A Course in Model Theory

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Contents

1 The Basics

1.1 Structures

Definition 1.1. Let $\mathfrak{A}, \mathfrak{B}$ be L-structures. A map $h : A \to B$ is called a **homomorphism** if for all $a_1, \ldots, a_n \in A$

$$\begin{aligned} h(c^{\mathfrak{A}}) &= c^{\mathfrak{B}} \\ h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) &= f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) \\ R^{\mathfrak{A}}(a_1, \ldots, a_n) &\Rightarrow R^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) \end{aligned}$$

We denote this by

$$h : \mathfrak{A} \to \mathfrak{B}$$

If in addition h is injective and

$$R^{\mathfrak{A}}(a_1, \ldots, a_n) \Leftrightarrow R^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$$

for all $a_1, \dots, a_n \in A$, then h is called an (isomorphic) **embedding**. An **isomorphism** is a surjective embedding

Lemma 1.2. *Let $h : \mathfrak{A} \xrightarrow{\sim} \mathfrak{A}'$ be an isomorphism and $\mathfrak{B}$ an extension of $\mathfrak{A}$. Then there exists an extension $\mathfrak{B}'$ of $\mathfrak{A}'$ and an isomorphism $g : \mathfrak{B} \xrightarrow{\sim} \mathfrak{B}'$ extending h*

Definition 1.3. Let $(I, \leq)$ be a **directed partial order**. This means that for all $i, j \in I$ there exists a $k \in I$ s.t. $i \leq k$ and $j \leq k$. A family $(\mathfrak{A}_i)_{i \in I}$ of L-structures is called **directed** if

$$i \leq j \Rightarrow \mathfrak{A}_i \subseteq \mathfrak{A}_j$$

If I is linearly ordered, we call $(\mathfrak{A}_i)_{i \in I}$ a **chain**

If a structure $\mathfrak{A}_1$ is isomorphic to a substructure $\mathfrak{A}_0$ of itself,

$$h_0 : \mathfrak{A}_0 \xrightarrow{\sim} \mathfrak{A}_1$$

then Lemma 1.2 gives an extension

$$h_1 : \mathfrak{A}_1 \xrightarrow{\sim} \mathfrak{A}_2$$

Continuing in this way we obtain a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \dots$ and an increasing sequence $h_i : \mathfrak{A}_i \xrightarrow{\sim} \mathfrak{A}_{i+1}$ of isomorphism

Lemma 1.4. *Let $(\mathfrak{A}_i)_{i \in I}$ be a directed family of L-structures. Then $A = \bigcup_{i \in I} A_i$ is the universe of a (uniquely determined) L-structure*

$$\mathfrak{A} = \bigcup_{i \in I} \mathfrak{A}_i$$

which is an extension of all $\mathfrak{A}_i$

A subset K of L is called a **sublanguage**. An L-structure becomes a K-structure, the **reduct**.

$$\mathfrak{A} \restriction K = (A, (Z^{\mathfrak{A}})_{Z \in K})$$

Conversely we call $\mathfrak{A}$ an **expansion** of $\mathfrak{A} \restriction K$.

1. Let $B \subseteq A$, we obtain a new language

   $$L(B) = L \cup B$$

   and the $L(B)$-structure

   $$\mathfrak{A}_B = (\mathfrak{A}, b)_{b \in B}$$

   Note that **Aut**$(\mathfrak{A}_B)$ is the group of automorphisms of $\mathfrak{A}$ fixing B elementwise. We denote this group by **Aut**$(\mathfrak{A}/B)$

Let S be a set, which we call the set of sorts. An S-sorted language L is given by a set of constants for each sort in S, and typed function and relations. For any tuple $(s_1, \dots, s_n)$ and $(s_1, \dots, s_n, t)$ there is a set of relation symbols and function symbols respectively. An S-sorted structure is a pair $\mathfrak{A} = (A, (Z^{\mathfrak{A}})_{Z \in L})$, where

A	if a family $(A_s)_{s \in S}$ of non-empty sets
$Z^{\mathfrak{A}} \in A_s$	if Z is a constant of sort $s \in S$
$Z^{\mathfrak{A}} : A_{s_1} \times \dots \times A_{s_n} \to A_t$	if Z is a function symbol of type $(s_1, \dots, s_n, t)$
$Z^{\mathfrak{A}} \subseteq A_{s_1} \times \dots \times A_{s_n}$	if Z is a relation symbol of type $(s_1, \dots, s_n)$

