# A Course in Model Theory

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# 1 The Basics

## 1.1 Structures

**Definition 1.1.** Let $\mathfrak{A}, \mathfrak{B}$ be $L$-structures. A map $h : A \to B$ is called a **homomorphism** if for all $a_1, \dots, a_n \in A$

$$\begin{array}{rcl} h(c^{\mathfrak{A}}) & = & c^{\mathfrak{B}} \\ h(f^{\mathfrak{A}}(a_1, \dots, a_n)) & = & f^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \\ R^{\mathfrak{A}}(a_1, \dots, a_n) & \Rightarrow & R^{\mathfrak{B}}(h(a_1), \dots, h(a_n)) \end{array}$$

We denote this by

$$h : \mathfrak{A} \to \mathfrak{B}$$

If in addition $h$ is injective and

$$R^{\mathfrak{A}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \dots, h(a_n))$$

for all $a_1, \dots, a_n \in A$, then $h$ is called an (isomorphic) **embedding**. An **isomorphism** is a surjective embedding

**Lemma 1.2.** *Let $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ be an isomorphism and $\mathfrak{B}$ an extension of $\mathfrak{A}$. Then there exists an extension $\mathfrak{B}'$ of $\mathfrak{A}'$ and an isomorphism $g : \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}'$ extending $h$*

For any family $\mathfrak{A}_i$ of substructures of $\mathfrak{B}$, the intersection of the $A_i$ is either empty or a substructure of $\mathfrak{B}$. Therefore if $S$ is any non-empty subset of $\mathfrak{B}$, then there exists a smallest substructure $\mathfrak{A} = \langle S \rangle^{\mathfrak{B}}$ which contains $S$. We call the $\mathfrak{A}$ the substructure **generated** by $S$

**Lemma 1.3.** *If $\mathfrak{a} = \langle S \rangle$, then every homomorphism $h : \mathfrak{A} \to \mathfrak{B}$ is determined by its values on $S$*

**Definition 1.4.** Let $(I, \leq)$ be a **directed partial order**. This means that for all $i, j \in I$ there exists a $k \in I$ s.t. $i \leq k$ and $j \leq k$. A family $(\mathfrak{A}_i)_{i \in I}$ of $L$-structures is called **directed** if

$$i \leq j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j$$

If $I$ is linearly ordered, we call $(\mathfrak{A}_i)_{i \in I}$ a **chain**

If a structure $\mathfrak{A}_1$ is isomorphic to a substructure $\mathfrak{A}_0$ of itself,

$$h_0 : \mathfrak{A}_0 \xrightarrow{\sim} \mathfrak{A}_1$$

then Lemma 1.2 gives an extension

$$h_1 : \mathfrak{A}_1 \xrightarrow{\sim} \mathfrak{A}_2$$

Continuing in this way we obtain a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$ and an increasing sequence $h_i : \mathfrak{A}_i \xrightarrow{\sim} \mathfrak{A}_{i+1}$ of isomorphism

**Lemma 1.5.** *Let $(\mathfrak{A}_i)_{i \in I}$ be a directed family of L-structures. Then $A = \bigcup_{i \in I} A_i$ is the universe of a (uniquely determined) L-structure*

$$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$$

*which is an extension of all $\mathfrak{A}_i$*

A subset $K$ of $L$ is called a **sublanguage**. An $L$-structure becomes a $K$-structure, the **reduct**.

$$\mathfrak{A} \restriction K = (A, (Z^{\mathfrak{A}})_{Z \in K})$$

Conversely we call $\mathfrak{A}$ an **expansion** of $\mathfrak{A} \restriction K$.

1. Let $B \subseteq A$ , we obtain a new language

   $$L(B) = L \cup B$$

   and the $L(B)$-structure

   $$\mathfrak{A}_B = (\mathfrak{A}, b)_{b \in B}$$

   Note that **Aut**$(\mathfrak{A}_B)$ is the group of automorphisms of $\mathfrak{A}$ fixing $B$ elementwise. We denote this group by **Aut**$(\mathfrak{A}/B)$

Let $S$ be a set, which we call the set of sorts. An $S$-sorted language $L$ is given by a set of constants for each sort in $S$, and typed function and relations. For any tuple $(s_1, \dots, s_n)$ and $(s_1, \dots, s_n, t)$ there is a set of relation symbols and function symbols respectively. An $S$-sorted structure is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$, where

| | |
|---|---|
| $A$ | if a family $(A_s)_{s \in S}$ of non-empty sets |
| $Z^{\mathfrak{A}} \in A_s$ | if $Z$ is a constant of sort $s \in S$ |
| $Z^{\mathfrak{A}} : A_{s_1} \times \dots \times A_{s_n} \to A_t$ | if $Z$ is a function symbol of type $(s_1, \dots, s_n, t)$ |
| $Z^{\mathfrak{A}} \subseteq A_{s_1} \times \dots \times A_{s_n}$ | if $Z$ is a relation symbol of type $(s_1, \dots, s_n)$ |

**Example 1.1.** Consider the two-sorted language $L_{Perm}$ for permutation groups with a sort $x$ for the set and a sort $g$ for the group. The constants and function symbols for $L_{Perm}$ are those of $L_{Group}$ restricted to the sort $g$ and an additional function symbol $\varphi$ of type $(x, g, x)$. Thus an $L_{Perm}$-structure $(X, G)$ is given by a set $X$ and an $L_{Group}$-structure $G$ together with a function $X \times G \to X$

## 1.2 Language

**Lemma 1.6.** *Suppose $\overrightarrow{b}$ and $\overrightarrow{c}$ agree on all variables which are free in $\varphi$. Then*

$$\mathfrak{A} \models \varphi[\overrightarrow{b}] \Leftrightarrow \mathfrak{A} \models \varphi[\overrightarrow{c}]$$

We define

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n]$$

by $\mathfrak{A} \models \varphi[\overrightarrow{b}]$, where $\overrightarrow{b}$ is an assignment satisfying $\overrightarrow{b}(x_i) = a_i$. Because of Lemma 1.6 this is well defined.

Thus $\varphi(x_1, \dots, x_n)$ defines an $n$-ary relation

$$\varphi(\mathfrak{A}) = \{\bar{a} \mid \mathfrak{A} \models \varphi[\bar{a}]\}$$

on $A$, the **realisation set** of $\varphi$. Such realisation sets are called **0-definable subsets** of $A^n$, or 0-definable relations

Let $B$ be a subset of $A$. A $B$**-definable** subset of $\mathfrak{A}$ is a set of the form $\varphi(\mathfrak{A})$ for an $L(B)$-formula $\varphi(x)$. We also say that $\varphi$ are defined **over** $B$ and that the set $\varphi(\mathfrak{A})$ is defined by $\varphi$. We call two formulas **equivalent** if in every structure they define the same set.

Atomic formulas and their negations are called **basic**. Formulas without quantifiers are Boolean combinations of basic formulas. It is convenient to allow the empty conjunction and the empty disjunction. For that we introduce two new formulas: the formula $\top$, which is always true, and the formula $\bot$, which is always false. We define

$$\bigwedge_{i<0} \pi_i = \top$$
$$\bigvee_{i<0} \pi_i = \bot$$

A formula is in **negation normal form** if it is built from basic formulas using $\wedge, \vee, exists, \forall$

**Definition 1.7.** A formula in negation normal form which does not contain any existential quantifier is called **universal**. Formulas in negation normal form without universal quantifiers are called **existential**

Let $\mathfrak{A}$ be an $L$-structure. The **atomic diagram** of $\mathfrak{A}$ is

$$\text{Diag}(\mathfrak{A}) = \{\varphi \text{ basic } L(A)\text{-sentence} \mid \mathfrak{A}_A \models \varphi\}$$

**Lemma 1.8.** *The models of* $\mathrm{Diag}(\mathfrak{a})$ *are precisely those structures* $(\mathfrak{B}, h(a))_{a\in A}$ *for embeddings* $h : \mathfrak{A} \to \mathfrak{B}$

*Exercise* 1.2.1. Every formula is equivalent to a formula in prenex normal form:

$$Q_1 x_1 \dots Q_n x_n \varphi$$

The $Q_i$ are quantifiers and $\varphi$ is quantifier-free

*Proof.*

$$(\forall x)\phi \wedge \psi \vDash\dashv \forall x(\phi \wedge \psi) \text{ if } \exists x\top(\text{at least one individual exists})$$
$$(\forall x\phi) \vee \psi \vDash\dashv \forall x(\phi \vee \psi)$$
$$(\exists x\phi) \wedge \psi \vDash\dashv \exists x(\phi \wedge \psi)$$
$$(\exists x\phi) \vee \psi \vDash\dashv \exists x(\phi \vee \psi) \text{ if } \exists x\top$$
$$\neg\exists x\phi \vDash\dashv \forall x\neg\phi$$
$$\neg\forall x\phi \vDash\dashv \exists x\neg\phi$$
$$(\forall x\phi) \to \psi \vDash\dashv \exists x(\phi \to \psi) \text{ if } \exists x\top$$
$$(\exists x\phi) \to \psi \vDash\dashv \forall x(\phi \to \psi)$$
$$\phi \to (\exists x\psi) \vDash\dashv \exists x(\phi \to \psi) \text{ if } \exists x\top$$
$$\phi \to (\forall x\psi) \vDash\dashv \forall x(\phi \to \psi)$$

