Math 315 Lab 7

The following lab compares the convergence rates of Gauss and Clenshaw-Curtis quadrature.

Contents

Gauss Quadrature Code

The following code approximates the integral of f on the interval -1 to 1 using Gauss Quadrature.

```
disp(fileread("gauss.m"));
```

```
function I = gauss(f, n)
    beta = .5./sqrt(1-(2*(1:n)).^(-2));
    T = diag(beta,1) + diag(beta,-1);
    [V,D] = eig(T);
    x = diag(D); [x,i] = sort(x);
    w = 2*V(1,i).^2;
    I = w*feval(f,x);
```

Clenshaw-Curtis Quadrature Code

The following code approximates the integral of f on the interval -1 to 1 using Clenshaw-Curtis Quadrature.

```
disp(fileread("clenshawcurtis.m"));
```

```
function I = clenshaw_curtis(f,n)
    x = cos(pi*(0:n)'/n);
    fx = feval(f,x)/(2*n);
    g = real(fft(fx([1:n+1 n:-1:2])));
    a = [g(1); g(2:n)+g(2*n:-1:n+2); g(n+1)];
    w = 0*a'; w(1:2:end) = 2./(1-(0:2:n).^2);
    I = w*a;
```

x^20

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of x^20 on the interval -1 to 1.

```
f = @(x) x.^20;
true_I = 2/21;
N = 30;
gauss_I_1 = zeros(1, N);
clenshawcurtis_I_1 = zeros(1, N);

for n = 1:N
    gauss_I_1(n) = gauss(f, n);
    clenshawcurtis_I_1(n) = clenshawcurtis(f, n);
end

gauss_err_1 = abs(gauss_I_1 - true_I);
clenshawcurtis_err_1 = abs(clenshawcurtis_I_1 - true_I);
```

e^x

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of e^x on the interval -1 to 1.

```
f = @(x) exp(x);
true_I = exp(1) - 1/exp(1);
N = 30;
gauss_I_2 = zeros(1, N);
clenshawcurtis_I_2 = zeros(1, N);

for n = 1:N
    gauss_I_2(n) = gauss(f, n);
    clenshawcurtis_I_2(n) = clenshawcurtis(f, n);
end

gauss_err_2 = abs(gauss_I_2 - true_I);
clenshawcurtis_err_2 = abs(clenshawcurtis_I_2 - true_I);
```

e^(-x^2)

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of e^(-x^2) on the interval -1 to 1.

```
f = @(x) exp(-x.^2);
true_I = integral(f, -1, 1);
N = 30;
gauss_I_3 = zeros(1, N);
clenshawcurtis_I_3 = zeros(1, N);

for n = 1:N
    gauss_I_3(n) = gauss(f, n);
    clenshawcurtis_I_3(n) = clenshawcurtis(f, n);
end

gauss_err_3 = abs(gauss_I_3 - true_I);
clenshawcurtis_err_3 = abs(clenshawcurtis_I_3 - true_I);
```

1/(1+16x^2)

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of 1/(1+16x^2) on the interval -1 to 1.

```
f = @(x) 1./(1+16*x.^2);
true_I = 0.5*atan(4);
N = 30;
gauss_I_4 = zeros(1, N);
clenshawcurtis_I_4 = zeros(1, N);

for n = 1:N
    gauss_I_4(n) = gauss(f, n);
    clenshawcurtis_I_4(n) = clenshawcurtis(f, n);
end

gauss_err_4 = abs(gauss_I_4 - true_I);
clenshawcurtis_err_4 = abs(clenshawcurtis_I_4 - true_I);
```

e^(-x^-2)

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of e^(-x^-2) on the interval -1 to 1.

```
f = @(x) exp(-x.^-2);
true_I = integral(f, -1, 1);
```

```
N = 30;
gauss_I_5 = zeros(1, N);
clenshawcurtis_I_5 = zeros(1, N);

for n = 1:N
    gauss_I_5(n) = gauss(f, n);
    clenshawcurtis_I_5(n) = clenshawcurtis(f, n);
end

gauss_err_5 = abs(gauss_I_5 - true_I);
clenshawcurtis_err_5 = abs(clenshawcurtis_I_5 - true_I);
```

| x |^3

Using both Gauss and Clenshaw-Curtis Quadrature, I will approximate the integral of x^3 on the interval -1 to 1.

```
f = @(x) abs(x).^3;
true_I = 0.5;
N = 30;
gauss_I_6 = zeros(1, N);
clenshawcurtis_I_6 = zeros(1, N);

for n = 1:N
    gauss_I_6(n) = gauss(f, n);
    clenshawcurtis_I_6(n) = clenshawcurtis(f, n);
end

gauss_err_6 = abs(gauss_I_6 - true_I);
clenshawcurtis_err_6 = abs(clenshawcurtis_I_6 - true_I);
```

