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Modal Logic

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Contents

1 Basic Concepts

1.1 Modal Languages

Definition 1.1. The **basic modal language** is defined using a set of **proposition letters** Φ whose elements are usually denoted p, q, r and so on, and a unary modal operator $\Diamond$. The well-formed **formulas** ϕ of the basic modal language are given by the rule

$$\phi \; ::= p \mid \bot \mid \neg\phi \mid \psi \vee \phi \mid \Diamond\phi$$

Definition 1.2. A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called **modal operators**; we use $\triangle$, $\triangle_0, \triangle_1, \ldots$ to denote elements of O. The function ρ assigns to each operator $\delta \in O$ a finite **arity**

Definition 1.3. A **modal language** $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ. The set $Form(\tau, \Phi)$ of **modal formulas** over τ and Φ is given by the rule

$$\phi \; := p \mid \bot \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)})$$

where p ranges over elements of Φ

Definition 1.4. For each $\triangle \in O$ the **dual** ∇ of $\triangle$ is defined as $\nabla(\phi_1, \ldots, \phi_n) := \neg\triangle(\neg\phi_1, \ldots, \neg\phi_n)$

Example 1.1 (The Basic Temporal Language)**.** The basic temporal language is built using a set of unary operators $O = \{\langle F \rangle, \langle P \rangle\}$. The intended interpretation of a formula $\langle F \rangle\phi$ is ' ϕ will be true at some Future time' and the intended interpretation of $\langle P \rangle\phi$ is ' ϕ was true at some Past time.' This language is called the **basic temporal language**. Their duals are written as G and H ('it is Going to be the case' and 'it always Has been the case')

Let's denote the converse of a relation R by $R^{\smile}$. We will call a frame of the form $(T, R, R^{\smile})$ a **bidirectional frame**, and a model built over such a frame a **bidirectional model**. From now on, we will only interpret basic temporal language in bidirectional models. That is, if $\mathfrak{M} = (T, R, R^{\smile}, V)$ is a bidirectional model then

$$\begin{aligned} \mathfrak{M}, t \Vdash F\phi &\quad \text{iff} \quad \exists s(Rts \wedge \mathfrak{M}, s \Vdash \phi) \\ \mathfrak{M}, t \Vdash P\phi &\quad \text{iff} \quad \exists s(R^{\smile}ts \wedge \mathfrak{M}, s \Vdash \phi) \end{aligned}$$

Example 1.2 (An Arrow Language)**.** The type $\tau_{\rightarrow}$ of **arrow logic** is a similarity type with modal operators other than diamonds. The language of arrow logic is designed to talk about the objects in arrow structures. The well-formed formulas ϕ are given by

$$\phi \ := p \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid \phi \circ \psi \mid \otimes\phi \mid 1'$$

1′ ('identity') is a nullary modality, the 'converse' operator $\otimes$ is a diamond, and the 'composition' operator $\circ$ is a dyadic operator. Possible readings of these operators are:

$1'$	identity	'skip'
$\otimes\,\phi$	converse	'ϕ conversely'
$\phi \circ \psi$	composition	'first ϕ, then ψ'

1.2 Models and Frames

Definition 1.5. A **frame** for the basic modal language is a pair $\mathfrak{F} = (W, R)$ s.t.

1. W is a non-empty set
2. R is a binary relation on W

A **model** for the basic modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame for the basic modal language and V is a function assigning to each proposition letter p in Φ a subset $V(p)$ of W. The function V is called a **valuation**. $\mathfrak{M}$ is **based on** the frame $\mathfrak{F}$

Definition 1.6. Suppose w is a state in a model $\mathfrak{M} = (W, R, V)$. Then ϕ is **satisfied** in $\mathfrak{M}$ at state w if

$$\begin{array}{rll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p),\ \text{where } p \in \Phi \\
\mathfrak{M}, w \Vdash \bot & \text{iff} & \text{never} \\
\mathfrak{M}, w \Vdash \neg\phi & \text{iff} & \text{not } \mathfrak{M}, w \Vdash \phi \\
\mathfrak{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \Diamond\phi & \text{iff} & \text{for some } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi
\end{array}$$

It follows that $\mathfrak{M}, w \Vdash \Box\phi$ iff for all $v \in W$ s.t. Rwv, we have $\mathfrak{M}, v \Vdash \phi$

Definition 1.7. Let τ be a modal similarity type. A **τ-frame** is a tuple $\mathfrak{F}$ consisting of the following ingredients

1. a non-empty set W

2. for each $n \geq 0$, and each n-ary modal operator $\triangle$ in the similarity type τ, an $(n+1)$-ary relation $R_\triangle$

ϕ is **satisfied at a state** w in a model $\mathfrak{M} = (W, \{R_\triangle \mid \triangle \in \tau\}, V)$ when $\rho(\triangle) > 0$ if

$$\mathfrak{M}, w \Vdash \triangle(\phi_1, \dots, \phi_n) \quad \text{iff} \quad \text{for some } v_1, \dots, v_n \in W \text{ with } R_\triangle w v_1 \dots v_n \text{ we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i$$

