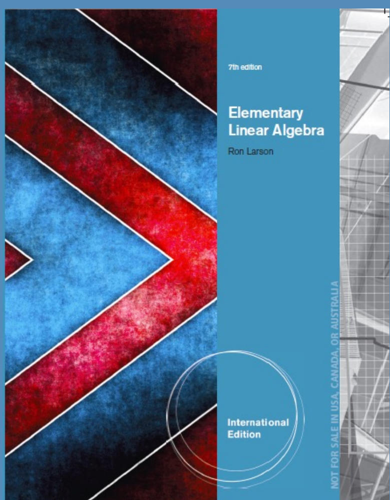
# CHAPTER 1
# SYSTEMS OF LINEAR EQUATIONS



# Course description

- Text book
  - Elementary Linear Algebra, $7^{th}$ ed.
    - Ron Larson
    - 高立圖書公司
    - 02 2290 0318 ext. 222
- Scoring
  - Midterm 40%
  - Final 40%
  - Homework 10%
  - Presentation 10%

# Syllabus

1. Systems of Linear Equations (1 week)
2. Matrices (2 weeks)
3. Determinants (2 weeks)
4. Vector Spaces (4 weeks) ← midterm
5. Inner Product Spaces (2 weeks)
6. Linear Transformations (2 weeks)
7. Eigenvalues and Eigenvectors (2 weeks) ← final

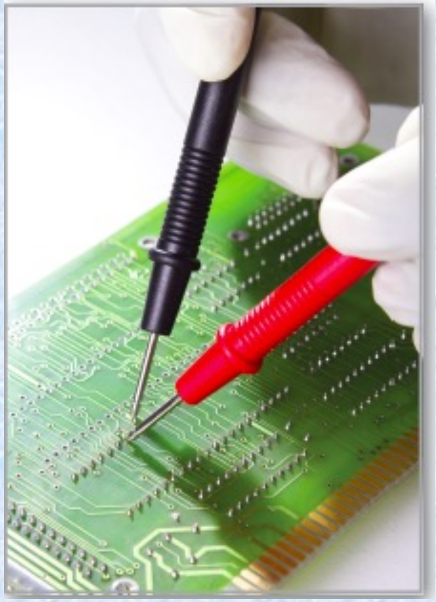
# CH 1 Linear Algebra Applied

Balancing Chemical Equations (p.4)

Airspeed of a Plane (p.11)

Global Positioning System (p.16)

Traffic Flow (p.28)



Electrical Network Analysis (p.30)

# 1.1 Introduction to Systems of Linear Equations

- a linear equation in $n$ variables:

$$a_1x_1+a_2x_2+a_3x_3+\cdots+a_nx_n=b$$

$a_1,a_2,a_3,\ldots,a_n$, $b$: real number

$a_1$: leading coefficient

$x_1$: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

trigonometric functions

exponential

Nonlinear $(e)\ xy + z = 2$

not the first power

$(f)\ e^x - 2y = 4$ Nonlinear

Nonlinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$

trigonometric functions

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ Nonlinear

not the first power

- a solution of a linear equation in $n$ variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

$$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \cdots,\ x_n = s_n$$

such that $a_1s_1 + a_2s_2 + a_3s_3 + \cdots + a_ns_n = b$

- Solution set:

  the set of all solutions of a linear equation

- Ex 2： (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4$$

a solution: (2, 1), i.e. $x_1 = 2, x_2 = 1$

If you solve for $x_1$ in terms of $x_2$, you obtain

$$x_1 = 4 - 2x_2,$$

By letting $x_2 = t$ you can represent the solution set as

$$x_1 = 4 - 2t$$

And the solutions are $\{(4-2t, t) \mid t \in R\}$ or $\{(s, 2-\frac{1}{2}s) \mid s \in R\}$

- a system of m linear equations in n variables:

$$
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

- Consistent:

having at least one common solution,
as of two or more equations or inequalities

A system of linear equations has <u>at least one solution</u>.

