Model Theory

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1 Models Constructed From Constants

1.1 Completeness and Compactness

Definition 1.1. Let T be a set of sentences of $\mathcal{L}$ and let C be a set of constant symbols of $\mathcal{L}$. We say that C is a **set of witnesses** for T iff for every formula φ of $\mathcal{L}$ with at most one free variable, say ,x, there is a constant $c \in C$ s.t.

$$T \vdash (\exists x)\varphi \to \varphi(c)$$

We say that T **has witnesses** in $\mathcal{L}$ iff T has some set C of witness in $\mathcal{L}$

Lemma 1.2. *Let T be a consistent set of sentences of $\mathcal{L}$. Let C be a set of new constant symbols of power $|C| = \|\mathcal{L}\|$, and let $\overline{\mathcal{L}} = \mathcal{L} \cup C$ be the simple extension of $\mathcal{L}$ formed by adding C. Then T can be extended to a consistent set of sentences $\overline{T}$ in $\overline{\mathcal{L}}$ which has C as a set of witnesses in $\overline{\mathcal{L}}$*

Proof. Let $\alpha = \|\mathcal{L}\|$. For each $\beta < \alpha$, let c_β be a constant symbol which does not occur in $\mathcal{L}$ and s.t. $\beta \neq c_\gamma$ if $\beta < \gamma < \alpha$. Let $C = \{c_\beta : \beta < \alpha\}$, $\overline{\mathcal{L}} = \mathcal{L} \cup C$. Clearly $\left\|\overline{\mathcal{L}}\right\| = \alpha$, so we may arrange all formulas of $\overline{\mathcal{L}}$ with at most one free variable in a sequence $\varphi_\xi, \xi < \alpha$. We now define an increasing sequence of sets of sentences of $\overline{\mathcal{L}}$:

$$T = T_0 \subset T_1 \subset \cdots \subset T_\xi \subset \ldots, \quad \xi < \alpha$$

and a sequence $d_\xi, \xi < \alpha$ of constants from C s.t.

1. each T_ξ is consistent in $\overline{\mathcal{L}}$
2. if $\xi = \xi + 1$, then $T_\xi = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$; ξ_ζ is the free variable in φ_ζ if it has one, otherwise $x_\xi = v_0$

3.if ξ is a limit ordinal different from 0, then $T_\xi = \bigcup_{\zeta<\xi} T_\zeta$

Let d_ζ be the first element of C which has not yet occurred in T_ζ. We show that

$$T_{\zeta+1} = T_\zeta \cup \{(\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta)\}$$

is consistent. If this were not the case, then

$$T_\zeta \vdash \neg((\exists x_\zeta)\varphi_\zeta \to \varphi_\zeta(d_\zeta))$$

By propositional logic

$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg\varphi_\zeta(d_\zeta)$$

As d_ζ does not occur in T_ζ, we have by predicate logic

$$T_\zeta \vdash (\forall x_\zeta)((\exists x_\zeta)\varphi_\zeta \wedge \neg\varphi_\zeta(x_\zeta))$$
$$T_\zeta \vdash (\exists x_\zeta)\varphi_\zeta \wedge \neg(\exists x_\zeta)\varphi_\zeta$$

which contradicts the consistency of T_ζ. If ξ is a nonzero limit ordinal, and each member of the increasing chain $T_\zeta, \zeta < \xi$ is consistent, then T_ξ is consistent.

Now let $\overline{T} = \bigcup_{\xi<\alpha} T_\xi$. Suppose φ is a formula of $\overline{\mathcal{L}}$ with at most the variable x free. Then we may assume that $\varphi = \varphi_x i$ and $x = x_\xi$ for some $\xi < \alpha$. Whence the sentence

$$(\exists x_\xi)\varphi_x i \to \varphi_\xi(d_\xi)$$

belongs to $T_{\xi+1}$ and so to $\overline{T}$ $\square$

Lemma 1.3. *Let T be a consistent set of sentences and C be a set of witnesses for T in $\mathcal{L}$. Then T has a model $\mathfrak{A}$ s.t. every element of $\mathfrak{A}$ is an interpretation of a constant $c \in C$*

Proof. If a set of sentences T has a set C of witnesses in $\mathcal{L}$, then C is also a set of witnesses for every extension of T. Second, if an extension of T has a model $\mathfrak{A}$, then fA is also a model of T. So we may assume that T is maximal consistent in $\mathcal{L}$

For two constants $c, d \in C$, define

$$c \sim d \quad \text{iff} \quad c \equiv d \in T$$

Because T is maximal consistent, we see that $\sim$ is an equivalence relation on C. For each $c \in C$, let

$$\tilde{c} = \{d \in C : d \sim c\}$$

be the equivalence class of c. We propose to construct a model $\mathfrak{A}$ whose set of elements A is the set of all these equivalence classes $\tilde{c}$, for $c \in C$; so we define

1. $A = \{\tilde{c} : c \in C\}$
2. For each n-placed relation symbol P in $\mathcal{L}$, we define an n-placed relation R' on the set C by: for all $c_1, \dots, c_n \in C$
   $R'(c_1, \dots, c_n)$ iff $P(c_1, \dots, c_n) \in T$
   By our axioms of identity, we have

$$\vdash P(c_1, \dots, c_n) \wedge c_1 \equiv d_1 \wedge \cdots \wedge c_n \equiv d_n \to P(d_1, \dots, d_n)$$

