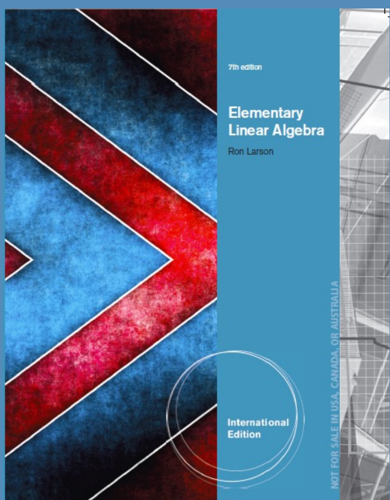
CHAPTER 4
VECTOR SPACES



CENGAGE Learning
Elementary Linear Algebra
R. Larson (7 Edition)

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4.7 Coordinates and Change of Basis

- Coordinate representation relative to a basis

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V and let $\mathbf{x}$ be a vector in V such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates of x relative to the basis *B***. The **coordinate matrix** (or **coordinate vector**) of $\mathbf{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\mathbf{x}$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Ex 1: (Coordinates and components in R^n)

Find the coordinate matrix of $\mathbf{x} = (-2, 1, 3)$ in R^3

relative to the standard basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

Sol:

$$\because \mathbf{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1),$$

$$\therefore [\mathbf{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

- Ex 3: (Finding a coordinate matrix relative to a nonstandard basis)

Find the coordinate matrix of $\mathbf{x}=(1, 2, -1)$ in R^3

relative to the (nonstandard) basis

$$B' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}=\{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

Sol: $\boldsymbol{x} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 \Rightarrow (1, 2, -1) = c_1(1, 0, 1) + c_2(0, -1, 2) + c_3(2, 3, -5)$

$$\Rightarrow \begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \text{i.e.} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{bmatrix} \xrightarrow{\text{G.J.E.}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

- Change of basis problem:

You were given the coordinates of a vector relative to one basis B and were asked to find the coordinates relative to another basis B'.

- Ex: (Change of basis)

Consider two bases for a vector space V

$$B = \{\mathbf{u}_1, \mathbf{u}_2\},\ B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$$

If $[\mathbf{u}'_1]_B = \begin{bmatrix} a \\ b \end{bmatrix},\ [\mathbf{u}'_2]_B = \begin{bmatrix} c \\ d \end{bmatrix}$

i.e., $\mathbf{u}'_1 = a\mathbf{u}_1 + b\mathbf{u}_2,\quad \mathbf{u}'_2 = c\mathbf{u}_1 + d\mathbf{u}_2$

Let $\mathbf{v} \in V$, $[\mathbf{v}]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

$$\begin{aligned} \Rightarrow \mathbf{v} &= k_1\mathbf{u}'_1 + k_2\mathbf{u}'_2 \\ &= k_1(a\mathbf{u}_1 + b\mathbf{u}_2) + k_2(c\mathbf{u}_1 + d\mathbf{u}_2) \\ &= (k_1a + k_2c)\mathbf{u}_1 + (k_1b + k_2d)\mathbf{u}_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow [\mathbf{v}]_B &= \begin{bmatrix} k_1a + k_2c \\ k_1b + k_2d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \\ &= \left[[\mathbf{u}'_1]_B \ [\mathbf{u}'_2]_B \right] [\mathbf{v}]_{B'} \end{aligned}$$

Lemma

- Transition matrix from *B*' to *B*:

Let $B=\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_n\}$ and $B'=\{\mathbf{u}'_1,\mathbf{u}'_2...,\mathbf{u}'_n\}$ be two bases for a vector space V

