# Logic Language And Meaning

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## Contents

# 1 The Theory of Types and Categorical Grammar

## 1.1 The Theory of Types

### 1.1.1 Type Distinctions in Natural Language

1. If John is self-satisfied, then there is at least one thing he has in common with Peter

Sentence (1) contains quantification over properties.

2. Santa Claus has all the attributes of a sadist

If we are to quantify not only over entities but also over properties of entities, then we need to extend predicate logic by introducing variables other than the ones we already have, which only range over entities. Besides predicate letters, we need **predicate variables**, so that we can quantify over this kind of variable in the syntax. Letting $X$ be such a variable, (1) and (2) may be represented as in (3) and (4):

3. $Zj \rightarrow \exists X(Xj \wedge Xp)$
4. $\forall X(\forall x(Sx \rightarrow Xx) \rightarrow Xs)$

But second-order predicate logic does not exhaust the expressive power of natural language. For not only are there natural language sentences which quantify over properties of entities, but there are also sentences which attribute properties to these properties of entities in turn.The predicate **red** expresses a property of individuals, so the predicate **color** expresses a property of properties of individuals. So in a sentence like **Red is a color**, which we represent as $\mathcal{C}(R)$, the second-order predicate **color** is applied to the first-order predicate **red**. We can also quantify over these properties of properties, as in **Red has something in common with green**.

Besides higher-order predicates, there are other kinds of expressions which for linguistic purposes may usefully be added to predicate logic.

Our first class of examples is formed by expressions with **predicate adverbials**

5. John is walking quickly

The expression **quickly** is, form a linguistic perspective, a modifier acting on the verb **is walking**. From a logical perspective, the property of walking quickly is attributed to an entity, John. This property cannot be seen as a conjunction of two properties, 'being quick' and 'walking'. For sentence (5) does not mean the same thing as sentence (6):

6. John is walking and John is quick

In logical terms, **quickly** is an expression which when applied to the first-order predicate **walking** result in a new first-order predicate **walking**

**quickly**. From a logical point of view, the **relative adjectives** are expressions of the same kind. Sentence (7) may be represented in first-order predicate logic as formula (8)

7. Jumbo is a pink elephant
8. $Ej \wedge Pj$

The adjective **pink** may, in other words, be represented as a standard first-order predicate. But the same does not apply to relative adjectives like **small**. Sentence (9) is the same kind of sentence as (7)

9. Jumbo is a small elephant

But sentence (9) cannot be analyzed as a conjunction of two first-oder predicates. The formula (10) which we would then obtain:

10. $Ej \wedge Sj$

expresses something which is generally false. The relative adjective **small** works the same way as the predicate adverbial **quickly**. When applied to the predicate **elephant**, it results in a new predicate **small elephant**

### 1.1.2 Syntax

As our two basic types we have $e$, which is the type of those expressions which refer to entities, and $t$, the type of those expressions which refer to truth values.

**Definition 1.1.** **T**, the set of types, is the smallest set s.t.

1. $e, t \in \mathbf{T}$
2. if $a, b \in \mathbf{T}$, then $\langle a, b\rangle \in \mathbf{T}$

An expression of type $\langle a, b\rangle$ is an expression which when applied to an expression of type $a$ results in an expression of type $b$. If $\alpha$ is an expression of type $\langle a, b\rangle$ and $\beta$ is an expression of type $a$, then $\alpha(\beta)$ will be an expression of type $b$. This process of applying an $\alpha$ of type $\langle a, b\rangle$ to a $\beta$ of type $a$ is called **(functional) application of** $\alpha$ **to** $\beta$

The **vocabulary** of a type-theoretical language L contains some symbols which are shared by all such languages and a number of symbols which are characteristic of $L$. The shared part consists of:

1. For every type $a$, an infinite set $\mathrm{VAR}_a$ of variables of type $a$
2. The usual connectives $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$
3. The quantifiers $\forall$ and $\exists$
4. Two brackets ( and )
5. The symbol for identity =

The part of the vocabulary which is characteristic of $L$ contains

6. for every type $a$, a (possibly empty) set $\mathrm{CON}_a^L$ of constants of type $a$

We will write $v_a$ for variables of type $a$ and $c_a$ for constants of type $a$.

