

# 考研题目本

五狗砸

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# 1 微积分

## 1.1 一元函数微分

**Example 1.1.** 设  $f'(x)$  连续,  $f(0) = 0, f'(0) \neq 0$ , 求  $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} f(x^2 - t) dt}{x^3 \int_0^1 f(xt) dt}$

令  $x^2 - t = u, xt = u$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\int_0^{x^2} f(x^2 - t) dt}{x^3 \int_0^1 f(xt) dt} &= \lim_{x \rightarrow 0} \frac{-\int_{x^2}^0 f(u) du}{x^3 \int_0^x f(u) \frac{du}{x}} = \lim_{x \rightarrow 0} \frac{\int_0^{x^2} f(u) du}{x^2 \int_0^x f(u) du} \\ &= \lim_{x \rightarrow 0} \frac{2xf(x^2)}{2x \int_0^x f(u) du + x^2 f(x)} \\ &= \lim_{x \rightarrow 0} \frac{2f(x^2)}{2 \int_0^x f(u) du + xf(x)} \\ &= \lim_{x \rightarrow 0} \frac{4xf'(x^2)}{3f(x) + xf'(x)} \\ &= \lim_{x \rightarrow 0} \frac{4f'(x^2)}{3 \frac{f(x) - f(0)}{x} + f'(x)} = 1 \end{aligned}$$

**Example 1.2.** 求  $\lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1+x^2}}{(\cos x - e^{x^2}) \sin x^2}$

利用泰勒展开,  $\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$ ,  $\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$ ,  $e^{x^2} = 1 + x^2 + o(x^2)$ , 因此

$$\lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1+x^2}}{(\cos x - e^{x^2}) \sin x^2} = \lim_{x \rightarrow 0} \frac{\frac{x^4}{8} + o(x^4)}{-\frac{3}{2}x^4 + o(x^4)} = -\frac{1}{12}$$

**Example 1.3.** 求  $\lim_{n \rightarrow \infty} \tan^n(\frac{\pi}{4} + \frac{2}{n})$

因为  $\lim_{x \rightarrow \infty} f(x) = A \Rightarrow \lim_{n \rightarrow \infty} f(n) = A$

**Example 1.4.** suppose  $y_n = \left[ \frac{(2n)!}{n!n^n} \right]^{\frac{1}{n+1}}$ . Compute  $\lim_{n \rightarrow \infty} y_n$

$$\begin{aligned} \ln y_n &= \frac{1}{n+1} \ln \frac{(2n)!}{n!n^n} = \frac{1}{n+1} \ln \frac{(2n)(2n-1) \dots (n+1)}{n^n} \\ &= \frac{1}{n+1} \sum_{k=1}^n \ln(1 + \frac{k}{n}) = \frac{n}{n+1} \left( \frac{1}{n} \sum_{k=1}^n \ln(1 + \frac{k}{n}) \right) \end{aligned}$$

Hence

$$\begin{aligned}\lim_{n \rightarrow \infty} y_n &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \left( \frac{1}{n} \sum_{k=1}^n \ln\left(1 + \frac{k}{n}\right) \right) \\ &= 1 \cdot \int_0^1 \ln(1+x) dx = x \ln(1+x) \Big|_0^1 - \int_0^1 \frac{x}{1+x} dx \\ &= \ln 2 - 1 + \ln 2 = \ln \frac{4}{e}\end{aligned}$$

**Example 1.5.** 已知  $x \rightarrow 0$  时,  $e^{-x^4} - \cos(\sqrt{2}x^2)$  与  $ax^n$  是等价无穷小, 试求  $a, n$

$$\begin{aligned}e^{-x^4} &= 1 - x^4 + \frac{x^8}{2} + o(x^8) \\ \cos(\sqrt{2}x^2) &= 1 - x^4 + \frac{x^8}{6} + o(x^8)\end{aligned}$$

Hence  $a = \frac{1}{3}, n = 8$

**Example 1.6.** 设  $f(x) = \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x}$ , 且点  $x = 0$  是  $f(x)$  的可去间断点, 求  $\alpha, \beta$

由极限存在可知,  $\alpha = 1$ , 泰勒展开

$$\begin{aligned}& \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{1 + \frac{1}{2}(\sin x + \sin^2 x) - \frac{1}{8}(\sin x + \sin^2 x)^2 - (1 + \beta \sin x) + o(\sin^2 x)}{\sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{(\frac{1}{2} - \beta) \sin x + \frac{3}{8} \sin^2 x}{\sin^2 x}\end{aligned}$$

故  $\beta = \frac{1}{2}$

**Example 1.7.** let  $f(x) = \lim_{n \rightarrow \infty} \frac{2x^n - 3x^{-n}}{x^n + x^{-n}} \sin \frac{1}{x}$

$$f(x) = \begin{cases} 2 \sin \frac{1}{x} & x < -1 \\ -\frac{1}{2} \sin \frac{1}{x} & x = -1 \\ -3 \sin \frac{1}{x} & -1 < x < 0 \\ -3 \sin \frac{1}{x} & 0 < x < 1 \\ -\frac{1}{2} \sin \frac{1}{x} & x = 1 \\ 2 \sin \frac{1}{x} & x > 1 \end{cases}$$

$x = 0$  是第二类间断点,  $x = \pm 1$  是第一类间断点

**Example 1.8.** 设  $f(1) = 0, f'(1) = a$ , 求极限  $\lim_{x \rightarrow 0} \frac{\sqrt{1+2f(e^{x^2})} - \sqrt{1+f(1+\sin^2 x)}}{\ln \cos x}$

由  $f(1) = 0, f'(1) = a$  可知,  $f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{t \rightarrow 0} \frac{f(1+t)}{t} = a$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+2f(e^{x^2})} - \sqrt{1+f(1+\sin^2 x)}}{\ln \cos x} &= \frac{2f(e^{x^2}) - f(1+\sin^2 x)}{-\frac{1}{2}x^2 \left[ \sqrt{1+2f(e^{x^2})} + \sqrt{1+f(1+\sin^2 x)} \right]} \\ &= \lim_{x \rightarrow 0} \frac{f(1+\sin^2 x) - f(e^{x^2})}{x^2} \\ &= \lim_{x \rightarrow 0} \left[ \frac{f(1+\sin^2 x) - f(1)}{\sin^2 x} \cdot \frac{\sin^2 x}{x^2} - \frac{f(e^{x^2}) - f(1)}{e^{x^2} - 1} \cdot \frac{e^{x^2} - 1}{x^2} \right] \\ &= -a \end{aligned}$$

**Example 1.9.** 设  $f(x)$  在  $x = 0$  的某邻域内二阶可导, 且  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0, f''(0) \neq 0$ ,

$0, \lim_{x \rightarrow 0^+} \frac{\int_0^x f(t) dt}{x^\alpha - \sin x} = \beta (\beta \neq 0)$ , 求  $\alpha, \beta$

因为  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0, f(0) = 0, f'(0) = 0$

因为  $\lim_{x \rightarrow 0^+} \int_0^x f(t) dt = 0$ , 因此  $\lim_{x \rightarrow 0^+} x^\alpha - \sin x = 0$ , 因此  $\alpha > 0$

1. 若  $0 < \alpha < 1$

2. 若  $\alpha > 1$

3. 若  $\alpha = 1$

$$\beta = f''(0)$$

**Example 1.10.** 设  $f(x)$  在  $(-\infty, +\infty)$  上有定义, 且  $f'(0) = 1, f(x+y) = f(x)e^y + f(y)e^x$ , 求  $f(x)$

$$f(0) = 0$$

$$\begin{aligned}
f'(x) &= \lim_{y \rightarrow 0} \frac{f(x+y) - f(x)}{y} \\
&= \lim_{y \rightarrow 0} \frac{f(x)e^y + f(y)e^x - f(x)}{y} \\
&= \lim_{y \rightarrow 0} \left[ f(x) \frac{e^y - 1}{y} + e^x \frac{f(y) - f(0)}{y} \right] \\
&= f(x) + e^x f'(0) = f(x) + e^x
\end{aligned}$$

即  $f'(x) - f(x) = e^x$ , 因此  $f(x) = e^x(x+C)$ , 又  $f(0) = 0, C = 0, f(x) = xe^x$

**Example 1.11.** 已知函数  $f(x) = \begin{cases} x & x \leq 0 \\ \frac{1}{n} & \frac{1}{n+1} < x \leq \frac{1}{n} \end{cases}$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{n}}{x} \left( \frac{1}{n+1} < x \leq \frac{1}{n} \right)$$

而  $1 \leq \frac{1}{n} < \frac{n+1}{n}$ , 由夹逼准则得  $f'_+(0) = 1$ , 因此  $f'(0) = 1$

**Example 1.12.** 设  $f(x)$  是可导的偶函数, 它在  $x = 0$  的某邻域内满足

$$f(e^{x^2}) - 3f(1 + \sin x^2) = 2x^2 + o(x^2)$$

求曲线  $y = f(x)$  在点  $(-1, f(-1))$  处的切线方程

由

$$\lim_{x \rightarrow 0} \frac{f(e^{x^2}) - 3f(1 + \sin x^2) - 2x^2}{x^2} = 0$$

得

$$f(0) - 3f(1) = 0 \Rightarrow f(1) = 0$$

变形

$$\lim_{x \rightarrow 0} \left( \frac{f(e^{x^2})}{e^{x^2} - 1} \cdot \frac{e^{x^2} - 1}{x^2} - \frac{3f(1 + \sin x^2)}{\sin x^2} \cdot \frac{\sin x^2}{x^2} - 2 \right) = 0$$

