

# A Course in Model Theory

Katin Tent & Martin Ziegler

November 4, 2020

## Contents

<b>1</b>	<b>The Basics</b>	<b>1</b>
1.1	Structures . . . . .	1
1.2	Language . . . . .	3
1.3	Theories . . . . .	5
<b>2</b>	<b>Elementary Extensions and Compactness</b>	<b>6</b>
2.1	Elementary substructures . . . . .	6
2.2	The Compactness Theorem . . . . .	7
<b>3</b>	<b>Quantifier Elimination</b>	<b>9</b>
3.1	Preservation theorems . . . . .	9
3.2	Quantifier elimination . . . . .	13
3.3	Examples . . . . .	19
<b>4</b>	<b>Countable Models</b>	<b>21</b>
4.1	The omitting types theorem . . . . .	21
4.2	The space of types . . . . .	22
<b>5</b>	<b>TODO Don't understand</b>	<b>22</b>

# 1 The Basics

## 1.1 Structures

**Definition 1.1.** Let  $\mathfrak{A}, \mathfrak{B}$  be  $L$ -structures. A map  $h : A \rightarrow B$  is called a **homomorphism** if for all  $a_1, \dots, a_n \in A$

$$\begin{aligned} h(c^{\mathfrak{A}}) &= c^{\mathfrak{B}} \\ h(f^{\mathfrak{A}}(a_1, \dots, a_n)) &= f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \\ R^{\mathfrak{A}}(a_1, \dots, a_n) &\Rightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \end{aligned}$$

We denote this by

$$h : \mathfrak{A} \rightarrow \mathfrak{B}$$

If in addition  $h$  is injective and

$$R^{\mathfrak{A}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n))$$

for all  $a_1, \dots, a_n \in A$ , then  $h$  is called an (isomorphic) **embedding**. An **isomorphism** is a surjective embedding

**Lemma 1.2.** Let  $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$  be an isomorphism and  $\mathfrak{B}$  an extension of  $\mathfrak{A}$ . Then there exists an extension  $\mathfrak{B}'$  of  $\mathfrak{A}'$  and an isomorphism  $g : \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}'$  extending  $h$

For any family  $\mathfrak{A}_i$  of substructures of  $\mathfrak{B}$ , the intersection of the  $A_i$  is either empty or a substructure of  $\mathfrak{B}$ . Therefore if  $S$  is any non-empty subset of  $\mathfrak{B}$ , then there exists a smallest substructure  $\mathfrak{A} = \langle S \rangle^{\mathfrak{B}}$  which contains  $S$ . We call the  $\mathfrak{A}$  the substructure **generated** by  $S$

**Lemma 1.3.** If  $\mathfrak{a} = \langle S \rangle$ , then every homomorphism  $h : \mathfrak{A} \rightarrow \mathfrak{B}$  is determined by its values on  $S$

**Definition 1.4.** Let  $(I, \leq)$  be a **directed partial order**. This means that for all  $i, j \in I$  there exists a  $k \in I$  s.t.  $i \leq k$  and  $j \leq k$ . A family  $(\mathfrak{A}_i)_{i \in I}$  of  $L$ -structures is called **directed** if

$$i \leq j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j$$

If  $I$  is linearly ordered, we call  $(\mathfrak{A}_i)_{i \in I}$  a **chain**

If a structure  $\mathfrak{A}_1$  is isomorphic to a substructure  $\mathfrak{A}_0$  of itself,

$$h_0 : \mathfrak{A}_0 \xrightarrow{\sim} \mathfrak{A}_1$$

then Lemma 1.2 gives an extension

$$h_1 : \mathfrak{A}_1 \xrightarrow{\sim} \mathfrak{A}_2$$

Continuing in this way we obtain a chain  $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$  and an increasing sequence  $h_i : \mathfrak{A}_i \xrightarrow{\sim} \mathfrak{A}_{i+1}$  of isomorphism

**Lemma 1.5.** Let  $(\mathfrak{A}_i)_{i \in I}$  be a directed family of  $L$ -structures. Then  $A = \bigcup_{i \in I} A_i$  is the universe of a (uniquely determined)  $L$ -structure

$$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$$

which is an extension of all  $\mathfrak{A}_i$

A subset  $K$  of  $L$  is called a **sublanguage**. An  $L$ -structure becomes a  $K$ -structure, the **reduct**.

$$\mathfrak{A} \upharpoonright K = (A, (Z^{\mathfrak{A}})_{Z \in K})$$

Conversely we call  $\mathfrak{A}$  an **expansion** of  $\mathfrak{A} \upharpoonright K$ .

1. Let  $B \subseteq A$ , we obtain a new language

$$L(B) = L \cup B$$

and the  $L(B)$ -structure

$$\mathfrak{A}_B = (\mathfrak{A}, b)_{b \in B}$$

Note that  $\mathbf{Aut}(\mathfrak{A}_B)$  is the group of automorphisms of  $\mathfrak{A}$  fixing  $B$  elementwise. We denote this group by  $\mathbf{Aut}(\mathfrak{A}/B)$

Let  $S$  be a set, which we call the set of sorts. An  $S$ -sorted language  $L$  is given by a set of constants for each sort in  $S$ , and typed function and relations. For any tuple  $(s_1, \dots, s_n)$  and  $(s_1, \dots, s_n, t)$  there is a set of relation symbols and function symbols respectively. An  $S$ -sorted structure is a pair  $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$ , where

$$\begin{array}{ll} A & \text{if a family } (A_s)_{s \in S} \text{ of non-empty sets} \\ Z^{\mathfrak{A}} \in A_s & \text{if } Z \text{ is a constant of sort } s \in S \\ Z^{\mathfrak{A}} : A_{s_1} \times \dots \times A_{s_n} \rightarrow A_t & \text{if } Z \text{ is a function symbol of type } (s_1, \dots, s_n, t) \\ Z^{\mathfrak{A}} \subseteq A_{s_1} \times \dots \times A_{s_n} & \text{if } Z \text{ is a relation symbol of type } (s_1, \dots, s_n) \end{array}$$

**Example 1.1.** Consider the two-sorted language  $L_{Perm}$  for permutation groups with a sort  $x$  for the set and a sort  $g$  for the group. The constants and function symbols for  $L_{Perm}$  are those of  $L_{Group}$  restricted to the sort  $g$  and an additional function symbol  $\varphi$  of type  $(x, g, x)$ . Thus an  $L_{Perm}$ -structure  $(X, G)$  is given by a set  $X$  and an  $L_{Group}$ -structure  $G$  together with a function  $X \times G \rightarrow X$

## 1.2 Language

**Lemma 1.6.** Suppose  $\vec{b}$  and  $\vec{c}$  agree on all variables which are free in  $\varphi$ . Then

$$\mathfrak{A} \models \varphi[\vec{b}] \Leftrightarrow \mathfrak{A} \models \varphi[\vec{c}]$$

We define

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n]$$

by  $\mathfrak{A} \models \varphi[\vec{b}]$ , where  $\vec{b}$  is an assignment satisfying  $\vec{b}(x_i) = a_i$ . Because of Lemma 1.6 this is well defined.

Thus  $\varphi(x_1, \dots, x_n)$  defines an  $n$ -ary relation

$$\varphi(\mathfrak{A}) = \{\vec{a} \mid \mathfrak{A} \models \varphi[\vec{a}]\}$$

on  $A$ , the **realisation set** of  $\varphi$ . Such realisation sets are called **0-definable subsets** of  $A^n$ , or 0-definable relations

Let  $B$  be a subset of  $A$ . A  **$B$ -definable** subset of  $\mathfrak{A}$  is a set of the form  $\varphi(\mathfrak{A})$  for an  $L(B)$ -formula  $\varphi(x)$ . We also say that  $\varphi$  are defined **over**  $B$  and that the set  $\varphi(\mathfrak{A})$  is defined by  $\varphi$ . We call two formulas **equivalent** if in every structure they define the same set.

