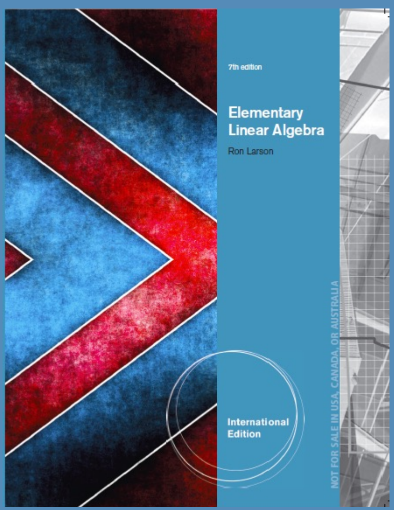
CHAPTER 7 EIGENVALUES AND EIGENVECTORS



7.2 Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP$ is **a diagonal matrix**.

  (P diagonalizes A)

- Notes:

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called **similar**.

  (2) The eigenvalue problem is related closely to the diagonalization problem.

- Thm 7.4: (Similar matrices have the same eigenvalues)

  If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
|\lambda \mathrm{I} - B| &= |\lambda \mathrm{I} - P^{-1}AP| = |P^{-1}\lambda \mathrm{I} P - P^{-1}AP| = |P^{-1}(\lambda \mathrm{I} - A)P| \\
&= |P^{-1}||\lambda \mathrm{I} - A||P| = |P^{-1}||P||\lambda \mathrm{I} - A| = |P^{-1}P||\lambda \mathrm{I} - A| \\
&= |\lambda \mathrm{I} - A|
\end{aligned}$$

A and B have the same characteristic polynomial.

Thus A and B have the same eigenvalues.

- Ex 1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{Eigenvalues}:\ \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\lambda=4\Rightarrow \text{Eigenvector}:\ p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}\ (\text{See p.403 Ex.5})$$

$$(2)\lambda = -2 \Rightarrow \text{Eigenvector}: p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ (See p.403 Ex.5)

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = [p_2 \quad p_1 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = [p_2 \quad p_3 \quad p_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Thm 7.5: (Condition for diagonalization)

An $n\times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Pf:

$(\Rightarrow) A$ is diagonalizable

there exists an invertible P s.t. $D = P^{-1}AP$ is diagonal

Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$ and $D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$PD = [p_1 \mid p_2 \mid \cdots \mid p_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n]$$

$$AP = A[p_1 \mid p_2 \mid \cdots \mid p_n] = [Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n]$$

$$\because AP = PD$$

$$\therefore Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$$

($i.e.$ the column vector p_i of P are eigenvectors of A)

$\because P$ is invertible $\Rightarrow p_1, p_2, \cdots, p_n$ are linearly independent.

$\therefore$ A has n linearly independent eigenvectors.

$(\Leftarrow)$ A has n linearly independent eigenvectors $p_1, p_2, \cdots p_n$

with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$

i.e. $Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$

Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

$$
\begin{aligned}
AP &= A[p_1 \mid p_2 \mid \cdots \mid p_n] \\
&= [Ap_1 \mid Ap_2 \mid \cdots \mid Ap_n] \\
&= [\lambda_1 p_1 \mid \lambda_2 p_2 \mid \cdots \mid \lambda_n p_n] \\
&= [p_1 \mid p_2 \mid \cdots \mid p_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD
\end{aligned}
$$

$\because\ p_1, p_1, \cdots, p_n$ are linearly independent $\Rightarrow P$ is invertible

$\therefore\ P^{-1}AP = D$

$\Rightarrow\ A$ is diagonalizable

Note: If n linearly independent vectors do not exist, then an $n\times n$ matrix A is not diagonalizable.

- Ex 4: (A matrix that is not diagonalizable)

Show that the following matrix is not diagonalizable.

$$A=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-1\end{vmatrix}=(\lambda-1)^2=0$$

Eigenvalue: $\lambda_1=1$

$$\lambda \mathrm{I}-A=I-A=\begin{bmatrix}0 & -2\\ 0 & 0\end{bmatrix}\sim\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\Rightarrow \text{Eigenvector}:\ p_1=\begin{bmatrix}1\\ 0\end{bmatrix}$$

A does not have two (n=2) linearly independent eigenvectors, so A is not diagonalizable.

