T4.

$\xi \sim p(x, \theta) = a\{(-1,1)\} + b\{0,2\}$

$\int\limits_{-\infty}^{\infty} p(x)dx = 1 \qquad \int\limits_{-1}^{1} a\,dx + \cancel{0 \cdot b + 2b}\ b = 1.$

$2a + b = 1$

$a = \theta, \quad b = 1 - 2\theta, \quad \theta \in (0, 2)$

$\xi \sim p(x, \theta) = \theta\{(-1,1)\} + (1-2\theta)\{0,2\}$

① ОММ

$\alpha_1 = M[\xi] = \int\limits_{-1}^{1} \theta x\,dx + 0(1-2\theta) + 2(1-2\theta) = \cancel{\tfrac{4}{4}}\theta + 2 - 4\theta =$

$= 2 - 3\theta$

$\alpha_2 = M[\xi^2] = \int\limits_{-1}^{1} \theta x^2 dx + 2^2(1-2\theta) = \frac{2}{3}\theta + 4 - 8\theta =$

$= \frac{2}{3}\cancel{\theta}\ 4 - \frac{22}{3}\theta$

$2 - 3\tilde{\theta} = \alpha_1 = \bar{x} \Rightarrow \quad \tilde{\theta} = \frac{2 - \bar{x}}{3}$

Несмещ. ✓

1 $M[\tilde{\theta}] = M\left[\frac{2}{3} - \frac{\bar{x}}{3}\right] = \frac{2}{3} - \frac{1}{3} M\left[\frac{1}{n}\sum x_i\right] = \frac{2}{3} - \frac{1}{3} M\xi =$

$= \frac{2}{3} - \frac{1}{3}(2 - 3\theta) = \theta$

Состоят. ✓

$D\tilde{\theta} = D\left[\frac{2}{3} - \frac{1}{3}\bar{x}\right] = \frac{1}{9} D\left[\frac{1}{n}\sum x_i\right] = \frac{n}{9n^2} D\xi =$

$$\frac{1}{9n}\left[4-\frac{22}{3}\theta-(2-3\theta)^2\right]=\frac{1}{9n}\left[4-\frac{22}{3}\theta-4+12\theta-9\theta^2\right]=$$

$$=\frac{1}{9n}\left[\frac{14}{3}\theta-9\theta^2\right]\longrightarrow 0,\quad n\to\infty$$

'' ЮЭ

Эффект.

а) Регул модел.

1) $p(x)\in C^1$ по θ ✓

2) $$\int_{-1}^{1}\frac{\partial}{\partial\theta}\theta\,dx+\frac{\partial}{\partial\theta}(1-2\theta)=2-2=0 \ ✓$$

3) $$I(\theta)=\int_{-\infty}^{\infty}\left(\frac{\partial\ln p(x,\theta)}{\partial\theta}\right)^2 p(x,\theta)dx=\int_{-1}^{1}\left(\frac{\partial\ln p(x,\theta)}{\partial\theta}\right)^2\cdot$$

$$p(x,\theta)dx+\left(\frac{\partial\ln p(\theta)}{\partial\theta}\right)^2 p(\theta)=$$

$$=\int_{-1}^{1}\frac{1}{\theta^2}\theta\,dx+\left(\frac{-2}{1-2\theta}\right)^2(1-2\theta)=\frac{2}{\theta}+\frac{4}{1-2\theta}=$$

$$=\frac{2-4\theta+4\theta}{\theta(1-2\theta)}=\frac{2}{\theta(1-2\theta)}$$

$I(\theta)>0$ и непрер. ✓

б) Регул оценки

1) Несмещ ✓

2) Модель регул ✓

3) $D[\tilde{\theta}]$ огран на $(0,\ 1/2)$

$$D[\tilde{\theta}] \geq \frac{1}{n I(\theta)} = \frac{\theta(1-2\theta)}{2n}$$

$$\frac{1}{9\cancel{n}}\left(\frac{14}{3}\theta - 9\theta^2\right) \geq \frac{\theta(1-2\theta)}{2\cancel{n}}$$

$$\frac{14}{27}\theta - \theta^2 \ngeq \frac{\theta}{2} - \theta$$

$$>$$

② ОМП

Пусть 0 и 2 m штук

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = (1-2\theta)^m \theta^{n-m}$$

$$\ln L = m \ln(1-2\theta) + (n-m)\ln\theta$$

$$(\ln L)'_\theta = \frac{-2m}{1-2\theta} + \frac{n-m}{\theta} = 0$$

$$\cancel{2m\theta} = (n-m)(1-2\theta) = n - m - 2\theta n + \cancel{2\theta m}$$

$$\tilde{\theta} = \frac{n-m}{2n} = \frac{1}{2} - \frac{\nu}{2} = \frac{1-\nu}{2}$$

