IDC402
Assignment - 2

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MS21113

Q1. Analyze the long-term behavior of the map $x_{n+1} = \frac{rx_n}{1+x_n^2}$, where $r > 0$. Find and classify all fixed points as a function of r and their stability.

For fixed points,

$$\begin{gathered} x = \frac{rx}{1+x^2} \\ \therefore x(1-r+x^2) = 0 \\ \therefore x = 0 \text{ or } x = \pm\sqrt{r-1} \end{gathered} \tag{1}$$

For $0 < r \leq 1$, $x = 0$ is the only fixed point. For $r > 1$, $x = 0$ and $x = \pm\sqrt{r-1}$ are fixed points.

For stability of these finxed points, we need to find the derivative of the map:

$$\begin{aligned} f(x) &= \frac{rx}{1+x^2} \\ f'(x) &= \frac{r(1+x^2) - 2rx^2}{(1+x^2)^2} = \frac{r(1-x^2)}{(1+x^2)^2} \end{aligned} \tag{2}$$

- For $x = 0$:

$$f'(0) = r > 0 \tag{3}$$

  - Therefore x = 0 is stable for $r < 1$ and unstable for $r > 1$.
  - At $r = 1$, $f'(0) = 1$, which is inconclusive, so

$$|x_{n+1}| = \left|\frac{x_n}{1+x_n^2}\right| < \left|\frac{x_n}{1}\right| = |x_n| \tag{4}$$

  So x_n is stable for $r = 1$.

- For $x = \pm\sqrt{r-1}$:

$$f'\left(\sqrt{r-1}\right) = \frac{r(1-(r-1))}{(1+(r-1))^2} = \frac{r(2-r)}{r^2} = \frac{2}{r} - 1 \tag{5}$$

  - Therefore $x = \pm\sqrt{r-1}$ is stable for $r > 1$.

To check the long-term behavior of the map, we see how x evolves for different values of r:

- For $r \leq 1$ and $x_0 \neq 0$

$$|x_{n+1}| = r\left|\frac{x_n}{1+x_n^2}\right| < r\left|\frac{x_n}{1}\right| = r|x_n| < |x_n| \tag{6}$$

We showed that x_n is monotonically decreasing and hence cant have periodic solutions.

- For $r > 1$

$$|x_{n+1}| = r\left|\frac{x_n}{1+x_n^2}\right| < r\left|\frac{x_n}{1}\right| = r|x_n| < |x_n| \tag{7}$$

So the sequence $|x_n|$ is monotonically increasing or monotonically decreasing, and constant only if the sequence starts at a fixed point.

So the solutions can never be periodic or chaotic.

Q2. Calculate the Lyapunov exponent of the map : $x_n + 1 = 10x_n \pmod 1$ with x belongs to $[0, 1]$. Can there be periodic solutions or chaos?

This function can also be written as

$$f(x) = \begin{cases} 10x & 0 \leq x < 0.1 \\ 10x - 1 & 0.1 \leq x < 0.2 \\ 10x - 2 & 0.2 \leq x < 0.3 \\ \vdots \end{cases} \tag{8}$$

We can see that the function is piecewise linear with slope 10. The Lyapunov exponent is given by

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(|f'(x_i)|) = 10 \tag{9}$$

Since the Lyapunov exponent is positive, we can conclude that the system is chaotic.

Q3. For the following Lorenz system with parameters $\sigma = 10, \beta = \frac{8}{3}$, plot $x(t), y(t)$ and x vs z. Find fixed points for each value of r and discuss the different behaviours one observes:

