T1.

$\xi \sim R(0, \theta)$ $\quad \theta > 0$ — вер. модель

Будет на к/р

$\vec{X}_n$ — выборка объема n.

$\tilde{\theta}_1 = 2\overline{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$

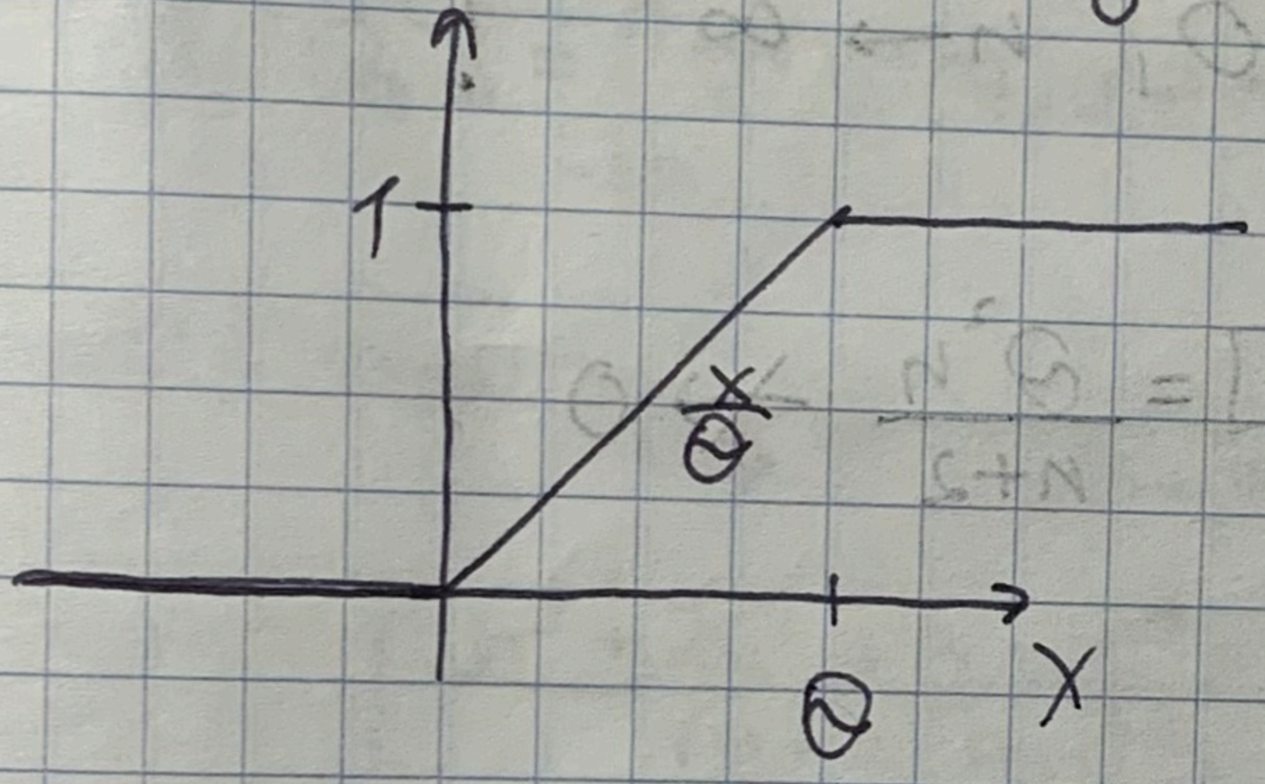
$\tilde{\theta}_2 = \min(x_i)$

$\tilde{\theta}_3 = \max(x_i)$

$\tilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$

Оценить оценки на состоят. и анесмен.

$M[\xi] = \int_{-\infty}^{\infty} x\, p(x)\,dx = \int_0^{\theta} \frac{x}{\theta}\,dx = \frac{\theta}{2}.$



$p(x) = \frac{1}{\theta}\{(0,\theta)\}$

$M[\xi^2] = \frac{\theta^2}{3}$

$D[\tilde{\theta}]$

$D[\xi] = \theta^2/3 - \theta^2/4 = \theta^2/12.$

- $\tilde{\theta}_1 = 2\overline{x}$

Несмещ: $\forall \theta > 0 \quad M[\tilde{\theta}_1] = \theta$

$M[\frac{2}{n}\sum_{i=1}^{n} x_i] = \frac{2}{n}\sum_{i=1}^{n} M[x_i] = 2M[\xi] = \theta$ ✓

Сост. $D[\theta_1] = D[\frac{2}{n}\sum_{i=1}^{n} x_i] = \frac{4}{n^2}\sum_{i=1}^{n} D[x_i] = \frac{4}{n}D[\xi] = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0$ ✓

по дост. усл.