- Steps for diagonalizing an $n\times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $p_1, p_2, \cdots, p_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$

Step 2: Let $P = [p_1 \mid p_2 \mid \cdots \mid p_n]$

Step 3:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \text{ where } Ap_i = \lambda_i p_i,\ i = 1, 2, \ldots, n$$

Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex 5: (Diagonalizing a matrix)

$$A=\begin{bmatrix}1 & -1 & -1\\ 1 & 3 & 1\\ -3 & 1 & -1\end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 1 & 1\\ -1 & \lambda-3 & -1\\ 3 & -1 & \lambda+1\end{vmatrix}=(\lambda-2)(\lambda+2)(\lambda-3)=0$$

$$\text{Eigenvalues}: \ \lambda_1=2,\ \lambda_2=-2,\ \lambda_3=3$$

$$\lambda_1 = 2$$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector}: \; p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2$$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} = \tfrac{1}{4}t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector}: \; p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$\lambda_3 = 3$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

$$(1)\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k=\begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D=P^{-1}AP$$

$$\begin{aligned} \Rightarrow\ D^k &= (P^{-1}AP)^k \\ &= (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP) \\ &= P^{-1}A(PP^{-1})A(PP^{-1})\cdots(PP^{-1})AP \\ &= P^{-1}AA\cdots AP \\ &= P^{-1}A^kP \end{aligned}$$

$$\therefore A^k=PD^kP^{-1}$$

- **Thm 7.6: (Sufficient conditions for diagonalization)**

  If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

Proof

- Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be n distinct eigenvalues of A corresponding to the eigenvectors $x_1, x_2, \cdots, x_n$.
- Assume the set of eigenvectors is linearly dependent.
- Moreover, consider the eigenvectors to be ordered so that the first m eigenvectors are linearly independent, but the first $m+1$ are dependent, where $m < n$.
- Then,

$$x_{m+1} = c_1 x_1 + c_2 x_2 + \cdots + c_m x_m, \qquad \text{Equation 1}$$

where the c_i's are not all zero.

(cont.)

- Multiplying both side by A yields

$$Ax_{m+1} = Ac_1x_1 + Ac_2x_2 + \cdots + Ac_mx_m.$$

- Because $Ax_i = \lambda_i x_i$,

Equation 2

$$\lambda_{m+1}x_{m+1} = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \cdots + c_m\lambda_mx_m.$$

- Multiplying Equation 1 by λ_{m+1} yields

Equation 3

$$\lambda_{m+1}x_{m+1} = c_1\lambda_{m+1}x_1 + c_2\lambda_{m+1}x_2 + \cdots + c_m\lambda_{m+1}x_m$$

- Subtract Equation 2 from Equation 3 produces

$$c_1(\lambda_{m+1} - \lambda_1)x_1 + \cdots + c_m(\lambda_{m+1} - \lambda_m)x_m = 0$$

(cont.)

- Because the first m eigenvectors are linearly independent, we have

$$c_1(\lambda_{m+1} - \lambda_1) = \cdots = c_m(\lambda_{m+1} - \lambda_m) = 0.$$

- Because all the eigenvalues are distinct, it follows that $c_i = 0$, $i = 1,2,\cdots,m$.

- But this results contradicts our assumption that x_{m+1} can be written as a linear combination of the first m eigenvectors.

- So, the set of eigenvectors is linearly independent, and from Theorem 7.5, A is diagonal.

- Ex 7: (Determining whether a matrix is diagonalizable)

$$A=\begin{bmatrix}1 & -2 & 1\\ 0 & 0 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol: Because A is a triangular matrix,

its eigenvalues are the main diagonal entries.

$$\lambda_1=1,\ \lambda_2=0,\ \lambda_3=-3$$

These three values are distinct, so A is diagonalizable. (Thm.7.6)

- Ex 8: (Finding a diagonalizing matrix for a linear transformation)

Let $T{:}R^3 \rightarrow R^3$ be the linear transformation given by

$$T(x_1,x_2,x_3) = (x_1 - x_2 - x_3,\ x_1 + 3x_2 + x_3,\ -3x_1 + x_2 - x_3)$$

Find a basis B for R^3 such that the matrix for T relative to B is diagonal.