有  $f'(1) - 3f'(1) - 2 = 0 \Rightarrow f'(1) = -1$

**Example 1.13.** 若  $y = f(x)$  存在单值反函数, 且  $y' \neq 0$ , 求  $\frac{d^2x}{dy^2}$

根据反函数的求导法则  $\frac{dx}{dy} = \frac{1}{y'}$ , 于是

$$\frac{d^2x}{dy^2} = \frac{d}{dy} \left( \frac{dx}{dy} \right) = \frac{d}{dx} \left( \frac{dx}{dy} \right) \frac{dx}{dy}$$

因为  $\frac{1}{y'}$  是以  $x$  为变量的函数

**Example 1.14.** 设函数  $f(x) = \arctan x - \frac{x}{1+ax^2}$ , 且  $f'''(0) = 1$ , 求  $a$  泰勒展开

$$\begin{aligned} f(x) &= \arctan x - \frac{x}{1+ax^2} = \left( x - \frac{x^3}{3} + \dots \right) - x(1 - ax^2 + \dots) \\ &= (a - \frac{1}{3})x^3 + \dots \end{aligned}$$

因此  $f'''(0)/3! = a - 1/3, a = 1/2$

**Example 1.15.** 设  $f(x)$  在  $[a, b]$  上连续且  $f(x) > 0$ , 证明存在  $\xi \in (a, b)$  使得

$$\int_a^\xi f(x)dx = \int_\xi^b f(x)dx = \frac{1}{2} \int_a^b f(x)dx$$

令  $F(x) = \int_a^x f(t)dt - \int_x^b f(t)dt$ , 则  $F(x)$  在  $[a, b]$  上连续, 且

$$F(a)F(b) = - \left[ \int_a^b f(t)dt \right]^2 < 0$$

故由连续函数的零点定理知: 在  $(a, b)$  内存在  $\xi$  使得  $F(\xi) = 0$ , 即  $\int_a^\xi f(x)dx = \int_\xi^b f(x)dx$

**Example 1.16.** 设  $f(x), g(x)$  在  $[a, b]$  上连续, 证明存在  $\xi \in (a, b)$  使得

$$g(\xi) \int_a^\xi f(x)dx = f(\xi) \int_\xi^b g(x)dx$$

令  $F'(x) = g(x) \int_a^x f(x)dx - f(x) \int_x^b g(x)dx = (\int_a^x f(t)dt \int_b^x g(t)dt)'$ , 可取辅助函数  $F(x) = \int_a^x f(t)dt \int_x^b g(t)dt$ . 则  $F(a) = F(b) = 0$ , 则存在  $\xi \in (a, b)$  使得  $F'(\xi) = 0$

**Example 1.17.** 设实数  $a_1, \dots, a_n$  满足关系式  $a_1 - \frac{a_2}{3} + \dots + (-1)^{n-1} \frac{a_n}{2n-1} = 0$ , 证明方程  $a_1 \cos x + a_2 \cos 3x + \dots + a_n \cos(2n-1)x = 0$  在  $(0, \frac{\pi}{2})$  内至少有一实根

令  $f(x) = a_1 \cos x + a_2 \cos 3x + \dots + a_n \cos(2n-1)x$ , 但  $f(x)$  在  $[0, \frac{\pi}{2}]$  内不满足零点定理, 因此考虑  $f'(x) = a_1 \cos x + a_2 \cos 3x + \dots + a_n \cos(2n-1)x$ , 则  $f(x) = a_1 \cos x + \frac{a_2}{3} \sin 3x + \dots + \frac{a_n}{2n-1} \sin(2n-1)x$ , 则  $f(0) = f(\pi/2) = 0$

**Example 1.18.** 试确定方程  $e^x = ax^2 (a > 0)$  的根的个数, 并指出每个根所在的范围

若直接令  $f(x) = e^x - ax^2$ ,  $f'(x)$  的符号不易判断。又  $x = 0$  不是方程的根, 于是方程可化为等价方程  $\frac{e^x}{x^2} = a$

令  $f(x) = \frac{e^x}{x^2} - a$ , 由  $f'(x) = \frac{x-2}{x^3}e^x = 0$  得  $x = 2$

**Example 1.19.** 已知方程  $\frac{1}{\ln(1+x)} - \frac{1}{x} = k$  在区间  $(0, 1)$  内有实根, 确定常数  $k$  的取值范围

令  $f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x} - k$ ,  $x \in (0, 1]$ , 则

$$f'(x) = \frac{(1+x)\ln^2(1+x) - x^2}{x^2(1+x)\ln^2(1+x)}$$

因为  $x^2(1+x)\ln^2(1+x) > 0$ , 因此只讨论  $g(x) = (1+x)\ln^2(1+x) - x^2$ .

$$g'(x) = \ln^2(1+x) + 2\ln(1+x) - 2x$$

$$g''(x) = \frac{2\ln(1+x)}{1+x} + \frac{2}{1+x} - 2 = \frac{2\ln(1+x) - 2x}{1+x}$$

因此当  $x \in (0, 1)$  时,  $g''(x) < 0$ , 而  $g'(0) = 0$ , 因此  $g(x)$  递减

**Example 1.20.** 设  $f(x)$  在  $[0, 3]$  上连续, 在  $(0, 3)$  内可导, 且  $f(0) + f(1) + f(2) = 3, f(3) = 1$ , 证明存在  $\xi \in (0, 3)$  使得  $f'(\xi) = 0$

因为  $f(x)$  在  $[0, 3]$  上连续, 所以在  $[0, 2]$  内必有最大值  $M$  和最小值  $m$ , 于是  $m \leq f(0) \leq M, m \leq f(1) \leq M, m \leq f(2) \leq M$ , 故

$$m \leq \frac{f(0) + f(1) + f(2)}{3} \leq M$$

由介值定理, 至少存在一点  $\eta \in [0, 2]$  使

$$f(\eta) = \frac{f(0) + f(1) + f(2)}{3} = 1$$

因此  $f(\eta) = f(3) = 1$ , 由罗尔定理知, 必存在  $\xi \in (\eta, 3) \subset (0, 3)$  使得  $f'(\xi) = 0$

**Example 1.21.** 设  $f(x)$  在  $[0, 2]$  上连续, 在  $(0, 2)$  内具有二阶导数且  $\lim_{x \rightarrow \frac{1}{2}} \frac{f(x)}{\cos \pi x} = 0, 2 \int_{1/2}^1 f(x)dx = f(2)$ , 证明存在  $\xi \in (0, 2)$  使得  $f''(\xi) = 0$

$f(0.5) = 0$ , 因此

$$f'(0.5) = \lim_{x \rightarrow 0.5} \frac{f(x) - f(0.5)}{x - 0.5} = \lim_{x \rightarrow 0.5} \frac{f(x)}{\cos \pi x} \frac{\cos \pi x}{x - 0.5} = \lim_{x \rightarrow 0.5} \frac{f(x)}{\cos \pi x} \lim_{x \rightarrow 0.5} \frac{\cos \pi x}{x - 0.5} = 0$$

再由  $2 \int_{0.5}^2 f(x) dx = f(2)$ , 用积分中值定理  $\exists \xi_1 \in [0.5, 1]$  使得  $2f(\xi_1)0.5 = f(2)$ , 即  $f(\xi) = f(2)$ , 在  $[\xi_1, 2]$  上应用罗尔定理,  $\exists \xi_2 \in (\xi_1, 2)$  使  $f'(\xi_2) = 0$   
 再在  $[0.5, \xi_2]$  上对  $f'(x)$  应用罗尔定理, 知  $\exists \xi \in (0.5, \xi_2)$ , 使  $f''(\xi) = 0$

**Example 1.22.** 设  $f(x)$  在  $[0, 1]$  上连续,  $(0, 1)$  内可导, 且

$$f(1) = k \int_0^{\frac{1}{k}} x e^{1-x} f(x) dx, k > 1$$

证明: 在  $(0, 1)$  内至少存在一点  $\xi$  使  $f'(\xi) = (1 - \xi^{-1})f(\xi)$

1.  $\xi$  换为  $x$ ,  $f'(x) = (1 - x^{-1})f(x)$
2. 变形  $\frac{f'(x)}{f(x)} = 1 - x^{-1}$
3. 两边积分  $\ln f(x) = x - \ln x + \ln C$
4. 分离常数  $\ln \frac{x f(x)}{e^x} = \ln C$ , 即  $x e^{-x} f(x) = C$ , 可令辅助函数  $F(x) = x e^{-x} f(x)$

由积分中值定理, 存在  $\xi_1 \in [0, \frac{1}{k}]$  使得  $f(1) = \xi_1 e^{1-\xi_1} f(\xi_1)$ , 即  $1 \times e^{-1} f(1) = \xi_1 e^{-\xi_1} f(\xi_1)$ 。因此  $F(x)$  满足在  $[\xi_1, 1]$  内的罗尔定理, 因此存在  $\xi$  使得  $f'(\xi) = (1 - \xi^{-1})f(\xi)$

**Example 1.23.** 设  $f(x)$  在  $[a, b]$  上连续, 在  $(a, b)$  内可导, 且  $f(a) = f(b) = \lambda$ , 证明存在  $\xi \in (a, b)$  使得  $f'(\xi) + f(\xi) = \lambda$

1.  $\xi$  换为  $x$ ,  $f'(x) + f(x) = \lambda$  这是关于  $f(x)$  的一阶线性微分方程
2. 解微分方程  $f(x) = e^{-x}(\lambda e^x + C)$
3. 分离常数  $[f(x) - \lambda]e^x = C$ , 可令辅助函数  $F(x) = [f(x) - \lambda]e^x$   
 $F(a) = F(b) = 0$ , 因此存在  $\xi \in [a, b]$  使得  $F'(\xi) = 0$