**Definition 1.2.** 1. If $\alpha$ is a variable or a constant of type $a$ in $L$, then $\alpha$ is an expression of type $a$ in $L$

2. If $\alpha$ is an expression of type $\langle a, b\rangle$ in $L$, and $\beta$ is an expression of type $a$ in $L$, then $(\alpha(\beta))$ is an expression of type $b$ in $L$
3. If $\phi$ and $\psi$ are expressions of type $t$ in $L$, then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$,$(\phi \rightarrow \psi)$ and $\phi \leftrightarrow \psi$
4. If $\phi$ is an expression of type $t$ in $L$ and $v$ is a variable (of arbitrary type $a$), then $\forall x\phi$ and $\exists v\phi$ are expressions of type $t$ in $L$
5. If $\alpha$ and $\beta$ are expressions in $L$ which belong to the same (arbitrary) type, then $(\alpha = \beta)$ is an expression of type $t$ in $L$
6. Every expression in $L$ is to be constructed by means of (1) - (5) in a finite number of steps

We refer to the set of all expressions in $L$ of type $a$ as $\mathrm{WE}_a^L$ or, if it is clear which $L$ is meant, as $\mathrm{WE}_a$. The **formulas** are the elements of $\mathrm{WE}_t$

### 1.1.3 Semantics

Given a domain $D$, one-place predicates are interpreted as the characteristic functions of subsets of that domain.

The domain of interpretation of expressions of type $a$, given a domain $D$, is written as $\mathbf{D}_{a,D}$ and is defined as follows

**Definition 1.3.** 1. $\mathbf{D}_{e,D} = D$

2. $\mathbf{D}_{t,D} = \{0, 1\}$
3. $\mathbf{D}_{\langle a,b\rangle,D} = \mathbf{D}_{b,D}^{\mathbf{D}_{a,D}}$

For example, in the theory of types, a two-place predicate $L(loves)$ is an expression of type $\langle e, \langle e, t\rangle\rangle$. The corresponding interpretation domain $\mathbf{D}_{\langle e,\langle e,t\rangle\rangle}$ is $(\{0, 1\}^D)^D$

Consider the second-order predicate $\mathcal{C}(color)$, which is of type $\langle\langle e, t\rangle, t\rangle$. The interpretation domain $\mathbf{D}_{\langle\langle e,t\rangle,t\rangle}$ is the set of functions $\{0, 1\}^{\{0,1\}^D}$

A model $\mathbf{M}$ for an language $L$ for the theory of types consists of a nonempty domain set $D$ together with an interpretation function $I$. For each type $a$, $I$ is a function from $\mathrm{CON}_a^L$ into $\mathbf{D}_{a,D}$.

We must define the concept of **the interpretation of $\alpha$ w.r.t. a model $\mathbf{M}$ and an assignment** $g$, to be written as $\llbracket\alpha\rrbracket_{\mathbf{M},g}$. The interpretation function$\llbracket\ \rrbracket_{\mathbf{M},g}$ can be seen as a function which for all types $a$, maps $\mathrm{WE}_a^L$ into $\mathbf{D}_{a,D}$.

**Definition 1.4.** 1. If $\alpha \in \mathrm{CON}_a^L$, then $[\![\alpha]\!]_{\mathbf{M},g} = I(\alpha)$
If $\alpha \in \mathrm{VAR}_a$, then $[\![\alpha]\!]_{\mathbf{M},g} = g(\alpha)$

