Introduction To Commutative Algebra

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1 Rings and Ideals

A **unit** is an element u with a **reciprocal** $1/u$ or the **multiplicative inverse**. The units form a multiplicative group, denoted $R^\times$

A ring **homomorphism**, or simply a **ring map**, $\varphi : R \to R'$ is a map preserving sum, products and 1

If there is an unspecified isomorphism between rings R and R', then we write $R = R'$ when it is **canonical**; that is, it does not depend on any artificial choices.

A subset $R'' \subset R$ is a **subring** if R'' is a ring and the inclusion $R'' \hookrightarrow R$ is a ring map. In this case, we call R a (**ring**) **extension**.

An R-**algebra** is a ring R' that comes equipped with a ring map $\varphi : R \to R'$, called the **structure map**, denoted by R'/R. For example, every ring is canonically a $\mathbb{Z}$-algebra. An R-**algebra homomorphism**, or R-**map**, $R' \to R''$ is a ring map between R-algebras.

A group G is said to **act** on R if there is a homomorphism given from G into the group of automorphism of R. The **ring of invariants** R^G is the subring defined by

$$R^G := \{x \in R \mid gx = g \text{ for all } g \in G\}$$

Similarly a group G is said to **act** on R'/R if G acts on R' and each $g \in G$ is an R-map. Note that R'^G is an R-subalgebra

Boolean rings

The simplest nonzero ring has two elements, 0 and 1. It's denoted $\mathbb{F}_2$

Given any ring R and any set X, let R^X denote the set of functions $f : X \to R$. Then R^X is a ring.

For example, take $R := \mathbb{F}_2$. Given $f : X \to R$, put $S := f^{-1}\{1\}$. Then $f(x) = 1$ if $x \in S$. In other words, f is the **characteristic function** χ_S. Thus *the characteristic functions form a ring, namely,* $\mathbb{F}_2^X$

Given $T \subset X$, clearly $\chi_S \cdot \chi_T = \chi_{S\cap T}$. $\chi_S + \chi_T = \chi_{S\triangle T}$, where $S\triangle T$ is the **symmetric difference**:

$$S\triangle T := (S \cup T) - (S \cap T)$$

Thus *the subsets of X form a ring: sum is symmetric difference, and product is intersection. This ring is canonically isomorphic to* $\mathbb{F}_2^X$

A ring B is called **Boolean** if $f^2 = f$ for all $f \in B$. If so, then $2f = 0$ as $2f = (f + f)^2 = f^2 + 2f + f^2 = 4f$

Suppose X is a topological space, and give $\mathbb{F}_2$ the **discrete** topology; that is, every subset is both open and closed. Consider the continuous functions $f : X \to \mathbb{F}_2$. Clearly, they are just the χ_S where S is both open and closed.

Polynomial rings

Let R be a ring, $P := R[X_1, \dots, X_n]$. P has this **Universal Mapping Property** (UMP): *given a ring map $\varphi : R \to R'$ and given an element x_i of R' for each i, there is a unique ring map $\pi : P \to R'$ with $\pi|R = \varphi$ and $\pi(X_i) = x_i$.* In fact, since π is a ring map, necessarily π is given by the formula:

$$\pi(\sum a_{(i_1,\dots,i_n)} X_1^{i_1} \dots X_n^{i_n}) = \sum \varphi(a_{(i_1,\dots,i_n)}) x_1^{i_1} \dots x_n^{i_n} \tag{1.0.1}$$

In other words, P is universal among R-algebras equipped with a list of n elements

Similarly let $\mathcal{X} := \{X_\lambda\}_{\lambda \in \Lambda}$ be any set of variables. Set $P' := R[\mathcal{X}]$; the elements of P' are the polynomials in any finitely many of the X_λ. P' has essentially the same UMP as P

Ideals

Let R be a ring. A subset $\mathfrak{a}$ is called an **ideal** if

1. $0 \in \mathfrak{a}$
2. whenever $a, b \in \mathfrak{a}$, also $a + b \in \mathfrak{a}$
3. whenever $x \in R$ and $a \in \mathfrak{a}$ also $xa \in \mathfrak{a}$

Given a subset $\mathfrak{a} \subset R$, by the ideal $\langle \mathfrak{a} \rangle$ that $\mathfrak{a}$ **generates**, we mean the smallest ideal containing $\mathfrak{a}$

All ideal containing all the a_λ contains any (finite) **linear combination** $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and almost all 0.

Given a single element a, we say that the ideal $\langle a \rangle$ is **principal**

Given a number of ideals $\mathfrak{a}_\lambda$, by their **sum** $\sum \mathfrak{a}_\lambda$ we mean the set of all finite linear combinations $\sum x_\lambda a_\lambda$ with $x_\lambda \in R$ and $a_\lambda \in \mathfrak{a}_\lambda$

Given two ideals $\mathfrak{a}$ and $\mathfrak{b}$, by the **transporter** of $\mathfrak{b}$ into $\mathfrak{a}$ we mean the set

$$(\mathfrak{a} : \mathfrak{b}) := \{x \in R \mid x\mathfrak{b} \subset \mathfrak{a}\}$$

$(\mathfrak{a} : \mathfrak{b})$ is an ideal. Plainly,

$$\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}, \quad \mathfrak{a}, \mathfrak{b} \subset \mathfrak{a} + \mathfrak{b}, \quad \mathfrak{a} \subset (\mathfrak{a} : \mathfrak{b})$$

Further, for any ideal $\mathfrak{c}$, the distributive law holds: $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$

Given an ideal fa, notice $\mathfrak{a} = R$ *if and only if* $1 \in \mathfrak{a}$. It follows that $\mathfrak{a} = R$ iff $\mathfrak{a}$ contains a unit.

Given a ring map $\varphi : R \to R'$, denote by $\mathfrak{a}R'$ or $\mathfrak{a}^e$ the ideal of R' generated by the set $\varphi(\mathfrak{a})$. We call it the **extension** of $\mathfrak{a}$

Given an ideal $\mathfrak{a}'$ of R', its preimage $\varphi^{-1}(\mathfrak{a}')$ is an ideal of R. We call $\varphi^{-1}(\mathfrak{a}')$ the **contraction** of $\mathfrak{a}'$ and sometimes denote it by $\mathfrak{a}'^c$

Residue rings

kernel $\ker(\varphi)$ is defined to be the ideal $\varphi^{-1}(0)$ of R

Let $\mathfrak{a}$ be an ideal of R. Form the set of cosets of $\mathfrak{a}$

$$R/\mathfrak{a} := \{x + \mathfrak{a} \mid x \in R\}$$

$R/\mathfrak{a}$ is called the **residure ring** or **quotient ring** or **factor ring** of R **modulo** $\mathfrak{a}$. From the **quotient map**

$$\kappa : R \to R/\mathfrak{a} \quad \text{by } \kappa x := x + \mathfrak{a}$$

The element $\kappa x \in R/\mathfrak{a}$ is called the **residue** of x.

If $\ker(\varphi) \supset \mathfrak{a}$, *then there is a ring map* $\psi : R/\mathfrak{a} \to R'$ *with* $\psi\kappa = \varphi$; that is, the following diagram is commutative

$$\begin{tikzcd} R \arrow[r, "\kappa"] \arrow[dr, "\varphi"'] & R/\mathfrak{a} \arrow[d, "\psi"] \\ & R' \end{tikzcd}$$

by $\psi(x\mathfrak{a}) = \varphi(x)$. Then we only need to verify that ψ is a map

Conversely, *if ψ exists, then* $\ker(\varphi) \supset \mathfrak{a}$, *or* $\varphi\mathfrak{a} = 0$, *or* $\mathfrak{a}R' = 0$, since $\kappa\mathfrak{a} = 0$

Further, *if ψ exists, then ψ is unique* as κ is surjective

Finally, as κ is surjective, *if ψ exists, then ψ is surjective iff ψ is so*. In addition, *ψ is injective iff* $\mathfrak{a} = \ker(\varphi)$. Hence *$\psi$ is an isomorphism iff φ is surjective and* $\mathfrak{a} = \ker(\varphi)$. Therefore,

$$R/\ker(\varphi) \xrightarrow{\sim} \operatorname{im}(\varphi)$$

$R/\mathfrak{a}$ has UMP: $\kappa(\mathfrak{a}) = 0$, and given $\varphi : R \to R'$ s.t. $\varphi : R \to R'$ s.t. $\varphi(\mathfrak{a}) = 0$, there is a unique ring map $\psi : R/\mathfrak{a} \to R'$ s.t. $\psi\kappa = \varphi$. In other words, $R/\mathfrak{a}$ is universal among R-algebras R' s.t. $\mathfrak{a}R' = 0$

If $\mathfrak{a}$ is the ideal generated by elements a_λ,then the UMP can be usefully rephrased as follows: $\kappa(a_\lambda) = 0$ for all λ, and given $\varphi : R \to R'$ s.t. $\varphi(a_\lambda) = 0$ for all λ, there is a unique ring map $\psi : R/\mathfrak{a} \to R'$ s.t. $\psi\kappa = \varphi$

The UMP serves to determine $R/\mathfrak{a}$ up to unique isomorphism. Say R', equipped with $\varphi : R \to R'$ has the UMP too. $\kappa(\mathfrak{a}) = 0$ so there is a unique $\psi' : R' \to R/\mathfrak{a}$ with $\psi'\varphi = \kappa$. Then $\psi'\psi\kappa = \kappa$. Hence $\psi'\psi = 1$ by uniqueness. Thus ψ and ψ' are inverse isomorphism

$$\begin{tikzcd}
 & & R/\mathfrak{a} \arrow[dl, "\psi"] \arrow[dd, "1"] \\
R \arrow[urr, "\kappa"] \arrow[r, "\varphi"] \arrow[drr, "\kappa"'] & R' \arrow[dr, "\psi'"] & \\
 & & R/\mathfrak{a}
\end{tikzcd}$$

Proposition 1.1. *Let R be a ring, $P := R[X]$, $a \in R$ and $\pi : P \to R$ the R-algebra map defined by $\pi(X) := a$. Then*

1. $\ker(\pi) = \{F(X) \in P \mid F(a) = 0\} = \langle X - a \rangle$
2. $R/\langle X - a \rangle \simeq R$

Proof. Set $G := X - a$. Given $F \in P$, let's show $F = GH + r$ with $H \in P$ and $r \in R$. By linearity, we may assume $F := X^n$. If $n \geq 1$, then $F = (G + a)X^{n-1}$, so $F = GH + aX^{n-1}$ with $H := X^{n-1}$.

Then $\pi(F) = \pi(G)\pi(H) + \pi(r) = r$. Hence $F \in \ker(\pi)$ iff $F = GH$. But $\pi(F) = F(a)$ by 1.0.1 □

Degree of a polynomial

Let R be a ring, P the polynomial ring in any number of variables. If F is a monomial $\mathbf{M}$, then its degree $\deg(\mathbf{M})$ is the sum of its exponents; in general, $\deg(F)$ is the largest $\deg(\mathbf{M})$ of all monomials $\mathbf{M}$ in F

Given any $G \in P$ with FG nonzero, notice that

$$\deg(FG) \leq \deg(F) + \deg(G)$$

Order of a polynomial

Let R be a ring, P the polynomial ring in variable X_λ for $\lambda \in \Lambda$, and $(x_\lambda) \in R^\Lambda$ a vector. Let $\varphi_{(x_\lambda)} : P \to P$ denote the R-algebra map defined by $\varphi_{(x_\lambda)} X_\mu := X_\mu + x_\mu$ for all $\mu \in \Lambda$. Fix a nonzero $F \in P$

The **order** of F at the zero vector (0), denoted $\operatorname{ord}_{(0)} F$, is defined as the smallest $\deg(\mathbf{M})$ of all the monomials $\mathbf{M}$ in F. In general, the **order** of F at the vector (x_λ), denoted $\operatorname{ord}_{(x_\lambda)} F$ is defined by the formula: $\operatorname{ord}_{(x_\lambda)} F := \operatorname{ord}_{(0)}(\varphi_{(x_\lambda)} F)$

Notice that $\operatorname{ord}_{(x_\lambda)} F = 0$ iff $F(x_\lambda) \neq 0$ as $(\varphi_{x_\lambda} F)(0) = F(x_\lambda)$

Given μ and $x \in R$, form $F_{\mu,x}$ by substituting x for X_μ in F. If $F_{\mu,x_\mu} \neq 0$, then

$$\operatorname{ord}_{(x_\lambda)} F \leq \operatorname{ord}_{(x_\lambda)} F_{\mu,x_\mu}$$

If $x_\mu = 0$, then F_{μ,x_μ} is the sum of the terms without x_μ in F. Hence if $(x_\lambda) = (0)$, then 1 holds. But substituting 0 for X_μ in $\varphi_{(x_\lambda)} F$ is the same as substituting x_μ for X_μ in F and then applying $\varphi_{(x_\lambda)}$ to the result; that is, $(\varphi_{(x_\mu)} F)_{\mu,0} = \varphi_{(x_\lambda)} F_{\mu,x_\mu}$

Given any $G \in P$ with FG nonzero,

$$\operatorname{ord}_{(x_\lambda)} FG \geq \operatorname{ord}_{(x_\lambda)} F + \operatorname{ord}_{(x_\lambda)} G$$

Nested ideals

Let R be a ring, $\mathfrak{a}$ an ideal, and $\kappa : R \to R/\mathfrak{a}$ the quotient map. Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the corresponding set of cosets of $\mathfrak{a}$

$$\mathfrak{b}/\mathfrak{a} := \{b + \mathfrak{a} \mid b \in \mathfrak{b}\} = \kappa(\mathfrak{b})$$

Clearly, $\mathfrak{b}/\mathfrak{a}$ is an ideal of $R/\mathfrak{a}$. Also $\mathfrak{b}/\mathfrak{a} = \mathfrak{b}(R/\mathfrak{a})$

The operation $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$ and $\mathfrak{b}' \mapsto \kappa^{-1}(\mathfrak{b}')$ are inverse to each other, and establish a bijective correspondence between the set of ideals $\mathfrak{b}$ of R containing $\mathfrak{a}$ and the set of all ideals $\mathfrak{b}'$ of $R/\mathfrak{a}$. Moreover, this correspondence preserves inclusions

Given an ideal $\mathfrak{b} \supset \mathfrak{a}$, form the composition of the quotient maps

$$\varphi : R \to R/\mathfrak{a} \to (R/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$$

φ is surjective and $\ker(\varphi) = \mathfrak{b}$. Hence φ factors

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R/\mathfrak{b} \arrow[d, "\psi", "\simeq"'] \\ R/\mathfrak{a} \arrow[r] & (R/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \end{tikzcd}$$

Idempotents

Let R be a ring. Let $e \in R$ be an **idempotent**; that is, $e^2 = e$. Then Re is a ring with e as 1.

Set $e' := 1 - e$. Then e' is idempotent and $e \cdot e' = 0$. We call e and e' **complementary idempotents**. Conversely, if two elements $e_1, e_2 \in R$ satisfy $e_1 + e_2 = 1$ and $e_1 e_2 = 0$, then they are complementary idempotents, as for each i,

$$e_i = e_i \cdot 1 = e_i(e_1 + e_2) = e_i^2$$

We denote the set of all idempotents by $\mathrm{Idem}(R)$. Let $\varphi : R \to R'$ be a ring map. Then $\varphi(e)$ is idempotent. So the restriction of φ to $\mathrm{Idem}(R)$ is a map

$$\mathrm{Idem}(\varphi) : \mathrm{Idem}(R) \to \mathrm{Idem}(R')$$

Example 1.1. Let $R := R' \times R''$ be a **product** of two rings. Set $e' := (1, 0)$ and $e'' := (0, 1)$. Then e' and e'' are complementary idempotents.

Proposition 1.2. *Let R be a ring, and e', e'' complementary idempotents. Set $R' := Re'$ and $R'' := Re''$. Define $\varphi : R \to R' \times R''$ by $\varphi(x) := (xe', xe'')$. Then φ is a ring isomorphism. Moreover, $R' = R/Re''$ and $R'' = R/Re'$*

Proof. Define a surjection $\varphi' : R \to R'$ by $\varphi'(x) := xe'$. Then φ' is a ring map, since $xye' = xye'^2 = (xe')(ye')$. Moreover, $\ker(\varphi') = Re''$ since $x = x \cdot 1 = xe' + xe'' = xe''$. Thus $R' = R/Re''$

Since φ is a ring map. It's surjective since $(xe', x'e'') = \varphi(xe' + x'e'')$ □

Exercise

Exercise 1.0.1. Let $\varphi : R \to R'$ be a map of rings, $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}$ ideals of R, $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}$ ideals of R'. Prove

1. $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$
2. $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c + \mathfrak{b}_2^c$
3. $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subset \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$
4. $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$
5. $(\mathfrak{a}_1 \mathfrak{a}_2)^e = \mathfrak{a}_1^e \mathfrak{a}_2^e$
6. $(\mathfrak{b}_1 \mathfrak{b}_2)^c \supset \mathfrak{b}_1^c \mathfrak{b}_2^c$
7. $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subset (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$
8. $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subset (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$

Exercise 1.0.2. Let $\varphi : R \to R'$ be a map of rings, $\mathfrak{a}$ an ideal of R, and $\mathfrak{b}$ an ideal of R'. Prove the following statements:

1. $\mathfrak{a}^{ec} \supset \mathfrak{a}$ and $\mathfrak{b}^{ce} \subset \mathfrak{b}$
2. $\mathfrak{a}^{ece} = \mathfrak{a}^e$ and $\mathfrak{b}^{cec} = \mathfrak{b}^c$
3. If $\mathfrak{b}$ is an extension, then $\mathfrak{b}^c$ is the largest ideal of R with extension $\mathfrak{b}$
4. If two extensions have the same contraction, then they are equal

Exercise 1.0.3. Let R be a ring, $\mathfrak{a}$ an ideal, $\mathcal{X}$ a set of variables. Prove:
1. The extension $\mathfrak{a}(R[\mathcal{X}])$ is the set $\mathfrak{a}[\mathcal{X}]$
2. $\mathfrak{a}(R[\mathcal{X}]) \cap R = \mathfrak{a}$

Exercise 1.0.4. Let R be a ring, $\mathfrak{a}$ an ideal, and $\mathcal{X}$ a set of variables. Set $P := R[\mathcal{X}]$. Prove $P/\mathfrak{a}P = (R/\mathfrak{a})[\mathcal{X}]$

Exercise 1.0.5. Let R be a ring, $P := R[\{X_\lambda\}]$ the polynomial ring in variables X_λ for $\lambda \in \Lambda$ a vector. Let $\pi_{(x_\lambda)} : P \to R$ denote the R-algebra map defined by $\pi_{(x_\lambda)} X_\mu := x_\mu$ for all $\mu \in \Lambda$. Show:
1. Any $F \in P$ has the form $F = \sum a_{(i_1,\dots,i_n)}(X_{\lambda_1}^{i_1} - x_{\lambda_1}) \dots (X_{\lambda_n} - x_{\lambda_n})^{i_n}$ for unique $a_{(i_1,\dots,i_n)} \in R$
2. $\ker(\pi_{(x_\lambda)}) = \{F \in P \mid F((x_\lambda)) = 0\} = \langle\{X_\lambda - x_\lambda\}\rangle$
3. π induces an isomorphism $P/\langle\{X_\lambda - x_\lambda\}\rangle \simeq R$
4. Given $F \in P$, its residue in $P/\langle\{X_\lambda - x_\lambda\}\rangle$ is equal to $F((x_\lambda))$
5. Let $\mathcal{Y}$ be a second set of variables. Then $P[\mathcal{Y}]/\langle\{X_\lambda - x_\lambda\}\rangle \simeq R[\mathcal{Y}]$

Proof.
1. Let $\varphi_{(x_\lambda)}$ be the R-automorphism of P. Say $\varphi_{(x_\lambda)} F = \sum a_{(i_1,\dots,i_n)} X_{\lambda_1}^{i_1} \dots X_{\lambda_n}^{i_n}$. And $\varphi_{(x_\lambda)}^{-1} \varphi_{(x_\lambda)} F = F$
2. Note that $\pi_{(x_\lambda)} F = F((x_\lambda))$. Hence $F \in \ker(\pi_{(x_\lambda)})$ iff $F((x_\lambda)) = 0$. If $F((x_\lambda)) = 0$, then $a_{(0,\dots,0)} = 0$, and so $F \in \langle\{X_\lambda - x_\lambda\}\rangle$
5. Set $R' := R[\mathcal{Y}]$

□

Exercise 1.0.6. Let R be a ring, $P := R[X_1, \dots, X_n]$ the polynomial ring in variables X_i. Given $F = \sum a_{(i_1,\dots,i_n)} X_1^{i_1} \dots X_n^{i_n} \in P$, formally set

$$\partial F/\partial X_j := \sum i_j a_{(i_1,\dots,i_n)} X_1^{i_i} \dots X_n^{i_n}/X_j \in P$$

Given $(x_1, \dots, x_n) \in R^n$, set $\mathbf{x} := (x_1, \dots, x_n)$, set $a_j := (\partial F/\partial X_j)(\mathbf{x})$, and set $\mathfrak{M} := \langle X_1 - x_1, \dots, X_n - x_n\rangle$. Show $F = F(\mathbf{x}) + \sum a_j(X_j - x_j) + G$ with $G \in \mathfrak{M}^2$. First show that if $F = (X_1 - x_1)^{i_1} \dots (X_n - x_n)^{i_n}$, then $\partial F/\partial X_j = i_j F/(X_j - x_j)$

Proof. $(\partial F/\partial X_j)(\mathbf{x}) = b_{(\delta_{1j},\dots,\delta_{nj})}$ where δ_{ij} is the Kronecker delta □

Exercise 1.0.7. Let R be a ring, X a variable, $F \in P := R[x]$, and $a \in R$. Set $F' := \partial F/\partial X$. We call a a **root** of F if $F(a) = 0$, a **simple root** if also $F'(a) \neq 0$, and a **supersimple root** if also $F'(a)$ is a unit.

