Modal Logic

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1 Basic Concepts

1.1 Modal Languages

Definition 1.1. The **basic modal language** is defined using a set of **proposition letters** Φ whose elements are usually denoted p, q, r and so on, and a unary modal operator $\Diamond$. The well-formed **formulas** ϕ of the basic modal language are given by the rule

$$\phi \ ::= p \mid \bot \mid \neg\phi \mid \psi \vee \phi \mid \Diamond\phi$$

Definition 1.2. A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called **modal operators**; we use $\triangle$, $\triangle_0, \triangle_1, ...$ to denote elements of O. The function ρ assigns to each operator $\delta \in O$ a finite **arity**

Definition 1.3. A **modal language** $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ. The set $Form(\tau, \Phi)$ of **modal formulas** over τ and Φ is given by the rule

$$\phi \ := p \mid \bot \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)})$$

where p ranges over elements of Φ

Definition 1.4. For each $\triangle \in O$ the **dual** $\triangledown$ of $\triangle$ is defined as $\triangledown(\phi_1, \ldots, \phi_n) := \neg\triangle(\neg\phi_1, \ldots, \neg\phi_n)$

Example 1.1 (The Basic Temporal Language)**.** The basic temporal language is built using a set of unary operators $O = \{\langle F \rangle, \langle P \rangle\}$. The intended interpretation of a formula $\langle F \rangle\phi$ is ' ϕ will be true at some Future time' and the intended interpretation of $\langle P \rangle\phi$ is ' ϕ was true at some Past time.' This language is called the **basic temporal language**. Their duals are written as G and H ('it is Going to be the case' and 'it always Has been the case')

1.2 Models and Frames

Definition 1.5. A **frame** for the basic modal language is a pair $\mathfrak{F} = (W, R)$ s.t.

1. W is a non-empty set
2. R is a binary relation on W

A **model** for the basic modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame for the basic modal language and V is a function assigning to each proposition letter p in Φ a subset $V(p)$ of W. The function V is called a **valuation**. $\mathfrak{M}$ is **based on** the frame $\mathfrak{F}$

Definition 1.6. Suppose w is a state in a model $\mathfrak{M} = (W, R, V)$. Then ϕ is **satisfied** in $\mathfrak{M}$ at state w if

$$\begin{array}{rll} \mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \text{ where } p \in \Phi \\ \mathfrak{M}, w \Vdash \bot & \text{iff} & \text{never} \\ \mathfrak{M}, w \Vdash \neg\phi & \text{iff} & \text{not } \mathfrak{M}, w \Vdash \phi \\ \mathfrak{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \Diamond\phi & \text{iff} & \text{for some } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi \end{array}$$

It follows that $\mathfrak{M}, w \Vdash \Box\phi$ iff for all $v \in W$ s.t. Rwv, we have $\mathfrak{M}, v \Vdash \phi$

Definition 1.7. Let τ be a modal similarity type. A **τ-frame** is a tuple $\mathfrak{F}$ consisting of the following ingredients

1. a non-empty set W
2. for each $n \geq 0$, and each n-ary modal operator $\triangle$ in the similarity type τ, an $(n+1)$-ary relation $R_\triangle$

ϕ is **satisfied at a state** w in a model $\mathfrak{M} = (W, \{R_\triangle \mid \triangle \in \tau\}, V)$ when $\rho(\triangle) > 0$ if

$$\begin{array}{rll} \mathfrak{M}, w \Vdash \triangle(\phi_1, \ldots, \phi_n) & \text{iff} & \text{for some } v_1, \ldots, v_n \in W \text{ with } R_\triangle w v_1 \ldots v_n \\ & & \text{we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i \end{array}$$

When $\rho(\triangle) = 0$ we define

$$\mathfrak{M}, w \Vdash \triangle \quad \text{iff} \quad w \in R_\triangle$$

Definition 1.8. The set of all formulas that are valid in a class of frames $\mathsf{F}$is called the **logic** of F (notation: Λ_{F})