Convergence of Errors

The convergence of errors for the integral approximations from Gauss and Clenshaw-Curtis Quadrature are plotted below for increasing n. The rate of convergence for the first 5 integrals are of order O(e^-an) and the last integral is of order O(x^-b). The exact coefficients a and b for the rate of convergence are shown below. This is because the first 5 functions are infinitely differentiable, whereas the last function is not infinitely differentiable.

```
close all;
figure1 = figure('Position', [100 100 800 1200]);
x = linspace(1, 30, 30);
gauss_x_fit = x(1:9);
gauss_fit = gauss_err_1(1:9);
```

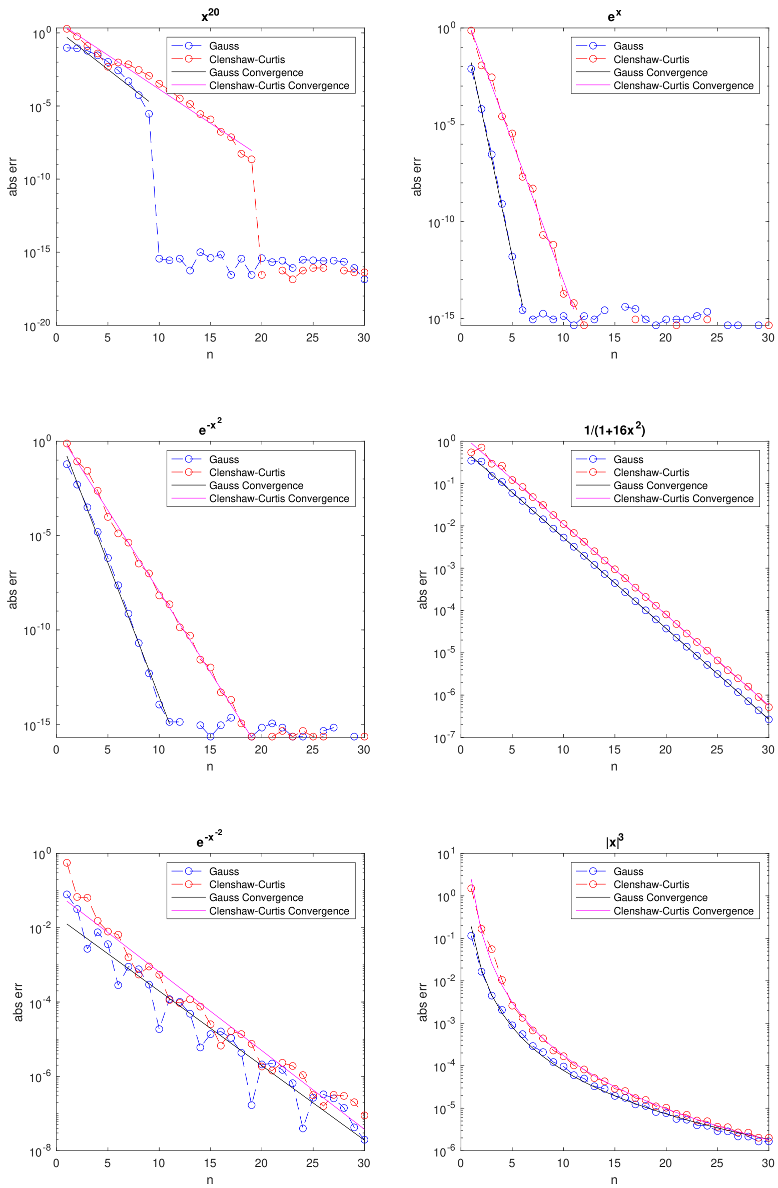
```
gauss_p = polyfit(gauss_x_fit, log(gauss_fit), 1);
gauss_y_fit = exp(gauss_p(2)) * exp(gauss_x_fit .* gauss_p(1));
fprintf('Gauss Convergence for x^20: y = %g * e^%gx\n', exp(gauss_p(2)), ...
    gauss_p(1));
clenshawcurtis_x_fit = x(1:19);
clenshawcurtis_fit = clenshawcurtis_err_1(1:19);
clenshawcurtis_p = polyfit(clenshawcurtis_x_fit, log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * exp(clenshawcurtis_x_fit ...
    .* clenshawcurtis_p(1));
fprintf('Clenshaw-Curtis Convergence for x^20: y = %g * e^%gx\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,1);
semilogy(x, gauss_err_1, 'b--o', x, clenshawcurtis_err_1, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, ...
    clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('x^{20}')

gauss_x_fit = x(1:6);
gauss_fit = gauss_err_2(1:6);
gauss_p = polyfit(gauss_x_fit, log(gauss_fit), 1);
gauss_y_fit = exp(gauss_p(2)) * exp(gauss_x_fit .* gauss_p(1));
fprintf('Gauss Convergence for e^x: y = %g * e^%gx\n', exp(gauss_p(2)), ...
    gauss_p(1));
clenshawcurtis_x_fit = x(1:11);
clenshawcurtis_fit = clenshawcurtis_err_2(1:11);
clenshawcurtis_p = polyfit(clenshawcurtis_x_fit, log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * exp(clenshawcurtis_x_fit ...
    .* clenshawcurtis_p(1));
fprintf('Clenshaw-Curtis Convergence for e^x: y = %g * e^%gx\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,2);
semilogy(x, gauss_err_2, 'b--o', x, clenshawcurtis_err_2, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('e^x')

gauss_x_fit = x(1:11);
gauss_fit = gauss_err_3(1:11);
gauss_p = polyfit(gauss_x_fit, log(gauss_fit), 1);
```

```
gauss_y_fit = exp(gauss_p(2)) * exp(gauss_x_fit .* gauss_p(1));
fprintf('Gauss Convergence for e^-x^2: y = %g * e^%gx\n', exp(gauss_p(2)), ...
    gauss_p(1));
clenshawcurtis_x_fit = x(1:19);
clenshawcurtis_fit = clenshawcurtis_err_3(1:19);
clenshawcurtis_p = polyfit(clenshawcurtis_x_fit, log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * exp(clenshawcurtis_x_fit ...
    .* clenshawcurtis_p(1));
fprintf('Clenshaw-Curtis Convergence for e^-x^2: y = %g * e^%gx\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,3);
semilogy(x, gauss_err_3, 'b--o', x, clenshawcurtis_err_3, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('e^{-x^2}')

gauss_x_fit = x(1:30);
gauss_fit = gauss_err_4(1:30);
gauss_p = polyfit(gauss_x_fit, log(gauss_fit), 1);
gauss_y_fit = exp(gauss_p(2)) * exp(gauss_x_fit .* gauss_p(1));