Show that a is a root of F iff $F = (X - a)G$ for some $G \in P$, and if so, then G is unique; that a is a simple root iff also $G(a) \neq 0$; and that a is a supersimple root iff also $G(a)$ is a unit

Exercise 1.0.8. Let R be a ring, $P := R[X_1, \dots, X_n]$, $F \in P$ of degree d and $F_i := X_i^{d_i} + a_1 X_i^{d_i - 1} + \dots$ a monic polynomial in X_i aloen for all i. Find $G, G_i \in P$ s.t. $F = \sum_{i=1}^{n} F_i G_i + G$ where $G_i = 0$ or $\deg(G_i) \leq d - d_i$ and where the highest power of X_i in G is less than d_i

Proof. By linearity, we may assume $F := X_1^{m_1} \dots X_n^{m_n}$. If $m_i < d_i$ for all i, set $G_i := 0$ and $G := F$ and we're done. Else, fix i with $m_i \geq d_i$, and set $G_i := F/X_i^{d_i}$ and $G := (-a_1 X_i^{d_i - 1} - \dots)G_i$ □

Exercise 1.0.9 (Chinese Remainder Theorem). Let R be a ring

1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be **comaximal** ideals; that is, $\mathfrak{a} + \mathfrak{b} = R$. Show
   (a) $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$
   (b) $R/\mathfrak{a}\mathfrak{b} = (R/\mathfrak{a}) \times (R/\mathfrak{b})$
2. Let $\mathfrak{a}$ be comaximal to both $\mathfrak{b}$ and $\mathfrak{b}'$. Show $\mathfrak{a}$ is also comaximal to $\mathfrak{b}\mathfrak{b}'$
3. Given $m, n \geq 1$, show $\mathfrak{a}$ and $\mathfrak{b}$ are comaximal iff $\mathfrak{a}^m$ and $\mathfrak{b}^n$ are.
4. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be pairwise comaximal. Show
   (a) $\mathfrak{a}_1$ and $\mathfrak{a}_2 \dots \mathfrak{a}_n$ are comaximal
   (b) $\mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_n = \mathfrak{a}_1 \dots \mathfrak{a}_n$
   (c) $R/(\mathfrak{a}_1 \dots \mathfrak{a}_n) \simeq \prod (R/\mathfrak{a}_i)$
5. Find an example where $\mathfrak{a}$ and $\mathfrak{b}$ satisfy 1.1 but aren't comaximal

Proof.
1. $\mathfrak{a} + \mathfrak{b} = R$ implies $x + y = 1$ with $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. So given $z \in \mathfrak{a} \cap \mathfrak{b}$, we have $z = xz + yz \in \mathfrak{a}\mathfrak{b}$
2. $R = (\mathfrak{a} + \mathfrak{b})(\mathfrak{a} + \mathfrak{b}') = (\mathfrak{a}^2 + \mathfrak{b}\mathfrak{a} + \mathfrak{a}\mathfrak{b}') + \mathfrak{b}\mathfrak{b}' \subseteq \mathfrak{a} + \mathfrak{b}\mathfrak{b}' \subseteq R$
3. Build with $\mathfrak{a} + \mathfrak{b}^2 = R$. Conversely, note that $\mathfrak{a}^n \subset \mathfrak{a}$
4. Induction
5. Let k be a field. Take $R := k[X, Y]$ and $\mathfrak{a} := \langle X \rangle$ and $\mathfrak{b} := \langle Y \rangle$. Given $f \in \langle X \rangle \cap \langle Y \rangle$, note that every monomial of f contains both X and Y, and so $f \in \langle X \rangle \langle Y \rangle$. But $\langle X \rangle$ and $\langle Y \rangle$ are not comaximal

□

Exercise 1.0.10. First given a prime number p and a $k \geq 1$, find the idempotents in $\mathbb{Z}/\langle p^k \rangle$. Second, find the idempotents in $\mathbb{Z}/\langle 12 \rangle$. Third, find the number of idempotents in $\mathbb{Z}/\langle n \rangle$ where $n = \prod_{i=1}^{N} p_i^{n_i}$ with p_i distinct prime numbers

Proof. $x = 0, 1$

Since $-3 + 4 = 1$, the Chinese Remainder Theorem yields

$$\mathbb{Z}/\langle 12 \rangle = \mathbb{Z}/\langle 3 \rangle \times \mathbb{Z}/\langle 4 \rangle$$

m is idempotent in $\mathbb{Z}/\langle 12 \rangle$ iff it's idempotent in $\mathbb{Z}/\langle 3 \rangle$ and $\mathbb{Z}/\langle 4 \rangle$

$p_i^{n_i}$ has a linear combination equal to 1. Hence 2^N $\square$

Exercise 1.0.11. Let $R := R' \times R''$ be a product of rings, $\mathfrak{a} \subset R$ an ideal. Show $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{a}''$ with $\mathfrak{a}' \subset R$ and $\mathfrak{a}'' \subset R''$ ideals. Show $R/\mathfrak{a} = (R'/\mathfrak{a}') \times (R''/\mathfrak{a}'')$

Exercise 1.0.12. Let R be a ring; e, e' idempotents. Show

1. Set $\mathfrak{a} := \langle e \rangle$. Then $\mathfrak{a}$ is idempotent; that is, $\mathfrak{a}^2 = \mathfrak{a}$
2. Let $\mathfrak{a}$ be a principal idempotent ideal. Then $\mathfrak{a} = \langle f \rangle$ with f idempotent
3. Set $e'' := e + e' - ee'$. Then $\langle e, e' \rangle = \langle e'' \rangle$ and e'' is idempotent
4. Let $e_1, \ldots, e_r$ be idempotents. Then $\langle e_1, \ldots, e_r \rangle = \langle f \rangle$ with f idempotent
5. Assume R is Boolean. Then every finitely generated ideal is principal

Proof. 3. $ee'' = e^2 = e$

$\square$

Exercise 1.0.13. Let L be a **lattice**, that is, a partially ordered set in which every pair $x, y \in L$ has a sup $x \vee y$ and an inf $x \wedge y$. Assume L is **Boolean**; that is:

1. L has a least element 0 and a greatest element 1
2. The operations $\vee$ and $\wedge$ **distribute** over each other

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

3. Each $x \in L$ has a unique **complement** x'; that is, $x \wedge x' = 0$ and $x \vee x' = 1$
.

Show that the following six laws obeyed

$x \wedge x = x$	and	$x \vee x = x$	(**idempotent**)
$x \wedge 0 = 0, x \wedge 1 = x$	and	$x \vee 1 = 1, x \vee 0 = x$	(**unitary**)
$x \wedge y = y \wedge x$	and	$x \vee y = y \vee x$	(**commutative**)
$x \wedge (y \wedge z) = (x \wedge y) \wedge z$	and	$x \vee (y \vee z) = (x \vee y) \vee z$	(**associative**)
$x'' = x$	and	$0' = 1, 1' = 0$	(**involutory**)
$(x \wedge y)' = x' \vee y'$	and	$(x \vee y)' = x' \wedge y'$	(**De Morgan's**)

Moreover, show that $x \leq y$ iff $x = x \wedge y$

Exercise 1.0.14. Let L be a Boolean lattice. For all $x, y \in L$, set

$$x + y := (x \wedge y') \vee (x' \wedge y) \quad \text{and} \quad xy := x \wedge y$$

Show

1. $x + y = (x \vee y)(x' \vee y')$
2. $(x + y)' = (x'y') \vee (xy)$
3. L is a Boolean ring

Exercise 1.0.15. Given a Boolean ring R, order R by $x \le y$ if $x = xy$. Show R is thus a Boolean lattice. Viewing this construction as a map ρ from the set of Boolean-ring structures on the set R to the set of Boolean-lattice structures on R, show ρ is bijective with inverse the map λ associated to the construction in 1.0.14

Proof. First check R is partially ordered.

Given $x, y \in R$, set $x \vee y := x + y + xy$ and $x \wedge y := xy$. Then $x \le x \vee y$ as $x(x + y + xy) = x^2 + xy + x^2 y = x + 2xy = x$. If $z \le x$ and $z \le y$, then $z = zx$ and $z = zy$, and so $z(x \vee y) = z$; thus $z \le x \vee y$ □

Exercise 1.0.16. Let X be a set, and L the set of all subsets of X, partially ordered by inclusion. Show that L is a Boolean lattice and that the ring structure on L constructed in 1 coincides with that constructed in 1.0.14

Assume X is a topological space, and let M be the set of all its open and closed subsets. Show that M is a sublattice of L, and that the subring structure on M of 1 coincides with the ring structure of 1.0.14 with M for L

2 Prime Ideals

Zerodivisors

Let R be a ring. An element x is called a **zerodivisor** if there is a nonzero y with $xy = 0$; otherwise x is called a **nonzerodivisor**. Denote the set of zerodivisors by $\mathrm{z.div}(R)$ and the set of nonzerodivisor by S_0

Multiplicative subsets, prime ideals

Let R be a ring. A subset S is called **multiplicative** if $1 \in S$ and if $x, y \in S$ implies $xy \in S$

An ideal $\mathfrak{p}$ is called **prime** if its complement $R - \mathfrak{p}$ is multiplicative, or equivalently, if $1 \notin \mathfrak{p}$ and if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$

Fields, domains

A ring is called a **field** if $1 \neq 0$ and if every nonzero element is a unit.

A ring is called an **integral domain**, or simply a **domain**, if $\langle 0 \rangle$ is prime, or equivalently, if R is nonzero and has no nonzero zerodivisors.

Every domain R is a subring of its **fraction field** $\mathrm{Frac}(R)$. Conversely, any subring R of a field K, including K itself, is a domain. Further, $\mathrm{Frac}(R)$ has this UMP: the inclusion of R into any field L extends uniquely to an inclusion of $\mathrm{Frac}(R)$ into L.

Polynomials over a domain

Let R be a domain, $\mathcal{X} := \{X_\lambda\}_{\lambda\in\Lambda}$ a set of variables. Set $P := R[\mathcal{X}]$. Then P is a domain too. In fact, given nonzero $F, G \in P$, not only is their product FG nonzero, but also given a well ordering of the variables, the grlex leading term of FG is the product of the grlex leading terms of F and G, and

$$\deg(FG) = \deg(F) + \deg(G)$$

Using the given ordering of the variables, well order all the monomials $\mathbf{M}$ of the same degree via the lexicographic order on exponents. Among the $\mathbf{M}$ in F with $\deg(\mathbf{M}) = \deg(F)$, the largest is called the **grlex leading monomial** (graded lexicographic) of F. Its **grlex leading term** is the product $a\,\mathbf{M}$ whre $a \in R$ is the coefficient of $\mathbf{M}$ in F, and a is called the **grlex leading coefficient**

The grlex leading term of FG is the product of those $a\,\mathbf{M}$ and $b\,\mathbf{N}$ of F and G. and 2 holds, for the following reasons. First, $ab \neq 0$ as R is domain. Second

$$\deg(\mathbf{M}\,\mathbf{N}) = \deg(\mathbf{M}) + \deg(\mathbf{N}) = \deg(F) + \deg(G)$$

Third, $\deg(\mathbf{M}\,\mathbf{N}) \geq \deg(\mathbf{M}'\,\mathbf{N}')$ for every pair of monomials $\mathbf{M}'$ and $\mathbf{N}'$ in F and G.

The grlex hind term of FG is the product of the grlex hind terms of F and G. Further, given a vector $(x_\lambda) \in R^\Lambda$, then

$$\mathrm{ord}_{(x_\lambda)} FG = \mathrm{ord}_{(x_\lambda)} F + \mathrm{ord}_{(x_\lambda)} G$$

Among the monomials $\mathbf{M}$ in F with $\mathrm{ord}(\mathbf{M}) = \mathrm{ord}(F)$, the smallest is called the **grlex hind monomial** of F. The **grlex hind term** of F os the product $a\,\mathbf{M}$ where $a \in R$ is the coefficient of $\mathbf{M}$ in F

The fraction field $\mathrm{Frac}(P)$ is called the field of **rational functions**, and is also denoted by $K(\mathcal{X})$ where $K := \mathrm{Frac}(R)$

Unique factorization

Let R be a domain, p a nonzero nonunit. We call p **prime** if whenever $p \mid xy$, either $p \mid x$ or $p \mid y$. *p is prime iff $\langle p\rangle$ is prime*

We call p **irreducible** if whenever $p = yz$, either y or z is a unit. We call R a **Unique Factorization Domain** (UFD) if

1. every nonzero nonunit factors into a product of irreducibles

2. the factorization is unique up to order and units.

If R is a UFD, then $\gcd(x, y)$ always exists

Lemma 2.1. *Let $\varphi : R \to R'$ be a ring map, and $T \subset R'$ a subset. If T is multiplicative, then $\varphi^{-1}T$ is multiplicative; the converse holds if φ is surjective*

Proposition 2.2. *Let $\varphi : R \to R'$ be a ring map, and $\mathfrak{q} \subset R'$ an ideal. Set $\mathfrak{p} := \varphi^{-1}\mathfrak{q}$. If $\mathfrak{q}$ is prime, then $\mathfrak{p}$ is prime; the converse holds if φ is surjective*

Corollary 2.3. *Let R be a ring, $\mathfrak{p}$ an ideal. Then $\mathfrak{p}$ is prime iff $R/\mathfrak{p}$ is a domain*

Proof. By Proposition 2.2, $\mathfrak{p}$ is prime iff $\langle 0 \rangle \subset R/\mathfrak{p}$ is □

Exercise 2.0.1. Let R be a ring, $P := R[\mathcal{X}, \mathcal{Y}]$ the polynomial ring in two sets of variables $\mathcal{X}$ and $\mathcal{Y}$. Set $\mathfrak{p} := \langle \mathcal{X} \rangle$. Show $\mathfrak{p}$ is prime iff R is a domain

Proof. $\mathfrak{p}$ is prime iff $R[\mathcal{Y}]$ is a domain □

Definition 2.4. Let R be a ring. An ideal $\mathfrak{m}$ is said to be **maximal** if $\mathfrak{m}$ is proper and if there is no proper ideal $\mathfrak{a}$ with $\mathfrak{m} \subsetneq \mathfrak{a}$

Example 2.1. Let R be a domain, $R[X, Y]$ the polynomial ring. Then $\langle X \rangle$ is prime. However, $\langle X \rangle$ is not maximal since $\langle X \rangle \subsetneq \langle X, Y \rangle$

Proposition 2.5. *A ring R is a field iff $\langle 0 \rangle$ is a maximal ideal*

Proof. If $\langle 0 \rangle$ is maximal. Take $x \neq 0$, then $\langle x \rangle \neq 0$. So $\langle x \rangle = R$ and x is a unit. □

Corollary 2.6. *Let R be a ring, $\mathfrak{m}$ an ideal. Then $\mathfrak{m}$ is maximal iff $R/\mathfrak{m}$ is a field.*

Proof. $\mathfrak{m}$ is maximal iff $\langle 0 \rangle$ is maximal in $R/\mathfrak{m}$ by Correspondence Theorem. □

Example 2.2. Let R be a ring, P the polynomial ring in variable X_λ, and $x_\lambda \in R$ for all λ. Set $\mathfrak{m} := \langle \{X_\lambda - x_\lambda\} \rangle$. Then $P/\mathfrak{m} = R$ by Exercise **??**. Thus $\mathfrak{m}$ is maximal iff R is a field

Corollary 2.7. *In a ring, every maximal ideal is prime*

Coprime elements

Let R be a ring and $x, y \in R$. We say x and y are **(strictly) coprime** if their ideals $\langle x\rangle$ and $\langle y\rangle$ are comaximal

Plainly, x and y are coprime iff there are $a, b \in R$ s.t. $ax + by = 1$

Plainly, x and y are coprime iff there is $b \in R$ with $by \equiv 1 \mod \langle x\rangle$ iff the residue of y is a unit in $R/\langle x\rangle$

Fix $m, n \geq 1$. By Exercise 1.0.9, x and y are coprim eiff x^m and x^n are.

If x and y are coprime, then their images in algebra R' too.

PIDs

A domain R is called a **Principal Ideal Domain** (PID) if every ideal is principal. A PID is a UFD

Let R be a PID, $\mathfrak{p}$ a nonzero prime ideal. Say $\mathfrak{p} = \langle p\rangle$. Then p is prime, so irreducible. Now let $q \in R$ be irreducible. Then $\langle q\rangle$ is maximal for: if $\langle q\rangle \subsetneq \langle x\rangle$, then $q = xy$ for some nonunit y; so x must be a unit as q is irreducible. So $R/\langle q\rangle$ is a field. Also $\langle q\rangle$ is prime; so q is prime Thus every irreducible element is prime, and every nonzero prime ideal is maximal

Exercise 2.0.2. Show that, in a PID, nonzero elements x and y are **relatively prime** (share no prime factor) iff they are coprime

Proof. Say $\langle x\rangle + \langle y\rangle = \langle d\rangle$. Then $d = \gcd(x, y)$ □

Example 2.3. Let R be a PID, and $p \in R$ a prime. Set $k := R/\langle p\rangle$. Let X be a variable, and set $P := R[X]$. Take $G \in P$; let G' be its image in $k[X]$; assume G' is irreducible. Set $\mathfrak{m} := \langle p, G\rangle$. Then $P/\mathfrak{m} \simeq k[X]/\langle G'\rangle$ by **??** and 1 and $k[X]/\langle G'\rangle$ is a field; hence $\mathfrak{m}$ is maximal

Theorem 2.8. *Let R be a PID. Let $P := R[X]$ and $\mathfrak{p}$ a nonzero prime ideal of P*

1. *$\mathfrak{p} = \langle F\rangle$ with F prime or $\mathfrak{p}$ is maximal*
2. *Assume $\mathfrak{p}$ is maximal. Then either $\mathfrak{p} = \langle F\rangle$ with F prime, or $\mathfrak{p} = \langle p, G\rangle$ with $p \in R$ prime, $pR = \mathfrak{p} \cap R$ and $G \in P$ prime with image $G' \in (R/pR)[X]$ prime*

Proof. P is a UFD.

If $\mathfrak{p} = \langle F\rangle$ for some $F \in P$, then F is prime. Assume $\mathfrak{p}$ isn't principal

Take a nonzero $F_1 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $\mathfrak{p}$ contains a prime factor F_1' of F_1. Replace F_1 by F_1'. As $\mathfrak{p}$ isn't principal, $\mathfrak{p} \neq \langle F_1\rangle$. So there is a prime $F_2 \in \mathfrak{p} - \langle F_1\rangle$. Set $K := \mathrm{Frac}(R)$, Gauss's lemma implies that F_1 and F_2 are also prime in $K[X]$. So F_1 and F_2 are relatively prime in $K[X]$. So 2.0.2 yield $G_1, G_2 \in P$ and $c \in P$ with $(G_1/c)F_1 + (G_2/c)F_2 = 1$. So $c = G_1F_1 + G_2F_2 \in R \cap \mathfrak{p}$.

Hence $R \cap \mathfrak{p} \neq 0$. But $R \cap \mathfrak{p}$ is prime, and R is a PID; so $R \cap \mathfrak{p} = pR$ where p is prime. Also pR is maximal.

Set $k := R/pR$. Then k is a field. Set $\mathfrak{q} := \mathfrak{p}/pR \subset k[X]$. Then $k[X]/\mathfrak{q} = P/\mathfrak{p}$ by 1. But $\mathfrak{p}$ is prime, so $P/\mathfrak{p}$ is a domain. So $k[X]/\mathfrak{q}$ is a domain too. So $\mathfrak{q}$ is prime. So $\mathfrak{q}$ is maximal. So $\mathfrak{p}$ is maximal.

Since $k[X]$ is a PID and $\mathfrak{q}$ is prime, $\mathfrak{q} = \langle G' \rangle$ where G' is prime in $k[X]$. Take $G \in \mathfrak{p}$ with image G' □

Theorem 2.9. *Every proper ideal $\mathfrak{a}$ is contained in some maximal ideal*

Proof. Set $\mathcal{S} := \{\text{ideals } \mathfrak{b} \mid \mathfrak{b} \supset \mathfrak{a} \text{ and } \mathfrak{b} \not\ni 1\}$. Then $\mathfrak{a} \in \mathcal{S}$ and $\mathcal{S}$ is partially ordered by inclusion. By Zorn's Lemma □

Corollary 2.10. *Let R be a ring, $x \in R$. Then x is a unit iff x belongs to no maximal ideal*

Exercise

Exercise 2.0.3. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals, and $\mathfrak{p}$ a prime ideal. Prove that these conditions are equivalent

1. $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$
2. $\mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p}$
3. $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$

Exercise 2.0.4. Let R be a ring, $\mathfrak{p}$ a prime ideal, and $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ maximal ideals. Assume $\mathfrak{m}_1 \ldots \mathfrak{m}_n = 0$. Show $\mathfrak{p} = \mathfrak{m}_i$ for some i

Proof. Note $\mathfrak{p} \supset 0 = \mathfrak{m}_1 \ldots \mathfrak{m}_n$. So $\mathfrak{p} \supset \mathfrak{m}_1$ or $\mathfrak{p} \supset \mathfrak{m}_2 \ldots \mathfrak{m}_n$ by 2.0.3 □

Exercise 2.0.5. Let R be a ring, and $\mathfrak{p}, \mathfrak{a}_1, \ldots, \mathfrak{a}_n$ ideals with $\mathfrak{p}$ prime

1. Assume $\mathfrak{p} \supset \bigcap_{i=1}^n \mathfrak{a}_i$. Show $\mathfrak{p} \supset \mathfrak{a}_j$ for some j
2. Assume $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$. Show $\mathfrak{p} = \mathfrak{a}_j$ for some j

Exercise 2.0.6. Let R be a ring, $\mathcal{S}$ the set of all ideals that consist entirely of zerodivisors. Show that $\mathcal{S}$ has maximal elements and they're prime. Conclude that $\operatorname{z.div}(R)$ is a union of primes.

Proof. Order $\mathcal{S}$ by inclusion. $\mathcal{S}$ is not empty. $\mathcal{S}$ consists of a maximal element $\mathfrak{p}$.

Given $x, x' \in R$ with $xx' \in \mathfrak{p}$, but $x, x' \notin \mathfrak{p}$. Hence $\langle x \rangle + \mathfrak{p}, \langle x' \rangle + \mathfrak{p} \notin \mathcal{S}$. So there are $a, a' \in R$ and $p, p' \in \mathfrak{p}$ s.t. $y := ax + p$ and $y' := a'x' + p'$ are not zerodivisors. Then $yy' \in \mathfrak{p}$. So $yy' \in \operatorname{z.div}(R)$, a contradiction. Thus $\mathfrak{p}$ is prime.