□

## 1.3 Theories

**Definition 1.9.** An $L$**-theory** $T$ is a set of $L$-sentences

A theory which has a model is a **consistent** theory. We call a set $\Sigma$ of $L$-formulas **consistent** if there is an $L$-structure and **an assignment** $\vec{b}$ **s.t.** $\mathfrak{A} \vDash [\vec{b}]$ for all $\varphi \in \Sigma$

**Lemma 1.10.** *Let $T$ be an $L$-theory and $L'$ be an extension of $L$. Then $T$ is consistent as an $L$-theory iff $T$ is consistent as a $L'$-theory*

**Lemma 1.11.**

1. *If $T \vDash \varphi$ and $T \vDash (\varphi \to \psi)$, then $T \vDash \psi$*
2. *If $T \vDash \varphi(c_1, \dots, c_n)$ and the constants $c_1, \dots, c_n$ occur neither in $T$ nor in $\varphi(x_1, \dots, x_n)$, then $T \vDash \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$*

*Proof.* 2. Let $L' = L \setminus \{c_1, \dots, c_n\}$. If the $L'$-structure is a model of $T$ and $a_1, \dots, a_n$ are arbitrary elements, then $(\mathfrak{A}, a_1, \dots, a_n) \models \varphi(c_1, \dots, c_n)$. This means $\mathfrak{A} \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$.

□

$S$ and $T$ are called **equivalent**, $S \equiv T$, if $S$ and $T$ have the same models

**Definition 1.12.** A consistent $L$-theory $T$ is called **complete** if for all $L$-sentences $\varphi$

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi$$

**Definition 1.13.** For a complete theory $T$ we define

$$|T| = \max(|L|, \aleph_0)$$

The typical example of a complete theory is the theory of a structure $\mathfrak{A}$

$$\mathrm{Th}(\mathfrak{A}) = \{\varphi \mid \mathfrak{A} \models \varphi\}$$

**Lemma 1.14.** *A consistent theory is complete iff it is maximal consistent, i.e., if it is equivalent to every consistent extension*

**Definition 1.15.** Two $L$-structures $\mathfrak{A}$ and $\mathfrak{B}$ are called **elementary equivalent**

$$\mathfrak{A} \equiv \mathfrak{B}$$

if they have the same theory

**Lemma 1.16.** *Let $T$ be a consistent theory. Then the following are equivalent*

1. *$T$ is complete*
2. *All models of $T$ are elemantarily equivalent*
3. *There exists a structure $\mathfrak{A}$ with $T \equiv \mathrm{Th}(\mathfrak{A})$*

*Proof.* $1 \to 3 \to 2 \to 1$ □

# 2 Elementary Extensions and Compactness

## 2.1 Elementary substructures

Let $\mathfrak{A}, \mathfrak{B}$ be two $L$-structures. A map $h : A \to B$ is called **elementary** if for all $a_1, \dots, a_n \in A$ we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

We write

$$h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$$

**Lemma 2.1.** *The models of* $\mathrm{Th}(\mathfrak{A}_A)$ *are exactly the structures of the form* $(\mathfrak{B}, h(a))_{a \in A}$ *for elementary embeddings* $h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$

We call $\mathrm{Th}(\mathfrak{A}_A)$ the **elemantary diagram** of $\mathfrak{A}$

A substructure $\mathfrak{A}$ of $\mathfrak{B}$ is called **elementary** if the inclusion map is elementary. In this case we write

$$\mathfrak{A} \prec \mathfrak{B}$$

**Theorem 2.2** (Tarski's Test)**.** *Let* $\mathfrak{B}$ *be an* $L$*-structure and* $A$ *a subset of* $B$*. Then* $A$ *is the universe of an elementary substructure iff every* $L(A)$*-formula* $\varphi(x)$ *which is satisfiable in* $\mathfrak{B}$ *can be satisfied by an element of* $A$

We use Tarski's Test to construct small elementary substructures

**Corollary 2.3.** *Suppose* $S$ *is a subset of the* $L$*-structure* $\mathfrak{B}$*. Then* $\mathfrak{B}$ *has a elementary substructure* $\mathfrak{A}$ *containing* $S$ *and of cardinality at most*

$$\max(|S|, |L|, \aleph_0)$$

*Proof.* We construct $A$ as the union of an ascending sequence $S_0 \subseteq S_1 \subseteq \ldots$ of subsets of $B$. We start with $S_0 = S$. If $S_i$ is already defined, we choose an element $a_\varphi \in B$ for every $L(S_i)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ and define $S_{i+1}$ to be $S_i$ together with these $a_\varphi$.

An $L$-formula is a finite sequence of symbols from $L$, auxiliary symbols and logical symbols. These are $|L| + \aleph_0 = \max(|L|, \aleph_0)$ many symbols and there are exactlymax$(|L|, \aleph_0)$ many $L$-formulas

Let $\kappa = \max(|S|, |L|, \aleph_0)$. There are $\kappa$ many $L(S)$-formulas: therefore $|S_1| \leq \kappa$. Inductively it follows for every $i$ that $|S_i| \leq \kappa$. Finally we have $|A| \leq \kappa \cdot \aleph_0 = \kappa$ □

A directed family $(\mathfrak{A}_i)_{i \in I}$ of structures is **elementary** if $\mathfrak{A}_i \prec \mathfrak{A}_j$ for all $i \leq j$

**Theorem 2.4** (Tarski's Chain Lemma)**.** *The union of an elementary directed family is an elementary extension of all its members*

*Proof.* Let $\mathfrak{A} = \bigcup_{i \in I} (\mathfrak{A}_i)_{i \in I}$. We prove by induction on $\varphi(\bar{x})$ that for all $i$ and $\bar{a} \in \mathfrak{A}_i$

$$\mathfrak{A}_i \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \models \varphi(\bar{a})$$

□

## 2.2 The Compactness Theorem

**Theorem 2.5** (Compactness Theorem)**.** *Finitely satisfiable theories are consistent*

Let $L$ be a language and $C$ a set of new constants. An $L(C)$-theory $T'$ is called a **Henkin theory** if for every $L(C)$-formula $\varphi(x)$ there is a constant $c \in C$ s.t.

$$\exists x \varphi(x) \rightarrow \varphi(c) \in T'$$

**Lemma 2.6.** *Every finitely satisfiable $L$-theory $T$ can be extended to a finitely complete Henkin Theory $T^*$*

**Lemma 2.7.** *Every finitely satisfiable $L$-theory $T$ can be extended to a finitely complete Henkin theory $T^*$*

**Lemma 2.8.** *Every finitely complete Henkin theory $T^*$ has a model $\mathfrak{A}$ (unique up to isomorphism) consisting of constants; i.e.,*

$$(\mathfrak{A}, a_c)_{c \in C} \vDash T^*$$

*with $A = \{a_c \mid c \in C\}$*

**Corollary 2.9.** *A set of formulas $\Sigma(x_1, \dots, x_n)$ is consistent with $T$ if and only if every finite subset of $\Sigma$ is consistent with $T$*

*Proof.* Introduce new constants $c_1, \dots, c_n$. Then $\Sigma$ is consistent with $T$ is and only if $T \cup \Sigma(c_1, \dots, c_n)$ is consistent. Now apply the Compactness Theorem □

**Definition 2.10.** Let $\mathfrak{A}$ be an $L$-structure and $B \subseteq A$. Then $a \in A$ **realises** a set of $L(B)$-formulas $\Sigma(x)$ if $a$ satisfied all formulas from $\Sigma$. We write

$$\mathfrak{A} \vDash \Sigma(a)$$

We call $\Sigma(x)$ **finitely satisfiable** in $\mathfrak{A}$ if every finite subset of $\Sigma$ is realised in $\mathfrak{A}$

**Lemma 2.11.** *The set $\Sigma(x)$ is finitely satisfiable in $\mathfrak{A}$ iff there is an elementary extension of $\mathfrak{A}$ in which $\Sigma(x)$ is realised*

*Proof.* By Lemma 2.1 $\Sigma$ is realised in an elementary extension of $\mathfrak{A}$ iff $\Sigma$ is consistent with $\mathrm{Th}(\mathfrak{A}_A)$. So the lemma follows from the observation that a finite set of $L(A)$-formulas is consistent with $\mathrm{Th}(\mathfrak{A}_A)$ iff it is realised in $\mathfrak{A}$ □

**Definition 2.12.** Let $\mathfrak{A}$ be an $L$-structure and $B$ a subset of $A$. A set $p(x)$ of $L(B)$-formulas is a **type** over $B$ if $p(x)$ is maximal finitely satisfiable in $\mathfrak{A}$. We call $B$ the **domain** of $p$. Let

$$S(B) = S^{\mathfrak{A}}(B)$$

denote the set of types over $B$.