Sol: The standard matrix for T is given by

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

From Ex. 5, there are three distinct eigenvalues

$$\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$$

so A is diagonalizable. (Thm. 7.6)

Thus, the three linearly independent eigenvectors found in Ex. 5

$$p_1 = (-1, 0, 1), p_2 = (1, -1, 4), p_3 = (-1, 1, 1)$$

can be used to form the basis B. That is

$$B = \{p_1, p_2, p_3\} = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}$$

The matrix for T relative to this basis is

$$\begin{aligned} D &= \left[[T(p_1)]_B \;\; [T(p_2)]_B \;\; [T(p_3)]_B \right] \\ &= \left[[Ap_1]_B \;\; [Ap_2]_B \;\; [Ap_3]_B \right] \\ &= \left[[\lambda_1 p_1]_B \;\; [\lambda_2 p_2]_B \;\; [\lambda_3 p_3]_B \right] \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

Key Learning in Section 7.2

- Find the eigenvalues of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.
- Find, for a linear transformation $T: V \rightarrow V$ a basis B for V such that the matrix T for B relative to is diagonal.

Keywords in Section 7.2

- diagonalization problem: 對角化問題
- diagonalization: 對角化
- diagonalizable matrix: 可對角化矩陣

7.3 Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix:

  A square matrix A is **symmetric** if it is equal to its transpose:

$$A = A^T$$

- Ex 1: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Thm 7.7: (Eigenvalues of symmetric matrices)

If A is an $n \times n$ symmetric matrix, then the following properties are true.

(1) A is diagonalizable.

(2) All eigenvalues of A are real.

(3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

Proof

- A proof of Theorem 7.7 is beyond the scope of this text.

- Ex 2:

Prove that a symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= \underline{(a-b)^2 + 4c^2} \geq 0\end{aligned}$$

$$(1)\ \ (a-b)^2+4c^2=0$$

$$\Rightarrow\ \ a=b,\ \ c=0$$

$$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$ is a matrix of diagonal.

$$(2)\ \ (a-b)^2+4c^2>0$$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. Thus, A is diagonalizable.

- Orthogonal matrix:

A square matrix P is called **orthogonal** if it is invertible and

$$P^{-1} = P^T$$

- Ex 4: (Orthogonal matrices)

$$\text{(a) } P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\text{(b) } P = \begin{bmatrix} \frac{3}{5} & 0 & \frac{-4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{-4}{5} & 0 & \frac{3}{5} \end{bmatrix}.$$

- Thm 7.8: (Properties of orthogonal matrices)

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthogonormal set.

Proof

- Suppose the column vectors of $\boldsymbol{P}$ form an orthonormal set:

$$\boldsymbol{P} = [\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \cdots \quad \boldsymbol{p}_n] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

- Then the product $P^T P$ has the form

$$\boldsymbol{P}^T\boldsymbol{P} = \begin{bmatrix} \boldsymbol{p}_1 \cdot \boldsymbol{p}_1 & \boldsymbol{p}_1 \cdot \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_1 \cdot \boldsymbol{p}_n \\ \boldsymbol{p}_2 \cdot \boldsymbol{p}_1 & \boldsymbol{p}_2 \cdot \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_2 \cdot \boldsymbol{p}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{p}_n \cdot \boldsymbol{p}_1 & \boldsymbol{p}_n \cdot \boldsymbol{p}_2 & \cdots & \boldsymbol{p}_n \cdot \boldsymbol{p}_n \end{bmatrix}$$

(cont.)

- Because the set $\{\boldsymbol{p}_1 \quad \boldsymbol{p}_2 \quad \cdots \quad \boldsymbol{p}_n\}$ is orthonormal, you have $\boldsymbol{p}_i \cdot \boldsymbol{p}_j = 0,\ i \neq j$ and $\boldsymbol{p}_i \cdot \boldsymbol{p}_i = \|\boldsymbol{p}_i\|^2 = 1.$

- So, the matrix composed of dot products has the form

$$\boldsymbol{P}^T\boldsymbol{P} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \boldsymbol{I}_n.$$

- This implies that $\boldsymbol{P}^T = \boldsymbol{P}^{-1}$, so $\boldsymbol{P}$ is orthogonal.

- Ex 5: (An orthogonal matrix)

$$P=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1}=P^T \Rightarrow PP^T=\mathrm{I}$

$$PP^T=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}}\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}}\\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}}\\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathrm{I}$$

$$\text{Let } p_1=\begin{bmatrix}\frac{1}{3}\\ \frac{-2}{\sqrt{5}}\\ \frac{-2}{3\sqrt{5}}\end{bmatrix},\ p_2=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{\sqrt{5}}\\ \frac{-4}{3\sqrt{5}}\end{bmatrix},\ p_3=\begin{bmatrix}\frac{2}{3}\\ 0\\ \frac{5}{3\sqrt{5}}\end{bmatrix}$$

produces

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$$

$$\|p_1\| = \|p_2\| = \|p_3\| = 1$$

$\{p_1, p_2, p_3\}$ is an orthonormal set.