When $\rho(\triangle) = 0$ we define

$$\mathfrak{M}, w \Vdash \triangle \quad \text{iff} \quad w \in R_\triangle$$

Definition 1.8. The set of all formulas that are valid in a class of frames $\mathsf{F}$is called the **logic** of F (notation: Λ_{F})

1.3 General Frames

Definition 1.9. Given an $(n+1)$-ary relation R on a set W, we define the following n-ary operation m_R on the power set $\mathcal{P}(W)$ of W:

$$m_R(X_1, \dots, X_n) = \{w \in W \mid R w w_1 \dots w_n \text{ for some } w_1 \in X_1, \dots, w_n \in X_n\}$$

2 Models

2.1 Invariance Results

Definition 2.1. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of the same modal similarity type τ, and let w and w' be states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The τ**-theory** (or τ**-type**) **of** w is the set of all τ-formulas satisfied at w: that is, $\{\phi \mid \mathfrak{M}, w \Vdash \phi\}$. We say that w and w' are (**modally**) **equivalent** ($w \leftrightsquigarrow w'$) if they have the same τ-theories

The τ**-theory** of the model $\mathfrak{M}$ is the set of all τ-formulas satisfied by all states in fM; that is, $\{\phi \mid \mathfrak{M} \Vdash \phi\}$ Models $\mathfrak{M}$ and $\mathfrak{M}'$ are called (**modally**) **equivalent** ($\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$) if their theories are identical

2.1.1 Disjoint Unions

2.1.2 Generated submodels

Definition 2.2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models; we say that $\mathfrak{M}'$ is a **submodel** of $\mathfrak{M}$ if $W' \subseteq W$, R' is the restriction of R to W', and V' is the restriction of V to $\mathfrak{M}'$. We say that $\mathfrak{M}'$ is a **generated submodel** of $\mathfrak{M}$ ($\mathfrak{M}' \rightarrowtail \mathfrak{M}$) if $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$ and for all points w the following closure condition holds

$$\text{if } w \text{ is in } \mathfrak{M}' \text{ and } Rwv, \text{ then } v \text{ is in } \mathfrak{M}'$$

Let fM be a model, and X a subset of the domain of $\mathfrak{M}$; the **submodel generated by** X is the smallest generated submodel of $\mathfrak{M}$ whose domain contains X. A **rooted** or **point generated** model is a model that is generated by a singleton set, the element of which is called the **root** of the frame

2.1.3 Morphism for modalities

Definition 2.3 (Homomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$, if $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$, if $(w_0, \dots, w_n) \in R_\triangle$, then $(f(w_0), \dots, f(w_n)) \in R'_\triangle$ (the **homomorphic condition**)

Definition 2.4 (Strong Homomorphisms, Embeddings and Isomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **strong homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$ iff $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$ iff $(w_0, \dots, w_n) \in R_\triangle$, iff $(f(w_0), \dots, f(w_n)) \in R'_\triangle$ (the **strong homomorphic condition**)

An **embedding** of $\mathfrak{M}$ into $\mathfrak{M}'$ is a strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ which is injective. An **isomorphism** is a bijective strong homomorphism

Proposition 2.5. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. Then the following holds*

1. *for all elements w and w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, if there exists a surjective strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ with $f(w) = w'$, then w and w are modally equivalent*
2. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$*

Definition 2.6 (Bounded Morphisms - the Basic Case)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for the basic modal language. A mapping $f : \mathfrak{M} = (W, R, V) \to \mathfrak{M}' = (W', R', V')$ is a **bounded morphsim** if it satisfies

1. w and $f(w)$ satisfy the same proposition letters
2. f is a homomorphism w.r.t. the relation R (if Rwv then $R'f(w)f(v)$)
3. If $R'f(w)v'$ then there exists v s.t. Rwv and $f(v) = v'$ (the **back condition**)

If there is a **surjective** bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then we say that $\mathfrak{M}'$ is a **bounded morphic image** of $\mathfrak{M}$, and write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$

Proposition 2.7. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models s.t. $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded morphism. Then for each modal formula ϕ, and each element w of $\mathfrak{M}$ we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}', f(w) \Vdash \phi$.*

Let τ be a modal similarity type containing only diamonds (thus if $\mathfrak{M}$ is a τ-model, it has the form $(W, R_1, \dots, V)$ where each R_i is a binary relation on W) . In this context we will call a τ-model $\mathfrak{M}$ **tree-like** if the structure $(W, \bigcup_i R_i, V)$ is a tree

Proposition 2.8. *Assume that τ is a modal similarity type containing only diamonds. Then for any rooted τ-models $\mathfrak{M}$ there exists a tree-like τ-models $\mathfrak{M}'$ s.t. $\mathfrak{M}' \twoheadrightarrow \mathfrak{M}$. Hence any satisfiable τ-formula is satisfiable in a tree-like model*