Every element $a$ of $\mathfrak{A}$ determines a type

$$\text{tp}(a/B) = tp^{\mathfrak{A}}(a/B) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a), \varphi \text{ an } L(B)\text{-formula}\}$$

So an element $a$ realises the type $p \in S(B)$ exactly if $p = \text{tp}(a/B)$. If $\mathfrak{A}'$ is an elementary extension of $\mathfrak{A}$, then

$$S^{\mathfrak{A}}(B) = S^{\mathfrak{A}'}(B) \quad \text{and} \quad \text{tp}^{\mathfrak{A}'}(a/B) = \text{tp}^{\mathfrak{A}}(a/B)$$

If $\mathfrak{A}' \models p(x)$ then $\mathfrak{A}' \models \exists x p(x)$, so $\mathfrak{A} \models \exists x p(x)$.

We use the notation tp$(a)$ for tp$(a/\emptyset)$

Maximal finitely satisfiable sets of formulas in $x_1, \dots, x_n$ are called $n$-**types** and

$$S_n(B) = S_N^{\mathfrak{A}}(B)$$

denotes the set of $n$-types over $B$.

$$\text{tp}(C/B) = \{\varphi(x_{c_1}, \dots, x_{c_n}) \mid \mathfrak{A} \models \varphi(c_1, \dots, c_n), \varphi \text{ an } L(B)\text{-formula}\}$$

**Corollary 2.13.** *Every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ in which all types over $A$ are realised*

*Proof.* We choose for every $p \in S(A)$ a new constant $c_p$. We have to find a model of

$$\text{Th}(\mathfrak{A}_A) \cup \bigcup_{p \in S(A)} p(c_p)$$

This theory is finitely satisfiable since every $p$ is finitely satisfiable in $\mathfrak{A}$.

Or use Lemma 2.11. Let $(p_\alpha)_{\alpha<\lambda}$ be an enumeration of $S(A)$. Construct an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \dots \prec \mathfrak{A}_\beta \prec \dots (\beta \leq \lambda)$$

s.t. each $p_\alpha$ is realised in $\mathfrak{A}_{\alpha+1}$ (by recursion theorem on ordinal numbers)

Suppose that the elementary chain $(\mathfrak{A}_{\alpha'})_{\alpha'<\beta}$ is already constructed. If $\beta$ is a limit ordinal, we let $\mathfrak{A}_\beta = \bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$, which is elementary by Lemma 2.4. If $\beta = \alpha + 1$ we first note that $p_\alpha$ is also finitely satisfiable in $\mathfrak{A}_\alpha$, therefore we can realise $p_\alpha$ in a suitable elementary extension $\mathfrak{A}_\beta \succ \mathfrak{A}_\alpha$ by Lemma 2.11. Then $\mathfrak{B} = \mathfrak{A}_\lambda$ is the model we were looking for $\square$

## 2.3 The Löwenheim-Skolem Theorem

**Theorem 2.14** (Löwenheim-Skolem)**.** *Let $\mathfrak{B}$ be an $L$-structure, $S$ a subset of $B$ and $\kappa$ an infinite cardinal*

1. *If*
$$\max(|S|, |L|) \leq \kappa \leq |B|$$
*then $\mathfrak{B}$ has an elementary substructure of cardinality $\kappa$ containing $S$*

2. *If $\mathfrak{B}$ is infinite and*
$$\max(|\mathfrak{B}|, |L|) \leq \kappa$$
*then $\mathfrak{B}$ has an elementary extension of cardinality $\kappa$*

**Corollary 2.15.** *A theory which has an infinite model has a model in every cardinality $\kappa \geq \max(|L|, \aleph_0)$*

**Definition 2.16.** Let $\kappa$ be an infinite cardinal. A theory $T$ is called **$\kappa$-categorical** if for all models of $T$ of cardinality $\kappa$ are isomorphic

**Theorem 2.17** (Vaught's Test)**.** *A $\kappa$-categorical theory $T$ is complete if the following conditions are satisfied*

1. *$T$ is consistent*
2. *$T$ has no finite model*
3. *$|L| \leq \kappa$*

*Proof.* We have to show that all models $\mathfrak{A}$ and $\mathfrak{B}$ of $T$ are elemantarily equivalent. As $\mathfrak{A}$ and $\mathfrak{B}$ are infinite, Th($\mathfrak{A}$) and Th($\mathfrak{B}$) have models $\mathfrak{A}'$ and $\mathfrak{B}'$ of cardinality $\kappa$. By assumption $\mathfrak{A}'$ and $\mathfrak{B}'$ are isomorphic, and it follows that

$$\mathfrak{A} \equiv \mathfrak{A}' \equiv \mathfrak{B}' \equiv \mathfrak{B}$$

□

**Example 2.1.** 1. The theory DLO of dense linear orders without endpoints is $\aleph_0$-categorical and by Vaught's test complete. Let $A = \{a_i \mid i \in \omega\}$, $B = \{b_i \mid i \in \omega\}$. We inductively define sequences $(c_i)_{i<\omega}$, $(d_i)_{i<\omega}$ exhausting $A$ and $B$. Assume that $(c_i)_{i<m}, (d_i)_{i<m}$ have defined so that $c_i \mapsto d_i, i < m$ is an order isomorphism. If $m = 2k$ let $c_m = a_j$ where $a_j$ is the element with minimal index in $\{a_i \mid i \in \omega\}$ not occurring in $(c_i)_{i<m}$. Since $\mathfrak{B}$ is a dense linear order without endpoints there is some element $d_m \in \{b_i \mid i \in \omega\}$ s.t. $(c_i)_{i\leq m}$ and $(d_i)_{i\leq m}$ are order isomorphic. If $m = 2k+1$ we interchange the roles of $\mathfrak{A}$ and $\mathfrak{B}$

2.

Consider the Theorem 2.17 we strengthen our definition

**Definition 2.18.** Let $\kappa$ be an infinite cardinal. A theory $T$ is called **$\kappa$-categorical** if it is complete, $|T| \leq \kappa$ and, up to isomorphism, has exactly one model of cardinality $\kappa$

# 3 Quantifier Elimination

## 3.1 Preservation theorems

**Lemma 3.1** (Separation Lemma)**.** *Let $T_1, T_2$ be two theories. Assume $\mathcal{H}$ is a set of sentences which is closed under $\wedge, \vee$ and contains $\bot$ and $\top$. Then the following are equivalent*

1. *There is a sentence $\varphi \in \mathcal{H}$ which separates $T_1$ from $T_2$. This means*

$$T_1 \models \varphi \quad \text{and} \quad T_2 \models \neg\varphi$$

2. *All models $\mathfrak{A}_1$ of $T_1$ can be separated from all models $\mathfrak{A}_2$ of $T_2$ by a sentence $\varphi \in \mathcal{H}$. This means*

$$\mathfrak{A}_1 \models \varphi \quad \text{and} \quad \mathfrak{A}_2 \models \neg\varphi$$

*Proof.* $2 \to 1$. For any model $\mathfrak{A}_1$ of $T_1$ let $\mathcal{H}_{\mathfrak{A}_1}$ be the set of all sentences from $\mathcal{H}$ which are true in $\mathfrak{A}_1$. (2) implies that $\mathcal{H}_{\mathfrak{A}_1}$ and $T_2$ cannot have a common model. By the Compactness Theorem there is a finite conjunction $\varphi_{\mathfrak{A}_1}$ of sentences from $\mathcal{H}_{\mathfrak{A}_1}$ inconsistent with $T_2$. Clearly

$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$

is inconsistent. Again by compactness $T_1$ implies a disjunction $\varphi$ of finitely many of the $\varphi_{\mathfrak{A}_1}$ □

For structures $\mathfrak{A}, \mathfrak{B}$ and a map $f : A \to B$ preserving all formulas from a set of formulas $\Delta$, we use the notation

$$f : \mathfrak{A} \to_{\Delta} \mathfrak{B}$$

We also write

$$\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$$

to express that all sentences from $\Delta$ true in $\mathfrak{A}$ are also true in $\mathfrak{B}$

**Lemma 3.2.** *Let $T$ be a theory, $\mathfrak{A}$ a structure and $\Delta$ a set of formulas, closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent*

1. *All sentences $\varphi \in \Delta$ which are true in $\mathfrak{A}$ are consistent with $T$ (There is a model $\mathfrak{B} \models \mathrm{Th}_\Delta(\mathfrak{A}_A) \cup T$ and $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$)*

2. *There is a model $\mathfrak{B} \models T$ and a map $f : \mathfrak{A} \to_\Delta \mathfrak{B}$*