- Thm 7.9: (Properties of symmetric matrices)

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors x_1 and x_2 are orthogonal.

Proof

- Let λ_1 and λ_2 be distinct eigenvalues of $\boldsymbol{A}$ with corresponding eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

- So, $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$.

- $\because \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = \boldsymbol{x}_1^T \boldsymbol{x}_2$.

$$
\begin{aligned}
\lambda_1(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2) &= (\lambda_1 \boldsymbol{x}_1) \cdot \boldsymbol{x}_2 = (\boldsymbol{A}\boldsymbol{x}_1) \cdot \boldsymbol{x}_2 = (\boldsymbol{A}\boldsymbol{x}_1)^T \boldsymbol{x}_2 \\
&= (\boldsymbol{x}_1^T \boldsymbol{A}^T)\boldsymbol{x}_2 = (\boldsymbol{x}_1^T \boldsymbol{A})\boldsymbol{x}_2 = \boldsymbol{x}_1^T(\boldsymbol{A}\boldsymbol{x}_2) \\
&= \boldsymbol{x}_1^T(\lambda_2 \boldsymbol{x}_2) = \boldsymbol{x}_1 \cdot (\lambda_2 \boldsymbol{x}_2) = \lambda_2(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2)
\end{aligned}
$$

- This implies that $(\lambda_1 - \lambda_2)(\boldsymbol{x}_1 \cdot \boldsymbol{x}_2) = 0$.

- Because $\lambda_1 \neq \lambda_2$, $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = 0$.

- Ex 6: (Eigenvectors of a symmetric matrix)

Show that any two eigenvectors of $A=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ corresponding to distinct eigenvalues are orthogonal.

- Sol: Characteristic function

$$|\lambda I-A|=\begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{vmatrix}=\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0$$

$$\Rightarrow \text{Eigenvalues: } \lambda_1=2, \lambda_2=4$$

$$(1)\ \lambda_1=2\Rightarrow \lambda_1 I-A=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\quad\Rightarrow \mathrm{x}_1=s\begin{bmatrix}-1\\ 1\end{bmatrix},\ s\neq 0$$

$$(2)\ \lambda_2=4\Rightarrow \lambda_2 I-A=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\quad\Rightarrow \mathrm{x}_2=t\begin{bmatrix}1\\ 1\end{bmatrix},\ t\neq 0$$

$$\mathrm{x}_1\cdot\mathrm{x}_2=\begin{bmatrix}-s\\ s\end{bmatrix}\cdot\begin{bmatrix}t\\ t\end{bmatrix}=st-st=0\ \Rightarrow\ \mathrm{x}_1 \text{ and } \mathrm{x}_2 \text{ are orthogonal.}$$

- **Thm 7.10: (Fundamental theorem of symmetric matrices)**

  Let A be an $n{\times}n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalue if and only if A is symmetric.

- **Orthogonal diagonalization of a symmetric matrix:**

  Let A be an $n{\times}n$ symmetric matrix.

  (1) Find all eigenvalues of A and determine the multiplicity of each.

  (2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

  (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

  (4) The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

Proof

- (→) Assume A is orthogonally diagonalizable, then there exists an orthogonal matrix P such that $D = P^{-1}AP$.

- Moreover, because $P^{-1} = P^T$, you have
$$A = PDP^{-1} = PDP^T$$

- This implies that
$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

- So, A is symmetric.

(cont.)

- Assume A is symmetric.
- If A has an eigenvalue λ of multiplicity k, then by Theorem 7.7, λ has k linearly independent eigenvectors.
- Through the Gram-Schmidt orthonormalization process, use this set of k vectors to form an orthonormal basis of eigenvectors for the eigenspace corresponding to λ.
- The collection of all resulting eigenvectors is orthonormal by Theorem 7.9, and you know from the orthonormalization process that the collection is also orthonormal.
- By Theorem 7.8, P is an orthogonal matrix.

(cont.)

- Finally, by Theorem 7.5, you can conclude that $P^{-1}AP$ is diagonal.

- So, A is orthogonally diagonalizable.

