag at 0.250 s and 1.00 s after its release (remember: position is not the same as displacement!).

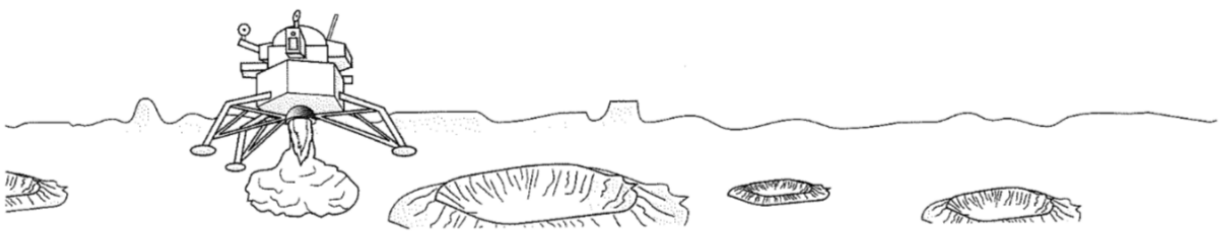
(b) How many seconds after its release will the bag strike the ground?

(c) With what magnitude of velocity does it strike the ground?

(d) What is the greatest height above the ground that the sandbag reaches?

(e) Sketch a vs. t, v vs. t, and x vs. t motions graphs for the motion.

7. A lunar landing craft descends vertically towards the surface of the Moon with a constant speed of 2.0 m/s. The craft and crew have a total mass of 15×10^3 kg. Assume that the gravitational field strength on the Moon is 1.6 N/kg.



(a) During the first part of the descent the upward thrust of the rocket engines is 24×10^3 N. Show that this results in the craft moving with a constant speed.

(b) The upward thrust of the engine is increased to 25.5×10^3 N for the last 18 seconds of the descent.

i. Calculate the acceleration of the craft during this time.
ii. What is the speed of the craft just before it lands?
iii. How far is the craft above the surface of the Moon when the engine thrust is increased to 25.5×10^3 N?

8. Go through the steps of eliminating time and acceleration from the suvat equations to derive the other two useful equations shown at the bottom of page 58.

9. Perform the following experiment: *Investigate how the mass of an object falling through air affects the time it takes to fall a given distance*. Use cupcake holders – the mass can easily be changed by adding more.

10. Solve the Ocean Park challenge. This is from a Year 10 Science trip I once organised. Answers are in the document.

Answers to all the questions.

Lesson 7 Quiz

Check your understanding of this lesson: Here is a quiz.

A little book about motion

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Lesson 16: Projectile motion

As we saw in earlier lessons, position, velocity, acceleration and force are all vector quantities. They have a magnitude and a direction. This isn't that obvious in one-dimensional motion (e.g a tennis ball falling straight down), but when describing motion in two and three dimensions it's clear to see these quantities have directions. **Projectile motion** is the term we use when describing the two-dimensional motion of a particle that is launched close to the surface of the Earth[1]. It is actually very easy to describe projectile motion because it is just an application of the suvat equations that we already derived in lesson 7. The key is to keep in mind that the motion takes place in two directions: There is motion in the horizontal x-direction and there is motion in the vertical y-direction and they are each described by independent equations of motion.

[1] It doesn't have to be on Earth, of course. As long as the gravitational field strength, g, is constant (as it is in a *uniform gravitational field*) then the equations we are about to derive will apply. For example, we could go to the surface of the Moon, with $g = 1.6\,\mathrm{m/s^2}$ and observe projectile motion.

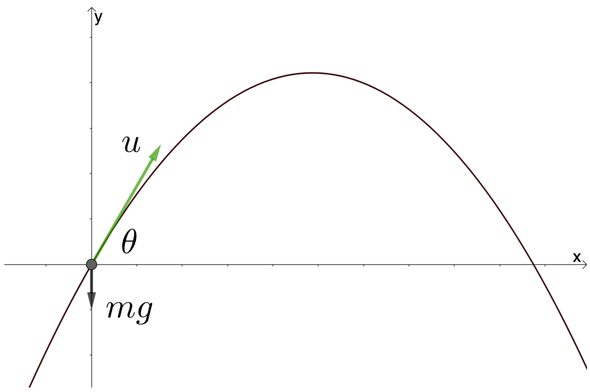


Figure 1: A particle is launched with an initial velocity u from the origin at an angle of θ to the horizontal. We are only interested in describing the motion as it proceeds under the influence of gravity alone.

Here is the usual situation we are looking at: Consider a particle launched from a point that we take to be the origin of a two-dimensional Cartesian plane (see figure 1). Let the particle be launched at an initial velocity u that makes an angle θ with the horizontal (the **launch angle**). In order to launch the particle in the first place, some sort of force has to be applied to it, but here we are only interested in describing the motion *after this force has done its job*. So we assume the only remaining force on the particle is its weight, mg (we are also assuming that air resistance is negligible, but see question 7 for more on that). Since the weight points straight down (in the vertical direction), the acceleration will also point straight down, and it has a constant magnitude that we label g as usual.

Let's first consider motion in the *horizontal direction*: Since there is no acceleration in the horizontal direction $a_x = 0$, the horizontal velocity, v_x, will remain constant. The horizontal velocity is the horizontal **component** of the velocity vector which we find using trigonometry (see figure 2):

$$v_x = v\cos\theta$$

This will always be equal to the initial horizontal velocity u_x. For example, if the initial speed is 8 m/s and the launch angle is 60°, then the initial horizontal velocity is

$$u_x = u\cos\theta = 8\cdot\cos 60^\circ = 4\,\mathrm{m/s}$$

and it will continue to have this value throughout the entire motion. So the horizontal velocity at any point along its trajectory is

$$v_x = 4\,\mathrm{m/s}$$

Given a constant velocity, we know that the position will be changing at a constant rate, which means the horizontal position just steadily increases as time passes. Since we set the initial position to be at the origin, the horizontal position (the x-coordinate of the position) is given by the suvat equation[2]

$$x = u_x t = 4t$$

The position and velocity vs. time motion graphs for this horizontal direction are shown below.

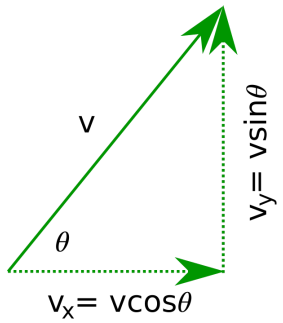


Figure 2: Any vector can be **resolved** into component vectors that are parallel to the axes of a reference frame. Simple right-angled trigonometry allows you to calculate the magnitudes. Note that components are actual vectors so they add up to the main vector by the usual tail-to-tip method, so e.g.

$$\vec{v} = \vec{v}_x + \vec{v}_y$$

[2] This is just the position suvat equation

$$x = \frac{1}{2}at^2 + ut + x_0$$

with zero acceleration and zero initial position.

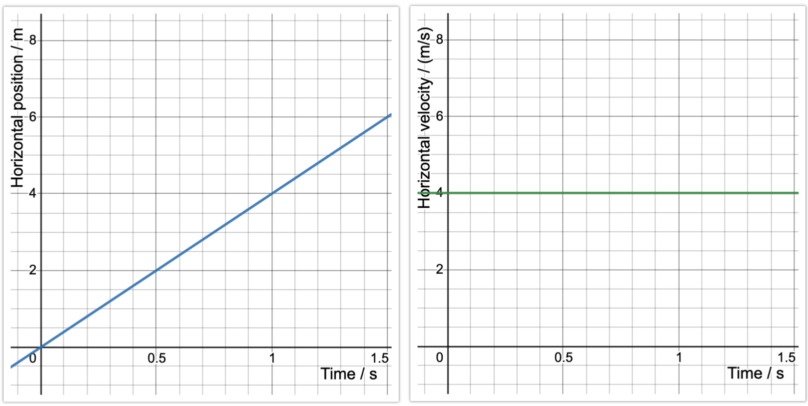


Figure 3: The position and velocity vs. time motion graphs in the horizontal direction.

Now what about the *vertical direction*? Since there is a constant vertical acceleration of g downwards[3]

$$a_y = -g$$

the vertical velocity will be changing at a constant rate. The initial vertical velocity is also found using trigonometry:

$$u_y = u\sin\theta$$

so the vertical velocity at any given point along the trajectory is given by the suvat equation[4]:

$$v_y = a_y t + u_y = -gt + u\sin\theta$$

[3] We will assume the letter g always stands for the *magnitude* of the acceleration, so direction must be indicated by a plus or minus sign. In this case our reference frame has a vertical axis defining up as the positive direction, so we need to put a negative sign in front of g.

[4] This is just the velocity suvat equation

$$v = at + u$$

with values of the vertical acceleration and vertical initial velocity inserted.

The numerical example given above, $u = 8\,\mathrm{m/s}, \theta = 60°$, and $g = 10\,\mathrm{m/s}^2$, results in the vertical velocity equation

$$v_y = -10t + 4\sqrt{3}$$

We can see that the vertical velocity decreases at a constant rate (as expected).

