A little book about motion

Energy

On a CIS Project Week trip to New Zealand, I once tried the Skyswing in Rotorua (I don't recommend it :). You sit in a little cage attached to a few long cables and after being pulled 90 degrees sideways, the cage is suddenly released. At the site there was a sign claiming that you will reach a speed of 150 km/h at the bottom 2 s after being released 50 m above the ground. That seemed a bit too good to be true, and I wanted to fact check these claims. How would you go about doing that? You could, of course, simply measure these quantities precisely and you have the answer – *never forget that the experiment is the logic of science!* But if taking measurements isn't an option, how can we make a theoretical prediction? How can we calculate the speed at the lowest point using only the methods that we have learnt so far? And how can we calculate the time it takes to reach this lowest point?

Figure 1: The Skyswing in Rotorua.

As a start, we can work out these quantities for a straight free fall: Falling straight down for 50 m with a constant acceleration of $g = 9.8\,\mathrm{m\,s^{-2}}$ results in a final velocity of (neglecting air resistance)

$$2as = v^2 - u^2 \quad \Rightarrow \quad v = \sqrt{2 \cdot 9.8 \cdot 50} \approx 31\,\mathrm{ms^{-1}} \approx 113\,\mathrm{km\,h^{-1}}$$

and it takes the time

$$s = \frac{1}{2}gt^2 \quad \Rightarrow \quad t = \sqrt{\frac{2 \cdot 50}{9.8}} \approx 3.2\,\mathrm{s}$$

This seems quite different from the stated 150 km/h in 2 s (if you include air resistance the difference is even greater), but obviously the Skyswing isn't exactly a free fall, it moves along a circular path. Will this make it go faster than a free fall? Something to think about. Let me already now reveal the point of this lesson: By introducing the concepts of **work** and **energy**, we'll be able to (at least partially) solve this problem very quickly without having to do a lot of complicated calculations. But let's first try to analyse the Skyswing using the methods we have learnt so far: Free-body diagrams and Newton's 2nd law of motion.

Figure 2: Physics is far less terrifying than a ride on the Skyswing!

A closer look at the Skyswing

Let's model the Skyswing as a simple pendulum, see figure 3. I have shown four instants of the motion with corresponding free-body diagrams (we are neglecting air resistance). At positions (1) and (2) the weight has been resolved into a component in the *radial* direction, W_r, and a component in the *tangential* direction, W_θ (the cross on W indicates it has been replaced by the two components).

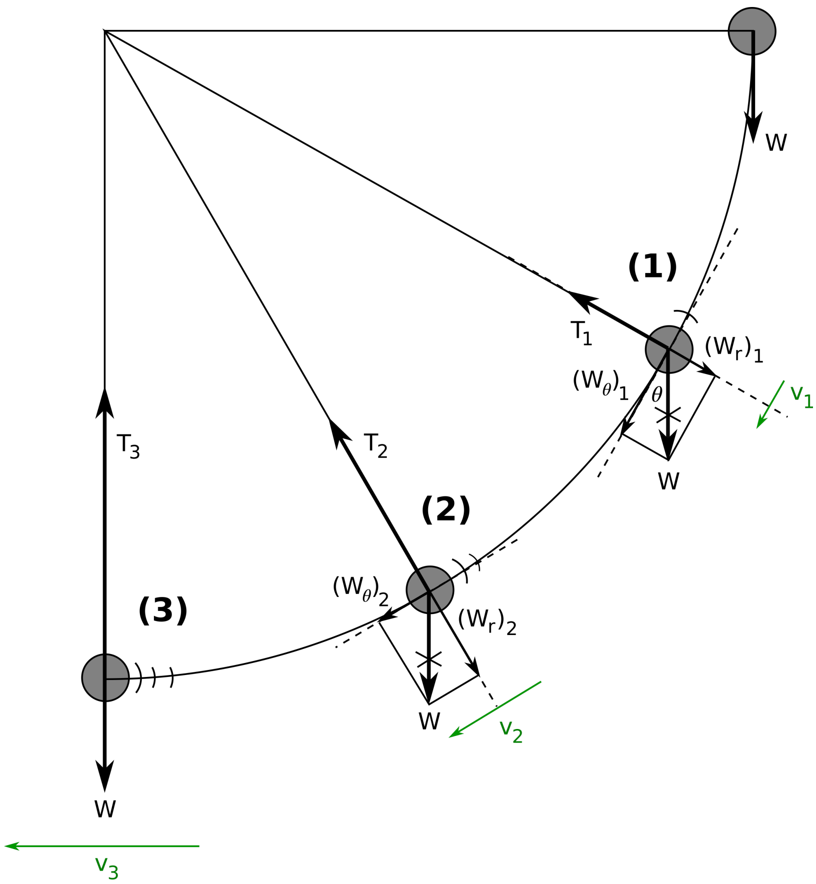


Figure 3: The Skyswing modelled as a simple pendulum. At positions (1) and (2) the weight has been resolved along the radial and tangential directions (the cross on W indicates the weight has been replaced by these two components).

At the initial position, weight W is the only force acting on the particle so it will accelerate straight down. Since the string is assumed to be inextensible, this will produce tension in the cable which will force the particle into a circular path. There must, therefore, be a net force in the radial direction pointing towards the center, and this centripetal force is given by

$$T_1 - (W_r)_1 > 0$$

which causes a centripetal acceleration. There is, however, also a net force in the *tangential* direction which is the component of weight pointing along the tangent of the circle

$$(W_\theta)_1$$

This unbalanced force creates a tangential acceleration which speeds up the particle in the tangential direction of motion. Hence it is continuously increasing its speed along the circular path, so we can conclude that this is *not* uniform circular motion. This is true for all pendulums.