1.3 General Frames

Definition 1.9. Given an $(n+1)$-ary relation R on a set W, we define the following n-ary operation m_R on the power set $\mathcal{P}(W)$ of W:

$$m_R(X_1, \ldots, X_n) = \{w \in W \mid Rww_1 \ldots w_n \text{ for some } w_1 \in X_1, \ldots, w_n \in X_n\}$$

2 Models

2.1 Invariance Results

Definition 2.1. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of the same modal similarity type τ, and let w and w' be states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The **τ-theory** (or

τ**-type**) **of** w is the set of all τ-formulas satisfied at w: that is, $\{\phi \mid \mathfrak{M}, w \Vdash \phi\}$. We say that w and w' are (**modally**) **equivalent** ($w \leftrightsquigarrow w'$) if they have the same τ-theories

The τ**-theory** of the model $\mathfrak{M}$ is the set of all τ-formulas satisfied by all states in fM; that is, $\{\phi \mid \mathfrak{M} \Vdash \phi\}$ Models $\mathfrak{M}$ and $\mathfrak{M}'$ are called (**modally**) **equivalent** ($\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$) if their theories are identical

2.1.1 Disjoint Unions

2.1.2 Generated submodels

Definition 2.2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models; we say that $\mathfrak{M}'$ is a **submodel** of $\mathfrak{M}$ if $W'W$, R' is the restriction of R to W', and V' is the restriction of V to $\mathfrak{M}'$. We say that $\mathfrak{M}'$ is a **generated submodel** of $\mathfrak{M}$ ($\mathfrak{M}' \rightarrowtail \mathfrak{M}$) if $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$ and for all points w the following closure condition holds

$$\text{if } w \text{ is in } \mathfrak{M}' \text{ and } Rwv, \text{ then } v \text{ is in } \mathfrak{M}'$$

Let fM be a model, and X a subset of the domain of $\mathfrak{M}$; the **submodel generated by** X is the smallest generated submodel of $\mathfrak{M}$ whose domain contains X. A **rooted** or **point generated** model is a model that is generated by a singleton set, the element of which is called the **root** of the frame

2.1.3 Morphism for modalities

Definition 2.3 (Homomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$, if $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$, if $(w_0, \ldots, w_n) \in R_\triangle$, then $(f(w_0), \ldots, f(w_n)) \in R'_\triangle$ (the **homomorphic condition**)

Definition 2.4 (Strong Homomorphisms, Embeddings and Isomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **strong homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$ iff $w \in V(p)$, then $f(w) \in V'(p)$

2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$ iff $(w_0, \ldots, w_n) \in R_\triangle$, then $(f(w_0), \ldots, f(w_n)) \in R'_\triangle$ (the **strong homomorphic condition**)

An **embedding** of $\mathfrak{M}$ into $\mathfrak{M}'$ is a strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ which is injective. An **isomorphism** is a bijective strong homomorphism

Proposition 2.5. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. Then the following holds*

1. *for all elements w and w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, if there exists a surjective strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ with $f(w) = w'$, then w and w are modally equivalent*
2. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$*

Definition 2.6 (Bounded Morphisms - the Basic Case)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for the basic modal language. A mapping $f : \mathfrak{M} = (W, R, V) \to \mathfrak{M}' = (W', R', V')$ is a **bounded morphsim** if it satisfies

1. w and $f(w)$ satisfy the same proposition letters
2. f is a homomorphism w.r.t. the relation R (if Rwv then $R'f(w)f(v)$)
3. If $R'f(w)v'$ then there exists v s.t. Rwv and $f(v) = v'$ (the **back condition**)

If there is a **surjective** bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then we say that $\mathfrak{M}'$ is a **bounded morphic image** of $\mathfrak{M}$, and write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$

Proposition 2.7. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models s.t. $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded morphism. Then for each modal formula ϕ, and each element w of $\mathfrak{M}$ we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}', f(w) \Vdash \phi$.*

Proposition 2.8. *Assume that τ is a modal similarity type containing only diamonds. Then for any rooted τ-models $\mathfrak{M}$ there exists a tree-like τ-models $\mathfrak{M}'$ s.t. $\mathfrak{M}' \twoheadrightarrow \mathfrak{M}$. Hence any satisfiable τ-formula is satisfiable in a tree-like model*