Given $x \in \text{z.div}(R)$, note $\langle x\rangle \in \mathcal{S}$. So $\langle x\rangle$ lies in a maximal element $\mathfrak{p}$ of $\mathcal{S}$. Thus $x \in \mathfrak{p}$ and $\mathfrak{p}$ is prime $\square$

Exercise 2.0.7. Given a prime number p and an integer $n \geq 2$, prove that the residue ring $\mathbb{Z}/\langle p^n\rangle$ does not contain a domain as a subring

Proof. Any subring of $\mathbb{Z}/\langle p^n\rangle$ must contain 1, and 1 generates $\mathbb{Z}/\langle p^n\rangle$ as an Abelian group. So $\mathbb{Z}/\langle p^n\rangle$ contains no proper subrings. $\square$

Exercise 2.0.8. Let $R := R' \times R''$ be a product of two rings. Show that R is a domain if and only if either R' or R'' is a domain and the other 0

Proof. Assume R is a domain. As $(1,0)\cdot(0,1) = (0,0)$, either R' or R'' is 0. $\square$

Exercise 2.0.9. Let $R := R' \times R''$ be a product of rings, $\mathfrak{p} \subset R$ an ideal. Show $\mathfrak{p}$ is prime iff either $\mathfrak{p} = \mathfrak{p}' \times R''$ with $\mathfrak{p}' \subset R'$ prime or $\mathfrak{p} = R' \times \mathfrak{p}''$ with $\mathfrak{p}'' \subset R''$ prime

Proof. $1 \in \mathfrak{p}$. $(1,0)(0,1) \in \mathfrak{p}$. Hence $(1,0) \in \mathfrak{p}$ or $(0,1) \in \mathfrak{p}$. $\square$

Exercise 2.0.10. Let R be a domain, and $x, y \in R$. Assume $\langle x\rangle = \langle y\rangle$. Show $x = uy$ for some unit u

Proof. $(1 - tu)y = 0$ and domain $\square$

Exercise 2.0.11. Let k be a field, R a nonzero ring, $\varphi : k \to R$ a ring map. Prove φ is injective

Proof. Since $1 \neq 0$, $\ker(\varphi) \neq k$. And by 2.5, $\ker(\varphi) = 0$ and hence φ is injective $\square$

Exercise 2.0.12. Let R be a ring, $\mathfrak{p}$ a prime, $\mathcal{X}$ a set of variables. Let $\mathfrak{p}[\mathcal{X}]$ denote the set of polynomials with coefficients in $\mathfrak{p}$. Prove

1. $\mathfrak{p}R[\mathcal{X}]$ and $\mathfrak{p}[\mathcal{X}]$ and $\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle$ are primes of $R[\mathcal{X}]$, which contract to $\mathfrak{p}$
2. Assume $\mathfrak{p}$ is maximal. Then $\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle$ is maximal

Proof. 1. $R/\mathfrak{p}$ is a domain. $\mathfrak{p}R[\mathcal{X}] = \mathfrak{p}[\mathcal{X}]$ by 1.0.3.
$(\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle/\mathfrak{p}R[\mathcal{X}])$ is equal to $\langle\mathcal{X}\rangle \subset (R/\mathfrak{p})[\mathcal{X}]$. $(R/\mathfrak{p})\langle\mathcal{X}\rangle/\langle\mathcal{X}\rangle$ is equal to $R/\mathfrak{p}$. Hence $R[X]/(\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle) = (R[x]/\mathfrak{p}R[X])/((\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle)/\mathfrak{p}R[X]) = R/\mathfrak{p}$
Since the canonical map $R/\mathfrak{p} \to R[\mathcal{X}]/(\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle)$ is bijective, it's injective.

2. $R/\mathfrak{p} \simeq R[\mathcal{X}]/(\mathfrak{p}R[\mathcal{X}] + \langle\mathcal{X}\rangle)$

□

Exercise 2.0.13. Let R be a ring, X a variable, $H \in P := R[X]$ and $a \in R$.Given $n \geq 1$, show $(X-a)^n$ and H are coprime iff $H(a)$ is a unit.

Proof. $(X-a)^n$ and H are coprime iff $X-a$ and H are coprime. $R[x]/\langle X-a\rangle = \langle H\rangle/\langle X-a\rangle$, which implies the residue of H modulo $X-a$ is a unit. Hence $H(a)$ is a unit. □

Exercise 2.0.14. Let R be a ring, X a variable, $F \in P := R[X]$, and $a \in R$. Set $F' := \partial F/\partial X$. Show the following statements are equivalent

1. a is a supersimple root of F
2. a is a root of F, and $X-a$ and F' are coprime
3. $F = (X-a)G$ for some G in P coprime to $X-a$

Show that if (3) holds, then G is unique

Exercise 2.0.15. Let R be a ring, $\mathfrak{p}$ a prime; $\mathcal{X}$ a set of variables; $F, G \in R[\mathcal{X}]$. Let $c(F)$, $c(G)$, $c(FG)$ be the ideals of R generated by the coefficients of F, G, FG

1. Assume $\mathfrak{p}$ doesn't contain either $c(F)$ or $c(G)$. Show $\mathfrak{p}$ doesn't contain $c(FG)$
2. Assume $c(F) = R$ and $c(G) = R$. Show $c(FG) = R$

Proof.
1. Denote the residues of F, G, FG in $(R/\mathfrak{p})[\mathcal{X}]$ by $\bar{F}, \bar{G}$ and $\bar{FG}$. Since $\mathfrak{p} \not\supset c(F), c(G)$, $\bar{F}, \bar{G} \neq 0$. Since $R/\mathfrak{p}$ is a domain, so is $(R/\mathfrak{p})[\mathcal{X}]$ and we have $\bar{F}\bar{G} \neq 0$. Note that $\bar{F}\bar{G} = \bar{FG}$, we have $\bar{FG} \neq 0$.
2. Assume $c(F) = c(G) = R$, since $\mathfrak{p} \not\supset c(F), c(G)$ we have $\mathfrak{p} \not\supset c(FG)$ for any prime ideals $\mathfrak{p}$. Hence $c(FG) = R$.
   If $c(FG) = R$, $c(FG) \subset c(F)$

□

Exercise 2.0.16. Let B be a Boolean ring. Show that every prime $\mathfrak{p}$ is maximal, and that $B/\mathfrak{p} = \mathbb{F}_2$

Proof. $x(x-1) = 0$ in $B/\mathfrak{p}$. Since $B/\mathfrak{p}$ is a domain, $x = 0$ or $x = 1$. □

Exercise 2.0.17. Let R be a ring. Assume that, given any $x \in R$, there is an $n \geq 2$ with $x^n = x$. Show that every prime $\mathfrak{p}$ is maximal

Proof. Same. Every element has an inverse □

Exercise 2.0.18. Prove the following statements or give a counterexample

1. The complement of a multiplicative subset is a prime ideal

2. Given two prime ideals, their intersection is prime
3. Given two prime ideals, their sum is prime
4. Given a ring map $\varphi : R \to R'$, the operation φ^{-1} carries maximal ideals of R' to maximal ideals of R
5. An ideal $\mathfrak{m}' \subset R/\mathfrak{a}$ is maximal iff $\kappa^{-1}\mathfrak{m}' \subset R$ is maximal in 1

Proof. 1. 0 can be belongs to the multiplicative subset
2. False. In $\mathbb{Z}$, $\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle$
3. False. In $\mathbb{Z}$, $\langle 2\rangle + \langle 3\rangle = \mathbb{Z}$
4. False. Consider $\varphi : \mathbb{Z} \to \mathbb{Q}$. $\varphi^{-1}(\langle 0\rangle) = \langle 0\rangle$
5.

□

3 Radicals

Definition 3.1. Let R be a ring. Its (Jacobson) **radical** $\mathrm{rad}(R)$ is defined to be the intersection of all its maximal ideals

Proposition 3.2. *Let R be a ring, $\mathfrak{a}$ an ideal, $x \in R, u \in R^\times$. Then $x \in \mathrm{rad}(R)$ iff $u - xy \in R^\times$ for all $x \in R$. In particular, the sum of an element of $\mathrm{rad}(R)$ and a unit is a unit, and $\mathfrak{a} \subset \mathrm{rad}(R)$ if $1 - \mathfrak{a} \in R^\times$*

Proof. Assume $x \in \mathrm{rad}(R)$. Given a maximal ideal $\mathfrak{m}$, suppose $u - xy \in \mathfrak{m}$. Since $x \in \mathfrak{m}$ too, also $u \in \mathfrak{m}$, a contradiction. Thus $u - xy$ is a unit by 2.10. In particular, tkaing $y := -1$ yields $u + x \in R^\times$

Conversely, assume $x \notin \mathrm{rad}(R)$. Then there is a maximal ideal $\mathfrak{m}$ with $x \notin \mathfrak{m}$. So $\langle x\rangle + \mathfrak{m} = R$. Hence there exists $y \in R$ and $m \in \mathfrak{m}$ s.t. $xy + m = u$. Then $u - xy = m \in \mathfrak{m}$. A contradiction

In particular, given $y \in R$, set $a := u^{-1}xy$. Then $u - xy = u(1 - a) \in R^\times$ if $1 - a \in R^\times$ □

Corollary 3.3. *Let R be a ring, $\mathfrak{a}$ an ideal, $\kappa : R \to R/\mathfrak{a}$ the quotient map. Assume $\mathfrak{a} \subset \mathrm{rad}(R)$. Then $\mathrm{Idem}(\kappa)$ is injective*

Proof. Given $e, e' \in \mathrm{Idem}(R)$ with $\kappa(e) = \kappa(e')$, set $x := e - e'$. Then

$$x^3 = e - e' = x$$

Hence $x(1 - x^2) = 0$. But $\kappa(x) = 0$; so $x \in \mathfrak{a}$. But $\mathfrak{a} \subset \mathrm{rad}(R)$. Hence $1 - x^2$ is a unit by 3.2. Thus $x = 0$. Thus $\mathrm{Idem}(\kappa)$ is injective □

Definition 3.4. A ring is called **local** if it has exactly one maximal ideal, and **semilocal** if it has at least one and at most finitely many

By the **residue field** of a local ring A, we mean the field $A/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of A

Lemma 3.5 (Nonunit Criterion)**.** *Let A be a ring, $\mathfrak{n}$ the set of nonunits. Then A is local iff $\mathfrak{n}$ is an ideal; if so, then $\mathfrak{n}$ is the maximal ideal*

Proof. Assume A is local with maximal ideal $\mathfrak{m}$. Then $A-\mathfrak{n}=A-\mathfrak{m}$ by 2.10. Thus $\mathfrak{n}$ is an ideal □

Example 3.1. The product ring $R' \times R''$ is not local by 3.5 if both R' and R'' are nonzero. $(1,0)$ and $(0,1)$ are nonunits, but their sum is a unit.

Example 3.2. Let R be a ring. A **formal power series** in the n variables $X_1,\dots,X_n$ is a formal *infinite* sum of the form $\sum a_{(i)}X_1^{i_1}\dots X_n^{i_n}$ where $a_{(i)}\in R$ and where $(i) := (i_1,\dots,i_n)$ with each $i_j\ge 0$. The term $a_{(0)}$ where $(0) := (0,\dots,0)$ is called the **constant term**. Addition and multiplication are performed as for polynomials; with these operations, these series form a ring $R[[X_1,\dots,X_n]]$

Set $P := R[[X_1,\dots,X_n]]$ and $\mathfrak{a} := \langle X_1,\dots,X_n\rangle$. Then $\sum a_{(i)}X_1^{i_1}\dots X_n^{i_n}\mapsto a_{(0)}$ is a canonical surjective ring map $P\to R$ with kernel $\mathfrak{a}$; hence $P/\mathfrak{a}=R$

Given an ideal $\mathfrak{m}\subset R$, set $\mathfrak{n} := \mathfrak{a}+\mathfrak{m}P$. Then 1 yields $P/\mathfrak{n}=R/\mathfrak{m}$

A power series F is a unit iff its constant term is a unit. If $a_{(0)}$ is a unit, then $F=a_{(0)}(1-G)$ with $G\in\mathfrak{a}$. Set $F' := a_{(0)}^{-1}(1+G+G^2+\dots)$;

Suppose R is a local ring with maximal ideal $\mathfrak{m}$. Given a power series $F\notin\mathfrak{n}$, its constant term lies outside $\mathfrak{m}$, so is a unit. So F is itself a unit. Hence the nonunits constitutes $\mathfrak{n}$. Thus P is local.

Example 3.3. Let k be a ring, and $A := k[[X]]$ the formal power series ring in one variables. A **formal Laurent series** is a formal sum of the form $\sum_{i=-m}^{\infty} a_iX^i$ with $a_i\in k$ and $m\in\mathbb{Z}$. Plainly, these seires form a ring $k\{\{X\}\}$. Set $K := k\{\{X\}\}$

Set $F := \sum_{i=-m}^{\infty} a_iX^i$. If $a_{-m}\in k^\times$, then $F\in K^\times$; indeed, $F=a_{-m}X^{-m}(1-G)$ where $G\in A$ and

Assume k is a field. If $F\neq 0$, then $F=X^{-m}H$ with $H := a_{-m}(1-G)\in A^\times$. Let $\mathfrak{a}\subset A$ be a nonzero ideal. Suppose $F\in\mathfrak{a}$. Then $X^{-m}\in\mathfrak{a}$. Let n be the smallest integer s.t. $X^n\in\mathfrak{a}$. Then $-m\ge n$. Set $E := X^{-m-n}H$. Then $E\in A$ and $F=X^nE$. Hence $\mathfrak{a}=\langle X^n\rangle$. Thus A **is a** PID

Further, K is a field. In fact, $K=\mathrm{Frac}(A)$.

Let $A[Y]$ be the polynomial ring in one variable, and $\iota : A\hookrightarrow K$ the inclusion. Define $\varphi : A[Y]\to K$ by $\varphi|A=\iota$ and $\varphi(Y)=X^{-1}$. Then φ is surjective. Set $\mathfrak{m} := \ker(\varphi)$. Then $\mathfrak{m}$ is maximal. So by 2.8 $\mathfrak{m}$ has the form $\langle F\rangle$ with F irreducible, or the form $\langle p, G\rangle$ with $p \in A$ irreducible and $G \in A[Y]$. But $\mathfrak{m} \cap A = \langle 0\rangle$ as ι is injective. So $\mathfrak{m} = \langle F\rangle$. But $XY - 1$ belongs to $\mathfrak{m}$, and is clearly irreducible; hence $XY - 1 = FH$ with H a unit. Thus $\langle XY - 1\rangle$ is maximal

In addition, $\langle X, Y\rangle$ is maximal. Indeed, $A[Y]/\langle X, Y\rangle = A/\langle X\rangle = k$. Howevery ,$\langle X, Y\rangle$ is not principal, as no nonunit of $A[Y]$ divides both X and Y. Thus $A[Y]$ *has both principal and nonprincipal maximal ideals, two types allows by 2.8*

Proposition 3.6. *Let R be a ring, S a multiplicative subset, and $\mathfrak{a}$ an ideal with $\mathfrak{a} \cap S = \emptyset$. Set $\mathcal{S} := \{\text{ideals } \mathfrak{b} \mid \mathfrak{b} \supset \mathfrak{a} \text{ and } \mathfrak{b} \cap S = \emptyset\}$. Then $\mathcal{S}$ has a maximal element $\mathfrak{p}$, and every such $\mathfrak{p}$ is prime*

Proof. Take $x, y \in R - \mathfrak{p}$. Then $\mathfrak{p} + \langle x\rangle$ and $\mathfrak{p} + \langle y\rangle$ are strictly larger than $\mathfrak{p}$. So there are $p, q \in \mathfrak{p}$ and $a, b \in R$ with $p + ax, q + by \in S$. Hence $pq + pby + qax + abxy \in S$. But $pq + pby + qax \in \mathfrak{p}$, so $xy \notin \mathfrak{p}$. Thus $\mathfrak{p}$ is prime □

Exercise 3.0.1. Let $\varphi : R \to R'$ be a ring map, $\mathfrak{p}$ an ideal of R. Show

1. there is an ideal $\mathfrak{q}$ of R' with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$ iff $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$
2. if $\mathfrak{p}$ is prime with $\varphi^{-1}(\mathfrak{p}R') = \mathfrak{p}$, then there is a prime $\mathfrak{q}$ of R' with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$

Saturated multiplicative subsets

Let R be a ring, and S a multiplicative subset. We say S is **saturated** if given $x, y \in R$ with $xy \in S$, necessarily $x, y \in S$

Lemma 3.7 (Prime Avoidance). *Let R be a ring, $\mathfrak{a}$ a subset of R that is stable under addition and multiplication, and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ideals s.t. $\mathfrak{p}_3, \dots, \mathfrak{p}_n$ are prime. If $\mathfrak{a} \not\subset \mathfrak{p}_j$ for all j, then there is an $x \in \mathfrak{a}$ s.t. $x \notin \mathfrak{p}_j$ for all j; or equivalently, if $\mathfrak{a} \subset \bigcup_{i=1}^{n} \mathfrak{p}_i$, then $\mathfrak{a} \subset \mathfrak{p}_i$ for some i*

Proof. Assume there is an $x_i \in \mathfrak{a}$ s.t. $x_i \notin \mathfrak{p}_j$ for all $i \neq j$and $x_i \in \mathfrak{p}_i$ for every i. If $n = 2$ then clearly $x_1 + x_2 \notin \mathfrak{p}_j$ for $j = 1, 2$. If $n \geq 3$, then $(x_1 \dots x_{n-1}) + x_n \notin \mathfrak{p}_j$ for all j as, if $j = n$, then $x_n \in \mathfrak{p}_n$ and $\mathfrak{p}_n$ is prime. □

Other radicals

Let R be a ring, $\mathfrak{a}$ a subset. Its **radical** $\sqrt{\mathfrak{a}}$ is the set

$$\sqrt{\mathfrak{a}} := \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \geq 1\}$$

If $\mathfrak{a}$ is an ideal and $\mathfrak{a} = \sqrt{\mathfrak{a}}$, then $\mathfrak{a}$ is said to be **radical**. For example, suppose $\mathfrak{a} = \bigcap \mathfrak{p}_\lambda$ with all $\mathfrak{p}_\lambda$ prime. If $x^n \in \mathfrak{a}$ for some $n \geq 1$, then $x \in \mathfrak{p}_\lambda$. Thus $\mathfrak{a}$ is radical. Hence two radicals coincide

We call $\sqrt{\langle 0 \rangle}$ the **nilradical**, and sometimes denote it by $\operatorname{nil}(R)$. We call an element $x \in R$ **nilpotent** if x belongs to $\sqrt{\langle 0 \rangle}$. We call an ideal $\mathfrak{a}$ **nilpotent** if $\mathfrak{a}^n = 0$ for some $n \geq 1$

$\langle 0 \rangle \subset \operatorname{rad}(R)$. So $\sqrt{\langle 0 \rangle} \subset \sqrt{\operatorname{rad}(R)}$. Thus

$$\operatorname{nil}(R) \subset \operatorname{rad}(R)$$

We call R **reduced** if $\operatorname{nil}(R) = \langle 0 \rangle$

Theorem 3.8 (Scheinnullstellensatz)**.** *Let R be a ring, $\mathfrak{a}$ an ideal. Then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$$

where $\mathfrak{p}$ runs through all the prime ideals containing $\mathfrak{a}$. (By convention, the empty intersection is equal to R)

Proof. Take $x \notin \sqrt{\mathfrak{a}}$. Set $S := \{1, x, x^2, \dots\}$. Then S is multiplicative, and $\mathfrak{a} \cap S = \emptyset$. By 3.6 there is a $\mathfrak{p} \supset \mathfrak{a}$, but $x \notin \mathfrak{p}$, but $x \notin \mathfrak{p}$. So $x \notin \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. Thus $\sqrt{\mathfrak{a}} \supset \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. □

Proposition 3.9. *Let R be a ring, $\mathfrak{a}$ an ideal. Then $\sqrt{\mathfrak{a}}$ is an ideal*

Proof. Assume $x^n, y^m \in \mathfrak{a}$. Then

$$(x+y)^{m+n-1} = \sum_{i+j=m+n-1} \binom{n+m-1}{j} x^i y^j$$

Thus $x + y \in \mathfrak{a}$

Alternatively by 3.8 □

Exercise 3.0.2. Use Zorn's lemma to prove that any prime ideal $\mathfrak{p}$ contains a prime ideal $\mathfrak{q}$ that is minimal containing any given subset $\mathfrak{s} \subset \mathfrak{p}$

Minimal primes

Let R be a ring, $\mathfrak{a}$ an ideal, $\mathfrak{p}$ a prime. We call $\mathfrak{p}$ a **minimal prime** of $\mathfrak{a}$, or over $\mathfrak{a}$, if $\mathfrak{p}$ is minimal in the set of primes containing $\mathfrak{a}$. We call $\mathfrak{p}$ a **minimal prime** of R if $\mathfrak{p}$ is a minimal prime of $\langle 0 \rangle$

Owing to 3.0.2, every prime of R containing $\mathfrak{a}$ contains a minimal prime of $\mathfrak{a}$. So owing to the Scheinnullstellensatz 3.8, the radical $\sqrt{\mathfrak{a}}$ is the intersection of all the minimal primes of $\mathfrak{a}$.