If $[\mathbf{v}]_B$ is the coordinate matrix of **v** relative to *B*

$[\mathbf{v}]_{B'}$ is the coordinate matrix of **v** relative to *B*'

then $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$

$$=\left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B\right] [v]_{B'}$$

where

$$P=\left[[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B\right]$$

is called **the transition matrix from *B*' to *B***

Proof of Lemma

- Let $v = d_1u'_1 + d_2u'_2 + \cdots + d_nu'_n$ be an arbitrary vector in V.
- The coordinate matrix of v with respect to the basis B' is

$$[v]_{B'} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

- If

$$u'_1 = c_{11}u_1 + \cdots + c_{n1}u_n$$
$$\vdots$$
$$u'_n = c_{1n}u_1 + \cdots + c_{nn}u_n$$

and

$$P = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

- Then you have

$$P[v]_{B'} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_{11}d_1 + \cdots + c_{1n}d_n \\ \vdots \\ c_{n1}d_1 + \cdots + c_{nn}d_n \end{bmatrix}$$

- On the other hand, you can write

$$v = d_1u'_1 + d_2u'_2 + \cdots + d_nv'_n$$

$$= d_1(c_{11}u_1 + \cdots + c_{n1}u_n) + \cdots + d_n(c_{1n}u_1 + \cdots + c_{nn}u_n)$$

$$= (c_{11}d_1 + \cdots + c_{1n}d_n)u_1 + \cdots + (c_{n1}d_1 + \cdots + c_{nn}d_n)u_n$$

- Thm 4.20: (The inverse of a transition matrix)

If P is the transition matrix from a basis B' to a basis B in R^n,

then

(1) P is invertible

(2) The transition matrix from B to B' is P^{-1}

- Notes:

$$B=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}, \qquad B'=\{\mathbf{u}'_1,\mathbf{u}'_2,\ldots,\mathbf{u}'_n\}$$

$$\left[\mathbf{v}\right]_B=\left[[\mathbf{u}'_1]_B,[\mathbf{u}'_2]_B,\ldots,[\mathbf{u}'_n]_B\right]\left[\mathbf{v}\right]_{B'}=P\left[\mathbf{v}\right]_{B'}$$

$$\left[\mathbf{v}\right]_{B'}=\left[[\mathbf{u}_1]_{B'},[\mathbf{u}_2]_{B'},\ldots,[\mathbf{u}_n]_{B'}\right]\left[\mathbf{v}\right]_B=P^{-1}\left[\mathbf{v}\right]_B$$

Proof of Theorem 4.20

- From the preceding lemma, let P be the transition matrix from B' to B.

- Then $[v]_B = P[v]_{B'}$ and $[v]_{B'} = Q[v]_B$, which implies that $[v]_B = PQ[v]_B$ for every vector v in R^n.

- From this it follows that $PQ = I$.

- So, P is invertible and P^{-1} is equal to Q, the transition matrix from B to B'.

- Thm 4.21: (Transition matrix from B to B')

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \vdots B]$ as follows.

$$[B' \vdots B] \longrightarrow [I_n \vdots P^{-1}]$$

Proof (***)

- To begin, let

$$v_1 = c_{11}u_1 + \cdots + c_{n1}u_n$$
$$\vdots$$
$$v_n = c_{1n}u_1 + \cdots + c_{nn}u_n$$

which implies (其中第i個式子)

$$c_{1i}\begin{bmatrix} u_{11} \\ \vdots \\ u_{n1} \end{bmatrix} + \cdots + c_{ni}\begin{bmatrix} u_{1n} \\ \vdots \\ u_{nn} \end{bmatrix} = \begin{bmatrix} v_{1i} \\ \vdots \\ v_{ni} \end{bmatrix}$$

- From these vector equations you can write the n systems of linear equations

$$u_{11}c_{1i} + \cdots + u_{1n}c_{ni} = v_{1i}$$
$$\vdots$$
$$u_{n1}c_{1i} + \cdots + u_{nn}c_{ni} = v_{ni}$$

- Because each of the n systems has the same coefficient matrix, you can reduce all n systems simultaneously using the following augmented matrix,

$$\begin{bmatrix} u_{11} & \cdots & u_{1n} & v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} & v_{n1} & \cdots & v_{nn} \end{bmatrix}$$

- Applying Gauss-Jordan elimination to this matrix produces

$$\begin{bmatrix} 1 & \cdots & 0 & c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

- By the lemma, the right-hand side of this matrix is $Q = P^{-1}$, which implies that the matrix has the form $[I \quad P^{-1}]$.

