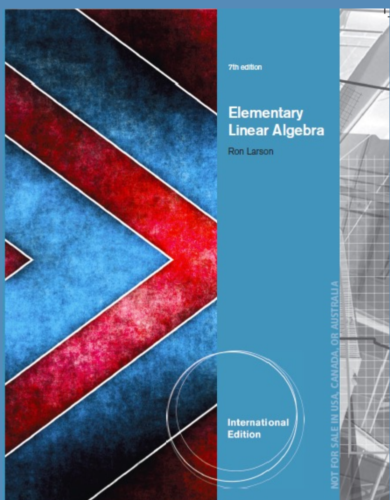
CHAPTER 3
DETERMINANTS

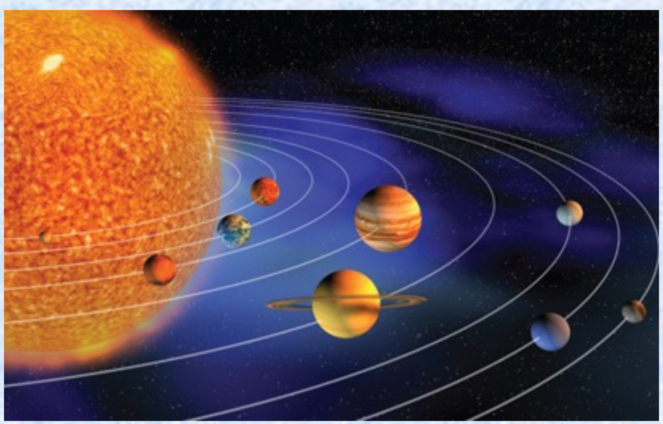


CH 3 Linear Algebra Applied

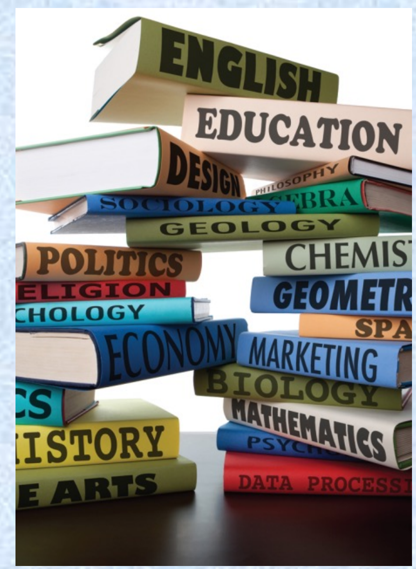
Volume of a Solid (p.108)

Sudoku (p.114)

Engineering and Control (p.124)



Planetary Orbits (p.135)



Textbook Publishing (p.137)

3.1 The Determinant of a Matrix

- the determinant of a 2 × 2 matrix:

行列式

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Note:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

- Ex. 1: (The determinant of a matrix of order 2)

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(2) - 4(1) = 4 - 4 = 0$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- Note: The determinant of a matrix can be positive, zero, or negative.

子行列式

- Minor of the entry a_{ij} :

The determinant of the matrix determined by deleting the *i*th row and *j*th column of A

$$M_{ij}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & & & \vdots & \vdots & & \\ a_{(i-1)1} & & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ a_{n1} & & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

餘因子

- Cofactor of a_{ij} :

$$C_{ij}=(-1)^{i+j}M_{ij}$$

- Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21} \qquad C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

- Notes: Sign pattern for cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

3×3 matrix $\qquad$ 4×4 matrix $\qquad$ $n \times n$ matrix

- Notes:

  Odd positions (where $i+j$ is odd) have negative signs, and even positions (where $i+j$ is even) have positive signs.

- Ex 2: Find all the minors and cofactors of A.

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of A.

$$\Rightarrow M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,\ M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5,\ M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2,\quad M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,\ M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,\quad M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3,\quad M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

Sol: (2) All the cofactors of A.

$$\because \quad C_{ij} = (-1)^{i+j} M_{ij}$$

$$\Rightarrow C_{11} = +\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1, \; C_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5, \quad C_{13} = +\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2, \quad C_{22} = +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4, \quad C_{23} = -\begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5, \quad C_{32} = -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3, \quad C_{33} = +\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

- Thm 3.1: (Expansion by cofactors)

Let A is a square matrix of order n.

