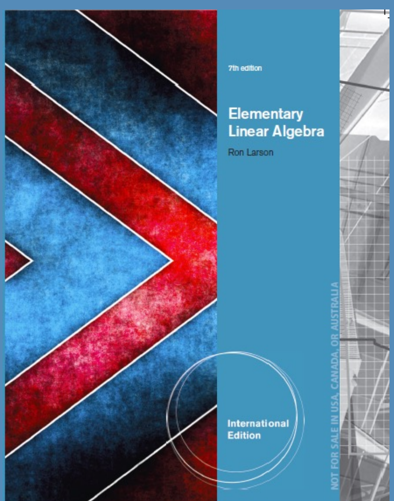
CHAPTER 3
DETERMINANTS



3.3 Properties of Determinants

- Thm 3.5: (Determinant of a matrix product)

$$\det(AB) = \det(A)\det(B)$$

$$|AB| = |E_k \cdots E_2 E_1 B|$$
$$= |E_k| \cdots |E_1||B|$$
$$= |E_k \cdots E_1||B|$$
$$= |A||B|$$

- Notes:

(1) $\det(EA) = \det(E)\det(A)$

(2) $\det(A+B) \neq \det(A) + \det(B)$

(3)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex 1: (The determinant of a matrix product)

$$A=\begin{bmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{bmatrix} \qquad B=\begin{bmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{bmatrix}$$

Find $|A|$, $|B|$, and $|AB|$

Sol:

$$|A|=\begin{vmatrix}1 & -2 & 2\\ 0 & 3 & 2\\ 1 & 0 & 1\end{vmatrix}=-7 \qquad |B|=\begin{vmatrix}2 & 0 & 1\\ 0 & -1 & -2\\ 3 & 1 & -2\end{vmatrix}=11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

- Check:

$$|AB| = |A|\,|B|$$

- Thm 3.6: (Determinant of a scalar multiple of a matrix)

If A is an $n \times n$ matrix and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

Proof: From Property 3 of Theorem 3.3.

- Ex 2:

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5$$

Find $|A|$.

Sol:

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- Thm 3.7: (Determinant of an invertible matrix)

  A square matrix A is invertible (nonsingular) if and only if

  $\det(A) \neq 0$

- Ex 3: (Classifying square matrices as singular or nonsingular)

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

Sol:

$|A| = 0 \quad \Rightarrow$ A has no inverse (it is singular).

$|B| = -12 \neq 0 \quad \Rightarrow$ B has an inverse (it is nonsingular).

Proof of Theorem 3.7

- Assume A is invertible. We have $AA^{-1} = I$ and $|A||A^{-1}| = 1$. Neither determinant on the left is zero. Thus, $|A| \neq 0$.

- Assume $|A| \neq 0$.
- Using Gaussian-Jordan elimination, find a matrix B, in reduced row-echelon form, that is row-equivalent to A.
- B must be the identity matrix or it must have at least one row that consists entirely of zeros.
- If B has a row of all zeros, $|B| = 0$, which implies that $|A| = 0$.
- Since $|A| \neq 0$ is assumed, $B = I$. A is row-equivalent to the identity matrix, and by Theorem 2.15, A is invertible.

- **Thm 3.8: (Determinant of an inverse matrix)**

$$\text{If } A \text{ is invertible then } \det(A^{-1}) = \frac{1}{\det(A)}.$$

- **Thm 3.9: (Determinant of a transpose)**

$$\text{If } A \text{ is a square matrix, then } \det(A^{T}) = \det(A).$$

- **Ex 4:**

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{(a)} \ \left|A^{-1}\right| = ? \qquad \text{(b)} \ \left|A^{T}\right| = ?$$

Sol:

$$\because \ |A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \qquad \therefore \ \left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{4}$$

$$\left|A^{T}\right| = |A| = 4$$

Proof of Theorem 3.8, 3.9

- A is invertible. $|A||A^{-1}| = 1$

- Calculating $|A|$ by expanding along the first row is equivalent to calculating $|A^T|$ by expanding along the first column.
- Because the determinant of a matrix can be found by expanding any row or column.
- Thus, $|A| = |A^T|$.

