Some mathematical tools

These notes will review some of the basic results of first-year analysis. We will also introduce the notion of complex numbers.

Complex numbers

The principal number systems in mathematics are the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. The complex numbers are the set of ordered pairs of real numbers with addition and multiplication defined as below. Given real numbers x and y, the corresponding complex number z is written $z = x + i\,y$. If $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$ are two complex numbers, then we define

1. $z_1 + z_2 \stackrel{\text{def}}{=} (x_1 + x_2) + i\,(y_1 + y_2)$
2. $z_1 \cdot z_2 \stackrel{\text{def}}{=} (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$.

There are several things to observe. The number $i = 0 + i\,(1)$ has the property that

$$i^2 = -1 = -1 + i\,(0)$$

by the definition of multiplication. If the complex number $z = x$ $(y = 0)$, then z is said to be **real**; If the complex number $z = i\,y$ $(x = 0)$, then z is said to be **imaginary**. If $z = x + i\,y$, then $x = \Re(z)$ is called the **real part** of z and $y = \Im(z)$ is called the **imaginary part** of z. (Note that the imaginary part is a real number, the coefficient of y.) Also the complicated formula in 2 for multiplication follows from the ordinary distributive law familiar from algebra:

$$\begin{aligned} z_1z_2 &= z_1(x_2 + i\,y_2) = z_1x_2 + z_1 i\,y_2 \\ &= (x_1 + i\,y_1)x_2 + (x_1 + i\,y_1)\,i\,y_2 = (x_1x_2 + i\,y_1x_2) + (x_1\,i\,y_2 + i^2\,y_1y_2) \\ &= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)\,. \end{aligned}$$

Division may be done by the usual rationalization of radicals procedure learned in school (treating i as the radical $\sqrt{-1}$):

$$\begin{aligned} z_1/z_2 &= \frac{x_1 + i\,y_1}{x_2 + i\,y_2} \equiv \frac{x_1 + i\,y_1}{x_2 + i\,y_2} \cdot \frac{x_2 - i\,y_2}{x_2 - i\,y_2} \\ &= \frac{(x_1x_2 + y_1y_2) + i\,(-x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}\,. \end{aligned}$$

Given $z = x + i\,y$, the number $\bar{z} = x - i\,y$ is called the **conjugate** of z. The **absolute value** of z is defined to be $|z| = \sqrt{x^2 + y^2}$. An important relationship is that

$$|z| = \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}\,.$$

Bounds

Theorem

Let the set $S \subset \mathbb{R}$ be bounded above, that is, there is a real number M such that for all $x \in S$, $x \leq M$. Then there is a least upper bound, called the ***supremum*** *and denoted* $\operatorname{Sup} S$.

Thus by definition M is any upper bound, $\operatorname{Sup} S \leq M$.

Corollary

Likewise, if $S \in \mathbb{R}$ be bounded below, then there is a greatest lower bound, called the ***infimum*** *and denoted* $\operatorname{Inf} S$.

Remark

If there is a number $x \in S$ such that $x = \operatorname{Sup} S$, then x is called the **maximum** of S and may be denoted $\operatorname{Max} S$. Likewise, if there is a number $x \in S$ such that $x = \operatorname{Inf} S$, then x is called the **minimum** of S and may be denoted $\operatorname{Min} S$.

Examples

Note how a function and domain may be used to indicate implicitly the set S.

1. $\operatorname{Sup}[0,1) = 1$. $\quad\operatorname{Inf}[0,1) = 0$.
2. $\displaystyle\operatorname*{Sup}_{0<x<\infty} \frac{1}{x}$ does not exist (or one could say infinite). $\quad\displaystyle\operatorname*{Inf}_{0<x<\infty} \frac{1}{x} = 0$.
3. $\displaystyle\operatorname*{Sup}_{-1<x<2} \sin x = 1$. $\quad\displaystyle\operatorname*{Inf}_{-1<x<2} \sin x = \sin(-1) = -\sin 1$.

