Logic Language And Meaning

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1 The Theory of Types and Categorical Grammar

1.1 The Theory of Types

1.1.1 Type Distinctions in Natural Language

1. If John is self-satisfied, then there is at least one thing he has in common with Peter

Sentence (1) contains quantification over properties.

2. Santa Claus has all the attributes of a sadist

If we are to quantify not only over entities but also over properties of entities, then we need to extend predicate logic by introducing variables other than the ones we already have, which only range over entities. Besides predicate letters, we need **predicate variables**, so that we can quantify over this kind of variable in the syntax. Letting X be such a variable, (1) and (2) may be represented as in (3) and (4):

3. $Zj \to \exists X(Xj \wedge Xp)$
4. $\forall X(\forall x(Sx \to Xx) \to Xs)$

But second-order predicate logic does not exhaust the expressive power of natural language. For not only are there natural language sentences which quantify over properties of entities, but there are also sentences which attribute properties to these properties of entities in turn.The predicate **red** expresses a property of individuals, so the predicate **color** expresses a property of properties of individuals. So in a sentence like **Red is a color**, which we represent as $\mathcal{C}(R)$, the second-order predicate **color** is applied to the first-order predicate **red**. We can also quantify over these properties of properties, as in **Red has something in common with green**.

Besides higher-order predicates, there are other kinds of expressions which for linguistic purposes may usefully be added to predicate logic.

Our first class of examples is formed by expressions with **predicate adverbials**

5. John is walking quickly

The expression **quickly** is, form a linguistic perspective, a modifier acting on the verb **is walking**. From a logical perspective, the property of walking quickly is attributed to an entity, John. This property cannot be seen as a conjunction of two properties, 'being quick' and 'walking'. For sentence (5) does not mean the same thing as sentence (6):

6. John is walking and John is quick

In logical terms, **quickly** is an expression which when applied to the first-order predicate **walking** result in a new first-order predicate **walking**

quickly. From a logical point of view, the **relative adjectives** are expressions of the same kind. Sentence (7) may be represented in first-order predicate logic as formula (8)

7. Jumbo is a pink elephant
8. $Ej \wedge Pj$

The adjective **pink** may, in other words, be represented as a standard first-order predicate. But the same does not apply to relative adjectives like **small**. Sentence (9) is the same kind of sentence as (7)

9. Jumbo is a small elephant

But sentence (9) cannot be analyzed as a conjunction of two first-oder predicates. The formula (10) which we would then obtain:

10. $Ej \wedge Sj$

expresses something which is generally false. The relative adjective **small** works the same way as the predicate adverbial **quickly**. When applied to the predicate **elephant**, it results in a new predicate **small elephant**

1.1.2 Syntax

As our two basic types we have e, which is the type of those expressions which refer to entities, and t, the type of those expressions which refer to truth values.

Definition 1.1. **T**, the set of types, is the smallest set s.t.

1. $e, t \in \mathbf{T}$
2. if $a, b \in \mathbf{T}$, then $\langle a, b\rangle \in \mathbf{T}$

An expression of type $\langle a, b\rangle$ is an expression which when applied to an expression of type a results in an expression of type b. If α is an expression of type $\langle a, b\rangle$ and β is an expression of type a, then $\alpha(\beta)$ will be an expression of type b. This process of applying an α of type $\langle a, b\rangle$ to a β of type a is called **(functional) application of α to β**

The **vocabulary** of a type-theoretical language L contains some symbols which are shared by all such languages and a number of symbols which are characteristic of L. The shared part consists of:

1. For every type a, an infinite set VAR_a of variables of type a
2. The usual connectives $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$
3. The quantifiers $\forall$ and $\exists$
4. Two brackets (and)
5. The symbol for identity =

The part of the vocabulary which is characteristic of L contains

6. for every type a, a (possibly empty) set CON_a^L of constants of type a

We will write v_a for variables of type a and c_a for constants of type a.

Definition 1.2.
1. If α is a variable or a constant of type a in L, then α is an expression of type a in L
2. If α is an expression of type $\langle a, b\rangle$ in L, and β is an expression of type a in L, then $(\alpha(\beta))$ is an expression of type b in L
3. If ϕ and ψ are expressions of type t in L, then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$,$(\phi \rightarrow \psi)$ and $\phi \leftrightarrow \psi$
4. If ϕ is an expression of type t in L and v is a variable (of arbitrary type a), then $\forall x\phi$ and $\exists v\phi$ are expressions of type t in L
5. If α and β are expressions in L which belong to the same (arbitrary) type, then $(\alpha = \beta)$ is an expression of type t in L
6. Every expression in L is to be constructed by means of (1) - (5) in a finite number of steps

We refer to the set of all expressions in L of type a as WE_a^L or, if it is clear which L is meant, as WE_a. The **formulas** are the elements of WE_t

1.1.3 Semantics

Given a domain D, one-place predicates are interpreted as the characteristic functions of subsets of that domain.

