#1

$$\|Ax_Z - b\| = \|A(Z+\delta)b - b\|$$
$$= \|(AZ-I)b + A\delta b\|$$
$$\le \|AZ-I\|\|b\| + \|A\|\|\delta\|\|b\| = O(\text{cond}(A)\cdot\varepsilon_{DP})\|b\| + \|A\|\|\delta\|\|b\|$$
$$= O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|) + \|A\|\|Z\|O(\varepsilon_{DP})\|b\|$$
$$= O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|) + \|A\|\|A^{-1}\|\varepsilon_{DP}\,O(\varepsilon_{DP})\|b\|$$
$$= O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|) + O(\|A\|\|A^{-1}\|\cdot\varepsilon_{DP})\|b\|$$
$$= O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|) + O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|)$$
$$= O(\text{cond}(A)\cdot\varepsilon_{DP}\cdot\|b\|)$$

#2

$$\frac{\|Ax_Z-b\|}{\|A\|\|x_Z\|+\|b\|} \le \frac{\|Ax_Z-b\|}{\|A\|\|x_Z\|} = \frac{O(\|A\|\|A^{-1}\|\|b\|\varepsilon_{DP})}{\|A\|\|x_Z\|} = O\left(\frac{\|A^{-1}\|\|b\|\varepsilon_{DP}}{\|x_Z\|}\right)$$

#3

Discussion in lab.

#4

Discussion in lab.

#5

$$\text{rel err} = O\left(\frac{\|A^{-1}\|\|b\|}{\|x_Z\|}\varepsilon_{DP}\right) = O\left(\frac{\|A^{-1}\|\|u_k\|}{\|x_Z\|}\varepsilon_{DP}\right)$$

$$\|A^{-1}\| = \frac{1}{\sigma_r} \text{ or } \frac{1}{\text{min singular value}}$$

$$x_Z = \sum_{j=1}^{n}\frac{u_j^T b}{\sigma_j}v_j = \sum_{j=1}^{n}\frac{u_j^T u_k}{\sigma_j}v_j = \frac{u_k^T u_k}{\sigma_k}v_k$$

since u_j and u_k are orthogonal for $j \neq k$ as U is an orthogonal matrix and $u_j^T u_k = 0$ for $j \neq k$

$$\|x_Z\| = \left|\frac{u_k^T u_k}{\sigma_k}\right|\|v_k\| = \left|\frac{u_k^T u_k}{\sigma_k}\right| = \frac{\|u_k\|}{\sigma_k}$$

since u_j and v_j are both normalized to length 1

$$\text{rel err} = O\left(\frac{\frac{1}{\sigma_r}\|u_k\|}{\frac{\|u_k\|}{\sigma_k}}\varepsilon_{DP}\right) = O\left(\frac{\sigma_k}{\sigma_r}\varepsilon_{DP}\right) = O\left(\frac{\text{corresponding singular value}}{\text{min singular value}}\cdot\varepsilon_{DP}\right)$$

This bound for relative backward error of inverse Z increases as the corresponding singular value increases.