IDC402
Assignment - 1

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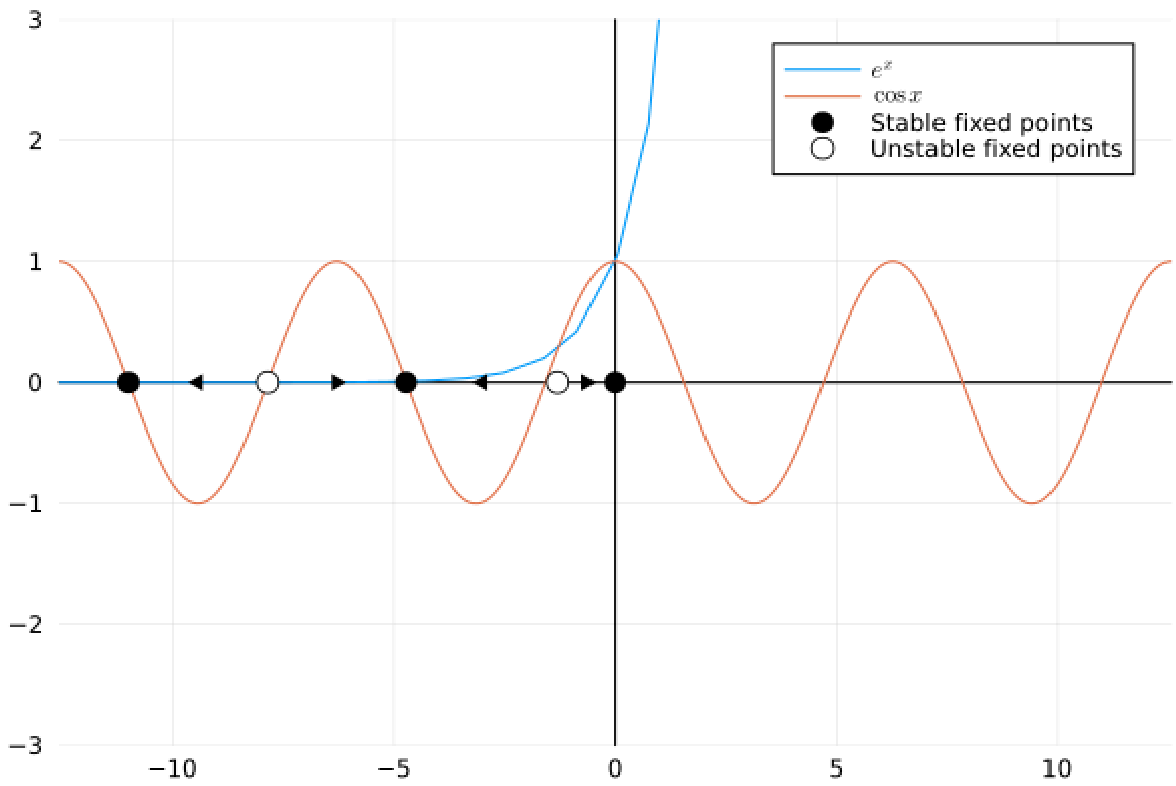
Q1. Find an approximate fixed points and discuss their stability graphically for the following dynamical system

$$\dot{x} = e^{x} - \cos x \tag{1}$$

where $\dot{x} = dx/dt$.

The exact solutions are hard to find analytically, so we plot e^{x} and $\cos x$ to see where they intersect, to graphically get the roots x^* of the equation $f(x^*) = e^{x^*} - \cos x^* = 0$. The regions having $e^{x} > \cos x$ have a flow to the right whereas the ones with $e^{x} < \cos x$ have a flow to the left.



Q2. Identify the dynamical system that has following fixed points with respective stabilities on the real axis: Clearly, there are multiple answers. You need to identify one possible system.

