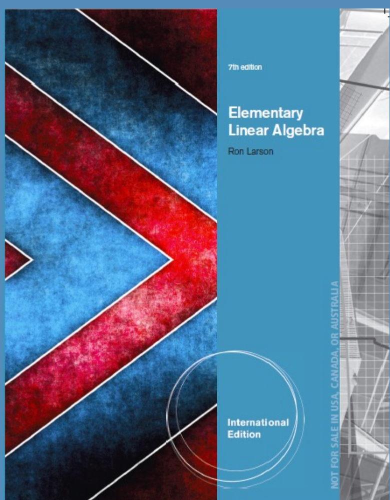
CHAPTER 6
LINEAR TRANSFORMATIONS



6.1 Introduction to Linear Transformations

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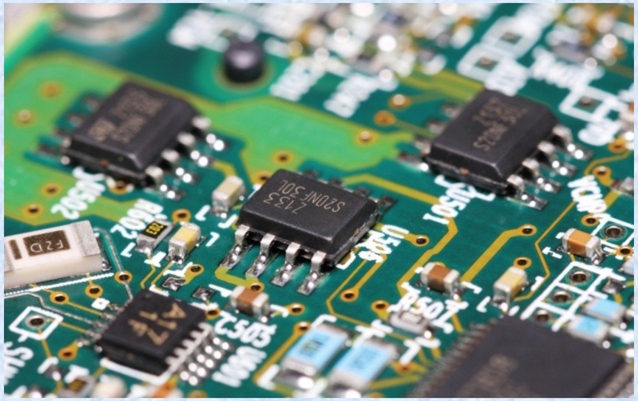

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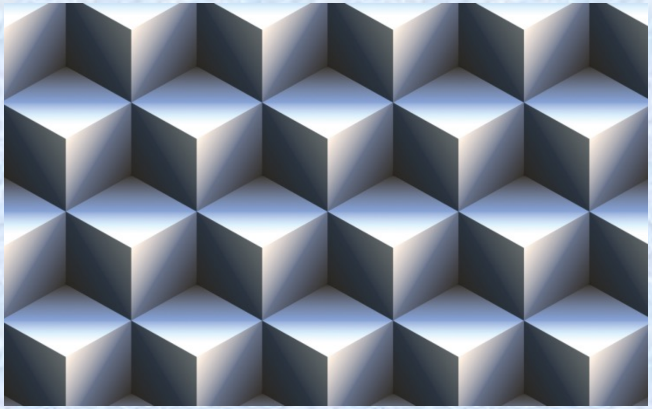

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6.1 Introduction to Linear Transformations

- Function T that maps a vector space V into a vector space W:

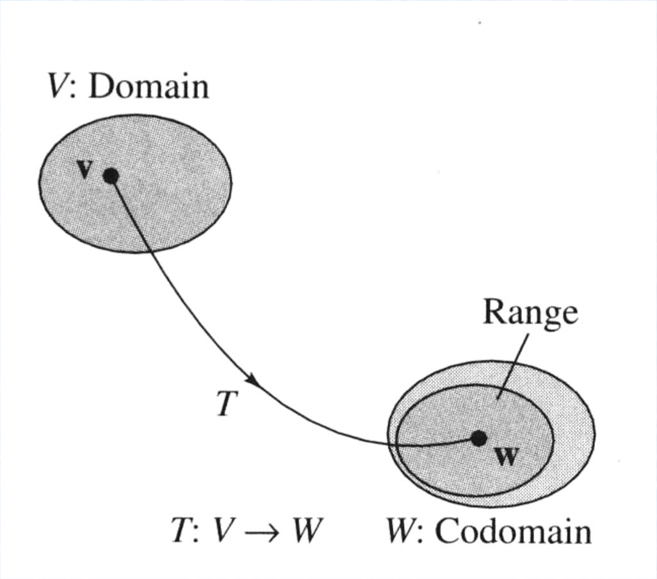
$$T: V \xrightarrow{\text{mapping}} W, \quad V, W: \text{vector space}$$

論域

V: the domain of T

W: the codomain of T

對應論域



- Image of **v** under *T*:

If **v** is in *V* and **w** is in *W* such that

$$T(\mathbf{v}) = \mathbf{w}$$ 像

Then **w** is called the image of **v** under T.