Proof. Let w be the root of $\mathfrak{M}$. Define the model $\mathfrak{M}'$ as follows. Its domain W' consist of all finite sequences $(w, u_1, \dots, u_n)$ s.t. $n \geq 0$ and for some modal operators $\langle a_1 \rangle, \dots, \langle a_n \rangle \in \tau$ there is a path $wR_{a_1}u_1 \cdots R_{a_n}u_n$ in $\mathfrak{M}$. Define $(w, u_1, \dots, u_n)R'_a(w, v_1, \dots, w_m)$ to hold if $m = n + 1, u_i = v_i$ for $i = 1, \dots, n$ and $R_a u_n v_m$ holds in $\mathfrak{M}$. That is, R'_a relates two sequences iff the second is an extension of the first with a state from $\mathfrak{M}$ that is a sucessor of the last element of the first sequence. Finally, V' is defined by putting $(w, u_1, \dots, u_n) \in V'(p)$ iff $u_n \in V(p)$. The mapping $f : (w, u_1, \dots, u_n) \mapsto u_n$ defines a surjective bounded morphism from $\mathfrak{M}'$ to $\mathfrak{M}$ □

2.2 Bisimulations

Definition 2.9 (Bisimulation - the Basic Case)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M} = (W', R', V')$ be two models

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation between** $\mathfrak{M}$ and $\mathfrak{M}'$ (notation: $Z : \mathfrak{M} \leftrightarrows \mathfrak{M}'$) if

1. If wZw' then w and w' satisfy the same proposition letters
2. If wZw' and Rwv, then there exists v' (in $\mathfrak{M}'$) s.t. vZv' and $R'w'v'$ (the **forth condition**)
3. The converse of (2): if wZw' and $R'w'v'$, then there exists v (in $\mathfrak{M}$) s.t. vZv' and Rwv (the **back condition**)

When Z is a bisimulation linking two states w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$ we say that w and w' are **bisimilar**, and we write $Z : \mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$. If there is a bisimulation, we sometimes write $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$ or $w \leftrightarrows w'$

Definition 2.10 (Bisimulation - the General Case)**.** Let τ be a modal similarity type, and let $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$ be τ-models. A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ ($Z : \mathfrak{M} \leftrightarrows \mathfrak{M}'$) if the above condition 1 is satisfied and

2. If wZw' and $R_\triangle wv_1 \dots v_n$ then there are $v'_1, \dots, v'_n \in W'$ s.t. $R'_\triangle w'v'_1 \dots v'_n$ and for all i $(1 \leq i \leq n)$ $v_iZv'_i$ (the **forth** condition)
3. If wZw' and $R'_\triangle w'v'_1 \dots v'_n$ then there are $v_1, \dots, v_n \in W$ s.t. $R_\triangle wv_1 \dots v_n$ and for all i $(1 \leq i \leq n)$ $v_iZv'_i$ (the **back** condition)

Proposition 2.11. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ and $\mathfrak{M}_i$ $(i \in I)$ be τ-models*

1. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarrows \mathfrak{M}'$*
2. *For every $i \in I$, and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarrows \biguplus_i \mathfrak{M}_i, w$*
3. *If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarrows \mathfrak{M}, w$ for all w in $\mathfrak{M}'$*
4. *If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$*

Proof. Suppose $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$ $\mathfrak{M}_i \subseteq \biguplus_i \mathfrak{M}_i$

1. Suppose $f : \mathfrak{M} \cong \mathfrak{M}'$, then we define wZw' iff $w' = f(w)$ where $w \in W, w' \in W'$. Bisimulation comes from the definition of the isomorphism
2. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}_i\} \subseteq \mathfrak{M}_i \times \biguplus \mathfrak{M}_i$. The first condition comes from the invariance. The forth condition is obvious. For the back condition, if $R'_\triangle w'v'_1 \dots v'_n$ and $w' \in W$, then $v'_1, \dots, v'_n \in W$ since each $R_{\triangle,i}$ is disjoint and we have $R_{\triangle,i}w'v'_1 \dots v'_n$
3. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}'\} \subseteq \mathfrak{M}' \times \mathfrak{M}$. The first condition comes from the invariance. Forth condition is obvious. For the back condition, suppose wZw and $R'_\triangle wv'_1 \dots v'_n$, by the definition, $v'_1, \dots, v'_n \in W$ and $R_\triangle wv'_1 \dots v'_n$
4. Define $Z = \{(w, f(w) \mid w \in W)\}$. The first condition comes from the definition. If wZw' and $R_\triangle wv_1 \dots v_n$, then $R'_\triangle f(w)f(v_1) \dots f(v_n)$. If wZw' and $R'_\triangle w'v'_1 \dots v_n$, then there is $v_1, \dots, v_n$ s.t. $R_\triangle wv_1, \dots, v_n$ and $f(v_i) = v'_i$ for $1 \leq i \leq n$

$\square$

Theorem 2.12. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ be τ-models. Then, for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ implies that $w \leftrightsquigarrow w'$. In other words, modal formulas are invariant under bisimulation*

Proof. Induction on the complexity of ϕ.