Example 1.1. Consider the two-sorted language L_{Perm} for permutation groups with a sort x for the set and a sort g for the group. The constants and function symbols for L_{Perm} are those of L_{Group} restricted to the sort g and an additional function symbol φ of type (x, g, x). Thus an L_{Perm}-structure (X, G) is given by a set X and an L_{Group}-structure G together with a function $X \times G \to X$

1.2 Language

Lemma 1.5. *Suppose $\overrightarrow{b}$ and $\overrightarrow{c}$ agree on all variables which are free in φ. Then*

$$\mathfrak{A} \models \varphi[\overrightarrow{b}] \Leftrightarrow \mathfrak{A} \models \varphi[\overrightarrow{c}]$$

We define

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n]$$

by $\mathfrak{A} \models \varphi[\overrightarrow{b}]$, where $\overrightarrow{b}$ is an assignment satisfying $\overrightarrow{b}(x_i) = a_i$. Because of Lemma 1.5 this is well defined.

Thus $\varphi(x_1, \dots, x_n)$ defines an n-ary relation

$$\varphi(\mathfrak{A}) = \{\bar{a} \mid \mathfrak{A} \models \varphi[\bar{a}]\}$$

on A, the **realisation set** of φ. Such realisation sets are called **0-definable subsets** of A^n, or 0-definable relations

Let B be a subset of A. A B**-definable** subset of $\mathfrak{A}$ is a set of the form $\varphi(\mathfrak{A})$ for an $L(B)$-formula $\varphi(x)$. We also say that φ are defined **over** B and that the set $\varphi(\mathfrak{A})$ is defined by φ. We call two formulas **equivalent** if in every structure they define the same set.

Atomic formulas and their negations are called **basic**. Formulas without quantifiers are Boolean combinations of basic formulas. It is convenient to allow the empty conjunction and the empty disjunction. For that we introduce two new formulas: the formula $\top$, which is always true, and the formula $\bot$, which is always false. We define

$$\bigwedge_{i<0} \pi_i = \top$$
$$\bigvee_{i<0} \pi_i = \bot$$

A formula is in **negation normal form** if it is built from basic formulas using $\wedge, \vee, exists, \forall$

Definition 1.6. A formula in negation normal form which does not contain any existential quantifier is called **universal**. Formulas in negation normal form without universal quantifiers are called **existential**

Let $\mathfrak{A}$ be an L-structure. The **atomic diagram** of $\mathfrak{A}$ is

$$\text{Diag}(\mathfrak{A}) = \{\varphi \text{ basic } L(A)\text{-sentence} \mid \mathfrak{A}_A \models \varphi\}$$

Lemma 1.7. *The models of* $\text{Diag}(\mathfrak{a})$ *are precisely those structures* $(\mathfrak{B}, h(a))_{a\in A}$ *for embeddings* $h : \mathfrak{A} \to \mathfrak{B}$

1.3 Theories

Definition 1.8. An L**-theory** T is a set of L-sentences

A theory which has a model is a **consistent** theory. We call a set Σ of L-formulas **consistent** if there is an L-structure and an assignment $\vec{b}$ s.t. $\mathfrak{A} \models [\vec{b}]$ for all $\varphi \in \Sigma$

Lemma 1.9. *1. If $T \models \varphi$ and $T \models (\varphi \to \psi)$, then $T \models \psi$*

2. If $T \models \varphi(c_1, \dots, c_n)$ and the constants $c_1, \dots, c_n$ occur neither in T nor in $\varphi(x_1, \dots, x_n)$, then $T \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$

Proof. 2. Let $L' = L \setminus \{c_1, \dots, c_n\}$. If the L'-structure is a model of T and $a_1, \dots, a_n$ are arbitrary elements, then $(\mathfrak{A}, a_1, \dots, a_n) \models \varphi(c_1, \dots, c_n)$. This means $\mathfrak{A} \models \forall x_1 \dots x_n \varphi(x_1, \dots, x_n)$.

