DAY 1F: PROPERTIES OF MEASUREABLE SETS

LECTURER: CHARLIE SMART

Definition 1.F.1. The *(Lebesgue) Outer Measure* of a set $A \subseteq \mathbb{R}^d$ is

$$\mu(A) = \inf \left\{ \sum_{k \geq 1} |R_k| : R_k \text{ rects such that } A \subset \bigcup_{k \geq 1} R_k \right\}.$$

Properties of μ are as follows

(1) If R is a rectangle, then $\mu(R) = R$.
(2) (Monotonicity) If $A \subseteq B \subset \mathbb{R}^d$ then $\mu(A) \leq \mu(B)$.
(3) (Subadditivity) If $\{A_k\}$ are subsets of $\mathbb{R}^d$ then $\mu(\bigcup_{k \geq 1} A_k) \leq \sum_{k \geq 1} \mu(A_k)$.
(4) (Translation Invariance) If $A \subseteq R^d$ and $x \in \mathbb{R}^d$ then $\mu(A) = \mu(A + x)$.

Definition 1.F.2. A set $A \subseteq \mathbb{R}^d$ is *Lebesgue Measureable* if for every rectangle R

$$\mu(A \cap R) + \mu(A^c \cap R) = \mu(R).$$

Lemma 1.F.3. *If $A \subseteq \mathbb{R}^d$ is measureable, then for any set $B \subseteq \mathbb{R}^d$ we have*

$$\mu(A \cap B) + \mu(A^c \cap B) = \mu(B).$$

Proof. Suppose $B \subseteq \mathbb{R}^d$ and fix $\epsilon > 0$. Choose a cover of rectangles $\{R_k\}$ of B such that $\sum_{k \geq 1} |R_k| < \mu(B) + \epsilon$.

$$\begin{aligned} \mu(A \cap B) &\leq \mu(A \cap \bigcup_{k \geq 1} R_k) && \text{By monotonicity} \\ &\leq \sum_{k \geq 1} \mu(A \cap R_k) && \text{By subadditivity.} \end{aligned}$$

Similarly

$$\mu(A^c \cap B) \leq \sum_{k \geq 1} \mu(A^c \cap R_k)$$

So we have

$$\begin{aligned} \mu(B) &\leq \mu(A \cap B) + \mu(A^c \cap B) && \text{By subadditivity} \\ &\leq \sum_{k \geq 1} \mu(A \cap R_k) + \mu(A \cap R_k) && \text{By prior steps} \\ &= \sum_{k \geq 1} \mu(R_k) && \text{By measurability of } A. \\ &< \mu(B) + \epsilon. \end{aligned}$$

Taking ϵ to 0 gives us the desired equality.

□

Theorem 1.F.4 (Properties of Measureable Sets). *(1) If $A \subseteq \mathbb{R}^d$ is measureable then A^c is measureable.*

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(2) *If* $\mu(A) = 0$ *then* A *is measureable.*
(3) *Any rectangle is measureable.*
(4) *If* A_k *is a collection measureable sets then* $\bigcup_{k\geq 1} A_k$ *is measureable. Moreover, if the* A_k *are dijoint then*

$$\mu(\bigcup_{k\geq 1} A_k) = \sum_{k\geq 1} \mu(A_k)$$

Proof of 1. This is immediate from the definition and the observation that $A = (A^c)^c$.

□

Proof of 2. Suppose $A \subseteq \mathbb{R}^d$ with $\mu(A) = 0$. Let R be any rectangle. Then

$$\begin{aligned} \mu(R) &\leq \mu(A \cap R) + \mu(A^c \cap R) && \text{By subadditivity} \\ &\leq \mu(A) + \mu(R) && \text{By monotonicity} \\ &= \mu(R) && \text{Because } \mu(A) = 0. \end{aligned}$$

So by definition A is measureable.

□

Proof of 3. Suppose R is a rectangle. Let $\widetilde{R}$ be any other rectangle. We note that $R \cap \widetilde{R}$ is a rectangle. We can split $\widetilde{R}$ into at most 9 other rectanlges $\{\widetilde{R}_1, \dots, \widetilde{R}_k\}$ along the boundary lines of $R \cap \widetilde{R}$. Let $\widetilde{R}_1 = R \cap \widetilde{R}$ without loss of generality. Then,

$$\begin{aligned} \mu(\widetilde{R}) &\leq \mu(R \cap \widetilde{R}) + \mu(R^c \cap \widetilde{R}) && \text{By subaditivity} \\ &= \mu(\widetilde{R}_1) + \mu(\widetilde{R}_2 \cup \cdots \cup \widetilde{R}_k) \\ &= \mu(\widetilde{R}) && \text{By “algebra”.} \end{aligned}$$

□

Proof of 4. Step 1: Suppose A_1 and A_2 are measureable subsets of $\mathbb{R}^d$. Let B be any subset of $\mathbb{R}^d$. A_2 is measureable so

$$\begin{aligned} \mu(B) &= \mu(A_2 \cap B) + \mu(A_2^c \cap B) && \text{By the lemma.} \\ &= \mu(A_1 \cap A_2 \cap B) + \mu(A_1^c \cap A_2 \cap B) \\ &+ \mu(A_1 \cap A_2^c \cap B) + \mu(A_1^c \cap A_2^c \cap B) && \text{Because } A_1 \text{ is measureable.} \\ &= (*) \end{aligned}$$