**Example 1.24.** 设  $f(x)$  在  $[a, b]$  上连续, 在  $(a, b)$  上可导, 求证: 存在  $\xi \in (a, b)$  使得  $f(b) - f(a) = \xi \ln \frac{b}{a} f'(\xi)$

可变形为

$$\frac{f(b) - f(a)}{\ln b - \ln a} = \xi f'(\xi)$$

令  $F(x) = \ln x$ , 由柯西中值定理, 存在  $\xi \in (a, b)$  使得

$$\frac{f(b) - f(a)}{\ln b - \ln a} = \frac{f'(\xi)}{F'(\xi)} = \xi f'(\xi)$$

**Example 1.25.** 设  $f(x)$  在  $[-1, 1]$  上具有三阶连续导数, 且  $f(-1) = 0, f(1) = 1, f'(0) = 0$ , 证明: 在  $(-1, 1)$  内存在一点  $\xi$  使得  $f'''(\xi) = 3$



泰勒展开  $f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(\xi)x^3, \xi \in (0, x)$ , 则

$$0 = f(-1) = f(0) + \frac{1}{2}f''(0) - \frac{1}{6}f'''(\xi_1), -1 < \xi_1 < 0$$

$$1 = f(1) = f(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(\xi_2), 0 < \xi_2 < 1$$

两式相减得

$$\frac{f'''(\xi_1) + f'''(\xi_2)}{2} = 3$$

由介值定理可证存在  $\xi \in [\xi_1, \xi_2]$  有  $f'''(\xi) = \frac{f'''(\xi_1) + f'''(\xi_2)}{2} = 3$

**Example 1.26.** 设  $f(x)$  在  $[a, b]$  上连续, 在  $(a, b)$  内可导,  $0 < a < b$ , 求证存在  $\xi, \eta \in (a, b)$  使得  $f'(\xi) = \frac{f'(\eta)}{2\eta}(a+b)$

根据拉格朗日中值定理至少存在一个  $\xi \in (a, b)$  使得

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

只要再证存在  $\eta \in (a, b)$  使得  $\frac{f(b) - f(a)}{b - a} = \frac{f'(\eta)}{2\eta}(a + b)$  即

$$\frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(\eta)}{2\eta}$$

只要用柯西中值定理

**Example 1.27.** 已知函数  $f(x)$  在  $[0, 1]$  上连续, 在  $(0, 1)$  内可导, 且  $f(0) = 0, f(1) = 1$ , 证明

1. 存在  $\xi \in (0, 1)$  使得  $f(\xi) = 1 - \xi$
2. 存在两个不同的点  $\eta, \zeta \in (0, 1)$  使得  $f'(\eta)f'(\zeta) = 1$

令  $F(x) = f(x) - 1 + x$ , 则  $F(0) = -1, F(1) = 1$

对  $[0, \xi], [\xi, 1]$  分别用拉格朗日中值定理, 则

$$f'(\eta)f'(\zeta) = \frac{f(\xi) - f(0)}{\xi - 0} \frac{f(1) - f(\xi)}{1 - \xi} = \frac{f(\xi)}{\xi} \frac{1 - f(\xi)}{1 - \xi} = \frac{1 - \xi}{\xi} \frac{\xi}{1 - \xi} = 1$$

**Example 1.28.** 求证  $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

$$f(x) = \sin x \tan x - x^2$$

$$f'(x) = \sin x + \tan x \sec x - 2x$$

$$f''(x) = \cos x + \sec^3 x + \tan^2 x \sec x - 2$$

$$f'''(x) = -\sin x + 5 \sec^3 x \tan x + \tan^3 x \sec x = \sin x(5 \sec^4 x - 1) + \tan^3 x \sec x > 0$$

**Example 1.29.** 设  $a > 0, b > 0$ , 证明不等式

$$a \ln a + b \ln b \geq (a+b)[\ln(a+b) - \ln 2]$$

令  $f(x) = x \ln x$ , 则  $f'(x) = \ln x + 1, f''(x) = \frac{1}{x} > 0$ , 即曲线  $y = f(x)$  在  $(0, +\infty)$  是凹的, 故对任意  $a > 0, b > 0$ , 有

$$\frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right)$$

代入得

$$\frac{a \ln a + b \ln b}{2} \geq \frac{a+b}{2} \ln \frac{a+b}{2}$$

**Example 1.30.** 证明: 对任意正整数  $n$ , 都有  $\frac{1}{n+1} \leq \ln(1 + \frac{1}{n}) < \frac{1}{n}$

由拉格朗日定理, 存在  $\xi \in (n, n+1)$

$$\begin{aligned} \ln(1 + \frac{1}{n}) &= \ln(n+1) - \ln n = \frac{1}{\xi} \\ \frac{1}{n+1} &< \frac{1}{\xi} < \frac{1}{n} \end{aligned}$$

**Example 1.31.** 设  $f(x)$  在  $[0, 1]$  上二阶可导, 且  $f(0) = f(1) = 0, f(x)$  在  $[0, 1]$  上的最小值等于  $-1$ , 证明: 至少存在一点  $\xi \in (0, 1)$  使  $f''(\xi) \geq 8$

存在  $a \in (0, 1), f'(a) = 0, f(a) = -1$ , 将  $f(x)$  在  $x = a$  泰勒展开

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi)}{2!}(x-a)^2 = -1 + \frac{f''(\xi)}{2}(x-a)^2 (\xi \in (a, x) \text{ or } (x, a))$$

令  $x = 0, x = 1$  得

$$\begin{aligned} f(0) = 0 &= -1 + \frac{f''(\xi_1)}{2}a^2, 0 < \xi_1 < a \\ f(1) = 0 &= -1 + \frac{f''(\xi_2)}{2}(1-a)^2, a < \xi_2 < 1 \end{aligned}$$

若  $0 < a < \frac{1}{2}$ , 则  $f''(\xi_1) > 8$

若  $\frac{1}{2} < a < 1$ , 则  $f''(\xi_2) > 8$

**Example 1.32.** 设函数  $f(x)$  在  $[0, 1]$  上二阶可导, 且  $\int_0^1 f(x)dx = 0$ , 则当  $f''(x) > 0$  时

$$f(x) = f(0.5) + f'(0.5)(x-0.5) + \frac{f''(\xi)}{2}(x-0.5)^2$$

积分

$$\begin{aligned} 0 &= f(0.5) + f'(0.5) \int_0^1 (x-0.5)dx + \frac{f''(\xi)}{2} \int_0^1 (x-0.5)^2 dx \\ &= f(0.5) + \frac{1}{2} f''(\xi) \int_0^1 (x-0.5)^2 dx \end{aligned}$$

因此  $f(0.5) < 0$

**Example 1.33.** 设函数  $f(x)$  在点  $x=0$  可导, 且  $f(0)=0$ , 求  $\lim_{x \rightarrow 0} \frac{f(1-\cos x)}{\tan^2 x}$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(1-\cos x)}{\tan^2 x} &= \lim_{x \rightarrow 0} \frac{f(1-\cos x) - f(0)}{1-\cos x} \cdot \frac{1-\cos x}{\tan^2 x} \\ &= f'(0) \cdot \frac{1}{2} \end{aligned}$$

**Example 1.34.** 设  $f(x)$  在  $[a, b]$  上连续, 在  $(a, b)$  内可导, 且  $f(a) \cdot f(b) > 0$ ,  $f(a) \cdot f(\frac{a+b}{2}) < 0$ , 证明: 对任意实数  $k$ , 存在  $\xi \in (a, b)$  使得  $(f'(\xi) - kf(\xi)) = 0$

**Example 1.35.** 设  $f(x)$  在  $[a, b]$  上连续, 在  $(a, b)$  内可导, 且  $f(a) = f(b) = 1$ , 证明: 存在两点  $\xi, \eta \in (a, b)$  使

$$(e^{2a} + e^{a+b} + e^{2b})[f(\xi) + f'(\xi)] = 3e^{3\eta-\xi}$$

$$\begin{aligned} (e^{2a} + e^{a+b} + e^{2b})[f(\xi) + f'(\xi)] &= 3e^{3\eta-\xi} \\ \Leftrightarrow (e^{2a} + e^{a+b} + e^{2b})[f(\xi) + f'(\xi)]e^\xi &= 3e^{3\eta} \\ \Leftrightarrow (e^{2a} + e^{a+b} + e^{2b})[e^x f(x)]'|_{x=\xi} &= e^{3\eta}|_{x=\eta} \end{aligned}$$

令  $g(x) = e^{3x}$ , 则由拉格朗日中值定理

$$g'(\eta) = \frac{g(b) - g(a)}{b - a}$$

即  $3e^{3\eta} = \frac{e^{3b} - e^{3a}}{b - a}$ . 令  $f(x) = e^x f(x)$ , 由拉格朗日中值定理, 存在  $\xi \in (a, b)$  使得

$$\frac{e^b f(b) - e^a f(a)}{b - a} = e^\xi [f(\xi) + f'(\xi)] = \frac{e^b - e^a}{b - a}$$

两边同乘  $e^{2a} + e^{a+b} + e^{2b}$  得

$$\frac{e^{3b} - e^{3a}}{b - a} = (e^{2a} + e^{a+b} + e^{2b})e^\xi [f(\xi) + f'(\xi)]$$

## 1.2 一元函数积分

**Example 1.36.** 求不定积分  $\int \frac{2^x \cdot 3^x}{9^x - 4^x} dx$

$$\begin{aligned} \int \frac{2^x \cdot 3^x}{9^x - 4^x} dx &= \int \frac{\left(\frac{3}{2}\right)^x}{\left(\frac{3}{2}\right)^{2x} - 1} dx = \frac{1}{\ln \frac{3}{2}} \int \frac{d\left[\left(\frac{3}{2}\right)^x\right]}{\left[\left(\frac{3}{2}\right)^{2x}\right] - 1} \\ &= \frac{1}{2(\ln 3 - \ln 2)} \ln \left| \frac{\left(\frac{3}{2}\right)^x - 1}{\left(\frac{3}{2}\right)^x + 1} \right| \end{aligned}$$