$\square$

2 Elementary Extensions and Compactness

2.1 Elementary substructures

Let $\mathfrak{A}, \mathfrak{B}$ be two L-structures. A map $h : A \to B$ is called **elementary** if for all $a_1, \dots, a_n \in A$ we have

$$\mathfrak{A} \models \varphi[a_1, \dots, a_n] \Leftrightarrow \mathfrak{B} \models \varphi[h(a_1), \dots, h(a_n)]$$

We write

$$h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$$

Lemma 2.1. *The models of* $\mathrm{Th}(\mathfrak{A}_A)$ *are exactly the structures of the form* $(\mathfrak{B}, h(a))_{a \in A}$ *for elementary embeddings* $h : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$

We call $\mathrm{Th}(\mathfrak{A}_A)$ the **elemantary diagram** of $\mathfrak{A}$

A substructure $\mathfrak{A}$ of $\mathfrak{B}$ is called **elementary** if the inclusion map is elementary. In this case we write

$$\mathfrak{A} \prec \mathfrak{B}$$

Theorem 2.2 (Tarski's Test)**.** *Let $\mathfrak{B}$ be an L-structure and A a subset of B. Then A is the universe of an elementary substructure iff every $L(A)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ can be satisfied by an element of A*

We use Tarski's Test to construct small elementary substructures

Corollary 2.3. *Suppose S is a subset of the L-structure $\mathfrak{B}$. Then $\mathfrak{B}$ has a elementary substructure $\mathfrak{A}$ containing S and of cardinality at most*

$$\max(|S|, |L|, \aleph_0)$$

Proof. We construct A as the union of an ascending sequence $S_0 \subseteq S_1 \subseteq \dots$ of subsets of B. We start with $S_0 = S$. If S_i is already defined, we choose an element $a_\varphi \in B$ for every $L(S_i)$-formula $\varphi(x)$ which is satisfiable in $\mathfrak{B}$ and define S_{i+1} to be S_i together with these a_φ.

An L-formula is a finite sequence of symbols from L, auxiliary symbols and logical symbols. These are $|L| + \aleph_0 = \max(|L|, \aleph_0)$ many symbols and there are exactlymax$(|L|, \aleph_0)$ many L-formulas

Let $\kappa = \max(|S|, |L|, \aleph_0)$. There are κ many $L(S)$-formulas: therefore $|S_1| \leq \kappa$. Inductively it follows for every i that $|S_i| \leq \kappa$. Finally we have $|A| \leq \kappa \cdot \aleph_0 = \kappa$ □

A directed family $(\mathfrak{A}_i)_{i\in I}$ of structures is **elementary** if $\mathfrak{A}_i \prec \mathfrak{A}_j$ for all $i \leq j$

Theorem 2.4 (Tarski's Chain Lemma)**.** *The union of an elementary directed family is an elementary extension of all its members*

Proof. Let $\mathfrak{A} = \bigcup_{i\in I}(\mathfrak{A}_i)_{i\in I}$. We prove by induction on $\varphi(\bar{x})$ that for all i and $\bar{a} \in \mathfrak{A}_i$

$$\mathfrak{A}_i \vDash \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \vDash \varphi(\bar{a})$$

□

2.2 The Compactness Theorem

Theorem 2.5 (Compactness Theorem)**.** *Finitely satisfiable theories are consistent*

Let L be a language and C a set of new constants. An $L(C)$-theory T' is called a **Henkin theory** if for every $L(C)$-formula $\varphi(x)$ there is a constant $c \in C$ s.t.

$$\exists x\varphi(x) \to \varphi(c) \in T'$$

Lemma 2.6. *Every finitely satisfiable L-theory T can be extended to a finitely complete Henkin theory T^**

Lemma 2.7. *Every finitely complete Henkin theory T^* has a model $\mathfrak{A}$ (unique up to isomorphism) consisting of constants; i.e.,*

$$(\mathfrak{A}, a_c)_{c\in C} \vDash T^*$$

with $A = \{a_c \mid c \in C\}$

Corollary 2.8. *A set of formulas $\Sigma(x_1, \dots, x_n)$ is consistent with T if and only if every finite subset of Σ is consistent with T*

Proof. Introduce new constants $c_1, \dots, c_n$. Then Σ is consistent with T is and only if $T \cup \Sigma(c_1, \dots, c_n)$ is consistent. Now apply the Compactness Theorem □