- the range of *T*:

The set of all images of vectors in V.

反像

- the preimage of **w**:

The set of all **v** in V such that $T(\mathbf{v})=\mathbf{w}$.

- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of **v**=(-1,2). (b) Find the preimage of **w**=(-1,11)

Sol:

$$(a) \;\; \mathbf{v} = (-1, 2)$$

$$\Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1 - 2, -1 + 2(2)) = (-3, 3)$$

$$(b) \;\; T(\mathbf{v}) = \mathbf{w} = (-1, 11)$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$\Rightarrow v_1 = 3, \; v_2 = 4$ Thus {(3, 4)} is the preimage of **w**=(-1, 11).

- Linear Transformation (L.T.):

V, W : vector space

$T: V \rightarrow W$: V to W linear transformation

$$(1)\ \ T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ \ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Notes:

(1) A linear transformation is said to be operation preserving.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

Addition in V | Addition in W | Scalar multiplication in V | Scalar multiplication in W

(2) A linear transformation $T: V \to V$ from a vector space into itself is called a **linear operator**.

- Ex 2: (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Pf:

$$\mathbf{u} = (u_1, u_2),\ \ \mathbf{v} = (v_1, v_2) : \text{vector in } R^2,\ \ c : \text{any real number}$$

(1)Vector addition :

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

$$\begin{aligned}
T(\mathbf{u} + \mathbf{v}) &= T(u_1 + v_1, u_2 + v_2) \\
&= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2)) \\
&= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2)) \\
&= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) \\
&= T(\mathbf{u}) + T(\mathbf{v})
\end{aligned}$$

(2) Scalar multiplication

$$c\mathbf{u} = c(u_1, u_2) = (cu_1, cu_2)$$

$$\begin{aligned} T(c\mathbf{u}) &= T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2) \\ &= c(u_1 - u_2, u_1 + 2u_2) \\ &= cT(\mathbf{u}) \end{aligned}$$

Therefore, T is a linear transformation.

- Ex 3: (Functions that are not linear transformations)

$(a)\ f(x) = \sin x$

$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$ $\Leftarrow$ $f(x) = \sin x$ is not linear transformation

$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$

$(b)\ f(x) = x^2$

$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$ $\Leftarrow$ $f(x) = x^2$ is not linear transformation

$(1+2)^2 \neq 1^2 + 2^2$

$(c)\ f(x) = x + 1$

$f(x_1 + x_2) = x_1 + x_2 + 1$

$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$

$f(x_1 + x_2) \neq f(x_1) + f(x_2)$ $\Leftarrow$ $f(x) = x + 1$ is not linear transformation

- Notes: Two uses of the term "linear".

(1) $f(x) = x + 1$ is called a linear function because its graph is a line.

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication.

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = 0, \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

- Thm 6.1: (Properties of linear transformations)

$$T: V \to W, \quad \mathbf{u}, \mathbf{v} \in V$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\mathbf{v}) = -T(\mathbf{v})$$

$$(3)\ T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$$

$$(4)\ \text{If } \mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

$$\text{Then } T(\mathbf{v}) = T(c_1 v_1 + c_2 v_2 + \cdots + c_n v_n)$$

$$= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

Proof of Theorem 6.1

1. $T(0) = T(0v) = 0T(v) = 0$.

2. $T(-v) = -T(v)$.

3. $T(u - v) = T\big(u + (-v)\big) = T(u) + T(-v) = T(u) - T(v)$.

4. $T(v) = T(c_1v_1 + \cdots + c_nv_n) = T(c_1v_1) + \cdots + T(c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)$

- Ex 4: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4)$$
$$T(0,1,0) = (1,5,-2)$$
$$T(0,0,1) = (0,3,1)$$

Find T(2, 3, -2).

