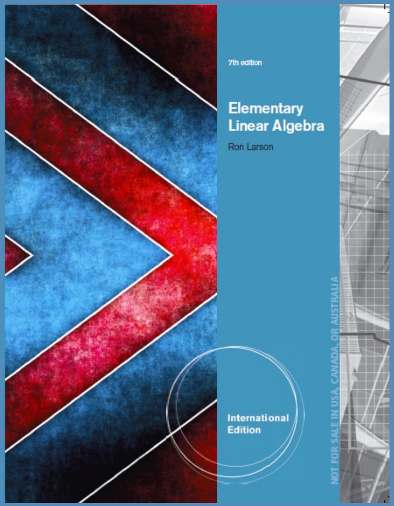
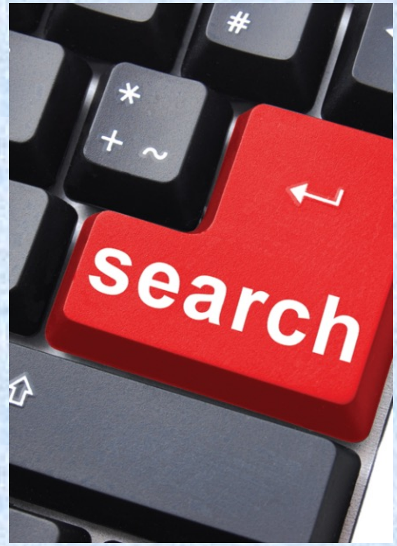
CHAPTER 2
MATRICES



CH 2 Linear Algebra Applied

Flight Crew Scheduling (p.47)



Information Retrieval (p.58)

Beam Deflection (p.64)

Computational Fluid Dynamics (p.79)

Data Encryption (p.87)

2.1 Operations with Matrices

- Matrix:

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \in M_{m\times n}$$

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m \times n$

- *i*-th row vector

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}$$

row matrix

- *j*-th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- Diagonal matrix:
  對角矩陣

$$A = diag(d_1, d_2, \cdots, d_n) = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n\times n}$$

- Trace: 跡數

If $A = [a_{ij}]_{n\times n}$

Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

$$\Rightarrow \quad r_1 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$$

$$\Rightarrow \quad c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \; c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- Equal matrix:

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

Then $A = B$ if and only if $a_{ij} = b_{ij}$ $\forall\, 1 \le i \le m,\ 1 \le j \le n$

- Ex 1: (Equal matrix)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$

Then $a = 1,\ b = 2,\ c = 3,\ d = 4$

- Matrix addition:

If $A=[a_{ij}]_{m\times n},\ B=[b_{ij}]_{m\times n}$

Then $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

- Ex 2: (Matrix addition)

$$\begin{bmatrix}-1 & 2\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1+1 & 2+3\\ 0-1 & 1+2\end{bmatrix}=\begin{bmatrix}0 & 5\\ -1 & 3\end{bmatrix}$$

$$\begin{bmatrix}1\\ -3\\ -2\end{bmatrix}+\begin{bmatrix}-1\\ 3\\ 2\end{bmatrix}=\begin{bmatrix}1-1\\ -3+3\\ -2+2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

- Scalar multiplication:

If $A=[a_{ij}]_{m\times n}$, c : scalar 純量

Then $cA=[ca_{ij}]_{m\times n}$

- Matrix subtraction:

$$A-B=A+(-1)B$$

- Ex 3: (Scalar multiplication and matrix subtraction)

$$A=\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (*a*) 3*A*, (*b*) –*B*, (*c*) 3*A* – *B*

Sol:

(a)

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication:

If $A = [a_{ij}]_{m\times n}, \ B = [b_{ij}]_{n\times p}$

Then $AB = [a_{ij}]_{m\times n}[b_{ij}]_{n\times p} = [c_{ij}]_{m\times p}$

Size of AB

where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1n} \\ b_{21} & \vdots & \boxed{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & \boxed{\vdots} & & \vdots \\ b_{n1} & \cdots & \boxed{b_{nj}} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ & & & & & \end{bmatrix}$$

- Notes: (1) $A+B = B+A$, (2) $AB \neq BA$

- Ex 4: (Find AB)

$$A=\begin{bmatrix}-1 & 3\\ 4 & -2\\ 5 & 0\end{bmatrix} \qquad B=\begin{bmatrix}-3 & 2\\ -4 & 1\end{bmatrix}$$