fprintf('Gauss Convergence for 1/(1+16x^2): y = %g * e^%gx\n', ...
    exp(gauss_p(2)), gauss_p(1));
clenshawcurtis_x_fit = x(1:30);
clenshawcurtis_fit = clenshawcurtis_err_4(1:30);
clenshawcurtis_p = polyfit(clenshawcurtis_x_fit, log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * exp(clenshawcurtis_x_fit ...
    .* clenshawcurtis_p(1));
fprintf('Clenshaw-Curtis Convergence for 1/(1+16x^2): y = %g * e^%gx\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,4);
semilogy(x, gauss_err_4, 'b--o', x, clenshawcurtis_err_4, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('1/(1+16x^2)')

gauss_x_fit = x(1:30);
gauss_fit = gauss_err_5(1:30);
gauss_p = polyfit(gauss_x_fit, log(gauss_fit), 1);
gauss_y_fit = exp(gauss_p(2)) * exp(gauss_x_fit .* gauss_p(1));
fprintf('Gauss Convergence for e^-x^-2: y = %g * e^%gx\n', exp(gauss_p(2)), ...
```

```
    gauss_p(1));
clenshawcurtis_x_fit = x(1:30);
clenshawcurtis_fit = clenshawcurtis_err_5(1:30);
clenshawcurtis_p = polyfit(clenshawcurtis_x_fit, log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * exp(clenshawcurtis_x_fit ...
    .* clenshawcurtis_p(1));
fprintf('Clenshaw-Curtis Convergence for e^-x^-2: y = %g * e^%gx\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,5);
semilogy(x, gauss_err_5, 'b--o', x, clenshawcurtis_err_5, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('e^{-x^{-2}}')

gauss_x_fit = x(1:30);
gauss_fit = gauss_err_6(1:30);
gauss_p = polyfit(log(gauss_x_fit), log(gauss_fit), 1);
gauss_y_fit = exp(gauss_p(2)) * gauss_x_fit .^ gauss_p(1);
fprintf('Gauss Convergence for |x|^3: y = %g * x^%g\n', exp(gauss_p(2)), ...
    gauss_p(1));
clenshawcurtis_x_fit = x(1:30);
clenshawcurtis_fit = clenshawcurtis_err_6(1:30);
clenshawcurtis_p = polyfit(log(clenshawcurtis_x_fit), log(clenshawcurtis_fit), 1);
clenshawcurtis_y_fit = exp(clenshawcurtis_p(2)) * clenshawcurtis_x_fit .^ ...
    clenshawcurtis_p(1);
fprintf('Clenshaw-Curtis Convergence for |x|^3: y = %g * x^%g\n\n', ...
    exp(clenshawcurtis_p(2)), clenshawcurtis_p(1));
subplot(3,2,6);
semilogy(x, gauss_err_6, 'b--o', x, clenshawcurtis_err_6, 'r--o', ...
    gauss_x_fit, gauss_y_fit, 'k', clenshawcurtis_x_fit, clenshawcurtis_y_fit, 'm');
xlabel('n');
ylabel('abs err');
legend('Gauss', 'Clenshaw-Curtis', 'Gauss Convergence', ['Clenshaw-Curtis ' ...
    'Convergence'])
title('|x|^3')
```

```
Gauss Convergence for x^20: y = 1.74504 * e^-1.2615x
Clenshaw-Curtis Convergence for x^20: y = 6.37064 * e^-1.07263x

Gauss Convergence for e^x: y = 5.29503 * e^-5.77107x
Clenshaw-Curtis Convergence for e^x: y = 21.7437 * e^-3.31919x

Gauss Convergence for e^-x^2: y = 4.33867 * e^-3.26612x
```

```
Clenshaw-Curtis Convergence for e^-x^2: y = 4.26161 * e^-1.9786x

Gauss Convergence for 1/(1+16x^2): y = 0.726662 * e^-0.493698x
Clenshaw-Curtis Convergence for 1/(1+16x^2): y = 1.4876 * e^-0.492467x

Gauss Convergence for e^-x^-2: y = 0.0200237 * e^-0.461993x
Clenshaw-Curtis Convergence for e^-x^-2: y = 0.0845879 * e^-0.487507x

Gauss Convergence for |x|^3: y = 0.191174 * x^-3.3939
Clenshaw-Curtis Convergence for |x|^3: y = 2.47692 * x^-4.15563
```



