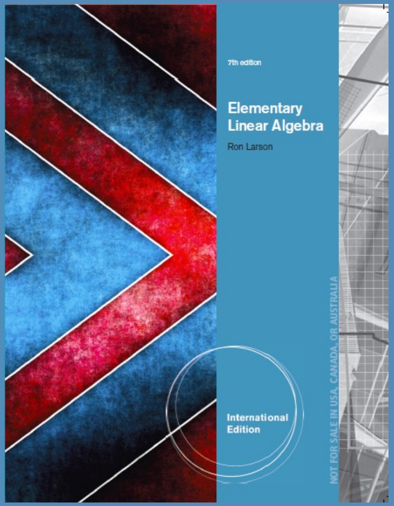
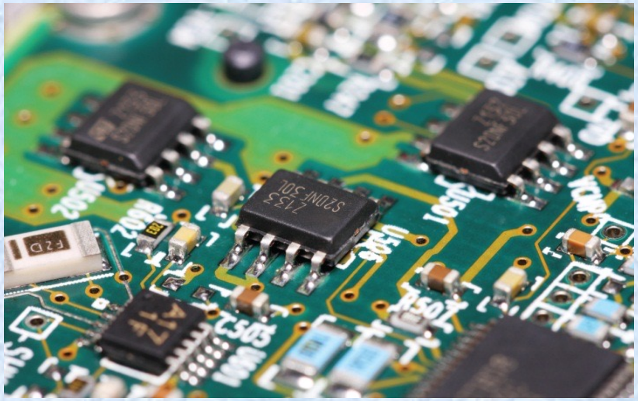
CHAPTER 6 LINEAR TRANSFORMATIONS



CH 6 Linear Algebra Applied

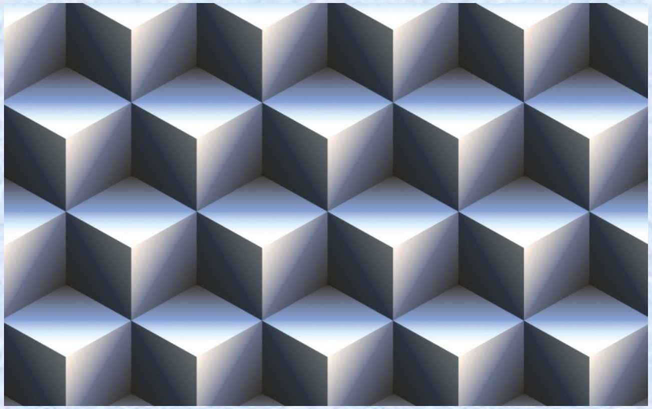
Multivariate Statistics (p.298)

Control Systems (p.308)



Circuit Design (p.316)

Weather (p.325)



Computer Graphics (p.332)

6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \to R^3$:

$$(1) T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2) T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- Thm 6.10: (Standard matrix for a linear transformation)

Let $T: R^n \to R^m$ be a linear trtansformation such that

$$T(e_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\quad T(e_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix},\quad \cdots,\quad T(e_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix whose n columns correspond to $T(e_i)$

$$A = \left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n)\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n.

A is called the standard matrix for T.

Pf:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

$$T \text{ is a L.T.} \Rightarrow T(\boldsymbol{v}) = T(v_1 e_1 + v_2 e_2 + \cdots + v_n e_n)$$
$$= T(v_1 e_1) + T(v_2 e_2) + \cdots + T(v_n e_n)$$
$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

$$= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

- Ex 1: (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T : R^3 \rightarrow R^2$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation | Matrix Notation

$$T(e_1) = T(1, 0, 0) = (1, 2) \qquad T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = T(0, 1, 0) = (-2, 1) \qquad T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(e_3) = T(0, 0, 1) = (0, 0) \qquad T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A=\left[T(e_1)\mid T(e_2)\mid T(e_3)\right]$$

$$=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}$$

- Check:

$$A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}x-2y\\ 2x+y\end{bmatrix}$$

$$\text{i.e. } T(x,y,z)=(x-2y, 2x+y)$$

- Note:

$$A=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}\begin{matrix}\leftarrow & 1x-2y+0z\\ \leftarrow & 2x+1y+0z\end{matrix}$$

- Ex 2: (Finding the standard matrix of a linear transformation)

The linear transformation $T: R^2 \to R^2$ is given by projecting each point in R^2 onto the x-axis. Find the standard matrix for T.

