Problem Set 2

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Problem 1. Let X be a finite set with cardinality d. Denote by

$$\Delta(X) = \{p \in \mathbf{R}^X : p_x \geq 0 \text{ for all } x, \text{ and } \sum_{x\in X} p_x = 1\}$$

the set of all *lotteries* over X. Interpret p_x as the probability of receiving $x \in X$ in lottery p.

For any $v \in \mathbf{R}^d$, let $u_v : \Delta(X) \to \mathbf{R}$ be defined by

$$u_v(p) \coloneqq \sum_{x\in X} v_x p_x = v \cdot p.$$

If v_x is the expected utility of receiving, or consuming, $x \in X$; then $u_v(p)$ is the *expected utility* of lottery p.

In an abuse of notation, we denote by $x \in X$ the *degenerate lottery* $\mathbf{1}_x \in \Delta(X)$ that places probability one on x.

1. A *probability measure* on X is a function $\mu\colon 2^X \to \mathbf{R}$ with the properties that $\mu(A) \geq 0$ for all $A \subseteq X$, $\mu(X) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A, B \subseteq X$ are disjoint. Show that μ is a probability measure iff there is a lottery $p \in \Delta(X)$ with $\mu(A) = \sum_{x\in A} p(x)$ for all $A \subseteq X$.[1]

   In this case, we define the *mathematical expectation* of a function $f\colon X \to \mathbf{R}$ by $\mathbf{E}_\mu f \coloneqq \sum_{x\in X} p(x)f(x) \eqqcolon f \cdot p$.

2. Let $\succeq_v$ be the binary relation on $\Delta(X)$ defined by $p \succeq_v q$ iff $u_v(p) \geq u_v(q)$. Show that $\succeq_v$ is a preference relation (so it is complete and transitive).

[1] The subsets $A \subseteq X$ are called *events*, and $\mu(A)$ is the probability of event A.

3. Let $\succeq$ be a binary relation on $\Delta(X)$ and consider the following property: for all $p, q, r \in \Delta(X)$ and $\lambda \in (0,1)$

$$p \succeq q \text{ iff } \lambda p + (1-\lambda)r \succeq \lambda q + (1-\lambda)r.$$

This property is known as the *independence axiom*. Show that $\succeq_v$ defined in Part 1 satisfies the independence axiom.

4. Consider an example in which X is a set of monetary quantities: 0, one million, and 5 million dollars. Consider four lotteries:

X	0	1.000.000	5.000.000
p_A	0	1	0
p_B	1/100	89/100	10/100
$p_{A'}$	89/100	11/100	0
$p_{B'}$	90/100	0	10/100

Consider choosing one lottery from the set $\{p_A, p_B\}$. Which one would you prefer? If, instead, you had to chose one from the set $\{p_{A'}, p_{B'}\}$, which would you then prefer?

Many people choose p_A from $\{p_A, p_B\}$ and $p_{B'}$ from $\{p_{A'}, p_{B'}\}$. Show that this is inconsistent with any preference of the form $\succeq_v$.

5. Consider a fixed strict preference $\geq$ on X (not on $\Delta(X)$). The rest of this problem concerns this fixed preference $\geq$.

Let $L(\geq)$ be the set of preferences over $\Delta(X)$ that satisfy two properties: a) For all $x, y \in X$, $x \geq y$ iff $x \succeq y$ (regarding x and y as degenerate lotteries in the latter case), and b) there exists some $v \in \mathbf{R}^X$ so that $\succeq = \succeq_v$. Show that $L(\geq)$ is nonempty and that

$$\text{if } \succeq_v, \succeq_w \in L(\geq) \text{ then } \succeq_{\alpha v + \beta w} \in L(\geq),$$

for any $\alpha, \beta > 0$.

6. Find an example of $\succeq_v$ and $\succeq_w$ in $L(\geq)$, and $p, q \in \Delta(X)$ such that $p \succ_v q$ while $q \succ_w p$. Prove that such an example must involve $|X| \geq 3$ (so there's no such example when X has two elements).

7. Define $U(x) = \{y \in X : y \geq x\}$, for all $x \in X$. Show that $p, q \in \Delta(X)$ have the property that $p \succeq q$ for all $\succeq \in L(\geq)$ if and only if $p(U(x)) \geq q(U(x))$ for all $x \in X$.[2]

[2]Here we're using the event notation, so $p(U(x)) = \sum_{z \in U(x)} p_z$.

Problem 2. Consider now a problem with an outside option: $X \cup \{\varnothing\}$. Let $\Delta_-(X) = \{p \in \mathbf{R}^X_+ : \sum_{x \in X} p_x \leq 1\}$. We still call the elements of $\Delta_-(X)$ lotteries, but the understanding now is that if an agent holds lottery $p \in \Delta_-(X)$ then they obtain object $x \in X$ with probability p_x, and $\varnothing$ with probability $1 - \sum_x p_x$.

