

# Modal Logic

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September 26, 2020

## Contents

<b>1</b>	<b>Basic Concepts</b>	<b>2</b>
1.1	Modal Languages . . . . .	2
1.2	Models and Frames . . . . .	3
1.3	General Frames . . . . .	4
<b>2</b>	<b>Models</b>	<b>4</b>
2.1	Invariance Results . . . . .	4
2.1.1	Disjoint Unions . . . . .	4
2.1.2	Generated submodels . . . . .	4
2.1.3	Morphism for modalities . . . . .	5
2.2	Bisimulations . . . . .	6
2.3	Finite Models . . . . .	9
2.3.1	Selecting a finite submodel . . . . .	10
2.3.2	Finite models via filtrations . . . . .	12
2.4	The Standard Translation . . . . .	14
2.5	Modal Saturation via Ultrafilter Extensions . . . . .	16
2.5.1	M-saturation . . . . .	16
2.5.2	Ultrafilter extensions . . . . .	17
2.6	Characterization and Definability . . . . .	25
2.6.1	The van Benthem Characterization Theorem . . . . .	25
2.6.2	Ultraproducts . . . . .	27
2.6.3	Definability . . . . .	29

# 1 Basic Concepts

## 1.1 Modal Languages

**Definition 1.1.** The **basic modal language** is defined using a set of **proposition letters**  $\Phi$  whose elements are usually denoted  $p, q, r$  and so on, and a unary modal operator  $\Diamond$ . The well-formed **formulas**  $\phi$  of the basic modal language are given by the rule

$$\phi := p \mid \perp \mid \neg\phi \mid \psi \vee \phi \mid \Diamond\phi$$

$$\mathfrak{M}, w \Vdash \phi$$

**Definition 1.2.** A **modal similarity type** is a pair  $\tau = (O, \rho)$  where  $O$  is a non-empty set, and  $\rho$  is a function  $O \rightarrow \mathbb{N}$ . The elements of  $O$  are called **modal operators**; we use  $\Delta, \Delta_0, \Delta_1, \dots$  to denote elements of  $O$ . The function  $\rho$  assigns to each operator  $\delta \in O$  a finite **arity**

**Definition 1.3.** A **modal language**  $ML(\tau, \Phi)$  is built up using a modal similarity type  $\tau = (O, \rho)$  and a set of proposition letters  $\Phi$ . The set  $Form(\tau, \Phi)$  of **modal formulas** over  $\tau$  and  $\Phi$  is given by the rule

$$\phi := p \mid \perp \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \Delta(\phi_1, \dots, \phi_{\rho(\Delta)})$$

where  $p$  ranges over elements of  $\Phi$

**Definition 1.4.** For each  $\Delta \in O$  the **dual**  $\nabla$  of  $\Delta$  is defined as  $\nabla(\phi_1, \dots, \phi_n) := \neg\Delta(\neg\phi_1, \dots, \neg\phi_n)$

**Example 1.1** (The Basic Temporal Language). The basic temporal language is built using a set of unary operators  $O = \{\langle F \rangle, \langle P \rangle\}$ . The intended interpretation of a formula  $\langle F \rangle\phi$  is ‘ $\phi$  will be true at some Future time’ and the intended interpretation of  $\langle P \rangle\phi$  is ‘ $\phi$  was true at some Past time.’ This language is called the **basic temporal language**. Their duals are written as  $G$  and  $H$  (‘it is Going to be the case’ and ‘it always Has been the case’)

Let’s denote the converse of a relation  $R$  by  $R^\sim$ . We will call a frame of the form  $(T, R, R^\sim)$  a **bidirectional frame**, and a model built over such a frame a **bidirectional model**. From now on, we will only interpret basic temporal language in bidirectional models. That is, if  $\mathfrak{M} = (T, R, R^\sim, V)$  is a bidirectional model then

$$\begin{aligned} \mathfrak{M}, t \Vdash F\phi & \quad \text{iff} \quad \exists s(Rts \wedge \mathfrak{M}, s \Vdash \phi) \\ \mathfrak{M}, t \Vdash P\phi & \quad \text{iff} \quad \exists s(R^\sim ts \wedge \mathfrak{M}, s \Vdash \phi) \end{aligned}$$

**Example 1.2** (An Arrow Language). The type  $\tau_{\rightarrow}$  of **arrow logic** is a similarity type with modal operators other than diamonds. The language of arrow logic is designed to talk about the objects in arrow structures. The well-formed formulas  $\phi$  are given by

$$\phi := p \mid \perp \mid \neg\phi \mid \phi \vee \psi \mid \phi \circ \psi \mid \otimes\phi \mid 1'$$

$1'$  ('identity') is a nullary modality, the 'converse' operator  $\otimes$  is a diamond, and the 'composition' operator  $\circ$  is a dyadic operator. Possible readings of these operators are:

$1'$	identity	'skip'
$\otimes\phi$	converse	' $\phi$ conversely'
$\phi \circ \psi$	composition	'first $\phi$ , then $\psi$ '

## 1.2 Models and Frames

**Definition 1.5.** A **frame** for the basic modal language is a pair  $\mathfrak{F} = (W, R)$  s.t.

1.  $W$  is a non-empty set
2.  $R$  is a binary relation on  $W$

A **model** for the basic modal language is a pair  $\mathfrak{M} = (\mathfrak{F}, V)$ , where  $\mathfrak{F}$  is a frame for the basic modal language and  $V$  is a function assigning to each proposition letter  $p$  in  $\Phi$  a subset  $V(p)$  of  $W$ . The function  $V$  is called a **valuation**.  $\mathfrak{M}$  is **based on** the frame  $\mathfrak{F}$

**Definition 1.6.** Suppose  $w$  is a state in a model  $\mathfrak{M} = (W, R, V)$ . Then  $\phi$  is **satisfied** in  $\mathfrak{M}$  at state  $w$  if

$$\begin{aligned} \mathfrak{M}, w \Vdash p & \text{ iff } w \in V(p), \text{ where } p \in \Phi \\ \mathfrak{M}, w \Vdash \perp & \text{ iff } \text{never} \\ \mathfrak{M}, w \Vdash \neg\phi & \text{ iff } \text{not } \mathfrak{M}, w \Vdash \phi \\ \mathfrak{M}, w \Vdash \phi \vee \psi & \text{ iff } \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \\ \mathfrak{M}, w \Vdash \Diamond\phi & \text{ iff for some } v \in W \text{ with } R w v \text{ we have } \mathfrak{M}, v \Vdash \phi \end{aligned}$$

It follows that  $\mathfrak{M}, w \Vdash \Box\phi$  iff for all  $v \in W$  s.t.  $R w v$ , we have  $\mathfrak{M}, v \Vdash \phi$

**Definition 1.7.** Let  $\tau$  be a modal similarity type. A  $\tau$ -**frame** is a tuple  $\mathfrak{F}$  consisting of the following ingredients

1. a non-empty set  $W$

2. for each  $n \geq 0$ , and each  $n$ -ary modal operator  $\Delta$  in the similarity type  $\tau$ , an  $(n + 1)$ -ary relation  $R_\Delta$

$\phi$  is **satisfied at a state**  $w$  in a model  $\mathfrak{M} = (W, \{R_\Delta \mid \Delta \in \tau\}, V)$  when  $\rho(\Delta) > 0$  if

$$\mathfrak{M}, w \Vdash \Delta(\phi_1, \dots, \phi_n) \quad \text{iff} \quad \text{for some } v_1, \dots, v_n \in W \text{ with } R_\Delta w v_1 \dots v_n \\ \text{we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i$$

When  $\rho(\Delta) = 0$  we define

$$\mathfrak{M}, w \Vdash \Delta \quad \text{iff} \quad w \in R_\Delta$$

**Definition 1.8.** The set of all formulas that are valid in a class of frames  $\mathbf{F}$  is called the **logic** of  $\mathbf{F}$  (notation:  $\Lambda_{\mathbf{F}}$ )

### 1.3 General Frames

**Definition 1.9.** Given an  $(n + 1)$ -ary relation  $R$  on a set  $W$ , we define the following  $n$ -ary operation  $m_R$  on the power set  $\mathcal{P}(W)$  of  $W$ :

$$m_R(X_1, \dots, X_n) = \{w \in W \mid R w w_1 \dots w_n \text{ for some } w_1 \in X_1, \dots, w_n \in X_n\}$$

## 2 Models

### 2.1 Invariance Results

**Definition 2.1.** Let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be models of the same modal similarity type  $\tau$ , and let  $w$  and  $w'$  be states in  $\mathfrak{M}$  and  $\mathfrak{M}'$  respectively. The  $\tau$ -**theory** (or  $\tau$ -**type**) of  $w$  is the set of all  $\tau$ -formulas satisfied at  $w$ : that is,  $\{\phi \mid \mathfrak{M}, w \Vdash \phi\}$ . We say that  $w$  and  $w'$  are **(modally) equivalent** ( $w \leftrightarrow w'$ ) if they have the same  $\tau$ -theories

The  $\tau$ -**theory** of the model  $\mathfrak{M}$  is the set of all  $\tau$ -formulas satisfied by all states in  $fM$ ; that is,  $\{\phi \mid \mathfrak{M} \Vdash \phi\}$ . Models  $\mathfrak{M}$  and  $\mathfrak{M}'$  are called **(modally) equivalent** ( $\mathfrak{M} \leftrightarrow \mathfrak{M}'$ ) if their theories are identical

#### 2.1.1 Disjoint Unions

#### 2.1.2 Generated submodels

**Definition 2.2.** Let  $\mathfrak{M} = (W, R, V)$  and  $\mathfrak{M}' = (W', R', V')$  be two models; we say that  $\mathfrak{M}'$  is a **submodel** of  $\mathfrak{M}$  if  $W' \subseteq W$ ,  $R'$  is the restriction of  $R$

to  $W'$ , and  $V'$  is the restriction of  $V$  to  $\mathfrak{M}'$ . We say that  $\mathfrak{M}'$  is a **generated submodel** of  $\mathfrak{M}$  ( $\mathfrak{M}' \succrightarrow \mathfrak{M}$ ) if  $\mathfrak{M}'$  is a submodel of  $\mathfrak{M}$  and for all points  $w$  the following closure condition holds

$$\text{if } w \text{ is in } \mathfrak{M}' \text{ and } R w v, \text{ then } v \text{ is in } \mathfrak{M}'$$

Let  $fM$  be a model, and  $X$  a subset of the domain of  $\mathfrak{M}$ ; the **submodel generated by  $X$**  is the smallest generated submodel of  $\mathfrak{M}$  whose domain contains  $X$ . A **rooted** or **point generated** model is a model that is generated by a singleton set, the element of which is called the **root** of the frame

### 2.1.3 Morphism for modalities

**Definition 2.3** (Homomorphisms). Let  $\tau$  be a modal similarity type and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be  $\tau$ -models. By a **homomorphism**  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ , we mean a function  $f : W \rightarrow W'$  satisfying

1. For each proposition letter  $p$  and each element  $w$  from  $\mathfrak{M}$ , if  $w \in V(p)$ , then  $f(w) \in V'(p)$
2. For each  $n \geq 0$  and each  $n$ -ary  $\triangle \in \tau$  and  $(n+1)$ -tuple  $\bar{w}$  from  $\mathfrak{M}$ , if  $(w_0, \dots, w_n) \in R_\triangle$ , then  $(f(w_0), \dots, f(w_n)) \in R'_\triangle$  (the **homomorphic condition**)

**Definition 2.4** (Strong Homomorphisms, Embeddings and Isomorphisms). Let  $\tau$  be a modal similarity type and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be  $\tau$ -models. By a **strong homomorphism**  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ , we mean a function  $f : W \rightarrow W'$  satisfying

1. For each proposition letter  $p$  and each element  $w$  from  $\mathfrak{M}$  iff  $w \in V(p)$ , then  $f(w) \in V'(p)$
2. For each  $n \geq 0$  and each  $n$ -ary  $\triangle \in \tau$  and  $(n+1)$ -tuple  $\bar{w}$  from  $\mathfrak{M}$  iff  $(w_0, \dots, w_n) \in R_\triangle$ , iff  $(f(w_0), \dots, f(w_n)) \in R'_\triangle$  (the **strong homomorphic condition**)

An **embedding** of  $\mathfrak{M}$  into  $\mathfrak{M}'$  is a strong homomorphism  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$  which is injective. An **isomorphism** is a bijective strong homomorphism

**Proposition 2.5.** *Let  $\tau$  be a modal similarity type and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be  $\tau$ -models. Then the following holds*

1. *for all elements  $w$  and  $w'$  of  $\mathfrak{M}$  and  $\mathfrak{M}'$ , respectively, if there exists a surjective strong homomorphism  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$  with  $f(w) = w'$ , then  $w$  and  $w'$  are modally equivalent*
2. *If  $\mathfrak{M} \cong \mathfrak{M}'$ , then  $\mathfrak{M} \leftrightarrow \mathfrak{M}'$*

**Definition 2.6** (Bounded Morphisms - the Basic Case). Let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be models for the basic modal language. A mapping  $f : \mathfrak{M} = (W, R, V) \rightarrow \mathfrak{M}' = (W', R', V')$  is a **bounded morphism** if it satisfies