Sol:

$$T(x, y) = (x, 0)$$

$$A = \left[T(e_1) \mid T(e_2)\right] = \left[T(1, 0) \mid T(0, 1)\right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Notes:

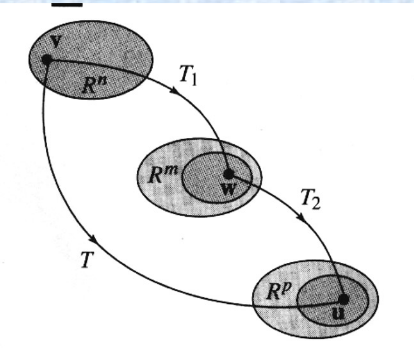
(1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

(2) The standard matrix for the identity transformation from R^n into R^n is I_n

- Composition of $T_1:R^n \to R^m$ with $T_2:R^m \to R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1, \quad \text{domain of } T = \text{domain of } T_1$$



Composition of Transformations

- Thm 6.11: (Composition of linear transformations)

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be L.T.
with standard matrices A_1 and A_2 , then

(1)The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L.T.

(2) The standard matrix A for T is given by the matrix product $A = A_2 A_1$

Pf:

(1)(T is a L.T.)

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n and let c be any scalar then

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1(\mathbf{v}))$$
$$= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = T(\mathbf{u}) + T(\mathbf{v})$$

$$T(c\mathbf{v}) = T_2(T_1(c\mathbf{v})) = T_2(cT_1(\mathbf{v})) = cT_2(T_1(\mathbf{v})) = cT(\mathbf{v})$$

(2)(A_2A_1 is the standard matrix for T)

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = T_2(A_1\mathbf{v}) = A_2A_1\mathbf{v} = (A_2A_1)\mathbf{v}$$

- Note:

$$T_1 \circ T_2 \neq T_2 \circ T_1$$

- Ex 3: (The standard matrix of a composition)

Let T_1 and T_2 be L.T. from R^3 into R^3 s.t.

$$T_1(x, y, z) = (2x + y, 0, x + z)$$

$$T_2(x, y, z) = (x - y, z, y)$$

Find the standard matrices for the compositions

$T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$,

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(standard matrix for } T_1\text{)}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(standard matrix for } T_2\text{)}$$

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Inverse linear transformation:

$$\text{If } T_1 : R^n \to R^n \text{ and } T_2 : R^n \to R^n \text{ are L.T. s.t. for every } \mathbf{v} \text{ in } R^n$$

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- Note:

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}.

- Thm 6.12: (Existence of an inverse transformation)

  Let $T: R^n \to R^n$ be a L.T. with standard matrix A,

  Then the following condition are equivalent.

  (1) T is invertible.

  (2) T is an isomorphism.

  (3) A is invertible.

- Note:

  If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

Proof of Theorem 6.12

- (1→2) Let T be invertible.
- If $T(v_1) = T(v_2)$, then $T^{-1}(T(v_1)) = T^{-1}(T(v_2))$ and $v_1 = v_2$, so T is one-to-one.
- T is onto because for any $w \in R^n$, $T^{-1}(w) = v$ satisfies $T(v) = w$.
- So, T is isomorphism.

Proof of Theorem 6.12

- (2→1) Let T be an isomorphism.
- Because T is onto, for any $w \in R^n$, there exists $v \in R^n$ such that $T(v) = w$.
- Because T is one-to-one, thus v is unique.
- So, define the inverse of T by $T^{-1}(w) = v$ if and only if $T(v) = w$.

Proof of Theorem 6.12

- (2 $\leftrightarrow$3) Finally, the corollaries to Theorems 6.3 and 6.4 show that 2 and 3 are equivalent.

  (The kernel of T is equal to the solution space of $Ax = 0$.)