Proposition 3.10. *A ring R is reduced and has only one minimal prime if and only if R is a domain*

Proof. 3 implies $\langle 0\rangle = \mathfrak{q}$ □

Exercise 3.0.3. Let R be a ring, $\mathfrak{a}$ an ideal, X a variable, $R[[X]]$ the formal power series ring, $\mathfrak{M} \subset R[[X]]$ an ideal, $F := \sum a_n X_n \in R[[X]]$. Set $\mathfrak{m} := \mathfrak{M} \cap R$ and $\mathfrak{A} := \{\sum b_n X^n \mid b_n \in \mathfrak{a}\}$. Prove the following statements:

1. If F is a nilpotent, then a_n is nilpotent for all n. The converse is false
2. $F \in \operatorname{rad}(R[[X]])$ iff $a_0 \in \operatorname{rad}(R)$
3. Assume $X \in \mathfrak{M}$. Then X and $\mathfrak{m}$ generate $\mathfrak{M}$
4. Assume $\mathfrak{M}$ is maximal. Then $X \in \mathfrak{M}$ and $\mathfrak{m}$ is maximal
5. If $\mathfrak{a}$ is finitely generated, then $\mathfrak{a}R[[X]] = \mathfrak{A}$. However, there's an example of an R with a prime ideal $\mathfrak{a}$ s.t. $\mathfrak{a}R[[X]] \neq \mathfrak{A}$

Proof.
1. Assume F and a_i for $i < n$ nilpotent. Set $G := \sum_{i \geq n} a_i X^i$. Then $G = F - \sum_{i<n} a_i X^i$. So G is nilpotent by 3.9; say $G^m = 0$ for some $m \geq 1$. Then $a_n^m = 0$
   Set $P := \mathbb{Z}[X_2, X_3, \dots]$. Set $R := P/\langle X_2^2, X_3^3, \dots\rangle$. Let a_n be the residue of X_n. Then $a_n^n = 0$, but $\sum a_n X^n$ is not nilpotent.
2. By 3.2, suppose $G = \sum b_i X^i$

$$F \in \operatorname{rad}(R[[X]]) \iff 1 + FG \in R[[X]]^\times \iff 1 + a_0 b_0 \in R^\times \iff a_0 \in \operatorname{rad}(R)$$

5. Take $R := \mathbb{Z}[a_1, a_2, \dots]$ and $\mathfrak{a} := \langle a_1, \dots\rangle$. Then $R/\mathfrak{a} = \mathbb{Z}$ and $\mathfrak{a}$ is prime. Given $G \in \mathfrak{a}R[[X]]$, say $G = \sum_{i=1}^m b_i G_i$ with $b_i \in \mathfrak{a}$ and $G_i = \sum_{n \geq 0} b_{in} X^n$ and $F \neq G$ for any m

□

Example 3.4. Let R be a ring, $R[[X]]$ the formal power series ring. Then every prime $\mathfrak{p}$ of R is the contraction of a prime of $R[[X]]$. Indeed $\mathfrak{p}R[[X]] \cap R = \mathfrak{p}$. So by 3.0.1 there is a prime $\mathfrak{q}$ of $R[[X]]$ with $\mathfrak{q} \cap R = \mathfrak{p}$. In fact ,a specific choice for $\mathfrak{q}$ is the set of series $\sum a_n X^n$ with $a_n \in \mathfrak{q}$. Indeed, the canonical map $R \to R/\mathfrak{p}$ induces a surjection $R[[X]] \to (R/\mathfrak{p})[[X]]$ with kernel $\mathfrak{q}$; so $R[[X]]/\mathfrak{q} = (R/\mathfrak{p})[[X]]$. But 3.0.3 shows $\mathfrak{q}$ may not be equal to $\mathfrak{p}R[[X]]$

Exercise

Exercise 3.0.4. Let R be a ring, $\mathfrak{a} \subset \operatorname{rad}(R)$ an ideal, $w \in R$ and $w' \in R/\mathfrak{a}$ its residue. Prove that $w \in R^\times$ iff $w' \in (R/\mathfrak{a})^\times$. What if $\mathfrak{a} \not\subset \operatorname{rad}(R)$?

Proof. Assume $\mathfrak{a} \subset \text{rad}(R)$. $\mathfrak{m} \mapsto \mathfrak{m}/\mathfrak{a}$ is a bijection for maximal ideal $\mathfrak{m}$. So w belongs to a maximal ideal of R iff w' belongs to one of $R/\mathfrak{a}$

Assume $\mathfrak{a} \not\subset \text{rad}(R)$, then there is a maximal ideal $\mathfrak{m}$ s.t. $\mathfrak{a} \not\subset \mathfrak{m}$. So $\mathfrak{a} + \mathfrak{m} = R$. So there are $a \in \mathfrak{a}$ and $v \in \mathfrak{m}$ s.t. $a + v = w$. Then $v \notin R^\times$ but the residue of v is w', even if $w' \in (R/\mathfrak{a})^\times$. For example, take $R := \mathbb{Z}$ and $\mathfrak{a} = \langle 2 \rangle$ and $w := 3$. Then $w \notin R^\times$ but the residue of w is $1 \in (R/\mathfrak{a})^\times$ □

Exercise 3.0.5. Let A be a local ring, e an idempotent. Show $e = 1$ or $e = 0$

Proof. $1 - e + e = 1$. Since $1 \notin \mathfrak{m}$, at least one of $1 - e$ and e doesn't belong to $\mathfrak{m}$ □

Exercise 3.0.6. Let A be a ring, $\mathfrak{m}$ a maximal ideal s.t. $1 + m$ is a unit for every $m \in \mathfrak{m}$. Prove A is local. Is this assertion still true if $\mathfrak{m}$ is not maximal?

Proof. Let $y \in A - \mathfrak{m}$. Then $\langle y \rangle + \mathfrak{m} = A$ and there is a $x \in A$ s.t. $xy + m = 1$. Hence xy is a unit and $\langle xy \rangle = \langle y \rangle$. y is a unit. □

Exercise 3.0.7. Let R be a ring, and S a subset. Show that S is saturated multiplicative iff $R - S$ is a union of primes.

Proof. Assume S is saturated multiplicative. Take $x \in R - S$. Then $xy \notin S$ for all $y \in R$; in other words, $\langle x \rangle \cap S = \emptyset$. Then 3.6 gives a prime $\mathfrak{p} \supset \langle x \rangle$ with $\mathfrak{p} \cap S = \emptyset$. Thus $R - S$ is a union of primes. □

Exercise 3.0.8. Let R be a ring, and S a multiplicative subset. Define its **saturation** to be the subset

$$\bar{S} := \{x \in R \mid \text{there is } y \in R \text{ with } xy \in S\}$$

1. Show that $\bar{S} \supset S$ and that $\bar{S}$ is saturated multiplicative and that any saturated multiplicative subset T containing S also contains $\bar{S}$
2. Set $U := \bigcup_{\mathfrak{p} \cap S = \emptyset} \mathfrak{p}$. Show that $R - \bar{S} = U$
3. Let $\mathfrak{a}$ an ideal; assume $S = 1 + \mathfrak{a}$; set $W := \bigcup_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$. Show $R - \bar{S} = W$
4. Given $f, g \in R$, show that $\bar{S_f} \subset \bar{S_g}$ iff $\sqrt{\langle f \rangle} \supset \sqrt{\langle g \rangle}$, where $S_f = \{f^n \mid n \geq 0\}$

Proof. 3. First take a prime $\mathfrak{p}$ with $\mathfrak{p} \cap S = \emptyset$. Then $1 \notin \mathfrak{p} + \mathfrak{a}$; else, $1 = p + a$ and $p = 1 - a \in \mathfrak{p} \cap S$. So $\mathfrak{p} + \mathfrak{a}$ lies in a maximal ideal $\mathfrak{m}$. Then $\mathfrak{a} \subset \mathfrak{m}$; so $\mathfrak{m} \subset W$. But also $\mathfrak{p} \subset W$. So $U \subset W$

Conversely, take $\mathfrak{p} \supset \mathfrak{a}$. Then $1 + \mathfrak{p} \supset 1 + \mathfrak{a} = S$. But $\mathfrak{p} \cap (1 + \mathfrak{p}) = \emptyset$. So $\mathfrak{p} \cap S = \emptyset$. Thus $U \subset W$. Thus $U = W$. Thus 2 implies (3)

4. $\overline{S_f} \subset \overline{S_g}$ iff $f \in \overline{S_g}$ iff $hf = g^n$ iff $g \in \sqrt{\langle f \rangle}$ iff $\sqrt{\langle g \rangle} \subset \sqrt{\langle f \rangle}$

□

Exercise 3.0.9. Let R be a nonzero ring, S a subset. Show S is maximal in the $\mathfrak{S}$ of multiplicative subsets T of R with $0 \notin T$ iff $R - S$ is a minimal prime

Proof. First assume S is maximal. Then $S = \overline{S}$. So $R - S$ is a union of primes $\mathfrak{p}$. Fix a $\mathfrak{p}$. Then 3.0.2 yields in $\mathfrak{p}$ a minimal prime ideal $\mathfrak{q}$. Then $S \subset R - \mathfrak{q}$. But $R - \mathfrak{q} \in \mathfrak{S}$. $S = R - \mathfrak{q}$

If $R - S$ is a minimal prime. Then $S \in \mathfrak{S}$. Given $T \in \mathfrak{S}$ with $S \subset T$, note $R - \overline{T} = \bigcup \mathfrak{p}$ with $\mathfrak{p}$ prime. Fix a $\mathfrak{p}$, then $S \subset T \subset \overline{T}$. So $\mathfrak{q} \supset \mathfrak{p}$. But $\mathfrak{q}$ is minimal and hence $\mathfrak{q} = \mathfrak{p}$. Hence $\mathfrak{q} = R - \overline{T}$. So $S = \overline{T}$ □

Exercise 3.0.10. Let k be a field, X_λ for $\lambda \in \Lambda$ variables, and Λ_π for $\pi \in \Pi$ disjoint subsets of Λ. Set $P := k[\{X_\lambda\}_{\lambda\in\Lambda}]$ and $\mathfrak{p}_\pi := \langle \{X_\lambda\}_{\lambda\in\Lambda_\pi} \rangle$for all $\pi \in \Pi$. Let $F, G \in P$ be nonzero, and $\mathfrak{a} \subset P$ a nonzero ideal. Set $U := \bigcup_{\pi\in\Pi} \mathfrak{p}_\pi$. Show

1. Assume $F \in \mathfrak{p}_\pi$ for some $\pi \in \Pi$, then every monomial of F is in $\mathfrak{p}_\pi$
2. Assume there are $\pi, \rho \in \Pi$ s.t. $F + G \in \mathfrak{p}_\pi$ and $G \in \mathfrak{p}_\rho$ but $\mathfrak{p}_\rho$ contains no monomial of F. Then $\mathfrak{p}_\pi$ contains every monomial of F and of G
3. Assume $\mathfrak{a} \subset U$. Then $\mathfrak{a} \subset \mathfrak{p}_\pi$ for some $\pi \in \Pi$

4 Modules

Modules

Let R be a ring. Recall that an R**-module** M is an abelian group, written additively, with a **scalar multiplication**, $R \times M \to M$, written $(x, m) \mapsto xm$, which is

1. **distributive**, $x(m + n) = xm + xn$ and $(x + y)m = xm + xn$
2. **associative**, $x(ym) = (xy)m$
3. **unitary**, $1 \cdot m = m$

For example, if R is a field, then an R-module is a vector space. A $\mathbb{Z}$-module is just an abelian group

A **submodule** N of M is a subgroup that is closed under multiplication.; that is, $xn \in N$ for all $x \in R$ and $n \in N$. For example, the ring R is itself an R-module, and the submodules are just the ideals. Given an ideal $\mathfrak{a}$, let $\mathfrak{a}N$ denote the smallest submodule containing all products an with $a \in \mathfrak{a}$ and $n \in N$. $\mathfrak{a}N$ is equal to the set of finite sums $\sum a_i n_i$.

Given $m \in M$, we call the set of $x \in R$ with $xm = 0$ the **annihilator** of m, and denote it $\text{Ann}(m)$. We call the set of $x \in R$ with $xm = 0$ for all $m \in M$ the **annihilator** of M, and denote it $\text{Ann}(M)$

Homomorphisms

Let R be a ring, M and N modules. A **homomorphism**, or **module map** is a map $\alpha : M \to N$ that is R**-linear**:

$$\alpha(xm + yn) = x(\alpha m) + y(\alpha n)$$

Note that f is injective iff it has a left inverse. f is surjective iff it has a right inverse

A homomorphism α is an isomorphism iff there is a set map $\beta : N \to M$ s.t. $\beta\alpha = 1_M$ and $\alpha\beta = 1_N$, and then $\beta = \alpha^{-1}$.

The set of homomorphisms α is denoted by $\text{Hom}_R(M, N)$ or simply $\text{Hom}(M, N)$. It is an R-module with addition and scalar multiplication defined by

$$(\alpha + \beta)m := \alpha m + \beta m \quad \text{and} \quad (x\alpha)m := x(\alpha m) = \alpha(xm)$$

Homomorphisms $\alpha : L \to M$ and $\beta : N \to P$ induce, via composition, a map

$$\text{Hom}(\alpha, \beta) : \text{Hom}(M, N) \to \text{Hom}(L, P)$$

When α is the identity map 1_M, we write $\text{Hom}(M, \beta)$ for $\text{Hom}(1_M, \beta)$

Exercise 4.0.1. Let R be a ring, M a module. Consider the map

$$\theta : \text{Hom}(R, M) \to M \quad \text{defined by} \quad \theta(\rho) := \rho(1)$$

Show that θ is an isomorphism, and describe its inverse

Proof. First, θ is R-linear. Set $H := \text{Hom}(R, M)$. Define $\eta : M \to H$ by $\eta(m)(x) := xm$. It is easy to check that $\eta\theta = 1_H$ and $\theta\eta = 1_M$. Thus θ and η are inverse isomorphism □

Endomorphisms

Let R be a ring, M a module. An **endomorphism** of M is a homomorphism $\alpha : M \to M$. The module of endomorphism $\text{Hom}(M, M)$ is also denoted $\text{End}_R(M)$. Further, $\text{End}_R(M)$ is a subring of $\text{End}_{\mathbb{Z}}(M)$

Given $x \in R$, let $\mu_x : M \to M$ denote the map of **multiplication** by x, defined by $\mu_x(m) := xm$. It is an endomorphism. Further, $x \mapsto \mu_x$ is a ring map

$$\mu_R : R \to \mathrm{End}_R(M) \subset \mathrm{End}_{\mathbb{Z}}(M)$$

(Thus we may view μ_R as representing R as a ring of operators on the abelian gorup). Note that $\ker(\mu_R) = \mathrm{Ann}(M)$

Conversely, given an abelian group N and a ring map

$$\nu : R \to \mathrm{End}_{\mathbb{Z}}(N)$$

we obtain a module structure on N by setting $xn := (\nu x)(n)$. Then $\mu_R = \nu$

We call M **faithful** if $\mu_R : R \to \mathrm{End}_R(M)$ is injective, or $\mathrm{Ann}(M) = 0$. For example, R is a faithful R-module for $x \cdot 1 = 0$ implies

Algebras

Fix two rings R and R'. Suppose R' is an R-algebra with structure map φ. Let M' be an R'-module. Then M' is also an R-module by **restriction on scalars**: $xm := \varphi(x)m$. In other words, the R-module structure on M' corresponds to the composition

$$R \xrightarrow{\varphi} R' \xrightarrow{\mu_{R'}} \mathrm{End}_{\mathbb{Z}}(M')$$

In particular, R' is an R-module; further, for all $x \in R$ and $y, z \in R'$

$$(xy)z = x(yz)$$

by restriction on scalars

Conversely, suppose R' is an R-module s.t. $(xy)z = x(yz)$. Then R' has an R-algebra structure that is compatible with the given R-module structure.. Indeed, define $\varphi : R \to R'$ by $\varphi(x) := x \cdot 1$. Then $\varphi(x)z = xz$ as $(x \cdot 1)z = x(1 \cdot z)$. So the composition $\mu_{R'}\varphi : R \to R' \to \mathrm{End}_{\mathbb{Z}}(R')$ is equal to μ_R. Hence φ is a ring map. Thus R' is an R-algebra, and restriction of scalars recovers its given R-module structure

Suppose that $R' = R/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Then an R-module M has a compatible R'-module structure iff $\mathfrak{a}M = 0$; if so, then the R'-structure is unique. Indeed, the ring map $\mu_R : R \to \mathrm{End}_{\mathbb{Z}}(M)$ factors through R' iff $\mu_R(\mathfrak{a}) = 0$, so iff $\mathfrak{a}M = 0$

Again suppose R' is an arbitrary R-algebra with structure map φ. A **subalgebra** R'' of R' is a subring s.t. φ maps into R''. The subalgebra **generated** by $x_1, \ldots, x_n \in R'$ is the smallest R-subalgebra that contains them. We denote it by $R[x_1, \ldots, x_n]$.

We say R' is a **finitely generated R-subalgebra** or is **algrbra finite over R** if there exist $x_1, \dots, x_n \in R'$ s.t. $R' = R[x_1, \dots, x_n]$

Residue modules

Let R be a ring, M a module, $M' \subset M$ a submodule. Form the set of cosets

$$M/M' := \{m + M' \mid m \in M\}$$

M/M' inherits a module structure, and is called the **residue module** or **quotient of M modulo M'**. Form the **quotient map**

$$\kappa : M \to M/M' \quad \text{by} \quad \kappa(m) := m + M'$$

Clearly κ is surjective, κ is linear, and κ has kernel M'

Let $\alpha : M \to N$ be linear. Note that $\ker(\alpha') \supset M'$ iff $\alpha(M') = 0$

If $\ker(\alpha) \supset M'$, then there exists a homomorphism $\beta : M/M' \to N$ s.t. $\beta\kappa = \alpha$

$$\begin{tikzcd} M \arrow[r, "\kappa"] \arrow[dr, "\alpha"'] & M/M' \arrow[d, "\beta"] \\ & N \end{tikzcd}$$

Always

$$M/\ker(\alpha) \xrightarrow{\sim} \operatorname{im}(\alpha)$$

M/M' has the following UMP: $\kappa(M') = 0$, and given $\alpha : M \to N$ s.t. $\alpha(M') = 0$, there is a unique homomorphism $\beta : M/M' \to N$ s.t. $\beta\kappa = \alpha$

Cyclic modules

Let R be a ring. A module M is said to be **cyclic** if there exists $m \in M$ s.t. $M = Rm$. If so, form $\alpha : R \to M$ by $x \mapsto xm$; then α induces an isomorphism $R/\operatorname{Ann}(m) \xrightarrow{\sim} M$. Note that $\operatorname{Ann}(m) = \operatorname{Ann}(M)$. Conversely, given any ideal $\mathfrak{a}$, the R-module $R/\mathfrak{a}$ is cyclic, generated by the coset of 1, and $\operatorname{Ann}(R/\mathfrak{a}) = \mathfrak{a}$

Noether Isomorphisms

Let R be a ring, N a module, and L and M submodules.

First, assume $L \subset M \subset N$. Form the following composition of quotient maps:

$$\alpha : N \to N/L \to (N/L)/(M/L)$$

α is surjective and $\ker(\alpha) = M$. Hence

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N/M \arrow[d, "\beta", "\simeq"'] \\ N/L \arrow[r] & (N/L)/(M/L) \end{tikzcd}$$

Second, let $L + M$ denote the set of all sums $l + m$ with $l \in L$ and $m \in M$. Clearly $L + M$ is a submodule of N. It is called the **sum** of L and M

Form the composition α' of the inclusion map $L \to L+M$ and the quotient map $L+M \to (L+M)/M$. Clearly α' is surjective and $\ker(\alpha') = L \cap M$. Hence

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L/(L \cap M) \arrow[d, "\beta'", "\simeq"'] \\ L + M \arrow[r] & (L+M)/M \end{tikzcd}$$

Cokernels, coimages

Let R be a ring, $\alpha : M \to N$ a linear map. Associated to α are its **cokernel** and its **coimage**

$$\operatorname{coker}(\alpha) := N/\operatorname{im}(\alpha) \quad \text{and} \quad \operatorname{coim}(\alpha) := M/\ker(\alpha)$$

they are quotient modules, and their quotient maps are both denoted by κ.

UMP of the cokernel: $\kappa\alpha = 0$ and given a map $\beta : N \to P$ with $\beta : N \to P$ with $\beta\alpha = 0$, there is a unique map $\gamma : \operatorname{coker}(\alpha) \to P$ with $\gamma\kappa = \beta$

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr] & N \arrow[r, "\kappa"] \arrow[d, "\beta"] & \operatorname{coker}(\alpha) \arrow[dl, "\gamma"] \\ & P & \end{tikzcd}$$

Further, $\operatorname{coim}(\alpha) \xrightarrow{\sim} \operatorname{im}(\alpha)$

Free modules

Let R be a ring, Λ a set, M a module. Given elements $m_\lambda \in M$ for $\lambda \in \Lambda$, by the submodule they **generate**, we mean the smallest submodule that contains then all. Clearly, any submodule that contains them all contains any (finite) linear combination $\sum x_\lambda m_\lambda$ with $x_\lambda \in R$

m_λ are said to be **free** or **linearly independent** if whenever $\sum x_\lambda m_\lambda = 0$, also $x_\lambda = 0$ for all λ. Finally, the m_λ are said to form a **free basis** of M if they are free and generate M; if so, then we say M is **free** on the m_λ

We say M is **free** if it has a free basis. Any two free bases have the same number l of elements, and we say M is **free of rank** l

For example, form the set of **restricted vectors**

$$R^{\oplus\Lambda} := \{(x_\lambda) \mid x_\lambda \in R \text{ with } x_\lambda = 0 \text{ for almost all } \lambda\}$$

It's a module under componentwise addition and scalar multiplication. It has a **standard basis**, which consists of the vectors e_μ whose λth component is the value of the **Kronecker delta function**

If Λ has a finite number l of elements, then $R^{\oplus\Lambda}$ is often written R^l and called the **direct sum of** l **copies** of R

The free module $R^{\oplus\Lambda}$ has the following UMP: given a module M and elements $m_\lambda \in M$ for $\lambda \in \Lambda$, there is a unique homomorphism

$$\alpha : R^{\oplus\Lambda} \to M \text{ with } \alpha(e_\lambda) = m_\lambda \text{ for each } \lambda \in \Lambda$$

namely, $\alpha((x_\lambda)) = \alpha(\sum x_\lambda e_\lambda) = \sum x_\lambda m_\lambda$. Note the following obvious statements:

1. α is surjective iff m_λ generate M
2. α is injective iff m_λ are linearly independent
3. α is an isomorphism iff m_λ for a free basis

Thus M is free of rank l iff $M \simeq R^l$

Exercise 4.0.2. Take $R := \mathbb{Z}$ and $M := \mathbb{Q}$. Then any two $x, y \in M$ are not free. Aso M is not finitely generated. Indeed, given any $m_1/n_1, \dots, m_r/n_r \in M$, let d be a common multiple of $n_1, \dots, n_r$. Then $(1/d)\mathbb{Z}$ contains every linear combination but $(1/d)\mathbb{Z} \neq \mathbb{Q}$

Exercise 4.0.3. Let R be a domain, and $x \in R$ nonzero. Let M be the submodule of $\mathrm{Frac}(R)$ generated by $1, x^{-1}, x^{-2}, \dots$. Suppose that M is finitely generated. Prove that $x^{-1} \in R$ and conclude that $M = R$