- Equivalent conditions for a nonsingular matrix:

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n

(5) A can be written as the product of elementary matrices.

(6) $\det(A) \neq 0$

- Ex 5: Which of the following system has a unique solution?

$$(a)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

$$(b)\quad \begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & -4 \end{array}$$

Sol:

$(a)\quad A\mathbf{x} = \mathbf{b}$

$\because \quad |A| = 0$

$\therefore$ This system does not have a unique solution.

$(b)\quad B\mathbf{x} = \mathbf{b}$

$\because \quad |B| = -12 \neq 0$

$\therefore$ This system has a unique solution.

Key Learning in Section 3.3

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors.

Keywords in Section 3.3

- determinant: 行列式
- matrix multiplication: 矩陣相乘
- scalar multiplication: 純量積
- invertible matrix: 可逆矩陣
- inverse matrix: 反矩陣
- nonsingular matrix: 非奇異矩陣
- transpose matrix: 轉置矩陣

3.4 Applications of Determinants

- Matrix of cofactors of A:

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} \qquad C_{ij} = (-1)^{i+j} M_{ij}$$

伴隨矩陣

- Adjoint matrix of A:

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

- Thm 3.10: (The inverse of a matrix given by its adjoint)

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \det(A) = ad - bc$$

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof of Theorem 3.10

- Consider the product

$$A[adj(A)] = \begin{bmatrix} & & \\ \boxed{a_{i1} & \cdots & a_{in}} \\ & & \end{bmatrix} \begin{bmatrix} & C_{j1} & \\ & \vdots & \\ & C_{jn} & \end{bmatrix}$$

- The entry in the ith row and jth column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

- If $i = j$, this sum is simply the determinant of A; otherwise, the sum is zero.

$$A[adj(A)] = \begin{bmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{bmatrix} = \det(A)\, I$$

$$\frac{1}{\det(A)} A[adj(A)] = I$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

- Ex 1 & Ex 2:

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

(a) Find the adjoint of A.

(b) Use the adjoint of A to find A^{-1}

Sol: $\because \quad C_{ij} = (-1)^{i+j} M_{ij}$

$$\Rightarrow C_{11} = +\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4, \quad C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1, \quad C_{13} = +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} = 6, \quad C_{22} = +\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$C_{31} = +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 7, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \quad C_{33} = +\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

$\Rightarrow$ cofactor matrix of A $\Rightarrow$ adjoint matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix}$$

$$adj(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)} adj(A) \qquad \because \det(A) = 3$$

$$= \tfrac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

- Check: $AA^{-1} = I$

- Thm 3.11: (Cramer's Rule)

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$A\mathbf{x} = \mathbf{b} \qquad A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}, A^{(2)}, \cdots, A^{(n)}\right] \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

(this system has a unique solution)

$$A_j = \left[A^{(1)}, A^{(2)}, \cdots, A^{(j-1)}, b, A^{(j+1)}, \cdots, A^{(n)}\right]$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & & \ddots & & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

$$(\text{ i.e. } \det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj})$$

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \qquad j = 1, 2, \cdots, n$$

- Pf:

$$A\mathbf{x} = \mathbf{b}, \qquad \det(A) \neq 0$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)} adj(A)\mathbf{b}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} & + & b_2C_{21} & + & \cdots & + & b_nC_{n1} \\ b_1C_{12} & + & b_2C_{22} & + & \cdots & + & b_nC_{n2} \\ & \vdots & & \vdots & & & \\ b_1C_{1n} & + & b_2C_{2n} & + & \cdots & + & b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow x_j = \frac{1}{\det(A)}(b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj})$$

$$= \frac{\det(A_j)}{\det(A)} \qquad j = 1, 2, \cdots, n$$

- Ex 4: Use Cramer's rule to solve the system of linear equations.

$$\begin{aligned} -x &+ 2y - 3z = 1 \\ 2x & + z = 0 \\ 3x &- 4y + 4z = 2 \end{aligned}$$

Sol:

