Model Theory

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1 Models Constructed From Constants

1.1 Completeness and Compactness

Definition 1.1. Let T be a set of sentences of $\mathcal{L}$ and let C be a set of constant symbols of $\mathcal{L}$. We say that C is a **set of witnesses** for T iff for every formula φ of $\mathcal{L}$ with at most one free variable, say ,x, there is a constant $c \in C$ s.t.

$$T \vdash (\exists x)\varphi \to \varphi(c)$$

We say that T **has witnesses** in $\mathcal{L}$ iff T has some set C of witness in $\mathcal{L}$

Lemma 1.2. *Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$, and let $\overline{\mathcal{L}} = \mathcal{L} \cup C$ be the simple extension of $\mathcal{L}$ formed by adding C. Then T can be extended to a consistent set of sentences $\overline{T}$ in $\overline{\mathcal{L}}$ which has C as a set of witnesses in $\overline{\mathcal{L}}$*

Proof. Let $\alpha = \|\mathcal{L}\|$. For each $\beta < \alpha$, let c_β be a constant symbol which does not occur in $\mathcal{L}$ and s.t. $\beta \neq c_\gamma$ if $\beta < \gamma < \alpha$. Let $C = \{c_\beta : \beta < \alpha\}$, $\overline{\mathcal{L}} = \mathcal{L} \cup C$. Clearly$\left\|\overline{\mathcal{L}}\right\| = \alpha$, so we may arrange all formulas of $\overline{\mathcal{L}}$ with at most one free variable in a sequence $\varphi_\xi, \xi < \alpha$. We now define an increasing sequence of sets of sentences of $\overline{\mathcal{L}}$:

$$T = T_0 \subset T_1 \subset \dots \subset T_\xi \subset \dots, \quad \xi < \alpha$$

and a sequence $d_\xi, \xi < \alpha$ of constants from C s.t.

1. each T_ξ is consistent in $\overline{\mathcal{L}}$
2. if $\xi = \xi + 1$, then $T_\xi = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$; ξ_ζ is the free variable in φ_ζ if it has one, otherwise $x_\xi = v_0$

3.if ξ is a limit ordinal different from 0, then $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$

Let d_ζ be the first element of C which has not yet occurred in T_ζ. We show that

$$T_{\zeta+1} = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$$

is consistent. If this were not the case, then

$$T_\zeta \vdash \neg((\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta))$$

By propositional logic

$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg\varphi_\zeta(d_\zeta)$$

As d_ζ does not occur in T_ζ, we have by predicate logic

$$T_\zeta \vdash (\forall x_\zeta)((\exists x_\zeta)\varphi_\zeta \wedge \neg\varphi_\zeta(x_\zeta))$$
$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg(\exists x_\zeta)\varphi_\zeta$$

which contradicts the consistency of T_ζ. If ξ is a nonzero limit ordinal, and each member of the increasing chain $T_\zeta, \zeta < \xi$ is consistent, then T_ξ is consistent.

Now let $\overline{T} = \bigcup_{\xi<\alpha} T_\xi$. Suppose φ is a formula of $\overline{\mathcal{L}}$ with at most the variable x free. Then we may assume that $\varphi = \varphi_x i$ and $x = x_\xi$ for some $\xi < \alpha$. Whence the sentence

$$(\exists x_\xi)\varphi_x i \to \varphi_\xi(d_\xi)$$

belongs to $T_{\xi+1}$ and so to $\overline{T}$ □

Lemma 1.3. *Let T be a consistent set of sentences and C be a set of witnesses for T in $\mathcal{L}$. Then T has a model $\mathfrak{A}$ s.t. every element of $\mathfrak{A}$ is an interpretation of a constant $c \in C$*

Proof. If a set of sentences T has a set C of witnesses in $\mathcal{L}$, then C is also a set of witnesses for every extension of T. Second, if an extension of T has a model $\mathfrak{A}$, then fA is also a model of T. So we may assume that T is maximal consistent in $\mathcal{L}$

For two constants $c, d \in C$, define

$$c \sim d \quad \text{iff} \quad c \equiv d \in T$$

Because T is maximal consistent, we see that $\sim$ is an equivalence relation on C. For each $c \in C$, let

$$\tilde{c} = \{d \in C : d \sim c\}$$

be the equivalence class of c. We propose to construct a model $\mathfrak{A}$ whose set of elements A is the set of all these equivalence classes $\tilde{c}$, for $c \in C$; so we define

1. $A = \{\tilde{c} : c \in C\}$
2. For each n-placed relation symbol P in $\mathcal{L}$, we define an n-placed relation R' on the set C by: for all $c_1, \dots, c_n \in C$
   $R'(c_1, \dots, c_n)$ iff $P(c_1, \dots, c_n) \in T$
   By our axioms of identity, we have

$$\vdash P(c_1, \dots, c_n) \wedge c_1 \equiv d_1 \wedge \dots \wedge c_n \equiv d_n \rightarrow P(d_1, \dots, d_n)$$