Given that the vertical acceleration is constant and the velocity is changing at a constant rate, we know that the vertical position must be changing as a quadratic function. The vertical position, the y-coordinate of the position, is given by the suvat equation

$$y = \frac{1}{2}a_y t^2 + u_y t + y_0 = -\frac{1}{2}gt^2 + (u \sin\theta)t$$

where we set the initial vertical position to zero since it starts at the origin. Inserting numbers from our example, $u = 8\,\mathrm{m/s}, \theta = 60°$, and $g = 10\,\mathrm{m/s}^2$, we get

$$y = -5t^2 + 4\sqrt{3}t$$

The position and velocity vs. time motion graphs for this vertical direction are shown below.

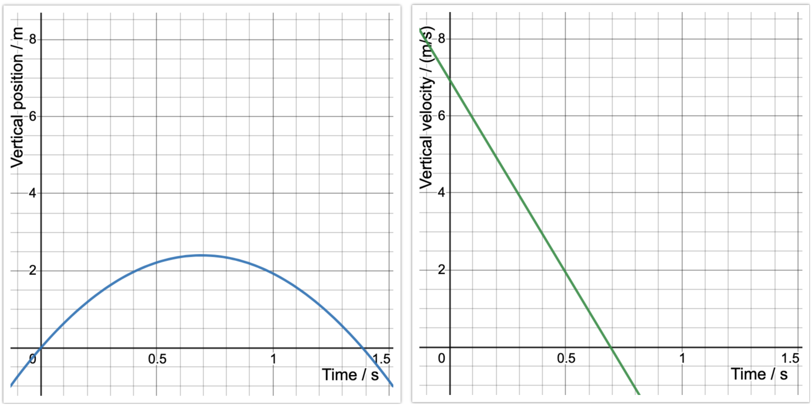


Figure 4: The position and velocity vs. time motion graphs in the vertical direction.

That is all there is to projectile motion! We can summarise our results by listing the general position and velocity functions of a particle being launched from the origin. First the velocity functions in the x and y direction,

$$\begin{aligned} v_x &= u\cos\theta \\ v_y &= -gt + u\sin\theta \end{aligned} \tag{1}$$

and then the position functions in the x and y direction,

$$\begin{aligned} x &= (u\cos\theta)t \\ y &= -\tfrac{1}{2}gt^2 + (u\sin\theta)t \end{aligned} \tag{2}$$

As you can see, these are all simply suvat equations. Note how the mass, m, is absent from all the equations which shouldn't be too surprising based on our work in lesson 6: When the air resistance is negligible, the motion does not depend on the mass of the particle.

Let's now derive a few results from these equations.

The trajectory is indeed a parabola

In figure 1, the trajectory of the particle in two-dimensional space resembles a parabola, but I haven't strictly proven that yet. Luckily that's easy to do now I have the position functions. I just need to eliminate time in equations (2) in order to get an equation of y in terms of x. From the first equation,

$$t = \frac{x}{u\cos\theta}$$

and inserting that in the second equation,

$$y = -\frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2 + (u\sin\theta)\left(\frac{x}{u\cos\theta}\right)$$

we can simplify that slightly to

$$y = -\left(\frac{g}{2u^2\cos^2\theta}\right)x^2 + (\tan\theta)x \tag{3}$$

which is indeed a quadratic expression in x, so it must have the shape of a parabola opening downwards and going through the origin.[5] An an example, if $u = 8\,\mathrm{m/s}$, $\theta = 60°$, and $g = 10\,\mathrm{m/s^2}$, you can check that (3) becomes

$$y = -\frac{5}{16}x^2 + \sqrt{3}x$$

which is shown in figure 5.

[5] If x is very small, meaning the particle is very close to the origin, then x^2 is much smaller than x. For example, if $x = 0.01\,\mathrm{m}$ then $x^2 = 10^{-4} \ll x$. This implies that the first term in (3) is negligible compared to the second term, and the graph is close to being just $y = \tan\theta x$. This is a straight line through the origin with a slope of θ as it should be. Making approximations like this and drawing sensible conclusions from them play a huge role in more advanced physics.

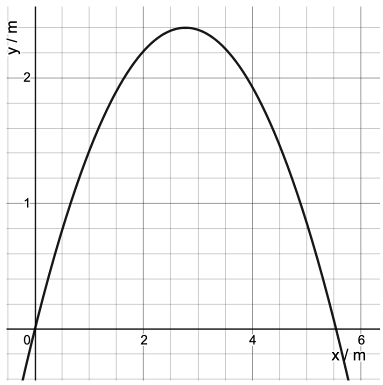


Figure 5: The actually trajectory of the launched particle in 2D space.

Maximum height

We can work out an expression for the maximum height by knowing that the *vertical velocity at that point is zero*[6] (the horizontal velocity is *not* zero). Take a minute to digest that and make sure you understand why (it's because it reaches the turning point for the vertical position). From this condition

$$v_y = 0 \quad \text{(at max height)} \quad \Rightarrow \quad -gt + u\sin\theta = 0$$

we get[7]

$$t = \frac{u\sin\theta}{g} \tag{4}$$

which can be inserted into the y-position function as follows

$$y_{\max} = -\frac{1}{2}g\left(\frac{u\sin\theta}{g}\right)^2 + (u\sin\theta)\left(\frac{u\sin\theta}{g}\right) = \frac{u^2\sin^2\theta}{2g}$$

Check what happens to this expression when you insert $\theta = 0°$ and $\theta = 90°$. Does it make sense? If $u = 8\,\mathrm{m/s}$, $\theta = 60°$, and $g = 10\,\mathrm{m/s^2}$, then

$$y_{\max} = \frac{u^2\sin^2\theta}{2g} = 2.4\,\mathrm{m}$$

which can be checked against figure 4 and 5.

[6] Maybe it's worth reminding you that we are of course talking about *instantaneous* velocity.

[7] This also follows easily by finding the first coordinate of the vertex of the parabola in (2).

Maximum range

We can work out an expression for the **range**, i.e. the total horizontal distance that it travels. We can do this in different ways, here's a short way (the other ways are explored in the questions): Due to the symmetry of the motion[8], it will take the particle twice as long to reach the ground as it does to reach the maximum height (4), hence

[8] If air resistance is included, the motion is *not* symmetric, see question 7 below.

$$t = 2\frac{u\sin\theta}{g} \quad \text{(to reach ground again)} \tag{5}$$

We can check this against our numerical example: If $u = 8\,\text{m/s}$, $\theta = 60°$, and $g = 10\,\text{m/s}^2$, then

$$t = 2\frac{u\sin\theta}{g} = 2\frac{8\cdot\sin 60°}{10} = \frac{4\sqrt{3}}{5} \approx 1.386\,\text{s}$$

which seems about right when looking at figure 4. We can then find the range from the x-position function (2)

$$x_{\text{range}} = u\cos\theta\left(2\frac{u\sin\theta}{g}\right) = \frac{2u^2\cos\theta\sin\theta}{g}$$

which can be reduced to

$$x_{\text{range}} = \frac{u^2\sin(2\theta)}{g} \tag{6}$$

by using the double angle identity for sine. Not surprisingly, the range depends on the initial speed, u, and the launch angle, θ (and g of course). Our number example gives us a range of

$$x_{\text{range}} = \frac{8^2\sin(2\cdot 60°)}{10} = \frac{16\sqrt{3}}{5} \approx 5.542\,\text{m}$$

which checks out when looking at figure 5.

Which launch angle results in the largest range? Since the sine function has a maximum value of one when its argument is 90 degrees, the answer is 45°:

$$2\theta = 90° \quad \Rightarrow \quad \theta = 45°$$

At this launch angle, the range obtains its maximum value of

$$x_{\text{range, max}} = \frac{u^2}{g}$$

Figure 6 shows a variety of launch angles for a constant launch speed. Can you see which one has the largest range?

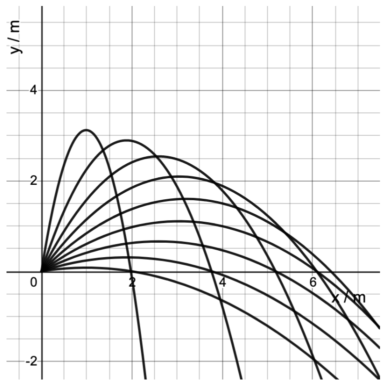


Figure 6: Nine different trajectories. The speed is constant, but the launch angle varies. One of them corresponds to a launch angle of 45° – which one is it?

Vector notation

The suvat equations (1) and (2) constitute the component functions of the velocity and position vectors, so we could also write these using vector notation. The velocity vector would be

$$\vec{v}(t) = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} u\cos\theta \\ -gt + u\sin\theta \end{pmatrix}$$

and the position vector

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (u\cos\theta)t \\ -\frac{1}{2}gt^2 + (u\sin\theta)t \end{pmatrix}$$

Lesson 16: Questions and activities

1. Go to this desmos simulation and play around with changing the parameters. Do you recognise the suvat equations?

2. Consider the projectile motion given by $g = 10\,\mathrm{m/s^2}$, $u = 2\,\mathrm{m/s}$, and $\theta = 30°$.

   (a) Write down the exact equations of motion (for position and velocity) and draw the motion graphs.