**Example 1.37.** 求  $\int \frac{dx}{\cos x \sqrt{\sin x}}$

$$\begin{aligned} \int \frac{dx}{\cos x \sqrt{\sin x}} &= \int \frac{\cos x dx}{(1 - \sin^2 x) \sqrt{\sin x}} = 2 \int \frac{d(\sqrt{\sin x})}{1 - (\sqrt{\sin x})^4} = 2 \int \frac{dt}{1 - t^4} \\ &= \int \left( \frac{1}{1 + t^2} + \frac{1}{1 - t^2} \right) dt \end{aligned}$$

**Example 1.38.** 求  $\int \frac{dx}{\sqrt{x(4-x)}}$

$$\int \frac{dx}{\sqrt{x(4-x)}} = \int \frac{2d(\sqrt{x})}{\sqrt{4-x}} = 2 \arcsin \frac{\sqrt{x}}{2} + C$$

**Example 1.39.** 求  $\int \frac{1}{1+e^x} dx$

$$\int \frac{1}{1+e^x} dx = \int \frac{e^x}{e^x(1+e^x)} dx = \int \left( \frac{1}{e^x} - \frac{1}{e^x+1} \right) de^x$$

**Example 1.40.** 求  $\int \frac{xe^x}{\sqrt{e^x-1}} dx$

$$\text{令 } \sqrt{e^x-1} = t, x = \ln(1+t^2)$$

$$\int \frac{xe^x}{\sqrt{e^x-1}} = 2 \int \ln(1+t^2) dt$$

**Example 1.41.** 求  $\int \frac{dx}{x^4(1+x^2)}$

$$\int \frac{dx}{x^4(1+x^2)} = \int \frac{1+x^2-x^2}{x^4(1+x^2)} dx$$

**Example 1.42.** 求  $\int \frac{3x^2 - x + 4}{x^3 - x^2 + 2x - 2} dx$   
 $x^3 - x^2 + 2x - 2 = (x^2 + 2)(x - 1)$ , 令

$$\frac{3x^2 - x + 4}{x^3 - x^2 + 2x - 2} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2}$$

**Example 1.43.** 求  $\int \frac{dx}{1 + \sin x}$

$$\int \frac{dx}{1 + \sin x} = \int \frac{1 - \sin x}{\cos^2 x} = \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x} = \tan x - \frac{1}{\cos x} + C$$

**Example 1.44.** 求  $I_n = \int \tan^n x dx$  的递推公式

$$\begin{aligned} I_n &= \int \tan^{n-2} x (\sec^2 x - 1) dx = \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \\ &= \frac{1}{n-1} \tan^{n-1} x - I_{n-2} \end{aligned}$$

**Example 1.45.** 求  $\lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx$

对于  $0 \leq x \leq 1$ , 有  $0 \leq \frac{x^n}{1+x} \leq x$ , 则

$$0 \leq \int_0^1 \frac{x^n}{1+x} dx \leq \int_0^1 x^n dx = \frac{1}{n+1}$$

因此由夹逼定理,  $\lim_{n \rightarrow \infty} \int_0^1 \frac{x^n}{1+x} dx = 0$

**Example 1.46.** 求  $\lim_{n \rightarrow \infty} n \left( \frac{1}{1+n^2} + \cdots + \frac{1}{n^2+n^2} \right)$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left( \frac{1}{1+n^2} + \cdots + \frac{1}{n^2+n^2} \right) &= \lim_{n \rightarrow \infty} \left[ \frac{1}{(\frac{1}{n})^2 + 1} + \cdots + \frac{1}{(\frac{n}{n})^2 + 1} \right] \cdot \frac{1}{n} \\ &= \int_0^1 \frac{1}{1+x^2} dx = \arctan \Big|_0^1 = \frac{\pi}{4} \end{aligned}$$

**Example 1.47.** 证明下列不等式

$$\frac{\sqrt{\pi}}{80} \pi^2 < \int_0^{\frac{\pi}{4}} x \sqrt{\tan x} dx < \frac{\pi^2}{32}$$

当  $0 < x < \frac{\pi}{4}$  时,  $0 < x < \tan x < 1$ , 则

$$\int_0^{\frac{\pi}{4}} x^{3/2} dx < \int_0^{\frac{\pi}{4}} x \sqrt{\tan x} dx < \int_0^{\frac{\pi}{4}} x dx$$

**Example 1.48.** 求  $\int_2^3 \frac{\sqrt{3+2x-x^2}}{(x-1)^2} dx$

$$\begin{aligned}\int_2^3 \frac{\sqrt{3+2x-x^2}}{(x-1)^2} dx &= \int_2^3 \frac{\sqrt{4-(x-1)^2}}{(x-1)^2} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\sqrt{4-4\sin^2 t}}{4\sin^2 t} 2\cos t dt \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\csc^2 t - 1) dt = -\cot t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \sqrt{3} - \frac{\pi}{3}\end{aligned}$$

**Example 1.49.** 求  $\int_0^{\ln 2} \sqrt{1-e^{-2x}} dx$

令  $e^{-x} = \sin t$ , 则

$$\begin{aligned}\int_0^{\ln 2} \sqrt{1-e^{-2x}} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos t \cdot \frac{\cos t}{\sin t} dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sin t} dt - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin t dt \\ &= -\ln(\csc t + \cot t) \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{\sqrt{3}}{2} = \ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{2}\end{aligned}$$

**Example 1.50.** 求  $\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx$

令  $\arcsin \sqrt{\frac{x}{1+x}} = t$ , 则  $\sin^2 u = \frac{x}{1+x}$ ,  $x \cos^2 u = \sin^2 u$ ,  $x = \tan^2 u$

$$\begin{aligned}\int_0^3 \arcsin \sqrt{\frac{x}{1+x}} dx &= \int_0^{\frac{\pi}{3}} u d(\tan^2 u) = (u \cdot \tan^2 u) \Big|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 1 \cdot \tan^2 u du \\ &= \pi - \int_0^{\frac{\pi}{3}} (\sec^2 u - 1) du = \pi - \tan u \Big|_0^{\frac{\pi}{3}} + \frac{\pi}{3} \\ &= \frac{4}{3}\pi - \sqrt{3}\end{aligned}$$

**Example 1.51.** 求  $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 x}{1+e^{-x}} dx$

令  $x = -t$ , 则  $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 x}{1+e^x} dx$ 。因此

$$\begin{aligned}I &= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \frac{\cos^2 x}{1+e^{-x}} + \frac{\cos^2 x}{1+e^x} \right) dx = \int_0^{\frac{\pi}{4}} \left( \frac{1+e^{-x}+1+e^x}{(1+e^{-x})(1+e^x)} \right) \cos^2 x dx \\ &= \int_0^{\frac{\pi}{4}} \cos^2 x dx = \frac{\pi}{8} + \frac{1}{4}\end{aligned}$$

*Remark.* 一般地, 有如下结论: 作变换  $x = a + b - t$

$$I = \int_a^b f(x)dx = \int_a^b f(a + b - t)dt$$

从而  $I = \frac{1}{2} \int_a^b [f(x) + f(a + b - x)]dx$

**Example 1.52.** 求  $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x} dx$

令  $x = \frac{\pi}{2} - t$ , 则

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin^2 x - \sin x \cos x + \cos^2 x) dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \frac{1}{2} \sin 2x) dx = \frac{\pi - 1}{4} \end{aligned}$$

*Remark.* 要求  $I = \int_0^{\frac{\pi}{2}} f(\sin x, \cos x) dx$ , 可作变换  $x = \frac{\pi}{2} - t$ , 则  $I =$

$$\int_0^{\frac{\pi}{2}} f(\cos x, \sin x) dx$$

**Example 1.53.** 求  $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

令  $x = \pi - t$ , 则

$$I = \int_0^{\pi} \frac{(\pi - t) \sin t}{1 + \cos^2 t} dt = \pi \int_0^{\pi} \frac{\sin t}{1 + \cos^2 t} dt - I$$

*Remark.* 一般地,  $I = \int_0^{\pi} x f(\sin x) dx = \int_0^{\pi} (\pi - t) f(\sin t) dt = \pi \int_0^{\pi} f(\sin t) dt - I$

**Example 1.54.** 求  $\int_0^1 \frac{x^b - x^a}{\ln x} dx, a, b > 0$

$$\begin{aligned} \int_0^1 \frac{x^b - x^a}{\ln x} dx, a, b > 0 &= \int_0^1 [f_a^b x^t dt] dx = \int_a^b \left[ \int_0^1 x^t dx \right] dt \\ &= \ln \frac{b+1}{a+1} \end{aligned}$$

**Example 1.55.** 设  $f(x) = \int_0^x \frac{\sin t}{\pi - t} dt$ , 求  $\int_0^{\pi} f(x) dx$

$$\begin{aligned}
\int_0^\pi f(x)dx &= \int_0^\pi f(x)d(x-\pi) \\
&= (x-\pi)f(x)|_0^\pi - \int_0^\pi (x-\pi)f'(x)dx \\
&= -\int_0^\pi (x-\pi)\frac{\sin x}{\pi-x}dx = 2
\end{aligned}$$