Theorem. (Extreme Value Theorem)

Let f be a continuous on $[a,b]$. Then there exist numbers c_1, c_2 in $[a,b]$ such that

$$f(c_1) \le f(x) \le f(c_2) \quad \text{for all } x \text{ in } [a,b].$$

Bounds and equations

The following theorems relate to values of functions and their roots. Since they can be used to determine bounds on the values being sought in a numerical problem, they are the principal tools we use to determine how well our methods approximate the solution. If, say, $\xi \in [a,b]$ and $|f(x)| \le M$ or $\left|f^{(k)}(x)\right| \le M$ for all x in $[a,b]$, then $f(\xi)$ or $f^{(k)}(\xi)$ found in the theorems are also bounded in absolute value by M.

Theorem. (Intermediate Value Theorem)

Let f be a continuous on $[a,b]$, and let

$$m = \operatorname*{Inf}_{a \le x \le b} f(x)\,, \qquad M = \operatorname*{Sup}_{a \le x \le b} f(x)\,.$$

Then for any real number y in the interval $[m,M]$, there is at least one solution $x = \xi$ in $[a,b]$ to

$$f(x) = y\,.$$

In particular there are points $x_{\min}$ and $x_{\max}$ in $[a,b]$ such that $f(x_{\min}) = m$ and $f(x_{\max}) = M$.

Theorem. (Mean Value Theorem)

Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there is a solution $x = \xi$ in the interval (a,b) to the equation

$$f(b) - f(a) = f'(x)\,(b-a)\,.$$

Theorem. (Cauchy Mean Value Theorem)

Let f, g be continuous on $[a, b]$ and differentiable on (a, b), and assume $g(a) \neq g(b)$. Then there is a solution $x = \xi$ in the interval (a, b) to the equation

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)} .$$

It follows from the Mean Value Theorem applied to $h(x) = (f(x) - f(a)(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$.

Theorem

Let f be bounded and integrable over $[a, b]$, and let

$$m = \operatorname*{Inf}_{a \le x \le b} f(x) , \qquad M = \operatorname*{Sup}_{a \le x \le b} f(x) .$$

Then

$$m\,(b - a) \le \int_a^b f(x)\, dx \ \le M\,(b - a) .$$

Theorem. (Mean Value Theorems for integrals)

Let f be continuous on $[a, b]$. Then we have the following:

1. *There is a solution $x = \xi$ in the interval (a, b) to the equation*

$$\int_a^b f(t)\, dt = f(x)\,(b - a) .$$

2. *If w is a nonnegative, integrable function, there is a solution $x = \xi$ in the interval (a, b) to the equation*

$$\int_a^b f(t)\, w(t)\, dt = f(x) \int_a^b w(t)\, dt .$$

Theorem. (Taylor)

Recall that a function f is of **class $\boldsymbol{C^r[a, b]}$** if f has r continuous derivatives on $[a, b]$. The class $C[a, b]$ is the class of all continuous functions on $[a, b]$.

Let f be of class $C^{n+1}[a, b]$ for some $n \ge 0$, and let $x, c \in [a, b]$. Then

$$\begin{aligned} f(x) &= p_n(x) + R_{n+1}(x) \\ p_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}\,(x - c)^k = f(c) + f'(c)\,(x - c) + \frac{f''(c)}{2}\,(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}\,(x - c)^n \\ R_{n+1}(x) &= \frac{1}{n!} \int_c^x (x - t)^n f^{(n+1)}(t)\, dt = \frac{f^{(n+1)}(\xi)}{(n + 1)!}\,(x - c)^{n+1} \end{aligned}$$

for some ξ between c and x.

The polynomial p_n is called the n^{th} order or n^{th} degree **Taylor approximant** to f at c. The term R_{n+1} is called the **remainder term**.

Norms

Norms measure size and in particular are used to measure the size of errors. In turns out that the best way to measure error depends on the application, and so we have different ways to defined a norm. Here are some common ways. The first example is the standard Euclidean norm that measures distance to the origin (or the length of a vector).