Suppose ϕ is $\diamond\psi$, we have $\mathfrak{M}, w \Vdash \diamond\psi$ iff there exists a v in $\mathfrak{M}$ s.t. Rwv and $\mathfrak{M}, v \Vdash \psi$. As $w \leftrightarrows w'$, there exists a v' in $\mathfrak{M}'$ s.t. $R'w'v'$ and $v \leftrightarrows v'$. By the I.H., $\mathfrak{M}', v' \Vdash \psi$, hence $\mathfrak{M}', w' \Vdash \diamond\psi$ □

Example 2.1 (Bisimulation and First-Order Logic).

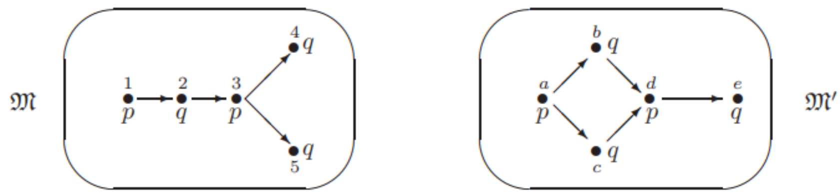

Fig. 2.4. Bisimilar models.

Example 2.2.

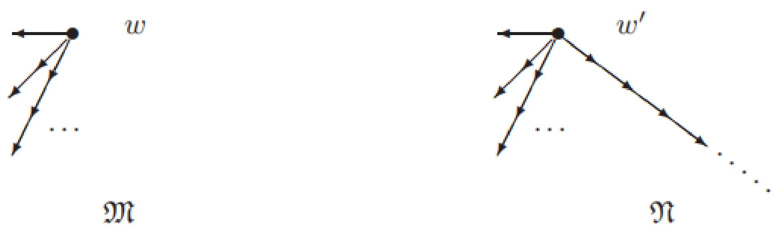

Fig. 2.5. Equivalent but not bisimilar.

$\mathfrak{M}$ is **image-finite** if for each state u in $\mathfrak{M}$ and each relation R in $\mathfrak{M}$, the set $\{(v_1, \dots, v_n) \mid Ruv_1 \dots v_n\}$ is finite

Theorem 2.13 (Hennessy-Milner Theorem). *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be two image-finite τ-models. Then for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ iff $w \leftrightsquigarrow w'$*

Proof. Assume that our similarity type τ only contains a single diamond. The direction from left to right follows from Theorem 2.12

Suppose $w \leftrightsquigarrow w'$. The first condition is immediate. If Rwv, assume there is no v' in $\mathfrak{M}'$ with $R'w'v'$ and $v \leftrightsquigarrow v'$. Let $S' = \{u' \mid R'w'u'\}$. Note that S' must be non-empty, for otherwise $\mathfrak{M}', w' \Vdash \Box\bot$, which would contradict $w \leftrightsquigarrow w'$ since $\mathfrak{M}, w \Vdash \diamond\top$. Furthermore, as $\mathfrak{M}'$ is image-finite, S' must be finite, say $S' = \{w'_1, \dots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i s.t. $\mathfrak{M}, v \Vdash \psi_i$, but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \dots \wedge \psi_n) \quad \text{and} \quad \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \dots \wedge \psi_n)$$

□

Exercise 2.2.1. Suppose that $\{Z_i \mid i \in I\}$ is a non-empty collection of bisimulations between $\mathfrak{M}$ and $\mathfrak{M}'$. Prove that the relation $\bigcup_{i\in I} Z_i$ is also a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Conclude that if $\mathfrak{M}$ and $\mathfrak{M}'$ are bisimilar, then there is a maximal bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

Proof.
1. If $(w, w') \in \bigcup_{i\in I} Z_i$, then $(w, w') \in Z_j$ for some $j \in I$ and hence they satisfy the same propositional letters
2. If $(w, w') \in \bigcup_{i\in I} Z_i$ and $R_\triangle w v_1 \dots v_n$, since $(w, w') \in Z_j$ for some $j \in I$, we have $R'_\triangle w' v'_1 \dots v'_n$ and $v_i Z_j v'_i$ for all $1 \leq i \leq n$, which means $(v_i, v'_i) \in \bigcup_{i\in I} Z_i$ for all $1 \leq i \leq n$
3. similarly

□

Remark (Bisimulations for the Basic Temporal Language and Arrow Logic). When working with the basic temporal language, we usually work with models (W, R, V) and implicitly take R_p to be $R^\smile$. Thus we need a notion of bisimulation between models (W, R, V) and (W', R', V') to be a relation Z between the states of the two models that satisfies the clauses of Definition 2.9, and in addition the following

