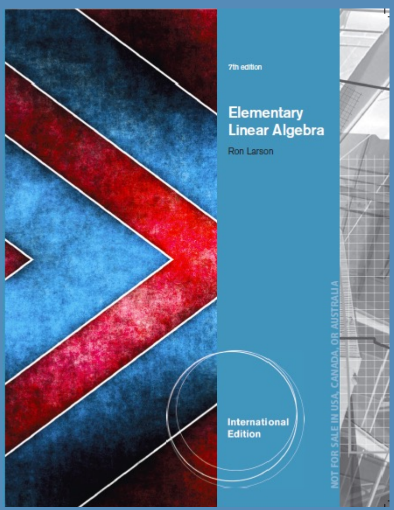
CHAPTER 2
MATRICES



Identity matrix

- $IA = A$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} & \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{nj} \end{matrix}} & \end{bmatrix} = \begin{bmatrix} & \boxed{\begin{matrix} a_{1j} \\ \vdots \\ a_{nj} \end{matrix}} & \end{bmatrix}$$

- $AI = A$

$$\begin{bmatrix} & & \\ \boxed{a_{i1} \quad \cdots \quad a_{in}} \\ & & \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = \begin{bmatrix} & & \\ \boxed{a_{i1} \quad \cdots \quad a_{in}} \\ & & \end{bmatrix}$$

Elementary matrix

- The elementary matrix for any row operation is obtained by executing the operation on the identity matrix.

- To apply the elementary row operation to a matrix A, one multiplies the elementary matrix on the left, i.e., EA.

2.3 The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n\times n}$

If there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- Note:

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- Thm 2.7: (The inverse of a matrix is unique)

If B and C are both inverses of the matrix A, then $B = C$.

Pf:
$$AB = I$$
$$C(AB) = CI$$
$$(CA)B = C$$
$$IB = C$$
$$B = C$$

Consequently, the inverse of a matrix is unique.

- Notes:

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$[A \mid I] \xrightarrow{\text{Gauss-Jordan Elimination}} [I \mid A^{-1}]$$

- Ex 2: (Find the inverse of the matrix)

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{rcrcr} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array} \quad (1)$$

$$\begin{array}{rcrcr} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array} \quad (2)$$

$$(1) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \quad (AX = I = AA^{-1})$$

- Note:

$$\begin{bmatrix} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{bmatrix} \xrightarrow[r_{12}^{(1)},\, r_{21}^{(-4)}]{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{bmatrix}$$

$$A \qquad I \qquad\qquad\qquad I \qquad A^{-1}$$

If A can't be row reduced to I, then A is singular.

- Ex 3: (Find the inverse of the following matrix)

$$A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ -6 & 2 & 3\end{bmatrix}$$

Sol:

$$\begin{bmatrix}A & \vdots & I\end{bmatrix}=\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ -6 & 2 & 3 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{r_{12}^{(-1)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ -6 & 2 & 3 & 0 & 0 & 1\end{array}\right]\xrightarrow{r_{13}^{(6)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & -4 & 3 & 6 & 0 & 1\end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & -1 & 2 & 4 & 1\end{array}\right]\xrightarrow{r_{3}^{(-1)}}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & -4 & -1\end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -1 & -4 & -1 \end{array}\right]$$

$$= [\, I \ \vdots \ A^{-1} \,]$$

So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

- Power of a square matrix:

$$(1)\, A^0 = I$$

$$(2)\, A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3)\, A^r \cdot A^s = A^{r+s} \qquad r, s : \text{integers}$$

$$(A^r)^s = A^{rs}$$

$$(4)\, D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- Thm 2.8： (Properties of inverse matrices)

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^{k} is invertible and $(A^{k})^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^{k} = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$

(4) A^{T} is invertible and $(A^{T})^{-1} = (A^{-1})^{T}$

Proof

1. $A(A^{-1}) = I_n$

2. $(A^{-1})^k A^k = (A^{-1})^{k-1} A^{-1} A A^{k-1} = (A^{-1})^{k-1} I_n A^{k-1} = (A^{-1})^{k-1} A^{k-1} = \cdots = I_n$

3. $\frac{1}{c} A^{-1}(cA) = A^{-1}A = I_n$

4. $(A^{-1})^T (A^T) = (AA^{-1})^T = I_n^T = I_n$

- Thm 2.9: (The inverse of a product)

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

If AB is invertible, then its inverse is unique.

So $(AB)^{-1} = B^{-1}A^{-1}$

- Note:

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Thm 2.10 (Cancellation properties)

If C is an invertible matrix, then the following properties hold:

(1) If $AC=BC$, then $A=B$ (Right cancellation property)

(2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf:

$$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1} \qquad \text{(C is invertible, so } C^{-1} \text{ exists)}$$

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

- Note:

If C is not invertible, then cancellation is not valid.

