Abstract and Concrete Categories
The Joy of Cats

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1.1 Categories and Functors

1.1.1 Categories

Definition 1.1. A **category** is a quadruple $\mathbf{A} = (\mathcal{O}, \hom, id, \circ)$ consisting

1. a class $\mathcal{O}$, whose members are called **A-objects**
2. for each pair (A, B) of **A**-objects, a set $\hom(A, B)$ whose members are called **A-morphisms from** A **to** B

Example 1.1. 1. The following **constructs**; i.e., categories of structured sets and structure-preserving functions between them

   (a) **Alg**(Ω) with objects all Ω**-algebras** and morphisms all Ω**-homomorphisms** between them. Here $\Omega = (n_i)_{i \in I}$ is a family of natural numbers n_i, indexed by a set I. An Ω-algebra is a pair $X, (\omega_i)_{i \in I}$ consisting of a set X and a family of functions $\omega_i : X^{n_i} \to X$, called n_i**-ary operations** on X. An Ω-homomorphism $f : (X, (\omega_i)_{i \in I} \to (\widehat{X}, (\hat{\omega}_i)_{i \in I})$ is a function $f : X \to \widehat{X}$ for which the diagram

$$\begin{tikzcd} X^{n_i} \arrow[r, "f^{n_i}"] \arrow[d, "\omega_i"'] & \widehat{X}^{n_i} \arrow[d, "\widehat{\omega}_i"] \\ X \arrow[r, "f"'] & \widehat{X} \end{tikzcd}$$

   commutes for each $i \in I$.

   (b) Σ**-Seq** with objects all (deterministic, sequential) Σ**-acceptor**, where Σ is a finite set of input symbols. Objects are quadruples (Q, δ, q_0, F) where Q is a finite set of states, $\delta : \Sigma \times Q \to Q$ is a transition map, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states. A morphism $f : (Q, \delta, q_0, F) \to (Q', \delta', q_0', F')$ (called a **simulation**) is a function $f : Q \to Q'$ that preserves

      i. transitions, i.e., $\delta'(\sigma, f(q)) = f(\delta(\sigma, q))$
      ii. the initial state, i.e., $f(q_0) = q_0'$
      iii. the final states, i.e., $f[F] \subseteq F'$

2. The following categories where the objects and morphisms are *not* constructed sets and structure-preserving functions:

   (a) Mat with objects all natural numbers, and for which $\hom(m, n)$ is the set of all real $m \times n$ matrices, $id_n : n \to n$ is the unit diagonal matrix, and composition is defined by $A \circ B = BA$

(b) Aut with objects all (deterministic, sequential, Moore) **automata**. Objects are sectuples $(Q, \Sigma, Y, \delta, q_0, y)$, where Q is the set of states, Σ and Y are the sets of input symbols and output symbols, respectively, $\delta : \Sigma \times Q \to Q$ is the transition map, $q_0 \in Q$ is the initial state, and $y : Q \to Y$ is the output map. Morphisms from an automaton $(Q, \Sigma, Y, \delta, q_0, y)$ to an automaton $(Q', \Sigma', Y', \delta', q_0', y')$ are triples (f_Q, f_Σ, f_Y) of functions satisfying the following conditions
   i. preservation of transitions: $\delta'(f_\Sigma(\sigma), f_Q(q)) = f_Q(\delta(\sigma, q))$
   ii. preservation of outputs: $f_Y(y(q)) = y'(f_Q(q))$
   iii. preservation of initial state: $f_Q(q_0) = q_0'$

1.1.2 The Dual Principle

Definition 1.2. For any category $\mathbf{A} = (\mathcal{O}, \hom_{\mathbf{A}}, id, \circ)$ the **dual** (or **opposite**) **category of A** is the category $\mathbf{A}^{\mathrm{op}} = (\mathcal{O}, \hom_{\mathbf{A}^{\mathrm{op}}}, id, \circ^{\mathrm{op}})$, where $\hom_{\mathbf{A}^{\mathrm{op}}}(A, B) = \hom_{\mathbf{A}}(B, A)$ and $f \circ^{\mathrm{op}} g = g \circ f$