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$T(2,3,-2) = 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \qquad (T \text{ is a L.T.})$$
$$= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1)$$
$$= (7,7,0)$$

- Ex 5: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2,-1)$

(b) Show that T is a linear transformation form R^2 into R^3

Sol: $(a)\mathbf{v} = (2,-1)$

$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}}$$

$$\therefore T(2,-1) = (6,3,0)$$

$(b)\ T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$ (vector addition)

$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$ (scalar multiplication)

- Thm 6.2: (The linear transformation given by a matrix)

  Let A be an $m\times n$ matrix. The function T defined by

  $$T(\mathbf{v}) = A\mathbf{v}$$

  is a linear transformation from R^n into R^m.

- Note:

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \overset{\substack{R^n \text{ vector} \\ \downarrow}}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} = \overset{\substack{R^m \text{ vector} \\ \downarrow}}{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T : R^n \longrightarrow R^m$$

- Ex 7: (Rotation in the plane)

Show that the L.T. $T : R^2 \rightarrow R^2$ given by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

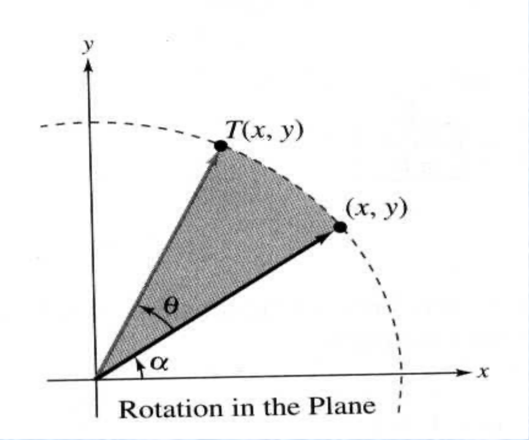
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$v = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

r : the length of v

α : the angle from the positive x-axis counterclockwise to the vector v



$$T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r : the length of $T(\mathbf{v})$

$\theta+\alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\mathbf{v})$

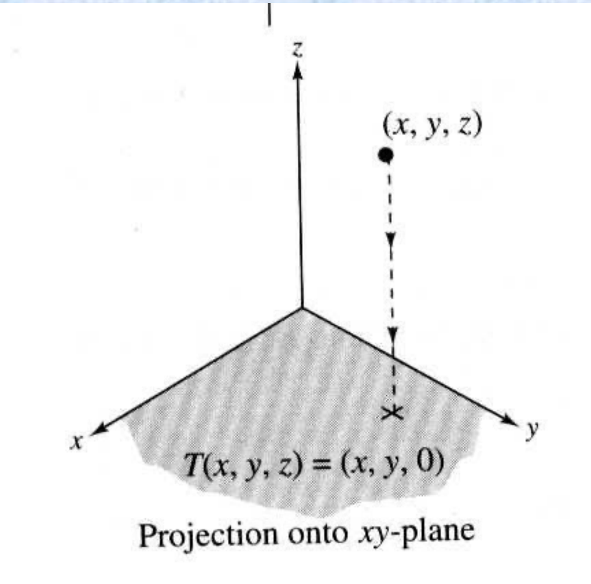
Thus, $T(\mathbf{v})$ is the vector that results from rotating the vector $\mathbf{v}$ counterclockwise through the angle θ.

- Ex 8: (A projection in R^3)

The linear transformation $T: R^3 \to R^3$ is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is called a projection in R^3.



- Ex 9: (A linear transformation from $M_{m\times n}$ into $M_{n\times m}$)

$$T(A) = A^T \quad (T: M_{m\times n} \to M_{n\times m})$$

Show that T is a linear transformation.

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

Key Learning in Section 6.1

- Find the image and preimage of a function.
- Show that a function is a linear transformation, and find a linear transformation.