- Thm 2.11: (Systems of equations with unique solutions)

If A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution given by

$$x = A^{-1}b$$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad (\ A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- Note:

  For square systems (those having the same number of equations as variables), Theorem 2.11 can be used to determine whether the system has a unique solution.

  - Note:

$$Ax = b \qquad \text{(A is an invertible matrix)}$$

$$\left[A \mid b\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}b\right] = \left[I \mid A^{-1}b\right]$$

$$\left[A \mid b_1 \mid b_2 \mid \cdots \mid b_n\right] \xrightarrow{A^{-1}} \left[I \mid A^{-1}b_1 \mid \cdots \mid A^{-1}b_n\right]$$

Key Learning in Section 2.3

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

Keywords in Section 2.3

- inverse matrix: 反矩陣
- invertible: 可逆
- nonsingular: 非奇異
- noninvertible: 不可逆
- singular: 奇異
- power: 冪次

2.4 Elementary Matrices

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

  (1) $R_{ij} = r_{ij}(I)$ Interchange two rows.

  (2) $R_i^{(k)} = r_i^{(k)}(I)$ $(k \neq 0)$ Multiply a row by a nonzero constant.

  (3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ Add a multiple of a row to another row.

- Note:

  Only do <u>a single</u> elementary row operation.

- Ex 1: (Elementary matrices and nonelementary matrices)

$$(a)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (b)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (c)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Yes $(r_2^{(3)}(I_3))$

(b) No (not square)

(c) No (Row multiplication must be by a nonzero constant)

$$(d)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad (e)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad (f)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) Yes $(r_{23}(I_3))$

(e) Yes $(r_{12}^{(2)}(I_2))$

(f) No (Use two elementary row operations)

- Thm 2.12: (Representing elementary row operations)

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 2 & 1\\1 & -3 & 6\\3 & 2 & -1\end{bmatrix}=\begin{bmatrix}1 & -3 & 6\\0 & 2 & 1\\3 & 2 & -1\end{bmatrix}(r_{12}(A)=R_{12}A)$$

$$(b)\begin{bmatrix}1 & 0 & 0\\0 & \frac{1}{2} & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -4 & 1\\0 & 2 & 6 & -4\\0 & 1 & 3 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & -4 & 1\\0 & 1 & 3 & -2\\0 & 1 & 3 & 1\end{bmatrix}(r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\-2 & -2 & 3\\0 & 4 & 5\end{bmatrix}=\begin{bmatrix}1 & 0 & -1\\0 & -2 & 1\\0 & 4 & 5\end{bmatrix}(r_{12}^{(2)}(A)=R_{12}^{(5)}A)$$

- Ex 3: (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A=\begin{bmatrix}0 & 1 & 3 & 5\\ 1 & -3 & 0 & 2\\ 2 & -6 & 2 & 0\end{bmatrix}$$

Sol:

$$E_1=r_{12}(I_3)=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$E_2=r_{13}^{(-2)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}$$

$$E_3=r_3^{(\frac{1}{2})}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}$$

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} = B$$

row-echelon form

$$\therefore\ B = E_3 E_2 E_1 A \quad \text{or} \quad B = r_3^{(\frac{1}{2})}(r_{13}^{(-2)}(r_{12}(A)))$$

- Row-equivalent:

Matrix B is **row-equivalent** to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Thm 2.13: (Elementary matrices are invertible)

  If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

  (1) $(R_{ij})^{-1} = R_{ij}$

  (2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

  (3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex:

Elementary Matrix | Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \qquad (R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12} \text{ (Elementary Matrix)}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)} \qquad (R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)} \text{ (Elementary Matrix)}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})} \qquad (R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)} \text{ (Elementary Matrix)}$$

- Thm 2.14: (A property of invertible matrices)

  A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Pf: (1) Assume that A is the product of elementary matrices.

(a) Every elementary matrix is invertible.

(b) The product of invertible matrices is invertible.

Thus A is invertible.

(2) If A is invertible, $A\mathbf{x} = 0$ has only the trivial solution. (Thm. 2.11)

$$\Rightarrow [A \vdots 0] \to [I \vdots 0]$$

$$\Rightarrow E_k \cdots E_3 E_2 E_1 A = I$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

Thus A can be written as the product of elementary matrices.

- Ex 4:

Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2^{(\frac{1}{2})}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $\quad R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

Thus $A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$

$= R_1^{(-1)} R_{12}^{(3)} R_2^{(2)} R_{21}^{(2)}$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible

Then $E_k \cdots E_3 E_2 E_1 A = I$

$A^{-1} = E_k \cdots E_3 E_2 E_1$

$E_k \cdots E_3 E_2 E_1 [A \vdots I] = [I \vdots A^{-1}]$

- Thm 2.15: (Equivalent conditions)

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.

(5) A can be written as the product of elementary matrices.