1.  $w$  and  $f(w)$  satisfy the same proposition letters
2.  $f$  is a homomorphism w.r.t. the relation  $R$  (if  $Rwv$  then  $R'f(w)f(v)$ )
3. If  $R'f(w)v'$  then there exists  $v$  s.t.  $Rwv$  and  $f(v) = v'$  (the **back condition**)

If there is a **surjective** bounded morphism from  $\mathfrak{M}$  to  $\mathfrak{M}'$ , then we say that  $\mathfrak{M}'$  is a **bounded morphic image** of  $\mathfrak{M}$ , and write  $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$

**Proposition 2.7.** *Let  $\tau$  be a modal similarity type and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be  $\tau$ -models s.t.  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$  is a bounded morphism. Then for each modal formula  $\phi$ , and each element  $w$  of  $\mathfrak{M}$  we have  $\mathfrak{M}, w \Vdash \phi$  iff  $\mathfrak{M}', f(w) \Vdash \phi$ .*

Let  $\tau$  be a modal similarity type containing only diamonds (thus if  $\mathfrak{M}$  is a  $\tau$ -model, it has the form  $(W, R_1, \dots, V)$  where each  $R_i$  is a binary relation on  $W$ ). In this context we will call a  $\tau$ -model  $\mathfrak{M}$  **tree-like** if the structure  $(W, \bigcup_i R_i, V)$  is a tree

**Proposition 2.8.** *Assume that  $\tau$  is a modal similarity type containing only diamonds. Then for any rooted  $\tau$ -models  $\mathfrak{M}$  there exists a tree-like  $\tau$ -models  $\mathfrak{M}'$  s.t.  $\mathfrak{M}' \twoheadrightarrow \mathfrak{M}$ . Hence any satisfiable  $\tau$ -formula is satisfiable in a tree-like model*

*Proof.* Let  $w$  be the root of  $\mathfrak{M}$ . Define the model  $\mathfrak{M}'$  as follows. Its domain  $W'$  consist of all finite sequences  $(w, u_1, \dots, u_n)$  s.t.  $n \geq 0$  and for some modal operators  $\langle a_1 \rangle, \dots, \langle a_n \rangle \in \tau$  there is a path  $wR_{a_1}u_1 \dots R_{a_n}u_n$  in  $\mathfrak{M}$ . Define  $(w, u_1, \dots, u_n)R'_a(w, v_1, \dots, v_m)$  to hold if  $m = n + 1, u_i = v_i$  for  $i = 1, \dots, n$  and  $R_a u_n v_m$  holds in  $\mathfrak{M}$ . That is,  $R'_a$  relates two sequences iff the second is an extension of the first with a state from  $\mathfrak{M}$  that is a successor of the last element of the first sequence. Finally,  $V'$  is defined by putting  $(w, u_1, \dots, u_n) \in V'(p)$  iff  $u_n \in V(p)$ . The mapping  $f : (w, u_1, \dots, u_n) \mapsto u_n$  defines a surjective bounded morphism from  $\mathfrak{M}'$  to  $\mathfrak{M}$   $\square$

## 2.2 Bisimulations

**Definition 2.9** (Bisimulation - the Basic Case). Let  $\mathfrak{M} = (W, R, V)$  and  $\mathfrak{M}' = (W', R', V')$  be two models

A non-empty binary relation  $Z \subseteq W \times W'$  is called a **bisimulation between**  $\mathfrak{M}$  and  $\mathfrak{M}'$  (notation:  $Z : \mathfrak{M} \rightleftharpoons \mathfrak{M}'$ ) if

1. If  $wZw'$  then  $w$  and  $w'$  satisfy the same proposition letters
2. If  $wZw'$  and  $Rwv$ , then there exists  $v'$  (in  $\mathfrak{M}'$ ) s.t.  $vZv'$  and  $R'w'v'$  (the **forth condition**)
3. The converse of (2): if  $wZw'$  and  $R'w'v'$ , then there exists  $v$  (in  $\mathfrak{M}$ ) s.t.  $vZv'$  and  $Rwv$  (the **back condition**)

When  $Z$  is a bisimulation linking two states  $w$  in  $\mathfrak{M}$  and  $w'$  in  $\mathfrak{M}'$  we say that  $w$  and  $w'$  are **bisimilar**, and we write  $Z : \mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$ . If there is a bisimulation, we sometimes write  $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$  or  $w \rightleftharpoons w'$

**Definition 2.10** (Bisimulation - the General Case). Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M} = (W, R_\Delta, V)_{\Delta \in \tau}$  and  $\mathfrak{M}' = (W', R'_\Delta, V')_{\Delta \in \tau}$  be  $\tau$ -models. A non-empty binary relation  $Z \subseteq W \times W'$  is called a **bisimulation** between  $\mathfrak{M}$  and  $\mathfrak{M}'$  ( $Z : \mathfrak{M} \rightleftharpoons \mathfrak{M}'$ ) if the above condition 1 is satisfied and

2. If  $wZw'$  and  $R_\Delta wv_1 \dots v_n$  then there are  $v'_1, \dots, v'_n \in W'$  s.t.  $R'_\Delta w'v'_1 \dots v'_n$  and for all  $i$  ( $1 \leq i \leq n$ )  $v_i Z v'_i$  (the **forth** condition)
3. If  $wZw'$  and  $R'_\Delta w'v'_1 \dots v'_n$  then there are  $v_1, \dots, v_n \in W$  s.t.  $R_\Delta wv_1 \dots v_n$  and for all  $i$  ( $1 \leq i \leq n$ )  $v_i Z v'_i$  (the **back** condition)

**Proposition 2.11.** Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}, \mathfrak{M}'$  and  $\mathfrak{M}_i$  ( $i \in I$ ) be  $\tau$ -models

1. If  $\mathfrak{M} \cong \mathfrak{M}'$ , then  $\mathfrak{M} \rightleftharpoons \mathfrak{M}'$
2. For every  $i \in I$ , and every  $w$  in  $\mathfrak{M}_i$ ,  $\mathfrak{M}_i, w \rightleftharpoons \biguplus_i \mathfrak{M}_i, w$
3. If  $\mathfrak{M}' \succrightarrow \mathfrak{M}$ , then  $\mathfrak{M}', w \rightleftharpoons \mathfrak{M}, w$  for all  $w$  in  $\mathfrak{M}'$
4. If  $f : \mathfrak{M} \rightarrow \mathfrak{M}'$ , then  $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', f(w)$  for all  $w$  in  $\mathfrak{M}$

*Proof.* Suppose  $\mathfrak{M} = (W, R_\Delta, V)_{\Delta \in \tau}$  and  $\mathfrak{M}' = (W', R'_\Delta, V')_{\Delta \in \tau}$   $\mathfrak{M}_i \subseteq \biguplus_i \mathfrak{M}_i$

1. Suppose  $f : \mathfrak{M} \cong \mathfrak{M}'$ , then we define  $wZw'$  iff  $w' = f(w)$  where  $w \in W, w' \in W'$ . Bisimulation comes from the definition of the isomorphism
2. Define the relation  $Z = \{(w, w) \mid w \in \mathfrak{M}_i\} \subseteq \mathfrak{M}_i \times \biguplus_i \mathfrak{M}_i$ . The first condition comes from the invariance. The forth condition is obvious. For the back condition, if  $R'_\Delta w'v'_1 \dots v'_n$  and  $w' \in W$ , then  $v'_1, \dots, v'_n \in W$  since each  $R_{\Delta, i}$  is disjoint and we have  $R_{\Delta, i} w'v'_1 \dots v'_n$
3. Define the relation  $Z = \{(w, w) \mid w \in \mathfrak{M}'\} \subseteq \mathfrak{M}' \times \mathfrak{M}$ . The first condition comes from the invariance. Forth condition is obvious. For the back condition, suppose  $wZw$  and  $R'_\Delta wv'_1 \dots v'_n$ , by the definition,  $v'_1, \dots, v'_n \in W$  and  $R_\Delta wv'_1 \dots v'_n$
4. Define  $Z = \{(w, f(w)) \mid w \in W\}$ . The first condition comes from the definition. If  $wZw'$  and  $R_\Delta wv_1 \dots v_n$ , then  $R'_\Delta f(w)f(v_1) \dots f(v_n)$ . If  $wZw'$  and  $R'_\Delta w'v'_1 \dots v'_n$ , then there is  $v_1, \dots, v_n$  s.t.  $R_\Delta wv_1, \dots, v_n$  and  $f(v_i) = v'_i$  for  $1 \leq i \leq n$

□

**Theorem 2.12.** *Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}, \mathfrak{M}'$  be  $\tau$ -models. Then, for every  $w \in W$  and  $w' \in W'$ ,  $w \simeq w'$  implies that  $w \leftrightarrow w'$ . In other words, modal formulas are invariant under bisimulation*

*Proof.* Induction on the complexity of  $\phi$ .

Suppose  $\phi$  is  $\diamond\psi$ , we have  $\mathfrak{M}, w \Vdash \diamond\psi$  iff there exists a  $v$  in  $\mathfrak{M}$  s.t.  $Rwv$  and  $\mathfrak{M}, v \Vdash \psi$ . As  $w \simeq w'$ , there exists a  $v'$  in  $\mathfrak{M}'$  s.t.  $R'w'v'$  and  $v \simeq v'$ . By the I.H.,  $\mathfrak{M}', v' \Vdash \psi$ , hence  $\mathfrak{M}', w' \Vdash \diamond\psi$   $\square$

**Example 2.1** (Bisimulation and First-Order Logic).

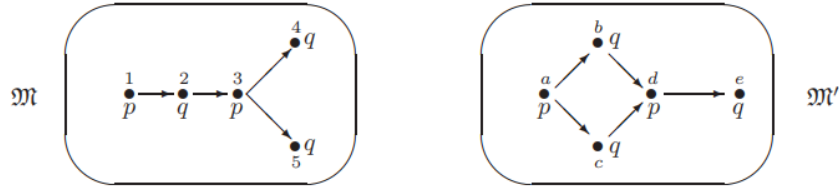


Fig. 2.4. Bisimilar models.

**Example 2.2.**



Fig. 2.5. Equivalent but not bisimilar.

$\mathfrak{M}$  is **image-finite** if for each state  $u$  in  $\mathfrak{M}$  and each relation  $R$  in  $\mathfrak{M}$ , the set  $\{(v_1, \dots, v_n) \mid Ruv_1 \dots v_n\}$  is finite

**Theorem 2.13** (Hennessy-Milner Theorem). *Let  $\tau$  be a modal similarity type and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be two image-finite  $\tau$ -models. Then for every  $w \in W$  and  $w' \in W'$ ,  $w \simeq w'$  iff  $w \leftrightarrow w'$*

*Proof.* Assume that our similarity type  $\tau$  only contains a single diamond. The direction from left to right follows from Theorem 2.12

Suppose  $w \leftrightarrow w'$ . The first condition is immediate. If  $Rwv$ , assume there is no  $v'$  in  $\mathfrak{M}'$  with  $R'w'v'$  and  $v \leftrightarrow v'$ . Let  $S' = \{u' \mid R'w'u'\}$ .



Note that  $S'$  must be non-empty, for otherwise  $\mathfrak{M}', w' \Vdash \Box \perp$ , which would contradict  $w \leftrightarrow w'$  since  $\mathfrak{M}, w \Vdash \Diamond \top$ . Furthermore, as  $\mathfrak{M}'$  is image-finite,  $S'$  must be finite, say  $S' = \{w'_1, \dots, w'_n\}$ . By assumption, for every  $w'_i \in S'$  there exists a formula  $\psi_i$  s.t.  $\mathfrak{M}, v \Vdash \psi_i$ , but  $\mathfrak{M}', w'_i \not\Vdash \psi_i$ . It follows that

$$\mathfrak{M}, w \Vdash \Diamond(\psi_1 \wedge \dots \wedge \psi_n) \quad \text{and} \quad \mathfrak{M}', w' \not\Vdash \Diamond(\psi_1 \wedge \dots \wedge \psi_n)$$

□

*Exercise 2.2.1.* Suppose that  $\{Z_i \mid i \in I\}$  is a non-empty collection of bisimulations between  $\mathfrak{M}$  and  $\mathfrak{M}'$ . Prove that the relation  $\bigcup_{i \in I} Z_i$  is also a bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$ . Conclude that if  $\mathfrak{M}$  and  $\mathfrak{M}'$  are bisimilar, then there is a maximal bisimulation between  $\mathfrak{M}$  and  $\mathfrak{M}'$ .

