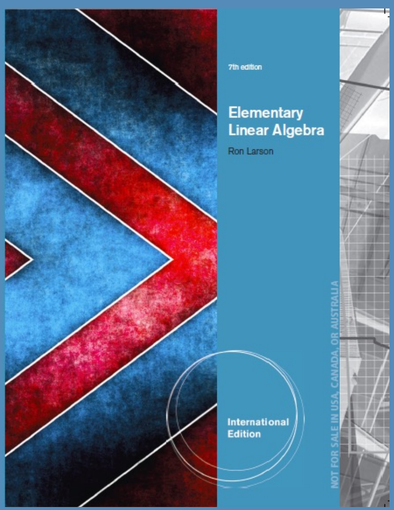
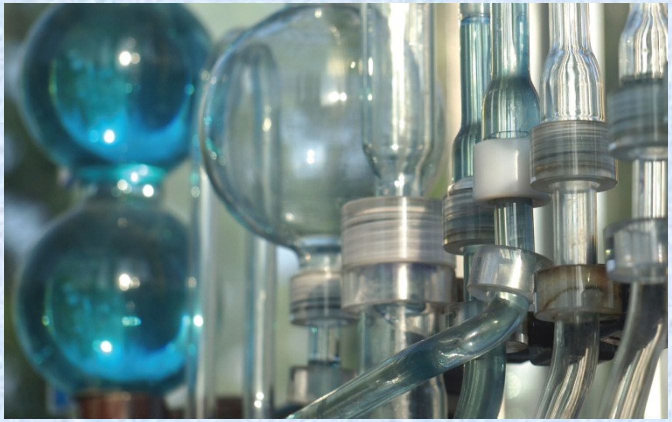
CHAPTER 7 EIGENVALUES AND EIGENVECTORS

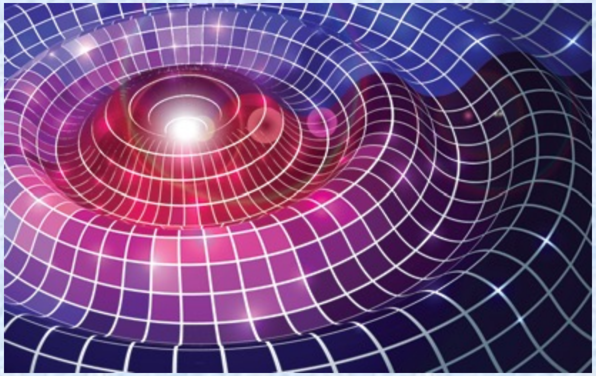


CH 7 Inner Product Space



Diffusion (p.348)

Genetics (p.359)



Relative Maxima and Minima (p.369)

Population of Rabbits (p.373)

Architecture (p.382)

7.1 Eigenvalues and Eigenvectors

- Eigenvalue problem:

  If A is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$？

- Eigenvalue and eigenvector:

  A：an $n \times n$ matrix

  λ：a scalar

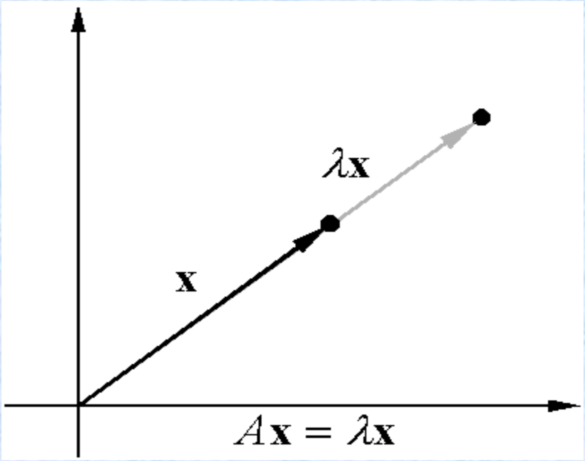
  $\boldsymbol{x}$： a nonzero vector in R^n

  Eigenvalue

  $$Ax = \lambda x$$

  Eigenvector

- Geometrical Interpretation



- Ex 1: (Verifying eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue

$$Ax_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2x_1$$

Eigenvector

Eigenvalue

$$Ax_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1)x_2$$

Eigenvector

- Thm 7.1: (The eigenspace of A corresponding to λ)

  If A is an $n\times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called **the eigenspace of λ** .

