№3.

$$p(x) = \begin{cases} \dfrac{e^{-\frac{x}{\theta}}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0, \quad n = 3.$$

$$\tilde{\theta}_1 = \overline{X}$$

$$\tilde{\theta}_2 = X_{(2)}$$

$$M[\xi] = \int_0^\infty x \frac{e^{-\frac{x}{\theta}}}{\theta} dx = -xe^{-\frac{x}{\theta}}\Big|_0^\infty + \int_0^\infty e^{-\frac{x}{\theta}} dx = -\theta e^{-\frac{x}{\theta}}\Big|_0^\infty = \theta$$

$$M[\xi^2] = \int_0^\infty x^2 \frac{e^{-\frac{x}{\theta}}}{\theta} dx = -x^2 e^{-\frac{x}{\theta}} \Big|_0^\infty + 2\int_0^\infty x e^{-\frac{x}{\theta}} dx =$$

$$= -2\theta \int_0^\infty x\, d e^{-\frac{x}{\theta}} = -2\theta x e^{-\frac{x}{\theta}} \Big|_0^\infty + 2\theta \int_0^\infty e^{-\frac{x}{\theta}} dx = -2\theta^2 e^{-\frac{x}{\theta}} \Big|_0^\infty =$$

$$= 2\theta^2$$

$$D\xi = 2\theta^2 - \theta^2 = \theta^2$$

1) $\tilde{\theta}_1 = \overline{x}$

$$M[\overline{x}] = M[\frac{1}{n}\sum x_i] = M[\xi] = \theta$$

Несмещ.

Из усл. предполагается, что нер-во Крамера-
-Рао выполняется

$$D[\overline{x}] = D[\frac{1}{n}\sum x_i] = \frac{1}{n^2}\sum D[x] = \frac{1}{n} D\xi = \frac{\theta^2}{n}.$$

$$\ln p = \ln \frac{e^{-\frac{x}{\theta}}}{\theta} = -\frac{x}{\theta} - \ln\theta$$

$$I(\theta) = \int_0^\infty \left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2 \frac{e^{-\frac{x}{\theta}}}{\theta} dx = \int_0^\infty \frac{x^2}{\theta^5} e^{-\frac{x}{\theta}} dx - 2\int_0^\infty \frac{x}{\theta^4} e^{-\frac{x}{\theta}} dx +$$

$$+ \int_0^\infty \frac{1}{\theta^3} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta^4} M[\xi^2] - \frac{2}{\theta^3} M[\xi] - \frac{1}{\theta^2}\int_0^\infty d e^{-\frac{x}{\theta}} =$$

$$= \frac{1}{\theta^4} 2\theta^2 - \frac{2}{\theta^3}\theta - \frac{1}{\theta^2} e^{-\frac{x}{\theta}} \Big|_0^\infty = \frac{1}{\theta^2}$$

$$D[\overline{x}] = \frac{\theta^2}{n} \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{n}$$

Эффект. по дост. усл.

2) $\tilde{\Theta}_2 = X_{(2)}$

$$p(x) = n\,p(x)\,C_{n-1}^{k-1}(1-F(x))^{n-k}(F(x))^{k} = n e^{-\frac{x}{\Theta}}\frac{1}{\Theta}\overset{2}{\underset{\shortparallel}{C}}{}_2^1 \cdot$$

$$\left\{ F(x) = \int_0^x e^{-\frac{t}{\Theta}}dt = -\int_0^x d e^{-\frac{t}{\Theta}} = -e^{-\frac{t}{\Theta}}\Big|_0^x = 1-e^{-\frac{x}{\Theta}} \right.$$

$$\cdot\left(1-(1-e^{-\frac{x}{\Theta}})\right)^{3-2}\cdot\left(1-e^{-\frac{x}{\Theta}}\right)^{2-1} = \frac{2n}{\Theta}e^{-2\frac{x}{\Theta}}\left(1-e^{-\frac{x}{\Theta}}\right)$$

$$M[X_{(2)}] = \int_0^\infty x\frac{2n}{\Theta}e^{-\frac{2x}{\Theta}}(1-e^{-\frac{x}{\Theta}})dx = \left\{ \frac{x}{\Theta} = t,\ dx = \Theta dt \right\} =$$