*Proof.* $1 \to 2$. Consider $\mathrm{Th}_\Delta(\mathfrak{A}_A)$, the set of all sentences $\delta(\bar{a})$ ($\delta(\bar{x}) \in \Delta$), which are true in $\mathfrak{A}_A$. The models $(\mathfrak{B}, f(a)_{a\in A})$ of this theory correspond to maps $f : \mathfrak{A} \to_\Delta \mathfrak{B}$. **This means that we have to find a model of** $T \cup \mathrm{Th}_\Delta(\mathfrak{A}_A)$. To show finite satisfiability it is enough to show that $T \cup D$ is consistent for every finite subset $D$ of $\mathrm{Th}_\Delta(\mathfrak{A}_A)$. Let $\delta(\bar{a})$ be the conjunction of the elements of $D$. Then $T$ has a model $\mathfrak{B}$ which is also a model of $\varphi = \exists \bar{x}\delta(\bar{x})$ □

Lemma 3.2 applied to $T = \mathrm{Th}(\mathfrak{B})$ shows that $\mathfrak{A} \Rightarrow_\Delta \mathfrak{B}$ iff there exists a map $f$ and a structure $\mathfrak{B}' \equiv \mathfrak{B}$ s.t. $f : \mathfrak{A} \to_\Delta \mathfrak{B}'$

**Theorem 3.3.** *Let $T_1$ and $T_2$ be two theories. Then the following are equivalent*

1. *There is a universal sentence which separates $T_1$ from $T_2$*

2. *No model of $T_2$ is a substructure of a model of $T_1$*

*Proof.* $2 \to 1$. If $T_1$ and $T_2$ cannot be separated by a universal sentence, then they have models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which cannot be separated by a universal sentence. This can be denoted by

$$\mathfrak{A}_2 \Rightarrow_\exists \mathfrak{A}_1$$

Now Lemma 3.2 implies that $\mathfrak{A}_2$ there is a map $\mathfrak{A}_2 \to_\exists \mathfrak{A}_1'$ where $\mathfrak{A}_1' \models T_1$ . Hence $\mathfrak{A}_2$ has an extension $\mathfrak{A}_2'$ s.t. $\mathfrak{A}_2' \equiv \mathfrak{A}_1'$. Then $\mathfrak{A}'$ is gain a model of $T_1$ contradicting (2) □

**Definition 3.4.** For any $L$-theory $T$, the formulas $\varphi(\bar{x}), \psi(\bar{x})$ are said to be **equivalent** modulo $T$ (or relative to $T$) if $T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

**Corollary 3.5.** *Let $T$ be a theory*

1. *Consider a formula $\varphi(x_1, \dots, x_n)$. The following are equivalent*

   (a) *$\varphi(x_1, \dots, x_n)$ is, modulo $T$, equivalent to a universal formula*

(b) *If $\mathfrak{A} \subseteq \mathfrak{B}$ are models of $T$ and $a_1, \ldots, a_n \in A$, then $\mathfrak{B} \models \varphi(a_1, \ldots, a_n)$ implies $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$*

2. *We say that a theory which consists of universal sentences is universal. Then $T$ is equivalent to a universal theory iff all substructures of models of $T$ are again models of $T$*

*Proof.* 1. Assume (2). We extend $L$ by an $n$-tuple $\bar{c}$ of new constants $c_1, \ldots, c_n$ and consider theory

$$T_1 = T \cup \{\varphi(\bar{c})\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi(\bar{c})\}$$

Then (2) says the substructures of models of $T_1$ cannot be models of $T_2$. By Theorem 3.3 $T_1$ and $T_2$ can be separated by a universal $L(\bar{c})$-sentence $\psi(\bar{c})$. By Lemma 1.11, $T_1 \models \psi(\bar{c})$ implies

$$T \models \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$$

and from $T_2 \models \neg\psi(\bar{c})$ we see

$$T \models \forall\bar{x}(\neg\varphi(\bar{x}) \to \neg\psi(\bar{x}))$$

2. Suppose a theory $T$ has this property. Let $\varphi$ be an axiom of $T$. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, it is not possible for $\mathfrak{B}$ to be a model of $T$ and for $\mathfrak{A}$ to be a model of $\neg\psi$ at the same time. By Theorem 3.3 there is a universal sentence $\psi$ with $T \models \psi$ and $\neg\varphi \models \neg\psi$. Hence all axioms of $T$ follow from

$$T_\forall = \{\psi \mid T \models \psi, \psi \text{ universal}\}$$

□

An $\forall\exists$-formula is of the form

$$\forall x_1 \ldots x_n \psi$$

where $\psi$ is existential

**Lemma 3.6.** *Suppose $\varphi$ is an $\forall\exists$-sentence, $(\mathfrak{A}_i)_{i \in I}$ is a directed family of models of $\varphi$ and $\mathfrak{B}$ the union of the $\mathfrak{A}_i$. Then $\mathfrak{B}$ is also a model of $\varphi$.*

*Proof.* Write

$$\varphi = \forall\bar{x}\psi(\bar{x})$$

where $\psi$ is existential. For any $\bar{a} \in B$ there is an $A_i$ containing $\bar{a}$, clearly $\psi(\bar{a})$ holds in $\mathfrak{A}_i$. As $\psi(\bar{a})$ is existential it must also hold in $\mathfrak{B}$ □

**Definition 3.7.** We call a theory $T$ **inductive** if the union of any directed family of models of $T$ is again a model

**Theorem 3.8.** *Let $T_1$ and $T_2$ be two theories. Then the following are equivalent*

1. *there is an $\forall\exists$-sentence which separates $T_1$ and $T_2$*
2. *No model of $T_2$ is the union of a chain (or of a directed family) of models of $T_1$*

*Proof.* $2 \to 1$. If (1) is not true, $T_1, T_2$ have models which cannot be separated by an $\forall\exists$-sentence. Since $\exists\forall$-formulas are equivalent to negated $\forall\exists$-formulas (since $\forall$ is too strong), we have

$$\mathfrak{B}^0 \Rightarrow_{\exists\forall} \mathfrak{A}$$

By Lemma 3.2 there is a map

$$f : \mathfrak{B}^0 \to_{\forall} \mathfrak{A}^0$$

with $\mathfrak{A}^0 \equiv \mathfrak{A}$ (since $\mathfrak{B}^0 \to_{\exists\forall} \mathfrak{A}^0$). We can assume that$\mathfrak{B}^0 \subseteq \mathfrak{A}^0$ and $f$ is the inclusion map. Then

$$\mathfrak{A}^0_B \Rightarrow_{\exists} \mathfrak{B}^0_B$$

Applying Lemma 3.2 again, we obtain an extension $\mathfrak{B}^1_B$ of $\mathfrak{A}^0_B$ with $\mathfrak{B}^1_B \equiv \mathfrak{B}^0_B$, i.e. $\mathfrak{B}^0 \prec \mathfrak{B}^1$. Hence we have an infinite chain

$$\mathfrak{B}^0 \subseteq \mathfrak{A}^0 \subseteq^1 \mathfrak{B}^1 \subseteq \mathfrak{A}^1 \subseteq \mathfrak{B}^2 \subseteq \cdots$$
$$\mathfrak{B}^0 \prec \mathfrak{B}^1 \prec \mathfrak{B}^2 \prec \cdots$$
$$\mathfrak{A}^i \equiv \mathfrak{A}$$

Let $\mathfrak{B}$ be the union of the $\mathfrak{A}^i$. Since $\mathfrak{B}$ is also the union of the elementary chain of the $\mathfrak{B}^i$, it is an elementary extension of $\mathfrak{B}^0$ and hence a model of $T_2$. But the $\mathfrak{A}^i$ are models of $T_1$, so (2) does not hold □

**Corollary 3.9.** *Let $T$ be a theory*

1. *For each sentence $\varphi$ the following are equivalent*
   (a) *$\varphi$ is, modulo $T$, equivalent to an $\forall\exists$-sentence*
   (b) *If*
   $$\mathfrak{A}^0 \subseteq \mathfrak{A}^1 \subseteq \cdots$$
   *and their union $\mathfrak{B}$ are models of $T$, then $\varphi$ holds in $\mathfrak{B}$ if it is true in all the $\mathfrak{A}^i$*

2. *$T$ is inductive iff it can be axiomatised by $\forall\exists$-sentences*

*Proof.* 1. Theorem 3.8 shows that $\forall\exists$-formulas are preserved by unions of chains. Hence (a)$\Rightarrow$(b). For the converse consider the theories

$$T_1 = T \cup \{\varphi\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi\}$$

Part (b) says that the union of a chain of models of $T_1$ cannot be a model of $T_2$. By Theorem 3.8 we can separate $T_1$ and $T_2$ by an $\forall\exists$-sentence $\psi$. Hence $T \cup \{\varphi\} \vDash \psi$ and $T \cup \{\neg\varphi\} \vDash \neg\psi$

2. Clearly $\forall\exists$-axiomatised theories are inductive. For the converse assume that $T$ is inductive and $\varphi$ is an axiom of $T$. If $\mathfrak{B}$ is a union of models of $T$, it cannot be a model of $\neg\varphi$. By Theorem 3.8 there is an $\forall\exists$-sentence $\psi$ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of $T$ follows from

$$T_{\forall\exists} = \{\psi \mid T \vDash \psi, \psi\ \forall\exists\text{-formula}\}$$