Consider the property of objects X in **A**:

$\mathcal{P}_{\mathbf{A}}(X) \equiv$ *For any* $\mathbf{A}$*-object* A *there exists exactly one* $\mathbf{A}$*-morphism* $f : A \to X$

Step1: In $\mathcal{P}_{\mathbf{A}}(X)$ replace all occurrences of **A** by $\mathbf{A}^{\mathrm{op}}$, thus obtaining the property

$\mathcal{P}_{\mathbf{A}^{\mathrm{op}}}(X) \equiv$ *For any* $\mathbf{A}^{\mathrm{op}}$*-object* A *there exists exactly one* $\mathbf{A}^{\mathrm{op}}$*-morphism* $f : A \to X$

Step2: Translate $\mathcal{P}_{\mathbf{A}^{\mathrm{op}}}(X)$ into the logically equivalent statement

$\mathcal{P}^{\mathrm{op}}_{\mathbf{A}}(X) \equiv$ *For any* $\mathbf{A}$*-object* A *there exists exactly one* $\mathbf{A}$*-morphism* $f : X \to A$

The **Duality Principle for Categories** states

Whenever a property $\mathcal{P}$ holds for all categories,
then the property $\mathcal{P}^{\mathrm{op}}$ holds for all categories.

1.1.3 Isomorphism

Definition 1.3. A morphism $f : A \to B$ in a category is called an **isomorphism** provided that there exists a morphism $g : B \to A$ with $g \circ f = id_A$ and $f \circ g = id_B$. Such a morphism g is called an **inverse** of f

Proposition 1.4. *If $f : A \to B, g : B \to A, h : B \to A$ are morphisms s.t. $g \circ f = id_A$ and $f \circ h = id_B$, then $g = h$*

Definition 1.5. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor

1. F is called an **embedding** provided that F is injective on morphisms
2. F is called **faithful** provided that all the hom-set restrictions

$$F : \hom_{\mathbf{A}}(A, A') \to \hom_{\mathbf{B}}(FA, FA')$$

are injective

3. F is called **full** provided that all hom-set restrictions are surjective
4. F is called **amnestic** provided that an **A**-isomorphism f is an identity whenever Ff is an identity

1.2 Subcategories

Definition 1.6. A category **A** is said to be a **subcategory** of a category **B** provided that the following conditions are satisfied

1. $Ob(\mathbf{A}) \subseteq Ob(\mathbf{B})$
2. for each $A, A' \in Ob(\mathbf{A})$, $\hom_{\mathbf{A}}(A, A') \subseteq \hom_{\mathbf{B}}(A, A')$
3. for each **A**-object A, the **B**-identity on A is the **A**-identity on A
4. the composition law in **A** is the restriction of the composition law in **B** to the morphisms of **A**

A is called a **full subcategory** of **B** if in addition to the above, for each $A, A' \in Ob(A)$, $\hom_{\mathbf{A}}(A, A') = \hom_{\mathbf{B}}(A, A')$

Proposition 1.7. 1. *A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is a (full) embedding if and only if there exists a (full) subcategory $\boldsymbol{C}$ of $\boldsymbol{B}$ with inclusion function $E : \boldsymbol{C} \to \boldsymbol{B}$ and an isomorphism $G : \boldsymbol{A} \to \boldsymbol{C}$ with $F = E \circ G$*

2. *A functor $F : \boldsymbol{A} \to \boldsymbol{B}$ is faithful iff there exists embeddings $E_1 : \boldsymbol{D} \to \boldsymbol{B}$ and $E_2 : \boldsymbol{A} \to \boldsymbol{C}$ and an equivalence $G : \boldsymbol{C} \to \boldsymbol{D}$ s.t. the diagram*

$$\begin{tikzcd} \boldsymbol{A} \arrow[r, "F"] \arrow[d, "E_2"] & \boldsymbol{B} \\ \boldsymbol{C} \arrow[r, "G"] & \boldsymbol{D} \arrow[u, "E_1"'] \end{tikzcd}$$