   (b) Draw the trajectory that the particle follows in 2D space.

   (c) At what time will it reach maximum height? When will it hit the ground again?

   (d) Draw position, velocity and acceleration vectors at the instants $t = 0.00\,\mathrm{s}, 0.05\,\mathrm{s}, 0.10\,\mathrm{s}, 0.15\,\mathrm{s}$, and $t = 0.20\,\mathrm{s}$.

3. If $\theta = 90°$, then what do the equations of motion reduce to? Describe the motion. Is the trajectory still a parabola?

Figure 7: Projectile motion is involved in quite a lot of sports events. Here is the field event called *shot put*.

4. A physics book slides off a horizontal tabletop with a speed of $1.10\,\mathrm{m/s}$. It strikes the floor in $0.350\,\mathrm{s}$. Ignore air resistance. Find

   (a) the height of the tabletop above the floor,

   (b) the horizontal distance from the edge of the table to the point where the book strikes the floor,

   (c) the horizontal and vertical components of the book's velocity, and the magnitude and direction of its velocity, just before the book reaches the floor,

   (d) draw x vs. t, y vs. t, v_x vs. t, and v_y vs. t graphs for the motion.

5. A daring swimmer dives off a cliff with a running horizontal leap, as shown in figure 8. What must her minimum speed be just as she leaves the top of the cliff so that she will miss the ledge at the bottom, which is 1.75 m wide and 9.00 m below the top of the cliff?

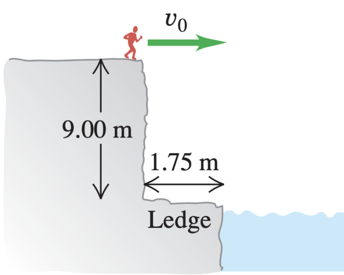


Figure 8: .

6. A stone is dropped from rest. At the same time (and almost from the same point), another stone is launched horizontally at speed $10\,\mathrm{m/s}$ from the same height of $1.5\,\mathrm{m}$.

   (a) Which one hits the ground first? (Here is a demo.)

   (b) How far apart are the stones when they hit the ground?

   (c) Watch this Mythbuster video. Why don't the bullets hit the ground at *exactly* the same time?

   (d) If there was no air resistance, would the bullets collide if they moved in exactly the same plane?

   (e) Watch me show you "The Hunter and The Monkey" demonstration in class!

7. When you add air resistance to the motion, the equations get much more complicated (a more complex reality requires more complex mathematics). The qualitative details of the motion are, however, relatively easily understood. Play around with this desmos simulation which includes air resistance (k is the parameter that controls the amount of air resistance).

   (a) What do you notice with regards to the location of the maximum height when we include air resistance?

   (b) What do you notice with regards to the time it takes to reach the peak height compared to the time it takes to go from the peak to the ground again?

   (c) What do you notice about the angle the velocity vector makes at launch compared to the angle it makes when it hits the ground again?

   (d) What happens when you change the mass?

8. Solve question 2 again, but this time let it be on the Moon ($g = 1.6\,\mathrm{m/s^2}$).

9. The maximum range formula can also be derived as follows:

   (a) What is the expression for the vertical velocity at launch?

   (b) When it hits the ground again, the vertical velocity has reversed completely. What is the expression for the vertical velocity at that time?

   (c) Use the expression from (b) and the vertical velocity suvat equation to find the time it takes to hit the ground again. Compare with (5).

   (d) Find the expression for the range.

10. The maximum range formula can also be derived as follows:

   (a) What is the y-position when the particle is on the ground?

   (b) Use (a) and equation (3) to solve for the two x-intercepts. Compare with (6).

Answers to all the questions.

Lesson 16 Quiz

Check your understanding of this lesson: Here is a quiz.

Lesson 17: Newton's laws of motion in more than one dimension

In the past many lessons the main focus was to understand the fundamental concept of acceleration and to see how Newton's 2nd law allowed us to calculate it by knowing the forces acting on a given particle. We mainly looked at one-dimensional motion to make it easier to understand this very significant idea. We will now expand our understanding to two and three dimensions and introduce different types of forces. The basic idea is still the same (forces cause accelerations), but now we need to keep track of multiple objects and directions. Projectile motion, which we covered in our last lesson, was the first real example of this, but it was a relatively easy case because the force (and therefore the acceleration) was constant and pointing in one direction. Now forces will be pointing in all sorts of directions.

This lesson is going to be a bit long, and we are not introducing any fundamentally new ideas – we are simply applying the tools and mindset of the Newtonian Paradigm. Let's start by writing up all three of Newton's laws of motion:

Newton's 1st law: "Every body perseveres in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it."

Newton's 2nd law: The resultant force on an object equals its mass times its acceleration[1],

$$\vec{F}_{\text{net}} = m\vec{a}$$

Newton's 3rd law: If object A exerts a force on object B, then object B will exert an equal and opposite force on object A.

The first law stresses that objects stay at rest or continue moving in a straight line with constant speed unless forces cause accelerations that change that. Remember this was very poorly understood before Galileo and Newton came along (people actually believed the opposite, that a force was required to maintain a constant speed), so establishing this fact was important back in 1687. This law is also referred to as the "law of inertia"[2].

The second law was introduced very carefully in chapter 6 and we will continue to explore it in this lesson.

The third law is the "action-reaction" law. It tells us that *forces always come in pairs*: If object A exerts a force on object B (for example if you push a wall 2 N to the right), then object B will exert a force on object A that is equal in magnitude and opposite in direction (the wall will push you back with 2 N to the left). We use this law a lot in identifying all the forces acting on an object. A very important point to notice is that the two forces making up an **N3**

[1] Newton himself expressed this law in a more general way using the concept called *momentum*. We will return to this important detail later on in lesson 19. As stated here, the second law is also only valid in an inertial reference frame, see the next sidenote.

[2] Although Newton's first law seems to be contained in his 2nd law (because if the net force is zero, then of course the acceleration is zero, and the object has a constant velocity meaning it's either at rest or moving with a non-zero constant velocity), it is an important stand-alone law that defines what we mean by an **inertial reference frame**. Think of his first law as being an experimental 'test': If the statement is accurate within the precision of your equipment, then you have established an inertial frame of reference and you are then allowed to proceed using his 2nd law. This detail becomes more relevant in advanced courses.

force pair act on *different objects*, so they *never* "cancel each other out". This is a very common mistake to make. It's also important to know that an N3 force pair is always due to the *same interaction*. So just because two forces happen to be equal and opposite it doesn't automatically mean they are an N3 force pair. Here's an example: A book lies at rest on a table. The weight of the book is 5 N. Since the book is at rest, the table must be pushing up on it with an equal and opposite force of 5 N. These two forces are *not* an N3 force pair! First, because they act on the same object (the book). Second, because they are not due to the same interaction. Here's the correct explanation: Earth's mass pulls in the book's mass with a force of gravity equal to 5 N. According to Newton's 3rd law, the book will therefore exert an equal and opposite force of gravity on the Earth, hence the book is pulling in the Earth with 5 N in the opposite direction[3]. Due to its weight, the book pushes into the table with a force of 5 N. This is a force *on the table*. The table, however, (due to its stiffness which deep down is an electromagnetic force, more on that later), will push back on the book with an equal and opposite force of 5 N. That was another N3 force pair, two forces originating in the same interaction and acting on different objects. See figure 1 and make sure to understand this subtle detail.

[3] The Earth just doesn't accelerate because it's mass is so enormous. If two objects have comparable masses though, then they both accelerate towards each other

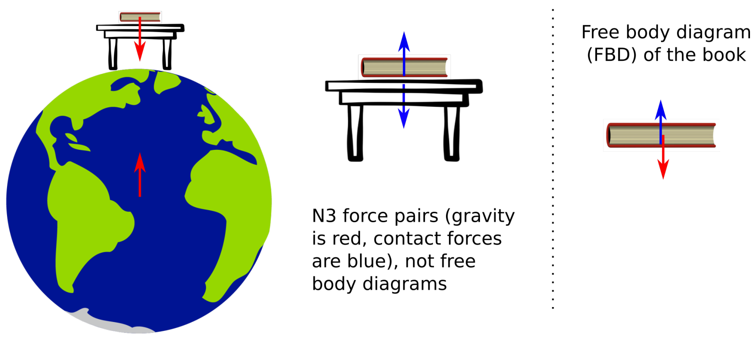


Figure 1: The diagrams on the left are *not* free body diagrams (FBDs) because they are showing N3 force pairs consisting of two equal and opposite forces acting on different objects. The diagram on the right is a FBD of the book. The force of gravity (red) and the push from the table (blue) are both forces acting *on the book*. They are indeed equal and opposite, but they are *not an N3 force pair* because they are due to two different interactions and they act on the same object. Nevertheless, they balance out and they result in the net force on the book being zero so it remains at rest.

Now you know all the basics and you are ready to dive into a more detailed study of various forces and motion in more than one dimension.

