

# **IDC402**

# **Assignment - 2**

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MS21113

**Q1. Analyze the long-term behavior of the map  $x_{n+1} = \frac{rx_n}{1+x_n^2}$ , where  $r > 0$ . Find and classify all fixed points as a function of  $r$  and their stability.**

For fixed points,

$$\begin{aligned} x &= \frac{rx}{1+x^2} \\ \therefore x(1-r+x^2) &= 0 \\ \therefore x &= 0 \text{ or } x = \pm\sqrt{r-1} \end{aligned} \tag{1}$$

For  $0 < r \leq 1$ ,  $x = 0$  is the only fixed point. For  $r > 1$ ,  $x = 0$  and  $x = \pm\sqrt{r-1}$  are fixed points.

For stability of these fixed points, we need to find the derivative of the map:

$$\begin{aligned} f(x) &= \frac{rx}{1+x^2} \\ f'(x) &= \frac{r(1+x^2) - 2rx^2}{(1+x^2)^2} = \frac{r(1-x^2)}{(1+x^2)^2} \end{aligned} \tag{2}$$

- For  $x = 0$ :

$$f'(0) = r > 0 \tag{3}$$

- Therefore  $x = 0$  is stable for  $r < 1$  and unstable for  $r > 1$ .
- At  $r = 1$ ,  $f'(0) = 1$ , which is inconclusive, so

$$|x_{n+1}| = \left| \frac{x_n}{1+x_n^2} \right| < \left| \frac{x_n}{1} \right| = |x_n| \tag{4}$$

So  $x_n$  is stable for  $r = 1$ .

- For  $x = \pm\sqrt{r-1}$ :

$$f'(\sqrt{r-1}) = \frac{r(1-(r-1))}{(1+(r-1))^2} = \frac{r(2-r)}{r^2} = \frac{2}{r} - 1 \tag{5}$$

- Therefore  $x = \pm\sqrt{r-1}$  is stable for  $r > 1$ .

To check the long-term behavior of the map, we see how  $x$  evolves for different values of  $r$ :

- For  $r \leq 1$  and  $x_0 \neq 0$

$$|x_{n+1}| = r \left| \frac{x_n}{1+x_n^2} \right| < r \left| \frac{x_n}{1} \right| = r|x_n| < |x_n| \tag{6}$$

We showed that  $x_n$  is monotonically decreasing and hence can't have periodic solutions.

- For  $r > 1$

$$|x_{n+1}| = r \left| \frac{x_n}{1+x_n^2} \right| < r \left| \frac{x_n}{1} \right| = r|x_n| < |x_n| \tag{7}$$

So the sequence  $|x_n|$  is monotonically increasing or monotonically decreasing, and constant only if the sequence starts at a fixed point.

So the solutions can never be periodic or chaotic.

**Q2. Calculate the Lyapunov exponent of the map :  $x_n + 1 = 10x_n \pmod{1}$  with  $x$  belongs to  $[0, 1]$ . Can there be periodic solutions or chaos?**

This function can also be written as

$$f(x) = \begin{cases} 10x & 0 \leq x < 0.1 \\ 10x - 1 & 0.1 \leq x < 0.2 \\ 10x - 2 & 0.2 \leq x < 0.3 \\ \vdots & \end{cases} \quad (8)$$

We can see that the function is piecewise linear with slope 10. The Lyapunov exponent is given by

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(|f'(x_i)|) = 10 \quad (9)$$

Since the Lyapunov exponent is positive, we can conclude that the system is chaotic.