Несмещ. ✓

$$M[\tilde{\theta}] = M\left[\frac{1}{2} - \frac{\nu}{2}\right] = \frac{1}{2} - \frac{1}{2}M[\nu] = \frac{1}{2} - \frac{1}{2}(1-2\theta) =$$

$$= \theta$$

Сост. ✓

$$D[\tilde{\theta}] = D\left[\frac{1}{2} - \frac{\nu}{2}\right] = \frac{1}{4}D[\nu] = \frac{1}{4}\frac{(1-2\theta)2\theta}{n} = \frac{(1-2\theta)\theta}{2n} \xrightarrow[n\to\infty]{} 0$$

а) Регул модель

по прошлому пункту

б) Регул оценки

1) Несмещ. V

2) Модель регул.

3) $D[\hat{\theta}]$ огран на $\forall$ комп. на $(0, \theta)$

$$D[\hat{\theta}] \geq \frac{1}{n I(\theta)}$$

$$\frac{(1-2\theta)\theta}{2n} = \frac{\theta(1-2\theta)}{2n}$$

Оценка эффективна

T5

$$\xi \sim R[\theta, 2\theta] \quad \frac{1}{\theta}\{(\theta, 2\theta)\} = p(\theta)$$

1. $$\alpha_1 = \int_{\theta}^{2\theta} x \frac{1}{\theta} dx = \frac{1}{\theta} \frac{x^2}{2}\Big|_{\theta}^{2\theta} = \frac{3}{2}\theta$$

$$\alpha_2 = \int_{\theta}^{2\theta} x^2 \frac{1}{\theta} dx = \frac{1}{\theta}\frac{x^3}{3}\Big|_{\theta}^{2\theta} = \frac{7}{3}\theta^2$$

$$D = \alpha_2 - \alpha_1^2 = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{1}{12}\theta^2$$

$$\frac{3}{2}\theta = \cancel{\frac{1}{n}} \overline{x} \Rightarrow \hat{\theta} = \frac{2}{3}\overline{x}$$

Несмещ. ✓

$$M[\hat{\theta}] = \frac{2}{3} M\left[\frac{1}{n}\sum x_i\right] = \frac{2}{3} M\xi = \frac{2}{3}\cdot\frac{3}{2}\theta = \theta$$

Сост. ✓

$$D[\hat{\theta}] = \frac{4}{9} D\left[\frac{1}{n}\sum x_i\right] = \frac{4}{9n} D\xi = \frac{\theta^2}{27n} \longrightarrow 0, \quad n\to\infty$$

Эффект.

а) Регул. модели

1) $p \in C^1[\theta, 2\theta]$ по θ ✓

2) $\int\limits_{\theta}^{2\theta} \frac{\partial}{\partial\theta}\frac{1}{\theta}\,dx = -\int\limits_{\theta}^{2\theta}\frac{1}{\theta^2}dx = -\frac{1}{\theta}$ ✗

???

2. $L(\theta) = \frac{1}{\theta^n}\{\forall i,\ \theta < x_i < 2\theta\} = \frac{1}{\theta^n}\{\theta < \max x_i < 2\theta\}$

$$\hat{\theta} = \max x_i = X_{max}$$

Несмещ:

$$M[\hat{\theta}] = M[X_{max}] = \int\limits_{-\infty}^{\infty} x\,\psi(x)\,dx = \int\limits_{\theta}^{2\theta} n\frac{x^n}{\theta^n}dx =$$

$$\left\{\psi = \Psi' = n(F)^{n-1}p = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\{(\theta, 2\theta)\}\right\}$$

$$= \frac{n}{n+1}\left((2\theta)^{n+1} - \theta^{n+1}\right)\frac{1}{\theta^n} = (2^{n+1}-1)\frac{n}{n+1}\theta$$

$$\hat{\theta}' = \frac{n+1}{n}\frac{1}{(2^{n+1}-1)}X_{max} \text{ — несмещ.}$$

Сост.