So $\sim$ is what is called a **congruence relation**. $R(\tilde{c}_1, \dots, \tilde{c}_n)$ iff $P(c_1, \dots, c_n) \in T$

3. Now consider a constant symbol d of $\mathcal{L}$. From predicate logic, we have

$$\vdash (\exists v_0)(d \equiv v_0)$$

So $(\exists v_0)(d \equiv v_0) \in T$, and because T has witnesses, there is a constant $c \in C$ s.t.

$$(d \equiv c) \in T$$

the constant c may not be unique, but its equivalence class is unique because

$$\vdash (d \equiv c \wedge d \equiv c' \to c \equiv c')$$

4. Let F be any m-placed function symbol of $\mathcal{L}$, and let $c_1, \dots, c_m \in C$. We have

$$(\exists v_0)(F(c_1, \dots, c_m) \equiv v_0) \in T$$

hence there is a constant $c \in C$ s.t.

$$(F(c_1, \dots, c_m) \equiv c) \in T$$

We use our axioms of identity to obtain

$$\vdash (F(c_1 \dots c_m) \equiv c \wedge c_1 \equiv d_1 \wedge \cdots \wedge c_m \equiv d_m \wedge c \equiv d) \to F(d_1 \dots d_m) \equiv d$$

Hence we define
$G(\tilde{c}_1 \dots \tilde{c}_m)$ iff $(F(c_1 \dots c_m) \equiv c) \in T$
By induction

$$\mathfrak{A} \vDash t \equiv c \quad \text{iff} \quad (t \equiv c) \in T$$

Since C is a set of witness for T, we have: for any terms t_1, t_2 of $\mathcal{L}$ with no free variables

$$\mathfrak{A} \vDash t_1 \equiv t_2 \quad \text{iff} \quad (t_1 \equiv t_2) \in T$$

for any atomic formula $P(t_1 \dots t_n)$ of $\mathcal{L}$ containing no free variables

$$\mathfrak{A} \vDash P(t_1 \dots t_n) \quad \text{iff} \quad P(t_1 \dots t_n) \in T$$

Hence for any sentence φ of $\mathcal{L}$

$$\mathfrak{A} \vDash \varphi \quad \text{iff} \quad \varphi \in T$$

Suppose $\varphi = (\exists x)\psi$. If $fA \vDash \varphi$, then for some $\tilde{c} \in A, \mathfrak{A} \vDash \psi[\tilde{c}]$. This means that $\mathfrak{A} \vDash \psi(c)$. So $\psi(c) \in T$ and because

$$\vdash \psi(c) \to (\exists x)\psi$$

we have $\varphi \in T$. On the other hand, if $\varphi \in T$, then because T has witnesses, there exists a constant $c \in C$ s.t. $\psi(c) \in T$, so $\mathfrak{A} \vDash \psi(c)$. This gives $\mathfrak{A} \vDash \psi[\tilde{c}]$ and $\mathfrak{A} \vDash \varphi$

□

Lemma 1.4. *Let C be a set of constant symbols of $\mathcal{L}$, and let T be a set of sentences of $\mathcal{L}$. If T has a model $\mathfrak{A}$ s.t. every element of $\mathfrak{A}$ is an interpretation of some constant $c \in C$, then T can be extended to a consistent $\overline{T}$ in $\mathcal{L}$ for which C is a set of witnesses*

Proof. Let $\overline{T}$ be the sentences of $\mathcal{L}$ true in $\mathfrak{A}$ □

Theorem 1.5 (Extended Completeness Theorem). *Let Σ be a set of sentences of $\mathcal{L}$. Then Σ is consistent iff Σ has a model*

Proof. Assume Σ is consistent. By Lemma 1.2 we consider extensions $\bar{\bar{\Sigma}}$ of Σ and $\overline{\mathcal{L}}$ of $\mathcal{L}$, so that $\bar{\bar{\Sigma}}$ has witnesses in $\overline{\mathcal{L}}$. By Lemma 1.3 let $\mathfrak{A}$ be the model of $\bar{\bar{\Sigma}}$. Let $\mathfrak{B}$ be the model for $\mathcal{L}$ which is the reduct of $\mathfrak{A}$ to $\mathcal{L}$. □

Corollary 1.6 (Downward Löwenheim–Skolem Theorem). *Every consistent theory T in $\mathcal{L}$ has a model of power at most $\|\mathcal{L}\|$*

Proof. Choose $\mathfrak{A}$ so that every element is a constant.
$|B| = |A| \leq \left\|\overline{\mathcal{L}}\right\| = \|\mathcal{L}\|$ □

Theorem 1.7 (Gödel's Completeness Theorem). *A sentence of $\mathcal{L}$ is a theorem of $\mathcal{L}$ iff it is valid*

Proof. If a sentence σ is not a theorem of $\mathcal{L}$, then $\{\neg\sigma\}$ is consistent in $\mathcal{L}$. By Theorem 1.5, $\{\neg\sigma\}$ will have a model where σ cannot hold. Hence σ is not valid □

Theorem 1.8 (Compactness Theorem). *A set of sentences Σ has a model iff every finite subset of Σ has a model*