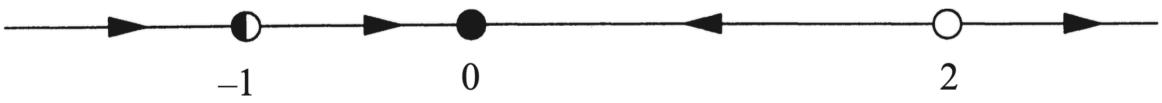
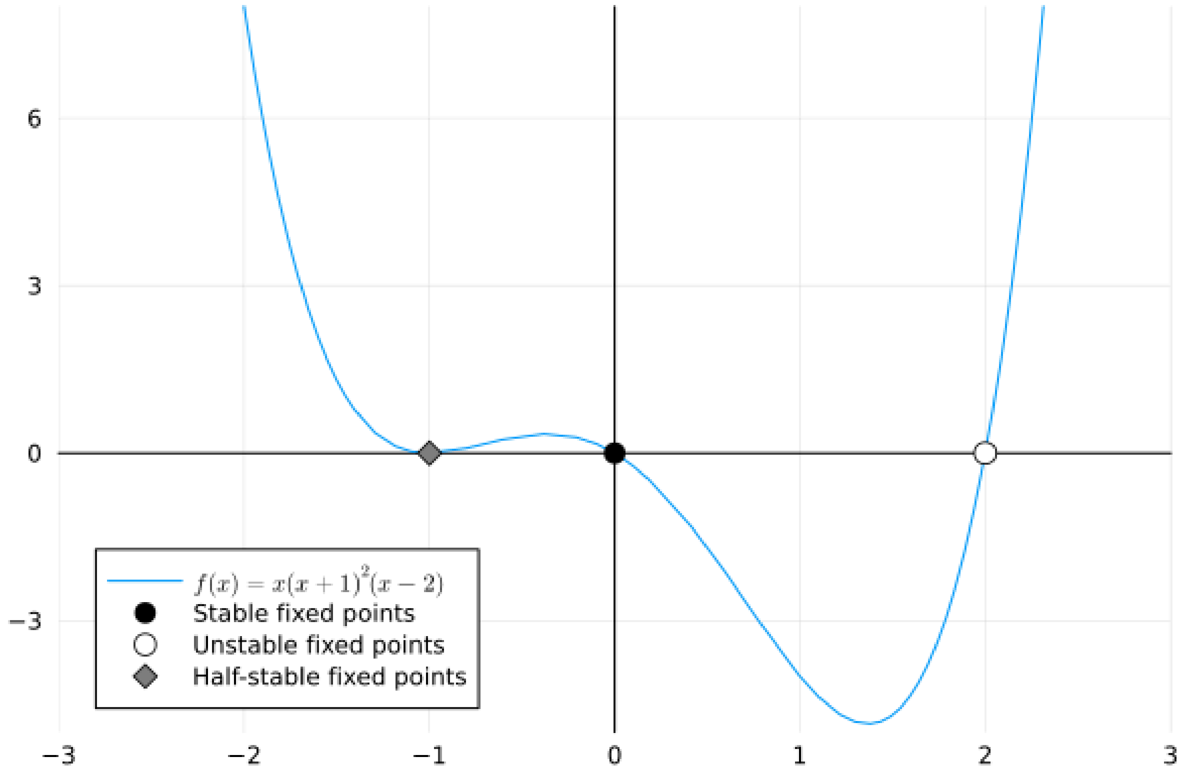


Figure 1: Fixed points with distinct stabilities

One possible system with these set of fixed points is

$$f(x) = x(x+1)^2(x-2) \tag{2}$$



Q3. Show that following dynamical system displays saddle-node bifurcation. Plot fixed plots and discuss their stabilities for various regions of control parameter r. What is the bifurcation point and also plot bifurcation diagram.

