Modal Logic

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November 2, 2020

Contents

1 Basic Concepts

1.1 Modal Languages

Definition 1.1. The **basic modal language** is defined using a set of **proposition letters** Φ whose elements are usually denoted p, q, r and so on, and a unary modal operator $\Diamond$. The well-formed **formulas** ϕ of the basic modal language are given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \psi \vee \phi \mid \Diamond\phi$$

$\mathfrak{M}, w \Vdash \phi$

Definition 1.2. A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called **modal operators**; we use $\triangle$, $\triangle_0, \triangle_1, \ldots$ to denote elements of O. The function ρ assigns to each operator $\delta \in O$ a finite **arity**

Definition 1.3. A **modal language** $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of proposition letters Φ. The set $Form(\tau, \Phi)$ of **modal formulas** over τ and Φ is given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \triangle(\phi_1, \ldots, \phi_{\rho(\triangle)})$$

where p ranges over elements of Φ

Definition 1.4. For each $\triangle \in O$ the **dual** $\triangledown$ of $\triangle$ is defined as $\triangledown(\phi_1, \ldots, \phi_n) := \neg\triangle(\neg\phi_1, \ldots, \neg\phi_n)$

Example 1.1 (The Basic Temporal Language)**.** The basic temporal language is built using a set of unary operators $O = \{\langle F\rangle, \langle P\rangle\}$. The intended interpretation of a formula $\langle F\rangle\phi$ is ' ϕ will be true at some Future time' and the intended interpretation of $\langle P\rangle\phi$ is ' ϕ was true at some Past time.' This language is called the **basic temporal language**. Their duals are written as G and H ('it is Going to be the case' and 'it always Has been the case')

Let's denote the converse of a relation R by $R^{\smile}$. We will call a frame of the form $(T, R, R^{\smile})$ a **bidirectional frame**, and a model built over such a frame a **bidirectional model**. From now on, we will only interpret basic temporal language in bidirectional models. That is, if $\mathfrak{M} = (T, R, R^{\smile}, V)$ is a bidirectional model then

$$\begin{aligned} \mathfrak{M}, t \Vdash F\phi \quad &\text{iff} \quad \exists s(Rts \wedge \mathfrak{M}, s \Vdash \phi) \\ \mathfrak{M}, t \Vdash P\phi \quad &\text{iff} \quad \exists s(R^{\smile}ts \wedge \mathfrak{M}, s \Vdash \phi) \end{aligned}$$

Example 1.2 (Propositional Dynamic Logic)**.** Each of these diamonds has the form $\langle\pi\rangle$, where π denotes a (non-deterministic) **program**. The intended interpretation of $\langle\pi\rangle$ is 'some terminating execution of π from the present state leas to a state bearing the information ϕ'. The dual assertion $[\pi]\phi$ states that 'every execution of π from the present state leads to a state bearing the information ϕ'

Suppose we have fixed some set of basic programs a, b, c and so on. Then we are allowed to define complex programs π over this base as follows

1. **Choice**. If π_1, π_2 are programs, then so is $\pi_1 \cup \pi_2$
2. **Composition**. so is $\pi_1; \pi_2$
3. **Iteration**. so is π^*

This is called **regular** PDL. There are also

1. **Intersection** if π_1, π_2 are programs, then so is $\pi_1 \cap \pi_2$

   execute both π_1, π_2, in parallel

2. **test** if ϕ is a formula, then so is $\phi?$

   test whether ϕ holds, and if so, continues; if not, it fails

The language of PDL has an infinite collection of diamonds, each indexed by a program π build from basic programs using the constructor $\cup, ;$ and *.

$$
\begin{aligned}
R_{\pi_1 \cup \pi_2} &= R_{\pi_1} \cup R_{\pi_2} \\
R_{\pi_1 ; \pi_2} &= R_{\pi_1} \circ R_{\pi_2} \\
R_{\pi_1^*} &= (R_{\pi_1})^*
\end{aligned}
$$

Suppose we have fixed a set of basic programs. Let Π be the set of programs containing the basic programs and all the programs constructed over them using the regular constructors $\cup, ; ,^*$. Then a **regular frame for** Π is a labeled transitive system $(W, \{R_\pi \mid \pi \in \Pi\})$ s..t R_a is an arbitrary binary relation for each basic program a, and for all complex programs π, R_π is the binary relation inductively constructed in accordance with the previous clauses. A **regular model** for Π is a model built over a regular frame.

Example 1.3 (An Arrow Language)**.** The type $\tau_\rightarrow$ of **arrow logic** is a similarity type with modal operators other than diamonds. The language of arrow logic is designed to talk about the objects in arrow structures. The well-formed formulas ϕ are given by

$$\phi := p \mid \bot \mid \neg\phi \mid \phi \vee \psi \mid \phi \circ \psi \mid \otimes\phi \mid 1'$$

$1'$ ('identity') is a nullary modality, the 'converse' operator $\otimes$ is a diamond, and the 'composition' operator $\circ$ is a dyadic operator. Possible readings of these operators are:

$1'$	identity	'skip'
$\otimes\,\phi$	converse	'ϕ conversely'
$\phi \circ \psi$	composition	'first ϕ, then ψ'

1.2 Models and Frames

Definition 1.5. A **frame** for the basic modal language is a pair $\mathfrak{F} = (W, R)$ s.t.

1. W is a non-empty set
2. R is a binary relation on W

A **model** for the basic modal language is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where $\mathfrak{F}$ is a frame for the basic modal language and V is a function assigning to each proposition letter p in Φ a subset $V(p)$ of W. The function V is called a **valuation**. $\mathfrak{M}$ is **based on** the frame $\mathfrak{F}$

Definition 1.6. Suppose w is a state in a model $\mathfrak{M} = (W, R, V)$. Then ϕ is **satisfied** in $\mathfrak{M}$ at state w if

$$\begin{array}{rll}
\mathfrak{M}, w \Vdash p & \text{iff} & w \in V(p), \text{ where } p \in \Phi \\
\mathfrak{M}, w \Vdash \bot & \text{iff} & \text{never} \\
\mathfrak{M}, w \Vdash \neg\phi & \text{iff} & \text{not } \mathfrak{M}, w \Vdash \phi \\
\mathfrak{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathfrak{M}, w \Vdash \phi \text{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \Diamond\phi & \text{iff} & \text{for some } v \in W \text{ with } Rwv \text{ we have } \mathfrak{M}, v \Vdash \phi
\end{array}$$

It follows that $\mathfrak{M}, w \Vdash \Box\phi$ iff for all $v \in W$ s.t. Rwv, we have $\mathfrak{M}, v \Vdash \phi$

Definition 1.7. Let τ be a modal similarity type. A τ**-frame** is a tuple $\mathfrak{F}$ consisting of the following ingredients

1. a non-empty set W

2. for each $n \geq 0$, and each n-ary modal operator $\triangle$ in the similarity type τ, an $(n+1)$-ary relation $R_\triangle$

ϕ is **satisfied at a state** w in a model $\mathfrak{M} = (W, \{R_\triangle \mid \triangle \in \tau\}, V)$ when $\rho(\triangle) > 0$ if

$$\mathfrak{M}, w \Vdash \triangle(\phi_1, \dots, \phi_n) \quad \text{iff} \quad \begin{array}{l} \text{for some } v_1, \dots, v_n \in W \text{ with } R_\triangle w v_1 \dots v_n \\ \text{we have, for each } i, \mathfrak{M}, v_i \Vdash \phi_i \end{array}$$

When $\rho(\triangle) = 0$ we define

$$\mathfrak{M}, w \Vdash \triangle \quad \text{iff} \quad w \in R_\triangle$$

Definition 1.8. The set of all formulas that are valid in a class of frames $\mathsf{F}$is called the **logic** of F (notation: Λ_F)

1.3 General Frames

Definition 1.9. Given an $(n+1)$-ary relation R on a set W, we define the following n-ary operation m_R on the power set $\mathcal{P}(W)$ of W:

$$m_R(X_1, \dots, X_n) = \{w \in W \mid Rww_1 \dots w_n \text{ for some } w_1 \in X_1, \dots, w_n \in X_n\}$$

2 Models

2.1 Invariance Results

Definition 2.1. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models of the same modal similarity type τ, and let w and w' be states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. The τ**-theory** (or τ**-type**) **of** w is the set of all τ-formulas satisfied at w: that is, $\{\phi \mid \mathfrak{M}, w \Vdash \phi\}$. We say that w and w' are **(modally) equivalent** ($w \leftrightsquigarrow w'$) if they have the same τ-theories

The τ**-theory** of the model $\mathfrak{M}$ is the set of all τ-formulas satisfied by all states in fM; that is, $\{\phi \mid \mathfrak{M} \Vdash \phi\}$ Models $\mathfrak{M}$ and $\mathfrak{M}'$ are called **(modally) equivalent** ($\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$) if their theories are identical

2.1.1 Disjoint Unions

2.1.2 Generated submodels

Definition 2.2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models; we say that $\mathfrak{M}'$ is a **submodel** of $\mathfrak{M}$ if $W' \subseteq W$, R' is the restriction of R

to W', and V' is the restriction of V to $\mathfrak{M}'$. We say that $\mathfrak{M}'$ is a **generated submodel** of $\mathfrak{M}$ ($\mathfrak{M}' \rightarrowtail \mathfrak{M}$) if $\mathfrak{M}'$ is a submodel of $\mathfrak{M}$ and for all points w the following closure condition holds

$$\text{if } w \text{ is in } \mathfrak{M}' \text{ and } Rwv, \text{ then } v \text{ is in } \mathfrak{M}'$$

Let fM be a model, and X a subset of the domain of $\mathfrak{M}$; the **submodel generated by** X is the smallest generated submodel of $\mathfrak{M}$ whose domain contains X. A **rooted** or **point generated** model is a model that is generated by a singleton set, the element of which is called the **root** of the frame

2.1.3 Morphism for modalities

Definition 2.3 (Homomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$, if $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$, if $(w_0, \ldots, w_n) \in R_\triangle$, then $(f(w_0), \ldots, f(w_n)) \in R'_\triangle$ (the **homomorphic condition**)

Definition 2.4 (Strong Homomorphisms, Embeddings and Isomorphisms)**.** Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. By a **strong homomorphism** $f : \mathfrak{M} \to \mathfrak{M}'$, we mean a function $f : W \to W'$ satisfying

1. For each proposition letter p and each element w from $\mathfrak{M}$ iff $w \in V(p)$, then $f(w) \in V'(p)$
2. For each $n \geq 0$ and each n-ary $\triangle \in \tau$ and $(n+1)$-tuple $\overline{w}$ from $\mathfrak{M}$ iff $(w_0, \ldots, w_n) \in R_\triangle$, iff $(f(w_0), \ldots, f(w_n)) \in R'_\triangle$ (the **strong homomorphic condition**)

An **embedding** of $\mathfrak{M}$ into $\mathfrak{M}'$ is a strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ which is injective. An **isomorphism** is a bijective strong homomorphism

Proposition 2.5. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models. Then the following holds*

1. *for all elements w and w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, if there exists a surjective strong homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ with $f(w) = w'$, then w and w are modally equivalent*

2. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightsquigarrow \mathfrak{M}'$*

Definition 2.6 (Bounded Morphisms - the Basic Case)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for the basic modal language. A mapping $f : \mathfrak{M} = (W, R, V) \to \mathfrak{M}' = (W', R', V')$ is a **bounded morphsim** if it satisfies

1. w and $f(w)$ satisfy the same proposition letters
2. f is a homomorphism w.r.t. the relation R (if Rwv then $R'f(w)f(v)$)
3. If $R'f(w)v'$ then there exists v s.t. Rwv and $f(v) = v'$ (the **back condition**)

If there is a **surjective** bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$, then we say that $\mathfrak{M}'$ is a **bounded morphic image** of $\mathfrak{M}$, and write $\mathfrak{M} \twoheadrightarrow \mathfrak{M}'$

Proposition 2.7. *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models s.t. $f : \mathfrak{M} \to \mathfrak{M}'$ is a bounded morphism. Then for each modal formula ϕ, and each element w of $\mathfrak{M}$ we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}', f(w) \Vdash \phi$.*

Let τ be a modal similarity type containing only diamonds (thus if $\mathfrak{M}$ is a τ-model, it has the form $(W, R_1, \dots, V)$ where each R_i is a binary relation on W) . In this context we will call a τ-model $\mathfrak{M}$ **tree-like** if the structure $(W, \bigcup_i R_i, V)$ is a tree

Proposition 2.8. *Assume that τ is a modal similarity type containing only diamonds. Then for any rooted τ-models $\mathfrak{M}$ there exists a tree-like τ-models $\mathfrak{M}'$ s.t. $\mathfrak{M}' \twoheadrightarrow \mathfrak{M}$. Hence any satisfiable τ-formula is satisfiable in a tree-like model*

Proof. Let w be the root of $\mathfrak{M}$. Define the model $\mathfrak{M}'$ as follows. Its domain W' consist of all finite sequences $(w, u_1, \dots, u_n)$ s.t. $n \geq 0$ and for some modal operators $\langle a_1 \rangle, \dots, \langle a_n \rangle \in \tau$ there is a path $wR_{a_1}u_1 \cdots R_{a_n}u_n$ in $\mathfrak{M}$. Define $(w, u_1, \dots, u_n)R'_a(w, v_1, \dots, w_m)$ to hold if $m = n + 1, u_i = v_i$ for $i = 1, \dots, n$ and $R_a u_n v_m$ holds in $\mathfrak{M}$. That is, R'_a relates two sequences iff the second is an extension of the first with a state from $\mathfrak{M}$ that is a sucessor of the last element of the first sequence. Finally, V' is defined by putting $(w, u_1, \dots, u_n) \in V'(p)$ iff $u_n \in V(p)$. The mapping $f : (w, u_1, \dots, u_n) \mapsto u_n$ defines a surjective bounded morphism from $\mathfrak{M}'$ to $\mathfrak{M}$ $\square$

