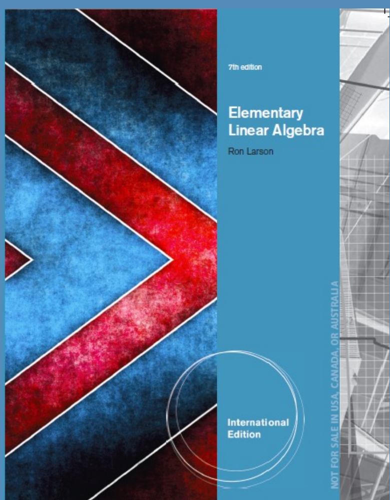
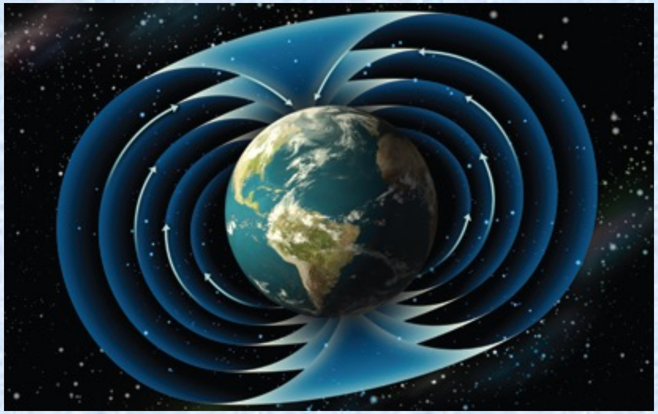
CHAPTER 5
INNER PRODUCT SPACES



CH 5 Inner Product Space


Electric/Magnetic Flux (p.234)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5.1 Length and Dot Product in R^n

- Length:

  The length of a vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ in R^n is given by

$$\|\mathbf{v}\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$$

- Notes: The length of a vector is also called its **norm**.

- Notes: Properties of length

  $(1)\quad \|\mathbf{v}\| \geq 0$

  $(2)\quad \|\mathbf{v}\| = 1 \Rightarrow \mathbf{v}$ is called a **unit vector**.

  $(3)\quad \|\mathbf{v}\| = 0 \text{ iff } \mathbf{v} = 0$

  $(4)\quad \|c\mathbf{v}\| = |c|\|\mathbf{v}\|$

- Ex 1:

(a) In R^5, the length of $\mathbf{v}=(0\,,-2\,,1\,,4\,,-2)$ is given by

$$\|\mathbf{v}\|=\sqrt{0^2+(-2)^2+1^2+4^2+(-2)^2}=\sqrt{25}=5$$

(b) In R^3 the length of $\mathbf{v}=(\frac{2}{\sqrt{17}}\,,\frac{-2}{\sqrt{17}}\,,\frac{3}{\sqrt{17}})$ is given by

$$\|\mathbf{v}\|=\sqrt{\left(\frac{2}{\sqrt{17}}\right)^2+\left(\frac{-2}{\sqrt{17}}\right)^2+\left(\frac{3}{\sqrt{17}}\right)^2}=\sqrt{\frac{17}{17}}=1$$

($\mathbf{v}$ is a unit vector)

- A standard unit vector in R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} = \{(1,0,\cdots,0), (0,1,\cdots,0), (0,0,\cdots,1)\}$$

- Ex:

the standard unit vector in R^2: $\{i, j\} = \{(1,0), (0,1)\}$

the standard unit vector in R^3: $\{i, j, k\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

- Notes: (Two nonzero vectors are parallel)

$$\mathbf{u} = c\mathbf{v}$$

(1) $c > 0 \Rightarrow$ **u** and **v** have the same direction

(2) $c < 0 \Rightarrow$ **u** and **v** have the opposite direction

- Thm 5.1: (Length of a scalar multiple)

Let $\mathbf{v}$ be a vector in R^n and c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

Pf:

$$\mathbf{v} = (v_1, v_2, \cdots, v_n)$$
$$\Rightarrow c\mathbf{v} = (cv_1, cv_2, \cdots, cv_n)$$
$$\|c\mathbf{v}\| = \|(cv_1, cv_2, \cdots, cv_n)\|$$
$$= \sqrt{(cv_1)^2 + (cv_2)^2 + \cdots + (cv_n)^2}$$
$$= \sqrt{c^2(v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2)}$$
$$= |c|\sqrt{v_1{}^2 + v_2{}^2 + \cdots + v_n{}^2}$$
$$= |c|\,\|\mathbf{v}\|$$

- Thm 5.2: (Unit vector in the direction of **v**)

  If **v** is a nonzero vector in R^n, then the vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

  has length 1 and has the same direction as **v**. This vector **u** is called the **unit vector in the direction of v**.