*Proof.* 1. If  $(w, w') \in \bigcup_{i \in I} Z_i$ , then  $(w, w') \in Z_j$  for some  $j \in I$  and hence they satisfy the same propositional letters  
 2. If  $(w, w') \in \bigcup_{i \in I} Z_i$  and  $R_\Delta w v_1 \dots v_n$ , since  $(w, w') \in Z_j$  for some  $j \in I$ , we have  $R'_\Delta w' v'_1 \dots v'_n$  and  $v_i Z_j v'_i$  for all  $1 \leq i \leq n$ , which means  $(v_i, v'_i) \in \bigcup_{i \in I} Z_i$  for all  $1 \leq i \leq n$   
 3. similarly

□

*Remark* (Bisimulations for the Basic Temporal Language and Arrow Logic). When working with the basic temporal language, we usually work with models  $(W, R, V)$  and implicitly take  $R_p$  to be  $R^\sim$ . Thus we need a notion of bisimulation between models  $(W, R, V)$  and  $(W', R', V')$  to be a relation  $Z$  between the states of the two models that satisfies the clauses of Definition 2.9, and in addition the following

4. If  $wZw'$  and  $Rvw$ , then there exists  $v'$  in  $\mathfrak{M}'$  s.t.  $vZv'$  and  $R'v'w'$
5. Converse of 4: if  $wZw'$  and  $R'v'w'$ , then there exists  $v$  in  $\mathfrak{M}$  s.t.  $vZv'$

### 2.3 Finite Models

**Definition 2.14** (Finite Model Property). Let  $\tau$  be a modal similarity type, and let  $M$  be a class of  $\tau$ -models. We say that  $\tau$  has the **finite model property w.r.t.**  $M$  if the following holds: if  $\phi$  is a formula of similarity type  $\tau$ , and  $\phi$  is satisfiable in some model in  $M$ , then  $\phi$  is satisfiable in a **finite** model in  $M$

### 2.3.1 Selecting a finite submodel

**Definition 2.15** (Degree). We define the **degree** of modal formulas as follows:

$$\begin{aligned}
 \deg(p) &= 0 \\
 \deg(\perp) &= 0 \\
 \deg(\neg\phi) &= \deg(\phi) \\
 \deg(\phi \vee \psi) &= \max\{\deg(\phi), \deg(\psi)\} \\
 \deg(\Delta(\phi_1, \dots, \phi_n)) &= 1 + \max\{\deg(\phi_1), \dots, \deg(\phi_n)\}
 \end{aligned}$$

**Proposition 2.16.** *Let  $\tau$  be a finite modal similarity type, and assume our collection of proposition letters is finite as well*

1. *for all  $n$ , up to logical equivalence there are only finitely many formulas of degree at most  $n$*
2. *for all  $n$ , and every  $\tau$ -model  $\mathfrak{M}$  and state  $w$  of  $\mathfrak{M}$ , the set of all  $\tau$ -formulas of degree at most  $n$  that are satisfied by  $w$ , is equivalent to a single formula*

**Definition 2.17** ( $n$ -Bisimulation). Let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be models, and let  $w$  and  $w'$  be states of  $\mathfrak{M}$  and  $\mathfrak{M}'$ , respectively. We say that  $w$  and  $w'$  are  $n$ -**bisimilar** ( $w \rightleftharpoons_n w'$ ) if there exists a sequence of binary relations  $Z_n \subseteq \dots \subseteq Z_0$  with the following properties (for  $i + 1 \leq n$ )

1.  $wZ_nw'$
2. if  $vZ_0v'$  then  $v$  and  $v'$  agree on all proposition letters
3. if  $vZ_{i+1}v'$  and  $Rvu$  then there exists  $u'$  with  $R'v'u'$  and  $uZ_iu'$
4. if  $vZ_{i+1}v'$  and  $R'v'u'$ , then there exists  $u$  with  $Rvu$  and  $uZ_iu'$

**Proposition 2.18.** *Let  $\tau$  be a finite modal similarity type,  $\Phi$  a finite set of proposition letters, and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be models for this language. Then for every  $w$  in  $\mathfrak{M}$  and  $w'$  in  $\mathfrak{M}'$ , the following are equivalent*

1.  $w \rightleftharpoons_n w'$
2.  $w$  and  $w'$  agree on all modal formulas of degree at most  $n$ .

*Proof.*  $2 \rightarrow 1$ . if  $n = 0$ , obvious.

If  $n = k$  and the proposition holds. Now suppose  $n = k + 1$ . Now  $w$  and  $w'$  agree on all modal formulas of degree at most  $n + 1$ . If there is not  $v, v'$  s.t.  $v$  and  $v'$  agree on all modal formulas of degree at most  $n$  and  $Rwv$  and  $Rwv'$ . Let  $S' = \{u' \mid R'w'u'\}$  and  $S'$  is finite, say  $S' = \{w'_1, \dots, w'_n\}$ . By assumption, for every  $w'_i \in S'$  there exists a formula  $\psi_i$  of degree at most  $n$  s.t.  $\mathfrak{M}, v \models \psi_i$  but  $\mathfrak{M}', w'_i \not\models \psi_i$ . It follows that

$$\mathfrak{M}, w \models \diamond(\psi_1 \wedge \dots \wedge \psi_n) \text{ and } \mathfrak{M}', w' \not\models \diamond(\psi_1 \wedge \dots \wedge \psi_n)$$

□

**Definition 2.19.** Let  $\tau$  be a modal similarity type containing only diamonds. Let  $\mathfrak{M} = (W, R_1, \dots, R_n, \dots, V)$  be a rooted  $\tau$ -model with root  $w$ . The notion of the **height** of states in  $\mathfrak{M}$  is defined by induction.

The only element of height 0 is the root of the model; the states of height  $n + 1$  are those immediate successors of elements of height  $n$  that have not yet assigned a height smaller than  $n + 1$ . The **height of a model**  $\mathfrak{M}$  is the maximum  $n$  s.t. there is a state of height  $n$  in  $\mathfrak{M}$ , if such a maximum exists; otherwise the height of  $\mathfrak{M}$  is infinite

For a natural number  $k$ , the **restriction** of  $\mathfrak{M}$  to  $k$  ( $\mathfrak{M} \upharpoonright k$ ) is defined as the submodel containing only states whose height is at most  $k$ .  $(\mathfrak{M} \upharpoonright k) = (W_k, R_{1k}, \dots, R_{nk}, \dots, V_k)$ , where  $W_k = \{v \mid \text{height}(v) \leq k\}$ ,  $R_{nk} = R_n \cap (W_k \times W_k)$ , and for each  $p$ ,  $V_k(p) = V(p) \cap W_k$

**Lemma 2.20.** Let  $\tau$  be a modal similarity type that contains only diamonds. Let  $\mathfrak{M}$  be a rooted  $\tau$ -model, and let  $k$  be a natural number. Then for every state  $w$  of  $(\mathfrak{M} \upharpoonright k)$ , we have  $(\mathfrak{M} \upharpoonright k), w \simeq_l \mathfrak{M}, w$ , where  $l = k - \text{height}(w)$

**Theorem 2.21** (Finite Model Property - via Selection). Let  $\tau$  be a modal similarity type containing only diamonds, and let  $\phi$  be a  $\tau$ -formula. If  $\phi$  is satisfiable, then it is satisfiable on a finite model

*Proof.* Fix a modal formula  $\phi$  with  $\deg(\phi) = k$ . We restrict our modal similarity type  $\tau$  and our collection of proposition letters to the modal operators and proposition letters actually occurring in  $\phi$ . Let  $\mathfrak{M}_1, w_1$  be s.t.  $\mathfrak{M}_1, w_1 \models \phi$ . By Proposition 2.8, there exists a tree-like model  $\mathfrak{M}_2$  with root  $w_2$  s.t.  $\mathfrak{M}_2, w_2 \models \phi$ . Let  $\mathfrak{M}_3 := (\mathfrak{M}_2 \upharpoonright k)$ . By Lemma 2.20 we have  $\mathfrak{M}_2, w_2 \simeq_k \mathfrak{M}_3, w_2$  and by Proposition 2.18 it follows that  $\mathfrak{M}_3, w_2 \models \phi$

By induction on  $n \leq k$  we define finite sets of states  $S_0, \dots, S_k$  and a (final) model  $\mathfrak{M}_4$  with domain  $S_0 \cup \dots \cup S_k$ ; the points in each  $S_n$  will have height  $n$

Define  $S_0$  to be the singleton  $\{w_2\}$ . Next, assume that  $S_0, \dots, S_n$  have already been defined. Fix an element  $v$  of  $S_n$ . By Proposition 2.16 there are only finitely many non-equivalent modal formulas whose degree is at most  $k - n$ , say  $\psi_1, \dots, \psi_m$ . For each formula that is of the form  $\langle a \rangle \chi$  and holds in  $\mathfrak{M}_3$  at  $v$ , select a state  $u$  from  $\mathfrak{M}_3$  s.t.  $R_a v u$  and  $\mathfrak{M}_3, u \models \chi$ . Add all these  $u$ s to  $S_{n+1}$ , and repeat this selection process for every state in  $S_n$ .  $S_{n+1}$  is defined as the set of all points that have been selected in this way

Finally, define  $\mathfrak{M}_4$  as follows. Its domain is  $S_0 \cup \dots \cup S_k$ ; as each  $S_i$  is finite,  $\mathfrak{M}_4$  is finite. The relations and valuation are obtained by restricting the relations and valuations of  $\mathfrak{M}_3$  to the domain of  $\mathfrak{M}_4$  □

### 2.3.2 Finite models via filtrations

**Definition 2.22.** A set of formulas  $\Sigma$  is **closed under subformulas** (or **subformula closed**) if for all formulas  $\phi, \phi'$ : if  $\phi \vee \phi' \in \Sigma$  then so are  $\phi$  and  $\phi'$ ; if  $\neg\phi \in \Sigma$  then so is  $\phi$ ; and if  $\triangle(\phi_1, \dots, \phi_n) \in \Sigma$  then so are  $\phi_1, \dots, \phi_n$ .

**Definition 2.23** (Filtrations). We work in the basic modal language. Let  $\mathfrak{M} = (W, R, V)$  be a model and  $\Sigma$  a subformula closed set of formulas. Let  $\leftrightarrow_\Sigma$  be the relation on the states of  $\mathfrak{M}$  defined by

$$w \leftrightarrow_\Sigma v \text{ iff for all } \phi \in \Sigma : (\mathfrak{M}, w \Vdash \phi \text{ iff } \mathfrak{M}, v \Vdash \phi)$$

Note that  $\leftrightarrow_\Sigma$  is an equivalence relation. We denote the equivalence class of a state  $w$  of  $\mathfrak{M}$  w.r.t.  $\leftrightarrow_\Sigma$  by  $|w|_\Sigma$ , or simply  $|w|$ . The mapping  $w \mapsto |w|$  is called the **natural map**

Let  $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$ . Suppose  $\mathfrak{M}_\Sigma^f$  is any model  $(W^f, R^f, V^f)$  s.t.

1.  $W^f = W_\Sigma$
2. if  $Rwv$  then  $R^f|w||v|$
3. if  $R^f|w||v|$  then for all  $\diamond\phi \in \Sigma$ , if  $\mathfrak{M}, v \Vdash \phi$  then  $\mathfrak{M}, w \Vdash \diamond\phi$
4.  $V^f(p) = \{|w| \mid \mathfrak{M}, w \Vdash p\}$ , for all proposition letters  $p$  in  $\Sigma$

$\mathfrak{M}_\Sigma^f$  is called a **filtration of  $fM$  through  $\Sigma$** ; we will often suppress subscripts and write  $\mathfrak{M}^f$  instead of  $\mathfrak{M}_\Sigma^f$

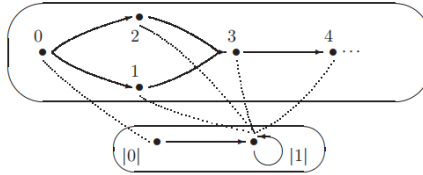


Fig. 2.6. A model and its filtration.

Let  $\mathfrak{M} = (\mathbb{N}, R, V)$ , where  $R = \{(0, 1), (0, 2), (1, 3)\} \cup \{(n, n+1) \mid n \geq 2\}$ , and  $V$  has  $V(p) = \mathbb{N} \setminus \{0\}$  and  $V(q) = \{2\}$

Further assume  $\Sigma = \{\diamond p, p\}$ .  $\Sigma$  is subformula closed. Then, the model  $\mathfrak{N} = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V')$ , where  $V'(p) = \{|1|\}$  is a filtration of  $\mathfrak{M}$  through  $\Sigma$ .  $\mathfrak{N}$  is not a bounded morphic image of  $\mathfrak{M}$ : any bounded morphism would have to preserve the formula  $q$

**Proposition 2.24.** Let  $\Sigma$  be a finite subformula closed set of basic modal formulas. For any model  $\mathfrak{M}$ , if  $\mathfrak{M}^f$  is a filtration of  $\mathfrak{M}$  through a subformula closed set  $\Sigma$ , then  $\mathfrak{M}^f$  contains at most  $2^n$  nodes (where  $n$  denotes the size of  $\Sigma$ )

*Proof.* The states of  $\mathfrak{M}^f$  are the equivalence classes in  $W_\Sigma$ . Let  $g$  be the function with domain  $W_\Sigma$  and range  $\mathcal{P}(\Sigma)$  defined by  $g(|w|) = \{\phi \in \Sigma \mid \mathfrak{M}, w \Vdash \phi\}$ . It follows from the definition of  $\leftrightarrow_\Sigma$  that  $g$  is well defined and injective. Thus  $|W_\Sigma| \leq 2^n, n = |\Sigma|$   $\square$

**Theorem 2.25** (Filtration Theorem). *Consider the basic modal language. Let  $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$  be a filtration of  $\mathfrak{M}$  through a subformula closed set  $\Sigma$ . Then for all formulas  $\phi \in \Sigma$ , and all nodes  $w$  in  $\mathfrak{M}$ , we have  $\mathfrak{M}, w \Vdash \phi$  iff  $\mathfrak{M}^f, |w| \Vdash \phi$*

*Proof.* Suppose  $\diamond\phi \in \Sigma$  and  $\mathfrak{M}, w \Vdash \diamond\phi$ . Then there is a  $v$  s.t.  $Rwv$  and  $\mathfrak{M}, v \Vdash \phi$ . As  $\mathfrak{M}^f$  is a filtration,  $R^f|w||v|$ . As  $\Sigma$  is a subformula closed,  $\phi \in \Sigma$ , thus by the inductive hypothesis  $\mathfrak{M}^f, |v| \Vdash \phi$ . Hence  $\mathfrak{M}^f, |w| \Vdash \diamond\phi$

Suppose  $\diamond\phi \in \Sigma$  and  $\mathfrak{M}^f, |w| \Vdash \diamond\phi$ . Thus there is a state  $|v|$  in  $\mathfrak{M}^f$  s.t.  $R^f|w||v|$  and  $\mathfrak{M}^f, |v| \Vdash \phi$ . As  $\phi \in \Sigma$ , we have  $\mathfrak{M}, v \Vdash \phi$ . By the definition, we have  $\mathfrak{M}, w \Vdash \diamond\phi$   $\square$

Note that clauses 2 and 3 of Definition 2.3.2 are designed to make the modal case of the inductive step go through.