- $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, $x \in \mathbb{R}^n$; or $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$, $x \in \mathbb{C}^n$.
- $\|x\|_\infty = \underset{1 \le i \le n}{\operatorname{Max}} |x_i|$, $x \in \mathbb{R}^n$ or $\mathbb{C}^n$.
- $\|f\|_2 = \left(\int_a^b |f(x)|^2 \, dx\right)^{1/2}$, $f \in C[a, b]$.
- $\|f\|_\infty = \underset{a \le x \le b}{\operatorname{Max}} f(x)$, $f \in C[a, b]$.
- Norms can be given for matrices, too, but these will be introduced later.

The function norms are just like the corresponding norms for vectors, in that vectors can be viewed as discrete functions $x\colon \{1, \ldots, n\} \to \mathbb{R}^n$ given by the mapping $x\colon i \mapsto x_i$, $i = 1, \ldots, n$.

All norms are characterized by the three properties:

1. (Positive definite.) $\|x\| \ge 0$ for all vectors x, with equality if and only if $x = 0$, the zero vector.
2. (Homogeneity.) $\|\alpha x\| = |\alpha| \, \|x\|$ for all vectors x and all real (complex) numbers α.
3. (Triangle inequality.) $\|x + y\| \le \|x\| + \|y\|$ for all vectors x, y.

Big O

Another method of measuring the size of functions is by their order of growth or vanishing. This is indicated by the **big-O** notation defined below.

We say

$$f(x) = O(g(x)) \quad \text{as} \quad x \to \infty$$

if there is a constant C and a number x_0 such that $|f(x)| \le C \, |g(x)|$ for all $x > x_0$.

For a finite number c, we say

$$f(x) = O(g(x)) \quad \text{as} \quad x \to c$$

if there is a constant C and a number $\delta > 0$ such that $|f(x)| \le C \, |g(x)|$ for all x with $|x - c| < \delta$.

Exercises

1. Show $\dfrac{(1 + 2i)(2 - i)}{3 + i} = \dfrac{3}{2} + \dfrac{i}{2}$.

2. Let $z = 3 - 6i$, $w = i - 2$. Show $|zw| = |z||w|$ and $|z \quad w| \le |z| + |w|$.

3. Assume f is continuous on the interval $[a, b]$, and consider the sum

$$S = \sum_{i=1}^{n} f(x_i)$$

with all $x_i \in [a, b]$. Use the Intermediate Value Theorem to show there is a number $\xi \in [a, b]$ such that $S = n \, f(\xi)$.

4. Let $x_0 < x_1 < x_2$ be equally spaced ($x_2 - x_1 = x_1 - x_0 = h$) and let $f(x)$ be a function of class C^2 over an interval containing x_0, x_1, x_2. Define the first two **divided differences** by

$$f[a,b] = \frac{f(b) - f(a)}{b - a}, \qquad f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

Show that there is a number ξ in $[x_0, x_2]$ such that $f[x_0, x_1, x_2] = \frac{1}{2} f''(\xi)$.

5. Construct a Taylor series for the following functions and bound the error when truncating after the degree-n term (series centered at zero are sufficient):

(a) $\displaystyle \frac{1}{x}\int_0^x e^{-t^2}\,dt$ **(b)** $\sin^{-1}(x)$, $|x| < 1$ **(c)** $\displaystyle \frac{1}{x}\int_0^x \frac{\tan^{-1}}{t}\,dt$ **(d)** $\cos x + \sin x$

6. Let f be of class C^{n+1} in an open interval containing a real number c. Let p_n be the n^{th} order Taylor approximant to f at c. Show $f(x) = p_n(x) + O((x-c)^{n+1})$ as $x \to c$.

7. Show:

(a) $\log x = (x-1) - \frac{1}{2}(x-1)^2 + O((x-1)^3)$ as $x \to 1$.

(b) $x^n = O(e^x)$ as $x \to +\infty$, for $n > 0$.

8. Show that $|f(x)| \leq |g(x)|$ for all x implies $f(x) = O(g(x))$. Is the converse true?

9. Let $f(x) = x^2 + x$ and $g(x) = 2x^2 - x$. Show that $f(x) = O(g(x))$ and $g(x) = O(f(x))$ as $x \to \infty$.