- Ex 5: (Finding a transition matrix)

$B=\{(-3, 2), (4,-2)\}$ and $B' =\{(-1, 2), (2,-2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B.

(b)Let $[\mathbf{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\mathbf{v}]_B$

(c) Find the transition matrix from B to B' .

Sol:

(a) $$\left[\begin{array}{cc:cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\qquad B \qquad\qquad B' \qquad\qquad\qquad I \qquad\quad P$$

$$\therefore P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \quad \text{(the transition matrix from } B' \text{ to } B\text{)}$$

(b)

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_B = P[\boldsymbol{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Check :

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{v} = (1)(-1,2) + (2)(2,-2) = (3,-2)$$

$$[\boldsymbol{v}]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = (-1)(3,-2) + (0)(4,-2) = (3,-2)$$

(c)

$$\left[\begin{array}{cc:cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G.J.E.}} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

$$B' \qquad\qquad B \qquad\qquad\qquad\qquad I \qquad\qquad P^{-1}$$

$$\therefore P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$$ (the transition matrix from B *to* B')

- Check:

$$PP^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- Ex 6: (Coordinate representation in $P_3(x)$)

(a) Find the coordinate matrix of $p = 3x^3-2x^2+4$ relative to the standard basis $S = \{1, x, x^2, x^3\}$ in $P_3(x)$.

(b) Find the coordinate matrix of $p = 3x^3-2x^2+4$ relative to the basis $S = \{1, 1+x, 1+x^2, 1+x^3\}$ in $P_3(x)$.

Sol:

$$\text{(a) } p = (4)(1) + (0)(x) + (\text{-}2)(x^2) + (3)(x^3) \quad \Rightarrow [p]_B = \begin{bmatrix} 4 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

$$\text{(b) } p = (3)(1) + (0)(1+x) + (\text{-}2)(1+x^2) + (3)(1+x^3) \quad \Rightarrow [p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 3 \end{bmatrix}$$

- Ex: (Coordinate representation in M_{2x2})

Find the coordinate matrix of $x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ relative to the standard basis in M_{2x2}.

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Sol:

$$x = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = 5\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 7\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow [x]_B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$$

Key Learning in Section 4.7

- Find a coordinate matrix relative to a basis in R^n
- Find the transition matrix from the basis to the basis B' in R^n.
- Represent coordinates in general n-dimensional spaces.

Keywords in Section 4.7

- coordinates of $\mathbf{x}$ relative to B: $\mathbf{x}$相對於B的座標
- coordinate matrix: 座標矩陣
- coordinate vector: 座標向量
- change of basis problem: 基底變換問題
- transition matrix from B' to B: 從 B' 到 B的轉移矩陣

Review exercises

61. Find the coordinate matrix of x in R^n relative to the basis B'.

$$B' = \{(1,2,3),(1,2,0),(0,-6,2)\}, \quad x = (3,-3,0)$$

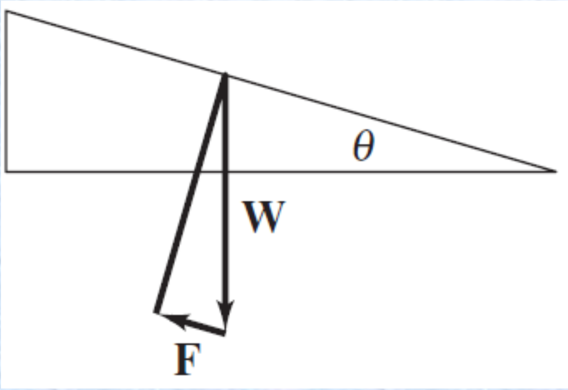
66. Find the transition matrix from B to B'.

$$B = \{(1,-1),(3,1)\}$$
$$B' = \{(1,2),(-1,0)\}$$

- Force

Vectors have a wide variety of applications in engineering and the physical sciences. For instance, to determine the amount of force required to pull an object up a ramp that has an angle of elevation θ, use the figure at the right.