The force of gravity

The main reason we can predict the motion of moons, planets, stars and galaxies to a high degree of accuracy is because of **Newton's universal law of gravity**. This is a law of physics that describes the gravitational interaction and it can be described in words as follows: *Every mass, m_1, exerts an attractive force of gravity on every other mass, say m_2, along the line joining their centers and this force is proportional to the product of the masses and inversely proportional to the square of the distance between their centers*. It's much easier to write as a formula, here is the expression for the magnitude of this force:

$$F = G\frac{m_1 m_2}{r^2}$$

where m_1 and m_2 are two masses, r is the distance between their centers and G is a constant called the gravitational constant with the following value:

$$G = 6.67 \times 10^{-11}\,\mathrm{Nm^2kg^{-2}}$$

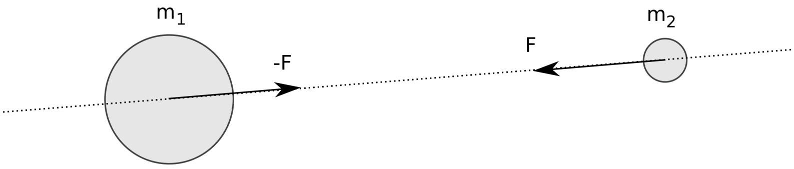


Figure 2: Newton's Universal Law of Gravity: Mass m_1 exerts an attractive force of gravity, $\vec{F}$, on mass m_2 along the line joining their centers and this force is proportional to the product of the masses and inversely proportional to the square of the distance between their centers. Due to Newton's 3rd law, m_2, will exert an equal but opposite force on m_1.

The force of gravity is responsible for the overall structure and development of the Universe. The Earth is held in orbit around the Sun (as are all the other planets) due to the Sun's gravitational pull, and the Moon is held in orbit around the Earth due to Earth's gravitational pull. The Sun shines because of gravity and we don't all fly off into space thanks to gravity!

When we stand on the Earth and only observe motion close to the surface, the formula for the gravitational attraction can be simplified significantly. Consider figure 3. Typical distances close to

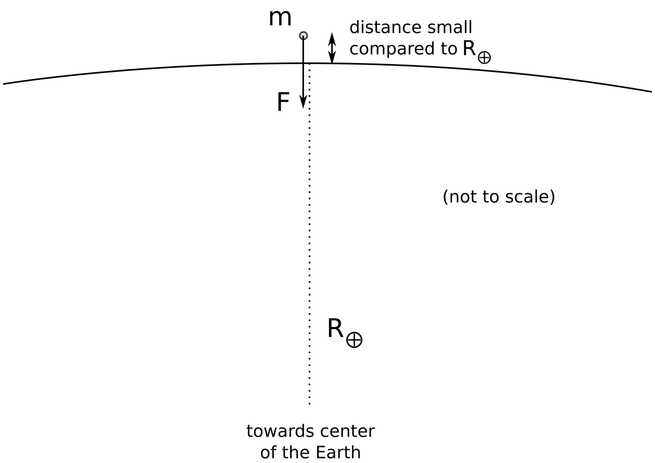


Figure 3: If we are only interested in calculating the force of gravity *close to the surface of the Earth*, then Newton's universal law of gravity formula simplifies to the usual $W = mg$ formula.

the surface of the Earth (hundreds of meters $\sim 0.1\,\mathrm{km}$) are much smaller than Earth's radius ($R_\oplus = 6370\,\mathrm{km}$) which means the distance between a mass m and the center of the Earth is approximately always equal to the Earth's radius $R_\oplus$. Hence the force of gravity on a mass m due to the mass of the Earth, $M_\oplus$, can be written as

$$F = G\frac{M_\oplus m}{R_\oplus^2} = m\left(G\frac{M_\oplus}{R_\oplus^2}\right) \equiv m g_{\mathrm{Earth}}$$

where we have defined the **gravitational field strength** at Earth's surface as g_{Earth} and the value of this can be calculated by inserting values for all the known quantities:

$$g_{\mathrm{Earth}} \equiv G\frac{M_\oplus}{R_\oplus^2} = 6.67 \cdot 10^{-11}\frac{5.97 \cdot 10^{24}}{(6370 \cdot 10^3)^2} \approx 9.8\,\mathrm{N/kg}$$

Hence, we can always just calculate the force of gravity on an object close to the surface of the Earth as[4]

$$F_{\text{gravity}} = m g_{\text{Earth}}$$

Recall this is the weight formula from earlier lessons and notice again how *weight* and *mass* are two different concepts although we often use those terms interchangeably in everyday life.

[4] Recall from an earlier lesson that the unit N/kg is in fact the same as $\mathrm{m/s^2}$, so gravitational field strength is really the same as acceleration due to gravity

$$g_{\text{Earth}} = 9.8\,\mathrm{N/kg} = 9.8\,\mathrm{m/s^2}$$

Tension (the string force)

Tension is a force that can be a bit confusing in abstract problem-solving. Let's see how it works: Consider a block of mass M attached to a string (or piece of rope) of mass m. Say we pull on the other end of the string with a force F, se figure 4. The combined system consisting of the block and the string will overall have a mass of $M + m$ and, provided the string stays attached to the mass, the system will accelerate with an acceleration given by Newton's 2nd law (we are assuming all other forces are negligible):

$$a = \frac{F}{M+m}$$

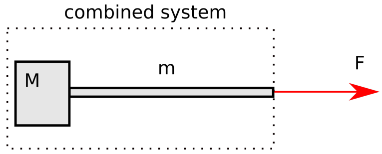


Figure 4: A block of mass M and a string of mass m. One end of the string is pulled with the force F. If we consider the block and the string to be one combined system, then we can write down Newton's 2nd law for that system.

The block and the string will both accelerate at this rate (they are attached to each other, one is not going faster than the other). If we look at the FBD of the block alone, see figure 5, the string exerts a force F' on it which is *not equal to* F because

$$F' = Ma = \frac{M}{M+m} F \neq F$$

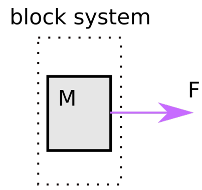


Figure 5: Here is the FBD of the block. The string is pulling in it with a force F'.

Indeed, F is larger than F' (as can be seen from the above equation) because it is accelerating more mass than F' is at the same rate. If we look at the FBD of the string alone, see figure 6, we see that the block is pulling the string to the left (N3 force pair: string pulls on block, so block pulls on string) with force $-F'$ and force F pulling on the string to the right. The net force on the string is therefore pointing to the right with magnitude

$$F - F' = (M+m)a - Ma = ma$$

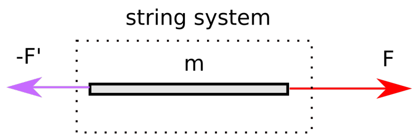


Figure 6: Here is the FBD of the string. It is being pulled in opposite directions but the right pull is larger than the left pull (otherwise it couldn't accelerate to the right).

which is simply Newton's 2nd law for the string. The string has a mass m that is also accelerating at rate a. OK, so far so good.

Now look closer at the string, let's consider a small segment of the string with mass μ at a certain position along the string, see figure 7. This little segment is accelerating to the right at rate a together with the whole string, hence there must be a net force on it given by the difference in pulls on either side of it. So there must be a pull F_1 to the right (due to the string on the right) that is a bit larger than the pull F_2 to the left (due to the string on the left). And Newton's 2nd law for this segment is

$$F_1 - F_2 = \mu a$$

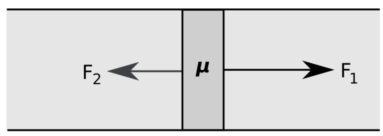


Figure 7: Here is a FBD of a small segment of the string. Since it is accelerating to the right at a rate of a, it must have a net force on it given by the difference of F_1 and F_2.

If μ becomes smaller and smaller, which means we look at a smaller and smaller segment of the string, then $F_1 - F_2$ must also become smaller and smaller. This is because a must remain constant in the above equation (the string is still accelerating). In fact, as μ approaches zero ($\mu \to 0$), $F_1 - F_2$ also approaches zero, so F_1 approaches the same value as F_2. Let's call that force T and that is what we mean by the **tension** in the string, see figure 8. *Tension arises from a material resisting being stretched*. At any given cross-section of the string there is a tension force T pulling one part of the string in one direction and the other part of the string in the other direction. This string tension, however, *varies along the string*. It is larger at the right end of the string ($T = F$) than at the left end ($T = F'$), see figure 6, and

$$F' \leq T \leq F$$

Again, this is because the tension on the right end is essentially responsible for accelerating a mass of $M + m$ whereas the tension on the left is only involved in accelerating a mass of M. It might be good to give a number example here, so assume $M = 1.00\,\text{kg}$, $m = 0.0100\,\text{kg}$, and $F = 5.00\,\text{N}$. Then

$$a = \frac{5.00}{1.0100} \approx 4.95\,\text{m/s}^2$$

and

$$F' = Ma \approx 4.95\,\text{N}.$$

Hence the string has a net force of around $0.05\,\text{N}$ on it and the tension, T, in the rope varies from $4.95\,\text{N}$ (on the left) to $5.00\,\text{N}$ (on the right). If the string is pulled with a greater and greater force, it will first break on the right rather than on the left due to the right end always having a slightly larger tension.