2. If $\alpha \in \mathrm{WE}^L_{\langle a,b\rangle}, \beta \in \mathrm{WE}^L_a$, then $[\![\alpha(\beta)]\!]_{\mathbf{M},g} = [\![\alpha]\!]_{\mathbf{M},g}([\![\beta]\!]_{\mathbf{M},g})$
3. If $\phi, \psi \in \mathrm{WE}^L_t$, then
$[\![\neg\phi]\!]_{\mathbf{M},g} = 1$ iff $[\![\phi]\!]_{\mathbf{M},g} = 0$
$[\![\phi \wedge \psi]\!]_{\mathbf{M},g} = 1$ iff $[\![\phi]\!]_{\mathbf{M},g} = [\![\psi]\!]_{\mathbf{M},g} = 1$
$[\![\phi \to \psi]\!]_{\mathbf{M},g} = 0$ iff $[\![\phi]\!]_{\mathbf{M},g} = 1$ and $[\![\psi]\!]_{\mathbf{M},g} = 0$
$[\![\phi \leftrightarrow \psi]\!]_{\mathbf{M},g}$ iff $[\![\phi]\!]_{\mathbf{M},g} = [\![\psi]\!]_{\mathbf{M},g}$
4. if $\phi \in \mathrm{WE}^L_t, v \in \mathrm{VAR}_a$, then
$[\![\forall v\phi]\!]_{\mathbf{M},g} = 1$ iff for all $d \in \mathbf{D}_{a,D}$: $[\![\phi]\!]_{M,g[v/d]} = 1$
$[\![\exists v\phi]\!]_{\mathbf{M},g} = 1$ iff there is at least one $d \in \mathbf{D}_{a,D}$ s.t.: $[\![\phi]\!]_{\mathbf{M},g[v/d]} = 1$
5. If $\alpha, \beta \in \mathrm{WE}^L_a$, then $[\![\alpha = \beta]\!]_{\mathbf{M},g} = 1$ iff $[\![\alpha]\!]_{\mathbf{M},g} = [\![\beta]\!]_{\mathbf{M},g}$

A sentence $\phi$ is said to be **true with respect to M** just in case $[\![\phi]\!]_{\mathbf{M}} = 1$ . A sentence $\phi$ is said to be **universally valid** just in case $[\![\phi]\!]_{\mathbf{M}} = 1$ for every appropriate **M** and once again the notation is $\vDash \phi$. We also say that two sentences $\phi$ and $\psi$ are **equivalent** iff $\vDash \phi \leftrightarrow \psi$ that is to say iff $[\![\phi]\!]_{\mathbf{M}} = [\![\psi]\!]_{\mathbf{M}}$ for every **M**.

Consider the formula $W(j)$, our representation of the sentence *John is walking*. Here $W$ is a constant of type $\langle e, t\rangle$, while $j$ is a constant of type $e$. Hence $[\![W(j)]\!]_{\mathbf{M},g} = [\![W]\!]_{\mathbf{M},g}([\![j]\!]_{\mathbf{M},g})$. And we have $[\![W]\!]_{\mathbf{M},g} = I(W)$ and $[\![j]\!]_{\mathbf{M},g} = I(j)$. According to the definition of the interpretation function $I$, we know that $I(W)$ is an element of the set of functions $\mathbf{D}_t^{\mathbf{D}_e} = \{0,1\}^D$. And the interpretation of the constant $I(j)$ is an element of $\mathbf{D}_e$

As an example of **higher-order quantification**, consider the formula $\exists\mathcal{X}(\mathcal{X}(R) \wedge \mathcal{X}(G))$, the representation of the sentence *Red and green have something in common*. $R$ and $G$ are constants of type $\langle e, t\rangle$ and $\mathcal{X}$ is a variable of type $\langle\langle e, t\rangle, t\rangle$. The interpretation of $\exists\mathcal{X}(\mathcal{X}(R) \wedge \mathcal{X}(G))$ runs as follows: $[\![\exists\mathcal{X}(\mathcal{X}(R) \wedge \mathcal{X}(G))]\!]_{\mathbf{M},g} = 1$ iff there is a $d \in \mathbf{D}_{\langle\langle e,t\rangle,t\rangle}$ s.t. $[\![\mathcal{X}(R) \wedge \mathcal{X}(G)]\!]_{\mathbf{M},g[\mathcal{X}/d]} = 1$. $\mathbf{D}_{\langle\langle e,t\rangle,t\rangle}$ is the set of functions $\mathbf{D}_t^{(\mathbf{D}_t^{\mathbf{D}_e})}$. $[\![\mathcal{X}(R)]\!]_{\mathbf{M},g[\mathcal{X}/d]} = 1$ iff $[\![\mathcal{X}]\!]_{\mathbf{M},g[\mathcal{X}/d]}([\![R]\!]_{\mathbf{M},g[\mathcal{X}/d]}) = 1$. Note that $[\![R]\!]_{\mathbf{M},g[\mathcal{X}/d]} = I(R)$ and that $[\![\mathcal{X}]\!]_{\mathbf{M},g[\mathcal{X}/d]} = g[\mathcal{X}/d](\mathcal{X}) = d$

