Modal Logic

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Contents

1 Basic Concepts

1.1 Modal Languages

Definition 1.1. The **basic modal language** is defined using a set of **proposition letters** Φ whose elements are usually denoted p, q, r and so on, and a unary modal operator $\Diamond$. The well-formed **formulas** ϕ of the basic modal language are given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \psi \vee \phi \mid \Diamond\phi$$

$\mathfrak{M}, w \Vdash \phi$

Definition 1.2. A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called **modal operators**; we use $\triangle$, $\triangle_0, \triangle_1, \ldots$ to denote elements of O. The function ρ assigns to each operator $\delta \in O$ a finite **arity**

Definition 1.3. A **modal language** $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ. The set $Form(\tau, \Phi)$ of **modal formulas** over τ and Φ is given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)})$$

where p ranges over elements of Φ

Definition 1.4. For each $\triangle \in O$ the **dual** $\triangledown$ of $\triangle$ is defined as $\triangledown(\phi_1, \ldots, \phi_n) := \neg\triangle(\neg\phi_1, \ldots, \neg\phi_n)$

Example 1.1 (The Basic Temporal Language)**.** The basic temporal language is built using a set of unary operators $O = \{\langle F\rangle, \langle P\rangle\}$. The intended interpretation of a formula $\langle F\rangle\phi$ is $'\ \phi$ will be true at some Future time' and the intended interpretation of $\langle P\rangle\phi$ is $'\ \phi$ was true at some Past time.' This language is called the **basic temporal language**. Their duals are written as G and H ('it is Going to be the case' and 'it always Has been the case')

Let's denote the converse of a relation R by $R^{\smile}$. We will call a frame of the form $(T, R, R^{\smile})$ a **bidirectional frame**, and a model built over such a frame a **bidirectional model**. From now on, we will only interpret basic temporal language in bidirectional models. That is, if $\mathfrak{M} = (T, R, R^{\smile}, V)$ is a bidirectional model then

$$\begin{aligned} \mathfrak{M}, t \Vdash F\phi \quad &\text{iff} \quad \exists s(Rts \wedge \mathfrak{M}, s \Vdash \phi) \\ \mathfrak{M}, t \Vdash P\phi \quad &\text{iff} \quad \exists s(R^{\smile}ts \wedge \mathfrak{M}, s \Vdash \phi) \end{aligned}$$

Example 1.2 (An Arrow Language)**.** The type $\tau_{\rightarrow}$ of **arrow logic** is a similarity type with modal operators other than diamonds. The language of arrow logic is designed to talk about the objects in arrow structures. The well-formed formulas ϕ are given by

$$\phi := p \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid \phi \circ \psi \mid \otimes\phi \mid 1'$$

$1'$ ('identity') is a nullary modality, the 'converse' operator $\otimes$ is a diamond, and the 'composition' operator $\circ$ is a dyadic operator. Possible readings of these operators are:

$1'$	identity	'skip'
$\otimes\,\phi$	converse	'ϕ conversely'
$\phi \circ \psi$	composition	'first ϕ, then ψ'

1.2 Models and Frames

Definition 1.5. A **frame** for the basic modal language is a pair $\mathfrak{F} = (W, R)$ s.t.

1. W is a non-empty set
2. R is a binary relation on W

A **model** for the basic modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame for the basic modal language and V is a function assigning to each proposition letter p in Φ a subset $V(p)$ of W. The function V is called a **valuation**. $\mathfrak{M}$ is **based on** the frame $\mathfrak{F}$

Definition 1.6. Suppose w is a state in a model $\mathfrak{M} = (W, R, V)$. Then ϕ is **satisfied** in $\mathfrak{M}$ at state w if

$$\begin{array}{rll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \text{ where } p \in \Phi \\
\mathfrak{M}, w \Vdash \bot & \text{iff} & \text{never} \\
\mathfrak{M}, w \Vdash \neg\phi & \text{iff} & \text{not } \mathfrak{M}, w \Vdash \phi \\
\mathfrak{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \Diamond\phi & \text{iff} & \text{for some } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi
\end{array}$$

It follows that $\mathfrak{M}, w \Vdash \Box\phi$ iff for all $v \in W$ s.t. Rwv, we have $\mathfrak{M}, v \Vdash \phi$

Definition 1.7. Let τ be a modal similarity type. A **τ-frame** is a tuple $\mathfrak{F}$ consisting of the following ingredients

1. a non-empty set W

2. for each $n \geq 0$, and each n-ary modal operator $\triangle$ in the similarity type τ, an $(n+1)$-ary relation $R_\triangle$

ϕ is **satisfied at a state** w in a model $\mathfrak{M} = (W, \{R_\triangle \mid \triangle \in \tau\}, V)$ when $\rho(\triangle) > 0$ if

$$\mathfrak{M}, w \Vdash \triangle(\phi_1, \dots, \phi_n) \quad \text{iff} \quad \begin{array}{l} \text{for some } v_1, \dots, v_n \in W \text{ with } R_\triangle w v_1 \dots v_n \\ \text{we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i \end{array}$$

When $\rho(\triangle) = 0$ we define

$$\mathfrak{M}, w \Vdash \triangle \quad \text{iff} \quad w \in R_\triangle$$

Definition 1.8. The set of all formulas that are valid in a class of frames $\mathsf{F}$is called the **logic** of F (notation: Λ_F)

1.3 General Frames

Definition 1.9. Given an $(n+1)$-ary relation R on a set W, we define the following n-ary operation m_R on the power set $\mathcal{P}(W)$ of W:

$$m_R(X_1, \dots, X_n) = \{w \in W \mid Rww_1 \dots w_n \text{ for some } w_1 \in X_1, \dots, w_n \in X_n\}$$

2 Models

2.1 Invariance Results

Definition 2.1. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of the same modal similarity type τ, and let w and w' be states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The τ**-theory** (or τ-**type**) **of** w is the set of all τ-formulas satisfied at w: that is, $\{\phi \mid \mathfrak{M}, w \Vdash \phi\}$. We say that w and w' are **(modally) equivalent** ($w \leftrightsquigarrow w'$) if they have the same τ-theories

The τ**-theory** of the model $\mathfrak{M}$ is the set of all τ-formulas satisfied by all states in fM; that is, $\{\phi \mid \mathfrak{M} \Vdash \phi\}$ Models $\mathfrak{M}$ and $\mathfrak{M}'$ are called **(modally) equivalent** ($\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$) if their theories are identical

2.1.1 Disjoint Unions

2.1.2 Generated submodels

Definition 2.2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models; we say that $\mathfrak{M}'$ is a **submodel** of $\mathfrak{M}$ if $W' \subseteq W$, R' is the restriction of R

to W', and V' is the restriction of V to $\mathfrak{M}'$. We say that $\mathfrak{M}'$ is a **generated submodel** of $\mathfrak{M}$ ($\mathfrak{M}' \rightarrowtail \mathfrak{M}$) if $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$ and for all points w the following closure condition holds

$$\text{if } w \text{ is in } \mathfrak{M}' \text{ and } Rwv, \text{ then } v \text{ is in } \mathfrak{M}'$$

Let fM be a model, and X a subset of the domain of $\mathfrak{M}$; the **submodel generated by** X is the smallest generated submodel of $\mathfrak{M}$ whose domain contains X. A **rooted** or **point generated** model is a model that is generated by a singleton set, the element of which is called the **root** of the frame

2.1.3 Morphism for modalities

Definition 2.3 (Homomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$, if $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$, if $(w_0, \dots, w_n) \in R_\triangle$, then $(f(w_0), \dots, f(w_n)) \in R'_\triangle$ (the **homomorphic condition**)

Definition 2.4 (Strong Homomorphisms, Embeddings and Isomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **strong homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$ iff $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$ iff $(w_0, \dots, w_n) \in R_\triangle$, iff $(f(w_0), \dots, f(w_n)) \in R'_\triangle$ (the **strong homomorphic condition**)

An **embedding** of $\mathfrak{M}$ into $\mathfrak{M}'$ is a strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ which is injective. An **isomorphism** is a bijective strong homomorphism

Proposition 2.5. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. Then the following holds*

1. *for all elements w and w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, if there exists a surjective strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ with $f(w) = w'$, then w and w are modally equivalent*
2. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$*

Definition 2.6 (Bounded Morphisms - the Basic Case)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for the basic modal language. A mapping $f : \mathfrak{M} = (W, R, V) \to \mathfrak{M}' = (W', R', V')$ is a **bounded morphsim** if it satisfies

1. w and $f(w)$ satisfy the same proposition letters
2. f is a homomorphism w.r.t. the relation R (if Rwv then $R'f(w)f(v)$)
3. If $R'f(w)v'$ then there exists v s.t. Rwv and $f(v) = v'$ (the **back condition**)