Pf:

$$\mathbf{v} \text{ is nonzero} \Rightarrow \|\mathbf{v}\| \neq 0 \Rightarrow \frac{1}{\|\mathbf{v}\|} > 0$$

$$\Rightarrow \mathbf{u} = \frac{\mathbf{1}}{\|\mathbf{v}\|}\mathbf{v} \qquad (\mathbf{u} \text{ has the same direction as } \mathbf{v})$$

$$\|\mathbf{u}\| = \left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1 \qquad (\text{u has length 1 })$$

- Notes:

(1) The vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is called the unit vector in the direction of $\mathbf{v}$.

(2) The process of finding the unit vector in the direction of $\mathbf{v}$ is called **normalizing** the vector $\mathbf{v}$.

- Ex 2: (Finding a unit vector)

Find the unit vector in the direction of $\mathbf{v}=(3\,,-1\,,2)$, and verify that this vector has length 1.

Sol:

$$\mathbf{v}=(3\,,-1\,,2) \quad \Rightarrow \|\mathbf{v}\|=\sqrt{3^2+(-1)^2+2^2}=\sqrt{14}$$

$$\Rightarrow \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{(3\,,-1\,,2)}{\sqrt{3^2+(-1)^2+2^2}}=\frac{1}{\sqrt{14}}(3\,,-1\,,2)=\left(\frac{3}{\sqrt{14}}\,,\frac{-1}{\sqrt{14}}\,,\frac{2}{\sqrt{14}}\right)$$

$$\because \quad \sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2}=\sqrt{\frac{14}{14}}=1$$

$\therefore \quad \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector.

- Distance between two vectors:

The **distance** between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

- Notes: (Properties of distance)

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Ex 3: (Finding the distance between two vectors)

The distance between **u**=(0, 2, 2) and **v**=(2, 0, 1) is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(0-2, 2-0, 2-1)\|$$

$$= \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- Dot product in R^n:

  The **dot product** of $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ is the scalar quantity

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex 4: (Finding the dot product of two vectors)

  The dot product of **u**=(1, 2, 0, -3) and **v**=(3, -2, 4, 2) is

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Thm 5.3: (Properties of the dot product)

If **u**, **v**, and **w** are vectors in R^n and c is a scalar, then the following properties are true.

(1) $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

(2) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

(3) $c(\mathbf{u}\cdot\mathbf{v})=(c\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(c\mathbf{v})$

(4) $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

(5) $\mathbf{v}\cdot\mathbf{v}\geq 0$ and $\mathbf{v}\cdot\mathbf{v}=0$ if and only if $\boldsymbol{v}=0$

- Euclidean n-space:

  R^n was defined to be the *set* of all order n-tuples of real numbers. When R^n is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called **Euclidean *n*-space**.

- Ex 5: (Finding dot products)

$$\mathbf{u}=(2\,,-2)\,,\ \mathbf{v}=(5\,,8),\mathbf{w}=(-4\,,3)$$

(a) $\mathbf{u}\cdot\mathbf{v}$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (c) $\mathbf{u}\cdot(2\mathbf{v})$ (d) $\|\mathbf{w}\|^2$ (e) $\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})$

Sol:

(a) $\mathbf{u}\cdot\mathbf{v}=(2)(5)+(-2)(8)=-6$

(b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}=-6\mathbf{w}=-6(-4\,,3)=(24\,,-18)$

(c) $\mathbf{u}\cdot(2\mathbf{v})=2(\mathbf{u}\cdot\mathbf{v})=2(-6)=-12$

(d) $\|\mathbf{w}\|^2=\mathbf{w}\cdot\mathbf{w}=(-4)(-4)+(3)(3)=25$

(e) $\mathbf{v}-2\mathbf{w}=(5-(-8)\,,8-6)=(13\,,2)$

$\mathbf{u}\cdot(\mathbf{v}-2\mathbf{w})=(2)(13)+(-2)(2)=26-4=22$

- Ex 6: (Using the properties of the dot product)

