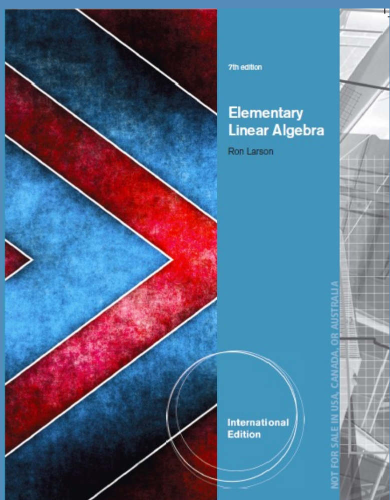
CHAPTER 4
VECTOR SPACES



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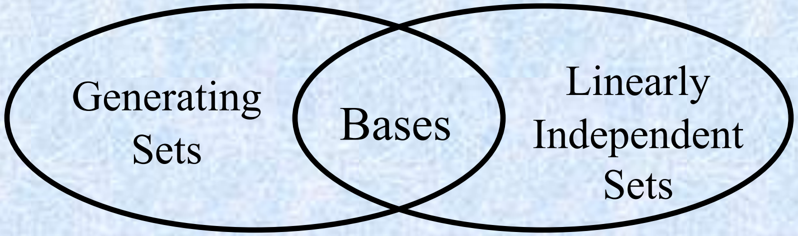
4.5 Basis and Dimension

- **Basis:**



V : a vector space

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$

$$\begin{cases} (a)\ \ S \text{ spans } V \ \ (\text{i.e., } \ span(S) = V\,) \\ (b)\ \ S \text{ is linearly independent} \end{cases}$$

$\Rightarrow$ S is called a **basis** for V

- **Notes:**

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{i, j, k\} \quad i = (1, 0, 0),\ j = (0, 1, 0),\ k = (0, 0, 1)$

(3) the standard basis for R^n :

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ $\mathbf{e}_1=(1,0,\ldots,0)$, $\mathbf{e}_2=(0,1,\ldots,0)$, $\mathbf{e}_n=(0,0,\ldots,1)$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

(4) the standard basis for $m\times n$ matrix space:

$\{ E_{ij} \mid 1\le i\le m , 1\le j\le n \}$

Ex: 2×2 matrix space:

$$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

(5) the standard basis for $P_n(x)$:

$\{1, x, x^2, \ldots, x^n\}$

Ex: $P_3(x)$ $\{1, x, x^2, x^3\}$

- Thm 4.9: (Uniqueness of basis representation)

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a basis for a vector space *V*, then every vector in *V* can be written in one and only one way as a linear combination of vectors in *S*.

Pf:

$$\because S \text{ is a basis} \Rightarrow \begin{cases} 1. & span(S)=V \\ 2. & S \text{ is linearly independent} \end{cases}$$

$\because span(S)=V$ Let $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$

$\mathbf{v}=b_1\mathbf{v}_1+b_2\mathbf{v}_2+\ldots+b_n\mathbf{v}_n$

$\Rightarrow \mathbf{0}=(c_1-b_1)\mathbf{v}_1+(c_2-b_2)\mathbf{v}_2+\ldots+(c_n-b_n)\mathbf{v}_n$

$\because S$ is linearly independent

$\Rightarrow c_1=b_1,\ c_2=b_2,\ldots,\ c_n=b_n$ (i.e., uniqueness)

- Thm 4.10: (Bases and linear dependence)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

Pf:

Let $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, $m > n$

$\because span(S) = V$

$$\mathbf{u}_i \in V \quad \Rightarrow \quad \begin{aligned} \mathbf{u}_1 &= c_{11}\mathbf{v}_1 + c_{21}\mathbf{v}_2 + \cdots + c_{n1}\mathbf{v}_n \\ \mathbf{u}_2 &= c_{12}\mathbf{v}_1 + c_{22}\mathbf{v}_2 + \cdots + c_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{u}_m &= c_{1m}\mathbf{v}_1 + c_{2m}\mathbf{v}_2 + \cdots + c_{nm}\mathbf{v}_n \end{aligned}$$

Let $k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$

$\Rightarrow d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_n\mathbf{v}_n=\mathbf{0}$ (where $d_i=c_{i1}k_1+c_{i2}k_2+\ldots+c_{im}k_m$)

$\because S$ is L.I.