Proof. 1. Let $E_1 : \mathbf{D} \to \mathbf{B}$ be the inclusion of the full subcategory **D** of **B** that has as objects all images of **A**-objects. Let **C** be the category with $Ob(\mathbf{C}) = Ob(\mathbf{A})$, with

$$\hom_{\mathbf{C}}(A, A') = \hom_{\mathbf{B}}(FA, FA')$$

Now define functors $E_2 : \mathbf{A} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{D}$ by

$$E_2(A \xrightarrow{f} A') = A \xrightarrow{Ff} A' \quad \text{and} \quad G(C \xrightarrow{g} C') = FC \xrightarrow{g} FC'$$

Then E_2 is an embedding, G is an equivalence and $F = E_1 \circ G \circ E_2$ □

Definition 1.8. A category **A** is said to be **fully embeddable** into **B** provided that there exists a full embedding $\mathbf{A} \to \mathbf{B}$

Definition 1.9. A full subcategory **A** of a category **B** is called

1. **isomorphism-closed** provided that every **B**-object that is isomorphic to some **A**-object is itself an **A**-object
2. **isomorphism-dense** provided that every **B**-object is isomorphic to some **A**-object

Remark. If **A** is a full subcategory of **B**, then the following conditions are equivalent

1. **A** is an isomorphism-dense subcategory of **B**
2. the inclusion functor $\mathbf{A} \hookrightarrow \mathbf{B}$ is isomorphism-dense
3. the inclusion functor $\mathbf{A} \hookrightarrow \mathbf{B}$ is an equivalence

Example 1.2. The full subcategory of **Set** with the single object $\mathbb{N}$ is neither isomorphism-closed nor isomorphism-dense in **Set**. It is equivalent to the isomorphism-closed full subcategory of **Set** consisting of all countable infinite sets.

Definition 1.10. A **skeleton** of a category is a full, isomorphism-dense subcategory in which no two distinct objects are isomorphic

Example 1.3.
1. The full subcategory of all cardinal numbers is a skeleton for **Set**

Proposition 1.11.
1. *Every category has a skeleton*
2. *Any two skeletons of a category are isomorphic*
3. *Any skeleton of a category **C** is equivalent to **C***

Proof.
1. This follows from the Axiom of Choice applied to the equivalence relation "is isomorphic to" on the class of objects of the category

□

Corollary 1.12. *Two categories are equivalent iff they have isomorphic skeletons*

Definition 1.13. Let **A** be a subcategory of **B**, and let B be a **B**-object

1. An **A-reflection** (or **A-reflection arrow**) for B is a **B**-morphism $B \xrightarrow{r} A$ from B to an **A**-object A with the following universal property:

   for any **B**-morphism $B \xrightarrow{f} A'$ from B into some **A**-object A', there exists a unique **A**-morphism $f' : A \to A$ s.t. the triangle

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, "f"'] & A \arrow[d, "f'"] \\ & A' \end{tikzcd}$$

commutes

2. **A** is called a **reflective subcategory** of **B** provided that each **B**-object has an **A**-reflection

Example 1.4. 1. **Modifications of the Structure**

(a) Making a relation symmetric: $\mathbf{B} = \mathbf{Rel}, \mathbf{A} = \mathbf{Sym}$, the full subcategory of symmetric relations, $(X, \rho) \xrightarrow{id_X} (X, \rho \cup \rho^{-1})$ is an **A**-reflection for (X, ρ)

2. **Improving Objects by Forming Quotients**

(a) Making a reachable acceptor minimal: $\mathbf{B} =$ the full subcategory of Σ**-Seq** consisting of all **reachable acceptors** (i.e., those for which each state can be reached from the initial one by an input word), $\mathbf{A} =$ the full subcategory of **B** consisting of all **minimal acceptors** (i.e. those reachable acceptors with the property that no two different states are **observably equivalent**. The observability equivalence $\approx$ on a reachable acceptor B is given by: $q \approx q'$ provided that whenever the initial state of B is changed to q, the resulting acceptor recognizes the same language as it does when the initial state is changed to q'). Then the canonical map $B \to B/\approx$ is an **A**-reflection for B