So $\sim$ is what is called a **congruence relation**. $R(\tilde{c}_1, \dots, \tilde{c}_n)$ iff $P(c_1, \dots, c_n) \in T$

3. Now consider a constant symbol d of $\mathcal{L}$. From predicate logic, we have

$$\vdash (\exists v_0)(d \equiv v_0)$$

So $(\exists v_0)(d \equiv v_0) \in T$, and because T has witnesses, there is a constant $c \in C$ s.t.

$$(d \equiv c) \in T$$

the constant c may not be unique, but its equivalence class is unique because

$$\vdash (d \equiv c \wedge d \equiv c' \rightarrow c \equiv c')$$

4. Let F be any m-placed function symbol of $\mathcal{L}$, and let $c_1, \dots, c_m \in C$. We have

$$(\exists v_0)(F(c_1, \dots, c_m) \equiv v_0) \in T$$

hence there is a constant $c \in C$ s.t.

$$(F(c_1, \dots, c_m) \equiv c) \in T$$

We use our axioms of identity to obtain

$$\vdash (F(c_1 \dots c_m) \equiv c \wedge c_1 \equiv d_1 \wedge \dots \wedge c_m \equiv d_m \wedge c \equiv d) \rightarrow F(d_1 \dots d_m) \equiv d$$

Hence we define
$G(\tilde{c}_1 \dots \tilde{c}_m)$ iff $(F(c_1 \dots c_m) \equiv c) \in T$
By induction

$$\mathfrak{A} \vDash t \equiv c \quad \text{iff} \quad (t \equiv c) \in T$$

Since C is a set of witness for T, we have: for any terms t_1, t_2 of $\mathcal{L}$ with no free variables

$$\mathfrak{A} \vDash t_1 \equiv t_2 \quad \text{iff} \quad (t_1 \equiv t_2) \in T$$

for any atomic formula $P(t_1 \dots t_n)$ of $\mathcal{L}$ containing no free variables

$$\mathfrak{A} \vDash P(t_1 \dots t_n) \quad \text{iff} \quad P(t_1 \dots t_n) \in T$$

Hence for any sentence φ of $\mathcal{L}$

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \varphi \in T$$

Suppose $\varphi = (\exists x)\psi$. If $fA \vDash \varphi$, then for some $\tilde{c} \in A, \mathfrak{A} \vDash \psi[\tilde{c}]$. This means that $\mathfrak{A} \vDash \psi(c)$. So $\psi(c) \in T$ and because

$$\vdash \psi(c) \rightarrow (\exists x)\psi$$

we have $\varphi \in T$. On the other hand, if $\varphi \in T$, then because T has witnesses, there exists a constant $c \in C$ s.t. $\psi(c) \in T$, so $\mathfrak{A} \vDash \psi(c)$. This gives $\mathfrak{A} \vDash \psi[\tilde{c}]$ and $\mathfrak{A} \vDash \varphi$

□

Lemma 1.4. *Let C be a set of constant symbols of $\mathcal{L}$, and let T be a set of sentences of $\mathcal{L}$. If T has a model $\mathfrak{A}$ s.t. every element of $\mathfrak{A}$ is an interpretation of some constant $c \in C$, then T can be extended to a consistent $\overline{T}$ in $\mathcal{L}$ for which C is a set of witnesses*

Proof. Let $\overline{T}$ be the sentences of $\mathcal{L}$ true in $\mathfrak{A}$ □

Theorem 1.5 (Extended Completeness Theorem)**.** *Let Σ be a set of sentences of $\mathcal{L}$. Then Σ is consistent iff Σ has a model*

Proof. Assume Σ is consistent. By Lemma 1.2 we consider extensions $\overline{\Sigma}$ of Σ and $\overline{\mathcal{L}}$ of $\mathcal{L}$, so that $\overline{\Sigma}$ has witnesses in $\overline{\mathcal{L}}$. By Lemma 1.3 let $\mathfrak{A}$ be the model of $\overline{\Sigma}$. Let $\mathfrak{B}$ be the model for $\mathcal{L}$ which is the reduct of $\mathfrak{A}$ to $\mathcal{L}$. □

Corollary 1.6 (Downward Löwenheim–Skolem Theorem)**.** *Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$*

Proof. Choose $\mathfrak{A}$ so that every element is a constant.
$|B| = |A| \leq \left\|\overline{\mathcal{L}}\right\| = \|\mathcal{L}\|$ □

Theorem 1.7 (Gödel's Completeness Theorem)**.** *A sentence of $\mathcal{L}$ is a theorem of $\mathcal{L}$ iff it is valid*