- $\tilde{\theta}_2 = \min(x_i)$

$M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} y\,\varphi(y)\,dy = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}\,y\,dy = \left\{t = 1-\frac{y}{\theta}\right\} =$

$\Phi(y) = 1-(1-F(y))^n$

$\varphi(y) = \Phi'(y) = n(1-\overbrace{F(y)}^{y/\theta})^{n-1}\underbrace{p(y)}_{\frac{1}{\theta}\{(0,\theta)\}}$

$$= -\int_1^0 n t^{n-1}(1-t)\theta\,dt = \int_0^1 n\theta t^{n-1}dt - \int_0^1 n\theta t^n dt =$$

$$= \theta\left[1-\frac{n}{n+1}\right] = \frac{\theta}{n+1}$$ — смещ.ное

$\tilde\theta_2' = (n+1)X_{min}$ — несмещ.

$M[\tilde\theta_2'] = \theta$

$$M[\tilde\theta_2^2] = \int_0^\theta n\left(1-\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta}y^2dy = -\int_1^0 n t^{n-1}(1-t)^2\theta^2 dt =$$

$$= n\theta^2\left[\int_0^1 (t^{n-1}-2t^n+t^{n+1})dt\right] = n\theta^2\left[\frac{1}{n} - 2\frac{1}{n+1}+\frac{1}{n+2}\right] =$$

$$= \cancel{n}\theta^2\left[\frac{(n+1)(n+2)-2n(n+2)+n(n+1)}{\cancel{n}(n+1)(n+2)}\right] = \theta^2\frac{\cancel{n^2}+\cancel{3n}+2-\cancel{2n^2}-\cancel{4n}+\cancel{n^2}+\cancel{n}}{(n+1)(n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

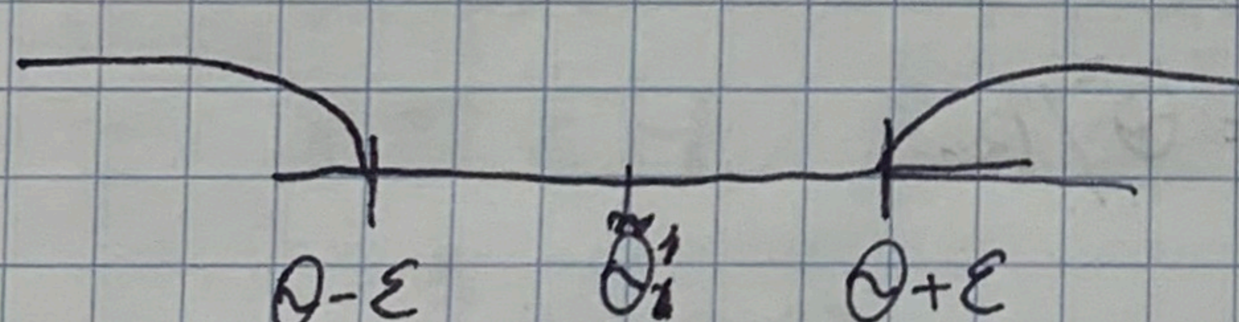
$$D[\tilde\theta_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left[\frac{2(n+1)-(n+2)}{(n+1)^2(n+2)}\right] =$$

$$= \theta^2\left[\frac{n}{(n+1)^2(n+2)}\right] \to 0,\ n\to\infty \quad ?$$

$$D[\tilde\theta_2'] = (n+1)^2 D[\tilde\theta_2] = \frac{\theta^2 n}{n+2} \not\to 0$$

$\tilde\theta_2'$ по определ.

$\forall\theta>0\ \forall\varepsilon>0\quad P(|\tilde\theta_2'-\theta|\geq\varepsilon)\to 0,\ n\to\infty$



$$P(|\tilde\theta_2'-\theta|\geq\varepsilon) \geq P(\tilde\theta_2' > \theta+\varepsilon) = P((n+1)X_{min}\geq\theta+\varepsilon) =$$

$$= P\left(X_{min}\geq\frac{\theta+\varepsilon}{n+1}\right) = 1-\underbrace{P\left(X_{min}<\frac{\theta+\varepsilon}{n+1}\right)}_{\Phi\left(\frac{\theta+\varepsilon}{n+1}\right)} = 1-\left(1-\left(1-F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n\right)$$

$$= \left(1-\left(\frac{\theta+\varepsilon}{\theta(n+1)}\right)\right)^n \to e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

Значит, не сост.

$\tilde\theta_2$ по опред.

$P(|\tilde\theta_2-\theta|\geq\varepsilon)\to 0,\ n\to\infty$

$$\underbrace{P(\tilde\theta_2<\theta-\varepsilon)}_{=0} + P(\underbrace{\tilde\theta_2}_{=X_{min}} > \theta+\varepsilon)$$

$\forall \theta > 0 \quad \forall \varepsilon > 0.$

$$P(x_{min} < \theta - \varepsilon) = \Phi(\theta - \varepsilon) \overset{\varepsilon < \theta}{=} 1 - \left(1 - \frac{\theta - \varepsilon}{\theta}\right)^n =$$

$$= 1 - \left(\frac{\varepsilon}{\theta}\right)^n \longrightarrow 1, \quad n \to \infty.$$

Не сост.