Define

1.  $R^s|w||v|$  iff  $\exists w' \in |w| \exists v' \in |v| R w' v'$
2.  $R^l|w||v|$  iff for all formulas  $\diamond\phi \in \Sigma$ :  $\mathfrak{M}, v \Vdash \phi$  implies  $\mathfrak{M}, w \Vdash \diamond\phi$

These relations give rise to the **smallest** and **largest** filtrations respectively

**Lemma 2.26.** *Consider the basic modal language. Let  $\mathfrak{M}$  be any model,  $\Sigma$  any subformula closed set of formulas,  $W_\Sigma$  the set of equivalence classes induced by  $\leftrightarrow_\Sigma$ , and  $V^f$  the standard valuation on  $W_\Sigma$ . Then both  $(W_\Sigma, R^s, V^f)$  and  $(W_\Sigma, R^l, V^f)$  are filtrations of  $\mathfrak{M}$  through  $\Sigma$ . Furthermore, if  $(W_\Sigma, R^f, V^f)$  is any filtration of  $\mathfrak{M}$  through  $\Sigma$ , then  $R^s \subseteq R^f \subseteq R^l$*

*Proof.* If  $Rwv$ , if  $\mathfrak{M}, v \Vdash \phi$ , then  $\mathfrak{M}, w \Vdash \diamond\phi$ , hence  $R^l|w||v|$

For any  $(W_\Sigma, R^f, V^f)$ .  $R^s \subseteq R^f$  by clause 2.  $R^f \subseteq R^l$  by clause 2  $\square$

**Theorem 2.27** (Finite Model Property - via Filtrations). *Let  $\phi$  be a basic modal formula. if  $\phi$  is satisfiable, then it is satisfiable on a finite model. Indeed, it is satisfiable on a finite model containing at most  $2^m$  nodes, where  $m$  is the number of subformulas of  $\phi$*

*Proof.* Assume that  $\phi$  is satisfiable on a model  $\mathfrak{M}$ ; take any filtration of  $\mathfrak{M}$  through the set of subformulas.  $\square$

**Lemma 2.28.** *Let  $\mathfrak{M}$  be a model,  $\Sigma$  a subformula closed set of formulas, and  $W_\Sigma$  the set of equivalence classes induced on  $\mathfrak{M}$  by  $\leftrightarrow_\Sigma$ . Let  $R^t$  be the binary relation on  $W_\Sigma$  defined by*

$$R^t|w||v| \text{ iff for all } \phi, \text{ if } \diamond \phi \in \Sigma \text{ and } \mathfrak{M}, v \Vdash \phi \vee \diamond \phi \text{ then } \mathfrak{M}, w \Vdash \diamond \phi$$

*If  $R$  is transitive then  $(W_\Sigma, R^t, V^f)$  is a filtration and  $R^t$  is transitive*

**Definition 2.29.** Let  $(W, R, V)$  be a transitive frame. A **cluster** on  $(W, R, V)$  is a maximal, nonempty equivalence class under  $R$ . That is,  $C \subseteq W$  is a cluster if the restriction of  $R$  to  $C$  is an equivalence relation

A cluster is **simple** if it consists of a single reflexive point, and **proper** if it consists more than one point

## 2.4 The Standard Translation

**Definition 2.30.** For  $\tau$  a modal similarity type and  $\Phi$  a collection of proposition letters, let  $\mathcal{L}_\tau^1(\Phi)$  be the first-order language (with equality) which has unary predicates  $P_0, P_1, \dots$  corresponding to the proposition letters  $p_0, p_1, \dots$  in  $\Phi$ , and an  $(n+1)$ -ary relation symbol  $R_\Delta$  for each  $(n\text{-ary})$  modal operator  $\Delta$  in our similarity type. We write  $\alpha(x)$  to denote a first-order formula  $\alpha$  with one free variable,  $x$

**Definition 2.31** (Standard Translation). Let  $x$  be a first-order variable. The **standard translation**  $ST_x$  taking modal formulas to first-order formulas in  $\mathcal{L}_\tau^1(\Phi)$  is defined as

$$\begin{aligned} ST_x(p) &= Px \\ ST_x(\perp) &= x \neq x \\ ST_x(\neg\phi) &= \neg ST_x(\phi) \\ ST_x(\phi \vee \psi) &= ST_x(\phi) \vee ST_x(\psi) \\ ST_x(\Delta(\phi_1, \dots, \phi_n)) &= \exists y_1 \dots \exists y_n (R_\Delta x y_1 \dots y_n \wedge \\ &\quad ST_{y_1}(\phi_1) \wedge \dots \wedge ST_{y_n}(\phi_n)) \end{aligned}$$

where  $y_1, \dots, y_n$  are fresh variables.

$$\begin{aligned} ST_x(\diamond\phi) &= \exists y (Rxy \wedge ST_y(\phi)) \\ ST_x(\Box\phi) &= \forall y (Rxy \rightarrow ST_y(\phi)) \end{aligned}$$

**Proposition 2.32** (Local and Global Correspondence on Models). *Fix a modal similarity type  $\tau$ , and let  $\phi$  be a  $\tau$ -formula. Then*

1. *For all  $\mathfrak{M}$  and all states  $w$  of  $\mathfrak{M}$ :  $\mathfrak{M}, w \Vdash \phi$  iff  $\mathfrak{M} \models ST_x(\phi)[w]$*
2. *For all  $\mathfrak{M}$ :  $\mathfrak{M} \Vdash \phi$  iff  $\mathfrak{M} \models \forall x ST_x(\phi)$*

**Proposition 2.33.** 1. *Let  $\tau$  be a modal similarity type that only contains diamonds. Then, every  $\tau$ -formula  $\phi$  is equivalent to a first-order formula containing at most two variables*

2. *If  $\tau$  does not contain modal operators  $\triangle$  whose arity exceeds  $n$ , all  $\tau$ -formulas are equivalent to first-order formulas containing at most  $(n + 1)$  variables*

*Proof.* Assume  $\tau$  contains only diamonds  $\langle a \rangle, \langle b \rangle$ . Fix two distinct variables  $x$  and  $y$ . Define two variants  $ST_x$  and  $ST_y$  of the standard translation as follows

$$\begin{aligned}
 ST_x(p) &= Px & ST_y(p) &= Py \\
 ST_x(\perp) &= x \neq x & ST_y(\perp) &= y \neq y \\
 ST_x(\neg\phi) &= \neg ST_x(\phi) & ST_y(\neg\phi) &= \neg ST_y(\phi) \\
 ST_x(\phi \vee \psi) &= ST_x(\phi) \vee ST_x(\psi) & ST_y(\phi \vee \psi) &= ST_y(\phi) \vee ST_y(\psi) \\
 ST_x(\langle a \rangle \phi) &= \exists y (R_a xy \wedge ST_y(\phi)) & ST_y(\langle a \rangle \phi) &= \exists x (R_a yx \wedge ST_x(\phi))
 \end{aligned}$$

Then for any  $\tau$ -formula  $\phi$ , its  $ST_x$ -translation contains at most the two variables  $x$  and  $y$ , and  $ST_x(\phi)$  is equivalent to the original standard translation of  $\phi$   $\square$

**Example 2.3.**

$$\begin{aligned}
 ST_x(\diamond(\Box p \rightarrow q)) &= \exists y (Rxy \wedge ST_y(\Box p \rightarrow q)) \\
 &= \exists y (Rxy \wedge (\forall x (Ryx \rightarrow ST_x(p)) \rightarrow Qy)) \\
 &= \exists y (Rxy \wedge (\forall x (Ryx \rightarrow Px) \rightarrow Qy))
 \end{aligned}$$

$Rxx$  is not equivalent to any modal formula. Suppose  $\phi$  is a modal formula s.t.  $ST_x(\phi)$  is equivalent to  $Rxx$ . Let  $\mathfrak{M}$  be a singleton reflexive model and let  $w$  be the unique state in  $\mathfrak{M}$ ; obviously  $\mathfrak{M} \models Rxx[w]$ . Let  $\mathfrak{N}$  be a model based on the strict ordering of the integers; for every integer  $v$ ,  $\mathfrak{N} \models \neg Rxx[v]$ . Let  $Z$  be the relation which links every integer with the unique state in  $fM$ , and assume that the valuations in  $\mathfrak{N}$  and  $\mathfrak{M}$  are s.t.  $Z$  is a bisimulation.

$$\mathfrak{M} \models Rxx[w] \Rightarrow \mathfrak{M}, w \Vdash \phi \Rightarrow \mathfrak{N}, v \Vdash \phi \Rightarrow \mathfrak{N} \models Rxx[v]$$

**Definition 2.34.** Let  $\tau$  be a modal similarity type,  $C$  a class of  $\tau$ -models, and  $\Gamma$  a set of formulas over  $\tau$ . We say that  $\Gamma$  **defines** or **characterizes** a class  $K$  of models **within**  $C$  if for all models  $\mathfrak{M}$  in  $C$  we have that  $\mathfrak{M}$  is in  $K$  iff  $\mathfrak{M} \models \Gamma$ . If  $C$  is the class of all  $\tau$ -models, we simply say that  $\Gamma$  defines or characterizes  $K$ ; we omit brackets whenever  $\Gamma$  is a singleton. We say that a formula  $\phi$  defines a **property** whenever  $\phi$  defines the class of models satisfying the property

## 2.5 Modal Saturation via Ultrafilter Extensions

### 2.5.1 M-saturation

**Definition 2.35** (Hennessy-Milner Classes). Let  $\tau$  be a modal similarity type, and  $K$  a class of  $\tau$ -models.  $K$  is a **Hennessy-Milner class**, or **has the Hennessy-Milner property**, if for every two models  $\mathfrak{M}$  and  $\mathfrak{M}'$  in  $K$  and any two states  $w, w'$  of  $\mathfrak{M}$  and  $\mathfrak{M}'$ , respectively,  $w \leftrightarrow w'$  implies  $\mathfrak{M}, w \rightleftharpoons \mathfrak{M}', w'$

For example, by Theorem 2.13 the class of image-finite models has the Hennessy-Milner property.

Suppose we are working in the basic modal language. Let  $\mathfrak{M} = (W, R, V)$  be a model, let  $w$  be a state in  $W$  and let  $\Sigma = \{\phi_0, \phi_1, \dots\}$  be an infinite set of formulas. Suppose that  $w$  has successors  $v_0, v_1, \dots$ , where respectively  $\phi_0, \phi_0 \wedge \phi_1, \phi_0 \wedge \phi_1 \wedge \phi_2, \dots$  hold. If there is no successor  $v$  of  $w$  where **all** formulas from  $\Sigma$  hold **at the same time**, then the model is in some sense incomplete. A model is called **m-saturated** if incompleteness of this kind does not occur

Suppose that we are looking for a successor of  $w$  at which every formula  $\phi_i$  of the infinite set of formulas  $\Sigma = \{\phi_0, \phi_1, \dots\}$  holds. M-saturation is a kind of compactness property, according to which it suffices to find satisfying successors of  $w$  for arbitrary finite approximations of  $\Sigma$

**Definition 2.36** (M-saturation). Let  $\mathfrak{M} = (W, R, V)$  be a model of the basic modal similarity type,  $X$  a subset of  $W$  and  $\Sigma$  a set of modal formulas.  $\Sigma$  is **satisfiable** in the set  $X$  if there is a state  $x \in X$  s.t.  $\mathfrak{M}, x \models \phi$  for all  $\phi \in \Sigma$ .  $\Sigma$  is **finitely satisfiable** in  $X$  if every finite subset of  $\Sigma$  is satisfiable in  $X$

The model  $\mathfrak{M}$  is called **m-saturated** if it satisfies the following condition for every state  $w \in W$  and every set  $\Sigma$  of modal formulas:

If  $\Sigma$  is finitely satisfiable in the set of successors of  $w$ ,  
then  $\Sigma$  is satisfiable in the set of successors of  $w$

Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}$  be a  $\tau$ -model.  $\mathfrak{M}$  is called **m-saturated** if for every state  $w$  of  $\mathfrak{M}$  and every  $(n$ -ary) modal operator  $\Delta \in \tau$  and sequence  $\Sigma_1, \dots, \Sigma_n$  of sets of modal formulas, we have the following:



If for every sequence of finite subsets  $\Delta_1 \subset \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$ , there are states  $v_1, \dots, v_n$  s.t.  $Rwv_1 \dots v_n$  and  $v_1 \Vdash \Delta_1, \dots, v_n \Vdash \Delta_n$ , then there are states  $v_1, \dots, v_n$  in  $\mathfrak{M}$  s.t.  $Rwv_1 \dots v_n$  and  $v_1 \Vdash \Sigma_1, \dots, v_n \Vdash \Sigma_n$

**Proposition 2.37.** *Let  $\tau$  be a modal similarity type. Then the class of  $m$ -saturated  $\tau$ -models has the Hennessy-Milner property*

*Proof.* Let  $\mathfrak{M} = (W, R, V)$  and  $\mathfrak{M}' = (W', R', V')$  be two  $m$ -saturated models.