2.2 Bisimulations

Definition 2.9 (Bisimulation - the Basic Case)**.** Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M} = (W', R', V')$ be two models

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation between** $\mathfrak{M}$ and $\mathfrak{M}'$ (notation: $Z : \mathfrak{M} \leftrightarrows \mathfrak{M}'$) if

1. If wZw' then w and w' satisfy the same proposition letters

2. If wZw' and Rwv, then there exists v' (in $\mathfrak{M}'$) s.t. vZv' and $R'w'v'$ (the **forth condition**)

3. The converse of (2): if wZw' and $R'w'v'$, then there exists v (in $\mathfrak{M}$) s.t. vZv' and Rwv (the **back condition**)

When Z is a bisimulation linking two states w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$ we say that w and w' are **bisimilar**, and we write $Z : \mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$. If there is a bisimulation, we sometimes write $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$ or $w \leftrightarroweq w'$

Definition 2.10 (Bisimulation - the General Case)**.** Let τ be a modal similarity type, and let $\mathfrak{M} = (W, R_\triangle, V)_{\triangle \in \tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle \in \tau}$ be τ-models. A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ ($Z : \mathfrak{M} \leftrightarroweq \mathfrak{M}'$) if the above condition 1 is satisfied and

2. If wZw' and $R_\triangle wv_1 \dots v_n$ then there are $v'_1, \dots, v'_n \in W'$ s.t. $R'_\triangle w'v'_1 \dots v'_n$ and for all i ($1 \le i \le n$) $v_iZv'_i$ (the **forth** condition)

3. If wZw' and $R'_\triangle w'v'_1 \dots v'_n$ then there are $v_1, \dots, v_n \in W$ s.t. $R_\triangle wv_1 \dots v_n$ and for all i ($1 \le i \le n$) $v_iZv'_i$ (the **back** condition)

Proposition 2.11. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ and $\mathfrak{M}_i$ ($i \in I$) be τ-models*

1. *If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$*

2. *For every $i \in I$, and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarroweq \biguplus_i \mathfrak{M}_i, w$*

3. *If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarroweq \mathfrak{M}, w$ for all w in $\mathfrak{M}'$*

4. *If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$*

Proof. Suppose $\mathfrak{M} = (W, R_\triangle, V)_{\triangle \in \tau}$ and $\mathfrak{M}' = (W', R'_\triangle, V')_{\triangle \in \tau}$ $\mathfrak{M}_i \subseteq \biguplus_i \mathfrak{M}_i$

1. Suppose $f : \mathfrak{M} \cong \mathfrak{M}'$, then we define wZw' iff $w' = f(w)$ where $w \in W, w' \in W'$. Bisimulation comes from the definition of the isomorphism

2. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}_i\} \subseteq \mathfrak{M}_i \times \biguplus \mathfrak{M}_i$. The first condition comes from the invariance. The forth condition is obvious. For the back condition, if $R'_\triangle w'v'_1 \dots v'_n$ and $w' \in W$, then $v'_1, \dots, v'_n \in W$ since each $R_{\triangle,i}$ is disjoint and we have $R_{\triangle,i} w'v'_1 \dots v'_n$

3. Define the relation $Z = \{(w, w) \mid w \in \mathfrak{M}'\} \subseteq \mathfrak{M}' \times \mathfrak{M}$. The first condition comes from the invariance. Forth condition is obvious. For the back condition, suppose wZw and $R'_{\triangle}wv'_1 \ldots v'_n$, by the definition, $v'_1, \ldots, v'_n \in W$ and $R_{\triangle}wv'_1 \ldots v'_n$

4. Define $Z = \{(w, f(w) \mid w \in W)\}$. The first condition comes from the definition. If wZw' and $R_{\triangle}wv_1 \ldots v_n$, then $R'_{\triangle}f(w)f(v_1) \ldots f(v_n)$. If wZw' and $R'_{\triangle}w'v'_1 \ldots v_n$, then there is $v_1, \ldots, v_n$ s.t. $R_{\triangle}wv_1, \ldots, v_n$ and $f(v_i) = v'_i$ for $1 \leq i \leq n$

$\square$

Theorem 2.12. *Let τ be a modal similarity type, and let $\mathfrak{M}, \mathfrak{M}'$ be τ-models. Then, for every $w \in W$ and $w' \in W'$, $w \Leftrightarrow w'$ implies that $w \leftrightsquigarrow w'$. In other words, modal formulas are invariant under bisimulation*

Proof. Induction on the complexity of ϕ.

Suppose ϕ is $\diamond\psi$, we have $\mathfrak{M}, w \Vdash \diamond\psi$ iff there exists a v in $\mathfrak{M}$ s.t. Rwv and $\mathfrak{M}, v \Vdash \psi$. As $w \Leftrightarrow w'$, there exists a v' in $\mathfrak{M}'$ s.t. $R'w'v'$ and $v \Leftrightarrow v'$. By the I.H., $\mathfrak{M}', v' \Vdash \psi$, hence $\mathfrak{M}', w' \Vdash \diamond\psi$ $\square$

Example 2.1 (Bisimulation and First-Order Logic)**.**

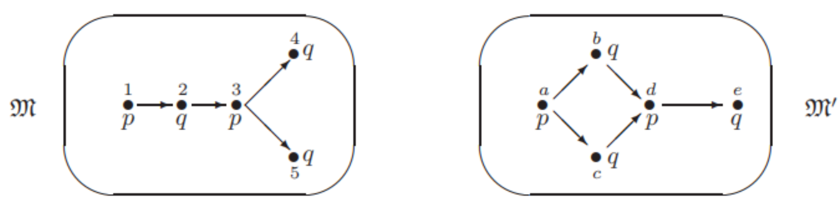


Fig. 2.4. Bisimilar models.

Example 2.2.

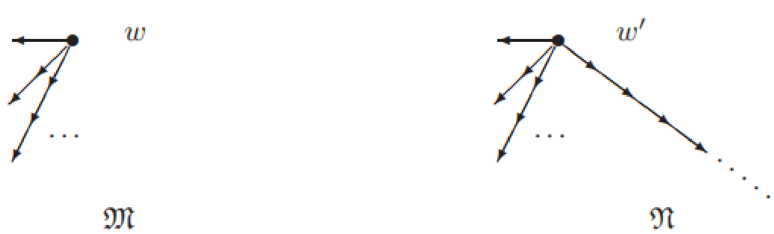


Fig. 2.5. Equivalent but not bisimilar.

$\mathfrak{M}$ is **image-finite** if for each state u in $\mathfrak{M}$ and each relation R in $\mathfrak{M}$, the set $\{(v_1, \dots, v_n) \mid Ruv_1 \dots v_n\}$ is finite

Theorem 2.13 (Hennessy-Milner Theorem). *Let τ be a modal similarity type and let $\mathfrak{M}$ and $\mathfrak{M}'$ be two image-finite τ-models. Then for every $w \in W$ and $w' \in W'$, $w \leftrightarrows w'$ iff $w \leftrightsquigarrow w'$*

Proof. Assume that our similarity type τ only contains a single diamond. The direction from left to right follows from Theorem 2.12

Suppose $w \leftrightsquigarrow w'$. The first condition is immediate. If Rwv, assume there is no v' in $\mathfrak{M}'$ with $R'w'v'$ and $v \leftrightsquigarrow v'$. Let $S' = \{u' \mid R'w'u'\}$. Note that S' must be non-empty, for otherwise $\mathfrak{M}', w' \Vdash \Box\bot$, which would contradict $w \leftrightsquigarrow w'$ since $\mathfrak{M}, w \Vdash \diamond\top$. Furthermore, as $\mathfrak{M}'$ is image-finite, S' must be finite, say $S' = \{w'_1, \dots, w'_n\}$. By assumption, for every $w'_i \in S'$ there exists a formula ψ_i s.t. $\mathfrak{M}, v \Vdash \psi_i$, but $\mathfrak{M}', w'_i \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n) \quad \text{and} \quad \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \cdots \wedge \psi_n)$$

□

Exercise 2.2.1. Suppose that $\{Z_i \mid i \in I\}$ is a non-empty collection of bisimulations between $\mathfrak{M}$ and $\mathfrak{M}'$. Prove that the relation $\bigcup_{i \in I} Z_i$ is also a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. Conclude that if $\mathfrak{M}$ and $\mathfrak{M}'$ are bisimilar, then there is a maximal bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

Proof.

1. If $(w, w') \in \bigcup_{i \in I} Z_i$, then $(w, w') \in Z_j$ for some $j \in I$ and hence they satisfy the same propositional letters
2. If $(w, w') \in \bigcup_{i \in I} Z_i$ and $R_\triangle wv_1 \dots v_n$, since $(w, w') \in Z_j$ for some $j \in I$, we have $R'_\triangle w'v'_1 \dots v'_n$ and $v_i Z_j v'_i$ for all $1 \leq i \leq n$, which means $(v_i, v'_i) \in \bigcup_{i \in I} Z_i$ for all $1 \leq i \leq n$
3. similarly

□

Remark (Bisimulations for the Basic Temporal Language and Arrow Logic). When working with the basic temporal language, we usually work with models (W, R, V) and implicitly take R_p to be $R^\smile$. Thus we need a notion of bisimulation between models (W, R, V) and (W', R', V') to be a relation Z between the states of the two models that satisfies the clauses of Definition 2.9, and in addition the following

4. If wZw' and Rvw, then there exists v' in $\mathfrak{M}'$ s.t. vZv' and $R'v'w'$
5. Converse of 4: if wZw' and $R'v'w'$, then there exists v in $\mathfrak{M}$ s.t. vZv'

2.3 Finite Models

Definition 2.14 (Finite Model Property)**.** Let τ be a modal similarity type, and let M be a class of τ-models. We say that τ has the **finite model property w.r.t.** M if the following holds: if ϕ is a formula of similarity type τ, and ϕ is satisfiable in some model in M, then ϕ is satisfiable in a **finite** model in M

2.3.1 Selecting a finite submodel

Definition 2.15 (Degree)**.** We define the **degree** of modal formulas as follows:

$$
\begin{array}{rcl}
\deg(p) & = & 0 \\
\deg(\bot) & = & 0 \\
\deg(\neg\phi) & = & \deg(\phi) \\
\deg(\phi \vee \psi) & = & \max\{\deg(\phi), \deg(\psi)\} \\
\deg(\triangle(\phi_1, \ldots, \phi_n)) & = & 1 + \max\{\deg(\phi_1), \ldots, \deg(\phi_2)\}
\end{array}
$$

Proposition 2.16. *Let τ be a finite modal similarity type, and assume our collection of proposition letters is finite as well*

1. *for all n, up to logical equivalence there are only finitely many formulas of degree at most n*
2. *for all n, and every τ-model $\mathfrak{M}$ and state w of $\mathfrak{M}$, the set of all τ-formulas of degree at most n that are satisfied by w, is equivalent to a single formula*

Definition 2.17 (n-Bisimulation)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models, and let w and w' be states of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. We say that w and w' are n**-bisimilar** ($w \leftrightarrows_n w'$) if there exists a sequence of binary relations $Z_n \subseteq \cdots \subseteq Z_0$ with the following properties (for $i + 1 \leq n$)

1. wZ_nw'
2. if vZ_0v' then v and v' agree on all proposition letters
3. if $vZ_{i+1}v'$ and Rvu then there exists u' with $R'v'u'$ and uZ_iu'
4. if $vZ_{i+1}v'$ and $R'v'u'$, then there exists u with Rvu and uZ_iu'

Proposition 2.18. *Let τ be a finite modal similarity type, Φ a finite set of proposition letters, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for this language. Then for every w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$, the following are equivalent*

1. $w \leftrightarrows_n w'$

2. *w and w' agree on all modal formulas of degree at most n.*

Proof. $2 \to 1$. if $n = 0$, obvious.

If $n = k$ and the proposition holds. Now suppose $n = k+1$. Now w and w' agree on all modal formulas of degree at most $n+1$. If there is not v, v' s.t. v and v' agree on all modal formulas of degree at most n and Rwv and Rwv'. Let $S' = \{u' \mid R'w'u'\}$ and S' is finite, say $S' == \{w_1', \dots, w_n'\}$. By assumption, for every $w_i' \in S'$ there exists a formula ψ_i of degree at most n s.t. $\mathfrak{M}, v \Vdash \psi_i$ but $\mathfrak{M}', w_i' \nVdash \psi_i$. It follows that

$$\mathfrak{M}, w \Vdash \diamond(\psi_1 \wedge \dots \wedge \psi_n) \text{ and } \mathfrak{M}', w' \nVdash \diamond(\psi_1 \wedge \dots \wedge \psi_n)$$

□

Definition 2.19. Let τ be a modal similarity type containing only diamonds. Let $\mathfrak{M} = (W, R_1, \dots, R_n, \dots, V)$ be a rooted τ-model with root w. The notion of the **height** of states in $\mathfrak{M}$ is defined by induction.