In the figure, the vector labeled **W** represents the weight of the object, and the vector labeled **F** represents the required force. Using similar triangles and some trigonometry, the required force is $\mathbf{F} = \mathbf{W} \sin \theta$. Try verifying this.



In a mass-spring system, motion is assumed to occur in only the vertical direction. That is, the system has one *degree of freedom*. When the mass is pulled downward and then released, the system will oscillate. If the system is undamped, meaning that there are no forces present to slow or stop the oscillation, then the system will oscillate indefinitely. Applying Newton's Second Law of Motion to the mass yields the second order differential equation

$$x'' + \omega^2 x = 0$$

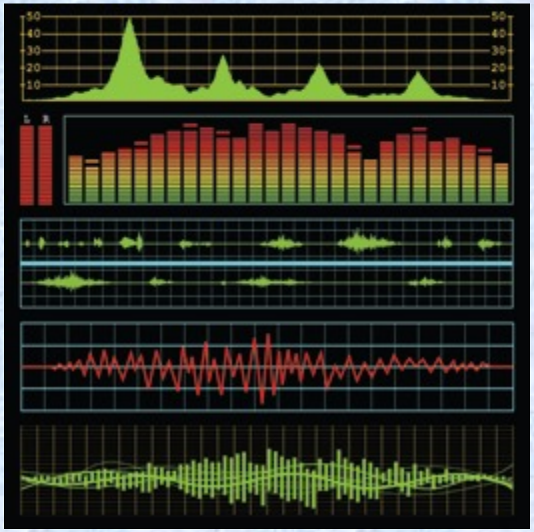
where x is the displacement at time t and ω is a fixed constant called the *natural frequency* of the system. The general solution of this differential equation is

$$x(t) = a_1 \sin \omega t + a_2 \cos \omega t$$

where a_1 and a_2 are arbitrary constants. (Try verifying this.) In Exercise 41, you are asked to show that the set of all functions $x(t)$ is a vector space.

4.3 Linear Algebra Applied

- Digital Sampling



Digital signal processing depends on sampling, which converts continuous signals into discrete sequences that can be used by digital devices. Traditionally, sampling is uniform and pointwise, and is obtained from a single vector space. Then, the resulting sequence is reconstructed into a continuous-domain signal. Such a process, however, can involve a significant reduction in information, which could result in a low-quality reconstructed signal. In applications such as radar, geophysics, and wireless communications, researchers have determined situations in which sampling from a union of vector subspaces can be more appropriate.