Assume that  $w, v \in W$  and  $w' \in W'$  are s.t.  $Rwv$  and  $w \rightsquigarrow w'$ . Let  $\Sigma$  be the set of formulas true at  $v$ . It is clear that for every finite subset  $\Delta$  of  $\Sigma$  we have  $\mathfrak{M}, v \Vdash \bigwedge \Delta$ , hence  $\mathfrak{M}, w \Vdash \diamond \bigwedge \Delta$ . As  $w \rightsquigarrow w'$ , it follows that  $\mathfrak{M}', w' \Vdash \diamond \bigwedge \Delta$ , so  $w'$  has an  $R'$ -successor  $v_\Delta$  s.t.  $\mathfrak{M}', v_\Delta \Vdash \bigwedge \Delta$ . In other words,  $\Sigma$  is finitely satisfiable in the set of successors of  $w'$ ; but then, by  $m$ -saturation,  $\Sigma$  itself is satisfiable in a successor  $v'$  of  $w'$ . Thus  $v \rightsquigarrow v'$   $\square$

### 2.5.2 Ultrafilter extensions

**Definition 2.38** (Filters and Ultrafilters). Let  $W$  be a non-empty set. A **filter**  $F$  over  $W$  is a set  $F \subseteq \mathcal{P}(W)$  s.t.

1.  $W \in F$
2. If  $X, Y \in F$ , then  $X \cap Y \in F$
3. If  $X \in F$  and  $X \subseteq Z \subseteq W$ , then  $Z \in F$

An **ultrafilter** over  $W$  is a proper filter s.t. for all  $X \in \mathcal{P}(W)$ ,  $X \in U$  iff  $(W \setminus X) \notin U$

**Definition 2.39.** Let  $W$  be a non-empty set, and let  $E$  be a subset of  $\mathcal{P}(W)$ . By the **filter generated by  $E$**  we mean the intersection  $F$  of the collection of all filters over  $W$  which include  $E$

$$F = \bigcap \{G \mid E \subseteq G \text{ and } G \text{ is a filter over } W\}$$

$E$  has the **finite intersection property** if the intersection of any finite number of elements of  $E$  is non-empty

**Lemma 2.40** (Zorn's Lemma). *Whenever  $<$  is a strict partial order of a set  $A$  satisfying for all chains  $C \subseteq A$  there is some  $b \in A$  s.t.  $x \leq b$  for all  $x \in C$  then for all  $a \in A$ , there is a maximal  $b \in A$  with  $b \geq a$*

**Theorem 2.41** (Ultrafilter Theorem). *Fix a non-empty set  $W$ . Any proper filter over  $W$  can be extended to an ultrafilter over  $W$ . As a corollary, any subset of  $\mathcal{P}(W)$  with the finite intersection property can be extended to an ultrafilter over  $W$*

**Definition 2.42.** Let  $W$  be a non-empty set. Given an element  $w \in W$ , the **principal ultrafilter**  $\pi_w$  generated by  $w$  is the filter generated by the singleton set  $\{w\}$

Suppose  $U$  is an ultrafilter over a non-empty set  $I$ , and that for each  $i \in I$ ,  $A_i$  is a non-empty set. Let  $C = \prod_{i \in I} A_i$ . That is,  $C$  is the set of all functions  $f$  with domain  $I$  s.t. for each  $i \in I$ ,  $f(i) \in A_i$ . For two functions  $f, g \in C$  we say that  $f$  and  $g$  are  **$U$ -equivalent** ( $f \sim_U g$ ) if  $\{i \in I \mid f(i) = g(i)\} \in U$

**Proposition 2.43.** *The relation  $\sim_U$  is an equivalence relation on the set  $C$*

*Proof.* Suppose  $\{i \mid f(i) = g(i)\} \in U$ ,  $\{i \mid g(i) = h(i)\} \in U$ , then  $\{i \mid f(i) = g(i) = h(i)\} = \{i \mid f(i) = g(i)\} \cap \{i \mid g(i) = h(i)\} \in U$ . And  $\{i \mid f(i) = g(i) = h(i)\} \subseteq \{i \mid f(i) = h(i)\}$   $\square$

**Definition 2.44.** Let  $f_U$  be the equivalence class of  $f$  modulo  $\sim_U$ , that is:  $f_U = \{g \in C \mid g \sim_U f\}$ . The **ultraproduct** of the sets  $A_i$  **modulo**  $U$  is the set of all equivalence classes of  $\sim_U$ . It is denoted by  $\prod_U A_i$ . So

$$\prod_U A_i = \{f_U \mid f \in \prod_{i \in I} A_i\}$$

**Definition 2.45.** Fix a first-order language  $\mathcal{L}^1$ , and let  $\mathfrak{A}_i (i \in I)$  be  $\mathcal{L}^1$ -models. The **ultraproduct**  $\prod_U \mathfrak{A}_i$  **of**  $\mathfrak{A}_i$  **modulo**  $U$  is the model described as follows:

1. The universe  $A_U$  is the set  $\prod_U A_i$ , where  $A_i$  is the universe of  $\mathfrak{A}_i$
2. Let  $R$  be an  $n$ -place relation symbol, and  $R_i$  its interpretation in the model  $\mathfrak{A}_i$ . The relation  $R_U$  in  $\prod_U \mathfrak{A}_i$  is given by

$$R_U f_U^1 \dots f_U^n \quad \text{iff} \quad \{i \in I \mid R_i f^1(i) \dots f^n(i)\} \in U$$

3. Let  $F$  be an  $n$ -place function symbol, and  $F_i$  its interpretation in  $\mathfrak{A}_i$ . The function  $F_U$  in  $\prod_U \mathfrak{A}_i$  is given by

$$F_U(f_U^1, \dots, f_U^n) = \{(i, F_i(f^1(i), \dots, f^n(i))) \mid i \in I\}_U$$

4. Let  $c$  be a constant, and  $a_i$  its interpretation in  $\mathfrak{A}_i$ . Then  $c$  is interpreted by the element  $c' \in \prod_U A_i$  where  $c' = \{(i, a_i) \mid i \in I\}_U$

In the case where all the structures are the same, say  $\mathfrak{A}_i = \mathfrak{A}$  for all  $i$ , we speak of the **ultrapower** of  $\mathfrak{A}$  modulo  $U$ , notation  $\prod_U \mathfrak{A}$

**Theorem 2.46** (Łoś's Theorem). *Let  $U$  be an ultrafilter over a non-empty set  $I$ . For each  $i \in I$ , let  $\mathfrak{A}_i$  be a model*

1. For every term  $t(x_1, \dots, x_n)$  and all elements  $f_U^1, \dots, f_U^n$  of  $\mathfrak{B} = \prod_U \mathfrak{A}_i$  we have

$$t^{\mathfrak{B}}[x_1 \mapsto f_U^1, \dots, x_n \mapsto f_U^n] = \{(i, t^{\mathfrak{A}_i}[f^1(i), \dots, f^n(i)]) \mid i \in I\}_U$$

2. Given any first-order formula  $\alpha(x_1, \dots, x_n)$  in  $\mathcal{L}_\tau^1$  and  $f_U^1, \dots, f_U^n$  in  $\prod_U \mathfrak{A}_i$  we have

$$\prod_U \mathfrak{A}_i \models \alpha[f_U^1, \dots, f_U^n] \quad \text{iff} \quad \{i \in I \mid \mathfrak{A}_i \models \alpha[f^1(i), \dots, f^n(i)]\} \in U$$

*Proof.* 1.

2. Induction on  $\alpha$ . The atomic case holds by definition. Suppose that  $\alpha \equiv \neg\beta(x_1, \dots, x_n)$ , then

$$\begin{aligned} \prod_U \mathfrak{A}_i \models \alpha[f_U^1, \dots, f_U^n] & \quad \text{iff} \quad \prod_U \mathfrak{A}_i \not\models \beta[f_U^1, \dots, f_U^n] \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{A}_i \models \beta[f_U^1, \dots, f_U^n]\} \notin U \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{A}_i \not\models \beta[f^1(i), \dots, f^n(i)]\} \in U \\ & \quad \text{iff} \quad \{i \in I \mid \mathfrak{A}_i \models \alpha[f^1(i), \dots, f^n(i)]\} \in U \end{aligned}$$

The second equivalence follows from the inductive hypothesis, and the third from the fact that  $U$  is an ultrafilter

Suppose that  $\alpha(x_1, \dots, x_n) \equiv \exists x_0 \beta(x_0, \dots, x_n)$ , then

$$\begin{aligned} \prod_U \mathfrak{A}_i \models \alpha[f_U^1, \dots, f_U^n] & \quad \text{iff} \quad \exists f_U^0 \in \prod_U \mathfrak{A}_i, \prod_U \mathfrak{A}_i \models \beta[f_U^0, \dots, f_U^n] \\ & \quad \text{iff} \quad \exists f_U^0 \in \prod_U \mathfrak{A}_i, \{i \in I \mid \mathfrak{A}_i \models \beta[f^0(i), \dots, f^n(i)]\} \in U \end{aligned} \tag{2.5.1}$$

As  $\mathfrak{A}_i \models \beta[f^0(i), \dots, f^n(i)]$  implies  $\mathfrak{A}_i \models \alpha[f^1(i), \dots, f^n(i)]$ , which means

$$\{i \in I \mid \mathfrak{A}_i \models \beta[f^0(i), \dots, f^n(i)]\} \subseteq \{i \in I \mid \mathfrak{A}_i \models \alpha[f^1(i), \dots, f^n(i)]\}$$

Hence

$$\{i \in I \mid \mathfrak{A}_i \models \alpha[f^1(i), \dots, f^n(i)]\} \in U \tag{2.5.2}$$

Conversely, if (2.5.2) holds, then we can select a function  $f^0 \in \prod_{i \in I} \mathfrak{A}_i$  s.t. (2.5.1) holds. So (2.5.1) is equivalent to (2.5.2)

□

**Corollary 2.47.** *Let  $\prod_U \mathfrak{A}$  be an ultrapower of  $\mathfrak{A}$ . Then for all first-order sentences  $\alpha$ ,  $\mathfrak{A} \models \alpha$  iff  $\prod_U \mathfrak{A} \models \alpha$*

There is a natural embedding of a model  $\mathfrak{A}$  in each of its ultrapowers. Define the **diagonal mapping**  $d$  of  $\mathfrak{A}$  into  $\prod_U \mathfrak{A}$  to be the function

$$\alpha \mapsto (f_\alpha)_U, \text{ where } f_\alpha(i) = a \text{ for all } i \in I$$

**Corollary 2.48.** *Let  $\prod_U \mathfrak{A}$  be an ultrapower of  $\mathfrak{A}$ . Then the diagonal mapping of  $\mathfrak{A}$  into  $\prod_U \mathfrak{A}$  is an elementary embedding*

*Proof.*

$$\begin{aligned} \prod_U \mathfrak{A} \models \alpha[d(a_1), \dots, d(a_n)] & \quad \text{iff} \quad \{i \in I \mid \mathfrak{A} \models \alpha[a_1, \dots, a_n]\} \in U \\ & \quad \text{iff} \quad \mathfrak{A} \models \alpha[a_1, \dots, a_n] \end{aligned}$$

□

$$V(\phi) = \{w \mid \mathfrak{M}, w \Vdash \phi\}$$

**Definition 2.49.** Given an  $(n+1)$ -ary relation  $R$  on a set  $W$ , we define the following two  $n$ -ary operations  $m_R$  and  $l_R$  on the power set  $\mathcal{P}(W)$  of  $W$ :

$$\begin{aligned} m_R(X_1, \dots, X_n) &:= \{w \in W \mid \exists w_1, \dots, w_n (Rww_1 \dots w_n \bigwedge \forall i (w_i \in X_i))\} \\ l_R(X_1, \dots, X_n) &:= \{w \in W \mid \forall w_1, \dots, w_n (Rww_1 \dots w_n \rightarrow \exists i (w_i \in X_i))\} \\ m_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\bigtriangleup(\phi_1, \dots, \phi_n)) \\ l_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\nabla(\phi_1, \dots, \phi_n)) \end{aligned}$$

It follows that for any model  $\mathfrak{M} = (W, R, V)$  we have

$$V(\diamond\phi) = m_R(V(\phi)) \quad \text{and} \quad V(\Box\phi) = l_R(V(\phi))$$

**Proposition 2.50.** *Let  $R$  be a relation of arity  $n+1$  on the set  $W$ . Then for every  $n$ -tuple  $X_1, \dots, X_n$  of subsets of  $W$  we have*

$$l_R(X_1, \dots, X_n) = W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n)$$

*Proof.* This is actually  $\nabla = \neg\bigtriangleup\neg$

$$\begin{aligned} W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n) &= \{w \mid \neg \exists w_1, \dots, w_n (Rww_1 \dots w_n \bigwedge \forall i (w_i \in W \setminus X_i))\} \\ &= \{w \mid \forall w_1, \dots, w_n (\neg Rww_1 \dots w_n \bigvee \neg \forall i (w_i \in W \setminus X_i))\} \\ &= \{w \mid \forall w_1, \dots, w_n (Rww_1 \dots w_n \rightarrow \exists i (w_i \notin W \setminus X_i))\} \\ &= l_R(X_1, \dots, X_n) \end{aligned}$$

□

**Definition 2.51** (Ultrafilter Extension). Let  $\tau$  be a modal similarity type, and  $\mathfrak{F} = (W, R_\Delta)_{\Delta \in \tau}$  is a  $\tau$ -frame. The **ultrafilter extension**  $ue\mathfrak{F}$  of  $\mathfrak{F}$  is defined as the frame  $(Uf(W), R_\Delta^{ue})_{\Delta \in \tau}$ . Here  $Uf(W)$  is the set of ultrafilters over  $W$  and  $R_\Delta^{ue}u_0u_1 \dots u_n$  holds for a tuple  $u_0, \dots, u_n$  of ultrafilters over  $W$  if we have that  $m_{R_\Delta}(X_1, \dots, X_n) \in u_0$  whenever  $X_i \in u_i$  for all  $i$  with  $1 \leq i \leq n$ .