$$\cancel{D[\hat{\theta}'] = \left(\frac{n+1}{n}\frac{1}{2^{n+1}-1}\right)^2 D[x_{max}]}$$

$$M[\xi^2] = \int_{\theta}^{2\theta} n\frac{x^{n+1}}{\theta^n}dx = \frac{n}{n+2}\frac{1}{\theta^n}\left((2\theta)^{n+2} - \theta^{n+2}\right) =$$

$$= \frac{n}{n+2}\left(2^{n+2}-1\right)\theta^2$$

$$D[\hat{\theta}] = \frac{n}{n+2}\left(2^{n+2}-1\right)\theta^2 - \left(\frac{n}{n+1}\right)^2\left(2^{n+1}-1\right)^2\theta^2 =$$

$$= \frac{(n+1)^2 n\left(2^{n+2}-1\right) - (n+2)n^2\left(2^{n+1}-1\right)^2}{(n+2)(n+1)^2}\theta^2 \not\to 0$$

Несост.

$$D[\hat{\theta}'] = \frac{(n+1)^2}{n^2}\frac{1}{\left(2^{n+1}-1\right)^2}D[\hat{\theta}] = \frac{(n+1)^2 n\left(2^{n+2}-1\right) - (n+2)n^2\left(2^{n+1}-1\right)^2}{n^2(n+2)\left(2^{n+1}-1\right)^2}\theta^2$$

$$= \left[\frac{(n+1)^2\left(2^{n+2}-1\right)}{n(n+2)\left(2^{n+1}-1\right)^2} - 1\right]\theta^2$$

$$\cancel{27}\ \frac{1}{27n}\theta^2 \leq \left[\frac{(n+1)^2\left(2^{n+2}-1\right)}{n(n+2)\left(2^{n+1}-1\right)^2} - 1\right]\theta^2$$

ОММ эффективнее

Асимп. дов. интерв.

по ОММ

$$\tilde{\theta} = \frac{2}{3}\bar{x} \quad \Rightarrow \quad g(\tilde{\alpha}_1) = \frac{2}{3}\tilde{\alpha}_1$$

$$\nabla g = \frac{2}{3} \qquad K = \alpha_2 - \alpha_1^2$$

$$\mu_2 = \tilde{\alpha}_2 - \tilde{\alpha}_1^2$$

$$\tilde{\mu}_2 = S^2 \frac{n-1}{n}$$

$$\frac{\sqrt{n}(\hat{\theta}-\theta)}{\sqrt{\frac{2}{3}\cdot\frac{2}{3}(\tilde{\alpha}_2-\tilde{\alpha}_1^2)}} \rightsquigarrow N(0,1)$$

~~$t_1 = \rho_{0{,}025} \Rightarrow \quad \frac{x}{\theta} - 1 = 0{,}025$~~

~~$t_1 = x = 1{,}025\theta$~~

~~$t_2 = \rho_{0{,}975} \Rightarrow \quad \frac{x}{\theta} - 1 = 0{,}975$~~

~~$t_2 = x = 1{,}975\theta$~~

$$t_1 < \frac{30\left(\frac{2}{3}\bar{x}-\theta\right)}{\sqrt{19S^2}} < t_2$$

$$0{,}025\theta < \frac{\sqrt{100}\,(\hat{\theta}-\theta)}{\sqrt{4/9\, S^2 \frac{19}{20}}} < 0{,}975\theta.$$

$$0{,}025\theta < \frac{30(\hat{\theta}-\theta)}{\sqrt{19S^2}} < 0{,}975\theta$$

$$F(x) = \int_0^x \frac{1}{\theta}\,dt = \frac{x}{\theta}.$$

$$\left.\begin{array}{ll} t_1 = u_{0{,}025} \Rightarrow & \frac{x}{\theta} = 0{,}025 \\ & x = 0{,}025\theta \\ t_2 = u_{0{,}975} \Rightarrow & \frac{x}{\theta} = 0{,}975 \\ & x = 0{,}975\theta \end{array}\right\}$$

Если ищем квантили для норм. распред.

то $t_1 = -t_2 = -1,96$, как на семинаре.

Но я уже не понимаю, что надо...

по ОМП.