The only element of height 0 is the rot of the model; the states of height $n+1$ are those immediate successors of elements of height n that have not yet assigned a height smaller than $n+1$. The **height of a model** $\mathfrak{M}$ is the maximum n s.t. there is a state of height n in $\mathfrak{M}$, if such a maximum exists; otherwise the height of $\mathfrak{M}$ is infinite

For a natural number k, the **restriction** of $\mathfrak{M}$ to k ($\mathfrak{M}\restriction k$) is defined as the submodel containing only states whose height is at most k. $(\mathfrak{M}\restriction k) = (W_k, R_{1k}, \dots, R_{nk}, \dots, V_k)$, where $W_k = \{v \mid \text{height}(v) \leq k\}$, $R_{nk} = R_n \cap (W_k \times W_k)$, and for each p, $V_k(p) = V(p) \cap W_k$

Lemma 2.20. *Let τ be a modal similarity type that contains only diamonds. Let $\mathfrak{M}$ be a rooted τ-models, and let k be a natural number. Then for every state w of $(\mathfrak{M}\restriction k)$, we have $(\mathfrak{M}\restriction k), w \leftrightarrows_l \mathfrak{M}, w$, where $l = k - \mathit{height}(w)$*

Theorem 2.21 (Finite Model Property - via Selection)**.** *Let τ be a modal similarity type containing only diamonds, and let ϕ be a τ-formula. If ϕ is satisfiable, then it is satisfiable on a finite model*

Proof. Fix a modal formula ϕ with $\deg(\phi) = k$. We restrict our modal similarity type τ and our collection of proposition letters to the modal operators and proposition letters actually occurring in ϕ. Let $\mathfrak{M}_1, w_1$ be s.t. $\mathfrak{M}_1, w_1 \Vdash \phi$. By Proposition 2.8, there exists a tree-like model $\mathfrak{M}_2$ with root w_2 s.t. $\mathfrak{M}_2, w_2 \Vdash \phi$. Let $\mathfrak{M}_3 := (\mathfrak{M}_2\restriction k)$. By Lemma 2.20 we have $\mathfrak{M}_2, w_2 \leftrightarrows_k \mathfrak{M}_3, w_2$ and by Proposition 2.18 it follows that $\mathfrak{M}_3, w_2 \Vdash \phi$

By induction on $n \leq k$ we define finite sets of states $S_0, \dots, S_k$ and a (final) model $\mathfrak{M}_4$ with domain $S_0 \cup \dots \cup S_k$; the points in each S_n will have height n

Define S_0 to be the singleton $\{w_2\}$. Next, assume that $S_0, \dots, S_n$ have already been defined. Fix an element v of S_n. By Proposition 2.16 there are only finitely many non-equivalent modal formulas whose degree is at most $k - n$, say $\psi_1, \dots, \psi_m$. For each formula that is of the form $\langle a \rangle \chi$ and holds in $\mathfrak{M}_3$ at v, select a state u from $\mathfrak{M}_3$ s.t. $R_a vu$ and $\mathfrak{M}_3, u \Vdash \chi$. Add all these us to S_{n+1}, and repeat this selection process for every state in S_n. S_{n+1} is defined as the set of all points that have been selected in this way

Finally, define $\mathfrak{M}_4$ as follows. Its domain is $S_0 \cup \dots \cup S_k$; as each S_i is finite, $\mathfrak{M}_4$ is finite. The relations and valuation are obtained by restricting the relations and valuations of $\mathfrak{M}_3$ to the domain of $\mathfrak{M}_4$ $\square$

2.3.2 Finite models via filtrations

Definition 2.22. A set of formulas Σ is **closed under subformulas** (or **subformula closed**) if for all formulas ϕ, ϕ': if $\phi \vee \phi' \in \Sigma$ then so are ϕ and ϕ'; if $\neg\phi \in \Sigma$ then so is ϕ; and if $\triangle(\phi_1, \dots, \phi_n) \in \Sigma$ then so are $\phi_1, \dots, \phi_n$

Definition 2.23 (Filtrations)**.** We work in the basic modal language. Let $\mathfrak{M} = (W, R, V)$ be a model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma$ be the relation on the states of $\mathfrak{M}$ defined by

$$w \leftrightsquigarrow_\Sigma v \text{ iff for all } \phi \in \Sigma : (\mathfrak{M}, w \Vdash \phi \text{ iff } \mathfrak{M}, v \Vdash \phi)$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation. We denote the equivalence class of a state w of $\mathfrak{M}$ w.r.t. $\leftrightsquigarrow_\Sigma$ by $|w|_\Sigma$, or simply $|w|$. The mapping $w \mapsto |w|$ is called the **natural map**

Let $W_\Sigma = \{|w|_\Sigma \mid w \in W\}$. Suppose $\mathfrak{M}^f_\Sigma$ is any model (W^f, R^f, V^f) s.t.

1. $W^f = W_\Sigma$
2. if Rwv then $R^f|w||v|$
3. if $R^f|w||v|$ then for all $\diamond\phi \in \Sigma$, if $\mathfrak{M}, v \Vdash \phi$ then $\mathfrak{M}, w \Vdash \diamond\phi$
4. $V^f(p) = \{|w| \mid \mathfrak{M}, w \Vdash p\}$, for all proposition letters p in Σ

$\mathfrak{M}^f_\Sigma$ is called a **filtration of** fM **through** Σ; we will often suppress subscripts and write $\mathfrak{M}^f$ instead of $\mathfrak{M}^f_\Sigma$

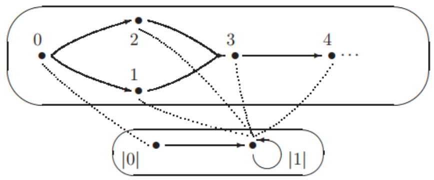


Fig. 2.6. A model and its filtration.

Let $\mathfrak{M} = (\mathbb{N}, R, V)$, where $R = \{(0,1), (0,2), (1,3)\} \cup \{(n, n+1) \mid n \geq 2\}$, and V has $V(p) = \mathbb{N} \setminus \{0\}$ and $V(q) = \{2\}$

Further assume $\Sigma = \{\diamond p, p\}$. Σ is subformula closed. Then, the model $\mathfrak{N} = (\{|0|, |1|\}, \{(|0|, |1|), (|1|, |1|)\}, V')$, where $V'(p) = \{|1|\}$ is a filtration of $\mathfrak{M}$ through Σ. $\mathfrak{N}$ is not a bounded morphic image of $\mathfrak{M}$: any bounded morphism would have to preserve the formula q

Proposition 2.24. *Let Σ be a finite subformula closed set of basic modal formulas. For any model $\mathfrak{M}$, if $\mathfrak{M}^f$ is a filtration of $\mathfrak{M}$ through a subformula closed set Σ, then $\mathfrak{M}^f$ contains at most 2^n nodes (where n denotes the size of Σ)*

Proof. The states of $\mathfrak{M}^f$ are the equivalence classes in W_Σ. Let g be the function with domain W_Σ and range $\mathcal{P}(\Sigma)$ defined by $g(|w|) = \{\phi \in \Sigma \mid \mathfrak{M}, w \Vdash \phi\}$. It follows from the definition of $\leftrightsquigarrow_\Sigma$ that g is well defined and injective. Thus $|W_\Sigma| \leq 2^n, n = |\Sigma|$ $\square$

Theorem 2.25 (Filtration Theorem)**.** *Consider the basic modal language. Let $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$ be a filtration of $\mathfrak{M}$ through a subformula closed set Σ. Then for all formulas $\phi \in \Sigma$, and all nodes w in $\mathfrak{M}$, we have $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M}^f, |w| \Vdash \phi$*

Proof. Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}, w \Vdash \diamond\phi$. Then there is a v s.t. Rwv and $\mathfrak{M}, v \Vdash \phi$. As $\mathfrak{M}^f$ is a filtration, $R^f|w||v|$. As Σ is a subformula closed, $\phi \in \Sigma$, thus by the inductive hypothesis $\mathfrak{M}^f, |v| \Vdash \phi$. Hence $\mathfrak{M}^f, |\Vdash| \diamond \phi$

Suppose $\diamond\phi \in \Sigma$ and $\mathfrak{M}^f, |w| \Vdash \diamond\phi$. Thus there is a state $|v|$ in $\mathfrak{M}^f$ s.t. $R^f|w||v|$ and $\mathfrak{M}^f, |v| \Vdash \phi$. As $\phi \in \Sigma$, we have $\mathfrak{M}, v \Vdash \phi$. By the definition, we have $\mathfrak{M}, w \Vdash \diamond\phi$ $\square$

Note that clauses 2 and 3 of Definition 2.3.2 are designed to make the modal case of the inductive step go through.

Define

1. $R^s|w||v|$ iff $\exists w' \in |w| \exists v' \in |v| Rw'v'$
2. $R^l|w||v|$ iff for all formulas $\diamond\phi \in \Sigma$: $\mathfrak{M}, v \Vdash \phi$ implies $\mathfrak{M}, w \Vdash \diamond\phi$

These relations give rise to the **smallest** and **largest** filtrations respectively

Lemma 2.26. *Consider the basic modal language. Let $\mathfrak{M}$ be any model, Σ any subformula closed set of formulas, W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$, and V^f the standard valuation on W_Σ. Then both (W_Σ, R^s, V^f) and (W_Σ, R^l, V^f) are filtrations of $\mathfrak{M}$ through Σ. Furthermore, if (W_Σ, R^f, V^f) is any filtration of $\mathfrak{M}$ through Σ, then $R^s \subseteq R^f \subseteq R^l$*

Proof. If Rwv, if $\mathfrak{M}, v \Vdash \phi$, then $\mathfrak{M}, w \Vdash \diamond\phi$, hence $R^l|w||v|$

For any (W_Σ, R^f, V^f). $R^s \subseteq R^f$ by clause 2. $R^f \subseteq R^l$ by clause 2 □

Theorem 2.27 (Finite Model Property - via Filtrations)**.** *Let ϕ be a basic modal formula. if ϕ is satisfiable, then it is satisfiable on a finite model. Indeed, it is satisfiable on a finite model containing at most 2^m nodes, where m is the number of subformulas of ϕ*

Proof. Assume that ϕ is satisfiable on a model $\mathfrak{M}$; take any filtration of $\mathfrak{M}$ through the set of subformulas . □

Lemma 2.28. *Let $\mathfrak{M}$ be a model, Σ a subformula closed set of formulas, and W_Σ the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Sigma$. Let R^t be the binary relation on W_Σ defined by*

$$R^t|w||v| \text{ iff for all } \phi,\ \text{if } \diamond\phi \in \Sigma \text{ and } \mathfrak{M}, v \Vdash \phi \vee \diamond\phi \text{ then } \mathfrak{M}, w \Vdash \diamond\phi$$

If R is transitive then (W_Σ, R^t, V^f) is a filtration and R^t is transitive

Definition 2.29. Let (W, R, V) be a transitive frame. A **cluster** on (W, R, V) is a maximal, nonempty equivalence class under R. That is, $C \subseteq W$ is a cluster if the restriction of R to C is an equivalence relation

A cluster is **simple** if it consists of a single reflexive point, and **proper** if it consists more than one point

2.4 The Standard Translation

Definition 2.30. For τ a modal similarity type and Φ a collection of proposition letters, let $\mathcal{L}^1_\tau(\Phi)$ be the first-order language (with equality) which has unary predicates $P_0, P_1, \ldots$ corresponding to the proposition letters $p_0, p_1, \ldots$ in Φ, and an $(n+1)$-ary relation symbol $R_\triangle$ for each (n-ary) modal operator $\triangle$ in our similarity type. We write $\alpha(x)$ to denote a first-order formula α with one free variable, x

Definition 2.31 (Standard Translation)**.** Let x be a first-order variable. The **standard translation** ST_x taking modal formulas to first-order formulas in $\mathcal{L}^1_\tau(\Phi)$ is defined as

$$\begin{array}{rcl} ST_x(p) & = & Px \\ ST_x(\bot) & = & x \neq x \\ ST_x(\neg\phi) & = & \neg ST_x(\phi) \\ ST_x(\phi \vee \psi) & = & ST_x(\phi) \vee ST_x(\psi) \\ ST_x(\triangle(\phi_1, \dots, \phi_n)) & = & \exists y_1 \dots \exists y_n (R_\triangle x y_1 \dots y_n \wedge \\ & & \quad ST_{y_1}(\phi_1) \wedge \dots \wedge ST_{y_n}(\phi_n)) \end{array}$$

where $y_1, \dots, y_n$ are fresh variables.

$$ST_x(\diamond\phi) = \exists y(Rxy \wedge ST_y(\phi))$$
$$ST_x(\Box\phi) = \forall y(Rxy \to ST_y(\phi))$$

Proposition 2.32 (Local and Global Correspondence on Models)**.** *Fix a modal similarity type τ, and let ϕ be a τ-formula. Then*

1. *For all $\mathfrak{M}$ and all states w of $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{M} \models ST_x(\phi)[w]$*
2. *For all $\mathfrak{M}$: $\mathfrak{M} \Vdash \phi$ iff $\mathfrak{M} \models \forall x ST_x(\phi)$*

Proposition 2.33. 1. *Let τ be a modal similarity type that only contains diamonds. Then, every τ-formula ϕ is equivalent to a first-order formula containing at most two variables*

2. *If τ does not contain modal operators $\triangle$ whose arity exceeds n, all τ-formulas are equivalent to first-order formulas containing at most $(n+1)$ vairables*

Proof. Assume τ contains only diamonds $\langle a \rangle, \langle b \rangle$. Fix two distinct variables x and y. Define two variants ST_x and ST_y of the standard translation as follows

$$\begin{array}{ll} ST_x(p) = Px & ST_y(p) = Py \\ ST_x(\bot) = x \neq x & ST_y(\bot) = y \neq y \\ ST_x(\neg\phi) = \neg ST_x(\phi) & ST_y(\neg\phi) = \neg ST_y(\phi) \\ ST_x(\phi \vee \psi) = ST_x(\phi) \vee ST_x(\psi) & ST_y(\phi \vee \psi) = ST_y(\phi) \vee ST_y(\psi) \\ ST_x(\langle a \rangle\phi) = \exists y(R_a xy \wedge ST_y(\phi)) & ST_y(\langle a \rangle\phi) = \exists x(R_a yx \wedge ST_x(\phi)) \end{array}$$

Then for any τ-formula ϕ, its ST_x-translation contains at most the two variables x and y, and $ST_x(\phi)$ is equivalent to the original standard translation of ϕ $\square$

Example 2.3.