If there is a **surjective** bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then we say that $\mathfrak{M}'$ is a **bounded morphic image** of $\mathfrak{M}$, and write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$

Proposition 2.7. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models s.t. $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded morphism. Then for each modal formula ϕ, and each element w of $\mathfrak{M}$ we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}', f(w) \Vdash \phi$.*

Let τ be a modal similarity type containing only diamonds (thus if $\mathfrak{M}$ is a τ-model, it has the form $(W, R_1, \dots, V)$ where each R_i is a binary relation on W) . In this context we will call a τ-model $\mathfrak{M}$ **tree-like** if the structure $(W, \bigcup_i R_i, V)$ is a tree

Proposition 2.8. *Assume that τ is a modal similarity type containing only diamonds. Then for any rooted τ-models $\mathfrak{M}$ there exists a tree-like τ-models $\mathfrak{M}'$ s.t. $\mathfrak{M}' \twoheadrightarrow \mathfrak{M}$. Hence any satisfiable τ-formula is satisfiable in a tree-like model*

Proof. Let w be the root of $\mathfrak{M}$. Define the model $\mathfrak{M}'$ as follows. Its domain W' consist of all finite sequences $(w, u_1, \dots, u_n)$ s.t. $n \geq 0$ and for some modal operators $\langle a_1 \rangle, \dots, \langle a_n \rangle \in \tau$ there is a path $wR_{a_1}u_1 \cdots R_{a_n}u_n$ in $\mathfrak{M}$. Define $(w, u_1, \dots, u_n)R'_a(w, v_1, \dots, w_m)$ to hold if $m = n + 1, u_i = v_i$ for $i = 1, \dots, n$ and $R_a u_n v_m$ holds in $\mathfrak{M}$. That is, R'_a relates two sequences iff the second is an extension of the first with a state from $\mathfrak{M}$ that is a sucessor of the last element of the first sequence. Finally, V' is defined by putting $(w, u_1, \dots, u_n) \in V'(p)$ iff $u_n \in V(p)$. The mapping $f : (w, u_1, \dots, u_n) \mapsto u_n$ defines a surjective bounded morphism from $\mathfrak{M}'$ to $\mathfrak{M}$ ☐

2.2 Bisimulations

Definition 2.9 (Bisimulation - the Basic Case)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M} = (W', R', V')$ be two models

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation between** $\mathfrak{M}$ and $\mathfrak{M}'$ (notation: $Z : \mathfrak{M} \leftrightarrows \mathfrak{M}'$) if

1. If wZw' then w and w' satisfy the same proposition letters
2. If wZw' and Rwv, then there exists v' (in $\mathfrak{M}'$) s.t. vZv' and $R'w'v'$ (the **forth condition**)
3. The converse of (2): if wZw' and $R'w'v'$, then there exists v (in $\mathfrak{M}$) s.t. vZv' and Rwv (the **back condition**)

When Z is a bisimulation linking two states w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$ we say that w and w' are **bisimilar**, and we write $Z : \mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$. If there is a bisimulation, we sometimes write $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$ or $w \leftrightarroweq w'$

Definition 2.10 (Bisimulation - the General Case)**.** Let τ be a modal similarity type, and let $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$ be τ-models. A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ ($Z : \mathfrak{M} \leftrightarroweq \mathfrak{M}'$) if the above condition 1 is satisfied and

2. If wZw' and $R_\triangle wv_1 \ldots v_n$ then there are $v'_1, \ldots, v'_n \in W'$ s.t. $R'_\triangle w'v'_1 \ldots v'_n$ and for all i $(1 \leq i \leq n)$ $v_i Z v'_i$ (the **forth** condition)
3. If wZw' and $R'_\triangle w'v'_1 \ldots v'_n$ then there are $v_1, \ldots, v_n \in W$ s.t. $R_\triangle wv_1 \ldots v_n$ and for all i $(1 \leq i \leq n)$ $v_i Z v'_i$ (the **back** condition)

Proposition 2.11. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ and $\mathfrak{M}_i$ $(i \in I)$ be τ-models*

1. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$*
2. *For every $i \in I$, and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarroweq \biguplus_i \mathfrak{M}_i, w$*
3. *If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarroweq \mathfrak{M}, w$ for all w in $\mathfrak{M}'$*
4. *If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$*

Proof. Suppose $\mathfrak{M} = (W, R_\triangle, V)_{\triangle\in\tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle\in\tau}$ $\mathfrak{M}_i \subseteq \biguplus_i \mathfrak{M}_i$

1. Suppose $f : \mathfrak{M} \cong \mathfrak{M}'$, then we define wZw' iff $w' = f(w)$ where $w \in W, w' \in W'$. Bisimulation comes from the definition of the isomorphism
2. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}_i\} \subseteq \mathfrak{M}_i \times \biguplus \mathfrak{M}_i$. The first condition comes from the invariance. The forth condition is obvious. For the back condition, if $R'_\triangle w'v'_1 \ldots v'_n$ and $w' \in W$, then $v'_1, \ldots, v'_n \in W$ since each $R_{\triangle,i}$ is disjoint and we have $R_{\triangle,i} w'v'_1 \ldots v'_n$
3. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}'\} \subseteq \mathfrak{M}' \times \mathfrak{M}$. The first condition comes from the invariance. Forth condition is obvious. For the back condition, suppose wZw and $R'_\triangle wv'_1 \ldots v'_n$, by the definition, $v'_1, \ldots, v'_n \in W$ and $R_\triangle wv'_1 \ldots v'_n$
4. Define $Z = \{(w, f(w) \mid w \in W)\}$. The first condition comes from the definition. If wZw' and $R_\triangle wv_1 \ldots v_n$, then $R'_\triangle f(w)f(v_1) \ldots f(v_n)$. If wZw' and $R'_\triangle w'v'_1 \ldots v_n$, then there is $v_1, \ldots, v_n$ s.t. $R_\triangle wv_1, \ldots, v_n$ and $f(v_i) = v'_i$ for $1 \leq i \leq n$

$\square$

Theorem 2.12. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ be τ-models. Then, for every $w \in W$ and $w' \in W'$, $w \Leftrightarrow w'$ implies that $w \leftrightsquigarrow w'$. In other words, modal formulas are invariant under bisimulation*

Proof. Induction on the complexity of ϕ.

Suppose ϕ is $\diamond\psi$, we have $\mathfrak{M}, w \Vdash \diamond\psi$ iff there exists a v in $\mathfrak{M}$ s.t. Rwv and $\mathfrak{M}, v \Vdash \psi$. As $w \Leftrightarrow w'$, there exists a v' in $\mathfrak{M}'$ s.t. $R'w'v'$ and $v \Leftrightarrow v'$. By the I.H., $\mathfrak{M}', v' \Vdash \psi$, hence $\mathfrak{M}', w' \Vdash \diamond\psi$ □

Example 2.1 (Bisimulation and First-Order Logic)**.**

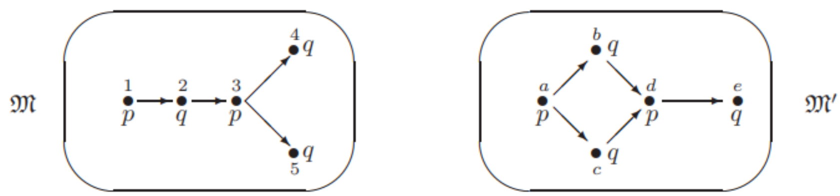

Fig. 2.4. Bisimilar models.

Example 2.2.

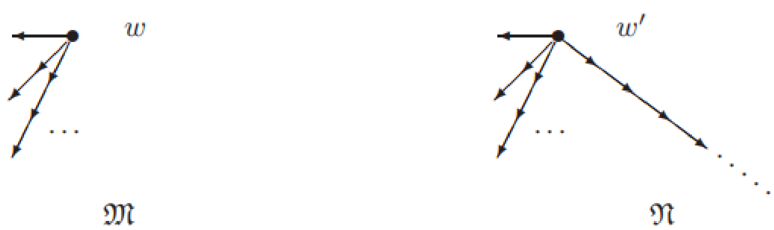

Fig. 2.5. Equivalent but not bisimilar.