Then the determinant of A is given by

$$(a)\quad \det(A)=|A|=\sum_{j=1}^{n} a_{ij}C_{ij}=a_{i1}C_{i1}+a_{i2}C_{i2}+\cdots+a_{in}C_{in}$$

(Cofactor expansion along the i-th row, i=1, 2,…, n)

or

$$(b)\quad \det(A)=|A|=\sum_{i=1}^{n} a_{ij}C_{ij}=a_{1j}C_{1j}+a_{2j}C_{2j}+\cdots+a_{nj}C_{nj}$$

(Cofactor expansion along the j-th row, j=1, 2,…, n)

- Ex: The determinant of a matrix of order 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$

- Ex 3: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \quad \overset{\text{Ex 2}}{\Rightarrow} \begin{matrix} C_{11} = -1, & C_{12} = 5, & C_{13} = 4 \\ C_{21} = -2, & C_{22} = -4, & C_{23} = 8 \\ C_{31} = 5, & C_{32} = 3, & C_{33} = -6 \end{matrix}$$

Sol:

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14 \end{aligned}$$

- Ex 5: (The determinant of a matrix of order 3)

$$A=\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A)=?$$

Sol:

$$\boxed{C_{11}=(-1)^{1+1}\begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix}=7} \qquad C_{12}=(-1)^{1+2}\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}=(-1)(-5)=5$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix}=-8$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13} \\ &= (0)(7)+(2)(5)+(1)(-8) \\ &= 2 \end{aligned}$$

- Notes:

The row (or column) containing the most zeros is the best choice for expansion by cofactors .

- Ex 4: (The determinant of a matrix of order 4)

$$A = \begin{bmatrix} 1 & -2 & \boxed{3} & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\begin{aligned}
\det(A) &= (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) \\
&= 3C_{13} \\
&= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \\
&= 3\left[(0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right] \\
&= 3[0 + (2)(1)(-4) + (3)(-1)(-7)] \\
&= (3)(13) \\
&= 39
\end{aligned}$$

- The determinant of a matrix of order 3:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Subtract these three products.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Add these three products.

$$\Rightarrow \det(A)=|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}$$
$$-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

- Ex 5:

$$\begin{array}{cccccc} & & & -4 & 0 & 6 \\ A = & \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} & & \begin{matrix} 0 \\ 3 \\ 4 \end{matrix} & \begin{matrix} 2 \\ -1 \\ -4 \end{matrix} & \\ & & & 0 & 16 & -12 \end{array}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4) - 0 - 6 = 2$$

- Upper triangular matrix:

  All the entries below the main diagonal are zeros.

- Lower triangular matrix:

  All the entries above the main diagonal are zeros.

- Diagonal matrix:

  All the entries above and below the main diagonal are zeros.

- Note:

  A matrix that is both upper and lower triangular is called diagonal.

- Ex:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular　　lower triangular　　diagonal

- Thm 3.2: (Determinant of a Triangular Matrix)

  If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then its determinant is the product of the entries on the main diagonal. That is

$$\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$$

■ Ex 6: Find the determinants of the following triangular matrices.

$$(a)\quad A=\begin{bmatrix}2 & 0 & 0 & 0\\ 4 & -2 & 0 & 0\\ -5 & 6 & 1 & 0\\ 1 & 5 & 3 & 3\end{bmatrix}\qquad (b)\quad B=\begin{bmatrix}-1 & 0 & 0 & 0 & 0\\ 0 & 3 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & -2\end{bmatrix}$$

Sol:

$$(a)\quad |A| = (2)(-2)(1)(3) = -12$$

$$(b)\quad |B| = (-1)(3)(2)(4)(-2) = 48$$

Key Learning in Section 3.1

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors..

Keywords in Section 3.1

- determinant : 行列式
- minor : 子行列式
- cofactor : 餘因子
- expansion by cofactors : 餘因子展開
- upper triangular matrix: 上三角矩陣
- lower triangular matrix: 下三角矩陣
- diagonal matrix: 對角矩陣

3.2 Evaluation of a determinant using elementary operations

- Thm 3.3: (Elementary row operations and determinants)

Let A and B be square matrices.

$$(a)\quad B = r_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b)\quad B = r_i^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c)\quad B = r_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

Proof of (a)

1. Assume that A and B are 2×2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}.$$

Then, you have $|B| = -|A|$.

