DAY 1M: LEBESGUE MEASURE

LECTURER: CHARLIE SMART

Administrivia

Teacher Charlie Smart (smart@math.uchicago.edu)
TA Peter Morfe (pmorfe@math.uchicago.edu)

Measure Theory

Measure theory is the abstract or algebraic theory of size. We will be starting with arbitrary sets $A \subset \mathbb{R}^d$ using the *Lebesgue Measure*. To do so we will be slightly generalizing the Riemann integral. The volume of set can be seen as the sum of volumes of the rectangles which most efficiently cover the set.

Definition 1.M.1. A *rectangle* is a product of intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$$

which has volume

$$|R| = \prod_{i=1}^{d} (b_i - a_i).$$

Definition 1.M.2. If $A \subset \mathbb{R}^d$, then the *Lebesgue outer measure* is

$$\mu(A) = \inf \left\{ \sum_{k \geq 1} |R_k| : \{R_k\}_{n \in \mathbb{N}} \text{ are rectangles s.t. } A \subset \bigcup_{k \geq 1} R_k \right\}.$$

This is like the Riemann integral in that it is a sum of squares where the indicator function is one somewhere, but it allows infinitely many rectangles as oposed to finitely many.

Example 1.M.3. As an example to show why we need infinitely many rectangles, we will show that $\mu(\mathbb{Q} \cap [-1, 1]) = 0$.

Proof. Let $\{r_1, r_2, \ldots\} = \mathbb{Q} \cap [-1, 1]$. Fix $\epsilon > 0$. Then, we note that

$$\mathbb{Q} \cap [-1, 1] \subseteq \bigcup_{k \geq 1} [r_k - \epsilon 2^{-k}, r_k + \epsilon 2^{-k}].$$

Furthermore,

$$\sum_{k \geq 1} |[r_k - \epsilon 2^{-k}, r_k + \epsilon 2^{-k}]| = \sum_{k \geq 1} \epsilon 2^{1-k} = 2\epsilon.$$

Thus, $\mu(A) \leq \epsilon$ for all $\epsilon > 0$ and therefore is 0.

□

Do 1.1. Show that if $\mathbb{Q} \cap [-1, 1] \subset I_1 \cup I_2 \cup \cdots \cup I_n$ then

$$|I_1| + \cdots + |I_n| \geq 2.$$

And therefore, to measure $\mathbb{Q} \cap [-1, 1]$ we need infinitely many rectangles.

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The question is then whether or not μ is a reasonable measure.

Some thing which contains another thing should be bigger:

Theorem 1.M.4. *If $A \subset B \subset \mathbb{R}^d$ then $\mu(A) \leq \mu(B)$.*

Proof. Suppose $\{R_k\}$ covers B. Then it will also cover A. So,

$$\left\{\sum_{k\geq 1} |R_k| : \{R_k\}_{n\in\mathbb{N}} \text{ are rectangles s.t. } A \subset \bigcup_{k\geq 1} R_k\right\} \subset \left\{\sum_{k\geq 1} |R_k| : \{R_k\}_{n\in\mathbb{N}} \text{ are rectangles s.t. } B \subset \bigcup_{k\geq 1} R_k\right\}.$$

Therefore, by definition of infimum, $\mu(A) \leq \mu(B)$.

□

Ideally, the volume of a bunch of things fused together will be smaller than the sum of its composite parts.

Theorem 1.M.5. *Suppose $\{A_k\}_{k\in\mathbb{N}}$ are all subsets of $\mathbb{R}^d$. Then,*

$$\mu(\bigcup_{k\geq 1} A_k) \leq \sum_{k\geq 1} \mu(A_k).$$

Proof. Suppose $\sum_{k\geq 1} \mu(A_k) = \infty$. Then this is clearly true. Now we suppose that sum is finite.

