Notes for "What is Numerical Analysis?"

What "analysis" signifies

Algebra and its extension calculus study relationships between numbers. You may be familiar only with algebra and calculus over the field of real numbers $\mathbb{R}$, but algebra over the complex numbers $\mathbb{C}$ is important and (suprisingly!) needed for a complete understanding of the relationships between real numbers. The fundamental object in modern mathematics used to model these relationships is the function. In first-year calculus one studies functions of the form $f\colon U \subset \mathbb{R} \to \mathbb{R}$. function evaluation and equations as well as operations on functions themselves that produce new functions (differentiation and integration) or associate numbers to functions (integration).

What "numerical" signifies

The numbers used in workaday calculations may be divided into two classes exact and approximate. Integers and fractions of integers such as 2 and 11/10 are said to be known **exactly**. In a calculator and often in computer, the machine stores numbers in an approximate form. For instance, the number 1.1 in the floating-point standard IEEE 754-2008, stored in the 64-bit double-precision form, is equivalent to the exact number

$$\frac{2476979795053773}{2251799813685248}$$

which differs from 11/10 by -8.88178×10^{-17}. In human work the notion of significant figures is used to represent approximate numbers. In this convention 1.1 represents an approximate number which is known to two digits and differs from the true number it represents by at most 0.05.

The qualifier "numerical" in its broadest sense means analyzing how functions work on numbers and breaking it down into basic arithmetical operations on numbers. Many functions are defined in terms of an infinite process (for example, limits, series, or integration), and one numerical approach is to truncate this process after a finite number of steps and study the result as an approximation of the exact value. In practice, using a computer often means using approximate numbers in the steps. Although this is not strictly a requirement, the common meaning of numerical tends to imply computing with approximate numbers.

What "analytical" signifies

In the context of scientific computation, the long-standing convention is that "analytical" or "analytic" means exact or symbolic computation. The term distinguishes it from numerical or approximate computation. It is perhaps best understood to refer to computations and formulas appropriate to (exact) real or complex *analysis). It is unfortunate in that approximate computation is derived by numerical* analysis). Care should be taken to distinguish the term from *analytic function*, which is a function represented by a power series in an open interval. It is an important class of common functions that have important properties (for example, the rate of convergence of its Chebyshev series on a closed interval).

Numerics and Liberal Education

On the face of it, the study of the computations that form the backbone of the solutions to daily, specific problems in engineering applications, statistical analyses, and checking models against experimental measurements in the mathematical sciences might seem too close to practical, professional training to serve the aims of liberal education, which might be glossed as developing the arts of using one's *mind*. Investigation of the problems of computation will raise questions of what it means to know the solution to a problem.

Lloyd Trefethen [`<Trefethen:2005>`] argues that almost nothing scientific is known *exactly*. Being able to compute an answer quickly and accurately represents some sort of effective knowledge. He somewhat arbitrarily chooses five seconds and ten digits as the criteria. The ten-digit criterion is chosen by comparison with other fields. From a practical engineering point of view, one rarely needs more than a few digits, and from a scientific point of view, practically no physical constants or measurements are known to ten digits. Being able to get to ten digits or more is a greater degree of knowledge. So, for instance, if you can write a program that quickly generates ten digits of $\sin x$, for any x (representable on a computer), that ability represents a significant amount of knowledge about the sine function. That we can in fact compute the value to many more digits reflects something about how knowable the sine function is.

Consider two numbers you might think you know, $\sqrt{2}$ and π. Well, you probably certainly know things about them, but there are many limitations on your knowledge, limitations that are significant in the light of computing. We had to invent symbols, such as the radical sign or simply a letter, to represent the numbers. You cannot write all their digits in either base 10 or base 2 (they're irrational). Depending on what you retain from calculus and algebra, you might struggle to compute the first few digits (*compute*, not recall from memory). A **computable number** is defined to be a number whose digits can be computed to any preassigned length in a finite number of steps using the four basic arithmetic operations. It turns out that not all numbers are computable; most of them are not, in some sense. "Computable" then serves to distinguish numbers that are better knowable; some might say that uncomputable numbers are unknowable. We will see that $\sqrt{2}$ and π are in fact computable.

The view from Numerical Analysis

The viewpoint in numerical analysis is that numerics is the true foundation of knowledge about numbers and functions on numbers. This parochial self-importance is common to many other disciplines as well. Within their own limited boundaries, each is probably correct.

What is Numerical Analysis

Numerical analysis is the study of algorithms for solving the problems of mathematical analysis.

References