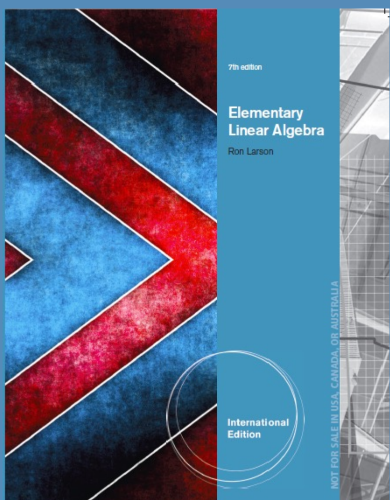
CHAPTER 4
VECTOR SPACES



4.3 Subspaces of Vector Spaces

- Subspace:

$(V, +, \bullet)$: a vector space

$\left.\begin{array}{l} W \neq \phi \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W, +, \bullet)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow$ W is a subspace of V

- Trivial subspace:

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

Definition of Subspace of a Vector Space

- A nonempty subset W of a vector space V is called a **subspace** of V when W is a vector space under the operation of addition and scalar multiplication defined in V.

To establish that a set W is a vector space, you must verify all ten vector space axioms.

If W is a nonempty subset of a larger vector space V, then most of the ten properties are inherited from V and need no verification.

- Thm 4.5: (Test for a subspace)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold.

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

Because a subspace of a vector space is a vector space, it must contain the zero vector.

Proof of Theorem 4.5

- The proof in one direction is straightforward. If W is a subspace of V, then W is a vector space and must be closed under addition and scalar multiplication.
- To prove the theorem in the other direction, assume that W is closed under addition and scalar multiplication. Note that if **u**, **v**, and **w** are in W, then they are also in V.
- Consequently, axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically. Because W is closed under addition and scalar multiplication, it follows that for any **v** in W and scalar c=0, cv=**0** and (-1)**v**=-**v** both lie in W, which satisfies axioms 4 and 5.

- Ex: Subspace of R^2

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0)$

  (2) Lines through the origin

  (3) R^2

- Ex: Subspace of R^3

  (1) $\{\mathbf{0}\}$ $\quad \mathbf{0}=(0,0,0)$

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

- Ex 2: (A subspace of $M_{2\times 2}$)

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Sol: You need to show that W satisfies the conditions of Theorem 4.5.

$\because W \subseteq M_{2\times 2} \qquad M_{2\times 2}$: vector sapces

Let $A_1, A_2 \in W \quad (A_1^T = A_1, A_2^T = A_2)$

$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \qquad (A_1 + A_2 \in W)$

$k \in R, A \in W \Rightarrow (kA)^T = kA^T = kA \qquad (kA \in W)$

$\therefore W$ is a subspace of $M_{2\times 2}$

- Ex 3: (The set of singular matrices is not a subspace of $M_{2\times 2}$)

  Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times 2}$ with the standard operations.

Sol:

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\in W$$

$$\therefore A+B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\notin W$$

$$\therefore W_2 \text{ is not a subspace of } M_{2\times 2}$$

- Ex 4: (The set of first-quadrant vectors is not a subspace of R^2)

  Show that $W=\{(x_1,x_2): x_1\geq 0 \text{ and } x_2\geq 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\mathbf{u}=(1,1)\in W$

$\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1)\notin W$

(not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

- Ex 6: (Determining subspaces of R^2)

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x+2y=0$.

(b) The set of points on the line given by $x+2y=1$.

Sol:

(a) $W=\{(x,y)\mid x+2y=0\}=\{(-2t,t)\mid t\in R\}$

Let $v_1=(-2t_1,t_1)\in W \quad v_2=(-2t_2,t_2)\in W$

$\because v_1+v_2=(-2(t_1+t_2),t_1+t_2)\in W$ (closed under addition)

$kv_1=(-2(kt_1),kt_1)\in W$ (closed under scalar multiplication)

$\therefore W$ is a subspace of R^2

(b) $W = \{(x, y) \mid x + 2y = 1\}$ (Note: the zero vector is not on the line)

Let $v = (1,0) \in W$

$\because (-1)v = (-1,0) \notin W$

$\therefore W$ is not a subspace of R^2

- Ex 8: (Determining subspaces of R^3)

Which of the following subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$

(b) $W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$

Sol:

(a) Let $\mathbf{v} = (0,0,1) \in W$

$\Rightarrow (-1)\mathbf{v} = (0,0,-1) \notin W$

$\therefore W$ is not a subspace of R^3

(b) Let $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_1 + \mathrm{v}_3, \mathrm{v}_3) \in W$, $\mathbf{u} = (\mathrm{u}_1, \mathrm{u}_1 + \mathrm{u}_3, \mathrm{u}_3) \in W$

