A little book about motion

Momentum

When Newton developed his theory of motion, one of the first concepts he defined was **momentum**. It seems strange not to have mentioned this at all yet in our course, but there are pedagogical reasons for that. Luckily, at this point, momentum is a rather straightforward concept to define: It is simply *the mass of an object multiplied by its velocity*,

$$\vec{p} = m\vec{v}$$

Momentum, $\vec{p}$, is the vector quantity that best quantifies the 'essence of motion' because it takes into account both the mass and the velocity of a moving object[1]. It points in the same direction as velocity and has the magnitude $p = |\vec{p}| = m|\vec{v}| = mv$. A slowly moving freight train can have less momentum (less 'motion') than a fast moving bullet, since the larger velocity can make up for the smaller mass. A typical unit of momentum is kg m/s which is also often expressed as N s (as we will seen soon).

As we have seen in our earlier lessons, a net force is the cause of a change in velocity of an object. This was the content of Newton's 2nd law. We can rewrite this law in terms of momentum as follows:

$$\vec{F}_{\text{net}} = m\vec{a} = \frac{m\Delta\vec{v}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}$$

where we used the fact that

$$\Delta\vec{p} \equiv \vec{p}_2 - \vec{p}_1 \equiv m\vec{v}_2 - m\vec{v}_1 \equiv m\Delta v$$

It might not seem like much has happened, but this small change is in fact a generalisation of Newton's 2nd law and it was the way he originally stated his law. Hence from now on, we will always refer to the 2nd law of motion as: *The (average) net force on an object is equal to its rate of change of momentum,*[2]

$$\vec{F}_{\text{net}} = \frac{\Delta\vec{p}}{\Delta t} \tag{1}$$

This more general version can solve more complicated problems of motion, for example when the mass of a system changes. The classic example is a rocket: When a rocket accelerates it loses mass (fuel) at the same time, so m is not constant and we cannot directly apply $F = ma$.

If we multiply by Δt on both sides of (1) and omit the 'net' subscript we get

$$\vec{F}\Delta t = \Delta\vec{p} \tag{2}$$

and although not much has happened, we call the left hand side the **impulse of the force** F and we notice that *the impulse is equal to the change in momentum* of the object (due to that force). Impulse is

[1] Whenever people use the term "inertia", they are often referring to momentum: When an object has a lot of momentum it's difficult to change its motion, hence it has a lot of 'inertia'.

[2] Forces are not always constant, in fact they often change continuously. In those more complicated cases, Newton's 2nd law becomes the instantaneous ('calculus') version:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$$

Figure 1: Rockets lose mass as they burn fuel, so $F = ma$ is not adequate in these cases.

a vector quantity that we often associate with brief collisions and impacts (although, as a general concept, it can be applied to any situation). From equation (2) we can see how a given change in momentum Δp can either be due to a LARGE force (F) acting for a short time (Δt) or a small force (F) acting for a LONG time (Δt). As an example, consider an egg moving with a certain velocity. If the egg needs to be stopped, it's momentum mv has to be reduced to zero. It would be better to have a small force act for a long time (to avoid cracking it) rather than a large force act for a short time (much more messy). Hence, throw an egg into a loose bed sheet rather than a wall if you still want to be able to cook it! (We'll do that demonstration in class :)

From the impulse-momentum equation, $\vec{F}\Delta t = \Delta\vec{p}$, we also see that *the area in a force vs. time graph is equal to the change in momentum*. This should be clear for a constant force, see figure 2. In cases where the force is not constant, we can still estimate the area in order to find the change in momentum. For example, figure 3 shows how the normal force on a tennis ball varies with time during an impact with a wall. The force is small when the impact starts, large in the middle of the impact when the tennis ball is compressed the most, and then small again as it leaves the wall. The total area will be equal to the impulse of the force, in other words, the total change in momentum of the tennis ball. Try to estimate the area below the graph by counting squares! One square corresponds to the momentum $F \cdot \Delta t = (1\,\mathrm{N})(0.1\,\mathrm{s}) = 0.1\,\mathrm{kg\,m/s}$.