$$\begin{aligned} ST_x(\diamond(\Box p \to q)) &= \exists y(Rxy \wedge ST_y(\Box p \to q)) \\ &= \exists y(Rxy \wedge (\forall x(Ryx \to ST_x(p)) \to Qy)) \\ &= \exists y(Rxy \wedge (\forall x(Ryx \to Px) \to Qy)) \end{aligned}$$

Rxx is not equivalent to any modal formula. Suppose ϕ is a modal formula s.t. $ST_x(\phi)$ is equivalent to Rxx. Let $\mathfrak{M}$ be a singleton reflexive model and let w be the unique state in $\mathfrak{M}$; obviously $\mathfrak{M} \models Rxx[w]$. Let $\mathfrak{N}$ be a model based on the strict ordering of the integers; for every integer v, $\mathfrak{N} \models \neg Rxx[v]$. Let Z be the relation which links every integer with the unique state in fM, and assume that the valuations in $\mathfrak{N}$ and $\mathfrak{M}$ are s.t. Z is a bisimulation.

$$\mathfrak{M} \models Rxx[w] \Rightarrow \mathfrak{M}, w \Vdash \phi \Rightarrow \mathfrak{N}, v \Vdash \phi \Rightarrow \mathfrak{N} \models Rxx[v]$$

Definition 2.34. Let τ be a modal similarity type, C a class of τ-models, and Γ a set of formulas over τ. We say that Γ **defines** of **characterizes** a class K of models **within** C if for all models $\mathfrak{M}$ in C we have that $\mathfrak{M}$ is in K iff $\mathfrak{M} \Vdash \Gamma$. If C is the class of all τ-models, we simply say that Γ defines or characterizes K; we omit brackets whenever Γ is a singleton. We say that a formula ϕ defines a **property** whenever ϕ defines the class of models satisfying the property

2.5 Modal Saturation via Ultrafilter Extensions

2.5.1 M-saturation

Definition 2.35 (Hennessy-Milner Classes)**.** Let τ be a modal similarity type, and K a class of τ-models. K is a **Hennessy-Milner** class, or **has the Hennessy-Milner property**, if for every two models $\mathfrak{M}$ and $\mathfrak{M}'$ in K and any two states w, w' of $\mathfrak{M}$ and $\mathfrak{M}'$, respectively, $w \leftrightsquigarrow w'$ implies $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$

For example, by Theorem 2.13 the class of image-finite models has the Hennessy-Milner property.

Suppose we are working in the basic modal language. Let $\mathfrak{M} = (W, R, V)$ be a model, let w be a state in W and let $\Sigma = \{\phi_0, \phi_1, \dots\}$ be an infinite set of formulas. Suppose that w has successors $v_0, v_1, \dots$, where respectively $\phi_0, \phi_0 \wedge \phi_1, \phi_0 \wedge \phi_1 \wedge \phi_2, \dots$ hold. If there is no successor v of w where **all** formulas from Σ hold **at the same time**, then the model is in some sense incomplete. A model is called m-saturated if incompleteness of this kind does not occur

Suppose that we are looking for a successor of w at which every formula ϕ_i of the infinite set of formulas $\Sigma = \{\phi_0, \phi_1, \dots\}$ holds. M-saturation is a kind of compactness property, according to which it suffices to find satisfying successors of w for arbitrary finite approximations of Σ

Definition 2.36 (M-saturation)**.** Let $\mathfrak{M} = (W, R, V)$ be a model of the basic modal similarity type, X a subset of W and Σ a set of modal formulas. Σ is **satisfiable** in the set X if there is a state $x \in X$ s.t. $\mathfrak{M}, x \Vdash \phi$ for all $\phi \in \Sigma$. Σ is **finitely satisfiable** in X if every finite subset of Σ is satisfiable in X

The model $\mathfrak{M}$ is called m**-saturated** if it satisfies the following condition for every state $w \in W$ and every set Σ of modal formulas:

If Σ is finitely satisfiable in the set of successors of w,
then Σ is satisfiable in the set of successors of w

Let τ be a modal similarity type, and let $\mathfrak{M}$ be a τ-model. $\mathfrak{M}$ is called m**-saturated** if for every state w of $\mathfrak{M}$ and every (n-ary) modal operator $\triangle \in \tau$ and sequence $\Sigma_1, \dots, \Sigma_n$ of sets of modal formulas, we have the following:

If for every sequence of finite subsets $\Delta_1 \subset \Sigma_1, \dots, \Delta_n \subseteq \Sigma_n$, there are
states $v_1, \dots, v_n$ s.t. $Rwv_1 \dots v_n$ and $v_1 \Vdash \Delta_1, \dots, v_n \Vdash \Delta_n$,
then there are states $v_1, \dots, v_n$ in $\mathfrak{M}$ s.t. $Rwv_1 \dots v_n$ and $v_1 \Vdash \Sigma_1, \dots v_n \Vdash \Sigma_n$

Proposition 2.37. *Let τ be a modal similarity type. Then the class of m-saturated τ-models has the Hennessy-Milner property*

Proof. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two m-saturated models.

Assume that $w, v \in W$ and $w' \in W'$ are s.t. Rwv and $w \leftrightsquigarrow w'$. Let Σ be the set of formulas true at v. It is clear that for every finite subset Δ of Σ we have $\mathfrak{M}, v \Vdash \bigwedge \Delta$, hence $\mathfrak{M}, w \Vdash \diamond \bigwedge \Delta$. As $w \leftrightsquigarrow w'$, it follows that $\mathfrak{M}', w' \Vdash \diamond \bigwedge \Delta$, so w' has an R'-successor v_Δ s.t. $\mathfrak{M}', v_\Delta \Vdash \bigwedge \Delta$. In other words, Σ is finitely satisfiable in the set of successors of w'; but then, by m-saturation, Σ itself is satisfiable in a successor v' of w'. Thus $v \leftrightsquigarrow v'$ $\square$

2.5.2 Ultrafilter extensions

Definition 2.38 (Filters and Ultrafilters)**.** Let W be a non-empty set. A **filter** F **over** W is a set $F \subseteq \mathcal{P}(W)$ s.t.

1. $W \in F$
2. If $X, Y \in F$, then $X \cap Y \in F$
3. If $X \in F$ and $X \subseteq Z \subseteq W$, then $Z \in F$

An **ultrafilter over** W is a proper filter s.t. for all $X \in \mathcal{P}(W)$, $X \in U$ iff $(W \setminus X) \in U$

Definition 2.39. Let W be a non-empty set, and let E be a subset of $\mathcal{P}(W)$. By the **filter generated by** E we mean the intersection F of the collection of all filters over W which include E

$$F = \bigcap \{G \mid E \subseteq G \text{ and } G \text{ is a filter over } W\}$$

E has the **finite intersection property** if the intersection of any finite number of elements of E is non-empty

Lemma 2.40 (Zorn's Lemma)**.** *Whenever $<$ is a strict partial order of a set A satisfying for all chains $C \subseteq A$ there is some $b \in A$ s.t. $x \leq b$ for all $x \in C$ then for all $a \in A$, there is a maximal $b \in A$ with $b \geq a$*

Theorem 2.41 (Ultrafilter Theorem)**.** *Fix a non-empty set W. Any proper filter over W can be extended to an ultrafilter over W. As a corollary, any subset of $\mathcal{P}(W)$ with the finite intersection property can be extended to an ultrafilter over W*

Definition 2.42. Let W be a non-empty set. Given an element $w \in W$, the **principal ultrafilter** π_w generated by w is the filter generated by the singleton set $\{w\}$

Suppose U is an ultrafilter over a non-empty set I, and that for each $i \in I$, A_i is a non-empty set. Let $C = \prod_{i \in I} A_i$. That is, C is the set of all functions f with domain I s.t. for each $i \in I$, $f(i) \in A_i$. For two functions $f, g \in C$ we say that f and g are U**-equivalent** ($f \sim_U g$) if $\{i \in I \mid f(i) = g(i)\} \in U$

Proposition 2.43. *The relation $\sim_U$ is an equivalence relation on the set C*

Proof. Suppose $\{i \mid f(i) = g(i)\} \in U, \{i \mid g(i) = h(i)\} \in U$, then $\{i \mid f(i) = g(i) = h(i)\} = \{i \mid f(i) = g(i)\} \cap \{i \mid g(i) = h(i)\} \in U$. And $\{i \mid f(i) = g(i) = h(i)\} \subseteq \{i \mid f(i) = h(i)\}$ □

Definition 2.44. Let f_U be the equivalence class of f modulo $\sim_U$, that is: $f_U = \{g \in C \mid g \sim_U f\}$. The **ultraproduct of** the sets A_i **modulo** U is the set of all equivalence classes of $\sim_U$. It is denoted by $\prod_U A_i$. So

$$\prod_U A_i = \{f_U \mid f \in \prod_{i \in I} A_i\}$$

Definition 2.45. Fix a first-order language $\mathcal{L}^1$, and let $\mathfrak{A}_i (i \in I)$ be $\mathcal{L}^1$-models. The **ultraproduct** $\prod_U \mathfrak{A}_i$ **of** $\mathfrak{A}_i$ **modulo** U is the model described as follows:

1. The universe A_U is the set $\prod_U A_i$, where A_i is the universe of $\mathfrak{A}_i$
2. Let R be an n-place relation symbol, and R_i its interpretation in the model $\mathfrak{A}_i$. The relation R_U in $\prod_U \mathfrak{A}_i$ is given by
$$R_U f_U^1 \dots f_U^n \quad \text{iff} \quad \{i \in I \mid R_i f^1(i) \dots f^n(i)\} \in U$$
3. Let F be an n-place function symbol, and F_i its interpretation in $\mathfrak{A}_i$. The function F_U in $\prod_U \mathfrak{A}_i$ is given by
$$F_U(f_U^1, \dots, f_U^n) = \{(i, F_i(f^1(i), \dots, f^n(i))) \mid i \in I\}_U$$
4. Let c be a constant, and a_i its interpretation in $\mathfrak{A}_i$. Then c is interpreted by the element $c' \in \prod_U A_i$ where $c' = \{(i, a_i) \mid i \in I\}_U$

In the case where all the structures are the same, say $\mathfrak{A}_i = \mathfrak{A}$ for all i, we speak of the **ultrapower** of $\mathfrak{A}$ modulo U, notation $\prod_U \mathfrak{A}$

Theorem 2.46 (Łoś's Theorem). *Let U be an ultrafilter over a non-empty set I. For each $i \in I$, let $\mathfrak{A}_i$ be a model*

1. *For every term $t(x_1, \dots, x_n)$ and all elements $f_U^1, \dots, f_U^n$ of $\mathfrak{B} = \prod_U \mathfrak{A}_i$ we have*
$$t^{\mathfrak{B}}[x_1 \mapsto f_U^1, \dots, x_n \mapsto f_U^n] = \{(i, t^{\mathfrak{A}_i}[f^1(i), \dots, f^n(i)]) \mid i \in I\}_U$$
2. *Given any first-order formula $\alpha(x_1, \dots, x_n)$ in $\mathcal{L}_\tau^1$ and $f_U^1, \dots, f_U^n$ in $\prod_U \mathfrak{A}_i$ we have*
$$\prod_U \mathfrak{A}_i \vDash \alpha[f_U^1, \dots, f_U^n] \quad \textit{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U$$

Proof. 1.

2. Induction on α. The atomic case holds by definition. Suppose that $\alpha \equiv \neg\beta(x_1, \dots, x_n)$, then
$$\begin{aligned}
\prod_U \mathfrak{A}_i \vDash \alpha[f_U^1 \dots f_U^n] \quad &\text{iff} \quad \prod_U \mathfrak{A}_i \nvDash \beta[f_U^1, \dots, f_U^n] \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \beta[f_U^1, \dots, f_U^n]\} \notin U \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \nvDash \beta[f^1(i), \dots, f^n(i)]\} \in U \\
&\text{iff} \quad \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U
\end{aligned}$$

The second equivalence follows from the inductive hypothesis, and the third from the fact that U is an ultrafilter

Suppose that $\alpha(x_1, \dots, x_n) \equiv \exists x_0 \beta(x_0, \dots, x_n)$, then

$$\begin{aligned}
\prod_U \mathfrak{A}_i \vDash \alpha[f_U^1, \dots, f_U^n] \quad & \text{iff} \quad \exists f_U^0 \in \prod_U \mathfrak{A}_i, \prod_U \mathfrak{A}_i \vDash \beta[f_U^0, \dots, f_U^n] \\
& \text{iff} \quad \exists f_U^0 \in \prod_U \mathfrak{A}_i, \{i \in I \mid \mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]\} \in U
\end{aligned} \tag{1}$$

As $\mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]$ implies $\mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]$, which means

$$\{i \in I \mid \mathfrak{A}_i \vDash \beta[f^0(i), \dots, f^n(i)]\} \subseteq \{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\}$$

Hence

$$\{i \in I \mid \mathfrak{A}_i \vDash \alpha[f^1(i), \dots, f^n(i)]\} \in U \tag{2}$$

Conversely, if (2) holds, then we can select a function $f^0 \in \prod_{i \in I} A_i$ s.t. (1) holds. So (1) is equivalent to (2)

□

Corollary 2.47. *Let $\prod_U \mathfrak{A}$ be an ultrapower of $\mathfrak{A}$. Then for all first-order sentences α, $\mathfrak{A} \vDash \alpha$ iff $\prod_U \mathfrak{A} \vDash \alpha$*

There is a natural embedding of a model $\mathfrak{A}$ in each of its ultrapowers. Define the **diagonal mapping** d of $\mathfrak{A}$ into $\prod_U \mathfrak{A}$ to be the function

$$\alpha \mapsto (f_\alpha)_U, \text{ where } f_a(i) = a \text{ for all } i \in I$$