$\because \mathbf{v} + \mathbf{u} = \left(\mathrm{v}_1 + \mathrm{u}_1, \left(\mathrm{v}_1 + \mathrm{u}_1\right) + \left(\mathrm{v}_3 + \mathrm{u}_3\right), \mathrm{v}_3 + \mathrm{u}_3\right) \in \mathrm{W}$

$k\mathbf{v} = \left(k\mathrm{v}_1, \left(k\mathrm{v}_1\right) + \left(k\mathrm{v}_3\right), k\mathrm{v}_3\right) \in \mathrm{W}$

$\therefore W$ is a subspace of R^3

- Thm 4.6: (The intersection of two subspaces is a subspace)

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap U$) is also a subspace of U.

Proof of Theorem 4.6

- Both V and W contain the zero vector, which means $V \cap W$ is nonempty.

- Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two vectors in $V \cap W$. Then, because V and W are both subspaces of U, both are closed under addition. That is, $\mathbf{v}_1 + \mathbf{v}_2$ must be in V; similarly, $\mathbf{v}_1 + \mathbf{v}_2$ must be in W. This implies $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$.

- Similarly, it can be shown that $V \cap W$ is closed under scalar multiplication.

Key Learning in Section 4.3

- Determine whether a subset W of a vector space V is a subspace of V.
- Determine subspaces of R^n.

Keywords in Section 4.3:

- subspace：子空間
- trivial subspace：顯然子空間

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k \qquad c_1, c_2, \cdots, c_k : \text{scalars}$$

- Ex 2-3: (Finding a linear combination)

$$\mathbf{v}_1=(1,2,3) \quad \mathbf{v}_2=(0,1,2) \quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol:

(a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1)&=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)\\ &=(c_1-c_3,\,2c_1+c_2,\,3c_1+2c_2+c_3)\end{aligned}$$

$$\Rightarrow \begin{aligned} c_1 \qquad\quad -c_3 &= 1\\ 2c_1+\;\; c_2 \qquad\quad &= 1\\ 3c_1+2c_2+c_3 &= 1\end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Guass}-\text{Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Guass−Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(\because 0 \neq 7)$

$$\Rightarrow \mathbf{w} \neq c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

- the span of a set: span (S)

If $S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$span(S)=\{c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k \mid \forall c_i \in R\}$$

(the set of all linear combinations of vectors in S)

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space.

- Notes:

$$\text{span}(S) = V$$
$$\Rightarrow S \text{ spans (generates) } V$$
$$V \text{ is spanned (generated) by } S$$
$$S \text{ is a spanning set of } V$$

- Notes:

(1) $span(\phi) = \{\mathbf{0}\}$

(2) $S \subseteq span(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$$

- Ex 5: (A spanning set for R^3)

Show that the set $S=\{(1,2,3),(0,1,2),(-2,0,1)\}$ sapns R^3

Sol:

We must determine whether an arbitrary vector $\mathbf{u}=(u_1,u_2,u_3)$ in R^3 can be as a linear combination of $\mathbf{v}_1,\mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{u}\in R^3 \Rightarrow \mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$$

$$\begin{aligned}\Rightarrow\ c_1 \qquad\quad -2c_3 &= u_1\\ 2c_1+\ c_2 \qquad\quad &= u_2\\ 3c_1+2c_2+c_3 &= u_3\end{aligned}$$

The problem thus reduces to determining whether this system is consistent for all values of u_1,u_2, and u_3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\mathbf{x} = \mathbf{b}$ has exactly one solution for every u.

$$\Rightarrow span(S) = R^3$$

- Thm 4.7: (Span(S) is a subspace of V)

If $S=\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V,

then

(a) span (S) is a subspace of V.

(b) span (S) is the smallest subspace of V that contains S.

(Every other subspace of V that contains S must contain span (S).)

Proof of Theorem 4.7

- To show that span(S) is a subspace of V, show that it is closed under addition and scalar multiplication.

- Consider two vectors **u** and **v** in span(S),

$$\mathbf{u} = c_1\mathbf{v}_1 + \mathrm{c}_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_\mathrm{k}$$

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_\mathrm{k}$$

Then

$$(\mathbf{u} + \mathbf{v}) = (c_1 + d_1)\mathbf{v}_1 + \cdots + (c_k + d_k)\mathbf{v}_k$$

$$c\mathbf{u} = (cc_1)\mathbf{v}_1 + \cdots + (cc_k)\mathbf{v}_k$$

which means (u+v) and cu are also in span(S). So, span(S) is a subspace of V.

- The following proves that span(S) is the smallest subspace of V that contains S.

- Let U be another subspace of V that contains S. To show that span(S) $\subset$ U, let u$\in$span(S).