1. **$r = 10$**
2. **$r = 24.5$**
3. **$r = 126.5$**

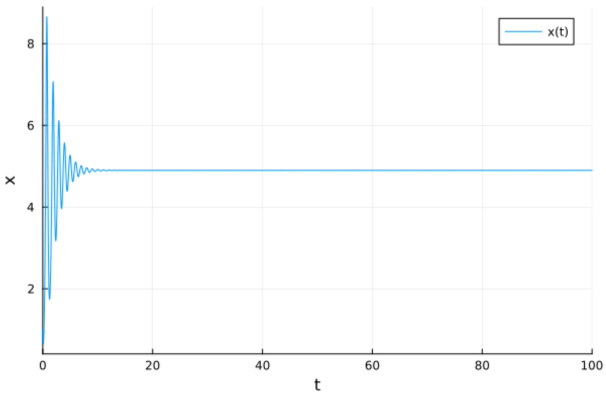

Figure 1: $x(t)$ for $r = 10.0$

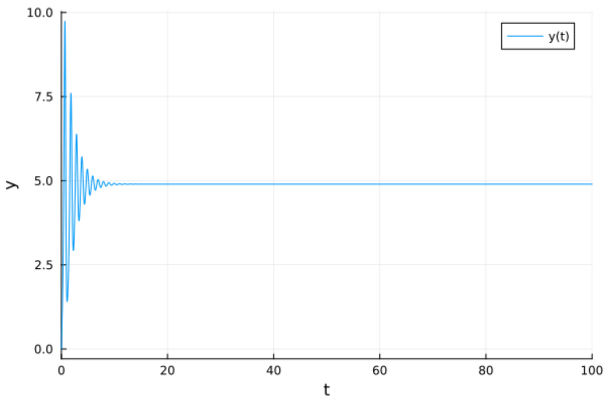

Figure 2: $y(t)$ for $r = 10.0$

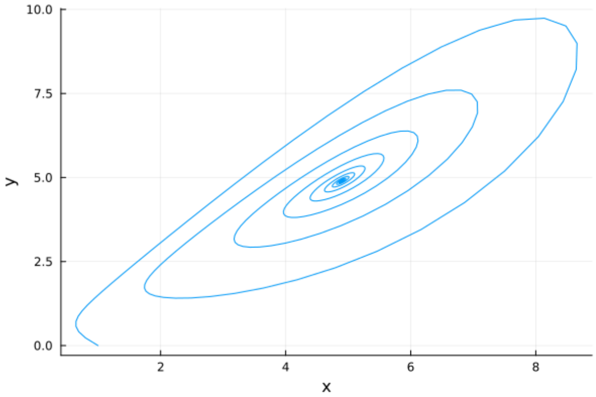

Figure 3: $y(x)$ for $r = 10.0$

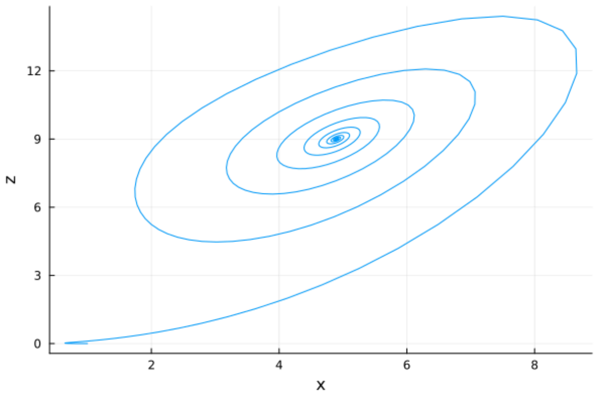

Figure 4: $z(x)$ for $r = 10.0$

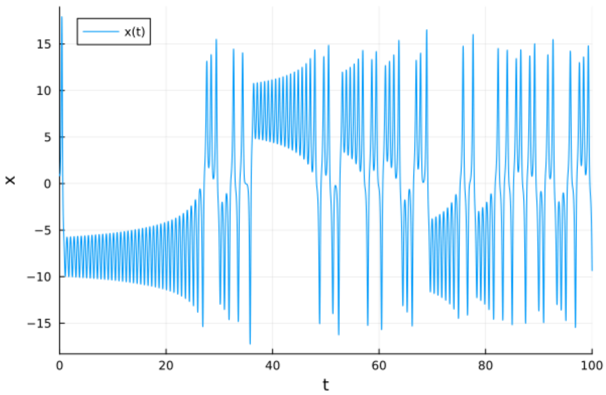

Figure 5: $x(t)$ for $r = 24.5$

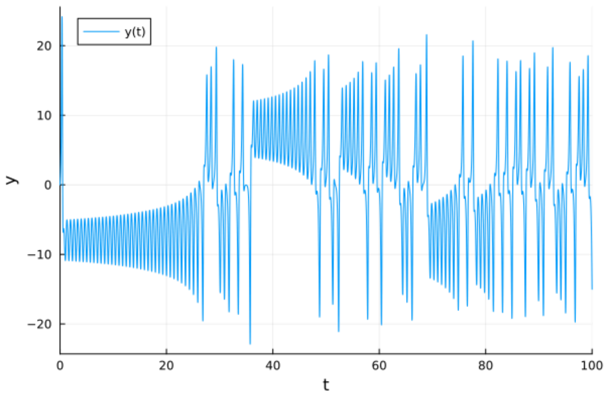

Figure 6: $y(t)$ for $r = 24.5$

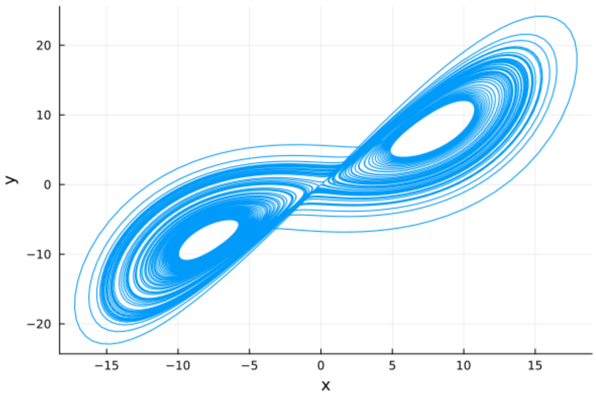

Figure 7: $y(x)$ for $r = 24.5$

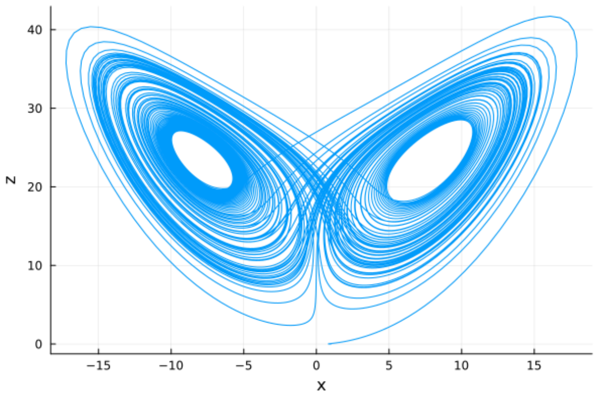

Figure 8: $z(x)$ for $r = 24.5$

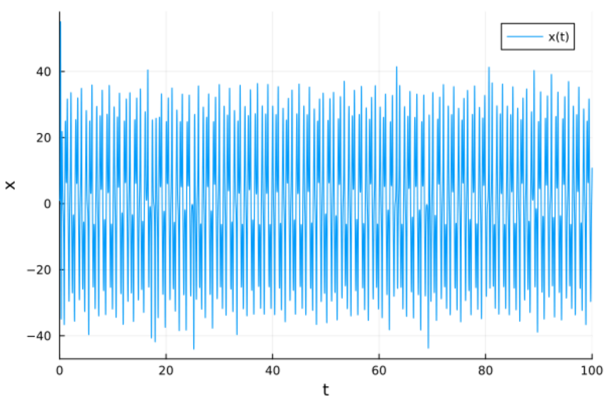

Figure 9: $x(t)$ for $r = 126.5$

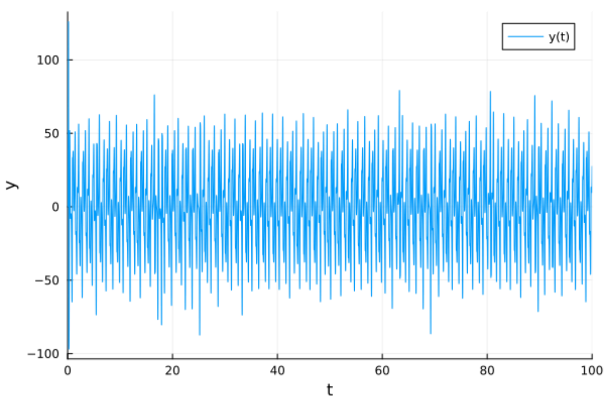

Figure 10: $y(t)$ for $r = 126.5$

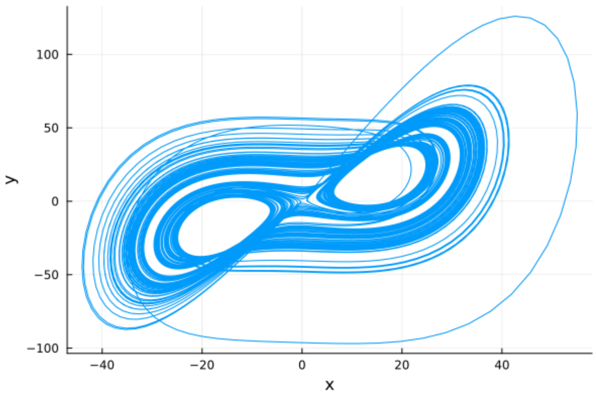

Figure 11: $y(x)$ for $r = 126.5$

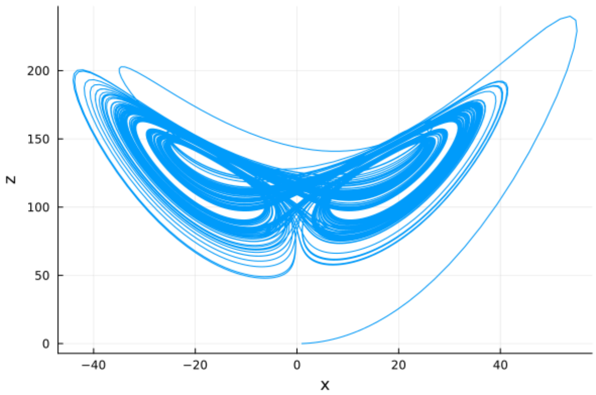

Figure 12: $z(x)$ for $r = 126.5$