Atomic formulas and their negations are called **basic**. Formulas without quantifiers are Boolean combinations of basic formulas. It is convenient to allow the empty conjunction and the empty disjunction. For that we introduce two new formulas: the formula  $\top$ , which is always true, and the formula  $\perp$ , which is always false. We define

$$\bigwedge_{i < 0} \pi_i = \top$$

$$\bigvee_{i < 0} \pi_i = \perp$$

A formula is in **negation normal form** if it is built from basic formulas using  $\wedge, \vee, \text{exists}, \forall$

**Definition 1.7.** A formula in negation normal form which does not contain any existential quantifier is called **universal**. Formulas in negation normal form without universal quantifiers are called **existential**

Let  $\mathfrak{A}$  be an  $L$ -structure. The **atomic diagram** of  $\mathfrak{A}$  is

$$\text{Diag}(\mathfrak{A}) = \{\varphi \text{ basic } L(A)\text{-sentence} \mid \mathfrak{A}_A \models \varphi\}$$

**Lemma 1.8.** *The models of  $\text{Diag}(\mathfrak{A})$  are precisely those structures  $(\mathfrak{B}, h(a))_{a \in A}$  for embeddings  $h : \mathfrak{A} \rightarrow \mathfrak{B}$*

*Exercise 1.2.1.* Every formula is equivalent to a formula in prenex normal form:

$$Q_1 x_1 \dots Q_n x_n \varphi$$

The  $Q_i$  are quantifiers and  $\varphi$  is quantifier-free

*Proof.*

$$\begin{aligned} (\forall x)\phi \wedge \psi &\models \forall x(\phi \wedge \psi) \text{ if } \exists x \top \text{ (at least one individual exists)} \\ (\forall x\phi) \vee \psi &\models \forall x(\phi \vee \psi) \\ (\exists x\phi) \wedge \psi &\models \exists x(\phi \wedge \psi) \\ (\exists x\phi) \vee \psi &\models \exists x(\phi \vee \psi) \text{ if } \exists x \top \\ \neg \exists x\phi &\models \forall x\neg\phi \\ \neg \forall x\phi &\models \exists x\neg\phi \\ (\forall x\phi) \rightarrow \psi &\models \exists x(\phi \rightarrow \psi) \text{ if } \exists x \top \\ (\exists x\phi) \rightarrow \psi &\models \forall x(\phi \rightarrow \psi) \\ \phi \rightarrow (\exists x\psi) &\models \exists x(\phi \rightarrow \psi) \text{ if } \exists x \top \\ \phi \rightarrow (\forall x\psi) &\models \forall x(\phi \rightarrow \psi) \end{aligned}$$

□

### 1.3 Theories

**Definition 1.9.** An  $L$ -theory  $T$  is a set of  $L$ -sentences

A theory which has a model is a **consistent** theory. We call a set  $\Sigma$  of  $L$ -formulas **consistent** if there is an  $L$ -structure and **an assignment**  $\vec{b}$  s.t.  $\mathfrak{A} \models [\vec{b}]$  for all  $\varphi \in \Sigma$

**Lemma 1.10.** *Let  $T$  be an  $L$ -theory and  $L'$  be an extension of  $L$ . Then  $T$  is consistent as an  $L$ -theory iff  $T$  is consistent as a  $L'$ -theory*

**Lemma 1.11.** 1. *If  $T \models \varphi$  and  $T \models (\varphi \rightarrow \psi)$ , then  $T \models \psi$*

2. *If  $T \models \varphi(c_1, \dots, c_n)$  and the constants  $c_1, \dots, c_n$  occur neither in  $T$  nor in  $\varphi(x_1, \dots, x_n)$ , then  $T \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$*

*Proof.* 2. Let  $L' = L \setminus \{c_1, \dots, c_n\}$ . If the  $L'$ -structure is a model of  $T$  and  $a_1, \dots, a_n$  are arbitrary elements, then  $(\mathfrak{A}, a_1, \dots, a_n) \models \varphi(c_1, \dots, c_n)$ . This means  $\mathfrak{A} \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$ . □

$S$  and  $T$  are called **equivalent**,  $S \equiv T$ , if  $S$  and  $T$  have the same models

**Definition 1.12.** A consistent  $L$ -theory  $T$  is called **complete** if for all  $L$ -sentences  $\varphi$

$$T \models \varphi \quad \text{or} \quad T \models \neg \varphi$$

**Definition 1.13.** For a complete theory  $T$  we define

$$|T| = \max(|L|, \aleph_0)$$

The typical example of a complete theory is the theory of a structure  $\mathfrak{A}$

$$\text{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$$

**Lemma 1.14.** A consistent theory is complete iff it is maximal consistent, i.e., if it is equivalent to every consistent extension

**Definition 1.15.** Two  $L$ -structures  $\mathfrak{A}$  and  $\mathfrak{B}$  are called **elementary equivalent**

$$\mathfrak{A} \equiv \mathfrak{B}$$

if they have the same theory

**Lemma 1.16.** Let  $T$  be a consistent theory. Then the following are equivalent

1.  $T$  is complete
2. All models of  $T$  are elementarily equivalent
3. There exists a structure  $\mathfrak{A}$  with  $T \equiv \text{Th}(\mathfrak{A})$

*Proof.*  $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$  □

## 2 Elementary Extensions and Compactness

### 2.1 Elementary substructures

Let  $\mathfrak{A}, \mathfrak{B}$  be two  $L$ -structures. A map  $h : A \rightarrow B$  is called **elementary** if for all  $a_1, \dots, a_n \in A$  we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

We write

$$h : \mathfrak{A} \xrightarrow{\text{el}} \mathfrak{B}$$

**Lemma 2.1.** *The models of  $\text{Th}(\mathfrak{A}_A)$  are exactly the structures of the form  $(\mathfrak{B}, h(a))_{a \in A}$  for elementary embeddings  $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$*

We call  $\text{Th}(\mathfrak{A}_A)$  the **elementary diagram** of  $\mathfrak{A}$

A substructure  $\mathfrak{A}$  of  $\mathfrak{B}$  is called **elementary** if the inclusion map is elementary. In this case we write

$$\mathfrak{A} \prec \mathfrak{B}$$

**Theorem 2.2** (Tarski's Test). *Let  $\mathfrak{B}$  be an  $L$ -structure and  $A$  a subset of  $B$ . Then  $A$  is the universe of an elementary substructure iff every  $L(A)$ -formula  $\varphi(x)$  which is satisfiable in  $\mathfrak{B}$  can be satisfied by an element of  $A$*

We use Tarski's Test to construct small elementary substructures

**Corollary 2.3.** *Suppose  $S$  is a subset of the  $L$ -structure  $\mathfrak{B}$ . Then  $\mathfrak{B}$  has a elementary substructure  $\mathfrak{A}$  containing  $S$  and of cardinality at most*

$$\max(|S|, |L|, \aleph_0)$$

*Proof.* We construct  $A$  as the union of an ascending sequence  $S_0 \subseteq S_1 \subseteq \dots$  of subsets of  $B$ . We start with  $S_0 = S$ . If  $S_i$  is already defined, we choose an element  $a_\varphi \in B$  for every  $L(S_i)$ -formula  $\varphi(x)$  which is satisfiable in  $\mathfrak{B}$  and define  $S_{i+1}$  to be  $S_i$  together with these  $a_\varphi$ .