Proof. If every finite subset of Σ has a model, then every finite subset of Σ is consistent. So Σ is consistent and has a model by Theorem 1.5 □

Corollary 1.9. *If a theory T has arbitrarily large finite models, then it has an infinite model*

Proof. Consider the expansion $\mathcal{L}' = \mathcal{L} \cup \{c_n : n \in \omega\}$ where c_n is a list of distinct constant symbols not in $\mathcal{L}$. Consider the set Σ of $\mathcal{L}'$ defined by

$$\Sigma = T \cup \{\neg(c_n \equiv c_m) : n < m < \omega\}$$

Any finite subset Σ' of Σ will involve at most the constants $c_0, \dots, c_m$ for some m. Let $\mathfrak{A}$ be a model of T with at least $m+1$ elements, and let $a_0, \dots, a_m$ be a list of $m+1$ distinct elements of $\mathfrak{A}$. The model $(\mathfrak{A}, a_0, \dots, a_m)$ for the finite expansion $\mathcal{L}'' = \mathcal{L} \cup \{c_0, \dots, c_m\}$ of $\mathcal{L}$ is a model of (Σ'). So by Theorem 1.8 Σ has a model. □

Corollary 1.10 (Upward Löwenheim–Skolem-Tarski Theorem)**.** *If T has infinite models, then it has infinite models of any given power $\alpha \geq \|\mathcal{L}\|$*

Method of diagrams. Let $\mathfrak{A}$ be a model of $\mathcal{L}$. We expand the language $\mathcal{L}$ to a new language

$$\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$$

by If $a \neq b$ and c_a, c_b are different symbols, we may then expand $\mathfrak{A}$ to the model

$$\mathfrak{A}_A = (\mathfrak{A}, a)_{a \in A}$$

The **diagram of** $\mathfrak{A}$, denote by $\varDelta_{\mathfrak{A}}$, is the set of all atomic sentences and negations of atomic sentences of $\mathcal{L}_A$ which hold in the model $\mathfrak{A}_A$

If X is a subset of A, then we let $\mathcal{L}_X = \mathcal{L} \cup \{c_a : a \in X\}$ and $\mathfrak{A}_X = (\mathfrak{A}, a)_{a \in X}$. If f is a mapping from X into the set of elements B of a model $\mathfrak{B}$ for $\mathcal{L}$, then $(\mathfrak{B}, fa)_{a \in X}$ is the expansion of $\mathfrak{B}$ to a model for $\mathcal{L}_X$

Proposition 1.11. *Let $\mathfrak{A}, \mathfrak{B}$ be models for $\mathcal{L}$ and let $f : A \to B$. Then the following are equivalent:*

1. *f is an isomorphic embedding of $\mathfrak{A}$ into $\mathfrak{B}$*
2. *There is an extension $\mathfrak{C} \supset \mathfrak{A}$ and an isomorphism $g : \mathfrak{C} \cong \mathfrak{B}$ s.t. $g \supset f$*
3. *$(\mathfrak{B}, fa)_{a \in A}$ is a model of the diagram of $\mathfrak{A}$*

Proof. $1 \to 2$. Extend the set A to a set C and extend the function f to a one-to-one function g from C onto B. Then define the relations

$$\mathfrak{C} \vDash R[c_1 \dots c_n] \quad \text{iff} \quad \mathfrak{B} \vDash R[gc_1 \dots gc_n]$$

$1 \leftrightarrow 2$. For each formula $\varphi(x_1 \dots x_n)$ and all $a_1, \dots, a_n \in A$

$$\mathfrak{A} \vDash \varphi[a_1 \dots a_n] \quad \text{iff} \quad \mathfrak{A}_A \vDash \varphi(a_1 \dots a_n)$$

and

$$\mathfrak{B} \vDash \varphi[fa_1 \dots fa_n] \quad \text{iff} \quad (\mathfrak{B}, fa)_{a \in A} \vDash \varphi(a_1 \dots a_n)$$

□

Corollary 1.12. *Suppose that $\mathcal{L}$ has no function or constant symbols. Let T be a theory in $\mathcal{L}$ and $\mathfrak{A}$ be a model for $\mathcal{L}$. Then $\mathfrak{A}$ is isomorphically embedded in some model of T iff every finite submodel of $\mathfrak{A}$ is isomorphically embedded in some model of T*

Proof. Suppose every finite submodel of $\mathfrak{A}$ is isomorphically embedded in some model of T. We show that the set $\Sigma = T \cup \Delta_{\mathfrak{A}}$ is consistent. Every finite subset Σ' of Σ contains at most a finite number of the new constants, say $c_{a_1}, \dots, c_{a_m}$. Because the language $\mathcal{L}$ has no function or constant symbols, the finite set $A' = \{a_1, \dots, a_m\}$ generates a finite submodel $\mathfrak{A}'$ of $\mathfrak{A}$. Let $\mathfrak{B}'$ be a model of T where $\mathfrak{A}'$ is isomorphically embedded. Since $\Sigma' \subset \Sigma$, by Proposition 1.11 $\mathfrak{B}'$ can be extended to a model of Σ', and hence Σ' has a model. By campactness, Σ has a model $\mathfrak{B}$. By Proposition 1.11 the reduct of $\mathfrak{B}$ to $\mathcal{L}$ gives a mode lof T □

1.2 Refinements of the method. Omitting types and interpolation theorems

$$\mathfrak{A} \vDash \Sigma[a_1 \dots a_n]$$

for every $\sigma \in \Sigma, a_1, \dots, a_n$satisfies σ in $\mathfrak{A}$; in this case we say that $a_1, \dots, a_n$ **satisfies**, or **realizes** Σ in $\mathfrak{A}$.