Corollary 2.48. *Let $\prod_U \mathfrak{A}$ be an ultrapower of $\mathfrak{A}$. Then the diagonal mapping of $\mathfrak{A}$ into $\prod_U \mathfrak{A}$ is an elementary embedding*

Proof.

$$\begin{aligned}
\prod_U \mathfrak{A} \vDash \alpha[d(a_1), \dots, d(a_n)] \quad & \text{iff} \quad \{i \in I \mid \mathfrak{A} \vDash \alpha[a_1, \dots, a_n]\} \in U \\
& \text{iff} \quad \mathfrak{A} \vDash \alpha[a_1, \dots, a_n]
\end{aligned}$$

□

$V(\phi) = \{w \mid \mathfrak{M}, w \Vdash \phi\}$

Definition 2.49. Given an $(n+1)$-ary relation R on a set W, we define the following two n-ary operations m_R and l_R on the power set $\mathcal{P}(W)$ of W:

$$\begin{aligned}
m_R(X_1, \dots, X_n) &:= \{w \in W \mid \exists w_1, \dots, w_n(Rww_1 \dots w_n \bigwedge \forall i(w_i \in X_i))\} \\
l_R(X_1, \dots, X_n) &:= \{w \in W \mid \forall w_1, \dots, w_n(Rww_1 \dots w_n \to \exists i(w_i \in X_i))\} \\
m_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\triangle(\phi_1, \dots, \phi_n)) \\
l_R(V(\phi_1), \dots, V(\phi_n)) &:= V(\triangledown(\phi_1, \dots, \phi_n))
\end{aligned}$$

It follows that for any model $\mathfrak{M} = (W, R, V)$ we have

$$V(\diamond\phi) = m_R(V(\phi)) \quad \text{and} \quad V(\Box\phi) = l_R(V(\phi))$$

Proposition 2.50. *Let R be a relation of arity $n+1$ on the set W. Then for every n-tuple $X_1, \dots, X_n$ of subsets of W we have*

$$l_R(X_1, \dots, X_n) = W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n)$$

Proof. This is actually $\triangledown = \neg\triangle\neg$

$$\begin{aligned}
W \setminus m_R(W \setminus X_1, \dots, W \setminus X_n) &= \{w \mid \neg\exists w_1, \dots, w_n(Rww_1 \dots w_n \bigwedge \forall i(w_i \in W \setminus X_i))\} \\
&= \{\forall w_1, \dots, w_n(\neg Rww_1 \dots w_n \bigvee \neg\forall i(w_i \in W \setminus X_i))\} \\
&= \{\forall w_1, \dots, w_n(Rww_1 \dots w_n \to \exists i(w_i \notin W \setminus X_i))\} \\
&= l_R(X_1, \dots, X_n)
\end{aligned}$$

□

Definition 2.51 (Ultrafilter Extension)**.** Let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\triangle)_{\triangle\in\tau}$ is a τ-frame. The **ultrafilter extension** $\mathfrak{ue}\mathfrak{F}$ of $\mathfrak{F}$ is defined as the frame $(Uf(W), R^{ue}_\triangle)_{\triangle\in\tau}$. Here $Uf(W)$ is the set of ultrafilters over W and $R^{ue}_\triangle u_0 u_1 \dots u_n$ holds for a tuple $u_0, \dots, u_n$ of ultrafilters over W if we have that $m_{R_\triangle}(X_1, \dots, X_n) \in u_0$ whenever $X_i \in u_i$ for all i with $1 \leq i \leq n$

The **ultrafilter extension** of a τ-model $\mathfrak{M} = (\mathfrak{F}, V)$ is the model $\mathfrak{ue}\mathfrak{M} = (\mathfrak{ue}\mathfrak{F}, V^{ue})$ where $V^{ue}(p_i)$ is the set of ultrafilters of which $V(p_i)$ is a member

Any subset of a frame can be viewed as a **proposition**. A filter over the universe of the frame can thus be seen as a **theory**, in fact as a logically closed theory, since filters are both closed under intersection (conjunction) and upward closed (entailment). Viewed this way, a proper filter is a **consistent** theory, or **state of affairs**, for it does not contain the empty set (falsum). Finally an ultrafilter is a **complete** theory.

In a given frame $\mathfrak{F}$ not every state not every state of affairs needs to 'realized', in the sense that there is a state satisfying all and only the propositions belonging to the state of affairs; only the states of affairs that correspond to the **principal** ultrafilters are realized. We build $\mathfrak{ue}\mathfrak{F}$ by adding every state of affairs for $\mathfrak{F}$ as a new element of the domain - that is, $\mathfrak{ue}\mathfrak{F}$ realizes every proposition in $\mathfrak{F}$

Stipulate that $R^{ue}_{\triangle}u_0u_1 \dots u_n$ if u_0 'sees' the n-tuple $u_1, \dots, u_n$. That is, whenever $X_1, \dots, X_n$ are propositions of $u_1, \dots, u_n$ respectively, then u_0 'sees' this combination: that is, the proposition $m_{R_\triangle}(X_1, \dots, X_n)$ is a member of u_0.

Principal ultrafilters over W plays a special role. By identifying a state w of a frame $\mathfrak{F}$ with the principal ultrafilter $\pi_w = \{X \subset W \mid w \in X\}$, it is easily seen that any frame $\mathfrak{F}$ is (isomorphic to) a **submodel** (but in general not a **generated** submodel) of its ultrafilter extension. For we have the following equivalences

$$\begin{aligned} Rwv \quad & \text{iff} \quad w \in m_R(X) \text{ for all } X \subseteq W \text{ s.t. } v \in X \\ & \text{iff} \quad m_R(X) \in \pi_w \text{ for all } X \subseteq W \text{ s.t. } X \in \pi_v \\ & \text{iff} \quad R^{ue}\pi_w\pi_v \end{aligned}$$

since

$$Rwv \quad \text{iff} \quad \forall X \subseteq W(v \in X \to w \in m_R(X))$$

Example 2.4. Consider the frame $\mathfrak{N} = (\mathbb{N}, <)$

$$0 \to 1 \to 2 \to 3 \to 4 \to \quad \cdots$$

What is the ultrafilter extension of $\mathfrak{N}$? There are two kinds of ultrafilter over an infinite set: the principal ultrafilter that are in one-to-one correspondence with the points of the set, and the non-principal ones which contain all cofinite sets and only infinite sets, cf Exercise 2.5.1. The principal ultrafilters form an isomorphic copy of the frame $\mathfrak{N}$ inside $\mathfrak{ue}\mathfrak{N}$. For any pair u, u' of ultrafilters, if u' is non-principal, then $R^{ue}uu'$. To set this, let $X \in u'$. As X is infinite, for any $n \in \mathbb{N}$ there is an m s.t. $n < m$ and $m \in X$. This show that $m_<(X) = \mathbb{N}$. But $\mathbb{N}$ is an element of every ultrafilter

The shows that the ultrafilter extension of $\mathfrak{N}$ consists of a copy of $\mathfrak{N}$ followed by a uncountable cluster consisting of all the non-principal ultrafilters

Proposition 2.52. *Let τ be a modal similarity type, and $\mathfrak{M}$ a τ-model. Then for any formula ϕ and any ultrafilter u over W, $V(\phi) \in u$ iff $\mathfrak{ue}\mathfrak{M}, u \vDash \phi$. Hence for every state w of $\mathfrak{M}$ we have $w \leftrightsquigarrow \pi_w$*

Proof. The second claim of the proposition is immediate from the first one by the observation that $w \Vdash \phi$ iff $w \in V(\phi)$ iff $V(\phi) \in \pi_w$

Induction on ϕ. The basic case is immediate from the definition of V^{ue}. Suppose ϕ is of the form $\neg\psi$, then

$$\begin{array}{lll} V(\neg\psi) \in u & \text{iff} & W \setminus V(\psi) \in u \\ & \text{iff} & V(\psi) \notin u \\ & \text{iff} & \mathfrak{ue}\mathfrak{M}, u \not\Vdash \psi \quad \text{IH} \\ & \text{iff} & \mathfrak{ue}\mathfrak{M}, u \Vdash \neg\psi \end{array}$$

Now consider the case where ϕ is of the form $\diamond\psi$. Assume first that $\mathfrak{ue}\mathfrak{M}, u \Vdash \diamond\psi$. Then there is an ultrafilter u' s.t. $R^{ue}uu'$ and $\mathfrak{ue}\mathfrak{M}, u' \Vdash \psi$. The induction hypothesis implies that $V(\psi) \in u'$, so by the definition of R^{ue}, $m_R(V(\psi)) \in u$. Now the result follows immediately from the observation that $m_R(V(\psi)) = V(\diamond\psi)$

Assume that $V(\diamond\psi) \in u$. We have to find an ultrafilter u' s.t. $V(\psi) \in u'$ and $R^{ue}uu'$. The latter constraint reduces to the condition that $m_R(X) \in u$ whenever $X \in u'$, or equivalently (see Exercise 2.5.2)

$$u'_0 := \{Y \mid l_R(Y) \in u\} \subseteq u'$$

We will first show that u'_0 is closed under intersection. Let $Y, Z \in u'_0$. By definition, $l_R(Y)$ and $l_R(Z)$ are in u. But then $l_R(Y \cap Z) \in u$ as $l_R(Y \cap Z) = l_R(Y) \cap l_R(Z)$. This proves that $Y \cap Z \in u'_0$

Next we make sure that for any $Y \in u'_0$, $Y \cap V(\psi) \neq \emptyset$. Let Y be an arbitrary element of u'_0, then by definition of u'_0, $l_R(Y) \in u$. As u is closed under intersection and does not contain the empty set, there must be an element $x \in l_R(Y) \cap V(\diamond\psi)$. But then x must have a successor y in $V(\psi)$. Finally, $x \in l_R(Y)$ implies $y \in Y$

From the fact that u'_0 is closed under intersection, and the fact that for any $Y \in u'_0$, $Y \cap V(\psi) \neq \emptyset$, it follows that the set $u'_0 \cup \{V(\psi)\}$ has the finite intersection property. So the Ultrafilter Theorem provides us with an ultrafilter u' s.t. $u'_0 \cup \{V(\psi)\} \subseteq u'$. This ultrafilter u' has the desired properties: it is clearly a successor of u, and the fact the $\mathfrak{ue}\mathfrak{M}, u' \Vdash \psi$ follows from $V(\psi) \in u'$ and the induction hypothesis $\square$

Example 2.5. Our new invariance result can be used to compare the relative expressive power of modal languages. Consider the modal constant $\circlearrowleft$ whose truth definition in a model for the basic modal language is

$$\mathfrak{M}, w \Vdash \circlearrowleft \quad \text{iff} \quad \mathfrak{M} \models Rxx[v] \text{ for some } v \text{ in } \mathfrak{M}$$

Comparing the pictures of the frame $(\mathbb{N}, <)$ and its ultrafilter extension given in Example 2.4 . The former is loop-free but the latter contains uncountably many loops

Proposition 2.53. *Let τ be a modal similarity type, and let $\mathfrak{M}$ be a τ-model. Then $\mathfrak{ue}\mathfrak{M}$ is m-saturated*

Proof. Let $\mathfrak{M} = (W, R, V)$ be a model. Consider an ultrafilter u over W, and a set Σ of modal formulas which is finitely satisfiable in the set of successors of u. We have to find an ultrafilter u' s.t. $R^{ue}uu'$ and $\mathfrak{ue}\mathfrak{M}, u' \Vdash \Sigma$. Define

$$\Delta = \{V(\phi) \mid \phi \in \Sigma'\} \cup \{Y \mid l_R(Y) \in u\}$$

where Σ' is the set of (finite) conjunctions of formulas in Σ. We claim that the set Δ has the finite intersection property. Since both $\{V(\phi) \mid \phi \in \Sigma'\}$ and $\{Y \mid l_R(Y) \in u\}$ are closed under taking intersections, it suffices to prove that for an arbitrary $\phi \in \Sigma'$ and an arbitrary set $Y \subseteq W$ for which $l_R(Y) \in u$, we have $V(\phi) \cap Y \neq \emptyset$. but if $\phi \in \Sigma'$, then by assumption, there is a successor u'' of u s.t. $\mathfrak{ue}\mathfrak{M}, u'' \Vdash \phi$, or in other words, $V(\phi) \in u''$. Then $l_R(Y) \in u$ implies $Y \in u''$ by Exercise 2.5.2 . Hence $V(\phi) \cap Y$ is an element of the ultrafilter u'' and therefore cannot be identical to the empty set.

It follows by the Ultrafilter Theorem that Δ can be extended to an ultrafilter u'. Clearly u' is the required successor □

Theorem 2.54. *Let τ be a modal similarity type, and let $\mathfrak{M}$ and $\mathfrak{M}'$ be τ-models, and w, w' two states in $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. Then*

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w' \quad \textit{iff} \quad \mathfrak{ue}\mathfrak{M}, \pi_w \leftrightarrows \mathfrak{ue}\mathfrak{M}', \pi_{w'}$$

Proof. From Propositions 2.52, 2.53 and 2.37 □

Exercise 2.5.1. Let W be an infinite set. Recall that $X \subseteq W$ is **co-finite** if $W \setminus X$ is finite

1. Prove that the collection of co-finite subsets of W has the finite intersection property
2. Show that there are ultrafilters over W that do not contain any finite set
3. Prove that an ultrafilter is non-principal iff it contains only infinite sets iff it contains all co-finite sets
4. Prove that any ultrafilter over W has uncountably many elements

Proof. Suppose $U = \{X \subseteq W \mid X \text{ is cofinite}\}$

1. For any $A, B \in U$, if $A \cap B =$, $A \subset \overline{B}$. But A is infinite and $\overline{B}$ is finite, this can't happen. Hence $A \cap B \neq \emptyset$

2. U can be extended to a ultrafilter $\mathcal{U}$. If A is finite, then $\overline{A} \in U \subseteq \mathcal{U}$. Hence $\mathcal{U}$ does not contain any finite set.