Degree of Precision

The degree of precision for Gauss Quadrature is 2N-1 and the degree of precision for Clenshaw-Curtis Quadrature is N, where N is the number of points. In the code, the n represents the degree of the underlying interpolant. Therefore the number of points N=n+1.

The degree of precision for Gauss and Clenshaw-Curtis are shown in the first error plot for the integral of xˆ20. For Gauss, between n=9 and n=10, the error jumps down to around epsilon. This is because at n=9, N=10 and the degree of precision is 2N-1 = 19. At n=10, N=11 and the degree of precision is 2N-1 = 21. Since a degree 21 polynomial can perfectly interpolate the integral of a degree 20 function, the error jumps to epsilon. For Clenshaw-Curtis, this happens at n=19 and n=20, where the degree of precision is 20 when n=19 and 21 when n=20. Similarly, the degree 21 interpolation can perfectly approximate the integral whereas the degree 20 interpolation can't.

In the second and third graph, the Gauss Quadrature converges almost twice as fast as the Clenshaw-Curtis Quadrature. This also reflects the difference in the degree of precision for Gauss and Clenshaw-Curtis as the DOP for Gauss is almost twice the DOP for Clenshaw-Curtis given the same number of points.

From the first three graphs, we can see that the Gauss Quadrature converges twice as fast as the Clenshaw-Curtis Quadrature if the function being integrated is analytic. In the last three graphs, the function being integrated is non-analytic because the 4th and 5th functions have a pole in the Bernstein ellipse and the 6th function is not infinitely differentiable. Here, Gauss and Clenshaw-Curtis Quadratures converge at a similar rate. The convergence for these three integrals are also all very slow, none of which converged prior to n=30. This is because these functions can't be represented by a power series and thus, a polynomial approximation will converge slower for functions that can't be represented by a power series than for functions that can.