Consider the formula $(Q(W))(j)$, which is the representation of the sentence **John walks quickly** in the theory of types. The adverbial **quickly** should be treated as an expression whose application to a predicate results in another predicate. It is represented by means of a constant $Q$, which is of type $\langle\langle e, t\rangle, \langle e, t\rangle\rangle$. The interpretation of this composite expression is $[\![Q(W)]\!]_{\mathbf{M},g} = [\![Q]\!]_{\mathbf{M},g}([\![W]\!]_{\mathbf{M},g})$, and that is $I(Q)I(W)$; $I(Q)$ is an element of

$\left(\mathbf{D}_t^{\mathbf{D}_e}\right)^{(\mathbf{D}_t^{\mathbf{D}_e})} = (\{0,1\}^D)^{(\{0,1\}^D)}$

Note that the above only says what **kind** of thing the interpretation of an adverbial like **quickly** is. But say nothing at all about the relation between the interpretation of the predicate to which $Q$ is applied and the interpretation of the composite predicate which is the result. For example, the validity of the argument

$$11.\quad \frac{\text{John walks quickly}}{\text{John walks}}$$

is then not yet guarenteed: in the theory of types, $(Q(W))(j)/W(j)$ is not a valid argument schema. Also consider the following argument

$$12.\quad \frac{\begin{array}{l}\text{Albert is taller than Bert}\\ \text{Bert is taller than Charley}\end{array}}{\text{Albert is taller than Charley}}$$

The predicate logic schema corresponding to this is $Tab, Tbc/Tac$. The validity of (12) depends essentially on the transitivity of the relation **is taller than**

Something similar applies to (11). There too an additional premise is needed. In this case it would say that whenever $x$ does $X$ quickly, $x$ does $X$. This premise can be expressed, in the formalism of the theory of types, as $\forall x \forall X((Q(X)(x)) \rightarrow X(x))$. The argument schema (13) is indeed valid in the theory of types:

1. $\forall x \forall X((Q(X))(x) \rightarrow X(x)), (Q(W))(j)/W(j)$

Such extra premises are known as 'meaning postulates'

## 1.2 Categorial Grammar

### 1.2.1 Characteristics of Categorial Grammar

A **pure categorial grammar** has the following four characteristics:

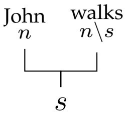
1. There is a finite set of **basic categories**
2. From these basic categories, a set of **derived categories** is constructed
3. There are either one or two **syntactic rules** describing the one syntactic operation of concatenation and determining the category of the result of operation
4. Every lexical element is assigned to a category

Here is a very simple example of a categorial grammar

1. The basic categories are $n$ (for 'noun') and $s$ (for 'sentence')
2. The derived categories may be obtained as follows: If $A$ and $B$ are categories, then $(A\backslash B)$ is a category too

3. The syntactic rule is: If $\alpha$ is an expression of category $A$, and $\beta$ is an expression of category $(A\backslash B)$, then $\alpha\beta$ is an expression of category $B$
4. *John* is of category $n$; *walks* is of category $n\backslash s$; and *quickly* is of category $(n\backslash s)\backslash(n\backslash s)$

According to this categorial grammar



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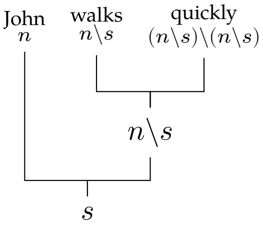


15.