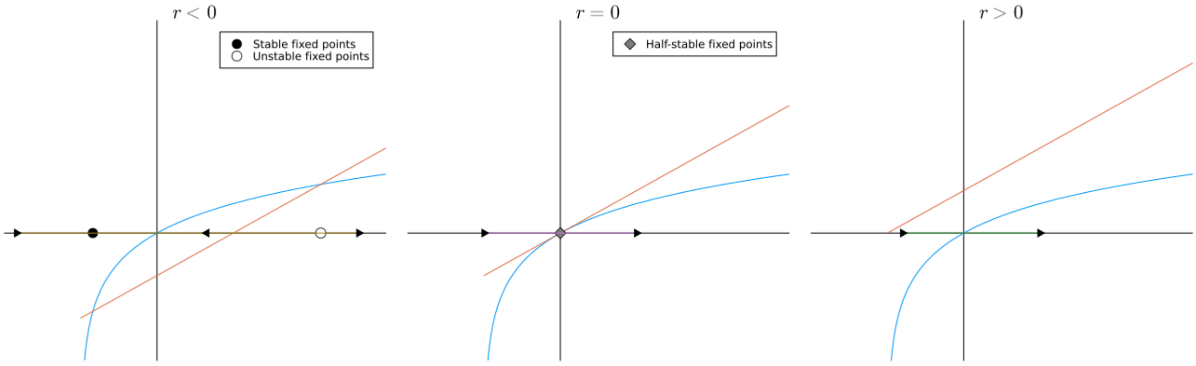
$$\dot{x} = r + x - \ln(1+x) \tag{3}$$

The ln term can be expanded to give

$$\begin{aligned} \dot{x} = f(x) &= r + x - \left(x - \frac{x^2}{2} + \ldots\right) \\ &= r + \frac{x^2}{2} \end{aligned} \tag{4}$$

This can be rescaled to the form $\dot{x} = r + x^2$ which is the normal form for saddle-node bifurcation. Thus, this system undergoes a saddle-node bifurcation.

We plot $r + x$ and $\ln(1+x)$ to see where they intersect, to graphically find the fixed points of the system. The regions having $r + x > \ln(1+x)$ have a flow to the right whereas the ones with $r + x < \ln(1+x)$ have a flow to the left. The bifurcation occurs when the curves both intersect tangentially.



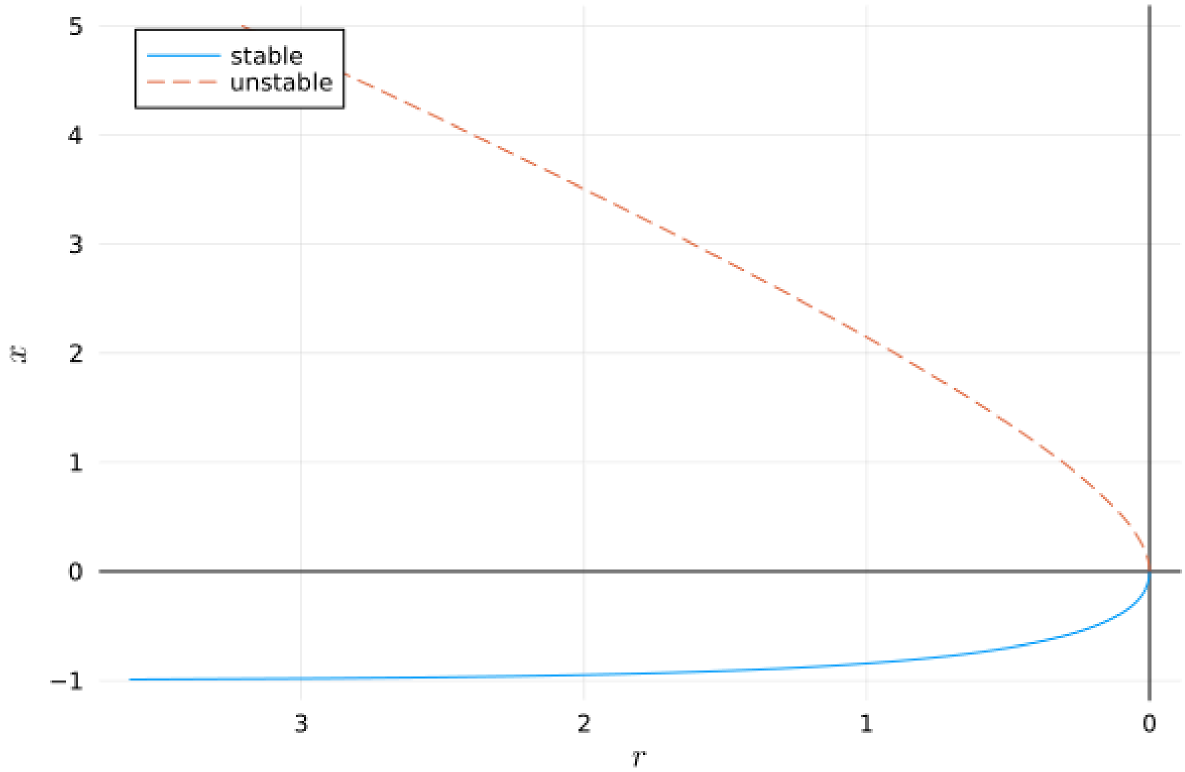
This value of r can be found by equating the curves and their slopes