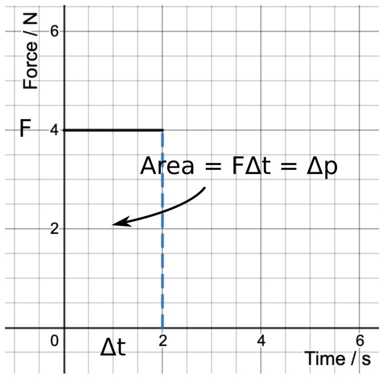


Figure 2: .

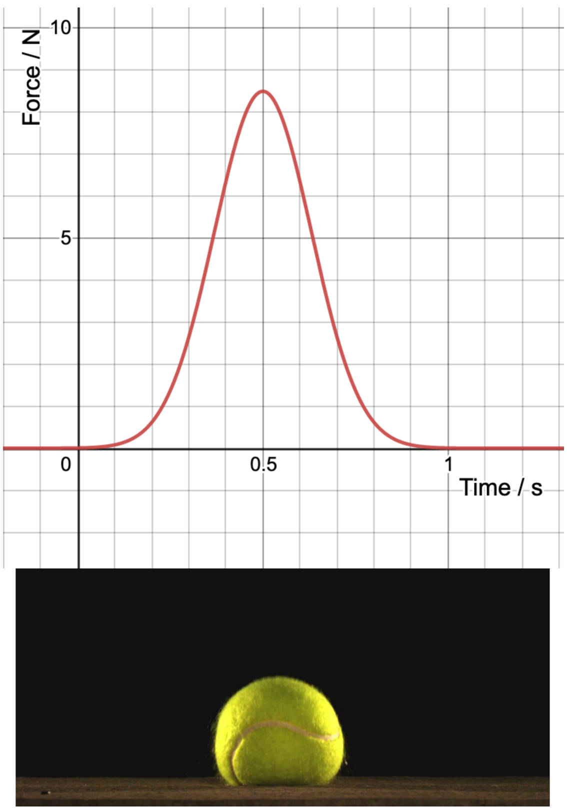


Figure 3: When a ball bounces off a surface, the normal force typically changes in the way shown in this diagram.

Conservation of momentum

The true usefulness (and therefore importance) of momentum, turns out to be that it is *conserved in isolated systems*. This is particularly useful when dealing with systems of objects that undergo collisions and explosions. Let's derive this important result: Assume two particles interact with each other during a time interval Δt. The interaction can take any form whatsoever and due to this interaction, their momenta change. One particle changes its momentum by

$$\Delta \vec{p} = \vec{p}_2 - \vec{p}_1$$

while the other particle changes its momentum by

$$\Delta \vec{q} = \vec{q}_2 - \vec{q}_1$$

see figure 4. Label the force on the first particle (due to the second

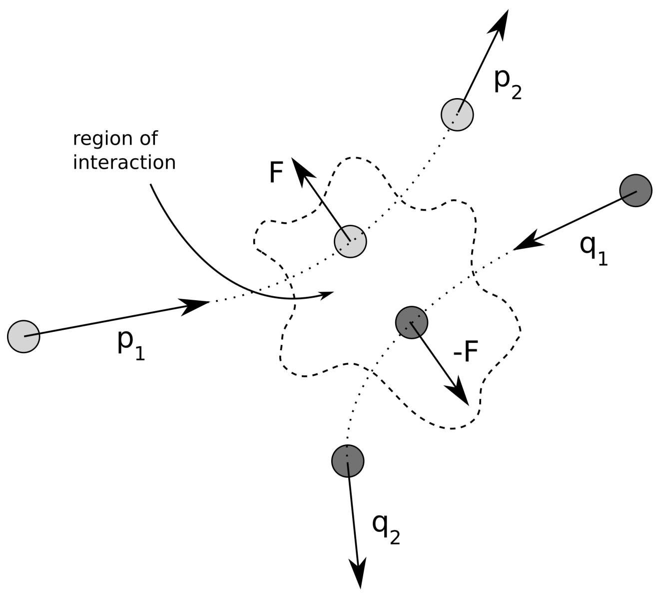


Figure 4: A general interaction between two particles.