Keywords in Section 6.1

- function: 函數
- domain: 論域
- codomain: 對應論域
- image of v under T: 在T映射下v的像
- range of T: T的值域
- preimage of w: w的反像
- linear transformation: 線性轉換
- linear operator: 線性運算子
- zero transformation: 零轉換
- identity transformation: 相等轉換

Review exercises

4. Find (a) the image of v and (b) the preimage of w for the linear transformation.

$$T: R^3 \to R^3, T(v_1, v_2, v_3) = (v_1 + v_2, v_2 + v_3, v_3),$$
$$v = (-2,1,2), w = (0,1,2).$$

16. Let T be a linear transformation from R^2 into R^2 such that $T(1,-1) = (2,-3)$ and $T(0,2) = (0,8)$. Find $T(2,4)$.

6.2 The Kernel and Range of a Linear Transformation

- Kernel of a linear transformation T:

Let $T : V \to W$ be a linear transformation

Then the set of all vectors **v** in V that satisfy $T(\mathbf{v}) = 0$ is called the kernel of T and is denoted by ker(T).

零向量

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = 0, \forall \mathbf{v} \in V\}$$

- Ex 1: (Finding the kernel of a linear transformation)

$$T(A) = A^T \quad (T : M_{3\times 2} \to M_{2\times 3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

- Ex 2: (The kernel of the zero and identity transformations)

(a) $T(\mathbf{v})=\mathbf{0}$ (the zero transformation $T:V \to W$)

$$\ker(T) = V$$

(b) $T(\mathbf{v})=\mathbf{v}$ (the identity transformation $T:V \to V$)

$$\ker(T) = \{\mathbf{0}\}$$

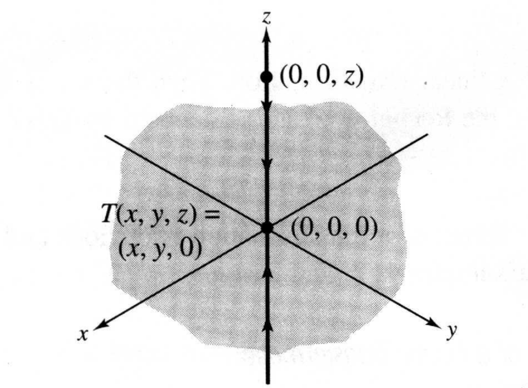
- Ex 3: (Finding the kernel of a linear transformation)

$$T(x, y, z) = (x, y, 0) \qquad (T: R^3 \to R^3)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(0,0,z) \mid z \text{ is a real number}\}$$



The kernel of T is the set of all vectors on the z-axis.

- Ex 5: (Finding the kernel of a linear transformation)

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T : R^3 \to R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0,0),\ x = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\}$$
$$= \operatorname{span}\{(1,-1,1)\}$$

- Thm 6.3: (The kernel is a subspace of V)

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

Pf: $\because T(\mathbf{0}) = \mathbf{0}$ (Theorem 6.1)

$\therefore \ker(T)$ is a nonempty subset of V

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the kernel of T. then

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0}+\mathbf{0} = \mathbf{0} \qquad \Rightarrow \mathbf{u}+\mathbf{v} \in \ker(T)$$

$$T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{0} = \mathbf{0} \qquad \Rightarrow c\mathbf{u} \in \ker(T)$$

Thus, $\ker(T)$ is a subspace of V.

- Note:

The kernel of T is sometimes called the **nullspace** of T.

- Ex 6: (Finding a basis for the kernel)

Let $T: R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is in R ^{5} and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5.

Sol:

$$[A \mid 0] =$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{G.J.E} \begin{array}{cccccc} \left[\begin{matrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right] \\ \qquad\qquad s \qquad\quad t \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$$B = \{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}: \text{one basis for the kernel of } T$$

- Corollary to Thm 6.3:

Let $T: R^n \to R^m$ be the L.T given by $T(\mathbf{x}) = A\mathbf{x}$

Then the kernel of T is equal to the solution space of $A\mathbf{x} = 0$

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{(a linear transformation } T: R^n \to R^m\text{)}$$

$$\Rightarrow Ker(T) = NS(A) = \left\{\mathbf{x} \mid A\mathbf{x} = 0, \forall \mathbf{x} \in R^m\right\} \text{ (subspace of } R^m\text{)}$$

- Range of a linear transformation T:

Let $T: V \to W$ be a L.T.