- Ex 7: (Determining whether a matrix is orthogonally diagonalizable)

	Symmetric matrix	Orthogonally diagonalizable
$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	○	○
$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$	×	×
$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	×	×
$A_4 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$	○	○

- Ex 9: (Orthogonal diagonalization)

Find an orthogonal matrix P that diagonalizes A.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$

$\lambda_1 = -6, \lambda_2 = 3$ (has a multiplicity of 2)

(2) $\lambda_1 = -6, \quad v_1 = (1, -2, 2) \Rightarrow u_1 = \dfrac{v_1}{\|v_1\|} = (\frac{1}{3}, \frac{-2}{3}, \frac{2}{3})$

(3) $\lambda_2 = 3, \quad v_2 = (2, 1, 0), \; v_3 = (-2, 0, 1)$

Linear Independent

Gram-Schmidt Process:

$$w_2 = v_2 = (2, 1, 0), \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (\tfrac{-2}{5}, \tfrac{4}{5}, 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = (\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0), \quad u_3 = \frac{w_3}{\|w_3\|} = (\tfrac{-2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}})$$

$$(4)\ P = [p_1\ p_2\ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$\Rightarrow \quad P^{-1}AP = P^T AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Key Learning in Section 7.3

- Find the eigenvalues of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.
- Find, for a linear transformation $T: V \rightarrow V$ a basis B for V such that the matrix T for B relative to is diagonal.

Keywords in Section 7.3

- symmetric matrix: 對稱矩陣
- orthogonal matrix: 正交矩陣
- orthonormal set: 單範正交集
- orthogonal diagonalization: 正交對角化

Key Learning in Section 7.4

- Recognize, and apply properties of, symmetric matrices.
- Recognize, and apply properties of, orthogonal matrices.
- Find an orthogonal matrix that orthogonally diagonalizes a symmetric matrix A.

- Diffusion

Eigenvalues and eigenvectors are useful for modeling real-life phenomena. For instance, suppose that in an experiment to determine the diffusion of a fluid from one flask to another through a permeable membrane and then out of the second flask, researchers determine that the flow rate between flasks is twice the volume of fluid in the first flask and the flow rate out of the second flask is three times the volume of fluid in the second flask. The following system of linear differential equations, where represents the volume of fluid in flask models this situation.

$$y_1' = -2y_1$$
$$y_2' = \quad 2y_1 - 3y_2$$

In Section 7.4, you will use eigenvalues and eigenvectors to solve such systems of linear differential equations. For now, verify that the solution of this system is

$$y_1 = C_1 e^{-2t}$$
$$y_2 = C_1 e^{-2t} + C_2 e^{-3t} .$$

Genetics

Genetics is the science of heredity. A mixture of chemistry and biology, genetics attempts to explain hereditary evolution and gene movement between generations based on the deoxyribonucleic acid (DNA) of a species. Research in the area of genetics called population genetics, which focuses on genetic structures of specific populations, is especially popular today. Such research has led to a better understanding of the types of genetic inheritance. For instance, in humans, one type of genetic inheritance is called *X–linked inheritance* (or *sex-linked inheritance*), which refers to recessive genes on the X chromosome. Males have one X and one Y chromosome, and females have two X chromosomes. If a male has a defective gene on the X chromosome, then its corresponding trait will be expressed because there is not a normal gene on the Y chromosome to suppress its activity. With females, the trait will not be expressed unless it is present on both X chromosomes, which is rare. This is why inherited diseases or conditions are usually found in males, hence the term sex-linked inheritance. Some of these include hemophilia A, Duchenne muscular dystrophy, red-green color blindness, and hereditary baldness. Matrix eigenvalues and diagonalization can be useful for coming up with mathematical models to describe X-linked inheritance in a given population.

- Relative Maxima and Minima

The *Hessian matrix* is a symmetric matrix that can be helpful in finding relative maxima and minima of functions of several variables. For a function f of two variables x and y—that is, a surface in R^3 —the Hessian matrix has the form

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The determinant of this matrix, evaluated at a point for which f_x and f_y are zero, is the expression used in the Second Partials Test for relative extrema.

- Architecture

Some of the world's most unusual architecture makes use of quadric surfaces. For instance, Catedral Metropolitana Nossa Senhora Aparecida, a cathedral located in Brasilia, Brazil, is in the shape of a hyperboloid of one sheet. It was designed by Pritzker Prize winning architect Oscar Niemeyer, and dedicated in 1970. The sixteen identical curved steel columns, weighing 90 tons each, are intended to represent two hands reaching up to the sky. Pieced together between the columns, in the 10-meter-wide and 30-meter-high triangular gaps formed by the columns, is semitransparent stained glass, which allows light inside for nearly the entire height of the columns.