A Course in Model Theory

Katin Tent & Martin Ziegler

October 26, 2020

Contents

1 The Basics

1.1 Structures

Definition 1.1. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. A map $h : A \to B$ is called a **homomorphism** if for all $a_1, \dots, a_n \in A \quad h(c^{\mathfrak{A}}) \quad = \quad c^{\mathfrak{B}}$

2 Elementary Extensions and Compactness

3 Quantifier Elimination

3.1 Preservation theorems

Lemma 3.1 (Separation Lemma). *Let T_1, T_2 be two theories. Assume $\mathcal{H}$ is a set of sentences which is closed under $\wedge, \vee$ and contains $\perp$ and $\top$. Then the following are equivalent*

1. *There is a sentence $\varphi \in \mathcal{H}$ which separates T_1 from T_2. This means*

$$T_1 \vdash \varphi \quad \textit{and} \quad T_2 \vdash \neg\varphi$$

2. *All models $\mathfrak{A}_1$ of T_1 can be separated from all models $\mathfrak{A}_2$ of T_2 by a sentence $\varphi \in \mathcal{H}$. This means*

$$\mathfrak{A}_1 \models \varphi \quad \textit{and} \quad \mathfrak{A}_2 \models \neg\varphi$$

Proof. $2 \to 1$. For any model $\mathfrak{A}_1$ of T_1 let $\mathcal{H}_{\mathfrak{A}_1}$ be the set of all sentences from $\mathcal{H}$ which are true in $\mathfrak{A}_1$. (2) implies that $\mathcal{H}_{\mathfrak{A}_1}$ and T_2 cannot have a common model. By the Compactness Theorem there is a finite conjunction $\varphi_{\mathfrak{A}_1}$ of sentences from $\mathcal{H}_{\mathfrak{A}_1}$ inconsistent with T_2. Clearly

$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$

is inconsistent $\square$