□

### 3.2 Quantifier elimination

**Definition 3.10.** A theory $T$ has **quantifier elimination** if every $L$-formula $\varphi(x_1, \dots, x_n)$ in the theory is equivalent modulo $T$ to some quantifier-free formula $\rho(x_1, \dots, x_n)$

It's easy to transform any theory $T$ into a theory with quantifier elimination if one is willing to expand the language: just enlarge $L$ by adding an $n$-place relation symbol $R_\varphi$ for every $L$-formula $\varphi(x_1, \dots, x_n)$ and $T$ by adding all axioms

$$\forall x_1, \dots, x_n (R_\varphi(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n))$$

The resulting theory, the **Morleyisation** $T^m$ of $T$, has quantifier elimination

A **prime structure** of $T$ is a structure which embeds into all models of $T$

**Lemma 3.11.** *A consistent theory $T$ with quantifier elimination which posseses a prime structure is complete*

*Proof.* If $\mathfrak{M}, \mathfrak{N} \vDash T$ and $\mathfrak{M} \vDash \varphi$ and $\mathfrak{N} \vDash \neg\varphi$. The prime structure is $\mathfrak{H}$. Then we have $h_1 : \mathfrak{H} \to \mathfrak{M}$ and $h_2 : \mathfrak{H} \to \mathfrak{N}$. If $\varphi$ doesn't contain existential quantification, then there is a contradiction. □

**Definition 3.12.** A **simple existential formula** has the form

$$\varphi = \exists y \rho$$

for a quantifier-free formula $\rho$. If $\rho$ is a conjunction of basic formulas, $\varphi$ is called **primitive existential**

**Lemma 3.13.** *The theory $T$ has quantifier elimination iff every primitive existential formula is, modulo $T$, equivalent to a quantifier-free formula*

*Proof.* We can write every simple existential formula in the form $\exists y \bigvee_{i<n} \rho_i$ for $\rho_i$ which are conjunctions of basic formulas. This shows that every simple existential formula is equivalent to a disjunction of primitive existential formulas, namely to $\bigvee_{i<n}(\exists y \rho_i)$. We can therefore assume that every simple existential formula is, modulo $T$, equivalent to a quantifier-free formula

We are now able to eliminate the quantifiers in arbitrary formulas in prenex normal form (Exercise 1.2.1)

$$Q_1 x_1 \dots Q_n x_n \rho$$

if $Q_n = \exists$, we choose a quantifier-free formula $\rho_0$ which, modulo $T$, is equivalent to $\exists x_n \rho$ and proceed with the formula $Q_1 x_1 \dots Q_{n-1} x_{n-1} \rho_0$. If $Q_n = \forall$, we find a quantifier-free $\rho_1$ which is, modulo $T$, equivalent to $\exists x_n \neg \rho$ and proceed with $Q_1 x_1 \dots Q_{n-1} x_{n-1} \neg \rho_1$ $\square$

**Theorem 3.14.** *For a theory $T$ the following are equivalent*

1. *$T$ has quantifier elimination*
2. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of $T$ with a common substructure $\mathfrak{A}$ we have*

$$\mathfrak{M}^1_A \equiv \mathfrak{M}^2_A$$

3. *For all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of $T$ with a common substructure $\mathfrak{A}$ and for all primitive existential formulas $\varphi(x_1, \dots, x_n)$ and parameter $a_1, \dots, a_n$ from $A$ we have*

$$\mathfrak{M}^1 \vDash \varphi(a_1, \dots, a_n) \Rightarrow \mathfrak{M}^2 \vDash \varphi(a_1, \dots, a_n)$$

*(this is exactly the equivalence relation)*

*If $L$ has no constants, $\mathfrak{A}$ is allowed to be the empty "structure"*

*Proof.* $3 \to 1$. Let $\varphi(\bar{x})$ be a primitive existential formula. In order to show that $\varphi(\bar{x})$ is equivalent, modulo $T$, to a quantifier-free formula $\rho(\bar{x})$ we extend $L$ by an $n$-tuple $\bar{c}$ of new constants $c_1, \dots, c_n$. **We have to show that we can separate $T \cup \{\varphi(\bar{c})\}$ and $T \cup \{\neg\varphi(\bar{c})\}$ by a quantifier free sentence $\rho(\bar{c})$.** We apply the Separation Lemma ($\mathcal{H}$ hear is the set of quantifier-free sentence). Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be two models of $T$ with two distinguished $n$-tuples $\bar{a}^1$ and $\bar{a}^2$. Suppose that $(\mathfrak{M}^1, \bar{a}^1)$ and $(\mathfrak{M}^2, \bar{a}^2)$ satisfy the same quantifier-free $L(\bar{c})$-sentences. We have to show that

$$\mathfrak{M}^1 \vDash \varphi(\bar{a}^1) \Rightarrow \mathfrak{M}^2 \vDash \varphi(\bar{a}^2)$$

then there is no $L(\bar{c})$-sentence that can separate the models of $T \cup \{\varphi(\bar{c})\}$ and the models of $T \cup \{\neg\varphi(\bar{c})\}$ Consider the substructure $\mathfrak{A}^i = \langle \bar{a}^i \rangle^{\mathfrak{M}^i}$, generated by $\bar{a}^i$. If we can show that there is an isomorphism

$$f : \mathfrak{A}^1 \to \mathfrak{A}^2$$

taking $\bar{a}$ to $\bar{a}$, we may assume that $\mathfrak{A}^1 = \mathfrak{A}^2 = \mathfrak{A}$ and $\bar{a}^1 = \bar{a}^2 = \bar{a}$.

Every element of $\mathfrak{A}^1$ has the form $t^{\mathfrak{M}^1}[\bar{a}^1]$ for an $L$-term $t(\bar{x})$. The isomorphism $f$to be constructed must satisfy

$$f(t^{\mathfrak{M}^1}[\bar{a}^1]) = t^{\mathfrak{M}^2}[\bar{a}^2]$$

We define $f$ by this equation and have to check that $f$ is well defined and injective. Assume

$$s^{\mathfrak{M}^1}[\bar{a}^1] = t^{\mathfrak{M}^1}[\overline{af^1}]$$

Then $\mathfrak{M}^1, \bar{a}^1 \vDash s(\bar{c}) \doteq t(\bar{c})$, and by out assumption it also holds in $(\mathfrak{M}^2, \bar{a}^2)$, which means

$$s^{\mathfrak{M}^2}[\bar{a}^2] = t^{\mathfrak{M}^2}[\bar{a}^2]$$

Swapping the two sides yields injectivity.

Surjectivity is clear. It remains to show that $f$ commutes with the interpretation of the relation symbols. Now

$$\mathfrak{M}^1 \vDash R\left[t_1^{\mathfrak{M}^1}[\bar{a}^1], \dots, t_m^{\mathfrak{M}^1}[\bar{a}^1]\right]$$

is equivalent to $(\mathfrak{M}^1, \bar{a}^1) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which is equivalent to $(\mathfrak{M}^2, \bar{a}^2) \vDash R(t_1(\bar{c}), \dots, t_m(\bar{c}))$, which in turn is equivalent to

$$\mathfrak{M}^2 \vDash R\left[t_1^{\mathfrak{M}^2}[\bar{a}^2], \dots, t_m^{\mathfrak{M}^2}[\bar{a}^2]\right]$$

□

Note that (2) of Theorem 3.14 is saying that $T$ is **substructure complete**; i.e., for any model $\mathfrak{M} \models T$ and substructure $\mathfrak{A} \subseteq \mathfrak{M}$ the theory $T \cup \mathrm{Diag}(\mathfrak{A})$ is complete

**Definition 3.15.** We call $T$ **model complete** if for all models $\mathfrak{M}^1$ and $\mathfrak{M}^2$ of $T$

$$\mathfrak{M}^1 \subseteq \mathfrak{M}^2 \Rightarrow \mathfrak{M}^1 \prec \mathfrak{M}^2$$

$T$ is model complete iff for any $\mathfrak{M} \models T$ the theory $T \cup \mathrm{Diag}(\mathfrak{M})$ is complete

**Lemma 3.16** (Robinson's Test)**.** *Let $T$ be a theory. Then the following are equivalent*

1. *$T$ is model complete*
2. *For all models $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ of $T$ and all existential sentences $\varphi$ from $L(M^1)$*

   $$\mathfrak{M}^2 \models \varphi \Rightarrow \mathfrak{M}^1 \models \varphi$$

3. *Each formula is, modulo $T$, equivalent to a universal formula*

*Proof.* $1 \leftrightarrow 3$. Corollary 3.5

(2) implies that every existential formula is, modulo $T$, equivalent to a universal formula ☐

If $\mathfrak{M}^1 \subseteq \mathfrak{M}^2$ satisfies (2), we call $\mathfrak{M}^1$ **existentially closed** in $\mathfrak{M}^2$. We denote this by

$$\mathfrak{M}^1 \prec_1 \mathfrak{M}^2$$

**Definition 3.17.** Let $T$ be a theory. A theory $T^*$ is a **model companion** of $T$ if the following three conditions are satisfied