Given $\mathbf{u}\cdot\mathbf{u}=39 \quad \mathbf{u}\cdot\mathbf{v}=-3 \qquad \mathbf{v}\cdot\mathbf{v}=79$

Find $(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v})$

Sol:

$$
\begin{aligned}
(\mathbf{u}+2\mathbf{v})\cdot(3\mathbf{u}+\mathbf{v}) &= \mathbf{u}\cdot(3\mathbf{u}+\mathbf{v})+2\mathbf{v}\cdot(3\mathbf{u}+\mathbf{v}) \\
&= \mathbf{u}\cdot(3\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+(2\mathbf{v})\cdot(3\mathbf{u})+(2\mathbf{v})\cdot\mathbf{v} \\
&= 3(\mathbf{u}\cdot\mathbf{u})+\mathbf{u}\cdot\mathbf{v}+6(\mathbf{v}\cdot\mathbf{u})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(\mathbf{u}\cdot\mathbf{u})+7(\mathbf{u}\cdot\mathbf{v})+2(\mathbf{v}\cdot\mathbf{v}) \\
&= 3(39)+7(-3)+2(79)=254
\end{aligned}
$$

- Thm 5.4: (The Cauchy - Schwarz inequality)

  If **u** and **v** are vectors in R^n, then

  $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$ ($|\mathbf{u}\cdot\mathbf{v}|$ denotes the absolute value of $\mathbf{u}\cdot\mathbf{v}$)

- Ex 7: (An example of the Cauchy - Schwarz inequality)

  Verify the Cauchy - Schwarz inequality for **u**=(1, -1, 3) and **v**=(2, 0, -1)

Sol: $\mathbf{u}\cdot\mathbf{v} = -1,\quad \mathbf{u}\cdot\mathbf{u} = 11,\quad \mathbf{v}\cdot\mathbf{v} = 5$

$$\Rightarrow |\mathbf{u}\cdot\mathbf{v}| = |-1| = 1$$

$$\|\mathbf{u}\|\|\mathbf{v}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}}\cdot\sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{11}\cdot\sqrt{5} = \sqrt{55}$$

$$\therefore |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$$

Proof

- Case 1

- If u=0, then $|u \cdot v| = |0 \cdot v| = 0$ and $\|u\|\|v\| = 0\|v\|$=0.
- So, the theorem is true when u=0.

- Case 2

- When $u \neq 0$, let t be any real number and consider $tu + v$.

- Because $(tu + v) \cdot (tu + v) \geq 0$, it follows that

$$(tu + v) \cdot (tu + v) = t^2(u \cdot u) + 2t(u \cdot v) + v \cdot v \geq 0$$

- Now, let $a = u \cdot u$, $b = 2(u \cdot v)$, and $c = v \cdot v$ to obtain the quadratic inequality $at^2 + bt + c \geq 0$.

- Because this quadratic is never negative, it has either no real roots or a single repeated real root, which implies

$$b^2 - 4ac \leq 0$$

$$b^2 \leq 4ac$$

$$4(u \cdot v)^2 \leq 4(u \cdot u)(v \cdot v)$$

$$(u \cdot v)^2 \leq (u \cdot u)(v \cdot v)$$

- Taking the square roots of both sides produces

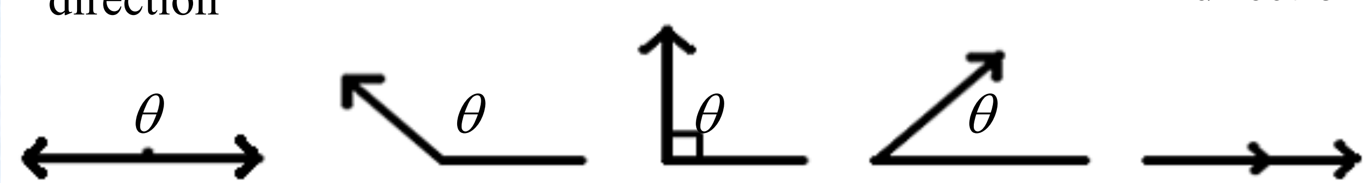
$$|u \cdot v| \leq \sqrt{u \cdot u}\sqrt{v \cdot v} = \|u\|\|v\|$$

Definition

- The angle between two vectors in R^n:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

Opposite direction	$\mathbf{u}\cdot\mathbf{v}<\mathbf{0}$	$\mathbf{u}\cdot\mathbf{v}=\mathbf{0}$	$\mathbf{u}\cdot\mathbf{v}>\mathbf{0}$	Same direction



$\theta=\pi$	$\frac{\pi}{2}<\theta<\pi$	$\theta=\frac{\pi}{2}$	$0<\theta<\frac{\pi}{2}$	$\theta=0$
$\cos=-1$	$\cos<0$	$\cos=0$	$\cos>0$	$\cos=1$

- Note:

  The angle between the zero vector and another vector is not defined.