At position (2), the speed is higher than at position (1), and since the radius hasn't changed it must have a greater centripetal force on it (remember mv^2/r). This is achieved by the tension increasing (the string needs to pull more in order to change the direction of a greater velocity). Notice how the radial component of weight is increasing while the tangential component is decreasing.[1] Finally, at position (3) there is no more tangential acceleration and it obtains the largest velocity (pointing to the left) and the largest centripetal acceleration (pointing towards the center).

[1] A good exercise is to try and draw the net force at each position shown in the figure.

From the above analysis it should be clear that we cannot solve this problem by a direct application of the suvat equations or the equations for uniform circular motion (neither the acceleration nor the speed are constant). But don't give up too easily! We can in fact still make progress with some clever thinking and **numerical analysis**. Here's what we could do[2]: First approximate the circular path to a series of straight lines connecting points on the circle, see figure 4. Now think of the particle as moving from point to point along those straight line segments rather than the circular path (as the number of line segments increases, the segment path will approach the circle). At each point we can work out the *component of weight along the straight line segment* and the corresponding acceleration that it causes. *Assuming this acceleration is constant* we can then use suvat to get the change in speed, Δv, between the two points and the travel time. At the next point we repeat the process and so on. By adding up all these contributions we can eventually find the final speed and the total amount of time it takes. Here are some more details of the math involved, and here is a spreadsheet using 200 line segments in which everything is calculated (you can experiment by changing the parameters in the spreadsheet). The bottom row in the spreadsheet gives us the answer: The final speed is around 31 m/s and the time it takes is 4.2 s. Note that *the final*

[2] This way of solving problems is not part of the DP syllabus so don't worry if you don't get all the details. But it's good to see an example of how a rather complex problem can be solved numerically. Never be too "proud" to solve a problem numerically – there is no "preferred" way to get a correct, useful answer!

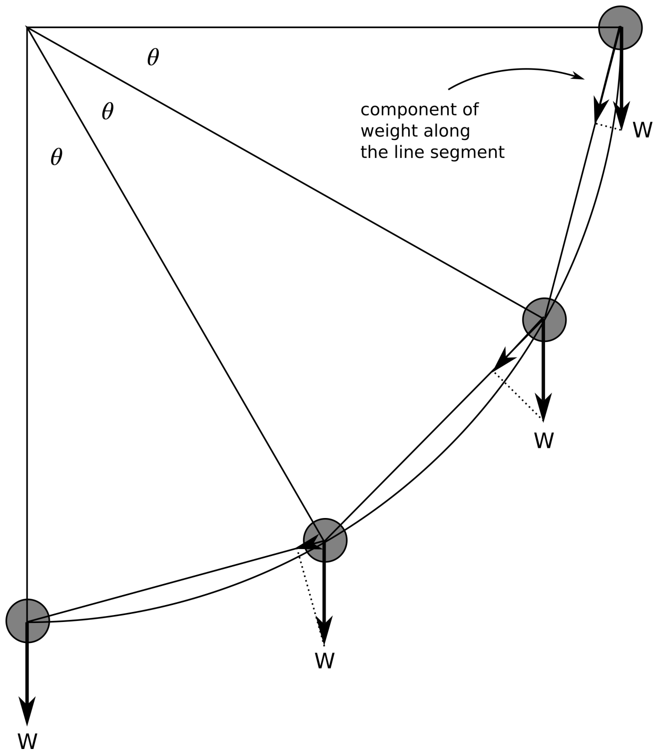


Figure 4: Pendulum motion approximated to motion along straight line segments. By finding the component of weight along the line segments and assuming that is constant for the whole line segment, then we can apply the suvat equations to each line segment and work out the time it takes to travel along a line segment and the increase in speed at the end of a line segment.

speed is exactly the same as when it falls straight down, which might be a bit surprising, whereas *the time is longer than a free fall*, which perhaps isn't that surprising after all since it travels a longer distance and on average has a smaller tangential acceleration compared to the constant value of $g = 9.8\,\mathrm{m/s^2}$ in a free fall.

OK, having solved the problem in a complicated way (and proving the Skyswing is giving its customers false information!), let's see how we can solve for the final speed in a much simpler way, it goes like this: A vertical drop of $h = 50\,\mathrm{m}$ in a uniform gravitational field will – *regardless of the path it takes and assuming there is no air resistance* – convert **gravitational potential energy**

$$mgh$$

at the highest point into **kinetic energy**

$$\frac{1}{2}mv^2$$

at the lowest point. This **conservation of energy** can be expressed as

$$mgh = \frac{1}{2}mv^2$$

which immediately gives us the final speed:

$$v = \sqrt{2gh} = \sqrt{2 \cdot 9.8 \cdot 50} \approx 31\,\mathrm{m/s}$$

That was much easier to do than draw free-body diagrams and work out 200 rows in a spreadsheet![3] In the next section I'll explain where the magical expressions mgh and $\frac{1}{2}mv^2$ come from. Although the final calculation above looks like a suvat equation, it's important

[3] Note that this energy approach only gives us the magnitude of the velocity vector (the speed), not the direction – although in this particular context we know what it is.

to know that it isn't – suvat cannot be used for a circular path with non-uniform acceleration.

Can we also use this energy approach to find out how long the Skyswing takes to reach the lowest point? It turns out that we *cannot* do that. So we managed to find part of the problem (the speed) very quickly, but at the expense of not being able to solve the rest of the problem (the time) – there are no free lunches in life!

The work done by a force

In order to understand the energy approach shown in the previous section, we first need to introduce a new concept called *the work done by a force*. This concept was introduced by the French mathematician Gustave-Gaspard Coriolis in 1826. It's worth mentioning that work and energy was completely unknown to Newton because all these ideas were first introduced in the 1800s.