An  $L$ -formula is a finite sequence of symbols from  $L$ , auxiliary symbols and logical symbols. These are  $|L| + \aleph_0 = \max(|L|, \aleph_0)$  many symbols and there are exactly  $\max(|L|, \aleph_0)$  many  $L$ -formulas

Let  $\kappa = \max(|S|, |L|, \aleph_0)$ . There are  $\kappa$  many  $L(S)$ -formulas: therefore  $|S_1| \leq \kappa$ . Inductively it follows for every  $i$  that  $|S_i| \leq \kappa$ . Finally we have  $|A| \leq \kappa \cdot \aleph_0 = \kappa$   $\square$

A directed family  $(\mathfrak{A}_i)_{i \in I}$  of structures is **elementary** if  $\mathfrak{A}_i \prec \mathfrak{A}_j$  for all  $i \leq j$

**Theorem 2.4** (Tarski's Chain Lemma). *The union of an elementary directed family is an elementary extension of all its members*

*Proof.* Let  $\mathfrak{A} = \bigcup_{i \in I} (\mathfrak{A}_i)_{i \in I}$ . We prove by induction on  $\varphi(\bar{x})$  that for all  $i$  and  $\bar{a} \in \mathfrak{A}_i$

$$\mathfrak{A}_i \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \models \varphi(\bar{a})$$

$\square$

## 2.2 The Compactness Theorem

**Theorem 2.5** (Compactness Theorem). *Finitely satisfiable theories are consistent*

Let  $L$  be a language and  $C$  a set of new constants. An  $L(C)$ -theory  $T'$  is called a **Henkin theory** if for every  $L(C)$ -formula  $\varphi(x)$  there is a constant  $c \in C$  s.t.

$$\exists x \varphi(x) \rightarrow \varphi(c) \in T'$$

**Lemma 2.6.** *Every finitely satisfiable  $L$ -theory  $T$  can be extended to a finitely complete Henkin Theory  $T^*$*

**Lemma 2.7.** *Every finitely satisfiable  $L$ -theory  $T$  can be extended to a finitely complete Henkin theory  $T^*$*

**Lemma 2.8.** *Every finitely complete Henkin theory  $T^*$  has a model  $\mathfrak{A}$  (unique up to isomorphism) consisting of constants; i.e.,*

$$(\mathfrak{A}, a_c)_{c \in C} \models T^*$$

with  $A = \{a_c \mid c \in C\}$

**Corollary 2.9.** *A set of formulas  $\Sigma(x_1, \dots, x_n)$  is consistent with  $T$  if and only if every finite subset of  $\Sigma$  is consistent with  $T$*

*Proof.* Introduce new constants  $c_1, \dots, c_n$ . Then  $\Sigma$  is consistent with  $T$  if and only if  $T \cup \Sigma(c_1, \dots, c_n)$  is consistent. Now apply the Compactness Theorem  $\square$

**Definition 2.10.** Let  $\mathfrak{A}$  be an  $L$ -structure and  $B \subseteq A$ . Then  $a \in A$  **realises** a set of  $L(B)$ -formulas  $\Sigma(x)$  if  $a$  satisfies all formulas from  $\Sigma$ . We write

$$\mathfrak{A} \models \Sigma(a)$$

We call  $\Sigma(x)$  **finitely satisfiable** in  $\mathfrak{A}$  if every finite subset of  $\Sigma$  is realised in  $\mathfrak{A}$

**Lemma 2.11.** *The set  $\Sigma(x)$  is finitely satisfiable in  $\mathfrak{A}$  iff there is an elementary extension of  $\mathfrak{A}$  in which  $\Sigma(x)$  is realised*

*Proof.* By Lemma 2.1  $\Sigma$  is realised in an elementary extension of  $\mathfrak{A}$  iff  $\Sigma$  is consistent with  $\text{Th}(\mathfrak{A}_A)$ . So the lemma follows from the observation that a finite set of  $L(A)$ -formulas is consistent with  $\text{Th}(\mathfrak{A}_A)$  iff it is realised in  $\mathfrak{A}$   $\square$



**Definition 2.12.** Let  $\mathfrak{A}$  be an  $L$ -structure and  $B$  a subset of  $A$ . A set  $p(x)$  of  $L(B)$ -formulas is a **type** over  $B$  if  $p(x)$  is maximal finitely satisfiable in  $\mathfrak{A}$ . We call  $B$  the **domain** of  $p$ . Let

$$S(B) = S^{\mathfrak{A}}(B)$$

denote the set of types over  $B$ .

Every element  $a$  of  $\mathfrak{A}$  determines a type

$$\text{tp}(a/B) = tp^{\mathfrak{A}}(a/B) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a), \varphi \text{ an } L(B)\text{-formula}\}$$

So an element  $a$  realises the type  $p \in S(B)$  exactly if  $p = \text{tp}(a/B)$ . If  $\mathfrak{A}'$  is an elementary extension of  $\mathfrak{A}$ , then

$$S^{\mathfrak{A}'}(B) = S^{\mathfrak{A}}(B) \quad \text{and} \quad \text{tp}^{\mathfrak{A}'}(a/B) = \text{tp}^{\mathfrak{A}}(a/B)$$

If  $\mathfrak{A}' \models p(x)$  then  $\mathfrak{A}' \models \exists x p(x)$ , so  $\mathfrak{A} \models \exists x p(x)$ .

We use the notation  $\text{tp}(a)$  for  $\text{tp}(a/\emptyset)$

Maximal finitely satisfiable sets of formulas in  $x_1, \dots, x_n$  are called  **$n$ -types** and

$$S_n(B) = S_N^{\mathfrak{A}}(B)$$

denotes the set of  $n$ -types over  $B$ .

$$\text{tp}(C/B) = \{\varphi(x_{c_1}, \dots, x_{c_n}) \mid \mathfrak{A} \models \varphi(c_1, \dots, c_n), \varphi \text{ an } L(B)\text{-formula}\}$$

**Corollary 2.13.** Every structure  $\mathfrak{A}$  has an elementary extension  $\mathfrak{B}$  in which all types over  $A$  are realised

*Proof.* We choose for every  $p \in S(A)$  a new constant  $c_p$ . We have to find a model of

$$\text{Th}(\mathfrak{A}_A) \cup \bigcup_{p \in S(A)} p(c_p)$$

This theory is finitely satisfiable since every  $p$  is finitely satisfiable in  $\mathfrak{A}$ .

Or use Lemma 2.11. Let  $(p_\alpha)_{\alpha < \lambda}$  be an enumeration of  $S(A)$ . Construct an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \dots \prec \mathfrak{A}_\beta \prec \dots (\beta \leq \lambda)$$

s.t. each  $p_\alpha$  is realised in  $\mathfrak{A}_{\alpha+1}$  (by recursion theorem on ordinal numbers)

Suppose that the elementary chain  $(\mathfrak{A}_{\alpha'})_{\alpha' < \beta}$  is already constructed. If  $\beta$  is a limit ordinal, we let  $\mathfrak{A}_\beta = \bigcup_{\alpha < \beta} \mathfrak{A}_\alpha$ , which is elementary by Lemma 2.4. If  $\beta = \alpha + 1$  we first note that  $p_\alpha$  is also finitely satisfiable in  $\mathfrak{A}_\alpha$ , therefore we can realise  $p_\alpha$  in a suitable elementary extension  $\mathfrak{A}_\beta \succ \mathfrak{A}_\alpha$  by Lemma 2.11. Then  $\mathfrak{B} = \mathfrak{A}_\lambda$  is the model we were looking for  $\square$

### 3 Quantifier Elimination

#### 3.1 Preservation theorems

**Lemma 3.1** (Separation Lemma). *Let  $T_1, T_2$  be two theories. Assume  $\mathcal{H}$  is a set of sentences which is closed under  $\wedge, \vee$  and contains  $\perp$  and  $\top$ . Then the following are equivalent*

1. *There is a sentence  $\varphi \in \mathcal{H}$  which separates  $T_1$  from  $T_2$ . This means*

$$T_1 \models \varphi \quad \text{and} \quad T_2 \models \neg\varphi$$

2. *All models  $\mathfrak{A}_1$  of  $T_1$  can be separated from all models  $\mathfrak{A}_2$  of  $T_2$  by a sentence  $\varphi \in \mathcal{H}$ . This means*

$$\mathfrak{A}_1 \models \varphi \quad \text{and} \quad \mathfrak{A}_2 \models \neg\varphi$$