Proof. Suppose M is generated by $m_1, \dots, m_k$. Say $m_i = \sum_{j=0}^{n_i} a_{ij} x^{-j}$ for some n_i and $a_{ij} \in R$. Set $n := \max\{n_i\}$. Then $1, x^{-1}, \dots, x^{-n}$ generate M. So

$$x^{-n+1} = a_n x^{-n} + \cdots + a_0$$

Thus

$$x^{-1} = a_n + \cdots + a_0 x^n$$

$\square$

Direct Products, Direct Sums

Let R be a ring, Γ a set, M_λ a module for $\lambda \in \Lambda$. The **direct product** of the M_λ is the set of arbitrary vectors:

$$\prod M_\lambda := \{(m_\lambda) \mid m_\lambda \in M_\lambda\}$$

The **direct sum** of the M_λ is the subset of **restricted vectors**:

$$\bigoplus M_\lambda := \{(m_\lambda) \mid m_\lambda = 0 \text{ for almost all } \lambda\} \subset \prod M_\lambda$$

The direct product comes equipped with projections

$$\pi_\kappa : \prod M_\lambda \to M_\kappa \quad \text{given by} \quad \pi_\kappa((m_\lambda)) := m_\kappa$$

$\prod M_\lambda$ has UMP: given homomorphisms $\alpha_\kappa : N \to M_\kappa$, there is a unique homomorphism $\alpha : N \to \prod M_\lambda$ satisfying $\pi_\kappa \alpha = \alpha_\kappa$ for all $\kappa \in \Lambda$; namely $\alpha(n) = (\alpha_\lambda(n))$. Often α is denoted (α_λ). In other words, the π_λ induce a bijection of sets

$$\mathrm{Hom}(N, \prod M_\lambda) \xrightarrow{\sim} \prod \mathrm{Hom}(N, M_\lambda)$$

Similarly, the direct sum comes equipped with injections

$$\iota_\kappa : M_\kappa \to \bigoplus M_\lambda \quad \text{given by} \quad \iota_\kappa(m) := (m_\lambda) \text{ where } m_\lambda := \begin{cases} m & \lambda = \kappa \\ 0 \end{cases}$$

UMP: given homomorphisms $\beta_\kappa : M_\kappa \to N$, there is a unique homomorphism $\beta : \bigoplus M_\lambda \to N$ satisfying $\beta\iota_\kappa = \beta_\kappa$ for all $\kappa \in \Lambda$ for all $\kappa \in \Lambda$; namely, $\beta((m_\lambda)) = \sum \beta_\lambda(m_\lambda)$. Often β is denoted $\sum \beta_\lambda$; often (β_λ). In other words, the ι_κ induce this bijection of sets:

$$\mathrm{Hom}(\bigoplus M_\lambda, N) \xrightarrow{\sim} \prod \mathrm{Hom}(M_\lambda, N) \tag{4.0.1}$$

For example, if $M_\lambda = R$ for all λ, then $\bigoplus M_\lambda = R^{\oplus\Lambda}$. Further, if $N_\lambda := N$ for all λ, then $\mathrm{Hom}(R^{\oplus\Lambda}, N) = \prod N_\lambda$ by (4.0.1) and 4.0.1

Exercise 4.0.4. Let Λ be an infinite set, R_λ a ring for $\lambda \in \Lambda$. Endow $\prod R_\lambda$ and $\bigoplus R_\lambda$ with componentwise addition and multiplication. Show that $\prod R_\lambda$ has a multiplicative identity (so is a ring), but $\bigoplus R_\lambda$ does not (so is not a ring)

Exercise 4.0.5. Let L, M, N be modules. Consider a diagram

$$L \underset{\rho}{\overset{\alpha}{\rightleftarrows}} M \underset{\sigma}{\overset{\beta}{\rightleftarrows}} N$$

where α, β, ρ and σ are homomorphisms. Prove that

$$M = L \oplus N \quad \text{and} \quad \alpha = \iota_L, \beta = \pi_N, \sigma = \iota_N, \rho = \pi_L$$

iff the following relations holds

$$\beta\alpha = 0, \beta\sigma = 1, \rho\sigma = 0, \rho\alpha = 1, \alpha\rho + \sigma\beta = 1$$

Proof. Consider the map $\varphi : M \to L \oplus N$ and $\theta : L \oplus N \to M$ given by $\varphi m := (\rho m, \rho m)$ and $\theta(l, n) := \alpha l + \sigma n$. They are inverse isomorphism since

$$\varphi\theta(l, n) = (\rho\alpha l + \rho\sigma n, \beta\alpha l + \beta\sigma n) = (l, n) \quad \text{and} \quad \theta\varphi m = \alpha\rho m + \sigma\beta m = m$$

□

Exercise 4.0.6. Let N be a module, Λ a nonempty set, M_λ a module for $\lambda \in \Lambda$. Prove that the injections $\iota_\kappa : M_\kappa \to \bigoplus M_\lambda$ induce an injection

$$\bigoplus \operatorname{Hom}(N, M_\lambda) \hookrightarrow \operatorname{Hom}(N, \bigoplus M_\lambda)$$

and that it is an isomorphism if N is finitely generated

Proof. For $(\beta_\kappa) \in \bigoplus \operatorname{Hom}(N, M_\lambda)$

$$\beta(n) = \begin{cases} \iota_\kappa\beta_\kappa & \text{if } \beta_\kappa \neq 0 \\ 0 & \beta_\kappa = 0 \end{cases} \in \operatorname{Hom}(N, \bigoplus M_\lambda)$$

If N is fintitely generated, suppose $a_1, \dots, a_n$ generates N and $\beta(a_i) = b_i \in \bigoplus M_\lambda$, which means $\beta(N)$ is a finite direct subsum of $\bigoplus M_\lambda$. then we have $\beta_\kappa = \pi_\kappa\beta$ and almost □

Exercise 4.0.7. Let $\mathfrak{a}$ be an ideal, Λ a nonempty set, M_λ a module for $\lambda \in \Lambda$. Prove $\mathfrak{a}(\bigoplus M_\lambda) = \bigoplus \mathfrak{a}M_\lambda$. Prove $\mathfrak{a}(\prod M_\lambda) = \prod \mathfrak{a}M_\lambda$ if $\mathfrak{a}$ is finitely generated

5 Exact Sequence

Definition 5.1. A (finite or infinite) sequence of module homomorphisms

$$\cdots \to M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \to \cdots$$

is said to be **exact at** M_i if $\ker(\alpha_i) = \operatorname{im}(\alpha_{i-1})$.. The sequence is said to be **exact** if it is exact at every M_i, except an initial source or final target

Example 5.1. 1. A sequence $0 \to L \xrightarrow{\alpha} M$ is exact iff α is injective. If so, then we often identify L with its image $\alpha(L)$

Dually - a sequence $M \xrightarrow{\beta} N \to 0$ is exact iff β is surjective

2. A sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact iff $L = \ker(\beta)$, where '=' means "canocially isomorphic". Dually, a sequence $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is exact iff $N = \operatorname{coker}(\alpha)$

Short exact sequences

A sequence $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is exact iff α is injective and $N = \operatorname{coker}(\alpha)$, or dually, iff β is surjective and $L = \ker(\beta)$. If so, then the sequence is called **short exact**, and often we regard L as a submodule of M, and N as the quotient M/L

For example, the following sequence is shor t exact

$$0 \to L \xrightarrow{\iota_L} L \oplus N \xrightarrow{\pi_N} N \to 0$$

Proposition 5.2. *For $\lambda \in \Lambda$, let $M'_\lambda \to M_\lambda \to M''_\lambda$ be a sequence of module homomorphisms. If every sequence is exact, then so are the two induced sequences*

$$\bigoplus M'_\lambda \to \bigoplus M_\lambda \to \bigoplus M''_\lambda \quad \textit{and} \quad \prod M'_\lambda \to \prod M_\lambda \to \prod M''_\lambda$$

Conversely, if either induced sequence is exact then so is every original one

Exercise 5.0.1. Let M' and M'' be modules, $N \subset M'$ a submodule. Set $M := M' \oplus M''$. Prove $M/N = M'/N \oplus M''$

Proof. $N = N \oplus 0$

The two sequence $0 \to M'' \to M'' \to 0$ and $0 \to N \to M' \to M'/N \to 0$ are exact. So by 5.2, the sequence

$$0 \to N \to M' \oplus M'' \to (M'/N) \oplus M'' \to 0$$

is exact □

Exercise 5.0.2. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence. Prove the if M' and M'' are finitely generated, then so is M

Lemma 5.3. *Let $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ be a short exact sequence, and $N \subset M$ a submodule. Set $N' := \alpha^{-1}$ and $N'' := \beta(N)$. Then the induced sequence $0 \to N' \to N \to N'' \to 0$ is short exact*

Definition 5.4. We say that a short exact sequence

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

splits if there is an isomorphism $\varphi : M \xrightarrow{\sim} M' \oplus M''$ with $\varphi\alpha = \iota_{M'}$ and $\beta = \pi_{M''}\varphi$

We call a homomorphism $\rho : M \to M'$ a **retraction** of α if $\rho\alpha = 1_{M'}$

Dually, we call a homomorphism $\sigma : M'' \to M$ a **section** of β if $\beta\sigma = 1_{M''}$

Proposition 5.5. *Let* $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ *be a short exact sequence. Then the following conditions are equivalent*

1. *The sequence splits*
2. *There exists a retraction*
3. *There exists a section*

Proof. Assume (2). Set $\sigma' := 1_M - \alpha\rho$. Then $\sigma'\alpha = 0$. So there exists $\sigma : M'' \to M$ with $\sigma\beta = \sigma'$ by 5.1 and UMP. So $1_M = \alpha\rho + \sigma\beta$. Since $\beta\sigma\beta = \beta$ and β is surjective, $\beta\sigma = 1_{M''}$. Hence $\alpha\rho\sigma = 0$. Since α is injective, $\rho\sigma = 0$. Thus 4.0.5 yields (1) and also (3) □

Exercise 5.0.3. Let M', M'' be modules, and set $M := M' \oplus M''$. Let N be a submodule of M containing M', and set $N'' := N \cap M''$. Prove $N = M' \oplus N''$

Proof. Form the sequence $0 \to M' \to N \to \pi_{M''}N \to 0$. It splits by 5.5 as $(\pi_{M'}|N) \circ \iota_{M'} = 1_{M'}$. Finally if $(m', m'') \in N$, then $(0, m'') \in N$ as $M' \subset N$; hence $\pi_{M''}N = N''$ □

Exercise 5.0.4. Criticize the following misstatement of 5.5: given a short exactg sequence $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$, there is an isomorphism $M \simeq M' \oplus M''$ iff there is a section $\sigma : M'' \to M$ of β

Proof. We have $\alpha : M' \to M$ and $\iota_{M'} : M' \oplus M''$, but 5.5 requires that they be compatible with the isomorphism $M \simeq M' \oplus M''$.

Let's construct a counterexample. For each integer $n \geq 2$, let M_n be the direct sum of countably many copies of $\mathbb{Z}/\langle n\rangle$. Set $M := \bigoplus M_n$

First let us check these two statements:

1. For any finite abelian group G, we have $G \oplus M \simeq M$
2. For any finite abelian subgroup $G \subset M$, we have $M/G \simeq M$

Statement (1) holds since G is isomorphic to a direct sum of copies of $\mathbb{Z}/\langle n\rangle$

To prove (2), write $M = B \oplus M'$, where B contains G and involes only finitely many components of M. Then $M' \simeq M$. Therefore, 5.0.3 yields

$$M/G \simeq (B/G) \oplus M' \simeq M$$

To construct the counterexample, let p be a prime number. Take one of the $\mathbb{Z}/\langle p^2\rangle$ components of M, and let $M' \subset \mathbb{Z}/\langle p^2\rangle$ be the cyclic subgroup of order p. There is no retraction $\mathbb{Z}/\langle p^2\rangle \to M'$, so there is no traction $M \to M'$ either, since the latter would induce the former. Finally take $M'' := M/M'$. Then (1) and (2) yield $M \simeq M' \oplus M''$ $\square$

Lemma 5.6 (Snake). *Consider this commutative diagram with exact rows:*

$$\begin{tikzcd}
 & M' \arrow[r, "\alpha"] \arrow[d, "\gamma'"] & M \arrow[r, "\beta"] \arrow[d, "\gamma"] & M'' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\alpha'"] & N \arrow[r, "\beta'"] & N'' &
\end{tikzcd}$$

It yields the following exact sequence

$$\ker(\gamma') \xrightarrow{\varphi} \ker(\gamma) \xrightarrow{\psi} \ker(\gamma'') \xrightarrow{\partial} \operatorname{coker}(\gamma') \xrightarrow{\varphi'} \operatorname{coker}(\gamma) \xrightarrow{\psi'} \operatorname{coker}(\gamma'')$$

Moreover, if α is injective, then so is φ; dually, if β' is surjective, then so is ψ'

Proof. Clearly, α yields a unique compatible homomorphism $\ker(\gamma') \to \ker(\gamma)$ since $\gamma\alpha(\ker(\gamma')) = 0$. By the UMP in 4, α' yields a unique compatible homomorphism φ' because M' goes to 0 in $\operatorname{coker}(\gamma)$.

$$\begin{tikzcd}
M' \arrow[r, "\gamma'"] & N' \arrow[r] \arrow[dl, "\alpha'"] \arrow[d] & \operatorname{coker}(\gamma') \arrow[dl, "\varphi''"] \\
N \arrow[r] & \operatorname{coker}(\gamma) &
\end{tikzcd}$$

Similarly, β and β' induce corresponding homomorphisms ψ and ψ'

To define ∂, **chase** an $m'' \in \ker(\gamma'')$ through the diagram. Since β is surjective, there is $m \in M$ s.t. $\beta(m) = m''$. By commutativity, $\gamma''\beta(m) = \beta'\gamma(m)$. So $\beta'\gamma(m) = 0$. By exactness of the bottom row, there is a unique $n' \in N'$ s.t. $\alpha'(n') = \gamma(m)$. Define $\partial(m'')$ to be the image of n' in $\operatorname{coker}(\gamma')$

To see ∂ is well defined, choose another $m_1 \in M$ with $\beta(m_1) = m''$. Let $n'_1 \in N'$ be the unique element with $\alpha'(n'_1) = \gamma(m_1)$. Since $\beta(m - m_1) = 0$, there is an $m' \in M'$ with $\alpha(m') = m - m_1$. But $\alpha'\gamma' = \gamma\alpha$. So $\alpha'\gamma'(m') = \alpha'(n' - n'_1)$. Hence $\gamma'(m') = n' - n'_1$ since α' is injective. So n' and n'_1 have the same image in $\operatorname{coker}(\gamma')$. Thus ∂ is well defined

Exact at $\ker(\gamma'')$. Take $m'' \in \ker(\gamma'')$. As in the construction of ∂, take $m \in M$ s.t. $\beta(m) = m''$ and take $n' \in N'$ s.t. $\alpha'(n') = \gamma(m)$. Suppose $m'' \in \ker(\partial)$. Then the image of n' in $\operatorname{coker}(\gamma')$ is equal to 0; so there is $m' \in M'$ s.t. $\gamma'(m') = n'$. Clearly $\gamma\alpha(m') = \alpha'\gamma'(m')$. So $\gamma\alpha(m') = \alpha'(n') = \gamma(m)$. Hence $m - \alpha(m') \in \ker(\gamma)$. Since $\beta(m - \alpha(m')) = m''$, clearly $m'' = \psi(m - \alpha(m'))$; so $m'' \in \operatorname{im}(\psi)$. Hence $\ker(\partial) \subset \operatorname{im}(\psi)$

Conversely, suppose $m'' \in \operatorname{im}(\psi)$. We may assume $m \in \ker(\gamma)$. So $\gamma(m) = 0$ and $\alpha'(n') = 0$. Since α' is injective, $n' = 0$. Thus $\partial(m'') = 0$ and so $\operatorname{im}(\psi) \subset \ker(\partial)$.Thus $\ker(\partial) = \operatorname{im}(\psi)$ □

Exercise 5.0.5. Referring to 4, give an alternative proof that β is an isomorphism by applying the Snake Lemma to the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "\kappa"] & N/M \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & M/L \arrow[r] & N/L \arrow[r, "\lambda"] & (N/L)/(M/L) \arrow[r] & 0
\end{tikzcd}$$

Proof. The Snake Lemma yields an exact sequence

$$L \xrightarrow{1} L \longrightarrow \ker(\beta) \longrightarrow 0$$

hence $\ker(\beta) = 0$ and β is injective. Moreover, β is surjective because κ and λ are □

Exercise 5.0.6 (Five Lemma). Consider this commutative diagram

$$\begin{tikzcd}
M_4 \arrow[r, "\alpha_4"] \arrow[d, "\gamma_4"] & M_3 \arrow[r, "\alpha_3"] \arrow[d, "\gamma_3"] & M_2 \arrow[r, "\alpha_2"] \arrow[d, "\gamma_2"] & M_1 \arrow[r, "\alpha_1"] \arrow[d, "\gamma_1"] & M_0 \arrow[d, "\gamma_0"] \\
N_4 \arrow[r, "\beta_4"] & N_3 \arrow[r, "\beta_3"] & N_2 \arrow[r, "\beta_2"] & N_1 \arrow[r, "\beta_1"] & N_0
\end{tikzcd}$$

Assume it has exact rows. Via a chase, prove these two statements
1. If γ_3 and γ_1 are surjective and if γ_0 is injective, then γ_2 is surjective
2. If γ_3 and γ_1 are injective and if γ_4 is surjective, then γ_2 is injective

Proof. Take $n_2 \in N_2$. Since γ_1 is surjective, there is $m_1 \in M_1$ s.t. $\gamma_1(m_1) = \beta_2(n_2)$. Then $\gamma_0\alpha_1(m_1) = \beta_1\gamma_1(m_1) = \beta_1\beta_2(n_1) = 0$. Since γ_0 is injective, $\alpha_1(m_1) = 0$. Hence exactness yields $m_2 \in M_2$ with $\alpha_2(m_2) = m_1$. So $\beta_2(\gamma_2(m_2) - n_2) = \gamma_1\alpha_2(m_2) - \beta_2(n_2) = 0$.

Hence exactness yields $n_3 \in N_3$ with $\beta_3(n_3) = \gamma_2(m_2) - n_2$. Since γ_3 is surjective, there is $m_3 \in M_3$ with $\gamma_3(m_3) = n_3$. Then $\gamma_2\alpha_3(m_3) = \beta_3\gamma_3(m_3) = \gamma_2(m_2) - n_2$. Hence $\gamma_2(m_2 - \alpha_3(m_3)) = n_2$. Thus γ_2 is surjective □

Exercise 5.0.7 (Nine Lemma). Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Assume all the columns are exact and the middle row is exact. Apply the Snake Lemma, prove that the first row is exact iff the third is

Exercise 5.0.8. Consider this commutative diagram with exact rows

$$\begin{tikzcd}
M' \arrow[r, "\beta"] \arrow[d, "\alpha'"] & M \arrow[r, "\gamma"] \arrow[d, "\alpha"] & M'' \arrow[d, "\alpha''"] \\
N' \arrow[r, "\beta'"] & N \arrow[r, "\gamma'"] & N''
\end{tikzcd}$$

Assume α' and γ are surjective. Given $n \in N$ and $m'' \in M''$ with $\alpha''(m'') = \gamma'(n)$, show that there is $m \in M$ s.t. $\alpha(m) = n$ and $\gamma(m) = m''$

Theorem 5.7 (Left exactness of Hom). *1. Let $M' \to M \to M'' \to 0$ be a sequence of module homomorphisms. Then it is exact iff for all modules N, the following induced sequence is exact*

$$0 \to \mathrm{Hom}(M'', N) \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M', N) \tag{5.0.1}$$

2. Let $0 \to N' \to N \to N''$ be a sequence of module homomorphisms. Then it is exact iff for all modules M, the following induced sequence is exact:

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'')$$

Proof. The exactness of $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ means simply that $M'' = \operatorname{coker}(\alpha)$. On the other hand, the exactness of (5.0.1) means that a $\varphi \in \operatorname{Hom}(M, N)$ maps to 0, or equivalently, $\varphi\alpha = 0$ iff there is a unique $\gamma : M'' \to N$ s.t. $\gamma\beta = \varphi$. So (5.0.1) is exact iff M'' has the UMP of coker(α), discussed in 4

$$\begin{tikzcd} M' \arrow[r, "\alpha"] & M \arrow[r] \arrow[d] & \operatorname{coker}(\alpha) \arrow[dl, dashed] \\ & N & \end{tikzcd}$$

□

Definition 5.8. A (free) **presentation** of a module M is an exact sequence

$$G \to F \to M \to 0$$

with G and F free. If G and F are free of finite rank, then the presentation is called **finite**. If M has a finite presentation, then M is said to be **finitely presented**

Proposition 5.9. *Given a module Mand a set of generators* $\{m_\lambda\}_{\lambda\in\Lambda}$, *there is an exact sequence* $0 \to K \to R^{\oplus\Lambda} \xrightarrow{\alpha} M \to 0$ *with* $\alpha(e_\lambda) = m_\lambda$, *where* $\{e_\lambda\}$ *is the standard basis; further, there is a presentation* $R^{\oplus\Sigma} \to R^{\oplus\Lambda} \xrightarrow{\alpha} M \to 0$

Proof. By 4, there is a surjection $\alpha : R^{\oplus\Lambda} \twoheadrightarrow M$ with $\alpha(e_\lambda) = m_\lambda$. Set $K := \ker(\alpha)$. Then $0 \to K \to R^{\oplus\Lambda} \to M \to 0$ is exact. Take a generators $\{k_\sigma\}_{\sigma\in\Sigma}$ of K, and repeat the process to obtain a surjection $R^{\oplus\Sigma} \twoheadrightarrow K$. Then $R^{\oplus\Sigma} \to R^{\oplus\Lambda} \to M \to 0$ is a presentation □

Definition 5.10. A module P is called **projective** if, given any surjective homomorphism $\beta : M \twoheadrightarrow N$, every homomorphism $\alpha : P \to N$ **lifts** to a homomorphism $\gamma : P \to M$; that is, $\alpha = \beta\gamma$

Exercise 5.0.9. Show that a free module $R^{\oplus\Lambda}$ is projective

Theorem 5.11. *The following conditions on a module P are equivalent:*

1. *The module P is projective*
2. *Every short exact sequence* $0 \to K \to M \to P \to 0$ *splits*
3. *There is a module K s.t.* $K \oplus P$ *is free*
4. *Every exact sequence* $N' \to N \to N''$ *induces an exact sequence*

$$\operatorname{Hom}(P, N') \to \operatorname{Hom}(P, N) \to \operatorname{Hom}(P, N'')$$

5. *Every surjective homomorphism $\beta : M \twoheadrightarrow N$ induces a surjection*

$$\mathrm{Hom}(P,\beta) : \mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$$

Proof. Assume (1). The surjection $M \twoheadrightarrow P$ and the identity $P \to P$ yield a section $P \to M$. So the sequence splits by 5.5

Assume (2). By 5.9 there is an exact sequence $0 \to K \to R^{\oplus\Lambda} \to P \to 0$. Then $K \oplus P \simeq R^{\oplus\Lambda}$.