The fixed points of the Lorenz system are $(0, 0, 0)$ (for all paramater values) and $\left(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1\right)$ (for $r > 1$). So

- For $r = 10$, the fixed points are $(0, 0, 0)$ and $(\pm 3.162, \pm 3.162, 9)$.
- For $r = 24.5$, the fixed points are $(0, 0, 0)$ and $(\pm 4.123, \pm 4.123, 23.5)$.
- For $r = 126.5$, the fixed points are $(0, 0, 0)$ and $(\pm 8.165, \pm 8.165, 125.5)$.

The system exhibits chaotic behavior for $r = 24.5$ and $r = 126.5$.

Q4. Consider the Lienard equation

$$\ddot{x} - (\mu - x^2)\dot{x} + x = 0 \tag{10}$$

Show that the system exhibits supercritical Hopf bifurcation around the only fixed point of the system? Create phase portraits for few μ values to show that bifurcation.

Converting this to a system of first order equations, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\mu - x^2)y - x \end{aligned} \tag{11}$$

The fixed point of the system is $(0, 0)$ for all values of μ. The Jacobian matrix at the fixed point is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \tag{12}$$

The eigenvalues of the Jacobian matrix are given by

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \tag{13}$$

For $\mu < 0$, the eigenvalues are complex conjugates with negative real part, so the fixed point is stable. For $\mu = 0$, the eigenvalues are purely imaginary. For $\mu > 0$, the eigenvalues are complex conjugates with positive real part, so the fixed point is unstable. Thus, we can conclude that the system Hopf bifurcates at $\mu = 0$.

To check if this a supercritical bifurcation, we look at the original equation again. The term $-(\mu - x^2)\dot{x}$ acts as damping.

- When $\mu > 0$ and x is very small ($x^2 < \mu$), the damping term is negative and pushing the trajectories away from the origin.
- When $\mu > 0$ and x is larger ($x^2 > \mu$), the damping term is positive and pulling the trajectories towards the origin.