*Proof.*  $2 \rightarrow 1$ . For any model  $\mathfrak{A}_1$  of  $T_1$  let  $\mathcal{H}_{\mathfrak{A}_1}$  be the set of all sentences from  $\mathcal{H}$  which are true in  $\mathfrak{A}_1$ . (2) implies that  $\mathcal{H}_{\mathfrak{A}_1}$  and  $T_2$  cannot have a common model. By the Compactness Theorem there is a finite conjunction  $\varphi_{\mathfrak{A}_1}$  of sentences from  $\mathcal{H}_{\mathfrak{A}_1}$  inconsistent with  $T_2$ . Clearly

$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$

is inconsistent. Again by compactness  $T_1$  implies a disjunction  $\varphi$  of finitely many of the  $\varphi_{\mathfrak{A}_1}$  □

For structures  $\mathfrak{A}, \mathfrak{B}$  and a map  $f : A \rightarrow B$  preserving all formulas from a set of formulas  $\Delta$ , we use the notation

$$f : \mathfrak{A} \rightarrow_{\Delta} \mathfrak{B}$$

We also write

$$\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$$

to express that all sentences from  $\Delta$  true in  $\mathfrak{A}$  are also true in  $\mathfrak{B}$

**Lemma 3.2.** *Let  $T$  be a theory,  $\mathfrak{A}$  a structure and  $\Delta$  a set of formulas, closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent*

1. *All sentences  $\varphi \in \Delta$  which are true in  $\mathfrak{A}$  are consistent with  $T$  (There is a model  $\mathfrak{B} \models \text{Th}_{\Delta}(\mathfrak{A}_A) \cup T$  and  $\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$ )*

2. There is a model  $\mathfrak{B} \models T$  and a map  $f : \mathfrak{A} \rightarrow_{\Delta} \mathfrak{B}$

*Proof.*  $1 \rightarrow 2$ . Consider  $\text{Th}_{\Delta}(\mathfrak{A}_A)$ , the set of all sentences  $\delta(\bar{a})$  ( $\delta(\bar{x}) \in \Delta$ ), which are true in  $\mathfrak{A}_A$ . The models  $(\mathfrak{B}, f(a)_{a \in A})$  of this theory correspond to maps  $f : \mathfrak{A} \rightarrow_{\Delta} \mathfrak{B}$ . **This means that we have to find a model of  $T \cup \text{Th}_{\Delta}(\mathfrak{A}_A)$ .** To show finite satisfiability it is enough to show that  $T \cup D$  is consistent for every finite subset  $D$  of  $\text{Th}_{\Delta}(\mathfrak{A}_A)$ . Let  $\delta(\bar{a})$  be the conjunction of the elements of  $D$ . Then  $T$  has a model  $\mathfrak{B}$  which is also a model of  $\varphi = \exists \bar{x} \delta(\bar{x})$   $\square$

Lemma 3.2 applied to  $T = \text{Th}(\mathfrak{B})$  shows that  $\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$  iff there exists a map  $f$  and a structure  $\mathfrak{B}' \equiv \mathfrak{B}$  s.t.  $f : \mathfrak{A} \rightarrow_{\Delta} \mathfrak{B}'$

**Theorem 3.3.** Let  $T_1$  and  $T_2$  be two theories. Then the following are equivalent

1. There is a universal sentence which separates  $T_1$  from  $T_2$
2. No model of  $T_2$  is a substructure of a model of  $T_1$

*Proof.*  $2 \rightarrow 1$ . If  $T_1$  and  $T_2$  cannot be separated by a universal sentence, then they have models  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  which cannot be separated by a universal sentence. This can be denoted by

$$\mathfrak{A}_2 \Rightarrow_{\exists} \mathfrak{A}_1$$

Now Lemma 3.2 implies that  $\mathfrak{A}_2$  there is a map  $\mathfrak{A}_2 \rightarrow_{\exists} \mathfrak{A}'_1$  where  $\mathfrak{A}'_1 \models T_1$ . Hence  $\mathfrak{A}_2$  has an extension  $\mathfrak{A}'_2$  s.t.  $\mathfrak{A}'_2 \equiv \mathfrak{A}'_1$ . Then  $\mathfrak{A}'_2$  is a model of  $T_1$  contradicting (2)  $\square$

**Definition 3.4.** For any  $L$ -theory  $T$ , the formulas  $\varphi(\bar{x}), \psi(\bar{x})$  are said to be **equivalent** modulo  $T$  (or relative to  $T$ ) if  $T \models \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

**Corollary 3.5.** Let  $T$  be a theory

1. Consider a formula  $\varphi(x_1, \dots, x_n)$ . The following are equivalent
  - (a)  $\varphi(x_1, \dots, x_n)$  is, modulo  $T$ , equivalent to a universal formula
  - (b) If  $\mathfrak{A} \subseteq \mathfrak{B}$  are models of  $T$  and  $a_1, \dots, a_n \in A$ , then  $\mathfrak{B} \models \varphi(a_1, \dots, a_n)$  implies  $\mathfrak{A} \models \varphi(a_1, \dots, a_n)$
2. We say that a theory which consists of universal sentences is **universal**. Then  $T$  is equivalent to a universal theory iff all substructures of models of  $T$  are again models of  $T$

*Proof.* 1. Assume (2). We extend  $L$  by an  $n$ -tuple  $\bar{c}$  of new constants  $c_1, \dots, c_n$  and consider theory

$$T_1 = T \cup \{\varphi(\bar{c})\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi(\bar{c})\}$$

Then (2) says the substructures of models of  $T_1$  cannot be models of  $T_2$ . By Theorem 3.3  $T_1$  and  $T_2$  can be separated by a universal  $L(\bar{c})$ -sentence  $\psi(\bar{c})$ . By Lemma 1.11,  $T_1 \models \psi(\bar{c})$  implies

$$T \models \forall \bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$$

and from  $T_2 \models \neg\psi(\bar{c})$  we see

$$T \models \forall \bar{x}(\neg\varphi(\bar{x}) \rightarrow \neg\psi(\bar{x}))$$

2. Suppose a theory  $T$  has this property. Let  $\varphi$  be an axiom of  $T$ . If  $\mathfrak{A}$  is a substructure of  $\mathfrak{B}$ , it is not possible for  $\mathfrak{B}$  to be a model of  $T$  and for  $\mathfrak{A}$  to be a model of  $\neg\psi$  at the same time. By Theorem 3.3 there is a universal sentence  $\psi$  with  $T \models \psi$  and  $\neg\varphi \models \neg\psi$ . Hence all axioms of  $T$  follow from

$$T_\forall = \{\psi \mid T \models \psi, \psi \text{ universal}\}$$

□

An  $\forall\exists$ -formula is of the form

$$\forall x_1 \dots x_n \psi$$

where  $\psi$  is existential

**Lemma 3.6.** Suppose  $\varphi$  is an  $\forall\exists$ -sentence,  $(\mathfrak{A}_i)_{i \in I}$  is a directed family of models of  $\varphi$  and  $\mathfrak{B}$  the union of the  $\mathfrak{A}_i$ . Then  $\mathfrak{B}$  is also a model of  $\varphi$ .