  (The column space of A is equal to the range of T.)

- If T is invertible, $T(x) = Ax$ implies that $T^{-1}(T(x)) = x = A^{-1}(Ax)$ and the standard matrix of T^{-1} is A^{-1}.

- Ex 4: (Finding the inverse of a linear transformation)

The L.T. $T: R^3 \rightarrow R^3$ is defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3, 3x_1 + 3x_2 + x_3, 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{G.J.E}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 0 & 1 & 6 & -2 & -3\end{array}\right]=\left[I \mid A^{-1}\right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}

$$A^{-1}=\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 6 & -2 & -3\end{bmatrix}$$

$$T^{-1}(\mathbf{v})=A^{-1}\mathbf{v}=\begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 6 & -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-x_1+x_2\\ -x_1+x_3\\ 6x_1-2x_2-3x_3\end{bmatrix}$$

In other words,

$$T^{-1}(x_1,x_2,x_3)=(-x_1+x_2,-x_1+x_3,6x_1-2x_2-3x_3)$$

- the matrix of T relative to the bases B *and* B':

$$T : V \rightarrow W \qquad \text{(a L.T.)}$$

$$B = \{v_1, v_2, \cdots, v_n\} \qquad \text{(a basis for } V\text{)}$$

$$B' = \{w_1, w_2, \cdots, w_m\} \qquad \text{(a basis for } W\text{)}$$

Thus, the matrix of T relative to the bases B *and* B' is

$$A = \left[[T(v_1)]_{B'}, [T(v_2)]_{B'}, \cdots, [T(v_n)]_{B'} \right] \in M_{m \times n}$$

- Transformation matrix for nonstandard bases:

Let V and W be finite - dimensional vector spaces with basis B and B', respectively, where $B = \{v_1, v_2, \cdots, v_n\}$

If $T : V \to W$ is a L.T. s.t.

$$[T(v_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(v_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad [T(v_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the $m \times n$ matrix whose n columns correspond to $[T(v_i)]_{B'}$

$$A = \left[T(e_1) \mid T(e_2) \mid \cdots \mid T(e_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\left[T(\mathbf{v}) \right]_{B'} = A[\mathbf{v}]_B$ for every $\mathbf{v}$ in V.

- Ex 5: (Finding a matrix relative to nonstandard bases)

Let $T: R^2 \rightarrow R^2$ be a L.T. defined by

$$T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2)$$

Find the matrix of T relative to the basis

$B = \{(1, 2), (-1, 1)\}$ and $B' = \{(1, 0), (0, 1)\}$

Sol:

$$T(1, 2) = (3, 0) = 3(1, 0) + 0(0, 1)$$

$$T(-1, 1) = (0, -3) = 0(1, 0) - 3(0, 1)$$

$$\left[T(1, 2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad \left[T(-1, 1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

the matrix for T relative to B and B'

$$A = \left[\left[T(1, 2)\right]_{B'} \quad \left[T(1, 2)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex 6:

For the L.T. $T: R^2 \rightarrow R^2$ given in Example 5, use the matrix A to find $T(\mathbf{v})$, where $\mathbf{v} = (2, 1)$

Sol:

$$\mathbf{v} = (2, 1) = 1(1, 2) - 1(-1, 1) \qquad B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\mathbf{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\mathbf{v})]_{B'} = A[\mathbf{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\mathbf{v}) = 3(1, 0) + 3(0, 1) = (3, 3) \qquad B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$$

- Notes:

(1) In the special case where $V = W$ and $B = B'$,

the matrix A is called the matrix of T relative to the basis B

(2) $T : V \to V$: the identity transformation

$B = \{v_1, v_2, \cdots, v_n\}$: a basis for V

$\Rightarrow$ the matrix of T relative to the basis B

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

Key Learning in Section 6.3

- Find the standard matrix for a linear transformation.
- Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation.
- Find the matrix for a linear transformation relative to a nonstandard basis.