- Ex 8: (Finding the angle between two vectors)

$$\mathbf{u} = (-4, 0, 2, -2) \quad \mathbf{v} = (2, 0, -1, 1)$$

Sol:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{2^2 + (0)^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\mathbf{u}\cdot\mathbf{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = -\frac{12}{\sqrt{144}} = -1$$

$$\Rightarrow \theta = \pi \qquad \therefore\ \mathbf{u} \text{ and } \mathbf{v} \text{ have opposite directions.} \quad (\mathbf{u} = -2\mathbf{v})$$

- Orthogonal vectors:

  Two vectors **u** and **v** in R^n are orthogonal if

  $$\mathbf{u} \cdot \mathbf{v} = 0$$

- Note:

  The vector **0** is said to be orthogonal to every vector.

- Ex 10: (Finding orthogonal vectors)

Determine all vectors in R^n that are orthogonal to $\mathbf{u}$=(4, 2).

Sol:

$$\mathbf{u}=(4\,,2) \quad \text{Let} \quad \mathbf{v}=(v_1\,,v_2)$$

$$\Rightarrow \quad \mathbf{u}\cdot\mathbf{v}=(4\,,2)\cdot(v_1\,,v_2)$$
$$=4v_1+2v_2$$
$$=0$$

$$\begin{bmatrix}4 & 2 & 0\end{bmatrix}\rightarrow\begin{bmatrix}1 & \frac{1}{2} & 0\end{bmatrix}$$

$$\Rightarrow \quad v_1=\frac{-t}{2} \quad , \quad v_2=t$$

$$\therefore \quad \mathbf{v}=\left(\frac{-t}{2},t\right), \qquad t\in R$$

- Thm 5.5: (The triangle inequality)

  If **u** and **v** are vectors in R^n, then $\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$

Pf:

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$$

$$= \mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) + \mathbf{v}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2 \quad \le \|\mathbf{u}\|^2 + 2\,|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|^2$$

$$\le \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2$$

$$= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

$$\therefore \|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

- Note:

  Equality occurs in the triangle inequality if and only if the vectors **u** and **v** have the same direction.

- Thm 5.6: (The Pythagorean theorem)

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

- Dot product and matrix multiplication:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

(A vector $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ in R^n is represented as an $n \times 1$ column matrix)

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = [u_1 \quad u_2 \quad \cdots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = [u_1 v_1 + u_2 v_2 + \cdots + u_n v_n]$$

Key Learning in Section 5.1

- Find the length of a vector and find a unit vector.
- Find the distance between two vectors.
- Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwarz Inequality, the triangle inequality, and the Pythagorean Theorem.
- Use a matrix product to represent a dot product.

Keywords in Section 5.1

- length: 長度
- norm: 範數
- unit vector: 單位向量
- standard unit vector : 標準單位向量
- normalizing: 單範化
- distance: 距離
- dot product: 點積
- Euclidean *n*-space: 歐基里德*n*維空間
- Cauchy-Schwarz inequality: 科西-舒瓦茲不等式
- angle: 夾角
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理

Review exercise

22. Determine all vectors that are orthogonal to u.

$$u = (1, -1, 2)$$

5.2 Inner Product Spaces

- Inner product:

Let **u**, **v**, and **w** be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number <**u**, **v**> with each pair of vectors **u** and **v** and satisfies the following axioms.

(1) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

(2) $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

(3) $c \langle \mathbf{u}, \mathbf{v} \rangle = \langle c\mathbf{u}, \mathbf{v} \rangle$

(4) $\langle \mathbf{v}, \mathbf{v} \rangle \ge 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

- Note:

$$\mathbf{u} \cdot \mathbf{v} = \text{dot product (Euclidean inner product for } R^n)$$

$$< \mathbf{u}\, , \mathbf{v} >= \text{general inner product for vector space } V$$

- Note:

A vector space V with an inner product is called an inner product space.