**Example 1.56.** 证明  $\int_1^a f(x^2 + \frac{a^2}{x^2})\frac{dx}{x} = \int_1^a f(x + \frac{a^2}{x})\frac{dx}{x}$

$$\begin{aligned}
\int_1^a f(x^2 + \frac{a^2}{x^2})\frac{dx}{x} &= \frac{1}{2} \int_1^{a^2} f(t + \frac{a^2}{t})\frac{dt}{t} \\
&= \frac{1}{2} \int_1^a f(t + \frac{a^2}{t})\frac{dt}{t} + \frac{1}{2} \int_a^{a^2} f(t + \frac{a^2}{t})\frac{dt}{t}
\end{aligned}$$

令  $t = \frac{a^2}{u}$

$$\begin{aligned}
\frac{1}{2} \int_a^{a^2} f(t + \frac{a^2}{t})\frac{dt}{t} &= \int_a^1 f(\frac{a^2}{u} + u)\frac{u}{a^2} \left(-\frac{a^2}{u^2}\right) du \\
&= \int_1^a f(u + \frac{a^2}{u})\frac{1}{u} du
\end{aligned}$$

**Example 1.57.** 设  $f(x)$  在  $[a, b]$  上有二阶连续导数, 又  $f(a) = f'(a) = 0$ , 证明:

$$\int_a^b f(x)dx = \frac{1}{2} \int_a^b f''(x)(x-b)^2 dx$$

利用分部积分

$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^b f(x)d(x-b) = -\int_a^b f'(x)(x-b)d(x-b) \\
&= -\frac{1}{2} \int_a^b f''(x)d(x-b)^2 = \frac{1}{2} \int_a^b f''(x)(x-b)^2 dx
\end{aligned}$$

**Example 1.58.** 设  $f(x)$  在  $[a, b]$  上有二阶连续导数且  $f(a) = f(b) = 0, M = \max_{[a,b]} |f''(x)|$ , 证明  $\left| \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{12} M$



$$\begin{aligned}
\int_a^b f(x)dx &= \int_a^b f(x)d(x-a) = -\int_a^b f'(x)(x-a)d(x-b) \\
&= \int_a^b f''(x)(x-a)(x-b)dx + \int_a^b f'(x)(x-b)dx \\
&= \int_a^b f''(x)(x-a)(x-b)dx + \int_a^b (x-b)df(x) \\
&= \int_a^b f''(x)(x-a)(x-b)dx - \int_a^b f(x)dx
\end{aligned}$$

则

$$\int_a^b f(x)dx = \frac{1}{2} \int_a^b f''(x)(x-a)(x-b)dx$$

因此

$$\begin{aligned}
\left| \int_a^b f(x)dx \right| &\leq \frac{1}{2}M \int_a^b (x-a)(b-a)dx \\
&= \frac{1}{4}M \int_a^b (x-a)^2dx = \frac{(b-a)^3}{12}M
\end{aligned}$$

**Example 1.59.** 设  $f(x)$  在  $[a, b]$  上连续且严格单调增, 证明:

$$(a+b) \int_a^b f(x)dx < 2 \int_a^b xf(x)dx$$

$$\text{令 } F(x) = (a+x) \int_a^x f(t)dt - 2 \int_a^x tf(t)dt, (a < x \leq b)$$

**Example 1.60.** 求  $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{|x-x^2|}}dx$

$$\begin{aligned}
\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{|x-x^2|}}dx &= \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{x-x^2}}dx + \int_1^{\frac{3}{2}} \frac{1}{\sqrt{x^2-x}}dx \\
&= \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{\frac{1}{4} - (x-\frac{1}{2})^2}}dx + \int_1^{\frac{3}{2}} \frac{1}{\sqrt{(x-\frac{1}{2})^2 - \frac{1}{4}}}dx \\
&= \arcsin(2x-1) \Big|_{\frac{1}{2}}^1 + \ln \left[ (x-\frac{1}{2}) + \sqrt{(x-\frac{1}{2})^2 - \frac{1}{4}} \right] \Big|_1^{\frac{3}{2}}
\end{aligned}$$

**Example 1.61.** 求  $\int e^x \frac{1+\sin x}{1+\cos x}dx$

$$\begin{aligned}\int e^x \frac{1 + \sin x}{1 + \cos x} dx &= \int e^x (1 + \sin x) \frac{1}{2 \cos^2 \frac{x}{2}} dx = \int e^x d \tan \frac{x}{2} + \int e^x \tan \frac{x}{2} dx \\ &= e^x \tan \frac{x}{2} + C\end{aligned}$$

**Example 1.62.** 设  $f(x)$  为非负连续函数, 当  $x \geq 0$  时, 有  $\int_0^x f(x)f(x-t)dt = e^{2x} - 1$ , 求  $f(x)$

$f(x) \int_0^x f(u)du = e^{2x-1}$ , 令  $F(x) = \int_0^x f(t)dt$ , 则有  $F'(x)F(x) = e^{2x-1}$ ,  $F(0) = 0$ , 两边积分, 得

$$\frac{1}{2}F^2(x) = \frac{1}{2}e^{2x} - x + C$$

由  $F(0) = 0$  得,  $C = -\frac{1}{2}$ . 因此  $F^2(x) = e^{2x} - x - 1$ , 故

$$f(x) = F'(x) = \frac{e^{2x} - 1}{\sqrt{e^{2x} - 2x - 1}}$$

**Example 1.63.** 设  $f(x) = \int_1^x \frac{\ln t}{1+t} dt (x > 0)$ ,  $g(x)$  连续, 且  $f(x) + f(\frac{1}{x}) = \int_0^1 g(xt)dt$ , 求  $g(x)$

$$\int_0^1 g(xt)dt = \frac{1}{x} \int_0^x g(t)dt, \text{ 又}$$

$$f(\frac{1}{x}) = \int_0^{\frac{1}{x}} \frac{\ln t}{1+t} dt = \int_0^x \frac{\ln \frac{1}{u}}{1+\frac{1}{u}} (-\frac{1}{u^2}) du = \int_1^x \frac{\ln u}{u(1+u)} du$$

因此  $f(x) + f(\frac{1}{x}) = \int_1^x \frac{\ln t}{t} dt$ , 于是  $\int_0^x g(t)dt = x \int_1^x \frac{\ln t}{t} dt$ ,

$$g(x) = \int_1^x \frac{\ln t}{t} dt + \ln x = \frac{1}{2} \ln^2 x + \ln x$$

**Example 1.64.** 设  $f(x)$  在  $[0, +\infty)$  上连续且单调增加, 证明: 对任意  $a, b > 0$ , 恒有

$$\int_a^b xf(x)dx \geq \frac{1}{2} \left[ b \int_0^b f(x)dx - a \int_0^a f(x)dx \right]$$

令  $F(x) = x \int_0^x f(t)dt$ , 则  $F'(x) = \int_0^x f(t)dt + xf(x)$

$$\begin{aligned}F(b) - F(a) &= \int_a^b F'(x)dx = \int_a^b \left[ \int_0^x f(t)dt + xf(x) \right] dx \\ &\leq \int_a^b [xf(x) + xf(x)]dx = 2 \int_a^b xf(x)dx\end{aligned}$$

### 1.3 多元函数微积分学

**Example 1.65.** 求极限  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}}$

$x^2 y^2 \leq (\frac{x^2 + y^2}{2})^2$ , 因而

$$0 \leq \frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}} \leq \frac{1}{4} \sqrt{x^2 + y^2}$$

**Example 1.66.** 讨论极限  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy^2}{x^2 + y^4}$  的存在性

当点  $P(x, y)$  沿曲线  $x = ky^2$  趋于点  $(0, 0)$  时

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy^2}{x^2 + y^4} = \lim_{y \rightarrow 0} \frac{ky^4}{k^2 y^4 + y^4} = \frac{k}{k^2 + 1}$$

不是一个确定的常数, 因此极限不存在

**Example 1.67.** 讨论函数

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

在  $(0, 0)$  处的连续性

令  $x = r \cos \theta, y = r \sin \theta$ , 则

$$0 \leq |f(x, y)| = \left| \frac{r^2 \sin 4\theta}{4} \right| \leq \frac{r^2}{4}$$

因此连续

**Example 1.68.** 设  $z = (s \in y^3 + x^3)(x + y^4)^{\frac{y}{x} + e^{y^3 x^2}}$ , 求  $\frac{\partial z}{\partial x} \Big|_{(1,0)}$

$$\frac{\partial z}{\partial x} \Big|_{(1,0)} = \frac{\partial z(x, 0)}{\partial x} \Big|_{x=1} = (x^4)' \Big|_{x=1} = 4$$

**Example 1.69.** 已知函数  $f(x, y)$  在点  $(0, 0)$  的某邻域内有定义, 且  $f(0, 0) = 0$ ,  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y)}{x^2 + y^2} = 1$ , 则  $f(x, y)$  在点  $(0, 0)$  处

由于  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y)}{x^2 + y^2} = 1$ ,  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} (x^2 + y^2) = 0$ , 于是  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) = 0$ , 又

$f(0, 0) = 0$ , 所以  $f(x, y)$  在  $(0, 0)$  处极限存在且连续, 又由  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y)}{x^2 + y^2} = 1$ , 得

$$\lim_{x \rightarrow 0} \frac{f(x, 0)}{x^2} = 1, \lim_{y \rightarrow 0} \frac{f(0, y)}{y^2} = 1$$

所以

$$f'_x(0,0) = \lim_{x \rightarrow 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \rightarrow 0} \frac{f(x,0)}{x} = \lim_{x \rightarrow 0} \frac{f(x,0)}{x^2} x = 0$$