4. If wZw' and Rvw, then there exists v' in $\mathfrak{M}'$ s.t. vZv' and $R'v'w'$
5. Converse of 4: if wZw' and $R'v'w'$, then there exists v in $\mathfrak{M}$ s.t. vZv'

2.3 Finite Models

Definition 2.14 (Finite Model Property)**.** Let τ be a modal similarity type, and let M be a class of τ-models. We say that τ has the **finite model property w.r.t.** M if the following holds: if ϕ is a formula of similarity type τ, and ϕ is satisfiable in some model in M, then ϕ is satisfiable in a **finite** model in M

2.3.1 Selecting a finite submodel

Definition 2.15 (Degree)**.** We define the **degree** of modal formulas as follows:

$$\begin{array}{rcl}
\deg(p) & = & 0 \\
\deg(\bot) & = & 0 \\
\deg(\neg\phi) & = & \deg(\phi) \\
\deg(\phi \vee \psi) & = & \max\{\deg(\phi), \deg(\psi)\} \\
\deg(\triangle(\phi_1, \dots, \phi_n)) & = & 1 + \max\{\deg(\phi_1), \dots, \deg(\phi_2)\}
\end{array}$$

Proposition 2.16. *Let τ be a finite modal similarity type, and assume our collection of proposition letters is finite as well*

1. *for all n, up to logical equivalence there are only finitely many formulas of degree at most n*
2. *for all n, and every τ-model $\mathfrak{M}$ and state w of $\mathfrak{M}$, the set of all τ-formulas of degree at most n that are satisfied by w, is equivalent to a single formula*

Definition 2.17 (n-Bisimulation)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models, and let w and w' be states of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. We say that w and w' are **n-bisimilar** ($w \leftrightarrows_n w'$) if there exists a sequence of binary relations $Z_n \subseteq \cdots \subseteq Z_0$ with the following properties (for $i + 1 \leq n$)

1. wZ_nw'
2. if vZ_0v' then v and v' agree on all proposition letters
3. if $vZ_{i+1}v'$ and Rvu then there exists u' with $R'v'u'$ and uZ_iu'
4. if $vZ_{i+1}v'$ and $R'v'u'$, then there exists u with Rvu and uZ_iu'

Proposition 2.18. *Let τ be a finite modal similarity type, Φ a finite set of proposition letters, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for this language. Then for every w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$, the following are equivalent*

1. *$w \leftrightarrows_n w'$*
2. *w and w' agree on all modal formulas of degree at most n.*

Proof. $2 \to 1$. if $n = 0$, obvious.

If $n = k$ and the proposition holds. Now suppose $n = k + 1$. Now w and w' agree on all modal formulas of degree at most $n+1$. If there is not v, v' s.t. v and v' agree on all modal formulas of degree at most n and Rwv and Rwv'. Let $S' = \{u' \mid R'w'u'\}$ and S' is finite, say $S' == \{w'_1, \dots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i of degree at most n s.t. $\mathfrak{M}, v \Vdash \psi_i$ but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n) \text{ and } \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n)$$

□

Definition 2.19. Let τ be a modal similarity type containing only diamonds. Let $\mathfrak{M} = (W, R_1, \dots, R_n, \dots, V)$ be a rooted τ-model with root w. The notion of the **height** of states in $\mathfrak{M}$ is defined by induction.

The only element of height 0 is the rot of the model; the states of height $n + 1$ are those immediate successors of elements of height n that have not yet assigned a height smaller than $n + 1$. The **height of a model** $\mathfrak{M}$ is the maximum n s.t. there is a state of height n in $\mathfrak{M}$, if such a maximum exists; otherwise the height of $\mathfrak{M}$ is infinite

For a natural number k, the **restriction** of $\mathfrak{M}$ to k ($\mathfrak{M}{\restriction}k$) is defined as the submodel containing only states whose height is at most k. $(\mathfrak{M}{\restriction}k) = (W_k, R_{1k}, \ldots, R_{nk}, \ldots, V_k)$, where $W_k = \{v \mid \text{height}(v) \leq k\}$, $R_{nk} = R_n \cap (W_k \times W_k)$, and for each p, $V_k(p) = V(p) \cap W_k$

Lemma 2.20. *Let τ be a modal similarity type that contains only diamonds. Let $\mathfrak{M}$ be a rooted τ-models, and let k be a natural number. Then for every state w of $(\mathfrak{M}{\restriction}k)$, we have $(\mathfrak{M}{\restriction}k), w \leftrightarrows_l \mathfrak{M}, w$, where $l = k - \text{height}(w)$*

Theorem 2.21 (Finite Model Property - via Selection)**.** *Let τ be a modal similarity type containing only diamonds, and let ϕ be a τ-formula. If ϕ is satisfiable, then it is satisfiable on a finite model*