**Q3. For the following Lorenz system with parameters  $\sigma = 10$ ,  $\beta = \frac{8}{3}$ , plot  $x(t)$ ,  $y(t)$  and  $x$  vs  $z$ . Find fixed points for each value of  $r$  and discuss the different behaviours one observes:**

1.  $r = 10$
2.  $r = 24.5$
3.  $r = 126.5$

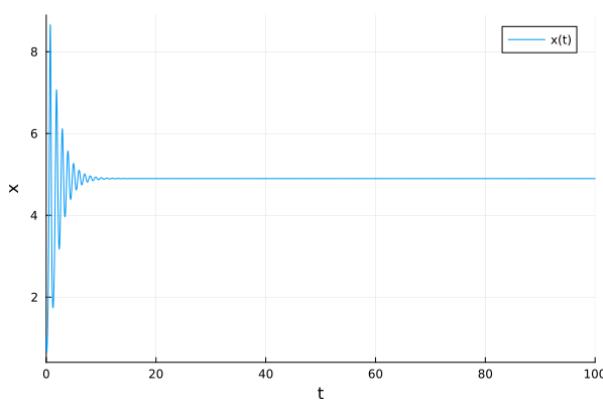


Figure 1:  $x(t)$  for  $r = 10.0$

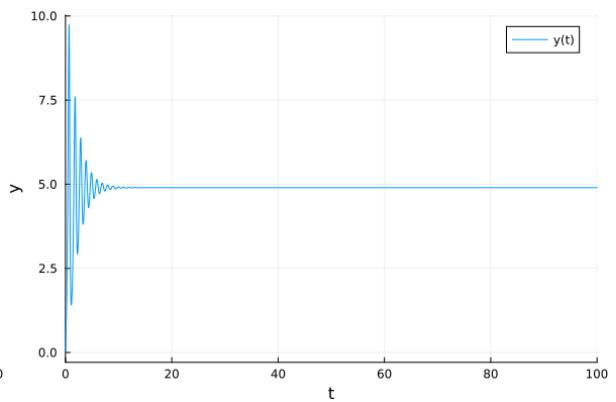


Figure 2:  $y(t)$  for  $r = 10.0$

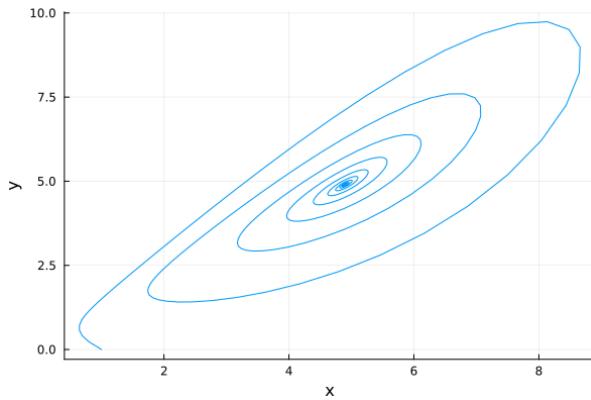


Figure 3:  $y(x)$  for  $r = 10.0$

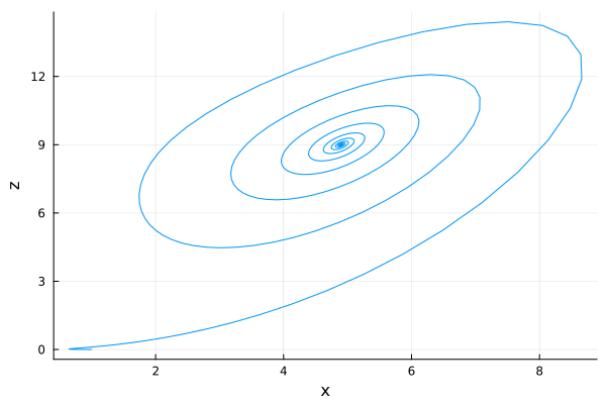


Figure 4:  $z(x)$  for  $r = 10.0$

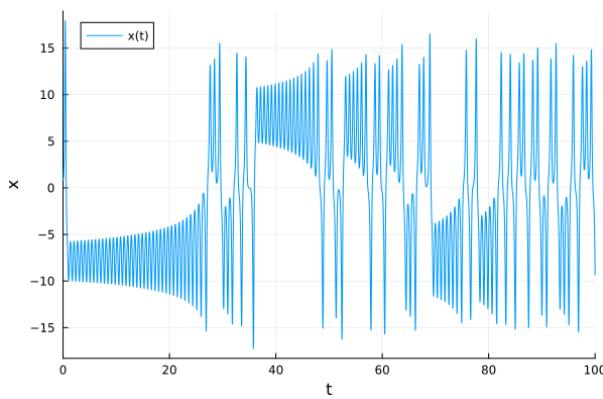


Figure 5:  $x(t)$  for  $r = 24.5$

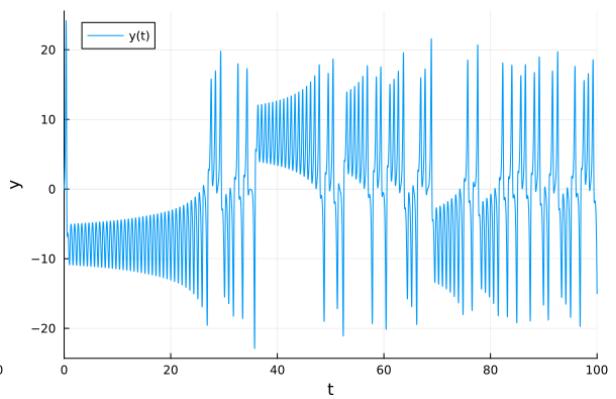


Figure 6:  $y(t)$  for  $r = 24.5$

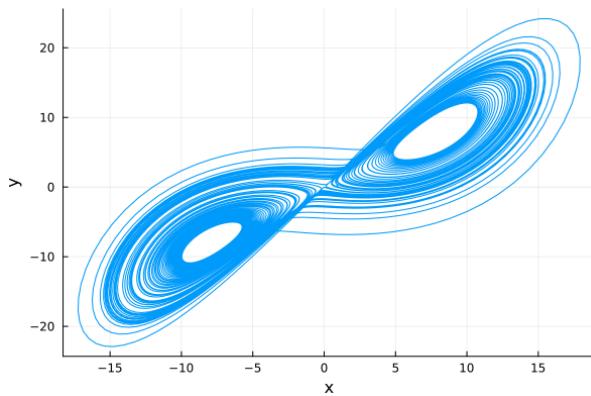


Figure 7:  $y(x)$  for  $r = 24.5$

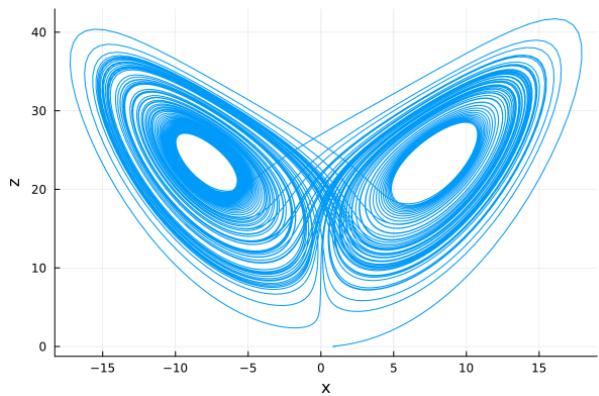


Figure 8:  $z(x)$  for  $r = 24.5$

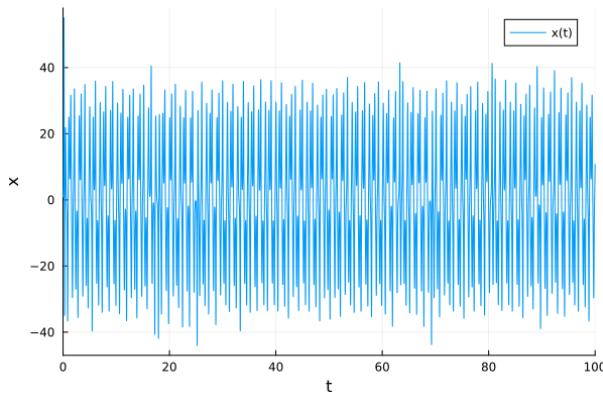