3. **Completions**

Proposition 1.14. *Reflections are essentially unique, i.e.*

1. *if $B \xrightarrow{r} A$ and $B \xrightarrow{\hat{r}} \hat{A}$ are **A**-reflections for B, then there exists an **A**-isomorphism $k : A \to \hat{A}$ s.t. the triangle*

$$\begin{tikzcd} B \arrow[r, "r"] \arrow[dr, "\hat{r}"'] & A \arrow[d, "k"] \\ & \hat{A} \end{tikzcd}$$

commutes

2. *if $B \xrightarrow{r} A$ is an **A**-reflection for B and $A \xrightarrow{k} \hat{A}$ is an **A**-isomorphism, then $B \xrightarrow{k \circ r} \hat{A}$ is an **A**-reflection for B*

Proposition 1.15. *If **A** is reflective subcategory of **B**, then the following conditions are equivalent*

1. ***A*** *is a full subcategory of* ***B***
2. *for each* ***A****-object* A*,* $A \xrightarrow{id_A} A$ *is an* ***A****-reflection*
3. *for each* ***A****-object* A*,* ***A****-reflection arrows* $A \xrightarrow{r_A} A^*$ *are* ***A****-isomorphism*
4. *for each* ***A****-object* A*,* ***A****-reflection arrows* $A \xrightarrow{r_A} A^*$ *are* ***A****-morphisms*

Proof. $2 \to 3$.

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[ddr, "r_A"'] & A^* \arrow[d, "f"] \\ & A \arrow[d, "r_A"] \\ & A^* \end{tikzcd}$$

□

Proposition 1.16. *Let* ***A*** *be a reflective subcategory of* ***B****, and for each* ***B****-object* B *let* $r_B : B \to A_B$ *be an* ***A****-reflection arrow. Then there exists a unique functor* $R : \mathbf{B} \to \mathbf{A}$ *s.t. the following conditions are satisfied*

1. $R(B) = A_B$ *for each* ***B****-object* B
2. *for each* ***B****-morphism* $f : B \to B'$ *the diagram*

$$\begin{tikzcd} B \arrow[r, "r_B"] \arrow[d, "f"'] & R(B) \arrow[d, "R(f)"] \\ B' \arrow[r, "r_{B'}"'] & R(B') \end{tikzcd}$$

commutes

Proof. Show that functor is well-defined and preserves identities and compositions □

Definition 1.17. A functor $R : \mathbf{B} \to \mathbf{A}$ constructed according to the above proposition is called a **reflector for A**

Definition 1.18. Let **A** be a subcategory of **B** and let B be a **B**-object

1. An **A-coreflection** (or A**-coreflection arrow**) for B is a **B**-morphism $A \xrightarrow{c} B$ form an **A**-object A to B with the following universal property: for any **B**-morphism $A' \xrightarrow{f} B$ from some **A**-object A' to B there exists a unique **A**-morphism $f' : A' \to A$ s.t. the triangle

$$\begin{tikzcd} A' \arrow[d, "f'"'] \arrow[dr, "f"] & \\ A \arrow[r, "c"'] & B \end{tikzcd}$$

commutes

2. **A** is called a **coreflective subcategory** of **B** provided that each **B**-object has an **A**-coreflection

Proposition 1.19. *If **A** is a coreflective subcategory of **B** and for each **B**-object B, $A_B \xrightarrow{c_B} B$ is an **A**-coreflection arrow, then there exists a unique functor $C : \mathbf{B} \to \mathbf{A}$ (called a **coreflector for A**) s.t. the following conditions are satisfied*