Sol:

$$AB=\begin{bmatrix}(-1)(-3)+(3)(-4) & (-1)(2)+(3)(1)\\ (4)(-3)+(-2)(-4) & (4)(2)+(-2)(1)\\ (5)(-3)+(0)(-4) & (5)(2)+(0)(1)\end{bmatrix}$$

$$=\begin{bmatrix}-9 & 1\\ -4 & 6\\ -15 & 10\end{bmatrix}$$

- Matrix form of a system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

m linear equations

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{\parallel \atop A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\parallel \atop x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\parallel \atop b}$$

$$\downarrow$$

Single matrix equation

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}$$

- Partitioned matrices:

submatrix

$$A=\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} \\ \hdashline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$$A=\left[\begin{array}{c:c:c:c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

- Linear combination of column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}_{m\times 1} = x_1 \underbrace{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}_{c_1} + x_2 \underbrace{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}_{c_2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{c_n}$$

- **Ex 7：(Solve a system of linear equations)**

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 0 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 3 \\ 7x_1 & + & 8x_2 & + & 9x_3 & = & 6 \end{array}$$

(infinitely many solutions)

$$\Rightarrow A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}, c_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, c_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, c_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\Rightarrow Ax = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = b$$

$$\Rightarrow 1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \quad (\text{one solution}: \ x = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \text{i.e.} \ x_1 = 1, x_2 = 1, x_3 = -1)$$

Key Learning in Section 2.1

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors.

Keywords in Section 2.1

- row vector: 列向量
- column vector: 行向量
- diagonal matrix: 對角矩陣
- trace: 跡數
- equality of matrices: 相等矩陣
- matrix addition: 矩陣相加
- scalar multiplication: 純量乘法(純量積)
- matrix subtraction: 矩陣相減
- matrix multiplication: 矩陣相乘
- partitioned matrix: 分割矩陣
- linear combination: 線性組合

2.2 Properties of Matrix Operations

- Three basic matrix operators:

  (1) matrix addition

  (2) scalar multiplication

  (3) matrix multiplication

- Zero matrix: $0_{m \times n}$

- Identity matrix of order *n*: I_n

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m\times n}$, c, d : scalar

Then (1) $A+B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $(cd)\, A = c\, (dA)$

(4) $1A = A$

(5) $c(A+B) = cA + cB$

(6) $(c+d)\, A = cA + dA$

- Properties of zero matrices:

If $A \in M_{m \times n}$, $\quad c : scalar$

Then (1) $A + 0_{m \times n} = A$

(2) $A + (-A) = 0_{m \times n}$

(3) $cA = 0_{m \times n} \Rightarrow c = 0 \; or \; A = 0_{m \times n}$

- Notes:

加法單位元素

(1) $0_{m \times n}$: **the additive identity** for the set of all $m \times n$ matrices

(2) $-A$: **the additive inverse** of A

加法反元素

- Properties of matrix multiplication:

(1) A(BC) = (AB)C

(2) A(B+C) = AB + AC

(3) (A+B)C = AC + BC

(4) c (AB) = (cA) B = A (cB)

- Properties of identity matrix:

If $A \in M_{m \times n}$

Then (1) $AI_n = A$

(2) $I_m A = A$

Proof of property 1

- For A(BC):

$$
\begin{aligned}
&a_{i1}\left(b_{11}c_{1j}+b_{12}c_{2j}+\cdots+b_{1n}c_{nj}\right)\\
&+a_{i2}\left(b_{21}c_{1j}+b_{22}c_{2j}+\cdots+b_{2n}c_{nj}\right)\\
&+\cdots\\
&+a_{in}\left(b_{n1}c_{1j}+b_{n2}c_{2j}+\cdots+b_{nn}c_{nj}\right)
\end{aligned}
$$

$$
\boxed{=\left(a_{i1}b_{11}+a_{i2}b_{21}+\cdots+a_{in}b_{n1}\right)c_{1j}}
$$

- For (AB)C

Proof of property 2

- For (A+B)C:

$$a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{in}(b_{nj} + c_{nj})$$

- For (AB+AC):

$$(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj})$$

Proof of property 3

- For (A+B)C:

$$(a_{i1} + b_{i1})c_{1j} + (a_{i2} + b_{i2})c_{2j} + \cdots + (a_{in} + b_{in})c_{nj}$$

- For (AC+BC):

$$\left(a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{in}c_{nj}\right) + \left(b_{i1}c_{1j} + b_{i2}c_{2j} + \cdots + b_{in}c_{nj}\right)$$

Proof of property 4

- For c(AB):

$$c(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj})$$

- For (cA)B:

$$(ca_{i1} \cdot b_{1j} + ca_{i2} \cdot b_{2j} + \cdots + ca_{in} \cdot b_{nj})$$

- For A(cB):

$$(a_{i1} \cdot cb_{1j} + a_{i2} \cdot cb_{2j} + \cdots + a_{in} \cdot cb_{nj})$$