Then the set of all vectors w in W that are images of vector in V is called the range of T and is denoted by $range(T)$

$$range(T) = \{T(\mathbf{v}) \mid \forall \mathbf{v} \in V\}$$

- Thm 6.4: (The range of T is a subspace of W)

The range of a linear transformation $T: V \rightarrow W$ is a subspace of W.

Pf:

$\because T(\mathbf{0}) = \mathbf{0}$ (Thm.6.1)

$\therefore range(T)$ is a nonempty subset of W

Let $T(\mathbf{u})$ and $T(\mathbf{v})$ be vector in the range of T

$T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}) \in range(T)$ $\quad (\mathbf{u} \in V, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V)$

$cT(\mathbf{u}) = T(c\mathbf{u}) \in range(T)$ $\quad (\mathbf{u} \in V \Rightarrow c\mathbf{u} \in V)$

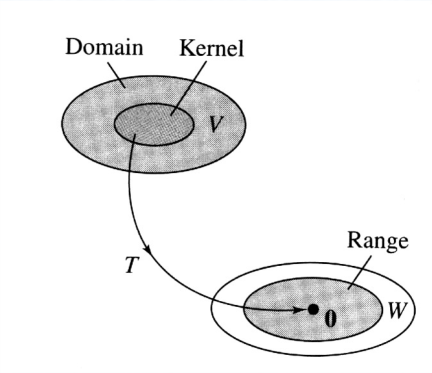
Therefore, $range(T)$ is W subspace.

- Notes:

$$T: V \to W \text{ is a L.T.}$$

$(1) Ker(T)$ is subspace of V

$(2) range(T)$ is subspace of W



- Corollary to Thm 6.4:

Let $T: R^n \to R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$

Then the range of T is equal to the column space of A

$\Rightarrow range(T) = CS(A)$

- Ex 7: (Finding a basis for the range of a linear transformation)

Let $T : R^5 \rightarrow R^4$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T.

Sol:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \xrightarrow{G.J.E} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \qquad \begin{array}{ccccc} w_1 & w_2 & w_3 & w_4 & w_5 \end{array}$$

$\Rightarrow \{w_1, w_2, w_4\}$ is a basis for $CS(B)$

$\{c_1, c_2, c_4\}$ is a basis for $CS(A)$

$\Rightarrow \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T

- Rank of a linear transformation $T: V \rightarrow W$:

$$rank(T) = \text{the dimension of the range of } T$$

- Nullity of a linear transformation $T: V \rightarrow W$:

$$nullity(T) = \text{the dimension of the kernel of } T$$

- Note:

Let $T : R^n \rightarrow R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$, then

$$rank(T) = rank(A)$$

$$nullity(T) = nullity(A)$$

Definition of Rank and Nullity of a Linear Transformation

- Let $T: V \rightarrow W$ be a linear transformation.

- The dimension of the kernel of T is called the nullity of T and is denoted by nullity(T).

- The dimension of the range of T is called the rank of T and is denoted by rank(T).

- Thm 6.5: (Sum of rank and nullity)

Let $T: V \to W$ be a L.T. from an n-dimensional vector space V into a vector space W. then

$$rank(T) + nullity(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

Pf:

Let T is represented by an $m \times n$ matrix A

Assume $rank(A) = r$

(Thm4.15, Ch4_3)

$$(1)\ rank(T) = \dim(\text{range of } T) = \dim(\text{column space of } A)$$
$$= rank(A) = r$$

$$(2)\ nullity(T) = \dim(\text{kernel of } T) = \dim(\text{solution space of } Ax = 0)$$
$$= n - r$$

$$\Rightarrow rank(T) + nullity(T) = r + (n - r) = n$$

- Ex 8: (Finding the rank and nullity of a linear transformation)

Find the rank and nullity of the L.T. $T: R^3 \rightarrow R^3$ define by

$$A=\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}$$

Sol:

$$rank(T)=rank(A)=2$$

$$nullity(T)=\dim(\text{domain of } T)-rank(T)=3-2=1$$

- Ex 9: (Finding the rank and nullity of a linear transformation)

Let $T: R^5 \rightarrow R^7$ be a linear transformation.