$\mathfrak{M}$ is **image-finite** if for each state u in $\mathfrak{M}$ and each relation R in $\mathfrak{M}$, the set $\{(v_1, \ldots, v_n) \mid Ruv_1 \ldots v_n\}$ is finite

Theorem 2.13 (Hennessy-Milner Theorem)**.** *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be two image-finite τ-models. Then for every $w \in W$ and $w' \in W'$, $w \Leftrightarrow w'$ iff $w \leftrightsquigarrow w'$*

Proof. Assume that our similarity type τ only contains a single diamond. The direction from left to right follows from Theorem 2.12

Suppose $w \leftrightsquigarrow w'$. The first condition is immediate. If Rwv, assume there is no v' in $\mathfrak{M}'$ with $R'w'v'$ and $v \leftrightsquigarrow v'$. Let $S' = \{u' \mid R'w'u'\}$.

Note that S' must be non-empty, for otherwise $\mathfrak{M}', w' \Vdash \Box\bot$, which would contradict $w \leftrightarrow w'$ since $\mathfrak{M}, w \Vdash \diamond\top$. Furthermore, as $\mathfrak{M}'$ is image-finite, S' must be finite, say $S' = \{w'_1, \ldots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i s.t. $\mathfrak{M}, v \Vdash \psi_i$, but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n) \quad \text{and} \quad \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n)$$

□

Exercise 2.2.1. Suppose that $\{Z_i \mid i \in I\}$ is a non-empty collection of bisimulations between $\mathfrak{M}$ and $\mathfrak{M}'$. Prove that the relation $\bigcup_{i\in I} Z_i$ is also a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Conclude that if $\mathfrak{M}$ and $\mathfrak{M}'$ are bisimilar, then there is a maximal bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

Proof.
1. If $(w, w') \in \bigcup_{i\in I} Z_i$, then $(w, w') \in Z_j$ for some $j \in I$ and hence they satisfy the same propositional letters
2. If $(w, w') \in \bigcup_{i\in I} Z_i$ and $R_\triangle wv_1 \ldots v_n$, since $(w, w') \in Z_j$ for some $j \in I$, we have $R'_\triangle w'v'_1 \ldots v'_n$ and $v_i Z_j v'_i$ for all $1 \leq i \leq n$, which means $(v_i, v'_i) \in \bigcup_{i\in I} Z_i$ for all $1 \leq i \leq n$
3. similarly

□

Remark (Bisimulations for the Basic Temporal Language and Arrow Logic). When working with the basic temporal language, we usually work with models (W, R, V) and implicitly take R_p to be $R^\smile$. Thus we need a notion of bisimulation between models (W, R, V) and (W', R', V') to be a relation Z between the states of the two models that satisfies the clauses of Definition 2.9, and in addition the following

4. If wZw' and Rvw, then there exists v' in $\mathfrak{M}'$ s.t. vZv' and $R'v'w'$
5. Converse of 4: if wZw' and $R'v'w'$, then there exists v in $\mathfrak{M}$ s.t. vZv'

2.3 Finite Models

Definition 2.14 (Finite Model Property)**.** Let τ be a modal similarity type, and let M be a class of τ-models. We say that τ has the **finite model property w.r.t.** M if the following holds: if ϕ is a formula of similarity type τ, and ϕ is satisfiable in some model in M, then ϕ is satisfiable in a **finite** model in M

2.3.1 Selecting a finite submodel

Definition 2.15 (Degree)**.** We define the **degree** of modal formulas as follows:

$$\begin{aligned}
\deg(p) &= 0 \\
\deg(\bot) &= 0 \\
\deg(\neg\phi) &= \deg(\phi) \\
\deg(\phi \vee \psi) &= \max\{\deg(\phi), \deg(\psi)\} \\
\deg(\triangle(\phi_1, \dots, \phi_n)) &= 1 + \max\{\deg(\phi_1), \dots, \deg(\phi_2)\}
\end{aligned}$$

Proposition 2.16. *Let τ be a finite modal similarity type, and assume our collection of proposition letters is finite as well*

1. *for all n, up to logical equivalence there are only finitely many formulas of degree at most n*
2. *for all n, and every τ-model $\mathfrak{M}$ and state w of $\mathfrak{M}$, the set of all τ-formulas of degree at most n that are satisfied by w, is equivalent to a single formula*

Definition 2.17 (n-Bisimulation)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models, and let w and w' be states of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. We say that w and w' are n**-bisimilar** ($w \leftrightarroweq_n w'$) if there exists a sequence of binary relations $Z_n \subseteq \cdots \subseteq Z_0$ with the following properties (for $i + 1 \leq n$)

1. wZ_nw'
2. if vZ_0v' then v and v' agree on all proposition letters
3. if $vZ_{i+1}v'$ and Rvu then there exists u' with $R'v'u'$ and uZ_iu'
4. if $vZ_{i+1}v'$ and $R'v'u'$, then there exists u with Rvu and uZ_iu'

Proposition 2.18. *Let τ be a finite modal similarity type, Φ a finite set of proposition letters, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for this language. Then for every w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$, the following are equivalent*

1. *$w \leftrightarroweq_n w'$*
2. *w and w' agree on all modal formulas of degree at most n.*

Proof. $2 \to 1$. if $n = 0$, obvious.

If $n = k$ and the proposition holds. Now suppose $n = k+1$. Now w and w' agree on all modal formulas of degree at most $n + 1$. If there is not v, v' s.t. v and v' agree on all modal formulas of degree at most n and Rwv and Rwv'. Let $S' = \{u' \mid R'w'u'\}$ and S' is finite, say $S' == \{w'_1, \dots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i of degree at most n s.t. $\mathfrak{M}, v \Vdash \psi_i$ but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n) \text{ and } \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n)$$

□

Definition 2.19. Let τ be a modal similarity type containing only diamonds. Let $\mathfrak{M} = (W, R_1, \dots, R_n, \dots, V)$ be a rooted τ-model with root w. The notion of the **height** of states in $\mathfrak{M}$ is defined by induction.

The only element of height 0 is the rot of the model; the states of height $n+1$ are those immediate successors of elements of height n that have not yet assigned a height smaller than $n+1$. The **height of a model** $\mathfrak{M}$ is the maximum n s.t. there is a state of height n in $\mathfrak{M}$, if such a maximum exists; otherwise the height of $\mathfrak{M}$ is infinite

For a natural number k, the **restriction** of $\mathfrak{M}$ to k $(\mathfrak{M}{\restriction}k)$ is defined as the submodel containing only states whose height is at most k. $(\mathfrak{M}{\restriction}k) = (W_k, R_{1k}, \dots, R_{nk}, \dots, V_k)$, where $W_k = \{v \mid \text{height}(v) \leq k\}$, $R_{nk} = R_n \cap (W_k \times W_k)$, and for each p, $V_k(p) = V(p) \cap W_k$

Lemma 2.20. *Let τ be a modal similarity type that contains only diamonds. Let $\mathfrak{M}$ be a rooted τ-models, and let k be a natural number. Then for every state w of $(\mathfrak{M}{\restriction}k)$, we have $(\mathfrak{M}{\restriction}k), w \leftrightarroweq_l \mathfrak{M}, w$, where $l = k - \text{height}(w)$*

Theorem 2.21 (Finite Model Property - via Selection)**.** *Let τ be a modal similarity type containing only diamonds, and let ϕ be a τ-formula. If ϕ is satisfiable, then it is satisfiable on a finite model*

Proof. Fix a modal formula ϕ with $\deg(\phi) = k$. We restrict our modal similarity type τ and our collection of proposition letters to the modal operators and proposition letters actually occurring in ϕ. Let $\mathfrak{M}_1, w_1$ be s.t. $\mathfrak{M}_1, w_1 \Vdash \phi$. By Proposition 2.8, there exists a tree-like model $\mathfrak{M}_2$ with root w_2 s.t. $\mathfrak{M}_2, w_2 \Vdash \phi$. Let $\mathfrak{M}_3 := (\mathfrak{M}_2{\restriction}k)$. By Lemma 2.20 we have $\mathfrak{M}_2, w_2 \leftrightarroweq_k \mathfrak{M}_3, w_2$ and by Proposition 2.18 it follows that $\mathfrak{M}_3, w_2 \Vdash \phi$

By induction on $n \leq k$ we define finite sets of states $S_0, \dots, S_k$ and a (final) model $\mathfrak{M}_4$ with domain $S_0 \cup \dots \cup S_k$; the points in each S_n will have height n

Define S_0 to be the singleton $\{w_2\}$. Next, assume that $S_0, \dots, S_n$ have already been defined. Fix an element v of S_n. By Proposition 2.16 there are only finitely many non-equivalent modal formulas whose degree is at most $k-n$, say $\psi_1, \dots, \psi_m$. For each formula that is of the form $\langle a \rangle \chi$ and holds in $\mathfrak{M}_3$ at v, select a state u from $\mathfrak{M}_3$ s.t. $R_a vu$ and $\mathfrak{M}_3, u \Vdash \chi$. Add all these us to S_{n+1}, and repeat this selection process for every state in S_n. S_{n+1} is defined as the set of all points that have been selected in this way