*Proof.* Write

$$\varphi = \forall \bar{x} \psi(\bar{x})$$

where  $\psi$  is existential. For any  $\bar{a} \in B$  there is an  $A_i$  containing  $\bar{a}$ , clearly  $\psi(\bar{a})$  holds in  $\mathfrak{A}_i$ . As  $\psi(\bar{a})$  is existential it must also hold in  $\mathfrak{B}$  □

**Definition 3.7.** We call a theory  $T$  **inductive** if the union of any directed family of models of  $T$  is again a model

**Theorem 3.8.** Let  $T_1$  and  $T_2$  be two theories. Then the following are equivalent

1. there is an  $\forall\exists$ -sentence which separates  $T_1$  and  $T_2$

2. No model of  $T_2$  is the union of a chain (or of a directed family) of models of  $T_1$

*Proof.*  $2 \rightarrow 1$ . If (1) is not true,  $T_1, T_2$  have models which cannot be separated by an  $\forall\exists$ -sentence. Since  $\exists\forall$ -formulas are equivalent to negated  $\forall\exists$ -formulas (since  $\forall$  is too strong), we have

$$\mathfrak{B}^0 \Rightarrow_{\exists\forall} \mathfrak{A}$$

By Lemma 3.2 there is a map

$$f : \mathfrak{B}^0 \rightarrow_{\forall} \mathfrak{A}^0$$

with  $\mathfrak{A}^0 \equiv \mathfrak{A}$  (since  $\mathfrak{B}^0 \rightarrow_{\exists\forall} \mathfrak{A}^0$ ). We can assume that  $\mathfrak{B}^0 \subseteq \mathfrak{A}^0$  and  $f$  is the inclusion map. Then

$$\mathfrak{A}_B^0 \Rightarrow_{\exists} \mathfrak{B}_B^0$$

Applying Lemma 3.2 again, we obtain an extension  $\mathfrak{B}_B^1$  of  $\mathfrak{A}_B^0$  with  $\mathfrak{B}_B^1 \equiv \mathfrak{B}_B^0$ , i.e.  $\mathfrak{B}^0 \prec \mathfrak{B}^1$ . Hence we have an infinite chain

$$\begin{aligned} \mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq^1 \mathfrak{B}^1 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{B}^2 \subseteq \dots \\ \mathfrak{B}^0 \prec \mathfrak{B}^1 \prec \mathfrak{B}^2 \prec \dots \\ \mathfrak{A}^i \equiv \mathfrak{A} \end{aligned}$$

Let  $\mathfrak{B}$  be the union of the  $\mathfrak{A}^i$ . Since  $\mathfrak{B}$  is also the union of the elementary chain of the  $\mathfrak{B}^i$ , it is an elementary extension of  $\mathfrak{B}^0$  and hence a model of  $T_2$ . But the  $\mathfrak{A}^i$  are models of  $T_1$ , so (2) does not hold  $\square$

**Corollary 3.9.** *Let  $T$  be a theory*

1. *For each sentence  $\varphi$  the following are equivalent*

- (a)  *$\varphi$  is, modulo  $T$ , equivalent to an  $\forall\exists$ -sentence*
- (b) *If*

$$\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \dots$$

*and their union  $\mathfrak{B}$  are models of  $T$ , then  $\varphi$  holds in  $\mathfrak{B}$  if it is true in all the  $\mathfrak{A}^i$*

2.  *$T$  is inductive iff it can be axiomatised by  $\forall\exists$ -sentences*

*Proof.* 1. Theorem 3.8 shows that  $\forall\exists$ -formulas are preserved by unions of chains. Hence (a) $\Rightarrow$ (b). For the converse consider the theories

$$T_1 = T \cup \{\varphi\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi\}$$

Part (b) says that the union of a chain of models of  $T_1$  cannot be a model of  $T_2$ . By Theorem 3.8 we can separate  $T_1$  and  $T_2$  by an  $\forall\exists$ -sentence  $\psi$ . Hence  $T \cup \{\varphi\} \models \psi$  and  $T \cup \{\neg\varphi\} \models \neg\psi$

2. Clearly  $\forall\exists$ -axiomatised theories are inductive. For the converse assume that  $T$  is inductive and  $\varphi$  is an axiom of  $T$ . If  $\mathfrak{B}$  is a union of models of  $T$ , it cannot be a model of  $\neg\varphi$ . By Theorem 3.8 there is an  $\forall\exists$ -sentence  $\psi$  with  $T \models \psi$  and  $\neg\varphi \models \neg\psi$ . Hence all axioms of  $T$  follows from

$$T_{\forall\exists} = \{\psi \mid T \models \psi, \psi \text{ } \forall\exists\text{-formula}\}$$

□

### 3.2 Quantifier elimination

**Definition 3.10.** A theory  $T$  has **quantifier elimination** if every  $L$ -formula  $\varphi(x_1, \dots, x_n)$  in the theory is equivalent modulo  $T$  to some quantifier-free formula  $\rho(x_1, \dots, x_n)$

It's easy to transform any theory  $T$  into a theory with quantifier elimination if one is willing to expand the language: just enlarge  $L$  by adding an  $n$ -place relation symbol  $R_\varphi$  for every  $L$ -formula  $\varphi(x_1, \dots, x_n)$  and  $T$  by adding all axioms

$$\forall x_1, \dots, x_n (R_\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n))$$

The resulting theory, the **Morleyisation**  $T^m$  of  $T$ , has quantifier elimination

A **prime structure** of  $T$  is a structure which embeds into all models of  $T$

**Lemma 3.11.** A consistent theory  $T$  with quantifier elimination which possesses a prime structure is complete

*Proof.* If  $\mathfrak{M}, \mathfrak{N} \models T$  and  $\mathfrak{M} \models \varphi$  and  $\mathfrak{N} \models \neg\varphi$ . The prime structure is  $\mathfrak{H}$ . Then we have  $h_1 : \mathfrak{H} \rightarrow \mathfrak{M}$  and  $h_2 : \mathfrak{H} \rightarrow \mathfrak{N}$ . If  $\varphi$  doesn't contain existential quantification, then there is a contradiction. □

**Definition 3.12.** A **simple existential formula** has the form

$$\varphi = \exists y \rho$$

for a quantifier-free formula  $\rho$ . If  $\rho$  is a conjunction of basic formulas,  $\varphi$  is called **primitive existential**

**Lemma 3.13.** *The theory  $T$  has quantifier elimination iff every primitive existential formula is, modulo  $T$ , equivalent to a quantifier-free formula*

*Proof.* We can write every simple existential formula in the form  $\exists y \bigvee_{i < n} \rho_i$  for  $\rho_i$  which are conjunctions of basic formulas. This shows that every simple existential formula is equivalent to a disjunction of primitive existential formulas, namely to  $\bigvee_{i < n} (\exists y \rho_i)$ . We can therefore assume that every simple existential formula is, modulo  $T$ , equivalent to a quantifier-free formula

We are now able to eliminate the quantifiers in arbitrary formulas in prenex normal form (Exercise 1.2.1)

$$Q_1 x_1 \dots Q_n x_n \rho$$

if  $Q_n = \exists$ , we choose a quantifier-free formula  $\rho_0$  which, modulo  $T$ , is equivalent to  $\exists x_n \rho$  and proceed with the formula  $Q_1 x_1 \dots Q_{n-1} x_{n-1} \rho_0$ . If  $Q_n = \forall$ , we find a quantifier-free  $\rho_1$  which is, modulo  $T$ , equivalent to  $\exists x_n \neg \rho$  and proceed with  $Q_1 x_1 \dots Q_{n-1} x_{n-1} \neg \rho_1$   $\square$

**Theorem 3.14.** *For a theory  $T$  the following are equivalent*

1.  $T$  has quantifier elimination
2. For all models  $\mathfrak{M}^1$  and  $\mathfrak{M}^2$  of  $T$  with a common substructure  $\mathfrak{A}$  we have

$$\mathfrak{M}_A^1 \equiv \mathfrak{M}_A^2$$

3. For all models  $\mathfrak{M}^1$  and  $\mathfrak{M}^2$  of  $T$  with a common substructure  $\mathfrak{A}$  and for all primitive existential formulas  $\varphi(x_1, \dots, x_n)$  and parameter  $a_1, \dots, a_n$  from  $A$  we have

$$\mathfrak{M}^1 \models \varphi(a_1, \dots, a_n) \Rightarrow \mathfrak{M}^2 \models \varphi(a_1, \dots, a_n)$$

(this is exactly the equivalence relation)

If  $L$  has no constants,  $\mathfrak{A}$  is allowed to be the empty "structure"