Proof. Fix a modal formula ϕ with $\deg(\phi) = k$. We restrict our modal similarity type τ and our collection of proposition letters to the modal operators and proposition letters actually occurring in ϕ. Let $\mathfrak{M}_1, w_1$ be s.t. $\mathfrak{M}_1, w_1 \Vdash \phi$. By Proposition 2.8, there exists a tree-like model $\mathfrak{M}_2$ with root w_2 s.t. $\mathfrak{M}_2, w_2 \Vdash \phi$. Let $\mathfrak{M}_3 := (\mathfrak{M}_2{\restriction}k)$. By Lemma 2.20 we have $\mathfrak{M}_2, w_2 \leftrightarrows_k \mathfrak{M}_3, w_2$ and by Proposition 2.18 it follows that $\mathfrak{M}_3, w_2 \Vdash \phi$

By induction on $n \leq k$ we define finite sets of states $S_0, \ldots, S_k$ and a (final) model $\mathfrak{M}_4$ with domain $S_0 \cup \cdots \cup S_k$; the points in each S_n will have height n

Define S_0 to be the singleton $\{w_2\}$. Next, assume that $S_0, \ldots, S_n$ have already been defined. Fix an element v of S_n. By Proposition 2.16 there are only finitely many non-equivalent modal formulas whose degree is at most $k - n$, say $\psi_1, \ldots, \psi_m$. For each formula that is of the form $\langle a \rangle \chi$ and holds in $\mathfrak{M}_3$ at v, select a state u from $\mathfrak{M}_3$ s.t. $R_a vu$ and $\mathfrak{M}_3, u \Vdash \chi$. Add all these us to S_{n+1}, and repeat this selection process for every state in S_n. S_{n+1} is defined as the set of all points that have been selected in this way

Finally, define $\mathfrak{M}_4$ as follows. Its domain is $S_0 \cup \cdots \cup S_k$; as each S_i is finite, $\mathfrak{M}_4$ is finite. The relations and valuation are obtained by restricting the relations and valuations of $\mathfrak{M}_3$ to the domain of $\mathfrak{M}_4$ $\square$

2.3.2 Finite models via filtrations

Definition 2.22. A set of formulas Σ is **closed under subformulas** (or **subformula closed**) if for all formulas ϕ, ϕ': if $\phi \vee \phi' \in \Sigma$ then so are ϕ and ϕ'; if $\neg\phi \in \Sigma$ then so is ϕ; and if $\triangle(\phi_1, \ldots, \phi_n) \in \Sigma$ then so are $\phi_1, \ldots, \phi_n$

Definition 2.23 (Filtrations)**.** We work in the basic modal language. Let $\mathfrak{M} = (W, R, V)$ be a model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma$ be the relation on the states of $\mathfrak{M}$ defined by

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \phi \in \Sigma : (\mathfrak{M}, w \Vdash \phi \text{ iff } \mathfrak{M}, v \Vdash \phi)$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. We denote the equivalence class of a state w of $\mathfrak{M}$ w.r.t. $\leftrightsquigarrow_\Sigma$ by $|w|_\Sigma$, or simply $|w|$. The mapping $w \mapsto |w|$ is called the **natural map**

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model (W^f, R^f, V^f) s.t.

1. $W^f = W_\Sigma$
2. if Rwv then $R^f|w||v|$
3. if $R^f|w||v|$ then for all $\diamond\phi \in \Sigma$, if $\mathfrak{M}, v \Vdash \phi$ then $\mathfrak{M}, w \Vdash \diamond\phi$
4. $V^f(p) = \{|w| \mid \mathfrak{M}, w \Vdash p\}$, for all proposition letters p in Σ

$\mathfrak{M}^f_\Sigma$ is called a **filtration of** fM **through** Σ; we will often suppress subscripts and write $\mathfrak{M}^f$ instead of $\mathfrak{M}^f_\Sigma$

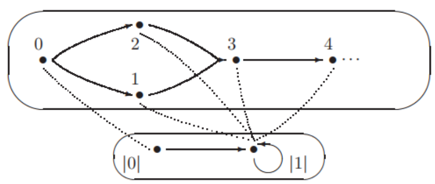

Fig. 2.6. A model and its filtration.