$\mathfrak{A}$ **realizes** Σ iff some n-tuple of elements of A satisfies Σ in $\mathfrak{A}$. $\mathfrak{A}$ **omits** Σ iff $\mathfrak{A}$ does not realize Σ. Σ is **satisfiable in** $\mathfrak{A}$ iff $\mathfrak{A}$ realizes Σ. Σ is **consistent** iff its satisfiable

By a **type** $\Gamma(x_1 \dots x_n)$ in the variables $x_1, \dots, x_n$ we mean a maximal consistent set of formulas of $\mathcal{L}$ in these variables. Given any model $\mathfrak{A}$ and n-tuple $a_1, \dots, a_n \in A$, the set $\Gamma(x_1 \dots x_n)$ of all formulas $\gamma(x_1 \dots x_n)$ satisfied by $a_1, \dots, a_n$ is a type and is the unique type realized by $a_1, \dots, a_n$. It is called the **type of** $a_1, \dots, a_n$ in $\mathfrak{A}$

Proposition 1.13. *Let T be a theory and let $\Sigma = \Sigma(x_1 \dots x_n)$. The following are equivalent*

1. *T has a model which realizes Σ*
2. *Every finite subset of Σ is realized in some model of T*
3. *$T \cup \{(\exists x_1 \dots x_n)(\sigma_1 \wedge \dots \wedge \sigma_m) : m < \omega, \sigma_1, \dots, \sigma_m \in \Sigma\}$ is consistent*

Let $\Sigma = \Sigma(x_1 \dots x_n)$ be a set of formulas of $\mathcal{L}$. A theory T in $\mathcal{L}$ is said to **locally realize** Σ iff there is a formula $\varphi(x_1 \dots x_n)$ in $\mathcal{L}$ s.t.

1. φ is consistent with T
2. For all $\sigma \in \Sigma, T \vDash \varphi \to \sigma$

That is, every n-tuple in a model of T which satisfies φ realizes Σ

T **locally omits** Σ iff T does not locally realize Σ. Thus T locally omits Σ iff for every formula $\varphi(x_1 \dots x_n)$ which is consistent with T, there exists $\sigma \in \Sigma$ s.t. $\varphi \wedge \neg\sigma$ is consistent with T

Proposition 1.14. *Let T be a complete theory in $\mathcal{L}$, and let $\Sigma = \Sigma(x_1 \dots x_n)$ be a set of formulas of $\mathcal{L}$. If T has a model which omits Σ, then T locally omits Σ*

Proof. If T locally realizes Σ, then every model of T realizes Σ □

Theorem 1.15 (Omitting Types Theorem)**.** *Let T be a consistent theory in a countable language $\mathcal{L}$, and let $\Sigma(x_1 \dots x_n)$ be a set of formulas. If T locally omits Σ, then T has a countable model which omits Σ*

Proof. Suppose T locally omits $\Sigma(x)$. Let $C = \{c_0, c_1, \dots\}$ be a countable set of new constant symbols not already in $\mathcal{L}$ and let $\mathcal{L}' = \mathcal{L} \cup C$. Then $\mathcal{L}'$ is countable. Arrange all the sentences of $\mathcal{L}'$ in a list $\varphi_0, \varphi_1, \dots$. We shall construct an increasing sequence of consistent theories

$$T = T_0 \subset T_0 \subset \dots \subset T_m \subset \dots$$

s.t.

1. Each T_m is a consistent theory of $\mathcal{L}'$ which is a finite extension of T
2. Either $\varphi_m \in T_{m+1}$ or $(\neg\varphi_m) \in T_{m+1}$
3. If $\varphi_m = (\exists x)\psi(x)$ and $\varphi_m \in T_{m+1}$, then $\psi(c_p) \in T_{m+1}$ where c_p is the first constant not occuring in T_m or φ_m
4. There is a formula $\sigma(x) \in \Sigma(x)$ s.t. $(\neg\sigma(c_m)) \in T_{m+1}$

Assuming we already have the theory T_m, we construct T_{m+1} as follows: Let $T_m = T \cup \{\theta_1, \dots, \theta_r\}, r > 0$ and let $\theta = \theta_1 \wedge \dots \wedge \theta_r$. Let $c_0, \dots, c_n$ contain all the constants from C occuring in θ. For the formula $\theta(x_m)$ of $\mathcal{L}$ by replacing each constant c_i by x_i (renaming bound variables if necessary) and prefixing by $\exists x_i, i \not\equiv m$. Then $\theta(x_m)$ is consistent with T. Therefore for some $\sigma(x) \in \Sigma(x)$, $\theta(x_m) \wedge \neg\sigma(x_m)$ is consistent with T. Put the sentence $\neg\sigma(c_m)$ into T_{m+1}. This makes (4) hold