*Proof.*  $3 \rightarrow 1$ . Let  $\varphi(\bar{x})$  be a primitive existential formula. In order to show that  $\varphi(\bar{x})$  is equivalent, modulo  $T$ , to a quantifier-free formula  $\rho(\bar{x})$  we extend  $L$  by an  $n$ -tuple  $\bar{c}$  of new constants  $c_1, \dots, c_n$ . **We have to show that we can separate  $T \cup \{\varphi(\bar{c})\}$  and  $T \cup \{\neg \varphi(\bar{c})\}$  by a quantifier free sentence  $\rho(\bar{c})$ .** We apply the Separation Lemma ( $\mathcal{H}$  hear is the set of quantifier-free sentence). Let  $\mathfrak{M}^1$  and  $\mathfrak{M}^2$  be two models of  $T$  with two distinguished  $n$ -tuples  $\bar{a}^1$  and  $\bar{a}^2$ . Suppose that  $(\mathfrak{M}^1, \bar{a}^1)$  and  $(\mathfrak{M}^2, \bar{a}^2)$  satisfy the same quantifier-free  $L(\bar{c})$ -sentences. We have to show that

$$\mathfrak{M}^1 \models \varphi(\bar{a}^1) \Rightarrow \mathfrak{M}^2 \models \varphi(\bar{a}^2)$$

then there is no  $L(\bar{c})$ -sentence that can separate the models of  $T \cup \{\varphi(\bar{c})\}$  and the models of  $T \cup \{\neg\varphi(\bar{c})\}$ . Consider the substructure  $\mathfrak{A}^i = \langle \bar{a}^i \rangle^{\mathfrak{M}^i}$ , generated by  $\bar{a}^i$ . If we can show that there is an isomorphism

$$f : \mathfrak{A}^1 \rightarrow \mathfrak{A}^2$$

taking  $\bar{a}$  to  $\bar{a}$ , we may assume that  $\mathfrak{A}^1 = \mathfrak{A}^2 = \mathfrak{A}$  and  $\bar{a}^1 = \bar{a}^2 = \bar{a}$ .

Every element of  $\mathfrak{A}^1$  has the form  $t^{\mathfrak{M}^1}[\bar{a}^1]$  for an  $L$ -term  $t(\bar{x})$ . The isomorphism  $f$  to be constructed must satisfy

$$f(t^{\mathfrak{M}^1}[\bar{a}^1]) = t^{\mathfrak{M}^2}[\bar{a}^2]$$

We define  $f$  by this equation and have to check that  $f$  is well defined and injective. Assume

$$s^{\mathfrak{M}^1}[\bar{a}^1] = t^{\mathfrak{M}^1}[\bar{a}^1]$$

Then  $\mathfrak{M}^1, \bar{a}^1 \models s(\bar{c}) \doteq t(\bar{c})$ , and by our assumption it also holds in  $(\mathfrak{M}^2, \bar{a}^2)$ , which means

$$s^{\mathfrak{M}^2}[\bar{a}^2] = t^{\mathfrak{M}^2}[\bar{a}^2]$$

Swapping the two sides yields injectivity.

Surjectivity is clear. It remains to show that  $f$  commutes with the interpretation of the relation symbols. Now

$$\mathfrak{M}^1 \models R[t_1^{\mathfrak{M}^1}[\bar{a}^1], \dots, t_m^{\mathfrak{M}^1}[\bar{a}^1]]$$

is equivalent to  $(\mathfrak{M}^1, \bar{a}^1) \models R(t_1(\bar{c}), \dots, t_m(\bar{c}))$ , which is equivalent to  $(\mathfrak{M}^2, \bar{a}^2) \models R(t_1(\bar{c}), \dots, t_m(\bar{c}))$ , which in turn is equivalent to

$$\mathfrak{M}^2 \models R[t_1^{\mathfrak{M}^2}[\bar{a}^2], \dots, t_m^{\mathfrak{M}^2}[\bar{a}^2]]$$

□

Note that (2) of Theorem 3.14 is saying that  $T$  is **substructure complete**; i.e., for any model  $\mathfrak{M} \models T$  and substructure  $\mathfrak{A} \subseteq \mathfrak{M}$  the theory  $T \cup \text{Diag}(\mathfrak{A})$  is complete

**Definition 3.15.** We call  $T$  **model complete** if for all models  $\mathfrak{M}^1$  and  $\mathfrak{M}^2$  of  $T$

$$\mathfrak{M}^1 \subseteq \mathfrak{M}^2 \Rightarrow \mathfrak{M}^1 \prec \mathfrak{M}^2$$

$T$  is model complete iff for any  $\mathfrak{M} \models T$  the theory  $T \cup \text{Diag}(\mathfrak{M})$  is complete

**Lemma 3.16** (Robinson's Test). *Let  $T$  be a theory. Then the following are equivalent*



1.  $T$  is model complete
2. For all models  $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$  of  $T$  and all existential sentences  $\varphi$  from  $L(M^1)$

$$\mathfrak{M}^2 \models \varphi \Rightarrow \mathfrak{M}^1 \models \varphi$$

3. Each formula is, modulo  $T$ , equivalent to a universal formula

*Proof.*  $1 \leftrightarrow 3$ . Corollary 3.5

(2) implies that every existential formula is, modulo  $T$ , equivalent to a universal formula  $\square$

If  $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$  satisfies (2), we call  $\mathfrak{M}^1$  **existentially closed** in  $\mathfrak{M}^2$ . We denote this by

$$\mathfrak{M}^1 \prec_1 \mathfrak{M}^2$$

**Definition 3.17.** Let  $T$  be a theory. A theory  $T^*$  is a **model companion** of  $T$  if the following three conditions are satisfied

1. Each model of  $T$  can be extended to a model of  $T^*$
2. Each model of  $T^*$  can be extended to a model of  $T$
3.  $T^*$  is model complete

**Theorem 3.18.** A theory  $T$  has, up to equivalence, at most one model companion  $T^*$

*Proof.* If  $T^+$  is another model companion of  $T$ , every model of  $T^+$  is contained in a model of  $T^*$  and conversely. Let  $\mathfrak{A}_0 \models T^+$ . Then  $\mathfrak{A}_0$  can be embedded in a model  $\mathfrak{B}_0$  of  $T^*$ . In turn  $\mathfrak{B}_0$  is contained in a model  $\mathfrak{A}_1$  of  $T^+$ . In this way we find two elementary chains  $(\mathfrak{A}_i)$  and  $(\mathfrak{B}_i)$ , which have a common union  $\mathfrak{C}$ . Then  $\mathfrak{A}_0 \prec \mathfrak{C}$  and  $\mathfrak{B}_0 \prec \mathfrak{C}$  implies  $\mathfrak{A}_0 \equiv \mathfrak{B}_0$  since  $T$  are all sentences. Thus  $\mathfrak{A}_0$  is a model of  $T^*$   $\square$

### Existentially closed structures and the Kaiser hull

Let  $T$  be an  $L$ -theory. It follows from 3.2 that the models of  $T_{\forall}$  are the substructures of models of  $T$ . The conditions (1) and (2) in the definition of "model companion" can therefore be expressed as

$$T_{\forall} = T_{\forall}^*$$

Hence the model companion of a theory  $T$  depends only on  $T_{\forall}$ . (Note that  $T_{\forall}$  is model complete)

**Definition 3.19.** An  $L$ -structure  $\mathfrak{A}$  is called  $T$ -**existentially closed** (or  $T$ -**ec**) if

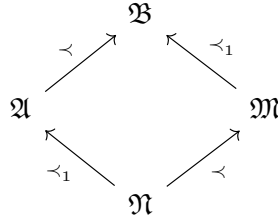
1.  $\mathfrak{A}$  can be embedded in a model of  $T$
2.  $\mathfrak{A}$  is existentially closed in every extension which is a model of  $T$

A structure  $\mathfrak{A}$  is  $T$ -ec exactly if it is  $T_\forall$ -ec. Since every model of  $\mathfrak{B}$  of  $T_\forall$  can be embedded in a model  $\mathfrak{M}$  of  $T$  and  $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$  and  $\mathfrak{A} \prec_1 \mathfrak{M}$  implies  $\mathfrak{A} \prec_1 \mathfrak{B}$