$$\begin{aligned} \frac{d}{dx}(r + x) &= \frac{d}{dx}\ln(1+x) \\ \therefore 1 &= \frac{1}{1+x} \\ \therefore x &= 0 \end{aligned} \tag{5}$$

which is put in

$$\begin{aligned} r + x &= \ln(1+x) \\ \therefore r &= 0 \end{aligned} \tag{6}$$

Hence, $r = 0$ is the bifurcation point. The bifurcation diagram is like



Q4. Consider the normal form of subcritical pitchfork bifurcation,

$$\dot{x} = rx + x^3 \tag{7}$$

where r is the control parameter. This system is discussed in class. When $r > 0$, there is no stable solution for this system. Now to stabilize the system, an additional term is added which results into following dynamical system,

$$\dot{x} = rx + x^3 - x^5. \tag{8}$$

Calculate all its fixed points and discuss their nature. Draw bifurcation diagram and also argue why there is a possible hysteresis effect in this system.

First we solve $f(x) = rx - x^3 + x^5 = 0$ by factoring out x, which gives $x(r - x^2 + x^4) = 0$. S0, $x = 0$ is a fixed point. For nonzero x, set $r - x^2 + x^4 = 0$. Substitute $u = x^2$ to obtain a quadratic in u, $u^2 - u + r = 0$. The discriminant is $\Delta = 1 - 4r$. Real solutions in u (and hence for x) exist when $\Delta \geq 0$, i.e. $r \leq \frac{1}{4}$.

The solutions for u are:

$$u_{1,2} = \frac{1 \pm \sqrt{1-4r}}{2}. \tag{9}$$

Since $x^2 = u$, the additional fixed points are $x = \pm\sqrt{\frac{1\pm\sqrt{1-4r}}{2}}$.

As for stability, the derivative

$$f'(x) = \frac{d}{dx}(rx - x^3 + x^5) = r - 3x^2 + 5x^4. \tag{10}$$

- At $x = 0$, $f'(0) = r$ so that:
  - If $r < 0$, $x = 0$ is locally attracting (stable).
  - If $r > 0$, $x = 0$ is repelling (unstable).
- For nonzero fixed points (with $x^2 = u$), substitute u into $f'(x)$:

$$f'(x) = r - 3u + 5u^2 \tag{11}$$

.

Noting that u satisfies $u^2 - u + r = 0$ (i.e. $u^2 = u - r$), we can simplify:

$f'(x) = r - 3u + 5(u - r) = 2u - 4r.$

For $u_1 = \frac{1+\sqrt{1-4r}}{2}$, $f'(x) = 1 + \sqrt{1-4r} - 4r$. For typical r-values where these fixed points exist ($r < \frac{1}{4}$) this derivative is positive, indicating that these fixed points are unstable.

For $u^2 = \frac{1-\sqrt{1-4r}}{2}$, $f'(x) = 1 - \sqrt{1-4r} - 4r$. Depending on r, this expression can be negative, suggesting stability.

- At $r = \frac{1}{4}$, the discriminant $\Delta = 0$ and the two branches of nonzero fixed points coalesce in a saddle-node bifurcation.
- At $r = 0$, the stability of $x = 0$ changes (a pitchfork bifurcation).

Q5. Suppose that our overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\theta$. Then the equation of motion becomes $b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta$.

a) Does this equation give a well-defined vector field on the circle?

To check if it is well defined, we check if $f(\theta + 2\pi) = f(\theta)$, which is not true in this case. So it does not give a well defined vector field on a circle.

b) Nondimensionalize the equation.

Rearranging the equation to get

$$\frac{b}{mgL}\dot{\theta} = \frac{\Gamma}{mgL} - \frac{k}{mgL}\theta \tag{12}$$

Taking

$$\tau = \frac{mgL}{b}, \quad \gamma = \frac{\Gamma}{mgL}, \quad \kappa = \frac{k}{mgL} \tag{13}$$

we get

$$\theta' = -\sin\theta + \gamma - \kappa\theta \tag{14}$$

c) What does the pendulum do in the long run?

Since γ and κ both should be ≥ 0, we can divide this into two major cases:

- If $b = 0$, then this the case of overdamped pendulum.
- If $b > 0$, then the graph shows that there will be many saddle-node bifurcations.