particle) as $\vec{F}$. Newton's 3rd law then tells us that the force on the second particle (due to the first particle) will be equal and opposite, $-\vec{F}$. This force pair will continuously change in size and direction during the interaction, but they are always equal and opposite at any given time. Let's further assume our two objects constitute a so-called **isolated system**. This means that any *external* interactions (interactions due to other objects besides the two in focus) are negligible (either because those interactions are too weak or too brief). We can then write down Newton's 2nd law for the two objects as follows:

$$\vec{F} = \frac{\Delta \vec{p}}{\Delta t} \quad \text{and} \quad -\vec{F} = \frac{\Delta \vec{q}}{\Delta t}$$

Adding these two equations, gives us

$$\vec{F} + (-\vec{F}) = 0 = \frac{\Delta \vec{p} + \Delta \vec{q}}{\Delta t}$$

which implies that

$$\Delta\vec{p} + \Delta\vec{q} = 0 \quad \Rightarrow \quad \Delta\vec{p} = -\Delta\vec{q} \tag{3}$$

This says that *the change in momentum of one particle is equal and opposite to the change in momentum of the other particle*. In other words, there is a complete transfer of an amount of momentum from one particle to the other. Figure 5 is a visualisation of this fact.[3]

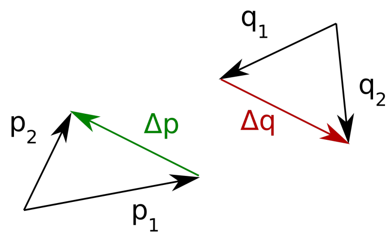


Figure 5: The change in momentum of one particle is equal and opposite to the change in momentum of the other particle.

[3] It's very important to understand the vector nature of momentum. This will be explored in the exercises so make sure to work on all those!

If we define the total momentum $\vec{P}_{\text{tot}}$ at any given time as the sum of all the momenta in our system,

$$\vec{P}_{\text{tot}} \equiv \vec{p} + \vec{q}$$

then equation (3) can be expressed as

$$\begin{aligned}
\Delta\vec{P}_{\text{tot}} &\equiv \vec{P}_{2,\text{tot}} - \vec{P}_{1,\text{tot}} \\
&= (\vec{p}_2 + \vec{q}_2) - (\vec{p}_1 + \vec{q}_1) \\
&= \vec{p}_2 - \vec{p}_1 + \vec{q}_2 - \vec{q}_1 \\
&= \Delta\vec{p} + \Delta\vec{q} \\
&= 0
\end{aligned}$$

In other words, *the change in total momentum is zero*. We could also have written this as

$$\vec{p}_1 + \vec{q}_1 = \vec{p}_2 + \vec{q}_2$$

which clearly shows that the total momentum is *conserved*: The total momentum before the interaction (total initial) is equal to the total momentum after the interaction (total final). This **conservation of momentum** is one of the most fundamental laws of physics so we better summarise it properly: *The total momentum in an isolated system is always conserved.* This is visualised in figure 6.

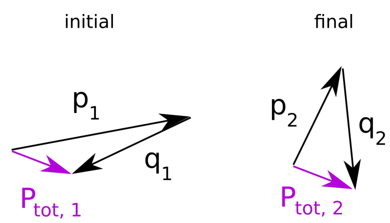


Figure 6: The total momentum, $\vec{P}_{\text{tot}}$, is conserved.

Collisions

Momentum is a very useful concept to use when dealing with collisions. Consider the following example: Say a block of mass m and initial velocity u strikes another block of mass $3m$ which is at rest, see figure 7. This is a one-dimensional problem, so we will omit the vector arrows, but remember that the $+$ and $-$ signs will indicate direction. What do you think the velocities of the two blocks will be after the collision? It's not hard to realise that the block at rest will definitely start moving to the right since the force applied to it will accelerate it from rest in that direction. But what about the moving block? The force applied to it will accelerate it to the left, but will that only slow it down a little and leave it moving to the right? Or will the force be enough to turn it around and make it move to the left (rebounding off the block at rest)? Let's figure it out!