Now, *if we assume the mass of the string is negligible (in other words "massless"), then the two forces F and F′ are very close to being equal, which means the tension force has approximately the same magnitude* $T \approx F \approx F'$ *everywhere in the string*. This is the infamous strange property of massless strings: They "transmit" a constant tension force through the whole string and they can accelerate at any given acceleration despite the net force on them always being zero! In this case, the FBDs of the whole system and the block would look like figure 9. In almost all our problems (unless otherwise specified), you can safely assume the strings are "massless".

Figure 8: The tension T at a given point in the string. The left part of the string is pulling on the right part and vice versa (imagine one layer of molecules pulling in another layer). If the string has a mass that is not negligible, then this force varies along the string.

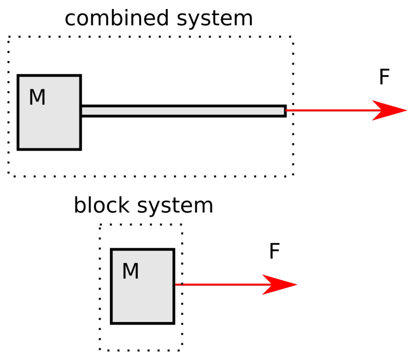


Figure 9: When a string has negligible mass the tension in it is constant everywhere and equal to the force applied on either side on it. This can sometimes be a bit confusing.

Contact forces (normal and friction)

A **contact** force is a force that appears when objects are in direct contact with each other. It's also sometimes called a **reaction** force because objects are "reacting" to each other. For example, the book and the table in figure 1 are in direct contact, so they will exert a contact (reaction) force on each other[5]. The contact force on the book due to the table (the blue force pointing up) is perpendicular

[5] This force is due to layers of molecules interacting with each other, so at a deeper level contact forces are electromagnetic in nature.

to the surface so we call that the **normal** force ("normal" is a term often used in mathematics to describe a perpendicular direction). *A normal force arises from a material resisting being compressed*.

Imagine pushing, very gently, the book sideways with a perfectly horizontal push. If your horizontal push is very small, then the book won't move (it won't accelerate out of rest) so there must be an opposing horizontal force balancing out your push. This force is called the **friction** force. Friction appears due to various interactions between the surfaces that are touching and it is always parallel to the surfaces. The complexity of these interactions is very difficult to understand in theory, so friction is mainly investigated using empirical methods.[6].

[6] A lot can be said about friction, Wikipedia has a rather good, detailed page about it.

Notice, by the way, how we have indirectly applied Newton's 2nd law independently to the vertical and horizontal directions (see figure 10). *This is the key point in doing physics in more than one dimension: Define a reference frame and apply Newton's 2nd law independently along each direction.* You get an equation for each direction which you then work with:

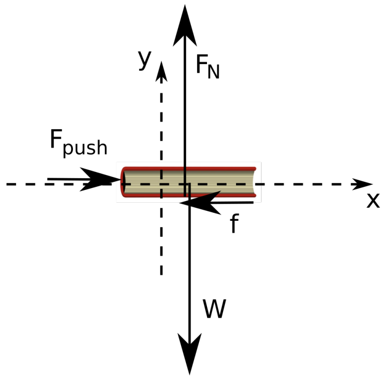


Figure 10: After defining a reference frame (a coordinate system), we can apply Newton's 2nd law along each direction independently. Here is shown the normal force, F_N, the friction force f, the weight W, and the push force. In the horizontal and vertical directions the forces balance out so the book is at rest.

$$\vec{F}_{\text{net}} = m\vec{a} \quad \Rightarrow \quad \begin{aligned} F_{\text{x net}} &= ma_x \\ F_{\text{y net}} &= ma_y \end{aligned}$$

The classic example of mechanics in two dimensions is a block on a slope which we will cover in our next section.

If the object is not moving, we refer to the friction force as a **static** friction force. If the object is moving we refer to the friction force as a **dynamic** (or **kinetic**) friction force. Experiments show that the static friction force, f_s can vary from zero to some maximum value:

$$0 \leq f_\text{s} \leq f_{\text{s max}}$$

The maximum value is found to be proportional to the normal force (the more the surfaces are pushed together, the larger the friction), so we model it as

$$f_{\text{s max}} = \mu_\text{s} F_\text{N}$$

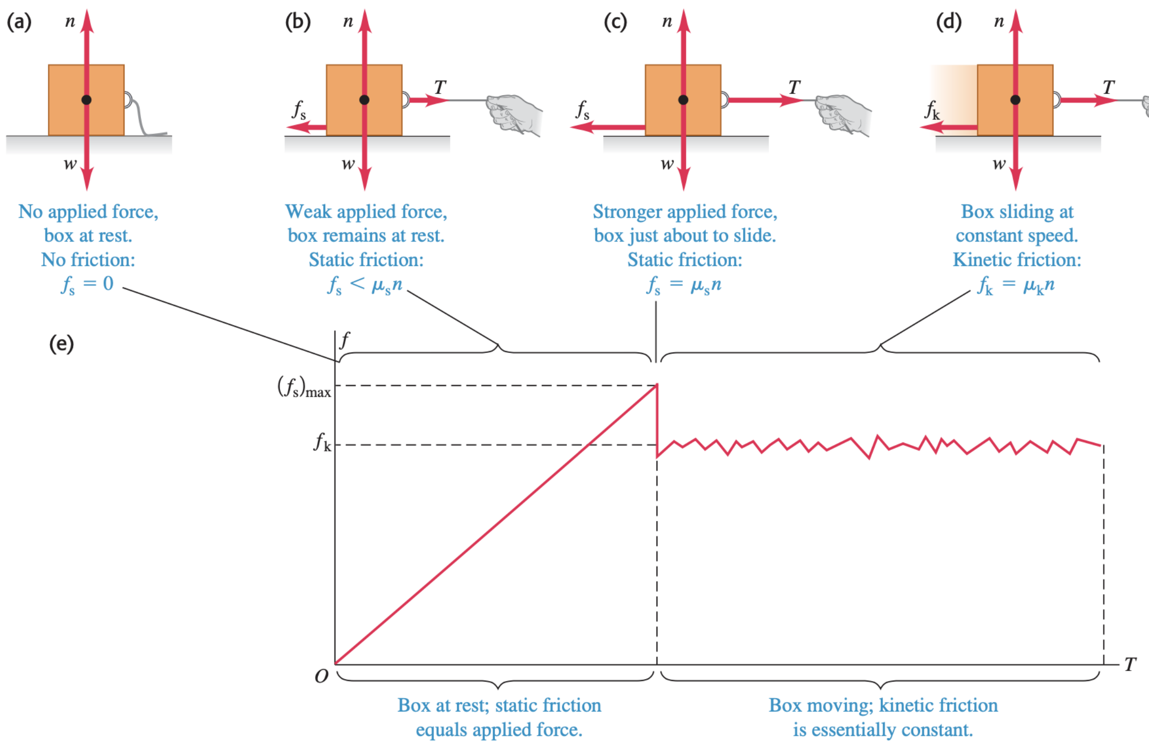
where F_N is the normal force and μ_s is a constant called the **coefficient of static friction** (or just static friction constant).

Experiments show that the dynamic friction force, f_d, always has a constant value that is also proportional to the normal force but is slightly less than the maximum static friction force:

$$f_\text{d} = \mu_\text{d} F_\text{N} < f_{\text{s max}}$$

where μ_d is a constant called the **coefficient of dynamic friction** (or just dynamics friction constant). The next figure shows a graph of how the friction force on a block changes as the applied force increases.

The normal force and the friction force are two force components of the overall contact force. The normal force is the component perpendicular to the surface and friction is the component parallel to the surface. For example, take a book of weight 3 N lying at rest on



a table, see figure 11. If I push down on the book with a force of 1 N that makes an angle 60° with the vertical, then what is the necessary contact force, F_{table}, required to keep it at rest? First look in the perpendicular direction. The vertical component of my push is 1 N cos 60° downwards and the weight is 3 N vertically downwards. So the normal force provided by the table must balance out these two forces added together

$$F_N = 1\cos 60^\circ + 3 = 3.50\,\mathrm{N}$$

The normal force is larger than the weight because the table has to support the weight of the book and the vertical component of the additional push. The horizontal component of my push is 1 N sin 30° sideways, so the static friction force must balance out this force

$$f_s = 1\sin 60^\circ \approx 0.87\,\mathrm{N}$$

The overall contact force, $\vec{F}_{table}$ will therefore have a magnitude of

$$|\vec{F}_{table}| = \sqrt{F_N^2 + f_s^2} \approx 3.62\,\mathrm{N}$$

and it will point upwards to the left at an angle of (see figure 13)

$$\theta = \tan^{-1}\frac{0.87}{3.50} \approx 14^\circ$$

to the vertical. All forces balance out nicely (the vectors add up to zero using the tail-to-tip method, see figure 12) and the book remains at rest.