Figure 9:  $x(t)$  for  $r = 126.5$

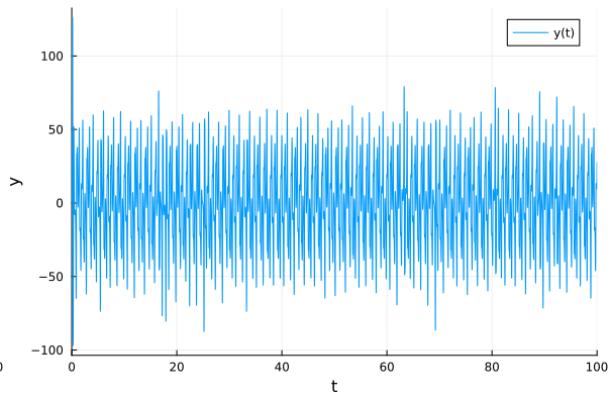


Figure 10:  $y(t)$  for  $r = 126.5$

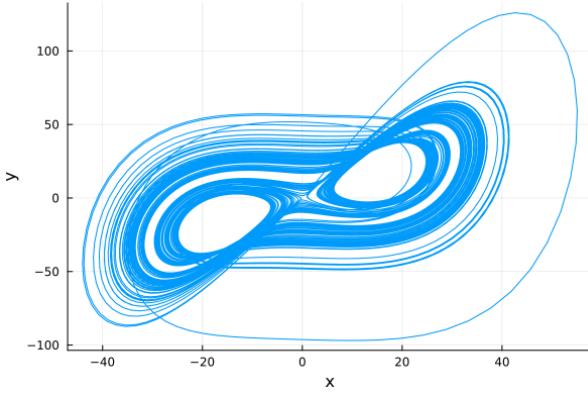


Figure 11:  $y(x)$  for  $r = 126.5$

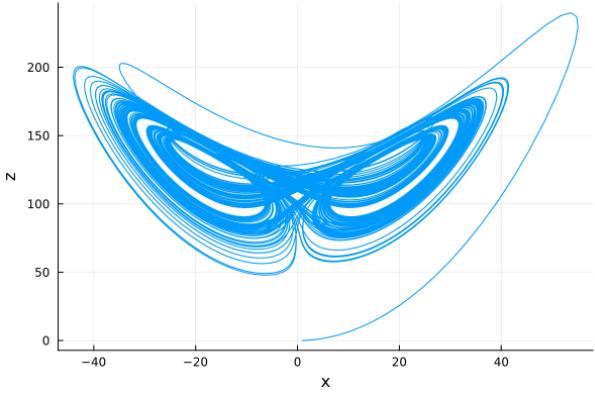


Figure 12:  $z(x)$  for  $r = 126.5$

The fixed points of the Lorenz system are  $(0, 0, 0)$  (for all parameter values) and  $(\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$  (for  $r > 1$ ). So

- For  $r = 10$ , the fixed points are  $(0, 0, 0)$  and  $(\pm 3.162, \pm 3.162, 9)$ .
- For  $r = 24.5$ , the fixed points are  $(0, 0, 0)$  and  $(\pm 4.123, \pm 4.123, 23.5)$ .
- For  $r = 126.5$ , the fixed points are  $(0, 0, 0)$  and  $(\pm 8.165, \pm 8.165, 125.5)$ .

The system exhibits chaotic behavior for  $r = 24.5$  and  $r = 126.5$ .

#### Q4. Consider the Lienard equation

$$\ddot{x} - (\mu - x^2)\dot{x} + x = 0 \quad (10)$$

Show that the system exhibits supercritical Hopf bifurcation around the only fixed point of the system? Create phase portraits for few  $\mu$  values to show that bifurcation.

Converting this to a system of first order equations, we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= (\mu - x^2)y - x \end{aligned} \quad (11)$$

The fixed point of the system is  $(0, 0)$  for all values of  $\mu$ . The Jacobian matrix at the fixed point is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \quad (12)$$

The eigenvalues of the Jacobian matrix are given by

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \quad (13)$$

For  $\mu < 0$ , the eigenvalues are complex conjugates with negative real part, so the fixed point is stable. For  $\mu = 0$ , the eigenvalues are purely imaginary. For  $\mu > 0$ , the eigenvalues are complex conjugates with positive real part, so the fixed point is unstable. Thus, we can conclude that the system Hopf bifurcates at  $\mu = 0$ .