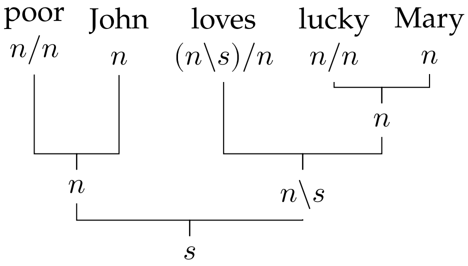
The above example is of a **undirectional grammar**. You can only work in one direction, in the sense that if you have an expression of category $A\backslash B$, then you have to write an expression of type $A$ on the *left-hand* side in order to obtain an expression of type $B$. There are many expressions which would result in a new expression if something were written to their right. Take, for example, an adjective like **poor**. Together with **John**, obtained from category $n$, this gives us **poor John**. The definition of derived categories can be modified in the following manner so as to allow for expressions like this:

5. If $A$ and $B$ are categories, then both $(A\backslash B)$ and $(A/B)$ are categories

Thus we obtain a **bidirectional** categorial grammar. Such a categorial grammar needs two syntactic rules:

6. (a) If $\alpha$ is in cateogry $A$ and $\beta$ is in cateogry $A\backslash B$, then $\alpha\beta$ is in cateogry $B$
   (b) If $\alpha$ is in category $A/B$ and $\beta$ is in category $B$, then $\alpha\beta$ is in category $A$

In the example given in (16), both kinds of derived categories are involved

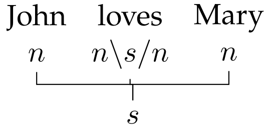


16.

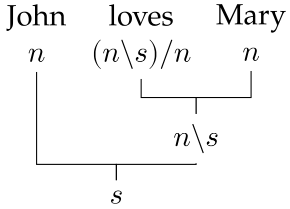
In other variant of categorical grammar, expressions in derived categories may be concatenated with several other expressions simultaneously.

7. If $A, B, C$ are categories, then $A\backslash B/C$ is a category
8. If $\alpha$ is in cateogry $A$, $\beta$ is in category $A\backslash B/C$, and $\gamma$ is in category $C$, then $\alpha\beta\gamma$ is in category $B$

In this way, the transitive verb **loves** may be categories as $n\backslash s/n$, instead of as $(n\backslash s)/n$

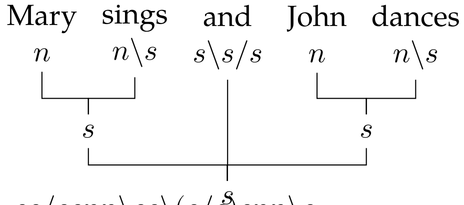


17.



18.

The analysis depicted in (17) attributes less structure to this example sentence than the analysis in figure (18). In (18), **loves Mary** is treated as a single constituent which is not so in (17). Generally speaking, an analysis like that in (18) will be preferred. An exception is, for example, formed by coordinative conjunctions, for which the categorization $s\backslash s/s$ is to be preferred over $s\backslash(s/s)$ or $(s\backslash s)/s$.



19. $ss/ssnn\backslash ss\backslash(s/s)snn\backslash s$

20. $s$

The difference between a bidirectional categorial grammar and a context-free grammar is in essence the following: A bidirectional categorial grammar always indicates which of a given pair of constituents is dependent on the other, whereas a context-free grammar need not always provide this information

### 1.2.2 The Descriptive Adequacy of Categorial Grammar

Compare sentences (22) and (23):

22. The job was quickly finished
23. The job was finished quickly

In (22), the constituent **was finished** occurs discontinuously, that is, it is interrupted by another expression. This contrasts with its continuity in (23). In a categorial grammar in its pure form, this presents a problem. Hence we are foreced to consider **was** and **finished** as separate lexical items and to place each of them in (at least) two different cateogries, so they can form both continuous and discontinuous constituents. Sentence (24) gives another example of this phenomenon

24. John never calls up Mary, so she calls him up instead

Here we have both a continuous and a discontinuous occurrence of the constituent **calls up**. Here too, in a categorial grammar there is no choice but to classify **calls up**, **calls** and **up** separately, as three distinct lexical items.