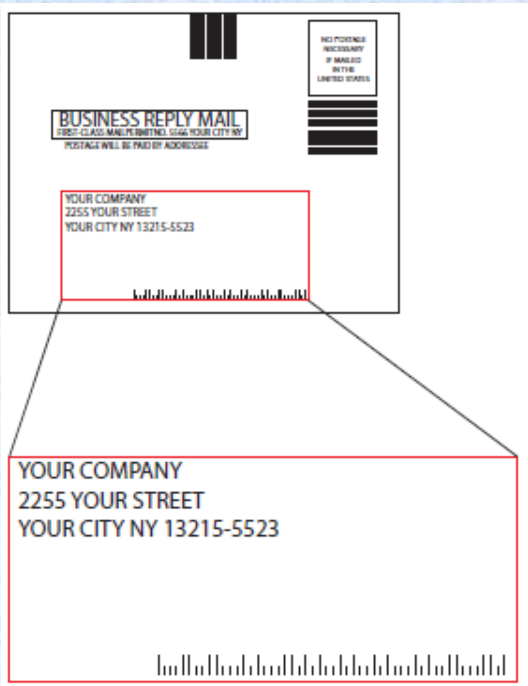
- Image Morphing

Image morphing is the process by which one image is transformed into another by generating a sequence of synthetic intermediate images. Morphing has a wide variety of applications, including movie special effects, wound healing and cosmetic surgery results simulation, and age progression software. Morphing an image makes use of a process called warping, in which a piece of an image is distorted. The mathematics behind warping and morphing can include forming a linear combination of the linearly independent vectors that bound a triangular piece of an image, and performing an *affine* transformation to form new vectors and an image piece that is distorted.

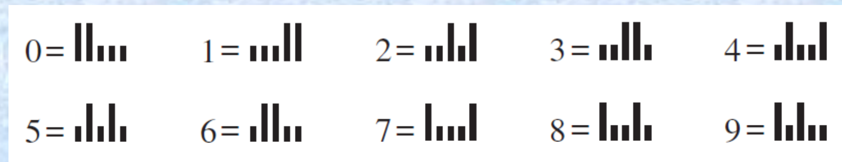
4.5 Linear Algebra Applied

The RGB color model uses the theory that all visible colors are combinations of the colors red (**r**), green (**g**), and blue (**b**), known as the primary additive colors. Using the standard basis for R^3, where $\mathbf{r} = (1, 0, 0)$, $\mathbf{g} = (0, 1, 0)$ and $\mathbf{b} = (0, 0, 1)$ any visible color can be represented as a linear combination $c_1\mathbf{r} + c_2\mathbf{g} + c_3\mathbf{b}$ of the primary additive colors. The coefficients c_i are values between 0 and a specified maximum a inclusive. When $c_1 = c_2 = c_3$ the color is *grayscale*, with $c_i = 0$ representing black and $c_i = a$ representing white. The RGB color model is commonly used in computer monitors, smart phones, televisions, and other electronic equipment.

4.6 Linear Algebra Applied



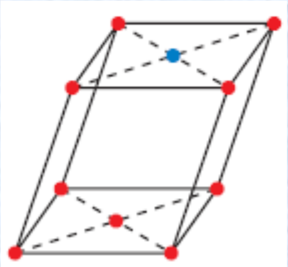
The U.S. Postal Service uses barcodes to represent such information as ZIP codes and delivery addresses. The ZIP + 4 barcode shown at the left starts with a long bar, has a series of short and long bars to represent each digit in the ZIP + 4 code and an additional digit for error checking, and ends with a long bar. The following is the code for the digits.



The error checking digit is such that when it is summed with the digits in the ZIP + 4 code, the result is a multiple of 10. (Verify this, as well as whether the ZIP + 4 code shown is coded correctly.) More sophisticated barcodes will also include error correcting digit(s). In an analogous way, matrices can be used to check for errors in transmitted messages. Information in the form of column vectors can be multiplied by an error detection matrix. When the resulting product is in the nullspace of the error detection matrix, no error in transmission exists. Otherwise, an error exists somewhere in the message. If the error detection matrix also has error correction, then the resulting matrix.

- Crystallography

Crystallography is the science of the forms and structures of crystals. In a crystal, atoms are in a repeating pattern called a lattice. The simplest repeating unit in a lattice is called a unit cell. Crystallographers can use bases and coordinate matrices in to designate the locations of atoms in a unit cell. For instance, the following figure shows the unit cell known as end-centered monoclinic.



The coordinate matrix for the top end-centered (blue) atom could be given as $[\mathbf{x}]_{B'} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T$.

- Satellite Dish

A satellite dish is an antenna that is designed to transmit or receive signals of a specific type. A standard satellite dish consists of a bowl-shaped surface and a feed horn that is aimed toward the surface. The bowl-shaped surface is typically in the shape of an elliptic paraboloid. (See Section 7.4.) The cross section of the surface is typically in the shape of a rotated parabola.