If φ_m is consistent with $T_m \cup \{\neg\sigma(c_m)\}$, put φ_m into T_{m+1}. Otherwise put $(\neg\varphi_m)$ into T_{m+1}. This take care of (2). If $\varphi_m = (\exists x)\psi(x)$ is consistent with $T_m \cup \{\neg(\sigma(c_m))\}$, put $\psi(c_p)$ into T_{m+1}. This take care of (3). The theory T_{m+1} is a consistent finite extension of T_m. Thus (1) - (4) hold for T_{m+1}

Let $T_\omega = \bigcup_{n<\omega} T_n$. From (1) and (2) we see that T_ω is a maximal consistent theory in $\mathcal{L}'$. Let $\mathfrak{B}' = (\mathfrak{B}, b_0, b_1, \dots)$ be a countable model of T_ω, and let $\mathfrak{A}' = (\mathfrak{A}, b_0, b_1, \dots)$ be the submodel of $\mathfrak{B}'$ generated by the constants $b_0, b_1, \dots$ We then see from (3) that

$$A = \{b_0, b_1, \dots\}$$

Moreover, using (3) and the completeness of T_ω, we can show by induction on the complexity of a sentence φ in $\mathcal{L}'$ that

$$\mathfrak{A}' \vDash \varphi, \quad \mathfrak{B}' \vDash \varphi, \quad T_\omega \vDash \varphi$$

are all equivalent. Thus $\mathfrak{A}'$ is a model of T_ω and hence $\mathfrak{A}$ is a model of T. Finally condition (4) ensures that $\mathfrak{A}$ omits Σ □

Corollary 1.16. *Let $\mathcal{L}$ be countable. A theory T has a (countable) model omitting $\Sigma(x_1 \dots x_n)$ iff some complete extension of T locally omits $\Sigma(x_1 \dots x_n)$*

Example 1.1. Consider the language $\mathcal{L} = \{+, \cdot, S, 0\}$. We abbreviate $1 = S0, 2 = SS0, 3 = SSS0, \dots$. By an ω-**model** we mean a model $\mathfrak{A}$ in which

$$A = \{0, 1, 2, 3, \dots\}$$

that is, $\mathfrak{A}$ omits the set $\{x \not\equiv 0, x \not\equiv 1, \dots\}$. A theory T in $\mathcal{L}$ is said to be ω-**consistent** iff there is no formula $\varphi(x)$ of $\mathcal{L}$ s.t.

$$T \vDash \varphi(0), \quad T \vDash \varphi(1), \quad T \vDash \varphi(2), \dots$$

and

$$T \vDash (\exists x)\neg\varphi(x)$$

T is said to be ω-**complete** iff for every formula $\varphi(x)$ of $\mathcal{L}$ we have

$$T \vDash \varphi(0), T \vDash \varphi(1), T \vDash \varphi(2), \dots \text{ implies } T \vDash (\forall x)\varphi(x)$$

If follows from the omitting types theorem that

Proposition 1.17. *Let T be a consistent theory in $\mathcal{L}$*

1. *If T is ω-complete, then T has an ω-model*
2. *If T has an ω-model, then T is ω-consistent*

Proof. 1. We show that T locally omits the set $\Sigma(x) = \{x \not\equiv 0, x \not\equiv 1, \dots\}$. Suppose $\theta(x)$ is consistent with T. Then $T \vDash (\forall x)\neg\theta(x)$ fails. By ω-completeness, there is a n s.t. not $T \vDash \neg\theta(n)$. Hence $\theta(n)$ is consistent with T, so $\theta(x) \wedge \neg x \not\equiv n$ is consistent with T. Thus T locally omits $\Sigma(x)$

□

The ω**-rule** is the following infinite rule of proof: From $\varphi(0), \varphi(1), ...$ infer $(\forall x)\varphi(x)$, where $\varphi(x)$ is any formula of $\mathcal{L}$. ω**-logic** is formed by adding the ω-rule to the axioms and rules of inference of the first-order logic $\mathcal{L}$ and allowing infinitely long proofs. We have the following completeness theorem for ω-logic

Proposition 1.18 (ω-Completeness Theorem)**.** *A theory T in $\mathcal{L}$ is consistent in ω-logic iff T has an ω-model*

Proof. Let T' be the set of all sentences of $\mathcal{L}$ provable from T in ω-logic. Then T is consistent in ω-logic iff T' is consistent in $\mathcal{L}$. Moreover, T' is ω-complete. Therefore T' has an ω-model iff T' is consistent $\square$

Example 1.2 (Continue)**.** Let $\mathcal{L}'$ be a countable language which has among its symbols a special unary relation symbol N and special constant symbols $0, 1, 2, ...$ By an ω**-model** for $\mathcal{L}'$ we mean a model $\mathfrak{A}$ for $\mathcal{L}'$ in which N is interpreted by the set ω of natural numbers, and $0, 1, 2, ...$ are interpreted by themselves. In an ω-model, ω is a subset of the universe A, but we allow A to contain elements outside of ω or even to be uncountable

Let T_N be the special set of sentences

$$T_N = \{N(m) : m < \omega\} \cup \{\neg m \equiv n : m < n < \omega\}$$

which state that the natural numbers are distinct and belong to N. T_N holds in every ω-model for $\mathcal{L}'$. A theory T in $\mathcal{L}'$ is said to be ω**-consistent** iff there is no formula $\varphi(x)$ of $\mathcal{L}'$ s.t.