Fix $\epsilon > 0$. For each k, choose a cover $\{R_j^k\}_{j\in\mathbb{N}}$ of A_k so that

$$\sum_{j\geq 1} |R_j^k| \leq \mu(A_k) + \epsilon 2^{-k}.$$

Such a cover exists by the definition of infimum. Now,

$$\begin{aligned}
\bigcup_{k\geq 1} A_k &\subset \bigcup_{k\geq 1} \bigcup_{j\geq 1} R_j^k & \text{so by definition,} \\
\mu(\bigcup_{k\geq 1} A_k) &\leq \sum_{k\geq 1} \sum_{j\geq 1} |R_j^k| \\
&\leq \sum_{k\geq 1} [\mu(A_k) + \epsilon 2^{-k}] \\
&= \left[\sum_{k\geq 1} \mu(A_k)\right] + \epsilon.
\end{aligned}$$

As this holds for all $\epsilon > 0$,

$$\mu(\bigcup_{k\geq 1} A_k) \leq \sum_{k\geq 1} \mu(A_k)$$

□

Reminder Definition. The *interior* of a set $A \subset \mathbb{R}^d$ is given by A° and is the unique open set contained in A which contains all other open subsets of A.

Reminder Definition. The *indicator function* of a set $A \subset \mathbb{R}^d$ is the function $\mathbb{1}_A : \mathbb{R}^d \to \mathbb{R}$ defined as

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Recall 1.M.6. If $E \subset \mathbb{R}^d$ then

$$\mathrm{Vol}(E) = \int_{\mathbb{R}} \mathbb{1}_E(x)\,\mathrm{d}x.$$

Hopefully, the thing we use to measure other things will have the same measure as its volume.

Theorem 1.M.7. *Suppose R is a rectangle. Then, $\mu(R) = |R|$.*

Proof. To show this, it suffices to show for a collection of rectangles $\{R_k\}$ which covers R, we have that $|R| \le \sum_{k\ge 1} |R_k|$.

First, for all k, we dilate R_k to $\widetilde{R}_k$ so that $R_k \subset \widetilde{R}_k{}^\circ$ and

$$|\widetilde{R}_k| \le |R_k| + \epsilon 2^{-k}.$$

Then, $R \subseteq \bigcup_{k\ge 1} R_k \subset \bigcup_{k\ge 1} \widetilde{R}_k{}^\circ$. However, R is compact, so we can choose a finite subcover $\{\widetilde{R}_1, \dots, \widetilde{R}_n\}$ for some finite n. Thus, $R \subset \bigcup_{k=1}^n \widetilde{R}_k$.

$$\begin{aligned}\sum_{k=1}^n |\widetilde{R}_k| &\le \sum_{k=1}^n (|R_k| + \epsilon 2^{-k}) \\ &\le \left(\sum_{k=1}^n |R_k|\right) + \epsilon \\ &\le \left(\sum_{k\ge 1} |R_k|\right) + \epsilon.\end{aligned}$$

(1) Therefore, it is enough to show that if

$$R \subset \bigcup_{k=1}^n R_k \text{ for some } |R| \le \sum_{k=1}^n |R_k|.$$

We claim that $\mathbb{1}_R \le \sum_{k=1}^n \mathbb{1}_{R_k}$. Suppose $x \in R$. Then, there exists some $j \in [n]$ such that $x \in R_j$. Thus,

$$\mathbb{1}_R(x) = 1 = \mathbb{1}_{R_j}(x) \le \sum_{k=1}^n \mathbb{1}_{R_k}(x).$$

Suppose $x \notin R$. Then, $\mathbb{1}_R(x) = 0 \le \sum_{k=1}^n \mathbb{1}_{R_k}(x)$ as those functions are bounded below by 0. Therefore we have our claim. This implies that

$$\begin{aligned}|R| = \int_{\mathbb{R}^d} \mathbb{1}_R(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^d} \sum_{k=1}^n \mathbb{1}_{R_k}(x)\,\mathrm{d}x \\ &= \sum_{k=1}^n \int_{\mathbb{R}^d} \mathbb{1}_{R_k}(x)\,\mathrm{d}x && \text{by linearity.} \\ &= \sum_{k=1}^n |R_k|.\end{aligned}$$

Therefore we have shown (1) and the theorem holds.

$\square$

Remark 1.M.8. The elementary proof in the text essentially rebuilds the Riemann integral from scratch. To fully prove this rigorously, one would need to use induction on total number of elements.