考研题目本

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目录

1 微积分

Example 1.1. 设 $f'(x)$ 连续，$f(0) = 0, f'(0) \neq 0$，求 $\lim_{x\to 0} \frac{\int_0^{x^2} f(x^2-t)dt}{x^3\int_0^1 f(xt)dt}$

令 $x^2 - t = u, xt = u$

$$\begin{aligned}
\lim_{x\to 0} \frac{\int_0^{x^2} f(x^2-t)dt}{x^3\int_0^1 f(xt)dt} &= \lim_{x\to 0} \frac{-\int_{x^2}^{0} f(u)du}{x^3\int_0^x f(u)\frac{du}{x}} = \lim_{x\to 0} \frac{\int_0^{x^2} f(u)du}{x^2\int_0^x f(u)du} \\
&= \lim_{x\to 0} \frac{2xf(x^2)}{2x\int_0^x f(u)du + x^2 f(x)} \\
&= \lim_{x\to 0} \frac{2f(x^2)}{2\int_0^x f(u)du + xf(x)} \\
&= \lim_{x\to 0} \frac{4xf'(x^2)}{3f(x) + xf'(x)} \\
&= \lim_{x\to 0} \frac{4f'(x^2)}{3\frac{f(x)-f(0)}{x} + f'(x)} = 1
\end{aligned}$$

Example 1.2. 求 $\lim_{x\to 0} \frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x - e^{x^2})\sin x^2}$

利用泰勒展开，$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$, $\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$,$e^{x^2} = 1 + x^2 + o(x^2)$，因此

$$\lim_{x\to 0} \frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x - e^{x^2})\sin x^2} = \lim_{x\to 0} \frac{\frac{x^4}{8} + o(x^4)}{-\frac{3}{2}x^4 + o(x^4)} = -\frac{1}{12}$$

Example 1.3. 求 $\lim_{n\to\infty} \tan^n(\frac{\pi}{4} + \frac{2}{n})$

因为 $\lim_{x\to\infty} f(x) = A \Rightarrow \lim_{n\to\infty} f(n) = A$

Example 1.4. suppose $y_n = \left[\frac{(2n)!}{n!n^n}\right]^{\frac{1}{n+1}}$. Compute $\lim_{n\to\infty} y_n$

$$\begin{aligned}
\ln y_n &= \frac{1}{n+1}\ln\frac{(2n)!}{n!n^n} = \frac{1}{n+1}\ln\frac{(2n)(2n-1)\dots(n+1)}{n^n} \\
&= \frac{1}{n+1}\sum_{k=1}^{n}\ln(1+\frac{k}{n}) = \frac{n}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n}\ln(1+\frac{k}{n})\right)
\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty}\frac{n}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n}\ln(1+\frac{k}{n})\right)\\ &= 1\cdot\int_0^1 \ln(1+x)dx = x\ln(1+x)|_0^1 - \int_0^1\frac{x}{1+x}dx\\ &= \ln 2 - 1 + \ln 2 = \ln\frac{4}{e}\end{aligned}$$

Example 1.5. 已知 $x\to 0$ 时，$e^{-x^4}-\cos(\sqrt{2}x^2)$ 与 ax^n 是等价无穷小，试求 a, n

$$\begin{aligned}e^{-x^4} &= 1 - x^4 + \frac{x^8}{2} + o(x^8)\\ \cos(\sqrt{2}x^2) &= 1 - x^4 + \frac{x^8}{6} + o(x^8)\end{aligned}$$

Hence $a = \frac{1}{3}, n = 8$

Example 1.6. 设 $f(x) = \dfrac{\sqrt{1+\sin x+\sin^2 x}-(\alpha+\beta\sin x)}{\sin^2 x}$，且点 $x=0$ 是 $f(x)$ 的可去间断点，求 α, β

由极限存在可知，$\alpha = 1$，泰勒展开

$$\begin{aligned}&\frac{\sqrt{1+\sin x+\sin^2 x}-(\alpha+\beta\sin x)}{\sin^2 x}\\ &= \lim_{x\to 0}\frac{1+\frac{1}{2}(\sin x+\sin^2 x)-\frac{1}{8}(\sin x+\sin^2 x)^2-(1+\beta\sin x)+o(\sin^2 x)}{\sin^2}\\ &= \lim_{x\to 0}\frac{(\frac{1}{2}-\beta)\sin x+\frac{3}{8}\sin^2 x}{\sin^2 x}\end{aligned}$$

故 $\beta = \frac{1}{2}$

Example 1.7. let $f(x) = \lim_{n\to\infty}\frac{2x^n-3x^{-n}}{x^n+x^{-n}}\sin\frac{1}{x}$

$$f(x) = \begin{cases} 2\sin\frac{1}{x}^x & x < -1\\ -\frac{1}{2}\sin\frac{1}{x} & x = -1\\ -3\sin\frac{1}{x} & -1 < x < 0\\ -3\sin\frac{1}{x} & 0 < x < 1\\ -\frac{1}{2}\sin\frac{1}{x} & x = 1\\ 2\sin\frac{1}{x}^x & x > 1\end{cases}$$

$x = 0$ 是第二类间断点，$x = \pm 1$ 是第一类间断点

Example 1.8. 设 $f(1) = 0, f'(1) = a$，求极限