Let $\mathfrak{M} = (\mathbb{N}, R, V)$, where $R = \{(0,1), (0,2), (1,3)\} \cup \{(n, n+1) \mid n \geq 2\}$, and V has $V(p) = \mathbb{N} \setminus \{0\}$ and $V(q) = \{2\}$

Further assume $\Sigma = \{\diamond p, p\}$. Σ is subformula closed. Then, the model $\mathfrak{N} = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V')$, where $V'(p) = \{|1|\}$ is a filtration of $\mathfrak{M}$ through Σ. $\mathfrak{N}$ is not a bounded morphic image of $\mathfrak{M}$: any bounded morphism would have to preserve the formula q

Proposition 2.24. *Let Σ be a finite subformula closed set of basic modal formulas. For any model $\mathfrak{M}$, if $\mathfrak{M}^f$ is a filtration of $\mathfrak{M}$ through a subformula closed set Σ, then $\mathfrak{M}^f$ contains at most 2^n nodes (where n denotes the size of Σ)*

Proof. The states of $\mathfrak{M}^f$ are the equivalence classes in W_Σ. Let g be the function with domain W_Σ and range $\mathcal{P}(\Sigma)$ defined by $g(|w|) = \{\phi \in \Sigma \mid \mathfrak{M}, w \Vdash \phi\}$. It follows from the definition of $\leftrightsquigarrow_\Sigma$ that g is well defined and injective. Thus $\left|W_\Sigma\right| \leq 2^n, n = |\Sigma|$ □

Theorem 2.25 (Filtration Theorem)**.** *Consider the basic modal language. Let $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes w in $\mathfrak{M}$, we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}^f, |w| \Vdash \phi$*

Proof. Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}, w \Vdash \diamond\phi$. Then there is a v s.t. Rwv and $\mathfrak{M}, v \Vdash \phi$. As $\mathfrak{M}^f$ is a filtration, $R^f|w||v|$. As Σ is a subformula closed, $\phi \in \Sigma$, thus by the inductive hypothesis $\mathfrak{M}^f, |v| \Vdash \phi$. Hence $\mathfrak{M}^f, |\Vdash| \diamond \phi$

Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}^f, |w| \Vdash \diamond\phi$. Thus there is a state $|v|$ in $\mathfrak{M}^f$ s.t. $R^f|w||v|$ and $\mathfrak{M}^f, |v| \Vdash \phi$. As $\phi \in \Sigma$, we have $\mathfrak{M}, v \Vdash \phi$. By the definition, we have $\mathfrak{M}, w \Vdash \diamond\phi$ □

Note that clauses 2 and 3 of Definition 2.3.2 are designed to make the modal case of the inductive step go through.

Define

1. $R^s|w||v|$ iff $\exists w' \in |w| \exists v' \in |v| Rw'v'$
2. $R^l|w||v|$ iff for all formulas $\diamond\phi \in \Sigma$: $\mathfrak{M}, v \Vdash \phi$ implies $\mathfrak{M}, w \Vdash \diamond\phi$

These relations give rise to the **smallest** and **largest** filtrations respectively

Lemma 2.26. *Consider the basic modal language. Let $\mathfrak{M}$ be any model, Σ any subformula closed set of formulas, W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$, and V^f the standard valuation on W_Σ. Then both (W_Σ, R^s, V^f) and (W_Σ, R^l, V^f) are filtrations of $\mathfrak{M}$ through Σ. Furthermore, if (W_Σ, R^f, V^f) is any filtration of $\mathfrak{M}$ through Σ, then $R^s \subseteq R^f \subseteq R^l$*

Proof. If Rwv, if $\mathfrak{M}, v \Vdash \phi$, then $\mathfrak{M}, w \Vdash \diamond\phi$, hence $R^l|w||v|$

For any (W_Σ, R^f, V^f). $R^s \subseteq R^f$ by clause 2. $R^f \subseteq R^l$ by clause 2 □

Theorem 2.27 (Finite Model Property - via Filtrations)**.** *Let ϕ be a basic modal formula. if ϕ is satisfiable, then it is satisfiable on a finite model. Indeed, it is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of ϕ*

Proof. Assume that ϕ is satisfiable on a model $\mathfrak{M}$; take any filtration of $\mathfrak{M}$ through the set of subformulas . □

Lemma 2.28. *Let $\mathfrak{M}$ be a model, Σ a subformula closed set of formulas, and W_Σ the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Sigma$. Let R^t be the binary relation on W_Σ defined by*

$$R^t|w||v| \textit{ iff for all } \phi, \textit{ if } \diamond \phi \in \Sigma \textit{ and } \mathfrak{M}, v \Vdash \phi \vee \diamond\phi \textit{ then } \mathfrak{M}, w \Vdash \diamond\phi$$

If R is transitive then (W_Σ, R^t, V^f) is a filtration and R^t is transitive

Definition 2.29. Let (W, R, V) be a transitive frame. A **cluster** on (W, R, V) is a maximal, nonempty equivalence class under R. That is, $C \subseteq W$ is a cluster if the restriction of R to C is an equivalence relation

A cluster is **simple** if it consists of a single reflexive point, and **proper** if it consists more than one point

2.4 The Standard Translation

Definition 2.30. For τ a modal similarity type and Φ a collection of proposition letters, let $\mathcal{L}^1_\tau(\Phi)$ be the first-order language (with equality) which has unary predicates $P_0, P_1, \ldots$ corresponding to the proposition letters $p_0, p_1, \ldots$ in Φ, and an $(n+1)$-ary relation symbol $R_\triangle$ for each (n-ary) modal operator $\triangle$ in our similarity type. We write $\alpha(x)$ to denote a first-order formula α with one free variable, x