This means that there exists a stable limit cycle for $\mu > 0$. Since the fixed point is stable for $\mu < 0$ and unstable for $\mu > 0$, and the system exhibits a stable limit cycle for $\mu > 0$, we can conclude that there is a supercritical Hopf bifurcation at $\mu = 0$.

This can be visualized by plotting the phase portraits for different values of μ through 0:

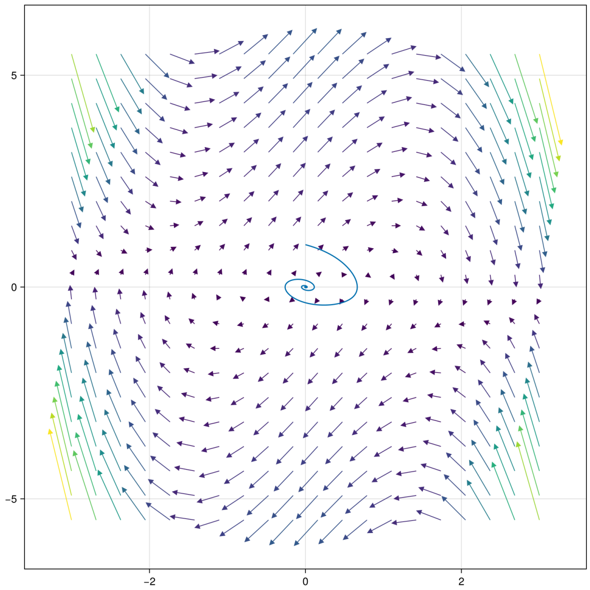

Figure 13: $\mu = -0.5$