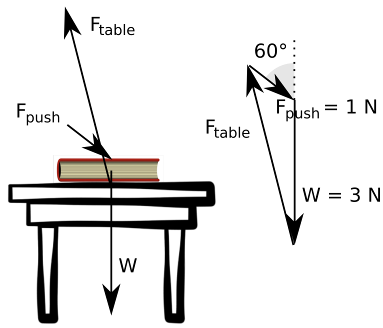


Figure 11: A book is being pushed into the table at an angle. It is at rest, so the contact force is balancing out the combined effect of the push and the weight. The vector sum of the forces must be zero.

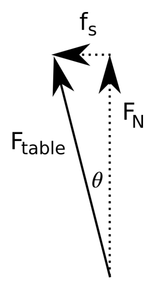


Figure 12: The normal force and the friction force are the vertical and horizontal components of the contact (reaction) force. The magnitude and direction of the contact force is found using simple right-angled trigonometry as shown on the left.

A block on a slope

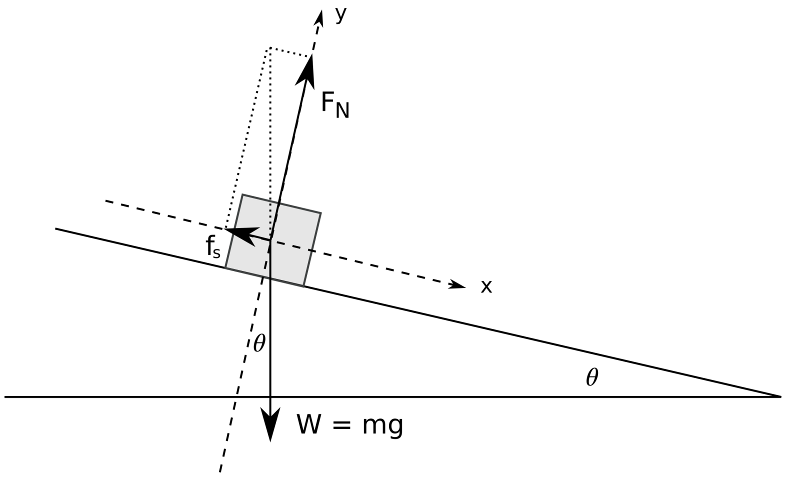


Figure 13: The classic two-dimensional motion problem: The famous block on a slope. The block is being pushed into the slope and that contact force can be resolved into a normal component perpendicular to the slope and a friction component parallel to the slope.

Consider a block on a slope that makes an angle θ to the horizontal. If the block is at rest, the weight is balanced out by a contact force from the slope, hence there is a friction force parallel to the slope and a normal force perpendicular to the slope as shown in figure 13. When describing situations like these, it is convenient to choose a reference frame that has an x-axis pointing down the slope, and a y-axis pointing perpendicular to the slope as shown in the figure. It's important to remember that you are always free to define your reference frame as you wish (see question 26).

Let's first write down Newton's 2nd law for the x-direction (parallel to the slope):

$$F_{\text{x net}} = ma_x \quad \Rightarrow \quad mg\sin\theta - f_s = 0$$

In the above equation, I used trigonometry to express the component of weight parallel to the slope as $mg\sin\theta$, see figure 14, and I set the acceleration to zero because the block is not accelerating in this direction (we assumed it's at rest). Hence from this application of Newton's 2nd law in the x-direction we find that friction can be expressed as

$$f_s = mg\sin\theta \tag{1}$$

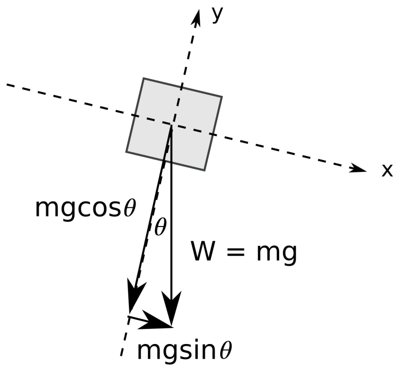


Figure 14: The normal force and the friction force are the vertical and horizontal components of the contact (reaction) force. The magnitude and direction of the contact force is found using simple right-angled trigonometry as shown on the left.

This is simply an equation stating that the friction force is balancing out the component of gravity down the slope. Now let's apply Newton's 2nd law in the y-direction:

$$F_{\text{y net}} = ma_y \quad \Rightarrow \quad -mg\cos\theta + F_N = 0$$

In the above equation, I used trigonometry to express the magnitude of the component of weight perpendicular to the slope as $mg\cos\theta$, see figure 14, and I set the acceleration to zero because the block is not accelerating in this direction (it is always *on* the slope). Notice the weight component is negative in the equation because it points in the negative direction of the y-axis. Hence from this application of Newton's 2nd law in the y-direction we find that the normal force is simple balancing out the component of gravity perpendicular to the slope:

$$F_{\mathrm{N}} = mg\cos\theta \tag{2}$$

Now imagine slowly increasing the angle of the slope. What will eventually happen? At some point it will start sliding down. Right before it starts sliding down the slope, the static friction force will have reached its maximum value $\mu_{\mathrm{s}} F_N$. Hence from (1) and (2) we get

$$\begin{aligned} \mu_{\mathrm{s}} F_N &= mg\sin\theta \\ \mu_{\mathrm{s}} mg\cos\theta &= mg\sin\theta \\ \mu_{\mathrm{s}} &= \tan\theta \end{aligned} \tag{3}$$

which gives a method to measure the static friction constant between the block and the slope: Just measure the angle at which the block starts sliding down!

If the angle is set to an angle greater than the angle given by (3), then the block will always slide down the slope. The friction force will in that case always be given by the dynamic friction force which has a constant value. In these situations the block will in fact accelerate down the slope, since there is a non-zero net force parallel to the slope. Here are the details: First, Newton's 2nd law parallel to the slope (in the x direction),

$$\begin{aligned} F_{\mathrm{x\ net}} &= ma_x \quad \Rightarrow \\ mg\sin\theta - f_{\mathrm{d}} &= ma_x \quad \Rightarrow \\ mg\sin\theta - \mu_{\mathrm{d}} F_N &= ma_x \end{aligned}$$

Newton's 2nd law perpendicular to the slope (in the y direction) *gives us the same as before, because there is still no acceleration in the y direction* (the block remains *on* the slope):

$$\begin{aligned} F_{\mathrm{y\ net}} &= ma_y \quad \Rightarrow \\ -mg\cos\theta + F_N &= 0 \quad \Rightarrow \\ F_N &= mg\cos\theta \end{aligned}$$

Combining the two results above, we get the acceleration of the block down the slope

$$\begin{aligned} mg\sin\theta - \mu_{\mathrm{d}} mg\cos\theta &= ma_x \\ a_x &= g(\sin\theta - \mu_{\mathrm{d}}\cos\theta) \end{aligned} \tag{4}$$

Knowing how to apply Newton's 2nd law to an object on a slope is a very important example to be familiar with, so make sure you really understand everything explained above.

Friction in fluids

A fluid is the term we often use for either a gas or a liquid and the friction experienced when objects move through such materials is a very complex matter to investigate. The complexity arises from the difficulty of finding solutions to the mathematical models that describe the interactions of billions and billions of particles in a fluid.[7]

In very simple cases, however, there is an easy way to model the drag experienced by an object moving through a fluid. Experiments show that the drag is proportional to the velocity at low speeds,

$$F_{\text{drag}} \propto v \quad \text{(at low speeds)}$$

and proportional to the velocity squared at high speeds,

$$F_{\text{drag}} \propto v^2 \quad \text{(at high speeds)}$$

I simulated drag for projectile motion in lesson 16, question 7, using the low-speed model and solved the equations exactly using calculus.

[7] The Navier-Stokes equations are complex equations that arise from applying Newton's 2nd law to the mechanics of fluids and they are very hard to solve. One of the seven most important open problems in mathematics today is actually related to solving these equations. The Clay Mathematics Institute has offered a US$1 million prize for a solution. Fluid mechanics has applications in a wide range of disciplines, including mechanical, civil, chemical and biomedical engineering, geophysics, oceanography, meteorology, astrophysics, and biology.

Buoyancy (Archimedes' law)

In a fluid at rest (a *static* fluid, for example a column of water at rest), the pressure in the fluid depends on the depth. Most of you will have experienced that when diving to the bottom of a swimming pool. This increase in pressure is due to the weight of the fluid above. One consequence of this is the force of **buoyancy** (also called **upthrust**) which is the force that appears when an object is put into a fluid. Archimedes (287-212 BCE) – often said to be the greatest thinker of antiquity – discovered the famous law that bears his name: *The buoyancy on an object is equal and opposite to the weight of the displaced fluid.*

Figure 15: It almost defies common sense to see heavy ships float on water. The large amount of displaced water is what creates the necessary buoyancy to balance out the huge weight.