Proof. If a sentence σ is not a theorem of $\mathcal{L}$, then $\{\neg\sigma\}$ is consistent in $\mathcal{L}$. By Theorem 1.5, $\{\neg\sigma\}$ will have a model where σ cannot hold. Hence σ is not valid □

Theorem 1.8 (Compactness Theorem)**.** *A set of sentences Σ has a model iff every finite subset of Σ has a model*

Proof. If every finite subset of Σ has a model, then every finite subset of Σ is consistent. So Σ is consistent and has a model by Theorem 1.5 □

Corollary 1.9. *If a theory T has arbitrarily large finite models, then it has an infinite model*

Proof. Consider the expansion $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \omega\}$ where c_n is a list of distinct constant symbols not in $\mathcal{L}$. Consider the set Σ of $\mathcal{L}'$ defined by

$$\Sigma = T \cup \{\neg(c_n \equiv c_m) : n < m < \omega\}$$

Any finite subset Σ' of Σ will involve at most the constants $c_0, \dots, c_m$ for some m. Let $\mathfrak{A}$ be a model of T with at least $m+1$ elements, and let $a_0, \dots, a_m$ be a list of $m+1$ distinct elements of $\mathfrak{A}$. The model $(\mathfrak{A}, a_0, \dots, a_m)$ for the finite expansion $\mathcal{L}'' = \mathcal{L} \cup \{c_0, \dots, c_m\}$ of $\mathcal{L}$ is a model of (Σ'). So by Theorem 1.8 Σ has a model. $\square$

Corollary 1.10 (Upward Löwenheim–Skolem-Tarski Theorem)**.** *If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$*

Method of diagrams. Let $\mathfrak{A}$ be a model of $\mathcal{L}$. We expand the language $\mathcal{L}$ to a new language

$$\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$$

by If $a \neq b$ and c_a, c_b are different symbols, we may then expand $\mathfrak{A}$ to the model

$$\mathfrak{A}_A = (\mathfrak{A}, a)_{a \in A}$$

The **diagram of** $\mathfrak{A}$, denote by $\triangle_{\mathfrak{A}}$, is the set of all atomic sentences and negations of atomic sentences of $\mathcal{L}_A$ which hold in the model $\mathfrak{A}_A$

If X is a subset of A, then we let $\mathcal{L}_X = \mathcal{L} \cup \{c_a : a \in X\}$ and $\mathfrak{A}_X = (\mathfrak{A}, a)_{a \in X}$. If f is a mapping from X into the set of elements B of a model $\mathfrak{B}$ for $\mathcal{L}$, then $(\mathfrak{B}, fa)_{a \in X}$ is the expansion of $\mathfrak{B}$ to a model for $\mathcal{L}_X$

Proposition 1.11. *Let $\mathfrak{A}, \mathfrak{B}$ be models for $\mathcal{L}$ and let $f : A \to B$. Then the following are equivalent:*

1. *f is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$*
2. *There is an extension $\mathfrak{C} \supset \mathfrak{A}$ and an isomorphism $g : \mathfrak{C} \cong \mathfrak{B}$ s.t. $g \supset f$*
3. *$(\mathfrak{B}, fa)_{a \in A}$ is a model of the diagram of $\mathfrak{A}$*

Proof. $1 \to 2$. Extend the set A to a set C and extend the function f to a one-to-one function g from C onto B. Then define the relations

$$\mathfrak{C} \vDash R[c_1 \dots c_n] \quad \text{iff} \quad \mathfrak{B} \vDash R[gc_1 \dots gc_n]$$

$1 \leftrightarrow 2$. For each formula $\varphi(x_1 \dots x_n)$ and all $a_1, \dots, a_n \in A$

$$\mathfrak{A} \vDash \varphi[a_1 \dots a_n] \quad \text{iff} \quad \mathfrak{A}_A \vDash \varphi(a_1 \dots a_n)$$

and

$$\mathfrak{B} \vDash \varphi[fa_1 \dots fa_n] \quad \text{iff} \quad (\mathfrak{B}, fa)_{a \in A} \vDash \varphi(a_1 \dots a_n)$$

$\square$

Corollary 1.12. *Suppose that $\mathcal{L}$ has no function or constant symbols. Let T be a theory in $\mathcal{L}$ and $\mathfrak{A}$ be a model for $\mathcal{L}$. Then $\mathfrak{A}$ is isomorphically embedded in some model of T iff every finite submodel of $\mathfrak{A}$ is isomorphically embedded in some model of T*

Proof. Suppose every finite submodel of $\mathfrak{A}$ is isomorphically embedded in some model of T. We show that the set $\Sigma = T \cup \Delta_{\mathfrak{A}}$ is consistent. Every finite subset Σ' of Σ contains at most a finite number of the new constants $\square$