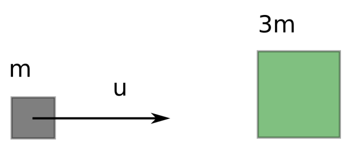


Figure 7: A collision is about to take place. Let the positive direction be to the right.

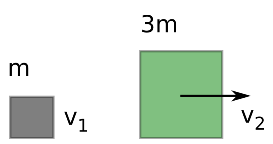


Figure 8: After the collision we know that $3m$ must be moving to the right. Since I'm not sure about the direction m is moving in, I haven't drawn v_1 yet.

Figure 8 shows the two blocks after the collision. Notice that I haven't drawn the direction of v_1, because I don't yet know what it is. The direction of v_1 will be revealed by the sign in the end. My end goal is to find an expression for v_1 in terms of u. First we use the conservation of momentum to write down

$$mu = mv_1 + (3m)v_2 \quad \Rightarrow \quad u = v_1 + 3v_2 \tag{4}$$

Since I have three variables here, I need another equation, and I can get one by considering the total energy. If the total kinetic energy is conserved in a collision we call the collision an **elastic collision**. Let's assume this is an elastic collision, so we can write down

$$\frac{1}{2}mu^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}(3m)v_2^2 \quad \Rightarrow \quad u^2 = v_1^2 + 3v_2^2 \tag{5}$$

I can now eliminate v_2 from equations (4) and (5) to arrive at the quadratic equation

$$4v_1^2 - 2uv_1 - 2u^2 = 0$$

which has the two solutions

$$v_1 = u \quad \text{or} \quad v_1 = -\frac{1}{2}u$$

The first solution is simply a description of the system before the collision: $v_1 = u$ and inserting this in (4) gives $v_2 = 0$, compare with figure 7. The second solution is a description of the system after the collision! We see that $v_1 = -\frac{1}{2}u$ which means it will move opposite to u with half the magnitude! So we predict it will rebound with half the speed. Inserting this in (4) we can also find the velocity of the big block:

$$u = -\frac{1}{2}u + 3v_2 \quad \Rightarrow \quad v_2 = \frac{1}{2}u$$

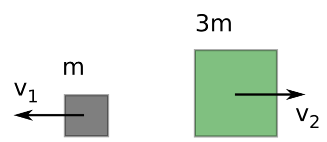


Figure 9: We have now solved for both velocities. They will be moving in opposite directions after the collision with half the initial speed of m.

and we see that it will move to the right with half the speed of u! Let's see if that really happens (I will show you this in class.) You can check yourself that in this case a magnitude $|\Delta p| = \frac{3}{2}mu$ of momentum has been exchanged between the two blocks.

In the above example we assumed the collision was elastic, i.e. the total kinetic energy was conserved in the collision. Sometimes (rather often) this is *not the case* and some kinetic energy is converted to other forms of energy (it is not really "lost" of course, because energy is never truly lost). These types of collisions are called **inelastic collisions**. Furthermore, a **totally inelastic collision** is defined as when the two colliding objects stick together after the collision. Let's predict the outcome of our experiment assuming the collision is totally inelastic. Figure 10 shows the two blocks after the collision. Since the total momentum before the collision is pointing to the right, the total momentum after the collision must also be pointing to the right, so I know they will move together to the right. *Momentum is still conserved*, but now the equation takes the form

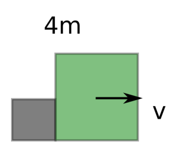


Figure 10: If the collision is totally inelastic the objects stick together after the collision.

$$mu = (m + 3m)v \quad \Rightarrow \quad v = \frac{1}{4}u$$

and the problem is solved! They will both move to the right with a velocity that is a quarter u. In this case $|\Delta p| = \frac{3}{4}mu$ of momentum has been transferred between the two masses. You can play around with collisions using this PhET simulation.