2.2 Bisimulations

Definition 2.9 (Bisimulation - the Basic Case)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M} = (W', R', V')$ be two models

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation between** $\mathfrak{M}$ and $\mathfrak{M}'$ (notation: $Z : \mathfrak{M} \leftrightarroweq \mathfrak{M}'$) if

1. If wZw' then w and w' satisfy the same proposition letters
2. If wZw' and Rwv, then there exists v' (in $\mathfrak{M}'$) s.t. vZv' and $R'w'v'$ (the **forth condition**)
3. The converse of (2): if wZw' and $R'w'v'$, then there exists v (in $\mathfrak{M}$) s.t. vZv' and Rwv (the **back condition**)

When Z is a bisimulation linking two states w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$ we say that w and w' are **bisimilar**, and we write $Z : \mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$. If there is a bisimulation, we sometimes write $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$ or $w \leftrightarrows w'$

Definition 2.10 (Bisimulation - the General Case)**.** Let τ be a modal similarity type, and let $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$ be τ-models. A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ $(Z : \mathfrak{M} \leftrightarrows \mathfrak{M}')$ if the above condition 1 is satisfied and

2. If wZw' and $R_\triangle wv_1 \dots v_n$ then there are $v'_1, \dots, v'_n \in W'$ s.t. $R'_\triangle w'v'_1 \dots v'_n$ and for all i $(1 \leq i \leq n)$ $v_iZv'_i$ (the **forth** condition)
3. If wZw' and $R'_\triangle w'v'_1 \dots v'_n$ then there are $v_1, \dots, v_n \in W$ s.t. $R_\triangle wv_1 \dots v_n$ and for all i $(1 \leq i \leq n)$ $v_iZv'_i$ (the **back** condition)

Proposition 2.11. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ and $\mathfrak{M}_i$ $(i \in I)$ be τ-models*

1. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarrows \mathfrak{M}'$*
2. *For every $i \in I$, and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarrows \biguplus_i \mathfrak{M}_i, w$*
3. *If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarrows \mathfrak{M}, w$ for all w in $\mathfrak{M}'$*
4. *If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$*

Proof. Suppose $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$

1. Suppose $f : \mathfrak{M} \cong \mathfrak{M}'$, then we define wZw' iff $w' = f(w)$ where $w \in W, w' \in W'$. Bisimulation comes from the definition of the isomorphism
2. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}_i\} \subseteq \mathfrak{M}_i \times \biguplus \mathfrak{M}_i$. The first condition comes from the invariance. The forth condition is obvious. For the back condition, if $R'_\triangle w'v'_1 \dots v'_n$ and $w' \in W$, then $v'_1, \dots, v'_n \in W$ since each $R_{\triangle,i}$ is disjoint and we have $R_{\triangle,i}w'v'_1 \dots v'_n$
3. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}'\} \subseteq \mathfrak{M}' \times \mathfrak{M}$. The first condition comes from the invariance. Forth condition is obvious. For the back condition, suppose wZw and $R'_\triangle wv'_1 \dots v'_n$, by the definition, $v'_1, \dots, v'_n \in W$ and $R_\triangle wv'_1 \dots v'_n$
4. Define $Z = \{(w, f(w) \mid w \in W)\}$. The first condition comes from the definition. If wZw' and $R_\triangle wv_1 \dots v_n$, then $R'_\triangle f(w)f(v_1) \dots f(v_n)$. If wZw' and $R'_\triangle w'v'_1 \dots v_n$, then there is $v_1, \dots, v_n$ s.t. $R_\triangle wv_1, \dots, v_n$ and $f(v_i) = v'_i$ for $1 \leq i \leq n$

□

Theorem 2.12. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ be τ-models. Then, for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ implies that $w \leftrightsquigarrow w'$. In other words, modal formulas are invariant under bisimulation*

Proof. Induction on the complexity of ϕ.