The **ultrafilter extension** of a  $\tau$ -model  $\mathfrak{M} = (\mathfrak{F}, V)$  is the model  $ue\mathfrak{M} = (ue\mathfrak{F}, V^{ue})$  where  $V^{ue}(p_i)$  is the set of ultrafilters of which  $V(p_i)$  is a member.

Any subset of a frame can be viewed as a **proposition**. A filter over the universe of the frame can thus be seen as a **theory**, in fact as a logically closed theory, since filters are both closed under intersection (conjunction) and upward closed (entailment). Viewed this way, a proper filter is a **consistent** theory, or **state of affairs**, for it does not contain the empty set (falsum). Finally an ultrafilter is a **complete** theory.

In a given frame  $\mathfrak{F}$  not every state not every state of affairs needs to 'realized', in the sense that there is a state satisfying all and only the propositions belonging to the state of affairs; only the states of affairs that correspond to the **principal** ultrafilters are realized. We build  $ue\mathfrak{F}$  by adding every state of affairs for  $\mathfrak{F}$  as a new element of the domain - that is,  $ue\mathfrak{F}$  realizes every proposition in  $\mathfrak{F}$ .

Stipulate that  $R_\Delta^{ue}u_0u_1 \dots u_n$  if  $u_0$  'sees' the  $n$ -tuple  $u_1, \dots, u_n$ . That is, whenever  $X_1, \dots, X_n$  are propositions of  $u_1, \dots, u_n$  respectively, then  $u_0$  'sees' this combination: that is, the proposition  $m_{R_\Delta}(X_1, \dots, X_n)$  is a member of  $u_0$ .

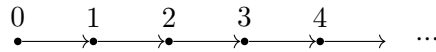
**Principal** ultrafilters over  $W$  plays a special role. By identifying a state  $w$  of a frame  $\mathfrak{F}$  with the principal ultrafilter  $\pi_w = \{X \subseteq W \mid w \in X\}$ , it is easily seen that any frame  $\mathfrak{F}$  is (isomorphic to) a **submodel** (but in general not a **generated** submodel) of its ultrafilter extension. For we have the following equivalences

$$\begin{aligned} R w v & \text{ iff } w \in m_R(X) \text{ for all } X \subseteq W \text{ s.t. } v \in X \\ & \text{ iff } m_R(X) \in \pi_w \text{ for all } X \subseteq W \text{ s.t. } X \in \pi_v \\ & \text{ iff } R^{ue} \pi_w \pi_v \end{aligned}$$

since

$$R w v \text{ iff } \forall X \subseteq W (v \in X \rightarrow w \in m_R(X))$$

**Example 2.4.** Consider the frame  $\mathfrak{N} = (\mathbb{N}, <)$



What is the ultrafilter extension of  $\mathfrak{M}$ ? There are two kinds of ultrafilter over an infinite set: the principal ultrafilter that are in one-to-one correspondence with the points of the set, and the non-principal ones which contain all co-finite sets and only infinite sets, cf Exercise 2.5.1. The principal ultrafilters form an isomorphic copy of the frame  $\mathfrak{M}$  inside  $ue\mathfrak{M}$ . For any pair  $u, u'$  of ultrafilters, if  $u'$  is non-principal, then  $R^{ue}uu'$ . To see this, let  $X \in u'$ . As  $X$  is infinite, for any  $n \in \mathbb{N}$  there is an  $m$  s.t.  $n < m$  and  $m \in X$ . This shows that  $m_{<}(X) = \mathbb{N}$ . But  $\mathbb{N}$  is an element of every ultrafilter

This shows that the ultrafilter extension of  $\mathfrak{M}$  consists of a copy of  $\mathfrak{M}$  followed by an uncountable cluster consisting of all the non-principal ultrafilters

**Proposition 2.52.** *Let  $\tau$  be a modal similarity type, and  $\mathfrak{M}$  a  $\tau$ -model. Then for any formula  $\phi$  and any ultrafilter  $u$  over  $W$ ,  $V(\phi) \in u$  iff  $ue\mathfrak{M}, u \models \phi$ . Hence for every state  $w$  of  $\mathfrak{M}$  we have  $w \leftrightarrow \pi_w$*

*Proof.* The second claim of the proposition is immediate from the first one by the observation that  $w \models \phi$  iff  $w \in V(\phi)$  iff  $V(\phi) \in \pi_w$

Induction on  $\phi$ . The basic case is immediate from the definition of  $V^{ue}$ . Suppose  $\phi$  is of the form  $\neg\psi$ , then

$$\begin{aligned} V(\neg\psi) \in u & \text{ iff } W \setminus V(\psi) \in u \\ & \text{ iff } V(\psi) \notin u \\ & \text{ iff } ue\mathfrak{M}, u \not\models \psi \quad \text{IH} \\ & \text{ iff } ue\mathfrak{M}, u \models \neg\psi \end{aligned}$$

Now consider the case where  $\phi$  is of the form  $\diamond\psi$ . Assume first that  $ue\mathfrak{M}, u \models \diamond\psi$ . Then there is an ultrafilter  $u'$  s.t.  $R^{ue}uu'$  and  $ue\mathfrak{M}, u' \models \psi$ . The induction hypothesis implies that  $V(\psi) \in u'$ , so by the definition of  $R^{ue}$ ,  $m_R(V(\psi)) \in u$ . Now the result follows immediately from the observation that  $m_R(V(\psi)) = V(\diamond\psi)$

Assume that  $V(\diamond\psi) \in u$ . We have to find an ultrafilter  $u'$  s.t.  $V(\psi) \in u'$  and  $R^{ue}uu'$ . The latter constraint reduces to the condition that  $m_R(X) \in u$  whenever  $X \in u'$ , or equivalently (see Exercise 2.5.2)

$$u'_0 := \{Y \mid l_R(Y) \in u\} \subseteq u'$$

We will first show that  $u'_0$  is closed under intersection. Let  $Y, Z \in u'_0$ . By definition,  $l_R(Y)$  and  $l_R(Z)$  are in  $u$ . But then  $l_R(Y \cap Z) \in u$  as  $l_R(Y \cap Z) = l_R(Y) \cap l_R(Z)$ . This proves that  $Y \cap Z \in u'_0$

Next we make sure that for any  $Y \in u'_0$ ,  $Y \cap V(\psi) \neq \emptyset$ . Let  $Y$  be an arbitrary element of  $u'_0$ , then by definition of  $u'_0$ ,  $l_R(Y) \in u$ . As  $u$  is closed

under intersection and does not contain the empty set, there must be an element  $x \in l_R(Y) \cap V(\diamond\psi)$ . But then  $x$  must have a successor  $y$  in  $V(\psi)$ . Finally,  $x \in l_R(Y)$  implies  $y \in Y$ .

From the fact that  $u'_0$  is closed under intersection, and the fact that for any  $Y \in u'_0$ ,  $Y \cap V(\psi) \neq \emptyset$ , it follows that the set  $u'_0 \cup \{V(\psi)\}$  has the finite intersection property. So the Ultrafilter Theorem provides us with an ultrafilter  $u'$  s.t.  $u'_0 \cup \{V(\psi)\} \subseteq u'$ . This ultrafilter  $u'$  has the desired properties: it is clearly a successor of  $u$ , and the fact the  $\text{ue}\mathfrak{M}, u' \Vdash \psi$  follows from  $V(\psi) \in u'$  and the induction hypothesis  $\square$

**Example 2.5.** Our new invariance result can be used to compare the relative expressive power of modal languages. Consider the modal constant  $\mathcal{O}$  whose truth definition in a model for the basic modal language is

$$\mathfrak{M}, w \Vdash \mathcal{O} \quad \text{iff} \quad \mathfrak{M} \models Rxx[v] \text{ for some } v \text{ in } \mathfrak{M}$$

Comparing the pictures of the frame  $(\mathbb{N}, <)$  and its ultrafilter extension given in Example 2.4. The former is loop-free but the latter contains uncountably many loops

**Proposition 2.53.** *Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}$  be a  $\tau$ -model. Then  $\text{ue}\mathfrak{M}$  is  $m$ -saturated*

*Proof.* Let  $\mathfrak{M} = (W, R, V)$  be a model. Consider an ultrafilter  $u$  over  $W$ , and a set  $\Sigma$  of modal formulas which is finitely satisfiable in the set of successors of  $u$ . We have to find an ultrafilter  $u'$  s.t.  $R^{ue}uu'$  and  $\text{ue}\mathfrak{M}, u' \Vdash \Sigma$ . Define

$$\Delta = \{V(\phi) \mid \phi \in \Sigma'\} \cup \{Y \mid l_R(Y) \in u\}$$

where  $\Sigma'$  is the set of (finite) conjunctions of formulas in  $\Sigma$ . We claim that the set  $\Delta$  has the finite intersection property. Since both  $\{V(\phi) \mid \phi \in \Sigma'\}$  and  $\{Y \mid l_R(Y) \in u\}$  are closed under taking intersections, it suffices to prove that for an arbitrary  $\phi \in \Sigma'$  and an arbitrary set  $Y \subseteq W$  for which  $l_R(Y) \in u$ , we have  $V(\phi) \cap Y \neq \emptyset$ . but if  $\phi \in \Sigma'$ , then by assumption, there is a successor  $u''$  of  $u$  s.t.  $\text{ue}\mathfrak{M}, u'' \Vdash \phi$ , or in other words,  $V(\phi) \in u''$ . Then  $l_R(Y) \in u$  implies  $Y \in u''$  by Exercise 2.5.2. Hence  $V(\phi) \cap Y$  is an element of the ultrafilter  $u''$  and therefore cannot be identical to the empty set.

It follows by the Ultrafilter Theorem that  $\Delta$  can be extended to an ultrafilter  $u'$ . Clearly  $u'$  is the required successor  $\square$

**Theorem 2.54.** *Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be  $\tau$ -models, and  $w, w'$  two states in  $\mathfrak{M}$  and  $\mathfrak{M}'$  respectively. Then*

$$\mathfrak{M}, w \rightsquigarrow \mathfrak{M}', w' \quad \text{iff} \quad \text{ue}\mathfrak{M}, \pi_w \cong \text{ue}\mathfrak{M}', \pi_{w'}$$

*Proof.* From Propositions 2.52, 2.53 and 2.37  $\square$

*Exercise 2.5.1.* Let  $W$  be an infinite set. Recall that  $X \subseteq W$  is **co-finite** if  $W \setminus X$  is finite

1. Prove that the collection of co-finite subsets of  $W$  has the finite intersection property
2. Show that there are ultrafilters over  $W$  that do not contain any finite set
3. Prove that an ultrafilter is non-principal iff it contains only infinite sets iff it contains all co-finite sets
4. Prove that any ultrafilter over  $W$  has uncountably many elements

*Proof.* Suppose  $U = \{X \subseteq W \mid X \text{ is cofinite}\}$

1. For any  $A, B \in U$ , if  $A \cap B = \emptyset$ ,  $A \subset \bar{B}$ . But  $A$  is infinite and  $\bar{B}$  is finite, this can't happen. Hence  $A \cap B \neq \emptyset$
2.  $U$  can be extended to a ultrafilter  $\mathcal{U}$ . If  $A$  is finite, then  $\bar{A} \in U \subseteq \mathcal{U}$ . Hence  $\mathcal{U}$  does not contain any finite set.
3.  $1 \rightarrow 2$ . If an ultrafilter contains a finite set. Then its a principal ultrafilter generated on the intersection of all finite sets.  
 $2 \rightarrow 3$  and  $3 \rightarrow 1$  are obvious.
4. Half of the  $\mathcal{P}(W)$  belongs to the ultrafilter and  $\mathcal{P}(W)$  is uncountable  $\square$

*Exercise 2.5.2.* Given a model  $\mathfrak{M} = (W, R, V)$  and two ultrafilters  $u$  and  $v$  over  $W$ , show that  $R^{ue}uv$  iff  $\{Y \mid l_R(Y) \in u\} \subseteq v$