Vector space: $(V,\ +,\ \bullet)$

Inner product space: $(V,\ +,\ \bullet,\ <,>)$

- Ex 1: (The Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\mathbf{u} = (u_1, u_2, \cdots, u_n) \quad , \quad \mathbf{v} = (v_1, v_2, \cdots, v_n)$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

- Ex 2: (A different inner product for R^n)

  Show that the function defines an inner product on R^2, where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2$$

Sol:

$$(a) \quad \mathbf{u}, \mathbf{v} \ = u_1 v_1 + 2u_2 v_2 = v_1 u_1 + 2v_2 u_2 = \ \mathbf{v}, \mathbf{u}$$

$$(b) \quad \mathbf{w} = (w_1, w_2)$$

$$\begin{aligned}
\Rightarrow \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle &= u_1(v_1 + w_1) + 2u_2(v_2 + w_2) \\
&= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2 \\
&= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2) \\
&= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle
\end{aligned}$$

$$(c)\quad c\ \ \mathbf{u}, \mathbf{v}\ \ = c(u_1v_1 + 2u_2v_2) = (cu_1)v_1 + 2(cu_2)v_2 = \ \ c\mathbf{u}, \mathbf{v}$$

$$(d)\quad \mathbf{v}, \mathbf{v}\ \ = {v_1}^2 + 2{v_2}^2 \geq 0$$

$$\mathbf{v}, \mathbf{v}\ \ = 0 \Rightarrow {v_1}^2 + 2{v_2}^2 = 0 \qquad \Rightarrow \quad v_1 = v_2 = 0 \quad (\mathbf{v} = 0)$$

- Note: (An inner product on R^n)

$$\langle\, \mathbf{u}, \mathbf{v}\, \rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n\ , \qquad c_i > 0$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3.

$$\langle \mathbf{u} \cdot \mathbf{v} \rangle = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$

Then $< \mathbf{v}, \mathbf{v} >= (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied.

Thus this function is not an inner product on R^3.

- Thm 5.7: (Properties of inner products)

  Let **u, v**, and **w** be vectors in an inner product space V, and let c be any real number.

  (1) $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$

  (2) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

  (3) $\langle \mathbf{u}, c\mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$

- Norm (length) of **u**:

$$\|\mathbf{u}\| = \sqrt{\ \mathbf{u}, \mathbf{u}\ }$$

- Note:

$$\|\mathbf{u}\|^2 = \ \mathbf{u}, \mathbf{u}$$

Proof

1. $\langle 0, v \rangle = \langle 0(v), v \rangle = 0\langle v, v \rangle = 0$

2. $\langle u + v, w \rangle = \langle w, u + v \rangle = \langle w, u \rangle + \langle w, v \rangle = \langle u, w \rangle + \langle v, w \rangle$

3. $\langle u, cv \rangle = \langle cv, u \rangle = c\langle v, u \rangle = c\langle u, v \rangle$

- Distance between **u** and **v**:

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \sqrt{\langle \mathbf{u}-\mathbf{v}, \mathbf{u}-\mathbf{v}\rangle}$$

- Angle between two nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|},\ 0 \le \theta \le \pi$$

- Orthogonal: $(\mathbf{u} \perp \mathbf{v})$

u and **v** are **orthogonal** if $\langle \mathbf{u},\mathbf{v}\rangle = 0$.

- Notes:

(1) If $\|\mathbf{v}\|=1$, then v is called a unit vector.

(2) $\|\mathbf{v}\| \neq 1$
$\mathbf{v} \neq 0$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ (the unit vector in the direction of **v**)

not a unit vector

- Ex 6: (Finding inner product)

$$\langle \mathrm{p}, \mathrm{q} \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n \quad \text{is an inner product}$$

Let $p(x) = 1 - 2x^2,\ q(x) = 4 - 2x + x^2$ be polynomials in $P_2(x)$

$$(a)\ \ \langle p\,, q\,\rangle = ? \qquad (b)\ \ \| q \| = ? \qquad (c)\ \ d(p\,, q) = ?$$