Imagine a block being dragged along the floor with a constant velocity, see figure 5. Say it undergoes a displacement of $\Delta\vec{x}$ along

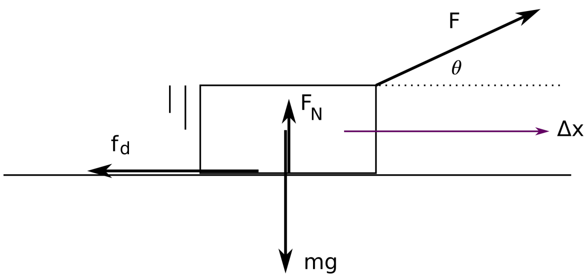


Figure 5: A block moving to the right with constant velocity.

the floor. There are four forces acting on the mass: Weight, a normal force, a (dynamic) friction force and a pull force $\vec{F}$. The pull force makes an angle θ with the displacement vector. Since the block has a constant velocity, the net force is zero and all forces balance out (concept check: why is the magnitude of the normal force less than the weight?). We define **the work done by a force** $\vec{F}$ as[4]

$$W \equiv |\vec{F}||\Delta\vec{x}|\cos\theta \tag{1}$$

where θ is the angle between the force vector and the displacement vector. You should think of this as *the component of the force along the direction of the displacement multiplied by the displacement*,

$$\left(|\vec{F}|\cos\theta\right)\cdot|\Delta\vec{x}|$$

as can be seen by comparing with figure 6. If force is measured in newtons, N, and displacement in meters, m, then work has the unit Nm which per definition is called the **joule**, J,

$$\mathrm{J} \equiv \mathrm{Nm}$$

Let's do some calculations: Say the pull force in our example has a magnitude of $|\vec{F}| = 2.0\,\mathrm{N}$, it makes an angle $\theta = 30°$ to the horizontal, and the displacement has a magnitude of $7.0\,\mathrm{m}$. How

[4] When you learn more about vectors in mathematics, you will be told that this definition is called the **scalar (or dot) product** and we write it as

$$\vec{F}\cdot\Delta\vec{x} \equiv |\vec{F}||\Delta\vec{x}|\cos\theta$$

Scalar products, and vectors in general, were quantities invented by physicists to help them better describe how the world works.

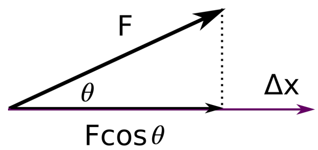


Figure 6: $F\cos\theta$ is the component of the force along the direction of the displacement.

much work does the pull force do? Just plugging into the formula we get

$$W \equiv |\vec{F}||\Delta\vec{x}|\cos\theta = 2.0\,\mathrm{N}\cdot 7.0\,\mathrm{m}\cdot\cos(30^\circ) \approx 12\,\mathrm{J}$$

What about the normal force, F_N? How much work does that do? Again, just plug into the definition and pay good attention to what the angle between the force and the displacement is. In this case the angle is 90 degrees so the work done by the normal force is zero:

$$W \equiv |\vec{F}_N||\Delta\vec{x}|\cos(90^\circ) = 0\,\mathrm{J}$$

The weight also does an amount of work equal to zero. Finally, what about the friction force? The magnitude of the friction force is equal to the horizontal component of the pull force which is $|\vec{f}_d| = F\cos\theta = (2\,\mathrm{N})\cos(30^\circ) \approx 1.7\,\mathrm{N}$. Since the friction force is pointing opposite to the displacement, the angle between those two vectors is 180 degrees, and the work done by friction will be the following negative amount,

$$W \equiv |\vec{f}_d||\Delta\vec{x}|\cos(180^\circ) = 1.7\,\mathrm{N}\cdot 7.0\,\mathrm{m}\cdot(-1) \approx -12\,\mathrm{J}$$

It's important to know that *the work done by a force can be positive, zero, or negative.*

What about the work done by the net force? The net force on the block is zero (since it is moving with constant velocity), hence the work done by the net force, something we also call the **net work** (or total work), is zero too. This can also be seen by simply adding up all the individual contributions of work:

$$W_{\text{net}} = W_F + W_{F_N} + W_{mg} + W_{f_d} = 12 + 0 + 0 + (-12) = 0$$

What would happen if the net force is not zero? Consider a mass moving in a straight line with a constant net force $\vec{F}_{\text{net}}$ acting on it along the direction of motion as shown below.

At first it has velocity $\vec{u}$ and after an amount of time it displaces itself $\Delta\vec{x}$ and has velocity $\vec{v}$. The work done by the net force can be expressed in the following way:

$$\begin{aligned}
W_{\text{net}} &\equiv |\vec{F}_{\text{net}}||\Delta\vec{x}|\cos\theta \\
\theta = 0^\circ \longrightarrow \quad &= (ma)\cdot\Delta x \\
\frac{1}{2}\cdot 2 = 1 \longrightarrow \quad &= \frac{1}{2}m(2a\Delta x) \\
\text{suvat formula} \longrightarrow \quad &= \frac{1}{2}m(v^2 - u^2) \\
&= \frac{1}{2}mv^2 - \frac{1}{2}mu^2
\end{aligned}$$

Defining the **kinetic energy** of a mass m moving with speed v as

$$\text{KE} \equiv \frac{1}{2}mv^2$$

we can express the work done by the net force as

$$W_{\text{net}} = \text{KE}_{\text{after}} - \text{KE}_{\text{before}} = \Delta\text{KE}$$

which is often referred to as the **work-energy theorem**: *The work done by the net force is always equal to the change in kinetic energy.*[5] This is sometimes also referred to as the work-energy principle. For our block being dragged along the floor, the net work was zero which means it won't be changing it's kinetic energy – which we already knew because the net force is zero.