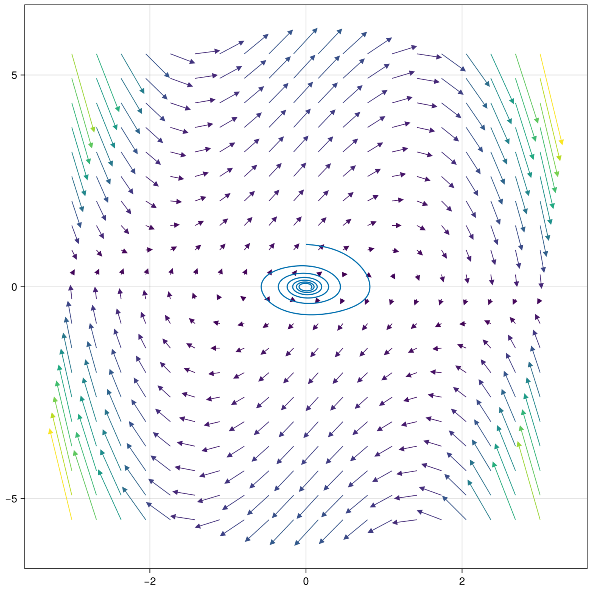

Figure 14: $\mu = -0.1$

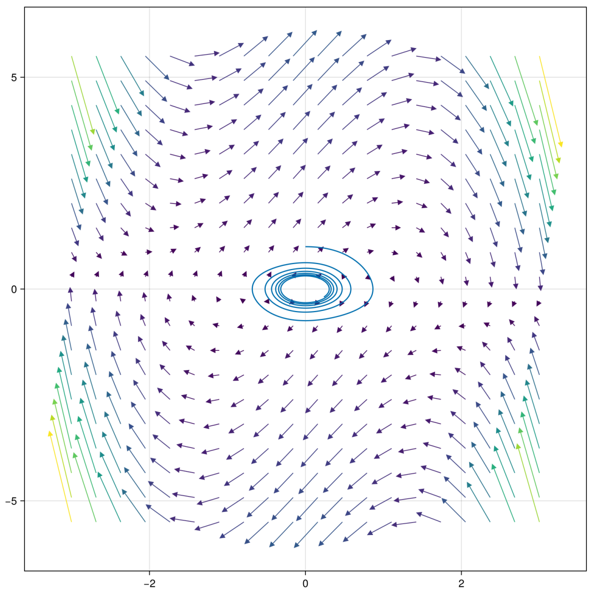

Figure 15: $\mu = 0$

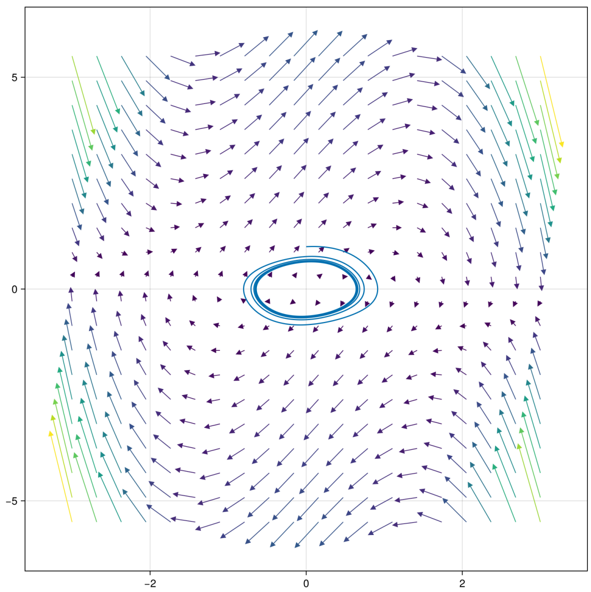

Figure 16: $\mu = 0.1$

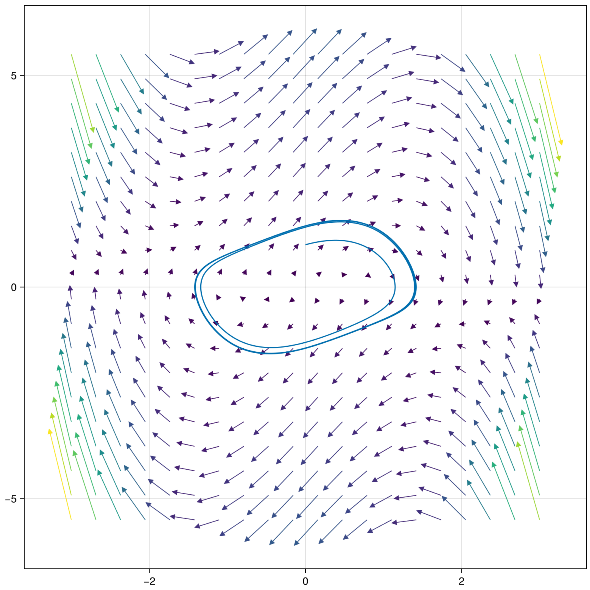

Figure 17: $\mu = 0.5$

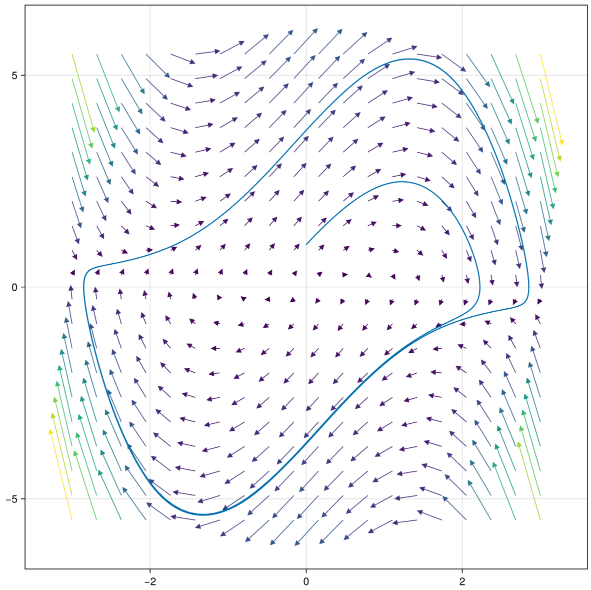

Figure 18: $\mu = 2$

The plots have been made in Julia° using DifferentialEquations.jl°, Plots.jl° and Makie.jl°

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