$\Rightarrow d_i=0 \quad \forall i$ i.e.

$$\begin{aligned} c_{11}k_1+c_{12}k_2+\cdots+c_{1m}k_m &= 0 \\ c_{21}k_1+c_{22}k_2+\cdots+c_{2m}k_m &= 0 \\ &\vdots \\ c_{n1}k_1+c_{n2}k_2+\cdots+c_{nm}k_m &= 0 \end{aligned}$$

$\because$ Thm 1.1: If the homogeneous system has fewer equations than variables, then it must have infinitely many solution.

$m>n \Rightarrow k_1\mathbf{u}_1+k_2\mathbf{u}_2+\ldots+k_m\mathbf{u}_m=\mathbf{0}$ has nontrivial solution

$\Rightarrow S_1$ is linearly dependent

- Thm 4.11: (Number of vectors in a basis)

  If a vector space V has one basis with n vectors, then every basis for V has n vectors. (All bases for a finite-dimensional vector space has the same number of vectors.)

Pf:

$$\left.\begin{array}{l} S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \\ S'=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \end{array}\right\} \text{ two bases for a vector space}$$

$$\left.\begin{array}{l} \left.\begin{array}{r} S \text{ is a basis} \\ S' \text{ is L.I.} \end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \geq m \\ \left.\begin{array}{r} S \text{ is L.I.} \\ S' \text{ is a basis} \end{array}\right\} \overset{Thm.4.10}{\Rightarrow} n \leq m \end{array}\right\} \Rightarrow n = m$$

- Finite dimensional:

  A vector space V is called **finite dimensional**,
  if it has a basis consisting of a finite number of elements.

- Infinite dimensional:

  If a vector space V is not finite dimensional,
  then it is called **infinite dimensional**.

- Dimension:

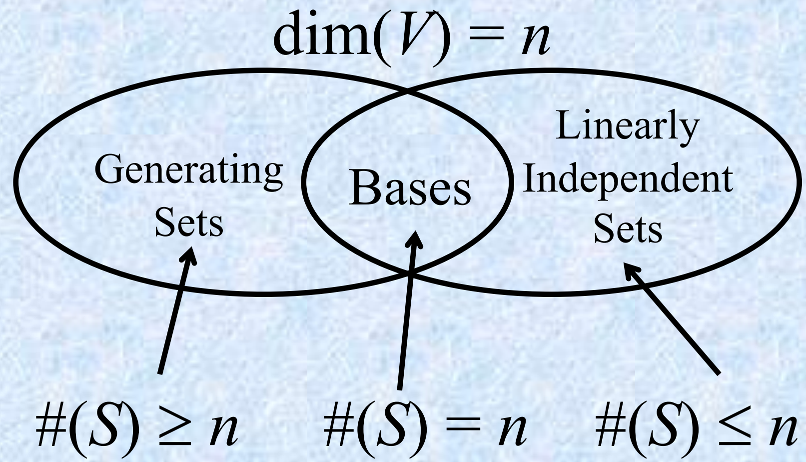
  The **dimension** of a finite dimensional vector space V is
  defined to be <u>the number of vectors in a basis</u> for V.

  V: a vector space $\qquad$ S: a basis for V

  $\Rightarrow \dim(V) = \#(S)$ $\qquad$ (the number of vectors in S)

- Notes:

(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\varnothing)$



(2) $\dim(V) = n$, $S \subseteq V$

S : a generating set $\Rightarrow$ $\#(S) \geq n$

S : a L.I. set $\Rightarrow$ $\#(S) \leq n$

S : a basis $\Rightarrow$ $\#(S) = n$

(3) $\dim(V) = n$, W is a subspace of V $\Rightarrow$ $\dim(W) \leq n$

- Ex:

(1) Vector space R^n $\Rightarrow$ basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$

$\Rightarrow \dim(R^n) = n$

(2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1\le i\le m, 1\le j\le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn$

(3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots, x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

(4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \ldots\}$

$\Rightarrow \dim(P(x)) = \infty$

- Ex 9: (Finding the dimension of a subspace)

  (a) $W=\{(d, c-d, c)$: c and d are real numbers$\}$

  (b) $W=\{(2b, b, 0)$: b is a real number$\}$

Sol: (Note: Find a set of L.I. vectors that spans the subspace)

(a) $(d, c-d, c) = c(0, 1, 1) + d(1, -1, 0)$

$\Rightarrow S = \{(0, 1, 1), (1, -1, 0)\}$ (S is L.I. and S *spans* W)

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 2$

(b) $\because (2b, b, 0) = b(2, 1, 0)$

$\Rightarrow S = \{(2, 1, 0)\}$ spans W and S is L.I.