$$T_N \cup T \vDash \varphi(0), T_N \cup T \vDash \varphi(1), T_N \cup T \vDash \varphi(2), ...$$

and

$$T_N \cup T \vDash (\exists x)(N(x) \wedge \neg\varphi(x))$$

T is said to be ω**-complete** iff for every formula $\varphi(x)$ of $\mathcal{L}'$ we have

$$T_N \cup T \vDash \varphi(0), T_N \cup T \vDash \varphi(1), T_N \cup T \vDash \varphi(2), ...$$

implies

$$T_N \cup T \vDash (\forall x)(N(x) \to \varphi(x))$$

The ω**-rule** for $\mathcal{L}'$ is the infinite rule: From $\varphi(0), \varphi(1), \varphi(2), ...$ infer $(\forall x)(N(x) \to \varphi(x))$. By **generalized** ω**-logic** we mean first order logic for the language $\mathcal{L}'$ with T_N added as an additional set of logical axioms and the ω-rule added as an additional rule of proof

Proposition 1.19. *Let T be a theory in $\mathcal{L}'$ s.t. $T_N \cup T$ is consistent*

1. *If T is ω-complete, then T has an ω-model*
2. *If T has an ω-model, then T is ω-consistent*

Proposition 1.20. *A theory T in $\mathcal{L}'$ is consistent in generalized ω-logic iff T has an ω-model*

Theorem 1.21 (Extended Omitting Types Theorem)**.** *Let T be a consistent theory in a countable language $\mathcal{L}$, and for each $r < \omega$ let $\Sigma_r(x_1, \dots, x_{n_r})$ be a set of formulas in n_r variables. If T locally omits each Σ_r, then T has a countable model which omits each Σ_r*

Let's consider the theory ZF, Zermelo-Fraenkel set theory. A model $\mathfrak{B} = \langle B, F \rangle$ of ZF is said to be an **end extension** of a model $\mathfrak{A} = \langle A, E \rangle$ of ZF iff $\mathfrak{B}$ is a proper extension of $\mathfrak{A}$ and no member of A gets a new element, that is

$$\text{if} \quad a \in A \quad \text{and} \quad b \in B, \quad \text{then} \quad bFa \quad \text{implies} \quad b \in A$$

Theorem 1.22. *Every countable model $\mathfrak{A} = \langle A, E \rangle$ of ZF has an end elementary extension*

Proof. Let $\mathcal{L}$ be the language with the symbol $\in$, a constant symbol $\bar{a}$ for each $a \in A$, and a new constant symbol c. Let T be the theory with the axioms

$$\begin{gathered} \mathrm{Th}((\mathfrak{A}, a)_{a \in A}) \\ c \notin \bar{a}, \quad \text{where } a \in A \end{gathered}$$

T is consistent because every finite subset of T has a model of the form $(\mathfrak{A}, a, c)_{a \in A}$. For each $a \in A$, let $\Sigma_a(x)$ be the set of formulas

$$\Sigma_a(x) = \{x \in \bar{a}\} \cup \{x \not\equiv \bar{b} : bEa\}$$

It suffices to show that T locally omits each set $\Sigma_a(x)$. For then T has a model $(\mathfrak{B}, a, c)_{a \in A}$ which omits each $\Sigma_a(x)$. We may also assume that $A \subset B$. $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$ because $\mathrm{Th}((\mathfrak{A}, a)_{a \in A}) \subset T$, whence $(\mathfrak{A}, a)_{a \in A} \equiv (\mathfrak{B}, a)_{a \in A}$. $\mathfrak{B}$ is a proper extension because $c \in B \setminus A$. Finally $\mathfrak{B}$ is an end extension because it omits each $\Sigma_a(x)$

A formula $\varphi(x, c)$ of $\mathcal{L}$ is consistent with T iff

$$(\mathfrak{A}, a)_{a \in A} \vDash (\forall y)(\exists z)(\exists x)[z \notin y \wedge \varphi(x, z)]$$

Suppose $\varphi(x,c)$ is consistent with T, but $\varphi(x,c) \wedge \neg x \in \bar{a}$ is not. Then $\varphi(x,c) \wedge x \in \bar{a}$ is consistent with T. Using the axiom of replacement in ZF, we see in turn that the following sentences hold in $(\mathfrak{A}, a)_{a \in A}$

$$(\forall y)(\exists z)(\exists x)[z \notin y \wedge \varphi(x,z) \wedge x \in \bar{a}]$$
$$(\exists x)(\forall y)(\exists z)[z \notin y \wedge \varphi(x,z) \wedge x \in \bar{a}]$$

Then for some $b \in A$, $\varphi(\bar{b},c) \wedge \bar{b} \in \bar{a}$ is consistent with T, whence $\varphi(x,c) \wedge x \equiv \bar{b}$ is consistent with T. Thus T locally omits $\Sigma_a(x)$ $\square$

The omitting types theorem is false for uncountable languages. For example, let T be the theory with the axioms

$$c_\alpha \neq c_\beta, \alpha < \beta < \omega_1$$

in the language $\mathcal{L}$ with constants

$$\{c_\alpha : \alpha < \omega_1\} \cup \{d_n : n < \omega\}$$

Let $\Gamma(x)$ be the set of formulas

$$\Gamma(x) = \{x \not\equiv d_n : n < \omega\}$$

Then T locally omits $\Gamma(x)$. However no model of T omits $\Gamma(x)$ because every model of T is uncountable but each model which omits $\Gamma(x)$ is countable

Let T be a theory and $\Sigma(x_1 \dots x_n)$ a set of formulas in a language $\mathcal{L}$ of power α. We say that T **α-realizes** Σ iff there is a set $\Phi(x_1 \dots x_n)$ of fewer than α formulas of $\mathcal{L}$ s.t.