- $\tilde{\theta}_3 = x_{max}$

$$M[\tilde{\theta}_3] = \int_{-\infty}^{\infty} z\, \psi(z)\, dz = \int_0^{\theta} n \frac{z^n}{\theta^n} dz = \cancel{\frac{n}{\theta^n}} \quad \frac{n}{n+1}\theta \quad \text{— смещ.}$$

$$\Psi(z) = (F(z))^n$$

$$\psi = \Psi'(z) = n(F(z))^{n-1} p(z) = n\left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta} \{(0, \theta)\}$$

$$\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \quad \text{— несмещ.}$$

$$M[\tilde{\theta}_3^2] = \int_{-\infty}^{\infty} z^2 \psi(z)\, dz = \int_0^{\theta} n \frac{z^{n+1}}{\theta^n} dz = \frac{n}{n+2}\theta^2$$

$$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2 \left[\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right] =$$

$$= \theta^2 \left[\frac{\cancel{n^3} + \cancel{2n^2} + n - \cancel{n^3} - \cancel{2n^2}}{(n+2)(n+1)^2}\right] = \frac{n}{(n+2)(n+1)^2}\theta^2 \longrightarrow 0$$

$$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \longrightarrow 0. \quad n \to \infty$$

Явл. сост.

$\tilde{\theta}_3$ по опред.

$\forall \theta > 0$, $\forall \varepsilon > 0$: $P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(x_{max} < \theta - \varepsilon) + \overset{0}{P(x_{max} > \theta + \varepsilon)} =$

$= (F(\theta - \varepsilon))^n$

$0 < \varepsilon < \theta: \quad \left(\frac{\theta - \varepsilon}{\theta}\right)^n \longrightarrow 0 \quad n \to \infty$

$\varepsilon \geq \theta \qquad 0^n \longrightarrow 0 \quad n \to \infty$

$\frac{n+1}{n} x_{max}$ док-ть сост. по опр

17.02.25

Занятие 3

Продолжаем T1

$\tilde{\Theta}_4 = x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i$

$M[\tilde{\Theta}_4] = M[x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i] + M[x_1] + \frac{1}{n-1}\sum_{i=2}^{n} M[x_i] =$

$= \frac{\Theta}{2} + \frac{\Theta}{2} = \Theta$ — несмещ. ✓

$D[\tilde{\Theta}_4] = D[x_1 + \frac{1}{n-1}\sum_{i=2}^{n} x_i] = D[\xi] + \frac{1}{(n-1)^2}(n-1)D\xi =$

$= \frac{\Theta^2}{12} + \frac{\Theta^2}{12}(n-1)^{-1} = \frac{\Theta^2 n}{12(n-1)} \not\to 0$ $\quad n \to \infty$

По опред. $\tilde{\Theta}_4 \xrightarrow{P} \Theta$

$\xi_n \xrightarrow{P} \xi$, $\eta_n \xrightarrow{P} \eta$

$\xi_n + \eta_n \xrightarrow{P} \xi + \eta$

$x_1 \xrightarrow{P} \xi$

$\frac{1}{n-1}\sum_{i=2}^{n} x_i \xrightarrow{P} M[\xi] = \frac{\Theta}{2}$ — несмещ.

ЗБЧ Хинчина

$\xi_1 \dots \xi_n$ независ., один. распр.

$\frac{1}{n}\sum_{i=1}^{n} \xi_i \xrightarrow{P} M[\xi]$

$\tilde{\Theta}_1 = 2\bar{x}$

$\tilde{\Theta}_3 = \frac{n+1}{n} x_{max}$

$D[\tilde{\Theta}_1] = \frac{\Theta^2}{3n}$

$D[\tilde{\Theta}_3] = \frac{\Theta^2}{n(n+2)}$

$\forall \Theta > 0 \quad \frac{\Theta^2}{n(n+2)} < \frac{\Theta^2}{3n}$ $\qquad 3n < n^2 + 2n$

$n^2 > n$

$n > 1$

№ 1.

$$\tilde{\Theta}_3' = \frac{n+1}{n} X_{max}$$

$$P(|\tilde{\Theta}_3' - \Theta| \geq \varepsilon) = P(\tilde{\Theta}_3' < \Theta - \varepsilon) + \overset{0}{\overset{\parallel}{P(\tilde{\Theta}_3' > \Theta + \varepsilon)}}$$

$\forall \varepsilon > 0 \quad \forall \Theta > 0$

$$P\left(\frac{n+1}{n} X_{max} \leq \Theta - \varepsilon\right) = P\left(X_{max} \leq \frac{(\Theta - \varepsilon)n}{n+1}\right) =$$

$$= \left(F\left(\frac{(\Theta-\varepsilon)n}{n+1}\right)\right)^n = \left(\frac{(\Theta-\varepsilon)n}{\Theta(n+1)}\right)^n$$

Переходим к пределу

$0 < \varepsilon < \Theta$: $\quad \left(\frac{\Theta - \varepsilon}{\Theta}\right)^n \longrightarrow 0, \quad n \to \infty$

$\varepsilon \geq \Theta$: $\quad 0^n \longrightarrow 0, \quad n \to \infty$

$\tilde{\Theta}_3'$ сост. по определению