- Inconsistent:

A system of linear equations has <u>no solution</u>.

- Notes:

Every system of linear equations has either

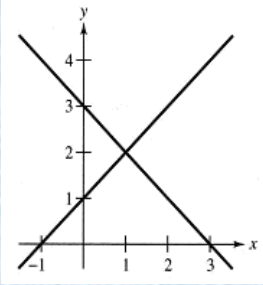
(1) exactly one solution,

(2) infinitely many solutions, or

(3) no solution.

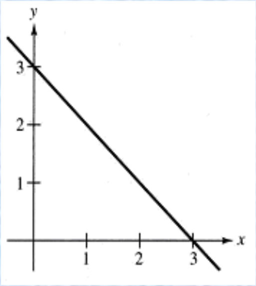
- Ex 4: (Solution of a system of linear equations)

(1)
$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$
two intersecting lines



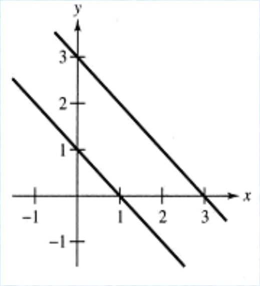

exactly one solution

(2)
$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$
two coincident lines




inifinite number

(3)
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$
two parallel lines




no solution

- Ex 5: (Using back substitution to solve a system in row echelon form)

梯形

$$\begin{aligned} x - 2y &= 5 \quad &(1) \\ y &= -2 \quad &(2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -2$

- Ex 6: (Using back substitution to solve a system in row echelon form)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ & & y & + & 3z & = & 5 & \quad (2) \\ & & & & z & = & 2 & \quad (3) \end{array}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{aligned} y + 3(2) &= 5 \\ y &= -1 \end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x &= 1 \end{aligned}$$

The system has exactly one solution:

$x = 1, y = -1, z = 2$

- Equivalent:

  Two systems of linear equations are called **equivalent** if they have precisely the same solution set.

- Notes:

  Each of the following operations on a system of linear equations produces an equivalent system.

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

- Ex 7: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \quad (1) \\ -x & + & 3y & & & = & -4 & \quad (2) \\ 2x & - & 5y & + & 5z & = & 17 & \quad (3) \end{array}$$

Sol: $(1)+(2)\rightarrow(2)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \quad (4) \\ 2x & - & 5y & + & 5z & = & 17 & \end{array}$$

$(1)\times(-2)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x & - & 2y & + & 3z & = & 9 & \\ & & y & + & 3z & = & 5 & \\ & & -y & - & z & = & -1 & \quad (5) \end{array}$$

$$(4)+(5)\rightarrow(5)$$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array} \qquad (6)$$

$$(6)\times\tfrac{1}{2}\rightarrow(6)$$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is $x=1,\ y=-1,\ z=2$ (only one solution)

- Ex 8: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & \quad (1) \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 & \quad (2) \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 & \quad (3) \end{array}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcrl} x_1 & - & 3x_2 & + & x_3 & = & 1 & \\ & & 5x_2 & - & 4x_3 & = & 0 & \quad (4) \\ & & 5x_2 & - & 4x_3 & = & -2 & \quad (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ \boxed{0 = -2} & \quad \text{(a false statement)} \end{aligned}$$

So the system has no solution (an inconsistent system).

- Ex 9: Solve a system of linear equations (infinitely many solutions)

$$\begin{array}{rcrcrcrl} & & x_2 & - & x_3 & = & 0 & \quad (1) \\ x_1 & & & - & 3x_3 & = & -1 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \quad (1) \\ & & x_2 & - & x_3 & = & 0 & \quad (2) \\ -x_1 & + & 3x_2 & & & = & 1 & \quad (3) \end{array}$$

$(1) + (3) \rightarrow (3)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \\ & & x_2 & - & x_3 & = & 0 & \\ & & 3x_2 & - & 3x_3 & = & 0 & \quad (4) \end{array}$$

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

let $x_3 = t$

then $x_1 = 3t - 1,$

$x_2 = t, \qquad t \in R$

$x_3 = t,$

So this system has infinitely many solutions.