We note that

$$\begin{aligned} A_1^c \cap A_2^c &= (A_1 \cup A_2)^c \\ \text{and } (A_1 \cap A_2) \cup (A_1^c \cap A_2) \cup (A_1 \cap A_2^c) &= A_1 \cup A_2. \end{aligned}$$

Therefore, we can apply subadditivity to $(*)$ to find

$$\begin{aligned} (*) &\geq \mu((A_1 \cup A_2) \cap B) + \mu((A_1 \cup A_2)^c \cap B) \\ &\geq \mu(B) && \text{BY subadditivity.} \end{aligned}$$

So $A_1 \cup A_2$ is measureable. Moreover, we note that all inequalities above are actually equalities. Suppose A_1 and A_2 were disjoint. Take $B = A_1 \cup A_2$. Then,

$$\begin{aligned} \mu(A_1 \cup A_2) = (*) &= \mu(\emptyset \cap (A_1 \cup A_2)) + \mu(A_2 \cap (A_1 \cup A_2)) \\ &+ \mu(A_1 \cap (A_1 \cup A_2)) + \mu((A_1^C \cap A_2^c) \cap (A_1 \cup A_2)) \\ &= 0 + \mu(A_2) + \mu(A_1) + 0. \end{aligned}$$

Therefore, we have finite, disjoint additivity.

Step 2: Suppose $A_1, A_2 \subseteq \mathbb{R}^d$ are measureable. Then $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$ is measureable by the prior step and (1).

Step 3: Suppose $\{A_k\}$ is a sequence of measureable subsets of $\mathbb{R}^d$. Note we can always reduct this to the case of disjiont unions by noticing

$$\bigcup_{k\geq 1} A_k = \bigcup_{k\geq 1}(A_k \cap A_{k-1}^c \cap \cdots \cap A_1^c)$$

and that sets in the left union are disjoint. So, we assume that A_k are disjoint. Let $B_k = A_1 \cup \cdots \cup A_k$. By part 1, B_k is measureable for all k.

Suppose $C \subseteq \mathbb{R}^d$.

$$\begin{aligned}\mu(B_k \cap C) &= \mu(A_k \cap B_k \cap C) + \mu(A_k^c \cap B_k \cap C)\\ &= \mu(A_k \cap C) + \mu(B_{k-1} \cap C)\\ &= \mu(A_1 \cap C) + \mu(A_2 \cap C) + \cdots + \mu(A_k \cap C).\end{aligned}$$

Let $B = \bigcup_{k\geq 1} A_k$. Then, for any $n > 1$,

$$\begin{aligned}\mu(C) = \mu(B_n \cap C) + \mu(B_n^c \cap C)&\\ &= \sum_{k=1}^{n} \mu(A_k \cap C) + \mu(B_n^c \cap C).\end{aligned}$$

We note that because $B_n \subseteq B$ that $B_n^c \supseteq B^c$. So monotonicity gives us that

$$\begin{aligned}\sum_{k=1}^{n} \mu(A_k \cap C) + \mu(B_n^c \cap C) &\geq \sum_{k=1}^{n} \mu(A_k \cap C) + \mu(B_n^c \cap C)\\ \text{send } n \to \infty&\\ \mu(C) &\geq \sum_{k\geq 1} \mu(A_k \cap C) + \mu(B^c \cap C)\\ &\geq \mu(B \cap C) + \mu(B^c \cap C) && \text{By subadditivity.}\\ &\geq \mu(C) && \text{By subadditivity again.}\end{aligned}$$

Therefore, $\bigcup_{k\geq 1} A_k$ is measureable. Moreover,

$$\sum_{k\geq 1} \mu(A_k \cap C) + \mu(B^c \cap C) = \mu(B \cap C) + \mu(B^c \cap C)$$

so

$$\mu(B \cap C) = \sum_{k\geq 1} \mu(A_k \cap C).$$

□

Due to these properties, if we restrict our attention to measureable sets, everything is nice.

Corollary 1.F.5. *Open and closed sets are measureable.*

Proof. Open sets in $\mathbb{R}^d$ are countable unions of rectangles as proven in quarter 1. So, by properties (3) and (4) we have that open sets are measureable. Complements of closed sets are open so by (1) closed sets are alsow measureable.

□

Corollary 1.F.6. *If A_k is a sequence of measureable sets then*

$$\bigcup_{k\geq 1} A_k \textit{ is measureable and } \mu\Big(\bigcup_{k\geq 1} A_k\Big) = \lim_{n\to\infty} \mu(A_1 \cup \cdots \cup A_n).$$

Additionally,

$$\bigcap_{k\geq 1} A_k \textit{ is measureable and } \mu(\bigcap_{k\geq 1} A_k) = \lim_{n\to\infty} \mu(A_1 \cap \cdots \cap A_n).$$

Proof. $\bigcup_{k\geq 1} A_k$ is measureable by (5). Let $B_k = A_k \cap A_{k-1}^c \cap \cdots \cap A_1^c$. Notice that the B_k are disjoint and by measureability we have

$$\mu(B_n) = \mu(A_1 \cup \cdots \cup A_n) - \mu(A_1 \cup \cdots \cup A_{n-1}).$$

Therefore, by properties of telescoping sums we have

$$\sum_{k=1}^{n} \mu(B_n) = \mu(A_1 \cup \cdots \cup A_n).$$

And then we have

$$\mu(\bigcup_{k\geq 1} A_k) = \mu(\bigcup_{k\geq 1} B_k) = \sum_{k\geq 1} \mu(B_k) = \lim_{n\to\infty} \mu(A_1 \cup \cdots \cup A_k).$$

$\square$

Do 1.1. Show the corollary for intersections.