[5] This result is always true, also for a net force that is not constant and not pointing in the direction of the displacement. In this more general case though, the work done by the net force has to be calculated as something called a **line integral**:

$$W_{\text{net}} = \int_C \vec{F}_{\text{net}} \cdot d\vec{x}$$

You will learn about line integrals when you take a course on vector calculus :)

Gravitational potential energy

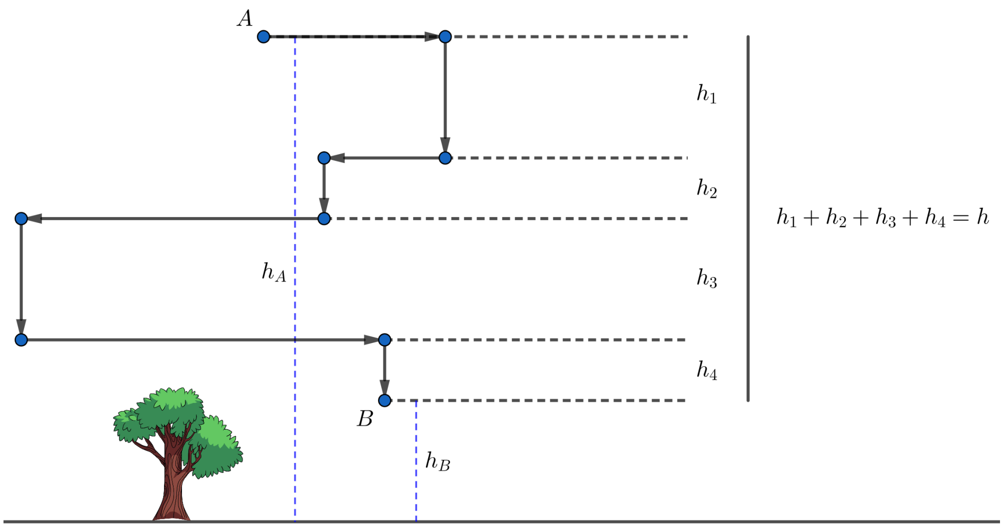


Figure 7: A mass being displaced from A to B along a particular path.

Figure 7 shows a mass being displaced from A to B along a path of horizontal and vertical sections. Let's calculate the total work done by the force of gravity along this path: The work done along the first horizontal section is zero because the weight is perpendicular to the displacement ($\cos 90° = 0$). The work done along the first vertical section is

$$W_1 \equiv |mg||h_1|\cos(0°) = mgh_1$$

This pattern continues so that along each horizontal section the force of gravity does zero work, whereas along each vertical section it does an amount of work given by

$$W_2 = mgh_2, \quad W_3 = mgh_3, \quad \text{and} \quad W_4 = mgh_4$$

The total amount of work done along the entire path is therefore:

$$W_{\text{total}} = W_1 + W_2 + W_3 + W_4 = mg(h_1 + h_2 + h_3 + h_4) = mgh$$

which is exactly the same work done as if it just fell straight down a height h. Because any path can be approximated to a series of horizontal and vertical sections in this way[6], we conclude that *the work done by gravity depends only on the height difference between two points and not the actual path taken*.

[6] Even smooth curves, since we can obtain those in the limit when the section lengths approach zero – the calculus trick again.

This important point of the work done by gravity being path-independent leads to the following definition: We define the **gravitational potential energy at** A to be

$$\mathrm{PE}_A = mgh_A$$

where h_A is the height of A relative to some fixed height (say the ground) and the potential energy at B to be

$$\mathrm{PE}_B = mgh_B$$

where h_B is the height of B relative to the same fixed height. Note that the potential energy at A is larger than at B, so as the mass goes from A to B the potential energy decreases. The work done by gravity as a mass goes from point A to point B can then be expressed as *the negative change in the potential energy*:

$$W_{\text{grav}} = mgh = mg(h_A - h_B) = -mg(h_B - h_A) = -\Delta\mathrm{PE} \tag{2}$$

It's important to understand what the above equation says: The work done by gravity when the mass goes from A to B is positive ($W_{\text{grav}} > 0$), but this corresponds to a decrease in potential energy $\Delta\mathrm{PE} < 0$). Conversely, the work done by gravity if the mass goes from B to A would be negative ($W_{\text{grav}} < 0$, we say "work is done against gravity"), but this corresponds to an increase in potential energy ($\Delta\mathrm{PE} > 0$). In other words, *gravitational potential energy is stored when we lift a mass up, and it is released again by letting the mass move down – and the path it takes doesn't matter at all.*[7]

[7] When the work done by a force is path-independent, we can always define a potential energy for that force. Such forces are called **conservative forces** and gravity is one example. The word "conservative" suggests that something is "conserved" and it is the potential energy associated with the force that is conserved. Friction is an example of a *non-conservative* force (can you see why?).

The conservation of energy

We'll now show how it was possible for us to partly solve the Skyswing problem so quickly. Let's consider, as an example, a projectile launched off a tall building, see figure 8. You should be able to solve for the final velocity, $\vec{v}$, using the suvat equations, but if we are only interested in finding the final speed, $|\vec{v}|$, then it's faster to use the work-energy theorem as follows: If we assume air resistance is negligible, then the mass will only move under the influence of gravity. Hence the force of gravity is the net force. We know from the work-energy theorem that

$$W_{\text{net}} = \Delta\mathrm{KE} \tag{3}$$

and we showed earlier, see equation (2), that we can express the work done by gravity (regardless of the path taken) as