- Then, $\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{v}_i$, where $\mathbf{v}_i \in$ S.

- Because U contains S, $\mathbf{v}_i \in$ U.

- Since U is a subspace, $\mathbf{u} \in$ U. (∵封閉性)

Definition

- Linear Independent (L.I.) and Linear Dependent (L.D.):

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\cdots,\mathbf{v}_k\} \quad \text{: a set of vectors in a vector space V}$$

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_k\mathbf{v}_k=\mathbf{0}$$

(1) If the equation has only the trivial solution $(c_1=c_2=\cdots=c_k=0)$
then S is called linearly independent.

(2) If the equation has a nontrivial solution (i.e., not all zeros),
then S is called linearly dependent.

- Notes:

(1) ϕ is linearly independent

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\mathbf{v} \neq \mathbf{0} \Rightarrow \{\mathbf{v}\}$ is linearly independent

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 8: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{(1,2,3)}, \underset{\mathbf{v}_2}{(0,1,2)}, \underset{\mathbf{v}_3}{(-2,0,1)}\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Rightarrow \begin{aligned} c_1 \quad\quad\quad - 2c_3 &= 0 \\ 2c_1 + c_2 + \quad\quad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = c_3 = 0 \quad (\text{only the trivial solution})$$

$$\Rightarrow S \text{ is linearly independent}$$

- Ex 9: (Testing for linearly independent)

Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\underset{\mathbf{v}_1}{1+x-2x^2},\ \underset{\mathbf{v}_2}{2+5x-x^2},\ \underset{\mathbf{v}_3}{x+x^2}\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3 = \mathbf{0}$$

P_n = set of all polynomials of degree ≤ n

i.e. $c_1(1+x-2x^2) + c_2(2+5x-x^2) + c_3(x+x^2) = 0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \qquad &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1-c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.J.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions.
(i.e., This system has nontrivial solutions.)

$\Rightarrow$ S is linearly dependent. (Ex: $c_1=2$, $c_2=-1$, $c_3=3$)

- Ex 10: (Testing for linearly independent)

  Determine whether the following set of vectors in 2×2 matrix space is L.I. or L.D.

$$S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}} \right\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} + c_3\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1+3c_2+\ c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2+2c_3 &= 0 \\ c_1+c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution.)

$\Rightarrow$ S is linearly independent.

- Thm 4.8: (A property of linearly dependent sets)

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\boldsymbol{v}_j$ in S can be written as a linear combination of the other vectors in S.

Pf:

$(\Rightarrow)$ $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$

$\because$ S is linearly dependent

$\Rightarrow c_i \neq 0$ for some i

$$\Rightarrow \mathbf{v}_i = \frac{c_1}{c_\mathrm{i}}\mathbf{v}_1 + \cdots + \frac{c_{i-1}}{c_i}\mathbf{v}_{i-1} + \frac{c_{i+1}}{c_i}\mathbf{v}_{i+1} + \cdots + \frac{c_k}{c_i}\mathbf{v}_k$$

(⇐)

Let $\quad \mathbf{v}_i = d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k$

$\Rightarrow d_1\mathbf{v}_1+\ldots+d_{i\text{-}1}\mathbf{v}_{i\text{-}1}-\mathbf{v}_i+d_{i+1}\mathbf{v}_{i+1}+\ldots+d_k\mathbf{v}_k = \mathbf{0}$

$\Rightarrow c_1=d_1,\ \ldots,c_{i\text{-}1}=d_{i\text{-}1},\ c_i=-1,c_{i+1}=d_{i+1},\ldots,\ c_k=d_k$ (nontrivial solution)

$\Rightarrow S$ is linearly dependent

- Corollary to Theorem 4.8:

  Two vectors **u** and **v** in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Key Learning in Section 4.4

- Write a linear combination of a set of vectors in a vector space *V*.
- Determine whether a set of vectors in a vector space *V* is a spanning set of *V*.
- Determine whether a set of vectors in a vector space *V* is linearly independent.

Keywords in Section 4.4:

- linear combination：線性組合
- spanning set：生成集合
- trivial solution：顯然解
- linear independent：線性獨立
- linear dependent：線性相依

Review exercises

19. Determine whether W is a subspace of the vector space V.

$$W = \{(x, y): y = ax, a \text{ } is \text{ } an \text{ } integer\}, \qquad V = R^2$$

24. Determine whether W is a subspace of the vector space V.

$$W = \{f: f(-1) = 0\}, \qquad V = C[-1,1]$$

C[a,b]=set of all continuous functions defined on a closed interval [a,b]

30. Determine whether $S = \{(2,0,1), (2,-1,1), (4,2,0)\}$ is linearly independent.