Finally, define $\mathfrak{M}_4$ as follows. Its domain is $S_0 \cup \dots \cup S_k$; as each S_i is finite, $\mathfrak{M}_4$ is finite. The relations and valuation are obtained by restricting the relations and valuations of $\mathfrak{M}_3$ to the domain of $\mathfrak{M}_4$ □

2.3.2 Finite models via filtrations

Definition 2.22. A set of formulas Σ is **closed under subformulas** (or **subformula closed**) if for all formulas ϕ, ϕ': if $\phi \vee \phi' \in \Sigma$ then so are ϕ and ϕ'; if $\neg\phi \in \Sigma$ then so is ϕ; and if $\triangle(\phi_1, \dots, \phi_n) \in \Sigma$ then so are $\phi_1, \dots, \phi_n$

Definition 2.23 (Filtrations)**.** We work in the basic modal language. Let $\mathfrak{M} = (W, R, V)$ be a model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma$ be the relation on the states of $\mathfrak{M}$ defined by

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \phi \in \Sigma : (\mathfrak{M}, w \Vdash \phi \text{ iff } \mathfrak{M}, v \Vdash \phi)$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. We denote the equivalence class of a state w of $\mathfrak{M}$ w.r.t. $\leftrightsquigarrow_\Sigma$ by $|w|_\Sigma$, or simply $|w|$. The mapping $w \mapsto |w|$ is called the **natural map**

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model (W^f, R^f, V^f) s.t.

1. $W^f = W_\Sigma$
2. if Rwv then $R^f|w||v|$
3. if $R^f|w||v|$ then for all $\diamond\phi \in \Sigma$, if $\mathfrak{M}, v \Vdash \phi$ then $\mathfrak{M}, w \Vdash \diamond\phi$
4. $V^f(p) = \{|w| \mid \mathfrak{M}, w \Vdash p\}$, for all proposition letters p in Σ

$\mathfrak{M}^f_\Sigma$ is called a **filtration of** fM **through** Σ; we will often suppress subscripts and write $\mathfrak{M}^f$ instead of $\mathfrak{M}^f_\Sigma$

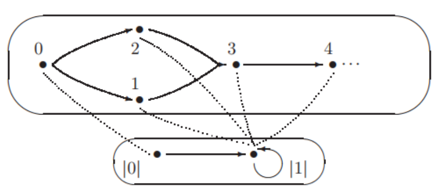

Fig. 2.6. A model and its filtration.

Let $\mathfrak{M} = (\mathbb{N}, R, V)$, where $R = \{(0,1), (0,2), (1,3)\} \cup \{(n, n+1) \mid n \geq 2\}$, and V has $V(p) = \mathbb{N} \setminus \{0\}$ and $V(q) = \{2\}$

Further assume $\Sigma = \{\diamond p, p\}$. Σ is subformula closed. Then, the model $\mathfrak{N} = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V')$, where $V'(p) = \{|1|\}$ is a filtration of $\mathfrak{M}$ through Σ. $\mathfrak{N}$ is not a bounded morphic image of $\mathfrak{M}$: any bounded morphism would have to preserve the formula q

Proposition 2.24. *Let Σ be a finite subformula closed set of basic modal formulas. For any model $\mathfrak{M}$, if $\mathfrak{M}^f$ is a filtration of $\mathfrak{M}$ through a subformula closed set Σ, then $\mathfrak{M}^f$ contains at most 2^n nodes (where n denotes the size of Σ)*

Proof. The states of $\mathfrak{M}^f$ are the equivalence classes in W_Σ. Let g be the function with domain W_Σ and range $\mathcal{P}(\Sigma)$ defined by $g(|w|) = \{\phi \in \Sigma \mid \mathfrak{M}, w \Vdash \phi\}$. It follows from the definition of $\leftrightsquigarrow_\Sigma$ that g is well defined and injective. Thus $|W_\Sigma| \leq 2^n, n = |\Sigma|$ $\square$

Theorem 2.25 (Filtration Theorem)**.** *Consider the basic modal language. Let $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes w in $\mathfrak{M}$, we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}^f, |w| \Vdash \phi$*

Proof. Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}, w \Vdash \diamond\phi$. Then there is a v s.t. Rwv and $\mathfrak{M}, v \Vdash \phi$. As $\mathfrak{M}^f$ is a filtration, $R^f|w||v|$. As Σ is a subformula closed, $\phi \in \Sigma$, thus by the inductive hypothesis $\mathfrak{M}^f, |v| \Vdash \phi$. Hence $\mathfrak{M}^f, |\Vdash| \diamond \phi$

Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}^f, |w| \Vdash \diamond\phi$. Thus there is a state $|v|$ in $\mathfrak{M}^f$ s.t. $R^f|w||v|$ and $\mathfrak{M}^f, |v| \Vdash \phi$. As $\phi \in \Sigma$, we have $\mathfrak{M}, v \Vdash \phi$. By the definition, we have $\mathfrak{M}, w \Vdash \diamond\phi$ $\square$

Note that clauses 2 and 3 of Definition 2.3.2 are designed to make the modal case of the inductive step go through.

Define

1. $R^s|w||v|$ iff $\exists w' \in |w| \exists v' \in |v| Rw'v'$
2. $R^l|w||v|$ iff for all formulas $\diamond\phi \in \Sigma$: $\mathfrak{M}, v \Vdash \phi$ implies $\mathfrak{M}, w \Vdash \diamond\phi$

These relations give rise to the **smallest** and **largest** filtrations respectively

Lemma 2.26. *Consider the basic modal language. Let $\mathfrak{M}$ be any model, Σ any subformula closed set of formulas, W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$, and V^f the standard valuation on W_Σ. Then both (W_Σ, R^s, V^f) and (W_Σ, R^l, V^f) are filtrations of $\mathfrak{M}$ through Σ. Furthermore, if (W_Σ, R^f, V^f) is any filtration of $\mathfrak{M}$ through Σ, then $R^s \subseteq R^f \subseteq R^l$*

Proof. If Rwv, if $\mathfrak{M}, v \Vdash \phi$, then $\mathfrak{M}, w \Vdash \diamond\phi$, hence $R^l|w||v|$

For any (W_Σ, R^f, V^f). $R^s \subseteq R^f$ by clause 2. $R^f \subseteq R^l$ by clause 2 $\square$

Theorem 2.27 (Finite Model Property - via Filtrations)**.** *Let ϕ be a basic modal formula. if ϕ is satisfiable, then it is satisfiable on a finite model. Indeed, it is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of ϕ*

Proof. Assume that ϕ is satisfiable on a model $\mathfrak{M}$; take any filtration of $\mathfrak{M}$ through the set of subformulas . $\square$

Lemma 2.28. *Let $\mathfrak{M}$ be a model, Σ a subformula closed set of formulas, and W_Σ the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Sigma$. Let R^t be the binary relation on W_Σ defined by*

$$R^t|w||v| \text{ iff for all } \phi, \text{ if } \diamond\phi \in \Sigma \text{ and } \mathfrak{M}, v \Vdash \phi \vee \diamond\phi \text{ then } \mathfrak{M}, w \Vdash \diamond\phi$$

If R is transitive then (W_Σ, R^t, V^f) is a filtration and R^t is transitive

Definition 2.29. Let (W, R, V) be a transitive frame. A **cluster** on (W, R, V) is a maximal, nonempty equivalence class under R. That is, $C \subseteq W$ is a cluster if the restriction of R to C is an equivalence relation

A cluster is **simple** if it consists of a single reflexive point, and **proper** if it consists more than one point

2.4 The Standard Translation

Definition 2.30. For τ a modal similarity type and Φ a collection of proposition letters, let $\mathcal{L}^1_\tau(\Phi)$ be the first-order language (with equality) which has unary predicates $P_0, P_1, \ldots$ corresponding to the proposition letters $p_0, p_1, \ldots$ in Φ, and an $(n+1)$-ary relation symbol $R_\triangle$ for each (n-ary) modal operator $\triangle$ in our similarity type. We write $\alpha(x)$ to denote a first-order formula α with one free variable, x

Definition 2.31 (Standard Translation)**.** Let x be a first-order variable. The **standard translation** ST_x taking modal formulas to first-order formulas in $\mathcal{L}^1_\tau(\Phi)$ is defined as

$$\begin{array}{rcl} ST_x(p) & = & Px \\ ST_x(\bot) & = & x \neq x \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \vee \psi) & = & ST_x(\phi) \vee ST_x(\psi) \\ ST_x(\triangle(\phi_1, \ldots, \phi_n)) & = & \exists y_1 \ldots \exists y_n (R_\triangle x y_1 \ldots y_n \wedge \\ & & \quad ST_{y_1}(\phi_1) \wedge \cdots \wedge ST_{y_n}(\phi_n)) \end{array}$$

where $y_1, \ldots, y_n$ are fresh variables.

$$ST_x(\diamond\phi) = \exists y(Rxy \wedge ST_y(\phi))$$
$$ST_x(\Box\phi) = \forall y(Rxy \to ST_y(\phi))$$

Proposition 2.32 (Local and Global Correspondence on Models)**.** *Fix a modal similarity type τ, and let ϕ be a τ-formula. Then*

1. *For all $\mathfrak{M}$ and all states w of $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M} \models ST_x(\phi)[w]$*
2. *For all $\mathfrak{M}$: $\mathfrak{M} \Vdash \phi$ iff $\mathfrak{M} \models \forall x ST_x(\phi)$*

Proposition 2.33.