$$W_{\text{grav}} = -\Delta\mathrm{PE} \tag{4}$$

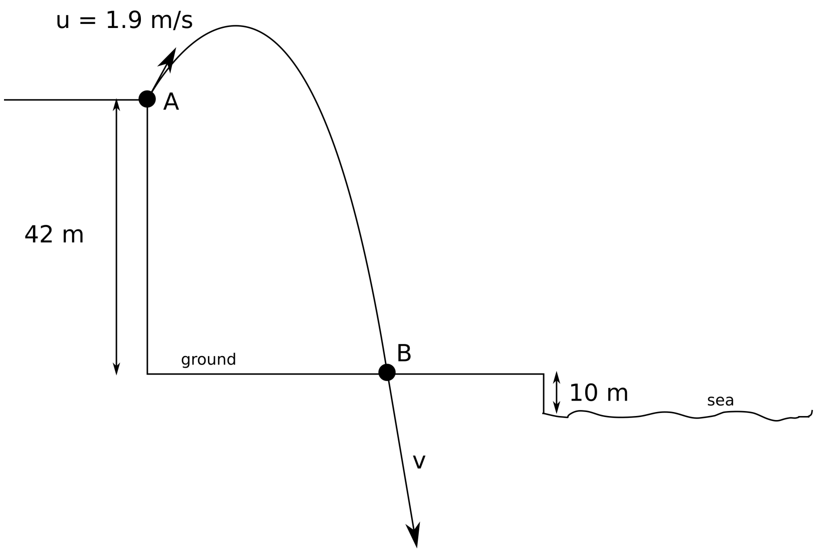


Figure 8: A projectile launched off a tall building. We can use the suvat approach, but if we only need to find the final speed, then the energy approach is much faster.

Since the net force is the force of gravity, equating the two equations (3) and (4), we get

$$-\Delta \text{PE} = \Delta \text{KE}$$

or by rearranging slightly:

$$0 = \Delta \text{KE} + \Delta \text{PE} \tag{5}$$

The above equation is a fundamental result often referred to as the **conservation of energy**. It states that *the sum of the changes in kinetic and potential energy is zero for an object only moving under the influence of gravity (no matter what path it takes)*. It's easy to show that (5) can also be written as

$$\text{KE}_{\text{before}} + \text{PE}_{\text{before}} = \text{KE}_{\text{after}} + \text{PE}_{\text{after}} \tag{6}$$

which is a useful equation to use in problem-solving. We also often define the **total energy**, TE, to be the sum of the kinetic and potential energies

$$\text{TE} \equiv \text{KE} + \text{PE}$$

which means equations (5) and (6) can be expressed as

$$0 = \Delta \text{TE} \quad \text{or} \quad \text{TE}_{\text{before}} = \text{TE}_{\text{after}}$$

Let's use this to solve for the speed of the mass in figure 8 right before it hits the ground: The first step is to define a reference height where $h = 0$ (PE $= 0$). Let's set that to be the *ground* level. Second, let's write down the total energy at position A:

$$\text{TE}_A = \text{KE}_A + \text{PE}_A = \frac{1}{2}mu^2 + mgh_A$$

Third, let's write down the total energy at position B:

$$\text{TE}_B = \text{KE}_B + \text{PE}_B = \frac{1}{2}mv^2 + mgh_B$$

These two expressions must be equal to each other according to the conservation of energy, and from the data given, we can isolate and calculate v:

$$\begin{aligned}\frac{1}{2}mv^2 + mgh_B &= \frac{1}{2}mu^2 + mgh_A \\ v^2 &= u^2 + 2gh_A \\ v &= \sqrt{u^2 + 2gh_A} \\ &= \sqrt{(1.9)^2 + 2(9.8)(42)} \approx 29\,\mathrm{m/s}\end{aligned}$$

This was much easier than using the projectile motion suvat equations in two dimensions!

How would the above calculations have been different if the reference height $h = 0$ had been set to *sea* level? Try and do the calculations and notice that you get the same answer. Hence your choice of reference level $h = 0$ ($\mathrm{PE} = 0$) doesn't affect the answer – you just end up adding the same quantity to both sides of equation (6).

The work done by a non-conservative force

In order to derive the conservation of energy equation from the previous section, we assumed that gravity was the only force acting on an object. Due to the nature of gravity (a *conservative* force) we could define a potential energy function and rewrite the work-energy theorem as a conservation of energy equation

$$0 = \Delta\mathrm{KE} + \Delta\mathrm{PE}$$

Very often, however, there are other *non-conservative* forces that act on our object, and since potential energies cannot be defined for such forces, the work done by them needs to "stay on the W side" in the work-energy theorem. Consider for example a resistive/friction force: Friction always does negative work on an object because it always points in the direction opposite to the motion. So if one path is longer than another path, more negative work is done along the long path. *We conclude that the work done by friction is not path independent, hence friction is a non-conservative force and we cannot define a potential energy function for it.*

The way to include this in the work-energy theorem is as follows: Imagine including air resistance (a resistive/friction force) in our projectile motion example. Since gravity and drag are the only forces in play, the net work is the sum of the work done by each force. According to the work-energy theorem

$$W_{\mathrm{grav}} + W_{\mathrm{drag}} = \Delta\mathrm{KE}$$

which we can also write as follows by introducing the gravitational potential energy, $W_{\mathrm{grav}} = -\Delta\mathrm{PE}$,

$$-\Delta\mathrm{PE} + W_{\mathrm{drag}} = \Delta\mathrm{KE}$$

which implies that

$$W_{\text{drag}} = \Delta\text{KE} + \Delta\text{PE} = \Delta\text{TE}$$

From this we can see that the total energy (the sum of kinetic and gravitational potential energy) is now *not* conserved – instead it changes by an amount that is equal to the work done by the resistive force. And since the work done by a resistive force is always negative, the total energy will always decrease due to friction. *This is the reason all moving objects eventually come to rest when friction is involved*. This loss of total energy (which is really a loss of kinetic energy) cannot be recovered (as opposed to losses in potential energy that can be recovered) since it turns into **heat** in the system – we often say energy is *dissipated as heat*. We'll discuss what really happens to that energy in topic 3 when we talk about thermal physics (spoiler alert: heat is just kinetic energy in the random movements of molecules making up a system, so the kinetic energy is in fact still there, it just got distributed to many small particles making up the system).

