Nonstandard Analysis

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目录

1　实数域和超结构的非标准扩张

设 $\mathbb{R}$ 为所有 (标准) 实数的集合而

$$\mathcal{R} := (\mathbb{R}; +, *, 0, 1, <)$$

表示 (标准) 实数域

1.1　非主超滤子存在性

Definition 1.1. X be an infinite set. $\mathcal{U} \subset \mathcal{P}(X)$. if any finite intersection is infinite, then $\mathcal{U}$ is a 非主滤子基 on X. 非主滤子基 $\mathcal{U}$ is a 非主滤子 on X if

1. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$
2. if $A \in \mathcal{U}$ and $A \subseteq B \subseteq X$,then $B \in \mathcal{U}$

   let $X \setminus A$ be A^C. if 非主滤子再满足

3. for any $A \subseteq X$ there is $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$

   then $\mathcal{U}$ is 非主超滤子 on X

Proposition 1.2 (Suppose Zorn's Lemma)**.** *Any nonprincipal filter on infinite set X can be extended to a nonprincipal ultrafilter*

1.2　实数域的非标准扩张

By we could fix a nonprincipal ultrafilter $\mathcal{U}$ on ω.

Definition 1.3. For any $\langle x_n \rangle, \langle y_n \rangle \in \mathbb{R}^\omega$, we define $\langle x_n \rangle \sim \langle y_n \rangle$ iff

$$\{n \in \omega : x_n = y_n\} \in \mathcal{U}$$

In other words, $\langle x_n \rangle$ and $\langle y_n \rangle$ is equivalent iff two sequences equal almost everywhere

Definition 1.4. $[\langle x_n \rangle] := \{\langle y_n \rangle \in \mathbb{R}^\omega : \langle y_n \rangle \sim \langle x_n \rangle\}$

$$^*\mathbb{R} := \{[\langle x_n \rangle] : \langle x_n \rangle \in \mathbb{R}^\omega\}$$

Definition 1.5. For any m-ary relation P on $\mathbb{R}$ we define

$$^*P := \{([\langle r_n^{(1)}\rangle], \dots, [\langle r_n^{(m)}\rangle]) : \{n \in \omega : (r_n^{(1)}, \dots, r_n^{(m)}) \in P\} \in \mathcal{U}\}$$

Hence

$$[\langle a_n\rangle] + [\langle b_n\rangle] = [\langle a_n + b_n\rangle] \quad \text{and} \quad [\langle a_n\rangle] * [\langle b_n\rangle] = [\langle a_n * b_n\rangle]$$

since $\{n \in \omega : (a_n, b_n, a_n + b_n) \in P_+\} = \omega$

$^*\mathbb{R} := (^*\mathbb{R}; +, *, 0, 1, <)$ is an ordered field

For any $r \in \mathbb{R}$, we have $[\langle r\rangle] \in {}^*\mathbb{R}$ and hence we could regard $\mathbb{R}$ as a subset of $^*\mathbb{R}$

if $[\langle r_n\rangle] \in {}^*\mathbb{R}$ is larger than every element of $\mathbb{R}$, then we call $[\langle r_n\rangle]$ 无穷大, and 无穷小 vice versa

Proposition 1.6. *In* $^*\mathbb{R}$, $[\langle n\rangle]$ 是无穷大, $[\langle 1/n\rangle]$ 是无穷小

1.3 超结构的非标准扩张

Definition 1.7. For any set X, let $V_0 := \mathbb{R} \cup X$. For any $m \in \omega$ define $V_{m+1} := V_m \cup \mathcal{P}(V_m)$. Choose a large enough natural number $\mathfrak{n}$, define

$$V := \bigcup_{m=0}^{\mathfrak{n}} V_m$$

let $\in$ be a 从属关系 on V. Then structure $\mathcal{V} = ⊠V; \in$ is called 超结构. If $a \in V$, $a \in V_m$ but $a \notin V_{m-1}$, then we call a $\mathcal{V}$ 中的第 m 层元素, written $l(a) = m$

Definition 1.8. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on ω, $m \leq \mathfrak{n}$

1. For any element sequence $\langle a_n\rangle, \langle b_n\rangle \in V_m$, define $\langle a_n\rangle \sim \langle b_n\rangle$ iff

$$\{n \in \omega : a_n = b_n\} \in \mathcal{U}$$

2. For any element sequence $\langle a_n\rangle \in V_m^\omega$, define $[\langle a_n\rangle] := \{\langle b_n\rangle \in V_m^\omega : \langle a_n\rangle \sim \langle b_n\rangle\}$
3. $^*V_m := \{[\langle a_n\rangle] : \langle a_n\rangle \in V_m^\omega\}$

4. ${}^*V := \bigcup_{m=0}^{\mathfrak{n}} {}^*V_m$
5. for $[\langle a_n \rangle], [\langle A_n \rangle] \in {}^*V$, define $[\langle a_n \rangle]^* \in [\langle A_n \rangle]$ iff

$$\{n \in \omega : a_n \in A_n\} \in \mathcal{U}$$

Lemma 1.9. *suppose* $\varphi(\overline{[\langle a_n \rangle]})$ *is a statement on* ${}^*\mathcal{V}$

1.4 exercise

Exercise 1.4.1. $A = \{[\langle 1 \rangle], [\langle 2 \rangle], \dots\}$

2 非标准分析和微积分

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4 非标准分析和随机过程

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