Review exercises

47. Find the inverse of the linear transformation

$$T: R^2 \to R^2, T(x, y) = (x, -y).$$

53. $T: R^2 \to R^3, \mathrm{T(x, y) = (-x, y, x + y), v = (0,1)}$,

$\mathrm{B} = \{(1,1), (1, -1)\}, B' = \{(0,1,0), (0,0,1), (1,0,0)\}$.

Find $T(v)$ by using

(a) the standard matrix and

(b) the matrix relative to B and B'

Keywords in Section 6.3

- standard matrix for T: T 的標準矩陣
- composition of linear transformations: 線性轉換的合成
- inverse linear transformation: 反線性轉換
- matrix of T relative to the bases B and B' : T對應於基底B到B'的矩陣
- matrix of T relative to the basis B: T對應於基底B的矩陣

6.4 Transition Matrices and Similarity

$T : V \to V$ (a L.T.)

$B = \{v_1, v_2, \cdots, v_n\}$ (a basis of V)

$B' = \{w_1, w_2, \cdots, w_n\}$ (a basis of V)

$A = [[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B]$ (matrix of T relative to B)

$A' = [[T(w_1)]_{B'}, [T(w_2)]_{B'}, \cdots, [T(w_n)]_{B'}]$ (matrix of T relative to B')

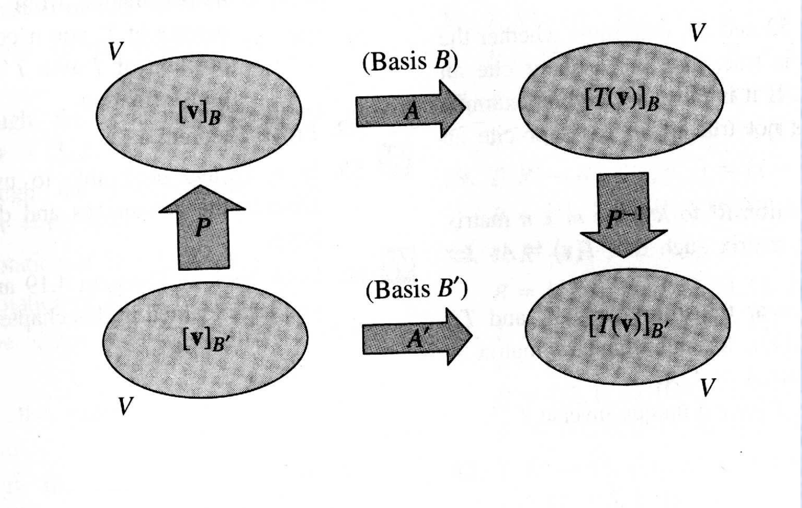
$P = [[w_1]_B, [w_2]_B, \cdots, [w_n]_B]$ (transition matrix from B' to B)

$P^{-1} = [[v_1]_{B'}, [v_2]_{B'}, \cdots, [v_n]_{B'}]$ (transition matrix from B to B')

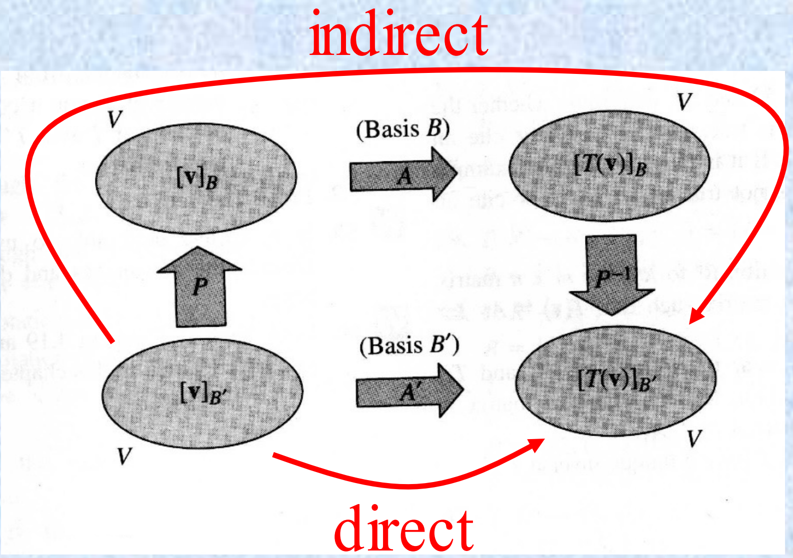
$\therefore [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \; [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$