$\Rightarrow S$ is a basis for W

$\Rightarrow \dim(W) = \#(S) = 1$

- Ex 11: (Finding the dimension of a subspace)

  Let W be the subspace of all symmetric matrices in $M_{2\times 2}$.

  What is the dimension of W?

Sol:

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle| a, b, c \in R \right\}$$

$$\because \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ spans } W \text{ and } S \text{ is L.I.}$$

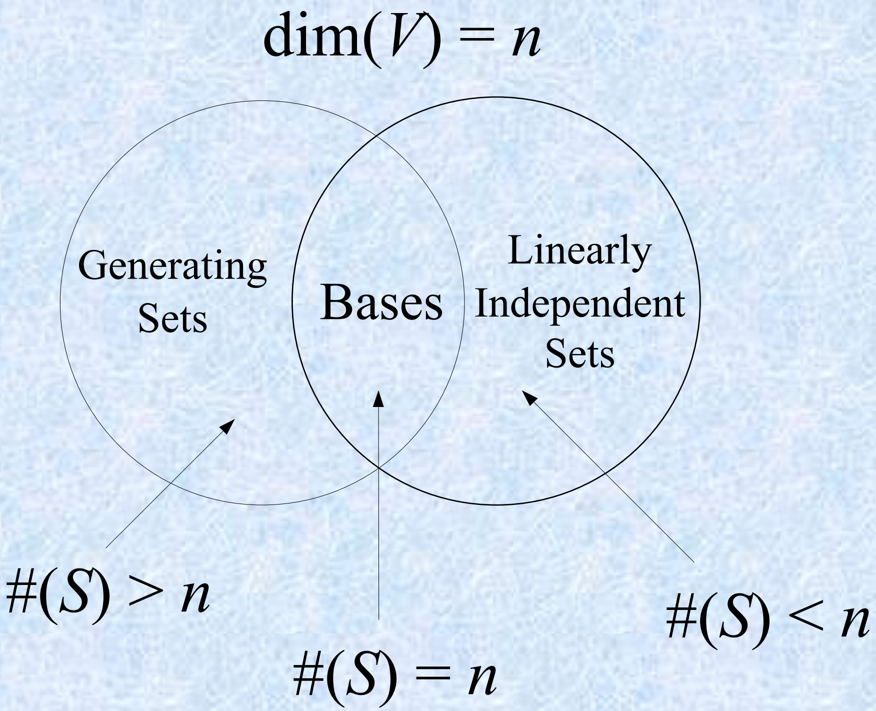
$$\Rightarrow S \text{ is a basis for } W \quad \Rightarrow \dim(W) = \#(S) = 3$$

- Thm 4.12: (Basis tests in an n-dimensional space)

Let V be a vector space of dimension n.

(1) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.

(2) If $S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_n\}$ spans V, then S is a basis for V.



Proof of Theorem 4.12 (1)

- Let $S = \{v_1, \cdots, v_n\}$ be an independent set.
- By way of contradiction, suppose that S does not span V.
- Then, there exists v $\in V$, such that $v \notin \{v_1, \cdots, v_n\}$.
- So, $\{v_1, \cdots, v_n, v\}$ is linearly independent, which is impossible by Theorem 4.10.
- Thus, S does span V, and therefore is a basis.

Proof of Theorem 4.12 (2)

- Let $S = \{v_1, \cdots, v_n\}$ spans V.
- By way of contradiction, suppose that S is linearly dependent.
- Then, $\exists v_i \in S$ is a linear combination of the other vectors in S.
- Without loss of generality, assume that v_n is a linear combination of $\{v_1, \cdots, v_{n-1}\}$, and therefore, $\{v_1, \cdots, v_{n-1}\}$ spans V.
- But, $n - 1$ vectors span a vector space of at most dimension $n - 1$, which is a contradiction.
- So, S is linearly independent, and therefore a basis.

Key Learning in Section 4.5

- Recognize bases in the vector spaces R^n, P_n and $M_{m,n}$
- Find the dimension of a vector space.