1. Each model of $T$ can be extended to a model of $T^*$
2. Each model of $T^*$ can be extended to a model of $T$
3. $T^*$ is model complete

**Theorem 3.18.** *A theory $T$ has, up to equivalence, at most one model companion $T^*$*

*Proof.* If $T^+$ is another model companion of $T$, every model of $T^+$ is contained in a model of $T^*$ and conversely. Let $\mathfrak{A}^0 \models T^+$ . Then $\mathfrak{A}_0$ can be embedded in a model $\mathfrak{B}_0$ of $T^*$. In turn $\mathfrak{B}_0$ is contained in a model $\mathfrak{A}^1$ of $T^+$. In this way we find two elementary chains $(\mathfrak{A}_i)$ and $(\mathfrak{B}_i)$, which have a common union $\mathfrak{C}$. Then $\mathfrak{A}_0 \prec \mathfrak{C}$ and $\mathfrak{B}_0 \prec \mathfrak{C}$ implies $\mathfrak{A}_0 \equiv \mathfrak{B}_0$ since $T$ are all sentences. Thus $\mathfrak{A}_0$ is a model of $T^*$ ☐

**Existentially closed structures and the Kaiser hull**

Let $T$ be an $L$-theory. It follows from 3.2 that the models of $T_\forall$ are the substructures of models of $T$. The conditions (1) and (2) in the definition of ″model companion″ can therefore be expressed as

$$T_\forall = T_\forall^*$$

Hence the model companion of a theory $T$ depends only on $T_\forall$. (Note that $T_\forall$ is model complete)

**Definition 3.19.** An $L$-structure $\mathfrak{A}$ is called **$T$-existentiallay closed** (or **$T$-ec**) if

1. $\mathfrak{A}$ can be embedded in a model of $T$
2. $\mathfrak{A}$ is existentially closed in every extension which is a model of $T$

A structure $\mathfrak{A}$ is $T$-ec exactly if it is $T_\forall$-ec. Since every model of $\mathfrak{B}$ of $T_\forall$ can be embedded in a model $\mathfrak{M}$ of $T$ and $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$ and $\mathfrak{A} \prec_1 \mathfrak{M}$ implies $\mathfrak{A} \prec_1 \mathfrak{B}$

**Lemma 3.20.** *Every model of a theory $T$ can be embedded in a $T$-ec structure*

*Proof.* Let $\mathfrak{A}$ be a model of $T_\forall$. We choose an enumeration $(\varphi_\alpha)_{\alpha<\kappa}$ of all existential $L(A)$-sentences and construct an ascending chain $(\mathfrak{A}_\alpha)_{\alpha\le\kappa}$ of models of $T_\forall$. We begin with $\mathfrak{A}_0 = \mathfrak{A}$. Let $\mathfrak{A}_\alpha$ be constructed. If $\varphi_\alpha$ holds in an extension of $\mathfrak{A}_\alpha$ which is a model of $T$ we let $\mathfrak{A}_{\alpha+1}$ be such a model. Otherwise we set $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$. For limit ordinals $\lambda$ we define $\mathfrak{A}_\lambda$ to be the union of all $\mathfrak{A}_\alpha$. $\mathfrak{A}_\lambda$ is again a model of $T_\forall$ □

Every elementary substructure $\mathfrak{N}$ of a $T$-ec structure $\mathfrak{M}$ is again $T$-ec. Let $\mathfrak{N} \subseteq \mathfrak{A}$ be a model of $T$. Since $\mathfrak{M}_N \Rrightarrow_\exists \mathfrak{A}_N$, there is an embedding of $\mathfrak{M}$ in an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ which is the identity on $N$. Since $\mathfrak{M}$ is existentially closed in $\mathfrak{B}$, it follows that $\mathfrak{N}$ is existentially closed in $\mathfrak{B}$ and therefore also in $\mathfrak{A}$

$$\begin{tikzcd}
 & \mathfrak{B} & \\
\mathfrak{A} \arrow[ur, "\prec"] & & \mathfrak{M} \arrow[ul, "\prec_1"'] \\
 & \mathfrak{N} \arrow[ul, "\prec_1"] \arrow[ur, "\prec"'] &
\end{tikzcd}$$

**Lemma 3.21.** *Let $T$ be a theory. Then there is a biggest inductive theory $T^{\mathrm{KH}}$ with $T_\forall = T^{\mathrm{KH}}_\forall$. We call $T^{\mathrm{KH}}$ the **Kaiser hull** of $T$*

*Proof.* Let $T^1$ and $T^2$ be two inductive theories with $T^1_\forall = T^2_\forall = T_\forall$. We have to show that $(T^1 \cup T^2)_\forall = T_\forall$. Let $\mathfrak{M}$ be a model of $T$, as in the proof of 3.18 we extend $\mathfrak{M}$ by a chain $\mathfrak{A}_0 \subseteq \mathfrak{B}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{B}_1 \subseteq \cdots$ of models of $T^1$ and $T^2$. The union of this chain is a model of $T^1 \cup T^2$

(Both of $T^1_\forall$ and $T^2_\forall$ and model companion and hence equivalent) □

**Lemma 3.22.** *The Kaiser hull $T^{KH}$ is the $\forall\exists$-part of the theory of all $T$-ec structures*

*Proof.* Let $T^*$ be the $\forall\exists$-part of the theory of all $T$-ec structures. Since $T$-ec structures are models of $T_\forall$, we have $T_\forall \subseteq T^*_\forall$. It follows from 3.20 that $T^*_\forall \subseteq T_\forall$. Hence $T^*$ is contained in the Kaiser Hull. □

This implies that $T$-ec strctures are models of $T_{\forall\exists}$

**Theorem 3.23.** *For any theory $T$ the following are equivalent*

1. *$T$ has a model companion $T^*$*
2. *All models of $K^{\mathrm{KH}}$ are $T$-ec*
3. *The $T$-ec structures form an elementary class.*

*If $T^*$ exists, we have*

$$T^* = T^{\mathrm{KH}} = \text{ theory of all } T\text{-ec structures}$$

*Exercise* 3.2.1. Let $L$ be the language containing a unary function $f$ and a binary relation symbol $R$ and consider the $L$-theory $T = \{\forall x \forall y (R(x,y) \to (R(x, f(y))))\}$. Showing the follow

1. For any $T$-structure $\mathfrak{M}$ and $a, b \in M$ with $b \notin \{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ we have $\mathfrak{M} \models \exists z (R(z,a) \wedge \neg R(z,b))$
2. Let $\mathfrak{M}$ be a model of $T$ and $a$ an element of $M$ s.t. $\{a, f^{\mathfrak{M}}(a), (f^{\mathfrak{M}})^2(a), \dots\}$ is infinite. Then in an elementary extension $\mathfrak{M}'$ there is an element $b$ with $\mathfrak{M}' \models \forall z (R(z,a) \to R(z,b))$
3. The class of $T$-ec structures is not elementary, so $T$ does not have a model companion

*Exercise* 3.2.2. A theory $T$ with quantifier elimination is axiomatisable by sentences of the form

$$\forall x_1 \dots x_n \psi$$

where $\psi$ is primitive existential formula

## 3.3 Examples

**Infinite sets**. The models of the theory Infset of **infinite sets** are all infinite sets without additional structure. The language $L_\emptyset$ is empty, the axioms are (for $n = 1, 2, \ldots$)

- $\exists x_0 \ldots x_{n-1} \bigwedge_{i<j<n} \neg x_i \doteq x_j$

**Theorem 3.24.** *The theory Infset of infinite sets has quantifier elimination and is complete*

*Proof.* Since the language is empty, the only basic formula is $x_i = x_j$ and $\neg(x_i = x_j)$. By Lemma 3.13 we only need to consider primitive existential formulas. □

**Dense linear orderings**.

$$\forall a, b(a \leq b \wedge b \leq a \rightarrow a \doteq b)$$
$$\forall a, b, c(a \leq b \wedge b \leq c \rightarrow a \leq c)$$
$$\forall a, b(a \leq b \vee b \leq a)$$
$$\forall a, b \exists c(a < b \rightarrow a < c < b)$$

**Theorem 3.25.** *DLO has quantifier elimination*

*Proof.* Let $A$ be a finite common substructure of the two models $O_1$ and $O_2$. We choose an ascending enumeration $A = \{a_1, \ldots, a_n\}$. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence, which is true in $O_1$ and assume $O_1 \vDash \rho(b_1)$. We want to extend the order preserving map $a_i \mapsto a_i$ to an order preserving map $A \cup \{b_1\} \rightarrow O_2$. For this we have an image $b_2$ of $b_1$. There are four cases