1. *Let τ be a modal similarity type that only contains diamonds. Then, every τ-formula ϕ is equivalent to a first-order formula containing at most two variables*
2. *If τ does not contain modal operators $\triangle$ whose arity exceeds n, all τ-formulas are equivalent to first-order formulas containing at most $(n+1)$ vairables*

Proof. Assume τ contains only diamonds $\langle a \rangle, \langle b \rangle$. Fix two distinct variables x and y. Define two variants ST_x and ST_y of the standard translation as follows

$$
\begin{aligned}
&ST_x(p) = Px && ST_y(p) = Py \\
&ST_x(\bot) = x \neq x && ST_y(\bot) = y \neq y \\
&ST_x(\neg\phi) = \neg ST_x(\phi) && ST_y(\neg\phi) = \neg ST_y(\phi) \\
&ST_x(\phi \vee \psi) = ST_x(\phi) \vee ST_x(\psi) && ST_y(\phi \vee \psi) = ST_y(\phi) \vee ST_y(\psi) \\
&ST_x(\langle a \rangle \phi) = \exists y(R_a xy \wedge ST_y(\phi)) && ST_y(\langle a \rangle \phi) = \exists x(R_a yx \wedge ST_x(\phi))
\end{aligned}
$$

Then for any τ-formula ϕ, its ST_x-translation contains at most the two variables x and y, and $ST_x(\phi)$ is equivalent to the original standard translation of ϕ $\square$

Example 2.3.

$$
\begin{aligned}
ST_x(\diamond(\Box p \to q)) &= \exists y(Rxy \wedge ST_y(\Box p \to q)) \\
&= \exists y(Rxy \wedge (\forall x(Ryx \to ST_x(p)) \to Qy)) \\
&= \exists y(Rxy \wedge (\forall x(Ryx \to Px) \to Qy))
\end{aligned}
$$

Rxx is not equivalent to any modal formula. Suppose ϕ is a modal formula s.t. $ST_x(\phi)$ is equivalent to Rxx. Let $\mathfrak{M}$ be a singleton reflexive model and let w be the unique state in $\mathfrak{M}$; obviously $\mathfrak{M} \models Rxx[w]$. Let $\mathfrak{N}$ be a model based on the strict ordering of the integers; for every integer v, $\mathfrak{N} \models \neg Rxx[v]$. Let Z be the relation which links every integer with the unique state in fM, and assume that the valuations in $\mathfrak{N}$ and $\mathfrak{M}$ are s.t. Z is a bisimulation.

$$\mathfrak{M} \models Rxx[w] \Rightarrow \mathfrak{M}, w \Vdash \phi \Rightarrow \mathfrak{N}, v \Vdash \phi \Rightarrow \mathfrak{N} \models Rxx[v]$$

Definition 2.34. Let τ be a modal similarity type, C a class of τ-models, and Γ a set of formulas over τ. We say that Γ **defines** of **characterizes** a class K of models **within** C if for all models $\mathfrak{M}$ in C we have that $\mathfrak{M}$ is in K iff $\mathfrak{M} \Vdash \Gamma$. If C is the class of all τ-models, we simply say that Γ defines or characterizes K; we omit brackets whenever Γ is a singleton. We say that a formula ϕ defines a **property** whenever ϕ defines the class of models satisfying the property

2.5 Modal Saturation via Ultrafilter Extensions

2.5.1 M-saturation

Definition 2.35 (Hennessy-Milner Classes)**.** Let τ be a modal similarity type, and K a class of τ-models. K is a **Hennessy-Milner** class, or **has the Hennessy-Milner property**, if for every two models $\mathfrak{M}$ and $\mathfrak{M}'$ in K and any two states w, w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, $w \leftrightsquigarrow w'$ implies $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$

For example, by Theorem 2.13 the class of image-finite models has the Hennessy-Milner property.

Suppose we are working in the basic modal language. Let $\mathfrak{M} = (W, R, V)$ be a model, let w be a state in W and let $\Sigma = \{\phi_0, \phi_1, \dots\}$ be an infinite set of formulas. Suppose that w has successors $v_0, v_1, \dots$, where respectively $\phi_0, \phi_0 \wedge \phi_1, \phi_0 \wedge \phi_1 \wedge \phi_2, \dots$ hold. If there is no successor v of w where **all** formulas from Σ hold **at the same time**, then the model is in some sense incomplete. A model is called m-saturated if incompleteness of this kind does not occur

Suppose that we are looking for a successor of w at which every formula ϕ_i of the infinite set of formulas $\Sigma = \{\phi_0, \phi_1, \dots\}$ holds. M-saturation is a kind of compactness property, according to which it suffices to find satisfying successors of w for arbitrary finite approximations of Σ

Definition 2.36 (M-saturation)**.** Let $\mathfrak{M} = (W, R, V)$ be a model of the basic modal similarity type, X a subset of W and Σ a set of modal formulas. Σ is **satisfiable** in the set X if there is a state $x \in X$ s.t. $\mathfrak{M}, x \Vdash \phi$ for all $\phi \in \Sigma$. Σ is **finitely satisfiable** in X if every finite subset of Σ is satisfiable in X

The model $\mathfrak{M}$ is called m**-saturated** if it satisfies the following condition for every state $w \in W$ and every set Σ of modal formulas:

If Σ is finitely satisfiable in the set of successors of w,
then Σ is satisfiable in the set of successors of w

Let τ be a modal similarity type, and let $\mathfrak{M}$ be a τ-model. $\mathfrak{M}$ is called m**-saturated** if for every state w of $\mathfrak{M}$ and every (n-ary) modal operator $\triangle \in \tau$ and sequence $\Sigma_1, \dots, \Sigma_n$ of sets of modal formulas, we have the following:

If for every sequence of finite subsets $\Delta_1 \subset \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$, there are states $v_1, \dots, v_n$ s.t. $Rwv_1 \dots v_n$ and $v_1 \Vdash \Delta_1, \dots, v_n \Vdash \Delta_n$, then there are states $v_1, \dots, v_n$ in $\mathfrak{M}$ s.t. $Rwv_1 \dots v_n$ and $v_1 \Vdash \Sigma_1, \dots v_n \Vdash \Sigma_n$

Proposition 2.37. *Let τ be a modal similarity type. Then the class of m-saturated τ-models has the Hennessy-Milner property*

Proof. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two m-saturated models.

Assume that $w, v \in W$ and $w' \in W'$ are s.t. Rwv and $w \leftrightsquigarrow w'$. Let Σ be the set of formulas true at v. It is clear that for every finite subset Δ of Σ we have $\mathfrak{M}, v \Vdash \bigwedge \Delta$, hence $\mathfrak{M}, w \Vdash \diamond \bigwedge \Delta$. As $w \leftrightsquigarrow w'$, it follows that $\mathfrak{M}', w' \Vdash \diamond \bigwedge \Delta$, so w' has an R'-successor v_Δ s.t. $\mathfrak{M}', v_\Delta \Vdash \bigwedge \Delta$. In other words, Σ is finitely satisfiable in the set of successors of w'; but then, by m-saturation, Σ itself is satisfiable in a successor v' of w'. Thus $v \leftrightsquigarrow v'$ $\square$

2.5.2 Ultrafilter extensions

Definition 2.38 (Filters and Ultrafilters)**.** Let W be a non-empty set. A **filter** F **over** W is a set $F \subseteq \mathcal{P}(W)$ s.t.