All this sounds awfully theoretical, so play around with this fun PhET simulation to gain an intuitive understanding of how friction "takes away energy" from a system.

Elastic potential energy

As another example of potential energy, let's consider the case of a spring, see figure 9. As we have learnt previously, when a spring is displaced from its equilibrium length it will exert a spring (or elastic) force given by Hooke's law

$$F_{\text{elastic}} = -kx$$

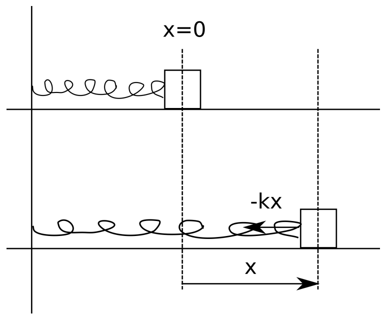


Figure 9: The spring force.

How can we calculate the work done by the elastic force as the spring is displaced a given amount? If we try to use the definition of work, we run into a slight problem: In equation (1) the force is assumed constant during the displacement, but the spring force varies as the mass is displaced, so how do we actually do the calculation? The proper answer is that we need to use calculus, but in the case of Hooke's law it is so simple that we can do it without really using calculus (very similar to what we did when we derived the suvat equations). First note that when force is graphed vs. displacement *the area under the graph has units of* $\text{N} \cdot \text{m} = \text{J}$ *so area represents the work done by the force.* Let's apply this to the special case of the elastic force: Imagine you stretch a spring from its equilibrium length. As you pull outwards, the magnitude of your pull force is kx at any given displacement x (hence it increases) and you are doing positive work on the spring since your pull is in the same direction as the displacement of the spring. The force vs. displacement graph for this situation looks like figure 10 and *the area represents the total work done by your pull force*. Since the area is a

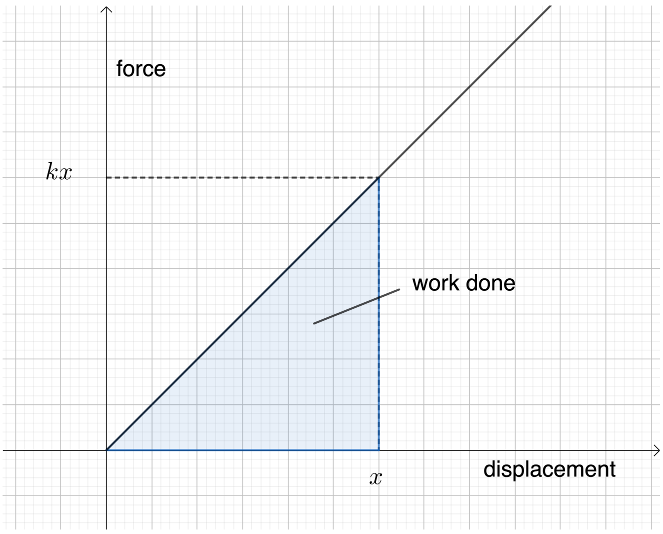


Figure 10: In a force vs. displacement diagram, the area below a graph represents the work done by the force. For a spring, this area is very easy to work out since the shape is a triangle.

simple triangle we don't need to use calculus to work it out:

$$W = \text{ area under the force vs. displacement graph}$$
$$= \frac{1}{2}(kx)(x) = \frac{1}{2}kx^2$$

This positive work, which you did on the spring, is now stored as **elastic potential energy**,

$$\text{PE}_{\text{elastic}} = \frac{1}{2}kx^2$$

hence the potential energy of the spring has increased. If you let go, this energy will be released (and converted to kinetic energy) as the spring contracts back to its equilibrium length where the potential energy is zero. Compressing a spring gives you the same result.

Power and efficiency

Work can be done at different rates. For example, in lifting a 1 kg mass 1 m above the ground, your lift force (and therefore you) will have done 10 J of work. But how fast did you do it? In 1 s? 10 s? 2 hours? **The rate at which work is being done is called power**, P. Since doing work can be thought of as being equivalent to transferring energy, we often also express power as **the rate at which energy is being transferred**:

$$P \equiv \frac{\Delta E}{\Delta t}$$

If work (= the energy transfer) is measured in joules and time in seconds, then the unit of power becomes joules per second which we define as one **watt**, W:

$$\text{W} \equiv \frac{\text{J}}{\text{s}}$$

A concept that is often used together with power is efficiency. It's a very simple concept that can be used in many different contexts, here's an example: Consider an electric motor used to lift a mass up in the air. If the mass is 2 kg and it moves up by 15 m in 5 s, then the motor would have had a useful output power of

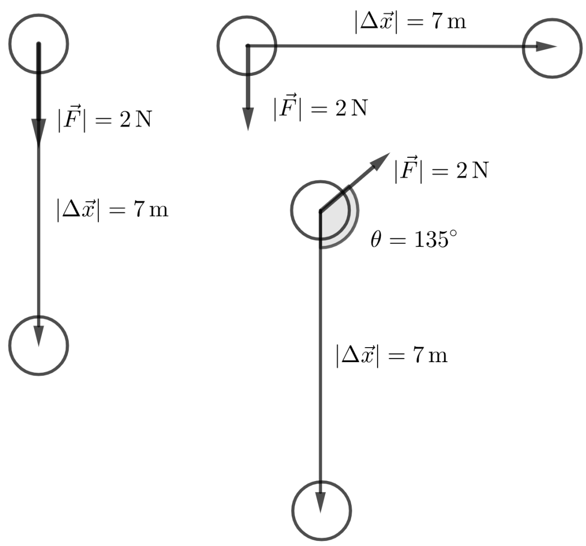
$$P = \frac{\Delta E}{\Delta t} = \frac{mgh}{\Delta t} = \frac{2\,\mathrm{kg} \cdot 10\,\mathrm{N/kg} \cdot 15\,\mathrm{m}}{5\,\mathrm{s}} = 60\,\mathrm{W}$$