$$\hat{\theta} = x_{max}$$

$$I(\theta) = \int_{\theta}^{2\theta} \left(\frac{\partial \ln p(x, \theta)}{\partial \theta} \right)^2 p(x, \theta)\, dx = \int_{\theta}^{2\theta} \left(\frac{1}{\frac{1}{\theta}} \left(-\frac{1}{\theta^2} \right) \right)^2 \frac{1}{\theta} dx =$$

$$= \frac{1}{\theta^2}$$

$$\frac{\sqrt{n}(\hat{\theta} - \theta)}{\sqrt{\hat{\theta}^2}} \rightsquigarrow N(0, 1)$$

$$0{,}025\theta < \frac{\sqrt{n}(x_{max} - \theta)}{\sqrt{\hat{\theta}^2}} < 0{,}975\,\theta$$

$$t_1 < \frac{\sqrt{n}(x_{max} - \theta)}{\sqrt{\hat{\theta}^2}} < t_2$$

сгенер. $\vec{x}_n$, $n = 100$

ОММ:

$$\tilde{\theta} = \cancel{1{,}1{,}5{,}6} \quad 4{,}917$$

$$S^2 = 2{,}041$$

$$0{,}123 < \overset{10{,}552}{\cancel{4{,}8{,}1{,}8{,}2}}(4{,}917 - \theta) < 3{,}5763$$

$$4{,}578 \quad \cancel{2{,}8{,}3{,}6} < \theta < \cancel{3{,}0{,}2{,}2} \quad 4{,}905$$

$$l = \cancel{0{,}486} \quad 0{,}327$$

ОМП:

$$\hat{\theta} = 9{,}99$$

$$0{,}25 < 1{,}001(9{,}99 - \theta) < 9{,}74$$

$$0{,}26 < \theta < 9{,}74$$

$$l = 9{,}48$$

Какой-то бред

Т 6.

$$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1$$

ОМП:

$$L(x_i, \theta) = \left(\frac{\theta-1}{x_i^\theta}\right)^n = \frac{(\theta-1)^n}{\prod\limits_{i=1}^n x_i^\theta}$$

$$\ln L = n\ln(\theta-1) - \theta\sum_{i=1}^n \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^n \ln x_i \longrightarrow \max_{\theta>1}.$$

$$\frac{n}{\theta-1} = \sum_{i=1}^n \ln x_i$$

$$\tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{(\theta-1)^2} < 0 \text{ — т. max.}$$

$$F(x) = \int_1^x \frac{\theta-1}{t^\theta}\,dt = (\theta-1)\frac{t^{1-\theta}}{1-\theta}\Big|_1^x = -x^{1-\theta} + 1.$$

медиана:

$$-x^{1-\theta} + 1 = \frac{1}{2}$$

$$x_{\frac{1}{2}} = 2^{\frac{1}{\theta-1}}$$

$$g(\tilde{\theta}) = 2^{\frac{1}{\tilde\theta-1}}$$

$$I(\theta)=\int_1^{+\infty}\left(\frac{\partial \ln p}{\partial\theta}\right)^2 p\,dx=\int_1^{+\infty}\left(\frac{\partial \ln\frac{\theta-1}{x^\theta}}{\partial\theta}\right)^2\frac{\theta-1}{x^\theta}dx=$$

$$=\int_1^{+\infty}\left(\frac{\cancel{x^\theta}}{\theta-1}\cdot\frac{\cancel{x^\theta}-(\theta-1)x^\theta\ln x}{\cancel{x^{2\theta}}}\right)^2\frac{\theta-1}{x^\theta}dx=$$

$$=\int_1^{+\infty}\left(\frac{1-(\theta-1)\ln x}{\theta-1}\right)^2\frac{\theta-1}{x^\theta}dx=\int_1^{+\infty}\left(\frac{1}{\theta-1}-\frac{\ln x}{\theta-1}\right)^2\frac{\theta-1}{x^\theta}dx=$$

$$=\int_1^{+\infty}(1-\ln x)^2\frac{1}{(\theta-1)x^\theta}dx=\frac{\theta^2-4\theta+5}{(\theta-1)^4}$$

$$\nabla g=-\frac{2^{\frac{1}{\theta-1}}\ln 2}{(\theta-1)^2}$$

$$\frac{\sqrt{n}\,(g(\alpha)-g(\theta))}{\sqrt{\nabla' g\, I^{-1}\,\nabla g}}\rightsquigarrow N(0,1)$$

Для норм. распред. $t_1=-t_2=-1{,}96$, как я понял

$$\Rightarrow\ -1{,}96<\frac{\sqrt{n}\left(2^{\frac{1}{\theta-1}}-g(\theta)\right)}{\sqrt{\frac{2^{\frac{2}{\theta-1}}\ln^2 2}{\cancel{(\theta-1)^4}}\cdot\frac{\cancel{(\theta-1)^4}}{\theta^2-4\theta+5}}}<1{,}96$$

θ:

$$-1{,}96<\frac{\sqrt{n}\left(\frac{n}{\sum\ln x_i}+1-\theta\right)}{\sqrt{\frac{(\theta-1)^4}{\theta^2-4\theta+5}}}<1{,}96$$