- *LU*-factorization: 分解

  If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then A=LU is an LU-factorization of A

$$A = LU$$

L is a lower triangular matrix

U is an upper triangular matrix

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using <u>only the row operation of adding a multiple of one row to another row below it</u>, then it is easy to find an LU-factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU$$

- Ex 5: (LU-factorization)

$$(a)\ A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \qquad (b)\ A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{r_{23}^{(4)}} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = U$$

$$\Rightarrow R_{23}^{(4)} R_{13}^{(-2)} A = U$$

$$\Rightarrow A = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} U = LU$$

$$\Rightarrow L = (R_{13}^{(-2)})^{-1} (R_{23}^{(4)})^{-1} = R_{13}^{(2)} R_{23}^{(-4)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

- Solving $Ax=b$ with an LU-factorization of A

$$Ax = b \qquad \text{If } A = LU\text{, then } LUx = b$$

$$\text{Let } y = Ux\text{, then } Ly = b$$

- Two steps:

(1) Write $y = Ux$ and solve $Ly = b$ for y

(2) Solve $Ux = y$ for x

- Ex 7: (Solving a linear system using *LU*-factorization)

$$\begin{aligned} x_1 - 3x_2 &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $y = Ux$, and solve $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 \end{aligned}$$

$$= -20 - 2(-5) + 4(-1) = -14$$

(2) Solve the following system $Ux = y$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $\quad x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Key Learning in Section 2.4

- Factor a matrix into a product of elementary matrices.
- Find and use an LU-factorization of a matrix to solve a system of linear equations.

Keywords in Section 2.4

- row elementary matrix: 列基本矩陣
- row equivalent: 列等價
- lower triangular matrix: 下三角矩陣
- upper triangular matrix: 上三角矩陣
- *LU*-factorization: *LU*分解

Key Learning in Section 2.5

- Write and use a stochastic matrix.
- Use matrix multiplication to encode and decode messages.
- Use matrix algebra to analyze an economic system (Leontief input-output model).
- Find the least squares regression line for a set of data.

2.1 Linear Algebra Applied

- Fight Crew Scheduling

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and more than 12,750,000 columns. This application of linear programming required that the problem be partitioned into smaller pieces and then solved on a Cray supercomputer.

- Information Retrieval 資料檢索
  搜索引擎

Information retrieval systems such as Internet search engines make use of matrix theory and linear algebra to keep track of, for instance, keywords that occur in a database. To illustrate with a simplified example, suppose you wanted to perform a search on some of the m available keywords in a database of n documents. You could represent the search with the $m\times 1$ column matrix $\mathbf{x}$ in which a 1 entry represents a keyword you are searching and 0 represents a keyword you are not searching. You could represent the occurrences of the m keywords in the n documents with A, an $m \times n$ matrix in which an entry is 1 if the keyword occurs in the document and 0 if it does not occur in the document. Then, the $n\times 1$ matrix product $A^T\mathbf{x}$ would represent the number of keywords in your search that occur in each of the n documents.

- Beam Deflection

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Recall Hooke's law, which states that for relatively small deformations of an elastic object, the amount of deflection is directly proportional to the force causing the deformation. In a simply supported elastic beam subjected to multiple forces, deflection **d** is related to force **w** by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where is a flexibility matrix whose entries depend on the material of the beam. The inverse of the flexibility matrix, F^{-1} is called the *stiffness matrix*. In Exercises 61 and 62, you are asked to find the stiffness matrix F^{-1} and the force matrix **w** for a given set of flexibility and deflection matrices.

- Computational Fluid Dynamics

Computational fluid dynamics (CFD) is the computer-based analysis of such real-life phenomena as fluid flow, heat transfer, and chemical reactions. Solving the conservation of energy, mass, and momentum equations involved in a CFD analysis can involve large systems of linear equations. So, for efficiency in computing, CFD analyses often use matrix partitioning and -factorization in their algorithms. Aerospace companies such as Boeing and Airbus have used CFD analysis in aircraft design. For instance, engineers at Boeing used CFD analysis to simulate airflow around a virtual model of their 787 aircraft to help produce a faster and more efficient design than those of earlier Boeing aircraft.

- Data Encryption

Because of the heavy use of the Internet to conduct business, Internet security is of the utmost importance. If a malicious party should receive confidential information such as passwords, personal identification numbers, credit card numbers, social security numbers, bank account details, or corporate secrets, the effects can be damaging. To protect the confidentiality and integrity of such information, the most popular forms of Internet security use data encryption, the process of encoding information so that the only way to decode it, apart from a brute force "exhaustion attack," is to use a key. Data encryption technology uses algorithms based on the material presented here, but on a much more sophisticated level, to prevent malicious parties from discovering the key.

Review exercises

34. Find a sequence of elementary matrices whose product is the given nonsingular matrix.

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

45. Use an *LU*-factorization of the coefficient matrix to solve the linear system.

$$\begin{aligned} x + z &= 3 \\ 2x + y + 2z &= 7 \\ 3x + 2y + 6z &= 8 \end{aligned}$$