Definition 2.9. Let $\mathfrak{A}$ be an L-structure and $B \subseteq A$. Then $a \in A$ **realises** a set of $L(B)$-formulas $\Sigma(x)$ if a satisfied all formulas from Σ. We write

$$\mathfrak{A} \vDash \Sigma(a)$$

We call $\Sigma(x)$ **finitely satisfiable** in $\mathfrak{A}$ if every finite subset of Σ is realised in $\mathfrak{A}$

Lemma 2.10. *The set $\Sigma(x)$ is finitely satisfiable in $\mathfrak{A}$ iff there is an elementary extension of $\mathfrak{A}$ in which $\Sigma(x)$ is realised*

Proof. By Lemma 2.1 Σ is realised in an elementary extension of $\mathfrak{A}$ iff Σ is consistent with Th($\mathfrak{A}_A$). So the lemma follows from the observation that a finite set of $L(A)$-formulas is consistent with Th($\mathfrak{A}_A$) iff it is realised in $\mathfrak{A}$ $\square$

Definition 2.11. Let $\mathfrak{A}$ be an L-structure and B a subset of A. A set $p(x)$ of $L(B)$-formulas is a **type** over B if $p(x)$ is maximal finitely satisfiable in $\mathfrak{A}$. We call B the **domain** of p. Let

$$S(B) = S^{\mathfrak{A}}(B)$$

denote the set of types over B.

Every element a of $\mathfrak{A}$ determines a type

$$\text{tp}(a/B) = tp^{\mathfrak{A}}(a/B) = \{\varphi(x) \mid \mathfrak{A} \models \varphi(a), \varphi \text{ an } L(B)\text{-formula}\}$$

So an element a realises the type $p \in S(B)$ exactly if $p = \text{tp}(a/B)$. If $\mathfrak{A}'$ is an elementary extension of $\mathfrak{A}$, then

$$S^{\mathfrak{A}}(B) = S^{\mathfrak{A}'}(B) \quad \text{and} \quad \text{tp}^{\mathfrak{A}'}(a/B) = \text{tp}^{\mathfrak{A}}(a/B)$$

If $\mathfrak{A}' \models p(x)$ then $\mathfrak{A}' \models \exists x p(x)$, so $\mathfrak{A} \models \exists x p(x)$.

We use the notation tp(a) for tp($a/\emptyset$)

Maximal finitely satisfiable sets of formulas in $x_1, \dots, x_n$ are called n-**types** and

$$S_n(B) = S_N^{\mathfrak{A}}(B)$$

denotes the set of n-types over B.

$$\text{tp}(C/B) = \{\varphi(x_{c_1}, \dots, x_{c_n}) \mid \mathfrak{A} \models \varphi(c_1, \dots, c_n), \varphi \text{ an } L(B)\text{-formula}\}$$

Corollary 2.12. *Every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ in which all types over A are realised*

Proof. We choose for every $p \in S(A)$ a new constant c_p. We have to find a model of

$$\text{Th}(\mathfrak{A}_A) \cup \bigcup_{p \in S(A)} p(c_p)$$

This theory is finitely satisfiable since every p is finitely satisfiable in $\mathfrak{A}$.

Or use Lemma 2.10. Let $(p_\alpha)_{\alpha<\lambda}$ be an enumeration of $S(A)$. Construct an elementary chain

$$\mathfrak{A} = \mathfrak{A}_0 \prec \mathfrak{A}_1 \prec \cdots \prec \mathfrak{A}_\beta \prec \ldots (\beta \leq \lambda)$$

s.t. each p_α is realised in $\mathfrak{A}_{\alpha+1}$ (by recursion theorem on ordinal numbers)

Suppose that the elementary chain $(\mathfrak{A}_{\alpha'})_{\alpha'<\beta}$ is already constructed. If β is a limit ordinal, we let $\mathfrak{A}_\beta = \bigcup_{\alpha<\beta} \mathfrak{A}_\alpha$, which is elementary by Lemma 2.4. If $\beta = \alpha + 1$ we first note that p_α is also finitely satisfiable in $\mathfrak{A}_\alpha$, therefore we can realise p_α in a suitable elementary extension $\mathfrak{A}_\beta \succ \mathfrak{A}_\alpha$ by Lemma 2.10. Then $\mathfrak{B} = \mathfrak{A}_\lambda$ is the model we were looking for □