同理  $f'_y(0,0) = 0$ , 故  $f(x,y)$  在  $(0,0)$  处偏导数存在

因为

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{\Delta z - [f'_x(0,0)\Delta x + f'_y(0,0)\Delta y]}{\rho} &= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x,y) - f(0,0)}{\sqrt{x^2 + y^2}} (\rho = \sqrt{x^2 + y^2}) \\ &= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x,y)}{x^2 + y^2} \sqrt{x^2 + y^2} = 0 \end{aligned}$$

所以  $f(x,y)$  在  $(0,0)$  处可微

**Remark.** 讨论二元函数  $f(x,y)$  在  $(x_0, y_0)$  的可微性, 可从如下几个方面考虑

1. 若二元函数  $f(x,y)$  在  $(x_0, y_0)$  的偏导数至少有一个不存在, 则函数不可微
2. 若二元函数  $f(x,y)$  在  $(x_0, y_0)$  不连续, 则函数不可微
3. 若二元函数  $f(x,y)$  在  $(x_0, y_0)$  连续, 两个偏导数存在, 则考虑

$$\lim_{\rho \rightarrow 0} \frac{\Delta z - [f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y]}{\rho}, \rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

若极限为 0, 则函数在  $(x_0, y_0)$  可微, 否则不可微

**Example 1.70.** 设  $z = (\frac{y}{x})^{\frac{x}{y}}$ , 求  $dz|_{(1,2)}$

取对数, 有

$$\ln z = \frac{x}{y} \ln \frac{y}{x} \Rightarrow y \ln z = x(\ln y - \ln x)$$

**Example 1.71.** 设  $u = f(\frac{x}{y}, \frac{y}{z}), u = f(s, t)$  有二阶连续偏导数, 求  $du, \frac{\partial^2 u}{\partial y \partial z}$

$$\begin{aligned} du &= f'_1 d(\frac{x}{y}) + f'_2 d(\frac{y}{z}) = f'_1 \frac{ydx - xdy}{y^2} + f'_2 \frac{zdy - ydz}{z^2} \\ &= \frac{1}{y} f'_1 dx + (-\frac{x}{y^2} f'_1 + \frac{1}{z} f'_2) dy - \frac{y}{z^2} f'_2 dz \end{aligned}$$

**Example 1.72.** 已知  $(axy^3 - y^2 \cos x)dx + (1 + by \sin x + 3x^2 y^2)dy$  为某一函数  $f(x,y)$  的全微分, 求  $a, b$

由题意知,  $\frac{\partial f}{\partial x} = axy^3 - y^2 \cos x, \frac{\partial f}{\partial y} = 1 + by \sin x + 3x^2 y^2$ , 从而有  $\frac{\partial^2 f}{\partial x \partial y} = 3axy^2 - 2y \cos x, \frac{\partial^2 f}{\partial y \partial x} = by \cos x + 6xy^2$ , 显然  $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$  均连续, 所以  $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ , 即  $by \cos x + 6xy^2 = 3axy^2 - 2y \cos x$ , 因此  $a = 2, b = -2$

**Example 1.73.** 设  $z = f(x, y)$  满足  $\frac{\partial^2 f}{\partial y^2} = 2x$ ,  $f(x, 1) = 0$ ,  $\frac{\partial f(x, 0)}{\partial y} = \sin x$ , 求  $f(x, y)$

$f(x, y) = xy^2 + \varphi(x)y + \psi(x)$ , 从  $\frac{\partial f(x, 0)}{\partial y} = \sin x$ , 即  $[2xy + \varphi(x)]\Big|_{y=0} = \sin x$ , 得  $\varphi(x) = \sin x$

**Example 1.74.** 设函数  $u = f(\ln \sqrt{x^2 + y^2})$ , 满足  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^2 + y^2)^{3/2}$ , 求函数  $f$  的表达式

设  $t = \ln \sqrt{x^2 + y^2}$ , 则  $x^2 + y^2 = e^{2t}$

$$\begin{aligned}\frac{\partial u}{\partial x} &= f'(t) \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = f'(t) \frac{y}{x^2 + y^2} \\ \frac{\partial^2 u}{\partial x^2} &= f''(t) \frac{x^2}{(x^2 + y^2)^2} + f'(t) \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{\partial^2 u}{\partial y^2} &= f''(t) \frac{y^2}{(x^2 + y^2)^2} + f'(t) \frac{x^2 - y^2}{(x^2 + y^2)^2}\end{aligned}$$

代入得  $f''(t) = (x^2 + y^2)^{5/2} = e^{5t}$ , 因此有

$$f(t) = \frac{1}{25}e^{5t} + C_1 t + C_2$$

**Example 1.75.** 已知函数  $f(x, y)$  在点  $(0, 0)$  的某个邻域内连续, 且  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y) - xy}{(x^2 + y^2)^2} = 1$ , 则

1. 点  $(0, 0)$  不是  $f(x, y)$  的极值点
2. 点  $(0, 0)$  是  $f(x, y)$  的极大值点
3. 点  $(0, 0)$  是  $f(x, y)$  的极小值点
4. 根据所给条件无法判断点  $(0, 0)$  是否为  $f(x, y)$  的极值点

分子的极限为 0, 从而有  $f(0, 0) = 0$ , 且由极限的性质知,  $\frac{f(x, y) - xy}{(x^2 + y^2)^2} = 1 + \alpha(x, y)$ , 这里  $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \alpha(x, y) = 0$ , 因而  $f(x, y) = xy + (x^2 + y^2)^2[1 + \alpha(x, y)]$ ,

在点  $(0, 0)$  的某充分小去心邻域内, 取  $y = x$  且  $|x|$  充分小时,  $f(x, y) = x^2 + 4x^4[1 + \alpha(x, x)] > 0 = f(0, 0)$ , 在点  $(0, 0)$  的某充分小去心邻域内, 取  $y = -x$  且  $|x|$  充分小时,  $f(x, y) = -x^2 + 4x^4[1 + \alpha(x, -x)] < 0 = f(0, 0)$ , 故点  $(0, 0)$  不是  $f(x, y)$  的极值点

**Example 1.76.** 讨论二元函数  $z = x^3 + y^3 - 2(x^2 + y^2)$  的极值

$$\begin{cases} \frac{\partial z}{\partial x} = 3x^2 - 4x = 0 \\ \frac{\partial z}{\partial y} = 3y^2 - 4y = 0 \end{cases}$$

得驻点  $(0, 0), (4/3, 0), (0, 4/3), (4/3, 4/3)$ . 进而

$$A = \frac{\partial^2 z}{\partial x^2} = 6x - 4, B = \frac{\partial^2 z}{\partial xy} = 0, C = \frac{\partial^2 z}{\partial y^2} = 6y - 4$$

$$AC - B^2 = 16 + 36xy - 24(x + y)$$

在点  $(0, 0)$  时  $AC - B^2 > 0$  且  $A < 0$  有极大值

在点  $(4/3, 4/3)$  时  $AC - B^2 > 0$  且  $A > 0$  有极小值

**Example 1.77.** 求椭圆  $x^2 + 2xy + 3y^2 - 8y = 0$  与直线  $x + y = 8$  之间的最短距离

椭圆上任意一点  $P(x, y)$  到直线  $x + y = 8$  的距离的平方为

$$d^2 = \frac{(x + y - 8)^2}{2}$$

令  $F(x, y) = \frac{1}{2}(x + y - 8)^2 + \lambda(x^2 + 2xy + 3y^2 - 8y)$  则有方程组

$$\begin{cases} F'_x = x + y - 8 + (2\lambda x + 2\lambda y) = 0 \\ F'_y = x + y - 8 + \lambda(2x + 6y - 8) = 0 \\ x^2 + 2xy + 3y^2 - 8y = 0 \end{cases}$$

解得

$$\begin{cases} x = -2 + 2\sqrt{2} \\ y = 2 \end{cases} \quad \text{or} \quad \begin{cases} x = -2 - 2\sqrt{2} \\ y = 2 \end{cases}$$

且  $d_1 = 4\sqrt{2} - 2, d_2 = 4\sqrt{2} + 2$ , 所以所求最短距离为  $4\sqrt{2} - 2$

**Example 1.78.** 求函数  $f(x, y) = x^2 + 2y^2 - x^2y^2$  在区域  $D = \{(x, y) \mid x^2 + y^2 \leq 4, y \geq 0\}$  上的最大值和最小值

解方程组

$$\begin{cases} f'_x = 2x - 2xy^2 = 0 \\ f'_y = 4y - 2x^2y = 0 \end{cases}$$

得开区域内的可能极值点为  $(\pm\sqrt{2}, 1)$ , 其对应函数值为  $f(\pm\sqrt{2}, 1) = 2$

当  $y = 0$  时,  $f(x, y) = x^2$  在  $-2 \leq x \leq 2$  上的最大值为 4, 最小值为 0

当  $x^2 + y^2 = 4, y > 0, -2 < x < 2$  时, 构造拉格朗日函数

$$F(x, y, \lambda) = x^2 + 2y^2 - x^2y^2 + \lambda(x^2 + y^2 - 4)$$

解方程组

$$\begin{cases} F'_x = 2x - 2xy^2 + 2\lambda x = 0 \\ F'_y = 4y - 2x^2y + 2\lambda y = 0 \\ F'_\lambda = x^2 + y^2 - 4 = 0 \end{cases}$$

得可能极值点:  $(0, 2), \left(\pm\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right)$ , 其对应函数值为  $f(0, 2) = 8, f\left(\pm\sqrt{\frac{5}{2}}, \sqrt{\frac{3}{2}}\right) = \frac{7}{4}$