**Lemma 3.20.** Every model of a theory  $T$  can be embedded in a  $T$ -ec structure

*Proof.* Let  $\mathfrak{A}$  be a model of  $T_\forall$ . We choose an enumeration  $(\varphi_\alpha)_{\alpha < \kappa}$  of all existential  $L(A)$ -sentences and construct an ascending chain  $(\mathfrak{A}_\alpha)_{\alpha \leq \kappa}$  of models of  $T_\forall$ . We begin with  $\mathfrak{A}_0 = \mathfrak{A}$ . Let  $\mathfrak{A}_\alpha$  be constructed. If  $\varphi_\alpha$  holds in an extension of  $\mathfrak{A}_\alpha$  which is a model of  $T$  we let  $\mathfrak{A}_{\alpha+1}$  be such a model. Otherwise we set  $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$ . For limit ordinals  $\lambda$  we define  $\mathfrak{A}_\lambda$  to be the union of all  $\mathfrak{A}_\alpha$ .  $\mathfrak{A}_\lambda$  is again a model of  $T_\forall$   $\square$

Every elementary substructure  $\mathfrak{N}$  of a  $T$ -ec structure  $\mathfrak{M}$  is again  $T$ -ec. Let  $\mathfrak{N} \subseteq \mathfrak{A}$  be a model of  $T$ . Since  $\mathfrak{M}_N \Rightarrow_{\exists} \mathfrak{A}_N$ , there is an embedding of  $\mathfrak{M}$  in an elementary extension  $\mathfrak{B}$  of  $\mathfrak{A}$  which is the identity on  $N$ . Since  $\mathfrak{M}$  is existentially closed in  $\mathfrak{B}$ , it follows that  $\mathfrak{N}$  is existentially closed in  $\mathfrak{B}$  and therefore also in  $\mathfrak{A}$



**Lemma 3.21.** Let  $T$  be a theory. Then there is a biggest inductive theory  $T^{\text{KH}}$  with  $T_\forall = T_\forall^{\text{KH}}$ . We call  $T^{\text{KH}}$  the **Kaiser hull** of  $T$

*Proof.* Let  $T^1$  and  $T^2$  be two inductive theories with  $T_\forall^1 = T_\forall^2 = T_\forall$ . We have to show that  $(T^1 \cup T^2)_\forall = T_\forall$ . Let  $\mathfrak{M}$  be a model of  $T$ , as in the proof of 3.18 we extend  $\mathfrak{M}$  by a chain  $\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \dots$  of models of  $T^1$  and  $T^2$ . The union of this chain is a model of  $T^1 \cup T^2$

(Both of  $T_\forall^1$  and  $T_\forall^2$  and model companion and hence equivalent)  $\square$

**Lemma 3.22.** *The Kaiser hull  $T^{KH}$  is the  $\forall\exists$ -part of the theory of all  $T$ -ec structures*

*Proof.* Let  $T^*$  be the  $\forall\exists$ -part of the theory of all  $T$ -ec structures. Since  $T$ -ec structures are models of  $T_\forall$ , we have  $T_\forall \subseteq T_\forall^*$ . It follows from 3.20 that  $T_\forall^* \subseteq T_\forall$ . Hence  $T^*$  is contained in the Kaiser Hull.  $\square$

This implies that  $T$ -ec structures are models of  $T_{\forall\exists}$

**Theorem 3.23.** *For any theory  $T$  the following are equivalent*

1.  $T$  has a model companion  $T^*$
2. All models of  $K^{KH}$  are  $T$ -ec
3. The  $T$ -ec structures form an elementary class.

*If  $T^*$  exists, we have*

$$T^* = T^{KH} = \text{theory of all } T\text{-ec structures}$$

**Exercise 3.2.1.** Let  $L$  be the language containing a unary function  $f$  and a binary relation symbol  $R$  and consider the  $L$ -theory  $T = \{\forall x \forall y (R(x, y) \rightarrow (R(x, f(y))))\}$ . Showing the follow

1. For any  $T$ -structure  $\mathfrak{M}$  and  $a, b \in M$  with  $b \notin \{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$  we have  $\mathfrak{M} \models \exists z (R(z, a) \wedge \neg R(z, b))$
2. Let  $\mathfrak{M}$  be a model of  $T$  and  $a$  an element of  $M$  s.t.  $\{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$  is infinite. Then in an elementary extension  $\mathfrak{M}'$  there is an element  $b$  with  $\mathfrak{M}' \models \forall z (R(z, a) \rightarrow R(z, b))$
3. The class of  $T$ -ec structures is not elementary, so  $T$  does not have a model companion

**Exercise 3.2.2.** A theory  $T$  with quantifier elimination is axiomatisable by sentences of the form

$$\forall x_1 \dots x_n \psi$$

where  $\psi$  is primitive existential formula

### 3.3 Examples

**Infinite sets.** The models of the theory *Infset* of **infinite sets** are all infinite sets without additional structure. The language  $L_\emptyset$  is empty, the axioms are (for  $n = 1, 2, \dots$ )

$$\bullet \exists x_0 \dots x_{n-1} \bigwedge_{i < j < n} \neg x_i = x_j$$

**Theorem 3.24.** *The theory *Infset* of infinite sets has quantifier elimination and is complete*

*Proof.* Since the language is empty, the only basic formula is  $x_i = x_j$  and  $\neg(x_i = x_j)$ . By Lemma ?? we only need to consider primitive existential formulas.  $\square$

**Dense linear orderings.**

$$\begin{aligned} \forall a, b (a \leq b \wedge b \leq a \rightarrow a = b) \\ \forall a, b, c (a \leq b \wedge b \leq c \rightarrow a \leq c) \\ \forall a, b (a \leq b \vee b \leq a) \\ \forall a, b \exists c (a < b \rightarrow a < c < b) \end{aligned}$$

**Theorem 3.25.** *DLO has quantifier elimination*

*Proof.* Let  $A$  be a finite common substructure of the two models  $O_1$  and  $O_2$ . We choose an ascending enumeration  $A = \{a_1, \dots, a_n\}$ . Let  $\exists y \rho(y)$  be a simple existential  $L(A)$ -sentence, which is true in  $O_1$  and assume  $O_1 \models \rho(b_1)$ . We want to extend the order preserving map  $a_i \mapsto a_i$  to an order preserving map  $A \cup \{b_1\} \rightarrow O_2$ . For this we have an image  $b_2$  of  $b_1$ . There are four cases

1.  $b_1 \in A$ , we set  $b_2 = b_1$
2.  $b_1 \in (a_i, a_{i+1})$ . We choose  $b_2$  in  $O_2$  with the same property
3.  $b_1$  is smaller than all elements of  $A$ . We choose a  $b_2 \in O_2$  of the same kind
4.  $b_1$  is bigger than all  $a_i$ . Choose  $b_2$  in the same manner

This defines an isomorphism  $A \cup \{b_1\} \rightarrow A \cup \{b_2\}$ , which show that  $O_2 \models \rho(b_2)$   $\square$

**Modules.** Let  $R$  be a (possibly non-commutative) ring with 1. An  $R$ -module

$$\mathfrak{M} = (\cdot, 0, +, -, r)_{r \in R}$$

is an abelian group  $(M, 0, +, -)$  together with operations  $r : M \rightarrow M$  for every ring element  $r \in R$ . We formulate the axioms in the language  $L_{Mod}(R) = L_{AbG} \cup \{r \mid r \in R\}$ . The theory  $Mod(R)$  of  $R$ -modules consists of

AbG

$$\forall x, y \ r(x + y) \doteq rx + ry$$

$$\forall x \ (r + s)x \doteq rx + sx$$

$$\forall x \ (rs)x \doteq r(sx)$$

$$\forall x \ 1x \doteq x$$

for all  $r, s \in R$ . Then  $Infset \cup Mod(R)$  is the theory of all infinite  $R$ -modules  
A module over fields is a vector space