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $Ker(T) = \{0\}$

Sol:

$(a)\ \text{dim(domain of } T) = 5$

$\text{dim(kernel of } T) = n - \text{dim(range of } T) = 5 - 2 = 3$

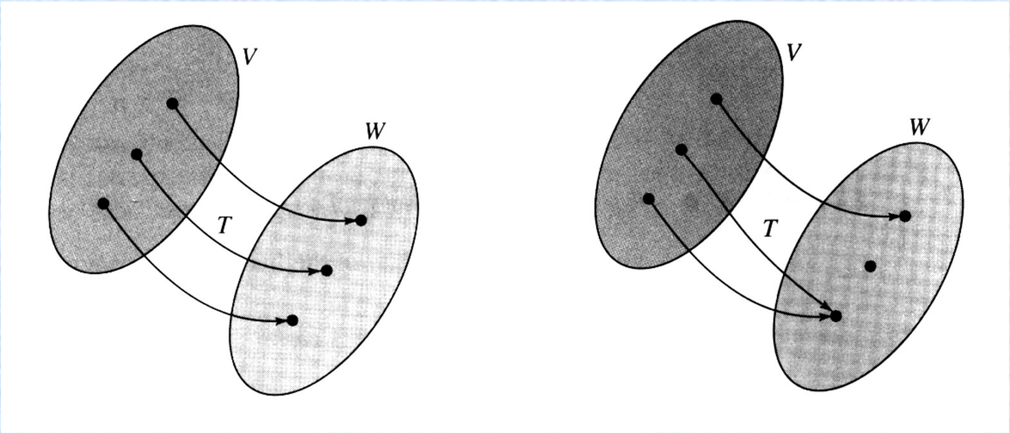
$(b)\ rank(T) = n - nullity(T) = 5 - 4 = 1$

$(c)\ rank(T) = n - nullity(T) = 5 - 0 = 5$

- One-to-one:

  A function $T: V \to W$ is called one-to-one if the preimage of every w in the range consists of a single vector.

  T is one-to-one iff for all u and v in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies that $\mathbf{u} = \mathbf{v}$.



one-to-one not one-to-one

- Onto:

A function $T : V \to W$ is said to be onto if every element in $\mathbf{w}$ has a preimage in V

(T is onto W when W is equal to the range of T.)

- Thm 6.6: (One-to-one linear transformation)

Let $T: V \to W$ be a L.T.

Then T is 1-1 iff $Ker(T) = \{0\}$

Pf:

Suppose T is 1-1

(Thm6.1)

Then $T(v) = 0$ can have only one solution : $v = 0$

i.e. $Ker(T) = \{0\}$

Suppose $Ker(T) = \{0\}$ and $T(u) = T(v)$

$T(u - v) = T(u) - T(v) = 0$

T is a L.T.

$\because u - v \in Ker(T) \Rightarrow u - v = 0$

$\Rightarrow T$ is 1-1

- Ex 10: (One-to-one and not one-to-one linear transformation)

(a) The L.T. $T: M_{m\times n} \rightarrow M_{n\times m}$ given by $T(A) = A^T$

is one-to-one.

Because its kernel consists of only the $m\times n$ zero matrix.

(b) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one.

Because its kernel is all of R^3.

- Thm 6.7: (Onto linear transformation)

Let $T: V \to W$ be a L.T., where W is finite dimensional.

Then T is onto iff the rank of T is equal to the dimension of W.

- Thm 6.8: (One-to-one and onto linear transformation)

Let $T: V \to W$ be a L.T. with vector space V and W both of dimension n. Then T is one-to-one if and only if it is onto.

Pf:

If T is one-to-one, then $Ker(T) = \{0\}$ and $\dim(Ker(T)) = 0$

$\dim(range(T)) = n - \dim(Ker(T)) = n = \dim(W)$

Consequently, T is onto.

If T is onto, then $\dim(\text{range of } T) = \dim(W) = n$

$\dim(Ker(T)) = n - \dim(\text{range of } T) = n - n = 0$

Therefore, T is one-to-one.

Proof of Theorem 6.7

- Let T: V→W be a linear transformation, where $\dim(W) = n$.

