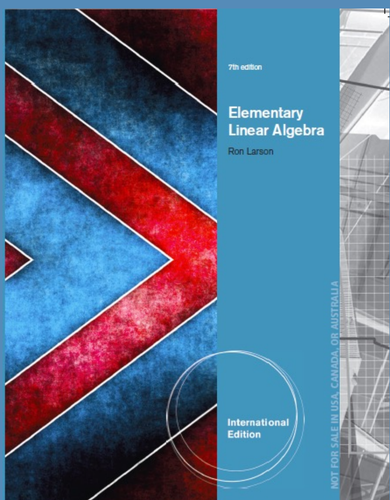
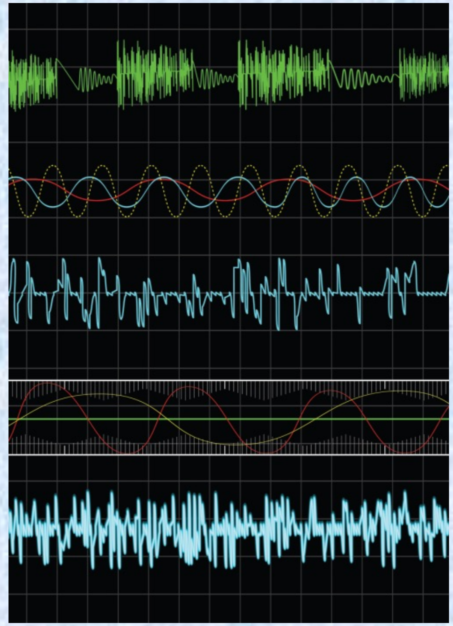
CHAPTER 4
VECTOR SPACES



CH 4 Linear Algebra Applied

Force (p.151)



Digital Sampling (p.166)

Image Morphing (p.174)

Crystallography (p.207)

Satellite Dish (p.217)

4.1 Vectors in R^n

- An ordered n-tuple:

  a sequence of n real number $(x_1, x_2, \cdots, x_n)$

- n-space: R^n

  the set of all ordered n-tuple

- Ex:

$n = 1 \qquad R^1 = \text{1-space}$

$\qquad\qquad = \text{set of all real number}$

$n = 2 \qquad R^2 = \text{2-space}$

$\qquad\qquad = \text{set of all ordered pair of real numbers } (x_1, x_2)$

$n = 3 \qquad R^3 = \text{3-space}$

$\qquad\qquad = \text{set of all ordered triple of real numbers } (x_1, x_2, x_3)$

$n = 4 \qquad R^4 = \text{4-space}$

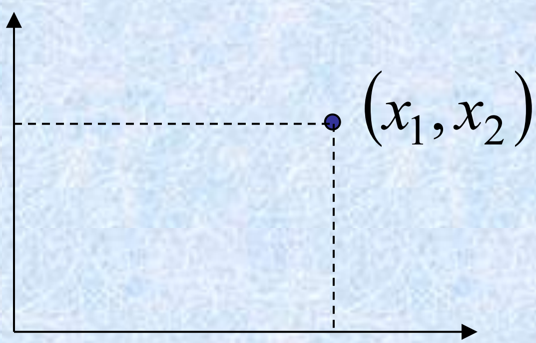
$\qquad\qquad = \text{set of all ordered quadruple of real numbers } (x_1, x_2, x_3, x_4)$

- Notes:

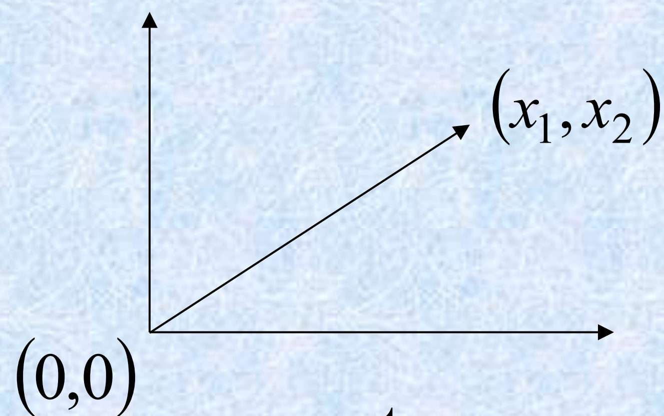
(1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a point</u> in R^n with the x_i's as its coordinates.