3. $1 \rightarrow 2$.If an ultrafilter contains a finite set. Then its a principal ultrafilter generated on the intersection of all finite sets.

   $2 \rightarrow 3$ and $3 \rightarrow 1$ are obvious.

4. Half of the $\mathcal{P}(W)$ belongs to the ultrafilter and $\mathcal{P}(W)$ is uncountable

□

Exercise 2.5.2. Given a model $\mathfrak{M} = (W, R, V)$ and two ultrafilters u and v over W, show that $R^{ue}uv$iff $\{Y \mid l_R(Y) \in u\} \subseteq v$

Proof.

$$\begin{aligned}
R^{ue}uv &\Leftrightarrow X \in v \rightarrow m_R(X) \in u \\
&\Leftrightarrow \neg m_R(X) \in u \rightarrow \neg X \in v \\
&\Leftrightarrow W - m_R(X) \in u \rightarrow W - X \in v \\
&\Leftrightarrow l_R(W - X) \in u \rightarrow W - X \in v \\
&(\text{Since } m_R(X) = W - l_R(W - X)) \\
&\Leftrightarrow \{Y \mid l_R(Y) \in u\} \subseteq v
\end{aligned}$$

□

2.6 Characterization and Definability

2.6.1 The van Benthem Characterization Theorem

Let $\Gamma(x)$ be a set of first-order formulas in which a single individual variable may occur free - such a set of formulas is called a **type**. A first-order model $\mathfrak{M}$ **realizes** $\Gamma(x)$ if there is an element w in $\mathfrak{M}$ s.t. for all $\gamma \in \Gamma, \mathfrak{M} \models \gamma[w]$

Let $\mathfrak{M}$ be a model for a given first-order language $\mathcal{L}^1$ with domain W. For a subset $A \subset W$, $\mathcal{L}^1[A]$ is the language obtained by extending $\mathcal{L}^1$ with new constant $\underline{a}$ for all elements $a \in A$. $\mathfrak{M}_A$ is the expansion of $\mathfrak{M}$ to a structure for $\mathcal{L}^1[A]$ in which each $\underline{a}$ is interpreted as a

Assume that A is of size at most α. Assume that $\alpha = 3$ and $A = \{\alpha_1, \alpha_2\}$. Let $\Gamma(\underline{a}_1, \underline{a}_2, x)$ be a type of the language $\mathcal{L}^1[A]$; $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is consistent with the first-order theory of $\mathfrak{M}_A$ iff $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is finitely realizable in $\mathfrak{M}_A$. So for this particular set $\Gamma(\underline{a}_1, \underline{a}_2, x)$, 3-saturation of $\mathfrak{M}$ means that if $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is finitely realizable in $\mathfrak{M}_A$, then $\Gamma(\underline{a}_1, \underline{a}_2, x)$ is realizable in $\mathfrak{M}_A$

Or consider a formula $\gamma(\underline{a}_1, \underline{a}_2, x)$ and let $\gamma(x_1, x_2, x)$ be the formula with the fresh variables x_1 and x_2 replacing each occurrence in γ of $\underline{a}_1$ and $\underline{a}_2$ respectively. Then we have the following equivalence

$$\mathfrak{M}_A \text{ realizes } \{\gamma(\underline{a}_1, \underline{a}_2, x)\} \text{ iff there is a } b \text{ s.t. } \mathfrak{M} \vDash \gamma(x_1, x_2, x)[a_1, a_2, b]$$

So a model is α-saturated iff the following holds for every $n < \alpha$ and every set Γ of formulas of the form $\gamma(x_1, \ldots, x_n, x)$

If $(a_1, \ldots, a_n)$ is an n-tuple s.t. for every finite $\Delta \subseteq \Gamma$ there is a b_Δ s.t. $\mathfrak{M} \vDash \gamma(x_1, \ldots, x_n, x)[a_1, \ldots, a_n, b_\Delta]$ for every $\gamma \in \Delta$
then we have that there is a b s.t. $\mathfrak{M} \vDash \gamma(x_1, \ldots, x_n, x)[a_1, \ldots, a_n, b]$ for every $\gamma \in \Gamma$

Definition 2.55. Let α be a natural number, or ω. A model $\mathfrak{M}$ is α**-saturated** if for every subset $A \subseteq W$ of size less than α, the expansion $\mathfrak{M}_A$ realizes every set $\Gamma(x)$ of $\mathcal{L}^1[A]$-formulas (with only x occurring free) that is *consistent* (a proof-theoretic notion, only finite deductions, hence this definition is consistent the definition above) with the first-order theory of $\mathfrak{M}_A$. An ω-saturated model is usually called **countably saturated**

Example 2.6.
1. Every finite model is countably saturated. For if $\mathfrak{M}$ is finite, and $\Gamma(x)$ is a set of first-order formulas consistent with the first-order theory of $\mathfrak{M}$, there exists a model $\mathfrak{N}$ that is elementarily equivalent to $\mathfrak{M}$ and that realizes $\Gamma(x)$. But as $\mathfrak{M}$ and $\mathfrak{N}$ are finite, elementary equivalence implies isomorphism (proof) , and hence $\Gamma(x)$ is realized in $\mathfrak{M}$

2. The ordering of the rational numbers $(\mathbb{Q}, <)$ is countably saturated as well. The relevant first-order language $\mathcal{L}^1$ has $<$ and $=$. Take a subset A of $\mathbb{Q}$ and let $\Gamma(x)$ be a set of formulas in the resulting expansion $\mathcal{L}^1[A]$ of the first-order language that is consistent with the theory of $(\mathbb{Q}, <, a)_{a\in A}$. Then there exists a model $\mathfrak{N}$ of the theory of $(\mathbb{Q}, <, a)_{a\in A}$ that realizes $\Gamma(x)$. $\star$. Now take a countable elementary submodel $\mathfrak{N}'$ of $\mathfrak{N}$ that contains at least one object realizing $\Gamma(x)$. Then $\mathfrak{N}'$ is a countable dense linear ordering without endpoints, and hence the ordering of $\mathfrak{N}'$ is isomorphic to $(\mathbb{Q}, <)$.

Theorem 2.56. *Let τ be a modal similarity type. Any countably saturated τ-model is m-saturated. It follows that the class of countably saturated τ-models has the Hennessy-Milner property*

Proof. Assume that $\mathfrak{M} = (W, R, V)$ viewed as a first-order model, is countably saturated. Let a be a state in W, and consider a set Σ of modal formulas which is finite satisfiable in the successor set of a. Define Σ' to be the set

$$\Sigma' = \{R\underline{a}x\} \cup ST_x(\Sigma)$$

where $ST_x(\Sigma) = \{ST_x(\phi) \mid \phi \in \Sigma\}$. Σ' is consistent with the first-order theory of $\mathfrak{M}_a$: $\mathfrak{M}_a$ realizes every finite subset of Σ', namely in some successor of a. So by the countable saturation of $\mathfrak{M}$, Σ' is realized in some state b. By $\mathfrak{M}_A \models R\underline{a}x[b]$ it follows that b is a successor of a. Then, by Proposition 2.32 and the fact that $\mathfrak{M}_a \models ST_x(\phi)[b]$ for all $\phi \in \Sigma$, it follows that $\mathfrak{M}, b \Vdash \Sigma$. Thus Σ is satisfiable in a successor of a $\square$

Lemma 2.57 (Detour Lemma). *Let τ be a modal similarity type, and let $\mathfrak{M}$ and $\mathfrak{N}$ be two models, and w and v states in $\mathfrak{M}$ and $\mathfrak{N}$, respectively. Then the following are equivalent:*

1. *For all modal formulas ϕ: $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{N}, v \Vdash \phi$*
2. *There exists a bisimulation $Z : \mathfrak{ue}\mathfrak{M}, \pi_w \leftrightarrows \mathfrak{ue}\mathfrak{N}, \pi_v$*
3. *There exist countably saturated models $\mathfrak{M}^*, w^*, \mathfrak{N}^*, v^*$ and elementary embeddings $f : \mathfrak{M} \preceq \mathfrak{M}^*$ and $g : \mathfrak{N} \preceq \mathfrak{N}^*$ s.t.*
   - *(a) $f(w) = w^*$ and $g(v) = v^*$*
   - *(b) $\mathfrak{M}^*, w^* \leftrightarrows \mathfrak{N}^*, v^*$*

Definition 2.58. A first-order formula $\alpha(x)$ in $\mathcal{L}^1_\tau$ is **invariant for bisimulations** if for all models $\mathfrak{M}$ and $\mathfrak{N}$, and all states w in $\mathfrak{M}$, v in $\mathfrak{N}$, and all bisimulations Z between $\mathfrak{M}$ and $\mathfrak{N}$ s.t. wZv, we have $\mathfrak{M} \models \alpha(x)[w]$ iff $\mathfrak{N} \models \alpha(x)[v]$

Theorem 2.59 (van Benthem Characterization Theorem). *Let $\alpha(x)$ be a first-order formula in $\mathcal{L}^1_\tau$. Then $\alpha(x)$ is invariant for bisimulations iff it is equivalent to the standard translation of a modal τ-formula*

Proof. Assume $\alpha(x)$ is invariant for bisimulations and consider the set of modal consequences of α:

$$MOC(\alpha) = \{ST_x(\phi) \mid \phi \text{ is a modal formula, and } \alpha(x) \models ST_x(\phi)\}$$

Our first claim is that if $MOC(\alpha) \vDash \alpha(x)$, then α is equivalent to the translation of a modal formula. Assume $MOC(\alpha) \vDash \alpha(x)$, then by the Compactness Theorem for first-order logic, for some finite subset $X \subseteq MOC(\alpha)$, we have $X \vDash \alpha(x)$. So $\vDash \bigwedge X \to \alpha(x)$. Trivially $\vDash \alpha(x) \to \bigwedge X$, thus $\vDash \alpha(x) \leftrightarrow \bigwedge X$. And as every $\beta \in X$ is the translation of a modal formula, so is $\bigwedge X$

So it suffices to show that $MOC(\alpha) \vDash \alpha(x)$. Assume $\mathfrak{M} \vDash MOC(\alpha)[w]$; we need to show that $\mathfrak{M} \vDash \alpha(x)[w]$. Let

$$T(x) = \{ST_x(\phi) \mid \mathfrak{M} \vDash ST_x(\phi)[w]\}$$

We claim that $T(x) \cup \{\alpha(x)\}$ is consistent. Assume that $T(x) \cup \{\alpha(x)\}$ is inconsistent. Then by compactness, for some finite subset $T_0(x) \subset T(x)$ we have $\vDash \alpha(x) \to \neg \bigwedge T_x(x)$. Hence $\neg \bigwedge T_0(x) \in MOC(\alpha)$. But this implies $\mathfrak{M} \vDash \neg \bigwedge T_0(x)[w]$, a contradiction

Let $\mathfrak{N}, v$ be s.t. $\mathfrak{N} \vDash T(x) \cup \{\alpha(x)\}[v]$. Observe that w and v are modally equivalent: $\mathfrak{M}, w \Vdash \phi$ implies $ST_x(\phi) \in T(x)$, which implies $\mathfrak{N}, v \Vdash \phi$; and likewise, if $\mathfrak{M}, w \nVdash \phi$ then $\mathfrak{M}, w \Vdash \neg\phi$ and $\mathfrak{N}, v \Vdash \neg\phi$.

We can use the Detour Lemma and make a detour through a Hennessy-Milner class where modal equivalence and bisimilarity do coincide.

$$\begin{array}{ccc}
\mathfrak{M}, w & & \mathfrak{N}, v \\
\Big| \preceq & & \Big| \preceq \\
\mathfrak{M}^*, w^* & \xrightarrow{\;\leftrightarrows\;} & \mathfrak{N}^*, v^*
\end{array}$$

$\mathfrak{N} \vDash \alpha(x)[v]$ implies $\mathfrak{N}^* \vDash \alpha(x)[v^*]$. As $\alpha(x)$ is invariant for bisimulations, we get $\mathfrak{M}^* \vDash \alpha(x)[w^*]$. By invariance under elementary embeddings, we have $\mathfrak{M} \vDash \alpha(x)[w]$ □

2.6.2 Ultraproducts

Suppose $I \neq \emptyset$, U is an ultrafilter over I.