Lesson 19: Questions and activities

1. A car with mass 1000 kg travels in a straight line at a speed of 10 m/s. A tennis ball with mass 100 g moves in a straight line at a speed of 100 km/s.

   (a) Which object is hardest to stop?

   (b) If the car is brought to rest in 0.1 s, what is the average force exerted on the car?

   (c) If the car is brought to rest in 1.0 s, what is the average force exerted on the car? Crumple zones are a structural safety feature used in vehicles to increase the time over which the momentum is changed. As a result the stopping force experienced by a passenger will be reduced.

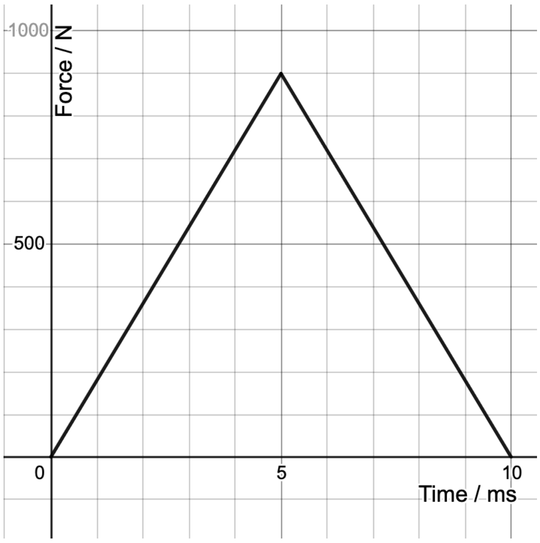
2. In a game of snooker a player hits a 0.20 kg billiard ball with the cue, exerting an average force of 40 N to the right on the ball for 12 milliseconds.

   (a) What is the impulse of the force exerted on the ball? (Direction and magnitude.)

   (b) What is the change in momentum of the ball? (Direction and magnitude.)

   (c) With what velocity does the ball leave the cue? (Direction and magnitude.)

3. The figure below shows an idealised graph of the force vs. time relationship for the interaction between a 50 g golf ball and the club head hitting it. Determine the magnitude of



   (a) the impulse of the force transferred to the ball.

   (b) the resulting change in momentum of the ball.

   (c) the velocity of the ball as it leaves the club head.

(d) the rate of change of momentum of the ball, $\Delta p/\Delta t$.

(e) the average force exerted by the club head on the golf ball.

4. A competitor in a women's shot put event drops the 4.00 kg ball and it falls vertically. Assume the air resistance is negligible. While it is falling:

   (a) what is the net force on the ball?

   (b) at what rate is its momentum changing?

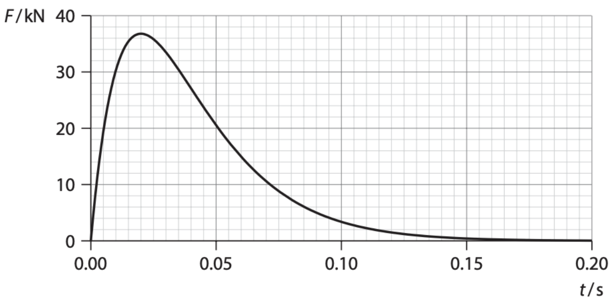
5. A truck of mass 4×10^3 kg moving at 3.5 m/s collides with a stationary truck to which it becomes automatically coupled. Immediately after the impact the trucks move along together at 2 m/s. Find the mass of the second truck.

6. A bag of sand of mass 9 kg which is suspended by a rope is used as as target for a pistol bullet of mass 30 g and which strikes the bag with a horizontal speed of 301 m/s. If the bullet remains embedded in the sand, determine the velocity of the bag immediately after impact.

7. A man of mass 80 kg stands on a trolley of mass 120 kg and throws a parcel of mass 10 kg with a velocity of 4.0 m/s horizontally away from the rear of the trolley. What is the speed with which the trolley and the man commence to move? (*Note: This is the basic principle of rockets: Hot gas (from burning fuel) is ejected in one direction, so the rocket moves in the opposite direction.*)

8. Solve the following problem:

It is proposed to launch projectiles of mass 8.0 kg from satellites in space in order to destroy incoming ballistic missiles. The launcher exerts a force on the projectile that varies with time according to the graph.