1. *$C(B) = A_B$ for each **B**-object B*
2. *for each **B**-morphism $f : B \to B'$ the diagram*

$$\begin{tikzcd} C(B) \arrow[r, "c_B"] \arrow[d, "C(f)"'] & B \arrow[d, "f"] \\ C(B') \arrow[r, "c_{B'}"'] & B' \end{tikzcd}$$

commutes

Exercise 1.2.1. A subcategory **A** of a category **B** is called **isomorphism-closed** provided that every **B**-isomorphism with domain in **A** belongs to **A**. Show that every subcategory **A** of **B** can be embedded into a smallest isomorphism-closed subcategory $\mathbf{A}'$ of **B** that contains **A**. The inclusion functor $\mathbf{A} \hookrightarrow \mathbf{A}'$ is an equivalence iff all **B**-isomorphisms between **A**-objects belong to **A**

Exercise 1.2.2. 1. Show that a category is discrete iff each of its subcategories is full
2. Show that in a poset, considered as a category
   - every subcategory is isomorphism-closed
   - every (co)reflective subcategory is full

1.3 Concrete categories and concrete functors

Definition 1.20. Let **X** be a category. A **concrete category** over **X** is a pair $(\mathbf{A}, U)$ where **A** is the category and $U : \mathbf{A} \to \mathbf{X}$ is a faithful functors. Sometimes U is called the **forgetful** (or **underlying**) **functor** of the concrete category and **X** is called the **base category** for $(\mathbf{A}, U)$

A concrete category over **Set** is called a **construct**

Remark. We adopt the following conventions:

1. Since faithful functors are injective on hom-sets, we usually assume that $\hom_{\mathbf{A}}(A, B)$ is a subset of $\hom_{\mathbf{X}}(UA, UB)$ for each pair (A, B) of **A**-objects. This allows one to express the property that "for bA-objects A and B and an **X**-morphism $UA \xrightarrow{f} UB$ there exists a (necessarily unique) **A**-morphism $A \to B$ with $U(A \to B) = UA \xrightarrow{f} UB$" much more succinctly, by stating

$$UA \xrightarrow{f} UB \text{ is an } \mathbf{A}\text{-morphism (from } A \text{ to } B)$$

Observe, however, that since U doesn't need to be injective on objects, the expression

$$UA \xrightarrow{id_X} UB \text{ is an } \mathbf{A}\text{-morphism (from } A \text{ to } B)$$

does not imply that $A = B$ or that $id_X = id_A$, although it does imply that $UA = UB = X$. We call an **A**-morphism $A \xrightarrow{f} B$ **identity-carried** if $Uf = id_X$
2. Sometimes we will write **A** for the concrete category $(\mathbf{A}, U)$ over **X**, when U is clear from the context. In these cases the underlying object of an **A**-object A will sometimes be denoted by $|A|$
3. If P is a property of categories (or of functors), then we will say that a concrete category $(\mathbf{A}, U)$ **has property** P provided that **A** (or U) has property P

Definition 1.21. Let $(\mathbf{A}, U)$ be a concrete category over **X**

1. The **fibre** of an **X**-object X is the preordered class consisting of all **A**-objects A with $U(A) = X$ ordered by

$$A \leq B \quad \text{iff} \quad id_X : UA \to UB \text{ is an } \mathbf{A}\text{-morphism}$$

2. **A**-objects A, B are **equivalent** if $A \leq B$ and $B \leq A$
3. $(\mathbf{A}, U)$ is said to be **amnestic** provided that its fibres are partially ordered classes; i.e., no two different **A**-objects are equivalent
4. $(\mathbf{A}, U)$ is said to be **fibre-small** provided that each of its fibres is small, i.e., a preordered set

Remark. A concrete category $(\mathbf{A}, U)$ is amnestic iff the functor U is amnestic. Most of the familiar concrete categories are both amnestic and fibre-small.