Sol:

$$(a)\ \ \langle p\,, q\,\rangle = (1)(4) + (0)(-2) + (-2)(1) = 2$$

$$(b)\ \ \| q \| = \sqrt{\langle\, q\,, q\,\rangle} = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{21}$$

$$(c)\ \ \because\ p - q = -3 + 2x - 3x^2$$

$$\therefore d(p\,, q) = \| p - q \| = \sqrt{\langle p - q, p - q \rangle}$$

$$= \sqrt{(-3)^2 + 2^2 + (-3)^2} = \sqrt{22}$$

- Properties of norm:

(1) $\|\mathbf{u}\| \geq 0$

(2) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(3) $\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

- Properties of distance:

(1) $d(\mathbf{u}, \mathbf{v}) \geq 0$

(2) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

(3) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

- Thm 5.8：

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

(1) Cauchy-Schwarz inequality:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\| \qquad \text{Theorem 5.4}$$

(2) Triangle inequality:

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\| \qquad \text{Theorem 5.5}$$

(3) Pythagorean theorem :

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \qquad \text{Theorem 5.6}$$

- Orthogonal projections in inner product spaces:

  Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the **orthogonal projection of u onto v** is given by

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}\,,\,\mathbf{v}\rangle}{\langle \mathbf{v}\,,\,\mathbf{v}\rangle}\mathbf{v}$$

- Note:

  If **v** is a init vector, then $\mathbf{v}\,,\,\mathbf{v} \quad = \|\mathbf{v}\|^2 = 1$.

  The formula for the orthogonal projection of **u** onto **v** takes the following simpler form.

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \langle \mathbf{u}\,,\,\mathbf{v}\rangle\mathbf{v}$$

- Ex 10: (Finding an orthogonal projection in R^3)

  Use the Euclidean inner product in R^3 to find the orthogonal projection of u=(6, 2, 4) onto v=(1, 2, 0).

Sol:

$$\because \langle \mathbf{u}\,,\,\mathbf{v}\rangle = (6)(1)+(2)(2)+(4)(0)=10$$

$$\langle \mathbf{v}\,,\,\mathbf{v}\rangle = 1^2+2^2+0^2=5$$

$$\therefore \operatorname{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \tfrac{10}{5}(1\,,2\,,0)=(2\,,4\,,0)$$

- Note:

$\boldsymbol{u}-\operatorname{proj}_{v}\boldsymbol{u}=(6, 2, 4)-(2, 4, 0)=(4, -2, 4)$ is orthogonal to $v=(1,2,0)$.

- Thm 5.9: (Orthogonal projection and distance)

Let **u** and **v** be two vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

$$d(\mathbf{u}, \mathrm{proj}_{\mathbf{v}}\mathbf{u}) < d(\mathbf{u}, c\mathbf{v}), \qquad c \neq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Proof

- Let $b = \langle u, v\rangle / \langle v, v\rangle$. Then
$$\|u - cv\|^2 = \|(u - bv) + (b - c)v\|^2$$
- where $(u - bv)$ and $(b - c)v$ are orthogonal. You can verify this using the inner product axioms to show that $\langle (u - bv), (b - c)v\rangle = 0$.

- Now, by the Pythagorean Theorem,
$$\|(u - bv) + (b - c)v\|^2 = \|u - bv\|^2 + \|(b - c)v\|^2$$
- which implies that
$$\|u - cv\|^2 = \|u - bv\|^2 + (b - c)^2\|v\|^2$$

- Because $b \neq c$ and $v \neq 0$, we have $(b-c)^2\|v\|^2 > 0$.

- So, $\|u - cv\|^2 > \|u - bv\|^2$.

- It follows that $d(u, cv) > d(u, bv)$.

Key Learning in Section 5.2

- Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n and C[a, b].
- Find an orthogonal projection of a vector onto another vector in an inner product space.

Keywords in Section 5.2

- inner product: 內積
- inner product space: 內積空間
- norm: 範數
- distance: 距離
- angle: 夾角
- orthogonal: 正交
- unit vector: 單位向量
- normalizing: 單範化
- Cauchy – Schwarz inequality: 科西 - 舒瓦茲不等式
- triangle inequality: 三角不等式
- Pythagorean theorem: 畢氏定理
- orthogonal projection: 正交投影

Review exercise

- 46. Let $f(x) = x + 2$ and $g(x) = 15x - 8$ be functions in the vector space C[0,1] with inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$.
  a) Find $\langle f, g \rangle$.
  b) Find $\langle -4f, g \rangle$.
  c) Find $\|f\|$.
  d) Orthonomalize the set $B = \{f, g\}$.