Suppose ϕ is $\diamond\psi$, we have $\mathfrak{M}, w \Vdash \diamond\psi$ iff there exists a v in $\mathfrak{M}$ s.t. Rwv and $\mathfrak{M}, v \Vdash \psi$. As $w \leftrightarrows w'$, there exists a v' in $\mathfrak{M}'$ s.t. $R'w'v'$ and $v \leftrightarrows v'$. By the I.H., $\mathfrak{M}', v' \Vdash \psi$, hence $\mathfrak{M}', w' \Vdash \diamond\psi$ □

Example 2.1 (Bisimulation and First-Order Logic)**.**

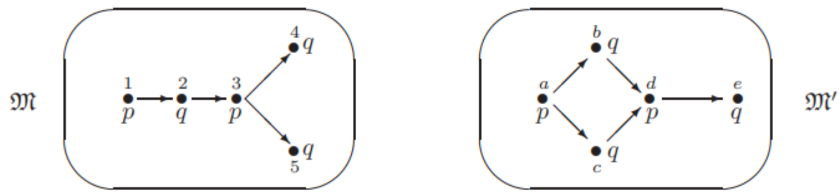

Fig. 2.4. Bisimilar models.

Example 2.2.

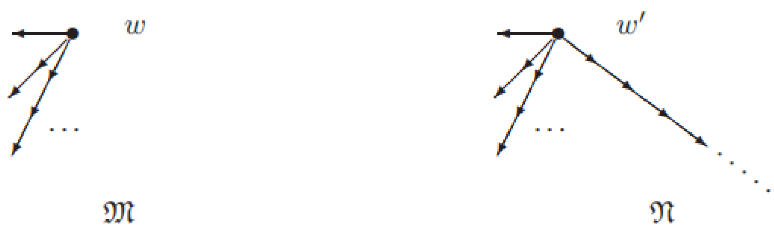

Fig. 2.5. Equivalent but not bisimilar.

$\mathfrak{M}$ is **image-finite** if for each state u in $\mathfrak{M}$ and each relation R in $\mathfrak{M}$, the set $\{(v_1, \dots, v_n) \mid Ruv_1 \dots v_n\}$ is finite

Theorem 2.13 (Hennessy-Milner Theorem)**.** *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be two image-finite τ-models. Then for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ iff $w \leftrightsquigarrow w'$*

Proof. Assume that our similarity type τ only contains a single diamond. The direction from left to right follows from Theorem 2.12

Suppose $w \leftrightsquigarrow w'$. The first condition is immediate. If Rwv, assume there is no v' in $\mathfrak{M}'$ with $R'w'v'$ and $v \leftrightsquigarrow v'$. Let $S' = \{u' \mid R'w'u'\}$. Note that S' must be non-empty, for otherwise $\mathfrak{M}', w' \Vdash \Box\bot$, which would contradict $w \leftrightsquigarrow w'$ since $\mathfrak{M}, w \Vdash \diamond\top$. Furthermore, as $\mathfrak{M}'$ is image-finite, S' must be finite, say $S' = \{w'_1, \dots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i s.t. $\mathfrak{M}, v \Vdash \psi_i$, but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \dots \wedge \psi_n) \quad \text{and} \quad \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \dots \wedge \psi_n)$$

□

Exercise 2.2.1. Suppose that $\{Z_i \mid i \in I\}$ is a non-empty collection of bisimulations between $\mathfrak{M}$ and $\mathfrak{M}'$. Prove that the relation $\bigcup_{i\in I} Z_i$ is also a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Conclude that if $\mathfrak{M}$ and $\mathfrak{M}'$ are bisimilar, then there is a maximal bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

Proof.

1. If $(w, w') \in \bigcup_{i\in I} Z_i$, then $(w, w') \in Z_j$ for some $j \in I$ and hence they satisfy the same propositional letters
2. If $(w, w') \in \bigcup_{i\in I} Z_i$ and $R_\triangle w v_1 \dots v_n$, since $(w, w') \in Z_j$ for some $j \in I$, we have $R'_\triangle w' v'_1 \dots v'_n$ and $v_i Z_j v'_i$ for all $1 \leq i \leq n$, which means $(v_i, v'_i) \in \bigcup_{i\in I} Z_i$ for all $1 \leq i \leq n$
3. similarly

□