**Theorem 3.26.** *Let  $K$  be a field. Then the theory of all infinite  $K$ -vector spaces has quantifier elimination and is complete*

*Proof.* Let  $A$  be a common finitely generated substructure (i.e., a subspace) of the two infinite  $K$ -vector spaces  $V_1$  and  $V_2$ . Let  $\exists y \rho(y)$  be a simple existential  $L(A)$ -sentence which holds in  $V_1$ . Choose a  $b_1$  from  $V_1$  which satisfies  $\rho(y)$ . If  $b_1$  belongs to  $A$ , we finished. If not, we choose a  $b_2 \in V_2 \setminus A$ . Possibly we have to replace  $V_2$  by an elementary extension. The vector spaces  $A + Kb_1$  and  $A + Kb_2$  are isomorphic by an isomorphism which maps  $b_1$  to  $b_2$  and fixes  $A$  elementwise. Hence  $V_2 \models \rho(b_2)$

The theory is complete since a quantifier-free sentence is true in a vector space iff it is true in the zero-vector space.  $\square$

**Definition 3.27.** An **equation** is an  $L_{Mod}(R)$ -formula  $\gamma(\bar{x})$  of the form

$$r_1x_1 + \cdots + r_mx_m = 0$$

A **positive primitive** formula (**pp**-formula) is of the form

$$\exists \bar{y} (\gamma_1 \wedge \cdots \wedge \gamma_n)$$

where the  $\gamma_i(\bar{x}\bar{y})$  are equations

**Theorem 3.28.** *For every ring  $R$  and any  $R$ -module  $M$ , every  $L_{Mod}(R)$ -formula is equivalent (modulo the theory of  $M$ ) to a Boolean combination of positive primitive formulas*

### Algebraically closed fields.

**Theorem 3.29** (Tarski). *The theory ACF of algebraically closed fields has quantifier elimination*

*Proof.* Let  $K_1$  and  $K_2$  be two algebraically closed fields and  $R$  a common subring. Let  $\exists y \rho(y)$  be a simple existential sentence with parameters in  $R$  which hold in  $K_1$ . We have to show that  $\exists y \rho(y)$  is also true in  $K_2$ .

Let  $F_1$  and  $F_2$  be the quotient fields of  $R$  in  $K_1$  and  $K_2$ , and let  $f : F_1 \rightarrow F_2$  be an isomorphism which is the identity on  $R$ . Then  $f$  extends to an isomorphism  $g : G_1 \rightarrow G_2$  between the relative algebraic closures  $G_i$  of  $F_i$  in  $K_i$ .  $\square$

## 4 Countable Models

### 4.1 The omitting types theorem

**Definition 4.1.** Let  $T$  be an  $L$ -theory and  $\Sigma(x)$  a set of  $L$ -formulas. A model  $\mathfrak{A}$  of  $T$  not realizing  $\Sigma(x)$  is said to **omit**  $\Sigma(x)$ . A formula  $\varphi(x)$  **isolates**  $\Sigma(x)$  if

1.  $\varphi(x)$  is consistent with  $T$
2.  $T \models \forall x(\varphi(x) \rightarrow \sigma(x))$  for all  $\sigma(x) \in \Sigma(x)$

A set of formulas is often called a **partial type**.

**Theorem 4.2** (Omitting Types). *If  $T$  is countable and consistent and if  $\Sigma(x)$  is not isolated in  $T$ , then  $T$  has a model which omits  $\Sigma(x)$*

If  $\Sigma(x)$  is isolated by  $\varphi(x)$  and  $\mathfrak{A}$  is a model of  $T$ , then  $\Sigma(x)$  is realised in  $\mathfrak{A}$  by all realisations  $\varphi(x)$ . Therefore the converse of the theorem is true for **complete** theories  $T$ : if  $\Sigma(x)$  is isolated in  $T$ , then it is realised in every model of  $T$

*Proof.* We choose a countable set  $C$  of new constants and extend  $T$  to a theory  $T^*$  with the following properties

1.  $T^*$  is a Henkin theory: for all  $L(C)$ -formulas  $\psi(x)$  there exists a constant  $c \in C$  with  $\exists x \psi(x) \rightarrow \psi(c) \in T^*$
2. for all  $c \in C$  there is a  $\sigma(x) \in \Sigma(x)$  with  $\neg \sigma(c) \in T^*$

We construct  $T^*$  inductively as the union of an ascending chain

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

of consistent extensions of  $T$  by finitely many axioms from  $L(C)$ , in each step making an instance of (1) or (2) true.

Enumerate  $C = \{c_i \mid i < \omega\}$  and let  $\{\psi_i(x) \mid i < \omega\}$  be an enumeration of the  $L(C)$ -formulas

Assume that  $T_{2i}$  is the already constructed. Choose some  $c \in C$  which doesn't occur in  $T_{2i} \cup \{\psi_i(x)\}$  and set  $T_{2i+1} = T_{2i} \cup \{\exists x \psi_i(x) \rightarrow \psi_i(c)\}$ .

Up to equivalence  $T_{2i+1}$  has the form  $T \cup \{\delta(c_i, \bar{c})\}$  for an  $L$ -formula  $\delta(x, \bar{y})$  and a tuple  $\bar{c} \in C$  which doesn't contain  $c_i$ . Since  $\exists \bar{y} \delta(x, \bar{y})$  doesn't isolate  $\Sigma(x)$ , for some  $\sigma \in \Sigma$  the formula  $\exists \bar{y} \delta(x, \bar{y}) \wedge \neg \sigma(x)$  is consistent with  $T$ . Thus  $T_{2i+2} = T_{2i+1} \cup \{\neg \sigma(c_i)\}$  is consistent

Take a model  $(\mathfrak{A}', a_c)_{c \in C}$  of  $T^*$ . Since  $T^*$  is a Henkin theory, Tarski's Test 2.2 shows that  $A = \{a_c \mid c \in C\}$  is the universe of an elementary substructure  $\mathfrak{A}$  (Lemma 2.7). By property (2),  $\Sigma(x)$  is omitted in  $\mathfrak{A}$   $\square$

**Corollary 4.3.** *Let  $T$  be countable and consistent and let*

$$\Sigma_0(x_0, \dots, x_{n_0}), \Sigma_1(x_1, \dots, x_{n_1}), \dots$$

*be a sequence of partial types. If all  $\Sigma_i$  are not isolated, then  $T$  has a model which omits all  $\Sigma_i$*

*Proof.* If  $\Sigma_0(x), \Sigma_1(x), \dots$ . Then  $T_{2i+2} = T_{2i+1} \cup \{\neg \sigma_m(c_{mn})\}$

If  $\Sigma(x_1, \dots, x_n)$ , then  $T_{2i+1} = T_{2i} \cup \{\exists \bar{x} \psi_i(\bar{x}) \rightarrow \psi_i(\bar{c})\}$ .

Combine the two case  $\square$

## 4.2 The space of types

Fix a theory  $T$ . An  $n$ -**type** is a maximal set of formulas  $p(x_1, \dots, x_n)$  consistent with  $T$ . We denote by  $S_n(T)$  the set of all  $n$ -types of  $T$ . We also write  $S(T)$  for  $S_1(T)$ .  $S_0(T)$  is all complete extensions of  $T$

If  $B$  is a subset of an  $L$ -structure  $\mathfrak{A}$ , we recover  $S_n^{\mathfrak{A}}(B)$  as  $S_n(\text{Th}(\mathfrak{A}_B))$ . In particular, if  $T$  is complete and  $\mathfrak{A}$  is any model of  $T$ , we have  $S^{\mathfrak{A}}(\emptyset) = S(T)$

For any  $L$ -formula  $\varphi(x_1, \dots, x_n)$ , let  $[\varphi]$  denote the set of all types containing  $\varphi$ .

## 5 TODO Don't understand

Lemma 3.22

Exercise 3.2.2