$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$

$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$



- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:



(1)(direct)

$$A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

(2)(indirect)

$$P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$$

$$\Rightarrow A' = P^{-1}AP$$

- Ex 1: (Finding a matrix for a linear transformation)

Find the matrix A' for $T: R^2 \rightarrow R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

reletive to the basis $B' = \{(1, 0), (1, 1)\}$

Sol:

(I) $A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right]$

$$T(1,0) = (2, -1) = 3(1,0) - 1(1,1) \quad \Rightarrow \quad [T(1,0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1,1) = (0, 2) = -2(1,0) + 2(1,1) \quad \Rightarrow \quad [T(1,1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \left[[T(1,0)]_{B'} \quad [T(1,1)]_{B'} \right] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) standard matrix for T (matrix of T relative to $B = \{(1, 0), (0, 1)\}$)

$$A = \begin{bmatrix} T(1, 0) & T(0, 1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

transition matrix from B' to B

$$P = \begin{bmatrix} [(1, 0)]_B & [(1, 1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

transition matrix from B to B'

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

matrix of T relative B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- Ex 2: (Finding a matrix for a linear transformation)

Let $B=\{(-3,2),(4,-2)\}$ and $B'=\{(-1,2),(2,-2)\}$ be basis for R^2,

and let $A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to B.

Find the matrix of T relative to B'.

Sol:

transition matrix from B' to B: $P=\left[[(-1,2)]_B \quad [(2,-2)]_B\right]=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$

transition matrix from B to B': $P^{-1}=\left[[(-3,2)]_{B'} \quad [(4,-2)]_{B'}\right]=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}$

matrix of T relative to B':

$$A'=P^{-1}AP=\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix}\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}$$

- Ex 3: (Finding a matrix for a linear transformation)

For the linear transformation $T: R^2 \rightarrow R^2$ given in Ex.2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$ and $[T(\mathbf{v})]_{B'}$, for the vector $\mathbf{v}$ whose coordinate matrix is

$$[\mathbf{v}]_{B'} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$\text{or } [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

- Similar matrix:

  For square matrices A and A' of order n, A' is said to be similar to A if there exist an invertible matrix P *s.t.* $A' = P^{-1}AP$

- Thm 6.13: (Properties of similar matrices)

  Let A, B, and C be square matrices of order n.
  Then the following properties are true.
  (1) A is similar to A.
  (2) If A is similar to B, then B is similar to A.
  (3) If A is similar to B and B is similar to C, then A is similar to C.

Pf:

$$(1)\, A = I_n A I_n$$

$$(2)\, A = P^{-1}BP \;\Rightarrow\; PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$PAP^{-1} = B \;\Rightarrow\; Q^{-1}AQ = B \;\; (Q = P^{-1})$$

- Ex 4: (Similar matrices)

$$(a)\, A=\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \text{ and } A'=\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(b)\, A=\begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \text{ and } A'=\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$

- Ex 5: (A comparison of two matrices for a linear transformation)

Suppose $A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$ is the matrix for $T: R^3 \to R^3$ relative to the standard basis. Find the matrix for T relative to the basis

$$B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Sol:

The transition matrix from B' to the standard matrix

$$P=\left[[(1,1,0)]_B \quad [(1,-1,0)]_B \quad [(0,0,1)]_B\right]=\begin{bmatrix}1 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow P^{-1}=\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}$$

matrix of T relative to B':

$$A' = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes: Computational advantages of diagonal matrices:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(1) D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2) D^T = D$$

$$(3) D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}, \quad d_i \neq 0$$

Key Learning in Section 6.4

- Find and use a matrix for a linear transformation.
- Show that two matrices are similar and use the properties of similar matrices.