The impulse delivered to the projectile is 2.0×10^3 N s. The projectile leaves the launcher in 0.20 s.

a Estimate:

i the area under the curve **[1]**

ii the average acceleration of the projectile **[3]**

iii the average speed of the projectile **[2]**

iv the length of the launcher. **[2]**

b Calculate, for the projectile as it leaves the launcher:

i the speed **[2]**

ii the kinetic energy. **[2]**

c Estimate the power delivered to the projectile by the launcher. **[2]**

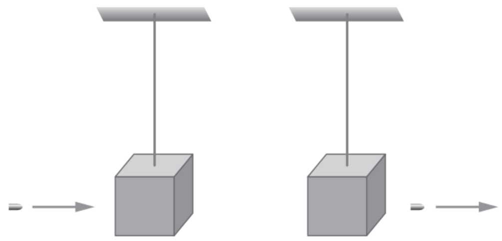
9. Solve the following problem:

A toy helicopter has mass $m = 0.30\,\mathrm{kg}$ and blade rotors of radius $R = 0.25\,\mathrm{m}$. It may be assumed that as the blades turn, the air exactly under the blades is pushed downwards with speed v. The density of air is $\rho = 1.2\,\mathrm{kg\,m^{-3}}$.

a **i** Show that the force that the rotor blades exert on the air is $\rho\pi R^2 v^2$. **[3]**
ii Hence estimate the speed v when the helicopter just hovers. **[2]**
b Determine the power generated by the helicopter's motor when it just hovers as in **a**. **[2]**
c The rotor blades now move faster pushing air downwards at a speed double that found in **a**. The helicopter is raised vertically a distance of 12 m.
Estimate:
i the time needed to raise the helicopter. **[2]**
ii the speed of the helicopter after it is raised 12 m. **[2]**
iii the work done by the rotor in raising the helicopter. **[1]**

10. Solve the following problem:

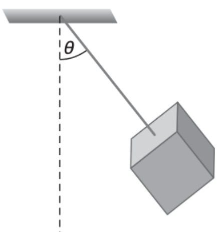
A bullet of mass 0.090 kg is shot at a wooden block of mass 1.20 kg that is hanging vertically at the end of a string.



The bullet enters the block with speed $130\,\mathrm{m\,s^{-1}}$ and leaves it with speed $90\,\mathrm{m\,s^{-1}}$. The mass of the block does not change appreciably as a result of the hole made by the bullet.

a **i** Calculate the change in the momentum of the bullet. **[2]**
ii Show that the initial velocity of the block is $3.0\,\mathrm{m\,s^{-1}}$. **[1]**
iii Estimate the loss of kinetic energy in the bullet–block system. **[2]**

As a result of the impact, the block is displaced. The maximum angle that the string makes with the vertical is θ. The length of the string is 0.80 m.



b Show that $\theta \approx 65°$. **[3]**
c **i** State and explain whether the block in **b** is in equilibrium. **[2]**
ii Calculate the tension in the string in **b**. **[3]**

11. Here's a classic brainteaser: A small cart of mass 300 g when empty can move freely and without friction on a horizontal surface. The cart contains 100 g of sand and moves initially with a speed of 18 cm/s.

(a) What would the speed of the cart be if a further 200 g of sand were dropped vertically into it from a stationary container? (*Hint: Imagine a little packet of sand, m, falling into the* *cart. After impact what is the motion of this packet? Did it change its horizontal motion?*)

(b) If, instead of adding sand to the cart, the original 100 g of sand were allowed to pour out through a hole in the bottom of the cart. What would be the final speed of the cart? (*Hint: Imagine a little packet of sand, m, is released at rest relative to the cart. Is the packet at rest relative to the ground?*)

Solutions to all the questions.

Lesson 19 Quiz

Check your understanding of this lesson: Here is a quiz.