因此  $f(x, y)$  在  $D$  上的最大值为 8, 最小值 0

**Example 1.79.** 设  $f(x, y)$  有二阶连续偏导数,  $g(x, y) = f(e^{xy}, x^2 + y^2)$ , 且

$$f(x, y) = 1 - x - y + o(\sqrt{(x-1)^2 + y^2})$$

证明  $g(x, y)$  在  $(0, 0)$  取得极值, 判断此极值是极大值还是极小值, 并求出此极值

由  $f(x, y) = -(x-1) - y + o(\sqrt{(x-1)^2 + y^2})$ , 由全微分的定义得

$$f(1, 0) = 0, f'_x(1, 0) = f'_y(1, 0) = -1$$

计算得  $g'_x = f'_1 \cdot e^{xy}y + f'_2 \cdot 2x, g'_y = f'_1 \cdot e^{xy}x + f'_2 \cdot 2y$ , 有

$$g'_x(0, 0) = 0, g'_y(0, 0) = 0$$

再求二阶导数

$$\begin{aligned} g''_{xx} &= (f''_{11} \cdot e^{xy}y + f''_{12} \cdot 2x)e^{xy}y + f'_1 \cdot e^{xy}y^2 + (f''_{21} \cdot e^{xy}y + f''_{22} \cdot 2x)2x + 2f'_2 \\ g''_{xy} &= (f''_{11} \cdot e^{xy}x + f''_{12} \cdot 2y)e^{xy}y + f'_1 \cdot (e^{xy}xy + e^{xy}) + (f''_{21} \cdot e^{xy}x + f''_{22} \cdot 2y)2x \\ g''_{yy} &= (f''_{11} \cdot e^{xy}x + f''_{12} \cdot 2y)e^{xy}x + f'_1 \cdot e^{xy}x^2 + (f''_{21} \cdot e^{xy}x + f''_{22} \cdot 2y)2x + 2f'_2 \end{aligned}$$

因此  $A = g''_{xx}(0, 0) = 2f'_2(1, 0) = -2, B = g''_{xy}(0, 0) = f'_1(1, 0) = -1, C = g''_{yy}(0, 0) = 2f'_2(1, 0) = -2$ , 进而  $AC - B^2 > 0$ , 因此  $g(0, 0) = f(1, 0) = 0$  是极大值

**Example 1.80.** 已知  $x, y, z$  为实数, 且  $e^x + y^2 + |z| = 3$ , 求证  $e^x y^2 |z| \leq 1$

证明 1。在  $e^x + y^2 + |z| = 3$  约束条件下求函数  $u = e^x y^2 |z|$  的最值问题, 转化为无条件极值  $u = e^x y^2 (3 - e^x - y^2)$

证明 2。可化为以下等价问题: 已知  $X > 0, Y \geq 0, Z \geq 0$ , 且  $X + Y + Z = 3$ , 求  $XYZ \leq 1$ 。因此用拉格朗日乘数法

**Example 1.81.** 设闭区域  $D: x^2 + y^2 \leq y, x \geq 0$ ,  $f(x, y)$  为  $D$  上的连续函数, 且

$$f(x, y) = \sqrt{1 - x^2 - y^2} - \frac{8}{\pi} \iint_D f(u, v) du dv$$

求  $f(u, v)$

设  $\iint_D f(u, v) du dv = A$ , 在已知等式两边求区域  $D$  的二重积分

$$\iint_D f(x, y) dx dy = \iint_D \sqrt{1 - x^2 - y^2} dx dy - \frac{8A}{\pi} \iint_D dx dy$$

$$A = \iint_D \sqrt{1 - x^2 - y^2} dx dy - A$$

$$2A = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\sin \theta} \sqrt{1 - r^2} r dr = \frac{1}{3} \left( \frac{\pi}{2} - \frac{2}{3} \right)$$

**Example 1.82.** 设区域  $D = \{(x, y) \mid x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$ ,  $f(x)$  为  $D$  上的正值连续函数,  $a, b$  为常数, 求  $\iint_D \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma$

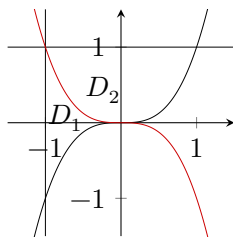
由轮换对称性

$$\begin{aligned} \iint_D \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(y)} + \sqrt{f(x)}} d\sigma &= \iint_D \frac{a\sqrt{f(y)} + b\sqrt{f(x)}}{\sqrt{f(x)} + \sqrt{f(y)}} d\sigma \\ &= \frac{1}{2} \iint_D \left[ \frac{a\sqrt{f(x)} + b\sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}} + \frac{a\sqrt{f(y)} + b\sqrt{f(x)}}{\sqrt{f(y)} + \sqrt{f(x)}} \right] \\ &= \frac{a+b}{2} \iint_D d\sigma = \frac{a+b}{2} \pi \end{aligned}$$

**Example 1.83.** 计算二重积分  $I = \iint_D x[1 + yf(x^2 + y^2)] dx dy$ , 其中积分区域  $D$  为  $y = x^3, y = 1, x = -1$  所围成的平面区域,  $f$  连续

补充曲线  $y = -x^3$ , 拆分积分区域  $D$  分别关于  $x, y$  坐标轴对称





$$\begin{aligned}
 I &= \iint_D = \iint_{D_1} + \iint_{D_2} \\
 &= \iint_{D_1} [x + xyf(x^2 + y^2)] dx dy + \iint_{D_2} x[1 + yf(x^2 + y^2)] dx dy \\
 &= \iint_{D_1} x dx dy = 2 \int_{-1}^0 dx \int_0^{-x^3} x dy \\
 &= -\frac{2}{5}
 \end{aligned}$$

**Example 1.84.** 设平面区域  $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$ , 计算

$$\iint_D \frac{x \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy$$

由轮换对称性

$$\begin{aligned}
 &\iint_D \frac{x \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy \\
 &= \frac{1}{2} \left[ \iint_D \frac{x \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy + \iint_D \frac{y \sin(\pi \sqrt{x^2 + y^2})}{x + y} dx dy \right] \\
 &= \frac{1}{2} \iint_D \sin(\pi \sqrt{x^2 + y^2}) dx dy = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta \int_1^2 r \sin \pi r dr \\
 &= -\frac{3}{4}
 \end{aligned}$$

**Example 1.85.** 计算二重积分  $\iint_D |x^2 + y^2 - 1| d\sigma$ , 其中  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$

记  $D_1 = \{(x, y) \mid x^2 + y^2 \leq 1, (x, y) \in D\}$ ,  $D_2 = \{(x, y) \mid x^2 + y^2 > 1, (x, y) \in D\}$ , 则

$$\begin{aligned}
\iint_D |x^2 + y^2 - 1| d\sigma &= - \iint_{D_1} (x^2 + y^2 - 1) dx dy + \iint_{D_2} (x^2 + y^2 - 1) dx dy \\
&= -2 \iint_{D_1} (x^2 + y^2 - 1) dx dy + \iint_D (x^2 + y^2 - 1) dx dy
\end{aligned}$$

**Example 1.86.** 设  $f(x)$  为连续函数,  $F(t) = \int_1^t dy \int_y^t f(x) dx$ , 求  $F'(2)$   
 交换积分次序得

$$F(t) = \int_1^t dy \int_y^t f(x) dx = \int_1^t \left[ \int_1^x f(x) dy \right] dx = \int_1^t f(x)(x-1) dx$$

因此  $F'(2) = f(2)(2-1) = f(2)$

**Example 1.87.** 计算二重积分  $\iint_D r^2 \sin \theta \sqrt{1-r^2 \cos 2\theta} dr d\theta$ , 其中  $D = \{(r, \theta) \mid 0 \leq r \leq \sec \theta, 0 \leq \theta \leq \frac{\pi}{4}\}$

直角坐标系下  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$

**Example 1.88.** 已知函数  $f(x, y)$  具有二阶连续偏导数, 且  $f(1, y) = 0, f(x, 1) = 0$ ,  $\iint_D f(x, y) dx dy = a$ , 其中  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ , 计算二重积分

$$\begin{aligned}
&\iint_D xy f''_{xy}(x, y) dx dy \\
&\iint_D xy f''_{xy}(x, y) dx dy = \int_0^1 x \left( \int_0^1 y f''_{xy}(x, y) dy \right) dx = \int_0^1 x \left( \int_0^1 y df'_x(x, y) \right) dx \\
&\int_0^1 y df'_x(x, y) = y f'_x(x, y) \Big|_0^1 - \int_0^1 f'_x(x, y) dy = - \int_0^1 f'_x(x, y) dy \\
&\int_0^1 x \left( \int_0^1 y df'_x(x, y) \right) dx = - \int_0^1 x \left( \int_0^1 f'_x(x, y) dy \right) dx = - \int_0^1 \left( \int_0^1 x f'_x(x, y) dx \right) dy \\
&\int_0^1 x f'_x(x, y) dx = \int_0^1 x df(x, y) = x f(x, y) \Big|_0^1 - \int_0^1 f(x, y) dx = - \int_0^1 f(x, y) dx \\
&\iint_D xy f''_{xy} dx dy = \int_0^1 dy \int_0^1 f(x, y) dx = a
\end{aligned}$$

**Example 1.89.** 求积分  $\int_0^1 dy \int_y^1 \left( \frac{e^{x^2}}{x} - e^{y^2} \right) dx$

$$\begin{aligned}
\int_0^1 dy \int_y^1 \left( \frac{e^{x^2}}{x} - e^{y^2} \right) dx &= \int_0^1 dx f_0^x \left( \frac{e^{x^2}}{x} - e^{y^2} \right) dy = \int_0^1 (e^{x^2} - \int_0^x e^{y^2} dy) dx \\
&= \int_0^1 e^{x^2} dx - \int_0^1 \left( \int_0^x e^{y^2} dy \right) dx \\
&= \int_0^1 e^{x^2} dx - x \int_0^x e^{y^2} dy \Big|_0^1 + \int_0^1 e^{x^2} \cdot x dx \\
&= \int_0^1 e^{x^2} dx - \int_0^1 e^{y^2} dy + \frac{1}{2} e^{x^2} \Big|_0^1 \\
&= \frac{e-1}{2}
\end{aligned}$$