Assume (3); say $K \oplus P \simeq R^{\oplus\Lambda}$. For each $\lambda \in \Lambda$, take a copy $N'_\lambda \to N_\lambda \to N''_\lambda$ of the exact sequence $N' \to N \to N''$. Then the induced sequence

$$\prod N'_\lambda \to \prod N_\lambda \to \prod N''_\lambda$$

is exact by 5.2. But at the end of 4, that sequence is equal to this one

$$\mathrm{Hom}(R^{\oplus\Lambda}, N') \to \mathrm{Hom}(R^{\oplus\Lambda}, N) \to \mathrm{Hom}(R^{\oplus\Lambda}, N'')$$

But $K \oplus P \simeq R^{\oplus\Lambda}$. So owing to (4.0.1), the latter sequence is also equal to

$$\mathrm{Hom}(K,N')\oplus\mathrm{Hom}(P,N') \to \mathrm{Hom}(K,N)\oplus\mathrm{Hom}(P,N) \to \mathrm{Hom}(K,N'')\oplus\mathrm{Hom}(P,N'')$$

hence by 5.2, it holds

Assume (4). Then every exact sequence $M \xrightarrow{\beta} N \to 0$ induces an exact sequence

$$\mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N) \to 0$$

Assume (5). By definition, $\mathrm{Hom}(P,\beta)(\gamma) = \beta\gamma$ □

Lemma 5.12 (Schanuel). *Given two short exact sequences*

$$0 \to L \xrightarrow{i} P \xrightarrow{\alpha} M \to 0 \quad \text{and} \quad 0 \to L' \xrightarrow{i'} P' \xrightarrow{\alpha'} M \to 0$$

with P and P' projective, there is an isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L \oplus P' \arrow[r, "i\oplus 1_{P'}"] \arrow[d, "\beta"', "\cong"] & P \oplus P' \arrow[r, "(\alpha\ 0)"] \arrow[d, "\gamma"', "\cong"] & M \arrow[r] \arrow[d, "1_M"', "="] & 0 \\
0 \arrow[r] & P \oplus L' \arrow[r, "1_P\oplus i'"] & P \oplus P' \arrow[r, "(0\ \alpha')"] & M \arrow[r] & 0
\end{tikzcd}$$

Proof. First, let's construct an intermediate isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & L \oplus P' \arrow[r, "i\oplus 1_{P'}"] & P \oplus P' \arrow[r, "(\alpha\ 0)"] & M \arrow[r] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[u, "\cong", "\lambda"'] & P \oplus P' \arrow[r, "(\alpha\ \alpha')"] \arrow[u, "\cong", "\theta"'] & M \arrow[r] \arrow[u, "=", "1_M"'] & 0
\end{tikzcd}$$

Take $K := \ker(\alpha\ \alpha')$. To form θ, recall that P' is projective and α is surjective. So there is a map $\pi : P' \to P$ s.t. $\alpha' = \alpha\pi$. Take $\theta := \left(\begin{smallmatrix} 1 & \pi \\ 0 & 1 \end{smallmatrix}\right)$

Then θ has $\left(\begin{smallmatrix} 1 & -\pi \\ 0 & 1 \end{smallmatrix}\right)$ as inverse. Further, the right-hand square is commutative

$$(\alpha\ 0)\theta = (\alpha\ 0)\left(\begin{smallmatrix} 1 & \pi \\ 0 & 1 \end{smallmatrix}\right) = (\alpha\ \alpha\pi) = (\alpha\ \alpha')$$

So θ induces the desired isomorphism $\lambda : K \xrightarrow{\sim} L \oplus P'$ since they are both kernels. $\alpha(\theta(\ker(\alpha\ \alpha'))) = 1_M(\alpha\ \alpha')(\ker(\alpha\ \alpha')) = 0$

Symmetrically, form an automorphism θ' of $P \oplus P'$, which induces an isomorphism $\lambda' : K \xrightarrow{\sim} P \oplus L'$. Finally, take $\gamma := \theta'\theta^{-1}$ and $\beta := \lambda'\lambda^{-1}$ □

Exercise 5.0.10. Let R be a ring, and $0 \to L \to R^n \to M \to 0$ an exact sequence. Prove M is finitely presented iff L is finitely generated

Proof. Assume M is finitely presented; say $R^l \to R^m \to M \to 0$ is a finite sequence. Let L' be the image of R^l. Then $L' \oplus R^n \simeq L \oplus R^m$ by Schanuel's Lemma since we can replace R^l by L'. Hence L is a quotient of $L' \oplus R^n$. Thus L is finitely generated

Conversely, suppose L is finitely generated by l elements. They yield a surjection $R^l \twoheadrightarrow L$ and a sequence $R^l \to R^n \to M \to 0$. □

Exercise 5.0.11. Let R be a ring, $X_1, X_2, \ldots$ infinitely many variables. Set $P := R[X_1, X_2, \ldots]$ and $M := P/\langle X_1, X_2, \ldots\rangle$. Is M finitely presented?

Proof. No. By 5.0.10 □

Proposition 5.13. *Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence with L finitely generated and M finitely generated. Then N is finitely generated*

Proof. Let R be the ground ring, $\mu : R^m \to M$ any surjection. Set $\nu := \beta\mu$, set $K := \ker(\nu)$ and set $\lambda := \mu|K$. Then the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\lambda"] & R^m \arrow[r, "\nu"] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "1_N"] & 0 \\
0 \arrow[r] & L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

The Snake Lemma yields an isomorphism $\ker(\lambda) \xrightarrow{\sim} \ker(\mu)$. But $\ker(\mu)$ is finitely generated by 5.0.10. So $\ker(\lambda)$ is finitely generated. Also the Snake Lemma implies $\operatorname{coker}(\lambda) = 0$ as $\operatorname{coker}(\mu) = 0$; so $0 \to \ker(\lambda) \to K \xrightarrow{\lambda} L \to 0$ is exact. Hence K is finitely generated by 5.0.2. Thus N is finitely generated by 5.0.10 □

Exercise 5.0.12. Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence with M finitely generated and N finitely presented. Prove L is finitely generated

Proposition 5.14. *Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence with L and N finitely generated. Prove M is finitely presented too.*

Proof. Let R be the ground ring, $\lambda : R^l \to L$ and $\nu : R^n \twoheadrightarrow N$ any surjections. Define $\gamma : R^l \to M$ by $\gamma := \alpha\lambda$. Note R^n is projective and define $\delta : R^n \to M$ by lifting ν along β. Define $\mu : R^l \oplus R^n \to M$ by $\mu := \gamma + \delta$. Then the following diagram is commutative, where $\iota := \iota_{R^l}$ and $\pi := \pi_{R^n}$

$$\begin{tikzcd}
0 \arrow[r] & R^l \arrow[r, "\iota"] \arrow[d, "\lambda"] & R^l \oplus R^n \arrow[r, "\pi"] \arrow[d, "\mu"] & R^n \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r] & 0
\end{tikzcd}$$

Since λ and ν are surjective, the Snake Lemma yields an exact sequence

$$0 \to \ker(\lambda) \to \ker(\mu) \to \ker(\nu) \to 0 \to \operatorname{coker}(\mu) \to 0$$

and implies $\operatorname{coker}(\mu) = 0$. Also $\ker(\lambda)$ and $\ker(\nu)$ are finitely generated. So $\ker(\mu)$ is finitely generated by 5.0.2. Thus M is finitely generated $\square$

6 Direct Limits

Categories

A **category** $\mathcal{C}$ is a collection of elements, called **objects**. Each pair of objects A, B is equipped with a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of elements, called **maps** or **morphisms**. We write $\alpha : A \to B$ or $A \xrightarrow{\alpha} B$ to mean $\alpha \in \mathrm{Hom}_{\mathcal{C}}(A, B)$

Given objects A, B, C, there is a **composition law**

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C), \quad \text{written} \quad (\alpha, \beta) \mapsto \beta\alpha$$

and there is a distinguished map $1_B \in \mathrm{Hom}_{\mathcal{C}}(B, B)$, called the **identity** s.t.

1. composition is **associative**
2. 1_B is **unitary**, or $1_B\alpha = \beta 1_B = \beta$

We say α is an **isomorphism** with **inverse** $\beta : B \to A$ if $\alpha\beta = 1_B$ and $\beta\alpha = 1_A$

Functors

A map of categories is known as a functor. Namely, given categories $\mathcal{C}$ and $\mathcal{C}'$, a (**covariant**) **functor** $F : \mathcal{C} \to \mathcal{C}'$ is a rule that assigns to each object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{C}'$ and to each map $\alpha : A \to B$ of $\mathcal{C}$ a map $F(\alpha) : F(A) \to F(B)$ of $\mathcal{C}'$ preserving composition and identiy; that is

1. $F(\beta\alpha) = F(\beta)F(\alpha)$ for maps $\alpha : A \to B$ and $\beta : B \to C$ of $\mathcal{C}$ and
2. $F(1_A) = 1_{F(A)}$

We also denote a functor F by $F(\bullet)$, by $A \mapsto F(A)$, or by $A \mapsto F_A$

Note that a functor preserves isomorphisms

For example, let R be a ring, M a module. Then clearly $\mathrm{Hom}_R(M, \bullet)$ is a functor from R-**Mod** to R-**Mod**. A second example is the **forgetful functor** from R-**Mod** to **Sets**; it sends a module to its underlying set and a homomorphism to its underlying set map

A map of functors is known as a natural transformation. Namely, given two functor $F, F' : \mathcal{C} \rightrightarrows \mathcal{C}'$, a **natural transformation** $\theta : F \to F'$ is a collection of maps $\theta(A) : F(A) \to F'(A)$ one for each object A of $\mathcal{C}$, s.t. $\theta(B)(F(\alpha) = F'(\alpha)\theta(A)$ for the map $\alpha : A \to B$

$$\begin{tikzcd}
F(A) \arrow[r, "F(\alpha)"] \arrow[d, "\theta(A)"] & F(B) \arrow[d, "\theta(B)"] \\
F'(A) \arrow[r, "F'(\alpha)"] & F'(B)
\end{tikzcd}$$

We call F and F' **isomorphic** if there are natural transformations $\theta : F \to F'$ and $\theta' : F' \to F$ with $\theta'\theta = 1_F$ and $\theta\theta' = 1_{F'}$

A **contravariant** functor G from $\mathcal{C}$ to $\mathcal{C}'$ is a rule similar to F, but G reverses the direction of maps. For example, fix a module N; then $\mathrm{Hom}(\bullet, N)$ is a contravariant functor

Exercise 6.0.1.
1. Show that the condition 6 (1) is equivalent to the commutativity of the corresponding diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B, C) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}'}(F(B), F(C)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(A, C) \arrow[r] & \mathrm{Hom}_{\mathcal{C}'}(F(A), F(C))
\end{tikzcd}$$

2. Given $\gamma : C \to D$, show 5.1 (1) yields the commutativity of this diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(B,C) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}'}(F(B),F(C)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}}(A,D) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}'}(F(A),F(D)) \end{array}$$

Adjoints

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, $F : \mathcal{C} \to \mathcal{C}'$ and $F' : \mathcal{C}' \to \mathcal{C}$ functors. We call (F, F') an **adjoint pair**, F the **left adjoint** of F', and F' the **right adjoint** of F if for each object $A \in \mathcal{C}$ and object $A' \in \mathcal{C}'$, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}'}(F(A), A') \simeq \mathrm{Hom}_{\mathcal{C}}(A, F'(A'))$$

Here **natural** means that maps $B \to A$ and $A' \to B'$ induce a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(F(A),A') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(A,F'(A')) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}'}(F(B),B') & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(B,F'(B')) \end{array}$$

Naturality serves to determine an adjoint up to canonical isomorphism. Indeed, let F and G be two left adjoints of F'. Given $A \in \mathcal{C}$, define $\theta(A) : G(A) \to F(A)$ to be the image of $1_{F(A)}$ under the adjoint bijections

$$\mathrm{Hom}_{\mathcal{C}'}(F(A), F(A)) \simeq \mathrm{Hom}_{\mathcal{C}}(A, F'F(A)) \simeq \mathrm{Hom}_{\mathcal{C}'}(G(A), F(A))$$

To see that $\theta(A)$ is natural in A, take a map $\alpha : A \to B$. It induces the following diagram, which is commutative owing to the naturality of the adjoint bijections:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}'}(F(A),F(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(A,F'F(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(G(A),F(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}'}(F(A),F(B)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(F(A),F'F(B)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(G(A),F(B)) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}_{\mathcal{C}'}(F(B),F(B)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(B,F'F(B)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(G(B),F(B)) \end{array}$$

Chase after $1_{F(A)}$ and $1_{F(B)}$. Both map to $F(\alpha) \in \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B))$. So both map to the same image in $\mathrm{Hom}_{\mathcal{C}'}(G(A), F(B))$. But clockwise, $1_{F(A)}$ maps to $F(\alpha)\theta(A)$; counterclockwise, $1_{F(B)}$ maps to $\theta(B)G(\alpha)$. So $\theta(B)G(\alpha) = F(\alpha)\theta(A)$. Thus the $\theta(A)$ form a natural transformation $\theta : G \to F$

Similearly, there is a natural transformation $\theta' : F \to G$. It remains to show $\theta'\theta = 1_G$ and $\theta\theta' = 1_F$. However, by naturality, this diagram is commutative

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}'}(F(A), F(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(A, F'F(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(G(A), F(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{C}'}(F(A), G(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}}(A, F'G(A)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathcal{C}'}(G(A), G(A)) \end{array}$$

Chase after $1_{F(A)}$. Clockwise, its image is $\theta'(A)\theta(A)$. Counterclockwise, its image is $1_{G(A)}$. Thus $\theta'\theta = 1_G$. And similarly, $\theta\theta' = 1_F$

For example, the "free module" functor is the left adjoint of the forgetful functor from R-**Mod** to **Sets**, since by 4

$$\mathrm{Hom}_{\text{R-}\mathbf{Mod}}(R^{\oplus\Lambda}, M) = \mathrm{Hom}_{\mathbf{Sets}}(\Lambda, M)$$

Similarly, the "polynomial ring" functor is the left adjoint of the forgetful functor from R-**Alg** to **Sets** since by **??**

$$\mathrm{Hom}_{\text{R-}\mathbf{Alg}}(R[X_1, \dots, X_n], R') = \mathrm{Hom}_{\mathbf{Sets}}(\{X_1, \dots, X_n\}, R')$$

Exercise 6.0.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, $F : \mathcal{C} \to \mathcal{C}'$ and $F' : \mathcal{C}' \to \mathcal{C}$ an adjoint pair. Let $\varphi_{A,A'} : \mathrm{Hom}_{\mathcal{C}'}(FA, A') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, F'A')$ denote the **natural** bijection, and set $\eta_A := \varphi_{A,FA}(1_{FA})$. Do the following

1. Prove η_A is natural in A; that is, given $g : A \to B$, the induced square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & F'FA \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F'Fg} \\ B & \xrightarrow{\eta_B} & F'FB \end{array}$$

   is commutative. We call the natural transformation $A \mapsto \eta_A$ the **unit** of (F, F')
2. Given $f' : FA \to A'$, prove $\varphi_{A,A'}(f') = F'f' \circ \eta_A$
3. Prove the natural map $\eta_A : A \to F'FA$ is **universal** from A to F'; that is, given $f : A \to F'A'$, there is a unique map $f' : FA \to A'$ with $F'f' \circ \eta_A = f$

4. Conversely, instead of assuming (F, F') is an adjoint pair, assume given a natural transformation $\eta : 1_{\mathcal{C}} \to F'F$ satisfying (1) and (3). Prove the equation in (2) defines a natural bijection making (F, F') an adjoint pair, whose unit is η
5. Identify the units in the two examples in 6.

(Dually, we can define a **counit** $\epsilon : FF' \to 1_{\mathcal{C}'}$)

Proof. 1.

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}'}(FA, FA) & \xrightarrow{(Fg)_*} & \mathrm{Hom}_{\mathcal{C}'}(FA, FB) & \xleftarrow{(Fg)^*} & \mathrm{Hom}_{\mathcal{C}'}(FB, FB) \\ \downarrow{\scriptstyle \varphi_{A,FA}} & & \downarrow{\scriptstyle \varphi_{A,FB}} & & \downarrow{\scriptstyle \varphi_{B,FB}} \\ \mathrm{Hom}_{\mathcal{C}}(A, F'FA) & \xrightarrow{(F'Fg)_*} & \mathrm{Hom}_{\mathcal{C}}(A, F'FB) & \xleftarrow{g^*} & \mathrm{Hom}_{\mathcal{C}}(B, F'FB) \end{array}$$

Follow 1_{FA} out of the upper left corner to find $F'Fg \circ \eta_A = \varphi_{A,FB}(Fg)$ in $\mathrm{Hom}_{\mathcal{C}}(A, F'FB)$. Follow 1_{FB} out of the upper right corner to find $\varphi_{A,FB}(Fg) = \eta_B \circ g$. Thus $F'Fg \circ \eta_A = \eta_B \circ g$.

2.
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}'}(FA, FA) & \xrightarrow{f'_*} & \mathrm{Hom}_{\mathcal{C}'}(FA, A') \\ \downarrow{\scriptstyle \varphi_{A,FA}} & & \downarrow{\scriptstyle \varphi_{A,A'}} \\ \mathrm{Hom}_{\mathcal{C}}(A, F'FA) & \xrightarrow{(F'f')_*} & \mathrm{Hom}_{\mathcal{C}}(A, F'A') \end{array}$$

4. Set $\psi_{A,A'}(f') := F'f' \circ \eta_A$. As η_A is universal, given $f : A \to F'A'$, there is a unique $f' : FA \to A'$ with $F'f' \circ \eta_A = f$. Thus $\psi_{A,A'}$ is a bijection
$$\psi_{A,A'} : \mathrm{Hom}_{\mathcal{C}'}(FA, A') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, F'A')$$
Also $\psi_{A,A'}$ is natural in A, as η_A is natural in A and F' is a functor. And $\psi_{A,A'}$ is natural in A', as F' is a functor. Clearly $\psi_{A,FA}(1_{FA}) = \eta_A$
5. $\eta_\Lambda : \Lambda \to R^{\oplus\Lambda}$ carries an element of Λ to the corresponding standard basis vector.
If F is the polynomial ring functor and if A is the set of variables $X_1, \ldots, X_n$, then $\eta_A(X_i)$ is just X_i viewed in $R[X_1, \ldots, X_n]$

□

Direct limits

Let $\Lambda, \mathcal{C}$ be categories. Assume Λ is **small**; that is, its objects form a set. Given a functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$, its **direct limit**, or **colimit**, denoted by $\varinjlim M_\lambda$ or $\varinjlim_{\lambda\in\Lambda} M_\lambda$, is defined as the universal example of an object P of $\mathcal{C}$ equipped with maps $\beta_\mu : M_\mu \to P$, called **insertions**, that are compatible with the **transition maps** $\alpha^\kappa_\mu : M_\kappa \to M_\mu$, which are the images of the maps of Λ. In other words, there is a unique map β s.t. all these diagrams commute

$$\begin{tikzcd} M_\kappa \arrow[r, "\alpha_\mu^\kappa"] \arrow[d, "\beta_\kappa"] & M_\mu \arrow[r, "\alpha_\mu"] \arrow[d, "\beta_\mu"] & \varinjlim M_\lambda \arrow[d, "\beta"] \\ P \arrow[r, "1_P"] & P \arrow[r, "1_P"] & P \end{tikzcd}$$

To indicate this context, the functor $\lambda \mapsto M_\lambda$ is often called a **direct system**

As usual, universaility implies that, once equipped with its insertions α_μ, the limit $\varinjlim M_\lambda$ is determined up to unique isomorphism, assuming it exists. In practice, there is usually a canonical choice for $\varinjlim M_\lambda$, given by a construction.