Definition 1.22. A concrete category is called

1. **fibre-complete** provided that its fibres are (possibly large) complete lattices
2. **fibre-discrete** provided that its fibres are ordered by equality

Proposition 1.23. *A concrete category $(\boldsymbol{A}, U)$ over $\boldsymbol{X}$ is fibre-discrete iff U **reflects identities** (i.e. if $U(k)$ is an $\boldsymbol{X}$-identity, then k must be an $\boldsymbol{A}$-identity)*

Definition 1.24. If $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ are concrete categories over $\mathbf{X}$, then a **concrete functor from** $(\mathbf{A}, U)$ **to** $(\mathbf{B}, V)$ is a functor $F : \mathbf{A} \to \mathbf{B}$ with $U = V \circ F$. We denote such a functor by $F : (\mathbf{A}, U) \to (\mathbf{B}, V)$

Proposition 1.25.
1. *Every concrete functor is faithful*
2. *Every concrete functor is completely determined by its values on objects*
3. *Objects that are identified by a full concrete functor are equivalent*
4. *Every full concrete functor with amnestic domain is an embedding*

Proof.
1. U and V are faithful
2. Suppose that $G : (\mathbf{A}, U) \to (\mathbf{B}, V)$ is a concrete functor with $G(A) = F(A)$ for each $\mathbf{A}$-object A. Then for any $\mathbf{A}$-morphism $A \xrightarrow{f} A'$ we have the $\mathbf{B}$-morphism

$$GA = FA \underset{Gf}{\overset{Ff}{\rightrightarrows}} FA' = GA'$$

with $V(Ff) = U(f) = V(Gf)$. Since V is faithful, $Ff = Gf$. Hence $F = G$
3. Let A and A' be $\mathbf{A}$-objects with $FA = FA'$. Then $id_B : FA \to FA'$ can be lifted to an $\mathbf{A}$-isomorphism $g : A \to A'$. Hence A and A' are equivalent

□

Remark. If $F : (\mathbf{A}, U) \to (\mathbf{B}, V)$ is a concrete isomorphism, then its inverse $F^{-1} : \mathbf{B} \to \mathbf{A}$ is concrete from $(\mathbf{B}, V)$ to $(\mathbf{A}, U)$. Unfortunately, the corresponding result does not hold for concrete equivalences. If $F : (\mathbf{A}, U) \to (\mathbf{B}, V)$ is a concrete equivalence from $(\mathbf{B}, V)$ to $(\mathbf{A}, U)$ even though there are equivalences from $\mathbf{B}$ to $\mathbf{A}$. For example, the embedding of the skeleton of cardinal numbers into **Set** is such a concrete categories over $\mathbf{X}$ that is not invertible

Proposition 1.26.
1. *The identity functor on a concrete category is a concrete isomorphism*
2. *Any composite of concrete functors over $\boldsymbol{X}$ is a concrete functor over $\boldsymbol{X}$*

Definition 1.27. The quansicategory that has as objects all concrete categories over $\mathbf{X}$ and as morphisms all concrete functors between them is denoted by $\mathbf{CAT}(\mathbf{X})$. In particular, $\mathbf{CONST} = \mathbf{CAT}(\mathbf{Set})$ is the quasicategory of all constructs.

Definition 1.28. If F and G are both concrete functors from $(\mathbf{A}, U)$ to $(\mathbf{B}, V)$, then F is **finer than** G (or G is **coaser than** F), denoted by $F \leq G$, provided that $F(A) \leq G(A)$ for each $\mathbf{A}$-object A

Example 1.5. 1. For order-preserving functions considered as concrete functors over $\mathbf{1}$, $f \leq g$ iff this relation holds pointwise

Remark. For every concrete category $(\mathbf{A}, U)$ over $\mathbf{X}$, its dual $(\mathbf{A}^{\text{op}}, U^{\text{op}})$ is a concrete category over $\mathbf{X}^{\text{op}}$. Moreover, for every concrete functor $F : (\mathbf{A}, U) \to (\mathbf{B}, V)$ over $\mathbf{X}$ its dual functor $F^{\text{op}} : (\mathbf{A}^{\text{op}}, U^{\text{op}}) \to (\mathbf{B}^{\text{op}}, V^{\text{op}})$ is a concrete functor over $\mathbf{X}^{\text{op}}$. However, unless $\mathbf{X} = \mathbf{X}^{\text{op}}$ there is **no** duality for concrete categories over a fixed base category $\mathbf{X}$. In particular, we don't have a duality principle for constructs. However, since $\mathbf{1} = \mathbf{1}^{\text{op}}$, there is a duality principle for concrete categories over $\mathbf{1}$ (i.e., for preordered classes)