3 Quantifier Elimination

3.1 Preservation theorems

Lemma 3.1 (Separation Lemma). *Let T_1, T_2 be two theories. Assume $\mathcal{H}$ is a set of sentences which is closed under $\wedge, \vee$ and contains $\perp$ and $\top$. Then the following are equivalent*

1. *There is a sentence $\varphi \in \mathcal{H}$ which separates T_1 from T_2. This means*

$$T_1 \models \varphi \quad \textit{and} \quad T_2 \models \neg\varphi$$

2. *All models $\mathfrak{A}_1$ of T_1 can be separated from all models $\mathfrak{A}_2$ of T_2 by a sentence $\varphi \in \mathcal{H}$. This means*

$$\mathfrak{A}_1 \models \varphi \quad \textit{and} \quad \mathfrak{A}_2 \models \neg\varphi$$

Proof. $2 \rightarrow 1$. For any model $\mathfrak{A}_1$ of T_1 let $\mathcal{H}_{\mathfrak{A}_1}$ be the set of all sentences from $\mathcal{H}$ which are true in $\mathfrak{A}_1$. (2) implies that $\mathcal{H}_{\mathfrak{A}_1}$ and T_2 cannot have a common model. By the Compactness Theorem there is a finite conjunction $\varphi_{\mathfrak{A}_1}$ of sentences from $\mathcal{H}_{\mathfrak{A}_1}$ inconsistent with T_2. Clearly

$$T_1 \cup \{\neg\varphi_{\mathfrak{A}_1} \mid \mathfrak{A}_1 \models T_1\}$$

is inconsistent. Again by compactness T_1 implies a disjunction φ of finitely many of the $\varphi_{\mathfrak{A}_1}$ □

For structures $\mathfrak{A}, \mathfrak{B}$ and a map $f : A \rightarrow B$ preserving all formulas from a set of formulas Δ, we use the notation

$$f : \mathfrak{A} \rightarrow_\Delta \mathfrak{B}$$

We also write

$$\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$$

to express that all sentences from Δ true in $\mathfrak{A}$ are also true in $\mathfrak{B}$

Lemma 3.2. *Let T be a theory, $\mathfrak{A}$ a structure and Δ a set of formulas, closed under existential quantification, conjunction and substitution of variables. Then the following are equivalent*

1. *All sentences $\varphi \in \Delta$ which are true in $\mathfrak{A}$ are consistent with T (There is a model $\mathfrak{B} \models \Delta \cup T$ and $\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$)*

2. *There is a model $\mathfrak{B} \models T$ and a map $f : \mathfrak{A} \to_{\Delta} \mathfrak{B}$*

Proof. $1 \to 2$. Consider $\mathrm{Th}_{\Delta}(\mathfrak{A}_A)$, the set of all sentences $\delta(\bar{a})$ ($\delta(\bar{x}) \in \Delta$), which are true in $\mathfrak{A}_A$. The models $(\mathfrak{B}, f(a)_{a \in A})$ of this theory correspond to maps $f : \mathfrak{A} \to_{\Delta} \mathfrak{B}$. **This means that we have to find a model of** $T \cup \mathrm{Th}_{\Delta}(\mathfrak{A}_A)$. To show finite satisfiability it is enough to show that $T \cup D$ is consistent for every finite subset D of $\mathrm{Th}_{\Delta}(\mathfrak{A}_A)$. Let $\delta(\bar{a})$ be the conjunction of the elements of D. Then $\mathfrak{A}$ is a model of $\varphi = \exists \bar{x} \delta(\bar{x})$ $\square$

Lemma 3.2 applied to $T = \mathrm{Th}(\mathfrak{B})$ shows that $\mathfrak{A} \Rightarrow_{\Delta} \mathfrak{B}$ iff there exists a map f and a structure $\mathfrak{B}' \equiv \mathfrak{B}$ s.t. $f : \mathfrak{A} \to_{\Delta} \mathfrak{B}'$

Theorem 3.3. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *There is a universal sentence which separates T_1 from T_2*