**Example 1.90.** 设  $f(x)$  在  $[a, b]$  上连续, 且  $f(x) > 0$ , 证明

$$\int_a^b f(x) dx \int_a^b \frac{1}{f(x)} dx \geq (b-a)^2$$

令  $D = \{(x, y) \mid a \leq x, y \leq b\}$

$$\begin{aligned}
\int_a^b f(x) dx \int_a^b \frac{1}{f(x)} dx &\geq (b-a)^2 = \int_a^b f(x) dx \int_a^b \frac{1}{f(y)} dy \\
&= \iint_D \frac{f(x)}{f(y)} dxdy = \iint_D \frac{f(y)}{f(x)} dxdy \\
&= \frac{1}{2} \iint_D \left[ \frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} \right] dxdy \\
&\geq \iint_D dxdy = (b-a)^2
\end{aligned}$$

**Example 1.91.** 证明  $\left( \int_0^1 e^{-x^2} dx \right)^2 > \frac{\pi}{4} \left( 1 - \frac{1}{e} \right)$

令  $D = \{(x, y) \mid 0 \leq x, y \leq 1\}$ ,  $D_1 = \{(x, y) \mid x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$

$$\begin{aligned}
\left( \int_0^1 e^{-x^2} dx \right)^2 &= \int_0^1 e^{-x^2} dx \int_0^1 e^{-y^2} dy = \iint_D e^{-x^2-y^2} dxdy \\
&> \iint_{D_1} e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 e^{-r^2} r dr = \frac{\pi}{4} \left( 1 - \frac{1}{e} \right)
\end{aligned}$$

**Example 1.92.** 设函数  $z = f(x, y)$  具有二阶连续偏导数, 且满足  $f''_{xx} = f''_{yy}$ , 又由  $f(x, 2x) = x$ ,  $f'_x(x, 2x) = x^2$ , 试求二阶偏导数  $f''_{xx}(x, 2x)$ ,  $f''_{xy}(x, 2x)$

因为  $f'_x \cdot 1 + f'_y \cdot 2 = 1$  所以  $2f'_y = 1 - x^2$ 。又因为  $2(f''_{yx} \cdot 1 + f''_{yy} \cdot 2) = -2x$ , 由条件知  $f'_x(x, 2x) = x^2$ , 则  $f''_{xx} \cdot 1 + f''_{xy} \cdot 2 = 2x$ , 解得  $f''_{xx}(x, 2x) = -\frac{4}{3}x$ ,  $f''_{xy}(x, 2x) = \frac{5}{3}x$

**Example 1.93.** 设函数  $u = u(x, y)$  由方程  $u = f(x, y, z, t), g(y, z, t) = 0, h(z, t) = 0$  所确定, 求  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$

由方程组

$$\begin{cases} f(x, y, z, t) - u = 0 \\ g(y, z, t) = 0 \\ h(z, t) = 0 \end{cases}$$

有

$$\begin{cases} \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial f}{\partial t} \frac{\partial t}{\partial y} - \frac{\partial u}{\partial y} = 0 \\ \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial g}{\partial t} \frac{\partial t}{\partial y} = 0 \\ \frac{\partial h}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial h}{\partial t} \frac{\partial t}{\partial y} = 0 \end{cases}$$

解

## 1.4 无穷级数

**Example 1.94.** 判断级数  $\sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx$  的敛散性

由于

$$0 < \sum_{n=1}^{\infty} \int_0^{\frac{1}{n}} \frac{\sqrt{x}}{1+x^2} dx < \int_0^{\frac{1}{n}} \sqrt{x} dx = \frac{2}{3} \frac{1}{n^{3/2}}$$

而级数  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  收敛, 因此级数收敛

**Example 1.95.** 判断级数  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \ln \frac{n+1}{n} \right)$  的收敛性

由泰勒公式  $\ln \frac{n+1}{n} = \frac{1}{n} - \frac{1}{2} \frac{1}{n^2} + o(\frac{1}{n^2})$ , 则  $\frac{1}{n} - \ln \frac{n+1}{n} \sim \frac{1}{2} \cdot \frac{1}{n^2}$ , 而  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  收敛

**Example 1.96.** 判定级数  $\sum_{n=1}^{\infty} \sin(n\pi + \frac{1}{n - \ln n})$  的敛散性

$\sum_{n=1}^{\infty} \sin(n\pi + \frac{1}{n - \ln n}) = \sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n - \ln n}$ , 令  $f(x) = \frac{1}{x - \ln x}$ , 则  $f'(x) < 0$  且  $\lim_{n \rightarrow \infty} u_n = 0$

**Example 1.97.** 判定级数  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n}$  的敛散性

$$\frac{(-1)^n}{\sqrt{n} + (-1)^n} = \frac{(-1)^n[\sqrt{n} - (-1)^n]}{n - 1} = \frac{(-1)^n\sqrt{n}}{n - 1} - \frac{1}{n - 1}$$

**Example 1.98.** 设  $a_1 = 2, a_{n+1} = \frac{1}{n}(a_n + \frac{1}{a_n}), n = 1, 2, \dots$ , 证明

1.  $\lim a_n$  存在
  2. 级数  $\sum_{n=1}^{\infty} (\frac{a_n}{a_{n+1}} - 1)$  收敛
1. 显然  $a_n \geq 0$ ,  $\{a_n\}$  单调减少且有下界, 因此  $\lim a_n$  存在
2. 由于数列单调减少, 所以有  $0 \leq \frac{a_n}{a_{n+1}} - 1 = \frac{a_n - a_{n+1}}{a_{n+1}} \leq a_n - a_{n+1}$ , 而
- $$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = a_1 - \lim_{n \rightarrow \infty} a_n \text{ 收敛}$$

**Example 1.99.** 设正项数列  $\{a_n\}$  单调减少, 且  $\sum_{n=1}^{\infty} (-1)^n a_n$  发散, 试问级数

$$\sum_{n=1}^{\infty} \left( \frac{1}{a_n + 1} \right)^n \text{ 是否收敛}$$

证明. 由已知正项数列  $\{a_n\}$  单调减少, 根据单调有界数列必有极限知, 极限  $\lim_{n \rightarrow \infty} a_n$  存在, 记  $a = \lim_{n \rightarrow \infty} a_n$ , 则有  $a_n \geq a \geq 0$ , 若  $a = 0$ , 则交错级数收敛, 矛盾, 因此  $a > 0$

又由于  $\left(\frac{1}{a_n+1}\right)^n \leq \left(\frac{1}{a+1}\right)^n$ , 而  $\frac{1}{a+1} < 1$ , 几何级数  $\sum_{n=1}^{\infty} \left(\frac{1}{a+1}\right)^n$  收敛, 因此级数收敛  $\square$

## 2 线性代数

### 2.1 行列式

**Example 2.1.** 计算行列式

$$D_n = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 & x & -1 & \cdots & 0 & 0 \\ a_3 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix}$$

$$\begin{aligned}
D_n &= \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 + a_1x & 0 & -1 & \cdots & 0 & 0 \\ a_3 & 0 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} & 0 & 0 & \cdots & x & -1 \\ a_n & 0 & 0 & \cdots & 0 & x \end{vmatrix} \\
&= \cdots = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 & 0 \\ a_2 + a_1x & 0 & -1 & \cdots & 0 & 0 \\ a_3 + a_2x + a_1x^2 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-1} + \cdots + a_1x^{n-2} & 0 & 0 & \cdots & 0 & -1 \\ a_n + \cdots + a_1x^{n-1} & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} \\
&= a_1x^{n-1} + \cdots + a_n
\end{aligned}$$

**Example 2.2.** 计算

$$\begin{vmatrix} a & b & c & d \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ d & c & b & a \end{vmatrix}$$

$$\begin{vmatrix} a & b & c & d \\ x & 0 & 0 & y \\ y & 0 & 0 & x \\ d & c & b & a \end{vmatrix} = - \begin{vmatrix} y & 0 & 0 & x \\ x & 0 & 0 & y \\ a & b & c & d \\ d & c & b & a \end{vmatrix} = \begin{vmatrix} y & x & 0 & 0 \\ x & y & 0 & 0 \\ a & d & c & b \\ d & a & b & c \end{vmatrix} = (x^2 - y^2)(b^2 - c^2)$$

**Example 2.3.** 已知  $\mathbf{A}, \mathbf{B}$  均为  $n$  阶矩阵, 若  $|\mathbf{A}| = 3, |\mathbf{B}| = 2, |\mathbf{A}^{-1} + \mathbf{B}| = 2$ , 求  $|\mathbf{A} + \mathbf{B}^{-1}|$

$$\begin{aligned}
|\mathbf{A} + \mathbf{B}^{-1}| &= |\mathbf{E}\mathbf{A} + \mathbf{B}^{-1}\mathbf{E}| = |(\mathbf{B}^{-1}\mathbf{B})\mathbf{A} + \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})| \\
&= |\mathbf{B}^{-1}(\mathbf{B} + \mathbf{A}^{-1})\mathbf{A}| = 3
\end{aligned}$$

**Example 2.4.** 已知 4 阶矩阵  $\mathbf{A}$  相似于  $\mathbf{B}$ ,  $\mathbf{A}$  的特征值为 2, 3, 4, 5,  $\mathbf{E}$  为 4 阶单位矩阵, 求  $|\mathbf{B} - \mathbf{E}|$