*Proof.*

$$\begin{aligned}
 R^{ue}uv &\Leftrightarrow X \in v \rightarrow m_R(X) \in u \\
 &\Leftrightarrow \neg m_R(X) \in u \rightarrow \neg X \in v \\
 &\Leftrightarrow W - m_R(X) \in u \rightarrow W - X \in v \\
 &\Leftrightarrow l_R(W - X) \in u \rightarrow W - X \in v \\
 &\text{(Since } m_R(X) = W - l_R(W - X)\text{)} \\
 &\Leftrightarrow \{Y \mid l_R(Y) \in u\} \subseteq v
 \end{aligned}$$

$\square$



## 2.6 Characterization and Definability

### 2.6.1 The van Benthem Characterization Theorem

Let  $\Gamma(x)$  be a set of first-order formulas in which a single individual variable may occur free - such a set of formulas is called a **type**. A first-order model  $\mathfrak{M}$  **realizes**  $\Gamma(x)$  if there is an element  $w$  in  $\mathfrak{M}$  s.t. for all  $\gamma \in \Gamma$ ,  $\mathfrak{M} \models \gamma[w]$

Let  $\mathfrak{M}$  be a model for a given first-order language  $\mathcal{L}^1$  with domain  $W$ . For a subset  $A \subset W$ ,  $\mathcal{L}^1[A]$  is the language obtained by extending  $\mathcal{L}^1$  with new constant  $\underline{a}$  for all elements  $a \in A$ .  $\mathfrak{M}_A$  is the expansion of  $\mathfrak{M}$  to a structure for  $\mathcal{L}^1[A]$  in which each  $\underline{a}$  is interpreted as  $a$

Assume that  $A$  is of size at most  $\alpha$ . Assume that  $\alpha = 3$  and  $A = \{\alpha_1, \alpha_2\}$ . Let  $\Gamma(\underline{a}_1, \underline{a}_2, x)$  be a type of the language  $\mathcal{L}^1[A]$ ;  $\Gamma(\underline{a}_1, \underline{a}_2, x)$  is consistent with the first-order theory of  $\mathfrak{M}_A$  iff  $\Gamma(\underline{a}_1, \underline{a}_2, x)$  is finitely realizable in  $\mathfrak{M}_A$ . So for this particular set  $\Gamma(\underline{a}_1, \underline{a}_2, x)$ , 3-saturation of  $\mathfrak{M}$  means that if  $\Gamma(\underline{a}_1, \underline{a}_2, x)$  is finitely realizable in  $\mathfrak{M}_A$ , then  $\Gamma(\underline{a}_1, \underline{a}_2, x)$  is realizable in  $\mathfrak{M}_A$

Or consider a formula  $\gamma(\underline{a}_1, \underline{a}_2, x)$  and let  $\gamma(x_1, x_2, x)$  be the formula with the fresh variables  $x_1$  and  $x_2$  replacing each occurrence in  $\gamma$  of  $\underline{a}_1$  and  $\underline{a}_2$  respectively. Then we have the following equivalence

$$\mathfrak{M}_A \text{ realizes } \{\gamma(\underline{a}_1, \underline{a}_2, x)\} \text{ iff there is a } b \text{ s.t. } \mathfrak{M} \models \gamma(x_1, x_2, x)[a_1, a_2, b]$$

So a model is  $\alpha$ -saturated iff the following holds for every  $n < \alpha$  and every set  $\Gamma$  of formulas of the form  $\gamma(x_1, \dots, x_n, x)$

If  $(a_1, \dots, a_n)$  is an  $n$ -tuple s.t. for every finite  $\Delta \subseteq \Gamma$  there is a  $b_\Delta$  s.t.

$$\mathfrak{M} \models \gamma(x_1, \dots, x_n, x)[a_1, \dots, a_n, b_\Delta] \text{ for every } \gamma \in \Delta$$

then we have that there is a  $b$  s.t.  $\mathfrak{M} \models \gamma(x_1, \dots, x_n, x)[a_1, \dots, a_n, b]$  for every  $\gamma \in \Gamma$

**Definition 2.55.** Let  $\alpha$  be a natural number, or  $\omega$ . A model  $\mathfrak{M}$  is  $\alpha$ -**saturated** if for every subset  $A \subseteq W$  of size less than  $\alpha$ , the expansion  $\mathfrak{M}_A$  realizes every set  $\Gamma(x)$  of  $\mathcal{L}^1[A]$ -formulas (with only  $x$  occurring free) that is *consistent* (a proof-theoretic notion, only finite deductions, hence this definition is consistent the definition above) with the first-order theory of  $\mathfrak{M}_A$ . An  $\omega$ -saturated model is usually called **countably saturated**

**Example 2.6.** 1. Every finite model is countably saturated. For if  $\mathfrak{M}$  is finite, and  $\Gamma(x)$  is a set of first-order formulas consistent with the first-order theory of  $\mathfrak{M}$ , there exists a model  $\mathfrak{N}$  that is elementarily equivalent to  $\mathfrak{M}$  and that realizes  $\Gamma(x)$ . But as  $\mathfrak{M}$  and  $\mathfrak{N}$  are finite, elementary equivalence implies isomorphism (proof), and hence  $\Gamma(x)$  is realized in  $\mathfrak{M}$

2. The ordering of the rational numbers  $(\mathbb{Q}, <)$  is countably saturated as well. The relevant first-order language  $\mathcal{L}^1$  has  $<$  and  $=$ . Take a subset  $A$  of  $\mathbb{Q}$  and let  $\Gamma(x)$  be a set of formulas in the resulting expansion  $\mathcal{L}^1[A]$  of the first-order language that is consistent with the theory of  $(\mathbb{Q}, <, a)_{a \in A}$ . Then there exists a model  $\mathfrak{N}$  of the theory of  $(\mathbb{Q}, <, a)_{a \in A}$  that realizes  $\Gamma(x)$ .  $\star$ . Now take a countable elementary submodel  $\mathfrak{N}'$  of  $\mathfrak{N}$  that contains at least one object realizing  $\Gamma(x)$ . Then  $\mathfrak{N}'$  is a countable dense linear ordering without endpoints, and hence the ordering of  $\mathfrak{N}'$  is isomorphic to  $(\mathbb{Q}, <)$ .

**Theorem 2.56.** *Let  $\tau$  be a modal similarity type. Any countably saturated  $\tau$ -model is  $m$ -saturated. It follows that the class of countably saturated  $\tau$ -models has the Hennessy-Milner property*

*Proof.* Assume that  $\mathfrak{M} = (W, R, V)$  viewed as a first-order model, is countably saturated. Let  $a$  be a state in  $W$ , and consider a set  $\Sigma$  of modal formulas which is finite satisfiable in the successor set of  $a$ . Define  $\Sigma'$  to be the set

$$\Sigma' = \{Rax\} \cup ST_x(\Sigma)$$

where  $ST_x(\Sigma) = \{ST_x(\phi) \mid \phi \in \Sigma\}$ .  $\Sigma'$  is consistent with the first-order theory of  $\mathfrak{M}_a$ :  $\mathfrak{M}_a$  realizes every finite subset of  $\Sigma'$ , namely in some successor of  $a$ . So by the countable saturation of  $\mathfrak{M}$ ,  $\Sigma'$  is realized in some state  $b$ . By  $\mathfrak{M}_A \models Rax[b]$  it follows that  $b$  is a successor of  $a$ . Then, by Proposition 2.32 and the fact that  $\mathfrak{M}_a \models ST_x(\phi)[b]$  for all  $\phi \in \Sigma$ , it follows that  $\mathfrak{M}, b \Vdash \Sigma$ . Thus  $\Sigma$  is satisfiable in a successor of  $a$   $\square$

**Lemma 2.57** (Detour Lemma). *Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}$  and  $\mathfrak{N}$  be two models, and  $w$  and  $v$  states in  $\mathfrak{M}$  and  $\mathfrak{N}$ , respectively. Then the following are equivalent:*

1. *For all modal formulas  $\phi$ :  $\mathfrak{M}, w \Vdash \phi$  iff  $\mathfrak{N}, v \Vdash \phi$*
2. *There exists a bisimulation  $Z : \text{ue}\mathfrak{M}, \pi_w \rightleftharpoons \text{ue}\mathfrak{N}, \pi_v$*
3. *There exist countably saturated models  $\mathfrak{M}^*, w^*, \mathfrak{N}^*, v^*$  and elementary embeddings  $f : \mathfrak{M} \preceq \mathfrak{M}^*$  and  $g : \mathfrak{N} \preceq \mathfrak{N}^*$  s.t.*
  - (a)  $f(w) = w^*$  and  $g(v) = v^*$
  - (b)  $\mathfrak{M}^*, w^* \rightleftharpoons \mathfrak{N}^*, v^*$

**Definition 2.58.** A first-order formula  $\alpha(x)$  in  $\mathcal{L}_\tau^1$  is **invariant for bisimulations** if for all models  $\mathfrak{M}$  and  $\mathfrak{N}$ , and all states  $w$  in  $\mathfrak{M}$ ,  $v$  in  $\mathfrak{N}$ , and all bisimulations  $Z$  between  $\mathfrak{M}$  and  $\mathfrak{N}$  s.t.  $wZv$ , we have  $\mathfrak{M} \models \alpha(x)[w]$  iff  $\mathfrak{N} \models \alpha(x)[v]$

**Theorem 2.59** (van Benthem Characterization Theorem). *Let  $\alpha(x)$  be a first-order formula in  $\mathcal{L}_\tau^1$ . Then  $\alpha(x)$  is invariant for bisimulations iff it is equivalent to the standard translation of a modal  $\tau$ -formula*

*Proof.* Assume  $\alpha(x)$  is invariant for bisimulations and consider the set of modal consequences of  $\alpha$ :

$$MOC(\alpha) = \{ST_x(\phi) \mid \phi \text{ is a modal formula, and } \alpha(x) \models ST_x(\phi)\}$$

Our first claim is that if  $MOC(\alpha) \models \alpha(x)$ , then  $\alpha$  is equivalent to the translation of a modal formula. Assume  $MOC(\alpha) \models \alpha(x)$ , then by the Compactness Theorem for first-order logic, for some finite subset  $X \subseteq MOC(\alpha)$ , we have  $X \models \alpha(x)$ . So  $\models \bigwedge X \rightarrow \alpha(x)$ . Trivially  $\models \alpha(x) \rightarrow \bigwedge X$ , thus  $\models \alpha(x) \leftrightarrow \bigwedge X$ . And as every  $\beta \in X$  is the translation of a modal formula, so is  $\bigwedge X$ .

So it suffices to show that  $MOC(\alpha) \models \alpha(x)$ . Assume  $\mathfrak{M} \models MOC(\alpha)[w]$ ; we need to show that  $\mathfrak{M} \models \alpha(x)[w]$ . Let

$$T(x) = \{ST_x(\phi) \mid \mathfrak{M} \models ST_x(\phi)[w]\}$$

We claim that  $T(x) \cup \{\alpha(x)\}$  is consistent. Assume that  $T(x) \cup \{\alpha(x)\}$  is inconsistent. Then by compactness, for some finite subset  $T_0(x) \subset T(x)$  we have  $\models \alpha(x) \rightarrow \neg \bigwedge T_0(x)$ . Hence  $\neg \bigwedge T_0(x) \in MOC(\alpha)$ . But this implies  $\mathfrak{M} \models \neg \bigwedge T_0(x)[w]$ , a contradiction.

Let  $\mathfrak{N}, v$  be s.t.  $\mathfrak{N} \models T(x) \cup \{\alpha(x)\}[v]$ . Observe that  $w$  and  $v$  are modally equivalent:  $\mathfrak{M}, w \Vdash \phi$  implies  $ST_x(\phi) \in T(x)$ , which implies  $\mathfrak{N}, v \Vdash \phi$ ; and likewise, if  $\mathfrak{M}, w \nVdash \phi$  then  $\mathfrak{M}, w \Vdash \neg \phi$  and  $\mathfrak{N}, v \Vdash \neg \phi$ .

We can use the Detour Lemma and make a detour through a Hennessy-Milner class where modal equivalence and bisimilarity do coincide.

$$\begin{array}{ccc} \mathfrak{M}, w & & \mathfrak{N}, v \\ \downarrow \preceq & & \downarrow \preceq \\ \mathfrak{M}^*, w^* & \xrightarrow{\cong} & \mathfrak{N}^*, v^* \end{array}$$

$\mathfrak{N} \models \alpha(x)[v]$  implies  $\mathfrak{N}^* \models \alpha(x)[v^*]$ . As  $\alpha(x)$  is invariant for bisimulations, we get  $\mathfrak{M}^* \models \alpha(x)[w^*]$ . By invariance under elementary embeddings, we have  $\mathfrak{M} \models \alpha(x)[w]$   $\square$

### 2.6.2 Ultraproducts

Suppose  $I \neq \emptyset$ ,  $U$  is an ultrafilter over  $I$ .