Keywords in Section 4.5

- basis：基底
- dimension：維度
- finite dimension：有限維度
- infinite dimension：無限維度

Review exercises

35. Determine whether S is a basis for $M_{2,2}$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

4.6 Rank of a Matrix and Systems of Linear Equations

- row vectors:

Row vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\begin{bmatrix} A_{(1)} \\ A_{(2)} \\ \vdots \\ A_{(m)} \end{bmatrix}$$

$$\begin{gathered} [a_{11}, a_{12}, \ldots, a_{1n}] = A_{(1)} \\ [a_{21}, a_{22}, \ldots, a_{2n}] = A_{(2)} \\ \vdots \\ [a_{m1}, a_{m2}, \ldots, a_{mn}] = A_{(n)} \end{gathered}$$

- column vectors:

Column vectors of A

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}=\left[A^{(1)} \vdots A^{(2)} \vdots \cdots \vdots A^{(n)}\right]$$

$$\underset{A^{(1)}}{\underset{\parallel}{\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}}} \underset{A^{(2)}}{\underset{\parallel}{\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}}} \cdots \underset{A^{(n)}}{\underset{\parallel}{\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}}}$$

Let A be an $m\times n$ matrix.

- Row space:

  The **row space** of A is the subspace of R^n spanned by the row vectors of A.

$$RS(A)=\{\alpha_1 A_{(1)}+\alpha_2 A_{(2)}+\ldots+\alpha_m A_{(m)}\mid \alpha_1,\alpha_2,\ldots,\alpha_m\in R\}$$

- Column space:

  The **column space** of A is the subspace of R^m spanned by the column vectors of A.

$$CS(A)=\{\beta_1 A^{(1)}+\beta_2 A^{(2)}+\cdots+\beta_n A^{(n)}\big|\beta_1,\beta_2,\cdots\beta_n\in R\}$$

- Null space:

  The **null space** of A is the set of all solutions of $A\mathbf{x}=\mathbf{0}$ and it is a subspace of R^n.

$$NS(A)=\{\mathbf{x}\in R^n\mid A\mathbf{x}=\mathbf{0}\}$$

- Thm 4.13: (Row-equivalent matrices have the same row space)

If an $m \times n$ matrix A is row equivalent to an $m \times n$ matrix B,

then the row space of A is equal to the row space of B.

  - Notes:

    (1) The row space of a matrix is not changed by elementary row operations.

    $RS(r(A)) = RS(A)$ $\quad$ r: elementary row operations

    (2) Elementary row operations can change the column space.

Proof of Theorem 4.13

- Because the rows of B can be obtained from the rows of A by elementary row operations, it follows that the row vectors of B can be written as linear combinations of the row vectors of A.
- The row vectors of B lie in the row space of A, and the subspace spanned by the row vectors of B is contained in the row space of A.
- But it is also true that the rows of A can be obtained from the rows of B by row elementary operations.
- So, the two row spaces are subspaces of each other, making them equal.

- Thm 4.14: (Basis for the row space of a matrix)

  If a matrix A is row equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

- Ex 2: (Finding a basis for a row space)

Find a basis of row space of $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix}$

Sol:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & 2 \end{bmatrix} \\ \begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{matrix} \end{array} \xrightarrow{\text{G.E.}} B = \begin{array}{cc} \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \\ \\ \\ \end{matrix} \\ \begin{matrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{matrix} & \end{array}$$

a basis for $RS(A)$ = {the nonzero row vectors of B} (Thm 4.14)
= $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ = $\{(1, 3, 1, 3), (0, 1, 1, 0), (0, 0, 0, 1)\}$

- Notes:

(1) $\mathbf{b}_3 = -2\mathbf{b}_1 + \mathbf{b}_2 \Rightarrow \mathbf{a}_3 = -2\mathbf{a}_1 + \mathbf{a}_2$

(2) $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is L.I. $\Rightarrow \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is L.I.

- Ex 3: (Finding a basis for a subspace)

Find a basis for the subspace of R^3 spanned by

$$S = \{(-1, \overset{\mathbf{v}_1}{2}, 5), (3, \overset{\mathbf{v}_2}{0}, 3), (5, \overset{\mathbf{v}_3}{1}, 8)\}$$

Sol:

$$A = \begin{array}{c} \begin{bmatrix} -1 & 2 & 5 \\ 3 & 0 & 3 \\ 5 & 1 & 8 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{array} \xrightarrow{\text{G.E.}} B = \begin{array}{c} \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \end{array}$$

a basis for $span(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\})$

$= $ a basis for $RS(A)$

$= $ {the nonzero row vectors of B} (Thm 4.14)

$= \{\mathbf{w}_1, \mathbf{w}_2\}$

$= \{(1, -2, -5), (0, 1, 3)\}$

- Ex 4-5: (Finding a basis for the column space of a matrix)

Find a basis for the column space of the matrix A given in Ex 2.