2. Using mathematical induction, assume the property is true for matrices of order (n-1).

3. Let A be an $n \times n$ matrix, such that B is obtained from A by interchanging two rows of A.

Then, to find $|A|$ and $|B|$, expand in a row other than the two interchanged rows.

By the induction assumption, the cofactors of B will be the negatives of the cofactors of A because the corresponding $(n-1)\times(n-1)$ matrices have two rows interchanged.

Finally, $|B| = -|A|$ and the proof is complete.

Proof of (b)

- Let A be an $n \times n$ matrix, such that B is obtained from A by multiplying the ith row of A by a nonzero k.

- To find $|A|$ and $|B|$, expand in the ith row.

- Then, since their cofactors C_{ij} are the same

$$|A| = \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$

$$|B| = \sum_{j=1}^{n} k \cdot a_{ij} \cdot C_{ij} = k|A|$$

Proof of (c)

- Let A be an $n \times n$ matrix, such that B is obtained from A by adding c times the jth row of A to the ith row of A.

- The determinant of B can be found by expanding in the ith row.

- That is,

$$|B| = \sum_{k=1}^{n} \left(c a_{jk} + a_{ik} \right) \cdot C_{ik} = \cancelto{0}{\sum_{k=1}^{n} c a_{jk} C_{ik}} + \sum_{k=1}^{n} a_{ik} C_{ik} = |A|$$

Theorem

- If two rows of a square matrix are the same, then its determinant is zero.

- Proof:

- Let $x = det(A)$, where A is a square matrix with two rows being identical.
- After swapping the two identical rows, the determinant of A changes from x to $-x$.
- But the matrix A remains the same.
- $\forall x \in R$, if $x = -x$, then $x = 0$.

- Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \qquad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \quad \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \quad \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \quad \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k\det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k}\det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$

Note:

A row-echelon form of a square matrix is always upper triangular.

- Ex 2: (Evaluation a determinant using elementary row operations)

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_{12}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_{12}^{(-2)}}{=} -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \overset{r_2^{(-\frac{1}{7})}}{=} (-1)\left(\frac{1}{\frac{-1}{7}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\overset{r_3^{(-\frac{1}{3})}}{=} (7)\left(\frac{1}{\frac{-1}{3}}\right)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = (7)(-3)(1)(1)(1) = -21$$

- Notes:

$$|EA| = |E||A|$$

$$(1) \quad E = R_{ij} \quad \Rightarrow |E| = |R_{ij}| = -1$$

$$\Rightarrow |EA| = |r_{ij}(A)| = -|A| = |R_{ij}||A| = |E||A|$$

$$(2) \quad E = R_i^{(k)} \quad \Rightarrow |E| = |R_i^{(k)}| = k$$

$$\Rightarrow |EA| = |r_i^{(k)}(A)| = k|A| = |R_i^{(k)}||A| = |E||A|$$

$$(3) \quad E = R_{ij}^{(k)} \quad \Rightarrow |E| = |R_{ij}^{(k)}| = 1$$

$$\Rightarrow |EA| = |r_{ij}^{(k)}(A)| = 1|A| = |R_{ij}^{(k)}||A| = |E||A|$$

- Determinants and elementary column operations

- Thm: (Elementary column operations and determinants)

  Let A and B be square matrices.

$$(a)\quad B = c_{ij}(A) \quad \Rightarrow \quad \det(B) = -\det(A) \qquad (\text{i.e.} \left|c_{ij}(A)\right| = -|A|)$$

$$(b)\quad B = c_{i}^{(k)}(A) \quad \Rightarrow \quad \det(B) = k\det(A) \qquad (\text{i.e.} \left|c_{i}^{(k)}(A)\right| = k|A|)$$

$$(c)\quad B = c_{ij}^{(k)}(A) \quad \Rightarrow \quad \det(B) = \det(A) \qquad (\text{i.e.} \left|c_{ij}^{(k)}(A)\right| = |A|)$$