Keywords in Section 6.4

- matrix of T relative to B: T 相對於B的矩陣
- matrix of T relative to B' : T 相對於B'的矩陣
- transition matrix from B' to B : 從B'到B的轉移矩陣
- transition matrix from B to B' : 從B到B'的轉移矩陣
- similar matrix: 相似矩陣

Key Learning in Section 6.5

- Identify linear transformations defined by reflections, expansions,
- contractions, or shears in R^2.
- Use a linear transformation to rotate a figure in R^3.

- ## Multivariate Statistics

Many multivariate statistical methods can use linear transformations. For instance, in a *multiple regression analysis*, there are two or more independent variables and a single dependent variable. A linear transformation is useful for finding weights to be assigned to the independent variables to predict the value of the dependent variable. Also, in a *canonical correlation analysis*, there are two or more independent variables and two or more dependent variables. Linear transformations can help find a linear combination of the independent variables to predict the value of a linear combination of the dependent variables.

- ## Control Systems

A control system, such as the one shown for a dairy factory, processes an input signal $\mathbf{x}_k$ and produces an output signal $\mathbf{x}_{k+1}$. Without external feedback, the **difference equation**

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

a linear transformation where $\mathbf{x}_i$ is an $n \times 1$ vector and A is an $n \times n$ matrix, can model the relationship between the input and output signals. Typically, however, a control system has external feedback, so the relationship becomes

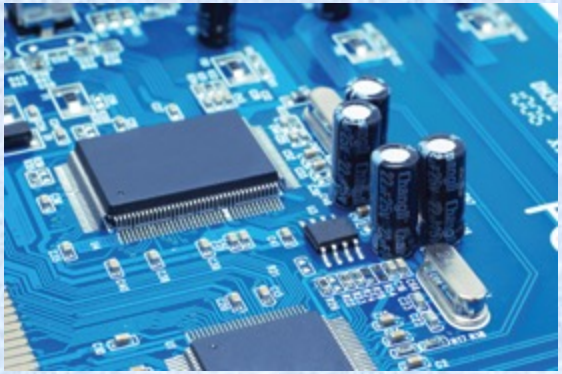
$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

where B is an $n \times m$ matrix and $\mathbf{u}_k$ is an $m \times 1$ input, or control, vector. A system is called *controllable* when it can reach any desired final state from its initial state in or fewer steps. If A and B make up a controllable system, then the rank of the *controllability matrix*

$$[B \quad AB \quad A^2B \quad . \ . \ . \quad A^{n-1}B]$$

is equal to n.

- Circuit Design



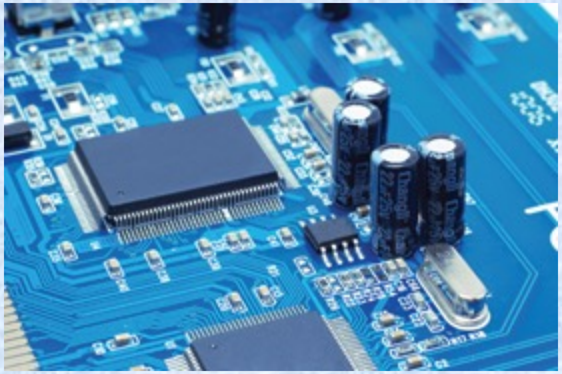
Ladder networks are useful tools for electrical engineers involved in circuit design. In a ladder network, the output voltage and current of one circuit are the input voltage and current of the circuit next to it. In the ladder network shown below, linear transformations can relate the input and output of an individual circuit (enclosed in a dashed box). Using Kirchhoff's Voltage and Current Laws and Ohm's Law,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/R_1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