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix}$$

Sol. (Method 1):

$$A^T = \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 3 & 1 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & -4 \\ 3 & 0 & -1 & 1 & -2 \end{bmatrix} \xrightarrow{G.E.} B = \begin{array}{c} \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \\ \\ \end{array}$$

$\because \quad CS(A) = RS(A^{\mathrm{T}})$

$\therefore \quad$ a basis for $CS(A)$

$=$ a basis for $RS(A^{\mathrm{T}})$

$=$ {the nonzero vectors of B}

$= \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$

$$= \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 9 \\ -5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$ (a basis for the column space of A)

- Note: This basis is not a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

- Sol. (Method 2):

$$A=\begin{array}{c}\left[\begin{array}{cccc}1 & 3 & 1 & 3\\ 0 & 1 & 1 & 0\\ -3 & 0 & 6 & -1\\ 3 & 4 & -2 & 1\\ 2 & 0 & -4 & -2\end{array}\right]\\ \begin{array}{cccc}\mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4\end{array}\end{array} \xrightarrow{G.E.} B=\begin{array}{c}\left[\begin{array}{cccc}1 & 3 & 1 & 3\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]\\ \begin{array}{cccc}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{array}\end{array}$$

Leading 1 => $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for $CS(B)$

$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_4\}$ is a basis for $CS(A)$

- Notes:

(1) This basis is a subset of $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$.

(2) $\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$, thus $\mathbf{c}_3 = -2\mathbf{c}_1 + \mathbf{c}_2$.

- Thm 4.16: (Solutions of a homogeneous system)

  If A is an $m\times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\mathbf{x} = \mathbf{0}$ is a subspace of R^n called the nullspace of A.

$$NS(A) = \{x \in R^n \mid Ax = 0\}$$

Pf:

$$NS(A) \in R^n$$

$$NS(A) \neq \phi \quad (\because A\mathbf{0} = \mathbf{0})$$

Let $\mathbf{x}_1, \mathbf{x}_2 \in NS(A)$ (i.e. $A\mathbf{x}_1 = \mathbf{0}, A\mathbf{x}_2 = \mathbf{0}$)

Then (1) $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$ Addition

(2) $A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(\mathbf{0}) = \mathbf{0}$ Scalar multiplication

Thus $NS(A)$ is a subspace of R^n

- Notes: The nullspace of A is also called the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

■ Ex 7: (Finding the solution space of a homogeneous system)

Find the nullspace of the matrix A.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}$$

Sol: The nullspace of A is the solution space of $A\mathbf{x} = \mathbf{0}$.

$$A=\begin{bmatrix}1 & 2 & -2 & 1\\ 3 & 6 & -5 & 4\\ 1 & 2 & 0 & 3\end{bmatrix}\xrightarrow{G.J.E}\begin{bmatrix}1 & 2 & 0 & 3\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow x_1 = -2s - 3t,\ x_2 = s,\ x_3 = -t,\ x_4 = t$$

$$\Rightarrow \mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=\begin{bmatrix}-2s-3t\\ s\\ -t\\ t\end{bmatrix}=s\begin{bmatrix}-2\\ 1\\ 0\\ 0\end{bmatrix}+t\begin{bmatrix}-3\\ 0\\ -1\\ 1\end{bmatrix}=s\mathbf{v}_1+t\mathbf{v}_2$$

$$\Rightarrow NS(A)=\{s\mathbf{v}_1+t\mathbf{v}_2 \mid s,t\in R\}$$

- Thm 4.15: (Row and column space have equal dimensions)

  If A is an $m \times n$ matrix, then the row space and the column space of A have the same dimension.

$$\dim(RS(A)) = \dim(CS(A))$$

- Rank:

  The dimension of the row (or column) space of a matrix A is called the **rank** of A and is denoted by rank(A).

$$\operatorname{rank}(A) = \dim(RS(A)) = \dim(CS(A))$$

Proof of Theorem 4.15

- Let $v_1, \cdots, v_m$ be the row vectors and $u_1, \cdots, u_n$ be the column vectors of

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

- Suppose the row space of A has dimension r and basis $S = \{b_1, \cdots, b_r\}$, where $b_i = (b_{i1}, \cdots, b_{in})$.