- Ex:

$$A=\begin{bmatrix}2 & 1 & -3\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad \det(A)=-8$$

$$A_1=\begin{bmatrix}1 & 1 & -3\\ 2 & 0 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad A_2=\begin{bmatrix}1 & 2 & -3\\ 0 & 4 & 1\\ 0 & 0 & 2\end{bmatrix} \qquad A_3=\begin{bmatrix}2 & 1 & 0\\ 4 & 0 & 1\\ 0 & 0 & 2\end{bmatrix}$$

$$A_1=c_1^{(\frac{1}{2})}(A) \quad \Rightarrow \det(A_1)=\det(c_1^{(4)}(A))=\frac{1}{2}\det(A)=(\frac{1}{2})(-8)=-4$$

$$A_2=c_{12}(A) \quad \Rightarrow \det(A_2)=\det(c_{12}(A))=-\det(A)=-(-8)=8$$

$$A_3=c_{23}^{(3)}(A) \quad \Rightarrow \det(A_3)=\det(c_{23}^{(3)}(A))=\det(A)=-8$$

- Thm 3.4: (Conditions that yield a zero determinant)

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$.

(a) An entire row (or an entire column) consists of zeros.

(b) Two rows (or two columns) are equal.

(c) One row (or column) is a multiple of another row (or column).

- Ex:

$$\begin{vmatrix} 1 & 2 & 3 \\ \boxed{0 \quad 0 \quad 0} \\ 4 & 5 & 6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Note:

Order n	Cofactor Expansion		Row Reduction	
	Additions	Multiplications	Additions	Multiplications
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

- Ex 5: (Evaluating a determinant)

$$A=\begin{bmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{C_{13}^{(2)}}{=}\begin{vmatrix}-3 & 5 & -4\\ 2 & -4 & 3\\ \boxed{-3 \quad 0 \quad 0}\end{vmatrix}$$

$$=(-3)(-1)^{3+1}\begin{vmatrix}5 & -4\\ -4 & 3\end{vmatrix}=(-3)(-1)=3$$

$$\det(A)=\begin{vmatrix}-3 & 5 & 2\\ 2 & -4 & -1\\ -3 & 0 & 6\end{vmatrix}\overset{r_{12}^{(\frac{4}{5})}}{=}\begin{vmatrix}-3 & 5 & 2\\ \frac{-2}{5} & 0 & \frac{3}{5}\\ -3 & 0 & 6\end{vmatrix}$$

$$=(5)(-1)^{1+2}\begin{vmatrix}\frac{-2}{5} & \frac{3}{5}\\ -3 & 6\end{vmatrix}=(-5)(-\frac{3}{5})=3$$

- Ex 6: (Evaluating a determinant)

$$A=\begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

Sol:

$$\det(A)=\begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \underset{\substack{r_{24}^{(1)} \\ r_{25}^{(-1)}}}{=} \begin{vmatrix} 2 & \boxed{0} & 1 & 3 & -2 \\ -2 & \boxed{1} & 3 & 2 & -1 \\ 1 & \boxed{0} & -1 & 2 & 3 \\ 1 & \boxed{0} & 5 & 6 & -4 \\ 3 & \boxed{0} & 0 & 0 & 1 \end{vmatrix}$$

$$=(1)(-1)^{2+2}\begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$\overset{C_{41}^{(-3)}}{=}\begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ \boxed{0 \quad 0 \quad 0 \quad 1} \end{vmatrix} = (1)(-1)^{4+4}\begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} \underset{r_{21}^{(1)}}{=} \begin{vmatrix} \boxed{0 \quad 0 \quad 5} \\ -8 \quad -1 \quad 2 \\ 13 \quad 5 \quad 6 \end{vmatrix}$$

$$= 5(-1)^{1+3}\begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

Key Learning in Section 3.2

- Determine whether two matrices are equal.
- Add and subtract matrices and multiply a matrix by a scalar.
- Multiply two matrices.
- Use matrices to solve a system of linear equations.
- Partition a matrix and write a linear combination of column vectors.

Keywords in Section 3.2

- determinant : 行列式
- elementary row operation: 基本列運算
- row equivalent: 列等價
- elementary matrix: 基本矩陣
- elementary column operation: 基本行運算
- column equivalent: 行等價

Review exercises

11. Find the determinant of the matrix.

$$\begin{bmatrix} 2 & 0 & -1 & 4 \\ -1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 \\ -2 & 0 & 3 & 1 \end{bmatrix}$$

18. Find the determinant of the matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$