The elastic force (Hooke's law)

When springs (and other elastic materials) are stretched or compressed, they apply a restoring force on their surroundings. For relatively small displacements, the restoring force (often called the spring force or elastic force) is *proportional and opposite to the displacement*, That is,

$$F_{\text{elastic}} = -kx$$

where x is the displacement from the spring's equilibrium position, and k is the **spring** (or **elastic**) **constant**. This equation is called **Hooke's law** after another great scientist Robert Hooke (1653-1703). A large value of k means a stiff spring; a small value means a weak spring. If x is positive, then the force is negative; and if x is negative, then the force is positive, so Hooke's law does indeed describe a restoring force, where the spring always tries to bring x back to zero. When graphing Hooke's law (see figure 16), we often only focus on the magnitude of the force, but when applying Newton's 2nd law it's important to remember the direction.

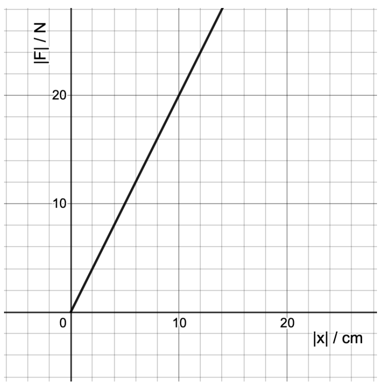


Figure 16: Hooke's law visualised in a force vs. displacement diagram (magnitudes only). From graphs like these, the spring constant k is the slope of the straight line. In this case $k = 2\,\text{N/cm}$, so it requires a force of 2 N to stretch the spring 1 cm.

Tension and normal forces (see earlier sections) are actually just special cases of spring forces. E.g. when you stand on a floor, the floor behaves like an extremely stiff spring: The floor compresses a tiny amount and that compression causes an upward elastic force that is enough to balance out your weight.

We will always assume springs are massless, otherwise it can get rather complicated.

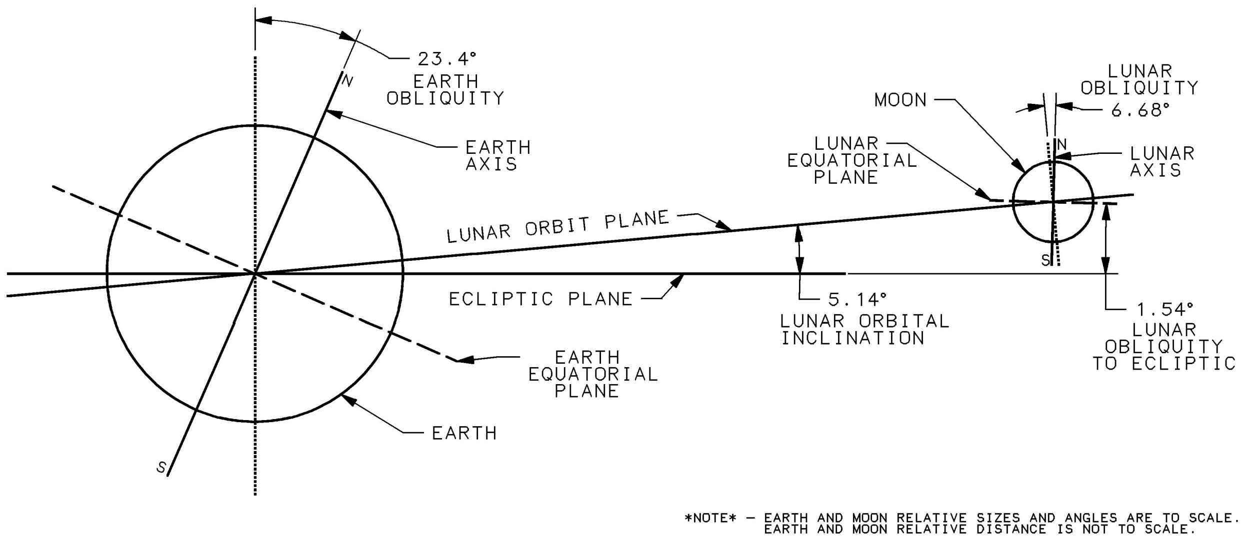
Lesson 17: Questions and activities

Whenever possible, try to experimentally verify your answers to the questions below. It helps to see these things unfold in real life – don't forget science is about understanding reality, it's not just about getting math problems right!

1. The mass of the Sun is $M_\odot = 2.0 \times 10^{30}$ kg and the mass of the Earth is $M_\odot = 6.0 \times 10^{24}$ kg. Calculate the amount of gravitational attraction between the two. The distance between the Sun and the Earth is 150 million kilometers. (*Hint: Don't forget to convert km to m so that units are consistent.*)

2. (a) Calculate, to one significant figure, the magnitude of Earth's pull in the Moon. The Moon has mass 7×10^{22} kg and the distance between (the centers of) the Earth and the Moon is 384000 km.

   (b) Draw on the diagram below the force of gravity acting on the Moon due to Earth's gravitational pull. Let 1 cm correspond to 4×10^{19} N and draw the length of the arrow according to that scale (note the relative distance between the Earth and the Moon is not to scale, but the relative sizes and angles are).



   (c) Imagine you place a 1 kg object exactly halfway between the Earth and the Moon (on the lunar orbit plane, see above).

      i. How much is Earth pulling in the object?

   ii. How much is the Moon pulling in the object?
   iii. What is the resultant force exerted on the object?
   iv. In which direction will it accelerate? What is the magnitude of the acceleration?
   v. At which distance from the center of the Earth would the net acceleration of the object be exactly zero?

3. Look up the mass and radius of Mars and use those quantities to calculate the gravitational field strength on its surface. Discuss how a Martian held Olympics would differ from the Earth held Olympics.

4. A heavy rope of mass $m = 2.0\,\text{kg}$ and length $6.0\,\text{m}$ is attached to the ceiling and the other end hangs freely. Let $g = 10\,\text{m/s}^2$.

   (a) What is the weight of the whole string?
   (b) Draw a FBD of the string.
   (c) What is the magnitude of the tension force at the top of the string?
   (d) Consider a point $x = 1.5\,\text{m}$ from the bottom of the rope. If the rope is completely uniform (which means the mass is evenly distributed and it has the same cross-sectional area), then what is the tension at the point x?
   (e) What is the tension at the bottom of the rope?
   (f) Can you write down a formula that shows how the tension force varies with x? What is the tension when $x = 0$?
   (g) Now a mass of $3\,\text{kg}$ is attached to the bottom of the rope. All quantities are unchanged. Answer all the above questions for this new situation.
   (h) If the rope is assumed massless, how do your answers to (g) change?

5. A force of $125\,\text{N}$ is required to extend a spring by $2.80\,\text{cm}$. What force is required to stretch the same spring by $3.20\,\text{cm}$?.

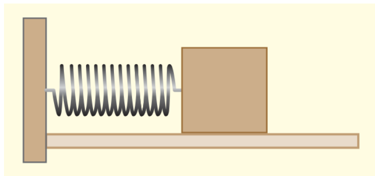


Figure 17: The spring is compressed and the mass is at rest on a rough table.

6. A spring is compressed a certain distance and a mass is attached to its free end as shown in figure 17. The table is rough and the mass is at rest. Draw a free-body diagram for the mass.

7. Evelyn claims: "When you step on a body scale to measure your mass, the force that is being measured is the normal force and not your weight." Is this a true statement?

Figure 18: Weightlessness is the rave of 2021!

8. Maxwell has a mass of 50 kg and he is on a body scale in an elevator. The body scale reads 50 kg when the elevator is at rest. Let $g = 10\,\text{m/s}^2$.

   (a) The elevator briefly accelerates upwards at a rate of $1\,\text{m/s}^2$. What is the normal force on Maxwell during the acceleration? This is sometimes referred to as "apparent weight" – why do you think that? What does the body scale reading show during this acceleration?

(b) Now the elevator briefly accelerates downwards at a rate of $1\,\mathrm{m/s^2}$. What is his apparent weight during this acceleration?

(c) What is his apparent weight if the elevator accelerates downwards at $10\,\mathrm{m/s^2}$? This is called being "weightless" – does he really have no weight?

(d) Test your results by taking a body scale into our school's elevator!

9. The upward normal force exerted by the floor is 620 N on an elevator passenger who weighs 650 N. What are the Newton's third law reaction forces to these two forces? Is the passenger accelerating? If so, what are the magnitude and direction of the acceleration?

10. A beach ball is fully submerged under water and it displaces $2000\,\mathrm{cm^3}$ of water. What is the buoyancy (= upthrust) on the beach ball according to Archimedes' law? Let $g = 10\,\mathrm{m/s^2}$ and water has a density of $1\,\mathrm{g/cm^3}$. If the beach ball has a mass of 0.3 kilograms, how much force do you need to apply on it to keep it at rest underwater?

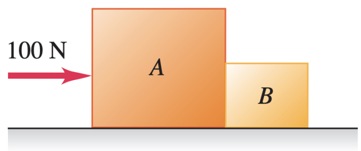


Figure 19: Two boxes. Good for practicing how to apply $F = ma$!