(2) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as <u>a vector</u> $x = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components.

- Ex:



a point



a vector

$\mathbf{u} = (u_1, u_2, \cdots, u_n),\ \mathbf{v} = (v_1, v_2, \cdots, v_n)$ (two vectors in R^n)

- Equal:

  $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2, \cdots, u_n = v_n$

- Vector addition (the sum of **u** and **v**):

  $$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

- Scalar multiplication (the scalar multiple of **u** by c):

  $$c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n.

- Negative:

$$-\mathbf{u} = (-u_1, -u_2, -u_3, \ldots, -u_n)$$

- Difference:

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3, \ldots, u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

- Notes:

加法單位元素

(1) The zero vector $\mathbf{0}$ in R^n is called the **additive identity** in R^n.

(2) The vector $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.

加法反元素

- Thm 4.2: (Properties of vector addition and scalar multiplication)

Let **u**, **v**, and **w** be vectors in R^n , and let c and d be scalars.

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ 交換律

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ 結合律

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(6) $c\mathbf{u}$ is a vector in R^n

(7) $\mathrm{c}(\mathbf{u}+\mathbf{v}) = \mathrm{c}\mathbf{u}+\mathrm{c}\mathbf{v}$ 分配律

(8) $(c+d)\mathbf{u} = \mathrm{c}\mathbf{u}+\mathrm{d}\mathbf{u}$

(9) $c(\mathrm{d}\mathbf{u}) = (\mathrm{cd})\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Ex 5: (Vector operations in R^4)

Let $\mathbf{u}=(2, -1, 5, 0)$, $\mathbf{v}=(4, 3, 1, -1)$, and $\mathbf{w}=(-6, 2, 0, 3)$ be vectors in R^4. Solve $\mathbf{x}$ for x in each of the following.

(a) $\mathbf{x} = 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w})$

(b) $3(\mathbf{x}+\mathbf{w}) = 2\mathbf{u} - \mathbf{v}+\mathbf{x}$

Sol: (a)
$$\begin{aligned}\mathbf{x} &= 2\mathbf{u} - (\mathbf{v} + 3\mathbf{w}) \\ &= 2\mathbf{u} - \mathbf{v} - 3\mathbf{w} \\ &= (4, -2, 10, 0) - (4, 3, 1, -1) - (-18, 6, 0, 9) \\ &= (4 - 4 + 18, -2 - 3 - 6, 10 - 1 - 0, 0 + 1 - 9) \\ &= (18, -11, 9, -8).\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\quad 3(\mathbf{x}+\mathbf{w}) &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}+3\mathbf{w} &= 2\mathbf{u}-\mathbf{v}+\mathbf{x} \\
3\mathbf{x}-\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
2\mathbf{x} &= 2\mathbf{u}-\mathbf{v}-3\mathbf{w} \\
\mathbf{x} &= \mathbf{u}-\tfrac{1}{2}\mathbf{v}-\tfrac{3}{2}\mathbf{w} \\
&= (2,1,5,0)+\left(-2,\tfrac{-3}{2},\tfrac{-1}{2},\tfrac{1}{2}\right)+\left(9,-3,0,\tfrac{-9}{2}\right) \\
&= \left(9,\tfrac{-11}{2},\tfrac{9}{2},-4\right)
\end{aligned}
$$

- Thm 4.3: (Properties of additive identity and additive inverse)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then the following is true.

(1) The additive identity is unique. That is, if $\mathbf{u}+\mathbf{v}=\mathbf{v}$, then $\mathbf{u} = \mathbf{0}$

(2) The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v}+\mathbf{u}=\mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$

(3) $0\mathbf{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\mathbf{v}=\mathbf{0}$

(6) $-(-\mathbf{v}) = \mathbf{v}$

- Linear combination:

The vector **x** is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_n$,

if it can be expressed in the form

$$\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v}_n \qquad c_1, c_2, \cdots, c_n : \text{scalar}$$

- Ex 6:

Given **x** = (– 1, – 2, – 2), **u** = (0,1,4), **v** = (– 1,1,2), and **w** = (3,1,2) in R^3, find a, b, and c such that **x** = a**u**+b**v**+c**w**.