1. $b_1 \in A$, we set $b_2 = b_1$
2. $b_1 \in (a_i, a_{i+1})$. We choose $b_2$ in $O_2$ with the same property
3. $b_1$ is smaller than all elements of $A$. We choose a $b_2 \in O_2$ of the same kind
4. $b_1$ is bigger than all $a_i$. Choose $b_2$ in the same manner

This defines an isomorphism $A \cup \{b_1\} \rightarrow A \cup \{b_2\}$, which show that $O_2 \vDash \rho(b_2)$ □

**Modules**. Let $R$ be a (possibly non-commutative) ring with 1. An $R$-module

$$\mathfrak{M} = (, 0, +, -, r)_{r \in R}$$

is an abelian group $(M, 0, +, -)$ together with operations $r : M \to M$ for every ring element $r \in R$. We formulate the axioms in the language $L_{Mod}(R) = L_{AbG} \cup \{r \mid r \in R\}$. The theory $\mathsf{Mod}(R)$ of $R$-modules consists of

$$\begin{aligned}
&\mathsf{AbG}\\
&\forall x, y \; r(x + y) \doteq rx + ry\\
&\forall x \; (r + s)x \doteq rx + sx\\
&\forall x \; (rs)x \doteq r(sx)\\
&\forall x \; 1x \doteq x
\end{aligned}$$

for all $r, s \in R$. Then $\mathsf{Infset} \cup \mathsf{Mod}(R)$ is the theory of all infinite $R$-modules

A module over fields is a vector space

**Theorem 3.26.** *Let $K$ be a field. Then the theory of all infinite $K$-vector spaces has quantifier elimination and is complete*

*Proof.* Let $A$ be a common finitely generated substructure (i.e., a subspace) of the two infinite $K$-vector spaces $V_1$ and $V_2$. Let $\exists y \rho(y)$ be a simple existential $L(A)$-sentence which holds in $V_1$. Choose a $b_1$ from $V_1$ which satisfies $\rho(y)$. If $b_1$ belongs to $A$, we finished. If not, we choose a $b_2 \in V_2 \setminus A$. Possibly we have to replace $V_2$ by an elementary extension. The vector spaces $A + Kb_1$ and $A + Kb_2$ are isomorphic by an isomophism which maps $b_1$ to $b_2$ and fixes $A$ elementwise. Hence $V_2 \models \rho(b_2)$

The theory is complete since a quantifier-free sentence is true in a vector space iff it is true in the zero-vector space. $\square$

**Definition 3.27.** An **equation** is an $L_{Mod}(R)$-formula $\gamma(\bar{x})$ of the form

$$r_1 x_1 + \cdots + r_m x_m = 0$$

A **positive primitive** formula (**pp**-formula) is of the form

$$\exists \bar{y} (\gamma_1 \wedge \cdots \wedge \gamma_n)$$

where the $\gamma_i(\overline{xy})$ are equations

**Theorem 3.28.** *For every ring $R$ and any $R$-module $M$, every $L_{Mod}(R)$-formula is equivalent (modulo the theory of $M$) to a Boolean combination of positive primitive formulas*

**Algebraically closed fields**.

**Theorem 3.29** (Tarski)**.** *The theory ACF of algebraically closed fields has quantifier elimination*

*Proof.* Let $K_1$ and $K_2$ be two algebraically closed fields and $R$ a common subring. Let $\exists y \rho(y)$ be a simple existential sentence with parameters in $R$ which hold in $K_1$. We have to show that $\exists y \rho(y)$ is also true in $K_2$.

Let $F_1$ and $F_2$ be the quotient fields of $R$ in $K_1$ and $K_2$, and let $f : F_1 \to F_2$ be an isomorphism which is the identity on $R$. Then $f$ extends to an isomorphism $g : G_1 \to G_2$ between the relative algebraic closures $G_i$ of $F_i$ in $K_i$. $\square$

# 4 Countable Models

## 4.1 The omitting types theorem

**Definition 4.1.** Let $T$ be an $L$-theory and $\Sigma(x)$ a set of $L$-formulas. A model $\mathfrak{A}$ of $T$ not realizing $\Sigma(x)$ is said to **omit** $\Sigma(x)$. A formula $\varphi(x)$ **isolates** $\Sigma(x)$ if

1. $\varphi(x)$ is consistent with $T$
2. $T \models \forall x(\varphi(x) \to \sigma(x))$ for all $\sigma(x) \in \Sigma(x)$

A set of formulas is often called a **partial type**.

**Theorem 4.2** (Omitting Types)**.** *If $T$ is countable and consistent and if $\Sigma(x)$ is not isolated in $T$, then $T$ has a model which omits $\Sigma(x)$*

If $\Sigma(x)$ is isolated by $\varphi(x)$ and $\mathfrak{A}$ is a model of $T$, then $\Sigma(x)$ is realised in $\mathfrak{A}$ by all realisations $\varphi(x)$. Therefore the converse of the theorem is true for **complete** theories $T$: if $\Sigma(x)$ is isolated in $T$, then it is realised in every model of $T$

*Proof.* We choose a countable set $C$ of new constants and extend $T$ to a theory $T^*$ with the following properties

1. $T^*$ is a Henkin theory: for all $L(C)$-formulas $\psi(x)$ there exists a constant $c \in C$ with $\exists x \psi(x) \to \psi(c) \in T^*$
2. for all $c \in C$ there is a $\sigma(x) \in \Sigma(x)$ with $\neg\sigma(c) \in T^*$

We construct $T^*$ inductively as the union of an ascending chain

$$T = T_0 \subseteq T_1 \subseteq T_1 \subseteq \ldots$$

of consistent extensions of $T$ by finitely many axioms from $L(C)$, in each step making an instance of (1) or (2) true.

Enumerate $C = \{c_i \mid i < \omega\}$ and let $\{\psi_i(x) \mid i < \omega\}$ be an enumeration of the $L(C)$-formulas

Assume that $T_{2i}$ is the already constructed. Choose some $c \in C$ which doesn't occur in $T_{2i} \cup \{\psi_i(x)\}$ and set $T_{2i+1} = T_{2i} \cup \{\exists x \psi_i(x) \to \psi_i(c)\}$.

Up to equivalence $T_{2i+1}$ has the form $T \cup \{\delta(c_i, \bar{c})\}$ for an $L$-formula $\delta(x, \bar{y})$ and a tuple $\bar{c} \in C$ which doesn't contain $c_i$. Since $\exists \bar{y} \delta(x, \bar{y})$ doesn't isolate $\Sigma(x)$, for some $\sigma \in \Sigma$ the formula $\exists \bar{y} \delta(x, \bar{y}) \wedge \neg \sigma(x)$ is consistent with $T$. Thus $T_{2i+2} = T_{2i+1} \cup \{\neg \sigma(c_i)\}$ is consistent

Take a model $(\mathfrak{A}', a_c)_{c \in C}$ of $T^*$. Since $T^*$ is a Henkin theory, Tarski's Test 2.2 shows that $A = \{a_c \mid c \in C\}$ is the universe of an elementary substructure $\mathfrak{A}$ (Lemma 2.7). By property (2), $\Sigma(x)$ is omitted in $\mathfrak{A}$ □

**Corollary 4.3.** *Let $T$ be countable and consistent and let*

$$\Sigma_0(x_0, \ldots, x_{n_0}), \Sigma_1(x_1, \ldots, x_{n_1}), \ldots$$

*be a sequence of partial types. If all $\Sigma_i$ are not isolated, then $T$ has a model which omits all $\Sigma_i$*

*Proof.* If $\Sigma_0(x), \Sigma_1(x), \ldots$. Then $T_{2i+2} = T_{2i+1} \cup \{\neg \sigma_m(c_{mn})\}$

If $\Sigma(x_1, \ldots, x_n)$, then $T_{2i+1} = T_{2i} \cup \{\exists \bar{x} \psi_i(\bar{x}) \to \psi_i(\bar{c})\}$.

Combine the two case □

## 4.2 The space of types

Fix a theory $T$. An $n$**-type** is a maximal set of formulas $p(x_1, \ldots, x_n)$ consistent with $T$. We denote by $S_n(T)$ the set of all $n$-types of $T$. We also write $S(T)$ for $S_1(T)$. $S_0(T)$ is all complete extensions of $T$

If $B$ is a subset of an $L$-structure $\mathfrak{A}$, we recover $S_n^{\mathfrak{A}}(B)$ as $S_n(\mathrm{Th}(\mathfrak{A}_B))$. In particular, if $T$ is complete and $\mathfrak{A}$ is any model of $T$, we have $S^{\mathfrak{A}}(\emptyset) = S(T)$

For any $L$-formula $\varphi(x_1, \ldots, x_n)$, let $[\varphi]$ denote the set of all types containing $\varphi$.