# Key Learning in Section 1.1

- Recognize a linear equation in *n* variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

Rewriting a system of linear equations in row-echelon form usually involves a chain of equivalent systems, each of which is obtained by using one of the three basic operations.

# Keywords in Section 1.1

- linear equation: 線性方程式
- system of linear equations: 線性方程式系統
- leading coefficient: 領先係數
- leading variable: 領先變數
- solution: 解
- solution set: 解集合
- parametric representation: 參數化表示
- consistent: 一致性(有解)
- inconsistent: 非一致性(無解、矛盾)
- equivalent: 等價

# 1.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & m \text{ rows} \\ n \text{ columns} & \end{array}$$

- Notes:

(1) Every entry $a_{ij}$ in a matrix is a number.

(2) A matrix with $\underline{m \text{ rows}}$ and $\underline{n \text{ columns}}$ is said to be of size $m \times n$ .

(3) If $m = n$ , then the matrix is called square of order $n$.

(4) For a square matrix, the entries $a_{11}$, $a_{22}$, …, $a_{nn}$ are called the main diagonal entries.

the inclusion or insertion of an item, as in a record

- Ex 1: Matrix Size

| Matrix | Size |
|---|---|
| $[2]$ | $1\times 1$ |
| $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ | $2\times 2$ |
| $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ | $1\times 4$ |
| $\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$ | $3\times 2$ |

- Note:

a plural of matrix

One very common use of matrices is to represent a system of linear equations.

- a system of $m$ equations in $n$ variables:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \dots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \dots & + & a_{3n}x_n & = & b_3 \\
 & & & & \vdots & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \dots & + & a_{mn}x_n & = & b_m
\end{array}$$

Matrix form: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ & \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right] = [A \mid b]$$

- Coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:

(1) Interchange two rows. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant. $r_i^{(k)} : (k)R_i \rightarrow R_i$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- Row equivalent:

Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex 2: (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex 3: Using elementary row operations to solve a system

| Linear System | Associated Augemented Matrix | Elementary Row Operation |
|---|---|---|
| $\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \quad\quad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ | |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$ | $r_{12}^{(1)} : (1)R_1 + R_2 \rightarrow R_2$ |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$ | $r_{13}^{(-2)} : (-2)R_1 + R_3 \rightarrow R_3$ |

| Linear System | Associated Augemented Matrix | Elementary Row Operation |
|---|---|---|
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ | $r_{23}^{(1)} : (1)R_2 + R_3 \rightarrow R_3$ |
| $\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$ | $\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ | $r_3^{(\frac{1}{2})} : (\frac{1}{2})R_3 \rightarrow R_3$ |

$$\longrightarrow \quad \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called **a leading 1**).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Ex 4: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \text{(row-echelon form)}$$

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(reduced row-echelon form)}$$

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{(row-echelon form)}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{(reduced row-echelon form)}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

- Ex: (Procedure of Gaussian elimination and Gauss-Jordan elimination)

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -6 & 4 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} 2 & 8 & -6 & 4 & 12 & 28 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix}$$

The first nonzero column

Produce leading 1

$$\xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & -2 & 0 & 8 & 12 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix}$$

leading 1

Zeros elements below leading 1

Produce leading 1

The first nonzero column

Submatrix

leading 1

$$\xrightarrow{r_2^{(-\frac{1}{2})}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 5 & 0 & -17 & -24 \end{bmatrix} \xrightarrow{r_{23}^{(-5)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 6 & 14 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & \boxed{3} & \boxed{6} \end{bmatrix}$$