Pf:

x_1 and x_2 are eigenvectors corresponding to λ

$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$

(1) $A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$

$(i.e.\ \ x_1 + x_2$ is an eigenvector corresponding to $\lambda)$

(2) $A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$

$(i.e.\ \ cx_1$ is an eigenvector corresponding to $\lambda)$

- Ex 3: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\quad \mathrm{v} = (x, y)$

$$A\mathrm{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

For a vector on the x-axis

Eigenvalue $\lambda_1 = -1$

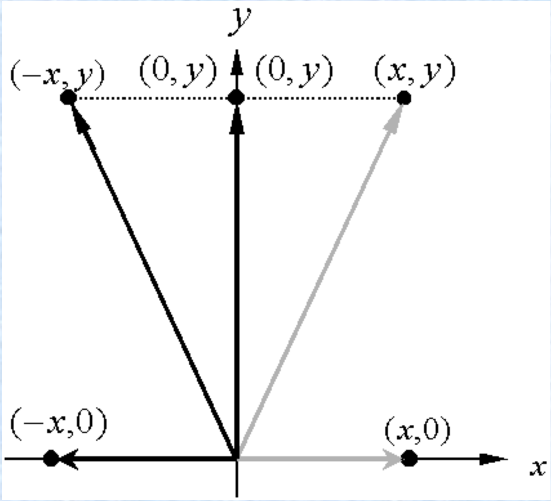
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$

For a vector on the *y*-axis

Eigenvalue $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the *y*-axis.



The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

- Thm 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

  Let A is an $n\times n$ matrix.

  (1) An eigenvalue of A is a scalar λ such that $\det(\lambda \mathrm{I} - A) = 0$.

  (2) The eigenvectors of A corresponding to λ are <u>the nonzero solutions</u> of $(\lambda \mathrm{I} - A)x = 0$.

- Note:

  $Ax = \lambda x \;\;\Rightarrow\;\; (\lambda \mathrm{I} - A)x = 0$ (homogeneous system)

  If $(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$.

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda \mathrm{I} - A) = \left|(\lambda \mathrm{I} - A)\right| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic equation of A:

$$\det(\lambda \mathrm{I} - A) = 0$$

- Ex 4: (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$\det(\lambda \mathrm{I} - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix}$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

$$\text{Eigenvalues}: \ \lambda_1 = -1, \lambda_2 = -2$$

$$(1)\lambda_1 = -1 \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix},\ t \neq 0$$

$$(2)\lambda_2 = -2 \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ t \neq 0$$

Check : $Ax = \lambda_i x$

- Ex 5: (Finding eigenvalues and eigenvectors)

  Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-2 & -1 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & \lambda-2\end{vmatrix}=(\lambda-2)^3=0$$

Eigenvalue: $\lambda=2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda \mathrm{I} - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\because \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; s, t \neq 0$$

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Notes:

(1) If an eigenvalue λ_1 occurs as a multiple root (*k times*) for the characteristic polynominal, then λ_1 has multiplicity *k*.

(2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex 6：Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda\mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

$$\text{Eigenvalues}:\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$$

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ s,t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\ \lambda_3 = 3 \quad \Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix},\ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Thm 7.3: (Eigenvalues of triangular matrices)

  If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 7: (Finding eigenvalues for diagonal and triangular matrices)

$$(a)\, A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b)\, A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ \left|\lambda \mathrm{I} - A\right| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b)\, \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Eigenvalues and eigenvectors of linear transformations:

A number λ is called an eigenvalue of a linear transformation $T : V \to V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of T corresponding to λ, and the setof all eigenvectors of λ (with the zero vector) is called the eigenspace of λ.

- Ex 8: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2 (\lambda - 4)$$

$$\text{eigenvalues}: \lambda_1 = 4, \lambda_2 = -2$$

The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) Let $T: R^3 \rightarrow R^3$ be the linear transformation whose standard matrix is A in Ex. 8, and let B' be the basis of R^3 made up of three linear independent eigenvectors found in Ex. 8. Then A', the matrix of T relative to the basis B', is diagonal.

$$B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Eigenvectors of A

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

(to be discussed in the next section)

Key Learning in Section 7.1

- Verify eigenvalues and corresponding eigenvectors.
- Find eigenvalues and corresponding eigenspaces.
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix.
- Find the eigenvalues and eigenvectors of a linear transformation.

Keywords in Section 7.1

- eigenvalue problem: 特徵值問題
- eigenvalue: 特徵值
- eigenvector: 特徵向量
- characteristic polynomial: 特徵多項式
- characteristic equation: 特徵方程式
- eigenspace: 特徵空間
- multiplicity: 重數