**Lemma 4.4.** 1. *$[\varphi] = [\psi]$ iff $\varphi$ and $\psi$ are equivalent modulo $T$*

2. *The sets $[\varphi]$ are closed under Boolean operations. In fact $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$, $[\varphi] \cup [\psi] = [\varphi \vee \psi]$, $S_n(T) \setminus [\varphi] = [\neg \varphi]$, $S_n(T) = [\top]$ and $\emptyset = [\bot]$*

It follows that the collection of sets of the form $[\varphi]$ is closed under finite intersection and includes $S_n(T)$. So these sets form a basis of a topology on $S_n(T)$

**Lemma 4.5.** *The space $S_n(T)$ is 0-dimensional and compact*

*Proof.* Being 0-dimensional means having a basis of clopen sets. Our basic open sets are clopen since their complements are also basic open

If $p$ and $q$ are two different types, there is a formula $\varphi$ contained in $p$ but not in $q$. It follows that $[\varphi]$ and $[\neg\varphi]$ are open sets which separate $p$ and $q$. This shows that $S_n(T)$ is Hausdorff

Consider a family $[\varphi_i]$, with the finite intersection property.This means that $\varphi_{i_i} \wedge \cdots \wedge \varphi_{i_k}$ are consistent with $T$. So Corollary 2.9 $\{\varphi_i \mid i \in I\}$ is consistent with $T$ and can be extended to a type $p$, which then belongs to all $[\varphi_i]$ □

**Lemma 4.6.** *All clopen subsets of $S_n(T)$ has the form $[\varphi]$*

*Proof.* It follows from Exercise 4.3 that we can separate any two disjoint closed subsets of $S_n(T)$ by a basic open set. □

The Stone duality theorem asserts that the map

$$X \mapsto \{C \mid C \text{ clopen subset of } X\}$$

yields an equivalence between the category of 0-dimensional compact spaces and the category of Boolean algebras. The inverse map assigns to every Boolean algebra to its **Stone space**

**Definition 4.7.** A map $f$ from a subset of a structure $\mathfrak{A}$ to a structure $\mathfrak{B}$ is **elementary** if it preserves the truth of formulas; i.e., $f : A_0 \to B$ is elementary if for every formula $\varphi(x_1, \dots, x_n)$ and $\bar{a} \in A_0$ we have

$$\mathfrak{A} \vDash \varphi(\bar{a}) \Rightarrow \mathfrak{B} \vDash \varphi(f(\bar{a}))$$

**Lemma 4.8.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures, $A_0$ and $B_0$ subsets of $A$ and $B$, respectively. Any elementary map $A_0 \to B_0$ induces a continuous surjective map $S_n(B_0) \to S_n(A_0)$*

*Proof.* If $q(x) \in S_n(B_0)$, we define

$$S(f)(q) = \{\varphi(x_1, \dots, x_n, \bar{a}) \mid \bar{a} \in A_0, \varphi(x_1, \dots, x_n, f(\bar{a})) \in q\}$$

$S(f)$ defines a map from $S_n(B_0)$ to $S_n(A_0)$. Moreover, it is surjective since $\{\varphi(x_1, \dots, x_n, f(\bar{a})) \mid \varphi(x_1, \dots, x_n, a) \in p\}$ is finitely satisfiable for all $p \in S_n(A_0)$. And $S(f)$ is continuous since $[\varphi[x_1, \dots, x_n, f(\bar{a})]]$ is the preimage of $[\varphi(x_1, \dots, x_n, \bar{a})]$ under $S(f)$ □

There are two main cases

1. An elementary bijection $f : A_0 \to B_0$ defines a homeomorphism $S_n(A_0) \to S_n(B_0)$. We write $f(p)$ for the image of $p$

2. If $\mathfrak{A} = \mathfrak{B}$ and $A_0 \subseteq B_0$, the inclusion map induces the **restriction** $S_n(B_0) \to S_n(A_0)$. We write $q\restriction A_0$ for the restriction of $q$ to $A_0$. We call $q$ an extension of $q\restriction A_0$)

**Lemma 4.9.** *A type $p$ is isolated in $T$ iff $p$ is an isolated point in $S_n(T)$. In fact, $\varphi$ isolates $p$ iff $[\varphi] = \{p\}$. That is, $[\varphi]$ is an **atom** in the Boolean algebra of clopen subsets of $S_n(T)$*

*Proof.* If $\varphi$ isolates $p$. Then $\varphi \in p$ and hence $[\varphi] = \{\varphi\}$.

If $[\varphi] = \{p\}$, then $\varphi \in p$. What's more, $\mathfrak{M} \vDash \varphi \Leftrightarrow \mathfrak{M} \vDash p$ in $T$

The set $[\varphi]$ is a singleton iff $[\varphi]$ is non-empty and cannot be divided into two non-empty clopen subsets $[\varphi \wedge \psi]$ and $\varphi \wedge \neg\psi$. This means that for all $\psi$ either $\psi$ or $\neg\psi$ follows from $\varphi$ modulo $T$. So $[\varphi]$ is a singleton iff $\varphi$ generates the type

$$\langle\varphi\rangle = \{\psi(\bar{x}) \mid T \vDash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$$

□

We call a formula $\varphi(x)$ **complete** if

$$\{\psi(\bar{x}) \mid T \vDash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))\}$$

is a type.

**Corollary 4.10.** *A formula isolates a type iff it is complete*

*Exercise* 4.2.1. 1. Closed subsets of $S_n(T)$ have the form $\{p \in S_n(T) \mid \Sigma \subseteq p\}$, where $\Sigma$ is any set of formulas

2. Let $T$ be countable and consistent. Then any meagre[1] subset $X$ of $S_n(T)$ can be omitted, i.e., there is a model which omits all $p \in X$

*Proof.* 1. The sets $[\varphi]$ are a basis for the closed subsets of $S_n(T)$. So the closed sets of $S_n(T)$ are exactly the intersections $\bigcap_{\varphi\in\Sigma}[\varphi] = \{p \in S_n(T) \mid \Sigma \subseteq p\}$

[1] A subset of a topological space is **nowhere dense** if its closure has no interior. A countable union of nowhere dense sets is meagre

2. The set $X$ is the union of a sequence of countable nowhere dense sets $X_i$. We may assume that $X_i$ are closed, i.e., of the form $\{p \in S_n(T) \mid \Sigma_i \subseteq p\}$. That $X_i$ has no interior means that $\Sigma_i$ is not isolated. The claim follows now from Corollary 4.3

□

*Exercise* 4.2.2. Consider the space $S_\omega(T)$ of all complete types in variables $v_0, v_1, \ldots$. Note that $S_\omega(T)$ is again a compact space and therefore not meagre by Baire's theorem

1. Show that $\{\mathrm{tp}(a_0, a_1, \ldots) \mid$ the $a_i$ enumerate a model of $T\}$ is comeagre in $S_\omega(T)$

#+END_#+BEGIN$_{\text{exercise}}$

### 4.3 $\aleph_0$-categorical theories

**Theorem 4.11.** *Let $T$ be a countable complete theory. Then $T$ is $\aleph_0$-categorical iff for every $n$ there are only finitely many formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence relative to $T$*

**Definition 4.12.** An $L$-structure $\mathfrak{A}$ is $\omega$**-saturated** if all types over finite subsets of $A$ are realised in $\mathfrak{A}$

The types in the definition are meant to be 1-types. On the other hand, it is not hard to see that an $\omega$-saturated structure realises all $n$-types over finite sets (Exercise **??**) for all $n \geq 1$. The following lemma is a generalisation of the $\aleph_0$-categoricity of DLO.

**Lemma 4.13.** *Two elementarily equivalent, countable and $\omega$-saturated structures are isomorphic*

*Proof.* Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are as in the lemma. We choose enumerations $A = \{a_0, a_1, \ldots\}$ and $B = \{b_0, b_1, \ldots\}$. Then we construct an ascending sequence $f_0 \subseteq f_1 \subseteq \cdots$ of finite elementary maps

$$f_i : A_i \to B_i$$

between finite subsets of $\mathfrak{A}$ and $\mathfrak{B}$. We will choose the $f_i$ in such a way that $A$ is the union of the $A_i$ and $B$ the union of the $B_i$. The union of the $f_i$ is then the desired isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$

The empty map $f_0 = \emptyset$ is elementary since $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. Assume that $f_i$ is already constructed. There are two cases:

$i = 2n$; We will extend $f_i$ to $A_{i+1} = A_i \cup \{a_n\}$. Consider the type

$$p(x) = \text{tp}(a_n/A_i) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a_n), \varphi(x) \text{ a } L(A_i)\text{-formula}\}$$

Since $f_i$ is elemantarily, $f_i(p)(x)$ is in $\mathfrak{B}$ a type over $B_i$. Since $\mathfrak{B}$ is $\omega$-saturated, there is a realisation $b'$ of this type. So for $\bar{a} \in A_i$

$$\mathfrak{A} \models \varphi(a_n, \bar{a}) \Rightarrow \mathfrak{B} \models \varphi(b', f_i(\bar{a}))$$

This shows that $f_{i+1}(a_n) = b'$ defines an elementary extension of $f_i$

$i = 2n + 1$; we exchange $\mathfrak{A}$ and $\mathfrak{B}$ □

*Proof of Theorem 4.11.* □

# 5 TODO Don't understand

Lemma 3.22

Exercise 3.2.2