This is the power provided by the motor. But since a real motor is a machine with moving parts, it is typically exposed to friction forces that generate some heat and therefore some energy/power is wasted. If the electrical input power to the motor was 80 W, then we can conclude that 20 W of power was not used in a useful way, and hence we can define the **efficiency** of the motor as

$$\eta \equiv \frac{P_{\text{useful output}}}{P_{\text{total input}}} = \frac{60\,\mathrm{W}}{80\,\mathrm{W}} = 0.75 = 75\%$$

Energy: Exercises

1. Calculate the work done by the force in the three cases shown below:



2. Work can be either positive, zero or negative.
   (a) Can you say anything general about the sign of the work done by a force of friction?
   (b) Can you say anything general about the work done by a normal force?
   (c) Does the centripetal force do any work on an object in uniform circular motion? Does the speed of the object increase, decrease or remain constant in this situation?
   (d) When a mass m moves freely downwards a vertical height of h, what is the work done by the force of gravity? Is it positive or negative? Does the speed of the object increase, decrease or remain constant in this situation?

(e) When a mass m moves freely upwards a vertical height of h, what is the work done by the force of gravity? Is it positive or negative? Does the speed of the object increase, decrease or remain constant in this situation?

3. You launch a tennis ball ($m = 0.20\,\mathrm{kg}$) straight up in the air with a speed of $3.0\,\mathrm{m/s}$. Assume air resistance is negligible, hence gravity is the only force acting on it as it goes up and down. Let $|\vec{g}| = 10\,\mathrm{m/s^2}$.

   (a) Use suvat to find the maximum height above the point of launch.

   (b) Calculate the work done by gravity during the *ascent*.

   (c) Calculate the change in kinetic energy from the point of launch to the maximum height and compare it with your answer to (b). Is the work-energy theorem satisfied?

   (d) Calculate the work done by gravity during the *descent* and compare it to the change in kinetic energy from the maximum height to the launch point. Is the work-energy theorem satisfied?

4. Consider a ball rolling down a frictionless slope shaped as shown in figure 11.

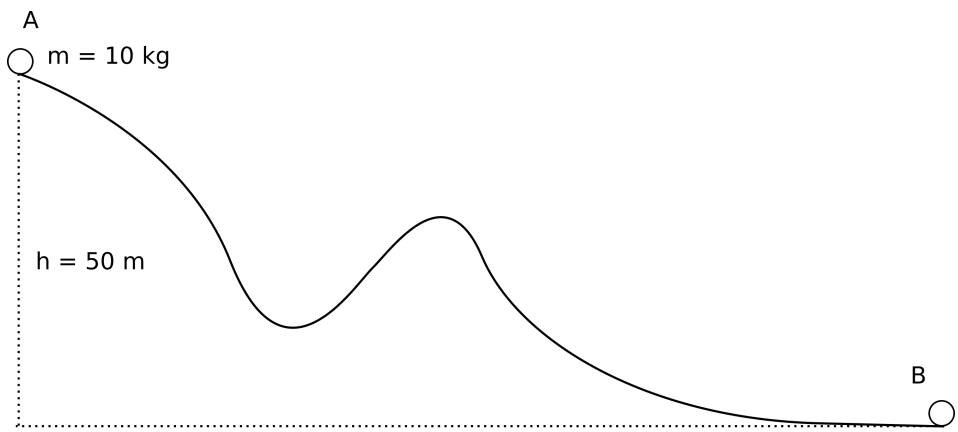


Figure 11: A ball rolling down a hill.

   (a) Assume the ball is always in contact with the surface. Calculate the work done by gravity as the ball rolls from A to B. Does this work depend on the shape of the slope?

   (b) Calculate the change in potential gravitational energy, ΔPE. Is it true that $W = -\Delta\mathrm{PE}$? Imagine the ball moving in the opposite direction from B to A. Is it still true that $W = -\Delta\mathrm{PE}$?

   (c) As the ball rolls down the slope, does the normal force do any work on the ball?

   (d) Do your answers to (a) - (c) depend on whether the ball was released from rest or with an initial velocity?

(e) Assume the slope is *not* frictionless. As the ball rolls down the slope does the friction force do any work on the ball? How do you think that would change the answer to (a)?

(f) Assume the ball is *not* always in contact with the surface (for example, it is launched into the air slightly after the first bump). How would this change the answer to (c)?

5. A stone is thrown from the top of a cliff of height 28 m above the sea. The stone is thrown at a speed of 14 m/s at an angle above the horizontal. Air resistance is negligible. By considering the energy of the stone, determine the speed with which the stone hits the sea.

6. Imagine a 1 kg block moving along a rough table. Initially it has a speed of 10 m/s, but due to a constant dynamic friction force of 2 N, it comes to rest after a while.

   (a) Calculate the change in kinetic energy, ΔKE.

   (b) Use the work-energy principle to calculate the magnitude of the displacement.

   (c) Check your answer by only using suvat equations to work out the displacement.

7. A bullet moving with a speed of 300 m/s and mass 30 g is shot into a tree. It buries itself 3 cm into the tree and comes to rest.

   (a) Assuming the tree applies a constant contact force on the bullet, use the work-energy principle to calculate the magnitude of this force.

   (b) Draw a force vs. displacement graph for when it is in contact with the tree. Interpret the area under the graph.

8. A child applies a force $\vec{F}$ parallel to the x-axis to a 10 kg sled moving on the frozen surface of a pond. As the child controls the speed of the sled, the x-component of the force she applies varies with the x-coordinate of the sled as shown in figure 12.