2. *No model of T_2 is a substructure of a model of T_1*

Proof. $2 \to 1$. If T_1 and T_2 cannot be separated by a universal sentence, then they have models $\mathfrak{A}_1$ and $\mathfrak{A}_2$ which cannot be separated by a universal sentence. This can be denoted by

$$\mathfrak{A}_2 \Rightarrow_{\exists} \mathfrak{A}_1$$

Now Lemma 3.2 implies that $\mathfrak{A}_2$ there is a map $\mathfrak{A}_2 \to_{\exists} \mathfrak{A}_1'$ where $\mathfrak{A}_1' \models T_1$. Hence $\mathfrak{A}_2$ has an extension $\mathfrak{A}_2'$ s.t. $\mathfrak{A}_2' \equiv \mathfrak{A}_1'$. Then $\mathfrak{A}'$ is gain a model of T_1 contradicting (2) $\square$

Definition 3.4. For any L-theory T, the formulas $\varphi(\bar{x}), \psi(\bar{x})$ are said to be **equivalent** modulo T (or relative to T) if $T \models \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$

Corollary 3.5. *Let T be a theory*

1. *Consider a formula $\varphi(x_1, \dots, x_n)$. The following are equivalent*

   (a) *$\varphi(x_1, \dots, x_n)$ is, modulo T, equivalent to a universal formula*

   (b) *If $\mathfrak{A} \subseteq \mathfrak{B}$ are models of T and $a_1, \dots, a_n \in A$, then $\mathfrak{B} \vDash \varphi(a_1, \dots, a_n)$ implies $\mathfrak{A} \vDash \varphi(a_1, \dots, a_n)$*

2. *We say that a theory which consists of universal sentences is universal. Then T is equivalent to a universal theory iff all substructures of models of T are again models of T*

Proof. 1. Assume (2). We extend L by an n-tuple $\bar{c}$ of new constants $c_1, \dots, c_n$ and consider theory

$$T_1 = T \cup \{\varphi(\bar{c})\} \quad \text{and} \quad T_2 = T \cup \{\neg\varphi(\bar{c})\}$$

Then (2) says the substructures of models of T_1 cannot be models of T_2. By Theorem 3.3 T_1 and T_2 can be separated by a universal $L(\bar{c})$-sentence $\psi(\bar{c})$. By Lemma 1.9, $T_1 \vDash \psi(\bar{c})$ implies

$$T \vDash \forall\bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$$

and from $T_2 \vDash \neg\psi(\bar{c})$ we see

$$T \vDash \forall\bar{x}(\neg\varphi(\bar{x}) \to \neg\psi(\bar{x}))$$

2. Suppose a theory T has this property. Let φ be an axiom of T. If $\mathfrak{A}$ is a substructure of $\mathfrak{B}$, it is not possible for $\mathfrak{B}$ to be a model of T and for $\mathfrak{A}$ to be a model of $\neg\psi$ at the same time. By Theorem 3.3 there is a universal sentence ψ with $T \vDash \psi$ and $\neg\varphi \vDash \neg\psi$. Hence all axioms of T follow from

$$T_\forall = \{\psi \mid T \vDash \psi, \psi \text{ universal}\}$$

□

An $\forall\exists$-formula is of the form

$$\forall x_1 \dots x_n \psi$$

where ψ is existential

Lemma 3.6. *Suppose φ is an $\forall\exists$-sentence, $(\mathfrak{A}_i)_{i\in I}$ is a directed family of models of φ and $\mathfrak{B}$ the union of the $\mathfrak{A}_i$. Then $\mathfrak{B}$ is also a model of φ.*

Proof. Write

$$\varphi = \forall \bar{x} \psi(\bar{x})$$

where ψ is existential. For any $\bar{a} \in B$ there is an A_i containing $\bar{a}$, clearly $\psi(\bar{a})$ holds in $\mathfrak{A}_i$. As $\psi(\bar{a})$ is existential it must also hold in $\mathfrak{B}$ $\square$

Definition 3.7. We call a theory T **inductive** if the union of any directed family of models of T is again a model

Theorem 3.8. *Let T_1 and T_2 be two theories. Then the following are equivalent*

1. *there is an $\forall\exists$-sentence which separates T_1 and T_2*
2. *No model of T_2 is the union of a chain (or of a directed family) of models of T_1*

Proof. $2 \to 1$. If (1) is not true, T_1, T_2 have models which cannot be separated by an $\forall\exists$-sentence. Since $\exists\forall$-formulas are equivalent to negated $\forall\exists$-formulas, we have $\square$