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta_P"] & \varinjlim M_\lambda \arrow[r, "\beta"] \arrow[d, "\beta_M"] & P \arrow[r, "\alpha"] \arrow[d, "\beta_P"] & \varinjlim M_\lambda \arrow[d, "\beta_M"] \\ P \arrow[r, "1_P"] & P \arrow[r, "1_P"] & P \arrow[r, "1_P"] & P \end{tikzcd}$$

Hence $\beta_P \beta\alpha = \beta_P$ and $\beta_M \alpha\beta = \beta_M$. Since the identity is unique, $\beta\alpha = 1_P$ and $\alpha\beta = 1_{\varinjlim M_\lambda}$. This induce an isomorphism

We say that $\mathcal{C}$ **has direct limits indexed by** Λ if, for every functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$, the direct limit $\varinjlim M_\lambda$ exists. We say that $\mathcal{C}$ **has direct limits** if it has direct limits indexed by every small category Λ. We say that a functor $F : \mathcal{C} \to \mathcal{C}'$ **preserves direct limits** if, given any direct limit $\varinjlim M_\lambda$ in $\mathcal{C}$, the direct limit $\varinjlim F(M_\lambda)$ exists, and is equal to $F(\varinjlim M_\lambda)$; more precisely, the maps $F(\alpha_\mu) : F(M_\mu) \to F(\varinjlim M_\lambda)$ induce a canonical map

$$\phi : \varinjlim F(M_\lambda) \to F(\varinjlim M_\lambda)$$

and ϕ is an isomorphism. Sometimes, we construct $\varinjlim F(M_\lambda)$ by showing that $F(\varinjlim M_\lambda)$ has the requisite UMP

Assuming $\mathcal{C}$ has direct limits by Λ. Then, given a natural transformation from $\lambda \mapsto M_\lambda$ to $\lambda \mapsto N_\lambda$, universaility yields unique commutative diagram

$$\begin{tikzcd} M_\mu \arrow[r] \arrow[d] & \varinjlim M_\lambda \arrow[d] \\ N_\mu \arrow[r] & \varinjlim N_\lambda \end{tikzcd}$$

To put it in another way, form the **functor category** $\mathcal{C}^\Lambda$: its objects are the functors $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$; its maps are the natural transformations. Then taking direct limits yields a functor $\varinjlim$ from $\mathcal{C}^\Lambda$ to $\mathcal{C}$

In fact, it is just a restatement of the definitions that the "direct limit" functor $\varinjlim$ is the left adjoint of the **diagonal functor**

$$\Delta : \mathcal{C} \to \mathcal{C}^{\Lambda}$$

By definition, Δ sends each object M to the **constant functor** ΔM, which has the same value M for every $\lambda \in \Lambda$ and has the same value 1_M at every map of Λ; further, Δ carries a map $\gamma : M \to N$ to the natural transformation $\Delta\gamma : \Delta M \to \Delta N$, which has the same value γ at every $\lambda \in \Lambda$

We have

$$\frac{\varinjlim M_\lambda \to M}{(\lambda \mapsto M_\lambda) \to (\lambda \mapsto M)}$$

Coproducts

Let $\mathcal{C}$ be a category, Λ a set, and M_λ an object of $\mathcal{C}$ for each $\lambda \in \Lambda$. The **coproduct** $\coprod_{\lambda\in\Lambda} M_\lambda$, or simply $\coprod M_\lambda$, is defined as the universal example of an object P equipped with a map $\beta_\mu : M_\mu \to P$ for each $\mu \in \Lambda$. The maps $\iota_\mu : M_\mu \to \coprod M_\lambda$ are called the **inclusions**. Thus, given an example P, there exists a unique map $\beta : \coprod M_\lambda \to P$ with $\beta\iota_\mu = \beta_\mu$ for all $\mu \in \Lambda$

If $\Lambda = \emptyset$, then the coproduct is an object B with a unique map β to every other object P. There are no μ in Λ, so no inclusions $\iota_\mu : M_\mu \to B$, so no equations $\beta\iota_\mu = \beta_\mu$ to restrict β. Such a B is called an **initial object**

For instance, suppose $\mathcal{C}$ = R-**Mod**. Then the zero module is an initial object. For any Λ, the coproducts $\coprod M_\lambda$ is just the direct sum $\oplus M_\lambda$. Further, suppose $\mathcal{C}$ = **Sets**. Then the empty set is an initial object. For any Λ, the coproduct $\coprod M_\lambda$ is the disjoint union $\bigsqcup M_\lambda$

Note that the coproduct is a special case of the direct limit. Indeed, regard Λ as a **discrete** category: its objects are the $\lambda \in \Lambda$ and it has just the required maps, namely, the 1_λ. Then $\varinjlim M_\lambda = \coprod M_\lambda$ with the insertions equal to the inclusions

Coequalizers

Let $\alpha, \alpha' : M \to N$ be a two maps in a category $\mathcal{C}$. Their **coequalizer** is defined as the universal example of an object P with a map $\eta : N \to P$ s.t. $\eta\alpha = \eta\alpha'$

$$\begin{tikzcd}
M \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\alpha'"'] & N \arrow[r, "\eta"] \arrow[dr, "\xi"'] & P \arrow[d, dashed, "u"] \\
& & Z
\end{tikzcd}$$

If $\mathcal{C}$ = R-**Mod**, then the coequalizer is $\operatorname{coker}(\alpha - \alpha')$. In particular, the coequalizer of α and 0 is just $\operatorname{coker}(\alpha)$

TODO Suppose $\mathcal{C}$ = **Sets**. Take the smallest equivalence relation $\sim$ on N with $\alpha(m) \sim \alpha'(m)$ for all $m \in M$; explicitly, $n \sim n'$ if there are elements $m_1, \dots, m_r$ with $\alpha(m_1) = n$ and $\alpha'(m_r) = n'$ and with $\alpha(m_i) = \alpha'(m_{i+1})$ for $1 \leq i < r$. Clearly, the coequalizer is the quotient $N/\sim$ equipped with the quotient group

Note that the coequalizer is a special case of the direct limit. Indeed, let Λ be the category consisting of two objects κ, μ and two nontrivial maps $\varphi, \varphi' : \kappa \to \mu$.

Exercise 6.0.3. Let $\alpha : L \to M$ and $\beta : L \to N$ be two maps. Their **pushout** is defined as the universal example of an object P equipped with a pair of maps $\gamma : M \to P$ and $\delta : N \to P$ s.t. $\gamma\alpha = \delta\beta$. Express the pushout as a direct limit. Show that, in **Sets**, the pushout is the disjoint union $M \uplus N$ modulo the smallest equivalence relation $\sim$ with $m \sim n$ if there is $\in L$ with $\alpha(l) = m$ and $\beta(l) = n$. Show that, in R-**Mod**, the pushout is equal to the direct sum $M \oplus N$ modulo the image of L under the map $\alpha, -\beta$

Proof.

$$\begin{tikzcd}
Q & & \\
& P \arrow[ul, dashed, "u"] & M \arrow[l, "\gamma"] \arrow[ull, bend right, "\gamma'"'] \\
& N \arrow[u, "\delta"] \arrow[uul, bend left, "\delta'"] & L \arrow[l, "\beta"] \arrow[u, "\alpha"']
\end{tikzcd}$$

Let Λ be the category with three objects λ, μ and ν and two nonidentity maps $\lambda \to \mu$ and $\lambda \to \nu$. Define a functor $\lambda \mapsto M_\lambda$ by $M_\lambda := L, M_\mu = M$, $M_\nu := N$, $\alpha_\mu^\lambda := \alpha$, $\alpha_\nu^\lambda := \beta$. Set $Q := \varinjlim M_\lambda$. Then writing

$$\begin{tikzcd}
N \arrow[d, "\eta_\nu"] & L \arrow[l, "\beta"'] \arrow[r, "\alpha"] \arrow[d, "\eta_\lambda"] & M \arrow[d, "\eta_\mu"] \\
Q & Q \arrow[l, "1_R"'] \arrow[r, "1_R"] & Q
\end{tikzcd}
\quad \text{as} \quad
\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\beta"'] & M \arrow[d, "\eta_\mu"] \\
N \arrow[r, "\eta_\nu"] & Q
\end{tikzcd}$$

In **Sets**, take γ and δ to be the inclusions followed by the quotient map. Clearly $\gamma\alpha = \delta\beta$. Further, given P and maps $\gamma' : M \to P$ and $\delta' : N \to P$, they define a unique map $M \coprod N \to P$, and it factors through the quotient iff $\gamma'\alpha = \delta'\beta$. Thus $(M \coprod N)/\sim$ is the pushout

In R-**Mod**, take γ and δ to be the inclusions followed by the quotient map. Then for all $l \in L$, clearly $\iota_M\alpha(l) - \iota_N(\beta(l)) = (\alpha(l), -\beta(l))$. So $\iota_M\alpha(l) - \iota_N\beta(l)$ is in im(L); hence $\iota_M\alpha(l)$ and $\iota_N\beta(l)$ has the same image in the quotient $\square$

Lemma 6.1. *A category $\mathcal{C}$ has direct limits iff $\mathcal{C}$ has coproducts and coequalizers. If a category $\mathcal{C}$ has direct limits, then a functor $F : \mathcal{C} \to \mathcal{C}'$ preserves then iff F preserves coproducts and coequalizers*

Proof. Assume $\mathcal{C}$ has coproducts and coequalizers. Let Λ be a small category, and $\lambda \mapsto M_\lambda$ a functor from Λ to $\mathcal{C}$. Let Σ be the set of transition maps $\alpha_\mu^\lambda : M_\lambda \to M_\mu$. For each $\sigma := \alpha_\mu^\lambda \in \Sigma$, set $M_\sigma := M_\lambda$. Set $M := \coprod_{\sigma\in\Sigma} M_\sigma$ and $N := \coprod_{\lambda\in\Lambda} M_\lambda$. For each σ, there are two maps $M_\sigma := M_\lambda \to N$: the inclusion ι_λ and the composition $\iota_\mu\alpha_\mu^\lambda$. Correspondingly, there are two maps $\alpha, \alpha' : M \to N$. Let C be their coequalizer, and $\eta : N \to C$

Given maps $\beta_\lambda : M_\lambda \to P$ with $\beta_\mu\alpha_\mu^\lambda = \beta_\lambda$, there is a unique map $\beta : N \to P$ with $\beta\iota_\lambda = \beta_\lambda$ by the UMP of the coproduct. Clearly, $\beta\alpha = \beta\alpha'$ (note the choice of β, just choose all M). so β factors uniquely through C by the UMP of the coequalizer. Thus $C = \varinjlim M_\lambda$

Further, if $F : \mathcal{C}' \to \mathcal{C}'$ preserves coproduct and coequalizers, then F preserves arbitrary direct limits as F preserves the above construction $\square$

Theorem 6.2. *The categories* R-***Mod*** *and* ***Sets*** *have direct limits*

Theorem 6.3. *Every left adjoint $F : \mathcal{C} \to \mathcal{C}'$ preserves direct limits*

Proof. Let Λ be a small category, $\lambda \mapsto M_\lambda$ a functor from Λ to $\mathcal{C}$ s.t. $\varinjlim M_\lambda$ exists. Given an object P' of $\mathcal{C}'$, consider all possible commutative diagrams

$$\begin{tikzcd}
F(M_\kappa) \arrow[r, "F(\alpha_\mu^\kappa)"] \arrow[d, "\beta'_\kappa"] & F(M_\mu) \arrow[r, "F(\alpha_\mu)"] \arrow[d, "\beta'_\mu"] & F(\varinjlim M_\lambda) \arrow[d, "\beta'"] \\
P' \arrow[r, "1"] & P \arrow[r, "1"] & P
\end{tikzcd}$$

given the β'_κ, we must show there is a unique β'

Say F is the left adjoint of $F' : \mathcal{C}' \to \mathcal{C}$. Then the above diagram is equivalent to

$$\begin{tikzcd}
M_\kappa \arrow[r, "\alpha_\mu^\kappa"] \arrow[d, "\beta_\kappa"] & M_\mu \arrow[r, "\alpha_\mu"] \arrow[d, "\beta_\mu"] & \varinjlim M_\lambda \arrow[d, "\beta"] \\
F'(P') \arrow[r, "1"] & F'(P') \arrow[r, "1"] & F'(P')
\end{tikzcd}$$

$\square$

Proposition 6.4. *Let $\mathcal{C}$ be a category, Λ and Σ small categories. Assume $\mathcal{C}$ has direct limits indexed by Σ. Then the functor category $\mathcal{C}^\Lambda$ does too.*

Proof. Let $\sigma \mapsto (\lambda \mapsto M_{\sigma\lambda})$ be a functor from Σ to $\mathcal{C}^\Lambda$. Then a map $\sigma \to \tau$ in Σ yields a natural transformation from $\lambda \mapsto M_{\sigma\lambda}$ to $\lambda \mapsto M_{\tau\lambda}$. So a map $\lambda \mapsto \mu$ in Λ yields a commutative square

$$\begin{tikzcd} M_{\sigma\lambda} \arrow[r] \arrow[d] & M_{\sigma\mu} \arrow[d] \\ M_{\tau\lambda} \arrow[r] & M_{\tau\mu} \end{tikzcd}$$

With λ fixed, the rule $\sigma \mapsto M_{\sigma\lambda}$ is a functor from Σ to $\mathcal{C}$

By hypothesis, $\varinjlim_{\sigma\in\Sigma} M_{\sigma\lambda}$ exists. So $\lambda \mapsto \varinjlim_{\sigma\in\Sigma} M_{\sigma\lambda}$ is a functor from Λ to $\mathcal{C}$. Further, as $\tau \in \Sigma$ varies, there are compatible natural transformation from the $\lambda \mapsto M_{\tau\lambda}$ to $\lambda \mapsto \varinjlim_{\sigma\in\Sigma} M_{\sigma\lambda}$ (definition of direct limit) Finally, the latter is the direct limit of the functor $\tau \mapsto (\lambda \mapsto M_{\tau\lambda})$ from Σ to $\mathcal{C}^\Lambda$, because given any functor $\lambda \mapsto P_\lambda$ from Λ to $\mathcal{C}$ equipped with, for $\tau \in \Sigma$, compatible natural transformations from the $\lambda \mapsto M_{\tau\lambda}$ to $\lambda \mapsto P_\lambda$, there are, for $\lambda \in \Lambda$, compatible unique maps $\varinjlim_{\sigma\in\Sigma} M_{\sigma\lambda} \to P_\lambda$ □

Theorem 6.5 (Direct limits commute)**.** *Let $\mathcal{C}$ be a category with direct limits indexed by small categories Σ and Λ. Let $\sigma \mapsto (\lambda \mapsto M_{\sigma\lambda})$ be a functor from Σ to $\mathcal{C}^\Lambda$. Then*

$$\varinjlim_{\sigma\in\Sigma} \varinjlim_{\lambda\in\Lambda} M_{\sigma,\lambda} = \varinjlim_{\lambda\in\Lambda} \varinjlim_{\sigma\in\Sigma} M_{\sigma,\lambda}$$

Proof. By 6, the functor $\varinjlim_{\lambda\in\Lambda} : \mathcal{C}^\Lambda \to \mathcal{C}$ is a left adjoint. By 6.4, the category $\mathcal{C}^\Lambda$ has direct limits indexed by Σ. So 6.3 yields the assertion □

Corollary 6.6. *Let Λ be a small category, R a ring, and $\mathcal{C}$ either **Sets** or* R-***Mod***. *Then the functor $\varinjlim : calc^\Lambda \to \mathcal{C}$ preserves coproducts and coequalizers*

Proof. They all have direct limits □

Exercise 6.0.4. Let $\mathcal{C}$ be a category, Σ and Λ small categories

1. Prove $\mathcal{C}^{\Sigma\times\Lambda} = (\mathcal{C}^\Lambda)^\Sigma$ with $(\sigma, \lambda) \mapsto M_{\sigma,\lambda}$ corresponding to $\sigma \mapsto (\lambda \mapsto M_{\sigma,\lambda})$
2. Assume $\mathcal{C}$ has direct limits indexed by Σ and by Λ. Prove that $\mathcal{C}$ has direct limits indexed by $\Sigma \times \Lambda$ and $\varinjlim_{\lambda\in\Lambda} \varinjlim_{\sigma\in\Sigma} = \varinjlim_{(\sigma,\lambda)\in\Sigma\times\Lambda}$

Proof. 1. In $\Sigma \times \Lambda$, a map $(\sigma, \lambda) \to (\tau, \mu)$ factors in two ways

$$(\sigma, \lambda) \to (\tau, \lambda) \to (\tau, \mu) \quad \text{and} \quad (\sigma, \lambda) \to (\sigma, \mu) \to (\tau, \mu)$$

So given a functor $(\sigma, \lambda) \mapsto M_{\sigma,\lambda}$, there is a diagram

$$\begin{tikzcd} M_{\sigma,\lambda} \arrow[r] \arrow[d] & M_{\sigma,\mu} \arrow[d] \\ M_{\tau,\lambda} \arrow[r] & M_{\tau,\mu} \end{tikzcd}$$

It shows that the map $\sigma \to \tau$ in Σ induces a natural transformation from $\lambda \mapsto M_{\sigma,\lambda}$ to $\lambda \mapsto M_{\tau,\lambda}$. Thus the rule $\sigma \mapsto (\lambda \mapsto M_{\sigma,\lambda})$ is a functor from Σ to $\mathcal{C}^{\Lambda}$

A map from $(\sigma, \lambda) \mapsto M_{\sigma,\lambda}$ to a second functor $(\sigma, \lambda) \mapsto N_{\sigma,\lambda}$ is a collection of maps $\theta_{\sigma,\lambda} : M_{\sigma,\lambda} \to N_{\sigma,\lambda}$ s.t., for every map $(\sigma, \lambda) \to (\tau, \mu)$, the square

$$\begin{tikzcd} M_{\sigma,\lambda} \arrow[r] \arrow[d, "\theta_{\sigma,\lambda}"] & M_{\tau,\mu} \arrow[d, "\theta_{\tau,\mu}"] \\ N_{\sigma,\lambda} \arrow[r] & N_{\tau,\mu} \end{tikzcd}$$

is commutative. Factoring $(\sigma, \lambda) \to (\tau, \mu)$ in two ways as above, we get a commutative cube. It shows that the $\theta_{\sigma,\lambda}$ define a map in $(\mathcal{C}^{\Lambda})^{\Sigma}$

2. $\mathcal{C}^{\Lambda}$ has direct limits indexed by Σ. So the functors $\varinjlim_{\lambda\in\Lambda} : \mathcal{C}^{\Lambda} \to \mathcal{C}$ and $\lim_{\sigma\in\Sigma} : (\mathcal{C}^{\Lambda})^{\Sigma} \to \mathcal{C}^{\Lambda}$ exists, and they are the left adjoints of the diagonal functors $\mathcal{C} \to \mathcal{C}^{\Lambda}$ and $\mathcal{C}^{\Lambda} \to (\mathcal{C}^{\Lambda})^{\Sigma}$. Hence the composition $\varinjlim_{\lambda\in\Lambda} \varinjlim_{\sigma\in\Sigma}$ is the left adjoint of the composition of the two diagonal functors. But the latter is just the diagonal $\mathcal{C} \to \mathcal{C}^{\Sigma\times\Lambda}$ owing to (1). So this diagonal has a left adjoint, which is necessarily $\varinjlim_{(\sigma,\lambda)\in\Sigma\times\Lambda}$ owing to the uniqueness of adjoints

□

Exercise 6.0.5. Let $\lambda \mapsto M_\lambda$ and $\lambda \mapsto N_\lambda$ be two functors from a small category Λ to R-**Mod**, and $\{\theta_\lambda : M_\lambda \to N_\lambda\}$ a natural transformation. Show

$$\varinjlim \operatorname{coker}(\theta_\lambda) = \operatorname{coker}(\varinjlim M_\lambda \mapsto \varinjlim N_\lambda)$$

Show that the analogous statement for kernel can be false by constructing a counterexample using the following commutative diagram with exact rows

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\mu_2"] \arrow[d, "\mu_2"] & \mathbb{Z} \arrow[r] \arrow[d, "\mu_2"] & \mathbb{Z}/\langle 2\rangle \arrow[r] \arrow[d, "\mu_2"] & 0 \\
\mathbb{Z} \arrow[r, "\mu_2"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/\langle 2\rangle \arrow[r] & 0
\end{tikzcd}$$

Proof. By 6, the cokernel is a direct limit, and by 6.5, direct limits commute;

To construct the desired counterexample. View its rows as expressing the cokernel $\mathbb{Z}/\langle 2\rangle$ as a direct limit over the category Λ of 6. View the left two columns as expressing a natural transformation $\{\theta_\lambda\}$ and view the third column as expressing the induced map between the two limits. The latter map is 0, so its kernel is $\mathbb{Z}/\langle 2\rangle$. However, $\ker(\theta_\lambda) = 0$ for $\lambda \in \Lambda$; so $\varinjlim \ker(\theta_\lambda) = 0$ □

7 Filtered Direct Limits

Filtered categories

We call a small category Λ **filtered** if

1. given objects κ and λ, for some μ there are maps $\kappa \to \mu$ and $\lambda \to \mu$
2. given two maps $\sigma, \tau : \eta \rightrightarrows \kappa$ with the same source and the same target, for some μ there is a map $\varphi : \kappa \to \mu$ s.t. $\varphi\sigma = \varphi\tau$

Given a category $\mathcal{C}$, we say a functor $\lambda \mapsto M_\lambda$ from Λ to $\mathcal{C}$ is **filtered** if Λ is filtered. It so, then we say that the direct limit $\varinjlim M_\lambda$ is **filtered** if it exists

For example, let Λ be a partially ordered set. Suppose Λ is **directed**; that is, given $\kappa, \lambda \in \Lambda$ there is a μ with $\kappa \le \mu$ and $\lambda \le \mu$. Regard Λ as a category whose objects are its elements and whose sets $\mathrm{Hom}(\kappa, \lambda)$ consist of a single element if $\kappa \le \lambda$ and are empty if not; morphisms can be composed as the ordering is transitive

Exercise 7.0.1. Let R be a ring, M a module, Λ a set, M_λ a submodule for each $\lambda \in \Lambda$. Assume $\bigcup M_\lambda = M$. Assume given $\lambda, \mu \in \Lambda$, there is $\nu \in \Lambda$ s.t. $M_\lambda, M_\mu \subset M_\nu$. Order Λ by inclusion: $\lambda \le \mu$ if $M_\lambda \subset M_\mu$. Prove $M = \varinjlim M_\lambda$

Proof. Let prove that M has the desired UMP. If $M_\lambda \not\subset M_\mu$ and $M_\mu \not\subset M_\lambda$.

$$\begin{tikzcd}
M_\mu \arrow[r, "i_\mu"] \arrow[d, "\beta_\mu"] & M \arrow[d, "\beta"] & M_\lambda \arrow[l, "i_\lambda"'] \arrow[d, "\beta_\lambda"] \\
P \arrow[r, "1_P"] & P & P \arrow[l, "1_P"']
\end{tikzcd}$$

If there is $x \in M_\mu \cap M_\lambda$, since the diagram should be commutative, we have $\beta_\mu(x) = \beta_\lambda(x)$. Hence $\beta = \bigcup \beta_\lambda$ satisfies the condition. And β is unique by the choice of β_λ. β is well-defined since for any β_λ, β_μ, there is a β_ν s.t. $\beta_\nu(m) = \beta_\lambda(m) = \beta_\nu(m)$ □

Exercise 7.0.2. Show that every module M is the filtered direct limit of its finitely generated submodules

Proof. M is the union of all its finitely generated submodules. Any two finitely generated submodules are contained in a third. So by 7.0.1 with Λ the set of all finite subsets of M □

Exercise 7.0.3. Show that every direct sum of modules is the filtered direct limit of its finite direct subsums

Example 7.1. Let Λ be the set of all positive integers, and for each $n \in \Lambda$, set $M_n := \{r/n \mid r \in \mathbb{Z}\} \subset \mathbb{Q}$. Then $\bigcup M_n = \mathbb{Q}$ and $M_m, M_n \subset M_{mn}$. Then 7.0.1 yields $\mathbb{Q} = \varinjlim M_n$ where Λ is ordered by inclusion of the M_n