11. Boxes A and B are in contact on a horizontal, frictionless surface, as shown in figure 19. Box A has mass 20.0 kg and box B has mass 5.0 kg. A horizontal force of 100 N is exerted on box A. What is the magnitude of the force that box A exerts on box B?

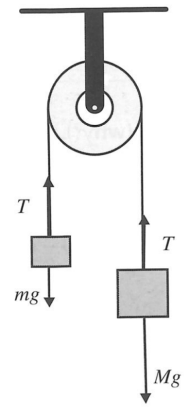


Figure 20: Atwood's machine - another good way of practicing how to apply $F = ma$!

12. Two masses of $m = 4.0\,\mathrm{kg}$ and $M = 6.0\,\mathrm{kg}$ are joined together by a string that passes over a pulley, see figure 20 (setups like these are referred to as an "Atwood's machine"). The masses are held stationary and suddenly released. What is the acceleration of each mass?

13. In a world without friction, which of the following activities could you do (or not do)? Explain your reasoning. (a) drive around an unbanked highway curve; (b) jump into the air; (c) start walking on a horizontal sidewalk; (d) climb a vertical ladder; (e) walk up stairs; (f) change lanes on the freeway.

14. Consider a frictionless slope (for example, a slope made of ice) making an angle θ with the horizontal. If a block is placed on the slope what is its acceleration down the slope?

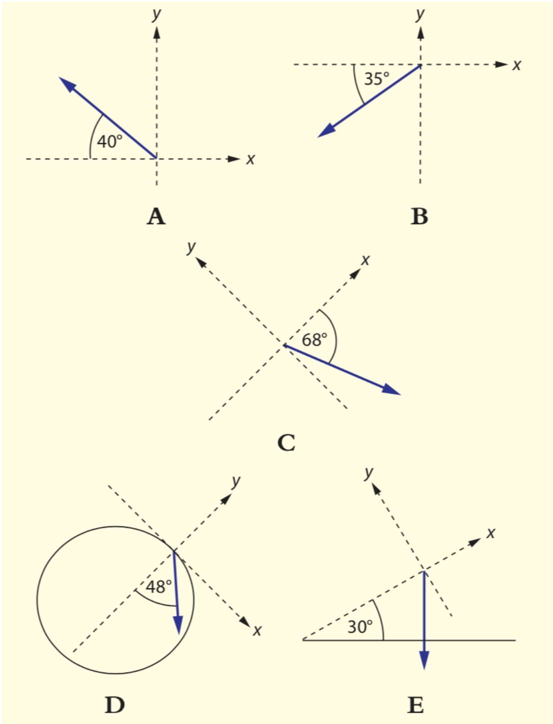
15. A block rests on an inclined plane with enough friction to prevent it from sliding down. To start the block moving, is it easier to push it up the plane or down the plane? Why?

16. A skier of mass 65.0 kg is pulled up a snow-covered slope at constant speed by a tow rope that is parallel to the ground. The ground slopes upward at a constant angle of 26.0° above the horizontal, and you can ignore friction.

(a) Draw a clearly labeled free-body diagram for the skier.

(b) Calculate the tension in the tow rope.

17. For each diagram shown below, find the components of the vectors along the axes shown. Take the magnitude of each vector to be 10.0 units.



18. In figure 21 a worker lifts a weight w by pulling down on a rope with a force F. The upper pulley is attached to the ceiling by a chain, and the lower pulley is attached to the weight by another chain. In terms of w, find the tension in each chain and the magnitude of the force F if the weight is lifted at constant speed. Include the free-body diagram or diagrams you used to determine your answers. Assume that the rope, pulleys, and chains all have negligible weights.

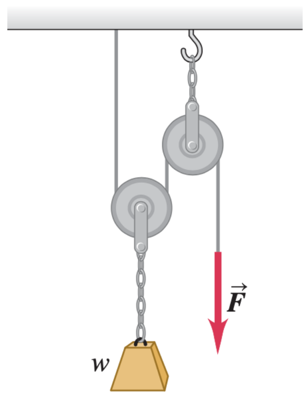
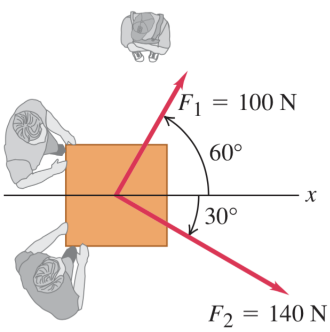


Figure 21: A pulley is a 'simple machine' that gives the user a mechanical advantage.

19. Two adults and a child want to push a wheeled cart in the direction marked x in the figure below. The two adults push



with horizontal forces F_1 and F_2 as shown in the figure. (a) Find the magnitude and direction of the smallest force that the child should exert. You can ignore the effects of friction. (b) If the child exerts the minimum force found in part (a), the cart accelerates at $2.0\,\mathrm{m/s^2}$ in the positive x-direction. What is the weight of the cart?

20. A block of mass M is on a frictionless slope. It is attached to another mass, $m < M$, via a string that goes over a pulley positioned at the top of the slope. The smaller mass is hanging freely from the string. At which angle are the two blocks at rest? Give your expression in terms of all the relevant quantities in the problem.

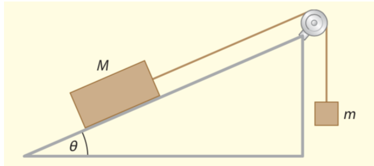


Figure 22: Determine the angle of the slope in terms of M and m in order to have equilibrium.

21. A floater of mass 8 kg is tied to the bottom of swimming pool by two strings, each making an angle of 25.0° to the vertical. The floater displaces $0.050\,\mathrm{m^3}$ of water.

    (a) Find the weight of water displaced by the floater.

    (b) Determine the vector sum of the upthrust on the floater and the weight of the floater.

    (c) Determine the tension in each of the strings.

22. A block of mass 2.0 kg rests on top of another block of mass 10.0 kg that itself rests on a frictionless table. The largest frictional force that can develop between the two blocks is 16 N.

    (a) Calculate the largest force with which the bottom block can be pulled so that both blocks move together without sliding on each other.

    (b) If a force of 110 N is applied to the 10 kg block, what is the acceleration of the 2 kg block *relative to the 10 kg block*?

23. A solid uniform 45.0 kg ball of diameter 32.0 cm is supported against a vertical, frictionless wall using a thin 30.0 cm wire of negligible mass, as shown in figure 23.

    (a) Draw a free-body diagram for the ball and use it to find the tension in the wire.

    (b) How hard does the ball push against the wall?

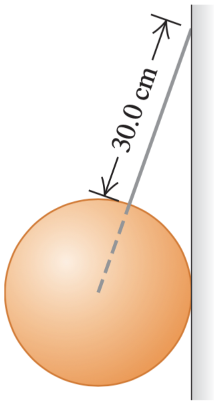


Figure 23: A ball resting against a wall.

24. A mass m is attached to two identical springs of spring constant k. The other end of each spring is attached to the ceiling so that each makes an angle θ with the vertical. If the mass is in equilibrium, what is the extension of each spring?

25. A student suspends a chain consisting of three links, each of mass $m = 0.250\,\mathrm{kg}$, from a light rope. She pulls upward on the rope, so that the rope applies an upward force of 9.00 N to the chain.

    (a) Draw a free-body diagram for the entire chain, considered as a body, and one for each of the three links.

(b) Use the diagrams of part (a) and Newton's laws to find (i) the acceleration of the chain, (ii) the force exerted by the top link on the middle link, and (iii) the force exerted by the middle link on the bottom link.

26. Write down the equations of motion for a block accelerating down a rough slope but use a reference frame consisting of a horizontal x-axis and a vertical y-axis (hence this time the axes are *not* parallel and perpendicular to the slope). You should find that the block is accelerating in both directions – does that make sense? Can you check if you got the correct result by comparing with (4)?

27. Here's a classic brainteaser: *The Monkey and Bananas Problem*. A 20 kg monkey has a firm hold on a light rope that passes over a frictionless pulley and is attached to a 20 kg bunch of bananas, see figure 24. The monkey looks up, sees the bananas, and starts to climb the rope to get them. (a) As the monkey climbs, do the bananas move up, down, or remain at rest? (b) As the monkey climbs, does the vertical distance between the monkey and the bananas decrease, increase, or remain constant? (c) The monkey releases her hold on the rope. What happens to the distance between the monkey and the bananas while she is falling? (d) Before reaching the ground, the monkey grabs the rope to stop her fall. What do the bananas do?

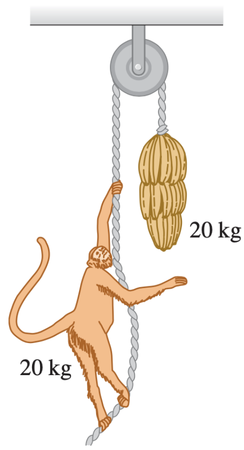


Figure 24: A classic brainteaser: *The Monkey and Bananas Problem*.

28. Solve question 20 again but this time when the slope is not frictionless.

Answers to all the questions.

Lesson 17 Quiz

Check your understanding of this lesson: Here is a quiz.