1. $W \in F$
2. If $X, Y \in F$, then $X \cap Y \in F$
3. If $X \in F$ and $X \subseteq Z \subseteq W$, then $Z \in F$

An **ultrafilter over** W is a proper filter s.t. for all $X \in \mathcal{P}(W)$, $X \in U$ iff $(W \setminus X) \in U$

Definition 2.39. Let W be a non-empty set, and let E be a subset of $\mathcal{P}(W)$. By the **filter generated by** E we mean the intersection F of the collection of all filters over W which include E

$$F = \bigcap \{G \mid E \subseteq G \text{ and } G \text{ is a filter over } W\}$$

E has the **finite intersection property** if the intersection of any finite number of elements of E is non-empty

Lemma 2.40 (Zorn's Lemma)**.** *Whenever $<$ is a strict partial order of a set A satisfying for all chains $C \subseteq A$ there is some $b \in A$ s.t. $x \leq b$ for all $x \in C$ then for all $a \in A$, there is a maximal $b \in A$ with $b \geq a$*

Theorem 2.41 (Ultrafilter Theorem)**.** *Fix a non-empty set W. Any proper filter over W can be extended to an ultrafilter over W. As a corollary, any subset of $\mathcal{P}(W)$ with the finite intersection property can be extended to an ultrafilter over W*

Definition 2.42. Let W be a non-empty set. Given an element $w \in W$, the **principal ultrafilter** π_w generated by w is the filter generated by the singleton set $\{w\}$

Suppose U is an ultrafilter over a non-empty set I, and that for each $i \in I$, A_i is a non-empty set. Let $C = \prod_{i\in I} A_i$. That is, C is the set of all functions f with domain I s.t. for each $i \in I$, $f(i) \in A_i$. For two functions $f, g \in C$ we say that f and g are U**-equivalent** $(f \sim_U g)$ if $\{i \in I \mid f(i) = g(i)\} \in U$

Proposition 2.43. *The relation $\sim_U$ is an equivalence relation on the set C*

Proof. Suppose $\{i \mid f(i) = g(i)\} \in U, \{i \mid g(i) = h(i)\} \in U$, then $\{i \mid f(i) = g(i) = h(i)\} = \{i \mid f(i) = g(i)\} \cap \{i \mid g(i) = h(i)\} \in U$. And $\{i \mid f(i) = g(i) = h(i)\} \subseteq \{i \mid f(i) = h(i)\}$ □

Definition 2.44. Let f_U be the equivalence class of f modulo $\sim_U$, that is: $f_U = \{g \in C \mid g \sim_U f\}$. The **ultraproduct of** the sets A_i **modulo** U is the set of all equivalence classes of $\sim_U$. It is denoted by $\prod_U A_i$. So

$$\prod_U A_i = \{f_U \mid f \in \prod_{i\in I} A_i\}$$

Definition 2.45. Fix a first-order language $\mathcal{L}^1$, and let $\mathfrak{A}_i (i \in I)$ be $\mathcal{L}^1$-models. The **ultraproduct** $\prod_U \mathfrak{A}_i$ **of** $\mathfrak{A}_i$ **modulo** U is the model described as follows:

1. The universe A_U is the set $\prod_U A_i$, where A_i is the universe of $\mathfrak{A}_i$
2. Let R be an n-place relation symbol, and R_i its interpretation in the model $\mathfrak{A}_i$. The relation R_U in $\prod_U \mathfrak{A}_i$ is given by

$$R_U f_U^1 \dots f_U^n \quad \text{iff} \quad \{i \in I \mid R_i f^1(i) \dots f^n(i)\} \in U$$

3. Let F be an n-place function symbol, and F_i its interpretation in $\mathfrak{A}_i$. The function F_U in $\prod_U \mathfrak{A}_i$ is given by

$$F_U(f_U^1, \dots, f_U^n) = \{(i, F_i(f^1(i), \dots, f^n(i))) \mid i \in I\}_U$$

4. Let c be a constant, and a_i its interpretation in $\mathfrak{A}_i$. Then c is interpreted by the element $c' \in \prod_U A_i$ where $c' = \{(i, a_i) \mid i \in I\}_U$

In the case where all the structures are the same, say $\mathfrak{A}_i = \mathfrak{A}$ for all i, we speak of the **ultrapower** of $\mathfrak{A}$ modulo U, notation $\prod_U \mathfrak{A}$

Theorem 2.46 (Łoś's Theorem)**.** *Let U be an ultrafilter over a non-empty set I. For each $i \in I$, let $\mathfrak{A}_i$ be a model*

1. *For every term* $t(x_1, \dots, x_n)$ *and all elements* $f^1_U, \dots, f^n_U$ *of* $\mathfrak{B} = \prod_U \mathfrak{A}_i$ *we have*

$$t^{\mathfrak{B}}[x_1 \mapsto f^1_U, \dots, x_n \mapsto f^n_U] = \{(i, t^{\mathfrak{A}_i}[f^1(i), \dots, f^n(i)]) \mid i \in I\}_U$$

2. *Given any first-order formula* $\alpha(x_1, \dots, x_n)$ *in* $\mathcal{L}^1_\tau$ *and* $f^1_U, \dots, f^n_U$ *in* $\prod_U \mathfrak{A}_i$ *we have*

$$\prod_U \mathfrak{A}_i \vDash \alpha[f^1_U, \dots, f^n_U] \quad \textit{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U$$

Proof. 1.

2. Induction on α. The atomic case holds by definition. Suppose that $\alpha \equiv \neg\beta(x_1, \dots, x_n)$, then

$$\begin{aligned}
\prod_U \mathfrak{A}_i \vDash \alpha[f^1_U \dots f^n_U] \quad &\text{iff} \quad \prod_U \mathfrak{A}_i \nvDash \beta[f^1_U, \dots, f^n_U] \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \beta[f^1_U, \dots, f^n_U]\} \notin U \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \nvDash \beta[f^1(i), \dots, f^n(i)]\} \in U \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U
\end{aligned}$$

The second equivalence follows from the inductive hypothesis, and the third from the fact that U is an ultrafilter
Suppose that $\alpha(x_1, \dots, x_n) \equiv \exists x_0 \beta(x_0, \dots, x_n)$, then

$$\begin{aligned}
\prod_U \mathfrak{A}_i \vDash \alpha[f^1_U, \dots, f^n_U] \quad &\text{iff} \quad \exists f^0_U \in \prod_U \mathfrak{A}_i, \prod_U \mathfrak{A}_i \vDash \beta[f^0_U, \dots, f^n_U] \\
&\text{iff} \quad \exists f^0_U \in \prod_U \mathfrak{A}_i, \{i \in I \mid \mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]\} \in U
\end{aligned} \tag{2.5.1}$$

As $\mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]$ implies $\mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]$, which means

$$\{i \in I \mid \mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]\} \subseteq \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\}$$

Hence

$$\{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U \tag{2.5.2}$$

Conversely, if (2.5.2) holds, then we can select a function $f^0 \in \prod_{i \in I} A_i$ s.t. (2.5.1) holds. So (2.5.1) is equivalent to (2.5.2)

□

Corollary 2.47. *Let $\prod_U \mathfrak{A}$ be an ultrapower of $\mathfrak{A}$. Then for all first-order sentences α, $\mathfrak{A} \vDash \alpha$ iff $\prod_U \mathfrak{A} \vDash \alpha$*

There is a natural embedding of a model $\mathfrak{A}$ in each of its ultrapowers. Define the **diagonal mapping** d of $\mathfrak{A}$ into $\prod_U \mathfrak{A}$ to be the function

$$\alpha \mapsto (f_\alpha)_U, \text{ where } f_a(i) = a \text{ for all } i \in I$$

Corollary 2.48. *Let $\prod_U \mathfrak{A}$ be an ultrapower of $\mathfrak{A}$. Then the diagonal mapping of $\mathfrak{A}$ into $\prod_U \mathfrak{A}$ is an elementary embedding*

Proof.

$$\begin{aligned}\prod_U \mathfrak{A} \vDash \alpha[d(a_1), \dots, d(a_n)] \quad &\text{iff} \quad \{i \in I \mid \mathfrak{A} \vDash \alpha[a_1, \dots, a_n]\} \in U \\ &\text{iff} \quad \mathfrak{A} \vDash \alpha[a_1, \dots, a_n]\end{aligned}$$

□

$V(\phi) = \{w \mid \mathfrak{M}, w \Vdash \phi\}$

Definition 2.49. Given an $(n+1)$-ary relation R on a set W, we define the following two n-ary operations m_R and l_R on the power set $\mathcal{P}(W)$ of W:

$$\begin{aligned}m_R(X_1, \dots, X_n) &:= \{w \in W \mid \exists w_1, \dots, w_n (Rww_1 \dots w_n \bigwedge \forall i (w_i \in X_i))\} \\ l_R(X_1, \dots, X_n) &:= \{w \in W \mid \forall w_1, \dots, w_n (Rww_1 \dots w_n \to \exists i (w_i \in X_i))\} \\ m_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\triangle(\phi_1, \dots, \phi_n)) \\ l_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\triangledown(\phi_1, \dots, \phi_n))\end{aligned}$$