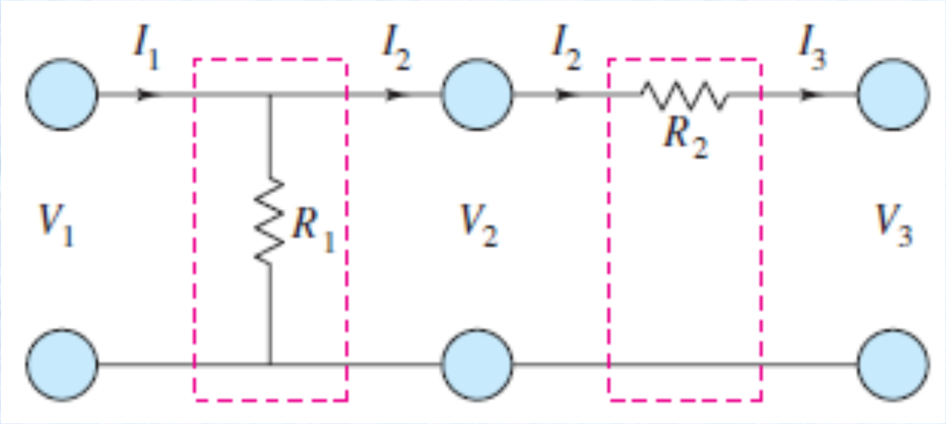
and

$$\begin{bmatrix} V_3 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & -1/R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

- Circuit Design



A composition can relate the input and output of the entire ladder network, that is, and to and to and Discussion on the composition of linear transformations begins on the following page.

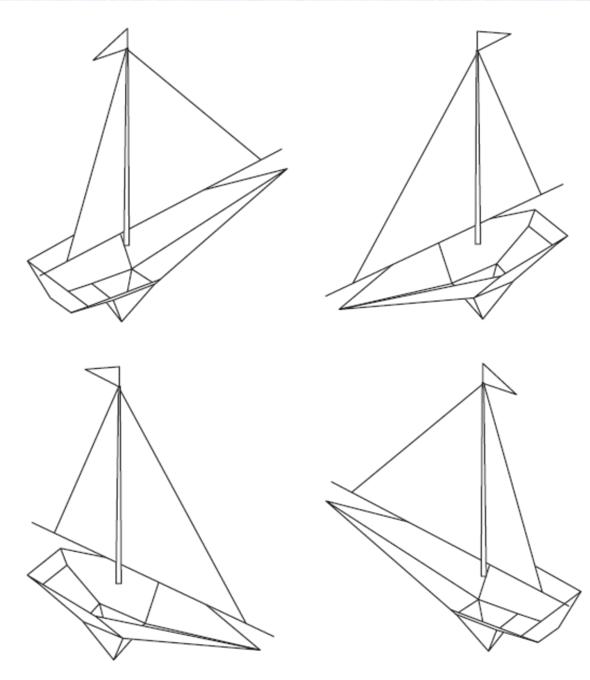


- Weather

In Section 2.5, you studied stochastic matrices and state matrices. A **Markov chain** is a sequence $\{\mathbf{x}_n\}$ of state matrices that are probability vectors related by the linear transformation $\mathbf{x}_{k+1} = P\mathbf{x}_k$, where P, the transition matrix from one state to the next, is a stochastic matrix $[p_{ij}]$. For instance, suppose that it has been established, through studying extensive weather records, that the probability p_{21} of a stormy day following a sunny day is 0.1 and the probability p_{22} of a stormy day following a stormy day is 0.2. The transition matrix can be written as

$$P = \begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.2 \end{bmatrix}.$$

- Computer Graphics



The use of computer graphics is common in many fields. By using graphics software, a designer can "see" an object before it is physically created. Linear transformations can be useful in computer graphics. To illustrate with a simplified example, only 23 points in R^3 make up the images of the toy boat shown in the figure at the left. Most graphics software can use such minimal information to generate views of an image from any perspective, as well as color, shade, and render as appropriate. Linear transformations, specifically those that produce rotations in R^3 can represent the different views. The remainder of this section discusses rotation in R^3.

Review exercise

55. Find the matrix A' for T relative to the basis B'.

$$T: R^2 \to R^2, T(x, y) = (x - 3y, y - x),$$
$$B' = \{(1, -1), (1,1)\}$$

57. Use the matrix P to show that the matrices A and A' are similar.

$$P = \begin{bmatrix} 3 & -5 \\ 1 & -4 \end{bmatrix}, \mathrm{A} = \begin{bmatrix} 18 & -19 \\ 11 & -12 \end{bmatrix}, A' = \begin{bmatrix} 5 & -3 \\ -4 & 1 \end{bmatrix}$$