- (→) If T is onto, then W is equal to the range of T. So, W and the range of T have the same dimension, and the rank of T is n.

- (←) If the rank of T is n, then there are n linearly independent vectors $T(v_1), \cdots, T(v_n)$ in the range of T.

- Since the range of T is a subspace of W, $T(v_1), \cdots, T(v_n)$ are linearly independent in W. Thus, they form a basis for W.

- So, any vector $w \in W$ can be written as a linear combination

$$w = c_1 T(v_1) + \cdots + c_n T(v_n) = T(c_1 v_1 + \cdots + c_n v_n)$$

- Since any $w \in W$ is in the range of T, T is onto.

- Ex 11:

The L.T. $T: R^n \rightarrow R^m$ is given by $T(\mathbf{x}) = A\mathbf{x}$, Find the nullity and rank of T and determine whether T is one-to-one, onto, or neither.

$$(a)\, A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (b)\, A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$(c)\, A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \qquad (d)\, A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T: R^n \rightarrow R^m$	dim(domain of T)	rank(T)	nullity(T)	1-1	onto
$(a)\, T: R^3 \rightarrow R^3$	3	3	0	Yes	Yes
$(b)\, T: R^2 \rightarrow R^3$	2	2	0	Yes	No
$(c)\, T: R^3 \rightarrow R^2$	3	2	1	No	Yes
$(d)\, T: R^3 \rightarrow R^3$	3	2	1	No	No

- Isomorphism:

  A linear transformation $T:V \rightarrow W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other.

- Thm 6.9: (Isomorphic spaces and dimension)

  Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension.

Pf:

Assume that V is isomorphic to W, where V has dimension n.

$\Rightarrow$ There exists a L.T. $T:V \rightarrow W$ that is one to one and onto.

$\because T$ is one-to-one

$\Rightarrow \dim(Ker(T)) = 0$

$\Rightarrow \dim(\text{range of } T) = \dim(\text{domain of } T) - \dim(Ker(T)) = n - 0 = n$

$\because T$ is onto.

$\Rightarrow \dim(\text{range of } T) = \dim(W) = n$

Thus $\dim(V) = \dim(W) = n$

Assume that V and W both have dimension n.

Let $\{v_1, v_2, \cdots, v_n\}$ be a basis of V, and

let $\{w_1, w_2, \cdots, w_n\}$ be a basis of W.

Then an arbitrary vector in V can be represented as

$\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

and you can define a L.T. $T : V \to W$ as follows.

$T(\mathbf{v}) = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n$

It can be shown that this L.T. is both 1-1 and onto.

Thus V and W are isomorphic.

- Ex 12: (Isomorphic vector spaces)

The following vector spaces are isomorphic to each other.

$(a) R^4 = 4\text{-space}$

$(b) M_{4\times 1} = \text{space of all } 4\times 1 \text{ matrices}$

$(c) M_{2\times 2} = \text{space of all } 2\times 2 \text{ matrices}$

$(d) P_3(x) = \text{space of all polynomials of degree 3 or less}$

$(e) V = \{(x_1, x_2, x_3, x_4, 0), x_i \text{ is a real number}\} (\text{subspace of } R^5)$

Key Learning in Section 6.2

- Find the kernel of a linear transformation.
- Find a basis for the range, the rank, and the nullity of a linear transformation.
- Determine whether a linear transformation is one-to-one or onto.
- Determine whether vector spaces are isomorphic.

Keywords in Section 6.2

- kernel of a linear transformation T: 線性轉換T的核空間
- range of a linear transformation T: 線性轉換T的值域
- rank of a linear transformation T: 線性轉換T的秩
- nullity of a linear transformation T: 線性轉換T的核次數
- one-to-one: 一對一
- onto: 映成
- isomorphism(one-to-one and onto): 同構
- isomorphic space: 同構的空間

Review exercises

31. Define the linear transformation T by $T(v) = Av$. Find

(a) $\ker(T)$,

(b) $nullity(T)$,

(c) $range(T)$, and

(d) $rank(T)$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$$

36. Given T: $P_5 \rightarrow P_3$ and $nullity(T) = 4$, find $rank(T)$.