Zeros elements below leading 1

Submatrix

Produce leading 1

Zeros elsewhere

$$\xrightarrow{r_3^{(\frac{1}{3})}} \begin{bmatrix} 1 & 4 & \boxed{-3} & 2 & \boxed{6} & 14 \\ 0 & 0 & 1 & 0 & \boxed{-4} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-6)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

leading 1

(row - echelon form) (row - echelon form)

$$\xrightarrow{r_{32}^{(4)}} \begin{bmatrix} 1 & 4 & -3 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(3)}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 & 8 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(row - echelon form) (reduced row - echelon form)

- Ex 7: Solve a system by Gauss-Jordan elimination method (only one solution)

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\, r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \underset{\text{(row-echelon form)}}{\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \xrightarrow{r_{21}^{(2)},\, r_{32}^{(-3)},\, r_{31}^{(-9)}} \underset{\text{(reduced row-echelon form)}}{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \longrightarrow \begin{array}{rcr} x & = & 1 \\ y & = & -1 \\ z & = & 2 \end{array}$$

- Ex 8：Solve a system by Gauss-Jordan elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + \ 4x_1 - \ 2x_3 &= \ 0 \\ 3x_1 + \ 5x_2 \qquad &= \ 1 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\, r_{12}^{(-3)},\, r_2^{(-1)},\, r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \text{(reduced row - echelon form)}$$

the corresponding system of equations is

$$\begin{aligned} x_1 \qquad + \quad 5x_3 &= \ \ 2 \\ x_2 \quad - 3x_3 &= -1 \end{aligned}$$

leading variable î $x_1, x_2$

free variable î $x_3$

$$\begin{aligned} x_1 &= \phantom{-}2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

Let $x_3 = t$

$$\begin{aligned} x_1 &= 2 - 5t, \\ x_2 &= -1 + 3t, \qquad t \in R \\ x_3 &= t, \end{aligned}$$

So this system has infinitely many solutions.

- Homogeneous systems of linear equations:

A system of linear equations is said to be homogeneous if all the constant terms are zero.

$$
\begin{array}{ccccccccc}
a_{11}x_1 + & a_{12}x_2 + & a_{13}x_3 + & \cdots + & a_{1n}x_n & = & \boxed{0} \\
a_{21}x_1 + & a_{22}x_2 + & a_{23}x_3 + & \cdots + & a_{2n}x_n & = & \boxed{0} \\
a_{31}x_1 + & a_{32}x_2 + & a_{33}x_3 + & \cdots + & a_{3n}x_n & = & \boxed{0} \\
 & \vdots & & & & & \\
a_{m1}x_1 + & a_{m2}x_2 + & a_{m3}x_3 + & \cdots + & a_{mn}x_n & = & \boxed{0}
\end{array}
$$

- Trivial solution:

$$x_1 = x_2 = x_3 = \cdots = x_n = 0$$

- Nontrivial solution:

  other solutions

- Notes:

(1) Every homogeneous system of linear equations is consistent.

(2) If the homogenous system has fewer equations than variables, then it must have an infinite number of solutions.

(3) For a homogeneous system, exactly one of the following is true.

($a$) The system has only the trivial solution.

($b$) The system has infinitely many nontrivial solutions in addition to the trivial solution.

- Ex 9: Solve the following homogeneous system

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\, r_2^{(\frac{1}{3})},\, r_{21}^{(1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$ (reduced row - echelon form)

leading variableî $x_1, x_2$

free variableî $x_3$

Let $x_3 = t$

$x_1 = -2t, x_2 = t, x_3 = t, t \in R$

When $t = 0, x_1 = x_2 = x_3 = 0$ (trivial solution)

# Key Learning in Section 1.2

- Determine the size of a matrix .
- Write an augmented or coefficient matrix from a system of linear equations.
- Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations.
- Use matrices and Gauss-Jordan elimination to solve a system of linear equations.
- Solve a homogeneous system of linear equations.