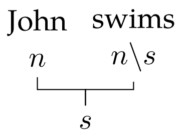
A second phenomenon which presents problems for pure categorial grammars and context-free grammars centers around the intuition that sentence (25) means the same as sentence (26):

25. John loves Mary, and Jack, Jill
26. John loves Mary, and Jack loves Jill

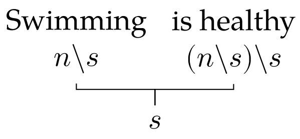
(25) is derived from (26) by leaving out the word **loves** in the second conjunct. Another conjecture is that the 'missing part' of (25) gets filled in during the process of interpretation. Either way, leaving out a constituent or filling one in introduces context-dependency into the picture, since the piece to be left out or filled in must always be present somewhere else in the structure.

A third phenomenon which illustrates that the limited generative capacity o context-free grammars and pure categorial grammars may lead to unituitive result is that of **word order**. Both kinds of grammar decree a fixed word order and hence seem to fail as adequate descriptive tools for languages in which it is not word order.

### 1.2.3 Categorial Grammar and the Theory of Types



$(S_{\langle e,t\rangle}(j_e))$



$H_{\langle\langle e,t\rangle,t\rangle}(S_{\langle e,t\rangle})$

27. *Categorial grammar* *Theory of types*

## 1.3 $\lambda$-Abstraction

### 1.3.1 The $\lambda$-Operator

29. Jogging is healthy

A translation of this sentence into the theory of types may be obtained as follows. Given that **jogging** expresses a property of individuals, the expression may be translated as a predicate constant $J$ of type $\langle e,t\rangle$. **Healthy** expresses, a property of properties of individuals and is as such to be rendered as a constant $\mathcal{H}$ of type $\langle\langle e,t\rangle,t$. The whole of (29) is then to be translated as the formula $\mathcal{H}(J)$

30. Not smoking is healthy
31. Drinking and driving is unwise
32. John admires John
33. John admires himself

In order to account for constructions like these, the following rule is added to the syntax of the theory of types as given in definition 2 in 1.2

7. If $\alpha$ is expression of type $a$ in $L$, and $v$ is a variable of type $b$, then $\lambda v\alpha$ is expression of type $\langle b,a\rangle$ in $L$

Let $W$ be a constant of type $\langle e,t\rangle$, and let $x$ be a variable of type $e$. Then $W(x)$ is a formula in which $x$ appears as a free variable. We can form the expression $\lambda x(W(x))$ of type $\langle e,t\rangle$. We say that the expression $\lambda x(W(x))$ has been formed from the expression $W(x)$ by **abstraction over** the free variable $x$. We say that the free occurrences of the variable $x$ in $\alpha$ are **bound** in $\lambda x\alpha$ by the **$\lambda$-operator** $\lambda x$

The interpretation of a $\lambda$-abstraction $\lambda x_b \alpha_a$ is a function. For this reason, $\lambda$-abstraction is also referred to as **functional abstraction**.

Now we add the following clause to Definition

6. If $\alpha \in \mathrm{WE}_a^L$ and $v \in \mathrm{VAR}_b$, then $[\![\lambda v \alpha]\!]_{\mathbf{M},g}$ is that function $h \in \mathbf{D}_a^{\mathbf{D}_b}$ s.t. for all $d \in \mathbf{D}_b$:$h(d) = [\![\alpha]\!]_{\mathbf{M},g[v/d]}$