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

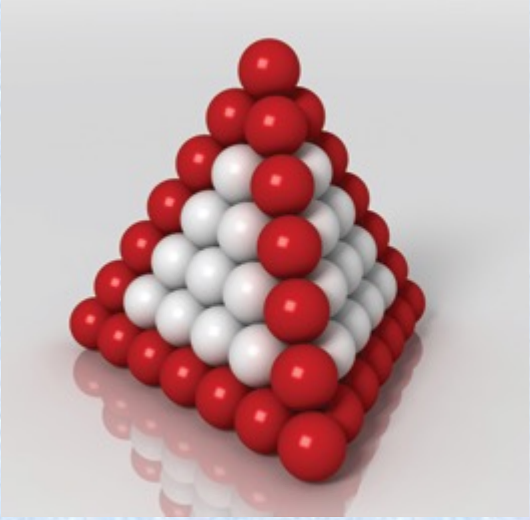
$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$

Keywords in Section 3.4

- matrix of cofactors : 餘因子矩陣
- adjoint matrix : 伴隨矩陣
- Cramer's rule : Cramer 法則

- Volume of a Solid



If x, y and z are continuous functions of u, v, and w with continuous first partial derivatives, then the **Jacobians** $J(u,\ v)$ and $J(u,\ v,\ w)$ are defined as the determinants

$$J(u,v)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix} \quad \text{and} \quad J(u,v,w)=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w}\\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w}\end{vmatrix}.$$

And one practical use of Jacobians is in finding the volume of a solid region. In Section 3.4, you will study a formula, which also uses determinants, for finding the volume of a tetrahedron. In the Chapter 3 Review, you are asked to find the Jacobian of a given set of functions. (See Review Exercises 49–52.)

- Sudoku

In the number-placement puzzle Sudoku, the object is to fill out a partially completed 9×9 grid of boxes with numbers from 1 to 9 so that each column, row, and 3×3 sub-grid contains each of these numbers without repetition. For a completed Sudoku grid to be valid, no two rows (or columns) will have the numbers in the same order. If this should happen in a row or column, then the determinant of the matrix formed by the numbers in the grid will be zero. This is a direct result of condition 2 of Theorem 3.4.

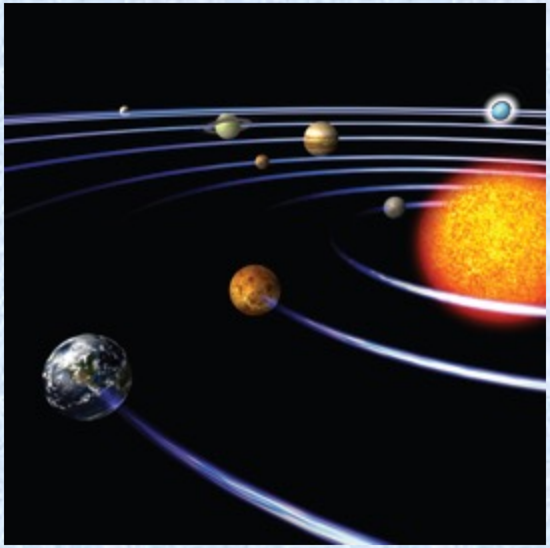
- Engineering and Control

Systems of linear differential equations often arise in engineering and control theory. For a function $f(t)$ that is defined for all positive values of t the **Laplace transform** of $f(t)$ is given by

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Laplace transforms and Cramer's Rule, which uses determinants to solve a system of linear equations, can often be used to solve a system of differential equations. You will study Cramer's Rule in the next section.

- Planetary Orbits



According to Kepler's First Law of Planetary Motion, the orbits of the planets are ellipses, with the sun at one focus of the ellipse. The general equation of a conic section (such as an ellipse) is

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

To determine the equation of the orbit of a planet, an astronomer can find the coordinates of the planet along its orbit at five different points (x_i, y_i) where i = 1, 2, 3, 4, and 5, and then use the determinant

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix}$$

Review exercises

26. Find $|A^T|, |A^3|, |A^T A|, and\ |5A|$.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$$

Review exercises

34. Solve the system of linear equations by each of the following methods.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer's Rule

$$
\begin{gathered}
2x_1 + x_2 + 2x_3 = 6 \\
-x_1 + 2x_2 - 3x_3 = 0 \\
3x_1 + 2x_2 - x_3 = 6
\end{gathered}
$$