- Transpose of a matrix:

$$\text{If } A = \begin{bmatrix} \boxed{a_{11} & a_{12} & \cdots & a_{1n}} \\ \boxed{a_{21} & a_{22} & \cdots & a_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{m1} & a_{m2} & \cdots & a_{mn}} \end{bmatrix} \in M_{m\times n}$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n\times m}$$

- Ex 8: (Find the transpose of the following matrix)

$$(a)\quad A=\begin{bmatrix}2\\8\end{bmatrix}\qquad (b)\quad A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\qquad (c)\quad A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}2\\8\end{bmatrix}\qquad \Rightarrow A^T=\begin{bmatrix}2&8\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\Rightarrow A^T=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$$

(c)

$$A=\begin{bmatrix}0&1\\2&4\\1&-1\end{bmatrix}\quad \Rightarrow A^T=\begin{bmatrix}0&2&1\\1&4&-1\end{bmatrix}$$

- Properties of transposes:

$$(1)\ \ (A^T)^T = A$$

$$(2)\ \ (A+B)^T = A^T + B^T$$

$$(3)\ \ (cA)^T = c(A^T)$$

$$(4)\ \ (AB)^T = B^T A^T$$

Proof of properties 1, 2, 3

1. $a_{ij} \to a_{ji} \to a_{ij}$

   $A \quad A^{\mathrm{T}} \quad (A^T)^T$

   $\Rightarrow \quad A = (A^T)^T$

2. $(a_{ij} + b_{ij}) \to (a_{ji} + b_{ji})$

   $A + B \qquad (A + B)^T$

   $\Rightarrow \quad (A + B)^T = A^T + B^T$

3. $ca_{ij} \to ca_{ji}$

   $cA \quad (cA)^T$

   $\Rightarrow \quad (cA)^T = cA^T$

Proof of property 4

4. For $(AB)^T$

$$\left(a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}\right)$$

For $B^T A^T$, the ith row of B^T is the ith column of B, i.e.

$$\begin{pmatrix} b_{1i} & \cdots & b_{ni} \end{pmatrix}$$

and the jth column of A^T is the jth row of A

$$\begin{pmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{pmatrix}$$

- Symmetric matrix:

  A square matrix A is **symmetric** if $A = A^T$

- Skew-symmetric matrix: 反對稱矩陣

  A square matrix A is **skew-symmetric** if $A^T = -A$

  歪、斜的

- Ex:

  If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix}$$

$$A = A^T$$
$$\Rightarrow a = 2, b = 3, c = 5$$

- Ex:

If $A=\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c?

Sol:

$$A=\begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix} \qquad -A^T=\begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A=-A^T \Rightarrow a=-1, b=-2, c=-3$$

- Note: AA^T is symmetric

Pf: $(AA^T)^T=(A^T)^T A^T = AA^T$

$\therefore AA^T$ is symmetric

- Real number:

$ab = ba$ (Commutative law for multiplication)

- Matrix: 交換律

$$\underset{m\times n}{A}\,\underset{n\times p}{B} \neq BA$$

Three situations:

If $m \neq p$, then AB is defined, BA is undefined.

If $m = p, m \neq n$, then $AB \in M_{m\times m}$, $BA \in M_{n\times n}$ (Sizes are not the same)

If $m = p = n$, then $AB \in M_{m\times m}$, $BA \in M_{m\times m}$

(Sizes are the same, but matrices are not equal)

- Ex 4:

Sow that AB and BA are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

- Note: $AB \neq BA$

- Real number:

$$ac = bc,\ c \neq 0$$

$$\Rightarrow\ a = b \qquad \text{(Cancellation law)}$$

- Matrix:

$$AC = BC \qquad C \neq 0$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex 5: (An example in which cancellation is not valid)

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\-1 & 2\end{bmatrix}$$

Sol:

$$AC=\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\-1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\-1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\-1 & 2\end{bmatrix}$$

So $\quad AC=BC$

But $\quad A\neq B$

Key Learning in Section 2.2

- Use the properties of matrix addition, scalar multiplication, and zero matrices.
- Use the properties of matrix multiplication and the identity matrix.
- Find the transpose of a matrix.

Keywords in Section 2.2

- zero matrix: 零矩陣
- identity matrix: 單位矩陣
- transpose matrix: 轉置矩陣
- symmetric matrix: 對稱矩陣
- skew-symmetric matrix: 反對稱矩陣

Review Exercises

- 4. Perform the matrix operation.

$$\begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 \\ 4 & 0 & 0 \end{bmatrix}$$

- 11. Find $A^T, A^T A, and\ AA^T$.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$