Sentence (30), *Not smoking is healthy*, can be translated as follows. Let $S$ be the translation of *smoking*. This is a constant of type $\langle e, t \rangle$. And let $x$ be a variable of type $x$. We obtain the expression $\lambda x \neg S(x)$ of type $\langle e, t \rangle$. The whole of (30) may now be obtained by applying the second-order predicate $\mathcal{H}$. The result is $\mathcal{H}(\lambda x \neg S(x))$

### 1.3.2 $\lambda$-Conversion

The general notation for the result of replacing all free occurrences of a variable $v$ in an expression $\beta$ by an expression $\gamma$ is $[\gamma/v]\beta$

**Definition 1.5.** A variable $v'$ is **free for** $v$ in the expression $\beta$ iff no free occurrence of $v$ in $\beta$ is within the scope of a quantifier $\exists v'$ or $\forall v'$ or a $\lambda$-operator $\lambda v'$

**Theorem 1.6.** *If all variables which occurs as free variables in $\gamma$ are free for $v$ in $\beta$, then $\lambda v \beta(\gamma)$ and $[\gamma/v]\beta$ are equivalent*

### 1.3.3 The $\lambda$-Operator and Compositionality

The main purpose of translating a natural language into a formal language is to obtain a semantic interpretation of the former via the semantics of the latter. For the meaning of a correct translation is the same as the meaning of what is translated. In order for the semantic interpretation to be satisfactory, it is necessary that the process of translation comply to certain requirements. Among these, two important requirements are that the process be **explicit** and that it can be **specified in a finite manner**. The requirement that is be explicit means that it may not in any way rely on the knowledge or creativity of the translator: it must be such that it could, at least in principle, be automated. Furthermore, the translation process, though essentially finite,must translate a potentially infinite number of sentences.

One way of doing this is to stay closed to the syntactic rules of the natural language in question, which are finite in number. Here we assume that translations are available for all of the lexical elements of the language, which are finite in number. The for each syntactic rule saying how expressions may be combined to form composite expressions we formulate a parallel rule, which says how the translations of these expressions may be combined to give the translations of the composite expressions.

Now it should be clear that the way we have translated natural language sentences into standard predicate logic up until now is neither explicit nor compositional.

35. John smokes and drinks

36. $Sj \wedge Dj$

The way of translating is not explicit, in that essential use is made of our knowledge of the meaning of (35), in particular our knowledge of the fact that (35) expressions a conjunction of two sentences. And it is not compositional, in that no account is given of how the translation of (35) is built up from the translations of **John** and **smokes and drinks**, or how the translation of the **smokes and drinks** is built up from the translations of **smokes** and **drinks**. Given that the lexical elements of (35) are rendered as follows: $John : j, smokes : S, drinks : D$, the phrase **smokes and drinks** can be rendered as (37), while the whole of (35) translates as (38)

37. $\lambda x(S(x) \wedge D(x))$

38. $\lambda x(S(x) \wedge D(x))(j)$

We shall introduce the first-order quantifiers $\exists$ and $\forall$ categorematically by treating them as second-order predicates, that is to say, as expressions of type $\langle\langle e,t\rangle,t\rangle$

$$I(\exists) \text{ is that function } f_\exists \in \{0,1\}^{(\{0,1\}^D)} \text{ s.t. if } h \in \{0,1\}^D\text{,then}$$
$$f_\exists(h) = 1 \text{ iff there is a } d \in D \text{ s.t. } h(d) = 1$$
$$I(\forall) \text{ is that function } f_\forall \in \{0,1\}^{(\{0,1\}^D)} \text{ s.t. if } h \in \{0,1\}^D\text{,then}$$
$$f_\exists(h) = 1 \text{ iff for all } d \in D : h(d) = 1$$

In other words, $I(\exists)$ is (the characteristic function of) the set of non-empty subsets of, that is $\{A \mid A \subseteq D\&A \neq \emptyset\}$. And $I(\forall)$ is (the characteristic function of) $\{D\}$

We have interpreted $\exists$ as the set of all nonempty subsets of $D$. This means that the quantifier $\exists$ is equivalent to the expression $\lambda Y \exists x(Y(x))$ in the theory of types.

## 2 The Intensional Theory of Types