# Key Learning in Section 1.3

- Set up and solve a system of equations to fit a polynomial function to a set of data points.
- Set up and solve a system of equations to represent a network.

# Keywords in Section 1.2

- matrix: 矩陣
- row: 列
- column: 行
- entry: 元素
- size: 大小
- square matrix: 方陣
- order: 階
- main diagonal: 主對角線
- augmented matrix: 增廣矩陣
- coefficient matrix: 係數矩陣

# Keywords in Section 1.2

- elementary row operation: 基本列運算
- row equivalent: 列等價
- row-echelon form: 列梯形形式
- reduced row-echelon form: 列簡梯形形式
- leading 1: 領先1
- Gaussian elimination: 高斯消去法
- Gauss-Jordan elimination: 高斯-喬登消去法
- free variable: 自由變數
- leading variable: 領先變數
- homogeneous system: 齊次系統
- trivial solution: 顯然解
- nontrivial solution: 非顯然解

- ## Balancing Chemical Equations

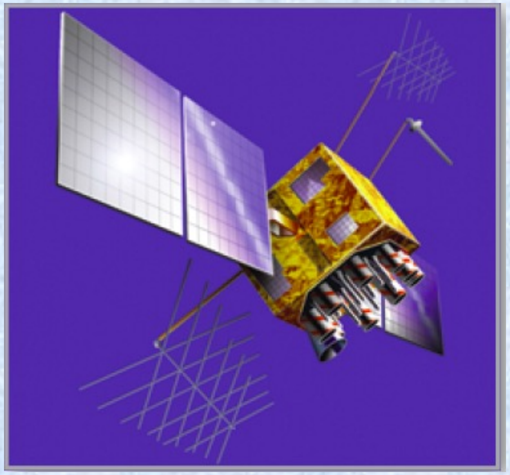
In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas($CH_4$) combines with oxygen($O_2$) and burns, carbon dioxide ($CO_2$) and water ($H_2O$) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O$$

Because a chemical reaction can neither create nor destroy atoms, all of the atoms represented on the left side of the arrow must be accounted for on the right side of the arrow. This is called balancing the chemical equation. In the given example, chemists can use a system of linear equations to find values of $x_1$, $x_2$, $x_3$, and $x_4$ that will balance the chemical equation.

# 1.2 Linear Algebra Applied

- Global Positioning System



The Global Positioning System (GPS) is a network of 24 satellites originally developed by the U.S. military as a navigational tool. Today, GPS receivers are used in a wide variety of civilian applications, such as determining directions, locating vessels lost at sea, and monitoring earthquakes. A GPS receiver works by using satellite readings to calculate its location. In three dimensions, the receiver uses signals from at least four satellites to "trilaterate" its position. In a simplified mathematical model, a system of three linear equations in four unknowns (three dimensions and time) is used to determine the coordinates of the receiver as functions of time.

- Traffic Flow

Researchers in Italy studying the acoustical noise levels from vehicular traffic at a busy three-way intersection on a college campus used a system of linear equations to model the traffic flow at the intersection. To help formulate the system of equations, “operators” stationed themselves at various locations along the intersection and counted the numbers of vehicles going by.

# Review Exercises

- 36. Solve the system using either Gaussian elimination with back-substitution or Gaussian-Jordan elimination

$$
\begin{aligned}
2x_1 + 5x_2 - 19x_3 &= 34 \\
3x_1 + 8x_2 - 31x_3 &= 54
\end{aligned}
$$

# Review Exercises

37. Solve the system using either Gaussian elimination with back-substitution or Gaussian-Jordan elimination

$$
\begin{aligned}
2x_1 + x_2 + x_3 + 2x_4 &= -1 \\
5x_1 - 2x_2 + x_3 - 3x_4 &= 0 \\
-x_1 + 3x_2 + 2x_3 + 2x_4 &= 1 \\
3x_1 + 2x_2 + 3x_3 - 5x_4 &= 12
\end{aligned}
$$