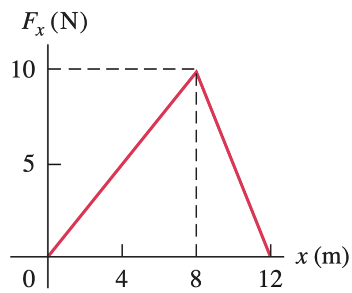


Figure 12:

   (a) Calculate the work done by the force $\vec{F}$ when the sled moves

      i. from $x = 0.0$ to 8.0 m,

      ii. from $x = 8.0$ to 12.0 m,

      iii. from $x = 0.0$ to 12.0 m.

   (b) Suppose the sled is initially at rest at $x = 0$. Ignoring any friction, use the work-energy principle to find the speed of the sled

      i. at $x = 8.0$ m,

      ii. at $x = 12.0$ m.

9. A 350 kg roller coaster starts from rest at point A and slides down the frictionless loop-the-loop shown in figure 13.

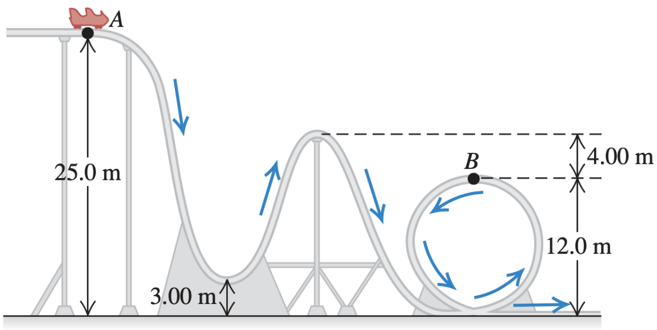


Figure 13: A roller coaster.

(a) How fast is this roller coaster moving at point B?

(b) How hard does it press against the track at point B?

10. A 2.00 kg block is pushed against a spring with negligible mass and force constant $k = 400\,\mathrm{N/m}$, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0°, see figure 14.

(a) What is the speed of the block as it slides along the horizontal surface after having left the spring?

(b) How far does the block travel up the incline before starting to slide back down?

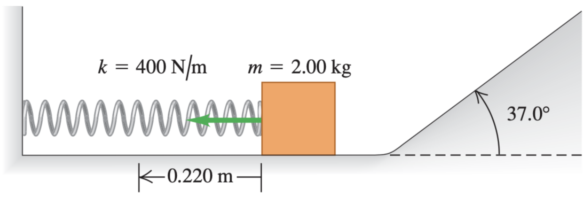


Figure 14: From elastic to kinetic to potential energy.

11. A rocket is about to be launched vertically off the ground on Earth. It has a mass of $m = 20 \cdot 10^3$ kg.

(a) What is its total kinetic and potential energy when it is at rest on the ground (call this point A)?

(b) It launches into the air and at a height of 500 m (call this point B) it is moving with a speed of 52 m/s. How much has the total energy changed from point A to B?

(c) Which force does positive work on the rocket as it moves from point A to B? How much work does this force do? What is the average magnitude of this force?

(d) Which force does negative work on the rocket? How much work does this force do?

(e) What is the net work done on the rocket? And does this equal to the change in kinetic energy as it should according to the work-energy theorem?

12. A 2.8 kg block slides over the smooth, icy hill shown in figure 15. The top of the hill is horizontal and 70 m higher than its base. What minimum speed must the block have at the base of the hill in order for it to pass over the pit at the far side of the hill?

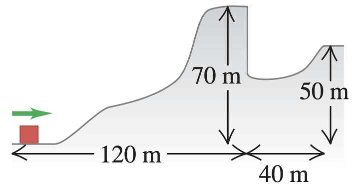


Figure 15: An icy hill.

13. A 2.0 kg piece of wood slides on the surface shown in figure 16. The curved sides are perfectly smooth, but the rough horizontal bottom is 30 m long and has a kinetic friction coefficient of 0.20 with the wood. The piece of wood starts from rest 4.0 m above the rough bottom.

Figure 16: Wood on a rough bottom.

    (a) Where will this wood eventually come to rest?

    (b) For the motion from the initial release until the piece of woos comes to rest, what is the total amount of work done by friction?

14. A car in an amusement park ride rolls without friction around the track shown in figure 17. It starts from rest at point A at a height h above the bottom of the loop. Treat the car as a particle.

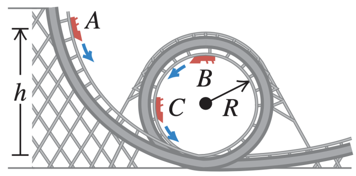
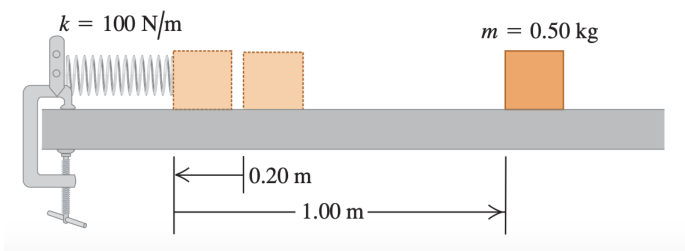


Figure 17: Another roller coaster.

    (a) What is the minimum value of h (in terms of R) such that the car moves around the loop without falling off at the top (point B)?

    (b) If $h = 3.5R$ and $R = 20.0\,\text{m}$, compute the speed, radial acceleration, and tangential acceleration of the passengers when the car is at a point C, which is at the end of a horizontal diameter. Show these acceleration components in a diagram, approximately to scale.

15. A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m. When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The spring constant k is 100 N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?



Solutions to all the questions.

Energy Quiz

Check your understanding of this lesson: Here is a quiz.