Problem Set 3

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Problem 1. Show that the Shapley value satisfies the axioms of marginality and substitute player.

Problem 2. Show that if (N, v_T) is a simple game, $T \subseteq N$, and $v = a \times v_T$ for some $a > 0$, then the Shapley value of v satisfies $\varphi_i(v) = 0$ for $i \notin T$, and $\varphi_i(v) = \frac{a}{|T|}$ for $i \in T$.

Problem 3. Let s be a solution that satisfies marginality. Show that if $\Delta_i^v(P) = \Delta_i^w(P)$ for all $P \subseteq N$ then $s_i(v) = s_i(w)$. Observe that this means that s is only a function of the players' marginal contributions.

Problem 4. Consider a market with three agents. Agent 1 is a seller who has one indivisible good to sell. The good is worth $w_1 > 0$ to the seller. Agents 2 and 3 are buyers. The good is worth w_i to buyer i, with $i = 2, 3$. Suppose that $w_3 > w_2 > w_1$.

The market defines a game $(\{1, 2, 3\}, v)$ as follows: $v(\{1\}) = w_1$, $v(\{1, i\}) = w_i$ for $i = 2, 3$; and $v(N) = w_3$. For all other coalitions, $v(S) = 0$.

1. Why does the definition of v make sense? Provide a short explanation.
2. Find the core of the game. What is the best imputation in the core from the viewpoint of the seller? What is the best from the viewpoint of the buyers?
3. Calculate the Shapley value. Is it in the core?

Problem 5. Consider a game (N, v) in which the coalitions are partitioned in two: $2^N = \mathcal{W} \cup \mathcal{L}$, with $\mathcal{W} \cap \mathcal{L}$ empty. The function $v(S)$ is 1 when $S \in \mathcal{W}$ and 0 when $S \in \mathcal{L}$. The coalitions in $\mathcal{W}$ are the "winning" coalitions and the others are the "losing" coalitions.[1] Suppose that:

- $\emptyset \in \mathcal{L}$.
- $S \in \mathcal{W}$ iff $S^c \in \mathcal{L}$.
- $S \in \mathcal{W}$ and $S \subseteq T$ implies that $T \in \mathcal{W}$.

Let $V = \{i \in N : i \in S \text{ for all } S \in \mathcal{W}\}$ be the set of *veto players*.

1. Show that if $V = \emptyset$ (there are no veto players) then the core of the game is empty.

2. Suppose that $V \neq \emptyset$. Show that x is a core imputation iff $\sum_{i \in V} x_i = 1$. In words, the veto players get everything.

Problem 6. Let (N, v) be a convex game (recall that this means that $V(A) + v(B) \leq V(A \cap B) + v(A \cup B)$ for any $A, B \subseteq N$).

1. Fix an ordering $\geq$ of the players in N and let $x_i = \Delta_i^v(S(\geq, i))$ be the marginal contribution of player i to the coalition of players who precede i in the ordering. Show that $x_i \geq 0$ and that $\sum_{i \in N} x_i = v(N)$.

2. Let $A \subseteq B$. Show that $\Delta_i^v(A) \leq \Delta_i^v(B)$.

3. Show that $(x_1, \ldots, x_n)$ from part (1) lies in the core of (N, v). Observe that this proves the theorem that we stated in lecture.

4. Show that the Shapley value of v (meaning the vector that gives each player their payoff in the Shapley value) is in the core of (N, v).

Problem 7. Let v be a simple game (these are the v_T games in lecture). Show that the core of a simple game is non-empty, and describe all the core imputations.

[1] The weighted majority games we talked about in class are examples of these games.

Problem 8. Consider a game (N, v) with $v(S) = f(|S|)$, for a monotone increasing function $f : \{0, 1, \ldots, n\} \to \mathbf{R}_+$ such that $f(0) = 0$. So the value of a coalition only depends on its size.

1. Show that if the core of the game is non-empty, then the imputation $(f(n)/n, \ldots, f(n)/n)$ is in the core.

2. Show that if $f(x)/x \leq f(n)/n$ for all $x \in \{1, \ldots, n\}$ then the core is non-empty.

3. Consider the "Nasty neighbor" game, where $N = \{1, \ldots, n\}$ and each person has a piece of garbage that they want to dispose of by throwing it in the gardens of their neighbors. One piece of garbage provides a disutility of 1. So the coalition S generates a value $v(S) = M - (n - |S|)$ for some large constant M. The iterpretation is that the members of S will throw their garbage into the garden of the members of $N \setminus S$, but receive the garbage of the remaining players The grand coalition has to consume its own garbage, so its value is $v(N) = M - n$. Show that the core of this game is empty.