It follows that for any model $\mathfrak{M} = (W, R, V)$ we have

$$V(\diamond\phi) = m_R(V(\phi)) \quad \text{and} \quad V(\Box\phi) = l_R(V(\phi))$$

Proposition 2.50. *Let R be a relation of arity $n+1$ on the set W. Then for every n-tuple $X_1, \dots, X_n$ of subsets of W we have*

$$l_R(X_1, \dots, X_n) = W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n)$$

Proof. This is actually $\triangledown = \neg\triangle\neg$

$$\begin{aligned}W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n) &= \{w \mid \neg\exists w_1, \dots, w_n (Rww_1 \dots w_n \bigwedge \forall i (w_i \in W \setminus X_i))\} \\ &= \{\forall w_1, \dots, w_n (\neg Rww_1 \dots w_n \bigvee \neg\forall i (w_i \in W \setminus X_i))\} \\ &= \{\forall w_1, \dots, w_n (Rww_1 \dots w_n \to \exists i (w_i \notin W \setminus X_i))\} \\ &= l_R(X_1, \dots, X_n)\end{aligned}$$

□

Definition 2.51 (Ultrafilter Extension)**.** Let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\triangle)_{\triangle\in\tau}$ is a τ-frame. The **ultrafilter extension** $\mathfrak{ue}\mathfrak{F}$ of $\mathfrak{F}$ is defined as the frame $(Uf(W), R^{ue}_\triangle)_{\triangle\in\tau}$. Here $Uf(W)$ is the set of ultrafilters over W and $R^{ue}_\triangle u_0 u_1 \dots u_n$ holds for a tuple $u_0, \dots, u_n$ of ultrafilters over W if we have that $m_{R_\triangle}(X_1, \dots, X_n) \in u_0$ whenever $X_i \in u_i$ for all i with $1 \leq i \leq n$

The **ultrafilter extension** of a τ-model $\mathfrak{M} = (\mathfrak{F}, V)$ is the model $\mathfrak{ue}\mathfrak{M} = (\mathfrak{ue}\mathfrak{F}, V^{ue})$ where $V^{ue}(p_i)$ is the set of ultrafilters of which $V(p_i)$ is a member

Any subset of a frame can be viewed as a **proposition**. A filter over the universe of the frame can thus be seen as a **theory**, in fact as a logically closed theory, since filters are both closed under intersection (conjunction) and upward closed (entailment). Viewed this way, a proper filter is a **consistent** theory, or **state of affairs**, for it does not contain the empty set (falsum). Finally an ultrafilter is a **complete** theory.

In a given frame $\mathfrak{F}$ not every state not every state of affairs needs to 'realized', in the sense that there is a state satisfying all and only the propositions belonging to the state of affairs; only the states of affairs that correspond to the **principal** ultrafilters are realized. We build $\mathfrak{ue}\mathfrak{F}$ by adding every state of affairs for $\mathfrak{F}$ as a new element of the domain - that is, $\mathfrak{ue}\mathfrak{F}$ realizes every proposition in $\mathfrak{F}$

Stipulate that $R^{ue}_\triangle u_0 u_1 \dots u_n$ if u_0 'sees' the n-tuple $u_1, \dots, u_n$. That is, whenever $X_1, \dots, X_n$ are propositions of $u_1, \dots, u_n$ respectively, then u_0 'sees' this combination: that is, the proposition $m_{R_\triangle}(X_1, \dots, X_n)$ is a member of u_0.

Principal ultrafilters over W plays a special role. By identifying a state w of a frame $\mathfrak{F}$ with the principal ultrafilter $\pi_w = \{X \subset W \mid w \in X\}$, it is easily seen that any frame $\mathfrak{F}$ is (isomorphic to) a **submodel** (but in general not a **generated** submodel) of its ultrafilter extension. For we have the following equivalences

$$\begin{array}{lll} Rwv & \text{iff} & w \in m_R(X) \text{ for all } X \subseteq W \text{ s.t. } v \in X \\ & \text{iff} & m_R(X) \in \pi_w \text{ for all } X \subseteq W \text{ s.t. } X \in \pi_v \\ & \text{iff} & R^{ue}\pi_w\pi_v \end{array}$$

since

$$Rwv \quad \text{iff} \quad \forall X \subseteq W (v \in X \to w \in m_R(X))$$

Example 2.4. Consider the frame $\mathfrak{N} = (\mathbb{N}, <)$

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{4}{\bullet} \longrightarrow \quad \dots$$

What is the ultrafilter extension of $\mathfrak{N}$? There are two kinds of ultrafilter over an infinite set: the principal ultrafilter that are in one-to-one correspondence with the points of the set, and the non-principal ones which contain all co-finite sets and only infinite sets, cf Exercise 2.5.1. The principal ultrafilters form an isomorphic copy of the frame $\mathfrak{N}$ inside $\mathfrak{ue}\mathfrak{N}$. For any pair u, u' of ultrafilters, if u' is non-principal, then $R^{ue}uu'$. To set this, let $X \in u'$. As X is infinite, for any $n \in \mathbb{N}$ there is an m s.t. $n < m$ and $m \in X$. This show that $m_<(X) = \mathbb{N}$. But $\mathbb{N}$ is an element of every ultrafilter

The shows that the ultrafilter extension of $\mathfrak{N}$ consists of a copy of $\mathfrak{N}$ followed by a uncountable cluster consisting of all the non-principal ultrafilters

Proposition 2.52. *Let τ be a modal similarity type, and $\mathfrak{M}$ a τ-model. Then for any formula ϕ and any ultrafilter u over W, $V(\phi) \in u$ iff $\mathfrak{ue}\mathfrak{M}, u \vDash \phi$. Hence for every state w of $\mathfrak{M}$ we have $w \leftrightsquigarrow \pi_w$*

Proof. The second claim of the proposition is immediate from the first one by the observation that $w \Vdash \phi$ iff $w \in V(\phi)$ iff $V(\phi) \in \pi_w$

Induction on ϕ. The basic case is immediate from the definition of V^{ue}. Suppose ϕ is of the form $\neg\psi$, then

$$\begin{array}{rll} V(\neg\psi) \in u & \text{iff} & W \setminus V(\psi) \in u \\ & \text{iff} & V(\psi) \notin u \\ & \text{iff} & \mathfrak{ue}\mathfrak{M}, u \nVdash \psi \quad \text{IH} \\ & \text{iff} & \mathfrak{ue}\mathfrak{M}, u \Vdash \neg\psi \end{array}$$

Now consider the case where ϕ is of the form $\diamond\psi$. Assume first that $\mathfrak{ue}\mathfrak{M}, u \Vdash \diamond\psi$. Then there is an ultrafilter u' s.t. $R^{ue}uu'$ and $\mathfrak{ue}\mathfrak{M}, u' \Vdash \psi$. The induction hypothesis implies that $V(\psi) \in u'$, so by the definition of R^{ue}, $m_R(V(\psi)) \in u$. Now the result follows immediately from the observation that $m_R(V(\psi)) = V(\diamond\psi)$

Assume that $V(\diamond\psi) \in u$. We have to find an ultrafilter u' s.t. $V(\psi) \in u'$ and $R^{ue}uu'$. The latter constraint reduces to the condition that $m_R(X) \in u$ whenever $X \in u'$, or equivalently (see Exercise 2.5.2)

$$u'_0 := \{Y \mid l_R(Y) \in u\} \subseteq u'$$

We will first show that u'_0 is closed under intersection. Let $Y, Z \in u'_0$. By definition, $l_R(Y)$ and $l_R(Z)$ are in u. But then $l_R(Y \cap Z) \in u$ as $l_R(Y \cap Z) = l_R(Y) \cap l_R(Z)$. This proves that $Y \cap Z \in u'_0$

Next we make sure that for any $Y \in u'_0$, $Y \cap V(\psi) \neq \emptyset$. Let Y be an arbitrary element of u'_0, then by definition of u'_0, $l_R(Y) \in u$. As u is closed under intersection and does not contain the empty set, there must be an element $x \in l_R(Y) \cap V(\diamond\psi)$. But then x must have a successor y in $V(\psi)$. Finally, $x \in l_R(Y)$ implies $y \in Y$

From the fact that u'_0 is closed under intersection, and the fact that for any $Y \in u'_0$, $Y \cap V(\psi) \neq \emptyset$, it follows that the set $u'_0 \cup \{V(\psi)\}$ has the finite intersection property. So the Ultrafilter Theorem provides us with an ultrafilter u' s.t. $u'_0 \cup \{V(\psi)\} \subseteq u'$. This ultrafilter u' has the desired properties: it is clearly a successor of u, and the fact the $\mathfrak{ue}\mathfrak{M}, u' \Vdash \psi$ follows from $V(\psi) \in u'$ and the induction hypothesis $\square$