The elements of $\Delta_-(X)$ are ordered as regular vectors in $\mathbf{R}^X$. So we write $p \geq q$ if $p_x \geq q_x$ for all $x \in X$. Say that a preference relation $\succeq$ on $\Delta_-(X)$ is *strictly monotonic* if $p \geq q$ implies that $p \succeq q$, and $p > q$ implies that $p \succ q$.

In the same spirit as the previous problem, we may consider preferences $\succeq_v$ over $\Delta_-(X)$ with $v \colon X \cup \{\varnothing\} \to \mathbf{R}$ and $p \succeq_v q$ iff

$$\sum_{x \in X} p_x v(x) + (1 - \sum_{x \in X} p_x) v(\varnothing) \geq \sum_{x \in X} q_x v(x) + (1 - \sum_{x \in X} q_x) v(\varnothing).$$

Show that $\succeq_v$ is a monotonic preference relation iff $v(\varnothing) < v(x)$ for all $x \in X$.

Problem 3. Consider a standard cake-cutting problem. Show that when $n = 2$ then proportionality and envy-freeness are equivalent.

Problem 4. Consider a cake-cutting protocol for $n = 3$ agents: Alice, Bob, and Carlos. Suppose that Alice first cuts the cake into three pieces. Then Bob gets to choose a piece of the three, followed by Carlos, who chooses one of the two pieces that remain after Bob's choice. Finally, Alice gets the piece that was not chosen by either Bob or Carlos.

Analyze the fairness of this protocol. Is any agent guaranteed to obtain at least a utility of 1/3 their value of the whole cake? Could any agent envy another?

Problem 5. Let $(X, A, \{u_i : i \in A\})$ be a cake-cutting problem. A division of the cake $(P_1, \dots, P_n)$ is ε-envy-free if

$$u_i(P_i) + \varepsilon \geq u_i(P_j),$$

for all $i, j \in A$.

Consider a protocol in which a knife moves from left to right in the cake. Whenever an agent thinks that the left piece is worth 1/3 of the value of the cake, they call out: receive the left piece and exit. Repeat this procedure until there are either no more agents left, or the cake is exhausted without anyone calling out. If we terminate because there are no more agents left, give the reminder of the cake to the last agent who got a piece. If we terminate because we ran out of cake without anyone calling out, give the remaining piece to a random agent who hasn't received a piece.

Show that this algorithm terminates with a 1/3-approximate envy-free division in which each piece P_i is an interval.

Problem 6. Consider a cake-cutting problem with $n = 3$. First Alice cuts the cake in two equal (to her) pieces. Then Bob picks one of the two pieces, and cuts this piece in three equal (to him) pieces. Alice cuts the piece that Bob didn't choose into three as well. So we have six pieces. Then Carlos chooses two pieces: one from Alice's sub-cake, and one from Bob's. Then Bob chooses one from his pile and one from Alice's. Finally, Alice gets the remainder.

1. Show that if all three have the same utilities then the partition satisfies proportionality.

2. Show that, even if they have different utilities, Carlos gets a utility of at least 1/3.

3. Consider now the following fact: for any piece of cake and any two agents, there is a partition of the cake into two or three pieces so that both agents regard all pieces as equally good [extra credit: prove this fact by means of a cake-cutting protocol; a hint is that you should use the intermediate value theorem]. Using the fact, show that it is possible to specify the above protocol so that it results in a proportional division.

Problem 7. Consider a cake-cutting problem with three agents and the following protocol. First, Alice cuts the cake into three pieces that she regards as equally valuable. Second, Bob picks the two most valuable pieces, according to him. If the two are equally valuable to Bob, then we finish simply by asking Carina to choose one of the three, followed by Bob, and then Alice gets the last piece.

If the two best pieces are not equally good to Bob, then we ask him to cut off a piece from the most valuable piece so as to make them equally valuable. Let's label these pieces: Q_1 is the piece that was just trimmed by Bob, Q_2 is the piece that was second-best to Bob, and is now just as good as Q_1; Q_3 is the last piece. Finally P is the piece that was cut off by Bob from the piece that is now Q_1.

Now we ask Carina to choose one piece. Then Bob picks one, but if Carina didn't pick Q_1, then Bob has to pick Q_1 (So Q_1 is either chosen by Carina or it goes to Bob). Finally, Alice gets the remaining piece. It remains to allocate P: Suppose that Bob received Q_1. Then we ask Carina to divide P into three equal pieces and we let Bob pick one of them: call this piece P_1. Then we ask Alice to pick a piece, call it P_2. Finally Carina gets the remaining piece, P_3. If it was Carina who picked Q_1, then proceed as just described with the remainder P, but with Carina and Bob's roles reversed.

Show that this procedure ends with an envy-free division.