Definition 2.60 (Ultraproducts of Sets)**.** Let f_U be the equivalence class of f modulo $\sim_U$, that is: $f_U = \{g \in C \mid g \sim_U f\}$. The **ultraproduct of** W_i **modulo** U, denoted as $\prod_U W_i$ is the set of all equivalence classes of $\sim_U$. So

$$\prod_U W_i = \{f_U \mid f \in \prod_{i \in I} W_i\}$$

In case $W_i = W$, the ultraproduct is called the **ultrapower of** W **modulo** U, and written $\prod_U W$

Definition 2.61 (Ultraproduct of Models)**.** Fix a modal similarity type τ, and let $\mathfrak{M}_i (i \in I)$ be τ-models. The **ultraproduct** $\prod_U \mathfrak{M}_i$ of $\mathfrak{M}_i$ modulo U is the model described as follows

1. The universe W_U of $\prod_U \mathfrak{M}_i$ is the set $\prod_U W_i$
2. Let V_i be the valuation of $\mathfrak{M}_i$. Then the valuation V_U of $\prod_U \mathfrak{M}_i$ is defined by
$$f_U \in V_U(p) \quad \text{iff} \quad \{i \in I \mid f(i) \in V_i(p)\} \in U$$
3. Let $\triangle$ be a modal operator in τ, and $R_{\triangle i}$ its associated relation in the model $\mathfrak{M}_i$. The relation $R_{\triangle U}$ in $\prod_U \mathfrak{M}_i$ is given by
$$R_{\triangle U} f_U^1 \dots f_U^{n+1} \quad \text{iff} \quad \{i \in I \mid R_{\triangle i} f^1(i) \dots f^{n+1}(i)\} \in U$$
In particular,
$$R_{\diamond U} f_U g_U \quad \text{iff} \quad \{i \in I \mid R_{\diamond i} f(i) g(i)\} \in U$$

Proposition 2.62. *Let $\prod_U \mathfrak{M}$ be an ultrapower of $\mathfrak{M}$. Then for al modal formulas ϕ we have $\mathfrak{M}, w \Vdash \phi$ iff $\prod_U \mathfrak{M}, (f_w)_U \Vdash \phi$, where f_w is the constant function s.t. $f_w(i) = w$ for all $i \in I$*

Proof. 1. $\phi = p$
$$\begin{aligned} \mathfrak{M}, w \Vdash \phi &\Leftrightarrow w \in V(\phi) \\ &\Leftrightarrow \{i \in I \mid f_w(i) \in V(p)\} = I \in U \\ &\Leftrightarrow \prod_U \mathfrak{M}, (f_w)_U \Vdash \phi \end{aligned}$$

2. $\phi = \diamond\psi$
$$Rwv \Leftrightarrow \{i \in I \mid R_{\diamond i} f_w(i) f_v(i)\} = I \in U \Leftrightarrow R_{\diamond U} f_w g_v$$

$\square$

An ultrafilter is **countably incomplete** if it is not closed under countably intersections

Example 2.7. Consider the set of natural numbers $\mathbb{N}$. Let U be an ultrafilter over $\mathbb{N}$ that does not contain any singletons $\{u\}$. Then for all n, $(\mathbb{N} \setminus \{n\}) \in U$. But
$$\emptyset = \bigcap_{n \in \mathbb{N}} (\mathbb{N} \setminus \{n\}) \notin U$$
So U is countably incomplete

Lemma 2.63. *Let $\mathcal{L}$ be a countable first-order language, U a countably incomplete ultrafilter over a non-empty set I, and $\mathfrak{M}$ an $\mathcal{L}$-model. The ultrapower $\prod_U \mathfrak{M}$ is countably saturated*

Theorem 2.64. *Let τ be a modal similarity type, and let $\mathfrak{M}$ and $\mathfrak{N}$ be τ-models, and w and v states in $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Then the following are equivalent*

1. *For all modal formulas ϕ: $\mathfrak{M}, w \Vdash \phi$ iff $\mathfrak{N}, v \Vdash \phi$*
2. *There exists ultrapowers $\prod_U \mathfrak{M}$ and $\prod_U \mathfrak{N}$ as well as a bisimulation $Z : \prod_U \mathfrak{M}, (f_w)_U \leftrightarrows \prod_U \mathfrak{N}, (f_v)_U$ linking $(f_w)_U$ and $(f_v)_U$, where $f_w(f_v)$ is the constant function mapping every index to $w(v)$*

Proof. $2 \to 1$. $\mathfrak{M}, w \Vdash \phi$ iff $\prod_U \mathfrak{M}, (f_w)_U \Vdash \phi$ iff $\prod_U \mathfrak{N}, (f_v)_U \Vdash \phi$ iff $\mathfrak{N}, v \Vdash \phi$

$1 \to 2$. Take $\mathbb{N}$ as our index set, and let U be a countably incomplete ultrafilter over $\mathbb{N}$. By Lemma 2.63 the ultrapowers $\prod_U \mathfrak{M}$ and $\prod_U \mathfrak{N}$ are countably saturated. Now $(f_w)_U$ and $(f_v)_U$ are modally equivalence. Next apply Theorem 2.56: as $(f_w)_U$ and $(f_v)_U$ are modally equivalent and $\prod_U \mathfrak{M}$ and $\prod_U \mathfrak{N}$ are countably saturated, there exists a bisimulation □

2.6.3 Definability

Given a modal similarity type τ, a pointed model is a pair $(\mathfrak{M}, w)$ where $\mathfrak{M}$ is a τ-model and w is a state of $\mathfrak{M}$. A class of pointed models K is said to be **closed under bisimulations** if $(\mathfrak{M}, w) \in \mathsf{K}$ and $\mathfrak{M}, w \leftrightarrows \mathfrak{N}, v$ implies $(\mathfrak{N}, v) \in \mathsf{K}$. K is **closed under ultraproducts** if any ultraproducts $\prod_U(\mathfrak{M}_i, w_i)$ of a family of pointed models $(\mathfrak{M}_i, w_i)$ in K belongs to K. If K is a class of pointed τ-models, $\overline{\mathsf{K}}$ denotes the complement of K within the class of all pointed τ-models. K is **definable by a set of modal formulas** if there is a set of modal formulas Γ s.t. for any pointed model $(\mathfrak{M}, w)$ we have $(\mathfrak{M}, w)$ in K iff for all $\gamma \in \Gamma, \mathfrak{M}, w \Vdash \gamma$. K is definable by a single modal formula iff it is definable by a singleton set

By Theorem 2.12 definable classes of pointed models must be closed under bisimulations, and by Proposition 2.32 and Corollary 2.47 they must be closed under ultraproducts as well.

Theorem 2.65. *Let τ be a modal similarity type, and K a class of pointed τ-models. Then the following are equivalent:*

1. *K is definable by a set of modal formulas*
2. *K is closed under bisimulations and ultraproducts, and $\overline{K}$ is closed under ultrapowers*

Proof. Assume K and $\bar{\mathsf{K}}$ satisfy the stated closure conditions. Observe that $\bar{\mathsf{K}}$ is closed under bisimulations as K is. Define

$$T = \{\phi \mid \forall(\mathfrak{M}, w) \in \mathsf{K} : \mathfrak{M}, w \Vdash \phi\}$$

We will show that T defines the class of K.

Assume $\mathfrak{M}, w \Vdash T$. Define $\Sigma = \{\phi \mid \mathfrak{M}, w \Vdash \phi\}$. It is obvious that Σ is finitely satisfiable in K; for suppose that the set $\{\sigma_1, \dots, \sigma_n\} \subseteq \Sigma$ is not satisfiable in K. Then the formula $\neg(\sigma_1 \wedge \dots \wedge \sigma_n)$ would be true on all pointed models in K, so it would belong to T, yet be false in $\mathfrak{M}, w$. But then the following claim shows that Σ is satisfiable in the ultraproduct of pointed models

Claim 1 . Let Σ be a set of modal formulas, and K a class of pointed models in which Σ is finitely satisfiable. Then Σ is satisfiable in some ultraproduct of models in K

Proof of Claim. Define in index set I as the collection of all finite subsets of Σ

$$I = \{\Sigma_0 \subset \Sigma \mid \Sigma_0 \text{ is finite}\}$$

By assumption, for each $i \in I$ there is a pointed model $(\mathfrak{N}_i, v_i)$ in K s.t. $\mathfrak{N}_i, v_i \Vdash i$. We now construct an ultrafilter U over I s.t. the ultraprodcut $\prod_U \mathfrak{N}_i$ has a state f_U with $\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$

For each $\sigma \in \Sigma$, let $\hat{\sigma}$ be the set of all $i \in I$ s.t. $\sigma \in i$. Then the set $E = \{\hat{\sigma} \mid \sigma \in \Sigma\}$ has the finite intersection property because

$$\{\sigma_1, \dots, \sigma_n\} \in \hat{\sigma}_1 \cap \dots \cap \hat{\sigma}_n$$

So E can be extended to an ultrafilter U over I. This defines $\prod_U \mathfrak{N}_i$; for the definition of f_U, let W_i denote the universe of the model $\mathfrak{N}_i$ and consider the function $f \in \prod_{i \in I} W_i$ s.t. $f(i) = v_i$. It is left to prove that

$$\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$$

Observe that for $i \in \hat{\sigma}$ we have $\sigma \in i$ and so $\mathfrak{N}_i, v_i \Vdash \sigma$. Therefore for each $\sigma \in \Sigma$

$$\{i \in I \mid \mathfrak{N}_i, v_i \Vdash \sigma\} \supseteq \hat{\sigma} \quad \text{and} \quad \hat{\sigma} \in U$$

since $\sigma \in i$ implies $\mathfrak{N}_i, v_i \Vdash \sigma$. It follows that $\{i \in I \mid \mathfrak{N}_i, v_i \Vdash \sigma\} \in U$, so by Theorem 2.46 $\prod_U \mathfrak{N}_i, f_U \Vdash \sigma$. Hence $\prod_U \mathfrak{N}_i, f_U \Vdash \Sigma$

It follows from Claim 1 and the closure of K under taking ultraproducts that Σ is satisfiable in some pointed model $(\mathfrak{N}, v) \in \mathsf{K}$. But $\mathfrak{N}, v \Vdash \Sigma$ implies that v and the state w from our original pointed model $(\mathfrak{M}, w)$ are modally equivalent. So by Theorem 2.64 there exists an ultrafilter U' s.t.

$$\prod_{U'}(\mathfrak{N}, v), (f_v)_U \leftrightarroweq \prod_{U'}(\mathfrak{M}, w), (f_w)_U$$

By closure under ultraproducts, the pointed model $(\prod_{U'}(\mathfrak{N}, v), (f_v)_U)$ belongs to K. Hence by closure under bisimulations, $(\prod_{U'}(\mathfrak{M}, w), (f_w)_U)$ is in K. By closure of $\overline{K}$ under ultrapowers, $(\mathfrak{M}, w) \in$ K □

Theorem 2.66. *Let τ be a modal similarity type, and K a class of pointed τ-models. Then the following are equivalent*

1. *K is definable by means of a single modal formula*
2. *Both K and $\overline{\mathsf{K}}$ are closed under bisimulations and ultraproducts*

Proof. Assume K, $\overline{\mathsf{K}}$ satisfy the stated conditions. Then both are closed under ultraproducts, hence by Theorem 2.65 there are set of modal formulas T_1, T_2 defining K and $\overline{\mathsf{K}}$ respectively. Observe their union is inconsistent in the sense that there is no pointed model $(\mathfrak{M}, w)$ s.t. $(\mathfrak{M}, w) \Vdash T_1 \cup T_2$. So by compactness there exists $\phi_1, \dots, \phi_n \in T_1$ and $\psi_1, \dots, \psi_m \in T_2$ s.t. for all pointed models $(\mathfrak{M}, w)$

$$\mathfrak{M}, w \Vdash \phi_1 \wedge \cdots \wedge \phi_n \rightarrow \neg\psi \vee \cdots \vee \neg\psi_m$$

By definition, for any $(\mathfrak{M}, w) \in$ K we have $\mathfrak{M}, w \Vdash \phi_1 \wedge \cdots \wedge \phi_n$. Conversely, if $\mathfrak{M}, w \Vdash \phi_1 \wedge \cdots \wedge \phi_n$, then $\mathfrak{M}, w \Vdash \neg\psi_1 \vee \cdots \vee \neg\psi_m$. Hence $\mathfrak{M}, w \nVdash T_2$. Therefore $(\mathfrak{M}, w) \notin \overline{\mathsf{K}}$, whence $(\mathfrak{M}, w) \in$ K □

3 Frames

3.1 Frame Definability

Definition 3.1 (Validity)**.** Let τ be a modal similarity type. A formula ϕ (of this similarity type) is **valid at a state** w **in a frame** $\mathfrak{F}$ (notation: $\mathfrak{F}, w \Vdash \phi$) if ϕ is true at w in every model $(\mathfrak{F}, V)$ based on $\mathfrak{F}$; ϕ is **valid on a frame** $\mathfrak{F}$ (notation: $\mathfrak{F} \Vdash \phi$) if it is valid at every state in $\mathfrak{F}$. A formula ϕ is **valid on a class of frames K** (notation: K $\Vdash \phi$) if it is valid on every frame $\mathfrak{F}$ in K. We denote the class of frames where ϕ is valid by Fr_ϕ

A set Γ of modal formulas (of type τ) is **valid on a frame** $\mathfrak{F}$ if every formula in Γ is valid on F; and Γ is **valid on a class K of frames** if Γ is valid on every member of K. We denote the class of frames where Γ is valid by Fr_Γ

Definition 3.2 (Definability)**.** Let τ be a modal similarity type, ϕ a modal formula of this type, and K a class of τ-frames. We say that ϕ **defines** (or **characterizes**) K if for all frames $\mathfrak{F}$, $\mathfrak{F}$ is in K iff $\mathfrak{F} \Vdash \phi$. Similarly, if Γ is a set of modal formulas of this type, we say that Γ **defines** $\mathfrak{F}$ is in K iff $\mathfrak{F} \Vdash \Gamma$

A class of frames is **(modally) definable** if there is some set of modal formulas that defines it

Definition 3.3 (Relative Definability)**.** Let τ be a modal similarity type, ϕ a modal formula of this type, and C a class of τ-frames. We say that ϕ **defines** (or **characterizes**) a class K of frames **within** C (or **relative to** C) if for all frames $\mathfrak{F}$ in C we have that $\mathfrak{F}$ is in K iff $\mathfrak{F} \Vdash \phi$

Similarly, if Γ is a set of modal formulas of this type, we say that Γ **defines** a class K of frames **within** C if for all frames $\mathfrak{F}$ in C we have that $\mathfrak{F}$ is in K iff $\mathfrak{F} \Vdash \Gamma$

Definition 3.4 (Frame Languages)**.** For any modal similarity type τ, the **first-order frame language** of τ is the first-order language that has the identity symbol = together with an $(n+1)$-ary relation symbol $R_\triangle$ for each n-ary modal operator $\triangle$ in τ. We denote this language by $\mathcal{L}^1_\tau$. We often call it the **first-order correspondence language** (for τ)

Let Φ be any set of proposition letters. The **monadic second-order frame language** of τ over Φ is the monadic second-order language obtained by augmenting $\mathcal{L}^1_\tau$ with a Φ-indexed collection of monadic predicate variables. (That is, this language has all the resources of $\mathcal{L}^1_\tau$, and in addition is capable of quantifying over subsets of frames). We denote this language by $\mathcal{L}^2_\tau(\Phi)$. We often simply call it the **second-order frame language** or the **second-order correspondence language** (for τ)

Definition 3.5 (Frame Correspondence)**.** If a class of frames (property) can be defined by a modal formula ϕ and by a formula α from one of these frame languages, then we say that ϕ and α are each others (global) frame **correspondents**

For example the basic modal formula $p \to \Diamond p$ and the first-order sentence $\forall x Rxx$ are correspondents

Example 3.1. Read $\Diamond\phi$ as 'it is **possibly** the case that ϕ' and $\Box\phi$ as '**necessarily** ϕ'.