Definition 2.31 (Standard Translation)**.** Let x be a first-order variable. The **standard translation** ST_x taking modal formulas to first-order formulas in $\mathcal{L}^1_\tau(\Phi)$ is defined as

$$\begin{array}{rcl} ST_x(p) & = & Px \\ ST_x(\bot) & = & x \neq x \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \vee \psi) & = & ST_x(\phi) \vee ST_x(\psi) \\ ST_x(\triangle(\phi_1, \ldots, \phi_n)) & = & \exists y_1 \ldots \exists y_n (R_\triangle x y_1 \ldots y_n \wedge \\ & & \quad ST_{y_1}(\phi_1) \wedge \cdots \wedge ST_{y_n}(\phi_n)) \end{array}$$

where $y_1, \ldots, y_n$ are fresh variables.

$$ST_x(\diamond\phi) = \exists y(Rxy \wedge ST_y(\phi))$$
$$ST_x(\phi) = \forall y(Rxy \to ST_y(\phi))$$

Proposition 2.32 (Local and Global Correspondence on Models)**.** *Fix a modal similarity type τ, and let ϕ be a τ-formula. Then*

1. *For all $\mathfrak{M}$ and all states w of $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M} \models ST_x(\phi)[w]$*
2. *For all $\mathfrak{M}$: $\mathfrak{M} \Vdash \phi$ iff $\mathfrak{M} \models \forall x ST_x(\phi)$*

Proposition 2.33.

1. *Let τ be a modal similarity type that only contains diamonds. Then, every τ-formula ϕ is equivalent to a first-order formula containing at most two variables*
2. *If τ does not contain modal operators $\triangle$ whose arity exceeds n, all τ-formulas are equivalent to first-order formulas containing at most $(n+1)$ vairables*

Proof. Assume τ contains only diamonds $\langle a \rangle, \langle b \rangle$. Fix two distinct variables x

and y. Define two variants ST_x and ST_y of the standard translation as follows

$$
\begin{aligned}
ST_x(p) &= Px & ST_y(p) &= Py \\
ST_x(\bot) &= x \neq x & ST_y(\bot) &= y \neq y \\
ST_x(\neg\phi) &= \neg ST_x(\phi) & ST_y(\neg\phi) &= \neg ST_y(\phi) \\
ST_x(\phi \vee \psi) &= ST_x(\phi) \vee ST_x(\psi) & ST_y(\phi \vee \psi) &= ST_y(\phi) \vee ST_y(\psi) \\
ST_x(\langle a \rangle \phi) &= \exists y(R_a xy \wedge ST_y(\phi)) & ST_y(\langle a \rangle \phi) &= \exists x(R_a yx \wedge ST_x(\phi))
\end{aligned}
$$

Then for any τ-formula ϕ, its ST_x-translation contains at most the two variables x and y, and $ST_x(\phi)$ is equivalent to the original standard translation of ϕ $\square$

Example 2.3.

$$
\begin{aligned}
ST_x(\diamond(p \to q)) &= \exists y(Rxy \wedge ST_y(p \to q)) \\
&= \exists y(Rxy \wedge (\forall x(Ryx \to ST_x(p)) \to Qy)) \\
&= \exists y(Rxy \wedge (\forall x(Ryx \to Px) \to Qy))
\end{aligned}
$$

Rxx is not equivalent to any modal formula. Suppose ϕ is a modal formula s.t. $ST_x(\phi)$ is equivalent to Rxx. Let $\mathfrak{M}$ be a singleton reflexive model and let w be the unique state in $\mathfrak{M}$; obviously $\mathfrak{M} \models Rxx[w]$. Let $\mathfrak{N}$ be a model based on the strict ordering of the integers; for every integer v, $\mathfrak{N} \models \neg Rxx[v]$. Let Z be the relation which links every integer with the unique state in fM, and assume that the valuations in $\mathfrak{N}$ and $\mathfrak{M}$ are s.t. Z is a bisimulation.

$$\mathfrak{M} \models Rxx[w] \Rightarrow \mathfrak{M}, w \Vdash \phi \Rightarrow \mathfrak{N}, v \Vdash \phi \Rightarrow \mathfrak{N} \models Rxx[v]$$

Definition 2.34. Let τ be a modal similarity type, C a class of τ-models, and Γ a set of formulas over τ. We say that Γ **defines** of **characterizes** a class K of models **within** C if for all models $\mathfrak{M}$ in C we have that $\mathfrak{M}$ is in K iff $\mathfrak{M} \Vdash \Gamma$. If C is the class of all τ-models, we simply say that Γ defines or characterizes K; we omit brackets whenever Γ is a singleton. We say that a formula ϕ defines a **property** whenever ϕ defines the class of models satisfying the property

2.5 Modal Saturation via Ultrafilter Extensions