If $(\mathbf{B}, U)$ is a concrete category over $\mathbf{X}$ and $\mathbf{A}$ is a subcategory of $\mathbf{B}$ with inclusion $E : \mathbf{A} \hookrightarrow \mathbf{B}$, then $\mathbf{A}$ will often be regarded (via the functor $U \circ E$) as a concrete category $(\mathbf{A}, U \circ E)$ over $\mathbf{X}$. In such cases we will call $(\mathbf{A}, U \circ E)$ a **concrete subcategory** of $(\mathbf{B}, U)$. In the case that the base category is **Set**, we will call $(\mathbf{A}, U \circ E)$ a **subconstruct** of $(\mathbf{B}, U)$

Definition 1.29. A concrete subcategory $(\mathbf{A}, U)$ of $(\mathbf{B}, V)$ is called **concretely reflective** in $(\mathbf{B}, V)$ provided that for each $(\mathbf{B})$-object there exists an identity-carried $\mathbf{A}$-reflection arrow

Relectors induced by identity-carried reflection arrows are called **concrete reflectors**

Example 1.6. 1. Let $\mathbf{X}$ be a category consisting of a single object X and two morphisms id_X and s with $s \circ s = id_X$. Let $\mathbf{A}$ be the concrete category over $\mathbf{X}$, consisting of two objects A_0 and A_1 and the morphism sets

$$\hom_{\mathbf{A}}(A_i, A_j) = \begin{cases} \{id_X\} & i = j \\ \{s\} & i \neq j \end{cases}$$

Consider $\mathbf{A}$ as a concretely reflective subcategory of itself. Then $id_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ is a concrete reflector, and the concrete functor $R : \mathbf{A} \to \mathbf{A}$, defined by $R(A_i) = A_{1-i}$ is a reflector that is not a concrete reflector

Proposition 1.30. *Every concretely reflective subcategory of an amnestic concrete category is a full subcategory*

Proof. Let $(\mathbf{A}, U)$ be a concretely reflective subcategory of an amnestic $(\mathbf{B}, V)$, let A be an $r : A \to A^*$ be an identity-carried $\mathbf{A}$-reflection arrow for A. We wish to show that $r = id_A$ so that Proposition 1.15 can be applied. By reflectivity there exists a unique $\mathbf{A}$-morphism $s : A^* \to A$ s.t. the diagram

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[dr, "id_A"'] & A^* \arrow[d, "s"] \\ & A \end{tikzcd}$$

commutes.

Since r is identity-carried, $V(r) = id_{VA}$. Since also $V(id_A) = id_{VA}$, we conclude that $V(s) = id_{VA}$. Faithfulness of V gives us $r \circ s = id_{A^*}$. Hence r is a $\mathbf{B}$-isomorphism with $V(r) = id_{VA}$. Amnesticity of $(\mathbf{B}, V)$ yields $r = id_A$. □

Proposition 1.31. *For a concrete full subcategory $(\boldsymbol{A}, U)$ of a concrete category $(\boldsymbol{B}, V)$ over $\boldsymbol{X}$, with inclusion functor $E : (\boldsymbol{A}, U) \hookrightarrow (\boldsymbol{B}, V)$, the following are equivalent*

1. *$(\boldsymbol{A}, U)$ is concretely reflective in $(\boldsymbol{B}, V)$*
2. *there exists a concrete functor $R : (\boldsymbol{B}, V) \to (\boldsymbol{A}, U)$ that is a reflector with $R \circ E = id_{\boldsymbol{A}}$ and $id_{\boldsymbol{B}} \leq E \circ R$*
3. *there exists a concrete functor $R : (\boldsymbol{B}, V) \to (\boldsymbol{A}, U)$ with $R \circ E \leq id_{\boldsymbol{A}}$ and $id_{\boldsymbol{B}} \leq E \circ R$*

Proof. $1 \to 2$. □

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