1. Φ is consistent with T
2. $T \cup \Phi(x_1 \dots x_n) \vDash \Sigma(x_1 \dots x_n)$

that is, in any model $\mathfrak{A}$ of T, any n-tuple which realizes Φ realizes Σ. Note that if Σ has power less than α, then T α-realizes Σ trivially.

Theorem 1.23 (α-Omitting Types Theorem)**.** *Let T be a consistent theory in a language $\mathcal{L}$ of power α and let $\Sigma(x_1 \dots x_n)$ be a set of formulas of $\mathcal{L}$. If T α-omits Σ, then T has a model of power $\leq \alpha$ which omits Σ*

Theorem 1.24 (Craig Interpolation Theorem)**.** *Let φ,ψ be sentences s.t. $\varphi \vDash \psi$. Then there exists a sentence θ s.t.*

1. *$\varphi \vDash \theta$ and $\theta \vDash \psi$*
2. *Every relation, function or constant symbol (excluding identity) which occurs in θ also occurs in both φ and ψ*

The sentence θ will be called a **Craig interpolation** of φ, ψ.

Example 1.3. In each of the following, φ and ψ are sentences s.t. the identity symbol occurs in at most one of them, and $\varphi \vDash \psi$; however, φ, ψ have no Craig interpolation in which the identity symbol does not occur

1. $\varphi = (\exists x)(P(x) \wedge \neg P(x))$,$\psi = (\exists x)Q(x)$
2. $\varphi = (\exists x)Q(x)$,$\psi = (\exists x)(P(x) \vee \neg P(x))$
3. $\varphi = (\forall xy)(x \equiv y)$,$\psi = (\forall xy)(P(x) \leftrightarrow P(y))$

Proof. We assume that there is no Craig interpolant θ of φ and ψ, and prove that it is note the case that $\varphi \vDash \psi$. To do this we construct a model of $\varphi \wedge \neg\psi$. We may assume without loss of generality that $\mathcal{L}$ is the language of all symbols which occur in either φ or ψ or both. Let $\mathcal{L}_1$ be the language of all symbols of φ, $\mathcal{L}_2$ be the language of all symbols of ψ, and $\mathcal{L}_0$ the language of all symbols occurring in both φ and ψ. Thus

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}_0, \quad \mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$$

Form an expansion $\mathcal{L}'$ of $\mathcal{L}$ by adding a countable set C of new constant symbols and let

$$\mathcal{L}_0' = \mathcal{L}_0 \cup C, \quad \mathcal{L}_1' = \mathcal{L}_1 \cup C, \quad \mathcal{L}_2' = \mathcal{L}_2 \cup C$$

Consider a pair of theories T in $\mathcal{L}_1'$ and U in $\mathcal{L}_2'$. A sentence θ of $\mathcal{L}_0'$ is said to **separate** T and U iff

$$T \vDash \theta \quad \text{and} \quad U \vDash \neg\theta$$

T and U are said to be **inseparable** iff no sentence θ of $\mathcal{L}_0'$ separates them.

1. $\{\varphi\}$ and $\{\neg\psi\}$ are inseparable

For if $\theta(c_1, \dots, c_n)$ separates $\{\varphi\}$ and $\{\neg\psi\}$ and $u_1, \dots, u_n$ are variables not occurring in $\theta(c_1, \dots, c_n)$, then $(\forall u_1 \dots u_n)\theta(u_1 \dots u_n)$ is a Craig interpolant of φ and ψ, contrary to our assumption

Now let

$$\varphi_0, \varphi_1, \dots, \psi_0, \psi_1, \dots$$

be enumerations of all sentences of $\mathcal{L}_1'$ and of $\mathcal{L}_2'$ respectively. We shall construct two increasing sequences of theories,

$$\{\varphi\} = T_0 \subset T_1 \subset \dots,$$
$$\{\neg\psi\} = U_0 \subset U_1 \subset \dots$$

in $\mathcal{L}_1'$ and $\mathcal{L}_2'$, respectively, s.t.

2. T_m and U_m are inseparable finite sets of sentences
3. If $U_m \cup \{\varphi_m\}$ and U_m are inseparable, then $\varphi_m \in T_{m+1}$
   If T_{m+1} and $U_m \cup \{\psi_m\}$ are inseparable, then $\psi_m \in U_{m+1}$
4. If $\varphi_m = (\exists x)\sigma(x)$ and $\varphi_m \in T_{m+1}$, then $\sigma(c) \in T_{m+1}$ for some $c \in C$
   If $\psi(m) = (\exists x)\delta(x)$ and $\psi_m \in U_{m+1}$, then $\delta(d) \in U_{m+1}$ for some $d \in C$