Sol:

$$\begin{array}{rcrcrcr} & & -b & + & 3c & = & -1 \\ a & + & b & + & c & = & -2 \\ 4a & + & 2b & + & 2c & = & -2 \end{array}$$

$$\Rightarrow a = 1,\ b = -2,\ c = -1$$

Thus $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$

- Notes:

A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be viewed as:

a $1 \times n$ row matrix (row vector): $\mathbf{u} = [u_1,\ u_2, \cdots, u_n]$

or

a $n \times 1$ column matrix (column vector): $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

(The matrix operations of addition and scalar multiplication give the same results as the corresponding vector operations)

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$\mathbf{u}+\mathbf{v}=[u_1,u_2,\cdots,u_n]+[v_1,v_2,\cdots,v_n]$$
$$=[u_1+v_1,u_2+v_2,\cdots,u_n+v_n]$$

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

$$c\mathbf{u}=c[u_1,\ u_2,\cdots,\ u_n]$$
$$=[cu_1,cu_2,\cdots,\ cu_n]$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Key Learning in Section 4.1

- Represent a vector as a directed line segment.
- Perform basic vector operations in R^n and represent them graphically.
- Perform basic vector operations in R^2.
- Write a vector as a linear combination of other vectors.

Keywords in Section 4.1

- ordered n-tuple：有序的n項
- n-space：n維空間
- equal：相等
- vector addition：向量加法
- scalar multiplication：純量乘法
- negative：負向量
- difference：向量差
- zero vector：零向量
- additive identity：加法單位元素
- additive inverse：加法反元素

4.2 Vector Spaces

- Vector spaces:

  Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every **u**, **v**, and **w** in V and every scalar (real number) c and d, then V is called a **vector space**.

  Addition:

  (1) $\mathbf{u}+\mathbf{v}$ is in V

  (2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

  (3) $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$

  (4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0}=\mathbf{u}$

  (5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V.

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

- Notes:

(1) A vector space consists of four entities:

a set of vectors, a set of scalars, and two operations

V : nonempty set

c : scalar

$+(\mathbf{u},\mathbf{v})=\mathbf{u}+\mathbf{v}$: vector addition

$\bullet(c,\mathbf{u})=c\mathbf{u}$: scalar multiplication

$(V, \quad +, \quad \bullet)$ is called a vector space

(2) $V=\{\mathbf{0}\}$: zero vector space

- Examples of vector spaces:

(1) n-tuple space: R^n

$$(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n)=(ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V=M_{m\times n}$ (the set of all $m\times n$ matrices with real values)

Ex: : $(m=n=2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

(3) *n*-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$

(4) Function space: $V = c(-\infty,\infty)$ (the set of all real-valued continuous functions defined on the entire real line.)

$$(f+g)(x)=f(x)+g(x)$$

$$(kf)(x)=kf(x)$$

- Thm 4.4: (Properties of scalar multiplication)

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

- Ex 6: The set of all integer is not a vector space.

Pf:

$$1 \in V, \tfrac{1}{2} \in R$$

$$\underset{\text{scalar}}{\underset{\uparrow}{(\tfrac{1}{2})}}\underset{\text{integer}}{\underset{\uparrow}{(1)}} = \underset{\text{noninteger}}{\underset{\uparrow}{\tfrac{1}{2}}} \notin V$$ (it is not closed under scalar multiplication)

- Ex 7: The set of all second-degree polynomials is not a vector space.

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 8:

$V=R^2$=the set of all ordered pairs of real numbers

vector addition: $(u_1, u_2)+(v_1, v_2)=(u_1+v_1, u_2+v_2)$

scalar multiplication: $c(u_1, u_2)=(cu_1, 0)$

Verify V is not a vector space.

Sol:

$\because 1(1, 1)=(1, 0)\neq(1, 1)$

$\therefore$the set (together with the two given operations) is not a vector space

Key Learning in Section 4.2

- Define a vector space and recognize some important vector spaces.
- Show that a given set is not a vector space.

Keywords in Section 4.2:

- vector space：向量空間
- *n*-space：*n*維空間
- matrix space：矩陣空間
- polynomial space：多項式空間
- function space：函數空間

Review exercises

10. Write v as a linear combination of u_1, u_2, and u_3, if possible.

$$v = (4,4,5), u_1 = (1,2,3), u_2 = (-2,0,1), u_3 = (1,0,0)$$

13. Determine the zero vector and the additive inverse of a vector in the vector space.

$$M_{3,4}$$