The domain of interpretation of expressions of type a, given a domain D, is written as $\mathbf{D}_{a,D}$ and is defined as follows

Definition 1.3.
1. $\mathbf{D}_{e,D} = D$
2. $\mathbf{D}_{t,D} = \{0, 1\}$
3. $\mathbf{D}_{\langle a,b\rangle,D} = \mathbf{D}_{b,D}^{\mathbf{D}_{a,D}}$

For example, in the theory of types, a two-place predicate $L(loves)$ is an expression of type $\langle e, \langle e, t\rangle\rangle$. The corresponding interpretation domain $\mathbf{D}_{\langle e,\langle e,t\rangle\rangle}$ is $(\{0, 1\}^D)^D$

Consider the second-order predicate $\mathcal{C}(color)$, which is of type $\langle\langle e, t\rangle, t\rangle$. The interpretation domain $\mathbf{D}_{\langle\langle e,t\rangle,t\rangle}$ is the set of functions $\{0, 1\}^{\{0,1\}^D}$

A model $\mathbf{M}$ for an language L for the theory of types consists of a nonempty domain set D together with an interpretation function I. For each type a, I is a function from CON_a^L into $\mathbf{D}_{a,D}$.

We must define the concept of **the interpretation of α w.r.t. a model $\mathbf{M}$ and an assignment** g, to be written as $[\![\alpha]\!]_{\mathbf{M},g}$. The interpretation function$[\![\ ]\!]_{\mathbf{M},g}$ can be seen as a function which for all types a, maps WE_a^L into $\mathbf{D}_{a,D}$.

Definition 1.4. 1. If $\alpha \in \mathrm{CON}^{l}_{a}$, then $[\![\alpha]\!]_{\mathbf{M},g} = I(\alpha)$
If $\alpha \in \mathrm{VAR}_a$, then $[\![\alpha]\!]_{\mathbf{M},g} = g(\alpha)$

2. If $\alpha \in \mathrm{WE}^{L}_{\langle a,b\rangle}, \beta \in \mathrm{WE}^{L}_{a}$, then $[\![\alpha(\beta)]\!]_{\mathbf{M},g} = [\![\alpha]\!]_{\mathbf{M},g}([\![\beta]\!]_{\mathbf{M},g})$
3. If $\phi, \psi \in \mathrm{WE}^{L}_{t}$, then
$[\![\neg\phi]\!]_{\mathbf{M},g} = 1$ iff $[\![\phi]\!]_{\mathbf{M},g} = 0$
$[\![\phi \wedge \psi]\!]_{\mathbf{M},g} = 1$ iff $[\![\phi]\!]_{\mathbf{M},g} = [\![\psi]\!]_{\mathbf{M},g} = 1$
$[\![\phi \rightarrow \psi]\!]_{\mathbf{M},g} = 0$ iff $[\![\phi]\!]_{\mathbf{M},g} = 1$ and $[\![\psi]\!]_{\mathbf{M},g} = 0$
$[\![\phi \leftrightarrow \psi]\!]_{\mathbf{M},g}$ iff $[\![\phi]\!]_{\mathbf{M},g} = [\![\psi]\!]_{\mathbf{M},g}$
4. if $\phi \in \mathrm{WE}^{l}_{t}, v \in \mathrm{VAR}_a$, then
$[\![\forall v\phi]\!]_{\mathbf{M},g} = 1$ iff for all $d \in \mathbf{D}_{a,D}$: $[\![\phi]\!]_{M,g[v/d]} = 1$
$[\![\exists v\phi]\!]_{\mathbf{M},g} = 1$ iff there is at least one $d \in \mathbf{D}_{a,D}$ s.t.: $[\![\phi]\!]_{\mathbf{M},g[v/d]} = 1$
5. If $\alpha, \beta \in \mathrm{WE}^{L}_{a}$, then $[\![\alpha = \beta]\!]_{\mathbf{M},g} = 1$ iff $[\![\alpha]\!]_{\mathbf{M},g} = [\![\beta]\!]_{\mathbf{M},g}$