- Using this basis, you can write the row vectors of A as

$$\begin{gathered} v_1 = c_{11}b_1 + \cdots + c_{1r}b_r \\ \vdots \\ v_m = c_{m1}b_1 + \cdots + c_{mr}b_r \end{gathered}$$

- Rewrite this system of vector equations as follows.

$$[a_{11} \cdots a_{1n}] = c_{11}[b_{11} \cdots b_{1n}] + \cdots + c_{1r}[b_{r1} \cdots b_{rn}]$$
$$\vdots$$
$$[a_{m1} \cdots a_{mn}] = c_{m1}[b_{11} \cdots b_{1n}] + \cdots + c_{mr}[b_{r1} \cdots b_{rn}]$$

- Now, for the entries of the jth column, you can obtain the following system.

$$a_{1j} = c_{11}b_{1j} + \cdots + c_{1r}b_{rj}$$
$$\vdots$$
$$a_{mj} = c_{m1}b_{1j} + \cdots + c_{mr}b_{rj}$$

- Let the vectors $c_i = [c_{1i} \cdots c_{mi}]^T$.
- Then, the jth column can be rewritten as

$$u_j = b_{1j}c_1 + \cdots + b_{rj}c_r.$$

- Put all column vectors together to obtain

$$u_1 = b_{11}c_1 + \cdots + b_{r1}c_r$$
$$\vdots$$
$$u_n = b_{1n}c_1 + \cdots + b_{rn}c_r$$

- Because each column vector of A is a linear combination of r vectors, the dimension of the column space of A is less than or equal to r.

- That is, dim(column space of A)≤dim(row space of A)

- Repeat this procedure for A^T, the dimension of the column space of A^T is less than or equal to the row space of A^T.

- This implies that the dimension of the row space of A is less than or equal to the dimension of the column space of A.

- That is, dim(row space of A)≤dim(column space of A).

- So, the two dimension must be equal.

- **Nullity:** 核次數

The dimension of the nullspace of A is called the nullity of A.

$$\text{nullity}(A) = \dim(NS(A))$$

- **Note:** $\text{rank}(A^T) = \text{rank}(A)$

Pf: $\text{rank}(A^T) = \dim(RS(A^T)) = \dim(CS(A)) = \text{rank}(A)$

- Thm 4.17: (Dimension of the solution space)

If A is an $m \times n$ matrix of rank r, then the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is $n - r$. That is

$$n = \text{rank}(A) + \text{nullity}(A)$$

- Notes:

(1) rank(A): The number of leading variables in the solution of $A\mathbf{x}=\mathbf{0}$.

(The number of nonzero rows in the row-echelon form of A)

(2) nullity (A): The number of free variables in the solution of $A\mathbf{x} = \mathbf{0}$.

- Notes:

If A is an $m \times n$ matrix and rank(A) = r, then

Fundamental Space	Dimension
$RS(A)=CS(A^T)$	r
$CS(A)=RS(A^T)$	r
$NS(A)$	$n-r$
$NS(A^T)$	$m-r$

- Ex 8: (Rank and nullity of a matrix)

Let the column vectors of the matrix A be denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, and $\mathbf{a}_5$.

$$A=\begin{array}{c} \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array}$$

(a) Find the rank and nullity of A.

(b) Find a subset of the column vectors of A that forms a basis for the column space of A .

(c) If possible, write the third column of A as a linear combination of the first two columns.

Sol: Let B be the reduced row-echelon form of A.

$$A = \begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{array}\right] \\ \begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array} \end{array} \qquad B = \begin{array}{c} \left[\begin{array}{ccccc} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \end{array} \end{array}$$

(a) $\operatorname{rank}(A) = 3$ (the number of nonzero rows in B)

$$\text{nuillity}(A) = n - \operatorname{rank}(A) = 5 - 3 = 2$$

(b) Leading 1

$\Rightarrow \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$ is a basis for $CS(B)$

$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is a basis for $CS(A)$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \text{ and } \mathbf{a}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix},$$

(c) $\mathbf{b}_3 = -2\mathbf{b}_1 + 3\mathbf{b}_2 \Rightarrow \ \mathbf{a}_3 = -2\mathbf{a}_1 + 3\mathbf{a}_2$