(T) $p \to \Box p$

(4) $\Diamond\Diamond p \to \Diamond p$

(5) $\Diamond p \to \Box\Diamond p$

Our first claim is that for any frame $\mathfrak{F} = (W, R)$, the axiom T corresponds to **reflexivity** of the relation R:

$$\mathfrak{F} \Vdash \mathrm{T} \quad \text{iff} \quad \mathfrak{F} \Vdash \forall x\ Rxx$$

Suppose that R is **not** reflexive. There exists a state w which is not accessible from itself. Now the valuation V has to satisfy two conditions

1. $w \in V(p)$
2. $\{x \in W \mid Rwx\} \cap V(p) = \emptyset$

Consider the **minimal** valuation V satisfying condition (1), that is, take

$$V(p) = \{w\}$$

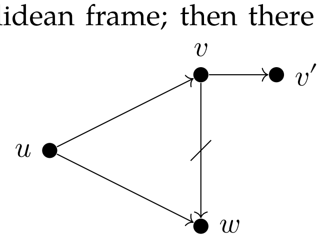
Now let v be an R-succesor of w. As Rww does not hold in $\mathfrak{F}$, v must be distinct from w, so $v \nVdash p$. As v was arbitrary, $w \nVdash p$

Likewise, one can prove that for any frame $\mathfrak{F} = (W, R)$

$$\begin{array}{lll} \mathfrak{F} \Vdash 4 & \text{iff} & R \text{ is transitive} \\ \mathfrak{F} \Vdash 5 & \text{iff} & R \text{ is euclidean} \end{array}$$

where a relation is **euclidean** if it satisfies $\forall xyz((Rxy \wedge Rxz) \to Ryz)$.

Assume $\mathfrak{F}$ is a non-euclidean frame; then there must be states u, v, w s.t. Ruv, Ruw but not Rvw:



We will try to falsify 5 in u; for this purpose we have to find a valuation V s.t. $(\mathfrak{F}, V), u \Vdash \Diamond p$ and $(\mathfrak{F}, V), u \nVdash \Box\Diamond p$. In other words, we have to make p **true** at some R-successor x of u, and **false** at all R-successors of some R-successor y of u. The constaints on V are

1. $w \in V(p)$
2. $\{z \mid Rvz\} \cap V(p) = \emptyset$

Let's take a **maximal** V satisfying condition (2), that is, define

$$V(p) = \{z \in W \mid \text{ it is not the case that } Rvz\}$$

Now $v \nVdash \Diamond p$, so $u \nVdash \Box\Diamond p$. On the other hand, we have $w \Vdash p$, so $u \Vdash \Diamond p$

Example 3.2. Suppose that we are working with the basic temporal language and that we are interested in **dense** bidirectional frames. This property can be defined using a first-order sentence (namely $\forall xy(x < y \to \exists z(x < z \land z < y))$) but can the basic temporal language define it too?

The following simple formula suffices: $Fp \to FFp$. Let $\mathfrak{T} = (T, <)$ be a frame s.t. $\mathfrak{T} \Vdash Fp \to FFp$. Suppose that a point $t \in T$ has a <-successor t'. Consider the following **minimal** valuation V_m guaranteeing that $(\mathfrak{T}, V_m), t \Vdash Fp$

$$V_m(p) = \{t'\}$$

Hence $t \Vdash FFp$. This means there is a point s s.t. $t < s$ and $s \Vdash Fp$. But as t' is the **only** states where p holds, this implies that $s < t'$

Example 3.3. Suppose we are working with a similarity type with three binary operators $\triangle_1, \triangle_2, \triangle_3$, and that we are interested in the class of frames in which the three ternary accessibility relations (denoted by R_1, R_2, R_3 respectively, if R_1stu and $s \Vdash p, t \Vdash q, s \Vdash r$, then $s \Vdash q\triangle_1 r$).

We want

$$R_1stu \quad \text{iff} \quad R_2tus \quad \text{iff} \quad R_3ust$$

to hold for all s, t, u in such frames. Can we define this class of frames?

We can. We will show that for all frames $\mathcal{F} = (W, R_1, R_2, R_3)$ we have

$$\mathcal{F} \Vdash p \land (q\triangle_1 r) \to (q \land r\triangle_2 p)\triangle_1 r \quad \text{iff} \quad \mathcal{F} \Vdash \forall xyz(R_1xyz \to R_2yzx)$$

Suppose that the modal formula $p \land (q\triangle_1 r) \to (q \land r\triangle_2 p)\triangle_1 r$ is valid in $\mathcal{F}$, and consider states s, t, u with R_1stu. Consider a valuation V with $V(p) = \{s\}, V(q) = \{t\}, V(r) = \{u\}$. Then $(\mathcal{F}, V), s \Vdash p \land q\triangle_1 r$, so $s \Vdash (q \land r\triangle_2 p)\triangle_1 r$. Hence there must be states t', u' with $R_1st'u'$, $t' \Vdash q \land r\triangle_2 p$ and $u' \Vdash r$.

Exercise 3.1.1. Consider a language with two diamonds $\langle 1\rangle$ and $\langle 2\rangle$. Show that $p \to [2]\langle 1\rangle p$ is valid on precisely those frame for the language satisfy the condition $\forall xy(R_2xy \to R_1yx)$. What sort of frames does $p \to [1]\langle 1\rangle p$ define?

Proof. Define $V(p) = \{w\}$ to prove left to right.
$R_1xy \to R_1yx$ □

Exercise 3.1.2. Consider a language with three diamonds $\langle 1\rangle, \langle 2\rangle, \langle 3\rangle$. Show that the modal formula $\langle 3\rangle p \leftrightarrow \langle 1\rangle\langle 2\rangle p$ is valid on a frame for this language iff the frame satisfies the condition $\forall xy(R_3xy \to \exists z(R_1xz \land R_2zy))$

3.2 Frame Definability and Second-Order Logic

Example 3.4. Consider the Löb formula $\Box(\Box p \to p) \to \Box p$, which we will call it L for brevity. We show that L defines the class of frames (W, R) s.t. R is transitive and R's converse is well-founded

We will then show that this is a class of frames that first-order frame languages **cannot** define; that is, we will show that this class is not elementary

Assume that $\mathfrak{F} = (W, R)$ is a frame with a transitive and conversely well-founded relation, and then suppose that L is not valid in $\mathfrak{F}$. This means that there is a valuation V and a state w s.t. $(\mathfrak{F}, V), w \nVdash \Box(\Box p \to p) \to \Box p$. In other words, $w \Vdash \Box(\Box p \to p)$ but $w \nVdash \Box p$. Then w must have a successor w_1 s.t. $w_1 \nVdash p$, and as $w_1 \Vdash \Box p \to p$, we have $w_1 \nVdash \Box p$. This in turn implies that w_1 have a successor w_2 where p is false; note that by the transitivity of R, w_2 is also a successor of w. Again, w_2 must have a p-falsifying successor w_3 . Hence we find an infinite path $wRw_1Rw_2R \dots$ contradicting the converse well-foundedness of R

For the other direction, assume that either R is not transitive or its converse is not well-founded; in both cases we have to find a valuation V and a state w s.t. $(\mathfrak{F}, V), w \nVdash L$. Assume that R is transitive, but not conversely well-founded. In other words, suppose we have a transitive frame containing an infinite sequence $w_0Rw_1Rw_2R \dots$. Define

$$V(p) = W \setminus \{x \in W \mid \text{ there is an infinite path starting from } x\}$$

$\Box p \to p$ is true **everywhere** in the model, whence certainly, $(\mathfrak{F}, V), w_0 \Vdash \Box(\Box p \to p)$. The claim then follows from the fact that $(\mathfrak{F}, V), w_0 \nVdash \Box p$

Assume that R is not transitive and its converse is well-founded. Since R is not transitive, there is Rw_1w_2 and Rw_2w_3 but $\neg Rw_1w_3$. Let $V(p) = \{w_2, w_3\}$. Then $(\mathfrak{F}, V), w_1 \Vdash \Diamond p$ and $(\mathfrak{F}, V), w_1 \Vdash \neg\Diamond(p \wedge \neg\Diamond p)$

Finally, to show that the class of frames defined by L is not elementary, an easy compactness argument suffices. Suppose for the sake of a contradiction that there is a first-order formula equivalent to L; call this formula λ. As λ is equivalent to L, and model making λ true must be transitive. Let $\sigma_n(x_0, \dots, x_n)$ be the first-order formula stating that there is an R-path of length n through $x_0, \dots, x_n$:

$$\sigma_n(x_0, \dots, x_n) = \bigwedge_{0 \le i < n} Rx_ix_{i+1}$$

Every **finite** subset of

$$\Sigma = \{\lambda\}\{\forall xyz((Rxy \wedge Ryz) \to Rxz)\} \cup \{\sigma_n \mid n \in \omega\}$$

is satisfiable in a finite linear order, and hence in the class of transitive, conversely well-founded frames. Thus Σ must have a model. But it is clear that Σ is **not** satisfiable in any conversely well-founded frame.

Example 3.5. PDL can be interpreted on any transition system of the form $\mathcal{F} = (W, R_\pi)_{\pi \in \Pi}$. Let's call such a frame $*$-**proper** if the transition relation R_{π^*} of each program π^* is the reflexive and transitive closure of the transitive relation R_π of π.

Consider the following set of formulas

$$\Delta = \{[\pi]^*(p \to [\pi]p) \to (p \to [\pi^*]p), \langle \pi^* \rangle p \leftrightarrow (p \vee \langle \pi \rangle \langle \pi^* \rangle p) \mid \pi \in \Pi\}$$

called **Segerberg's axiom**, or the **induction axiom**. We claim that for any PDL-frame $\mathfrak{F}$

$$\mathfrak{F} \Vdash \Delta \quad \text{iff} \quad \mathfrak{F} \text{ is *-proper}$$

A consequences is that PDL is strong enough to define the class of regular frames. The constraints on the relations interpreting $\cup$ and ; are simple first-order conditions, and

$$\Gamma = \{\langle \pi_1 ; \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p, \langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle p \vee \langle \pi_2 \rangle p \mid \pi \in \Pi\}$$

pins down what is required. So $\Delta \cup \Gamma$ defines the regular frames

Example 3.6. We will show that the McKinsey formula (M) $\Box\Diamond p \to \Diamond\Box p$ does not correspond to a first-order condition by show that it violates the Löwenheim-Skolem Theorem

View the predicate symbol P that corresponds to the propostion letter p as a monadic second-order variable that we can quantify over

Proposition 3.6. *Let τ be a modal similarity type, and ϕ a τ-formula. Then for any τ-frames and any state w in $\mathfrak{F}$*

$$\begin{array}{lll} \mathfrak{F}, w \Vdash \phi & \textit{iff} & \mathfrak{F} \models \forall P_1 \ldots \forall P_n ST_x(\phi)[w] \\ \mathfrak{F} \Vdash \phi & \textit{iff} & \mathfrak{F} \models \forall P_1 \ldots \forall P_n \forall x \; ST_x(\phi) \end{array}$$

Here, the second-order quantifier bind second-order variables P_i corresponding to the proposition letters p_i occurring in ϕ

Proof. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be any model based on $\mathfrak{F}$ □

Exercise 3.2.1. 1. Consider a modal language with two diamonds $\langle 1 \rangle$ and $\langle 2 \rangle$. Prove that the class of frames in which R_1 is the reflexive transitive closure of R_2 is defined by the conjunction of the formulas $\langle 1 \rangle p \to (p \vee \langle 1 \rangle(\neg p \wedge \langle 2 \rangle p))$ and $\langle 1 \rangle p \leftrightarrow (p \vee \langle 2 \rangle \langle 1 \rangle p)$

Proof. 1. **Right to left**. $p \vee \langle 2 \rangle \langle 1 \rangle p \to \langle 1 \rangle p$ means $p \to \langle 1 \rangle p$ and $\langle 2 \rangle \langle 1 \rangle p \to \langle 1 \rangle p$. Hence the frame must be reflexive.

By reflexivity and $\langle 2 \rangle \langle 1 \rangle p \to \langle 1 \rangle p$, we can show that $R_2 xy \to R_1 xy$. Consider

$$\begin{tikzcd} x \Vdash \neg p \arrow[r, "R_2"] & y \Vdash p \arrow[loop right, "R_1"] \end{tikzcd}$$

and $V(p) = \{y\}$ .Since $x \Vdash \langle 2 \rangle \langle 1 \rangle p$, we have $x \Vdash \langle 1 \rangle p$. Hence we must have $R_1 xy$

Now suppose

$$x \xrightarrow{R_2} y \xrightarrow{R_1} z$$

and $V(p) = \{z\}$. Since $x \Vdash \langle 2 \rangle \langle 1 \rangle p$, we have $x \Vdash \langle 1 \rangle p$, which means

$$\begin{tikzcd} x \arrow[rr, "R_1"] \arrow[dr, "R_2"'] & & z \\ & y \arrow[ur, "R_1"'] & \end{tikzcd}$$

what's more, $R_2 xy$ implies $R_1 xy$. Hence we have $R_1 xy \wedge R_1 yz \to R_1 xz$ $\square$