Given T_m and U_m, the theories T_{m+1} and U_{m+1} are constructed in the obvious way. For (4), use constants c and d which do not occur in $T_m, U_m, \varphi_m, \psi_m$. Then inseparability will be preserved. Let

$$T_\omega = \bigcup_{m<\omega} T_m, \quad U_\omega = \bigcup_{m<\omega} U_m$$

Then T_ω and U_ω are inseparable. It follows that T_ω and U_ω are each consistent. We must show that $T_\omega \cup U_\omega$ is consistent. We show first that

5. T_ω is a maximal consistent theory in $\mathcal{L}_1'$ and U_ω is a maximal consistent theory in $\mathcal{L}_2'$.

Suppose $\varphi_m \notin T_\omega$ and $(\neg\varphi_m) \notin T_\omega$. Since $T_m \cup \{\varphi_m\}$ is separable from U_m, there exists $\theta \in \mathcal{L}_0'$ s.t.

$$T_\omega \models \varphi_m \to \theta, \quad U_\omega \models \neg\theta$$

Also there exists $\theta' \in \mathcal{L}_0'$ s.t.

$$T_\omega \models \neg\varphi_m \to \theta', \quad U_\omega \models \neg\theta'$$

But then

$$T_\omega \models \theta \vee \theta', \quad U_\omega \models \neg(\theta \vee \theta')$$

contradicting the inseparability of T_ω and U_ω.

6. $T_\omega \cap U_\omega$ is a maximal consistent theory in $\mathcal{L}_0'$

Let σ be a sentence of $\mathcal{L}_0'$. By (5), either $\sigma \in T_\omega$ or $(\neg\sigma) \in T_\omega$ and either $\sigma \in U_\omega$ or $(\neg\sigma) \in U_\omega$. Therefore either $T_\omega \cap U_\omega \models \sigma$ or $T_\omega \cap U_\omega \models \neg\sigma$

Let $\mathfrak{B}_1' = (\mathfrak{B}_1, b_0, b_1, \dots)$ be a model of T_ω. Using (4) and (5) we see that the submodel $\mathfrak{A}_1' = (\mathfrak{A}_1, b_0, b_1, \dots)$ with universe $A_1 = \{b_0, b_1, \dots\}$is also a model of T_ω. Similarly U_ω has a model $\mathfrak{A}_2' = (\mathfrak{A}_2, d_0, d_1, \dots)$ with universe $A_2 = \{d_0, d_1, \dots\}$. By (6), the $\mathcal{L}_0'$ reducts of $\mathfrak{A}_1'$ and $\mathfrak{A}_2'$ are isomorphic, with b_n corresponding to d_n. We may therefore take $b_n = d_n$ for each n, where $\mathfrak{A}_1, \mathfrak{A}_2$ have the same $\mathcal{L}_0$ reduct. Let $\mathfrak{A}$ be the model for $\mathcal{L}$ with $\mathcal{L}_1$ reduct $\mathfrak{A}_1$ and $\mathcal{L}_2$ reduct $\mathfrak{A}_2$. Since $\varphi \in T_\omega$ and $(\neg\psi) \in U_\omega$, $\mathfrak{A}$ is a model of $\varphi \wedge \neg\psi$ $\square$

Let P and P' be two new n-placed relation symbols not in the language $\mathcal{L}$. Let $\Sigma(P)$ be a set of sentences of the language $\mathcal{L} \cup \{P\}$, and let $\Sigma(P')$ be the corresponding set of sentences $\mathcal{L} \cup \{P'\}$ formed by replacing P everywhere by P'. We say that $\Sigma(P)$ **defines** P **implicitly** iff

$$\Sigma(P) \cup \Sigma(P') \vDash (\forall x_1 \dots x_n)[P(x_1 \dots x_n) \leftrightarrow P'(x_1 \dots x_n)]$$

Equivalently, if $(\mathfrak{A}, R)$ and $(\mathfrak{A}, R')$ are models of $\Sigma(P)$, then $R = R'$, $\Sigma(P)$ is said to **define** P **explicitly** iff there exists a formula $\varphi(x_1 \dots x_n)$ of $\mathcal{L}$ s.t.

$$\Sigma(P) \vDash (\forall x_1 \dots x_n)[P(x_1 \dots x_n) \leftrightarrow \varphi(x_1 \dots x_n)]$$

If $\Sigma(P)$ defines P explicitly, then $\Sigma(P)$ defines P implicitly. Thus, to show that $\Sigma(P)$ does not define P explicitly, it suffices to find two model $(\mathfrak{A}, R)$ and $\mathfrak{A}, R'$ of $\Sigma(P)$, with the same reduct $\mathfrak{A}$ to $\mathcal{L}$ s.t. $R \neq R'$. This is **Padoa's method**

Theorem 1.25 (Beth's Theorem)**.** *$\Sigma(P)$ defines P implicitly iff $\Sigma(P)$ defines P explicitly*

Proof. Suppose that $\Sigma(P)$ defines P implicitly. Add new constants $c_1, \dots, c_n$ to $\mathcal{L}$. Then

$$\Sigma(P) \cup \Sigma(P') \vDash P(c_1 \dots c_n) \rightarrow P'(c_1 \dots c_n)$$

By the compactness theorem, there exists finite subsets $\Delta \subset \Sigma(P), \Delta' \subset \Sigma(P')$ s.t.

$$\Delta \subset \Delta' \vDash P(c_1 \dots c_n) \rightarrow P'(c_1 \dots c_n)$$

□