To check if this is a supercritical bifurcation, we look at the original equation again. The term  $-(\mu - x^2)\dot{x}$  acts as damping.

- When  $\mu > 0$  and  $x$  is very small ( $x^2 < \mu$ ), the damping term is negative and pushing the trajectories away from the origin.
- When  $\mu > 0$  and  $x$  is larger ( $x^2 > \mu$ ), the damping term is positive and pulling the trajectories towards the origin.

This means that there exists a stable limit cycle for  $\mu > 0$ . Since the fixed point is stable for  $\mu < 0$  and unstable for  $\mu > 0$ , and the system exhibits a stable limit cycle for  $\mu > 0$ , we can conclude that there is a supercritical Hopf bifurcation at  $\mu = 0$ .

This can be visualized by plotting the phase portraits for different values of  $\mu$  through 0:

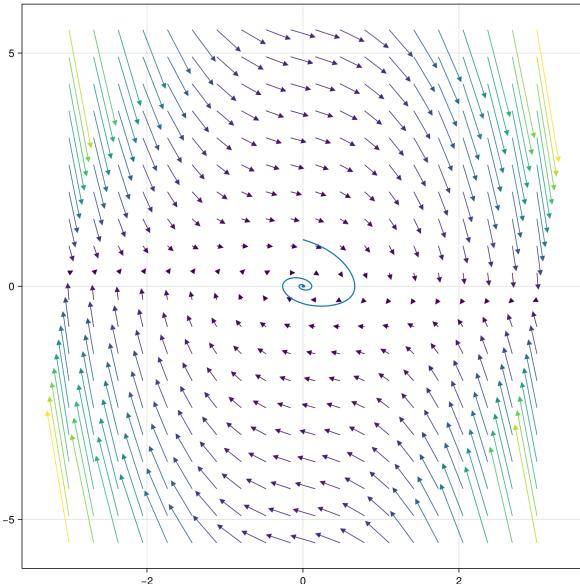


Figure 13:  $\mu = -0.5$

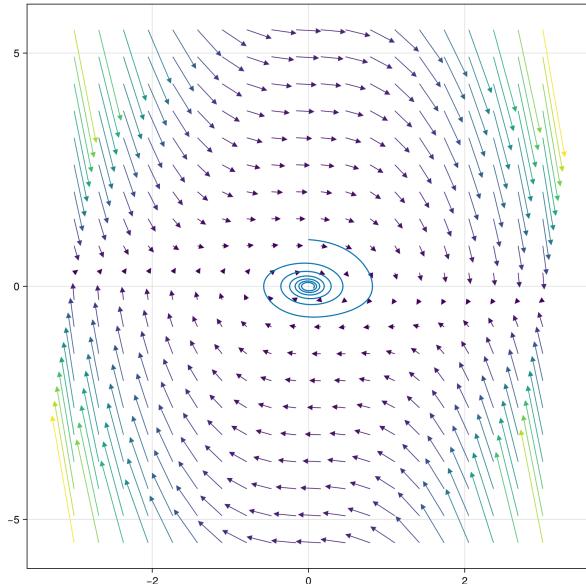


Figure 14:  $\mu = -0.1$

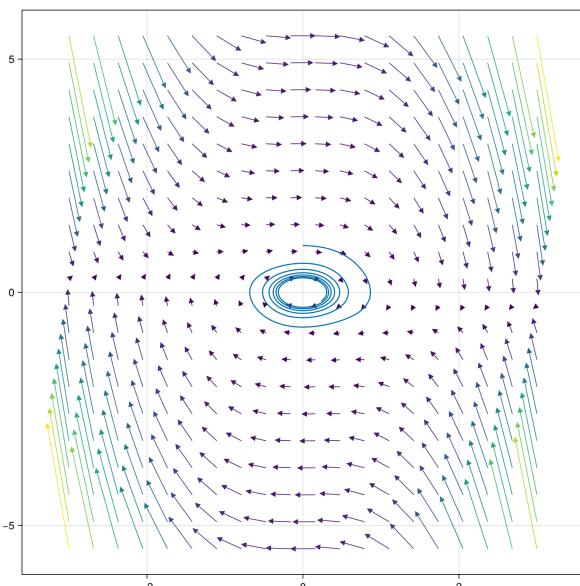


Figure 15:  $\mu = 0$

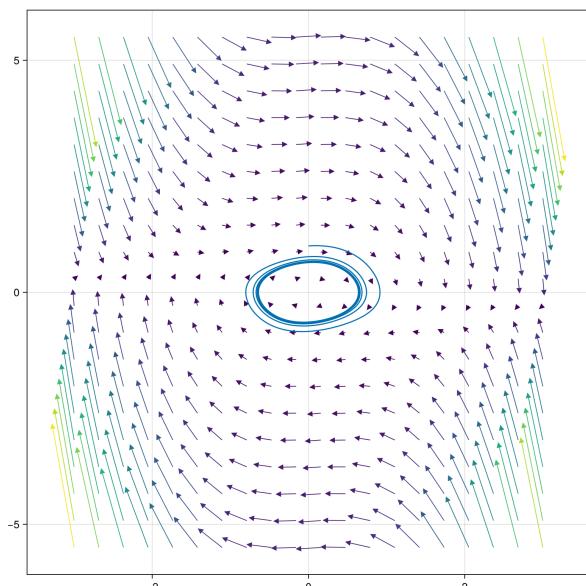


Figure 16:  $\mu = 0.1$

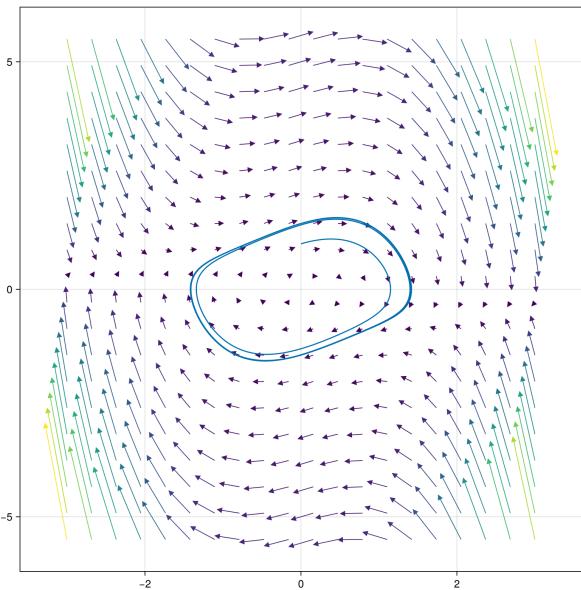


Figure 17:  $\mu = 0.5$

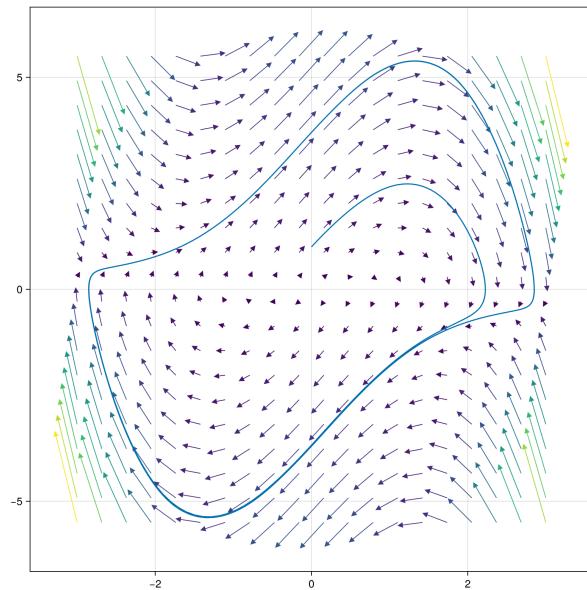


Figure 18:  $\mu = 2$

The plots have been made in [Julia](#)° using [DifferentialEquations.jl](#)°, [Plots.jl](#)° and [Makie.jl](#)°