Example 2.5. Our new invariance result can be used to compare the relative expressive power of modal languages. Consider the modal constant $\circlearrowleft$ whose truth definition in a model for the basic modal language is

$$\mathfrak{M}, w \Vdash \circlearrowleft \quad \text{iff} \quad \mathfrak{M} \models Rxx[v] \text{ for some } v \text{ in } \mathfrak{M}$$

Comparing the pictures of the frame $(\mathbb{N}, <)$ and its ultrafilter extension given in Example 2.4 . The former is loop-free but the latter contains uncountably many loops

Proposition 2.53. *Let τ be a modal similarity type, and let $\mathfrak{M}$ be a τ-model. Then $\mathfrak{ue}\mathfrak{M}$ is m-saturated*

Proof. Let $\mathfrak{M} = (W, R, V)$ be a model. Consider an ultrafilter u over W, and a set Σ of modal formulas which is finitely satisfiable in the set of successors of u. We have to find an ultrafilter u' s.t. $R^{ue}uu'$ and $\mathfrak{ue}\mathfrak{M}, u' \Vdash \Sigma$. Define

$$\Delta = \{V(\phi) \mid \phi \in \Sigma'\} \cup \{Y \mid l_R(Y) \in u\}$$

where Σ' is the set of (finite) conjunctions of formulas in Σ. We claim that the set Δ has the finite intersection property. Since both $\{V(\phi) \mid \phi \in \Sigma'\}$ and $\{Y \mid l_R(Y) \in u\}$ are closed under taking intersections, it suffices to prove that for an arbitrary $\phi \in \Sigma'$ and an arbitrary set $Y \subseteq W$ for which $l_R(Y) \in u$, we have $V(\phi) \cap Y \neq \emptyset$. but if $\phi \in \Sigma'$, then by assumption, there is a successor u'' of u s.t. $\mathfrak{ue}\mathfrak{M}, u'' \Vdash \phi$, or in other words, $V(\phi) \in u''$. Then $l_R(Y) \in u$ implies $Y \in u''$ by Exercise 2.5.2 . Hence $V(\phi) \cap Y$ is an element of the ultrafilter u'' and therefore cannot be identical to the empty set.

It follows by the Ultrafilter Theorem that Δ can be extended to an ultrafilter u'. Clearly u' is the required successor $\square$

Theorem 2.54. *Let τ be a modal similarity type, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models, and w, w' two states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. Then*

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w' \quad \textit{iff} \quad \mathfrak{ue}\mathfrak{M}, \pi_w \Leftrightarrow \mathfrak{ue}\mathfrak{M}', \pi_{w'}$$

Proof. From Propositions 2.52, 2.53 and 2.37 □

Exercise 2.5.1. Let W be an infinite set. Recall that $X \subseteq W$ is **co-finite** if $W \setminus X$ is finite

1. Prove that the collection of co-finite subsets of W has the finite intersection property
2. Show that there are ultrafilters over W that do not contain any finite set
3. Prove that an ultrafilter is non-principal iff it contains only infinite sets iff it contains all co-finite sets
4. Prove that any ultrafilter over W has uncountably many elements

Proof. Suppose $U = \{X \subseteq W \mid X \text{ is cofinite}\}$

1. For any $A, B \in U$, if $A \cap B =$, $A \subset \overline{B}$. But A is infinite and $\overline{B}$ is finite, this can't happen. Hence $A \cap B \neq \emptyset$
2. U can be extended to a ultrafilter $\mathcal{U}$. If A is finite, then $\overline{A} \in U \subseteq \mathcal{U}$. Hence $\mathcal{U}$ does not contain any finite set.
3. $1 \to 2$.If an ultrafilter contains a finite set. Then its a principal ultrafilter generated on the intersection of all finite sets.
   $2 \to 3$ and $3 \to 1$ are obvious.
4. Half of the $\mathcal{P}(W)$ belongs to the ultrafilter and $\mathcal{P}(W)$ is uncountable

□

Exercise 2.5.2. Given a model $\mathfrak{M} = (W, R, V)$ and two ultrafilters u and v over W, show that $R^{ue}uv$iff $\{Y \mid l_R(Y) \in u\} \subseteq v$

Proof.

$$\begin{aligned}
R^{ue}uv &\Leftrightarrow X \in v \to m_R(X) \in u \\
&\Leftrightarrow \neg m_R(X) \in u \to \neg X \in v \\
&\Leftrightarrow W - m_R(X) \in u \to W - X \in v \\
&\Leftrightarrow l_R(W - X) \in u \to W - X \in v \\
&\text{(Since } m_R(X) = W - l_R(W - X)) \\
&\Leftrightarrow \{Y \mid l_R(Y) \in u\} \subseteq v
\end{aligned}$$

□

2.6 Characterization and Definability

2.6.1 The van Benthem Characterization Theorem

Let $\Gamma(x)$ be a set of first-order formulas in which a single individual variable may occur free - such a set of formulas is called a **type**. A first-order model $\mathfrak{M}$ **realizes** $\Gamma(x)$ if there is an element w in $\mathfrak{M}$ s.t. for all $\gamma \in \Gamma, \mathfrak{M} \models \gamma[w]$

Let $\mathfrak{M}$ be a model for a given first-order language $\mathcal{L}^1$ with domain W. For a subset $A \subset W$, $\mathcal{L}^1[A]$ is the language obtained by extending $\mathcal{L}^1$ with new constant $\underline{a}$ for all elements $a \in A$. $\mathfrak{M}_A$ is the expansion of $\mathfrak{M}$ to a structure for $\mathcal{L}^1[A]$ in which each $\underline{a}$ is interpreted as a

Assume that A is of size at most α. Assume that $\alpha = 3$ and $A = \{\alpha_1, \alpha_2\}$. Let $\Gamma(\underline{a}_1, \underline{a}_2, x)$ be a type of the language $\mathcal{L}^1[A]$; $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is consistent with the first-order theory of $\mathfrak{M}_A$ iff $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is finitely realizable in $\mathfrak{M}_A$. So for this particular set $\Gamma(\underline{a}_1, \underline{a}_2, x)$, 3-saturation of $\mathfrak{M}$ means that if $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is finitely realizable in $\mathfrak{M}_A$, then $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is realizable in $\mathfrak{M}_A$

Or consider a formula $\gamma(\underline{a}_1, \underline{a}_2, x)$ and let $\gamma(x_1, x_2, x)$ be the formula with the fresh variables x_1 and x_2 replacing each occurrence in γ of $\underline{a}_1$ and $\underline{a}_2$ respectively. Then we have the following equivalence

$$\mathfrak{M}_A \text{ realizes } \{\gamma(\underline{a}_1, \underline{a}_2, x)\} \text{ iff there is a } b \text{ s.t. } \mathfrak{M} \models \gamma(x_1, x_2, x)[a_1, a_2, b]$$

So a model is α-saturated iff the following holds for every $n < \alpha$ and every set Γ of formulas of the form $\gamma(x_1, \dots, x_n, x)$

If $(a_1, \dots, a_n)$ is an n-tuple s.t. for every finite $\Delta \subseteq \Gamma$ there is a b_Δ s.t. $\mathfrak{M} \models \gamma(x_1, \dots, x_n, x)[a_1, \dots, a_n, b_\Delta]$ for every $\gamma \in \Delta$
then we have that there is a b s.t. $\mathfrak{M} \models \gamma(x_1, \dots, x_n, x)[a_1, \dots, a_n, b]$ for every $\gamma \in \Gamma$

Definition 2.55. Let α be a natural number, or ω. A model $\mathfrak{M}$ is **α-saturated** if for every subset $A \subseteq W$ of size less than α, the expansion $\mathfrak{M}_A$ realizes every set $\Gamma(x)$ of $\mathcal{L}^1[A]$-formulas (with only x occurring free) that is consistent with the first-order theory of $\mathfrak{M}_A$. An ω-saturated model is usually called **countably saturated**

Example 2.6. 1. Every finite model is countably saturated. For if $\mathfrak{M}$ is finite, and $\Gamma(x)$ is a set of first-order formulas consistent with the first-order theory of $\mathfrak{M}$, there exists a model $\mathfrak{N}$ that is elementarily equivalent to $\mathfrak{M}$ and that realizes $\Gamma(x)$. But as $\mathfrak{M}$ and $\mathfrak{N}$ are finite, elementary equivalence implies isomorphism (proof) , and hence $\Gamma(x)$ is realized in $\mathfrak{M}$

2.6.2 Ultraproducts

2.6.3 Definability