- Thm 4.18: (Solutions of a nonhomogeneous linear system)

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \boldsymbol{b}$, then every solution of this system can be written in the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, wher $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

Pf:

Let $\mathbf{x}$ be any solution of $A\mathbf{x} = \boldsymbol{b}$.

$$\Rightarrow A(\mathbf{x}-\mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b}-\mathbf{b} = \mathbf{0}.$$

$\Rightarrow (\mathbf{x}-\mathbf{x}_p)$ is a solution of $A\mathbf{x} = \mathbf{0}$

Let $\boldsymbol{x}_h = \boldsymbol{x} - \boldsymbol{x}_p$

$$\Rightarrow \boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_h$$

- Ex 9: (Finding the solution set of a nonhomogeneous system)

Find the set of all solution vectors of the system of linear equations.

$$\begin{aligned} x_1 \qquad\quad - 2x_3 + x_4 &= 5 \\ 3x_1 + x_2 - 5x_3 \qquad &= 8 \\ x_1 + 2x_2 \qquad - 5x_4 &= -9 \end{aligned}$$

Sol:

$$\begin{array}{c} \left[\begin{array}{cccc:c} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{array}\right] \xrightarrow{G.J.E} \begin{array}{c} \left[\begin{array}{cccc:c} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} & & s & t & \end{array} \end{array} \end{array}$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s & - & t & + & 5 \\ -s & + & 3t & - & 7 \\ s & + & 0t & + & 0 \\ 0s & + & t & + & 0 \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

$$= s\mathbf{u}_1 + t\mathbf{u}_2 + \mathbf{x}_p$$

i.e. $\mathbf{x}_p = \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution vector of $A\mathbf{x}=\mathbf{b}$.

$\mathbf{x}_h = s\mathbf{u}_1 + t\mathbf{u}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$

- Thm 4.19: (Solution of a system of linear equations)

  The system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Pf:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

be the coefficient matrix, the column matrix of unknowns, and the right-hand side, respectively, of the system $A\mathbf{x} = \mathbf{b}$.

Then

$$A\mathbf{x}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=\begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{bmatrix}$$

$$=x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}+x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}+\cdots+x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Hence, $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A. That is, the system is consistent if and only if $\mathbf{b}$ is in the subspace of R^m spanned by the columns of A.

- Note:

  If rank([A|**b**])=rank(A)

  Then the system A**x**=**b** is consistent.

- Ex 10: (Consistency of a system of linear equations)

$$\begin{aligned} x_1 &+ x_2 &- x_3 &= -1 \\ x_1 & &+ x_3 &= 3 \\ 3x_1 &+ 2x_2 &- x_3 &= 1 \end{aligned}$$

Sol:

$$A=\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix} \xrightarrow{G.J.E.} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A \vdots \mathbf{b}] = \begin{array}{c} \left[\begin{array}{ccc:c} 1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & -1 & 1 \end{array}\right] \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{b} \end{array} \end{array} \xrightarrow{G.J.E.} \begin{array}{c} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \mathbf{v} \end{array} \end{array}$$

$$\because \mathbf{v} = 3\mathbf{w}_1 - 4\mathbf{w}_2$$

$$\Rightarrow \mathbf{b} = 3\mathbf{c}_1 - 4\mathbf{c}_2 + 0\mathbf{c}_3 \quad (\mathbf{b} \text{ is in the column space of } A)$$

$\Rightarrow$ The system of linear equations is consistent.

- Check:

$$\operatorname{rank}(A) = \operatorname{rank}([A \vdots \mathbf{b}]) = 2$$

- Summary of equivalent conditions for square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent.

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $|A| \neq 0$

(6) $\operatorname{rank}(A) = n$

(7) The n row vectors of A are linearly independent.

(8) The n column vectors of A are linearly independent.

Key Learning in Section 4.6

- Find a basis for the row space, a basis for the column space, and the rank of a matrix.
- Find the nullspace of a matrix.
- Find the solution of a consistent system $Ax = b$ in the form $x_p + x_h$.

Keywords in Section 4.6:

- row space : 列空間
- column space : 行空間
- null space: 零空間
- solution space : 解空間
- rank: 秩
- nullity : 核次數

Review exercises

39. Find (a) a basis for the null space, (b) the nullity, and (c) the rank of the matrix A.

$$A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$$