$$= \int_0^\infty t\,2n e^{-2t}(1-e^{-t})\Theta dt = 2n\Theta\left[\int_0^\infty t e^{-2t}dt - \int_0^\infty t e^{-3t}dt\right] =$$

$$= 2n\Theta\left[-\frac{1}{2}\int_0^\infty t\,de^{-2t} + \frac{1}{3}\int_0^\infty t\,de^{-3t}\right] = 2n\Theta\left[-\frac{1}{2}te^{-2t}\Big|_0^\infty + \right.$$

$$\left. + \frac{1}{2}\int_0^\infty e^{-2t}dt + \frac{1}{3}te^{-3t}\Big|_0^\infty - \frac{1}{3}\int_0^\infty e^{-3t}dt\right] = 2n\Theta\left[-\frac{1}{4}e^{-2t}\Big|_0^\infty + \right.$$

$$\left. + \frac{1}{9}e^{-3t}\Big|_0^\infty\right] = 2n\Theta\left[\frac{1}{4}-\frac{1}{9}\right] = \frac{10}{36}n\Theta = \frac{10}{12}\Theta = \frac{5}{6}\Theta$$

Не несмещ.

$\tilde{\Theta}_2' = \cancel{\frac{9}{10n}X_{(2)}}$ — несмещ.

$= \frac{\cancel{3}\,6}{\cancel{10}\,5}X_{(2)}$

$M[\tilde{\Theta}_2'] = \Theta$

$$M[\underset{\tilde{\Theta}_2}{\cancel{X_{(2)}^2}}] = \int_0^\infty x^2\frac{6}{\Theta}e^{-\frac{2x}{\Theta}}(1-e^{-\frac{x}{\Theta}})dx = \left\{\frac{x}{\Theta}=t,\ dx=\Theta dt\right\} =$$

$$= \int_0^\infty \theta^2 t^2 \frac{6}{\cancel{\theta}} e^{-2t}(1-e^{-t}) \cancel{\theta}\, dt = 6\theta^2 \left[\int_0^\infty t^2 e^{-2t}\, dt - \int_0^\infty t^2 e^{-3t}\, dt\right] =$$

$$= 6\theta^2 \left[\frac{1}{\cancel{2}}\int_0^\infty \cancel{2} t e^{-2t}\, dt - \frac{1}{3}\int_0^\infty 2t e^{-3t}\, dt\right] = 6\theta^2\left[\frac{1}{2}\int_0^\infty e^{-2t}\, dt\right.$$

$$\left. - \frac{2}{9}\int_0^\infty e^{-3t}\, dt\right] = 6\theta^2\left[-\frac{1}{4} e^{-2t}\Big|_0^\infty + \frac{2}{27} e^{-3t}\Big|_0^\infty\right] = 6\theta^2\left[\frac{1}{4} - \frac{2}{27}\right] =$$

$$= 6\theta^2 \frac{19}{108} = \frac{19}{18}\theta^2$$

~~$D[\tilde{\theta}_2'] = \frac{19}{18}\theta^2 - \theta^2 = \frac{1}{18}\theta^2.$~~

~~$D[\tilde{\theta}_2'] = \frac{\theta^2}{18} \ngeq \frac{1}{nI(\theta)} = \frac{\theta^2}{n}$~~

вероятно, ошибка в вычислениях

~~Не эффективна~~

$$D[\tilde{\theta}_2] = \frac{19}{18}\theta^2 - \frac{25}{36}\theta^2 = \frac{13}{36}\theta^2$$

$$D[\tilde{\theta}_2'] = \frac{36}{25}\,\frac{13}{36}\theta^2 = \frac{13}{25}\theta^2$$

$$D[\tilde{\theta}_2'] = \frac{13}{25}\theta^2 \geq \frac{1}{nI(\theta)} = \frac{\theta^2}{3}$$

Не эффективна

$\forall\theta\quad \frac{\theta^2}{3} \leq \frac{13\theta^2}{25} \Rightarrow \quad \tilde{\theta}_1$ эффективнее $\tilde{\theta}_2'$