**Definition 2.60** (Ultraproducts of Sets). Let  $f_U$  be the equivalence class of  $f$  modulo  $\sim_U$ , that is:  $f_U = \{g \in C \mid g \sim_U f\}$ . The **ultraproduct** of  $W_i$  modulo  $U$ , denoted as  $\prod_U W_i$  is the set of all equivalence classes of  $\sim_U$ . So

$$\prod_U W_i = \{f_U \mid f \in \prod_{i \in I} W_i\}$$

In case  $W_i = W$ , the ultraproduct is called the **ultrapower of  $W$  modulo  $U$** , and written  $\prod_U W$

**Definition 2.61** (Ultraproduct of Models). Fix a modal similarity type  $\tau$ , and let  $\mathfrak{M}_i (i \in I)$  be  $\tau$ -models. The **ultraproduct**  $\prod_U \mathfrak{M}_i$  of  $\mathfrak{M}_i$  modulo  $U$  is the model described as follows

1. The universe  $W_U$  of  $\prod_U \mathfrak{M}_i$  is the set  $\prod_U W_i$
2. Let  $V_i$  be the valuation of  $\mathfrak{M}_i$ . Then the valuation  $V_U$  of  $\prod_U \mathfrak{M}_i$  is defined by

$$f_U \in V_U(p) \quad \text{iff} \quad \{i \in I \mid f(i) \in V_i(p)\} \in U$$

3. Let  $\Delta$  be a modal operator in  $\tau$ , and  $R_{\Delta i}$  its associated relation in the model  $\mathfrak{M}_i$ . The relation  $R_{\Delta U}$  in  $\prod_U \mathfrak{M}_i$  is given by

$$R_{\Delta U} f_U^1 \dots f_U^{n+1} \quad \text{iff} \quad \{i \in I \mid R_{\Delta i} f^1(i) \dots f^{n+1}(i)\} \in U$$

In particular,

$$R_{\diamond U} f_U g_U \quad \text{iff} \quad \{i \in I \mid R_{\diamond i} f(i) g(i)\} \in U$$

**Proposition 2.62.** Let  $\prod_U \mathfrak{M}$  be an ultrapower of  $\mathfrak{M}$ . Then for all modal formulas  $\phi$  we have  $\mathfrak{M}, w \Vdash \phi$  iff  $\prod_U \mathfrak{M}, (f_w)_U \Vdash \phi$ , where  $f_w$  is the constant function s.t.  $f_w(i) = w$  for all  $i \in I$

*Proof.* 1.  $\phi = p$

$$\begin{aligned} \mathfrak{M}, w \Vdash \phi &\Leftrightarrow w \in V(\phi) \\ &\Leftrightarrow \{i \in I \mid f_w(i) \in V(p)\} = I \in U \\ &\Leftrightarrow \prod_U \mathfrak{M}, (f_w)_U \Vdash \phi \end{aligned}$$

2.  $\phi = \diamond \psi$

$$R w v \Leftrightarrow \{i \in I \mid R_{\diamond i} f_w(i) f_v(i)\} = I \in U \Leftrightarrow R_{\diamond U} f_w g_v$$

□

An ultrafilter is **countably incomplete** if it is not closed under countably intersections

**Example 2.7.** Consider the set of natural numbers  $\mathbb{N}$ . Let  $U$  be an ultrafilter over  $\mathbb{N}$  that does not contain any singletons  $\{u\}$ . Then for all  $n, (\mathbb{N} \setminus \{n\}) \in U$ . But

$$\emptyset = \bigcap_{n \in \mathbb{N}} (\mathbb{N} \setminus \{n\}) \notin U$$

So  $U$  is countably incomplete

**Lemma 2.63.** *Let  $\mathcal{L}$  be a countable first-order language,  $U$  a countably incomplete ultrafilter over a non-empty set  $I$ , and  $\mathfrak{M}$  an  $\mathcal{L}$ -model. The ultrapower  $\prod_U \mathfrak{M}$  is countably saturated*

**Theorem 2.64.** *Let  $\tau$  be a modal similarity type, and let  $\mathfrak{M}$  and  $\mathfrak{N}$  be  $\tau$ -models, and  $w$  and  $v$  states in  $\mathfrak{M}$  and  $\mathfrak{N}$  respectively. Then the following are equivalent*

1. *For all modal formulas  $\phi$ :  $\mathfrak{M}, w \Vdash \phi$  iff  $\mathfrak{N}, v \Vdash \phi$*
2. *There exists ultrapowers  $\prod_U \mathfrak{M}$  and  $\prod_U \mathfrak{N}$  as well as a bisimulation  $Z : \prod_U \mathfrak{M}, (f_w)_U \rightleftharpoons \prod_U \mathfrak{N}, (f_v)_U$  linking  $(f_w)_U$  and  $(f_v)_U$ , where  $f_w(f_v)$  is the constant function mapping every index to  $w(v)$*

*Proof.*  $2 \rightarrow 1$ .  $\mathfrak{M}, w \Vdash \phi$  iff  $\prod_U \mathfrak{M}, (f_w)_U \Vdash \phi$  iff  $\prod_U \mathfrak{N}, (f_v)_U \Vdash \phi$  iff  $\mathfrak{N}, v \Vdash \phi$   
 $1 \rightarrow 2$ . Take  $\mathbb{N}$  as our index set, and let  $U$  be a countably incomplete ultrafilter over  $\mathbb{N}$ . By Lemma 2.63 the ultrapowers  $\prod_U \mathfrak{M}$  and  $\prod_U \mathfrak{N}$  are countably saturated. Now  $(f_w)_U$  and  $(f_v)_U$  are modally equivalent. Next apply Theorem 2.56: as  $(f_w)_U$  and  $(f_v)_U$  are modally equivalent and  $\prod_U \mathfrak{M}$  and  $\prod_U \mathfrak{N}$  are countably saturated, there exists a bisimulation  $\square$

### 2.6.3 Definability

Given a modal similarity type  $\tau$ , a pointed model is a pair  $(\mathfrak{M}, w)$  where  $\mathfrak{M}$  is a  $\tau$ -model and  $w$  is a state of  $\mathfrak{M}$ . A class of pointed models  $K$  is said to be **closed under bisimulations** if  $(\mathfrak{M}, w) \in K$  and  $\mathfrak{M}, w \rightleftharpoons \mathfrak{N}, v$  implies  $(\mathfrak{N}, v) \in K$ .  $K$  is **closed under ultraproducts** if any ultraproducts  $\prod_U (\mathfrak{M}_i, w_i)$  of a family of pointed models  $(\mathfrak{M}_i, w_i)$  in  $K$  belongs to  $K$ . If  $K$  is a class of pointed  $\tau$ -models,  $\bar{K}$  denotes the complement of  $K$  within the class of all pointed  $\tau$ -models.  $K$  is **definable by a set of modal formulas** if there is a set of modal formulas  $\Gamma$  s.t. for any pointed model  $(\mathfrak{M}, w)$  we have  $(\mathfrak{M}, w) \in K$  iff for all  $\gamma \in \Gamma$ ,  $\mathfrak{M}, w \Vdash \gamma$ .  $K$  is definable by a single modal formula iff it is definable by a singleton set

By Theorem 2.12 definable classes of pointed models must be closed under bisimulations, and by Proposition 2.32 and Corollary 2.47 they must be closed under ultraproducts as well.

**Theorem 2.65.** *Let  $\tau$  be a modal similarity type, and  $K$  a class of pointed  $\tau$ -models. Then the following are equivalent:*

1.  *$K$  is definable by a set of modal formulas*
2.  *$K$  is closed under bisimulations and ultraproducts, and  $\bar{K}$  is closed under ultrapowers*

*Proof.* Assume  $K$  and  $\bar{K}$  satisfy the stated closure conditions. Observe that  $\bar{K}$  is closed under bisimulations as  $K$  is. Define

$$T = \{\phi \mid \forall (\mathfrak{M}, w) \in K : \mathfrak{M}, w \Vdash \phi\}$$

We will show that  $T$  defines the class of  $K$ .

Assume  $\mathfrak{M}, w \Vdash T$ . Define  $\Sigma = \{\phi \mid \mathfrak{M}, w \Vdash \phi\}$ . It is obvious that  $\Sigma$  is finitely satisfiable in  $K$ ; for suppose that the set  $\{\sigma_1, \dots, \sigma_n\} \subseteq \Sigma$  is not satisfiable in  $K$ . Then the formula  $\neg(\sigma_1 \wedge \dots \wedge \sigma_n)$  would be true on all pointed models in  $K$ , so it would belong to  $T$ , yet be false in  $\mathfrak{M}, w$ . But then the following claim shows that  $\Sigma$  is satisfiable in the ultraproduct of pointed models

**Claim 1 .** Let  $\Sigma$  be a set of modal formulas, and  $K$  a class of pointed models in which  $\Sigma$  is finitely satisfiable. Then  $\Sigma$  is satisfiable in some ultraproduct of models in  $K$

*Proof of Claim.* Define in index set  $I$  as the collection of all finite subsets of  $\Sigma$

$$I = \{\Sigma_0 \subset \Sigma \mid \Sigma_0 \text{ is finite}\}$$

By assumption, for each  $i \in I$  there is a pointed model  $(\mathfrak{N}_i, v_i)$  in  $K$  s.t.  $\mathfrak{N}_i, v_i \Vdash i$ . We now construct an ultrafilter  $U$  over  $I$  s.t. the ultraproduct  $\prod_U \mathfrak{N}_i$  has a state  $f_U$  with  $\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$

For each  $\sigma \in \Sigma$ , let  $\hat{\sigma}$  be the set of all  $i \in I$  s.t.  $\sigma \in i$ . Then the set  $E = \{\hat{\sigma} \mid \sigma \in \Sigma\}$  has the finite intersection property because

$$\{\sigma_1, \dots, \sigma_n\} \in \hat{\sigma}_1 \cap \dots \cap \hat{\sigma}_n$$

So  $E$  can be extended to an ultrafilter  $U$  over  $I$ . This defines  $\prod_U \mathfrak{N}_i$ ; for the definition of  $f_U$ , let  $W_i$  denote the universe of the model  $\mathfrak{N}_i$  and consider the function  $f \in \prod_{i \in I} W_i$  s.t.  $f(i) = v_i$ . It is left to prove that

$$\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$$

Observe that for  $i \in \hat{\sigma}$  we have  $\sigma \in i$  and so  $\mathfrak{N}_i, v_i \Vdash \sigma$ . Therefore for each  $\sigma \in \Sigma$

$$\{i \in I \mid \mathfrak{N}_i, v_i \Vdash \sigma\} \supseteq \hat{\sigma} \quad \text{and} \quad \hat{\sigma} \in U$$

since  $\sigma \in i$  implies  $\mathfrak{N}_i, v_i \Vdash \sigma$ . It follows that  $\{i \in I \mid \mathfrak{N}_i, v_i \Vdash \sigma\} \in U$ , so by Theorem 2.46  $\prod_U \mathfrak{N}_i, f_U \Vdash \sigma$ . Hence  $\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$

It follows from Claim 1 and the closure of  $K$  under taking ultraproducts that  $\Sigma$  is satisfiable in some pointed model  $(\mathfrak{N}, v) \in K$ . But  $\mathfrak{N}, v \Vdash \Sigma$  implies

that  $v$  and the state  $w$  from our original pointed model  $(\mathfrak{M}, w)$  are modally equivalent. So by Theorem 2.64 there exists an ultrafilter  $U'$  s.t.

$$\prod_{U'} (\mathfrak{N}, v), (f_v)_U \Leftrightarrow \prod_{U'} (\mathfrak{M}, w), (f_w)_U$$

By closure under ultraproducts, the pointed model  $(\prod_{U'} (\mathfrak{N}, v), (f_v)_U)$  belongs to  $K$ . Hence by closure under bisimulations,  $(\prod_{U'} (\mathfrak{M}, w), (f_w)_U)$  is in  $K$ . By closure of  $\bar{K}$  under ultrapowers,  $(\mathfrak{M}, w) \in K$   $\square$

**Theorem 2.66.** *Let  $\tau$  be a modal similarity type, and  $K$  a class of pointed  $\tau$ -models. Then the following are equivalent*

1.  *$K$  is definable by means of a single modal formula*
2. *Both  $K$  and  $\bar{K}$  are closed under bisimulations and ultraproducts*

*Proof.* Assume  $K, \bar{K}$  satisfy the stated conditions. Then both are closed under ultraproducts, hence by Theorem 2.65 there are set of modal formulas  $T_1, T_2$  defining  $K$  and  $\bar{K}$  respectively. Observe their union is inconsistent in the sense that there is no pointed model  $(\mathfrak{M}, w)$  s.t.  $(\mathfrak{M}, w) \models T_1 \cup T_2$ . So by compactness there exists  $\phi_1, \dots, \phi_n \in T_1$  and  $\psi_1, \dots, \psi_m \in T_2$  s.t. for all pointed models  $(\mathfrak{M}, w)$

$$\mathfrak{M}, w \models \phi_1 \wedge \dots \wedge \phi_n \rightarrow \neg\psi_1 \vee \dots \vee \neg\psi_m$$

By definition, for any  $(\mathfrak{M}, w) \in K$  we have  $\mathfrak{M}, w \models \phi_1 \wedge \dots \wedge \phi_n$ . Conversely, if  $\mathfrak{M}, w \models \phi_1 \wedge \dots \wedge \phi_n$ , then  $\mathfrak{M}, w \models \neg\psi_1 \vee \dots \vee \neg\psi_m$ . Hence  $\mathfrak{M}, w \not\models T_2$ . Therefore  $(\mathfrak{M}, w) \notin \bar{K}$ , whence  $(\mathfrak{M}, w) \in K$   $\square$