We view Λ as ordered by divisibility

For each $n \in \Lambda$, set $R_n := \mathbb{Z}$, and define $\beta_n : R_n \to M_n$ by $\beta_n(r) := r/n$. Clearly β_n is a $\mathbb{Z}$-module isomorphism. And if $n = ms$, then this diagram is commutative

$$\begin{tikzcd} R_m \arrow[r, "\mu_s"] \arrow[d, "\beta_m", "\simeq"'] & R_n \arrow[d, "\beta_n", "\simeq"'] \\ M_m \arrow[r, hook, "\iota_n^m"] & M_n \end{tikzcd}$$

where ι_n^m is the inclusion. Hence $\mathbb{Q} = \varinjlim R_n$, where the transition maps are the multiplication maps μ_s

Exercise 7.0.4. Keep the setup of 7.1. For each $n \in \Lambda$, set $N_n := \mathbb{Z}/\langle n \rangle$; if $n = ms$, define $\alpha_n^m : N_m \to N_n$ by $\alpha_n^m(x) := xs \mod n$. Show $\varinjlim N_n = \mathbb{Q}/\mathbb{Z}$

Proof. For each $n \in \Lambda$, set $Q_n := M_n/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$, if $n = ms$, then

$$\begin{tikzcd} N_m \arrow[r, "\alpha_n^m"] \arrow[d, "\gamma_m", "\simeq"'] & N_n \arrow[d, "\gamma_n", "\simeq"'] \\ Q_m \arrow[r, hook, "\eta_n^m"] & Q_n \end{tikzcd}$$

□

Proposition 7.1. *Let Λ be a filtered category, R a ring, and $\mathcal{C}$ either **Sets** or R-**Mod** or R-**Alg**. Let $\lambda \mapsto M_\lambda$ be a functor from Λ to $\mathcal{C}$. Define a relation $\sim$ on the disjoint union $\bigsqcup M_\lambda$ as follows: $m_1 \sim m_2$ for $m_i \in M_{\lambda_i}$ if there are transitive maps $\alpha_\mu^{\lambda_i} : M_{\lambda_i} \to M_\mu$ s.t. $\alpha_\mu^{\lambda_1} m_1 = \alpha_\mu^{\lambda_2} m_2$. Then $\sim$ is an equivalence relation. Set $M := (\bigsqcup M_\lambda)/\sim$. Then $M = \varinjlim M_\lambda$ and for each μ, the canonical map $\alpha_\mu : M_\mu \to M$ is equal to the insertion map $M_\mu \to \varinjlim M_\lambda$*

Proof. $\sim$ is transitive. Suppose $\alpha_\mu^{\lambda_1} m_1 = \alpha_\mu^{\lambda_2} m_2$ and $\alpha_\nu^{\lambda_2} m_2 = \alpha_\nu^{\lambda_3} m_3$. Then since Λ is filtered. There is M_ρ with α_ρ^μ and α_ρ^ν. Hence there is α_σ^ρ with $\alpha_\sigma^\rho(\alpha_\rho^\mu \alpha_\mu^{\lambda_2}) = \alpha_\sigma^\rho(\alpha_\rho^\nu \alpha_\nu^{\lambda_2})$. Hence $(\alpha_\sigma^\rho \alpha_\rho^\mu)\alpha_\mu^{\lambda_1} m_1 = (\alpha_\sigma^\rho \alpha_\rho^\nu)\alpha_\nu^{\lambda_3} m_3$. Thus $m_1 \sim m_3$

If $\mathcal{C} = $ R-**Mod**, define addition in M as follows. Given $m_i \in M_{\lambda_i}$ for $i = 1, 2$, there are $\alpha_\mu^{\lambda_i}$. Set

$$\alpha_{\lambda_1} m_1 + \alpha_{\lambda_2} m_2 := \alpha_\mu(\alpha_\mu^{\lambda_1} m_1 + \alpha_\mu^{\lambda_2} m_2)$$

We must check that this addition is well defined

First consider μ. Suppose there are $\alpha_\nu^{\lambda_i}$ too. Then there are α_ρ^μ and α_ρ^ν. Furthermore, there is α_σ^ρ with $\alpha_\sigma^\rho(\alpha_\rho^\mu \alpha_\mu^{\lambda_1}) = \alpha_\sigma^\rho(\alpha_\rho^\nu \alpha_\nu^{\lambda_1})$ and then α_τ^σ with $\alpha_\tau^\sigma(\alpha_\sigma^\rho \alpha_\rho^\mu \alpha_\mu^{\lambda_2}) = \alpha_\tau^\sigma(\alpha_\sigma^\rho \alpha_\rho^\nu \alpha_\nu^{\lambda_2})$. Therefore

$$(\alpha_\tau^\sigma \alpha_\sigma^\rho \alpha_\rho^\mu)(\alpha_\mu^{\lambda_1} m_1 + \alpha_\mu^{\lambda_2} m_2) = (\alpha_\tau^\sigma \alpha_\sigma^\rho \alpha_\rho^\nu)(\alpha_\nu^{\lambda_1} m_1 + \alpha_\nu^{\lambda_2} m_2)$$

Thus both μ and ν yields the same value for $\alpha_{\lambda_1} m_1 + \alpha_{\lambda_2} m_2$

Second, suppose $m_1 \sim m_1' \in M_{\lambda_1'}$. Then $\alpha_{\lambda_1} m_1 + \alpha_{\lambda_2} m_2 = \alpha_{\lambda_1'} m_1' + \alpha_{\lambda_2} m_2$. Thus addition is well defined on M

Define scalar multiplication on M similarly

Finally, let $\beta_\lambda : M_\lambda \to N$ be maps with $\beta_\lambda \alpha_\lambda^\kappa = \beta_\kappa$ for all α_λ^κ. The β_λ induce a map $\bigsqcup M_\lambda \to N$. Suppose $m_1 \sim m_2$ for $m_1 \in M_{\lambda_i}$; that is, $\alpha_\mu^{\lambda_1} m_1 = \alpha_\mu^{\lambda_2} m_2$ for some $\alpha_\mu^{\lambda_i}$. Then $\beta_{\lambda_1} m_1 = \beta_{\lambda_2} m_2$ as $\beta_\mu \alpha_\mu^{\lambda_i} = \beta_{\lambda_i}$. So there is a unique map $\beta : M \to N$ with $\beta\alpha_\lambda = \beta_\lambda$ □

Corollary 7.2. *Preserve the conditions of 7.1*

1. *Given $m \in \varinjlim M_\lambda$ for some λ, there is $m_\lambda \in M_\lambda$ s.t. $m = a_\lambda m_\lambda$*
2. *Given $m_i \in M_{\lambda_i}$ for $i = 1, 2$ s.t. $\alpha_{\lambda_1} m_1 = \alpha_{\lambda_2} m_2$, there are $\alpha_\mu^{\lambda_i}$ s.t. $\alpha_\mu^{\lambda_1} m_1 = \alpha_\mu^{\lambda_2} m_2$*
3. *Suppose $\mathcal{C} = $ R-**Mod** or $\mathcal{C} = $ R-**Alg**. Then given $m_\lambda \in M_\lambda$ s.t. $\alpha_\lambda m_\lambda = 0$, there is α_μ^λ s.t. $\alpha_\mu^\lambda m_\lambda = 0$*

Exercise 7.0.5. Let R

Theorem 7.3 (Exactness of filtered direct limits)**.** *Let R be a ring, Λ a filtered category. Let $\mathcal{C}$ be the category of 3-term exact sequences of R-modules: its objects are the 3-term exact sequences and its maps are the commutative diagrams*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[d] \\ L' \arrow[r] & M' \arrow[r] & N' \end{tikzcd}$$

Then for any functor $\lambda \mapsto (L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda)$ from Λ to $\mathcal{C}$, the induced sequence $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact

8 Tensor Products

Bilinear maps

Let R be a ring, and M, N, P modules. We call a map

$$\alpha : M \times N \to P$$

bilinear if it is linear in each variable; that is, given $m \in M$ and $n \in N$, the maps

$$m' \mapsto \alpha(m', n) \quad \text{and} \quad n' \mapsto \alpha(m, n')$$

are R-linear. Denote the set of all these maps by $\mathrm{Bil}_R(M, N; P)$.

Tensor product

Let R be a ring, and M, N modules. Their **tensor product**, denoted by $M \otimes_R N$, or simply $M \otimes N$, is constructed as the quotient of the free module $R^{\oplus(M\times N)}$ modulo the submodule generated by the following elements, where (m, n) stands for the standard basis element $e_{(m,n)}$:

$$\begin{aligned} (m + m', n) - (m, n) - (m', n) \quad &\text{and} \quad (m, n + n') - (m, n) - (m, n') \\ (xm, n) - x(m, n) \quad &\text{and} \quad (m, xn) - x(m, n) \end{aligned} \tag{8.0.1}$$

for all $m, m' \in M$ and $n, n' \in N$ and $x \in R$. Hence we have distributivity and scalar multiplication

Note that $M \otimes N$ is the target of the canonical map with source $M \times N$

$$\beta : M \times N \to M \otimes N$$

which sends each (m, n) to its residue class $m \otimes n$. By construction, β is bilinear

Theorem 8.1 (UMP of tensor product)**.** *Let R be a ring, M, N modules. Then $\beta : M \times N \to M \otimes N$ is the universal example of a bilinear map with source $M \times N$; in fact, β induces a module isomorphism*

$$\theta : \mathrm{Hom}_R(M \otimes_R N, P) \xrightarrow{\sim} \mathrm{Bil}_R(M, N; P)$$

Proof. There is a obvious linear map $\psi : M \otimes N \to P$ with

$$\psi(r_1(m_1, n_1) + \cdots + (r_s(m_s, n_s))) = r_1\psi(m_1, n_1) + \cdots + r_s\psi(m_s, n_s)$$

Note that, if we follow any bilinear map with any linear map, then the composition is bilinear; hence θ is well-defined. Clearly, θ is a module homomorphism. Further, θ is injective since $M \otimes_R N$ is generated by the image of β (Suppose for $\alpha \in \mathrm{Hom}(M \otimes N, P)$, then $\theta : \alpha \mapsto \alpha\beta$. If $\alpha\beta = \alpha'\beta$, then $\alpha(x) = \alpha'(x)$ for all $x \in \mathrm{im}(\beta)$. But $M \otimes N$ is generated by $\mathrm{im}(\beta)$, hence $\alpha(x) = \alpha'(x)$ for all $x \in M \otimes N$). Finally, given any bilinear map $\alpha : M \times N \to P$, by 4, it extends to a map $\alpha' : R^{\oplus(M \times N)} \to P$, and α' carries all the elements in (8.0.1) to 0; hence α' factors through β. Thus β is also surjective □

Bifunctoriality

Let R be a ring, $\alpha : M \to M'$ and $\alpha' : N \to N'$ module homomorphisms. Then there is a canonical commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "\alpha\times\alpha'"] \arrow[d, "\beta"] & M' \times N' \arrow[d, "\beta'"] \\ M \otimes N \arrow[r, "\alpha\otimes\alpha'"] & M' \otimes N' \end{tikzcd}$$

Indeed, $\beta \circ (\alpha \times \alpha')$ is bilinear; so the UMP yields $\alpha \otimes \alpha'$. Thus $\bullet \otimes N$ and $M \otimes \bullet$ are commuting **linear** functors, that is, linear on maps

Proposition 8.2. *Let R be a ring, M and N modules*

1. *Then the switch map $M \times N \to N \times M$ induces an isomorphism*

$$M \otimes_R N = N \otimes_R M \qquad (\textbf{commutative law})$$

2. *Then multiplication of R on M induces an isomorphism*

$$R \otimes_R M = M \qquad \textbf{(unitary law)}$$

Proof. The switch map induces an isomorphism $R^{\oplus(M\times N)} \xrightarrow{\sim} R^{\oplus(N\times M)}$

Define $\beta : R \times M \to M$ by $\beta(x, m) := xm$. Clearly β is bilinear. Let's check β has the required UMP. Given a bilinear map $\alpha : R \times M \to P$, define $\gamma : M \to P$ by $\gamma(m) := \alpha(1, m)$. Then γ is linear as α is bilinear. Also $\alpha = \gamma\beta$ as

$$\alpha(x, m) = x\alpha(1, m) = \alpha(1, xm) = \gamma(xm) = \gamma\beta(x, m)$$

Further, γ is unique as β is surjective $\square$

Exercise 8.0.1. Let R be a domain, $\mathfrak{a}$ a nonzero ideal. Set $K := \mathrm{Frac}(R)$. Show that $\mathfrak{a} \otimes_R K = K$

Proof. Note that $1 \notin \mathfrak{a}$. $\square$

Bimodules

Let R and R' be rings. An abelian group N is an (R, R')**-bimodule** if it is both an R-module and an R'-module and if $x(x'n) = x'(xn)$ for all $x \in R$, all $x' \in R'$ and all $n \in N$. We can think of N as a left R-module with multiplication xn, and as a right R'-module, with multiplication nx'. Then the compatibility condition becomes the associative law: $x(nx') = (xn)x'$. A (R, R')**-homomorphism** of bimodule is a map that is both R-linear and R'-linear.

Let M be an R-module, and let N be an (R, R')-bimodule. Then $M \otimes_R N$ is an (R, R')-bimodule with R-structure as usual and with R'-structure defined by $x'(m \otimes n) := m \otimes (x'n)$

For instance, suppose R' is an R-algebra. Then R' is an (R, R')-bimodule. So $M \otimes_R R'$ is an R'-module. It is said to be obtained by **extension of scalars**

Exercise 8.0.2. Let R be a ring, R' an R-algebra, M, N two R'-modules. Show there is a canonical R-linear map $\tau : M \otimes_R N \to M \otimes_{R'} N$

Let $K \subset M \otimes_R N$ denote the R-submodule generated by all the differences $(x'm) \otimes n - m \otimes (x'n)$ for $x' \in R'$ and $m \in M$ and $n \in N$. Show $K = \ker(\tau)$. Show τ is surjective and is an isomorphism if R' is a quotient of R

Proof. The canonical map $\beta' : M \times N \to M \otimes_{R'} N$ is R'-bilinear, so R-bilinear

Set $Q := (M \otimes_R N)/K$. Then τ factors through a map $\tau' : Q \to M \otimes_{R'} N$ since $K \subset \ker(\tau)$

There is an R'-structure on $M \otimes_R N$ with $y'(m \otimes n) = m \otimes (y'n)$ and so by 8.2, another one with $y'(m \otimes n) = (y'm) \otimes n$. Clearly K is a submodule for each structure, so Q is too. But on Q the two structure coincide.

Further, the canonical map $M \times N \to Q$ is R'-bilinear. Hence the latter factors through $M \otimes_{R'} N$, furnishing an inverse to τ'. So $\tau' : Q \xrightarrow{\sim} M \otimes_{R'} N$. Hence $\ker(\tau) = K$ and τ is surjective

Finally, suppose R' is a quotient of R. Then every $x' \in R'$ is the residue of some $x \in R$. So each $(x'm) \otimes n - m \otimes (x'n)$ is equal to 0 in $M \otimes_R N$ as $x'm = xm$ and $x'n = xn$. Hence $\ker(\tau)$ vanishes. □

Theorem 8.3. *Let R and R' be rings, M an R-module, P an R'-module, N an (R, R')-bimodule. Then there is two canonical (R, R')-isomorphisms*

$$M \otimes_R (N \otimes_{R'} P) = (M \otimes_R N) \otimes_{R'} P \qquad (\textbf{associative law})$$

$$\mathrm{Hom}_{R'}(M \otimes_R N, P) = \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N, P)) \qquad (\textbf{adjoint associativity})$$

Proof. Note that $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ are (R, R')-bimodules. For each (R, R')-bimodule Q, call a map $\tau : M \times N \times P \to Q$ **trilinear** if it is R-bilinear in $M \times N$ and R'-bilinear in $N \times P$. Denote the set of all these τ by $\mathrm{Tril}(M, N, P; Q)$. It is an (R, R')-bimodule

A trilinear map τ yields an R-bilinear map $M \times (N \otimes_{R'} P) \to Q$, whence a map $M \otimes_R (N \otimes_{R'} P) \to Q$ which is both R-linear and R'-linear, and vice versa. Thus

$$\mathrm{Tril}_{(R,R')}(M, N, P; Q) = \mathrm{Hom}(M \otimes_R (N \otimes_{R'} P), Q)$$

Similarly, there is a canonical isomorphism of (R, R')-bimodules

$$\mathrm{Tril}_{(R,R')}(M, N, P; Q) = \mathrm{Hom}((M \otimes_R N) \otimes_{R'} P, Q)$$

Hence both $M \otimes_R (N \otimes_{R'} P)$ and $(M \otimes_R N) \otimes_{R'} P$ are universal examples of a target of a trilinear map with source $M \times N \times P$

To establish the isomorphism of adjoint associativity, define a map

$$\alpha : \mathrm{Hom}_{R'}(M \otimes_R N, P) \to \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N, P))$$
$$(\alpha(\gamma)(m))(n) := \gamma(m \otimes n)$$

Let's check that α is well defined. First, $\alpha(\gamma)(m)$ is R'-linear, because given $x' \in R'$

$$\gamma(m \otimes (x'n)) = \gamma(x'(m \otimes n)) = x'\gamma(m \otimes n)$$

since γ is R'-linear. Hence $\alpha(\gamma)(m) \in \mathrm{Hom}_{R'}(N,P)$. Further, $\alpha(\gamma)$ is R-linear, because given $x \in R$,

$$(xm) \otimes n = m \otimes (xn) \quad \text{and so} \quad (\alpha(\gamma)(xm))(n) = (\alpha(\gamma)(m))(xn)$$

Thus $\alpha(\gamma) \in \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N,P))$. Clearly α is an (R,R')-homomorphism

To obtain an inverse to α, given $\eta \in \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(N,P))$, define a map $\zeta : M \times N \to P$ by $\zeta(m,n) := (\eta(m))(n)$. Clearly, ζ is $\mathbb{Z}$-bilinear, so ζ induces a $\mathbb{Z}$-linear map $\delta : M \otimes_{\mathbb{Z}} N \to P$. Given $x \in R$, □

Corollary 8.4. *Let R and R' be rings, M an R-module, P an R'-module. If R' is an R-algebra, then there are two canonical (R,R')-isomorphisms*

$$(M \otimes_R R') \otimes_{R'} P = M \otimes_R P \qquad (\textbf{cancellation law})$$
$$\mathrm{Hom}_{R'}(M \otimes_R R', P) = \mathrm{Hom}_R(M,P) \qquad (\textbf{left adjoint})$$

Instead, if R is an R'-algebra, then there is another canonical (R,R')-isomorphism:

$$\mathrm{Hom}_{R'}(M,P) = \mathrm{Hom}_R(M, \mathrm{Hom}_{R'}(R,P)) \qquad (\textbf{right adjoint})$$

In other words, $\bullet \otimes_R R'$ is the left adjoint of restriction of scalars from R' to R, and $\mathrm{Hom}_{R'}(R, \bullet)$ is the right adjoint of restriction of scalars from R to R'

Corollary 8.5. *Let R, R' be rings, N a bimodule. Then the functor $\bullet \otimes_R N$ preserves direct limits, or equivalently, direct sums and cokernels*

Example 8.1. Tensor product does not preserve kernels, not even injections. Indeed, consider the injection $\mu_2 : \mathbb{Z} \to \mathbb{Z}$. Tensor it with $N := \mathbb{Z}/\langle 2 \rangle$, containing $\mu_2 : N \to N$. This map is zero, but not injective as $N \neq 0$

$\mu_2 : x \mapsto 2x$

Exercise 8.0.3. Let R be a ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals, and M a module

1. Use 8.5 to show that $(R/\mathfrak{a}) \otimes M = M/\mathfrak{a}M$
2. Use (1) to show that $(R/\mathfrak{a}) \otimes (R/\mathfrak{b}) = R/(\mathfrak{a} + \mathfrak{b})$

Proof.
1. View $R/\mathfrak{a}$ as the cokernel of the inclusion $\mathfrak{a} \to R$. Then 8.5 implies that $(R/\mathfrak{a}) \otimes M$ is the cokernel of $\mathfrak{a} \otimes M \to R \otimes M$. Now $R \otimes M = M$ and $x \otimes m = xm$ by 8.2. Correspondingly, $\mathfrak{a} \otimes M \to M$ has $\mathfrak{a}M$ as image. (Caution. $\mathfrak{a} \otimes M \to M$) needn't be injective; if it's not, then $\mathfrak{a} \otimes M \neq \mathfrak{a}M$. For example, take $R := \mathbb{Z}$, take $\mathfrak{a} := \langle 2 \rangle$, $M := \mathbb{Z}/\langle 2 \rangle$, then $\mathfrak{a}M = 0$)
2. Note that $\mathfrak{a}(R/\mathfrak{b}) = \mathfrak{a}/\mathfrak{b} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$. Hence

$$R/\mathfrak{a} \otimes R/\mathfrak{b} = (R/\mathfrak{b})/((\mathfrak{a} + \mathfrak{b})/\mathfrak{b})$$

□

Exercise 8.0.4. Let k be a field, M and N nonzero vector spaces. Prove that $M \otimes N \neq 0$

Proof. Since k is a filed, M and N are free; say $M = k^{\oplus \phi}$ and $N = k^{\oplus \psi}$. Then 8.5 yields $M \otimes N = k^{\oplus(\phi \times \psi)}$ as $k \otimes k = k$ □

Theorem 8.6 (Watts). *Let F : R-**Mod** → R-**Mod** be a linear functor. Then there is a natural transformation $\theta(\bullet) : \bullet \otimes F(R) \to F(\bullet)$ with $\theta(R) = 1$ and $\theta(\bullet)$ is an isomorphism iff F preserves direct sums and cokernels*

Proof. As F is linear, there is a natural R-linear map $\theta(M) : \mathrm{Hom}(R, M) \to \mathrm{Hom}(F(R), F(M))$. But $\mathrm{Hom}(R, M) = M$ by 4.0.1. Set $N := F(R)$. Then with $P := F(M)$ adjoint associativity yields the desired map

$$\theta(M) \in \mathrm{Hom}(M, \mathrm{Hom}(N, F(M))) = \mathrm{Hom}(M \otimes N, F(M))$$

Explicitly, $\theta(M)(m \otimes n) = F(\rho)(n)$ where $\rho : R \to M$ is defined by $\rho(1) = m$. Alternatively, this formula can be used to construct $\theta(M)$, as $(m, n) \mapsto F(\rho)(n)$ is clearly bilinear. □

9 TODO Problems

6 6.0.4